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Philippe Bergault

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UNIVERSITÉ PARIS 1
PANTHÉON SORBONNE

Université Paris 1 Panthéon Sorbonne
École doctorale 465 Economie Panthéon Sorbonne (EPS)

Thèse de Doctorat

Spécialité: Mathématiques appliquées

Philippe BERGAULT

MATHEMATICAL MODELING FOR MARKET MAKING AND
RELATED PROBLEMS OF FINANCIAL LIQUIDITY: A SONG OF
ASSETS AND TRADERS.

Sous la direction d'Olivier Guéant

Soutenance à Paris le 25/05/2021, devant le jury composé de:

Bernard DE MEYER	Président du jury
Bruno BOUCHARD	Rapporteur
Álvaro CARTEA	Rapporteur
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Olivier GUÉANT	Directeur de thèse
Sophie LARUELLE	Examinatrice
Mathieu ROSENBAUM	Examineur

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Paris, le 25 mars 2021
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- Bastien Baldacci, Philippe Bergault, Joffrey Derchu and Mathieu Rosenbaum, *On bid and ask side-specific tick sizes*, submitted, 2020.
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Contents

Introduction	1
1 Market making on OTC markets	2
1.1 The problem of the market maker	2
1.2 A first model: Avellaneda-Stoikov	3
1.3 Main existing results	5
1.4 Our contribution	5
2 Optimal tick sizes	15
2.1 The problem	15
2.2 The model with uncertainty zones	16
2.3 Our contribution	17
3 Optimal execution	20
3.1 The problem	20
3.2 A first model: Almgren-Chriss	21
3.3 Our contribution	22
I Optimal market making on OTC markets	26
1 Size matters for OTC market makers: general results and dimensionality reduction techniques	27
1.1 Introduction	27
1.2 Market making with marked point processes	29
1.2.1 Modeling framework and notations	30
1.2.2 Existence and uniqueness of a solution to (1.2)	33
1.2.3 Verification theorem	38
1.3 Solving the multi-asset market making problem with factors	41
1.3.1 A low-dimensional approximation	42
1.3.2 A Monte-Carlo method to take the residual risk into account	43
1.4 Numerical results	46
1.4.1 The case of two assets: one factor vs. two factors	46
1.4.2 Dealing with 30 assets	66
1.5 Appendix	73
1.5.1 A particular case: hedging the market factors	73
1.5.2 On the construction of the processes $J^{i,b}$ and $J^{i,a}$	77

2	From passive to active: when market makers are also market takers	78
2.1	Introduction	78
2.2	A model for active market makers	81
2.2.1	Modeling framework and notations	81
2.2.2	Preliminary results	87
2.3	Viscosity solution to (HJ)	88
2.3.1	Existence	88
2.3.2	Uniqueness	99
2.4	Numerical results	104
2.5	Appendix	108
2.5.1	Proof of Lemma 3	108
2.5.2	Proof of Proposition 4	108
3	Algorithmic market making for options	111
3.1	Introduction	111
3.2	Description of the problem	113
3.2.1	The market	113
3.2.2	The optimization problem of the market maker	115
3.2.3	Assumptions and approximations	117
3.3	An approximate solution to the problem	118
3.3.1	Change of variables: beating the curse of dimensionality	118
3.3.2	Hamilton-Jacobi-Bellman equation and optimal controls	119
3.4	Numerical results	119
3.4.1	Model parameters	119
3.4.2	Optimal quotes	121
3.4.3	Conclusion	128
3.5	Appendix	128
3.5.1	An alternative to the Δ -hedging assumption	128
3.5.2	Beyond the constant-vega assumption	129
4	Closed-form approximations in multi-asset market making	131
4.1	Introduction	131
4.2	The multi-asset market making model	134
4.2.1	Model setup	134
4.2.2	The optimization problems	135
4.2.3	The Hamilton-Jacobi-Bellman and Hamilton-Jacobi equations	136
4.2.4	Existing theoretical results	137
4.3	A quadratic approximation of the value function and its applications	138
4.3.1	Introduction	138
4.3.2	An approximation of the value function in closed form	139
4.3.3	From value functions to heuristics and quotes	145
4.4	Beyond the quadratic approximation: a perturbative approach	148
4.5	A multi-asset market making model with additional features	150
4.5.1	A more general model	150
4.5.2	The Hamilton-Jacobi equation	152
4.5.3	Quadratic approximation	153

4.5.4	From value functions to heuristics and quotes	156
4.6	Numerical results	158
4.6.1	Value function and optimal quotes	159
4.6.2	Closed-form approximations	164
II	Other problems in mathematical finance	170
5	On bid and ask side-specific tick sizes	171
5.1	Introduction	171
5.2	The model with uncertainty zones	173
5.3	High frequency market making under side-specific tick values and interaction with the exchange	175
5.3.1	The market maker's problem	175
5.3.2	The platform's problem	179
5.4	Numerical results	179
5.4.1	Similar tick values on both sides	180
5.4.2	Side-specific tick values: additional opportunities for the market maker	181
5.4.3	Side-specific tick values: effect of ϕ_-	182
5.5	Conclusion	185
5.6	Appendix	185
5.6.1	Proof of Proposition 7	185
5.6.2	Proof of Theorem 5	188
5.6.3	Effects of the uncertainty zones on h	195
6	Optimal execution and statistical arbitrage under Ornstein-Uhlenbeck dynamics	196
6.1	Introduction	196
6.2	The optimal liquidation problem	199
6.2.1	Modelling framework and notations	199
6.2.2	Hamilton-Jacobi-Bellman equation	201
6.2.3	Main mathematical results	205
6.3	Numerical results	210
6.3.1	Single-asset case	210
6.3.2	Multi-asset case	215
6.4	Appendix - Multi-asset optimal execution with correlated Brownian motions and execution costs	220
6.5	Appendix - Merton portfolio optimization problem under Ornstein-Uhlenbeck dynamics and exponential utility	221
6.5.1	Modelling framework	221
6.5.2	HJB equation	222
	Bibliography	228

Introduction

1 Market making on OTC markets

1.1 The problem of the market maker

A market maker is, in a broad sense, a liquidity supplier. It is a trader who provides immediacy to other market participants, by proposing prices at which she stands ready to buy and sell a given asset, on a regular and continuous basis. In exchange for this immediacy and for the risk she bears, the market maker charges a spread: she typically buys at a lower price and sells at a higher price, on average, than what the sellers and buyers could expect if they were ready to wait. With the electrification of most markets and the emergence of algorithmic trading in many of them, many market participants effectively act as market makers. This is particularly the case on order-driven markets, where “official” market makers (such as the Designated Market Makers on the NYSE) are now often competing with high-frequency market making companies, who are almost continuously present on both sides of the limit order book – even though they have no obligation to do so.

On quote-driven markets, where market makers provide liquidity by quoting bid and ask prices, the electrification process also brought many changes over the last ten years. In most investment banks, traders are now being replaced by algorithms to automate the market making business, at least for small tickets. This is particularly relevant on FX cash markets, where dealers now propose their own private electronic platforms, enabling clients to directly send them requests for stream (RFSs) and requests for quote (RFQs), but also on corporate bond markets, where Multi-dealer-to-client (MD2C) platforms now enable clients to send a given request for quote simultaneously to several dealers, in order to put them into competition.

The problem faced by a market maker is a complex one, as she must be able to adapt quickly to changing market conditions, and make as much profit as possible while controlling her level of risk. First, she faces a standard static trade-off: high margin and low volume versus low margin and high volume. A market maker quoting a large bid-ask spread (with no skew) trades indeed rarely, but each trade is associated with a large Mark-to-Market (MtM) gain. Conversely, a market maker who quotes a narrow bid-ask spread (with no skew) trades often, but each trade is associated with a small MtM gain. Market makers seldom buy and sell at the same time, and carry therefore an inventory (either long or short) over some periods of time. Hence, she is exposed to the risk that the price moves against them. This corresponds to a second – dynamic – problem: in a volatile market, quoting in a dynamic way is necessary as they need to mitigate her market risk exposure and, in particular, skew their quotes as a function of their inventory. For instance, a single-asset market maker with a long inventory should price in a conservative manner on the bid side and rather aggressively on the ask side, as she wants to increase her probability to sell and decrease her probability to buy.

The optimization problem faced by market makers has been addressed in a long list of academic papers. The first two references commonly cited in the market making literature are two economic papers: Grossman and Miller [59] and Ho and Stoll [71]. If the former is a classic from a theoretical point of view, the latter was revived in 2008 by Avellaneda and Stoikov [13] to build the first practical model of single-asset market making. Since then, many models have been proposed, mostly based on this work. For instance, Guéant et al. [65] provides a rigorous analysis of the stochastic optimal control problem introduced by Avellaneda and Stoikov and proves that, under inventory constraints, the problem boils down to a system of linear ordinary differential equations

(ODE) in the case of exponential intensity functions. Cartea et al. ([36, 39, 40]) contributed a lot to the literature and added many features to the initial models: alpha signals, ambiguity aversion, etc. They also considered a different objective function: a risk-adjusted expectation instead of the Von Neumann-Morgenstern expected utility of [13] and [65]. Multi-asset market making has been considered in Guéant [61, 62] for both kinds of objective functions and the author shows that the problem boils down, for general intensity functions, to solving a system of (*a priori* nonlinear) ODEs. Most of the above models are well suited to tackle market making in OTC markets or in order-driven markets when the tick/spread ratio is small. For major stock markets or for some foreign-exchange platforms, other models have been proposed that depart from the Avellaneda-Stoikov framework by taking into account the presence of the limit order book, such as those of Guilbaud and Pham (see [68, 69]), Fodra and Pham [49], or Kühn and Muhle-Karbe [77], for instance. The particular case of option market making, for which one needs to model the stochastic volatility, has also been studied, for instance in El Aoud and Abergel [46] or Stoikov and Saglam [98].

1.2 A first model: Avellaneda-Stoikov

In 2008, Avellaneda and Stoikov [13] revisited a paper by Ho and Stoll [71] to propose a model that is now the basis of almost every market making algorithm. In their model, they consider a market maker in charge of only one asset over a period of time $[0, T]$, whose mid-price $(S_t)_{t \in [0, T]}$ follows a simple arithmetic Brownian motion

$$S_t = S_0 + \sigma W_t,$$

with $S_0, \sigma > 0$ given. The market maker chooses at each time the bid and ask prices S_t^b and S_t^a . More precisely, she controls the mid-to-bid and ask-to-mid spreads δ_t^b and δ_t^a defined as

$$\delta_t^b = S_t - S_t^b \quad \text{and} \quad \delta_t^a = S_t^a - S_t \quad \forall t \in [0, T].$$

The dynamics of her inventory process $(q_t)_{t \in [0, T]}$ is governed by two point processes $(N_t^b)_{t \in [0, T]}$ and $(N_t^a)_{t \in [0, T]}$, modeling respectively the number of trades at the bid and at the ask

$$dq_t = dN_t^b - dN_t^a,$$

with q_0 given. Note that, here, the trade size is assumed to be constant. The intensity processes $(\lambda_t^b)_{t \in [0, T]}$ and $(\lambda_t^a)_{t \in [0, T]}$ of $(N_t^b)_{t \in [0, T]}$ and $(N_t^a)_{t \in [0, T]}$ respectively, are assumed to verify

$$\lambda_t^b = \Lambda^b(\delta_t^b) \quad \text{and} \quad \lambda_t^a = \Lambda^a(\delta_t^a),$$

for two functions $\Lambda^b, \Lambda^a : \mathbb{R} \rightarrow \mathbb{R}_+$. Some natural assumptions can be made on these two functions: they must be decreasing, and go to 0 as δ^b and δ^a go to $+\infty$. Avellaneda and Stoikov proposed intensity functions of the form

$$\Lambda^b(\delta) = \Lambda^a(\delta) = Ae^{-k\delta},$$

with $A, k > 0$ given.

Finally, the cash process $(X_t)_{t \in [0, T]}$ of the market maker has dynamics

$$dX_t = S_t^a dN_t^a - S_t^b dN_t^b,$$

with X_0 given.

For a given $\gamma > 0$ modeling the risk-aversion parameter, the agent's objective function is given by

$$\mathbb{E}[-\exp(-\gamma(X_T + q_T S_T))].$$

The Hamilton-Jacobi-Bellman (HJB) equation associated with this problem is given by

$$0 = \partial_t u(t, x, q, s) + \frac{1}{2} \sigma^2 \partial_{ss}^2 u(t, x, q, s) + \sup_{\delta^b} \Lambda^b(\delta^b) [u(t, x - s + \delta^b, q + 1, s) - u(t, x, q, s)] \\ + \sup_{\delta^a} \Lambda^a(\delta^a) [u(t, x + s + \delta^a, q - 1, s) - u(t, x, q, s)],$$

with terminal condition $u(T, x, q, s) = -\exp(-\gamma(x + qs))$. Using the ansatz

$$u(t, x, q, s) = -\exp(-\gamma(x + qs + \theta(t, q))),$$

we see that θ solves

$$0 = \partial_t \theta(t, q) - \frac{\gamma}{2} \sigma^2 q^2 + \sup_{\delta^b} \frac{\Lambda^b(\delta^b)}{\gamma} [1 - \exp(-\gamma(\delta^b + \theta(t, q + 1) - \theta(t, q)))] \\ + \sup_{\delta^a} \frac{\Lambda^a(\delta^a)}{\gamma} [1 - \exp(-\gamma(\delta^a + \theta(t, q - 1) - \theta(t, q)))] , \quad (1)$$

with terminal condition $\theta(T, q) = 0$.

Cartea et al. [40] later introduced a different objective function defined as

$$\mathbb{E} \left[X_T + q_T S_T - \frac{\gamma}{2} \int_t^T \sigma^2 q_s^2 ds \right].$$

In that case, the HJB equation is given by

$$0 = \partial_t v(t, x, q, s) - \frac{\gamma}{2} \sigma^2 q^2 + \frac{1}{2} \sigma^2 \partial_{ss}^2 v(t, x, q, s) + \sup_{\delta^b} \Lambda^b(\delta^b) [v(t, x - s + \delta^b, q + 1, s) - v(t, x, q, s)] \\ + \sup_{\delta^a} \Lambda^a(\delta^a) [v(t, x + s + \delta^a, q - 1, s) - v(t, x, q, s)],$$

with terminal condition $v(T, x, q, s) = x + qs$. Using the ansatz

$$v(t, x, q, s) = x + qs + \theta(t, q),$$

we see that θ solves

$$0 = \partial_t \theta(t, q) - \frac{1}{2} \gamma \sigma^2 q^2 + \sup_{\delta^b} \Lambda^b(\delta^b) [\delta^b + \theta(t, q + 1) - \theta(t, q)] \\ + \sup_{\delta^a} \Lambda^a(\delta^a) [\delta^a + \theta(t, q - 1) - \theta(t, q)], \quad (2)$$

with terminal condition $\theta(T, q) = 0$.

1.3 Main existing results

If we define for $\xi > 0$ the two functions

$$H_\xi^b(p) = \sup_{\delta^b} \frac{\Lambda^b(\delta^b)}{\xi} [1 - \exp(-\xi(\delta^b - p))],$$

$$H_\xi^a(p) = \sup_{\delta^a} \frac{\Lambda^a(\delta^a)}{\xi} [1 - \exp(-\xi(\delta^a - p))],$$

and the limit functions (for $\xi = 0$)

$$H_0^b(p) = \sup_{\delta^b} \Lambda^b(\delta^b) [\delta^b - p],$$

$$H_0^a(p) = \sup_{\delta^a} \Lambda^a(\delta^a) [\delta^a - p],$$

then Equations (1) and (2) can be written as

$$0 = \partial_t \theta(t, q) - \frac{1}{2} \gamma \sigma^2 q^2 + H_\xi^b(\theta(t, q) - \theta(t, q + 1)) + H_\xi^a(\theta(t, q) - \theta(t, q - 1)) \quad (3)$$

with terminal condition $\theta(T, q) = 0$, with $\xi = \gamma$ for Equation (1) and $\xi = 0$ for Equation (2).

Guéant et al. proved in [62] and [65] that, by adding risk limits to the inventory of the market maker, Equation (3) boils down to a finite system of ordinary differential equations (ODEs) that admits a unique solution C^1 in time.

A classical verification argument then proves that the optimal controls $\delta^{b,*}$ and $\delta^{a,*}$ are given by

$$\delta_t^{b,*} = \tilde{\delta}_\xi^b(\theta(t, q_{t-}) - \theta(t, q_{t-} + 1))$$

and

$$\delta_t^{a,*} = \tilde{\delta}_\xi^a(\theta(t, q_{t-}) - \theta(t, q_{t-} - 1))$$

with $\xi = \gamma$ in the first case, $\xi = 0$ in the second case, where $\tilde{\delta}_\xi^b, \tilde{\delta}_\xi^a$ are given by

$$\tilde{\delta}_\xi^b(p) = (\Lambda^b)^{-1} \left(\xi H_\xi^b(p) - H_\xi^{b'}(p) \right)$$

and

$$\tilde{\delta}_\xi^a(p) = (\Lambda^a)^{-1} \left(\xi H_\xi^a(p) - H_\xi^{a'}(p) \right).$$

1.4 Our contribution

Chapter 1: Request sizes and dimensionality reduction

Since the paper of Avellaneda and Stoikov, a lot of extensions of this model have been studied. In spite of this growing literature, many issues have not been addressed yet. First, every model so far assumes that the trade size is constant. Although this assumption is a reasonable way of simplifying the problem on most order-driven markets, it is limiting on markets organized around requests for quotes, for which quotes can and should depend on the size of each request. A second and more

general problem is that of the numerical approximation of the optimal quotes. Although optimal quotes can theoretically be computed through the solution of a system of ODEs, the size of that system (which grows exponentially with the number of assets) prevents any concrete computation with grid methods when it comes to portfolios with more than 4 or 5 assets. To our knowledge, the only attempt to beat this curse of dimensionality associated with market making models is Guéant and Manziuk [66] in which the authors propose a method – inspired by reinforcement learning techniques – that uses neural networks instead of grids.

In Chapter 1, we address these first two problems. We consider a model with $d \in \mathbb{N}^*$ assets, where the price of asset i is modeled by an arithmetic Brownian motion

$$S_t^i = S_0^i + \sigma^i W_t^i,$$

with $S_0^i, \sigma^i > 0$ given, and $((W_t^1, \dots, W_t^d)')_{t \in [0, T]}$ a d -dimensional Brownian motion with correlation matrix $(\rho^{i,j})_{1 \leq i, j \leq d}$. We denote by $\Sigma = (\rho^{i,j} \sigma^i \sigma^j)_{1 \leq i, j \leq d}$ the variance-covariance matrix of the process $(S_t)_{t \in [0, T]} = ((S_t^1, \dots, S_t^d)')$.

The market maker chooses the bid and ask prices for asset i at each time as functions of the size $z \in \mathbb{R}_+^*$ of the RFQ: they are modeled by maps $S^{i,b}, S^{i,a} : \Omega \times [0, T] \times \mathbb{R}_+^* \rightarrow \mathbb{R}$. As before, we introduce

$$\delta^{i,b}(t, z) = S_t^i - S^{i,b}(t, z) \quad \text{and} \quad \delta^{i,a}(t, z) = S^{i,a}(t, z) - S_t^i.$$

The transactions at the bid and at the ask for asset i are now modeled by two \mathbb{R}_+^* -marked point processes, that we denote by $J^{i,b}(dt, dz)$ and $J^{i,a}(dt, dz)$, with respective intensity kernels $(\nu_t^{i,b}(dz))_{t \in [0, T]}$, $(\nu_t^{i,a}(dz))_{t \in [0, T]}$ given by

$$\nu_t^{i,b}(dz) = \Lambda^{i,b}(\delta^{i,b}(t, z)) \mu^{i,b}(dz) \quad \text{and} \quad \nu_t^{i,a}(dz) = \Lambda^{i,a}(\delta^{i,a}(t, z)) \mu^{i,a}(dz),$$

where $\Lambda^{i,b}$ and $\Lambda^{i,a}$ are two functions satisfying the same assumptions as before, and $\mu^{i,b}$, $\mu^{i,a}$ are two probability measures modeling the distribution of request sizes at the bid and at the ask, respectively.

The inventory of the market maker in asset i , denoted by $(q_t^i)_{t \in [0, T]}$, has dynamics

$$dq_t^i = \int_{\mathbb{R}_+^*} z J^{i,b}(dt, dz) - \int_{\mathbb{R}_+^*} z J^{i,a}(dt, dz),$$

and we denote by $(q_t)_{t \in [0, T]}$ the vector $(q_t)_{t \in [0, T]} = ((q_t^1, \dots, q_t^d)')$.

The cash process $(X_t)_{t \in [0, T]}$ has dynamics

$$dX_t = \int_{\mathbb{R}_+^*} S^{i,a}(t, z) z J^{i,a}(dt, dz) - \int_{\mathbb{R}_+^*} S^{i,b}(t, z) z J^{i,b}(dt, dz),$$

and we want to maximize a risk-adjusted expectation of the PnL

$$\mathbb{E} \left[X_T + q_T' S_T - \ell_d(q_T) - \int_0^T \psi(q_t) dt \right],$$

where $\psi, \ell_d : \mathbb{R}^d \rightarrow \mathbb{R}_+$ are two given continuous penalty functions modeling the risk aversion of the market maker.

We prove that the value function θ associated with this problem is the unique solution of the following integro-differential Hamilton-Jacobi equation

$$0 = \frac{\partial \theta}{\partial t}(t, q) - \psi(q) + \sum_{i=1}^d \int_{\mathbb{R}_+^*} z H^{i,b} \left(\frac{\theta(t, q) - \theta(t, q + ze^i)}{z} \right) \mu^{i,b}(dz) \\ + \sum_{i=1}^d \int_{\mathbb{R}_+^*} z H^{i,a} \left(\frac{\theta(t, q) - \theta(t, q - ze^i)}{z} \right) \mu^{i,a}(dz), \quad \forall (t, q) \in [0, T] \times \mathbb{R}^d,$$

with terminal condition $\theta(T, q) = -\ell_d(q)$, where

$$H^{i,b}(p) = \sup_{\delta} \Lambda^{i,b}(\delta)(\delta - p) \quad \text{and} \quad H^{i,a}(p) = \sup_{\delta} \Lambda^{i,a}(\delta)(\delta - p),$$

and using a verification argument, we express the optimal quotes $\delta^{i,b}(t, z)$ and $\delta^{i,a}(t, z)$ for asset i as functions of $\frac{\theta(t, q_{t-}) - \theta(t, q_{t-} + ze^i)}{z}$ and $\frac{\theta(t, q_{t-}) - \theta(t, q_{t-} - ze^i)}{z}$, respectively.

We then propose a factorial approach in order to reduce the dimensionality of the problem. If, as is the case in most financial models, the prices of the d assets are modeled using a small number of factors k , the variance-covariance matrix Σ takes the form

$$\Sigma = \beta V \beta' + R,$$

where β is a d -by- k matrix of real coefficients, V is the k -by- k variance-covariance matrix of the factors, and R is the d -by- d variance-covariance matrix of the residuals. When the explanatory power of the factors is high, R should be small compared to Σ . It is then reasonable to ignore the residuals, by setting $R = 0$.

We then assume that the function ψ, ℓ_d have the forms $\psi(q) = \bar{\psi}(q' \Sigma q)$ and $\ell_d(q) = \bar{\ell}_d(q' \Sigma q)$ for some continuous functions $\bar{\psi}$ and $\bar{\ell}_d$. With $R = 0$, i.e. $\Sigma = \beta V \beta'$, the previous integro-differential equation can then be written as

$$0 = \frac{\partial \theta}{\partial t}(t, q) - \bar{\psi}((\beta' q)' V (\beta' q)) + \sum_{i=1}^d \int_{\mathbb{R}_+^*} z H^{i,b} \left(\frac{\theta(t, q) - \theta(t, q + ze^i)}{z} \right) \mu^{i,b}(dz) \\ + \sum_{i=1}^d \int_{\mathbb{R}_+^*} z H^{i,a} \left(\frac{\theta(t, q) - \theta(t, q - ze^i)}{z} \right) \mu^{i,a}(dz), \quad \forall (t, q) \in [0, T] \times \mathbb{R}^d,$$

with terminal condition $\theta(T, q) = -\bar{\ell}_d((\beta' q)' V (\beta' q))$.

Using the ansatz $\theta(t, q) = \tilde{\theta}(t, \beta' q)$, we get the following equation for the function $\tilde{\theta}$

$$\begin{aligned}
0 = & \frac{\partial \tilde{\theta}}{\partial t}(t, f) - \bar{\psi}(f'Vf) + \sum_{i=1}^d \int_{\mathbb{R}_+^*} z H^{i,b} \left(\frac{\tilde{\theta}(t, f) - \tilde{\theta}(t, f + z\tilde{e}^i)}{z} \right) \mu^{i,b}(dz) \\
& + \sum_{i=1}^d \int_{\mathbb{R}_+^*} z H^{i,a} \left(\frac{\tilde{\theta}(t, f) - \tilde{\theta}(t, f - z\tilde{e}^i)}{z} \right) \mu^{i,a}(dz), \quad \forall (t, f) \in [0, T] \times \mathbb{R}^k,
\end{aligned}$$

with terminal condition $\tilde{\theta}(T, f) = -\bar{\ell}_d(f'Vf)$, and where $\tilde{e}^i = \beta^i e^i$.

We can then express the optimal quotes in terms of the factors as a function of $\tilde{\theta}$, and the problem boils down to computing the solution to the above equation, which is numerically doable with grid methods when k is small.

Chapter 2: Active market makers

To the best of our knowledge, the market making literature on quote-driven markets has only dealt up until now with the case of a “passive” market maker, in the sense that in the models *à la* Avellaneda and Stoikov, the market maker cannot herself take the initiative to buy or sell an asset, and necessarily has to wait for a request for quote in order to trade. On many markets, this constraint does not exist, and the market maker can unwind part of her inventory on some liquidity pools, or trade continuously in some other assets correlated with the ones she is dealing, in order to mitigate the risk of her book. For instance, FX cash market makers frequently use the Dealer-to-Dealer (D2D) segment of the market to adjust their position without having to wait for a client.

In Chapter 2, we address this problem and propose a market making model authorizing the market maker to trade continuously in some assets. We consider again a model with $d \in \mathbb{N}^*$ assets, and write $I = \{1, \dots, d\}$. We now assume that there are two subsets $I_{\mathcal{M}}$ and $I_{\mathcal{T}}$ such that $I = I_{\mathcal{M}} \cup I_{\mathcal{T}}$, where the assets with index in $I_{\mathcal{M}}$ can be traded by the market maker with RFQs, as in the models *à la* Avellaneda-Stoikov, and the assets with index in $I_{\mathcal{T}}$ can be traded continuously by the market maker on some liquidity pools. Some assets may be traded in both ways: $I_{\mathcal{M}} \cap I_{\mathcal{T}}$ is not necessarily empty (and can even be I).

As before, the price of asset $i \in I$ is modeled by an arithmetic Brownian motion

$$S_t^i = S_0^i + \sigma^i W_t^i,$$

with $S_0^i, \sigma^i > 0$ given, and $((W_t^1, \dots, W_t^d)')_{t \in [0, T]}$ a d -dimensional Brownian motion with correlation matrix $(\rho^{i,j})_{1 \leq i, j \leq d}$. We denote by $\Sigma = (\rho^{i,j} \sigma^i \sigma^j)_{1 \leq i, j \leq d}$ the variance-covariance matrix of the process $(S_t)_{t \in [0, T]} = ((S_t^1, \dots, S_t^d)')$.

The market maker chooses the bid and ask prices for asset $i \in I_{\mathcal{M}}$ at each time as functions of the size $z \in \mathbb{R}_+^*$ of the RFQ: they are modeled by maps $S^{i,b}, S^{i,a} : \Omega \times [0, T] \times \mathbb{R}_+^* \rightarrow \mathbb{R}$. As before, we introduce

$$\delta^{i,b}(t, z) = S_t^i - S^{i,b}(t, z) \quad \text{and} \quad \delta^{i,a}(t, z) = S^{i,a}(t, z) - S_t^i.$$

We assume that the market maker wants her inventory to remain in a given compact subset \mathcal{Q} of \mathbb{R}^d . The transactions at the bid and at the ask for asset $i \in I_M$ are then modeled by two \mathbb{R}_+^* -marked point processes, that we denote by $J^{i,b}(dt, dz)$ and $J^{i,a}(dt, dz)$, with respective intensity kernels $(\nu_t^{i,b}(dz))_{t \in [0, T]}$ and $(\nu_t^{i,a}(dz))_{t \in [0, T]}$ given by

$$\nu_t^{i,b}(dz) = \mathbb{1}_{\{q_{t-} + ze^i \in \mathcal{Q}\}} \Lambda^{i,b}(\delta^{i,b}(t, z)) \mu^{i,b}(dz) \quad \text{and} \quad \nu_t^{i,a}(dz) = \mathbb{1}_{\{q_{t-} - ze^i \in \mathcal{Q}\}} \Lambda^{i,a}(\delta^{i,a}(t, z)) \mu^{i,a}(dz),$$

where $\{e^1, \dots, e^d\}$ denotes the canonical basis of \mathbb{R}^d , $\Lambda^{i,b}$ and $\Lambda^{i,a}$ are two functions satisfying the same assumptions as before, and $\mu^{i,b}$, $\mu^{i,a}$ are two probability measures modeling the distribution of request sizes at the bid and at the ask, respectively.

For the assets traded actively by the market maker, as is done in Almgren [7], we consider that she trades continuously in asset $i \in I_{\mathcal{T}}$ at a rate $(v_t^i)_{t \geq 0}$ (she buys when $v_t^i \geq 0$ and sells otherwise).

The inventory of the market maker, modeled by a d -dimensional inventory process $(q_t)_{t \geq 0} = (q_t^1, \dots, q_t^d)_{t \geq 0}$, has the following dynamics:

$$\begin{aligned} dq_t^i &= \int_{\mathbb{R}_+^*} z J^{i,b}(dt, dz) - \int_{\mathbb{R}_+^*} z J^{i,a}(dt, dz) \quad \forall i \in I_M \cap (I \setminus I_{\mathcal{T}}), \\ dq_t^i &= \int_{\mathbb{R}_+^*} z J^{i,b}(dt, dz) - \int_{\mathbb{R}_+^*} z J^{i,a}(dt, dz) + v_t^i dt \quad \forall i \in I_M \cap I_{\mathcal{T}}, \\ dq_t^i &= v_t^i dt \quad \forall i \in I_{\mathcal{T}} \cap (I \setminus I_M), \end{aligned} \tag{4}$$

with q_0 given.

Finally, the process $(X_t)_{t \geq 0}$ modeling the market maker's cash account has the dynamics:

$$\begin{aligned} dX_t &= \sum_{i \in I_M} \left(\int_{\mathbb{R}_+^*} z (S_t^i + \delta^{i,a}(t, z)) J^{i,a}(dt, dz) - \int_{\mathbb{R}_+^*} z (S_t^i - \delta^{i,b}(t, z)) J^{i,b}(dt, dz) \right) \\ &\quad + \sum_{i \in I_{\mathcal{T}}} (-v_t^i S_t^i dt - L^i(q_t, v_t^i) dt) \\ &= \sum_{i \in I_M} \left(\int_{\mathbb{R}_+^*} z \delta^{i,b}(t, z) J^{i,b}(dt, dz) + \int_{\mathbb{R}_+^*} z \delta^{i,a}(t, z) J^{i,a}(dt, dz) \right) \\ &\quad - \sum_{i \in I_{\mathcal{T}}} L^i(q_t, v_t^i) dt - \langle S_t, dq_t \rangle, \end{aligned} \tag{5}$$

where $\langle \cdot, \cdot \rangle$ denotes the standard scalar product in \mathbb{R}^d , and $\forall i \in I_{\mathcal{T}}$, the penalty function $L^i : \mathbb{R}^d \times \mathbb{R} \rightarrow \mathbb{R} \cup \{+\infty\}$ can be seen as the temporary price impact of the market maker when she chooses to be active, and satisfies some natural hypotheses: for $q \in \mathcal{Q}$, $L^i(q, 0) = 0$, $L^i(q, \cdot)$ is strictly convex, increasing on \mathbb{R}_+ and decreasing on \mathbb{R}_- , and $L^i(q, \cdot)$ is asymptotically superlinear. For $q \notin \mathcal{Q}$ however, we assume $L^i(q, \cdot) = +\infty$, to ensure that the market maker keeps her inventory within the risk limits, i.e. within \mathcal{Q} .

As before, we aim at maximizing a risk-neutral objective function with penalization

$$\mathbb{E} \left[X_T + q'_T S_T - \ell_d(q_T) - \int_0^T \psi(q_t) dt \right],$$

where $\psi, \ell_d : \mathbb{R}^d \rightarrow \mathbb{R}_+$ are two given continuous penalty functions modeling the risk aversion of the market maker.

We prove that the value function θ associated with this problem is the unique continuous viscosity solution on $[0, T] \times \mathcal{Q}$ to the following Hamilton-Jacobi (HJ) partial integro-differential equation:

$$\begin{cases} 0 = -\partial_t \theta(t, q) + \psi(q) - \sum_{i \in I_{\mathcal{M}}} \int_{\mathbb{R}_+^*} \mathbb{1}_{\{q+ze^i \in \mathcal{Q}\}} z H^{i,b} \left(\frac{\theta(t,q) - \theta(t,q+ze^i)}{z} \right) \mu^{i,b}(dz) \\ - \sum_{i \in I_{\mathcal{M}}} \int_{\mathbb{R}_+^*} \mathbb{1}_{\{q-ze^i \in \mathcal{Q}\}} z H^{i,a} \left(\frac{\theta(t,q) - \theta(t,q-ze^i)}{z} \right) \mu^{i,a}(dz) - \sum_{i \in I_{\mathcal{T}}} \mathcal{H}^i(q, \partial_{q^i} \theta(t, q)) \quad \forall t \in [0, T) \\ \theta(T, q) = -\ell_d(q), \end{cases}$$

where $\forall i \in I_{\mathcal{M}}$, $H^{i,b}$ and $H^{i,a}$ are defined as before, and $\forall i \in I_{\mathcal{T}}$:

$$\mathcal{H}^i : (q, p) \in \mathbb{R}^d \times \mathbb{R} \mapsto \sup_{v \in \mathbb{R}} (vp - L^i(q, v)),$$

i.e. $\mathcal{H}^i(q, \cdot)$ is the Legendre-Fenchel transform of $L^i(q, \cdot)$. We then perform Monte-Carlo simulations to show empirically that the “optimal” quotes and execution rates obtained with a numerical scheme correspond indeed to the optimal controls of the problem.

Chapter 3: Option market making

The problem of option market making raises the need for specific models. Indeed, the underlying asset needs to be represented accordingly by a stochastic volatility model if one wants to take into account the volatility risk, which is the main risk carried by option market makers as they usually manage to Δ -hedge their portfolio. To the best of our knowledge, the first paper dealing with the inventory risk of an option market maker is Stoikov and Saglam [98]. In this paper, the authors consider the case of a market maker in charge of only one option, and assume the intensity of orders arrival to be linear. Although those assumptions are quite restrictive and limit the usefulness of this model for traders dealing with dozens or even hundreds of options with different strikes and maturities, this paper initiated the academic research on the specific problem faced by option market makers. In El Aoud and Abergel [46], the authors still consider a market maker dealing with only one option, assuming that the position is always Δ -hedged. They treat the case of both a risk-neutral and a risk-averse market maker, and provide closed-form approximations for the optimal quotes.

In Chapter 3, we consider this problem with a slightly different approach. We study the case of a market maker in charge of N European options on the same underlying asset. Of course, this naturally yields an HJB equation in high dimension ($N+2$, in addition to the time), that is numerically untractable. In order to beat the curse of dimensionality, we assume that the vega is constant over the considered time period. This means that the time horizon T has to be small enough for this

approximation to make sense. Of course, one can use the model on a short period of time and then run it again with updated vega – although time-inconsistent, this is a common practice in applied optimal control.

We consider that the underlying price $(S_t)_{t \in [0, T]}$ follows a one-factor stochastic volatility model *à la* Heston

$$\begin{cases} dS_t = \mu S_t dt + \sqrt{\nu_t} S_t dW_t^S \\ d\nu_t = a_{\mathbb{P}}(t, \nu_t) dt + \xi \sqrt{\nu_t} dW_t^\nu, \end{cases}$$

where $\mu \in \mathbb{R}, \xi \in \mathbb{R}^{*+}, (W_t^S, W_t^\nu)_{t \in \mathbb{R}_+}$ is a couple of Brownian motions with quadratic covariation given by $\rho = \frac{d\langle W^S, W^\nu \rangle}{dt} \in (-1, 1)$, and $a_{\mathbb{P}}$ is such that the processes are well defined. Of course, one could also consider a one- or two-factor forward variance model *à la* Bergomi, and the results would be similar.

We assume that, under a risk-neutral probability measure \mathbb{Q} , the price and volatility processes become

$$\begin{cases} dS_t = \sqrt{\nu_t} S_t d\widehat{W}_t^S \\ d\nu_t = a_{\mathbb{Q}}(t, \nu_t) dt + \xi \sqrt{\nu_t} d\widehat{W}_t^\nu, \end{cases}$$

where $(\widehat{W}_t^S, \widehat{W}_t^\nu)_{t \in \mathbb{R}_+}$ is another couple of Brownian motions under \mathbb{Q} with quadratic covariation given by $\rho = \frac{d\langle \widehat{W}^S, \widehat{W}^\nu \rangle}{dt} \in (-1, 1)$, and where $a_{\mathbb{Q}}$ is such that the processes are well defined.

For each $i \in \{1, \dots, N\}$, option i has a maturity date $T^i > T$, and its price, denoted by $(O_t^i)_{t \in [0, T^i]} = (O^i(t, S_t, \nu_t))_{t \in [0, T^i]}$ is solution on $[0, T^i] \times \mathbb{R}_+^2$ to the following PDE:

$$\begin{aligned} 0 &= \partial_t O^i(t, S, \nu) + a_{\mathbb{Q}}(t, \nu) \partial_\nu O^i(t, S, \nu) \\ &\quad + \frac{1}{2} \nu S^2 \partial_{SS}^2 O^i(t, S, \nu) + \rho \xi \nu S \partial_{\nu S}^2 O^i(t, S, \nu) + \frac{1}{2} \xi^2 \nu \partial_{\nu\nu}^2 O^i(t, S, \nu). \end{aligned}$$

We denote by $(q_t^i)_{t \in [0, T]}$ the inventory of the market maker in option i , and its dynamics is given by

$$dq_t^i := \int_{\mathbb{R}_+^*} z (N^{i,b}(dt, dz) - N^{i,a}(dt, dz)),$$

where $N^{i,b}(dt, dz)$ and $N^{i,a}(dt, dz)$ are two right-continuous \mathbb{R}_+^* -marked point processes, modeling as before the transactions of the i -th option on the bid and ask side, whose respective intensity processes $(\lambda_t^{i,b}(dz))_{t \in \mathbb{R}_+}$ and $(\lambda_t^{i,a}(dz))_{t \in \mathbb{R}_+}$ are given by

$$\lambda_t^{i,b}(dz) := \Lambda^{i,b}(\delta_t^{i,b}(z)) \mathbb{1}_{\{q_t^- + ze^i \in \mathbb{Q}\}} \mu^{i,b}(dz) \quad \lambda_t^{i,a}(dz) := \Lambda^{i,a}(\delta_t^{i,a}(z)) \mathbb{1}_{\{q_t^- - ze^i \in \mathbb{Q}\}} \mu^{i,a}(dz)$$

where the bid and ask prices proposed by the market maker at time t for option i and for a transaction of size z are given by

$$\mathcal{O}_t^i - \delta_t^{i,b}(z) \quad \text{and} \quad \mathcal{O}_t^i + \delta_t^{i,a}(z),$$

respectively, and with $(e^i)_{i \in \{1, \dots, N\}}$ the canonical basis of \mathbb{R}^N , \mathcal{Q} the set of authorized inventories for the market maker, and $\mu^{i,b}$ and $\mu^{i,a}$ two probability measures on \mathbb{R}_+^* , modeling the distributions of transaction sizes. For $i \in \{1, \dots, N\}$, $\Lambda^{i,b}, \Lambda^{i,a}$ are positive functions satisfying the same hypotheses as before.

We denote by $(\Delta_t)_{t \in [0, T]}$ the Δ of the portfolio:

$$\Delta_t := \sum_{i=1}^N \partial_S \mathcal{O}^i(t, S_t, \nu_t) q_t^i.$$

The market maker continuously Δ -hedge her portfolio, hence the cash process $(X_t)_{t \in [0, T]}$ has dynamics

$$dX_t := \sum_{i=1}^N \left(\int_{\mathbb{R}_+^*} z \left(\delta_t^{i,b}(z) N^{i,b}(dt, dz) + \delta_t^{i,a}(z) N^{i,a}(dt, dz) \right) - \mathcal{O}_t^i dq_t^i \right) + S_t d\Delta_t + d\langle \Delta, S \rangle_t.$$

We denote by $(V_t)_{t \in [0, T]}$ the process for the Mark-to-Market (MtM) value of the market maker's portfolio (cash, shares, and options), i.e.,

$$V_t := X_t - \Delta_t S_t + \sum_{i=1}^N q_t^i \mathcal{O}_t^i.$$

Denoting by \mathcal{V}_t^i the vega of the i -th option

$$\mathcal{V}_t^i := \partial_{\sqrt{\nu}} \mathcal{O}^i(t, S_t, \nu_t) = 2\sqrt{\nu_t} \partial_\nu \mathcal{O}^i(t, S_t, \nu_t),$$

it is easy to see that the variance in V_t is driven by the term

$$\sum_{i=1}^N \frac{\xi}{2} q_t^i \mathcal{V}_t^i dW_t^\nu.$$

Hence, we want to maximize a risk-adjusted expectation of the form

$$\mathbb{E} \left[V_T - \frac{\gamma}{2} \int_0^T \left(\sum_{i=1}^N \frac{\xi}{2} q_t^i \mathcal{V}_t^i \right)^2 dt \right],$$

with $\gamma > 0$.

The value function of this problems depends on S, ν, q^1, \dots, q^N . Hence, to compute it, one needs to solve an HJB equation in dimension $N + 2$, which is numerically unfeasible if $N \geq 3$. To solve this issue, we assume

1. that the vega of each option is constant over $[0, T]$:

$$\mathcal{V}_t^i = \mathcal{V}_0^i =: \mathcal{V}^i \quad \forall t \in [0, T];$$

2. that the set of authorized inventories is associated with vega risk limits, i.e.

$$\mathcal{Q} = \left\{ q \in \mathbb{R}^N \left| \sum_{i=1}^N q^i \mathcal{V}^i \in [-\bar{\mathcal{V}}, \bar{\mathcal{V}}] \right. \right\},$$

where $\bar{\mathcal{V}} \in \mathbb{R}_+^*$ is the vega risk limit of the market maker.

Under those conditions, we introduce the portfolio vega $\mathcal{V}_t^\pi = \sum_{i=1}^N q_t^i \mathcal{V}^i$ and show that the value function now only depends on the variables ν and \mathcal{V}^π , and hence solve a simple HJB equation in dimension 2:

$$\begin{aligned} 0 = & \partial_t v(t, \nu, \mathcal{V}^\pi) + a_{\mathbb{P}}(t, \nu) \partial_\nu v(t, \nu, \mathcal{V}^\pi) + \frac{1}{2} \nu \xi^2 \partial_{\nu\nu}^2 v(t, \nu, \mathcal{V}^\pi) + \mathcal{V}^\pi \frac{a_{\mathbb{P}}(t, \nu) - a_{\mathbb{Q}}(t, \nu)}{2\sqrt{\nu}} - \frac{\gamma \xi^2}{8} \mathcal{V}^{\pi 2} \\ & + \sum_{i=1}^N \sum_{j=a, b} \int_{\mathbb{R}_+^*} z \mathbb{1}_{\{|\mathcal{V}^\pi - \psi(j)z\mathcal{V}^i| \leq \bar{\mathcal{V}}\}} H^{i,j} \left(\frac{v(t, \nu, \mathcal{V}^\pi) - v(t, \nu, \mathcal{V}^\pi - \psi(j)z\mathcal{V}^i)}{z} \right) \mu^{i,j}(dz), \end{aligned} \quad (6)$$

with final condition $v(T, \nu, \mathcal{V}^\pi) = 0$, where $\psi(a) = 1$, $\psi(b) = -1$, and $H^{i,b}$ and $H^{i,a}$ are defined as before. We are then able to derive the optimal quotes of the market maker.

Chapter 4: Closed-form approximations

In Chapter 1, we presented a dimensionality reduction technique based on a factorial approach to tackle the problem faced by a multi-asset market maker. Although this method is perfectly suited for the case of a group of highly correlated assets (for instance, we can think of a market maker in charge of a large number of corporate bonds from the same issuer), it can still be numerically untractable for heterogeneous portfolios in which the number of factors necessary to represent the assets is too large. Furthermore, even with only three factors, solving the associated HJB equation can sometimes be too much time consuming when the dealer is competing with other market makers.

In Chapter 4, we propose a perturbative approach to build a closed-form proxy of the value function and optimal quotes. More precisely, we see the HJB equation associated with our market making problem as a perturbation of a simpler HJB equation that we can solve analytically. Mathematically, this new PDE boils down to a system of Riccati equations, for which we compute a closed-form solution. From this, we obtain closed-form approximations of the optimal quotes. Those closed-form approximations have the nice advantage of being easily interpretable.

Those closed-form approximations can be used directly by practitioners, to – significantly – reduce the computation time while taking into account all the relevant financial effects of the initial market making model. They can also be used in reinforcement learning based algorithms. Indeed, a proxy of the value function can be used as a starting point for a value iteration algorithm, or for actor-critic approaches, for instance. Similarly, the proxy of the optimal quotes can be used as a starting point for a policy iteration algorithm.

Let us explain our approach in the simple case of Eq. (3) in multidimensional form, i.e.

$$0 = \partial_t \theta(t, q) - \frac{1}{2} \gamma q^\top \Sigma q \quad (7)$$

$$+ \sum_{i=1}^d H_\xi^{i,b} (\theta(t, q) - \theta(t, q + e^i)) + \sum_{i=1}^d H_\xi^{i,a} (\theta(t, q) - \theta(t, q - e^i)) \quad \forall q \in \mathbb{R}^d,$$

with terminal condition $\theta(T, q) = 0$ for all $q \in \mathbb{R}^d$.

For all $i \in \{1, \dots, d\}$, we then replace the Hamiltonian functions $H_\xi^{i,b}$ and $H_\xi^{i,a}$ by the quadratic functions

$$\check{H}^{i,b} : p \mapsto \alpha_0^{i,b} + \alpha_1^{i,b} p + \frac{1}{2} \alpha_2^{i,b} p^2 \quad \text{and} \quad \check{H}^{i,a} : p \mapsto \alpha_0^{i,a} + \alpha_1^{i,a} p + \frac{1}{2} \alpha_2^{i,a} p^2,$$

respectively. Of course, a natural choice for the $\alpha_j^{i,b}$ and $\alpha_j^{i,a}$ derives from the Taylor expansion around $p = 0$ of the Hamiltonian functions, i.e.

$$\alpha_j^{i,b} = H_\xi^{i,b(j)}(0) \quad \text{and} \quad \alpha_j^{i,a} = H_\xi^{i,a(j)}(0).$$

Eq. (7) then becomes

$$0 = \partial_t \check{\theta}(t, q) - \frac{1}{2} \gamma q^\top \Sigma q + \sum_{i=1}^d \left(\alpha_0^{i,b} + \alpha_0^{i,a} \right)$$

$$+ \sum_{i=1}^d \left(\alpha_1^{i,b} (\check{\theta}(t, q) - \check{\theta}(t, q + e^i)) + \alpha_1^{i,a} (\check{\theta}(t, q) - \check{\theta}(t, q - e^i)) \right)$$

$$+ \frac{1}{2} \sum_{i=1}^d \left(\alpha_2^{i,b} (\check{\theta}(t, q) - \check{\theta}(t, q + e^i))^2 + \alpha_2^{i,a} (\check{\theta}(t, q) - \check{\theta}(t, q - e^i))^2 \right),$$

with terminal condition $\check{\theta}(T, q) = 0$.

We then make the ansatz $\check{\theta}(t, q) = -q^\top A(t)q - q^\top B(t) - C(t) \quad \forall (t, q) \in [0, T] \times \mathbb{R}^d$, with $A : [0, T] \rightarrow S_d^{++}$, $B : [0, T] \rightarrow \mathbb{R}^d$ and $C : [0, T] \rightarrow \mathbb{R}$. By plugging this into the above equation and identifying the terms, the problem boils down to solving the following system of ordinary differential equations:

$$\begin{cases} A'(t) = 2A(t) (D_2^b + D_2^a) A(t) - \frac{1}{2} \gamma \Sigma \\ B'(t) = 2A(t) (V_1^b - V_1^a) + 2A(t) (D_2^b - D_2^a) \mathcal{D}(A(t)) + 2A(t) (D_2^b + D_2^a) B(t) \\ C'(t) = \text{Tr} (D_0^b + D_0^a) + \text{Tr} ((D_1^b + D_1^a) A(t)) + (V_1^b - V_1^a)^\top B(t) \\ \quad + \frac{1}{2} \mathcal{D}(A(t))^\top (D_2^b + D_2^a) \mathcal{D}(A(t)) + \frac{1}{2} B(t)^\top (D_2^b + D_2^a) B(t) + B(t)^\top (D_2^b - D_2^a) \mathcal{D}(A(t)), \end{cases}$$

with terminal conditions

$$A(T) = 0, B(T) = 0, \text{ and } C(T) = 0,$$

where \mathcal{D} is the linear operator mapping a matrix onto the vector of its diagonal coefficients, and with

$$\Delta_j^{i,b} = \alpha_j^{i,b} \quad \text{and} \quad \Delta_j^{i,a} = \alpha_j^{i,a},$$

$$V_j^b = \left(\Delta_j^{1,b}, \dots, \Delta_j^{d,b} \right)^\top \quad \text{and} \quad V_j^a = \left(\Delta_j^{1,a}, \dots, \Delta_j^{d,a} \right)^\top,$$

and

$$D_j^b = \text{diag}(\Delta_j^{1,b}, \dots, \Delta_j^{d,b}) \quad \text{and} \quad D_j^a = \text{diag}(\Delta_j^{1,a}, \dots, \Delta_j^{d,a}).$$

We are then able to derive closed-form solutions to this system, as well as the asymptotic behaviour of A , B and C , which is of particular interest for market makers:

$$\begin{aligned} A(0) &\xrightarrow{T \rightarrow +\infty} \frac{1}{2} \sqrt{\gamma} \Gamma, \\ B(0) &\xrightarrow{T \rightarrow +\infty} -D_+^{-\frac{1}{2}} \widehat{A} \widehat{A}^+ D_+^{-\frac{1}{2}} \left(V_- + \frac{1}{2} \sqrt{\gamma} D_- \mathcal{D}(\Gamma) \right), \\ \frac{C(0)}{T} &\xrightarrow{T \rightarrow +\infty} -\text{Tr} (D_0^b + D_0^a) - \frac{1}{2} \sqrt{\gamma} \text{Tr} \left((D_1^b + D_1^a) \Gamma \right) + V_-^\top D_+^{-\frac{1}{2}} \widehat{A} \widehat{A}^+ D_+^{-\frac{1}{2}} \left(V_- + \frac{1}{2} \sqrt{\gamma} D_- \mathcal{D}(\Gamma) \right) \\ &\quad - \frac{1}{8} \gamma \mathcal{D}(\Gamma)^\top (D_2^b + D_2^a) \mathcal{D}(\Gamma) - \frac{1}{2} \left(V_- + \frac{1}{2} \sqrt{\gamma} D_- \mathcal{D}(\Gamma) \right)^\top D_+^{-\frac{1}{2}} \widehat{A} \widehat{A}^+ D_+^{-\frac{1}{2}} \left(V_- + \frac{1}{2} \sqrt{\gamma} D_- \mathcal{D}(\Gamma) \right) \\ &\quad + \frac{1}{2} \sqrt{\gamma} \left(V_- + \frac{1}{2} \sqrt{\gamma} D_- \mathcal{D}(\Gamma) \right)^\top D_+^{-\frac{1}{2}} \widehat{A} \widehat{A}^+ D_+^{-\frac{1}{2}} D_- \mathcal{D}(\Gamma), \end{aligned}$$

with $\Gamma = D_+^{-\frac{1}{2}} \left(D_+^{\frac{1}{2}} \Sigma D_+^{\frac{1}{2}} \right)^{\frac{1}{2}} D_+^{-\frac{1}{2}}$ and \widehat{A}^+ is the Moore-Penrose generalized inverse of \widehat{A} , and where

$$D_+ = D_2^b + D_2^a, \quad D_- = D_2^b - D_2^a, \quad V_- = V_1^b - V_1^a, \quad \text{and} \quad \widehat{A} = \sqrt{\gamma} \left(D_+^{\frac{1}{2}} \Sigma D_+^{\frac{1}{2}} \right)^{\frac{1}{2}}.$$

This allows us to compute the following simple closed-form approximations of the optimal quotes:

$$\begin{aligned} \check{\delta}_t^{i,b} &= \check{\delta}_\xi^{i,b} \left(\sqrt{\gamma} q_t^\top \Gamma e^i + \frac{1}{2} \sqrt{\gamma} z^i e^{i^\top} \Gamma e^i - e^{i^\top} D_+^{-\frac{1}{2}} \widehat{A} \widehat{A}^+ D_+^{-\frac{1}{2}} \left(V_- + \frac{1}{2} \sqrt{\gamma} D_- \mathcal{D}(\Gamma) \right) \right), \\ \check{\delta}_t^{i,a} &= \check{\delta}_\xi^{i,a} \left(-\sqrt{\gamma} q_t^\top \Gamma e^i + \frac{1}{2} \sqrt{\gamma} z^i e^{i^\top} \Gamma e^i + e^{i^\top} D_+^{-\frac{1}{2}} \widehat{A} \widehat{A}^+ D_+^{-\frac{1}{2}} \left(V_- + \frac{1}{2} \sqrt{\gamma} D_- \mathcal{D}(\Gamma) \right) \right). \end{aligned}$$

Numerical experiments then show, in the 2-dimensional case for which we can compute the optimal quotes, that those approximations are very close to optimality and give very similar results to the optimal quotes in terms of average PnL and standard deviation of the PnL for the market maker.

2 Optimal tick sizes

2.1 The problem

On most electronic markets, the asset prices do not move continuously: they cannot take any value on the (positive) real line, but rather move on a discrete grid on which traders are allowed to place their prices. This grid typically depends on the price of the asset and the traded volume, and is fixed by the exchange and by the regulator. In most cases, the grid step is fixed, and represents the smallest interval between two values of the price. This step is usually known as the ‘‘tick size’’.

Over the last decades, the tick size has become one of the most important tools for the regulation of financial markets, especially to gain some control over high-frequency traders, which are the main liquidity providers on many electronic markets (as shown in Megarbane et al. [86]). The behavior of high-frequency traders is indeed directly impacted by the tick size, and Frino et al. showed in [54] that they are usually attracted to stocks with a lower relative tick size, as they can adjust their quotes rapidly to get priority. Moreover, Aitken and Comerton-Forde [4] observed that the improvement of liquidity linked to a reduction of the tick size is more significant for stocks with a larger relative tick size.¹

But fixing a tick size is not straightforward, and its optimal value is often subject to debates, as it can sometimes have unwanted effects on the market microstructure: if the exchange sets it too large, the price will not move freely according to the market participant’s views, whereas if it is set too small, the limit order book (LOB) becomes unstable as the price will change too quickly. In order to set the correct tick sizes, some exchanges (such as the Tokyo Stock Exchange in 2014-2015) have launched pilot programs during which they temporarily modified the tick sizes of some given assets to observe the impact. In Europe, the recent MiFID II regulation of 2018 introduced a new tick size regime on Euronext. Laruelle et al. studied this new regime in [80] and observed that the regulation improved on average the quality of liquidity.

To the best of our knowledge, the first quantitative academic paper dealing with the problem of optimal tick size is Dayri and Rosenbaum [44]. Their approach focuses on large-tick assets, i.e. assets for which “the bid-ask spread is almost always equal to one tick”, according to Eisler et al. [45]. Dayri and Rosenbaum link the tick value to the statistics of high-frequency returns and durations, and are then able to predict the evolution of those statistics when the tick value changes. This enables them to define a notion of optimal tick size.

2.2 The model with uncertainty zones

Dayri and Rosenbaum make use of the model with uncertainty zones developed by Robert and Rosenbaum in [93] – as we will. It is a statistical model for transaction prices and durations, in which only transaction leading to a price change are modeled. It aims at reproducing all the main stylized fact of prices observed on the market at any frequency. In this model, transaction prices are discrete, and change only when an underlying efficient price process enters or exits a predetermined zone. We detail a simplified version of the model introduced in Robert and Rosenbaum [93].

We define the efficient price $(S_t)_{t \in [0, T]}$ on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{F} = (\mathcal{F}_t)_{t \in [0, T]})$ where T is the time horizon. We assume that its logarithm $(Y_t)_{t \in [0, T]}$ is an \mathbb{F} -adapted continuous Brownian semi-martingale of the form

$$Y_t = \log(S_t) = \log(S_0) + \int_0^t a_s ds + \int_0^t \sigma_{s-} dW_s,$$

where $(W_t)_{t \in [0, T]}$ is a standard Brownian motion, $(\sigma_t)_{t \in [0, T]}$ is an \mathbb{F} -adapted *càdlàg* process and $(a_t)_{t \in [0, T]}$ is progressively measurable.

¹The relative tick size is defined here as the ratio between the tick size and the price level.

The transaction prices lie on a discrete tick grid, given by $\{k\alpha; k \in \mathbb{N}\}$, with $\alpha > 0$ denoting the tick size. Assume that the current transaction price is $k\alpha$. The natural idea would be to say that, as soon as the efficient price becomes closer to $(k+1)\alpha$ or $(k-1)\alpha$ than to $k\alpha$, the transaction price should jump to $(k+1)\alpha$ or $(k-1)\alpha$, respectively. However, this is not a very realistic approach, since there is always some uncertainty about the efficient price, and market participants are reluctant to price changes. Hence, the efficient price should not just be “closer”, but “significantly closer” to a new value of the tick grid in order to change.

For all $k \in \mathbb{N}$ and for a given $0 < \eta < 1$, we define the zone $U_k = [0, +\infty) \times (d_k, u_k)$ where

$$d_k = \left(k + \frac{1}{2} - \eta\right) \alpha \quad \text{and} \quad u_k = \left(k + \frac{1}{2} + \eta\right) \alpha.$$

Hence, U_k is a band around the mid-tick value $(k + \frac{1}{2})\alpha$. We assume that the transaction price may move from price $k\alpha$ to price $(k+1)\alpha$ only when the price exited up the zone U_k , and that it may move from price $k\alpha$ to price $(k-1)\alpha$ only when the price exited down the zone U_{k-1} .

We also define here the sequence of exit times from uncertainty zones leading to a price change as $(\tau_j)_{j \geq 0}$. The traded price process is characterized by the couples of exit times and transaction prices with price changes $(\tau_j, P_{\tau_j})_{j \geq 0}$. The efficient price at time τ_j can be retrieved as

$$S_{\tau_j} = P_{\tau_j} - \alpha \left(\frac{1}{2} - \eta\right) \text{sgn}(P_{\tau_j} - P_{\tau_{j-1}}).$$

2.3 Our contribution

High-frequency traders act most of the time as market makers, i.e. liquidity provider, as they are typically present on both sides of the book. They propose prices at which they are ready to buy (bid price) and sell (ask price) a given asset, and make money out of the bid-ask spread while trying to control their inventory risk. The main goal of Chapter 5 is to show that setting side-specific tick sizes, i.e. different tick sizes on the bid side and on the ask side, can benefit both the exchange and the high-frequency traders.

Indeed, in most market making model, the dealer is assumed to be indifferent between having a long or short inventory: she typically just penalizes $\int_0^T q_t^2 dt$, where $(q_t)_{t \in [0, T]}$ is her inventory process. But due to the overnight repo rate, and some regulatory constraints imposed by the exchange or by the regulator, this is not true in practice: most market makers would rather be long than short at the end of the day. We prove in our paper that this asymmetry should be taken into account in the market design.

To this end, we consider that market participants always have an efficient price in mind, and we make use of a slightly modified version of the model with uncertainty zones presented above to build the “fair” bid and the “fair” ask prices as seen by the traders. In our setting, those two prices are lying on different tick grids. The market maker (or high-frequency trader) then chooses whether or not to quote a constant volume at this fair bid and ask prices. This is of course just a stylized viewpoint, but we believe that this allows us to capture the main financial effects of an asymmetric tick size. We formulate the problem of the market maker as a stochastic control problem, and show

existence and uniqueness of a viscosity solution to the associated HJB equation.

We then solve the problem of the exchange, whose remuneration depends on the liquidity provided by the trader. It can then select tick sizes knowing the optimal response of the market maker.

We consider as before that the logarithm of the efficient price $(Y_t)_{t \in [0, T]}$ is an \mathbb{F} -adapted continuous Brownian semi-martingale of the form

$$Y_t = \log(S_t) = \log(S_0) + \int_0^t a_s ds + \int_0^t \sigma_{s-} dW_s,$$

where $(W_t)_{t \in [0, T]}$ is a standard Brownian motion, $(\sigma_t)_{t \in [0, T]}$ is an \mathbb{F} -adapted *càdlàg* process and $(a_t)_{t \in [0, T]}$ is progressively measurable.

In contrary to what we did above, we now assume that the transaction prices lie on two tick grids defined by $\{k\alpha^b; k \in \mathbb{N}\}$ and $\{k\alpha^a; k \in \mathbb{N}\}$, corresponding to the transaction price at the bid and at the ask respectively. For $i \in \{b, a\}$, we fix $0 \leq \eta^i \leq \frac{1}{2}$ and define the zones $U_k^i = [0, +\infty) \times (d_k^i, u_k^i)$ for $k \in \mathbb{N}$ where

$$d_k^i = \left(k + \frac{1}{2} - \eta^i\right) \alpha^i \quad \text{and} \quad u_k^i = \left(k + \frac{1}{2} + \eta^i\right) \alpha^i.$$

We consider a high-frequency trader that uses those uncertainty zones to determine her fair bid and ask prices. She changes her fair prices at the bid and/or at the ask when the price gets “far enough” from the current fair bid or ask prices. More precisely, she defines her fair bid and ask prices S^b and S^a as

$$\begin{aligned} S_t^a &= S_{t-}^a + \alpha^a \mathbf{1}_{\{S_t - S_{t-}^a > (\frac{1}{2} + \eta^a) \alpha^a\}} - \alpha^a \mathbf{1}_{\{S_t - S_{t-}^a < -(\frac{1}{2} + \eta^a) \alpha^a\}}, \\ S_t^b &= S_{t-}^b + \alpha^b \mathbf{1}_{\{S_t - S_{t-}^b > (\frac{1}{2} + \eta^b) \alpha^b\}} - \alpha^b \mathbf{1}_{\{S_t - S_{t-}^b < -(\frac{1}{2} + \eta^b) \alpha^b\}}. \end{aligned}$$

Her inventory process $(Q_t)_{t \in [0, T]}$ is given by

$$Q_t = N_t^b - N_t^a,$$

where N^b and N^a are two point processes corresponding to the number of transactions at the bid and at the ask, respectively. For $i \in \{b, a\}$, the intensity $(\lambda_t^i)_{t \in [0, T]}$ of N^i is given by

$$\lambda_t^i = \lambda^i(\ell_t^i) \mathbf{1}_{\{\varphi(i) Q_t > -\tilde{q}\}}, \quad \text{with} \quad \lambda^i(\ell^i) = \frac{\lambda \ell^i}{1 + (\kappa \alpha^i)^2}$$

with $\varphi(i) = \mathbf{1}_{\{i=a\}} - \mathbf{1}_{\{i=b\}}$, $\tilde{q} > 0$ corresponds to the risk limit of the market maker, $\lambda > 0$ is a scale parameter and κ controls the sensitivity to the tick size. The process $(\ell_t^i)_{t \in [0, T]}$ is the trader's control, which lies in the set of \mathbb{F} -predictable processes with values in $\{0, 1\}$ denoted by \mathcal{L} , i.e. at each time t the trader can either choose to be present on the bid or ask side, and set $\ell_t^b = 1$ or $\ell_t^a = 1$ respectively, or she can set $\ell_t^b = 0$ or $\ell_t^a = 0$ so that there are no incoming transaction at the bid or at the ask respectively.

Her cash process $(X_t)_{t \in [0, T]}$ naturally verifies

$$X_T = \int_0^T (S_t^a dN_t^a - S_t^b dN_t^b).$$

We then assume that the market maker wants to maximize

$$\mathbb{E} \left[X_T + Q_T(S_T - A Q_T) - \phi \int_0^T Q_t^2 dt - \phi_- \int_0^T Q_t^2 \mathbf{1}_{\{Q_t < 0\}} dt \right]$$

over the set of admissible controls \mathcal{L} , where A is the penalty for the remaining inventory at the end of the time period, $\phi > 0$ is the risk aversion parameter of the market maker and $\phi_- > 0$ is a parameter corresponding to the additional risk aversion toward short position.

We prove that this problem is characterized by the following HJB equation

$$\begin{aligned} 0 = & \partial_t h(t, S^a, S^b, S, q) - \phi q^2 - \phi_- q^2 \mathbf{1}_{q < 0} + \frac{1}{2} \sigma^2 \partial_{SS}^2 h(t, S^a, S^b, S, q) \\ & + \frac{\lambda}{1 + (\kappa \alpha^a)^2} \max_{\ell^a \in \{0, 1\}} \left\{ \ell^a \left(S^a + h(t, S^a, S^b, S, q - \ell^a) - h(t, S^a, S^b, S, q) \right) \right\} \\ & + \frac{\lambda}{1 + (\kappa \alpha^b)^2} \max_{\ell^b \in \{0, 1\}} \left\{ \ell^b \left((-S^b) + h(t, S^a, S^b, S, q + \ell^b) - h(t, S^a, S^b, S, q) \right) \right\}, \end{aligned}$$

for $(t, S^a, S^b, S, q) \in [0, T) \times \mathcal{D} \times \mathcal{Q}$, with terminal condition

$$h(T, S^a, S^b, S, q) = q(S - Aq),$$

and with the following boundary condition on $[0, T) \times \partial \mathcal{D} \times \mathcal{Q}$:

$$\begin{aligned} 0 = & \mathbf{1}_{\{S - S^a = (\frac{1}{2} + \eta^a) \alpha^a, S - S^b < (\frac{1}{2} + \eta^b) \alpha^b\}} \left(h(t, S^a + \alpha^a, S^b, S, q) - h(t, S^a, S^b, S, q) \right) \\ & + \mathbf{1}_{\{S - S^a < (\frac{1}{2} + \eta^a) \alpha^a, S - S^b = (\frac{1}{2} + \eta^b) \alpha^b\}} \left(h(t, S^a, S^b + \alpha^b, S, q) - h(t, S^a, S^b, S, q) \right) \\ & + \mathbf{1}_{\{S - S^a = (\frac{1}{2} + \eta^a) \alpha^a, S - S^b = (\frac{1}{2} + \eta^b) \alpha^b\}} \left(h(t, S^a + \alpha^a, S^b + \alpha^b, S, q) - h(t, S^a, S^b, S, q) \right) \\ & + \mathbf{1}_{\{S - S^a = -(\frac{1}{2} + \eta^a) \alpha^a, S - S^b > -(\frac{1}{2} + \eta^b) \alpha^b\}} \left(h(t, S^a - \alpha^a, S^b, S, q) - h(t, S^a, S^b, S, q) \right) \\ & + \mathbf{1}_{\{S - S^a > -(\frac{1}{2} + \eta^a) \alpha^a, S - S^b = -(\frac{1}{2} + \eta^b) \alpha^b\}} \left(h(t, S^a, S^b - \alpha^b, S, q) - h(t, S^a, S^b, S, q) \right) \\ & + \mathbf{1}_{\{S - S^a = -(\frac{1}{2} + \eta^a) \alpha^a, S - S^b = -(\frac{1}{2} + \eta^b) \alpha^b\}} \left(h(t, S^a - \alpha^a, S^b - \alpha^b, S, q) - h(t, S^a, S^b, S, q) \right), \end{aligned}$$

where

$$\begin{aligned} \mathcal{D} = & \left\{ (S^a, S^b, S) \in \alpha^a \mathbb{Z} \times \alpha^b \mathbb{Z} \times \mathbb{R} \text{ such that } -\left(\frac{1}{2} + \eta^a\right) \alpha^a < S - S^a < \left(\frac{1}{2} + \eta^a\right) \alpha^a \right. \\ & \left. \text{and } -\left(\frac{1}{2} + \eta^b\right) \alpha^b < S - S^b < \left(\frac{1}{2} + \eta^b\right) \alpha^b \right\} \end{aligned}$$

and

$$\begin{aligned} \partial \mathcal{D} = & \left\{ (S^a, S^b, S) \in \alpha^a \mathbb{Z} \times \alpha^b \mathbb{Z} \times \mathbb{R} \text{ such that } S - S^a = \pm \left(\frac{1}{2} + \eta^a\right) \alpha^a \right. \\ & \left. \text{and/or } S - S^b = \pm \left(\frac{1}{2} + \eta^b\right) \alpha^b \right\}. \end{aligned}$$

We prove existence and uniqueness of a viscosity solution to the above HJB equation, and are then able to derive the optimal controls as functions of the solution h .

We then move on to the platform's problem. We assume that the platform earns a fixed taker cost $c > 0$ for each market order. Its inventory process $(X_t^p)_{t \in [0, T]}$ then verifies

$$X_t^p = c(N_t^b + N_t^a),$$

and it wants to maximise

$$\mathbb{E}^{\ell^*} [X_T^p],$$

over the set of possible tick sizes $\alpha^b, \alpha^a \in \mathbb{R}_+^2$, where \mathbb{E}^{ℓ^*} denotes the expectation knowing the optimal response of the market maker to the chosen tick sizes.

It is easy to see that this problem boils down to maximizing the function v defined below over \mathbb{R}_+^2 :

$$v(\alpha^a, \alpha^b) := \mathbb{E} \left[\int_0^T c\lambda \left\{ \frac{\ell^{*a}(t, S_t^a, S_t^b, S_t, q_t)}{1 + (\kappa\alpha^a)^2} + \frac{\ell^{*b}(t, S_t^a, S_t^b, S_t, q_t)}{1 + (\kappa\alpha^b)^2} \right\} dt \right],$$

where ℓ^{*b} and ℓ^{*a} are the functions of optimal response of the market maker. We observe that a small tick size α^a increase the term $(1 + (\kappa\alpha^a)^2)^{-1}$ as it attracts more market orders. However, the optimal control $\ell^{*,a}$ is more often equal to zero: the gain of the market maker may be too small if he quotes at the price S^a , therefore he sets more often $\ell^{*,a} = 0$, which reduces the average gain of the platform. The problem is similar on the bid side.

We show numerically that, in general, both the market maker and the platform have an incentive to set a smaller tick size on the ask side than on the bid side.

3 Optimal execution

3.1 The problem

Optimal execution of a large block of shares has long been a major problem for cash traders and brokers. Indeed, a trader wishing to buy or sell a large amount of shares necessarily faces adverse price movements – generally called “market impact” in the literature – as a direct consequence of her trades. To minimize these costs, she has to find a way to “optimally” split her orders into smaller ones. Indeed, trading slowly will reduce her market impact, but it exposes her to the risk that the price moves over the course of the execution process. She therefore needs to find a trade-off between price risk and execution costs.

To our knowledge, the first attempt to solve this problem is due to Bertsimas and Lo [30] in 1998. However in this paper, the authors neglected the price risk by only minimizing the expected cost of execution. As a consequence, the optimal strategy in their model often boils down to a constant execution speed, which is sub-optimal when the price risk is taken into account. Moreover, their model does not produce different strategies for assets with different liquidity, all else equal, whereas it seems obvious, intuitively, that a trader should execute more rapidly when the asset is liquid than when it is not. In 1999, Almgren and Chriss [11] proposed a new framework which became the

basis of a lot of optimal execution algorithms. Instead of minimizing the cost of trading, they maximized a mean-variance objective function. This allows to take into account not only the expected execution cost but also the variance of the strategy, which is of major importance in the case of a volatile asset. Moreover, they obtain closed-form formula for the optimal execution strategy.

Since then, the so-called Almgren-Chriss model has been largely generalized, to incorporate random execution costs as in Almgren [7], to study stochastic liquidity and volatility [8, 9], or to better take into account the presence of the LOB (Guéant et al. [64], Obizhaeva and Wang [89], Predoiu et al. [92]). Various types of execution strategies have been built: maximizing a risk-adjusted function of the PnL as in Almgren and Chriss [11] corresponds to strategy of the Implementation Shortfall (IS) type, whereas Guéant and Royer [67] or Konishi [76] focus on Volume-Weighted Average Price (VWAP) execution, and Guéant [61] also studied Percentage of Volume (POV) and Target Close (TC) orders.

3.2 A first model: Almgren-Chriss

We present briefly here a continuous-time version of the Almgren-Chriss model. For more details, one can refer to Guéant [61] or Cartea et al. [39].

We consider the case of a trader with a single-stock portfolio. Her initial inventory (either long or short) is given by $q_0 \in \mathbb{R}$, and we denote by $(q_t)_{t \in [0, T]}$ its inventory process over the time period $[0, T]$, with $T > 0$. The trader controls the speed of execution, modeled by a process $(v_t)_{t \in [0, T]}$ that is progressively measurable, with $dq_t = v_t dt$, bounded and such that $\int_0^T v_t dt = -q_0$, i.e. the trader wishes to have an empty inventory at time T .

Almgren and Chriss separate two types of market impact. The first one is the permanent impact, which impacts durably the price of the asset (with no resilience). It is shown in Gatheral [55] that the absence of dynamic arbitrage implies a linear permanent market impact. Hence the price process $(S_t)_{t \in [0, T]}$ verifies

$$dS_t = \sigma dW_t + kv_t dt,$$

with the initial price $S_0 > 0$ given, $\sigma > 0$ the volatility of the asset, $(W_t)_{t \in [0, T]}$ a standard Brownian motion and $k \geq 0$ a parameter modeling the magnitude of the permanent impact.

The second type of market impact is the temporary impact. In the Almgren-Chriss framework, it represents the costs associated with the bid-ask spread or other execution costs, and it is instantaneous in the sense that the price process does not change. This impact is only modeled in the cash process $(X_t)_{t \in [0, T]}$ of the trader, and therefore everything works as if the price was impacted by the transaction but returned instantaneously to its original state. The dynamics of $(X_t)_{t \in [0, T]}$ is given by

$$dX_t = -v_t S_t dt - L(v_t) dt,$$

where L is the function modeling the temporary impact, which satisfies some natural assumptions ($L(0) = 0$, L is strictly convex with a minimum in 0, and L is asymptotically superlinear). In their paper, Almgren and Chriss choose a quadratic function for L .

Several optimization criteria have been studied in the literature. Let us consider here the case of a CARA utility function. For a given $\gamma > 0$ modeling her risk aversion, the trader wishes to maximize

$$\mathbb{E} \left[-e^{-\gamma X_T} \right]$$

over the set of admissible controls $(v_t)_{t \in [0, T]}$.

Schied et al. [97] proved that we can, without loss of generality, reduce our study to the case of deterministic execution strategies. It is then easy to prove that, in that case, the problem boils down to minimizing the function J given by

$$J(q) = \int_0^T \left(L(q'(t)) + \frac{1}{2} \gamma \sigma^2 q(t)^2 \right) dt,$$

over the set of absolutely continuous functions $q \in W^{1,1}(0, T)$ satisfying $q(0) = q_0$ and $q(T) = 0$. In particular, the permanent impact modeled by the parameter k plays no role in the resolution of the problem.

If the Legendre-Fenchel transform H of the function L is differentiable, the Hamiltonian characterization of the optimal strategy q^* is given by

$$\begin{cases} p'(t) = \gamma \sigma^2 q^*(t), \\ q^{*'}(t) = H'(p(t)), \\ q^*(0) = q_0, \\ q^*(T) = 0. \end{cases}$$

In the particular case of quadratic execution costs as proposed by Almgren and Chriss, i.e. $L(v) = \eta v^2$ for a given $\eta > 0$, q^* is then the unique solution to the ODE

$$q^{*''}(t) = \frac{\gamma \sigma^2}{2\eta} q^*(t),$$

with boundary conditions $q^*(0) = q_0$ and $q^*(T) = 0$. From this, we get the formula of Almgren and Chriss

$$q^*(t) = q_0 \frac{\sinh \left(\sqrt{\frac{\gamma \sigma^2}{2\eta}} (T - t) \right)}{\sinh \left(\sqrt{\frac{\gamma \sigma^2}{2\eta}} T \right)},$$

with the associated optimal control $v^*(t) = q^{*'}(t) = -q_0 \sqrt{\frac{\gamma \sigma^2}{2\eta}} \frac{\cosh \left(\sqrt{\frac{\gamma \sigma^2}{2\eta}} (T - t) \right)}{\sinh \left(\sqrt{\frac{\gamma \sigma^2}{2\eta}} T \right)}$. We observe in particular that q^* is convex if $q_0 > 0$, and concave if $q_0 < 0$, which differs from the constant execution speed obtained by Bertsimas and Lo [30].

3.3 Our contribution

The closed-form deterministic formula obtained by Almgren and Chriss relies mostly on the Brownian dynamics of the price. For more complex dynamics, in general, the optimal strategy depends on

the price and therefore becomes stochastic. Moreover, although multi-asset optimal execution was addressed in Almgren and Chriss [11], the literature on this topic remains quite limited. Liquidating (or buying) large orders of different assets simultaneously is however a major problem, for instance for funds facing large withdrawals (or subscriptions), and neglecting the cross-asset relations can lead to very sub-optimal strategies in terms of price and liquidity risk. Some extensions of the multi-asset Almgren-Chriss model have been proposed: for instance, Bismuth et al. [31] studied the problem of optimal liquidation of a portfolio with Bayesian learning techniques, and Lehalle [82] studied a similar problem with a constraint of balance between the portfolio lines over the course of the execution.

The multi-asset Almgren-Chriss model considers that the asset prices follow standard Brownian dynamics with a given correlation matrix. It ignores, however, the presence of co-integration between the prices, which can have an important impact on the optimal execution strategy. Optimal trading of a co-integrated pair of assets is presented in Cartea et al. [39]. More recently, Cartea et al. [37] considered a portfolio of assets following a multidimensional Ornstein-Uhlenbeck dynamics. This is a particularly interesting problem as, in addition to the simple optimal execution problem, it allows to build statistical arbitrage strategy to take advantage of the mean reversion. In their approach, they maximize the expected PnL of the trader minus a running risk penalty. Their problem boils down to a system of Riccati ODEs, for which they prove global existence and uniqueness, and they can therefore characterize the optimal strategy using a standard verification argument.

In Chapter 6, we consider a model similar to the one of Cartea et al. [37], but we maximize an expected utility function of the PnL. As in their paper, we characterize the value function by a system of Riccati ODEs. We prove existence and uniqueness of a solution to this system by a comparison argument.

We consider a trader in charge of $d \in \mathbb{N}^*$ assets over a period of time $[0, T]$, with $T > 0$. The trader controls the execution speed for each of her assets, represented by a process $(v_t)_{t \in [0, T]} = (v_t^1, \dots, v_t^d)_{t \in [0, T]}^\top$ in \mathbb{R}^d . Her inventory process $(q_t)_{t \in [0, T]} = (q_t^1, \dots, q_t^d)_{t \in [0, T]}^\top$ verifies therefore the dynamics

$$dq_t = v_t dt,$$

with $q_0 \in \mathbb{R}^d$ given.

The price process $(S_t)_{t \in [0, T]} = (S_t^1, \dots, S_t^d)_{t \in [0, T]}^\top$ of the assets verify a multidimensional Ornstein-Uhlenbeck dynamics given by

$$dS_t = R(\bar{S} - S_t)dt + VdW_t,$$

with $S_0 \in \mathbb{R}_+^{*d}$ given, $R \in \mathcal{M}_d(\mathbb{R})$, $V \in \mathcal{M}_{d,k}(\mathbb{R})$ and $(W_t)_{t \in [0, T]} = (W_t^1, \dots, W_t^k)_{t \in [0, T]}^\top$ is a k -dimensional standard Brownian motion (with independent coordinates), for some $k \in \mathbb{N}^*$. We denote by $\Sigma = VV^\top$ the quadratic covariation matrix of the price process.

Finally, the cash process $(X_t)_{t \in [0, T]}$ of the trader verifies the dynamics:

$$dX_t = -v_t^\top S_t dt - L(v_t) dt,$$

with $X_0 \in \mathbb{R}$ given, where $L : \mathbb{R}^d \rightarrow \mathbb{R}_+$ is a strictly convex function, decreasing on \mathbb{R}_- and increasing on \mathbb{R}_+ with $L(0) = 0$, asymptotically superlinear representing the temporary impact of

the trader.

We then aim at maximizing

$$\mathbb{E} \left[-e^{-\gamma(X_T + q_T^\top S_T - \ell(q_T))} \right],$$

over the set of admissible controls, where $\gamma > 0$ represents the risk aversion of the trader, and $\ell : \mathbb{R}^d \rightarrow \mathbb{R}_+$ is a continuous penalty function for the terminal inventory.

The HJB equation associated with the above problem is given by:

$$\begin{aligned} 0 = & \partial_t w(t, x, q, S) + \sup_{v \in \mathbb{R}^d} \{ -(v^\top S + L(v)) \partial_x w(t, x, q, S) + v^\top \partial_q w(t, x, q, S) \} \\ & + (\bar{S} - S)^\top R^\top \partial_S w(t, x, q, S) + \frac{1}{2} \text{Tr} (\Sigma \partial_{SS}^2 w(t, x, q, S)), \end{aligned}$$

for all $(t, x, q, S) \in [0, T] \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d$ with the terminal condition:

$$w(T, x, q, S) = -e^{-\gamma(x + q^\top S - \ell(q))} \quad \forall (x, q, S) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d.$$

We first make the following ansatz

$$w(t, x, q, S) = -e^{-\gamma(x + q^\top S + \theta(t, q, S))} \quad \forall (t, x, q, S) \in [0, T] \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d,$$

and get a new PDE for $\theta : [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$:

$$\begin{aligned} 0 = & \partial_t \theta(t, q, S) + \bar{S}^\top R^\top q + \bar{S}^\top R^\top \partial_S \theta(t, q, S) - S^\top R^\top q - S^\top R^\top \partial_S \theta(t, q, S) + \frac{1}{2} \text{Tr} (\Sigma \partial_{SS}^2 \theta(t, q, S)) \\ & - \frac{\gamma}{2} \text{Tr} (\Sigma (q + \nabla_S \theta(t, q, S))(q + \nabla_S \theta(t, q, S))^\top) + H(\partial_q \theta(t, q, S)), \end{aligned}$$

for all $(t, q, S) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^d$ with terminal condition:

$$\theta(T, q, S) = -\ell(q) \quad \forall (q, S) \in \mathbb{R}^d \times \mathbb{R}^d,$$

where $H : \mathbb{R}^d \rightarrow \mathbb{R}$ is the Legendre-Fenchel transform of the function L , i.e.

$$H(p) = \sup_{v \in \mathbb{R}^d} v^\top p - L(v).$$

If we then assume that the functions L and ℓ are of the form $L(v) = v^\top \eta v$ and $\ell(q) = q^\top \Gamma q$, for some $\Gamma \in \mathcal{S}_d^+(\mathbb{R})$, $\eta \in \mathcal{S}_d^{++}(\mathbb{R})$, we see that the supremum in the above definition is reached at a unique $v^* = \frac{1}{2} \eta^{-1} p$ and we get $\forall p \in \mathbb{R}^d$

$$H(p) = \frac{1}{4} p^\top \eta^{-1} p.$$

We can then use a second ansatz:

$$\theta(t, q, S) = q^\top A(t)q + q^\top B(t)S + S^\top C(t)S + D(t)^\top q + E(t)^\top S + F(t) \quad \forall (t, q, S) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^d, \quad (8)$$

which allows us to characterize the value function of the problem by the following system of ODEs

$$\begin{cases} A'(t) = \frac{\gamma}{2}(B(t) + I_d)\Sigma(B(t)^\top + I_d) - A(t)\eta^{-1}A(t) \\ B'(t) = (B(t) + I_d)R + 2\gamma(B(t) + I_d)\Sigma C(t) - A(t)\eta^{-1}B(t) \\ C'(t) = R^\top C(t) + C(t)R + 2\gamma C(t)\Sigma C(t) - \frac{1}{4}B(t)^\top \eta^{-1}B(t) \\ D'(t) = -(B(t) + I_d)R\bar{S} + \gamma(B(t) + I_d)\Sigma E(t) - A(t)\eta^{-1}D(t) \\ E'(t) = -2C(t)R\bar{S} + R^\top E(t) + 2\gamma C(t)\Sigma E(t) - \frac{1}{2}B(t)^\top \eta^{-1}D(t) \\ F'(t) = -\bar{S}^\top R^\top E(t) - \text{Tr}(\Sigma C(t)) + \frac{\gamma}{2}E(t)^\top \Sigma E(t) - \frac{1}{4}D(t)^\top \eta^{-1}D(t), \end{cases}$$

where I_d denotes the identity matrix in $\mathcal{M}_d(\mathbb{R})$, with terminal conditions:

$$A(T) = -\Gamma, \quad B(T) = C(T) = D(T) = E(T) = F(T) = 0.$$

Using comparison techniques, we prove existence and uniqueness of a solution to this system, and by a standard verification argument we are then able to characterize the optimal control $(v_t^*)_{t \in [0, T]}$ by

$$v_t^* = \frac{1}{2}\eta^{-1} (2A(t)q_t + B(t)S_t + D(t)).$$

We then illustrate this strategy with several numerical examples to show the importance of taking the non-brownian dynamics into account.

Part I

Optimal market making on OTC markets

Chapter 1

Size matters for OTC market makers: general results and dimensionality reduction techniques

Abstract

In most OTC markets, a small number of market makers provide liquidity to other market participants. More precisely, for a list of assets, they set prices at which they agree to buy and sell. Market makers face therefore an interesting optimization problem: they need to choose bid and ask prices for making money while mitigating the risk associated with holding inventory in a volatile market. Many market making models have been proposed in the academic literature, most of them dealing with single-asset market making whereas market makers are usually in charge of a long list of assets. The rare models tackling multi-asset market making suffer however from the curse of dimensionality when it comes to the numerical approximation of the optimal quotes. The goal of this paper is to propose a dimensionality reduction technique to address multi-asset market making by using a factor model. Moreover, we generalize existing market making models by the addition of an important feature: the existence of different transaction sizes and the possibility for the market makers in OTC markets to answer different prices to requests with different sizes.

Key words: Market making, Stochastic optimal control, Curse of dimensionality, Integro-differential equations, Risk factor models.

1.1 Introduction

The electronification of financial markets has changed the traditional role played by market makers. This is evident in the case of most order-driven markets, such as many stock markets, where the traditional market makers in charge of maintaining fair and orderly markets now often compete

with high-frequency market making companies. Surprisingly maybe, many OTC markets organized around dealers have also undergone upheaval linked to electronification over the last ten years. This is the case of the corporate bond markets on both sides of the Atlantic ocean where the electronification process is dominated by Multi-dealer-to-client (MD2C) platforms enabling clients to send the same request for quote (RFQ) to several dealers simultaneously and therefore instantly put them into competition with one another. Electronification is also in progress inside investment banks as most of them replace their traders by algorithms to be able to provide clients with quotes for a large set of assets and automate their market making business, at least for small tickets.

Building market making algorithms is a difficult task as the optimization problem faced by a market maker involves both static and dynamic components. A market maker faces indeed a first (static) trade-off: high margin and low volume versus low margin and high volume. A market maker quoting a large bid-ask spread (with no skew) trades indeed rarely, but each trade is associated with large Mark-to-Market (MtM) gain. Conversely, a market maker who quotes a narrow bid-ask spread (with no skew) trades often, but each trade is associated with a small MtM gain. In addition to this simple static trade-off, market makers face a dynamic problem: in a volatile market, they must quote in a dynamic way to mitigate their market risk exposure and, in particular, skew their quotes as a function of their inventory. For example, a single-asset market maker with a long inventory should price in a conservative manner on the bid side and rather aggressively on the ask side, if she wants – a reasonable behaviour – to decrease her probability to buy and increase her probability to sell.

The optimization problem faced by market makers has been addressed in a long list of academic papers. The first two references commonly cited in the market making literature are two economic papers: Grossman and Miller [59] and Ho and Stoll [71]. If the former is a classic from a theoretical point of view, the latter was revived in 2008 by Avellaneda and Stoikov [13] to build the first practical model of single-asset market making. Since then, many models have been proposed, most of them to tackle the same problem of single-asset market making. For instance, [65] provides a rigorous analysis of the stochastic optimal control problem introduced by Avellaneda and Stoikov and proves that the problem boils down to a system of linear ordinary differential equations (ODE) in the case of exponential intensity functions. Cartea et al. ([36, 39, 40]) contributed a lot to the literature and added many features to the initial models: alpha signals, ambiguity aversion, etc. They also considered a different objective function: a risk-adjusted expectation instead of the Von Neumann-Morgenstern expected utility of [13] and [65]. Multi-asset market making has been considered in [61, 62] for both kinds of objective functions and the author shows that the problem boils down, for general intensity functions, to solving a system of (*a priori* nonlinear) ODEs. Most of the above models are well suited to tackle market making in OTC markets or in order-driven markets when the tick/spread ratio is large. For major stock markets or for some foreign-exchange platforms, other models are better suited such as those of Guilbaud and Pham who really took the microstructure into account (see [68, 69]).¹

In spite of a large and growing literature on market making, several problems are rarely addressed. A first example is that of trade sizes: in markets organized around requests for quotes, quotes can and should depend on the size of the requests. A second and more general problem is that of the numerical approximation of the optimal quotes. If optimal quotes can theoretically be computed

¹Option market making has also been addressed, see for instance [15, 46, 98].

through the solution of a system of ODEs, the size of that system (which grows exponentially with the number of assets) prevents any concrete computation with grid methods when it comes to portfolios with more than 4 or 5 assets. To our knowledge, the only attempt to approximate the solution of the Hamilton-Jacobi equations associated with market making models in high dimension is [66] in which the authors propose a method – inspired by reinforcement learning techniques – that uses neural networks instead of grids.

In this paper, our goal is twofold. Our first goal is to generalize existing models to introduce a distribution of trade size. This extension is not straightforward as the optimal controls cannot be modeled anymore with real-valued stochastic processes, but must instead be modeled with predictable maps. A consequence, in terms of mathematics, is that the problem does not anymore boil down to a finite system of ODEs but instead to an integro-differential equation of the Hamilton-Jacobi type that can be regarded as an ordinary differential equation in an infinite-dimensional space. Our second goal is to propose a numerical method for approximating the optimal bid and ask quotes of a market maker over a large universe of assets. For that purpose, we show that the real dimension of the problem is not that of the number of assets, but rather that of the rank of the correlation matrix of asset prices. Then, by using a factor model, we show how to approximate the optimal quotes of a market maker. Indeed, if market risk is projected on a low-dimensional space of factors, solving the market making problem boils down to solving a low-dimensional Hamilton-Jacobi equation. In particular, if the number of factors is lower than 3, grid methods can be applied independently of the number of assets. Furthermore, we suggest a Monte-Carlo method to approximate the influence of the residual risk not taken into account when the risk is projected on the space of factors.

In Section 2, we present our market making model with distributed request sizes. We characterize the value function associated with the stochastic optimal control problem as the solution of an integro-differential equation of the Hamilton-Jacobi type by using ODE techniques in a well-chosen infinite-dimensional space and a verification argument. We subsequently provide expressions for the optimal quotes as a function of time, inventory, and request size. In Section 3, we show how the equations can be simplified when the dependence structure between the prices of the different assets can be modeled by risk factors. We then show how this simplification leads to an approximation that helps to tackle the curse of dimensionality by solving a low-dimensional equation on a grid. We also explain how Monte-Carlo simulations could be used to account for the part of the risk not accounted by the factors. We apply these techniques in Section 4 to portfolios of 2 and 30 bonds and discuss the results.

1.2 Market making with marked point processes

In all this paper, we consider a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{F} = (\mathcal{F}_t)_{t \geq 0})$ satisfying the usual conditions. We assume this probability space to be large enough to support all the processes we introduce.

1.2.1 Modeling framework and notations

We consider a market maker in charge of d assets. For $i \in \{1, \dots, d\}$, the reference price of asset i is modeled by a process $(S_t^i)_{t \geq 0}$ with the following dynamics:

$$dS_t^i = \sigma^i dW_t^i,$$

with S_0^i given, $\sigma^i > 0$, and $((W_t^1, \dots, W_t^d)')_{t \geq 0}$ a d -dimensional² Brownian motion with correlation matrix $(\rho^{i,j})_{1 \leq i, j \leq d}$, adapted to the filtration $(\mathcal{F}_t)_{t \geq 0}$. We denote by $\Sigma = (\rho^{i,j} \sigma^i \sigma^j)_{1 \leq i, j \leq d}$ the variance-covariance matrix associated with the process $(S_t)_{t \geq 0} = ((S_t^1, \dots, S_t^d)')$.

Those assets are traded with requests for quote (RFQ): the market maker first receives a request for quote from a client wishing to buy or sell a given asset and she then proposes a price to the client who finally decides whether she accepts to trade at that price or not.

At any time, the market maker must be ready to propose bid and ask quotes to buy and sell any of the d assets. These bid and ask quotes depend on the size $z \in \mathbb{R}_+$ of the RFQ (in all this paper, we use the notation $\mathbb{R}_+^* := (0, +\infty)$). For a given asset i , they are modeled by maps $S^{i,b}, S^{i,a} : \Omega \times [0, T] \times \mathbb{R}_+^* \rightarrow \mathbb{R}$ which are $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}_+^*)$ -measurable, where \mathcal{P} denotes the σ -algebra of \mathbb{F} -predictable subsets of $\Omega \times [0, T]$ and $\mathcal{B}(\mathbb{R}_+^*)$ denotes the Borelian sets of \mathbb{R}_+^* .

For each $i \in \{1, \dots, d\}$, we introduce $J^{i,b}(dt, dz)$ and $J^{i,a}(dt, dz)$ two càdlàg \mathbb{R}_+ -marked point processes.³

For $i \in \{1, \dots, d\}$, we denote by $(\nu_t^{i,b}(dz))_{t \geq 0}$ and $(\nu_t^{i,a}(dz))_{t \geq 0}$ the intensity kernels of $J^{i,b}(dt, dz)$ and $J^{i,a}(dt, dz)$, respectively. In addition, we assume that $(\nu_t^{i,b}(dz))_{t \geq 0}$ and $(\nu_t^{i,a}(dz))_{t \geq 0}$ verify:

$$\begin{aligned} \nu_t^{i,b}(dz) &= \Lambda^{i,b}(\delta^{i,b}(t, z)) \mu^{i,b}(dz), \\ \nu_t^{i,a}(dz) &= \Lambda^{i,a}(\delta^{i,a}(t, z)) \mu^{i,a}(dz), \end{aligned}$$

where for all $i \in \{1, \dots, d\}$, $(\mu^{i,b}, \mu^{i,a})$ is a couple of probability measures on \mathbb{R}_+^* , $\delta^{i,b}(t, z) = S_t^i - S^{i,b}(t, z)$, $\delta^{i,a}(t, z) = S^{i,a}(t, z) - S_t^i$, and $(\Lambda^{i,b}, \Lambda^{i,a})$ is a couple of functions satisfying the following hypotheses (H):

- $\Lambda^{i,b}$ and $\Lambda^{i,a}$ are twice continuously differentiable,
- $\Lambda^{i,b}$ and $\Lambda^{i,a}$ are decreasing, with $\forall \delta \in \mathbb{R}$, $\Lambda^{i,b'}(\delta) < 0$ and $\Lambda^{i,a'}(\delta) < 0$,
- $\lim_{\delta \rightarrow +\infty} \Lambda^{i,b}(\delta) = 0$ and $\lim_{\delta \rightarrow +\infty} \Lambda^{i,a}(\delta) = 0$,
- $\sup_{\delta \in \mathbb{R}} \frac{\Lambda^{i,b}(\delta) \Lambda^{i,b''}(\delta)}{(\Lambda^{i,b'}(\delta))^2} < 2$ and $\sup_{\delta \in \mathbb{R}} \frac{\Lambda^{i,a}(\delta) \Lambda^{i,a''}(\delta)}{(\Lambda^{i,a'}(\delta))^2} < 2$.

²The sign $'$ designates the transpose operator. It transforms here a line vector into a column vector.

³These processes are explicitly constructed in the Appendix. Note that in our model, as in most real OTC markets, there are no simultaneous RFQs in multiple assets.

For all $i \in \{1, \dots, d\}$, $J^{i,b}(dt, dz)$ and $J^{i,a}(dt, dz)$ model respectively the volumes of transactions at the bid and at the ask for asset i . The inventory of the market maker, modeled by the d -dimensional process $(q_t)_{t \geq 0} = ((q_t^1, \dots, q_t^d)')_{t \geq 0}$, has therefore the following dynamics:

$$\forall i \in \{1, \dots, d\}, dq_t^i = \int_{\mathbb{R}_+^*} z J^{i,b}(dt, dz) - \int_{\mathbb{R}_+^*} z J^{i,a}(dt, dz),$$

with q_0 given.

Remark 1. For a given asset i , $\Lambda^{i,\cdot}$ typically has the form $\Lambda^{i,\cdot}(\delta) = \lambda_{RFQ}^{i,\cdot} f^{i,\cdot}(\delta)$, where $\lambda_{RFQ}^{i,\cdot}$ is the (constant) intensity of arrival of requests for quote and $f^{i,\cdot}(\delta)$ gives the probability that a request will result in a transaction given the quote δ proposed by the market maker. Furthermore, $\mu^{i,\cdot}$ should be seen as the distribution of sizes.

Finally, the process $(X_t)_{t \geq 0}$ modeling the market maker's cash account has the dynamics

$$\begin{aligned} dX_t &= \sum_{i=1}^d \int_{\mathbb{R}_+^*} S^{i,a}(t, z) z J^{i,a}(dt, dz) - \sum_{i=1}^d \int_{\mathbb{R}_+^*} S^{i,b}(t, z) z J^{i,b}(dt, dz) \\ &= \sum_{i=1}^d \int_{\mathbb{R}_+^*} (S_t^i + \delta^{i,a}(t, z)) z J^{i,a}(dt, dz) - \sum_{i=1}^d \int_{\mathbb{R}_+^*} (S_t^i - \delta^{i,b}(t, z)) z J^{i,b}(dt, dz) \\ &= \sum_{i=1}^d \int_{\mathbb{R}_+^*} (\delta^{i,b}(t, z) z J^{i,b}(dt, dz) + \delta^{i,a}(t, z) z J^{i,a}(dt, dz)) - \sum_{i=1}^d S_t^i dq_t^i. \end{aligned}$$

We fix $\delta_\infty \geq 0$ and define the set \mathcal{A} of admissible controls⁴ by

$$\mathcal{A} = \left\{ \delta = (\delta^{i,b}, \delta^{i,a})_{1 \leq i \leq d} : \Omega \times [0, T] \times \mathbb{R}_+^* \mapsto \mathbb{R}^{2d} \mid \delta \text{ is } \mathcal{P} \otimes \mathcal{B}(\mathbb{R}_+^*) \text{ - measurable,} \right. \\ \left. \forall i \in \{1, \dots, d\}, \delta^{i,b}(t, z) \geq -\delta_\infty \mathbb{P} \otimes dt \otimes \mu^{i,b} \text{ a.e. and } \delta^{i,a}(t, z) \geq -\delta_\infty \mathbb{P} \otimes dt \otimes \mu^{i,a} \text{ a.e.} \right\}.$$

As proved in [62], under assumptions (H), the functions $\delta \in \mathbb{R} \mapsto \delta \Lambda^{i,b}(\delta)$ and $\delta \in \mathbb{R} \mapsto \delta \Lambda^{i,a}(\delta)$ have a unique maximum on \mathbb{R} . It is also easy to see that on $[-\delta_\infty, +\infty)$, they are bounded from below by $-\delta_\infty \Lambda^{i,b}(-\delta_\infty)$ and $-\delta_\infty \Lambda^{i,a}(-\delta_\infty)$, respectively.

For two given continuous penalty functions $\psi : \mathbb{R}^d \rightarrow \mathbb{R}_+$ and $\ell_d : \mathbb{R}^d \rightarrow \mathbb{R}_+$, modeling the risk aversion of the market maker, we aim at maximizing the objective function

$$\mathbb{E} \left[X_T + \sum_{i=1}^d q_T^i S_T^i - \ell_d(q_T) - \int_0^T \psi(q_t) dt \right], \quad (1.1)$$

over the set \mathcal{A} of admissible controls.

⁴We introduce here a unique lower bound for the quotes, independently of the asset, the side, the size, and the time. Generalizations are straightforward.

Remark 2. For instance, we can choose $\psi(q) = \frac{\gamma}{2}q'\Sigma q$ or $\psi(q) = \gamma\sqrt{q'\Sigma q}$ (for $\gamma > 0$) and $\ell_d(q) = 0$, $\ell_d(q) = \frac{\zeta}{2}q'\Sigma q$ or $\ell_d(q) = \zeta\sqrt{q'\Sigma q}$ (for $\zeta > 0$), as done in [39], [40], [66], and [62].

After applying Itô's formula to $\left(X_t + \sum_{i=1}^d q_t^i S_t^i\right)_{t \geq 0}$ between 0 and T , it is easy to see that the problem is equivalent to maximizing

$$\mathbb{E} \left[\int_0^T \left\{ \sum_{i=1}^d \int_{\mathbb{R}_+^*} (\delta^{i,b}(t, z) z \Lambda^{i,b}(\delta^{i,b}(t, z))) \mu^{i,b}(dz) + \delta^{i,a}(t, z) z \Lambda^{i,a}(\delta^{i,a}(t, z)) \mu^{i,a}(dz) - \psi(q_t) \right\} dt - \ell_d(q_T) \right],$$

over the set of admissible controls \mathcal{A} .

We introduce the function $\mathcal{J} : [0, T] \times \mathbb{R}^d \times \mathcal{A} \rightarrow \mathbb{R}$ such that, $\forall t \in [0, T]$, $\forall q = (q^1, \dots, q^d)' \in \mathbb{R}^d$, $\forall (\delta^i)_{i \in \{1, \dots, d\}} \in \mathcal{A}$,

$$\mathcal{J}(t, q, (\delta^i)_{i \in \{1, \dots, d\}}) = \mathbb{E} \left[\int_t^T \left\{ \sum_{i=1}^d \int_{\mathbb{R}_+^*} \left(\delta^{i,b}(s, z) z \Lambda^{i,b}(\delta^{i,b}(s, z)) \mu^{i,b}(dz) + \delta^{i,a}(s, z) z \Lambda^{i,a}(\delta^{i,a}(s, z)) \mu^{i,a}(dz) \right) - \psi(q_s^{t, q, (\delta^i)_{i \in \{1, \dots, d\}}}) \right\} ds - \ell_d(q_T^{t, q, (\delta^i)_{i \in \{1, \dots, d\}}}) \right],$$

where $(q_s^{t, q, (\delta^i)_{i \in \{1, \dots, d\}}})_{s \geq t}$ is the inventory process starting in state q at time t and controlled by $(\delta^i)_{i \in \{1, \dots, d\}}$.

The value function $\theta : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$ of the problem is then defined as follows:

$$\theta(t, q) = \sup_{(\delta^i)_{i \in \{1, \dots, d\}} \in \mathcal{A}} \mathcal{J}(t, q, (\delta^i)_{i \in \{1, \dots, d\}}), \forall (t, q) \in [0, T] \times \mathbb{R}^d.$$

We will show that θ is the unique (in a large class of functions) classical solution to the following integro-differential Hamilton-Jacobi (HJ) equation:

$$\begin{aligned} 0 = & \frac{\partial w}{\partial t}(t, q) - \psi(q) + \sum_{i=1}^d \int_{\mathbb{R}_+^*} z H^{i,b} \left(\frac{w(t, q) - w(t, q + ze^i)}{z} \right) \mu^{i,b}(dz) \\ & + \sum_{i=1}^d \int_{\mathbb{R}_+^*} z H^{i,a} \left(\frac{w(t, q) - w(t, q - ze^i)}{z} \right) \mu^{i,a}(dz), \quad \forall (t, q) \in [0, T] \times \mathbb{R}^d \end{aligned} \quad (1.2)$$

with terminal condition $w(T, q) = -\ell_d(q)$, $\forall q \in \mathbb{R}^d$, where

$$H^{i,b} : p \in \mathbb{R} \mapsto \sup_{\delta \geq -\delta_\infty} \Lambda^{i,b}(\delta)(\delta - p) \text{ and } H^{i,a} : p \in \mathbb{R} \mapsto \sup_{\delta \geq -\delta_\infty} \Lambda^{i,a}(\delta)(\delta - p),$$

and where (e^1, \dots, e^d) denotes the canonical basis of \mathbb{R}^d .

1.2.2 Existence and uniqueness of a solution to (1.2)

Lemma 1. $\forall i \in \{1, \dots, d\}$, $H^{i,b}$ and $H^{i,a}$ are two globally Lipschitz continuously differentiable decreasing functions. Moreover, the supremum in the definition of $H^{i,b}(p)$ (respectively $H^{i,a}(p)$) is reached at a unique $\delta^{i,b^*}(p)$ (respectively $\delta^{i,a^*}(p)$). Furthermore, δ^{i,b^*} and δ^{i,a^*} are continuous and nondecreasing functions.

Proof. We prove the result only for the ask side. The proof is similar for the bid side.

Let $i \in \{1, \dots, d\}$. For $p \in \mathbb{R}$, we define

$$h_p^i : \delta \in \mathbb{R} \mapsto \Lambda^{i,a}(\delta)(\delta - p).$$

h_p^i is a continuously differentiable function, positive for $\delta \in (p, +\infty)$ and nonpositive otherwise. It is easy to prove (see [62]) that there is a unique maximizer $\tilde{\delta}^{i,a^*}(p)$ of h_p^i on \mathbb{R} characterized by

$$p = \tilde{\delta}^{i,a^*}(p) + \frac{\Lambda^{i,a}(\tilde{\delta}^{i,a^*}(p))}{\Lambda^{i,a'}(\tilde{\delta}^{i,a^*}(p))}.$$

By the implicit function theorem, $p \in \mathbb{R} \mapsto \tilde{\delta}^{i,a^*}(p)$ is continuously differentiable and

$$\tilde{\delta}^{i,a^*'}(p) = \frac{1}{2 - \frac{\Lambda^{i,a}(\tilde{\delta}^{i,a^*}(p))\Lambda^{i,a'}(\tilde{\delta}^{i,a^*}(p))}{(\Lambda^{i,a'}(\tilde{\delta}^{i,a^*}(p)))^2}} > 0, \forall p \in \mathbb{R}.$$

In particular, $\tilde{\delta}^{i,a^*}$ is increasing.

We introduce $\tilde{H}^{i,a} : p \in \mathbb{R} \rightarrow \sup_{\delta \in \mathbb{R}} h_p^i(\delta)$. Then $\forall p \in \mathbb{R}$, we have $\tilde{H}^{i,a}(p) = h_p^i(\tilde{\delta}^{i,a^*}(p))$ and

$$\tilde{H}^{i,a'}(p) = -\Lambda^{i,a}(\tilde{\delta}^{i,a^*}(p)) < 0.$$

So $\tilde{H}^{i,a}$ is decreasing and

$$\tilde{\delta}^{i,a^*}(p) = (\Lambda^{i,a})^{-1} \left(-\tilde{H}^{i,a'}(p) \right).$$

Let us now recall that $\forall p \in \mathbb{R}$, $H^{i,a}(p) = \sup_{\delta \geq -\delta_\infty} h_p^i(\delta)$.

For all $p \in \mathbb{R}$ such that $-\delta_\infty \leq \tilde{\delta}^{i,a^*}(p)$, we clearly have

$$H^{i,a}(p) = h_p^i(\tilde{\delta}^{i,a^*}(p)).$$

Otherwise, if $\tilde{\delta}^{i,a^*}(p) < -\delta_\infty$, we can easily see that $h_p^i(\cdot)$ is increasing on $] -\infty, \tilde{\delta}^{i,a^*}(p)]$ and decreasing on $[\tilde{\delta}^{i,a^*}(p), +\infty[$, which implies

$$H^{i,a}(p) = h_p^i(-\delta_\infty).$$

This means that the supremum in $H^{i,a}(p)$ is reached at a unique $\delta^{i,a^*}(p)$ given by

$$\delta^{i,a^*}(p) = \max(\tilde{\delta}^{i,a^*}(p), -\delta_\infty).$$

In particular, δ^{i,a^*} is continuous and nondecreasing, so $H^{i,a}$ is continuous. Moreover, for all $p \in \mathbb{R}$ such that $\tilde{\delta}^{i,a^*}(p) > -\delta_\infty$, we have $H^{i,a}(p) = \tilde{H}^{i,a}(p)$ so $H^{i,a}$ is decreasing on $]\tilde{\delta}^{i,a^*-1}(-\delta_\infty), +\infty[$ and its derivative on this interval is

$$H^{i,a'}(p) = -\Lambda^{i,a}(\tilde{\delta}^{i,a^*}(p)) = -\Lambda^{i,a}(\delta^{i,a^*}(p)).$$

On $]-\infty, \tilde{\delta}^{i,a^*-1}(-\delta_\infty)[$, $H^{i,a}$ is affine and its derivative is

$$H^{i,a'}(p) = -\Lambda^{i,a}(-\delta_\infty) = -\Lambda^{i,a}(\delta^{i,a^*}(p)).$$

Thus, by continuity of δ^{i,a^*} , $H^{i,a}$ is continuously differentiable and decreasing on \mathbb{R} . In particular, $|H^{i,a'}(p)| \leq \Lambda^{i,a}(-\delta_\infty)$ for all $p \in \mathbb{R}$, so $H^{i,a}$ is Lipschitz. □

In what follows, we denote by $L^{i,a}$ the Lipschitz constant of $H^{i,a}$ for all $i \in \{1, \dots, d\}$, and we define similarly $L^{i,b}$ the Lipschitz constant of $H^{i,b}$ for all $i \in \{1, \dots, d\}$.

For $\pi \in C^0(\mathbb{R}^d, \mathbb{R}_+)$, let us consider \mathcal{C}_π the following vector space:

$$\mathcal{C}_\pi = \left\{ u \in C^0(\mathbb{R}^d, \mathbb{R}) \left| \sup_{q \in \mathbb{R}^d} \left| \frac{u(q)}{1 + \pi(q)} \right| < +\infty \right. \right\}.$$

Equipped with the norm $u \in \mathcal{C}_\pi \mapsto \|u\|_\pi = \sup_{q \in \mathbb{R}^d} \left| \frac{u(q)}{1 + \pi(q)} \right|$, \mathcal{C}_π is a Banach space.

We now consider for the rest of the paper that there exists $p \in \mathbb{N}^*$ and $C > 0$ such that:

- $\forall q \in \mathbb{R}^d, \pi(q) \leq C(1 + \|q\|^p),$
- $\forall q, y \in \mathbb{R}^d, \frac{1 + \pi(q+y)}{1 + \pi(q)} \leq C(1 + \|y\|^p),$
- $\forall i \in \{1, \dots, d\}, \int_{\mathbb{R}_+^*} (z^p \mu^{i,b}(dz) + z^p \mu^{i,a}(dz)) < +\infty,$

where $\|\cdot\|$ denotes the Euclidean norm on \mathbb{R}^d .

Moreover, we assume that $\psi, \ell_d \in \mathcal{C}_\pi$.⁵

Remark 3. For the examples of Remark 2, it is natural to choose a quadratic function π such that $\psi, \ell_d \leq \pi$. Then, the above assumptions are satisfied for $p = 2$ whenever $\mu^{i,b}$ and $\mu^{i,a}$ have a finite second moment.

⁵This assumption implies in particular that ψ and ℓ_d have, at most, polynomial growth at infinity.

Proposition 1. For all $u \in \mathcal{C}_\pi$, the function

$$F(u) : q \in \mathbb{R}^d \mapsto \psi(q) - \sum_{i=1}^d \int_{\mathbb{R}_+^*} z H^{i,b} \left(\frac{u(q) - u(q + ze^i)}{z} \right) \mu^{i,b}(dz) - \sum_{i=1}^d \int_{\mathbb{R}_+^*} z H^{i,a} \left(\frac{u(q) - u(q - ze^i)}{z} \right) \mu^{i,a}(dz)$$

is in \mathcal{C}_π .

Proof. Let $u \in \mathcal{C}_\pi$.

Let us consider $q \in \mathbb{R}^d$ and a sequence $(q_n)_n$ converging towards q .

From the continuity of ψ , we have $\lim_{n \rightarrow +\infty} \psi(q_n) = \psi(q)$.

Also, $\forall i \in \{1, \dots, d\}, \forall z \in \mathbb{R}_+^*$, from the continuity of $H^{i,b}$ and u , we have

$$\lim_{n \rightarrow +\infty} z H^{i,b} \left(\frac{u(q_n) - u(q_n + ze^i)}{z} \right) = z H^{i,b} \left(\frac{u(q) - u(q + ze^i)}{z} \right).$$

Now, we write $H^{i,b}(p) \leq H^{i,b}(0) + L^{i,b}|p|$ so that we get

$$\begin{aligned} z H^{i,b} \left(\frac{u(q_n) - u(q_n + ze^i)}{z} \right) &\leq z H^{i,b}(0) + L^{i,b} |u(q_n) - u(q_n + ze^i)| \\ &\leq z H^{i,b}(0) + L^{i,b} |u(q_n)| + CL^{i,b} \|u\|_\pi (1 + \pi(q_n)) (1 + z^p) \\ &\leq z H^{i,b}(0) + L^{i,b} \sup_n |u(q_n)| + CL^{i,b} \|u\|_\pi \left(1 + \sup_n \pi(q_n) \right) (1 + z^p), \end{aligned}$$

which is integrable by assumption. Using the same technique for the terms associated with the ask side and Lebesgue's dominated convergence theorem, we conclude that $\lim_{n \rightarrow +\infty} F(u)(q_n) = F(u)(q)$, hence the continuity of $F(u)$.

Moreover, for all $q \in \mathbb{R}^d$, we have

$$\begin{aligned}
\left| \frac{F(u)(q)}{1 + \pi(q)} \right| &= \left| \frac{\psi(q)}{1 + \pi(q)} - \sum_{i=1}^d \int_{\mathbb{R}_+^*} \frac{z}{1 + \pi(q)} H^{i,b} \left(\frac{u(q) - u(q + ze^i)}{z} \right) \mu^{i,b}(dz) \right. \\
&\quad \left. - \sum_{i=1}^d \int_{\mathbb{R}_+^*} \frac{z}{1 + \pi(q)} H^{i,a} \left(\frac{u(q) - u(q - ze^i)}{z} \right) \mu^{i,a}(dz) \right| \\
&\leq \|\psi\|_\pi + \sum_{i=1}^d \int_{\mathbb{R}_+^*} \frac{z}{1 + \pi(q)} H^{i,b} \left(\frac{u(q) - u(q + ze^i)}{z} \right) \mu^{i,b}(dz) \\
&\quad + \sum_{i=1}^d \int_{\mathbb{R}_+^*} \frac{z}{1 + \pi(q)} H^{i,a} \left(\frac{u(q) - u(q - ze^i)}{z} \right) \mu^{i,a}(dz) \\
&\leq \|\psi\|_\pi + \sum_{i=1}^d \int_{\mathbb{R}_+^*} \frac{1}{1 + \pi(q)} (zH^{i,b}(0) + L^{i,b} |u(q) - u(q + ze^i)|) \mu^{i,b}(dz) \\
&\quad + \sum_{i=1}^d \int_{\mathbb{R}_+^*} \frac{1}{1 + \pi(q)} (zH^{i,a}(0) + L^{i,a} |u(q) - u(q - ze^i)|) \mu^{i,a}(dz) \\
&\leq \|\psi\|_\pi + \sum_{i=1}^d \int_{\mathbb{R}_+^*} (zH^{i,b}(0) + L^{i,b} \|u\|_\pi + CL^{i,b} \|u\|_\pi (1 + z^p)) \mu^{i,b}(dz) \\
&\quad + \sum_{i=1}^d \int_{\mathbb{R}_+^*} (zH^{i,a}(0) + L^{i,a} \|u\|_\pi + CL^{i,a} \|u\|_\pi (1 + z^p)) \mu^{i,a}(dz).
\end{aligned}$$

We conclude that $\sup_{q \in \mathbb{R}^d} \left| \frac{F(u)(q)}{1 + \pi(q)} \right| < +\infty$ and therefore that $F(u) \in \mathcal{C}_\pi$.

□

We can therefore define a functional $F : \mathcal{C}_\pi \rightarrow \mathcal{C}_\pi$ such that, for all $u \in \mathcal{C}_\pi$ and for all $q \in \mathbb{R}^d$,

$$F(u)(q) = \psi(q) - \sum_{i=1}^d \int_{\mathbb{R}_+^*} zH^{i,b} \left(\frac{u(q) - u(q + ze^i)}{z} \right) \mu^{i,b}(dz) - \sum_{i=1}^d \int_{\mathbb{R}_+^*} zH^{i,a} \left(\frac{u(q) - u(q - ze^i)}{z} \right) \mu^{i,a}(dz).$$

We now come to the main property of the function F .

Proposition 2. *F is Lipschitz on \mathcal{C}_π .*

Proof. Let $u, v \in \mathcal{C}_\pi$. For all $q \in \mathbb{R}^d$, we have

$$\begin{aligned} |F(u)(q) - F(v)(q)| &\leq \sum_{i=1}^d \int_{\mathbb{R}_+^*} z \left| H^{i,b} \left(\frac{v(q) - v(q + ze^i)}{z} \right) - H^{i,b} \left(\frac{u(q) - u(q + ze^i)}{z} \right) \right| \mu^{i,b}(dz) \\ &\quad + \sum_{i=1}^d \int_{\mathbb{R}_+^*} z \left| H^{i,a} \left(\frac{v(q) - v(q - ze^i)}{z} \right) - H^{i,a} \left(\frac{u(q) - u(q - ze^i)}{z} \right) \right| \mu^{i,a}(dz). \end{aligned}$$

Therefore

$$\begin{aligned} |F(u)(q) - F(v)(q)| &\leq \sum_{i=1}^d \int_{\mathbb{R}_+^*} L^{i,b} |v(q) - v(q + ze^i) - u(q) + u(q + ze^i)| \mu^{i,b}(dz) \\ &\quad + \sum_{i=1}^d \int_{\mathbb{R}_+^*} L^{i,a} |v(q) - v(q - ze^i) - u(q) + u(q - ze^i)| \mu^{i,a}(dz). \\ &\leq \sum_{i=1}^d \int_{\mathbb{R}_+^*} L^{i,b} |v(q) - u(q)| \mu^{i,b}(dz) + \sum_{i=1}^d \int_{\mathbb{R}_+^*} L^{i,b} |v(q + ze^i) - u(q + ze^i)| \mu^{i,b}(dz) \\ &\quad + \sum_{i=1}^d \int_{\mathbb{R}_+^*} L^{i,a} |v(q) - u(q)| \mu^{i,a}(dz) + \sum_{i=1}^d \int_{\mathbb{R}_+^*} L^{i,a} |v(q - ze^i) - u(q - ze^i)| \mu^{i,a}(dz). \end{aligned}$$

We obtain therefore,

$$\begin{aligned} \frac{|F(u)(q) - F(v)(q)|}{1 + \pi(q)} &\leq \sum_{i=1}^d \int_{\mathbb{R}_+^*} L^{i,b} \|u - v\|_\pi \mu^{i,b}(dz) + \sum_{i=1}^d \int_{\mathbb{R}_+^*} CL^{i,b} \|u - v\|_\pi (1 + z^p) \mu^{i,b}(dz) \\ &\quad + \sum_{i=1}^d \int_{\mathbb{R}_+^*} L^{i,a} \|u - v\|_\pi \mu^{i,a}(dz) + \sum_{i=1}^d \int_{\mathbb{R}_+^*} CL^{i,a} \|u - v\|_\pi (1 + z^p) \mu^{i,a}(dz). \end{aligned}$$

By taking the supremum over q , we get that there exists a constant $K > 0$ such

$$\|F(u) - F(v)\|_\pi \leq K \|u - v\|_\pi.$$

We conclude that F is Lipschitz continuous. □

The Lipschitz property of F allows to obtain the following existence and uniqueness theorem:

Theorem 1. *There exists a unique function $\mathcal{W} \in C^1([0, T], \mathcal{C}_\pi)$ such that $w : (t, q) \in [0, T] \times \mathbb{R}^d \mapsto \mathcal{W}(t)(q)$ is solution to (1.2) with terminal condition $w(T, q) = -\ell_d(q), \forall q \in \mathbb{R}^d$.*

Proof. Let us observe that $\mathcal{W} \in C^1([0, T], \mathcal{C}_\pi)$ is solution of the Cauchy problem

$$\begin{cases} \mathcal{W}'(t) = F(\mathcal{W}(t)), \forall t \in [0, T] \\ \mathcal{W}(T) = -\ell_d \end{cases}$$

if and only if $w : (t, q) \in [0, T] \times \mathbb{R}^d \mapsto \mathcal{W}(t)(q)$ is solution to (1.2) with terminal condition $w(T, q) = -\ell_d(q), \forall q \in \mathbb{R}^d$.

As $(\mathcal{C}_\pi, \|\cdot\|_\pi)$ is a Banach space and $F : \mathcal{C}_\pi \rightarrow \mathcal{C}_\pi$ is Lipschitz continuous, we know by Cauchy-Lipschitz theorem that there exists a unique maximal solution \mathcal{W} to the above equation, and that this solution is in fact global, meaning in particular that \mathcal{W} is defined on $[0, T]$. \square

1.2.3 Verification theorem

We now want to prove that θ is in fact the function w defined in Theorem 1 and deduce the optimal controls associated with the problem (1.1) using a verification argument.

Theorem 2. *Let w be the function defined in Theorem 1.*

Let $(t, q) \in [0, T] \times \mathbb{R}^d$.

Let us define $(\bar{\delta}^i)_{i \in \{1, \dots, d\}} = (\bar{\delta}^{i,b}, \bar{\delta}^{i,a})_{i \in \{1, \dots, d\}} \in \mathcal{A}$ such that $\forall i \in \{1, \dots, d\}, \forall s \in [t, T], \forall z > 0$:

$$\begin{aligned}\bar{\delta}^{i,b}(s, z) &= \delta^{i,b*} \left(\frac{w(s, q_{s-}) - w(s, q_{s-} + ze^i)}{z} \right), \\ \bar{\delta}^{i,a}(s, z) &= \delta^{i,a*} \left(\frac{w(s, q_{s-}) - w(s, q_{s-} - ze^i)}{z} \right),\end{aligned}$$

where $\delta^{i,b*}$ and $\delta^{i,a*}$ are the functions defined in Lemma 1 and $(q_s)_{t \leq s \leq T} = (q_s^{t,q,(\bar{\delta}^1, \dots, \bar{\delta}^d)})_{t \leq s \leq T}$.

Then, $\theta(t, q) = w(t, q)$ and $(\bar{\delta}^1, \dots, \bar{\delta}^d)$ is an optimal control for our stochastic control problem starting at time t with $q_t = q$.

Proof. Let $(\delta^i)_{i \in \{1, \dots, d\}} = (\delta^{i,b}, \delta^{i,a})_{i \in \{1, \dots, d\}} \in \mathcal{A}$ be an arbitrary control and let us denote by $(q_s)_{s \in [t, T]}$ the process $(q_s^{t,q,(\delta^1, \dots, \delta^d)})_{s \in [t, T]}$.

Let us first prove that for all $i \in \{1, \dots, d\}$,

$$\mathbb{E} \left[\int_t^T \int_{\mathbb{R}_+^*} |w(s, q_{s-} + ze^i) - w(s, q_{s-})| \Lambda^{i,b}(\delta_s^{i,b}) \mu^{i,b}(dz) ds \right] < +\infty.$$

Denoting by M^w the quantity $\sup_{t \in [0, T]} \|w(t, \cdot)\|_\pi$, we have

$$\begin{aligned}
& \mathbb{E} \left[\int_t^T \int_{\mathbb{R}_+^*} |w(s, q_{s-} + ze^i) - w(s, q_{s-})| \Lambda^{i,b}(\delta_s^{i,b}) \mu^{i,b}(dz) ds \right] \\
& \leq \Lambda^{i,b}(-\delta_\infty) \mathbb{E} \left[\int_t^T \int_{\mathbb{R}_+^*} (|w(s, q_{s-} + ze^i)| + |w(s, q_{s-})|) \mu^{i,b}(dz) ds \right] \\
& \leq \Lambda^{i,b}(-\delta_\infty) M^w \mathbb{E} \left[\int_t^T \int_{\mathbb{R}_+^*} (1 + \pi(q_{s-} + ze^i) + 1 + \pi(q_{s-})) \mu^{i,b}(dz) ds \right] \\
& \leq \Lambda^{i,b}(-\delta_\infty) M^w \mathbb{E} \left[\int_t^T \int_{\mathbb{R}_+^*} (C(1+z^p)(1+\pi(q_{s-})) + 1 + \pi(q_{s-})) \mu^{i,b}(dz) ds \right].
\end{aligned}$$

Therefore

$$\begin{aligned}
& \mathbb{E} \left[\int_t^T \int_{\mathbb{R}_+^*} |w(s, q_{s-} + ze^i) - w(s, q_{s-})| \Lambda^{i,b}(\delta_s^{i,b}) \mu^{i,b}(dz) ds \right] \\
& \leq \Lambda^{i,b}(-\delta_\infty) M^w \mathbb{E} \left[\int_t^T \int_{\mathbb{R}_+^*} (C(1+z^p)(1+C(1+\|q_{s-}\|^p)) + 1 + C(1+\|q_{s-}\|^p)) \mu^{i,b}(dz) ds \right].
\end{aligned}$$

Subsequently, we just have to prove that

$$\mathbb{E} \left[\int_t^T \|q_{s-}\|^p ds \right] < +\infty.$$

Since $\|q_s\| \leq \|q\| + \|q_s - q\|$, $\|q_s\|^p \leq 2^{p-1} (\|q\|^p + \|q_s - q\|^p)$, and we need to prove that

$$\mathbb{E} \left[\int_t^T \|q_{s-} - q\|^p ds \right] < +\infty.$$

As we are working in \mathbb{R}^d , it is equivalent to prove that

$$\mathbb{E} \left[\int_t^T \|q_{s-} - q\|_p^p ds \right] < +\infty,$$

where $\|(x_1, \dots, x_d)'\|_p = \left(\sum_{i=1}^d |x_i|^p \right)^{1/p}$.

For that purpose, we introduce for each $j \in \{1, \dots, d\}$, two independent Poisson processes $N^{j,b}$ and $N^{j,a}$ with respective intensities $\Lambda^{j,b}(-\delta_\infty)$ and $\Lambda^{j,a}(-\delta_\infty)$, and $(\xi_k^{j,b})_{k \geq 1}$ and $(\xi_k^{j,a})_{k \geq 1}$ two

sequences of i.i.d. random variables with respective distributions $\mu^{j,b}$ and $\mu^{j,a}$. Then, we have

$$\begin{aligned}
\mathbb{E} \left[\int_t^T \|q_{s-} - q\|_p^p ds \right] &= \mathbb{E} \left[\int_t^T \sum_{j=1}^d \left| \int_{\mathbb{R}_+^*} z J^{j,b}(dt, dz) - \int_{\mathbb{R}_+^*} z J^{j,a}(dt, dz) \right|^p ds \right] \\
&\leq \mathbb{E} \left[\int_t^T \sum_{j=1}^d \left(\int_{\mathbb{R}_+^*} z J^{j,b}(dt, dz) + \int_{\mathbb{R}_+^*} z J^{j,a}(dt, dz) \right)^p \right] \\
&\leq \mathbb{E} \left[\int_t^T \sum_{j=1}^d \left(\sum_{k=1}^{N_s^{j,b}} \xi_k^{j,b} + \sum_{k=1}^{N_s^{j,a}} \xi_k^{j,a} \right)^p ds \right] \\
&\leq 2^{p-1} \mathbb{E} \left[\int_t^T \sum_{j=1}^d \left(\left(\sum_{k=1}^{N_s^{j,b}} \xi_k^{j,b} \right)^p + \left(\sum_{k=1}^{N_s^{j,a}} \xi_k^{j,a} \right)^p \right) ds \right] \\
&\leq 2^{p-1} \int_t^T \sum_{j=1}^d \left(\mathbb{E} \left[(N_s^{j,b})^{p-1} \sum_{k=1}^{N_s^{j,b}} (\xi_k^{j,b})^p \right] + \mathbb{E} \left[(N_s^{j,a})^{p-1} \sum_{k=1}^{N_s^{j,a}} (\xi_k^{j,a})^p \right] \right) ds \\
&\leq 2^{p-1} \int_t^T \sum_{j=1}^d \left(\mathbb{E} \left[(N_T^{j,b})^p \right] \mathbb{E} \left[(\xi_1^{j,b})^p \right] + \mathbb{E} \left[(N_T^{j,a})^p \right] \mathbb{E} \left[(\xi_1^{j,a})^p \right] \right) ds \\
&\leq 2^{p-1} T \sum_{j=1}^d \left(\mathbb{E} \left[(N_T^{j,b})^p \right] \int_{\mathbb{R}_+^*} z^p \mu^{j,b}(dz) + \mathbb{E} \left[(N_T^{j,a})^p \right] \int_{\mathbb{R}_+^*} z^p \mu^{j,a}(dz) \right) < +\infty.
\end{aligned}$$

Using the above, we have, for all $i \in \{1, \dots, d\}$,

$$\begin{aligned}
&\mathbb{E} \left[\int_t^T \int_{\mathbb{R}_+^*} (w(s, q_{s-} + ze^i) - w(s, q_{s-})) J^{i,b}(ds, dz) \right] \\
&= \mathbb{E} \left[\int_t^T \int_{\mathbb{R}_+^*} (w(s, q_{s-} + ze^i) - w(s, q_{s-})) \Lambda^{i,b}(\delta_s^{i,b}) \mu^{i,b}(dz) ds \right],
\end{aligned}$$

Of course, we can similarly prove that, for all $i \in \{1, \dots, d\}$,

$$\begin{aligned}
&\mathbb{E} \left[\int_t^T \int_{\mathbb{R}_+^*} (w(s, q_{s-} - ze^i) - w(s, q_{s-})) J^{i,a}(ds, dz) \right] \\
&= \mathbb{E} \left[\int_t^T \int_{\mathbb{R}_+^*} (w(s, q_{s-} - ze^i) - w(s, q_{s-})) \Lambda^{i,a}(\delta_s^{i,a}) \mu^{i,a}(dz) ds \right].
\end{aligned}$$

Now, by applying Itô's formula, we get

$$\begin{aligned} w(T, q_T) &= w(t, q) + \int_t^T \frac{\partial w}{\partial t}(s, q_s) ds + \sum_{i=1}^d \int_t^T \int_{\mathbb{R}_+^*} (w(s, q_{s-} + ze^i) - w(s, q_{s-})) J^{i,b}(ds, dz) \\ &\quad + \sum_{i=1}^d \int_t^T \int_{\mathbb{R}_+^*} (w(s, q_{s-} - ze^i) - w(s, q_{s-})) J^{i,a}(ds, dz). \end{aligned}$$

By taking expectation, we get

$$\begin{aligned} \mathbb{E}[w(T, q_T)] &= w(t, q) + \mathbb{E} \left[\int_t^T \left\{ \frac{\partial w}{\partial t}(s, q_s) + \sum_{i=1}^d \int_{\mathbb{R}_+^*} \Lambda^{i,b}(\delta^{i,b}(s, z)) (w(s, q_{s-} + ze^i) - w(s, q_{s-})) \mu^{i,b}(dz) \right. \right. \\ &\quad \left. \left. + \sum_{i=1}^d \int_{\mathbb{R}_+^*} \Lambda^{i,a}(\delta^{i,a}(s, z)) (w(s, q_{s-} - ze^i) - w(s, q_{s-})) \mu^{i,a}(dz) \right\} ds \right], \end{aligned}$$

which, by definition of w , gives us the following inequality:

$$\begin{aligned} \mathbb{E}[-\ell_d(q_T)] &\leq w(t, q) + \mathbb{E} \left[\int_t^T \left\{ \psi(q_s) - \sum_{i=1}^d \int_{\mathbb{R}_+^*} z \Lambda^{i,b}(\delta^{i,b}(s, z)) \delta^{i,b}(s, z) \mu^{i,b}(dz) \right. \right. \\ &\quad \left. \left. - \sum_{i=1}^d \int_{\mathbb{R}_+^*} z \Lambda^{i,a}(\delta^{i,a}(s, z)) \delta^{i,a}(s, z) \mu^{i,a}(dz) \right\} ds \right], \end{aligned}$$

with equality when $(\delta^i)_{i \in \{1, \dots, d\}} = (\bar{\delta}^i)_{i \in \{1, \dots, d\}}$.

In other words,

$$\begin{aligned} \mathbb{E} \left[\int_t^T \left\{ \sum_{i=1}^d \int_{\mathbb{R}_+^*} (z \Lambda^{i,b}(\delta^{i,b}(s, z)) \delta^{i,b}(s, z) \mu^{i,b}(dz) + z \Lambda^{i,a}(\delta^{i,a}(s, z)) \delta^{i,a}(s, z) \mu^{i,a}(dz)) \right. \right. \\ \left. \left. - \psi(q_s) \right\} ds - \ell_d(q_T) \right] \leq w(t, q), \end{aligned}$$

with equality when $(\delta^i)_{i \in \{1, \dots, d\}} = (\bar{\delta}^i)_{i \in \{1, \dots, d\}}$.

By taking the supremum over $(\delta^i)_{i \in \{1, \dots, d\}} \in \mathcal{A}$, we get $\theta(t, q) = w(t, q)$ and the fact that $(\bar{\delta}^i)_{i \in \{1, \dots, d\}}$ is optimal. \square

1.3 Solving the multi-asset market making problem with factors

Let us now consider the particular case of problem (1.1) where $\forall q \in \mathbb{R}^d$, $\psi(q) = \bar{\psi}(q' \Sigma q)$ and $\ell_d(q) = \bar{\ell}_d(q' \Sigma q)$ for some continuous functions $\bar{\psi}$ and $\bar{\ell}_d$ with, at most, polynomial growth at in-

finitly. This particular case covers the examples of the literature (see Remark 2).

If the prices of the d assets are modeled using a small number k of factors, as it is the case in most econometric models of financial asset prices, then the variance-covariance matrix Σ takes the form

$$\Sigma = \beta V \beta' + R,$$

where β is a d -by- k matrix of real coefficients, V the k -by- k variance-covariance matrix of the factors, and R the d -by- d variance-covariance matrix of the residuals.

If the explanatory power of the factors is high, R should be small compared to Σ (in Frobenius norm for instance). Our approach consists in ignoring the residuals, i.e. setting R to 0. In other words, we project the market risk on a space of factors of dimension k . As we shall see in Section 1.4, this approach provides very good results as measured by the objective function (1.1).

In what follows, we also discuss an approximation method based on Monte-Carlo simulations to account for the influence of R once one has computed the optimal quotes in the case with no residual risk. The advantages and drawbacks of this additional approximation method will be discussed in Section 1.4.

1.3.1 A low-dimensional approximation

Let us now assume that $\Sigma = \beta V \beta'$, i.e. $R = 0$. Under this assumption, we can write problem (1.1) as the maximization of

$$\mathbb{E} \left[X_T + \sum_{i=1}^d q_T^i S_T^i - \bar{\ell}_d ((\beta' q_T)' V (\beta' q_T)) - \int_0^T \bar{\psi} ((\beta' q_t)' V (\beta' q_t)) dt \right]. \quad (1.3)$$

Using the same ideas as in Section 1.2, this expression can be written as

$$\mathbb{E} \left[\int_0^T \left\{ \int_{\mathbb{R}_+^*} \sum_{i=1}^d (\delta^{i,b}(t, z) z \Lambda^{i,b}(\delta^{i,b}(t, z))) \mu^{i,b}(dz) + \delta^{i,a}(t, z) z \Lambda^{i,a}(\delta^{i,a}(t, z)) \mu^{i,a}(dz) - \bar{\psi} ((\beta' q_t)' V (\beta' q_t)) \right\} dt - \bar{\ell}_d ((\beta' q_T)' V (\beta' q_T)) \right].$$

Let us introduce $(f_t)_{t \in [0, T]} = (\beta' q_t)_{t \in [0, T]}$. Then, the problem of maximizing (1.3) is equivalent to that of maximizing

$$\mathbb{E} \left[\int_0^T \left\{ \int_{\mathbb{R}_+^*} \sum_{i=1}^d (\delta^{i,b}(t, z) z \Lambda^{i,b}(\delta^{i,b}(t, z))) \mu^{i,b}(dz) + \delta^{i,a}(t, z) z \Lambda^{i,a}(\delta^{i,a}(t, z)) \mu^{i,a}(dz) - \bar{\psi} (f_t' V f_t) \right\} dt - \bar{\ell}_d (f_T' V f_T) \right].$$

The state process of our problem is now the Markov process $(f_t)_{t \in [0, T]}$ instead of $(q_t)_{t \in [0, T]}$: we have reduced the dimension of the problem from d to k .

Let us introduce $\tilde{\mathcal{J}} : [0, T] \times \mathbb{R}^k \times \mathcal{A} \rightarrow \mathbb{R}$ such that, $\forall t \in [0, T]$, $\forall f = (f^1, \dots, f^k)' \in \mathbb{R}^k$, $\forall (\delta^i)_{i \in \{1, \dots, d\}} \in \mathcal{A}$

$$\begin{aligned} \tilde{\mathcal{J}}(t, f, (\delta^i)_{i \in \{1, \dots, d\}}) &= \mathbb{E} \left[\int_t^T \left\{ \int_{\mathbb{R}_+^*} \sum_{i=1}^d \left(\delta^{i,b}(s, z) z \Lambda^{i,b}(\delta^{i,b}(s, z)) \mu^{i,b}(dz) + \delta^{i,a}(s, z) z \Lambda^{i,a}(\delta^{i,a}(s, z)) \mu^{i,a}(dz) \right) \right. \right. \\ &\quad \left. \left. - \bar{\psi}(f'_s V f_s) \right\} ds - \bar{\ell}_d(f'_T V f_T) \right], \end{aligned}$$

where $(f_s)_{s \in [t, T]} = (f_s^{t, f, (\delta^i)_{i \in \{1, \dots, d\}}})_{s \in [t, T]}$ is here the state process starting in state f at time t and controlled by $(\delta^i)_{i \in \{1, \dots, d\}}$.

The value function $\tilde{\theta} : [0, T] \times \mathbb{R}^k \rightarrow \mathbb{R}$ of the problem is then defined as follows:

$$\tilde{\theta}(t, f) = \sup_{(\delta^i)_{i \in \{1, \dots, d\}} \in \mathcal{A}} \tilde{\mathcal{J}}(t, f, (\delta^i)_{i \in \{1, \dots, d\}}), \quad \forall (t, f) \in [0, T] \times \mathbb{R}^k.$$

By using the same arguments as in Section 1.2, we can show that $\tilde{\theta}$ is the unique (in a large class of functions) smooth solution to the following integro-differential Hamilton-Jacobi equation:

$$\begin{aligned} 0 &= \frac{\partial \tilde{\theta}}{\partial t}(t, f) - \bar{\psi}(f'_s V f_s) + \sum_{i=1}^d \int_{\mathbb{R}_+^*} z H^{i,b} \left(\frac{\tilde{\theta}(t, f) - \tilde{\theta}(t, f + z \tilde{e}^i)}{z} \right) \mu^{i,b}(dz) \\ &\quad + \sum_{i=1}^d \int_{\mathbb{R}_+^*} z H^{i,a} \left(\frac{\tilde{\theta}(t, f) - \tilde{\theta}(t, f - z \tilde{e}^i)}{z} \right) \mu^{i,a}(dz), \quad \forall (t, f) \in [0, T] \times \mathbb{R}^k, \end{aligned} \tag{1.4}$$

with terminal condition $\tilde{\theta}(T, f) = -\bar{\ell}_d(f'_T V f_T)$, $\forall f \in \mathbb{R}^k$, where $\forall i \in \{1, \dots, d\}$, $\tilde{e}^i = \beta' e^i$.

Furthermore, the optimal controls are now given by:

$$\begin{aligned} \bar{\delta}^{i,b}(s, z) &= \delta^{i,b*} \left(\frac{\tilde{\theta}(s, f_{s-}) - \tilde{\theta}(s, f_{s-} + z \tilde{e}^i)}{z} \right), \\ \bar{\delta}^{i,a}(s, z) &= \delta^{i,a*} \left(\frac{\tilde{\theta}(s, f_{s-}) - \tilde{\theta}(s, f_{s-} - z \tilde{e}^i)}{z} \right). \end{aligned}$$

When $R = 0$, the problem boils down therefore to finding the solution $\tilde{\theta}$ of (1.4) with the appropriate terminal condition. In particular, from a numerical point of view, we need to approximate the solution of an equation involving time plus k space dimensions, and this is doable with grid methods if k is small.

1.3.2 A Monte-Carlo method to take the residual risk into account

As we shall see in Section 1.4, the above approximation method provides very good results as measured by the value of the objective function (1.1). Nevertheless, when market risk is projected on

a low-dimensional space of factors, there are linear combinations of assets that falsely appear to be risk-free. To prevent trajectories of the inventory visiting too often regions that are falsely associated with low risk, it makes sense to look for methods that account for the residual risk measured by the matrix R .

In what follows, we propose an approximation method to take the residual risk into account. The idea consists in considering the first-order expansion in ε where

$$\Sigma = \beta V \beta' + \varepsilon R.$$

The rationale behind this idea is that, for a factor model with high explanatory power, R should be small and it makes sense therefore to use a perturbative approach.

When $\varepsilon = 0$, we know how to solve the problem, and the value function is given by $\tilde{\theta}$. To approximate the value function θ of the problem for $\varepsilon > 0$, we consider a first-order expansion of the form

$$\theta(t, q) = \tilde{\theta}(t, \beta' q) + \varepsilon \eta(t, q) + o(\varepsilon), \quad \forall (t, q) \in [0, T] \times \mathbb{R}^d.$$

By plugging this expression into equation (1.2), we formally get

$$\begin{aligned} 0 &= \frac{\partial \tilde{\theta}}{\partial t}(t, \beta' q) + \varepsilon \frac{\partial \eta}{\partial t}(t, q) + o(\varepsilon) - \bar{\psi} ((\beta' q)' V(\beta' q) + \varepsilon q' R q) \\ &+ \sum_{i=1}^d \int_{\mathbb{R}_+^*} z H^{i,b} \left(\frac{\tilde{\theta}(t, \beta' q) - \tilde{\theta}(t, \beta' q + z \tilde{e}^i)}{z} + \varepsilon \frac{\eta(t, q) - \eta(t, q + z e^i)}{z} + o(\varepsilon) \right) \mu^{i,b}(dz) \\ &+ \sum_{i=1}^d \int_{\mathbb{R}_+^*} z H^{i,a} \left(\frac{\tilde{\theta}(t, \beta' q) - \tilde{\theta}(t, \beta' q - z \tilde{e}^i)}{z} + \varepsilon \frac{\eta(t, q) - \eta(t, q - z e^i)}{z} + o(\varepsilon) \right) \mu^{i,a}(dz), \end{aligned}$$

and

$$\tilde{\theta}(T, \beta' q) + \varepsilon \eta(T, q) + o(\varepsilon) = -\bar{\ell}_d ((\beta' q)' V(\beta' q) + \varepsilon q' R q).$$

Assuming that $\bar{\psi}$ and $\bar{\ell}_d$ are C^1 and performing a Taylor expansion, we obtain

$$\begin{aligned} 0 &= \frac{\partial \tilde{\theta}}{\partial t}(t, \beta' q) + \varepsilon \frac{\partial \eta}{\partial t}(t, q) - \bar{\psi} ((\beta' q)' V(\beta' q)) - \varepsilon \bar{\psi}' ((\beta' q)' V(\beta' q)) q' R q \\ &+ \sum_{i=1}^d \int_{\mathbb{R}_+^*} z H^{i,b} \left(\frac{\tilde{\theta}(t, \beta' q) - \tilde{\theta}(t, \beta' q + z \tilde{e}^i)}{z} \right) \mu^{i,b}(dz) \\ &+ \varepsilon \sum_{i=1}^d \int_{\mathbb{R}_+^*} H^{i,b'} \left(\frac{\tilde{\theta}(t, \beta' q) - \tilde{\theta}(t, \beta' q + z \tilde{e}^i)}{z} \right) (\eta(t, q) - \eta(t, q + z e^i)) \mu^{i,b}(dz) \\ &+ \sum_{i=1}^d \int_{\mathbb{R}_+^*} z H^{i,a} \left(\frac{\tilde{\theta}(t, \beta' q) - \tilde{\theta}(t, \beta' q - z \tilde{e}^i)}{z} \right) \\ &+ \varepsilon \sum_{i=1}^d \int_{\mathbb{R}_+^*} H^{i,a'} \left(\frac{\tilde{\theta}(t, \beta' q) - \tilde{\theta}(t, \beta' q - z \tilde{e}^i)}{z} \right) (\eta(t, q) - \eta(t, q - z e^i)) \mu^{i,a}(dz) + o(\varepsilon), \end{aligned}$$

and

$$\tilde{\theta}(T, \beta'q) + \varepsilon\eta(T, q) + o(\varepsilon) = -\bar{\ell}'_d((\beta'q)'V(\beta'q)) - \varepsilon\bar{\ell}'_d((\beta'q)'V(\beta'q))q'Rq + o(\varepsilon).$$

As $\tilde{\theta}$ verifies (1.4), we get

$$\begin{aligned} 0 &= \frac{\partial\eta}{\partial t}(t, q) - \bar{\psi}'((\beta'q)'V(\beta'q))q'Rq + \sum_{i=1}^d \int_{\mathbb{R}_+^*} H^{i,b'} \left(\frac{\tilde{\theta}(t, \beta'q) - \tilde{\theta}(t, \beta'q + z\tilde{e}^i)}{z} \right) (\eta(t, q) - \eta(t, q + ze^i)) \mu^{i,b}(dz) \\ &\quad + \sum_{i=1}^d \int_{\mathbb{R}_+^*} H^{i,a'} \left(\frac{\tilde{\theta}(t, \beta'q) - \tilde{\theta}(t, \beta'q - z\tilde{e}^i)}{z} \right) (\eta(t, q) - \eta(t, q - ze^i)) \mu^{i,a}(dz). \end{aligned}$$

and $\eta(T, q) = -\bar{\ell}'_d((\beta'q)'V(\beta'q))q'Rq$.

This equation, although in space-dimension d , is linear. Therefore, by the Feynman-Kac representation theorem, we have the following formula:

$$\eta(t, q) = \mathbb{E}^{\tilde{\mathbb{P}}} \left[- \int_t^T \bar{\psi}'((\beta'q_s)'V(\beta'q_s))q'_s Rq_s ds - \bar{\ell}'_d((\beta'q_T)'V(\beta'q_T))q'_T Rq_T \Big| q_t = q \right],$$

where under $\tilde{\mathbb{P}}$, for all $i \in \{1, \dots, d\}$, $J^{i,b}$ and $J^{i,a}$ have their respective intensity kernels given by

$$\begin{aligned} \tilde{\nu}_t^{i,b}(dz) &= -H^{i,b'} \left(\frac{\tilde{\theta}(t, \beta'q_{t-}) - \tilde{\theta}(t, \beta'q_{t-} + z\tilde{e}^i)}{z} \right) \mu^{i,b}(dz), \\ \tilde{\nu}_t^{i,a}(dz) &= -H^{i,a'} \left(\frac{\tilde{\theta}(t, \beta'q_{t-}) - \tilde{\theta}(t, \beta'q_{t-} - z\tilde{e}^i)}{z} \right) \mu^{i,a}(dz). \end{aligned}$$

Remark 4. *It is noteworthy that the dynamics of $(q_s)_{s \in [t, T]}$ under $\tilde{\mathbb{P}}$ is that associated with the use of the optimal quotes when $R = 0$.*

Thanks to this probabilistic representation, we can easily compute $\eta(t, q)$ for a given time t and inventory q using a Monte-Carlo method, and therefore easily compute both an approximation of the value function and an approximation of the optimal quotes that account, to the first order, for the residual risk. Of course, in practice, it would be prohibitively expensive in terms of computation time to carry out a Monte-Carlo simulation for all possible values of the inventory, but this method can alternatively be used (online) upon receiving a request for quote for a specific asset and given the current time and inventory (this will be discussed in Section 1.4.1).

Remark 5. *In the computation of the optimal quotes associated with asset i , one relies on the approximation*

$$\theta(t, q_{t-}) - \theta(t, q_{t-} \pm ze^i) \simeq \tilde{\theta}(t, \beta'q_{t-}) - \tilde{\theta}(t, \beta'q_{t-} \pm z\tilde{e}^i) + \eta(t, q_{t-}) - \eta(t, q_{t-} \pm ze^i).$$

To compute $\eta(t, q_{t-}) - \eta(t, q_{t-} \pm ze^i)$, the same sample paths should be used in the estimations of $\eta(t, q_{t-})$ and $\eta(t, q_{t-} \pm ze^i)$. This is the same remark as for the computation of the Greeks of derivatives contracts with Monte-Carlo techniques.

1.4 Numerical results

1.4.1 The case of two assets: one factor vs. two factors

Model parameters

In this section, we apply our multi-asset market making model to the case of two highly-correlated assets (here bonds). Our goal is to show that, in this case, the reduced one-factor model gives very similar results to the complete two-factor model. For this purpose, we consider two assets with the following characteristics:

- Asset prices: $S_0^1 = S_0^2 = 100 \text{ €}$.
- Volatility of asset 1: $\sigma^1 = 1.2 \text{ €} \cdot \text{day}^{-\frac{1}{2}}$.
- Volatility of asset 2: $\sigma^2 = 0.6 \text{ €} \cdot \text{day}^{-\frac{1}{2}}$.
- Correlation: $\rho = 0.9$.
- Intensity functions:

$$\Lambda^{i,b}(\delta) = \Lambda^{i,a}(\delta) = \lambda_{RFQ} \frac{1}{1 + e^{\alpha_\Lambda + \beta_\Lambda \delta}}, \quad i \in \{1, 2\},$$

with $\lambda_{RFQ} = 30 \text{ day}^{-1}$, $\alpha_\Lambda = 0.7$, and $\beta_\Lambda = 30 \text{ €}^{-1}$. This corresponds to 30 RFQs per day for each asset, a probability of $\frac{1}{1+e^{0.7}} \simeq 33\%$ to trade when the answered quote is the reference price and a probability of $\frac{1}{1+e^{-0.2}} \simeq 55\%$ to trade when the answered quote is the reference price improved by 3 cents.

- Request sizes are distributed according to a Gamma distribution $\Gamma(\alpha_\mu, \beta_\mu)$ with $\alpha_\mu = 4$ and $\beta_\mu = 4 \cdot 10^{-4}$. This corresponds to an average request size of 10000 assets (i.e. approximately 1000000 €) and a standard deviation equal to half the average.

The variance-covariance matrix is therefore given by

$$\Sigma = \begin{bmatrix} 1.44 & 0.648 \\ 0.648 & 0.36 \end{bmatrix} = \Omega D \Omega' \simeq \begin{bmatrix} 0.906 & 0.424 \\ 0.424 & -0.906 \end{bmatrix} \begin{bmatrix} 1.744 & 0 \\ 0 & 0.056 \end{bmatrix} \begin{bmatrix} 0.906 & 0.424 \\ 0.424 & -0.906 \end{bmatrix}.$$

We can see that the second eigenvalue is very small in comparison to the first. This justifies that it is reasonable to approximate the two-dimensional problem with a one-dimensional problem using the result of Section 1.3, i.e. by considering $\beta \simeq \begin{bmatrix} 0.906 \\ 0.424 \end{bmatrix}$ and $V \simeq 1.744$.

Regarding the objective function, we consider the following:

- Time horizon given by $T = 12$ days. This horizon ensures convergence towards stationary quotes at time $t = 0$ – see Figure 1.4 below.
- $\psi : q \in \mathbb{R}^2 \mapsto \frac{\gamma}{2} q' \Sigma q$ with $\gamma = 8 \cdot 10^{-7} \text{ €}^{-1}$.
- $\ell_d = 0$.

Results with 2 factors

Since θ and $\tilde{\theta}$ are defined on $[0, T] \times \mathbb{R}^2$, a first step for approximating the value functions consists in restricting the state space to a compact set. A traditional way to proceed consists in setting boundary conditions. In what follows, we equivalently impose risk limits in the sense that no trade that would result in an inventory $q \in \mathbb{R}^2$ such that $q' \Sigma q > B$ is admitted, where $B = 2.4 \cdot 10^{10}$.⁶

We then approximate the solution $\tilde{\theta}$ to (1.4) with two factors using a monotone explicit Euler scheme with linear interpolation on a grid of size 141×141 for the factors and a discretization of the RFQ size distribution with 4 sizes: $z^1 = 6250$, $z^2 = 12500$, $z^3 = 18750$, and $z^4 = 25000$ assets – thereafter respectively designated by very small, small, large, and very large size – with respective probability $p^1 = 0.53$, $p^2 = 0.35$, $p^3 = 0.10$, and $p^4 = 0.02$.⁷

The value function (at time $t = 0$) as a function of the factors is plotted in Figure 1.1. From the value function, we obtain the optimal bid and ask quotes of the market maker as a function of inventories and request size. The optimal bid quotes (at time $t = 0$) for asset 1 and asset 2 (in the case of the smallest RFQ size) are plotted in Figures 1.2 and 1.3. The ask quotes are of course symmetric and are not plotted.

We see in Figures 1.2 and 1.3 that the optimal bid quotes for both assets are increasing functions of both the inventory in asset 1 and asset 2, as expected given the positive value of the correlation parameter ρ chosen in the example of this section.

As discussed above, we chose $T = 12$ days to ensure convergence of the optimal quotes to their stationary values. This is illustrated in Figure 1.4.

To see the impact of the RFQ size on the optimal quotes, we plot in Figure 1.5 the four functions

$$q^1 \mapsto \bar{\delta}^{1,b}(0, q^1, 0, z^k), k \in \{1, \dots, 4\}$$

and in Figure 1.6 the four functions

$$q^2 \mapsto \bar{\delta}^{1,b}(0, 0, q^2, z^k), k \in \{1, \dots, 4\}.$$

Likewise for asset 2: we plot in Figure 1.7 the four functions

$$q^2 \mapsto \bar{\delta}^{2,b}(0, 0, q^2, z^k), k \in \{1, \dots, 4\}$$

and in Figure 1.8 the four functions

$$q^1 \mapsto \bar{\delta}^{2,b}(0, q^1, 0, z^k), k \in \{1, \dots, 4\}.$$

We see that accounting for the size of RFQs significantly impacts the optimal quotes of asset 1. This is less the case for asset 2 (this difference is due to the fact that the volatility of asset 1 is twice that of asset 2). In all cases, the monotonicity is unsurprising.⁸

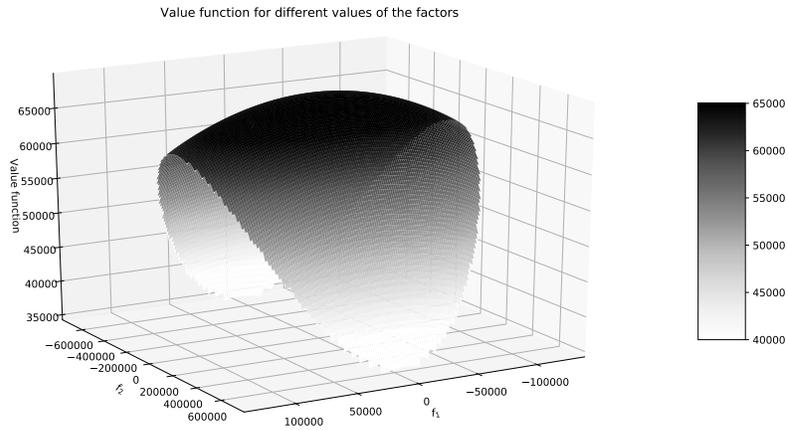


Figure 1.1: Value function for different values of the factors.

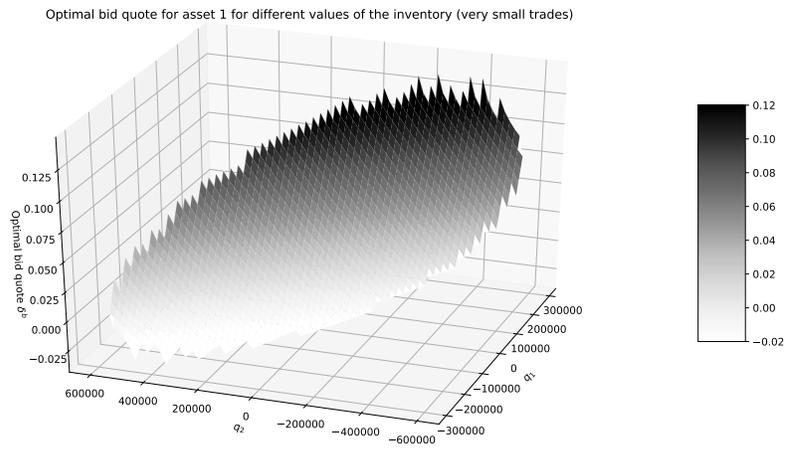


Figure 1.2: Optimal bid quote for asset 1 for different values of the inventory (very small trades).

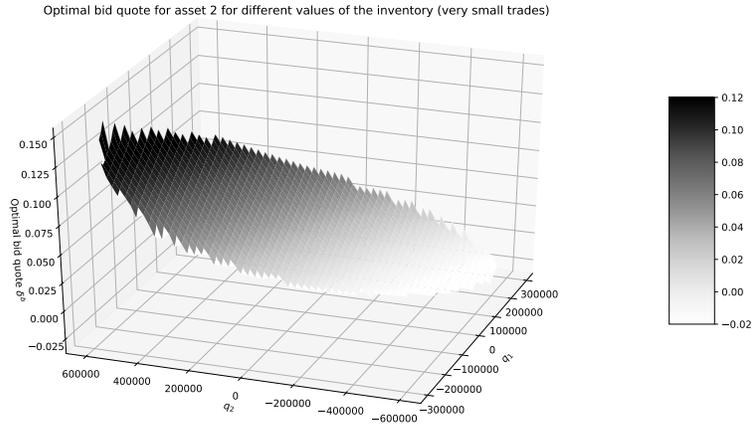


Figure 1.3: Optimal bid quote for asset 2 for different values of the inventory (very small trades).

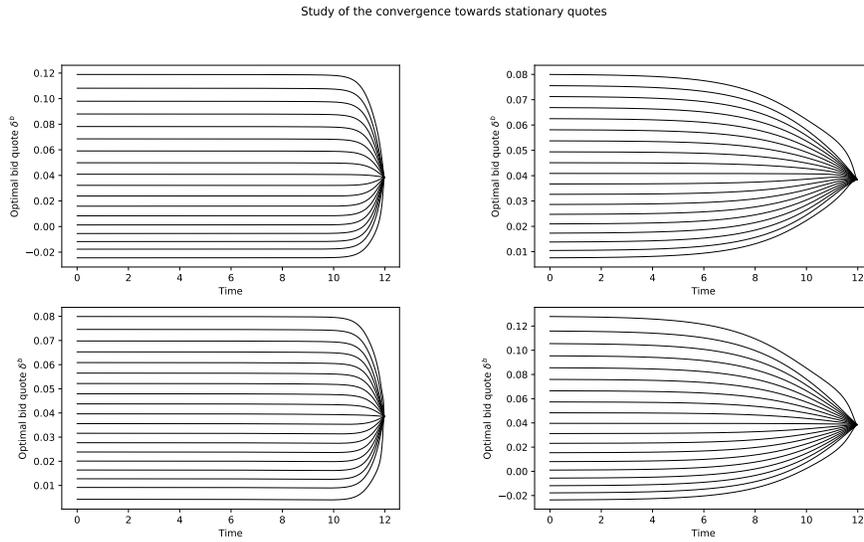


Figure 1.4: Optimal bid quotes as a function of time for various values of the factors (very small trades). Top left: Asset 1 when $f^2 = 0$. Top right: Asset 1 when $f^1 = 0$. Bottom left: Asset 2 when $f^2 = 0$. Bottom right: Asset 2 when $f^1 = 0$.

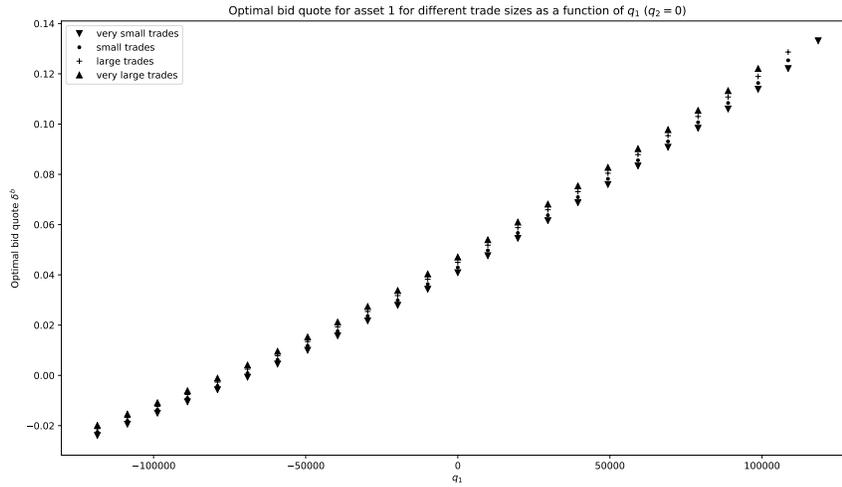


Figure 1.5: Optimal bid quote for asset 1 for different trade sizes as a function of q_1 ($q_2 = 0$).

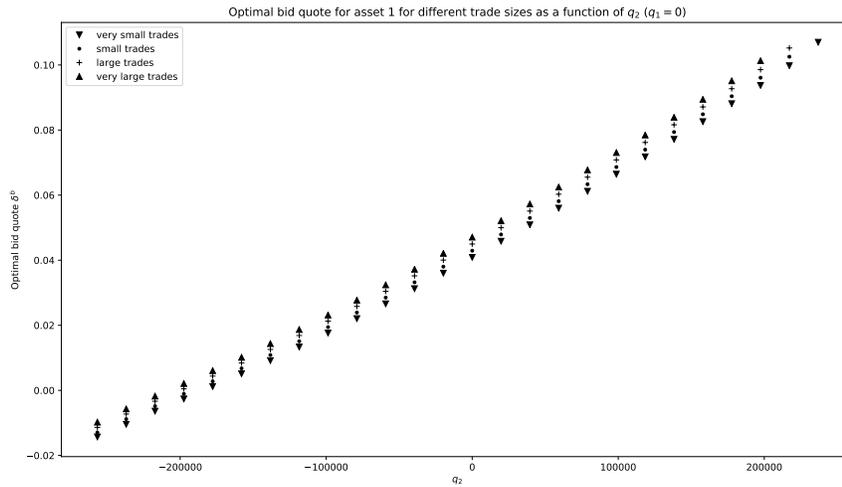


Figure 1.6: Optimal bid quote for asset 1 for different trade sizes as a function of q_2 ($q_1 = 0$).

⁶These risk limits can be expressed in the space of factors instead of being expressed with the inventory vector.

⁷When there are as many factors as assets, one could directly consider the problem with inventory variables.

⁸Boundary effects related to impossible interpolation explain the surprising position of some extreme points.

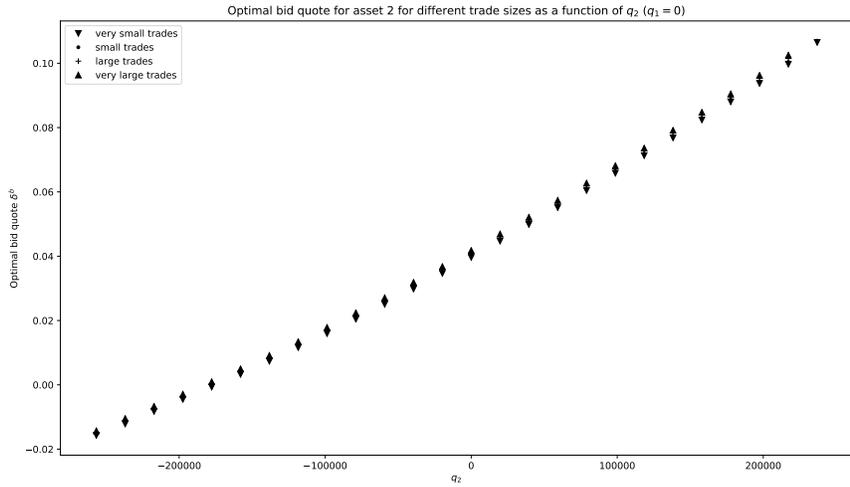


Figure 1.7: Optimal bid quote for asset 2 for different trade sizes as a function of q_2 ($q_1 = 0$).

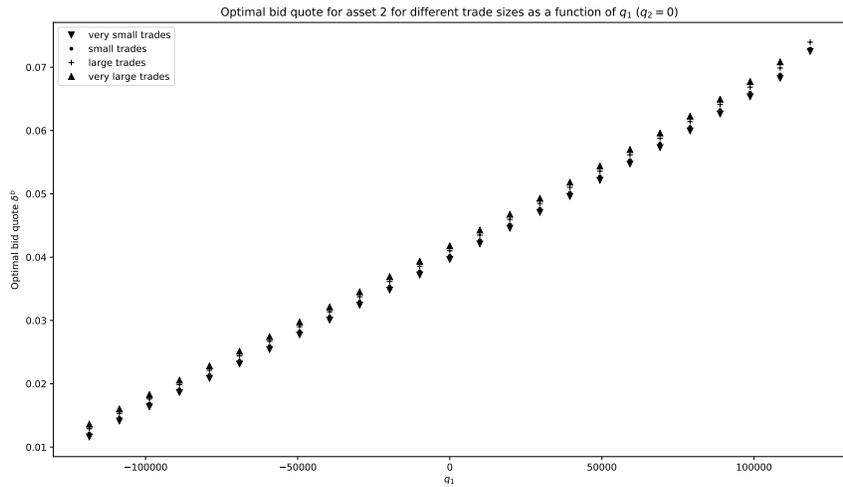


Figure 1.8: Optimal bid quote for asset 2 for different trade sizes as a function of q_1 ($q_2 = 0$).

In order to check that the value of B defining the risk limits does not have a significant impact on our numerical approximation, we carried out a Monte-Carlo simulation with 2000 trajectories starting from zero inventory, using the optimal quotes. The distribution of inventory is plotted in

Figure 1.9.⁹ We clearly see that the ellipse of authorized inventory is wide enough to have little influence on the outcome.

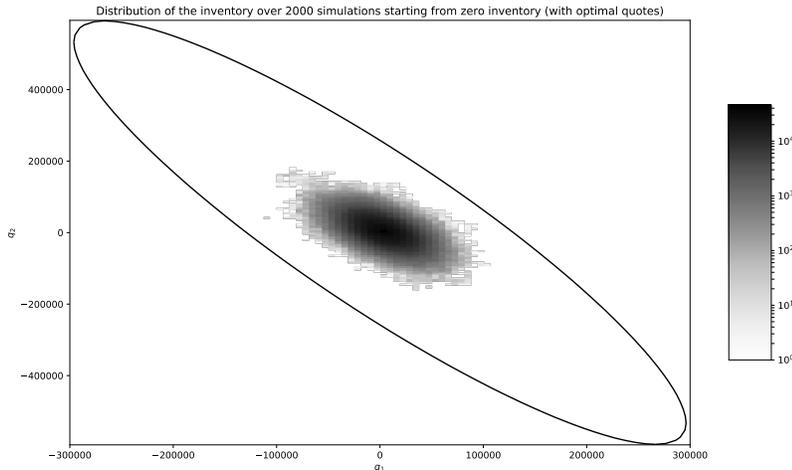


Figure 1.9: Distribution of the inventory over 2000 simulations starting from zero inventory (with optimal quotes).

The statistics associated with our simulations are documented in Table 1: the average PnL at time T , the standard deviation of that PnL at date T , the part of that standard deviation not related to market risk – i.e. only related to the randomness of RFQs –,¹⁰ and the estimated value of the objective function, i.e. the empirical mean of $\text{PnL}_T - \frac{\gamma}{2} \int_0^T q_t' \Sigma q_t dt$.

Mean PnL	Stdev PnL	Stdev coming from RFQs	Objective function
72081	80432	5959	69293

Table 1: Statistics associated with our 2000 simulations starting from zero inventory (with optimal quotes).

These figures have to be compared with those associated with a basic naive strategy. A basic strategy consists, for each asset and side, in always quoting the same “myopic” quote that maximizes the

⁹The shades of gray are in logarithmic scale.

¹⁰Using the law of total variance, it is easy to see that

$$\mathbb{V}(\text{PnL}_T) = \mathbb{E} \left[\int_0^T q_t' \Sigma q_t dt \right] + \mathbb{V} \left(\int_0^T \sum_{i=1}^2 \int_{\mathbb{R}_+^*} (\delta^{i,b}(t, z) z J^{i,b}(dt, dz) + \delta^{i,a}(t, z) z J^{i,a}(dt, dz)) \right).$$

This formula enables to distinguish the part of the variance of the PnL at time T coming from market risk (the first term) from that coming from the randomness of RFQs (the second term).

expected instantaneous PnL. In other words, these myopic quotes are defined, for all $i \in \{1, \dots, d\}$, by

$$\delta_{\text{myopic}}^{i,b} = \operatorname{argmax} \delta \Lambda^{i,b}(\delta) \quad \text{and} \quad \delta_{\text{myopic}}^{i,a} = \operatorname{argmax} \delta \Lambda^{i,a}(\delta).$$

In our case, the myopic quotes are all equal to 0.03854 €.

We carried out 2000 simulations using these myopic quotes with the same source of randomness as above. The distribution of inventory is plotted in Figure 1.10, and the statistics associated with our simulations are documented in Table 2.

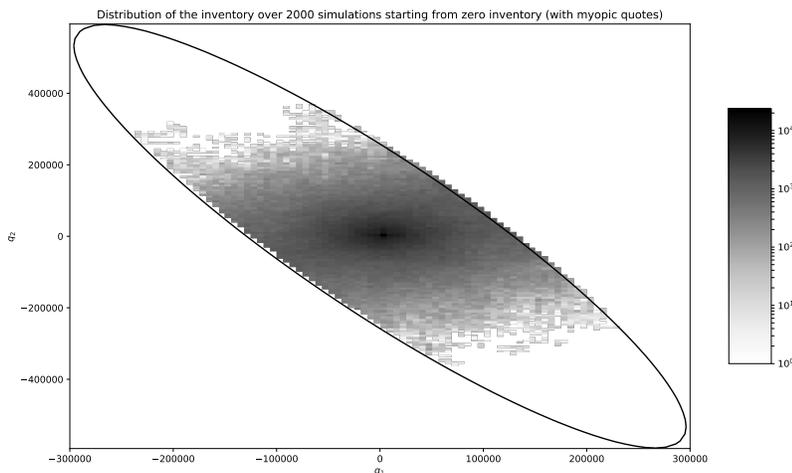


Figure 1.10: Distribution of the inventory over 2000 simulations starting from zero inventory (with myopic quotes).

Mean PnL	Stdev PnL	Stdev coming from RFQs	Objective function
73410	265906	6211	43953

Table 2: Statistics associated with our 2000 simulations starting from zero inventory (with myopic quotes).

With these figures, we clearly see that the main source of risk is market risk and not the risk associated with the randomness of RFQs. This justifies our choice of objective function that only penalizes the part of the variance coming from market risk.

We also clearly see that the use of the optimal quotes drastically reduces the variance of the PnL and results in a high value of the objective function. More precisely, although the use of optimal quotes reduces the average PnL from around 73410 to around 72081, it enables a reduction by a

factor 3 of the standard deviation of the PnL from around 265906 to around 80432, hence a major increase of the objective function from around 43953 to around 69293 (a figure consistent with the maximum of the value function plotted in Figure 1.1 which is approximately 69174).

Results with the one-factor model and comparison

Let us now compare the results with two factors to the results with one factor, i.e. when the smallest eigenvalue of Σ is replaced by 0.

As above, we start with an approximation of the solution $\tilde{\theta}$ to (1.4) with one factor. We used a monotone explicit Euler scheme on a grid of size 141 for the unique factor and the same discretization (with 4 sizes) as in the previous paragraphs for the RFQ size distribution. The set of authorized inventory $\{q \in \mathbb{R}^2 | q' \Sigma q \leq B\}$ is of course replaced by the set $\{q \in \mathbb{R}^2 | (\beta' q)' V (\beta' q) \leq B\}$ which corresponds, in terms of the unique factor, to the interval $[-\sqrt{\frac{B}{V}}, \sqrt{\frac{B}{V}}]$.

The value function (at time $t = 0$) as a function of the inventory, obtained through linear interpolation is plotted in Figure 1.11. The difference between the one-factor value function and the two-factor one is plotted in Figure 1.12.

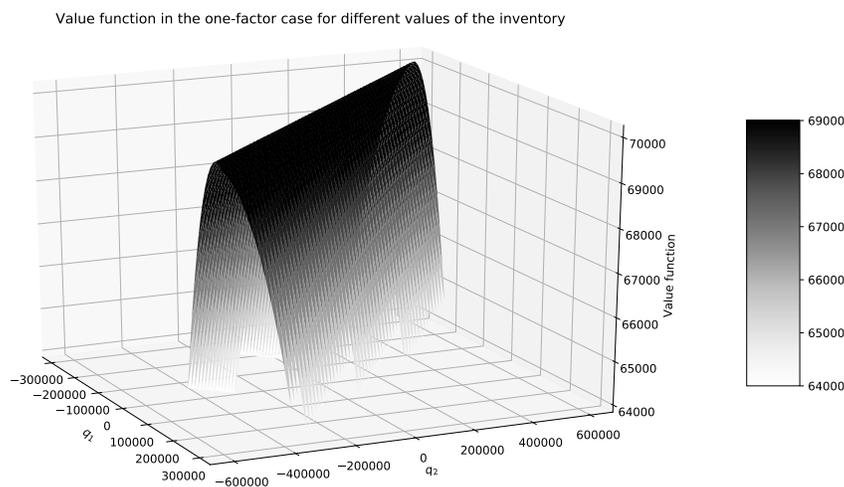


Figure 1.11: Value function in the one-factor case for different values of the inventory.

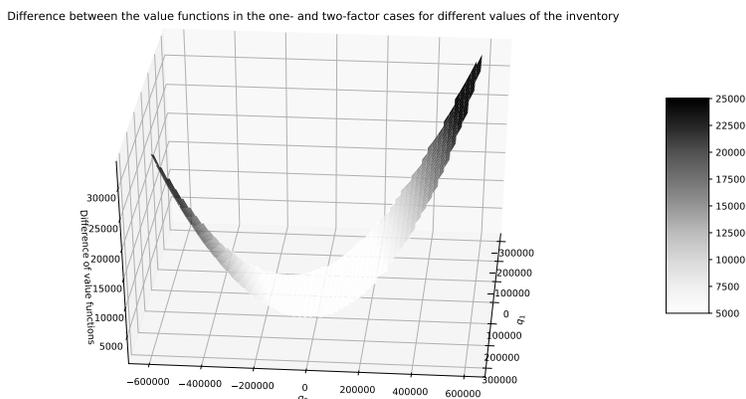


Figure 1.12: Difference between the value functions in the one- and two-factor cases for different values of the inventory.

We see that the value function in the one-factor case is above that of the two-factor case. This comes from the fact that not all the risk is taken into account in the one-factor case. We also see that the difference between the two value functions is very large at the two extremes of the major axis of the ellipse. This comes from the fact that the market maker using the one-factor model believes that positions close to the major axis of the ellipse are associated with low risk whereas this is less and less the case as the inventory in each asset increases in absolute value.

As in the two-factor case, we deduce from the value function the optimal bid and ask quotes of the market maker (at time $t = 0$) as a function of inventory and request size. The optimal bid quotes for asset 1 and asset 2 (in the case of the smallest RFQ size) are plotted in Figures 1.13 and 1.14. The differences between the optimal bid quotes in the one-factor case and two-factor case (for the smallest RFQ size) are plotted in Figure 1.15 and Figure 1.16.

We clearly see that the larger (in absolute value) the inventory in each asset, the larger the difference in optimal quotes between the exact model and the one-factor approximation. This is in line with expectation.

To better compare the quotes and see the impact of RFQ size, we plot in Figures 1.17, 1.18, and 1.19 the optimal bid quotes of asset 1 when $q_2 = 0$ for different values of q_1 , when $q_1 = 0$ for different values of q_2 , and when (q_1, q_2) spans the major axis of the ellipse of authorized inventory.

Likewise, we plot in Figures 1.20, 1.21, and 1.22 the optimal bid quotes of asset 2 when $q_1 = 0$ for different values of q_2 , when $q_2 = 0$ for different values of q_1 , and when (q_1, q_2) spans the major axis of the ellipse of authorized inventory.

Optimal bid quote in the one-factor case for asset 1 for different values of the inventory (very small trades)

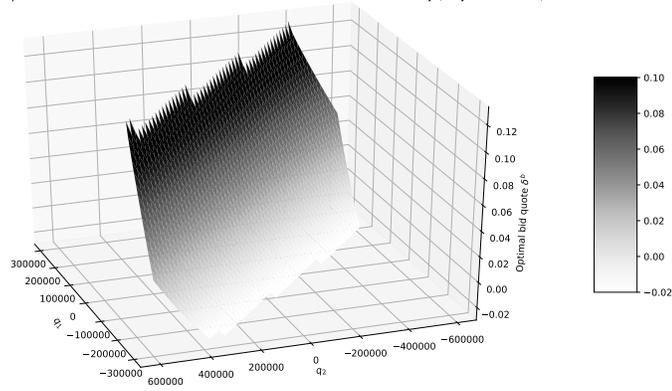


Figure 1.13: Optimal bid quote in the one-factor case for asset 1 for different values of the inventory (very small trades).

Optimal bid quote in the one-factor case for asset 2 for different values of the inventory (very small trades)

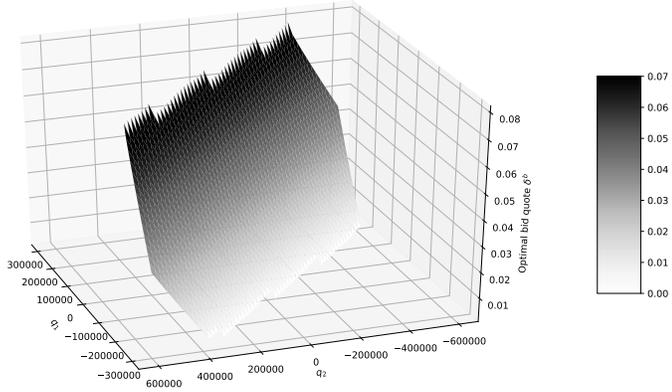


Figure 1.14: Optimal bid quote in the one-factor case for asset 2 for different values of the inventory (very small trades).

Difference between the optimal bid quotes of asset 1 in the one- and two-factor cases for different values of the inventory (very small trades)

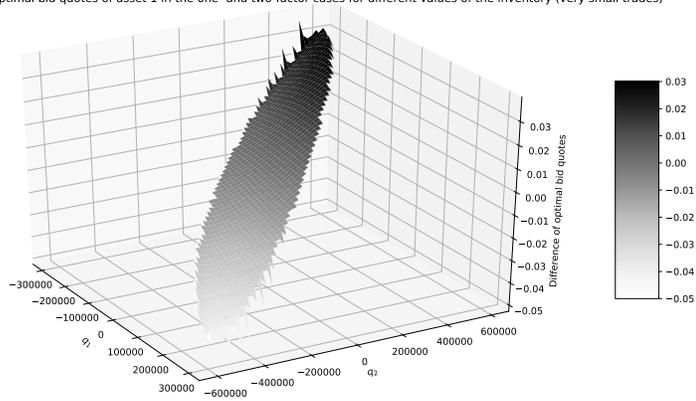


Figure 1.15: Difference between the optimal bid quotes of asset 1 in the one- and two-factor cases for different values of the inventory (very small trades).

Difference between the optimal bid quotes of asset 2 in the one- and two-factor cases for different values of the inventory (very small trades)

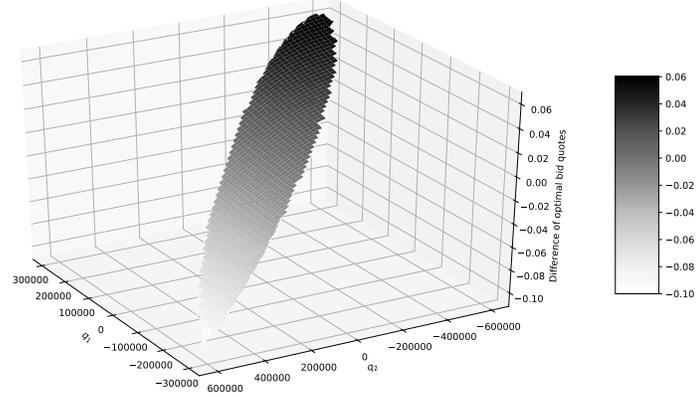


Figure 1.16: Difference between the optimal bid quotes of asset 2 in the one- and two-factor cases for different values of the inventory (very small trades).

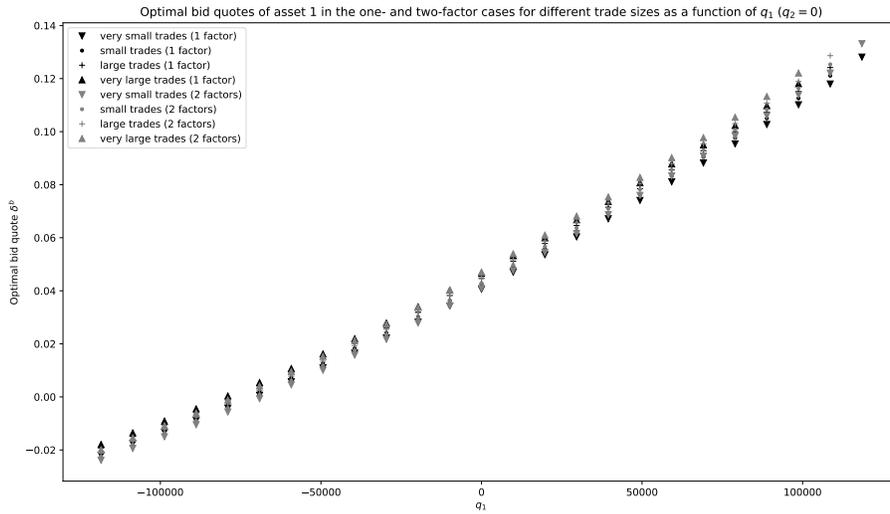


Figure 1.17: Optimal bid quote for asset 1 for different trade sizes as a function of q_1 ($q_2 = 0$).

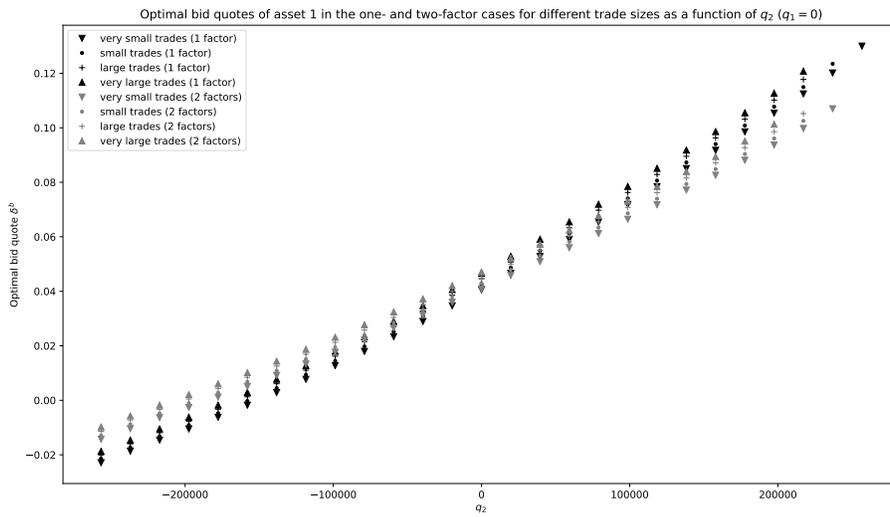


Figure 1.18: Optimal bid quote for asset 1 for different trade sizes as a function of q_2 ($q_1 = 0$).

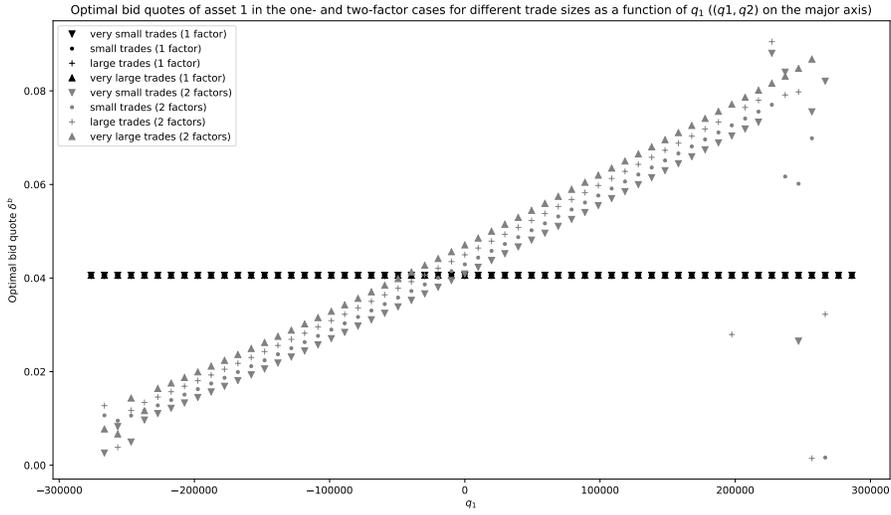


Figure 1.19: Optimal bid quote for asset 1 for different trade sizes as a function of q_1 ((q_1, q_2) on the major axis of the ellipse).

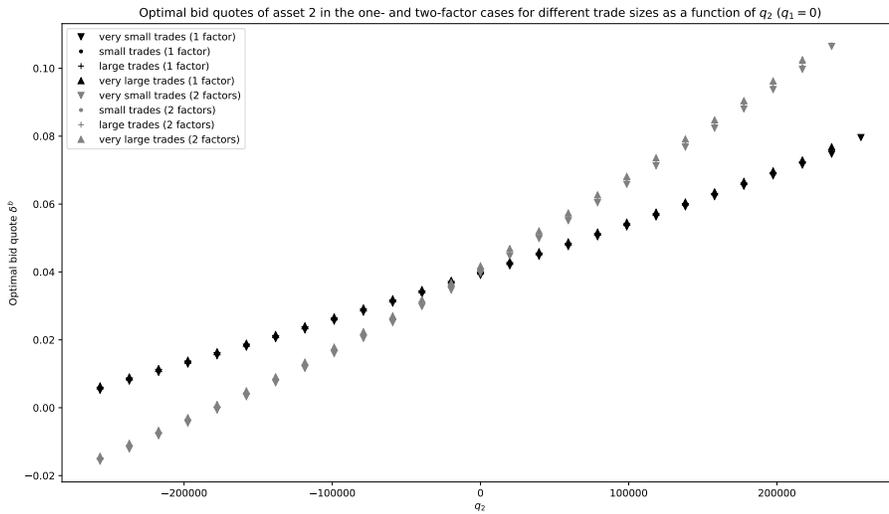


Figure 1.20: Optimal bid quote for asset 2 for different trade sizes as a function of q_2 ($q_1 = 0$).

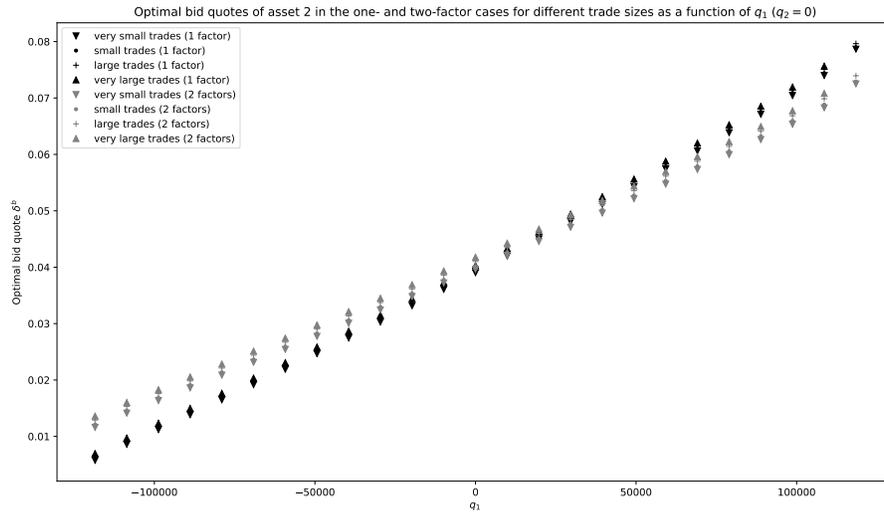


Figure 1.21: Optimal bid quote for asset 2 for different trade sizes as a function of q_1 ($q_2 = 0$).

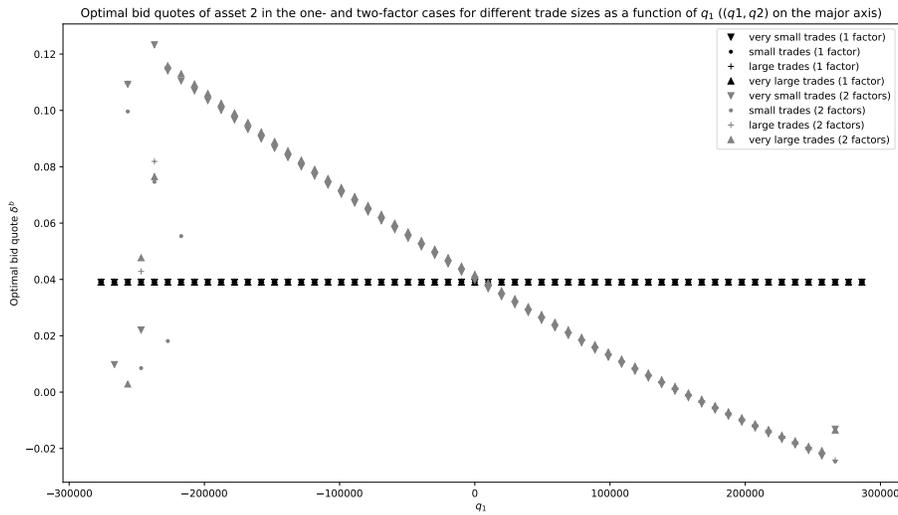


Figure 1.22: Optimal bid quote for asset 2 for different trade sizes as a function of q_1 ((q_1, q_2) on the major axis of the ellipse).

These plots confirm that the optimal quotes in the one-asset model are good approximations of the true optimal ones whenever the inventory in each asset is not too large, all the more for inventories that are not close to the major axis of the ellipse of authorized inventory. Moreover in our example, the one-factor model seems to return quotes closer to the true optimal ones for asset 1 than for asset 2: this is due to the fact that here, the factor obtained through PCA explains better the risk of asset 1 than that of asset 2 (the residual variance of the latter is four times that of the former).

Comparing quotes is important but what really matters is to compare the distribution of the PnL at time T when using the quotes obtained within the one-factor model with the distribution of the PnL at time T when using the optimal quotes (of the two-factor model). For that purpose, we carried out a Monte-Carlo with 2000 simulations using the same source of randomness as in Section 4.1.2. The distribution of inventory when using the optimal quotes in the one-factor model is plotted in Figure 1.23. The statistics associated with our simulations are documented in Table 3.

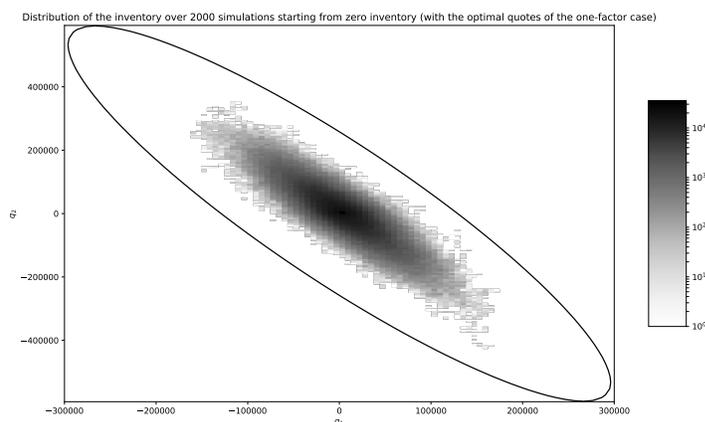


Figure 1.23: Distribution of the inventory over 2000 simulations starting from zero inventory (with the optimal quotes of the one-factor case).

Mean PnL	Stdev PnL	Stdev coming from RFQs	Objective function
72523	96746	6033	68567

Table 3: Statistics associated with our 2000 simulations starting from zero inventory (with the optimal quotes of the one-factor case).

We clearly see that the performance of the one-factor approximation is very good. The value of the objective function is indeed around 68567 when using the quotes obtained in the one-factor model, a bit smaller than the value of approximately 69293 obtained with the same source of randomness when using the optimal quotes. In fact, the average PnL is higher with the one-factor optimal quotes but, since the distribution of inventory is denser in areas that are falsely believed to be

risk-free or wrongfully associated with low risk, the standard deviation of the PnL is in fact higher (around 96746 versus around 80432), resulting in a lower value of the objective function.

Taking the residual risk into account with our Monte-Carlo method

We have seen above that the use of the optimal quotes of the one-factor model provides very good results in terms of the value of the objective function. Nevertheless, the distribution of the inventory plotted in Figure 1.23 differs from the distribution associated with the true optimal quotes plotted in Figure 1.9 because the major axis of the ellipse of authorized inventory is associated with zero risk in the one-factor model. In this section, we illustrate the Monte-Carlo method proposed in Section 1.3.2 in order to account for the residual risk in the approximation of the value function and the optimal quotes.

It is noteworthy that the Monte-Carlo method of Section 1.3.2, unlike the grid method of Section 1.3.1, does not allow to compute the optimal quotes for all assets, sides, sizes, and values of the inventory (we ignore time by focusing on $t = 0$) at once. Instead, it requires a different Monte-Carlo simulation for each desired quote. In particular, should it be used by practitioners, its use should be online. In other words, the computations should only be carried out upon receiving an RFQ or slightly beforehand if one wants to prepare the quotes (given the current inventory) for the most probable RFQs.

A related point is that, even for illustration and even with two assets, it is too time-consuming to compute the Monte-Carlo adjustment for all assets, sides, sizes, and possible inventories. As a consequence, it is too time-consuming to carry out simulations of the PnL with the quotes amended by the Monte-Carlo method of Section 1.3.2. Instead of a complete analysis, we focus on a sectional analysis by looking at the cases $q^2 = 0$, $q^1 = 0$, and (q^1, q^2) on the major axis of the ellipse of authorized inventory.

In Figures 1.24, 1.25, and 1.26, we compare the value function obtained in the two-factor case, i.e. the true value function, to the value function of the one-factor case and to its adjustment through the Monte-Carlo technique of Section 1.3.2 – we use 50 simulations for each point (with the same source of randomness for all points). Figure 1.24 deals with the comparison of the values on the section $\{q^2 = 0\}$, Figure 1.25 deals with the comparison of the values on the section $\{q^1 = 0\}$, and Figure 1.26 deals with the comparison of the values on the major axis of the ellipse of authorized inventory.

We clearly see that, unsurprisingly, the Monte-Carlo adjustment goes in the right direction. However, the Monte-Carlo method leads to (i) an overestimation of the gap between the value function of the one-factor case and the true value function and (ii) an overestimation of the degree of concavity of the value function (this is particularly the case for the third section).

It is noteworthy that the quality of the approximation is the best around 0. This point is interesting if one wants to estimate the degree of suboptimality of a quoting strategy in a scenario starting with zero inventory. The poor approximation of the concavity is however a limitation since quotes are based on finite differences of the value function. This is well illustrated by Figures 1.27, 1.28, and 1.29.

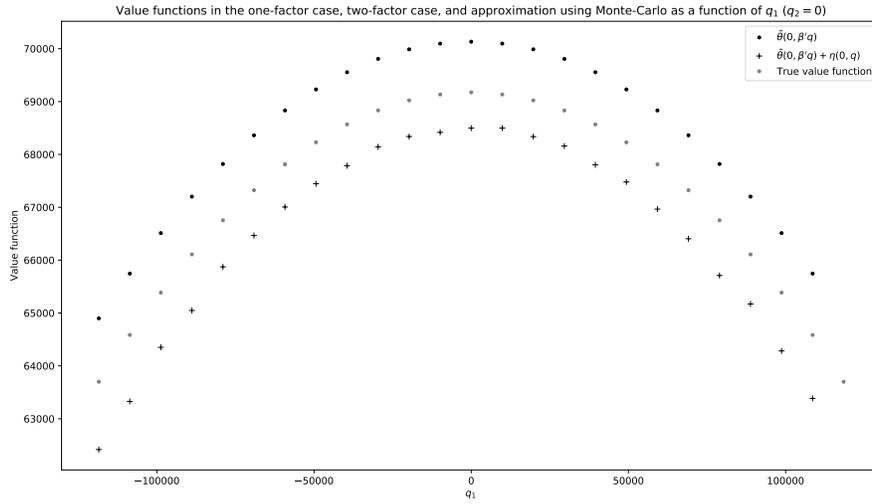


Figure 1.24: Value functions in the one-factor case, two-factor case, and approximation using Monte-Carlo as a function of q_1 ($q_2 = 0$).

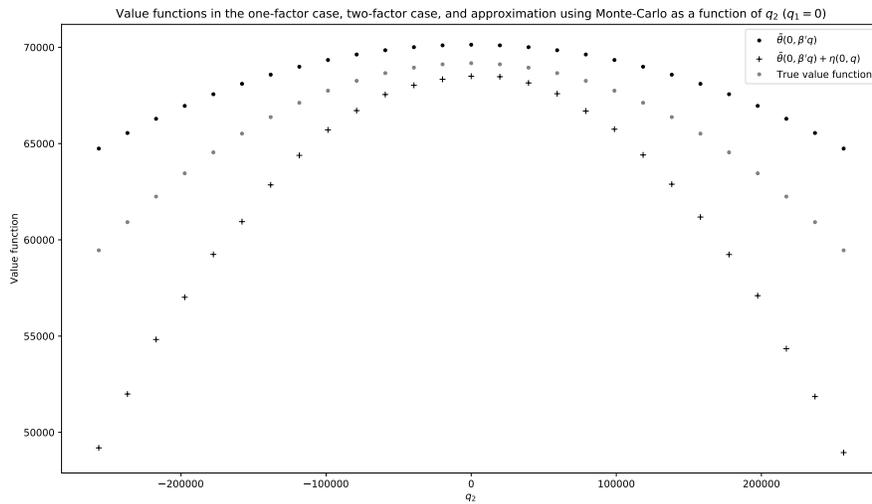


Figure 1.25: Value functions in the one-factor case, two-factor case, and approximation using Monte-Carlo as a function of q_2 ($q_1 = 0$).

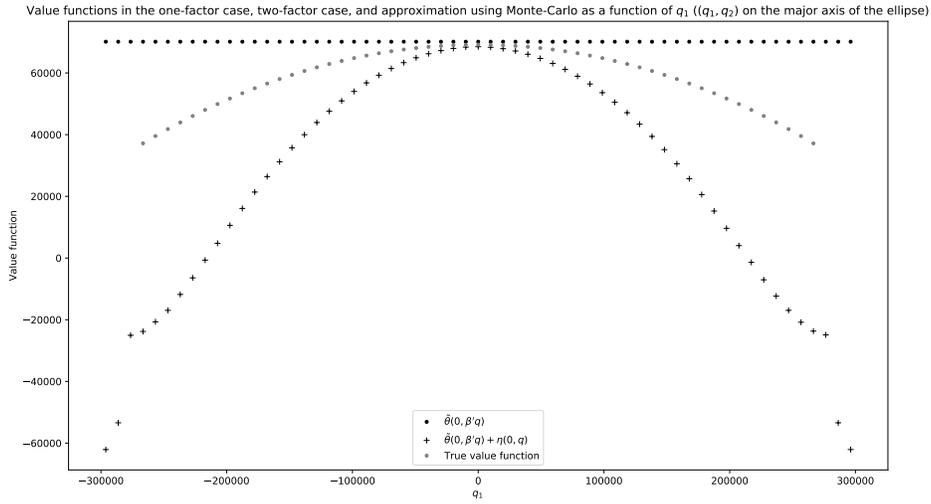


Figure 1.26: Value functions in the one-factor case, two-factor case, and approximation using Monte-Carlo as a function of q_1 ((q_1, q_2) on the major axis of the ellipse).

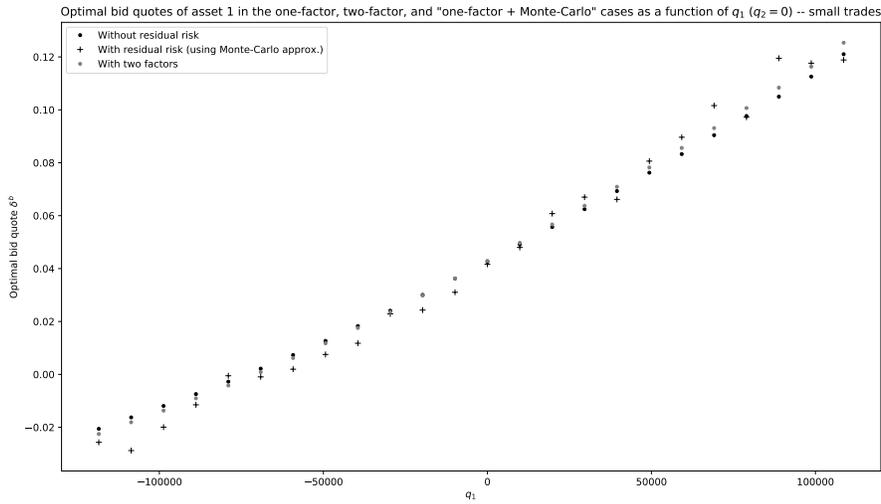


Figure 1.27: Optimal bid quotes of asset 1 in the one-factor, two-factor, and "one-factor + Monte-Carlo" cases as a function of q_1 ($q_2 = 0$) - small trades.

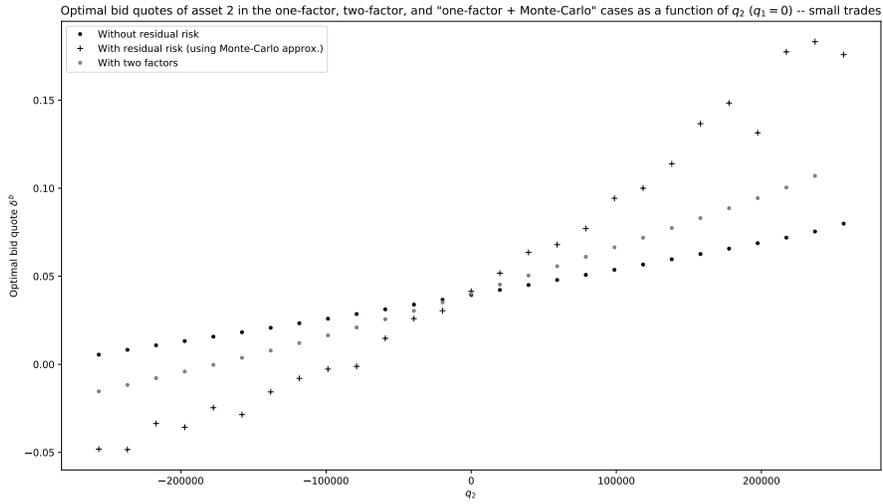


Figure 1.28: Optimal bid quotes of asset 2 in the one-factor, two-factor, and "one-factor + Monte-Carlo" cases as a function of q_2 ($q_1 = 0$) – small trades.

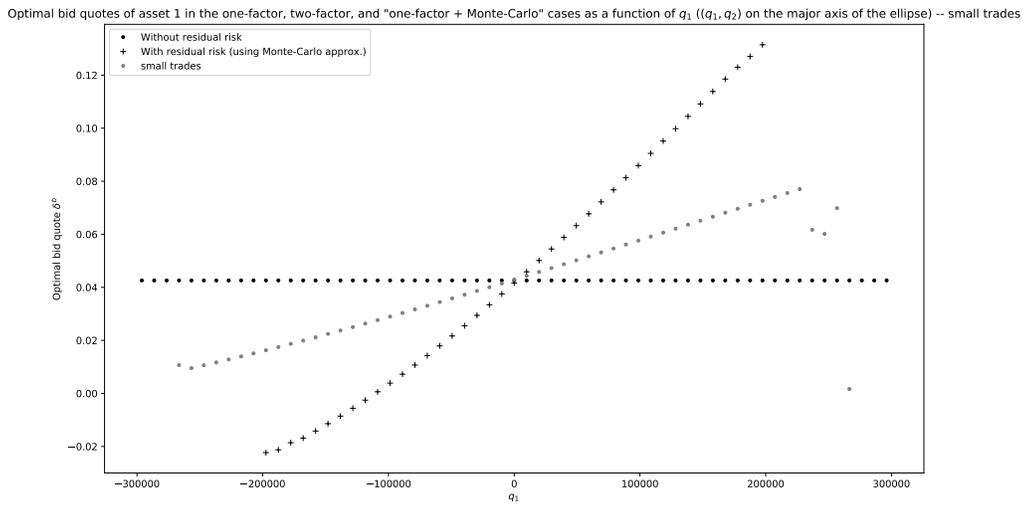


Figure 1.29: Optimal bid quotes of asset 1 in the one-factor, two-factor, and "one-factor + Monte-Carlo" cases as a function of q_1 ((q_1, q_2) on the major axis of the ellipse) – small trades.

Nevertheless, even though the Monte-Carlo adjustments of quotes are too large, especially on the major axis of the ellipse (see Figure 1.29), this drawback of the Monte-Carlo method of Section 1.3.2 should be qualified as the quotes obtained with the Monte-Carlo technique naturally lead – because of the overestimated slope of the quotes – to trajectories of the inventory more concentrated around 0 and therefore to a very rare use of the quotes that are too different from the optimal ones.

Before we go on with an example including 30 assets, let us conclude on the two-asset case. The method we propose to tackle the curse of dimensionality is based on the projection of market risk on a low-dimensional space of factors. It works very well in the two-asset case as the quotes permit to reach a value of the objective function close to the optimal one. The Monte-Carlo adjustment we suggested in Section 1.3.2 allows to approximate the true value function at the point of zero inventory, which is quite useful when one does not have access to the true value function, as is the case in high dimension. However, it overestimates the changes one must make to the quotes computed with the low-dimensional approximation.

1.4.2 Dealing with 30 assets

We now consider the more challenging case of a market maker in charge of 30 assets (here bonds) with the following characteristics:

- Asset prices: $S_0^i = 100 \text{ €}$, $\forall i \in \{1, \dots, 30\}$.
- Volatility of assets: $\sigma^i = 1.2 \text{ €} \cdot \text{day}^{-\frac{1}{2}}$, $\forall i \in \{1, \dots, 15\}$, and $\sigma^i = 0.6 \text{ €} \cdot \text{day}^{-\frac{1}{2}}$, $\forall i \in \{16, \dots, 30\}$.
- Correlation matrix: $\begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix}$, where

$$R_{11} = R_{22} = \begin{bmatrix} 1.0 & 0.9 & \dots & \dots & 0.9 \\ 0.9 & \ddots & \ddots & \ddots & 0.9 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0.9 & \ddots & \ddots & \ddots & 0.9 \\ 0.9 & \dots & \dots & 0.9 & 1.0 \end{bmatrix} \quad \text{and} \quad R_{12} = R_{21} = \begin{bmatrix} 0.2 & 0.2 & \dots & 0.2 & 0.2 \\ 0.2 & \ddots & \ddots & \ddots & 0.2 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0.2 & \ddots & \ddots & \ddots & 0.2 \\ 0.2 & 0.2 & \dots & 0.2 & 0.2 \end{bmatrix}.$$

- Intensity functions:

$$\Lambda^{i,b}(\delta) = \Lambda^{i,a}(\delta) = \lambda_{RFQ} \frac{1}{1 + e^{\alpha_\Lambda + \beta_\Lambda \delta}}, \quad \forall i \in \{1, \dots, 30\},$$

with $\lambda_{RFQ} = 10 \text{ day}^{-1}$, $\alpha_\Lambda = 0.7$, and $\beta_\Lambda = 30 \text{ €}^{-1}$. This corresponds to 10 RFQs per day for each asset, a probability of $\frac{1}{1+e^{0.7}} \simeq 33\%$ to trade when the answered quote is the reference price and a probability of $\frac{1}{1+e^{-0.2}} \simeq 55\%$ to trade when the answered quote is the reference price improved by 3 cents.

- Request sizes are distributed according to a Gamma distribution $\Gamma(\alpha, \beta)$ with $\alpha = 4$ and $\beta = 4 \cdot 10^{-4}$. This corresponds to an average request size of 10000 assets (i.e. approximately 1000000 €) and a standard deviation equal to half the average.

The variance-covariance matrix Σ has two eigenvalues equal to 19.895060 and 4.584941, and 28 eigenvalues below 0.15. The first eigenspace is spanned by a vector with all coordinates of the same sign. The associated factor – the first factor – represents an index of the 30 assets. The second eigenspace is spanned by a vector with the first 15 coordinates of the same sign and the next 15 of the same, but opposite, sign. The associated factor – the second factor – allows to separate the two groups of assets.

We can therefore legitimately approximate our 30-asset problem by a two-factor one and solve the corresponding Hamilton-Jacobi equation (1.4) to approximate the optimal quotes.

Regarding the objective function, we consider the following:

- Time horizon given by $T = 2$ days. This horizon ensures convergence towards stationary quotes at time $t = 0$ – see Figure 1.33.
- $\psi : q \in \mathbb{R}^2 \mapsto \frac{\gamma}{2} q' \Sigma q$ with $\gamma = 8 \cdot 10^{-7} \text{ €}^{-1}$.
- $\ell_d = 0$.

We approximate the solution $\tilde{\theta}$ to (1.4) with two factors using a monotone explicit Euler scheme with linear interpolation on a grid of size 71×71 for the factors and a discretization of the RFQ size distribution with the same 4 sizes as in the above two-asset example.¹¹

The value function (at time $t = 0$) is plotted in Figure 1.30. The associated optimal bid quotes for asset 1 and asset 16 (for the smallest RFQ size) are plotted in Figures 1.31 and 1.32. We see on these graphs that the optimal quotes depend monotonously on the two factors.¹²

As discussed above, we chose $T = 2$ days to ensure convergence of the quotes to their stationary values. This is illustrated in Figure 1.33.

To see the role of RFQ size on the optimal quotes, we plot in Figure 1.34 the four functions $f^1 \mapsto \bar{\delta}^{1,b}(0, f^1, 0, z^k)$, $k \in \{1, \dots, 4\}$ and in Figure 1.35 the four functions $f^2 \mapsto \bar{\delta}^{1,b}(0, 0, f^2, z^k)$, $k \in \{1, \dots, 4\}$.

Likewise, we plot in Figure 1.36 the four functions $f^1 \mapsto \bar{\delta}^{16,b}(0, f^1, 0, z^k)$, $k \in \{1, \dots, 4\}$ and in Figure 1.37 the four functions $f^2 \mapsto \bar{\delta}^{16,b}(0, 0, f^2, z^k)$, $k \in \{1, \dots, 4\}$.

We see, especially in Figures 1.35 and 1.36, that the size of the RFQ significantly impacts the quotes that should be answered (as computed with our two-factor approximation).

¹¹We considered risk limits similar to those of the above two-asset example. Here, no trade that would result in an inventory $q \in \mathbb{R}^{30}$ such that $q' \Sigma q > B$ was admitted, where $B = 5 \cdot 10^{10}$.

¹²Exceptions to this monotonicity property are related to boundary effects.

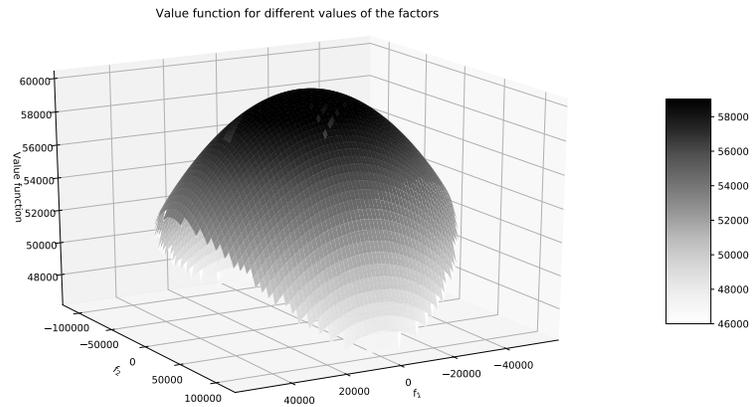


Figure 1.30: Value function for different values of the factors.

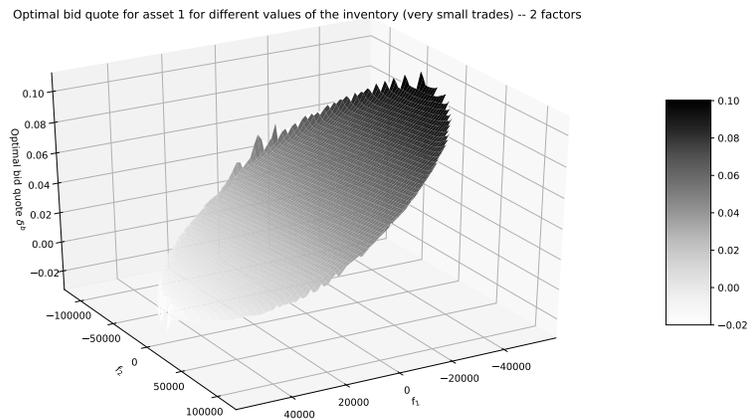


Figure 1.31: Optimal bid quote for asset 1 for different values of the inventory (very small trades) – 2 factors.

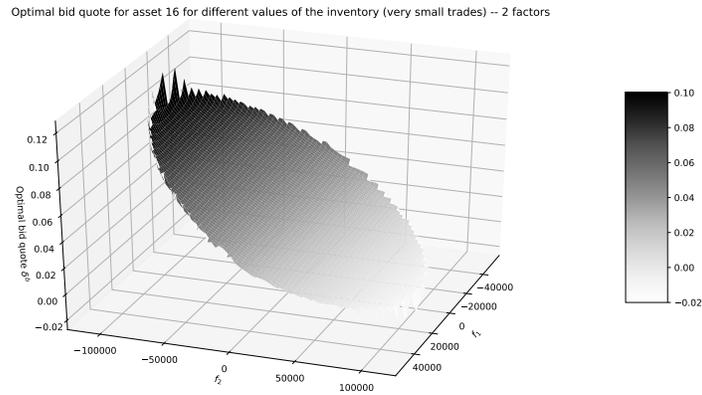


Figure 1.32: Optimal bid quote for asset 16 for different values of the inventory (very small trades) – 2 factors.

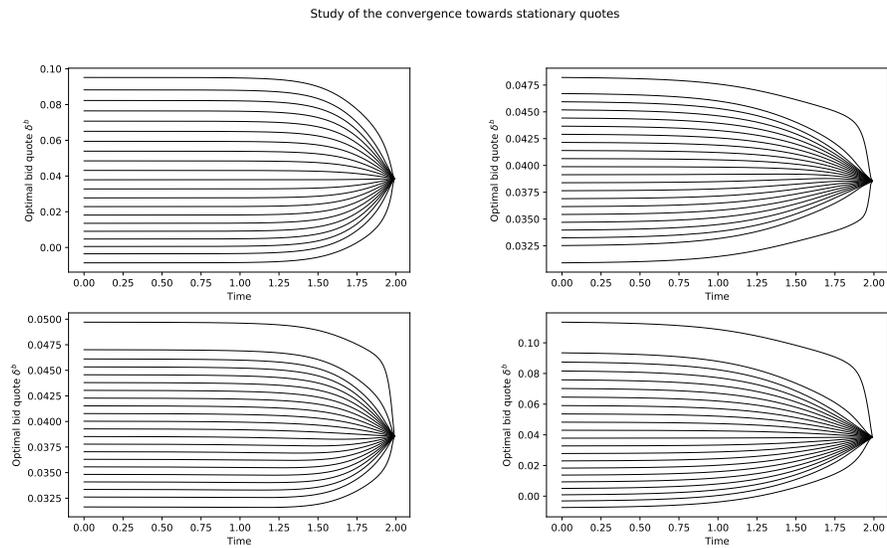


Figure 1.33: Optimal bid quotes as a function of time for various values of the two factors. Top left: Asset 1 when $f^2 = 0$. Top right: Asset 1 when $f^1 = 0$. Bottom left: Asset 16 when $f^2 = 0$. Bottom right: Asset 16 when $f^1 = 0$.

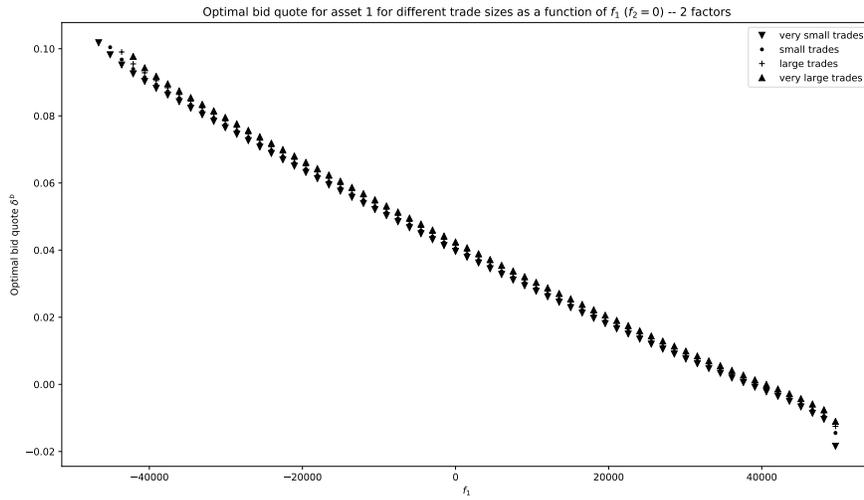


Figure 1.34: Optimal bid quote for asset 1 for different trade sizes as a function of f_1 ($f_2 = 0$) - 2 factors.

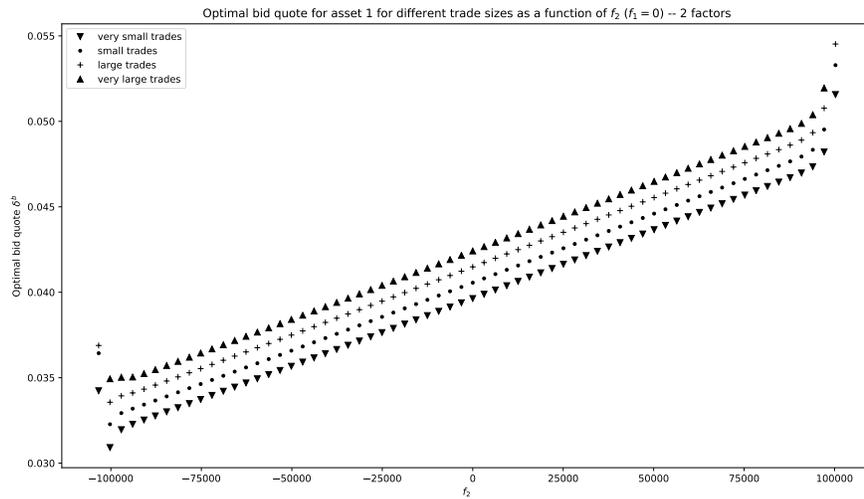


Figure 1.35: Optimal bid quote for asset 1 for different trade sizes as a function of f_2 ($f_1 = 0$) - 2 factors.

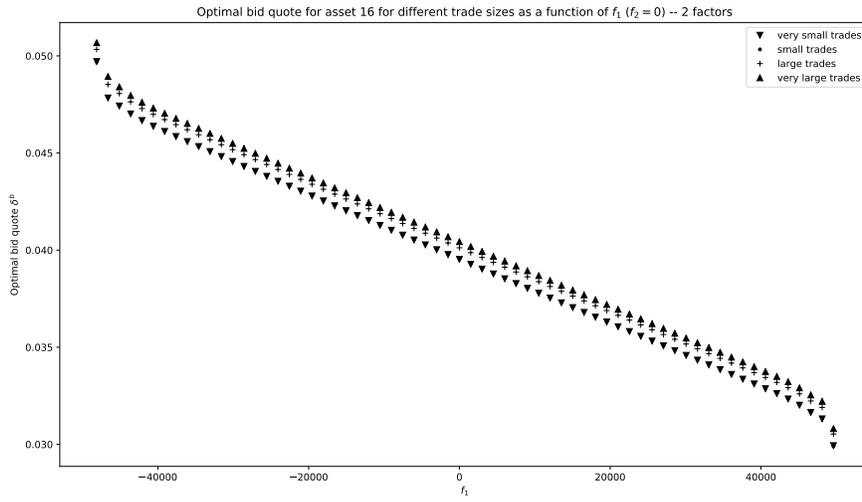


Figure 1.36: Optimal bid quote for asset 16 for different trade sizes as a function of f_1 ($f_2 = 0$) – 2 factors.

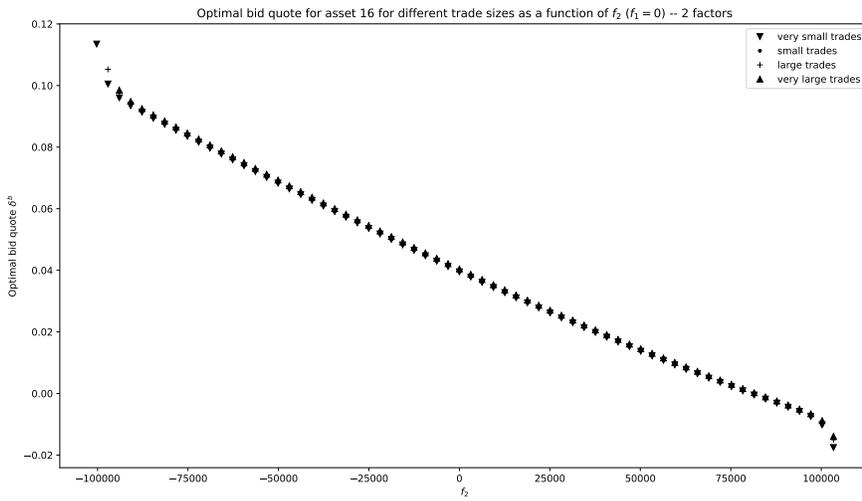


Figure 1.37: Optimal bid quote for asset 16 for different trade sizes as a function of f_2 ($f_1 = 0$) – 2 factors.

Unlike what we did in the two-asset case, it is impossible in our 30-asset case to know how far from real optimality are the optimal quotes computed with the two-factor approximation. Nevertheless, we can use Monte-Carlo simulations to estimate the value of the objective function associated with the optimal quotes computed with the two-factor approximation in a scenario starting from zero inventory, and compare such an estimation to an approximation of the value function at $(t, q) = (0, 0)$ computed through the Monte-Carlo approximation of Section 1.3.2.

We carried out 2000 trajectories starting from zero inventory, using the optimal quotes computed with the two-factor approximation. These 2000 simulations enable to illustrate the distribution of the PnL at time T . The statistics associated with our simulations are documented in Table 4.

Mean PnL	Stdev PnL	Stdev coming from RFQs	Objective function
61471	64911	5338	59765

Table 4: Statistics associated with our 2000 simulations starting from zero inventory (with the two-factor optimal quotes).

The value of the objective function documented in Table 4 has to be compared to an approximation of the true value function at $(t, q) = (0, 0)$. Given the value $\tilde{\theta}(0, 0) = 60156$ obtained with our numerical scheme and given an estimation of $\eta(0, 0)$ equal to -643 obtained using the Feynman-Kac representation of Section 1.3.2 – with 500 trajectories –, we obtain an approximation of the true value function at $(t, q) = (0, 0)$ equal to 59513. From the very small value of $\frac{\eta(0,0)}{\tilde{\theta}(0,0)}$, we deduce that our two-factor approximation is quite satisfactory. The near-optimality of the quotes obtained with our two-factor approximation is confirmed by the value 59765 obtained with our 2000 trajectories (see Table 4) which is even slightly above 59513.

Conclusion

In this paper, we generalized existing market making models to introduce trade size variability. This extension led to an integro-differential equation of the Hamilton-Jacobi type that can be solved using ODE techniques in an infinite-dimensional space. Then, we introduced a numerical method for approximating the optimal bid and ask quotes of a market maker over a large set of assets using a dimensionality reduction technique based on a factor decomposition of the risk. To exemplify our findings, and show that they contribute to beating the curse of dimensionality, we considered two cases of market making with respectively 2 and 30 assets. Our method scales linearly in the number of assets and exponentially in the number of factors, and can therefore be used on large markets driven by a few number of factors.

1.5 Appendix

1.5.1 A particular case: hedging the market factors

Let us now consider a particular case in which we have some common factors that can be hedged using indexes. We introduce $(W_t^1, \dots, W_t^M)_{t \geq 0}$ a M -dimensional brownian motion (W^1, \dots, W^M are assumed to be independents.) Let $K \in \mathbb{N}^*$. For each $k = 1, \dots, K$, we consider $n_k \in \mathbb{N}^*$ assets $(S^{k,1}, \dots, S^{k,n_k})$, with prices verifying, for each $i = 1, \dots, n_k$:

$$dS_t^{k,i} = \sum_{j=1}^M \sigma^{k,i,j} dW_t^j + \sum_{l=1}^{M_k} \bar{\sigma}^{k,i,l} d\bar{W}_t^{k,l},$$

where $S_0^{k,i} > 0$ is fixed, $\sigma^{k,i,j}, \bar{\sigma}^{k,i,l} > 0$ for all j, l , $M_k \in \mathbb{N}^*$, and $\bar{W}^{k,1}, \dots, \bar{W}^{k,M_k}$ are independent brownian motions (which are also independent from W^1, \dots, W^M and from $\bar{W}^{k',1}, \dots, \bar{W}^{k',M_{k'}}$ for $k' \neq k$). The intuition behind this is the following: we have K groups of assets, each depending on M market factors (common to all the groups) and M_k particular factors (different for each group). The dynamics of the inventory processes $(q_t^{k,i})_{t \geq 0}$ are given by

$$dq_t^{k,i} = \int_{\mathbb{R}_+^*} z(J^{k,i,b}(dt, dz) - J^{k,i,a}(dt, dz)),$$

where $J^{k,i,b}(dt, dz)$ and $J^{k,i,a}(dt, dz)$ are marked-point processes representing the request arrival for the asset on the bid and ask side, with respective intensity processes $(\nu_t^{k,i,b})_{t \geq 0}$ and $(\nu_t^{k,i,a})_{t \geq 0}$ satisfying:

$$\begin{aligned} \nu_t^{k,i,b}(dz) &= \Lambda^{k,i,b}(\delta^{k,i,b}(t, z))\mu^{k,i,b}(dz), \\ \nu_t^{k,i,a}(dz) &= \Lambda^{k,i,a}(\delta^{k,i,a}(t, z))\mu^{k,i,a}(dz), \end{aligned}$$

where $\Lambda^{k,i,b}, \Lambda^{k,i,a}$ satisfy hypotheses (H).

We also assume that for each $m = 1, \dots, M$, there exists a liquid index I^m reproducing exactly the m -th factor of the market that we can buy or sell with no transaction cost or market impact, verifying the dynamics:

$$dI_t^m = \sigma^m dW_t^m.$$

With those indexes, we can hedge perfectly the first M factors. For each $m = 1, \dots, M$, we denote by $(\Delta_t^m)_{t \geq 0}$ the trading strategy on I^m allowing us to hedge the m -th factor:

$$\Delta_t^m := - \sum_{k=1}^K \sum_{i=1}^{n_k} \frac{\sigma^{k,i,m}}{\sigma^m} q_t^{k,i}.$$

The resulting dynamics for the cash process $(X_t)_{t \geq 0}$ of the market maker is:

$$\begin{aligned} dX_t &= \sum_{k=1}^K \sum_{i=1}^{n_k} \left(\int_{\mathbb{R}_+^*} z (\delta^{k,i,b}(t, z) J^{k,i,b}(dt, dz) + \delta^{k,i,a}(t, z) J^{k,i,a}(dt, dz)) - S_t^{k,i} dq_t^{k,i} \right) \\ &\quad - \sum_{m=1}^M I_t^m d\Delta_t^m - \sum_{m=1}^M d\langle I_t^m, \Delta_t^m \rangle. \end{aligned}$$

We denote by $(V_t)_{t \geq 0}$ the process for the Mark-to-Market (MtM) value of the market maker's portfolio (cash, indexes and assets):

$$V_t := X_t + \sum_{m=1}^M \Delta_t^m I_t^m + \sum_{k=1}^K \sum_{i=1}^{n_k} q_t^{k,i} S_t^{k,i}.$$

We check easily that the dynamics of $(V_t)_{t \geq 0}$ is therefore given by

$$\begin{aligned} dV_t = & \sum_{k=1}^K \sum_{i=1}^{n_k} \left(\int_{\mathbb{R}_+^*} z (\delta^{k,i,b}(t,z) J^{k,i,b}(dt,dz) + \delta^{k,i,a}(t,z) J^{k,i,a}(dt,dz)) \right) \\ & + \sum_{m=1}^M \Delta_t^m dI_t^m + \sum_{k=1}^K \sum_{i=1}^{n_k} q_t^{k,i} dS_t^{k,i}, \end{aligned}$$

which can also be written as

$$\begin{aligned} dV_t = & \sum_{k=1}^K \sum_{i=1}^{n_k} \left(\int_{\mathbb{R}_+^*} z (\delta^{k,i,b}(t,z) \nu_t^{k,i,b}(dz) + \delta^{k,i,a}(t,z) \nu_t^{k,i,a}(dz)) \right) dt \\ & + \sum_{k=1}^K \sum_{i=1}^{n_k} \sum_{l=1}^{M_k} q_t^{k,i} \sigma^{k,i,l} d\bar{W}_t^{k,l} + \sum_{k=1}^K \sum_{i=1}^{n_k} \left(\int_{\mathbb{R}_+^*} z (\delta^{k,i,b}(t,z) \tilde{J}^{k,i,b}(dt,dz) + \delta^{k,i,a}(t,z) \tilde{J}^{k,i,a}(dt,dz)) \right), \end{aligned}$$

where for each k and i , $\tilde{J}^{k,i,b}$ and $\tilde{J}^{k,i,a}$ are the martingales associated with $J^{k,i,b}$ and $J^{k,i,a}$ respectively. We want to control the risk linked to the second term, i.e.

$$\sum_{k=1}^K \sum_{i=1}^{n_k} \sum_{l=1}^{M_k} q_t^{k,i} \sigma^{k,i,l} d\bar{W}_t^{k,l}.$$

The dynamics of the quadratic covariation of this term is given by

$$\sum_{k=1}^K \sum_{i,j=1}^{n_k} \sum_{l=1}^{M_k} q_t^{k,i} q_t^{k,j} \sigma^{k,i,l} \sigma^{k,j,l} dt = \sum_{k=1}^K \sum_{l=1}^{M_k} q_t^{k'} \Sigma^{k,l} q_t^k dt,$$

where for each k and l :

$$\Sigma^{k,l} = \begin{pmatrix} (\sigma^{k,1,l})^2 & \sigma^{k,1,l} \sigma^{k,2,l} & \dots & \dots & \sigma^{k,1,l} \sigma^{k,n_k,l} \\ \sigma^{k,2,l} \sigma^{k,1,l} & (\sigma^{k,2,l})^2 & \dots & \dots & \vdots \\ \vdots & \dots & \ddots & \dots & \vdots \\ \vdots & \dots & \dots & (\sigma^{k,n_k-1,l})^2 & \sigma^{k,n_k-1,l} \sigma^{k,n_k,l} \\ \sigma^{k,n_k,l} \sigma^{k,1,l} & \dots & \dots & \sigma^{k,n_k,l} \sigma^{k,n_k-1,l} & (\sigma^{k,n_k,l})^2 \end{pmatrix}, \quad (1.5)$$

and $(q_t^k)_{t \in [0,T]} = (q_t^{k,1}, \dots, q_t^{k,n_k})_{t \in [0,T]}$. Thus, we are interested in solving the problem

$$\sup_{((\delta^{k,i})_{i \in [1,n_k]})_{k \in [1,K]} \in \mathcal{A}_{\mathcal{K}}} \mathbb{E} \left[V_T - \sum_{k=1}^K \frac{\gamma^k}{2} \int_0^T \sum_{l=1}^{M_k} q_t^{k'} \Sigma^{k,l} q_t^k dt \right], \quad (1.6)$$

with $\gamma^1, \dots, \gamma^K > 0$, and

$$\mathcal{A}_{\mathcal{K}} = \mathcal{A}^{n_1} \times \dots \times \mathcal{A}^{n_K},$$

where for each $k \in \llbracket 1, K \rrbracket$,

$$\mathcal{A}^{n_k} = \left\{ \delta = (\delta^{k,i,b}, \delta^{k,i,a})_{1 \leq i \leq n_k} : \Omega \times [0, T] \times \mathbb{R}_+^* \mapsto \mathbb{R}^{2n_k} \delta \text{ is } \mathcal{P} \otimes \mathcal{B}(\mathbb{R}_+^*) \text{ - measurable,} \right. \\ \left. \forall i \in \{1, \dots, n_k\}, \delta^{k,i,b}(t, z) \geq -\delta_\infty \mathbb{P} \otimes dt \otimes \mu^{k,i,b} \text{ a.e. and } \delta^{k,i,a}(t, z) \geq -\delta_\infty \mathbb{P} \otimes dt \otimes \mu^{k,i,a} \text{ a.e.} \right\}.$$

From what precedes, this problem is equivalent to

$$\sup_{((\delta^{k,i})_{i \in \llbracket 1, n_k \rrbracket})_{k \in \llbracket 1, K \rrbracket} \in \mathcal{A}_{\mathcal{K}}} \mathbb{E} \left[\sum_{k=1}^K \int_0^T \left\{ \sum_{i=1}^{n_k} \left(\int_{\mathbb{R}_+^*} z \left(\delta^{k,i,b}(t, z) \Lambda^{k,i,b}(\delta^{k,i,b}(t, z)) \mu^{k,i,b}(dz) \right. \right. \right. \right. \\ \left. \left. \left. + \delta^{k,i,a}(t, z) \Lambda^{k,i,a}(\delta^{k,i,a}(t, z)) \mu^{k,i,a}(dz) \right) \right) - \frac{\gamma^k}{2} \int_0^T \sum_{l=1}^{M_k} q_t^{k'} \Sigma^{k,l} q_t^k \right\} dt \right]. \quad (1.7)$$

Finally, this problem boils down to

$$\sum_{k=1}^K \sup_{(\delta^{k,i})_{i \in \llbracket 1, n_k \rrbracket} \in \mathcal{A}^{n_k}} \mathbb{E} \left[\int_0^T \left\{ \sum_{i=1}^{n_k} \left(\int_{\mathbb{R}_+^*} z \left(\delta^{k,i,b}(t, z) \Lambda^{k,i,b}(\delta^{k,i,b}(t, z)) \mu^{k,i,a}(dz) \right. \right. \right. \right. \\ \left. \left. \left. + \delta^{k,i,a}(t, z) \Lambda^{k,i,a}(\delta^{k,i,a}(t, z)) \mu^{k,i,a}(dz) \right) \right) - \frac{\gamma^k}{2} \int_0^T \sum_{l=1}^{M_k} q_t^{k'} \Sigma^{k,l} q_t^k \right\} dt \right]. \quad (1.8)$$

We therefore have K simpler problems to solve. Let us fix $k \in \llbracket 1, K \rrbracket$, and consider the k -th problem:

$$\sup_{(\delta^{k,i})_{i \in \llbracket 1, n_k \rrbracket} \in \mathcal{A}^{n_k}} \mathbb{E} \left[\int_0^T \left\{ \sum_{i=1}^{n_k} \left(\int_{\mathbb{R}_+^*} z \left(\delta^{k,i,b}(t, z) \Lambda^{k,i,b}(\delta^{k,i,b}(t, z)) \mu^{k,i,b}(dz) \right. \right. \right. \right. \\ \left. \left. \left. + \delta^{k,i,a}(t, z) \Lambda^{k,i,a}(\delta^{k,i,a}(t, z)) \mu^{k,i,a}(dz) \right) \right) - \frac{\gamma^k}{2} \int_0^T \sum_{l=1}^{M_k} q_t^{k'} \Sigma^{k,l} q_t^k \right\} dt \right]. \quad (1.9)$$

The state process of this problem is $(q_t^k)_{t \geq 0}$, which is of dimension n_k . Notice from (1.5) that for each $l = 1, \dots, M_k$, we can write

$$\Sigma^{k,l} = u^{k,l} u^{k,l'}$$

where

$$u^{k,l} = \begin{pmatrix} \sigma^{k,1,l} \\ \sigma^{k,2,l} \\ \vdots \\ \sigma^{k,n_k,l} \end{pmatrix}.$$

Let us denote by $(f_t^{k,l})_{t \geq 0}$ the process defined by

$$f_t^{k,l} = q_t^{k'} u^{k,l} \quad \forall t \geq 0,$$

and let us denote $f_t^k = (f_t^{k,1}, \dots, f_t^{k,M_k}) \quad \forall t \geq 0$. Let us denote by U the matrix whose columns are given by the $(u^{k,l})_{l \in \llbracket 1, M_k \rrbracket}$:

$$U = (u^{k,1} \quad \dots \quad u^{k,M_k}).$$

Then $(f_t^k)_{t \geq 0} = (q_t^{k'} U)_{t \geq 0}$, and we see that problem (1.9) is equivalent to

$$\begin{aligned} \sup_{(\delta^{k,i})_{i \in \llbracket 1, n_k \rrbracket} \in \mathcal{A}^{n_k}} \mathbb{E} \left[\int_0^T \left\{ \sum_{i=1}^{n_k} \left(\int_{\mathbb{R}_+^*} z \left(\delta^{k,i,b}(t, z) \Lambda^{k,i,b}(\delta^{k,i,b}(t, z)) \mu^{k,i,b}(dz) \right. \right. \right. \right. \\ \left. \left. \left. + \delta^{k,i,a}(t, z) \Lambda^{k,i,a}(\delta^{k,i,a}(t, z)) \mu^{k,i,a}(dz) \right) \right) - \frac{\gamma^k}{2} \int_0^T \sum_{l=1}^{M_k} (f_t^{k,l})^2 \right\} dt \right]. \end{aligned} \quad (1.10)$$

The state process of this problem is $(f_t^k)_{t \geq 0}$, which is of dimension M_k . We typically have $M_k \ll n_k$ so this problem is more tractable than (1.9). Let us define the function $\mathcal{J}_k : [0, T] \times \mathbb{R}^{M_k} \times \mathcal{A}^{n_k} \rightarrow \mathbb{R}$ given by :

$$\begin{aligned} \mathcal{J}_k(t, f^k, (\delta^{k,i})_{i \in \llbracket 1, n_k \rrbracket}) = \mathbb{E} \left[\int_t^T \left\{ \sum_{i=1}^{n_k} \left(\int_{\mathbb{R}_+^*} z \left(\delta^{k,i,b}(t, z) \Lambda^{k,i,b}(\delta^{k,i,b}(t, z)) \mu^{k,i,b}(dz) \right. \right. \right. \right. \\ \left. \left. \left. + \delta^{k,i,a}(t, z) \Lambda^{k,i,a}(\delta^{k,i,a}(t, z)) \mu^{k,i,a}(dz) \right) \right) - \frac{\gamma^k}{2} \int_0^T \sum_{l=1}^{M_k} (f_t^{k,l})^2 \right\} dt \Big| f_t^k = f^k \right], \end{aligned} \quad (1.11)$$

$\forall t \in [0, T]$, $f^k \in \mathbb{R}^{M_k}$ and $\forall (\delta^{k,i})_{i \in \llbracket 1, n_k \rrbracket} \in \mathcal{A}^{n_k}$. Then the value function $\theta_k : [0, T] \times \mathbb{R}^{M_k} \rightarrow \mathbb{R}$ of (1.10) is given by :

$$\theta_k(t, f^k) = \sup_{(\delta^{k,i})_{i \in \llbracket 1, n_k \rrbracket} \in \mathcal{A}^{n_k}} \mathcal{J}_k(t, f^k, (\delta^{k,i})_{i \in \llbracket 1, n_k \rrbracket}) \quad \forall t \in [0, T], \forall f^k \in \mathbb{R}^{M_k}. \quad (1.12)$$

The problem (1.8) is equivalent to

$$\begin{aligned} \sum_{k=1}^K \sup_{(\delta^{k,i})_{i \in \llbracket 1, n_k \rrbracket} \in \mathcal{A}^{n_k}} \mathbb{E} \left[\int_0^T \left\{ \sum_{i=1}^{n_k} \left(\int_{\mathbb{R}_+^*} z \left(\delta^{k,i,b}(t, z) \Lambda^{k,i,b}(\delta^{k,i,b}(t, z)) \mu^{k,i,b}(dz) \right. \right. \right. \right. \\ \left. \left. \left. + \delta^{k,i,a}(t, z) \Lambda^{k,i,a}(\delta^{k,i,a}(t, z)) \mu^{k,i,a}(dz) \right) \right) - \frac{\gamma^k}{2} \int_0^T \sum_{l=1}^{M_k} (f_t^{k,l})^2 \right\} dt \right]. \end{aligned} \quad (1.13)$$

Therefore its value function $\theta : [0, T] \times \tilde{\mathcal{Q}}_1 \times \dots \times \tilde{\mathcal{Q}}_K \rightarrow \mathbb{R}$ is given by

$$\theta(t, f^1, \dots, f^K) = \sum_{k=1}^K \theta_k(t, f^k), \quad (1.14)$$

$\forall t \in [0, T]$ and $\forall f^1, \dots, f^K \in \mathbb{R}^{M_1} \times \dots \times \mathbb{R}^{M_K}$. For each $k = 1, \dots, K$, the function θ_k is the unique solution to

$$\begin{cases} 0 = -\frac{\partial \theta_k}{\partial t}(t, f^k) + \frac{\gamma^k}{2} \sum_{l=1}^{M_k} (f^{k,l})^2 - \sum_{i=1}^{n_k} \int_{\mathbb{R}_+^*} z H^{i,b} \left(\frac{\tilde{\theta}(t, f) - \tilde{\theta}(t, f + z e_k^i)}{z} \right) \mu^{k,i,b}(dz) \\ \quad - \sum_{i=1}^d \int_{\mathbb{R}_+^*} z H^{i,a} \left(\frac{\theta_k(t, f^k) - \theta_k(t, f^k - z e_k^i)}{z} \right) \mu^{k,i,a}(dz), & \forall f^k \in \mathbb{R}^{M_k}, \forall t \in [0, T]. \\ \theta_k(T, f^k) = 0 & \forall f^k \in \mathbb{R}^{M_k}, \end{cases} \quad (\text{HJ}_k)$$

1.5.2 On the construction of the processes $J^{i,b}$ and $J^{i,a}$

Let us consider a new filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathbb{R}_+}, \tilde{\mathbb{P}})$. For the sake of simplicity, assume that $d = 1$ and let us omit the superscript i (the generalization is straightforward). Let us introduce N^b and N^a two independent compound Poisson processes of intensity 1 whose increments follow respectively the distributions $\mu^b(dz)$ and $\mu^a(dz)$ with support on \mathbb{R}_+^* . We denote by $J^b(dt, dz)$ and $J^a(dt, dz)$ the associated random measures.

For each $\delta \in \mathcal{A}$, we introduce the probability measure $\tilde{\mathbb{P}}^\delta$ given by the Radon-Nikodym derivative

$$\frac{d\tilde{\mathbb{P}}^\delta}{d\tilde{\mathbb{P}}} \Big|_{\mathcal{F}_t} = L_t^\delta, \quad (1.15)$$

where $(L_t^\delta)_{t \geq 0}$ is the unique solution of the stochastic differential equation

$$dL_t^\delta = L_{t-}^\delta \left(\int_{\mathbb{R}_+^*} (\Lambda^b(\delta^b(t, z)) - 1) J^b(dt, dz) + \int_{\mathbb{R}_+^*} (\Lambda^a(\delta^a(t, z)) - 1) J^a(dt, dz) \right),$$

with $L_0^\delta = 1$.

We then know (see for instance [34]) that under $\tilde{\mathbb{P}}^\delta$, $J^b(dt, dz)$ and $J^a(dt, dz)$ have respective intensity kernels

$$\lambda_t^{\delta,b}(dz) = \Lambda^b(\delta^b(t, z)) \mu^b(dz) \quad \text{and} \quad \lambda_t^{\delta,a}(dz) = \Lambda^a(\delta^a(t, z)) \mu^a(dz)$$

as in the body of the paper.

Chapter 2

From passive to active: when market makers are also market takers

Abstract

In OTC markets, one of the main tasks of dealers / market makers consists in answering prices at which they agree to buy and sell the assets and securities they have in their scope. With trading automation, this quoting task has to be done algorithmically. Over the last ten years, many market making models have been designed that can be the basis of quoting algorithms in OTC markets. However, in the literature, most market making models adapted to OTC markets are general and do not focus on the different characteristics of markets for different asset classes. In particular, to the best of our knowledge, in every OTC market making model, the market maker only has a rather “passive” behaviour: that of waiting for clients wishing to trade with her. On some markets however, for instance on FX cash markets, market makers typically have access to liquidity pools where they can unwind part of their inventory. In this paper, we propose a model taking this feature into account, therefore allowing market makers to trade “actively” in the market.

Key words: Market making, Algorithmic trading, Stochastic optimal control, Viscosity solutions

2.1 Introduction

In all financial markets, liquidity has always been mostly provided by a specific category of agents who, on a continuous and regular basis, set prices at which they agree to buy or sell assets and securities. These agents, called market makers or dealers, play a key role in the price formation process in all markets but their exact role and behavior depend on the considered asset class.

In most order-driven markets, such as stock markets, where many traditional exchanges have converted from open outcry communications between human traders to electronic platforms organized around limit order books, and where computers handle almost all market activity, traditional market makers still provide liquidity to the market but they are in competition with other market

participants who can post liquidity-providing orders.

In quote-driven markets, electronification has also been one of the major upheavals of the last decade, with important consequences on market makers / dealers. New platforms have indeed emerged to allow clients to request several dealers for quotes in a simultaneous way in many bond markets. In FX cash markets, dealers propose their own private electronic platforms enabling clients to directly send them requests for quotes (RFQ) and to be connected to their stream (RFS). Many dealer-to-dealer or all-to-all platforms have also emerged, therefore blurring the frontier between OTC and organized markets.

In all financial markets, most market makers are now replaced by market making algorithms. This electronification of market making has gone along with the development of many market making models in the academic literature.

Economists interested in market microstructure have studied for a long time the behaviour of market makers / dealers with the aim of understanding market liquidity and the magnitude of bid-ask spreads. Since the 1980s, two types of models have been discussed in the literature: models where one or several risk-averse market makers optimize their pricing policy for managing their inventory risk models (see Amihud and Mendelson [12], Ho and Stoll [71, 72], and O'Hara and Oldfield [90]) and models focused on information asymmetries where bid-ask spreads derive from adverse selection (see for instance Copeland and Galai [42] or Glosten and Milgrom [58]).¹

In 2008, largely inspired by Ho and Stoll [71], Avellaneda and Stoikov [13] proposed a stochastic optimal control model to determine the optimal bid and ask quotes that a single-asset risk-averse market maker should set. The authors paved the way to a new literature on market making that goes far beyond the toy models of the economic literature. The resulting new models can be divided into two groups: those adapted to the problem of a market maker in a limit order book and those adapted to OTC markets.

To build a relevant market making model for order-driven markets, and especially stock markets, it is important to take the microstructure into account. For instance, Guilbaud and Pham [68] modeled the market bid-ask spread as a discrete Markov chain jumping according to a Poisson process, and studied the performance of a market maker submitting limit buy/sell order at the best bid/ask quotes, and at the best bid plus one tick and best ask minus one tick to get the priority. Guilbaud and Pham [69] studied a similar problem of optimal market making in a pro rata limit order book, where the dealer may post limit orders but also market orders – represented by impulse controls. Fodra and Pham [49, 50] considered a model in which market orders arrive in the limit order book according to a point process correlated with the stock price itself, modeled as a Markov renewal process. They modeled the market maker as an agent placing limit orders of constant size at the best bid and at the best ask, and solve the market making problem while taking the adverse selection risk into account. More recently, in Abergel et al. [2], the authors proposed a different model for the limit order book (first introduced in Abergel et al. [1]), in which the limit orders, market orders and cancel orders arrive according to Markov jump processes with intensities depending only on the state of the limit order book. They considered the case of a market maker trading in this

¹Other classic economic references on market making include Grossman and Miller [59] and the review paper of Stoll [99].

limit order book, and proposed a quantization-based algorithm to numerically solve the resulting high-dimensional problem. Finally, Capponi et al. [35] studied a discrete-time problem, assuming that the market maker can place bid and ask limit orders simultaneously on both sides at preset dates. In this framework, the number of filled orders during each period depends linearly on the distance between the fundamental price and the market maker’s limit order quotes, with random slope and intercept coefficients. Using standard tools of discrete-time optimal control, the authors managed to get an explicit characterization of the optimal strategy.

In the case of OTC markets, or for order-driven markets in which the spread/tick ratio is large, most market making models derive from the seminal work of Avellaneda and Stoikov [13].² For instance, Guéant et al. [65] provided a rigorous analysis of the stochastic optimal control problem formulated in Avellaneda and Stoikov [13] and showed that, by adding risk limits to the inventory of the market maker, the problem boils down to a system of ordinary differential equations (ODEs) – in particular, those ODEs are linear in the case of the exponential intensity functions proposed in [13]. Cartea, Jaimungal and Ricci considered in [40] a different kind of objective function: instead of the Von Neumann-Morgenstern expected utility of [13], they optimized a risk-adjusted expectation of the PnL. Cartea, Donnelly and Jaimungal studied in [36] the impact of uncertainty on some parameters of the model. Multi-asset market making models have been proposed by Guéant in [61, 62] for both kinds of objective function, and the author showed again that the problem boils down to a system of ODEs. All those models assumed that the trade size is constant: the same quantity of assets is bought/sold at each trade. In [26], Bergault and Guéant introduced a distribution of trade sizes in the model, with the quotes of the market maker depending on the size of each RFQ. This yields an integro-differential equation of the Hamilton-Jacobi type that does not boil down anymore to a finite system of ODEs (but this equation can be seen as an ODE in an infinite-dimensional space). They also proposed a dimensionality reduction technique in order to approximate numerically the optimal bid and ask quotes of the market maker in high dimension, by projecting the market risk on a low-dimensional space of factors. This problem of approximating the solution across a large universe of assets has been addressed using different approaches. In Bergault et al. [25], the authors regarded the Hamilton-Jacobi equation of the problem as a perturbation of another simpler equation whose solution can be computed explicitly and is used as a proxy of the value function. In Guéant and Manziuk [66], the authors used neural networks instead of grids to compute an approximate solution – a method inspired by approximate dynamic programming and reinforcement learning techniques.

In most models adapted to OTC markets, market makers are liquidity provider and only make money out of their bid-ask spread: they buy assets at the bid price they quote and sell them at the ask price they quote – ideally earning the difference between these two prices. Of course, market makers seldom buy and sell simultaneously and they therefore carry inventory; and hence have an exposure to price changes. The problem faced by market makers in these models is already a subtle dynamic optimization problem in which market makers must mitigate the risk associated with price changes by skewing their quotes as a function of their inventory. However, this problem is complexified on some markets like FX cash markets, as market makers have an additional way to manage their inventory risk: on those markets, traders typically have access to several liquidity pools – on the Dealer-to-Dealer (D2D) segment of the market, for instance – where they can unwind part of their position, without having to “wait” for clients sending requests and trading with them.

²See the books of Cartea et al. [39] and Guéant [61] for a detailed discussion.

To the best of our knowledge, this last feature (i.e. models with both requests for quotes / requests for stream and access to liquidity pools) has never been studied in the academic literature on market making on OTC markets. The main goal of this paper is in effect to include the possibility for the market makers to buy and sell “actively” (in continuous time) some assets on exchanges, in order to better mitigate their inventory risk. In comparison to previous models, we introduce a new form of control in addition to the bid and ask quotes. This control represents the trading rate of the market maker (for the assets she can trade “actively”), and is inspired by Almgren-Chriss-like models of optimal execution (see Almgren et al. [11, 7], and Guéant [61] for a general presentation).

In Section 2.2, we present our multi-asset market making model involving assets for which the market maker only has a “passive” market making role, (liquid) assets for which she can only trade “actively” by buying or selling in some liquidity pools, and assets for which she can do both. We then formulate the stochastic optimal control problem of the market maker. In Section 2.3, we characterize the associated value function as the unique continuous viscosity solution of a Partial Integro-Differential Equation (PIDE) of the Hamilton-Jacobi (HJ) type. We illustrate our model numerically in Section 2.4 and discuss the results.

2.2 A model for active market makers

In this paper, we consider a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{F} = (\mathcal{F}_t)_{t \geq 0})$ satisfying the usual conditions. We assume this probability space to be large enough to support all the processes we introduce.

2.2.1 Modeling framework and notations

We consider a market consisting of d assets, and denote by I the set $I = \{1, \dots, d\}$. We introduce $I_{\mathcal{M}}$ and $I_{\mathcal{T}}$ two subsets of I such that $I = I_{\mathcal{M}} \cup I_{\mathcal{T}}$. The assets with index in $I_{\mathcal{M}}$ can be traded by the market maker via requests for quote (RFQs)³: in that case, she first receives a request from a client wishing to buy or sell a given asset, and then proposes a price to the client who finally decides whether she accepts to trade at that price or not. The market maker can also choose to buy or sell actively the assets with index in $I_{\mathcal{T}}$ from the market. Note that $(I_{\mathcal{M}}, I_{\mathcal{T}})$ is not *a priori* a partition of I : some (and maybe all) assets may be traded both ways.

For $i \in I$, the price of the i -th asset is modeled by a process $(S_t^i)_{t \geq 0}$ with the following dynamics:

$$dS_t^i = \sigma^i dW_t^i,$$

with S_0^i given, $\sigma^i > 0$, and the process $(W_t^1, \dots, W_t^d)_{t \geq 0}$ is a d -dimensional Brownian motion adapted to the filtration \mathbb{F} and such that $\forall (i, j) \in I^2$,

$$\frac{d\langle W^i, W^j \rangle_t}{dt} =: \rho^{i,j} \in [-1, 1].$$

³Requests for stream (RFSs) can also be considered as their modelization does not differ from that of RFQs.

We denote by Σ the variance-covariance matrix associated with the process $(S_t^1, \dots, S_t^d)_{t \geq 0}$.

For the assets traded with requests, the bid and ask quotes proposed by the market maker depend on the size $z \in \mathbb{R}_+^*$ of the request. For a given asset $i \in I_{\mathcal{M}}$, they are modeled by maps $S^{i,b}, S^{i,a} : \Omega \times [0, T] \times \mathbb{R}_+^* \rightarrow \mathbb{R}_+$ which are $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}_+^*)$ -measurable, where \mathcal{P} denotes the σ -algebra of \mathbb{F} -predictable subsets of $\Omega \times [0, T]$ and $\mathcal{B}(\mathbb{R}_+^*)$ denotes the Borelian sets of \mathbb{R}_+^* .

For each $i \in I_{\mathcal{M}}$, we introduce $J^{i,b}(dt, dz)$ and $J^{i,a}(dt, dz)$ two *càdlàg* \mathbb{R}_+ -marked point processes corresponding to the number of transactions resulting from requests at the bid and at the ask, respectively, for asset i . We denote respectively by $\tilde{J}^{i,b}(dt, dz)$ and $\tilde{J}^{i,a}(dt, dz)$ the associated compensated processes.

We first present a “natural” formulation of the problem, that will then be relaxed.

For the assets traded actively by the market maker, we consider that she trades continuously in asset $i \in I_{\mathcal{T}}$ at a rate $(w_t^i)_{t \geq 0}$ (she buys when $w_t^i \geq 0$ and sells otherwise).⁴

Therefore the inventory of the market maker, modeled by a d -dimensional inventory process $(q_t)_{t \geq 0} = (q_t^1, \dots, q_t^d)_{t \geq 0}$, has the following dynamics:

$$\begin{aligned} dq_t^i &= \int_{\mathbb{R}_+^*} z J^{i,b}(dt, dz) - \int_{\mathbb{R}_+^*} z J^{i,a}(dt, dz) \quad \forall i \in I_{\mathcal{M}} \cap (I \setminus I_{\mathcal{T}}), \\ dq_t^i &= \int_{\mathbb{R}_+^*} z J^{i,b}(dt, dz) - \int_{\mathbb{R}_+^*} z J^{i,a}(dt, dz) + w_t^i dt \quad \forall i \in I_{\mathcal{M}} \cap I_{\mathcal{T}}, \\ dq_t^i &= w_t^i dt \quad \forall i \in I_{\mathcal{T}} \cap (I \setminus I_{\mathcal{M}}), \end{aligned} \tag{2.1}$$

with q_0 given. We assume that all the processes $J^{i,b}(dt, dz)$ and $J^{i,a}(dt, dz)$ for $i \in I_{\mathcal{M}}$ do not have simultaneous jumps almost surely. Moreover, for $i \in I_{\mathcal{M}}$, we denote by $(\nu_t^{i,b}(dz))_{t \geq 0}$ and $(\nu_t^{i,a}(dz))_{t \geq 0}$ the intensity kernels of $J^{i,b}(dt, dz)$ and $J^{i,a}(dt, dz)$, respectively.

We assume that the market maker wants her inventory to remain in a given compact set $\mathcal{Q} \subset \mathbb{R}^d$, and denote by $\text{int}(\mathcal{Q})$ the interior of the set \mathcal{Q} . We also need to assume that the boundary $\partial\mathcal{Q}$ of \mathcal{Q} is negligible according to the Lebesgue measure. The intensities $(\nu_t^{i,b}(dz))_{t \geq 0}$ and $(\nu_t^{i,a}(dz))_{t \geq 0}$ verify:

$$\begin{aligned} \nu_t^{i,b}(dz) &= \mathbb{1}_{\{q_{t-} + ze^i \in \mathcal{Q}\}} \Lambda^{i,b}(\delta^{i,b}(t, z)) \mu^{i,b}(dz), \\ \nu_t^{i,a}(dz) &= \mathbb{1}_{\{q_{t-} - ze^i \in \mathcal{Q}\}} \Lambda^{i,a}(\delta^{i,a}(t, z)) \mu^{i,a}(dz), \end{aligned} \tag{2.2}$$

where $\forall i \in I_{\mathcal{M}}$, $\mu^{i,b}$ and $\mu^{i,a}$ are probability measures on \mathbb{R}_+^* , absolutely continuous with respect to the Lebesgue measure such that $\int_{\mathbb{R}_+^*} z \mu^{i,b}(dz) =: \Delta^{i,b} < +\infty$, $\int_{\mathbb{R}_+^*} z \mu^{i,a}(dz) =: \Delta^{i,a} < +\infty$,

$$\delta^{i,b}(t, z) = S_t^i - S^{i,b}(t, z) \text{ and } \delta^{i,a}(t, z) = S^{i,a}(t, z) - S_t^i,$$

⁴This is similar to what was done in Almgren [7].

and $\Lambda^{i,b}, \Lambda^{i,a}$ are two functions satisfying the following hypotheses (H_Λ):

- $\Lambda^{i,b}$ and $\Lambda^{i,a}$ are twice continuously differentiable,
- $\Lambda^{i,b}$ and $\Lambda^{i,a}$ are decreasing, with $\forall \delta \in \mathbb{R}, \Lambda^{i,b'}(\delta) < 0$ and $\Lambda^{i,a'}(\delta) < 0$,
- $\lim_{\delta \rightarrow +\infty} \Lambda^{i,b}(\delta) = 0$ and $\lim_{\delta \rightarrow +\infty} \Lambda^{i,a}(\delta) = 0$,
- $\sup_{\delta \in \mathbb{R}} \frac{\Lambda^{i,b}(\delta)\Lambda^{i,b''}(\delta)}{(\Lambda^{i,b'}(\delta))^2} < 2$ and $\sup_{\delta \in \mathbb{R}} \frac{\Lambda^{i,a}(\delta)\Lambda^{i,a''}(\delta)}{(\Lambda^{i,a'}(\delta))^2} < 2$.

Remark 6. For a given asset $i \in I_{\mathcal{M}}$, $\Lambda^{i,\cdot}$ typically has the form $\Lambda^{i,\cdot}(\delta) = \lambda_R^{i,\cdot} f^{i,\cdot}(\delta)$, where $\lambda_R^{i,\cdot}$ can be seen as the (constant) intensity of arrival of requests. $\mu^{i,\cdot}$ can be seen as the distribution of the size of requests, and $f^{i,\cdot}$ is the probability that a request will result in a transaction knowing the quote that the market maker proposes.

Finally, the process $(X_t)_{t \geq 0}$ modeling the market maker's cash account has the dynamics:

$$\begin{aligned}
dX_t &= \sum_{i \in I_{\mathcal{M}}} \left(\int_{\mathbb{R}_+^*} z \left(S_t^i + \delta^{i,a}(t, z) \right) J^{i,a}(dt, dz) - \int_{\mathbb{R}_+^*} z \left(S_t^i - \delta^{i,b}(t, z) \right) J^{i,b}(dt, dz) \right) \\
&\quad + \sum_{i \in I_{\mathcal{T}}} \left(-w_t^i S_t^i dt - L^i(w_t^i) dt \right) \\
&= \sum_{i \in I_{\mathcal{M}}} \left(\int_{\mathbb{R}_+^*} z \delta^{i,b}(t, z) J^{i,b}(dt, dz) + \int_{\mathbb{R}_+^*} z \delta^{i,a}(t, z) J^{i,a}(dt, dz) \right) \\
&\quad - \sum_{i \in I_{\mathcal{T}}} L^i(w_t^i) dt - \langle S_t, dq_t \rangle,
\end{aligned} \tag{2.3}$$

where $\langle \cdot, \cdot \rangle$ denotes the standard scalar product in \mathbb{R}^d , and $\forall i \in I_{\mathcal{T}}$, the penalty function $L^i : \mathbb{R} \rightarrow \mathbb{R}$ can be seen as the temporary price impact of the market maker when she chooses to be active and satisfies the following hypotheses (H_L):

- $L^i(0) = 0$,
- L^i is strictly convex, increasing on \mathbb{R}_+ and decreasing on \mathbb{R}_- ,
- L^i is asymptotically superlinear, i.e., $\lim_{|e| \rightarrow +\infty} \frac{L^i(e)}{|e|} = +\infty$.

We define the set of admissible controls $\tilde{\mathcal{A}}$ as

$$\begin{aligned}
\tilde{\mathcal{A}} := & \left\{ ((\delta^i)_{i \in I_{\mathcal{M}}}, (w^i)_{i \in I_{\mathcal{T}}}) \mid \forall i \in I_{\mathcal{M}}, \delta^i = (\delta^{i,b}, \delta^{i,a}) : \Omega \times [0, T] \times \mathbb{R}_+^* \mapsto \mathbb{R}^2 \text{ is } \mathcal{P} \otimes \mathcal{B}(\mathbb{R}_+^*) \text{ - measurable,} \right. \\
& \forall i \in I_{\mathcal{T}}, w^i : \Omega \times [0, T] \mapsto \mathbb{R} \text{ is progressively measurable, and such that } \int_0^T |w_t^i| dt \in L^1(\Omega) \\
& \left. \text{and } q_t \in \mathcal{Q} \text{ } \mathbb{P} \otimes dt \text{ a.s.} \right\}.
\end{aligned} \tag{2.4}$$

For two given continuous penalty functions $\psi : \mathbb{R}^d \rightarrow \mathbb{R}_+$ and $\ell_d : \mathbb{R}^d \rightarrow \mathbb{R}_+$, modeling the risk aversion of the market maker and the cost of liquidity, we aim at maximizing:

$$\mathbb{E} \left[X_T + \langle q_T, S_T \rangle - \ell_d(q_T) - \int_0^T \psi(q_t) dt \right] \quad (2.5)$$

over the set $\tilde{\mathcal{A}}$ of admissible controls $((\delta^i)_{i \in I_{\mathcal{M}}}, (v^i)_{i \in I_{\mathcal{T}}})$. This is a problem with state constraints (as presented for instance in [19]).

In order to ease the mathematical analysis, we now relax the state constraints. We introduce a function $\chi : \mathbb{R}^d \rightarrow [0, 1]$ which is a Lipschitz approximation of the indicator function of $\text{int}(\mathcal{Q})$, with

$$\chi(q) = 0 \quad \forall q \in \mathbb{R}^d \setminus \text{int}(\mathcal{Q}).$$

We perform what heuristically corresponds to a change of variable “ $v_t^i = \frac{w_t^i}{\chi(q_t)}$ ”. For the assets traded actively by the market maker, we consider that she trades continuously in asset $i \in I_{\mathcal{T}}$ at a rate $(v_t^i)_{t \geq 0}$ (she buys when $v_t^i \geq 0$ and sells otherwise), and we assume from now on that the inventory of the market maker has the following dynamics:

$$\begin{aligned} dq_t^i &= \int_{\mathbb{R}_+^*} z J^{i,b}(dt, dz) - \int_{\mathbb{R}_+^*} z J^{i,a}(dt, dz) \quad \forall i \in I_{\mathcal{M}} \cap (I \setminus I_{\mathcal{T}}), \\ dq_t^i &= \int_{\mathbb{R}_+^*} z J^{i,b}(dt, dz) - \int_{\mathbb{R}_+^*} z J^{i,a}(dt, dz) + v_t^i \chi(q_t) dt \quad \forall i \in I_{\mathcal{M}} \cap I_{\mathcal{T}}, \\ dq_t^i &= \chi(q_t) v_t^i dt \quad \forall i \in I_{\mathcal{T}} \cap (I \setminus I_{\mathcal{M}}), \end{aligned} \quad (2.6)$$

with q_0 given.

Finally, the process $(X_t)_{t \geq 0}$ modeling the market maker’s cash account now has the dynamics:

$$\begin{aligned} dX_t &= \sum_{i \in I_{\mathcal{M}}} \left(\int_{\mathbb{R}_+^*} z (S_t^i + \delta^{i,a}(t, z)) J^{i,a}(dt, dz) - \int_{\mathbb{R}_+^*} z (S_t^i - \delta^{i,b}(t, z)) J^{i,b}(dt, dz) \right) \\ &\quad + \chi(q_t) \sum_{i \in I_{\mathcal{T}}} (-v_t^i S_t^i dt - L^i(v_t^i) dt) \\ &= \sum_{i \in I_{\mathcal{M}}} \left(\int_{\mathbb{R}_+^*} z \delta^{i,b}(t, z) J^{i,b}(dt, dz) + \int_{\mathbb{R}_+^*} z \delta^{i,a}(t, z) J^{i,a}(dt, dz) \right) \\ &\quad - \chi(q_t) \sum_{i \in I_{\mathcal{T}}} L^i(v_t^i) dt - \langle S_t, dq_t \rangle. \end{aligned} \quad (2.7)$$

This forces the market maker to trade continuously only in a way that would keep her inventory inside $\text{int}(\mathcal{Q})$.

We fix $\delta_\infty > 0$ to simplify the problem, and define the following set $\mathcal{A}_{\mathcal{M}}$ of admissible controls for

the quotes of an asset of group $I_{\mathcal{M}}$:

$$\mathcal{A}_{\mathcal{M}} = \left\{ \delta = (\delta^b, \delta^a) : \Omega \times [0, T] \times \mathbb{R}_+^* \mapsto \mathbb{R}^2 \mid \delta \text{ is } \mathcal{P} \otimes \mathcal{B}(\mathbb{R}_+^*) \text{ - measurable} \right. \\ \left. \text{and } \delta^b(t, z) \wedge \delta^a(t, z) \geq -\delta_\infty \mathbb{P} \otimes dt \otimes dz \text{ a.s.} \right\}.$$

We also fix $v_\infty > 0$ and define the set $\mathcal{A}_{\mathcal{T}}$ of admissible continuous trading strategies for an asset of group $I_{\mathcal{T}}$:

$$\mathcal{A}_{\mathcal{T}} = \left\{ v : \Omega \times [0, T] \mapsto \mathbb{R} \mid v \text{ is progressively measurable,} \right. \\ \left. \text{and such that } \forall i \in I_{\mathcal{T}}, |v_t^i| \leq v_\infty \text{ for all } t \in [0, T] \text{ a.s.} \right\}.$$

We now aim at maximizing:

$$\mathbb{E} \left[X_T + \langle q_T, S_T \rangle - \ell_d(q_T) - \int_0^T \psi(q_t) dt \right] \quad (2.8)$$

over the set $\mathcal{A}_{\mathcal{M}}^{\text{card}(I_{\mathcal{M}})} \times \mathcal{A}_{\mathcal{T}}^{\text{card}(I_{\mathcal{T}})}$ of admissible controls $((\delta^i)_{i \in I_{\mathcal{M}}}, (v^i)_{i \in I_{\mathcal{T}}})$. To ease the notations, in what follows, we denote by $N_{\mathcal{M}}$ and $N_{\mathcal{T}}$ the values $N_{\mathcal{M}} = \text{card}(I_{\mathcal{M}})$ and $N_{\mathcal{T}} = \text{card}(I_{\mathcal{T}})$.

Remark 7. For instance, for a constant $\gamma > 0$, we can choose $\psi(q) = \frac{\gamma}{2} q' \Sigma q$ or $\psi(q) = \gamma \sqrt{q' \Sigma q}$ and $\ell_d(q) = 0$, $\ell_d(q) = \frac{\zeta}{2} q' \Sigma q$ or $\ell_d(q) = \zeta \sqrt{q' \Sigma q}$ (for $\zeta > 0$), as done in [39], [40], [62], and [66].

After applying Itô's formula to $(X_t + \langle q_t, S_t \rangle)_{t \geq 0}$ between 0 and T , it is easy to see that the problem is equivalent to maximizing:

$$\mathbb{E} \left[\int_0^T \left\{ \int_{\mathbb{R}_+^*} \sum_{i \in I_{\mathcal{M}}} \left(z \delta^{i,b}(t, z) \mathbb{1}_{\{q_t^- + ze^i \in \mathcal{Q}\}} \Lambda^{i,b}(\delta^{i,b}(t, z)) \mu^{i,b}(dz) \right. \right. \right. \\ \left. \left. \left. + z \delta^{i,a}(t, z) \mathbb{1}_{\{q_t^- - ze^i \in \mathcal{Q}\}} \Lambda^{i,a}(\delta^{i,a}(t, z)) \mu^{i,a}(dz) \right) \right. \right. \\ \left. \left. - \chi(q_t) \sum_{i \in I_{\mathcal{T}}} L^i(v_t^i) - \psi(q_t) \right\} dt - \ell_d(q_T) \right],$$

over the set of admissible controls $\mathcal{A}_{\mathcal{M}}^{N_{\mathcal{M}}} \times \mathcal{A}_{\mathcal{T}}^{N_{\mathcal{T}}}$.

We therefore introduce the function $\mathcal{J} : [0, T] \times \mathcal{Q} \times \mathcal{A}_{\mathcal{M}}^{N_{\mathcal{M}}} \times \mathcal{A}_{\mathcal{T}}^{N_{\mathcal{T}}} \rightarrow \mathbb{R}$ such that, $\forall t \in [0, T]$, $\forall q = (q^1, \dots, q^d) \in \mathbb{R}^d$, $\forall ((\delta^i)_{i \in I_{\mathcal{M}}}, (v^i)_{i \in I_{\mathcal{T}}}) \in \mathcal{A}_{\mathcal{M}}^{N_{\mathcal{M}}} \times \mathcal{A}_{\mathcal{T}}^{N_{\mathcal{T}}}$, if we denote by $(q_s^{t,q,\delta,\bar{v}})_{s \geq t}$ the

inventory process starting in state q at time t and controlled by $(\bar{\delta}, \bar{v}) = ((\delta^i)_{i \in I_{\mathcal{M}}}, (v^i)_{i \in I_{\mathcal{T}}})$:

$$\begin{aligned} \mathcal{J}(t, q, \bar{\delta}, \bar{v}) = & \mathbb{E} \left[\int_t^T \left\{ \int_{\mathbb{R}_+^*} \sum_{i \in I_{\mathcal{M}}} \left(z \delta^{i,b}(s, z) \mathbb{1}_{\{q_{s-} + ze^i \in \mathcal{Q}\}} \Lambda^{i,b}(\delta^{i,b}(s, z)) \mu^{i,b}(dz) \right. \right. \right. \\ & \left. \left. + z \delta^{i,a}(s, z) \mathbb{1}_{\{q_{s-} - ze^i \in \mathcal{Q}\}} \Lambda^{i,a}(\delta^{i,a}(s, z)) \mu^{i,a}(dz) \right) \right. \\ & \left. - \chi(q_s) \sum_{i \in I_{\mathcal{T}}} L^i(v_s^i) - \psi(q_s^{t,q,\bar{\delta},\bar{v}}) \right\} ds - \ell_d(q_T^{t,q,\bar{\delta},\bar{v}}) \right]. \end{aligned}$$

The value function $\theta : [0, T] \times \mathcal{Q} \rightarrow \mathbb{R}$ of the problem is then defined as follows:

$$\theta(t, q) = \sup_{(\bar{\delta}, \bar{v}) \in \mathcal{A}_{\mathcal{M}}^{N_{\mathcal{M}}} \times \mathcal{A}_{\mathcal{T}}^{N_{\mathcal{T}}}} \mathcal{J}(t, q, \bar{\delta}, \bar{v}), \quad \forall (t, q) \in [0, T] \times \mathbb{R}^d.$$

We will show that θ is the unique continuous viscosity solution on $[0, T] \times \mathcal{Q}$ to the following Hamilton-Jacobi (HJ) partial integro-differential equation:

$$\begin{cases} 0 = -\partial_t \theta(t, q) + \psi(q) - \sum_{i \in I_{\mathcal{M}}} \int_{\mathbb{R}_+^*} \mathbb{1}_{\{q + ze^i \in \mathcal{Q}\}} z H^{i,b} \left(\frac{\theta(t, q) - \theta(t, q + ze^i)}{z} \right) \mu^{i,b}(dz) \\ \quad - \sum_{i \in I_{\mathcal{M}}} \int_{\mathbb{R}_+^*} \mathbb{1}_{\{q - ze^i \in \mathcal{Q}\}} z H^{i,a} \left(\frac{\theta(t, q) - \theta(t, q - ze^i)}{z} \right) \mu^{i,a}(dz) - \chi(q) \sum_{i \in I_{\mathcal{T}}} \mathcal{H}^i(\partial_{q^i} \theta(t, q)) \quad \forall t \in [0, T] \\ \theta(T, q) = -\ell_d(q), \end{cases} \quad (\text{HJ})$$

where $\forall i \in I_{\mathcal{M}}$:

$$H^{i,b} : p \in \mathbb{R} \mapsto \sup_{\delta \geq -\delta_\infty} \Lambda^{i,b}(\delta)(\delta - p) \text{ and } H^{i,a} : p \in \mathbb{R} \mapsto \sup_{\delta \geq -\delta_\infty} \Lambda^{i,a}(\delta)(\delta - p),$$

and $\forall i \in I_{\mathcal{T}}$:

$$\mathcal{H}^i : p \in \mathbb{R} \mapsto \sup_{|v| \leq v_\infty} (vp - L^i(v)).$$

Notice that for all $i \in I_{\mathcal{T}}$ and $p \in \mathbb{R}$,

$$\mathcal{H}^i(p) = \sup_{|v| \leq v_\infty} (vp - L^i(v)) = \sup_{v \in \mathbb{R}} (vp - \mathcal{L}^i(v)),$$

where

$$\mathcal{L}^i(v) := \begin{cases} L^i(v) & \text{if } |v| \leq v_\infty \\ +\infty & \text{otherwise,} \end{cases}$$

i.e. \mathcal{H}^i is the Legendre-Fenchel transform of \mathcal{L}^i where \mathcal{L}^i is strictly convex and lower semi-continuous.

2.2.2 Preliminary results

Let us start with a simple result:

Proposition 3. *The value function θ is bounded on $[0, T] \times \mathcal{Q}$.*

Proof. $\forall (t, q, (\delta^i)_{i \in I_{\mathcal{M}}}, (v^i)_{i \in I_{\mathcal{T}}}) \in [0, T] \times \mathcal{Q} \times \mathcal{A}_{\mathcal{M}}^{N_{\mathcal{M}}} \times \mathcal{A}_{\mathcal{T}}^{N_{\mathcal{T}}}$, letting $\bar{\delta} = (\delta^i)_{i \in I_{\mathcal{M}}}$ and $\bar{v} = (v^i)_{i \in I_{\mathcal{T}}}$, we have

$$\begin{aligned} \mathcal{J}(t, q, \bar{\delta}, \bar{v}) = & \mathbb{E} \left[\int_t^T \left\{ \int_{\mathbb{R}_+^*} \sum_{i \in I_{\mathcal{M}}} \left(z \delta^{i,b}(s, z) \mathbb{1}_{\{q_{s-} + ze^i \in \mathcal{Q}\}} \Lambda^{i,b}(\delta^{i,b}(s, z)) \mu^{i,b}(dz) \right. \right. \right. \\ & + z \delta^{i,a}(s, z) \mathbb{1}_{\{q_{s-} - ze^i \in \mathcal{Q}\}} \Lambda^{i,a}(\delta^{i,a}(s, z)) \mu^{i,a}(dz) \left. \left. \left. - \chi(q_s) \sum_{i \in I_{\mathcal{T}}} L^i(v_s^i) - \psi(q_s^{t,q,\bar{\delta},\bar{v}}) \right\} ds - \ell_d(q_T^{t,q,\bar{\delta},\bar{v}}) \right]. \end{aligned}$$

As ℓ_d , ψ , χ and L^i are nonnegative for all $i \in I_{\mathcal{T}}$, we get

$$\begin{aligned} \mathcal{J}(t, q, \bar{\delta}, \bar{v}) \leq & \mathbb{E} \left[\int_t^T \left\{ \int_{\mathbb{R}_+^*} \sum_{i \in I_{\mathcal{M}}} \left(z \delta^{i,b}(s, z) \mathbb{1}_{\{q_{s-} + ze^i \in \mathcal{Q}\}} \Lambda^{i,b}(\delta^{i,b}(s, z)) \mu^{i,b}(dz) \right. \right. \right. \\ & \left. \left. \left. + z \delta^{i,a}(s, z) \mathbb{1}_{\{q_{s-} - ze^i \in \mathcal{Q}\}} \Lambda^{i,a}(\delta^{i,a}(s, z)) \mu^{i,a}(dz) \right\} ds \right] \\ \leq & T \sum_{i \in I_{\mathcal{M}}} \left(\Delta^{i,b} \sup_{\delta \geq -\delta_\infty} \delta \Lambda^{i,b}(\delta) + \Delta^{i,a} \sup_{\delta \geq -\delta_\infty} \delta \Lambda^{i,a}(\delta) \right). \end{aligned}$$

The right-hand term is independent from t , q , $\bar{\delta}$ and \bar{v} , so it is clear that \mathcal{J} and θ are bounded from above.

We have

$$\begin{aligned} \mathcal{J}(t, q, \bar{\delta}, \bar{v}) \geq & T \sum_{i \in I_{\mathcal{M}}} \left(-\Delta^{i,b} \delta_\infty \Lambda^{i,b}(-\delta_\infty) - \Delta^{i,a} \delta_\infty \Lambda^{i,a}(-\delta_\infty) \right) \\ & - \mathbb{E} \left[\int_t^T \left\{ \chi(q_s) \sum_{i \in I_{\mathcal{T}}} L^i(v_s^i) + \psi(q_s^{t,q,\bar{\delta},\bar{v}}) \right\} ds + \ell_d(q_T^{t,q,\bar{\delta},\bar{v}}) \right]. \end{aligned}$$

This stays true in particular if we take the processes $(v_s^i)_{s \geq t}$ to be identically equal to 0, and as $L^i(0) = 0$, we have

$$\begin{aligned} \theta(t, q) \geq & T \sum_{i \in I_{\mathcal{M}}} \left(-\Delta^{i,b} \delta_\infty \Lambda^{i,b}(-\delta_\infty) - \Delta^{i,a} \delta_\infty \Lambda^{i,a}(-\delta_\infty) \right) - \mathbb{E} \left[\int_t^T \psi(q_s^{t,q,\bar{\delta},0}) ds + \ell_d(q_T^{t,q,\bar{\delta},0}) \right] \\ \geq & T \sum_{i \in I_{\mathcal{M}}} \left(-\Delta^{i,b} \delta_\infty \Lambda^{i,b}(-\delta_\infty) - \Delta^{i,a} \delta_\infty \Lambda^{i,a}(-\delta_\infty) \right) - T \left(\sup_{q \in \mathcal{Q}} \psi(q) + \sup_{q \in \mathcal{Q}} \ell_d(q) \right). \end{aligned}$$

As ψ and ℓ_d are continuous and \mathcal{Q} is compact, we get that θ is bounded from below. \square

Lemma 2 below is proved in [26], and Lemma 3 is proved in the appendix.

Lemma 2. $\forall i \in I_{\mathcal{M}}$, $H^{i,b}$ and $H^{i,a}$ are two continuously differentiable decreasing functions and the supremum in the definition of $H^{i,b}(p)$ (respectively $H^{i,a}(p)$) is reached at a unique $\delta^{i,b^*}(p)$ (respectively $\delta^{i,a^*}(p)$). Furthermore, δ^{i,b^*} and δ^{i,a^*} are continuous and nondecreasing in p .

Lemma 3. Let $\varphi : [0, T] \times \mathcal{Q} \mapsto \mathbb{R}$ be a bounded function. For all $i \in I_{\mathcal{M}}$, the functions

$$(t, q, z) \in [0, T] \times \mathcal{Q} \times \mathbb{R}_+^* \mapsto \mathbf{1}_{\{q+ze^i \in \mathcal{Q}\}} z H^{i,b} \left(\frac{\varphi(t, q) - \varphi(t, q + ze^i)}{z} \right)$$

and

$$(t, q, z) \in [0, T] \times \mathcal{Q} \times \mathbb{R}_+^* \mapsto \mathbf{1}_{\{q-ze^i \in \mathcal{Q}\}} z H^{i,a} \left(\frac{\varphi(t, q) - \varphi(t, q - ze^i)}{z} \right)$$

are nonnegative and bounded.

Given the properties of L^i for $i \in I_{\mathcal{T}}$, classical results of convex analysis imply the following lemma:

Lemma 4. $\forall i \in I_{\mathcal{T}}$, \mathcal{H}^i is continuously differentiable and Lipschitz, and the supremum in the definition of $\mathcal{H}^i(p)$ is uniquely reached at a $v^{i^*}(p) = \mathcal{H}^{i'}(p)$.

2.3 Viscosity solution to (HJ)

2.3.1 Existence

We denote by $C^1 := C^1([0, T] \times \mathbb{R}^d)$ the class of functions $\varphi : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$ that are continuously differentiable on $[0, T] \times \mathbb{R}^d$.

Definition 1. (i) If u is an upper semicontinuous (USC) function on $[0, T] \times \mathcal{Q}$, we say that u is a viscosity subsolution to (HJ) on $[0, T] \times \mathcal{Q}$ if $\forall (\bar{t}, \bar{q}) \in [0, T] \times \mathcal{Q}$, $\forall \varphi \in C^1$ such that $(u - \varphi)(\bar{t}, \bar{q}) = \max_{(t, q) \in [0, T] \times \mathcal{Q}} (u - \varphi)(t, q)$, we have:

$$\begin{aligned} -\frac{\partial \varphi}{\partial t}(\bar{t}, \bar{q}) + \psi(\bar{q}) - \sum_{i \in I_{\mathcal{M}}} \int_{\mathbb{R}_+^*} \mathbf{1}_{\{q+ze^i \in \mathcal{Q}\}} z H^{i,b} \left(\frac{\varphi(\bar{t}, \bar{q}) - \varphi(\bar{t}, \bar{q} + ze^i)}{z} \right) \mu^{i,b}(dz) \\ - \sum_{i \in I_{\mathcal{M}}} \int_{\mathbb{R}_+^*} \mathbf{1}_{\{q-ze^i \in \mathcal{Q}\}} z H^{i,a} \left(\frac{\varphi(\bar{t}, \bar{q}) - \varphi(\bar{t}, \bar{q} - ze^i)}{z} \right) \mu^{i,a}(dz) - \chi(q) \sum_{i \in I_{\mathcal{T}}} \mathcal{H}^i(\partial_{q^i} \varphi(\bar{t}, \bar{q})) \leq 0. \end{aligned}$$

(ii) If v is a lower semicontinuous (LSC) function on $[0, T] \times \mathcal{Q}$, we say that v is a viscosity supersolution to (HJ) on $[0, T] \times \mathcal{Q}$ if $\forall (\bar{t}, \bar{q}) \in [0, T] \times \mathcal{Q}$, $\forall \varphi \in C^1$ such that $(v - \varphi)(\bar{t}, \bar{q}) =$

$\min_{(t,q) \in [0,T] \times \mathcal{Q}} (v - \varphi)(t, q)$, we have:

$$\begin{aligned} & -\frac{\partial \varphi}{\partial t}(\bar{t}, \bar{q}) + \psi(\bar{q}) - \sum_{i \in I_{\mathcal{M}}} \int_{\mathbb{R}_+^*} \mathbf{1}_{\{q+ze^i \in \mathcal{Q}\}} z H^{i,b} \left(\frac{\varphi(\bar{t}, \bar{q}) - \varphi(\bar{t}, \bar{q} + ze^i)}{z} \right) \mu^{i,b}(dz) \\ & - \sum_{i \in I_{\mathcal{M}}} \int_{\mathbb{R}_+^*} \mathbf{1}_{\{q-ze^i \in \mathcal{Q}\}} z H^{i,a} \left(\frac{\varphi(\bar{t}, \bar{q}) - \varphi(\bar{t}, \bar{q} - ze^i)}{z} \right) \mu^{i,a}(dz) - \chi(q) \sum_{i \in I_{\mathcal{T}}} \mathcal{H}^i(\partial_{q^i} \varphi(\bar{t}, \bar{q})) \geq 0. \end{aligned}$$

(iii) If θ is locally bounded on $[0, T] \times \mathcal{Q}$, we say that θ is a viscosity solution to (HJ) if its upper semicontinuous envelope θ^* and its lower semicontinuous envelope θ_* are respectively subsolution on $[0, T] \times \mathcal{Q}$ and supersolution on $[0, T] \times \mathcal{Q}$ to (HJ).

The following result is proved in the appendix:

Proposition 4. (i) Let u be a USC function on $[0, T] \times \mathcal{Q}$. u is a viscosity subsolution to (HJ) if and only if $\forall (\bar{t}, \bar{q}) \in [0, T] \times \mathcal{Q}$, $\forall \varphi \in \mathcal{C}^1$ such that $\max_{(t,q) \in [0,T] \times \mathcal{Q}} (u - \varphi)(t, q) = (u - \varphi)(\bar{t}, \bar{q})$, we have:

$$\begin{aligned} & -\frac{\partial \varphi}{\partial t}(\bar{t}, \bar{q}) + \psi(\bar{q}) - \sum_{i \in I_{\mathcal{M}}} \int_{\mathbb{R}_+^*} \mathbf{1}_{\{q+ze^i \in \mathcal{Q}\}} z H^{i,b} \left(\frac{u(\bar{t}, \bar{q}) - u(\bar{t}, \bar{q} + ze^i)}{z} \right) \mu^{i,b}(dz) \\ & - \sum_{i \in I_{\mathcal{M}}} \int_{\mathbb{R}_+^*} \mathbf{1}_{\{q-ze^i \in \mathcal{Q}\}} z H^{i,a} \left(\frac{u(\bar{t}, \bar{q}) - u(\bar{t}, \bar{q} - ze^i)}{z} \right) \mu^{i,a}(dz) - \chi(q) \sum_{i \in I_{\mathcal{T}}} \mathcal{H}^i(\partial_{q^i} \varphi(\bar{t}, \bar{q})) \leq 0. \end{aligned}$$

(ii) Let v be a LSC function on $[0, T] \times \mathcal{Q}$. v is a viscosity supersolution to (HJ) if and only if $\forall (\bar{t}, \bar{q}) \in [0, T] \times \mathcal{Q}$, $\forall \varphi \in \mathcal{C}^1$ such that $\min_{(t,q) \in [0,T] \times \mathcal{Q}} (v - \varphi)(t, q) = (v - \varphi)(\bar{t}, \bar{q})$, we have:

$$\begin{aligned} & -\frac{\partial \varphi}{\partial t}(\bar{t}, \bar{q}) + \psi(\bar{q}) - \sum_{i \in I_{\mathcal{M}}} \int_{\mathbb{R}_+^*} \mathbf{1}_{\{q+ze^i \in \mathcal{Q}\}} z H^{i,b} \left(\frac{v(\bar{t}, \bar{q}) - v(\bar{t}, \bar{q} + ze^i)}{z} \right) \mu^{i,b}(dz) \\ & - \sum_{i \in I_{\mathcal{M}}} \int_{\mathbb{R}_+^*} \mathbf{1}_{\{q-ze^i \in \mathcal{Q}\}} z H^{i,a} \left(\frac{v(\bar{t}, \bar{q}) - v(\bar{t}, \bar{q} - ze^i)}{z} \right) \mu^{i,a}(dz) - \chi(q) \sum_{i \in I_{\mathcal{T}}} \mathcal{H}^i(\partial_{q^i} \varphi(\bar{t}, \bar{q})) \geq 0. \end{aligned}$$

We can now prove that θ is a viscosity solution to (HJ).

Proposition 5. θ is a viscosity subsolution to (HJ) on $[0, T] \times \mathcal{Q}$.

Proof. θ is bounded on $[0, T] \times \mathcal{Q}$ so we can define θ^* its upper semicontinuous envelope.

Let $(\bar{t}, \bar{q}) \in [0, T] \times \mathcal{Q}$ and $\varphi \in \mathcal{C}^1$ such that:

$$0 = (\theta^* - \varphi)(\bar{t}, \bar{q}) = \max_{(t,q) \in [0,T] \times \mathcal{Q}} (\theta^* - \varphi)(t, q).$$

We can classically assume this maximum to be strict. By definition of $\theta^*(\bar{t}, \bar{q})$, there exists $(t_m, q_m)_m$ a sequence of $[0, T) \times \mathcal{Q}$ such that:

$$\begin{aligned} (t_m, q_m) &\xrightarrow{m \rightarrow +\infty} (\bar{t}, \bar{q}), \\ \theta(t_m, q_m) &\xrightarrow{m \rightarrow +\infty} \theta^*(\bar{t}, \bar{q}). \end{aligned}$$

We prove the result by contradiction. Assume there exists $\eta > 0$ such that:

$$\begin{aligned} -\frac{\partial \varphi}{\partial t}(\bar{t}, \bar{q}) + \psi(\bar{q}) - \sum_{i \in I_{\mathcal{M}}} \int_{\mathbb{R}_+^*} \mathbf{1}_{\{q+ze^i \in \mathcal{Q}\}} z H^{i,b} \left(\frac{\varphi(\bar{t}, \bar{q}) - \varphi(\bar{t}, \bar{q} + ze^i)}{z} \right) \mu^{i,b}(dz) \\ - \sum_{i \in I_{\mathcal{M}}} \int_{\mathbb{R}_+^*} \mathbf{1}_{\{q-ze^i \in \mathcal{Q}\}} z H^{i,a} \left(\frac{\varphi(\bar{t}, \bar{q}) - \varphi(\bar{t}, \bar{q} - ze^i)}{z} \right) \mu^{i,a}(dz) - \chi(q) \sum_{i \in I_{\mathcal{T}}} \mathcal{H}^i(\partial_{q^i} \varphi(\bar{t}, \bar{q})) > 2\eta. \end{aligned}$$

Then, as φ is continuously differentiable, χ is continuous, the $H^{i,b}$, $H^{i,a}$ and \mathcal{H}^i are continuous, the boundary of \mathcal{Q} is negligible according to the Lebesgue measure, and the $\mu^{i,b}$ and $\mu^{i,a}$ are absolutely continuous with respect to the Lebesgue measure, we must have:

$$\begin{aligned} -\frac{\partial \varphi}{\partial t}(t, q) + \psi(q) - \sum_{i \in I_{\mathcal{M}}} \int_{\mathbb{R}_+^*} \mathbf{1}_{\{q+ze^i \in \mathcal{Q}\}} z H^{i,b} \left(\frac{\varphi(t, q) - \varphi(t, q + ze^i)}{z} \right) \mu^{i,b}(dz) \\ - \sum_{i \in I_{\mathcal{M}}} \int_{\mathbb{R}_+^*} \mathbf{1}_{\{q-ze^i \in \mathcal{Q}\}} z H^{i,a} \left(\frac{\varphi(t, q) - \varphi(t, q - ze^i)}{z} \right) \mu^{i,a}(dz) - \chi(q) \sum_{i \in I_{\mathcal{T}}} \mathcal{H}^i(\partial_{q^i} \varphi(t, q)) \geq 0 \end{aligned} \quad (2.9)$$

on $B := ((\bar{t} - r, \bar{t} + r) \cap [0, T)) \times (B(\bar{q}, r) \cap \mathcal{Q})$ for a sufficiently small $r \in (0, T - \bar{t})$, with $B(\bar{q}, r) = \{q \in \mathbb{R}^d \mid |q - \bar{q}| < r\}$. Without loss of generality, we can assume that B contains the sequence $(t_m, q_m)_m$.

Then, by potentially reducing the value of η , we have:

$$\theta \leq \theta^* \leq \varphi - \eta$$

on the parabolic boundary $\partial_p B$ of B , i.e. $\left(((\bar{t} - r, \bar{t} + r) \cap [0, T)) \times (\partial B(\bar{q}, r) \cap \mathcal{Q}) \right) \cup \left(\{\bar{t} + r\} \times \overline{B(\bar{q}, r) \cap \mathcal{Q}} \right)$. Without loss of generality we can assume that the above inequality holds on:

$$\tilde{B} := \{(t, q + z) \mid (t, q, z) \in B \times \mathbb{R}^d, q + z \in B(\bar{q}, r)^c \cap \mathcal{Q}\},$$

which is also bounded.

For all $i \in I_{\mathcal{M}}$ we introduce an arbitrary control $\delta^i = (\delta^{i,b}, \delta^{i,a}) \in \mathcal{A}_{\mathcal{M}}$ and define $\bar{\delta} = (\delta^i)_{i \in I_{\mathcal{M}}}$. For all $i \in I_{\mathcal{T}}$ we introduce an arbitrary control $v^i \in \mathcal{A}_{\mathcal{T}}$ and denote $\bar{v} = (v^i)_{i \in I_{\mathcal{T}}}$. We denote by π_m the first exit time of $(t, q_t^m)_{t \geq t_m}$ from B (where $q_t^m := q_t^{t_m, q_m, \bar{\delta}, \bar{v}}$):

$$\pi_m = \inf\{t \geq t_m \mid (t, q_t^m) \notin B\}.$$

By Itô's formula:

$$\begin{aligned} \varphi(\pi_m, q_{\pi_m}^m) &= \varphi(t_m, q_m) + \int_{t_m}^{\pi_m} \frac{\partial \varphi}{\partial t}(s, q_s^m) ds \\ &+ \int_{t_m}^{\pi_m} \sum_{i \in I_{\mathcal{M}}} \int_{\mathbb{R}_+^*} \mathbf{1}_{\{q_{s-}^m + ze^i \in \mathcal{Q}\}} \Lambda^{i,b}(\delta^{i,b}(s, z)) (\varphi(s, q_{s-}^m + ze^i) - \varphi(s, q_{s-}^m)) \mu^{i,b}(dz) ds \\ &+ \int_{t_m}^{\pi_m} \sum_{i \in I_{\mathcal{M}}} \int_{\mathbb{R}_+^*} \mathbf{1}_{\{q_{s-}^m - ze^i \in \mathcal{Q}\}} \Lambda^{i,a}(\delta^{i,a}(s, z)) (\varphi(s, q_{s-}^m - ze^i) - \varphi(s, q_{s-}^m)) \mu^{i,a}(dz) ds \\ &+ \int_{t_m}^{\pi_m} \sum_{i \in I_{\mathcal{M}}} \int_{\mathbb{R}_+^*} (\varphi(s, q_{s-}^m + ze^i) - \varphi(s, q_{s-}^m)) \tilde{J}^{i,b}(ds, dz) \\ &+ \int_{t_m}^{\pi_m} \sum_{i \in I_{\mathcal{M}}} \int_{\mathbb{R}_+^*} (\varphi(s, q_{s-}^m - ze^i) - \varphi(s, q_{s-}^m)) \tilde{J}^{i,a}(ds, dz) \\ &+ \int_{t_m}^{\pi_m} \chi(q_s^m) \sum_{i \in I_{\mathcal{T}}} \partial_{q^i} \varphi(s, q_s^m) v_s^i ds, \end{aligned}$$

which we can write:

$$\begin{aligned}
\varphi(\pi_m, q_{\pi_m}^m) &= \varphi(t_m, q_m) + \int_{t_m}^{\pi_m} \left\{ \frac{\partial \varphi}{\partial t}(s, q_s^m) \right. \\
&+ \sum_{i \in I_{\mathcal{M}}} \int_{\mathbb{R}_+^*} \mathbb{1}_{\{q_{s-}^m + ze^i \in \mathcal{Q}\}} z \Lambda^{i,b}(\delta^{i,b}(s, z)) \left(\delta^{i,b}(s, z) - \frac{\varphi(s, q_{s-}^m) - \varphi(s, q_{s-}^m + ze^i)}{z} \right) \mu^{i,b}(dz) \\
&+ \sum_{i \in I_{\mathcal{M}}} \int_{\mathbb{R}_+^*} \mathbb{1}_{\{q_{s-}^m - ze^i \in \mathcal{Q}\}} z \Lambda^{i,a}(\delta^{i,a}(s, z)) \left(\delta^{i,a}(s, z) - \frac{\varphi(s, q_{s-}^m) - \varphi(s, q_{s-}^m - ze^i)}{z} \right) \mu^{i,a}(dz) \\
&+ \chi(q_s^m) \sum_{i \in I_{\mathcal{T}}} (v_s^i \partial_{q^i} \varphi(s, q_s^m) - L^i(v_s^i)) - \psi(q_s^m) \left. \right\} ds \\
&+ \int_{t_m}^{\pi_m} \left\{ \psi(q_s^m) - \sum_{i \in I_{\mathcal{M}}} \int_{\mathbb{R}_+^*} \mathbb{1}_{\{q_{s-}^m + ze^i \in \mathcal{Q}\}} z \Lambda^{i,b}(\delta^{i,b}(s, z)) \delta^{i,b}(s, z) \mu^i(dz) \right. \\
&- \sum_{i \in I_{\mathcal{M}}} \int_{\mathbb{R}_+^*} \mathbb{1}_{\{q_{s-}^m - ze^i \in \mathcal{Q}\}} z \Lambda^{i,a}(\delta^{i,a}(s, z)) \delta^{i,a}(s, z) \mu^i(dz) + \chi(q_s^m) \sum_{i \in I_{\mathcal{T}}} L^i(v_s^i) \left. \right\} ds \\
&+ \int_{t_m}^{\pi_m} \sum_{i \in I_{\mathcal{M}}} \int_{\mathbb{R}_+^*} (\varphi(s, q_{s-}^m + ze^i) - \varphi(s, q_{s-}^m)) \tilde{J}^{i,b}(ds, dz) \\
&+ \int_{t_m}^{\pi_m} \sum_{i \in I_{\mathcal{M}}} \int_{\mathbb{R}_+^*} (\varphi(s, q_{s-}^m - ze^i) - \varphi(s, q_{s-}^m)) \tilde{J}^{i,a}(ds, dz).
\end{aligned}$$

From (2.9), and by definition of $H^{i,b}$, $H^{i,a}$, and \mathcal{H}^i , we then get:

$$\begin{aligned}
\varphi(\pi_m, q_{\pi_m}^m) &\leq \varphi(t_m, q_m) + \int_{t_m}^{\pi_m} \left\{ \psi(q_s^m) - \sum_{i \in I_{\mathcal{M}}} \int_{\mathbb{R}_+^*} \mathbb{1}_{\{q_{s-}^m + ze^i \in \mathcal{Q}\}} z \Lambda^{i,b}(\delta^{i,b}(s, z)) \delta^{i,b}(s, z) \mu^{i,b}(dz) \right. \\
&- \sum_{i \in I_{\mathcal{M}}} \int_{\mathbb{R}_+^*} \mathbb{1}_{\{q_{s-}^m - ze^i \in \mathcal{Q}\}} z \Lambda^{i,a}(\delta^{i,a}(s, z)) \delta^{i,a}(s, z) \mu^{i,a}(dz) + \chi(q_s^m) \sum_{i \in I_{\mathcal{T}}} L^i(v_s^i) \left. \right\} ds \\
&+ \int_{t_m}^{\pi_m} \sum_{i \in I_{\mathcal{M}}} \int_{\mathbb{R}_+^*} (\varphi(s, q_{s-}^m + ze^i) - \varphi(s, q_{s-}^m)) \tilde{J}^{i,b}(ds, dz) \\
&+ \int_{t_m}^{\pi_m} \sum_{i \in I_{\mathcal{M}}} \int_{\mathbb{R}_+^*} (\varphi(s, q_{s-}^m - ze^i) - \varphi(s, q_{s-}^m)) \tilde{J}^{i,a}(ds, dz).
\end{aligned}$$

The last two terms have expectations equal to zero and we obtain:

$$\begin{aligned} \varphi(t_m, q_m) &\geq \mathbb{E} \left[\varphi(\pi_m, q_{\pi_m}^m) + \int_{t_m}^{\pi_m} \left\{ \sum_{i \in I_{\mathcal{M}}} \int_{\mathbb{R}_+^*} \mathbb{1}_{\{q_{s-}^m + ze^i \in \mathcal{Q}\}} z \Lambda^{i,b}(\delta^{i,b}(s, z)) \delta^{i,b}(s, z) \mu^{i,b}(dz) \right. \right. \\ &\quad \left. \left. + \sum_{i \in I_{\mathcal{M}}} \int_{\mathbb{R}_+^*} \mathbb{1}_{\{q_{s-}^m - ze^i \in \mathcal{Q}\}} z \Lambda^{i,a}(\delta^{i,a}(s, z)) \delta^{i,a}(s, z) \mu^i(dz) - \chi(q_s^m) \sum_{i \in I_{\mathcal{T}}} L^i(v_s^i) - \psi(q_s^m) \right\} ds \right]. \end{aligned}$$

Thus:

$$\begin{aligned} \varphi(t_m, q_m) &\geq \eta + \mathbb{E} \left[\theta(\pi_m, q_{\pi_m}^m) + \int_{t_m}^{\pi_m} \left\{ \sum_{i \in I_{\mathcal{M}}} \int_{\mathbb{R}_+^*} \mathbb{1}_{\{q_{s-}^m + ze^i \in \mathcal{Q}\}} z \Lambda^{i,b}(\delta^{i,b}(s, z)) \delta^{i,b}(s, z) \mu^{i,b}(dz) \right. \right. \\ &\quad \left. \left. + \sum_{i \in I_{\mathcal{M}}} \int_{\mathbb{R}_+^*} \mathbb{1}_{\{q_{s-}^m - ze^i \in \mathcal{Q}\}} z \Lambda^{i,a}(\delta^{i,a}(s, z)) \delta^{i,a}(s, z) \mu^i(dz) - \chi(q_s^m) \sum_{i \in I_{\mathcal{T}}} L^i(v_s^i) - \psi(q_s^m) \right\} ds \right]. \end{aligned}$$

As $\varphi(t_m, q_m) \xrightarrow{m \rightarrow +\infty} \varphi(\bar{t}, \bar{q}) = \theta^*(\bar{t}, \bar{q})$ and $\theta(t_m, q_m) \xrightarrow{m \rightarrow +\infty} \theta^*(\bar{t}, \bar{q})$, we have for m large enough the inequality $\theta(t_m, q_m) + \frac{\eta}{2} \geq \varphi(t_m, q_m)$, from which we deduce:

$$\begin{aligned} \theta(t_m, q_m) &\geq \frac{\eta}{2} + \mathbb{E} \left[\theta(\pi_m, q_{\pi_m}^m) + \int_{t_m}^{\pi_m} \left\{ \sum_{i \in I_{\mathcal{M}}} \int_{\mathbb{R}_+^*} \mathbb{1}_{\{q_{s-}^m + ze^i \in \mathcal{Q}\}} z \Lambda^{i,b}(\delta^{i,b}(s, z)) \delta^{i,b}(s, z) \mu^{i,b}(dz) \right. \right. \\ &\quad \left. \left. + \sum_{i \in I_{\mathcal{M}}} \int_{\mathbb{R}_+^*} \mathbb{1}_{\{q_{s-}^m - ze^i \in \mathcal{Q}\}} z \Lambda^{i,a}(\delta^{i,a}(s, z)) \delta^{i,a}(s, z) \mu^i(dz) - \chi(q_s^m) \sum_{i \in I_{\mathcal{T}}} L^i(v_s^i) - \psi(q_s^m) \right\} ds \right]. \end{aligned}$$

By taking the sup over all the controls in $\mathcal{A}_{\mathcal{M}}^{N_{\mathcal{M}}} \times \mathcal{A}_{\mathcal{T}}^{N_{\mathcal{T}}}$ on the right-hand side, we contradict the dynamic programming principle.

Necessarily, we deduce:

$$\begin{aligned} -\frac{\partial \varphi}{\partial t}(\bar{t}, \bar{q}) + \psi(\bar{q}) - \sum_{i \in I_{\mathcal{M}}} \int_{\mathbb{R}_+^*} \mathbb{1}_{\{q + ze^i \in \mathcal{Q}\}} z H^{i,b} \left(\frac{\varphi(\bar{t}, \bar{q}) - \varphi(\bar{t}, \bar{q} + ze^i)}{z} \right) \mu^{i,b}(dz) \\ - \sum_{i \in I_{\mathcal{M}}} \int_{\mathbb{R}_+^*} \mathbb{1}_{\{q - ze^i \in \mathcal{Q}\}} z H^{i,a} \left(\frac{\varphi(\bar{t}, \bar{q}) - \varphi(\bar{t}, \bar{q} - ze^i)}{z} \right) \mu^{i,a}(dz) - \chi(q) \sum_{i \in I_{\mathcal{T}}} \mathcal{H}^i(\partial_{q^i} \varphi(\bar{t}, \bar{q})) \leq 0, \end{aligned}$$

and θ is a viscosity subsolution to (HJ) on $[0, T] \times \mathcal{Q}$. □

Proposition 6. θ is a viscosity supersolution to (HJ) on $[0, T] \times \mathcal{Q}$.

Proof. θ is bounded on $[0, T] \times \mathcal{Q}$, so we can define θ_* its lower semicontinuous envelope.

Let $(\bar{t}, \bar{q}) \in [0, T] \times \mathcal{Q}$ and $\varphi \in C^1$ such that:

$$0 = (\theta_* - \varphi)(\bar{t}, \bar{q}) = \min_{(t, q) \in [0, T] \times \mathcal{Q}} (\theta_* - \varphi)(t, q).$$

We can assume this minimum to be strict. By definition of $\theta_*(\bar{t}, \bar{q})$, there exists $(t_m, q_m)_m$ a sequence of $[0, T] \times \mathcal{Q}$ such that:

$$\begin{aligned} (t_m, q_m) &\xrightarrow{m \rightarrow +\infty} (\bar{t}, \bar{q}), \\ \theta(t_m, q_m) &\xrightarrow{m \rightarrow +\infty} \theta_*(\bar{t}, \bar{q}). \end{aligned}$$

Let us prove the proposition by contradiction. Assume there is $\eta > 0$ such that:

$$\begin{aligned} -\frac{\partial \varphi}{\partial t}(\bar{t}, \bar{q}) + \psi(\bar{q}) - \sum_{i \in I_{\mathcal{M}}} \int_{\mathbb{R}_+^*} \mathbf{1}_{\{q + ze^i \in \mathcal{Q}\}} z H^{i, b} \left(\frac{\varphi(\bar{t}, \bar{q}) - \varphi(\bar{t}, \bar{q} + ze^i)}{z} \right) \mu^{i, b}(dz) \\ - \sum_{i \in I_{\mathcal{M}}} \int_{\mathbb{R}_+^*} \mathbf{1}_{\{q - ze^i \in \mathcal{Q}\}} z H^{i, a} \left(\frac{\varphi(\bar{t}, \bar{q}) - \varphi(\bar{t}, \bar{q} - ze^i)}{z} \right) \mu^{i, a}(dz) - \chi(q) \sum_{i \in I_{\mathcal{T}}} \mathcal{H}^i(\partial_{q^i} \varphi(\bar{t}, \bar{q})) < -2\eta. \end{aligned}$$

Then, as φ is continuously differentiable, χ is continuous, the $H^{i, b}$, $H^{i, a}$ and \mathcal{H}^i are continuous, the boundary of \mathcal{Q} is negligible according to the Lebesgue measure, and the $\mu^{i, b}$ and $\mu^{i, a}$ are absolutely continuous with respect to the Lebesgue measure, we must have:

$$\begin{aligned} -\frac{\partial \varphi}{\partial t}(t, q) + \psi(q) - \sum_{i \in I_{\mathcal{M}}} \int_{\mathbb{R}_+^*} \mathbf{1}_{\{q + ze^i \in \mathcal{Q}\}} z H^{i, b} \left(\frac{\varphi(t, q) - \varphi(t, q + ze^i)}{z} \right) \mu^{i, b}(dz) \\ - \sum_{i \in I_{\mathcal{M}}} \int_{\mathbb{R}_+^*} \mathbf{1}_{\{q - ze^i \in \mathcal{Q}\}} z H^{i, a} \left(\frac{\varphi(t, q) - \varphi(t, q - ze^i)}{z} \right) \mu^{i, a}(dz) - \chi(q) \sum_{i \in I_{\mathcal{T}}} \mathcal{H}^i(\partial_{q^i} \varphi(t, q)) \leq 0 \end{aligned} \tag{2.10}$$

on $B := ((\bar{t} - r, \bar{t} + r) \cap [0, T]) \times (B(\bar{q}, r) \cap \mathcal{Q})$ for a sufficiently small $r \in (0, T - \bar{t})$, with $B(\bar{q}, r) = \{q \in \mathbb{R}^d \mid |q - \bar{q}| < r\}$. Without loss of generality, we can assume that B contains the sequence $(t_m, q_m)_m$.

Then, by potentially reducing η , we have:

$$\theta \geq \theta_* \geq \varphi + \eta$$

on $\partial_p B$. We can also without loss of generality assume that this inequality is true on:

$$\tilde{B} := \{(t, q + z) \mid (t, q, z) \in B \times \mathbb{R}^d, q + z \in B(\bar{q}, r)^c \cap \mathcal{Q}\},$$

which is also bounded.

$\forall i \in I_{\mathcal{M}}$, we introduce the controls $\delta^i = (\delta^{i,b}, \delta^{i,a}) \in \mathcal{A}_{\mathcal{M}}$ such that $\forall t \geq t_m, \forall z \in \mathbb{R}_+^*$,

$$\delta^{i,b}(t, z) = \delta^{i,b*} \left(\frac{\varphi(t, q_{t-}^m) - \varphi(t, q_{t-}^m + ze^i)}{z} \right) \quad \text{and} \quad \delta^{i,a}(t, z) = \delta^{i,a*} \left(\frac{\varphi(t, q_{t-}^m) - \varphi(t, q_{t-}^m - ze^i)}{z} \right),$$

where $\delta^{i,b*}$ and $\delta^{i,a*}$ are defined in Lemma 2. Similarly, $\forall i \in I_{\mathcal{T}}$, we introduce the control $v^i \in \mathcal{A}_{\mathcal{T}}$ such that $\forall t \geq t_m$,

$$v_{\mathcal{T}}^I = v^{i*} (\partial_{q^i} \varphi(t, q_t^m)),$$

where v^{i*} is defined in Lemma 4. As before, we denote by π_m the first exit time of $(t, q_t^m)_{t \geq t_m}$ from B (where $q_t^m := q_t^{t_m, q_m, \bar{\delta}, \bar{v}}$). By Itô's lemma, we obtain:

$$\begin{aligned} \varphi(\pi_m, q_{\pi_m}^m) &= \varphi(t_m, q_m) + \int_{t_m}^{\pi_m} \frac{\partial \varphi}{\partial t}(s, q_s^m) ds \\ &+ \int_{t_m}^{\pi_m} \sum_{i \in I_{\mathcal{M}}} \int_{\mathbb{R}_+^*} \mathbf{1}_{\{q_{s-}^m + ze^i \in \mathcal{Q}\}} \Lambda^{i,b}(\delta^{i,b}(s, z)) (\varphi(s, q_{s-}^m + ze^i) - \varphi(s, q_{s-}^m)) \mu^{i,b}(dz) ds \\ &+ \int_{t_m}^{\pi_m} \sum_{i \in I_{\mathcal{M}}} \int_{\mathbb{R}_+^*} \mathbf{1}_{\{q_{s-}^m - ze^i \in \mathcal{Q}\}} \Lambda^{i,a}(\delta^{i,a}(s, z)) (\varphi(s, q_{s-}^m - ze^i) - \varphi(s, q_{s-}^m)) \mu^{i,a}(dz) ds \\ &+ \int_{t_m}^{\pi_m} \sum_{i \in I_{\mathcal{M}}} \int_{\mathbb{R}_+^*} (\varphi(s, q_{s-}^m + ze^i) - \varphi(s, q_{s-}^m)) \tilde{J}^{i,b}(ds, dz) \\ &+ \int_{t_m}^{\pi_m} \sum_{i \in I_{\mathcal{M}}} \int_{\mathbb{R}_+^*} (\varphi(s, q_{s-}^m - ze^i) - \varphi(s, q_{s-}^m)) \tilde{J}^{i,a}(ds, dz) \\ &+ \int_{t_m}^{\pi_m} \chi(q_s^m) \sum_{i \in I_{\mathcal{T}}} \partial_{q^i} \varphi(s, q_s^m) v_s^i ds, \end{aligned}$$

which we can write:

$$\begin{aligned}
\varphi(\pi_m, q_{\pi_m}^m) &= \varphi(t_m, q_m) + \int_{t_m}^{\pi_m} \left\{ \frac{\partial \varphi}{\partial t}(s, q_s^m) \right. \\
&+ \sum_{i \in I_{\mathcal{M}}} \int_{\mathbb{R}_+^*} \mathbb{1}_{\{q_{s-}^m + ze^i \in \mathcal{Q}\}} z \Lambda^{i,b}(\delta^{i,b}(s, z)) \left(\delta^{i,b}(s, z) - \frac{\varphi(s, q_{s-}^m) - \varphi(s, q_{s-}^m + ze^i)}{z} \right) \mu^{i,b}(dz) \\
&+ \sum_{i \in I_{\mathcal{M}}} \int_{\mathbb{R}_+^*} \mathbb{1}_{\{q_{s-}^m - ze^i \in \mathcal{Q}\}} z \Lambda^{i,a}(\delta^{i,a}(s, z)) \left(\delta^{i,a}(s, z) - \frac{\varphi(s, q_{s-}^m) - \varphi(s, q_{s-}^m - ze^i)}{z} \right) \mu^{i,a}(dz) \\
&+ \chi(q_s^m) \sum_{i \in I_{\mathcal{T}}} (v_s^i \partial_{q^i} \varphi(s, q_s^m) - L^i(v_s^i)) - \psi(q_s^m) \left. \right\} ds \\
&+ \int_{t_m}^{\pi_m} \left\{ \psi(q_s^m) - \sum_{i \in I_{\mathcal{M}}} \int_{\mathbb{R}_+^*} \mathbb{1}_{\{q_{s-}^m + ze^i \in \mathcal{Q}\}} z \Lambda^{i,b}(\delta^{i,b}(s, z)) \delta^{i,b}(s, z) \mu^i(dz) \right. \\
&- \sum_{i \in I_{\mathcal{M}}} \int_{\mathbb{R}_+^*} \mathbb{1}_{\{q_{s-}^m - ze^i \in \mathcal{Q}\}} z \Lambda^{i,a}(\delta^{i,a}(s, z)) \delta^{i,a}(s, z) \mu^i(dz) + \chi(q_s^m) \sum_{i \in I_{\mathcal{T}}} L^i(v_s^i) \left. \right\} ds \\
&+ \int_{t_m}^{\pi_m} \sum_{i \in I_{\mathcal{M}}} \int_{\mathbb{R}_+^*} (\varphi(s, q_{s-}^m + ze^i) - \varphi(s, q_{s-}^m)) \tilde{J}^{i,b}(ds, dz) \\
&+ \int_{t_m}^{\pi_m} \sum_{i \in I_{\mathcal{M}}} \int_{\mathbb{R}_+^*} (\varphi(s, q_{s-}^m - ze^i) - \varphi(s, q_{s-}^m)) \tilde{J}^{i,a}(ds, dz).
\end{aligned}$$

By (2.10), we then get:

$$\begin{aligned}
\varphi(\pi_m, q_{\pi_m}^m) &\geq \varphi(t_m, q_m) + \int_{t_m}^{\pi_m} \left\{ \psi(q_s^m) - \sum_{i \in I_{\mathcal{M}}} \int_{\mathbb{R}_+^*} \mathbb{1}_{\{q_{s-}^m + ze^i \in \mathcal{Q}\}} z \Lambda^{i,b}(\delta^{i,b}(s, z)) \delta^{i,b}(s, z) \mu^{i,b}(dz) \right. \\
&- \sum_{i \in I_{\mathcal{M}}} \int_{\mathbb{R}_+^*} \mathbb{1}_{\{q_{s-}^m - ze^i \in \mathcal{Q}\}} z \Lambda^{i,a}(\delta^{i,a}(s, z)) \delta^{i,a}(s, z) \mu^{i,a}(dz) + \chi(q_s^m) \sum_{i \in I_{\mathcal{T}}} L^i(v_s^i) \left. \right\} ds \\
&+ \int_{t_m}^{\pi_m} \sum_{i \in I_{\mathcal{M}}} \int_{\mathbb{R}_+^*} (\varphi(s, q_{s-}^m + ze^i) - \varphi(s, q_{s-}^m)) \tilde{J}^{i,b}(ds, dz) \\
&+ \int_{t_m}^{\pi_m} \sum_{i \in I_{\mathcal{M}}} \int_{\mathbb{R}_+^*} (\varphi(s, q_{s-}^m - ze^i) - \varphi(s, q_{s-}^m)) \tilde{J}^{i,a}(ds, dz).
\end{aligned}$$

The last two terms have expectations equal to zero and we obtain:

$$\begin{aligned} \varphi(t_m, q_m) \leq & \mathbb{E} \left[\varphi(\pi_m, q_{\pi_m}^m) + \int_{t_m}^{\pi_m} \left\{ \sum_{i \in I_{\mathcal{M}}} \int_{\mathbb{R}_+^*} \mathbb{1}_{\{q_{s-}^m + ze^i \in \mathcal{Q}\}} z \Lambda^{i,b}(\delta^{i,b}(s, z)) \delta^{i,b}(s, z) \mu^{i,b}(dz) \right. \right. \\ & \left. \left. + \sum_{i \in I_{\mathcal{M}}} \int_{\mathbb{R}_+^*} \mathbb{1}_{\{q_{s-}^m - ze^i \in \mathcal{Q}\}} z \Lambda^{i,a}(\delta^{i,a}(s, z)) \delta^{i,a}(s, z) \mu^i(dz) - \chi(q_s^m) \sum_{i \in I_{\mathcal{T}}} L^i(v_s^i) - \psi(q_s^m) \right\} ds \right]. \end{aligned}$$

Thus:

$$\begin{aligned} \varphi(t_m, q_m) \leq & -\eta + \mathbb{E} \left[\theta(\pi_m, q_{\pi_m}^m) + \int_{t_m}^{\pi_m} \left\{ \sum_{i \in I_{\mathcal{M}}} \int_{\mathbb{R}_+^*} \mathbb{1}_{\{q_{s-}^m + ze^i \in \mathcal{Q}\}} z \Lambda^{i,b}(\delta^{i,b}(s, z)) \delta^{i,b}(s, z) \mu^{i,b}(dz) \right. \right. \\ & \left. \left. + \sum_{i \in I_{\mathcal{M}}} \int_{\mathbb{R}_+^*} \mathbb{1}_{\{q_{s-}^m - ze^i \in \mathcal{Q}\}} z \Lambda^{i,a}(\delta^{i,a}(s, z)) \delta^{i,a}(s, z) \mu^i(dz) - \chi(q_s^m) \sum_{i \in I_{\mathcal{T}}} L^i(v_s^i) - \psi(q_s^m) \right\} ds \right]. \end{aligned}$$

As $\varphi(t_m, q_m) \xrightarrow{m \rightarrow +\infty} \varphi(\bar{t}, \bar{q}) = \theta_*(\bar{t}, \bar{q})$ and moreover $\theta(t_m, q_m) \xrightarrow{m \rightarrow +\infty} \theta_*(\bar{t}, \bar{q})$, we have that for m sufficiently large, $\theta(t_m, q_m) - \frac{\eta}{2} \leq \varphi(t_m, q_m)$ and we deduce:

$$\begin{aligned} \theta(t_m, q_m) < & \mathbb{E} \left[\theta(\pi_m, q_{\pi_m}^m) + \int_{t_m}^{\pi_m} \left\{ \sum_{i \in I_{\mathcal{M}}} \int_{\mathbb{R}_+^*} \mathbb{1}_{\{q_{s-}^m + ze^i \in \mathcal{Q}\}} z \Lambda^{i,b}(\delta^{i,b}(s, z)) \delta^{i,b}(s, z) \mu^{i,b}(dz) \right. \right. \\ & \left. \left. + \sum_{i \in I_{\mathcal{M}}} \int_{\mathbb{R}_+^*} \mathbb{1}_{\{q_{s-}^m - ze^i \in \mathcal{Q}\}} z \Lambda^{i,a}(\delta^{i,a}(s, z)) \delta^{i,a}(s, z) \mu^i(dz) - \chi(q_s^m) \sum_{i \in I_{\mathcal{T}}} L^i(v_s^i) - \psi(q_s^m) \right\} ds \right], \end{aligned}$$

which contradicts the dynamic programming principle.

In conclusion, necessarily we have:

$$\begin{aligned} -\frac{\partial \varphi}{\partial t}(\bar{t}, \bar{q}) + \psi(\bar{q}) - \sum_{i \in I_{\mathcal{M}}} \int_{\mathbb{R}_+^*} \mathbb{1}_{\{q + ze^i \in \mathcal{Q}\}} z H^{i,b} \left(\frac{\varphi(\bar{t}, \bar{q}) - \varphi(\bar{t}, \bar{q} + ze^i)}{z} \right) \mu^{i,b}(dz) \\ - \sum_{i \in I_{\mathcal{M}}} \int_{\mathbb{R}_+^*} \mathbb{1}_{\{q - ze^i \in \mathcal{Q}\}} z H^{i,a} \left(\frac{\varphi(\bar{t}, \bar{q}) - \varphi(\bar{t}, \bar{q} - ze^i)}{z} \right) \mu^{i,a}(dz) - \chi(q) \sum_{i \in I_{\mathcal{T}}} \mathcal{H}^i(\partial_{q^i} \varphi(\bar{t}, \bar{q})) \geq 0, \end{aligned}$$

and θ is a viscosity supersolution to (HJ) on $[0, T] \times \mathcal{Q}$.

□

Proposition 7. $\forall q \in \mathcal{Q}$, we have $\theta_*(T, q) = \theta^*(T, q) = -\ell_d(q)$.

Proof. Let $q \in \mathbb{R}^d$ and let us take $(t_m, q_m)_{m \in \mathbb{N}}$ a sequence of $[0, T] \times \mathcal{Q}$ such that

$$(t_m, q_m) \xrightarrow{m \rightarrow +\infty} (T, q) \quad \text{and} \quad \theta(t_m, q_m) \xrightarrow{m \rightarrow +\infty} \theta_*(T, q).$$

For all $i \in I_{\mathcal{M}}$ we introduce arbitrary controls $\delta^i = (\delta^{i,b}, \delta^{i,a}) \in \mathcal{A}_{\mathcal{M}}$ and denote $\bar{\delta} = (\delta^i)_{i \in I_{\mathcal{M}}}$. For all $i \in I_{\mathcal{T}}$ we introduce an arbitrary control $v^i \in \mathcal{A}_{\mathcal{T}}$ and define $\bar{v} = (v^i)_{i \in I_{\mathcal{T}}}$. Then we have for all $m \in \mathbb{N}$, by denoting $q_t^m = q_t^{t_m, q_m, \bar{\delta}, \bar{v}}$ for all $t \in [t_m, T]$:

$$\begin{aligned} \theta(t_m, q_m) \geq & \mathbb{E} \left[\int_{t_m}^T \left\{ \int_{\mathbb{R}_+^*} \sum_{i \in I_{\mathcal{M}}} \left(z \delta^{i,b}(s, z) \mathbb{1}_{\{q_{s-}^m + ze^i \in \mathcal{Q}\}} \Lambda^{i,b}(\delta^{i,b}(s, z)) \mu^{i,b}(dz) \right. \right. \right. \\ & + z \delta^{i,a}(s, z) \mathbb{1}_{\{q_{s-}^m - ze^i \in \mathcal{Q}\}} \Lambda^{i,a}(\delta^{i,a}(s, z)) \mu^{i,a}(dz) \Big) \\ & \left. \left. - \chi(q_s^m) \sum_{i \in I_{\mathcal{T}}} L^i(v_s^i) - \psi(q_s^m) \right\} ds - \ell_d(q_T^m) \right]. \end{aligned}$$

For the sake of clarity, let us break down this expectation in four parts. We first have almost surely

$$\begin{aligned} \int_{t_m}^T \left\{ \int_{\mathbb{R}_+^*} \sum_{i \in I_{\mathcal{M}}} \left(z \delta^{i,b}(s, z) \mathbb{1}_{\{q_{s-}^m + ze^i \in \mathcal{Q}\}} \Lambda^{i,b}(\delta^{i,b}(s, z)) \mu^{i,b}(dz) \right. \right. \\ \left. \left. + z \delta^{i,a}(s, z) \mathbb{1}_{\{q_{s-}^m - ze^i \in \mathcal{Q}\}} \Lambda^{i,a}(\delta^{i,a}(s, z)) \mu^{i,a}(dz) \right) \right\} ds \xrightarrow{m \rightarrow +\infty} 0, \end{aligned}$$

and

$$\begin{aligned} & \left| \int_{t_m}^T \left\{ \int_{\mathbb{R}_+^*} \sum_{i \in I_{\mathcal{M}}} \left(z \delta^{i,b}(s, z) \mathbb{1}_{\{q_{s-}^m + ze^i \in \mathcal{Q}\}} \Lambda^{i,b}(\delta^{i,b}(s, z)) \mu^{i,b}(dz) \right. \right. \right. \\ & \left. \left. + z \delta^{i,a}(s, z) \mathbb{1}_{\{q_{s-}^m - ze^i \in \mathcal{Q}\}} \Lambda^{i,a}(\delta^{i,a}(s, z)) \mu^{i,a}(dz) \right) \right\} ds \Big| \\ & \leq T \sum_{i \in I_{\mathcal{M}}} \left(\Delta^{i,b} \sup_{\delta \geq -\delta_\infty} |\delta \Lambda^{i,b}(\delta)| + \Delta^{i,a} \sup_{\delta \geq -\delta_\infty} |\delta \Lambda^{i,a}(\delta)| \right), \end{aligned}$$

so by dominated convergence we get

$$\begin{aligned} & \mathbb{E} \left[\int_{t_m}^T \left\{ \int_{\mathbb{R}_+^*} \sum_{i \in I_{\mathcal{M}}} \left(z \delta^{i,b}(s, z) \mathbb{1}_{\{q_{s-}^m + ze^i \in \mathcal{Q}\}} \Lambda^{i,b}(\delta^{i,b}(s, z)) \mu^{i,b}(dz) \right. \right. \right. \\ & \left. \left. + z \delta^{i,a}(s, z) \mathbb{1}_{\{q_{s-}^m - ze^i \in \mathcal{Q}\}} \Lambda^{i,a}(\delta^{i,a}(s, z)) \mu^{i,a}(dz) \right) \right\} ds \Big] \xrightarrow{m \rightarrow +\infty} 0. \end{aligned}$$

We also have almost surely

$$- \int_{t_m}^T \chi(q_s^m) \sum_{i \in I_{\mathcal{T}}} L^i(v_s^i) ds \xrightarrow{m \rightarrow +\infty} 0,$$

and

$$\left| - \int_{t_m}^T \chi(q_s^m) \sum_{i \in I_\tau} L^i(v_s^i) ds \right| = \int_{t_m}^T \chi(q_s^m) \sum_{i \in I_\tau} L^i(v_s^i) ds \leq \int_0^T \sum_{i \in I_\tau} L^i(v_s^i) ds \leq T \sum_{i \in I_\tau} L^i(v_\infty).$$

We can again use dominated convergence to get

$$- \mathbb{E} \left[\int_{t_m}^T \chi(q_s^m) \sum_{i \in I_\tau} L^i(v_s^i) ds \right] \xrightarrow{m \rightarrow +\infty} 0.$$

For the third part, we have once again almost surely

$$- \int_{t_m}^T \psi(q_s^m) ds \xrightarrow{m \rightarrow +\infty} 0.$$

As ψ is continuous on \mathcal{Q} , it is bounded and we get

$$\mathbb{E} \left[\int_{t_m}^T \psi(q_s^m) ds \right] \xrightarrow{m \rightarrow +\infty} 0.$$

We finally show exactly the same way that $\mathbb{E}[\ell_d(q_T^m)] \xrightarrow{m \rightarrow +\infty} \ell_d(q)$, which gives us

$$\theta_*(T, q) \geq -\ell_d(q).$$

But as $\theta_*(T, q) \leq \theta(T, q) = -\ell_d(q)$, so $\theta_*(T, q) = -\ell_d(q)$. The proof for θ^* is similar, by taking ε -optimal controls and showing that $\theta^*(T, q) - \varepsilon \leq -\ell_d(q)$ for all $\varepsilon > 0$, $q \in \mathbb{R}^d$. \square

2.3.2 Uniqueness

Theorem 3. *Let u be a bounded USC subsolution and v be a bounded LSC supersolution to (HJ) such that $u \leq v$ on $\{T\} \times \mathcal{Q}$. Then $u \leq v$ on $[0, T] \times \mathcal{Q}$.*

Proof. We prove it by contradiction. Let us assume $\sup_{[0, T] \times \mathcal{Q}} u - v > 0$. Then this supremum cannot be reached on $\{T\} \times \mathcal{Q}$. $\forall n \geq 0$, for $\varepsilon > 0$, we introduce:

$$\phi_{n, \varepsilon}(t, s, q, y) = u(t, q) - v(s, y) - n\|q - y\|^2 - n(t - s)^2 - \varepsilon(2T - t - s).$$

We also introduce $(t_{n, \varepsilon}, s_{n, \varepsilon}, q_{n, \varepsilon}, y_{n, \varepsilon})$ such that:

$$\phi_{n, \varepsilon}(t_{n, \varepsilon}, s_{n, \varepsilon}, q_{n, \varepsilon}, y_{n, \varepsilon}) = \max_{[0, T]^2 \times \mathcal{Q}^2} \phi_{n, \varepsilon}(t, s, q, y).$$

Then for all $n \geq 0, \varepsilon > 0$ and for all $(t, q) \in [0, T] \times \mathcal{Q}$, we have

$$\phi_{n, \varepsilon}(t_{n, \varepsilon}, s_{n, \varepsilon}, q_{n, \varepsilon}, y_{n, \varepsilon}) \geq \phi_{n, \varepsilon}(t, t, q, q) = u(t, q) - v(t, q) - 2\varepsilon(T - t).$$

In particular,

$$\phi_{n, \varepsilon}(t_{n, \varepsilon}, s_{n, \varepsilon}, q_{n, \varepsilon}, y_{n, \varepsilon}) \geq \max_{[0, T] \times \mathcal{Q}} (u(t, q) - v(t, q)) - 2\varepsilon T, \quad (2.11)$$

and for $0 < \varepsilon < \frac{\sup_{[0,T] \times \mathcal{Q}} (u(t,q) - v(t,q))}{2T}$, the right-hand side is positive. We can now fix ε such that

$$0 < \varepsilon < \frac{\sup_{[0,T] \times \mathcal{Q}} (u(t,q) - v(t,q))}{4T},$$

to ensure that the right-hand side of (2.11) is larger than $\frac{\max_{[0,T] \times \mathcal{Q}} (u(t,q) - v(t,q))}{2}$, which is strictly positive by assumption. ε will remain fixed throughout the rest of the proof and, for ease of notation, we now write $\phi_n = \phi_{n,\varepsilon}$, $t_n = t_{n,\varepsilon}$, $s_n = s_{n,\varepsilon}$, $q_n = q_{n,\varepsilon}$, and $y_n = y_{n,\varepsilon}$.

From what precedes, we know that for all $n \geq 0$, we have

$$0 < \phi_n(t_n, s_n, q_n, y_n) \leq \sup_{[0,T]^2 \times (\mathcal{Q})^2} (u(t,q) - v(s,y)) < +\infty.$$

Then necessarily, $|t_n - s_n| \xrightarrow{n \rightarrow +\infty} 0$ and $|q_n - y_n| \xrightarrow{n \rightarrow +\infty} 0$: up to a subsequence, there exist $(\bar{t}, \bar{q}) \in [0, T] \times \mathcal{Q}$ such that $s_n, t_n \xrightarrow{n \rightarrow +\infty} \bar{t}$ and $q_n, y_n \xrightarrow{n \rightarrow +\infty} \bar{q}$.

Moreover, we know that

$$\phi_n(t_n, s_n, q_n, y_n) \geq \phi_n(\bar{t}, \bar{t}, \bar{q}, \bar{q}),$$

which implies

$$u(t_n, q_n) - v(s_n, y_n) - n\|q_n - y_n\|^2 - n(t_n - s_n)^2 - \varepsilon(2T - t_n - s_n) \geq u(\bar{t}, \bar{q}) - v(\bar{t}, \bar{q}) - 2\varepsilon(T - \bar{t}).$$

Hence we have

$$n\|q_n - y_n\|^2 + n(t_n - s_n)^2 \leq u(t_n, q_n) - u(\bar{t}, \bar{q}) + v(\bar{t}, \bar{q}) - v(s_n, y_n) + 2\varepsilon(T - \bar{t}) - \varepsilon(2T - t_n - s_n).$$

As u is USC and v is LSC, the lim sup when $n \rightarrow +\infty$ of the left-hand side is nonpositive, which implies $n\|q_n - y_n\|^2 + n(t_n - s_n)^2 \xrightarrow{n \rightarrow +\infty} 0$.

Let us assume $\bar{t} = T$. Then we have, as u is USC and v is LSC:

$$\limsup_{n \rightarrow +\infty} \phi_n(t_n, s_n, q_n, y_n) \leq \limsup_{n \rightarrow +\infty} u(t_n, q_n) - \liminf_{n \rightarrow +\infty} v(s_n, y_n) \leq u(T, \bar{q}) - v(T, \bar{q}) \leq 0,$$

which constitutes a contradiction. Necessarily, $\bar{t} < T$.

Hence, for n large enough we must have $t_n, s_n < T$, and we know that (t_n, q_n) is a maximum point of $u - \varphi_n$ where

$$\varphi_n(t, q) = v(s_n, y_n) + n\|q - y_n\|^2 + n(t - s_n)^2 + \varepsilon(2T - t - s_n).$$

By Proposition 4, we have:

$$\begin{aligned} \varepsilon - 2n(t_n - s_n) + \psi(q_n) - \sum_{i \in I_{\mathcal{M}}} \int_{\mathbb{R}_+^*} \mathbf{1}_{\{q_n + ze^i \in \mathcal{Q}\}} z H^{i,b} \left(\frac{u(t_n, q_n) - u(t_n, q_n + ze^i)}{z} \right) \mu^{i,b}(dz) \\ - \sum_{i \in I_{\mathcal{M}}} \int_{\mathbb{R}_+^*} \mathbf{1}_{\{q_n - ze^i \in \mathcal{Q}\}} z H^{i,a} \left(\frac{u(t_n, q_n) - u(t_n, q_n - ze^i)}{z} \right) \mu^{i,a}(dz) - \chi(q_n) \sum_{i \in I_{\mathcal{T}}} \mathcal{H}^i(2n(q_n^i - y_n^i)) \leq 0. \end{aligned}$$

Furthermore, (s_n, y_n) is a minimum point of $v - \xi_n$ where

$$\xi_n(s, y) = u(t_n, y_n) - n\|q_n - y\|^2 - n(t_n - s)^2 - \varepsilon(2T - t_n - s),$$

and by the same argument:

$$\begin{aligned} & -\varepsilon - 2n(t_n - s_n) + \psi(y_n) - \sum_{i \in I_{\mathcal{M}}} \int_{\mathbb{R}_+^*} \mathbb{1}_{\{y_n + ze^i \in \mathcal{Q}\}} z H^{i,b} \left(\frac{v(s_n, y_n) - v(s_n, y_n + ze^i)}{z} \right) \mu^{i,b}(dz) \\ & - \sum_{i \in I_{\mathcal{M}}} \int_{\mathbb{R}_+^*} \mathbb{1}_{\{y_n - ze^i \in \mathcal{Q}\}} z H^{i,a} \left(\frac{v(s_n, y_n) - v(s_n, y_n - ze^i)}{z} \right) \mu^{i,a}(dz) - \chi(y_n) \sum_{i \in I_{\mathcal{T}}} \mathcal{H}^i(2n(q_n^i - y_n^i)) \geq 0. \end{aligned}$$

Therefore by combining the two inequalities we get:

$$\begin{aligned} & \sum_{i \in I_{\mathcal{M}}} \int_{\mathbb{R}_+^*} z \left(\mathbb{1}_{\{y_n + ze^i \in \mathcal{Q}\}} H^{i,b} \left(\frac{v(s_n, y_n) - v(s_n, y_n + ze^i)}{z} \right) \right. \\ & \quad \left. - \mathbb{1}_{\{q_n + ze^i \in \mathcal{Q}\}} H^{i,b} \left(\frac{u(t_n, q_n) - u(t_n, q_n + ze^i)}{z} \right) \right) \mu^{i,b}(dz) \\ & + \sum_{i \in I_{\mathcal{M}}} \int_{\mathbb{R}_+^*} z \left(\mathbb{1}_{\{y_n - ze^i \in \mathcal{Q}\}} H^{i,a} \left(\frac{v(s_n, y_n) - v(s_n, y_n - ze^i)}{z} \right) \right. \\ & \quad \left. - \mathbb{1}_{\{q_n - ze^i \in \mathcal{Q}\}} H^{i,a} \left(\frac{u(t_n, q_n) - u(t_n, q_n - ze^i)}{z} \right) \right) \mu^{i,a}(dz) \\ & \leq -2\varepsilon + (\psi(q_n) - \psi(y_n)) - (\chi(q_n) - \chi(y_n)) \sum_{i \in I_{\mathcal{T}}} \mathcal{H}^i(2n(q_n^i - y_n^i)), \end{aligned}$$

and by rearranging the terms:

$$\begin{aligned}
& \sum_{i \in I_{\mathcal{M}}} \int_{\mathbb{R}_+^*} \mathbb{1}_{\{y_n + ze^i \in \mathcal{Q}\} \cap \{q_n + ze^i \in \mathcal{Q}\}} z \left(H^{i,b} \left(\frac{v(s_n, y_n) - v(s_n, y_n + ze^i)}{z} \right) \right. \\
& \quad \left. - H^{i,b} \left(\frac{u(t_n, q_n) - u(t_n, q_n + ze^i)}{z} \right) \right) \mu^{i,b}(dz) \\
& + \sum_{i \in I_{\mathcal{M}}} \int_{\mathbb{R}_+^*} \mathbb{1}_{\{y_n - ze^i \in \mathcal{Q}\} \cap \{q_n - ze^i \in \mathcal{Q}\}} z \left(H^{i,a} \left(\frac{v(s_n, y_n) - v(s_n, y_n - ze^i)}{z} \right) \right. \\
& \quad \left. - H^{i,a} \left(\frac{u(t_n, q_n) - u(t_n, q_n - ze^i)}{z} \right) \right) \mu^{i,a}(dz) \\
& \leq -2\varepsilon + (\psi(q_n) - \psi(y_n)) - (\chi(q_n) - \chi(y_n)) \sum_{i \in I_{\mathcal{T}}} \mathcal{H}^i(2n(q_n^i - y_n^i)) \tag{2.12} \\
& - \sum_{i \in I_{\mathcal{M}}} \int_{\mathbb{R}_+^*} \mathbb{1}_{\{y_n + ze^i \in \mathcal{Q}\} \cap \{q_n + ze^i \notin \mathcal{Q}\}} z H^{i,b} \left(\frac{v(s_n, y_n) - v(s_n, y_n + ze^i)}{z} \right) \mu^{i,b}(dz) \\
& + \sum_{i \in I_{\mathcal{M}}} \int_{\mathbb{R}_+^*} \mathbb{1}_{\{y_n + ze^i \notin \mathcal{Q}\} \cap \{q_n + ze^i \in \mathcal{Q}\}} z H^{i,b} \left(\frac{u(t_n, q_n) - u(t_n, q_n + ze^i)}{z} \right) \mu^{i,b}(dz) \\
& - \sum_{i \in I_{\mathcal{M}}} \int_{\mathbb{R}_+^*} \mathbb{1}_{\{y_n - ze^i \in \mathcal{Q}\} \cap \{q_n - ze^i \notin \mathcal{Q}\}} z H^{i,a} \left(\frac{v(s_n, y_n) - v(s_n, y_n - ze^i)}{z} \right) \mu^{i,a}(dz) \\
& + \sum_{i \in I_{\mathcal{M}}} \int_{\mathbb{R}_+^*} \mathbb{1}_{\{y_n - ze^i \notin \mathcal{Q}\} \cap \{q_n - ze^i \in \mathcal{Q}\}} z H^{i,a} \left(\frac{u(t_n, q_n) - u(t_n, q_n - ze^i)}{z} \right) \mu^{i,a}(dz).
\end{aligned}$$

We know that $q_n, y_n \xrightarrow[n \rightarrow +\infty]{} \bar{q}$. Therefore:

$$(\psi(q_n) - \psi(y_n)) \xrightarrow[n \rightarrow +\infty]{} 0.$$

Moreover, as χ and all the \mathcal{H}^i are Lipschitz, and as $\mathcal{H}^i(0) = 0$ for all $i \in I_{\mathcal{T}}$, there exists a constant C such that for all n ,

$$\left| (\chi(q_n) - \chi(y_n)) \sum_{i \in I_{\mathcal{T}}} \mathcal{H}^i(2n(q_n^i - y_n^i)) \right| \leq Cn \|q_n - y_n\|^2 \xrightarrow[n \rightarrow +\infty]{} 0.$$

We also have for all $i \in I_{\mathcal{M}}$ and for almost every $z > 0$:

$$\mathbb{1}_{\{y_n + ze^i \in \mathcal{Q}\} \cap \{q_n + ze^i \notin \mathcal{Q}\}} \xrightarrow[n \rightarrow +\infty]{} 0.$$

By Lemma 3, the term $z H^{i,b} \left(\frac{v(s_n, y_n) - v(s_n, y_n + ze^i)}{z} \right)$ is bounded uniformly in n and z , and by the absolute continuity of $\mu^{i,b}$ for all $i \in I_{\mathcal{M}}$, the dominated convergence theorem enables us to conclude that:

$$\sum_{i \in I_{\mathcal{M}}} \int_{\mathbb{R}_+^*} \mathbb{1}_{\{y_n + ze^i \in \mathcal{Q}\} \cap \{q_n + ze^i \notin \mathcal{Q}\}} z H^{i,b} \left(\frac{v(s_n, y_n) - v(s_n, y_n + ze^i)}{z} \right) \mu^{i,b}(dz) \xrightarrow[n \rightarrow +\infty]{} 0.$$

By the same reasoning, for all $i \in I_{\mathcal{M}}$:

$$\begin{aligned} & \sum_{i \in I_{\mathcal{M}}} \int_{\mathbb{R}_+^*} \mathbb{1}_{\{y_n + ze^i \notin \mathcal{Q}\} \cap \{q_n + ze^i \in \mathcal{Q}\}} z H^{i,b} \left(\frac{u(t_n, q_n) - u(t_n, q_n + ze^i)}{z} \right) \mu^{i,b}(dz) \xrightarrow{n \rightarrow +\infty} 0, \\ & \sum_{i \in I_{\mathcal{M}}} \int_{\mathbb{R}_+^*} \mathbb{1}_{\{y_n - ze^i \in \mathcal{Q}\} \cap \{q_n - ze^i \notin \mathcal{Q}\}} z H^{i,a} \left(\frac{v(s_n, y_n) - v(s_n, y_n - ze^i)}{z} \right) \mu^{i,a}(dz) \xrightarrow{n \rightarrow +\infty} 0, \\ & \sum_{i \in I_{\mathcal{M}}} \int_{\mathbb{R}_+^*} \mathbb{1}_{\{y_n - ze^i \notin \mathcal{Q}\} \cap \{q_n - ze^i \in \mathcal{Q}\}} z H^{i,a} \left(\frac{u(t_n, q_n) - u(t_n, q_n - ze^i)}{z} \right) \mu^{i,a}(dz) \xrightarrow{n \rightarrow +\infty} 0. \end{aligned}$$

We can thus choose n large enough so that the right-hand side of (2.12) is negative.

Moreover, on the left-hand side of (2.12), all the integrals are always nonnegative; indeed, $\forall i \in I_{\mathcal{M}}$, we have

$$\begin{aligned} & u(t_n, q_n - ze^i) - v(s_n, y_n - ze^i) - n|q_n - y_n|^2 - n|t_n - s_n|^2 - \varepsilon(2T - t_n - s_n) \\ & \leq u(t_n, q_n) - v(s_n, y_n) - n|q_n - y_n|^2 - n|t_n - s_n|^2 - \varepsilon(2T - t_n - s_n), \end{aligned} \quad (2.13)$$

therefore $v(s_n, y_n) - v(s_n, y_n - ze^i) \leq u(t_n, q_n) - u(t_n, q_n - ze^i)$ and as $H^{i,a}$ is nonincreasing, we get the result (the proof is identical for the integrals with $H^{i,b}$).

Therefore, the left-hand side is nonnegative for every n . But, as we said before, for n large enough, the right-hand side of (2.12) is negative, which yields a contradiction.

In conclusion, we necessarily have $\sup_{[0, T] \times \mathcal{Q}} u - v \leq 0$.

□

Theorem 4. θ is the only continuous viscosity solution to (HJ).

Proof. We know that θ is a bounded viscosity solution of (HJ), and in particular, θ_* is a bounded supersolution of (HJ), θ^* is a bounded subsolution of (HJ), and $\theta_*(T, \cdot) = \theta^*(T, \cdot) = -\ell_d$.

Hence θ_* and θ^* verify the assumptions of Theorem 3, and we get that $\theta_* \geq \theta^*$ on $[0, T] \times \mathcal{Q}$. But by definition of θ_* and θ^* , we have $\theta_* \leq \theta \leq \theta^*$. Thus we have $\theta_* = \theta = \theta^*$, and θ is continuous.

Let us now assume that we have another continuous viscosity solution $\tilde{\theta}$. In particular, $\tilde{\theta}$ is a subsolution to (HJ) and θ is a supersolution to (HJ), and as $\tilde{\theta}(T, q) = \theta(T, q) = -\ell_d(q) \forall q \in \mathcal{Q}$, we know by the comparison principle that $\tilde{\theta} \leq \theta$ on $[0, T] \times \mathcal{Q}$. But we also have that $\tilde{\theta}$ is a supersolution and θ is a subsolution, so by the same argument we have $\tilde{\theta} \geq \theta$ and finally $\tilde{\theta} = \theta$ on $[0, T] \times \mathcal{Q}$. Hence the uniqueness.

□

2.4 Numerical results

In this section, we apply our model to the case of a market maker dealing with only one asset, for which she has access to a liquidity pool where she can unwind part of her inventory. This is often the case for instance of foreign exchange markets. We consider an asset with the following characteristics:

- Asset price: $S_0 = 100 \text{ €}$.
- Volatility: $\sigma = 0.2 \text{ €} \cdot \text{day}^{-\frac{1}{2}}$.
- Intensity functions:

$$\Lambda^b(\delta) = \Lambda^a(\delta) = \lambda_R \frac{1}{1 + e^{\alpha_\Lambda + \beta_\Lambda \delta}},$$

with $\lambda_R = 140 \text{ day}^{-1}$, $\alpha_\Lambda = 0.7$, and $\beta_\Lambda = 100 \text{ €}^{-1}$. This corresponds to 140 requests per day, a probability of $\frac{1}{1+e^{0.7}} \simeq 33\%$ to trade when the answered quote is the reference price and a probability of $\frac{1}{1+e^{-0.3}} \simeq 57\%$ to trade when the answered quote is the reference price improved by 1 cent.

- Request sizes are distributed according to a Gamma distribution $\Gamma(\alpha_\mu, \beta_\mu)$ with $\alpha_\mu = 4$ and $\beta_\mu = 4 \cdot 10^{-4}$. This corresponds to an average request size of 10000 assets (i.e. approximately 1000000 €) and a standard deviation equal to half the average.

Regarding the objective function, we consider the following:

- Time horizon given by $T = 5$ days. This horizon ensures convergence towards stationary quotes at time $t = 0$.
- $L : v \in \mathbb{R} \mapsto \eta v^2 + \phi |v|$ with $\eta = 10^{-3}$ and $\phi = 5 \cdot 10^{-4}$.
- $\psi : q \in \mathbb{R} \mapsto \frac{\gamma}{2} \sigma^2 q^2$ with $\gamma = 5 \cdot 10^{-7} \text{ €}^{-1}$.
- $\ell_d = 0$.

We impose risk limits in the sense that no trade that would result in an inventory $q \in \mathbb{R}$ such that $q^2 \sigma^2 > B$ is admitted, where $B = 1.936 \cdot 10^9$ (this corresponds to 220000 assets, long or short). We then approximate the solution θ to (HJ) using a monotone implicit Euler scheme with linear interpolation on a grid with 313 points for the inventory and a discretization of the request size distribution with 4 sizes: $z^1 = 10000$, $z^2 = 20000$, $z^3 = 30000$, and $z^4 = 40000$ assets – thereafter denoted by very small, small, large, and very large size – with respective probability $p^1 = 0.8651$, $p^2 = 0.1268$, $p^3 = 0.0078$ and $p^4 = 0.0003$.

The value function (at time $t = 0$) is plotted in Figure 2.1. As discussed above, we choose $T = 5$ days to ensure convergence of the “optimal” quotes to their stationary values. This is illustrated in Figure 2.2.

To observe the impact of the request size on the “optimal” quotes, we plot in Figure 2.3 the four functions

$$q \mapsto \bar{\delta}^b(0, q, z^k), \quad k \in \{1, \dots, 4\},$$

where $\bar{\delta}^b$ represents the “optimal” quotes as a function of time, inventory and size of request. We see that accounting for the size of requests impacts the optimal bid quotes, especially for a large (long or short) inventory. The monotonicity of the quotes is of course unsurprising.

We then plot in Figure 2.4 the “optimal” trading rate of the market maker as a function of her inventory. Naturally, she will tend to buy the asset when her inventory is short, and sell it when her inventory is long. We observe two interesting effects: first, we see a plateau around zero. This is due to the proportional transaction cost (given by the parameter ϕ), that discourages the trader to buy or sell when her inventory is small enough (she prefers to bear this small risk than to pay the transaction cost). Second, we see some “waves” when the inventory gets close to the limit, which is essentially a numerical effect due to the discretization of the distribution of request size: the market maker has less incentive to trade fast in some positions because she wants to “wait” for a large trade that can get her inventory back to zero quickly.

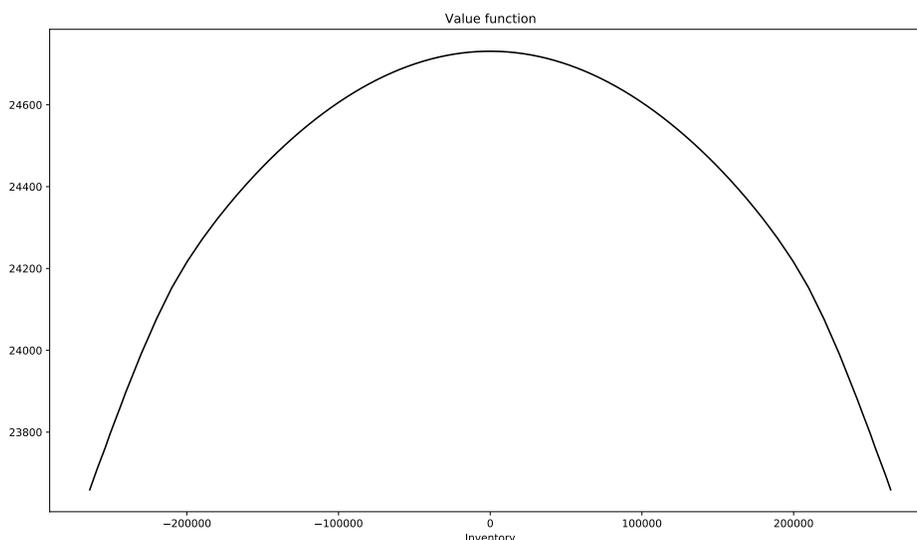


Figure 2.1: Value function of the problem for different values of the inventory.

We use quotation marks here around the word “optimal” because, as we could not prove the regularity of the value function – in fact, we have reason to think that it is not always C^1 in q – we were not able to obtain a verification theorem, and thus prove that the “optimal” quotes given by Lemma 2 and the “optimal” execution rates given by Lemma 4 are indeed optimal controls of the problem. To confirm empirically that the controls are indeed optimal, we perform Monte-Carlo simulations using those controls and compare the obtained expectation with the value function given above, and plot this in Figure 2.5. We see that the values coincide in our case.

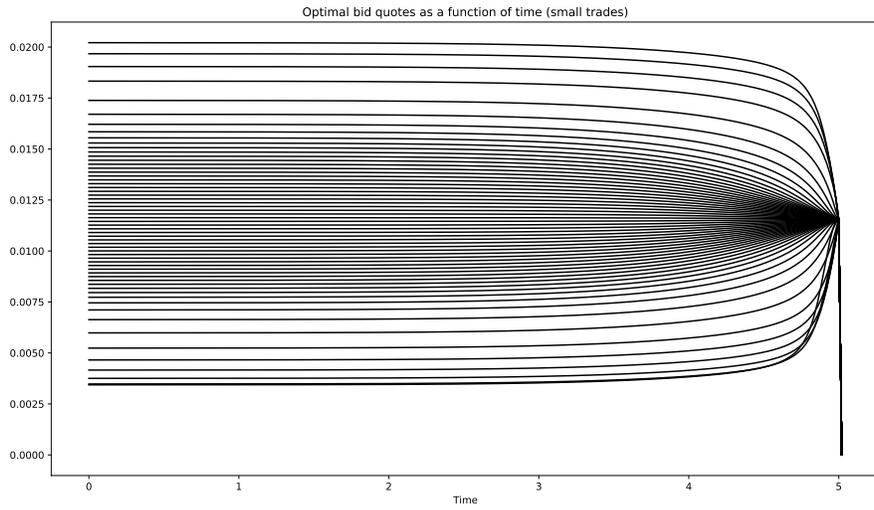


Figure 2.2: Optimal bid quotes as a function of time for different values of the inventory (very small trades).

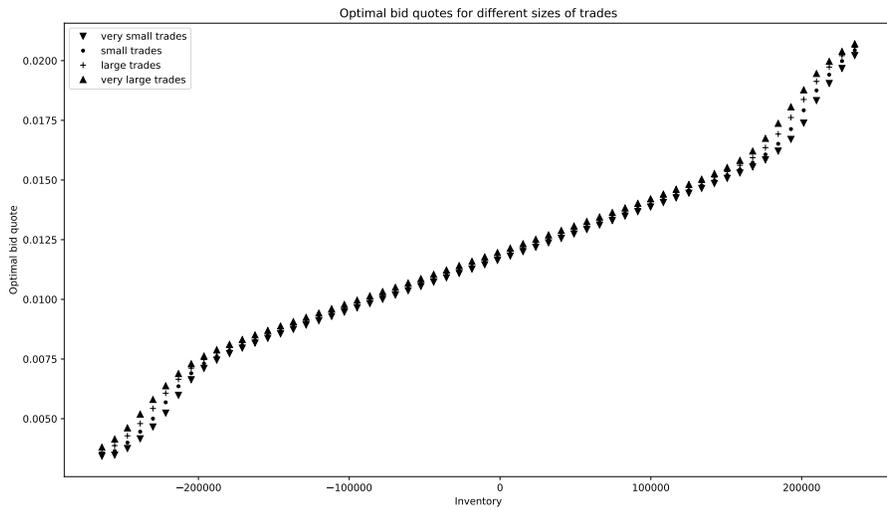


Figure 2.3: Optimal bid quotes for different trade sizes as a function of the inventory.

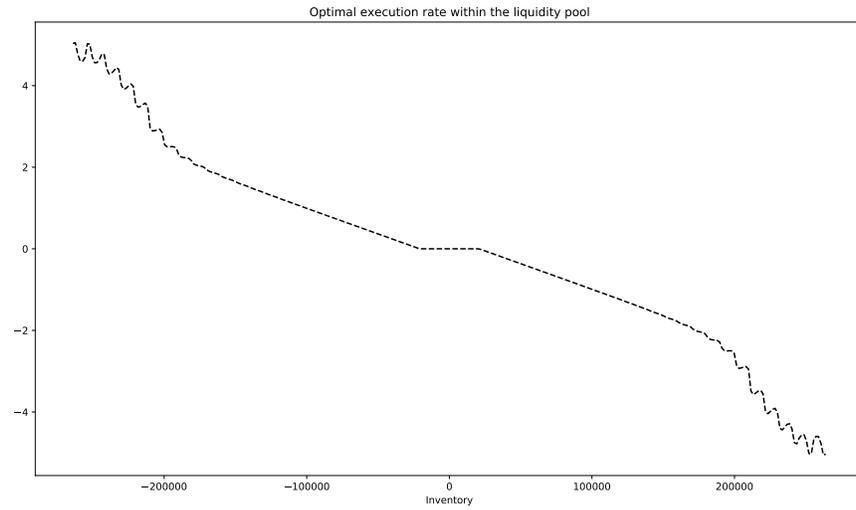


Figure 2.4: Optimal execution rate as a function of the inventory.

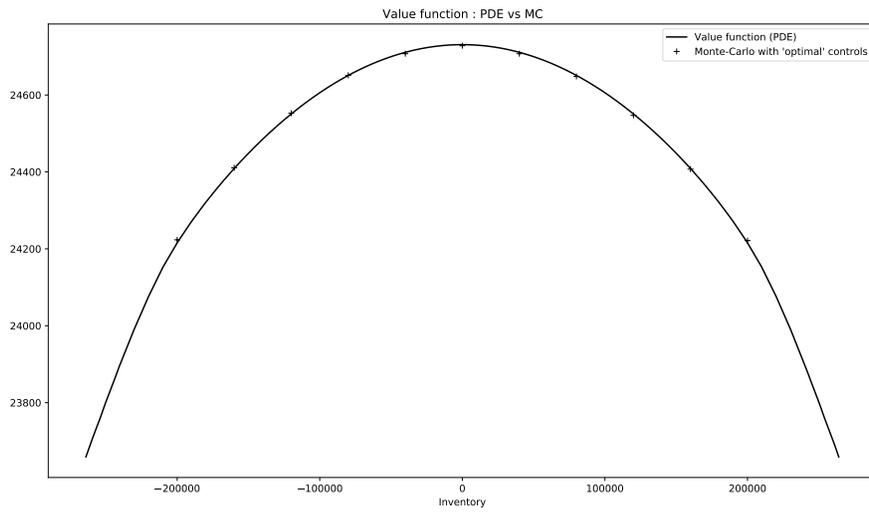


Figure 2.5: Value function obtained by playing the “optimal” quotes and execution rates (compared with the known value function of the problem).

Conclusion

In this paper, we generalized existing OTC market making models, and in particular the one presented in Bergault and Guéant [26], to introduce the possibility for the market maker to trade “actively” on some liquidity pools in order to unwind part of her inventory. This extension led to a partial integro-differential equation of the Hamilton-Jacobi type, to which we prove that the value function of the problem is the unique continuous viscosity solution. We illustrate our results numerically using an example in dimension one, and solving the equation on a grid using an implicit Euler scheme. We then compute the optimal quotes and trading rates. We use a Monte-Carlo method to show empirically that those controls are indeed optimal.

2.5 Appendix

2.5.1 Proof of Lemma 3

Let $i \in I_{\mathcal{M}}$. We only prove it for the ask side (the proof for the bid side is similar).

Let $t \in [0, T]$, $q \in \mathcal{Q}$ and $z \in \mathbb{R}_+^*$ such that $q - ze^i \in \mathcal{Q}$. We have:

$$\begin{aligned} zH^{i,a} \left(\frac{\varphi(t, q) - \varphi(t, q - ze^i)}{z} \right) &= z \sup_{\delta \geq -\delta_\infty} \Lambda^{i,a}(\delta) \left(\delta - \frac{\varphi(t, q) - \varphi(t, q - ze^i)}{z} \right) \\ &\leq z \sup_{\delta \geq -\delta_\infty} \Lambda^{i,a}(\delta) \delta + \sup_{\delta \geq -\delta_\infty} -\Lambda^{i,a}(\delta) \left(\varphi(t, q) - \varphi(t, q - ze^i) \right) \\ &\leq \text{diam}(\mathcal{Q}) \sup_{\delta \geq -\delta_\infty} \Lambda^{i,a}(\delta) \delta + 2 \left(\sup_{[0, T] \times \mathcal{Q}} |\varphi| \right) \sup_{\delta \geq -\delta_\infty} \Lambda^{i,a}(\delta), \end{aligned}$$

where $\text{diam}(\mathcal{Q})$ denotes the diameter of \mathcal{Q} . For the other bound, we have:

$$zH^{i,a} \left(\frac{\varphi(t, q) - \varphi(t, q - ze^i)}{z} \right) = \sup_{\delta \geq -\delta_\infty} \left\{ z\Lambda^{i,a}(\delta) \delta - \Lambda^{i,a}(\delta) \left(\varphi(t, q) - \varphi(t, q - ze^i) \right) \right\}.$$

We can just take $\delta \geq \frac{\varphi(t, q) - \varphi(t, q - ze^i)}{z}$ to see that:

$$zH^{i,a} \left(\frac{\varphi(t, q) - \varphi(t, q - ze^i)}{z} \right) \geq 0,$$

hence the result.

2.5.2 Proof of Proposition 4

We only prove the subsolution part (the proof for the supersolution part is identical).

Let us first assume that the following inequality holds:

$$\begin{aligned}
& -\frac{\partial \varphi}{\partial t}(\bar{t}, \bar{q}) + \psi(\bar{q}) - \sum_{i \in I_{\mathcal{M}}} \int_{\mathbb{R}_+^*} \mathbf{1}_{\{\bar{q} + ze^i \in \mathcal{Q}\}} z H^{i,b} \left(\frac{u(\bar{t}, \bar{q}) - u(\bar{t}, \bar{q} + ze^i)}{z} \right) \mu^{i,b}(dz) \\
& - \sum_{i \in I_{\mathcal{M}}} \int_{\mathbb{R}_+^*} \mathbf{1}_{\{\bar{q} - ze^i \in \mathcal{Q}\}} z H^{i,a} \left(\frac{u(\bar{t}, \bar{q}) - u(\bar{t}, \bar{q} - ze^i)}{z} \right) \mu^{i,a}(dz) - \chi(q) \sum_{i \in I_{\mathcal{T}}} \mathcal{H}^i(\partial_{q^i} \varphi(\bar{t}, \bar{q})) \leq 0.
\end{aligned}$$

We know that $\forall z > 0, \forall i \in \{1, \dots, d\}$:

$$u(\bar{t}, \bar{q} - ze^i) - \varphi(\bar{t}, \bar{q} - ze^i) \leq u(\bar{t}, \bar{q}) - \varphi(\bar{t}, \bar{q}).$$

Thus:

$$H^{i,a} \left(\frac{u(\bar{t}, \bar{q}) - u(\bar{t}, \bar{q} - ze^i)}{z} \right) \leq H^{i,a} \left(\frac{\varphi(\bar{t}, \bar{q}) - \varphi(\bar{t}, \bar{q} - ze^i)}{z} \right),$$

and the same holds for $H^{i,b}$:

$$H^{i,b} \left(\frac{u(\bar{t}, \bar{q}) - u(\bar{t}, \bar{q} + ze^i)}{z} \right) \leq H^{i,b} \left(\frac{\varphi(\bar{t}, \bar{q}) - \varphi(\bar{t}, \bar{q} + ze^i)}{z} \right).$$

So we get:

$$\begin{aligned}
& -\frac{\partial \varphi}{\partial t}(\bar{t}, \bar{q}) + \psi(\bar{q}) - \sum_{i=1}^d \int_{\mathbb{R}_+^*} \mathbf{1}_{\{\bar{q} + ze^i \in \mathcal{Q}\}} z H^{i,b} \left(\frac{\varphi(\bar{t}, \bar{q}) - \varphi(\bar{t}, \bar{q} + ze^i)}{z} \right) \mu^{i,b}(dz) \\
& - \sum_{i=1}^d \int_{\mathbb{R}_+^*} \mathbf{1}_{\{\bar{q} - ze^i \in \mathcal{Q}\}} z H^{i,a} \left(\frac{\varphi(\bar{t}, \bar{q}) - \varphi(\bar{t}, \bar{q} - ze^i)}{z} \right) \mu^{i,a}(dz) - \chi(q) \sum_{i \in I_{\mathcal{T}}} \mathcal{H}^i(\partial_{q^i} \varphi(\bar{t}, \bar{q})) \leq 0,
\end{aligned}$$

and u is a viscosity subsolution.

Let us now assume that u is a viscosity subsolution. Without loss of generality, we can assume $\varphi(\bar{t}, \bar{q}) = u(\bar{t}, \bar{q})$.

Let B_η be the open ball of center (\bar{t}, \bar{q}) and radius $\eta > 0$. Let (u_n) be a sequence of smooth functions uniformly (in n) bounded such that $u_n \geq u \forall n$ and $u_n \xrightarrow{n \rightarrow +\infty} u$ pointwise. Let ξ be a smooth nondecreasing function such that $\xi(x) = 1$ if $x > \eta/4$ and $\xi(x) = 0$ if $x < -\eta/4$. Let $d_{\eta/2}$ be the algebraic distance to $\partial B_{\eta/2}$ (with $d_{\eta/2} > 0$ on $B_{\eta/2}$ and $d_{\eta/2} \leq 0$ on $B_{\eta/2}^c$); this function is continuously differentiable. We introduce:

$$\varphi_\eta^n = \varphi \times (\xi \circ d_{\eta/2}) + u_n \times (1 - \xi \circ d_{\eta/2}).$$

Then (\bar{t}, \bar{q}) is still a max of $u - \varphi_\eta^n$ and $(u - \varphi_\eta^n)(\bar{t}, \bar{q}) = 0$. Furthermore we have $\frac{\partial \varphi_\eta^n}{\partial t}(\bar{t}, \bar{q}) = \frac{\partial \varphi}{\partial t}(\bar{t}, \bar{q})$ and $\partial_{q^i} \varphi_\eta^n(\bar{t}, \bar{q}) = \partial_{q^i} \varphi(\bar{t}, \bar{q})$ for all $i \in I_{\mathcal{T}}$. Thus:

$$\begin{aligned}
& -\frac{\partial\varphi}{\partial t}(\bar{t}, \bar{q}) + \psi(\bar{q}) - \sum_{i=1}^d \int_{\mathbb{R}_+^*} \mathbb{1}_{\{\bar{q} + ze^i \in \mathcal{Q}\}} z H^{i,b} \left(\frac{\varphi_\eta^n(\bar{t}, \bar{q}) - \varphi_\eta^n(\bar{t}, \bar{q} + ze^i)}{z} \right) \mu^{i,b}(dz) \\
& - \sum_{i=1}^d \int_{\mathbb{R}_+^*} \mathbb{1}_{\{\bar{q} - ze^i \in \mathcal{Q}\}} z H^{i,a} \left(\frac{\varphi_\eta^n(\bar{t}, \bar{q}) - \varphi_\eta^n(\bar{t}, \bar{q} - ze^i)}{z} \right) \mu^{i,a}(dz) - \chi(q) \sum_{i \in I_\tau} \mathcal{H}^i(\partial_{q^i} \varphi(\bar{t}, \bar{q})) \leq 0.
\end{aligned}$$

Plus we have $\varphi_\eta^n \xrightarrow{n \rightarrow +\infty} \varphi_\eta$ pointwise with $\varphi_\eta = \varphi \times (\xi \circ d_{\eta/2}) + u \times (1 - \xi \circ d_{\eta/2})$ which is smooth on $B_{\eta/4}$ and such that $\varphi_\eta = u$ on B_η^c and $\varphi_\eta(\bar{t}, \bar{q}) = u(\bar{t}, \bar{q})$.

By continuity of $H^{i,a}$ and $H^{i,b}$ and by dominated convergence (using the same argument than in Lemma 2 and the fact that the φ_η^n are bounded uniformly in n) we get:

$$\begin{aligned}
& -\frac{\partial\varphi}{\partial t}(\bar{t}, \bar{q}) + \psi(\bar{q}) - \sum_{i=1}^d \int_{\mathbb{R}_+^*} \mathbb{1}_{\{\bar{q} + ze^i \in \mathcal{Q}\}} z H^{i,b} \left(\frac{\varphi_\eta(\bar{t}, \bar{q}) - \varphi_\eta(\bar{t}, \bar{q} + ze^i)}{z} \right) \mu^{i,b}(dz) \\
& - \sum_{i=1}^d \int_{\mathbb{R}_+^*} \mathbb{1}_{\{\bar{q} - ze^i \in \mathcal{Q}\}} z H^{i,a} \left(\frac{\varphi_\eta(\bar{t}, \bar{q}) - \varphi_\eta(\bar{t}, \bar{q} - ze^i)}{z} \right) \mu^{i,a}(dz) - \chi(q) \sum_{i \in I_\tau} \mathcal{H}^i(\partial_{q^i} \varphi(\bar{t}, \bar{q})) \leq 0.
\end{aligned}$$

By then sending η to 0 and using again dominated convergence, we get the result:

$$\begin{aligned}
& -\frac{\partial\varphi}{\partial t}(\bar{t}, \bar{q}) + \psi(\bar{q}) - \sum_{i=1}^d \int_{\mathbb{R}_+^*} \mathbb{1}_{\{\bar{q} + ze^i \in \mathcal{Q}\}} z H^{i,b} \left(\frac{u(\bar{t}, \bar{q}) - u(\bar{t}, \bar{q} + ze^i)}{z} \right) \mu^{i,b}(dz) \\
& - \sum_{i=1}^d \int_{\mathbb{R}_+^*} \mathbb{1}_{\{\bar{q} - ze^i \in \mathcal{Q}\}} z H^{i,a} \left(\frac{u(\bar{t}, \bar{q}) - u(\bar{t}, \bar{q} - ze^i)}{z} \right) \mu^{i,a}(dz) - \chi(q) \sum_{i \in I_\tau} \mathcal{H}^i(\partial_{q^i} \varphi(\bar{t}, \bar{q})) \leq 0.
\end{aligned}$$

Chapter 3

Algorithmic market making for options

Abstract

In this article, we tackle the problem of a market maker in charge of a book of options on a single liquid underlying asset. By using an approximation of the portfolio in terms of its vega, we show that the seemingly high-dimensional stochastic optimal control problem of an option market maker is in fact tractable. More precisely, when volatility is modeled using a classical stochastic volatility model – e.g. the Heston model – the problem faced by an option market maker is characterized by a low-dimensional functional equation that can be solved numerically using a Euler scheme along with interpolation techniques, even for large portfolios. In order to illustrate our findings, numerical examples are provided.

3.1 Introduction

The electronification of financial markets started in the seventies with stock exchanges and now affects each and every asset class. For asset classes that are usually traded in a centralized way (stocks, futures, etc.), exchanges and other all-to-all trading platforms – based or not based on limit order books – are now fully automated. For assets that are still traded over the counter (OTC), electronification occurs through the introduction of new platforms, for instance single- and multi-dealer-to-client platforms.

This electronification is associated with a trend towards the automation of the trading process for many players of the financial industry: brokers, banks, but also systematic asset managers who often develop their own execution algorithms. For assets traded in centralized markets, trading automation is nowadays massive. For instance, in the cash equity world, a vast majority of the execution is now carried out using algorithms. For asset traded in dealer markets, the automation of the market making process has been at the agenda for a few years and more and more banks are developing market making algorithms for various asset classes (currencies, bonds, etc.).

In the academic literature, many market making models have been proposed since the eighties. In the early literature on market making, the two main references are the paper of Ho and Stoll [71] and the paper of Grossman and Miller [59]. Ho and Stoll introduced indeed a very relevant framework to tackle the main problem faced by market makers: inventory management. Grossman and Miller, who were more interested in capturing the essence of liquidity, proposed a very simple model with 3 periods that encompassed both market makers and final customers, enabled to understand what happens at equilibrium, and contributed to the important literature on the price formation process. If the latter paper belongs to a strand of literature that is extremely important to go beyond the simple Walrasian view of markets, it is of little help to build market making algorithms. The former paper however, after more than 25 years, has paved the way to a recent mathematical literature on algorithmic market making.

A seminal reference of the new literature on market making is the paper of Avellaneda and Stoikov [13] who revived the dynamic approach proposed by Ho and Stoll. They indeed showed how the quoting and inventory management problems of market makers could be addressed using the tools of stochastic optimal control. Since then, many models have been proposed, most of them to tackle the same problem of single-asset market making as that of Avellaneda and Stoikov. For instance, Guéant, Lehalle, and Fernandez-Tapia provided in [65] a rigorous analysis of the Avellaneda-Stoikov stochastic optimal control problem and proved that the problem could be simplified into a system of linear ordinary differential equations (ODE) in the case of exponential intensity functions. Cartea, Jaimungal, and coauthors contributed a lot to the literature and added many features to the initial models: market impact, alpha signals, ambiguity aversion, etc. (see [36, 39, 40]). They also considered a different objective function: a risk-adjusted expectation instead of a Von Neumann-Morgenstern expected utility.

The models proposed in the above papers all share the same characteristics: (i) they are agnostic with respect to the market structure, but are in fact more adapted to OTC markets or limit order book markets with small tick size, (ii) they only deal with single-asset market making, and (iii) they do not deal with the market making of options.

In fact, models have been specifically developed by Guilbaud and Pham (see [68, 69]) for assets traded through limit order books with large tick size (e.g. most stocks) and for assets traded on platforms with a pro-rata microstructure (e.g. some currency pairs). Interestingly, these models enable the use of aggressive orders by market makers, which is – surprisingly – a standard behavior on equity markets (see [95]).

As far as multi-asset market making is concerned, models have been developed recently to account for the correlation structure between asset price changes. Guéant extended to a multi-asset framework models *à la* Avellaneda-Stoikov and models *à la* Cartea-Jaimungal (see [61], [62], and [63]) and showed that the problem boils down, for general intensity functions, to solving a system of (*a priori* nonlinear) ODEs. The associated question of the numerical methods to approximate the solution of the equations characterizing the optimal quotes of a multi-asset market maker is addressed in [26] using a factorial approach and in [66] using reinforcement learning, both with applications to corporate bond markets.

Finally, as far as asset classes are concerned, there have been few attempts to address market mak-

ing problems outside of the cash world. Market making models for derivative contracts are indeed intrinsically more complicated because they must account for the strategies on both the market for the underlying asset and the market for the derivatives, and usually for numerous contracts (e.g. options for lots of strikes and maturities). Option market making is only addressed in a paper by El Aoud and Abergel (see [46]) and in a paper by Stoikov and Sağlam [98]. In the former, the authors consider a single-option market driven by a stochastic volatility model and assume that the position is always Δ -hedged. They provide optimal bid and ask quotes for the option and focus on the risk of model misspecification. In the latter, the authors consider three different settings, but all with only one option: (i) a market maker in a complete market where continuous trading in the perfectly liquid underlying stock is allowed, (ii) a market maker who may not trade continuously in the underlying stock, but rather sets bid and ask quotes in the option and the stock, and (iii) a market maker in an incomplete market with residual risks due to stochastic volatility and overnight jumps.

In this paper, we consider the case of a market maker in charge of a book of options whose prices are driven by a stochastic volatility model. We assume that trading in continuous time can be carried out in the underlying asset so that the residual risk is only that of the vega associated with the inventory. Using a constant-vega approximation, we show that the problem of an option market maker boils down to solving a low-dimensional functional equation of the Hamilton-Jacobi-Bellman type that can be tackled numerically using a simple Euler scheme along with interpolation techniques. In particular, in spite of the large number of assets, the market making problem is tractable.

In Section 1, we describe the model and present the optimization problem of the option market maker. In Section 2, we show how that problem can be simplified under the constant-vega approximation. In particular, we show that solving the high-dimensional stochastic optimal control problem of the market maker boils down to solving a low-dimensional functional equation. In Section 3, we consider the example of a book of options with several strikes and maturities and provide numerical results obtained through interpolation techniques and an explicit Euler scheme.

3.2 Description of the problem

We consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $(\mathcal{F}_t)_{t \in \mathbb{R}_+}$ satisfying the usual conditions. Throughout the paper, we assume that all stochastic processes are defined on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathbb{R}_+}, \mathbb{P})$.

3.2.1 The market

We consider an asset whose price dynamics is described by a one-factor stochastic volatility model of the form

$$\begin{cases} dS_t = \mu S_t dt + \sqrt{\nu_t} S_t dW_t^S \\ d\nu_t = a_{\mathbb{P}}(t, \nu_t) dt + \xi \sqrt{\nu_t} dW_t^\nu, \end{cases}$$

where $\mu \in \mathbb{R}, \xi \in \mathbb{R}^{*+}$, $(W_t^S, W_t^\nu)_{t \in \mathbb{R}_+}$ is a couple of Brownian motions with quadratic covariation given by $\rho = \frac{d\langle W^S, W^\nu \rangle}{dt} \in (-1, 1)$, and $a_{\mathbb{P}}$ is such that the processes are well defined (in particular,

we assume that the process $(\nu_t)_{t \in \mathbb{R}_+}$ stays positive almost surely).

Remark 8. A classical example for the function $a_{\mathbb{P}}$ is that of the Heston model (see [70]), i.e. $a_{\mathbb{P}} : (t, \nu) \mapsto \kappa_{\mathbb{P}}(\theta_{\mathbb{P}} - \nu)$ where $\kappa_{\mathbb{P}}, \theta_{\mathbb{P}} \in \mathbb{R}^+$ satisfy the Feller condition $2\kappa_{\mathbb{P}}\theta_{\mathbb{P}} > \xi^2$.

Remark 9. For the sake of simplicity, we consider throughout this paper a one-factor model where the instantaneous variance is the main variable of interest. Similar results could be obtained with a one-factor model focused on forward variances, such as the classical one-factor Bergomi model (see [28, 29]). Moreover, it is noteworthy that our approach can easily be extended to two-factor stochastic volatility models such as the celebrated two-factor Bergomi model (see [28, 29]), up to an increase – by 1 – in the dimension of the equation to solve.

Assuming interest rates are equal to 0, we introduce a risk-neutral / pricing probability measure¹ \mathbb{Q} under which the price and volatility processes become

$$\begin{cases} dS_t = \sqrt{\nu_t} S_t d\widehat{W}_t^S \\ d\nu_t = a_{\mathbb{Q}}(t, \nu_t) dt + \xi \sqrt{\nu_t} d\widehat{W}_t^\nu, \end{cases}$$

where $(\widehat{W}_t^S, \widehat{W}_t^\nu)_{t \in \mathbb{R}_+}$ is another couple of Brownian motions under \mathbb{Q} with quadratic covariation given by $\rho = \frac{d(\widehat{W}^S, \widehat{W}^\nu)}{dt} \in (-1, 1)$, and where $a_{\mathbb{Q}}$ is such that the processes are well defined.

We consider $N \geq 1$ European options written on the above asset (hereafter, the underlying asset). For each $i \in \{1, \dots, N\}$, the option maturity date is denoted by T^i and we denote by $(\mathcal{O}_t^i)_{t \in [0, T^i]}$ the price process associated with the i -th option.

Remark 10. In applications, the options under consideration will always be call and/or put options. However, our setting enables to consider any European payoff.

In the above one-factor model, we know that for all $i \in \{1, \dots, N\}$, and all $t \in [0, T^i]$, $\mathcal{O}_t^i = O^i(t, S_t, \nu_t)$ where O^i is solution on $[0, T^i] \times \mathbb{R}_+^2$ of the following partial differential equation (PDE):

$$\begin{aligned} 0 &= \partial_t O^i(t, S, \nu) + a_{\mathbb{Q}}(t, \nu) \partial_\nu O^i(t, S, \nu) \\ &\quad + \frac{1}{2} \nu S^2 \partial_{SS}^2 O^i(t, S, \nu) + \rho \xi \nu S \partial_{\nu S}^2 O^i(t, S, \nu) + \frac{1}{2} \xi^2 \nu \partial_{\nu\nu}^2 O^i(t, S, \nu). \end{aligned} \quad (3.1)$$

Remark 11. Options prices are also characterized by a terminal condition corresponding to the payoff. However, we will only consider short-term optimization problems for which the time horizon is before the maturity of all the options under consideration. Therefore, we shall never use the final condition associated with Eq. (3.1).

¹For references, see for instance [56].

3.2.2 The optimization problem of the market maker

We consider an option market maker in charge of providing bid and ask quotes for the N above options over the period $[0, T]$ where $T < \min_{i \in \{1, \dots, N\}} T^i$ (see Remark 11). For all $i \in \{1, \dots, N\}$ we denote by $\mathcal{O}_t^i - \delta_t^{i,b}(z)$ and $\mathcal{O}_t^i + \delta_t^{i,a}(z)$ the bid and ask prices proposed by the market maker for a transaction of a number z of i -th options, where $(\delta_t^i(\cdot))_{t \in [0, T]} := (\delta_t^{i,b}(\cdot), \delta_t^{i,a}(\cdot))_{t \in [0, T]}$ is \mathbb{F} -predictable and bounded from below by a given constant δ_∞ .² Hereafter, we denote by \mathcal{A} the set of such admissible control processes. The dynamics of the inventory process $(q_t)_{t \in [0, T]} := (q_t^1, \dots, q_t^N)'_{t \in [0, T]}$ of the market maker is given by

$$dq_t^i := \int_{\mathbb{R}_+^*} z(N^{i,b}(dt, dz) - N^{i,a}(dt, dz)), \quad i \in \{1, \dots, N\},$$

where $N^{i,b}(dt, dz), N^{i,a}(dt, dz)$ are two right-continuous \mathbb{R}_+^* -marked point processes, with almost surely no simultaneous jumps, modelling the transactions of the i -th option on the bid and ask side, whose respective intensity processes $(\lambda_t^{i,b}(dz))_{t \in \mathbb{R}_+}$ and $(\lambda_t^{i,a}(dz))_{t \in \mathbb{R}_+}$ are given by

$$\lambda_t^{i,b}(dz) := \Lambda^{i,b}(\delta_t^{i,b}(z)) \mathbb{1}_{\{q_{t-} + ze^i \in \mathcal{Q}\}} \mu^{i,b}(dz) \quad \lambda_t^{i,a}(dz) := \Lambda^{i,a}(\delta_t^{i,a}(z)) \mathbb{1}_{\{q_{t-} - ze^i \in \mathcal{Q}\}} \mu^{i,a}(dz)$$

with $(e^i)_{i \in \{1, \dots, N\}}$ the canonical basis of \mathbb{R}^N , \mathcal{Q} the set of authorized inventories³ for the market maker, and $\mu^{i,b}, \mu^{i,a}$ two probability measures on \mathbb{R}_+^* , modelling the distributions of transaction sizes. For $i \in \{1, \dots, N\}$, $\Lambda^{i,b}, \Lambda^{i,a}$ are positive functions satisfying the following classical hypotheses (see [61, 62] for similar assumptions):

- $\Lambda^{i,b}$ and $\Lambda^{i,a}$ are twice continuously differentiable.
- $\Lambda^{i,b}$ and $\Lambda^{i,a}$ are strictly decreasing, with $\Lambda^{i,b'} < 0$ and $\Lambda^{i,a'} < 0$.
- $\lim_{\delta \rightarrow +\infty} \Lambda^{i,b}(\delta) = \lim_{\delta \rightarrow +\infty} \Lambda^{i,a}(\delta) = 0$.
- $\sup_{\delta \in \mathbb{R}} \frac{\Lambda^{i,b}(\delta) \Lambda^{i,b''}(\delta)}{(\Lambda^{i,b'}(\delta))^2} < 2$ and $\sup_{\delta \in \mathbb{R}} \frac{\Lambda^{i,a}(\delta) \Lambda^{i,a''}(\delta)}{(\Lambda^{i,a'}(\delta))^2} < 2$.

The above conditions are sufficiently general to allow for several relevant forms of intensities: the exponential intensities initially introduced in [13] and used in most of the literature, logistic intensities as in [26], or many SU Johnson intensities as in [66].

In addition to quoting prices for the N options, the market maker can buy and sell the underlying asset. We assume that the market for that asset is liquid enough to ensure a perfect Δ -hedging.

Remark 12. *In practice, for a portfolio that is not vega-hedged, it is usually suboptimal to perfectly Δ -hedge the portfolio because of the correlation between the spot process and the instantaneous variance process. Nevertheless, we assume here for the sake of simplicity that Δ -hedging is carried out in continuous time. A study of the optimal position in the underlying asset and its consequence on our problem is carried out in Appendix 3.5.1.*

²In applications, we always choose δ_∞ negative enough so that this lower bound is never binding.

³The frontier of this set defines the risk limits of the market maker.

In what follows, we denote by $(\Delta_t)_{t \in [0, T]}$ the Δ of the portfolio:

$$\Delta_t := \sum_{i=1}^N \partial_S O^i(t, S_t, \nu_t) q_t^i \text{ for all } t \in [0, T].$$

The resulting dynamics for the cash process $(X_t)_{t \in [0, T]}$ of the market maker is:

$$dX_t := \sum_{i=1}^N \left(\int_{\mathbb{R}_+^*} z \left(\delta_t^{i,b}(z) N^{i,b}(dt, dz) + \delta_t^{i,a}(z) N^{i,a}(dt, dz) \right) - \mathcal{O}_t^i dq_t^i \right) + S_t d\Delta_t + d\langle \Delta, S \rangle_t.$$

We denote by $(V_t)_{t \in [0, T]}$ the process for the Mark-to-Market (MtM) value of the market maker's portfolio (cash, shares, and options), i.e.,

$$V_t := X_t - \Delta_t S_t + \sum_{i=1}^N q_t^i \mathcal{O}_t^i.$$

The dynamics of that process is given by

$$\begin{aligned} dV_t &= dX_t - S_t d\Delta_t - \Delta_t dS_t - d\langle \Delta, S \rangle_t + \sum_{i=1}^N \mathcal{O}_t^i dq_t^i + \sum_{i=1}^N q_t^i d\mathcal{O}_t^i \\ &= \sum_{i=1}^N \int_{\mathbb{R}_+^*} z \left(\delta_t^{i,b}(z) N^{i,b}(dt, dz) + \delta_t^{i,a}(z) N^{i,a}(dt, dz) \right) + q_t^i d\mathcal{O}_t^i - \Delta_t dS_t \\ &= \sum_{i=1}^N \int_{\mathbb{R}_+^*} z \left(\delta_t^{i,a}(z) N^{i,a}(dt, dz) + \delta_t^{i,b}(z) N^{i,b}(dt, dz) \right) + q_t^i \partial_\nu O^i(t, S_t, \nu_t) (a_{\mathbb{P}}(t, \nu_t) - a_{\mathbb{Q}}(t, \nu_t)) dt \\ &\quad + \sqrt{\nu_t} \xi q_t^i \partial_\nu O^i(t, S_t, \nu_t) dW_t^\nu. \end{aligned}$$

For all $i \in \{1, \dots, N\}$, the vega of the i -th option is defined as

$$\mathcal{V}_t^i := \partial_{\sqrt{\nu}} O^i(t, S_t, \nu_t) = 2\sqrt{\nu_t} \partial_\nu O^i(t, S_t, \nu_t) \text{ for all } t \in [0, T].$$

Hence, we can rewrite the dynamics of the portfolio as

$$dV_t = \sum_{i=1}^N \int_{\mathbb{R}_+^*} z \left(\delta_t^{i,b}(z) N_t^{i,b}(dt, dz) + \delta_t^{i,a}(z) N_t^{i,a}(dt, dz) \right) + q_t^i \mathcal{V}_t^i \frac{a_{\mathbb{P}}(t, \nu_t) - a_{\mathbb{Q}}(t, \nu_t)}{2\sqrt{\nu_t}} dt + \frac{\xi}{2} q_t^i \mathcal{V}_t^i dW_t^\nu.$$

Following the academic literature on market making, we can consider two objective functions: As in the initial Avellaneda and Stoikov setting [13] (see also [61, 62, 65]), we can consider the following expected utility objective function:

$$\sup_{\delta \in \mathcal{A}} \mathbb{E} \left[-\exp(-\gamma V_T) \right],$$

where $\gamma > 0$ is the risk-aversion parameter of the market maker. Instead, as in [36, 39, 40], but also in [62], we can consider a risk-adjusted expectation for the objective function, i.e.

$$\sup_{\delta \in \mathcal{A}} \mathbb{E} \left[V_T - \frac{\gamma}{2} \int_0^T \left(\sum_{i=1}^N \frac{\xi}{2} q_t^i \mathcal{V}_t^i \right)^2 dt \right].$$

The second objective function in our case writes

$$\sup_{\delta \in \mathcal{A}} \mathbb{E} \left[\int_0^T \sum_{i=1}^N \left(\left(\sum_{j=a,b} \int_{\mathbb{R}_+^*} z \delta_t^{i,j}(z) \Lambda^{i,j}(\delta_t^{i,j}(z)) \mathbb{1}_{\{q_t - \psi(j) z e^i \in \mathcal{Q}\}} \mu^{i,j}(dz) \right) + q_t^i \mathcal{V}_t^i \frac{a_{\mathbb{P}}(t, \nu_t) - a_{\mathbb{Q}}(t, \nu_t)}{2\sqrt{\nu_t}} \right) dt - \frac{\gamma \xi^2}{8} \int_0^T \left(\sum_{i=1}^N q_t^i \mathcal{V}_t^i \right)^2 dt \right], \quad (3.2)$$

where

$$\psi(j) := \begin{cases} +1 & \text{if } j = a \\ -1 & \text{if } j = b. \end{cases}$$

These two objective functions are close to one other in practice. Guéant showed in [62] that they give similar optimal quotes in practical examples. Furthermore, in many cases, the expected utility framework with exponential utility function can be reduced to the maximization of the expected PnL minus a quadratic penalty of the above form, up to a change in the intensity functions (see [85]).

In what follows, we consider the second framework. Therefore, we define the value function

$$u : (t, S, \nu, q) \in [0, T] \times \mathbb{R}_+^2 \times \mathcal{Q} \mapsto u(t, S, \nu, q)$$

associated with (3.2) as

$$\begin{aligned} u(t, S, \nu, q) = & \sup_{(\delta_s)_{s \in [t, T]} \in \mathcal{A}_t} \mathbb{E}_{(t, S, \nu, q)} \left[\int_t^T \sum_{i=1}^N \left(\left(\sum_{j=a,b} \int_{\mathbb{R}_+^*} z \delta_s^{i,j}(z) \Lambda^{i,j}(\delta_s^{i,j}(z)) \mathbb{1}_{\{q_t - \psi(j) z e^i \in \mathcal{Q}\}} \mu^{i,j}(dz) \right) \right. \right. \\ & \left. \left. + q_s^i \mathcal{V}_s^i \frac{a_{\mathbb{P}}(s, \nu_s) - a_{\mathbb{Q}}(s, \nu_s)}{2\sqrt{\nu_s}} \right) ds - \frac{\gamma \xi^2}{8} \int_t^T \left(\sum_{i=1}^N q_s^i \mathcal{V}_s^i \right)^2 ds \right], \quad (3.3) \end{aligned}$$

where \mathcal{A}_t is the set of admissible controls defined on $[t, T]$.

3.2.3 Assumptions and approximations

The above stochastic optimal control problem can be addressed from a theoretical point of view using an approach similar to that of [62]. However, when it comes to approximating the optimal quotes a market maker should set for the N options, classic numerical methods are of no help because the value function u has $N + 2$ variables (in addition to the time variable). In order to beat the curse of dimensionality and be able to approximate the solution of (3.3) we propose a method based on the following assumptions / approximations:

Assumption 1. *We approximate the vega of each option over $[0, T]$ by its value at time $t = 0$, namely*

$$\mathcal{V}_t^i = \mathcal{V}_0^i =: \mathcal{V}^i \in \mathbb{R}, \text{ for all } i \in \{1, \dots, N\}.$$

Assumption 2. *We assume that the set of authorized inventories is associated with vega risk limits, i.e.*

$$\mathcal{Q} = \left\{ q \in \mathbb{R}^N \mid \sum_{i=1}^N q^i \mathcal{V}^i \in [-\bar{\mathcal{V}}, \bar{\mathcal{V}}] \right\},$$

where $\bar{\mathcal{V}} \in \mathbb{R}_+^*$ is the vega risk limit of the market maker.

The first assumption is acceptable if T is not too large. This raises in fact the deep question of the reasonable value of T , as there is no natural choice for the horizon of the optimization problem. In practice, T has to be sufficient large to allow for several transactions in many options and small enough for the constant-vega approximation to be relevant (and smaller than the maturities of the options). It is also noteworthy, although it is time-inconsistent, that one can use the proceed of the model (with the constant-vega approximation) over a short period of time and then run the model again with updated vegas. This is a classical practice in applied optimal control when the parameters are estimated online.⁴

The second assumption states that risk limits are related to the only source of risk (as the portfolio is Δ -hedged). This is a natural assumption. The only drawback is that no risk limit can be set to individual options.

3.3 An approximate solution to the problem

3.3.1 Change of variables: beating the curse of dimensionality

Under the above assumptions, the $N + 2$ state variables can be replaced by only two: the instantaneous variance and the vega of the portfolio. This portfolio vega, defined by $\mathcal{V}_t^\pi := \sum_{i=1}^N q^i \mathcal{V}^i$, has the following dynamics

$$d\mathcal{V}_t^\pi = \sum_{i=1}^N \int_{\mathbb{R}_+^*} z \mathcal{V}^i (N^{i,b}(dt, dz) - N^{i,a}(dt, dz)).$$

It is clear indeed that the value function u verifies

$$\forall (t, S, \nu, q) \in [0, T] \times \mathbb{R}_+^2 \times \mathcal{Q}, u(t, S, \nu, q) = v \left(t, \nu, \sum_{i=1}^N q^i \mathcal{V}^i \right),$$

where

$$v(t, \nu, \mathcal{V}^\pi) = \sup_{(\delta_s)_{s \in [t, T]} \in \mathcal{A}_t} \mathbb{E}_{(t, \nu, \mathcal{V}^\pi)} \left[\int_t^T \left(\left(\sum_{i=1}^N \sum_{j=a, b} \int_{\mathbb{R}_+^*} z \delta_s^{i,j}(z) \Lambda^{i,j}(\delta_s^{i,j}(z)) \mathbf{1}_{\{|\mathcal{V}_s^\pi - \psi(j)z\mathcal{V}^i| \leq \bar{\mathcal{V}}\}} \mu^{i,j}(dz) \right) + \mathcal{V}_s^\pi \frac{a_{\mathbb{P}}(s, \nu_s) - a_{\mathbb{Q}}(s, \nu_s)}{2\sqrt{\nu_s}} - \frac{\gamma \xi^2}{8} \mathcal{V}_s^{\pi 2} \right) ds \right]. \quad (3.4)$$

In other words, the problem boils down, under the two above assumptions, to a low-dimensional optimal control problem where the two state variables are driven by $2N$ controlled point processes

⁴In Appendix 3.5.2 we propose a method to relax the constant-vega assumption. This method is based on a Taylor expansion around the constant-vega case. The curse of dimensionality is tamed by the reduction of the problem to a Monte-Carlo simulation.

and a standard Brownian motion.

3.3.2 Hamilton-Jacobi-Bellman equation and optimal controls

Following [91], the Hamilton-Jacobi-Bellman equation associated with (3.4) is given by

$$0 = \partial_t v(t, \nu, \mathcal{V}^\pi) + a_{\mathbb{P}}(t, \nu) \partial_\nu v(t, \nu, \mathcal{V}^\pi) + \frac{1}{2} \nu \xi^2 \partial_{\nu\nu}^2 v(t, \nu, \mathcal{V}^\pi) + \nu^\pi \frac{a_{\mathbb{P}}(t, \nu) - a_{\mathbb{Q}}(t, \nu)}{2\sqrt{\nu}} - \frac{\gamma \xi^2}{8} \nu^{\pi 2} \\ + \sum_{i=1}^N \sum_{j=a,b} \int_{\mathbb{R}_+^*} z \mathbb{1}_{\{|\mathcal{V}^\pi - \psi(j)z\mathcal{V}^i| \leq \bar{\nu}\}} H^{i,j} \left(\frac{v(t, \nu, \mathcal{V}^\pi) - v(t, \nu, \mathcal{V}^\pi - \psi(j)z\mathcal{V}^i)}{z} \right) \mu^{i,j}(dz), \quad (3.5)$$

with final condition $v(T, \nu, \mathcal{V}^\pi) = 0$, where

$$H^{i,j}(p) := \sup_{\delta^{i,j} \geq \delta_\infty} \Lambda^{i,j}(\delta^{i,j})(\delta^{i,j} - p), \quad i \in \{1, \dots, N\}, \quad j = a, b.$$

We end up therefore with a low-dimensional functional equation of the Hamilton-Jacobi-Bellman type.

Once the value function is known, the optimal controls, which are the optimal mid-to-bid and ask-to-mid associated with the N options, are given by the following formula (see [26, 62]):

$$\delta_t^{i,j*}(z) = \max \left(\delta_\infty, (\Lambda^{i,j})^{-1} \left(-H^{i,j'} \left(\frac{v(t, \nu_t, \mathcal{V}_{t-}^\pi) - v(t, \nu_t, \mathcal{V}_{t-}^\pi - \psi(j)z\mathcal{V}^i)}{z} \right) \right) \right), \quad i \in \{1, \dots, N\}, \quad j = a, b.$$

Remark 13. *In the case where $a_{\mathbb{P}} = a_{\mathbb{Q}}$, it is evident that v does not depend on ν . In that case $v(t, \nu, \mathcal{V}^\pi) = w(t, \mathcal{V}^\pi)$ where w is solution of the simpler Hamilton-Jacobi-Bellman*

$$0 = \partial_t w(t, \mathcal{V}^\pi) - \frac{\gamma \xi^2}{8} \nu^{\pi 2} + \sum_{i=1}^N \sum_{j=a,b} \int_{\mathbb{R}_+^*} z \mathbb{1}_{\{|\mathcal{V}^\pi - \psi(j)z\mathcal{V}^i| \leq \bar{\nu}\}} H^{i,j} \left(\frac{w(t, \mathcal{V}^\pi) - w(t, \mathcal{V}^\pi - \psi(j)z\mathcal{V}^i)}{z} \right) \mu^{i,j}(dz),$$

with final condition $w(T, \mathcal{V}^\pi) = 0$.

3.4 Numerical results

3.4.1 Model parameters

In this section we consider a book of options and derive the optimal quotes using the above approach.

For this purpose, we consider an underlying stock with the following characteristics:

- Stock price at time $t = 0$: $S_0 = 10$ €.
- Instantaneous variance at time $t = 0$: $\nu_0 = 0.0225$ year⁻¹.

- Heston model with $a_{\mathbb{P}}(t, \nu) = \kappa_{\mathbb{P}}(\theta_{\mathbb{P}} - \nu)$ where $\kappa_{\mathbb{P}} = 2 \text{ year}^{-1}$ and $\theta_{\mathbb{P}} = 0.04 \text{ year}^{-1}$, and $a_{\mathbb{Q}}(t, \nu) = \kappa_{\mathbb{Q}}(\theta_{\mathbb{Q}} - \nu)$ where $\kappa_{\mathbb{Q}} = 3 \text{ year}^{-1}$ and $\theta_{\mathbb{Q}} = 0.0225 \text{ year}^{-1}$.
- Volatility of volatility parameter: $\xi = 0.2 \text{ year}^{-1}$.
- Spot-variance correlation: $\rho = -0.5$.

We consider the case of a market maker dealing with 20 European call options written on that stock where the strike×maturity couples are the elements $(K^i, T^i)_{i=1, \dots, 20}$ of the set $\mathcal{K} \times \mathcal{T}$, where

$$\mathcal{K} = \{8 \text{ €}, 9 \text{ €}, 10 \text{ €}, 11 \text{ €}, 12 \text{ €}\} \quad \text{and} \quad \mathcal{T} = \{1 \text{ year}, 1.5 \text{ years}, 2 \text{ years}, 3 \text{ years}\}.$$

The associated implied volatility surface is plotted in Figure 3.1.⁵

The liquidity parameters of these options are the following:

- Intensity functions:

$$\Lambda^{i,j}(\delta) = \frac{\lambda^i}{1 + e^{\alpha + \frac{\beta}{\nu^i} \delta}}, \quad i \in \{1, \dots, N\}, \quad j = a, b.$$

where $\lambda^i = \frac{252 \times 30}{1 + 0.7 \times |S_0 - K^i|} \text{ year}^{-1}$, $\alpha = 0.7$, and $\beta = 150 \text{ year}^{\frac{1}{2}}$.

The choice of λ^i corresponds to 30 requests per day for at-the-money options, and decreases to 12.5 for more in- and out-the money options. The choice of α corresponds to a probability of $\frac{1}{1 + e^{0.7}} \approx 33\%$ to trade when the answered quote is the mid-price. The choice of β corresponds to a probability of $\frac{1}{1 + e^{-0.8}} \approx 69\%$ to trade when the answered quote corresponds to an implied volatility 1% better for the client and a probability of $\frac{1}{1 + e^{2.2}} \approx 10\%$ to trade when the answered quote corresponds to an implied volatility 1% worse for the client.

- Size of transactions: we assume constant sizes $z^i = \frac{5 \cdot 10^5}{\mathcal{O}_0^i}$ contracts for option i . This corresponds approximately to 500000 € per transaction.⁶ The measures $\mu^{i,b}$ and $\mu^{i,a}$ are here Dirac masses.

Regarding the risk limits and the objective function, we consider the following:

- Vega risk limit: $\bar{\nu} = 10^7 \text{ €} \cdot \text{year}^{\frac{1}{2}}$.
- Time horizon given by $T = 0.0012 \text{ year}$ (i.e. 0.3 days). This short time horizon surprisingly ensures convergence towards stationary quotes at time $t = 0$ (see Figure 3.3 below).
- Risk aversion given by $\gamma = 1 \cdot 10^{-3} \text{ €}^{-1}$.

⁵This plot has been computed using 10^5 Monte-Carlo simulations for each option.

⁶This is only an approximation as trade sizes are in number of options and option prices move.

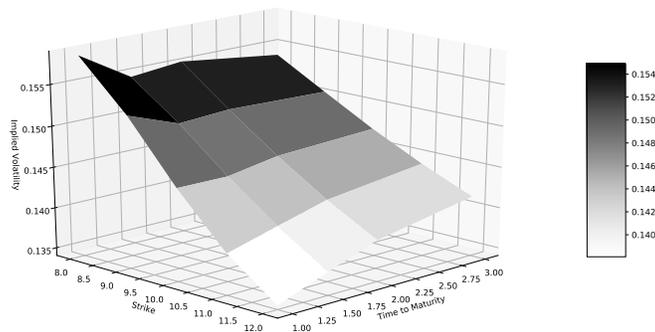


Figure 3.1: Implied volatility surface associated with the above parameters

3.4.2 Optimal quotes

Using a monotone explicit Euler scheme with linear interpolation on a grid of size $180 \times 30 \times 40$, we approximate the value function solution to (3.5) (with Neumann conditions at the boundaries in ν) on the domain $[0, T] \times [0.0144, 0.0324] \times [-\bar{\nu}, \bar{\nu}]$. This value function is plotted in Figure 3.2.

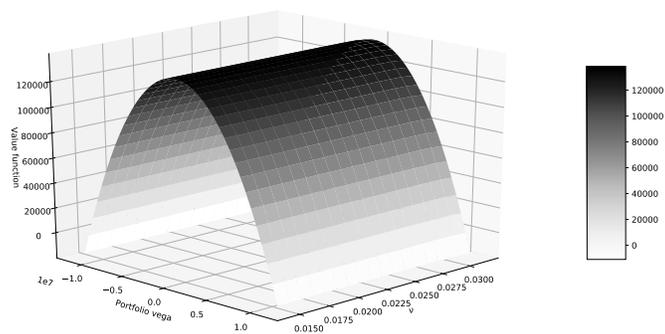


Figure 3.2: Value function as a function of instantaneous variance and portfolio vega.

From that value function, we deduce the optimal bid and ask quotes of the market maker for each option as a function of the portfolio vega. As mentioned above, we chose $T = 0.0008$ year (i.e. 0.2 days) – a choice that ensures convergence of the optimal quotes to their stationary values (see Figure 3.3).

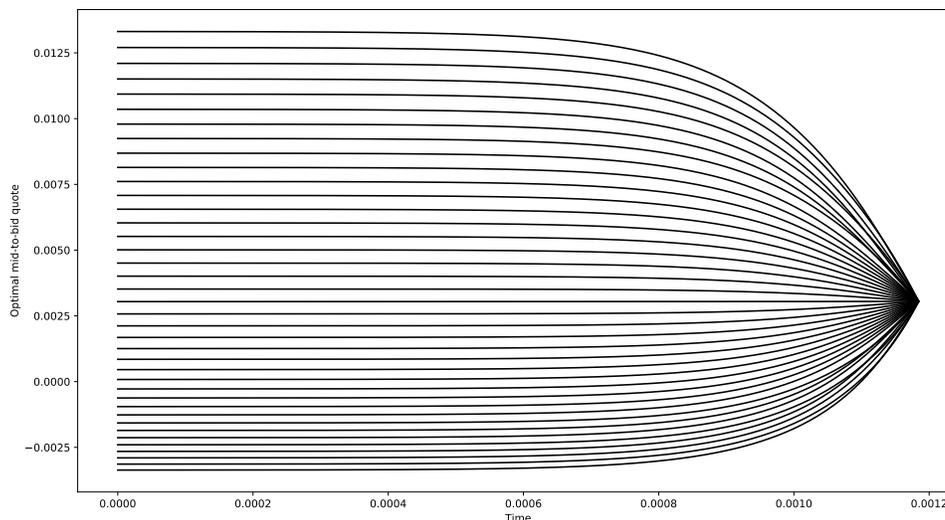


Figure 3.3: Optimal mid-to-bid quotes as a function of time for option 1: $(K^1, T^1) = (8, 1) - nu = 0.04$.

Focusing on the asymptotic values, we now present in Figures 3.4, 3.5, 3.6, 3.7, and 3.8, the optimal bid quotes as a function of the portfolio vega for each strike and maturity. More precisely, as the options we consider can have very different prices, we consider instead of the optimal bid quotes themselves the ratio between each optimal mid-to-bid quote and the price (at time $t = 0$) of the corresponding option. In Figures 3.9, 3.10, 3.11, 3.12, and 3.13, we plot the same optimal bid quotes for the 20 options in terms of implied volatility (divided by the implied volatility at time $t = 0$).

The results are in line with what was expected: the mid-to-bid quotes increase with the portfolio vega. The incentive to buy options with positive vega decreases indeed with the vega of the portfolio.

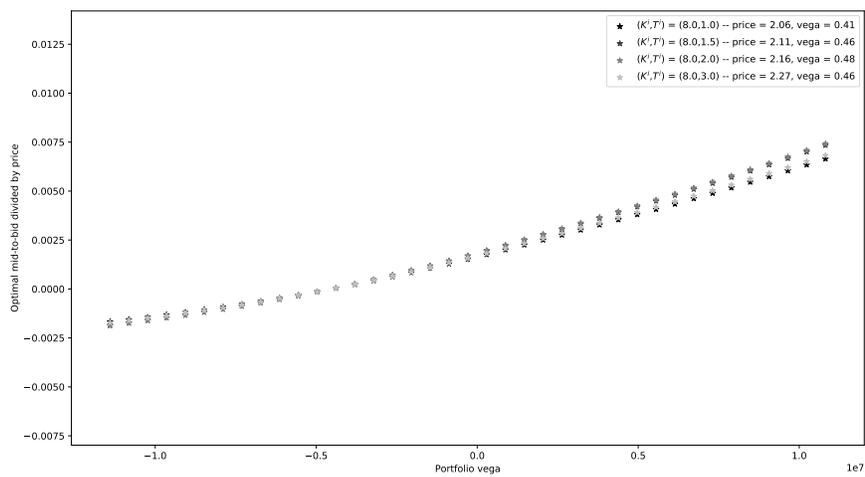


Figure 3.4: Optimal mid-to-bid quotes divided by option price as a function of the portfolio vega for $K=8$.

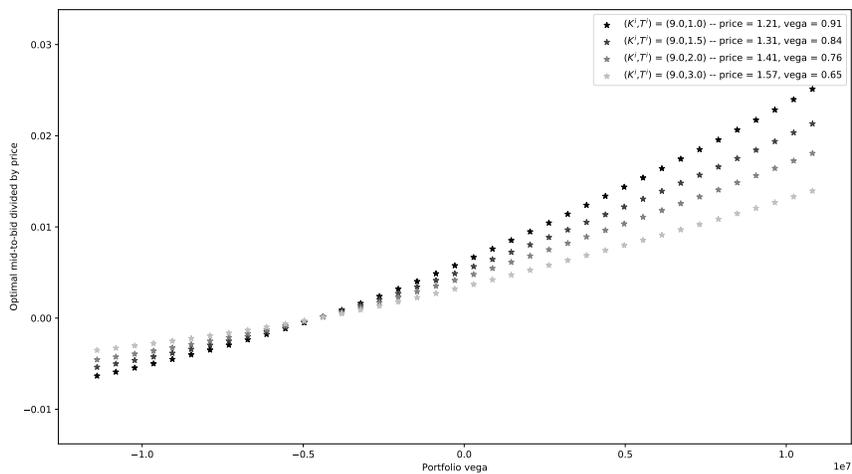


Figure 3.5: Optimal mid-to-bid quotes divided by option price as a function of the portfolio vega for $K=9$.

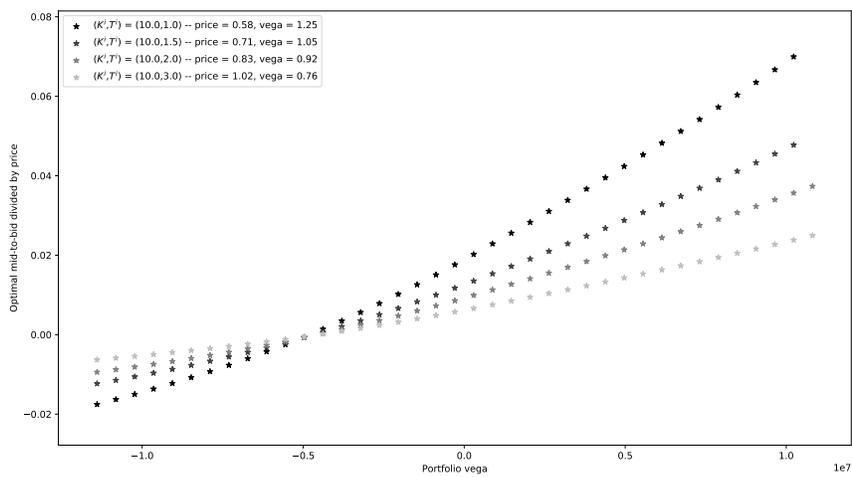


Figure 3.6: Optimal mid-to-bid quotes divided by option price as a function of the portfolio vega for $K=10$.

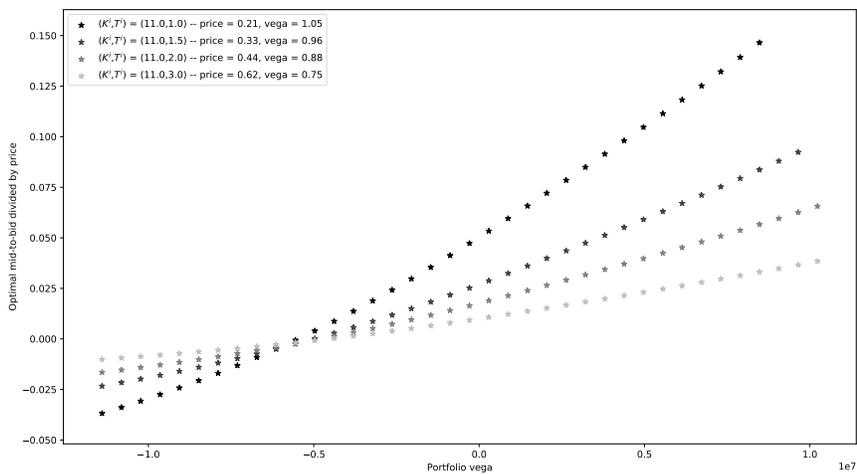


Figure 3.7: Optimal mid-to-bid quotes divided by option price as a function of the portfolio vega for $K=11$.

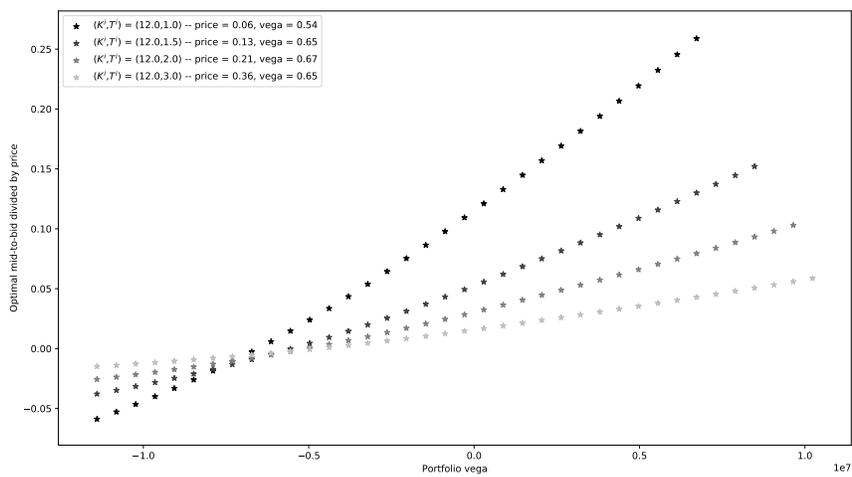


Figure 3.8: Optimal mid-to-bid quotes divided by option price as a function of the portfolio vega for K=12.

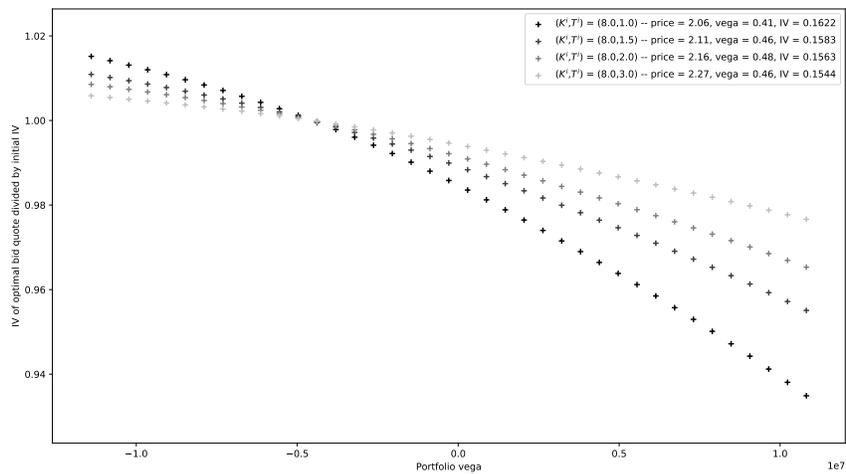


Figure 3.9: Optimal (relative) bid implied volatility as a function of the portfolio vega for K=8.

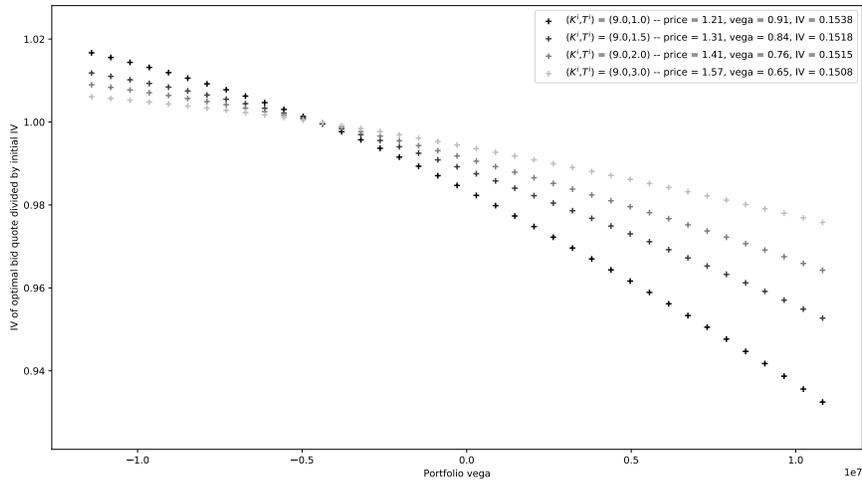


Figure 3.10: Optimal (relative) bid implied volatility as a function of the portfolio vega for K=9.

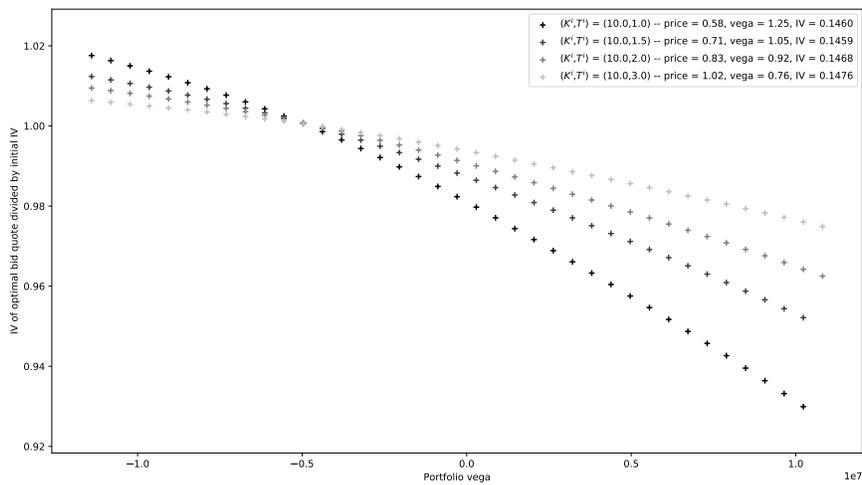


Figure 3.11: Optimal (relative) bid implied volatility as a function of the portfolio vega for K=10.

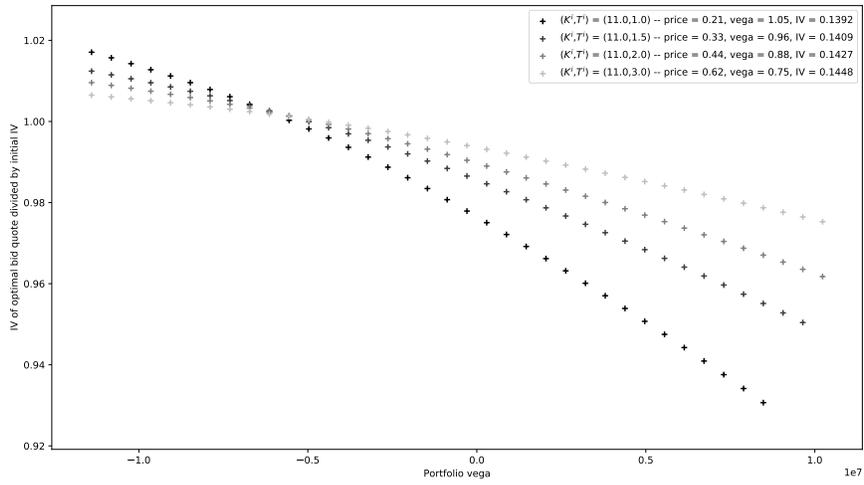


Figure 3.12: Optimal (relative) bid implied volatility as a function of the portfolio vega for $K=11$.

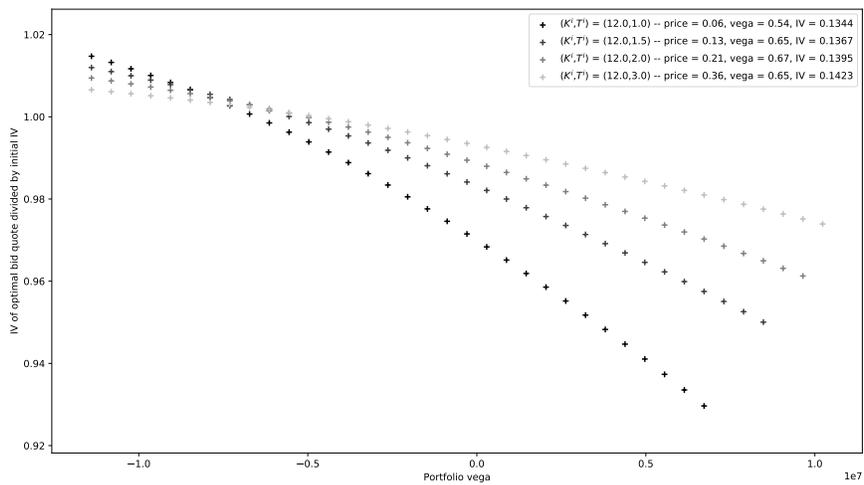


Figure 3.13: Optimal (relative) bid implied volatility as a function of the portfolio vega for $K=12$.

3.4.3 Conclusion

In this article, we tackled the problem of an option market maker dealing with options on a single underlying asset.⁷ Using a constant-vega approximation, we showed how to reduce the problem to a low-dimensional functional equation whose solution can easily be approximated using an explicit Euler scheme and linear interpolation. Furthermore, our method scales linearly in the number of options and can therefore be used with large books of options. Our method is illustrated by an example involving 20 European calls, but our model can be used with any European options.

3.5 Appendix

3.5.1 An alternative to the Δ -hedging assumption

Throughout the body of this paper, we assumed that the market maker ensured Δ -hedging. In this appendix, we show that this assumption can be relaxed without much change in the reasoning.

Let us introduce the process $(q_t^S)_{t \in [0, T]}$ representing the inventory of the market maker in the underlying asset. The dynamics of the cash process of the market maker $(X_t)_{t \in [0, T]}$ rewrites as

$$dX_t = \sum_{i=1}^N \left(\int_{\mathbb{R}_+^*} z \left(\delta_t^{i,b}(z) N_t^{i,b}(dt, dz) + \delta_t^{i,a}(z) N_t^{i,a}(dt, dz) \right) - \mathcal{O}_t^i dq_t^i \right) - q_t^S dS_t - d\langle q^S, S \rangle_t.$$

The Mark-to-Market value of the portfolio writes

$$V_t = X_t + q_t^S S_t + \sum_{i=1}^N q_t^i \mathcal{O}_t^i$$

and its dynamics is

$$\begin{aligned} dV_t &= \sum_{i=1}^N \left(\int_{\mathbb{R}_+^*} z \left(\delta_t^{i,b}(z) N_t^{i,b}(dt, dz) + \delta_t^{i,a}(z) N_t^{i,a}(dt, dz) \right) + q_t^i \mathcal{V}_t^i \frac{a_{\mathbb{P}}(t, \nu_t) - a_{\mathbb{Q}}(t, \nu_t)}{2\sqrt{\nu_t}} dt + \frac{\xi}{2} q_t^i \mathcal{V}_t^i dW_t^\nu \right) \\ &\quad + \sqrt{\nu_t} S_t \left(\sum_{i=1}^N q_t^i \partial_S \mathcal{O}^i(t, S_t, \nu_t) + q_t^S \right) dW_t^S. \end{aligned}$$

Denoting by $\Delta_t^\pi := \sum_{i=1}^N q_t^i \partial_S \mathcal{O}^i(t, S_t, \nu_t)$ the Δ of the market maker's portfolio at time t , our mean-variance optimization problem becomes

$$\sup_{(\delta, q^S) \in \mathcal{A}'} \mathbb{E}[V_T] - \frac{\gamma}{2} \mathbb{V} \left[\int_0^T \frac{\xi}{2} \mathcal{V}_t^\pi dW_t^\nu + \sqrt{\nu_t} S_t (\Delta_t^\pi + q_t^S) dW_t^S \right],$$

⁷As noted while publishing this paper, our method can easily be extended to the case of multiple underlying assets using the same method as in [26] if the instantaneous variance processes of the different assets are driven by a few factors.

where

$$\mathcal{A}' = \left\{ (\delta_t, q_t^S)_{t \in [0, T]} \mid \delta \text{ is an } \mathbb{R}^{2N}\text{-valued predictable process bounded from below by } \delta_\infty \right. \\ \left. \text{and } q^S \text{ is an } \mathbb{R}\text{-valued adapted process with } \mathbb{E} \left[\int_0^T \nu_t S_t^2 (\Delta_t^\pi + q_t^S)^2 dt \right] < +\infty \right\}.$$

Noticing that

$$\mathbb{V} \left(\int_0^T \frac{\xi}{2} \mathcal{V}_t^\pi dW_t^\nu + \sqrt{\nu_t} S_t (\Delta_t^\pi + q_t^S) dW_t^S \right) = \mathbb{E} \left[\int_0^T \left(\frac{\xi^2}{4} \mathcal{V}_t^{\pi^2} + \nu_t S_t^2 (\Delta_t^\pi + q_t^S)^2 + \rho \xi \mathcal{V}_t^\pi \sqrt{\nu_t} S_t (\Delta_t^\pi + q_t^S) \right) dt \right],$$

we easily see that the variance term is minimized for $q^S = q^{S^*}$ where

$$\forall t \in [0, T], \quad q_t^{S^*} = -\Delta_t^\pi - \frac{\rho \xi \mathcal{V}_t^\pi}{2\sqrt{\nu_t} S_t},$$

and that its minimum value is

$$(1 - \rho^2) \int_0^T \frac{\xi^2}{4} \mathcal{V}_t^{\pi^2} dt.$$

Therefore, the optimization problem boils down to

$$\sup_{\delta \in \mathcal{A}} \mathbb{E} \left[\int_0^T \left(\left(\sum_{i=1}^N \sum_{j=a, b} \int_{\mathbb{R}_+^*} z \delta_t^{i, j} (z) \Lambda^{i, j} (\delta_t^{i, j} (z)) \mathbb{1}_{|\mathcal{V}_t^\pi - \psi(j)z \mathcal{V}^i| \leq \bar{\nu}} \mu^{i, j} (dz) \right) \right. \right. \\ \left. \left. + \mathcal{V}_t^\pi \frac{a_{\mathbb{P}}(t, \nu_t) - a_{\mathbb{Q}}(t, \nu_t)}{2\sqrt{\nu_t}} - \frac{\gamma \xi^2}{8} (1 - \rho^2) \mathcal{V}_t^{\pi^2} \right) dt \right],$$

and we recover the same optimization problem as in the body of the paper, except that the risk aversion parameter is multiplied by $1 - \rho^2$ to account for the reduction of risk made possible by the optimal trading strategy in the underlying asset in presence of vol-spot correlation.

3.5.2 Beyond the constant-vega assumption

In this appendix we propose a method to relax our main assumption: the constant-vega approximation.

If, for all $i \in \{1, \dots, N\}$, the process $(\mathcal{V}_t^i)_{t \in [0, T]}$ stays close to its initial value \mathcal{V}_0^i , then it is reasonable to consider a perturbative approach around the constant-vega approximation. In particular, instead of assuming that $\sum_{i=1}^N q_t^i \partial_{\sqrt{\nu}} O^i(t, S_t, \nu_t) \simeq \sum_{i=1}^N q_t^i \partial_{\sqrt{\nu}} O^i(0, S_0, \nu_0) = \mathcal{V}_t^\pi$, we consider the expansion

$$\sum_{i=1}^N q_t^i \partial_{\sqrt{\nu}} O^i(t, S_t, \nu_t) = \mathcal{V}_t^\pi + \varepsilon \mathcal{W}(t, S_t, \nu_t, q_t)$$

and we consider an expansion of the value function u of the following form:

$$u(t, S, \nu, q) = v \left(t, \nu, \sum_{i=1}^N q^i \partial_{\sqrt{\nu}} O^i(0, S_0, \nu_0) \right) + \varepsilon \varphi(t, S, \nu, q).$$

Assuming that $Q = \mathbb{R}^N$ and noting that the Hamilton-Jacobi-Bellman equation associated with u is

$$\begin{aligned} 0 = & \partial_t u(t, S, \nu, q) + a_{\mathbb{P}}(t, \nu) \partial_\nu u(t, S, \nu, q) + \frac{1}{2} \nu S^2 \partial_{SS}^2 u(t, S, \nu, q) + \frac{1}{2} \nu \xi^2 \partial_{\nu\nu}^2 u(t, S, \nu, q) + \rho \nu S \xi \partial_{\nu S}^2 u(t, S, \nu, q) \\ & + \frac{a_{\mathbb{P}}(t, \nu) - a_{\mathbb{Q}}(t, \nu)}{2\sqrt{\nu}} \sum_{i=1}^N q^i \partial_{\sqrt{\nu}} O^i(t, S, \nu) - \frac{\gamma \xi^2}{8} \left(\sum_{i=1}^N q^i \partial_{\sqrt{\nu}} O^i(t, S, \nu) \right)^2 \\ & + \sum_{i=1}^N \sum_{j=a,b} \int_{\mathbb{R}_+^*} z H^{i,j} \left(\frac{u(t, S, \nu, q) - u(t, S, \nu, q - \psi(j) z e^i)}{z} \right) \mu^{i,j}(dz), \end{aligned}$$

with terminal condition equal to 0, the first-order term in ε in the Taylor expansion gives

$$\begin{aligned} 0 = & \partial_t \varphi(t, S, \nu, q) + a_{\mathbb{P}}(t, \nu) \partial_\nu \varphi(t, S, \nu, q) + \frac{1}{2} \nu S^2 \partial_{SS}^2 \varphi(t, S, \nu, q) + \frac{1}{2} \nu \xi^2 \partial_{\nu\nu}^2 \varphi(t, S, \nu, q) + \rho \nu S \xi \partial_{\nu S}^2 \varphi(t, s, \nu, q) \\ & + \frac{a_{\mathbb{P}}(t, \nu) - a_{\mathbb{Q}}(t, \nu)}{2\sqrt{\nu}} \mathcal{W}(t, S, \nu, q) - \frac{\gamma \xi^2}{4} \mathcal{W}(t, S, \nu, q) \sum_{i=1}^N q^i \mathcal{V}_0^i \\ & + \sum_{i=1}^N \sum_{j=a,b} \int_{\mathbb{R}_+^*} H^{i,j'} \left(\frac{v(t, \nu, \sum_{l=1}^N q^l \mathcal{V}_0^l) - v(t, \nu, \sum_{l=1}^N q^l \mathcal{V}_0^l - \psi(j) z \mathcal{V}_0^i)}{z} \right) \\ & \quad \times (\varphi(t, S, \nu, q) - \varphi(t, S, \nu, q - \psi(j) z e^i)) \mu^{i,j}(dz), \end{aligned}$$

with terminal condition equal to 0.

Although it involves $q \in \mathbb{R}^N$, this equation is linear and thus $\varphi(t, S, \nu, q)$ admits a Feynman-Kac representation that tames the curse of dimensionality for practical applications:

$$\varphi(t, S, \nu, q) = \mathbb{E}_{(t, S, \nu, q)} \left[\int_t^T \left(\frac{a_{\mathbb{P}}(s, \nu_s) - a_{\mathbb{Q}}(s, \nu_s)}{2\sqrt{\nu_s}} \mathcal{W}(s, S_s, \nu_s, q_s) - \frac{\gamma \xi^2}{4} \mathcal{W}(s, S_s, \nu_s, q_s) \sum_{i=1}^N q_s^i \mathcal{V}_0^i \right) ds \right]$$

where, for each $i \in \{1, \dots, N\}$, the processes $N^{i,b}$ and $N^{i,a}$ have respective intensities

$$\begin{aligned} \tilde{\lambda}_t^{i,b}(dz) &= -H^{i,b'} \left(\frac{v(t, \nu_t, \sum_{l=1}^N q_t^l \mathcal{V}_0^l) - v(t, \nu_t, \sum_{l=1}^N q_t^l \mathcal{V}_0^l - \psi(j) z \mathcal{V}_0^i)}{z} \right) \mu^{i,b}(dz), \\ \tilde{\lambda}_t^{i,a}(dz) &= -H^{i,a'} \left(\frac{v(t, \nu_t, \sum_{l=1}^N q_t^l \mathcal{V}_0^l) - v(t, \nu_t, \sum_{l=1}^N q_t^l \mathcal{V}_0^l - \psi(j) z \mathcal{V}_0^i)}{z} \right) \mu^{i,a}(dz), \end{aligned}$$

with as before $dq_t^i = \int_{\mathbb{R}_+^*} z (N^{i,b}(dt, dz) - N^{i,a}(dt, dz))$.

Subsequently, the function φ can be computed using a Monte-Carlo algorithm and quotes accounting for the variation of the vegas can therefore be computed (to the first order in ε).

Chapter 4

Closed-form approximations in multi-asset market making

Abstract

A large proportion of market making models derive from the seminal model of Avellaneda and Stoikov. The numerical approximation of the value function and the optimal quotes in these models remains a challenge when the number of assets is large. In this article, we propose closed-form approximations for the value functions of many multi-asset extensions of the Avellaneda-Stoikov model. These approximations or proxies can be used (i) as heuristic evaluation functions, (ii) as initial value functions in reinforcement learning algorithms, and/or (iii) directly to design quoting strategies through a greedy approach. Regarding the latter, our results lead to new and easily interpretable closed-form approximations for the optimal quotes, both in the finite-horizon case and in the asymptotic (ergodic) regime. Furthermore, we propose a perturbative approach to improve our closed-form approximations through Monte-Carlo simulations.

4.1 Introduction

Since the publication of the paper [13] by Avellaneda and Stoikov, who revisited the paper [71] by Ho and Stoll (see also [72]), there has been an extensive literature on optimal market making.¹ Guéant, Lehalle, and Fernandez-Tapia provided in [65] a rigorous analysis of the stochastic optimal control problem introduced by Avellaneda and Stoikov and proved that, under inventory constraints, the problem reduces to a system of linear ordinary differential equations in the case of exponential intensity functions suggested by Avellaneda and Stoikov. They also studied the asymptotics when the time horizon T tends to $+\infty$, proposed closed-form approximations, and introduced extensions to include a drift in the price dynamics and market impact / adverse selection. Cartea and Jaimungal, along with their various coauthors, contributed substantially to the literature and added many features to the initial models: alpha signals, ambiguity aversion, etc. (see [36, 40, 41] –

¹There is an economic literature on market making, for instance the seminal paper [59] by Grossman and Miller. The results in this literature are, however, more interesting for understanding the price formation process than for building market making algorithms.

see also their book [39]). They also considered a different objective function: the expected PnL minus a running penalty to avoid holding a large inventory instead of the Von Neumann-Morgenstern expected CARA (constant absolute risk aversion) utility of [13] and [65]. Many features have also been added by various authors: general dynamics for the price in [78], persistence of the order flow in [75], several requested sizes in [26], client tiering and access to a liquidity pool in [21], etc.

In spite of the focus of initial papers on stock markets,² the models derived from that of Avellaneda and Stoikov have been more useful to build market making algorithms in quote-driven markets: corporate bond markets based on requests for quotes, FX markets based on requests for quotes and requests for stream, etc. For stock markets or, more generally, order-driven markets with relatively low bid-ask spread to tick size ratio, many models have been proposed that depart from the original framework of Avellaneda and Stoikov in that the limit order book is modeled. Instances of papers proposing this type of models include those of Guilbaud and Pham [68, 69], that of Kühn and Muhle-Karbe [77], that of Fodra and Pham [49] or the more recent papers by Lu and Abergel [84] and Baradel, Bouchard, Evangelista, and Mounjid [18].

Most of the literature on optimal market making deals with single-asset models. However, because market making algorithms are typically built for entire portfolios, single-asset models are not sufficient to build operable algorithms, except under the unrealistic assumption that asset prices are uncorrelated. Multi-asset extensions of the Avellaneda-Stoikov model have been proposed. A paper by Guéant and Lehalle [63] touches upon this extension and a complete analysis for the various objective functions present in the literature can be found in [62] (see also the book [61]) or in [26] in which multiple trade sizes are also considered.

Although their mathematical characterization has been known for years, computing the value function and the optimal quotes is complicated in the multi-asset case whenever the prices of the assets are correlated. The grid methods that are classically used to tackle the single-asset case suffer indeed from the curse of dimensionality and do not scale up to many practical multi-asset cases. Bergault and Guéant proposed in [26] a factor method to reduce the dimensionality of the problem. Guéant and Manziuk proposed in [66] a numerical method based on reinforcement learning techniques (an actor-critic approach in fact³). In spite of these recent advances, the computational cost of most numerical schemes will still be prohibitive for practical use for some asset classes.

Instead of computing a numerical approximation of the value function (from which one traditionally deduces a numerical approximation of the optimal quotes), we propose in this paper a method for building a closed-form proxy for the value function. The idea behind the approach is that the value function associated with many market making problems is the solution of a Hamilton-Jacobi equation that can be seen as a perturbation of another Hamilton-Jacobi equation for which the solution can be computed in closed form. Of course, such closed-form formula does not define a solution to the initial Hamilton-Jacobi equation, but it has similar properties and should capture most of the relevant financial effects.

²There was also from the very beginning a focus on options markets – see for instance [98] (cf. [15] and [46] for more recent papers).

³Recent advances in the control of point processes (see [27]) might decrease significantly the computation time associated with reinforcement learning techniques for solving market making problems.

Having a proxy of a value function is known to be useful in the community of reinforcement learning (see [100] and [101] for a reference to the reinforcement learning terminology). An important use of a closed-form proxy of a value function is as a heuristic evaluation function. Heuristic evaluation functions are mainly used in game-playing computer programs as a proxy of the probability to win the game given the current state – usually the current board in board games – but they can be used as terminal values in many Monte-Carlo-based reinforcement learning techniques. Also, such a proxy can be used as a starting point for many iterative algorithms based on value functions: value iteration algorithm, actor-critic approaches, etc. The last application we highlight – which was also our initial motivation – is that one can build from a proxy of a value function a quoting strategy called the greedy strategy associated with that proxy. Having such a strategy in closed form has numerous advantages. First, it can be used directly by market practitioners as a quoting strategy. Second, it can be used as a starting point in iterative algorithms based on policy functions: policy iteration algorithm, actor-critic approaches, etc. Third, it has the advantage of being easily interpretable and gives insights on the true optimal strategy such as the identification of the leading factors and the sensitivity to changes in model parameters.

In addition to providing a closed-form approximation or proxy for the value function, we can move one step further in our perturbative approach and consider the first-order term of the Taylor expansion (in the perturbation parameter) of the true value function. By using this approach, we obtain another proxy of the value function. This additional approximation is not given in closed form, but it can be computed at any given point through a Monte-Carlo simulation (see [26] for a close idea in a different context).

The methods we propose are first applied to the multi-asset market making models of [62]. Then we generalize the framework in several directions to cover many important practical cases: (i) drift in prices, (ii) client tiering, (iii) several request sizes for each asset and each tier, and (iv) fixed transaction costs for each asset and each tier. The drift in prices models the views of the market maker. Client tiering is a common practice in OTC markets, justified by the large spectrum of needs and behaviors in the set of clients to be served. The introduction of several request sizes for each asset and each tier reflects the reality that request sizes are not in control of the market makers, but rather of their clients. The fixed transaction costs can model extra costs associated with the market making business, for instance related to trading platforms.

We end this introduction by outlining our paper. In Section 4.2 we recall the multi-asset extensions of the Avellaneda-Stoikov model proposed in [62], present the system of ordinary differential equations (the Hamilton-Jacobi equation) characterizing the value function, and state the main results regarding the optimal quotes. In Section 4.3, we present our perturbative approach and compute in closed form the zeroth-order term of the expansion, which is our proxy of the value function. We deduce from that proxy an approximation of the optimal quotes in closed form. In Section 4.4, we move one step further and show that the first-order term of the Taylor expansion (in the perturbation parameter) can easily be computed thanks to a probabilistic representation that enables the use of Monte-Carlo simulations. In Section 4.5, we extend our results to a more general multi-asset market making model with drift in prices, client tiering, several requested sizes for each asset and each tier, and fixed transaction costs for each asset and each tier.

4.2 The multi-asset market making model

4.2.1 Model setup

We fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with a filtration $(\mathcal{F}_t)_{t \in \mathbb{R}_+}$ satisfying the usual conditions. In what follows, we assume that all stochastic processes are defined on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathbb{R}_+}, \mathbb{P})$.

For $i \in \{1, \dots, d\}$, the reference price of asset i is modeled by a process $(S_t^i)_{t \in \mathbb{R}_+}$ with dynamics

$$dS_t^i = \sigma^i dW_t^i, \quad S_0^i \text{ given,}$$

where $(W_t^1, \dots, W_t^d)_{t \in \mathbb{R}_+}$ is a d -dimensional Brownian motion with correlation matrix $(\rho^{i,j})_{1 \leq i, j \leq d}$ adapted to the filtration $(\mathcal{F}_t)_{t \in \mathbb{R}_+}$ – hereafter we denote by $\Sigma = (\rho^{i,j} \sigma^i \sigma^j)_{1 \leq i, j \leq d}$ the variance-covariance matrix associated with the process $(S_t)_{t \in \mathbb{R}_+} = (S_t^1, \dots, S_t^d)_{t \in \mathbb{R}_+}$.

The market maker chooses at each point in time the price at which she is ready to buy/sell each asset: for $i \in \{1, \dots, d\}$, we let her bid and ask quotes for asset i be modeled by two stochastic processes, respectively denoted by $(S_t^{i,b})_{t \in \mathbb{R}_+}$ and $(S_t^{i,a})_{t \in \mathbb{R}_+}$.

For $i \in \{1, \dots, d\}$, we denote by $(N_t^{i,b})_{t \in \mathbb{R}_+}$ and $(N_t^{i,a})_{t \in \mathbb{R}_+}$ the two point processes modeling the number of transactions at the bid and at the ask, respectively, for asset i . We assume in this section that the transaction size for asset i is constant and denoted by z^i . The inventory process of the market maker for asset i , denoted by $(q_t^i)_{t \in \mathbb{R}_+}$, has therefore the dynamics

$$dq_t^i = z^i dN_t^{i,b} - z^i dN_t^{i,a}, \quad q_0^i \text{ given,}$$

and we denote by $(q_t)_{t \in \mathbb{R}_+}$ the (column) vector process $(q_t^1, \dots, q_t^d)_{t \in \mathbb{R}_+}^\top$.

For each $i \in \{1, \dots, d\}$, we denote by $(\lambda_t^{i,b})_{t \in \mathbb{R}_+}$ and $(\lambda_t^{i,a})_{t \in \mathbb{R}_+}$ the intensity processes of $(N_t^{i,b})_{t \in \mathbb{R}_+}$ and $(N_t^{i,a})_{t \in \mathbb{R}_+}$, respectively. We assume that the market maker stops proposing a bid (respectively ask) price for asset i when her position in asset i following the transaction would exceed a given threshold Q^i (respectively $-Q^i$).⁴

Formally, we assume that the intensities verify

$$\lambda_t^{i,b} = \Lambda^{i,b}(\delta_t^{i,b}) \mathbb{1}_{\{q_{t-}^i + z^i \leq Q^i\}} \quad \text{and} \quad \lambda_t^{i,a} = \Lambda^{i,a}(\delta_t^{i,a}) \mathbb{1}_{\{q_{t-}^i - z^i \geq -Q^i\}},$$

where the processes $(\delta_t^{i,b})_{t \in \mathbb{R}_+}$ and $(\delta_t^{i,a})_{t \in \mathbb{R}_+}$ are defined by

$$\delta_t^{i,b} = S_t^i - S_t^{i,b} \quad \text{and} \quad \delta_t^{i,a} = S_t^{i,a} - S_t^i, \quad \forall t \in \mathbb{R}_+.$$

Moreover, we assume that the functions $\Lambda^{i,b}$ and $\Lambda^{i,a}$ satisfy the following properties:

- $\Lambda^{i,b}$ and $\Lambda^{i,a}$ are twice continuously differentiable,
- $\Lambda^{i,b}$ and $\Lambda^{i,a}$ are decreasing, with $\forall \delta \in \mathbb{R}, \Lambda^{i,b'}(\delta) < 0$ and $\Lambda^{i,a'}(\delta) < 0$,

⁴ Q^i is assumed to be a multiple of z^i . It corresponds to the risk limit of the market maker for asset i .

- $\lim_{\delta \rightarrow +\infty} \Lambda^{i,b}(\delta) = \lim_{\delta \rightarrow +\infty} \Lambda^{i,a}(\delta) = 0$,
- $\sup_{\delta} \frac{\Lambda^{i,b}(\delta)\Lambda^{i,b''}(\delta)}{(\Lambda^{i,b'}(\delta))^2} < 2$ and $\sup_{\delta} \frac{\Lambda^{i,a}(\delta)\Lambda^{i,a''}(\delta)}{(\Lambda^{i,a'}(\delta))^2} < 2$.

Finally, the process $(X_t)_{t \in \mathbb{R}_+}$ modelling the amount of cash on the market maker's cash account has the following dynamics:

$$\begin{aligned}
dX_t &= \sum_{i=1}^d S_t^{i,a} z^i dN_t^{i,a} - S_t^{i,b} z^i dN_t^{i,b} \\
&= \sum_{i=1}^d (S_t^i + \delta_t^{i,a}) z^i dN_t^{i,a} - (S_t^i - \delta_t^{i,b}) z^i dN_t^{i,b} \\
&= \sum_{i=1}^d \left(\delta_t^{i,b} z^i dN_t^{i,b} + \delta_t^{i,a} z^i dN_t^{i,a} \right) - \sum_{i=1}^d S_t^i dq_t^i.
\end{aligned}$$

4.2.2 The optimization problems

We can consider two different optimization problems for the market maker. Following the initial model proposed by Avellaneda and Stoikov in [13], we can assume that she maximizes the expected value of a CARA utility function (with risk aversion parameter $\gamma > 0$) applied to the mark-to-market value of her portfolio at a given time T .⁵ More precisely, her optimization problem writes

$$\sup_{\substack{(\delta_t^{1,b})_t, \dots, (\delta_t^{d,b})_t \in \mathcal{A} \\ (\delta_t^{1,a})_t, \dots, (\delta_t^{d,a})_t \in \mathcal{A}}} \mathbb{E} \left[-\exp \left(-\gamma \left(X_T + \sum_{i=1}^d q_T^i S_T^i \right) \right) \right],$$

where \mathcal{A} is the set of predictable processes bounded from below. We call Model A our model with this first objective function.

Alternatively, as proposed by Cartea *et al.* in [40], we can consider a risk-adjusted expectation for the objective function of the market maker. In that case, the optimization problem writes

$$\sup_{\substack{(\delta_t^{1,b})_t, \dots, (\delta_t^{d,b})_t \in \mathcal{A} \\ (\delta_t^{1,a})_t, \dots, (\delta_t^{d,a})_t \in \mathcal{A}}} \mathbb{E} \left[X_T + \sum_{i=1}^d q_T^i S_T^i - \frac{1}{2} \gamma \int_0^T q_t^i \Sigma q_t dt \right].$$

We call Model B our model with this second objective function.

⁵In the literature there is sometimes a penalty function applied to the inventory at terminal time T . Here, as we shall focus on the asymptotic regime of the optimal quotes, there is no point considering such a penalty. However, it is noteworthy that most of our non-asymptotic results could be generalized to the case of a quadratic terminal penalty.

4.2.3 The Hamilton-Jacobi-Bellman and Hamilton-Jacobi equations

Let $\{e^i\}_{i=1}^d$ be the canonical basis of \mathbb{R}^d . The Hamilton-Jacobi-Bellman equation associated with Model A is

$$\begin{aligned}
0 &= \partial_t u(t, x, q, S) + \frac{1}{2} \sum_{i,j=1}^d \rho^{i,j} \sigma^i \sigma^j \partial_{S^i S^j}^2 u(t, x, q, S) \\
&+ \sum_{i=1}^d \mathbb{1}_{\{q^i + z^i \leq Q^i\}} \sup_{\delta^{i,b}} \Lambda^{i,b}(\delta^{i,b}) (u(t, x - z^i S^i + z^i \delta^{i,b}, q + z^i e^i, S) - u(t, x, q, S)) \\
&+ \sum_{i=1}^d \mathbb{1}_{\{q^i - z^i \geq -Q^i\}} \sup_{\delta^{i,a}} \Lambda^{i,a}(\delta^{i,a}) (u(t, x + z^i S^i + z^i \delta^{i,a}, q - z^i e^i, S) - u(t, x, q, S)), \quad (4.1)
\end{aligned}$$

with terminal condition $u(T, x, q, S) = -\exp\left(-\gamma\left(x + \sum_{i=1}^d q^i S^i\right)\right)$.

The Hamilton-Jacobi-Bellman equation associated with Model B is

$$\begin{aligned}
0 &= \partial_t v(t, x, q, S) - \frac{1}{2} \gamma q^\top \Sigma q + \frac{1}{2} \sum_{i,j=1}^d \rho^{i,j} \sigma^i \sigma^j \partial_{S^i S^j}^2 v(t, x, q, S) \\
&+ \sum_{i=1}^d \mathbb{1}_{\{q^i + z^i \leq Q^i\}} \sup_{\delta^{i,b}} \Lambda^{i,b}(\delta^{i,b}) (v(t, x - z^i S^i + z^i \delta^{i,b}, q + z^i e^i, S) - v(t, x, q, S)) \\
&+ \sum_{i=1}^d \mathbb{1}_{\{q^i - z^i \geq -Q^i\}} \sup_{\delta^{i,a}} \Lambda^{i,a}(\delta^{i,a}) (v(t, x + z^i S^i + z^i \delta^{i,a}, q - z^i e^i, S) - v(t, x, q, S)), \quad (4.2)
\end{aligned}$$

with terminal condition $v(T, x, q, S) = x + \sum_{i=1}^d q^i S^i$.

For each $i \in \{1, \dots, d\}$ and $\xi \geq 0$, let us define two Hamiltonian functions⁶ $H_\xi^{i,b}$ and $H_\xi^{i,a}$ by

$$H_\xi^{i,b}(p) = \begin{cases} \sup_{\delta} \frac{\Lambda^{i,b}(\delta)}{\xi z^i} (1 - \exp(-\xi z^i (\delta - p))) & \text{if } \xi > 0, \\ \sup_{\delta} \Lambda^{i,b}(\delta) (\delta - p) & \text{if } \xi = 0, \end{cases} \quad (4.3)$$

and

$$H_\xi^{i,a}(p) = \begin{cases} \sup_{\delta} \frac{\Lambda^{i,a}(\delta)}{\xi z^i} (1 - \exp(-\xi z^i (\delta - p))) & \text{if } \xi > 0, \\ \sup_{\delta} \Lambda^{i,a}(\delta) (\delta - p) & \text{if } \xi = 0. \end{cases} \quad (4.4)$$

Using the ansatz introduced in [62] for the two functions $u : [0, T] \times \mathbb{R} \times \prod_{i=1}^d (z^i \mathbb{Z} \cap [-Q^i, Q^i]) \times \mathbb{R}^d \rightarrow \mathbb{R}$ and $v : [0, T] \times \mathbb{R} \times \prod_{i=1}^d (z^i \mathbb{Z} \cap [-Q^i, Q^i]) \times \mathbb{R}^d \rightarrow \mathbb{R}$, i.e.

$$u(t, x, q, S) = -\exp\left(-\gamma\left(x + \sum_{i=1}^d q^i S^i + \theta(t, q)\right)\right) \quad \text{and} \quad v(t, x, q, S) = x + \sum_{i=1}^d q^i S^i + \theta(t, q),$$

⁶It is noteworthy that our definition of $H_\xi^{i,b}$ and $H_\xi^{i,a}$ differs from that of [62] (by a factor z^i). The alternative definition we use is also present in [26] for $\xi = 0$.

we see that solving the Hamilton-Jacobi-Bellman equations (4.1) and (4.2) boils down to finding the solution $\theta : [0, T] \times \prod_{i=1}^d (z^i \mathbb{Z} \cap [-Q^i, Q^i]) \rightarrow \mathbb{R}$ of the following Hamilton-Jacobi equation with $\xi = \gamma$ in the case of Model A and $\xi = 0$ in the case of Model B:

$$0 = \partial_t \theta(t, q) - \frac{1}{2} \gamma q^\top \Sigma q + \sum_{i=1}^d \mathbb{1}_{\{q^i + z^i \leq Q^i\}} z^i H_\xi^{i,b} \left(\frac{\theta(t, q) - \theta(t, q + z^i e^i)}{z^i} \right) + \sum_{i=1}^d \mathbb{1}_{\{q^i - z^i \geq -Q^i\}} z^i H_\xi^{i,a} \left(\frac{\theta(t, q) - \theta(t, q - z^i e^i)}{z^i} \right). \quad (4.5)$$

In both cases, the terminal condition simply boils down to

$$\theta(t, q) = 0. \quad (4.6)$$

4.2.4 Existing theoretical results

From [62, Theorem 5.1], for a given $\xi \geq 0$, there exists a unique $\theta : [0, T] \times \prod_{i=1}^d (z^i \mathbb{Z} \cap [-Q^i, Q^i]) \rightarrow \mathbb{R}$, C^1 in time, solution of Eq. (4.5) with terminal condition (4.6). Moreover (see [62, Theorems 5.2 and 5.3]), a classical verification argument enables to go from θ to optimal controls for both Model A and Model B. The optimal quotes as functions of θ are recalled in the following theorems (for details, see [62]).

In the case of Model A, the result is the following:

Theorem 1. *Let us consider the solution θ of Eq. (4.5) with terminal condition (4.6) for $\xi = \gamma$.*

Then, for $i \in \{1, \dots, d\}$, the optimal bid and ask quotes $S_t^{i,b} = S_t^i - \delta_t^{i,b}$ and $S_t^{i,a} = S_t^i + \delta_t^{i,a*}$ in Model A are characterized by*

$$\begin{aligned} \delta_t^{i,b*} &= \tilde{\delta}_\gamma^{i,b*} \left(\frac{\theta(t, q_{t-}) - \theta(t, q_{t-} + z^i e^i)}{z^i} \right) \quad \text{for } q_{t-} + z^i e^i \in \prod_{j=1}^d [-Q^j, Q^j], \\ \delta_t^{i,a*} &= \tilde{\delta}_\gamma^{i,a*} \left(\frac{\theta(t, q_{t-}) - \theta(t, q_{t-} - z^i e^i)}{z^i} \right) \quad \text{for } q_{t-} - z^i e^i \in \prod_{j=1}^d [-Q^j, Q^j], \end{aligned} \quad (4.7)$$

where the functions $\tilde{\delta}_\gamma^{i,b*}(\cdot)$ and $\tilde{\delta}_\gamma^{i,a*}(\cdot)$ are defined by

$$\begin{aligned} \tilde{\delta}_\gamma^{i,b*}(p) &= \Lambda^{i,b-1} \left(\gamma z^i H_\gamma^{i,b}(p) - H_\gamma^{i,b'}(p) \right), \\ \tilde{\delta}_\gamma^{i,a*}(p) &= \Lambda^{i,a-1} \left(\gamma z^i H_\gamma^{i,a}(p) - H_\gamma^{i,a'}(p) \right). \end{aligned}$$

For Model B, the result is the following:

Theorem 2. *Let us consider the solution θ of Eq. (4.5) with terminal condition (4.6) for $\xi = 0$.*

Then, for $i \in \{1, \dots, d\}$, the optimal bid and ask quotes $S_t^{i,b} = S_t^i - \delta_t^{i,b*}$ and $S_t^{i,a} = S_t^i + \delta_t^{i,a*}$ in Model B are characterized by

$$\begin{aligned} \delta_t^{i,b*} &= \tilde{\delta}_0^{i,b*} \left(\frac{\theta(t, q_{t-}) - \theta(t, q_{t-} + z^i e^i)}{z^i} \right) \quad \text{for } q_{t-} + z^i e^i \in \prod_{j=1}^d [-Q^j, Q^j], \\ \delta_t^{i,a*} &= \tilde{\delta}_0^{i,a*} \left(\frac{\theta(t, q_{t-}) - \theta(t, q_{t-} - z^i e^i)}{z^i} \right) \quad \text{for } q_{t-} - z^i e^i \in \prod_{j=1}^d [-Q^j, Q^j], \end{aligned} \quad (4.8)$$

where the functions $\tilde{\delta}_0^{i,b*}(\cdot)$ and $\tilde{\delta}_0^{i,a*}(\cdot)$ are defined by

$$\tilde{\delta}_0^{i,b*}(p) = \Lambda^{i,b^{-1}} \left(-H_0^{i,b'}(p) \right) \quad \text{and} \quad \tilde{\delta}_0^{i,a*}(p) = \Lambda^{i,a^{-1}} \left(-H_0^{i,a'}(p) \right).$$

In the following two sections, we propose new methods to find approximations of the solution to the system of ordinary differential equations (ODEs) (4.5) with terminal condition (4.6). Eqs. (4.7) and (4.8) can then serve to go from approximations of the value function to approximation of the optimal quotes. The resulting quotes correspond in fact to the greedy quoting strategy associated with the proxy of the value function.⁷

4.3 A quadratic approximation of the value function and its applications

4.3.1 Introduction

In the field of (stochastic) optimal control, finding value functions and optimal controls in closed form is the exception rather than the rule. One important exception goes with the class of Linear-Quadratic (LQ) and Linear-Quadratic-Gaussian (LQG) problems. Of course, the above market making problem does not belong to this class of control problems, for instance because the control of point processes is nonlinear by nature. Nevertheless, we see that price risk appears in both Model A and Model B through the quadratic term $\frac{1}{2} \gamma q^\top \Sigma q$ in the Hamilton-Jacobi equation (4.5). The main idea of this paper consists in replacing the Hamiltonian functions associated with our market making problem by quadratic functions that approximate them, and then regarding the real Hamiltonian functions as perturbations of the quadratic ones. The interest of quadratic Hamiltonian functions lies in that the resulting Hamilton-Jacobi equations can be solved in closed-form using the same tools as for LQ/LQG problems, i.e. Riccati equations.

At first sight, approximating the Hamiltonian functions involved in Eq. (4.5) by quadratic functions seems inappropriate. For all $i \in \{1, \dots, d\}$, the functions $H_\xi^{i,b}$ and $H_\xi^{i,a}$ are indeed positive and decreasing and approximating them with U-shaped functions can only be valid locally. However, one has to bear in mind that our goal is to approximate the solution of the Hamilton-Jacobi equations

⁷The true optimal quotes correspond to the greedy strategy with respect to the value function u (in Model A) or v (in Model B) deduced from the true θ .

and not the Hamiltonian functions. This remark is particularly important because the Hamiltonian terms involved in the Hamilton-Jacobi equations are (up to the indicator functions that we shall discard in what follows by considering the limit case where $\forall i \in \{1, \dots, d\}, Q^i = +\infty$) of the form

$$H_\xi^{i,b} \left(\frac{\theta(t, q) - \theta(t, q + z^i e^i)}{z^i} \right) + H_\xi^{i,a} \left(\frac{\theta(t, q) - \theta(t, q - z^i e^i)}{z^i} \right),$$

Assuming that $\frac{\theta(t, q) - \theta(t, q + z^i e^i)}{z^i} \simeq -\frac{\theta(t, q) - \theta(t, q - z^i e^i)}{z^i}$, we clearly see that, with respect to asset i , the function we need to approximate is $p \mapsto H_\xi^{i,b}(p) + H_\xi^{i,a}(-p)$ rather than $H_\xi^{i,b}$ and $H_\xi^{i,a}$ themselves, and it is natural to approximate the former function with a U-shaped one!

Let us replace for all $i \in \{1, \dots, d\}$ the Hamiltonian functions $H_\xi^{i,b}$ and $H_\xi^{i,a}$ by the quadratic functions

$$\check{H}^{i,b} : p \mapsto \alpha_0^{i,b} + \alpha_1^{i,b} p + \frac{1}{2} \alpha_2^{i,b} p^2 \quad \text{and} \quad \check{H}^{i,a} : p \mapsto \alpha_0^{i,a} + \alpha_1^{i,a} p + \frac{1}{2} \alpha_2^{i,a} p^2.$$

Remark 1. A natural choice for the functions $(\check{H}^{i,b})_{i \in \{1, \dots, d\}}$ and $(\check{H}^{i,a})_{i \in \{1, \dots, d\}}$ derives from Taylor expansions around $p = 0$. In that case,

$$\forall i \in \{1, \dots, d\}, \forall j \in \{0, 1, 2\}, \quad \alpha_j^{i,b} = H_\xi^{i,b(j)}(0) \quad \text{and} \quad \alpha_j^{i,a} = H_\xi^{i,a(j)}(0).$$

If we consider the limit case where $\forall i \in \{1, \dots, d\}, Q^i = +\infty$, then Eq. (4.5) becomes

$$\begin{aligned} 0 &= \partial_t \check{\theta}(t, q) - \frac{1}{2} \gamma q^\top \Sigma q + \sum_{i=1}^d z^i \left(\alpha_0^{i,b} + \alpha_0^{i,a} \right) \\ &+ \sum_{i=1}^d \left(\alpha_1^{i,b} (\check{\theta}(t, q) - \check{\theta}(t, q + z^i e^i)) + \alpha_1^{i,a} (\check{\theta}(t, q) - \check{\theta}(t, q - z^i e^i)) \right) \\ &+ \frac{1}{2} \sum_{i=1}^d \frac{1}{z^i} \left(\alpha_2^{i,b} (\check{\theta}(t, q) - \check{\theta}(t, q + z^i e^i))^2 + \alpha_2^{i,a} (\check{\theta}(t, q) - \check{\theta}(t, q - z^i e^i))^2 \right) \end{aligned} \quad (4.9)$$

and of course we consider the terminal condition

$$\check{\theta}(T, q) = 0. \quad (4.10)$$

4.3.2 An approximation of the value function in closed form

Eq. (4.9) with terminal condition (4.10) can be solved in closed form. To prove this point, we start with the following proposition:

Proposition 1. Let us introduce for $i \in \{1, \dots, d\}, j \in \{0, 1, 2\}, k \in \mathbb{N}$,

$$\begin{aligned} \Delta_{j,k}^{i,b} &= \alpha_j^{i,b} (z^i)^k \quad \text{and} \quad \Delta_{j,k}^{i,a} = \alpha_j^{i,a} (z^i)^k, \\ V_{j,k}^b &= \left(\Delta_{j,k}^{1,b}, \dots, \Delta_{j,k}^{d,b} \right)^\top \quad \text{and} \quad V_{j,k}^a = \left(\Delta_{j,k}^{1,a}, \dots, \Delta_{j,k}^{d,a} \right)^\top, \end{aligned}$$

and

$$D_{j,k}^b = \text{diag}(\Delta_{j,k}^{1,b}, \dots, \Delta_{j,k}^{d,b}) \quad \text{and} \quad D_{j,k}^a = \text{diag}(\Delta_{j,k}^{1,a}, \dots, \Delta_{j,k}^{d,a}).$$

Let us consider three differentiable functions $A : [0, T] \rightarrow S_d^{++}$, $B : [0, T] \rightarrow \mathbb{R}^d$, and $C : [0, T] \rightarrow \mathbb{R}$ solutions of the system of ordinary differential equations⁸

$$\begin{cases} A'(t) = 2A(t) (D_{2,1}^b + D_{2,1}^a) A(t) - \frac{1}{2}\gamma\Sigma \\ B'(t) = 2A(t) (V_{1,1}^b - V_{1,1}^a) + 2A(t) (D_{2,2}^b - D_{2,2}^a) \mathcal{D}(A(t)) + 2A(t) (D_{2,1}^b + D_{2,1}^a) B(t) \\ C'(t) = \text{Tr}(D_{0,1}^b + D_{0,1}^a) + \text{Tr}((D_{1,2}^b + D_{1,2}^a) A(t)) + (V_{1,1}^b - V_{1,1}^a)^\top B(t) \\ \quad + \frac{1}{2}\mathcal{D}(A(t))^\top (D_{2,3}^b + D_{2,3}^a) \mathcal{D}(A(t)) + \frac{1}{2}B(t)^\top (D_{2,1}^b + D_{2,1}^a) B(t) + B(t)^\top (D_{2,2}^b - D_{2,2}^a) \mathcal{D}(A(t)), \end{cases} \quad (4.11)$$

with terminal conditions

$$A(T) = 0, B(T) = 0, \text{ and } C(T) = 0, \quad (4.12)$$

where \mathcal{D} is the linear operator mapping a matrix onto the vector of its diagonal coefficients.

Then $\check{\theta} : (t, q) \in [0, T] \times \prod_{i=1}^d z^i \mathbb{Z} \mapsto -q^\top A(t)q - q^\top B(t) - C(t)$ is solution of Eq. (4.9) with terminal condition (4.10).

Proof. We have

$$\begin{aligned} & \partial_t \check{\theta}(t, q) - \frac{1}{2}\gamma q^\top \Sigma q + \sum_{i=1}^d z^i (\alpha_0^{i,b} + \alpha_0^{i,a}) \\ & + \sum_{i=1}^d (\alpha_1^{i,b} (\check{\theta}(t, q) - \check{\theta}(t, q + z^i e^i)) + \alpha_1^{i,a} (\check{\theta}(t, q) - \check{\theta}(t, q - z^i e^i))) \\ & + \frac{1}{2} \sum_{i=1}^d \frac{1}{z^i} (\alpha_2^{i,b} (\check{\theta}(t, q) - \check{\theta}(t, q + z^i e^i))^2 + \alpha_2^{i,a} (\check{\theta}(t, q) - \check{\theta}(t, q - z^i e^i))^2) \\ = & -q^\top A'(t)q - q^\top B'(t) - C'(t) - \frac{1}{2}\gamma q^\top \Sigma q + \sum_{i=1}^d z^i (\alpha_0^{i,b} + \alpha_0^{i,a}) \\ & + \sum_{i=1}^d \alpha_1^{i,b} (2z^i q^\top A(t)e^i + (z^i)^2 e^{i\top} A(t)e^i + z^i e^{i\top} B(t)) \\ & + \sum_{i=1}^d \alpha_1^{i,a} (-2z^i q^\top A(t)e^i + (z^i)^2 e^{i\top} A(t)e^i - z^i e^{i\top} B(t)) \\ & + \frac{1}{2} \sum_{i=1}^d \frac{1}{z^i} \alpha_2^{i,b} (2z^i q^\top A(t)e^i + (z^i)^2 e^{i\top} A(t)e^i + z^i e^{i\top} B(t))^2 \\ & + \frac{1}{2} \sum_{i=1}^d \frac{1}{z^i} \alpha_2^{i,a} (-2z^i q^\top A(t)e^i + (z^i)^2 e^{i\top} A(t)e^i - z^i e^{i\top} B(t))^2 \end{aligned}$$

⁸ S_d^{++} (resp. S_d^+) stands here for the set of positive definite (resp. semi-definite) symmetric d -by- d matrices.

$$\begin{aligned}
&= -q^\top A'(t)q - q^\top B'(t) - C'(t) - \frac{1}{2}\gamma q^\top \Sigma q + \text{Tr}(D_{0,1}^b + D_{0,1}^a) \\
&\quad + 2q^\top A(t)(V_{1,1}^b - V_{1,1}^a) + \text{Tr}((D_{1,2}^b + D_{1,2}^a)A(t)) + (V_{1,1}^b - V_{1,1}^a)^\top B(t) \\
&\quad + 2q^\top A(t)(D_{2,1}^b + D_{2,1}^a)A(t)q + \frac{1}{2}\mathcal{D}(A(t))^\top (D_{2,3}^b + D_{2,3}^a)\mathcal{D}(A(t)) + \frac{1}{2}B(t)^\top (D_{2,1}^b + D_{2,1}^a)B(t) \\
&\quad + 2q^\top A(t)(D_{2,2}^b - D_{2,2}^a)\mathcal{D}(A(t)) + 2q^\top A(t)(D_{2,1}^b + D_{2,1}^a)B(t) + B(t)^\top (D_{2,2}^b - D_{2,2}^a)\mathcal{D}(A(t)) \\
&= 0,
\end{aligned}$$

where the last equality comes from the definitions of (A, B, C) and the identification of the terms of degree 0, 1, and 2 in q .

As the terminal conditions are satisfied, the result is proved. \square

Proposition 2. *Assume $\alpha_2^{i,b} + \alpha_2^{i,a} > 0$ for all $i \in \{1, \dots, d\}$. The system of ODEs (4.11) with terminal conditions (4.12) admits the unique solution*

$$A(t) = \frac{1}{2}D_+^{-\frac{1}{2}}\widehat{A}\left(e^{\widehat{A}(T-t)} - e^{-\widehat{A}(T-t)}\right)\left(e^{\widehat{A}(T-t)} + e^{-\widehat{A}(T-t)}\right)^{-1}D_+^{-\frac{1}{2}}, \quad (4.13)$$

$$B(t) = -2e^{-2\int_t^T A(u)D_+ du} \int_t^T e^{2\int_s^T A(u)D_+ du} A(s)(V_- + D_- \mathcal{D}(A(s))) ds, \quad (4.14)$$

$$\begin{aligned}
C(t) &= -\text{Tr}(D_{0,1}^b + D_{0,1}^a)(T-t) - \text{Tr}\left((D_{1,2}^b + D_{1,2}^a) \int_t^T A(s) ds\right) - V_-^\top \int_t^T B(s) ds \\
&\quad - \frac{1}{2} \int_t^T \mathcal{D}(A(s))^\top (D_{2,3}^b + D_{2,3}^a) \mathcal{D}(A(s)) ds - \frac{1}{2} \int_t^T B(s)^\top D_+ B(s) ds \\
&\quad - \int_t^T B(s)^\top D_- \mathcal{D}(A(s)) ds.
\end{aligned} \quad (4.15)$$

where

$$D_+ = D_{2,1}^b + D_{2,1}^a, \quad D_- = D_{2,2}^b - D_{2,2}^a, \quad V_- = V_{1,1}^b - V_{1,1}^a, \quad \text{and} \quad \widehat{A} = \sqrt{\gamma} \left(D_+^{\frac{1}{2}} \Sigma D_+^{\frac{1}{2}}\right)^{\frac{1}{2}}.$$

Proof. The system of ODEs (4.11) being triangular – though not linear – we tackle the equations one by one, in order.

Solution for A First, we observe that $D_+ = \text{diag}((\alpha_2^{1,b} + \alpha_2^{1,a})z^1, \dots, (\alpha_2^{d,b} + \alpha_2^{d,a})z^d)$ is a positive diagonal matrix. Therefore $D_+^{\frac{1}{2}}$ is well defined. Then, since $D_+^{\frac{1}{2}} \Sigma D_+^{\frac{1}{2}} \in S_d^+$, \widehat{A} is well defined and in S_d^+ .

Now, by introducing the change of variables

$$\mathbf{a}(t) = 2D_+^{\frac{1}{2}}A(t)D_+^{\frac{1}{2}},$$

the terminal value problem for A in (4.11) becomes

$$\begin{cases} \mathbf{a}'(t) = \mathbf{a}(t)^2 - \widehat{A}^2 \\ \mathbf{a}(T) = 0. \end{cases} \quad (4.16)$$

To solve (4.16) let us introduce the function z defined by

$$z(t) = e^{\widehat{A}(T-t)} + e^{-\widehat{A}(T-t)},$$

that is a $C^2([0, T], S_d^{++})$ function verifying $z''(t) = \widehat{A}^2 z(t)$ and $z'(T) = 0$.

We have

$$\frac{d}{dt} (-z'(t)z(t)^{-1}) = -z''(t)z(t)^{-1} + z'(t)z(t)^{-1}z'(t)z(t)^{-1} = (z'(t)z(t)^{-1})^2 - \widehat{A}^2$$

and $-z'(T)z(T)^{-1} = 0$. Therefore, by Cauchy-Lipschitz theorem, we have $\mathbf{a} = -z'z^{-1}$.

Wrapping up, we obtain

$$\begin{aligned} A(t) &= \frac{1}{2} D_+^{-\frac{1}{2}} \mathbf{a}(t) D_+^{-\frac{1}{2}} \\ &= -\frac{1}{2} D_+^{-\frac{1}{2}} z'(t) z(t)^{-1} D_+^{-\frac{1}{2}} \\ &= \frac{1}{2} D_+^{-\frac{1}{2}} \widehat{A} \left(e^{\widehat{A}(T-t)} - e^{-\widehat{A}(T-t)} \right) \left(e^{\widehat{A}(T-t)} + e^{-\widehat{A}(T-t)} \right)^{-1} D_+^{-\frac{1}{2}}. \end{aligned}$$

Solution for B Let us notice that, by definition of the exponential of a matrix, for all $s, t \in [0, T]$, the matrices \widehat{A} , $\left(e^{\widehat{A}(T-s)} - e^{-\widehat{A}(T-s)} \right)$, $\left(e^{\widehat{A}(T-s)} + e^{-\widehat{A}(T-s)} \right)^{-1}$, $\left(e^{\widehat{A}(T-t)} - e^{-\widehat{A}(T-t)} \right)$, and $\left(e^{\widehat{A}(T-t)} + e^{-\widehat{A}(T-t)} \right)^{-1}$ commute. Therefore

$$\begin{aligned} &A(s)D_+A(t)D_+ \\ &= \frac{1}{4} D_+^{-\frac{1}{2}} \widehat{A} \left(e^{\widehat{A}(T-s)} - e^{-\widehat{A}(T-s)} \right) \left(e^{\widehat{A}(T-s)} + e^{-\widehat{A}(T-s)} \right)^{-1} \\ &\quad \times \widehat{A} \left(e^{\widehat{A}(T-t)} - e^{-\widehat{A}(T-t)} \right) \left(e^{\widehat{A}(T-t)} + e^{-\widehat{A}(T-t)} \right)^{-1} D_+^{\frac{1}{2}} \\ &= \frac{1}{4} D_+^{-\frac{1}{2}} \widehat{A} \left(e^{\widehat{A}(T-t)} - e^{-\widehat{A}(T-t)} \right) \left(e^{\widehat{A}(T-t)} + e^{-\widehat{A}(T-t)} \right)^{-1} \\ &\quad \times \widehat{A} \left(e^{\widehat{A}(T-s)} - e^{-\widehat{A}(T-s)} \right) \left(e^{\widehat{A}(T-s)} + e^{-\widehat{A}(T-s)} \right)^{-1} D_+^{\frac{1}{2}} \\ &= A(t)D_+A(s)D_+. \end{aligned}$$

Therefore, we can apply the method of Variation of Parameters to the linear ODE characterizing B to obtain

$$B(t) = -2e^{-2 \int_t^T A(u)D_+ du} \int_t^T e^{2 \int_s^T A(u)D_+ du} A(s) (V_- + D_- \mathcal{D}(A(s))) ds.$$

Solution for C We simply integrate the ODE characterizing C to obtain (4.15). □

From Eqs. (4.13), (4.14), and (4.15), we can deduce the asymptotic behaviour of (A, B, C) when T goes to infinity.

Proposition 3. Let (A, B, C) be the solution of the system of ODEs (4.11) with terminal conditions (4.12).

Then,

$$\begin{aligned}
A(0) &\xrightarrow{T \rightarrow +\infty} \frac{1}{2}\sqrt{\gamma}\Gamma, \\
B(0) &\xrightarrow{T \rightarrow +\infty} -D_+^{-\frac{1}{2}}\widehat{A}\widehat{A}^+D_+^{-\frac{1}{2}}\left(V_- + \frac{1}{2}\sqrt{\gamma}D_-\mathcal{D}(\Gamma)\right), \\
\frac{C(0)}{T} &\xrightarrow{T \rightarrow +\infty} -\text{Tr}(D_{0,1}^b + D_{0,1}^a) - \frac{1}{2}\sqrt{\gamma}\text{Tr}((D_{1,2}^b + D_{1,2}^a)\Gamma) + V_-^\top D_+^{-\frac{1}{2}}\widehat{A}\widehat{A}^+D_+^{-\frac{1}{2}}\left(V_- + \frac{1}{2}\sqrt{\gamma}D_-\mathcal{D}(\Gamma)\right) \\
&\quad - \frac{1}{8}\gamma\mathcal{D}(\Gamma)^\top(D_{2,3}^b + D_{2,3}^a)\mathcal{D}(\Gamma) - \frac{1}{2}\left(V_- + \frac{1}{2}\sqrt{\gamma}D_-\mathcal{D}(\Gamma)\right)^\top D_+^{-\frac{1}{2}}\widehat{A}\widehat{A}^+D_+^{-\frac{1}{2}}\left(V_- + \frac{1}{2}\sqrt{\gamma}D_-\mathcal{D}(\Gamma)\right) \\
&\quad + \frac{1}{2}\sqrt{\gamma}\left(V_- + \frac{1}{2}\sqrt{\gamma}D_-\mathcal{D}(\Gamma)\right)^\top D_+^{-\frac{1}{2}}\widehat{A}\widehat{A}^+D_+^{-\frac{1}{2}}D_-\mathcal{D}(\Gamma),
\end{aligned}$$

where $\Gamma = D_+^{-\frac{1}{2}}\left(D_+^{\frac{1}{2}}\Sigma D_+^{\frac{1}{2}}\right)^{\frac{1}{2}}D_+^{-\frac{1}{2}}$ and \widehat{A}^+ is the Moore-Penrose generalized inverse of \widehat{A} .

Proof. This proof is divided into three parts corresponding to the derivation of the asymptotic expression for A , B , and C , respectively.

Asymptotics for A Let us recall first that $\widehat{A} = \sqrt{\gamma}\left(D_+^{\frac{1}{2}}\Sigma D_+^{\frac{1}{2}}\right)^{\frac{1}{2}} \in S_d^+$. Therefore, there exists an orthogonal matrix P and there exists a diagonal matrix with nonnegative entries $\text{diag}(\lambda_1, \dots, \lambda_d)$ such that $\widehat{A} = P\text{diag}(\lambda_1, \dots, \lambda_d)P^\top$. From Eq. (4.13) we have

$$A(0) = \frac{1}{2}D_+^{-\frac{1}{2}}P\text{diag}(\lambda_1 \tanh(\lambda_1 T), \dots, \lambda_d \tanh(\lambda_d T))P^\top D_+^{-\frac{1}{2}}.$$

As $\lambda \tanh(\lambda T) \xrightarrow{T \rightarrow +\infty} \begin{cases} 0, & \text{if } \lambda = 0 \\ \lambda, & \text{if } \lambda > 0 \end{cases}$, we clearly have

$$A(0) \xrightarrow{T \rightarrow +\infty} \frac{1}{2}D_+^{-\frac{1}{2}}P\text{diag}(\lambda_1, \dots, \lambda_d)P^\top D_+^{-\frac{1}{2}} = \frac{1}{2}D_+^{-\frac{1}{2}}\widehat{A}D_+^{-\frac{1}{2}} = \frac{1}{2}\sqrt{\gamma}\Gamma.$$

Asymptotics for B From Eq. (4.14), we have

$$\begin{aligned}
B(0) &= -2e^{-2\int_0^T A(u)D_+ du} \int_0^T e^{2\int_s^T A(u)D_+ du} A(s) (V_- + D_-\mathcal{D}(A(s))) ds \\
&= -2e^{-2\int_0^T \tilde{A}(u)D_+ du} \int_0^T e^{2\int_s^T \tilde{A}(u)D_+ du} \tilde{A}(s) (V_- + D_-\mathcal{D}(\tilde{A}(s))) ds
\end{aligned}$$

where $\tilde{A} : t \mapsto \frac{1}{2}D_+^{-\frac{1}{2}}\widehat{A}\left(e^{\widehat{A}t} - e^{-\widehat{A}t}\right)\left(e^{\widehat{A}t} + e^{-\widehat{A}t}\right)^{-1}D_+^{-\frac{1}{2}}$.

Using the spectral decomposition of \hat{A} introduced in the above paragraph, we see that

$$2\tilde{A}(u)D_+ = D_+^{-\frac{1}{2}} P \text{diag}(\lambda_1 \tanh(\lambda_1 u), \dots, \lambda_d \tanh(\lambda_d u)) P^\top D_+^{\frac{1}{2}}$$

and therefore, after integration,

$$e^{2\int_0^T \tilde{A}(u)D_+ du} = D_+^{-\frac{1}{2}} P \text{diag}(\cosh(\lambda_1 T), \dots, \cosh(\lambda_d T)) P^\top D_+^{\frac{1}{2}}$$

and

$$e^{2\int_0^s \tilde{A}(u)D_+ du} \tilde{A}(s) = \frac{1}{2} D_+^{-\frac{1}{2}} P \text{diag}(\lambda_1 \sinh(\lambda_1 s), \dots, \lambda_d \sinh(\lambda_d s)) P^\top D_+^{-\frac{1}{2}}$$

Wrapping up, we get

$$\begin{aligned} & B(0) \\ &= - \int_0^T D_+^{-\frac{1}{2}} P \text{diag} \left(\lambda_1 \frac{\sinh(\lambda_1 s)}{\cosh(\lambda_1 T)}, \dots, \lambda_d \frac{\sinh(\lambda_d s)}{\cosh(\lambda_d T)} \right) P^\top D_+^{-\frac{1}{2}} \left(V_- + D_- \mathcal{D}(\tilde{A}(s)) \right) ds. \\ &= - \int_0^T D_+^{-\frac{1}{2}} P \text{diag} \left(\lambda_1 \frac{\sinh(\lambda_1 s)}{\cosh(\lambda_1 T)}, \dots, \lambda_d \frac{\sinh(\lambda_d s)}{\cosh(\lambda_d T)} \right) P^\top D_+^{-\frac{1}{2}} \left(V_- + \frac{1}{2} \sqrt{\gamma} D_- \mathcal{D}(\Gamma) \right) ds \\ &\quad + \int_0^T D_+^{-\frac{1}{2}} P \text{diag} \left(\lambda_1 \frac{\sinh(\lambda_1 s)}{\cosh(\lambda_1 T)}, \dots, \lambda_d \frac{\sinh(\lambda_d s)}{\cosh(\lambda_d T)} \right) P^\top D_+^{-\frac{1}{2}} \left(\frac{1}{2} \sqrt{\gamma} D_- \mathcal{D}(\Gamma) - D_- \mathcal{D}(\tilde{A}(s)) \right) ds \\ &= D_+^{-\frac{1}{2}} P \text{diag} \left(1 - \frac{1}{\cosh(\lambda_1 T)}, \dots, 1 - \frac{1}{\cosh(\lambda_d T)} \right) P^\top D_+^{-\frac{1}{2}} \left(V_- + \frac{1}{2} \sqrt{\gamma} D_- \mathcal{D}(\Gamma) \right) + J(T), \end{aligned}$$

where

$$J(T) = \int_0^T D_+^{-\frac{1}{2}} P \text{diag} \left(\lambda_1 \frac{\sinh(\lambda_1 s)}{\cosh(\lambda_1 T)}, \dots, \lambda_d \frac{\sinh(\lambda_d s)}{\cosh(\lambda_d T)} \right) P^\top D_+^{-\frac{1}{2}} \left(\frac{1}{2} \sqrt{\gamma} D_- \mathcal{D}(\Gamma) - D_- \mathcal{D}(\tilde{A}(s)) \right) ds.$$

Let us prove that $J(T) \xrightarrow{T \rightarrow +\infty} 0$. For that purpose, let us consider $\epsilon > 0$ and let us notice that there exists $\tau > 0$ such that $\forall s > \tau$, $\left\| \frac{1}{2} \sqrt{\gamma} D_+^{-1} D_- \mathcal{D}(\Gamma) - D_+^{-1} D_- \mathcal{D}(\tilde{A}(s)) \right\| \leq \epsilon$, where the norm used is the Euclidian norm on \mathbb{R}^d . Let us also denote by M the quantity $\sup_{s \geq 0} \left\| \frac{1}{2} \sqrt{\gamma} D_- \mathcal{D}(\Gamma) - D_- \mathcal{D}(\tilde{A}(s)) \right\|$.

Using the operator norm (still denoted by $\|\cdot\|$) associated with the Euclidian norm on \mathbb{R}^d and its well-known link with the spectral radius, we see that for $T > \tau$,

$$\begin{aligned} & \|J(T)\| \\ &\leq \int_0^T \left\| D_+^{-\frac{1}{2}} P \text{diag} \left(\lambda_1 \frac{\sinh(\lambda_1 s)}{\cosh(\lambda_1 T)}, \dots, \lambda_d \frac{\sinh(\lambda_d s)}{\cosh(\lambda_d T)} \right) P^\top D_+^{\frac{1}{2}} \right\| \left\| D_+^{-1} \frac{1}{2} \sqrt{\gamma} D_- \mathcal{D}(\Gamma) - D_+^{-1} D_- \mathcal{D}(\tilde{A}(s)) \right\| ds \\ &\leq \int_0^T \left(\max_i \lambda_i \frac{\sinh(\lambda_i s)}{\cosh(\lambda_i T)} \right) \left\| \frac{1}{2} \sqrt{\gamma} D_+^{-1} D_- \mathcal{D}(\Gamma) - D_+^{-1} D_- \mathcal{D}(\tilde{A}(s)) \right\| ds \\ &\leq M \int_0^\tau \max_i \lambda_i \frac{\sinh(\lambda_i s)}{\cosh(\lambda_i T)} ds + \epsilon \int_\tau^T \max_i \lambda_i \frac{\sinh(\lambda_i s)}{\cosh(\lambda_i T)} ds. \end{aligned}$$

By defining $\bar{\lambda} = \max\{\lambda_1, \dots, \lambda_d\}$ and $\underline{\lambda} = \min\{\lambda_i | \forall i \in \{1, \dots, d\}, \lambda_i > 0\}$, we have

$$\max_{i \in \{1, \dots, d\}} \lambda_i \frac{\sinh(\lambda_i s)}{\cosh(\lambda_i T)} \leq \max_{i \in \{1, \dots, d\}} \lambda_i \frac{e^{\lambda_i s}}{e^{\lambda_i T}} = \max_{i \in \{1, \dots, d\}, \lambda_i > 0} \lambda_i e^{-\lambda_i(T-s)} \leq \bar{\lambda} e^{-\underline{\lambda}(T-s)}.$$

Therefore,

$$\begin{aligned} & \limsup_{T \rightarrow \infty} \|J(T)\| \\ & \leq M \limsup_{T \rightarrow \infty} \bar{\lambda} \left(e^{-\underline{\lambda}(T-\tau)} - e^{-\underline{\lambda}T} \right) + \epsilon \limsup_{T \rightarrow \infty} \bar{\lambda} \left(1 - e^{-\underline{\lambda}(T-\tau)} \right) \\ & \leq \epsilon \end{aligned}$$

which allows to conclude that $J(T) \xrightarrow{T \rightarrow +\infty} 0$.

Now, as $P \text{diag} \left(1 - \frac{1}{\cosh(\lambda_1 T)}, \dots, 1 - \frac{1}{\cosh(\lambda_d T)} \right) P^\top$ converges toward the orthogonal projector on $\text{Im}(\widehat{A})$, which is also given by $\widehat{A}\widehat{A}^+$, we conclude that

$$B(0) \xrightarrow{T \rightarrow +\infty} -D_+^{-\frac{1}{2}} \widehat{A}\widehat{A}^+ D_+^{-\frac{1}{2}} \left(V_- + \frac{1}{2} \sqrt{\gamma} D_- \mathcal{D}(\Gamma) \right).$$

Asymptotics for C The asymptotic behavior of C is a straightforward consequence of that of A and B . □

4.3.3 From value functions to heuristics and quotes

Motivation for closed-form approximations

An approximation in closed form of the value function can be motivated by its numerous applications. In the following, we highlight three of them.

First, it can serve as a heuristic evaluation function in reinforcement learning algorithms. Indeed, in problems where the time horizon is too far away to consider full exploration in time, it is often useful, when using Monte-Carlo-based reinforcement learning techniques, to proxy the value of states in a tractable way – analogous to algorithms such as Deep Blue. The above closed-form approximations can be used for that purpose. Moreover, because the value of $C(t)$ is irrelevant for comparing two states, it is sometimes possible, especially when T is large, to consider the asymptotic expression

$$-\frac{1}{2} \sqrt{\gamma} q^\top \Gamma q + q^\top D_+^{-\frac{1}{2}} \widehat{A}\widehat{A}^+ D_+^{-\frac{1}{2}} \left(V_- + \frac{1}{2} \sqrt{\gamma} D_- \mathcal{D}(\Gamma) \right)$$

instead of $\check{\theta}(t, q)$.

Second, a closed-form approximation of the value function can be used as a starting point in iterative methods designed to compute the value function (value iteration algorithm, actor-critic algorithms,

etc.). Unlike for the above use, the value of $C(t)$ matters in that case.

A third important application, and the one that initially motivated our paper, is for computing policies (quotes, in our case). Indeed, a policy can be deduced from an approximation of the value function by computing the greedy strategy associated with that approximation. In our market making problem, the quotes obtained in this way are not only easy to compute, but also have the advantage of being easily interpretable. The qualitative analysis of such quotes can provide insights on the true optimal quotes such as the identification of the leading factors and sensitivity to changes in model parameters.

Quotes: the general case

The greedy quoting strategy associated with our closed-form proxy of the value function leads to the following quotes for all $i \in \{1, \dots, d\}$:

$$\begin{aligned}\check{\delta}_t^{i,b} &= \tilde{\delta}_\xi^{i,b*} \left(\frac{\check{\theta}(t, q_{t-}) - \check{\theta}(t, q_{t-} + z^i e^i)}{z^i} \right) \\ &= \tilde{\delta}_\xi^{i,b*} \left(2q_{t-}^\top A(t) e^i + z^i e^{i^\top} A(t) e^i + e^{i^\top} B(t) \right), \\ \check{\delta}_t^{i,a} &= \tilde{\delta}_\xi^{i,a*} \left(\frac{\check{\theta}(t, q_{t-}) - \check{\theta}(t, q_{t-} - z^i e^i)}{z^i} \right) \\ &= \tilde{\delta}_\xi^{i,a*} \left(-2q_{t-}^\top A(t) e^i + z^i e^{i^\top} A(t) e^i - e^{i^\top} B(t) \right),\end{aligned}$$

where $\tilde{\delta}_\xi^{i,b*}$ and $\tilde{\delta}_\xi^{i,a*}$ are given in Theorems 1 and 2 for $\xi = \gamma$ and $\xi = 0$ respectively (depending on whether one considers Model A or Model B).

The asymptotic regime exhibited in the above paragraphs can then serve to obtain the following simple closed-form approximations:

$$\check{\delta}_t^{i,b} = \tilde{\delta}_\xi^{i,b*} \left(\sqrt{\gamma} q_{t-}^\top \Gamma e^i + \frac{1}{2} \sqrt{\gamma} z^i e^{i^\top} \Gamma e^i - e^{i^\top} D_+^{-\frac{1}{2}} \hat{A} \hat{A}^+ D_+^{-\frac{1}{2}} \left(V_- + \frac{1}{2} \sqrt{\gamma} D_- \mathcal{D}(\Gamma) \right) \right), \quad (4.17)$$

$$\check{\delta}_t^{i,a} = \tilde{\delta}_\xi^{i,a*} \left(-\sqrt{\gamma} q_{t-}^\top \Gamma e^i + \frac{1}{2} \sqrt{\gamma} z^i e^{i^\top} \Gamma e^i + e^{i^\top} D_+^{-\frac{1}{2}} \hat{A} \hat{A}^+ D_+^{-\frac{1}{2}} \left(V_- + \frac{1}{2} \sqrt{\gamma} D_- \mathcal{D}(\Gamma) \right) \right) \quad (4.18)$$

In the case of symmetric intensities, i.e. when $\Lambda^{i,b} = \Lambda^{i,a}$ for all $i \in \{1, \dots, d\}$, the Hamiltonian functions $H^{i,b}$ and $H^{i,a}$ given in Eqs. (4.3) and (4.4) are identical and thus it is natural to set $\check{H}^{i,b} = \check{H}^{i,a}$ for all $i \in \{1, \dots, d\}$. In that case, (4.17) and (4.18) simplify into

$$\check{\delta}_t^{i,b} = \tilde{\delta}_\xi^{i,b*} \left(\sqrt{\gamma} q_{t-}^\top \Gamma e^i + \frac{1}{2} \sqrt{\gamma} z^i e^{i^\top} \Gamma e^i \right), \quad (4.19)$$

$$\check{\delta}_t^{i,a} = \tilde{\delta}_\xi^{i,a*} \left(-\sqrt{\gamma} q_{t-}^\top \Gamma e^i + \frac{1}{2} \sqrt{\gamma} z^i e^{i^\top} \Gamma e^i \right). \quad (4.20)$$

All these approximations of the optimal quotes can be used directly or as starting points in iterative methods designed to compute the optimal quotes (policy iteration algorithm, actor-critic

algorithms, etc.).

Quotes: the case of symmetric exponential intensities

Exponential intensity functions play an important role in the optimal market making literature and more generally in the algorithmic trading literature. This shape of intensity functions, initially proposed by Avellaneda and Stoikov in [13], leads indeed to simplification because of the form of the associated Hamiltonian functions.

If we assume that the intensity functions are given, for all $i \in \{1, \dots, d\}$, by

$$\Lambda^{i,b}(\delta) = \Lambda^{i,a}(\delta) = A^i e^{-k^i \delta}, \quad A^i, k^i > 0,$$

then (see [62]) the Hamiltonian functions are given, for all $i \in \{1, \dots, d\}$, by

$$H_\xi^{i,b}(p) = H_\xi^{i,a}(p) = \frac{A^i}{k^i} C_\xi^i \exp(-k^i p),$$

where

$$C_\xi^i = \begin{cases} \left(1 + \frac{\xi z^i}{k^i}\right)^{-\left(1 + \frac{k^i}{\xi z^i}\right)} & \text{if } \xi > 0 \\ e^{-1} & \text{if } \xi = 0, \end{cases}$$

and the functions $\tilde{\delta}_\xi^{i,b*}$ and $\tilde{\delta}_\xi^{i,a*}$ are given, for all $i \in \{1, \dots, d\}$, by

$$\tilde{\delta}_\xi^{i,b*}(p) = \tilde{\delta}_\xi^{i,a*}(p) = \begin{cases} p + \frac{1}{\xi z^i} \log\left(1 + \frac{\xi z^i}{k^i}\right) & \text{if } \xi > 0 \\ p + \frac{1}{k^i} & \text{if } \xi = 0. \end{cases}$$

Therefore, if we consider the quadratic approximation of the Hamiltonian functions based on their Taylor expansion around $p = 0$ (see Remark 1), then (4.19) and (4.20) become

$$\begin{aligned} \check{\delta}_t^{i,b} &= \begin{cases} \sqrt{\gamma} \left(q_{t-}^\top \Gamma e^i + \frac{1}{2} z^i e^{i^\top} \Gamma e^i \right) + \frac{1}{\gamma z^i} \log\left(1 + \frac{\gamma z^i}{k^i}\right) & \text{in Model A,} \\ \sqrt{\gamma} \left(q_{t-}^\top \Gamma e^i + \frac{1}{2} z^i e^{i^\top} \Gamma e^i \right) + \frac{1}{k^i} & \text{in Model B.} \end{cases} \\ \check{\delta}_t^{i,a} &= \begin{cases} -\sqrt{\gamma} \left(q_{t-}^\top \Gamma e^i - \frac{1}{2} z^i e^{i^\top} \Gamma e^i \right) + \frac{1}{\gamma z^i} \log\left(1 + \frac{\gamma z^i}{k^i}\right) & \text{in Model A,} \\ -\sqrt{\gamma} \left(q_{t-}^\top \Gamma e^i - \frac{1}{2} z^i e^{i^\top} \Gamma e^i \right) + \frac{1}{k^i} & \text{in Model B.} \end{cases} \end{aligned}$$

where $\Gamma = D_+^{-\frac{1}{2}} (D_+^{\frac{1}{2}} \Sigma D_+^{\frac{1}{2}})^{\frac{1}{2}} D_+^{-\frac{1}{2}}$ and $D_+ = \text{diag}(2A^1 C_\xi^1 k^1 z^1, \dots, 2A^d C_\xi^d k^d z^d)$.

It is noteworthy that these approximations of the optimal quotes are affine in the current inventory. In particular, in the case of Model A, when the number of assets is reduced to one (with unitary transaction size), they coincide with the affine closed-form approximations obtained in the paper [65] of Guéant, Lehalle, and Fernandez-Tapia. Their approximations, however, are obtained in a fundamentally different manner, by using spectral arguments and a continuous approximation of

the initial discrete problem.

Another useful point of view on the above quoting strategy is by observing the resulting approximations of the optimal (half) bid-ask spread and skew. The approximations of the optimal (half) bid-ask spread and skew for asset i are respectively given by

$$\frac{\check{\delta}_t^{i,a} + \check{\delta}_t^{i,b}}{2} = \begin{cases} \frac{1}{2}\sqrt{\gamma}z^i e^{i\top}\Gamma e^i + \frac{1}{\gamma z^i} \log\left(1 + \frac{\gamma z^i}{k^i}\right) & \text{in Model A,} \\ \frac{1}{2}\sqrt{\gamma}z^i e^{i\top}\Gamma e^i + \frac{1}{k^i} & \text{in Model B,} \end{cases}$$

and

$$\frac{\check{\delta}_t^{i,a} - \check{\delta}_t^{i,b}}{2} = -\sqrt{\gamma}q_t^\top \Gamma e^i \text{ in both Model A and Model B.}$$

These approximations give us a constant bid-ask spread and a skew linear in the inventory. This translates well the intuition that the skew has the role of inventory risk management, whereas the spread balances the trade-off between frequency of transactions and profit per round-trip trade (the term $\frac{1}{\gamma z^i} \log\left(1 + \frac{\gamma z^i}{k^i}\right)$ in Model A, which reduces to $\frac{1}{k^i}$ in the case of Model B), plus an additional risk aversion buffer (the term $\frac{1}{2}\sqrt{\gamma}z^i e^{i\top}\Gamma e^i$).

We can also observe that the subtle interplay between price risk and liquidity risk is summed up into a unique mathematical object: the matrix $\Gamma = D_+^{-\frac{1}{2}}(D_+^{\frac{1}{2}}\Sigma D_+^{\frac{1}{2}})^{\frac{1}{2}}D_+^{-\frac{1}{2}}$.

On the parameter sensitivity analysis, beyond the immediate conclusions of the above formulas, the reader is referred to [65] for a comprehensive analysis in the single-asset case.

4.4 Beyond the quadratic approximation: a perturbative approach

In Section 4.3, we approximated the Hamiltonian functions by quadratic functions in order to “approximate” the Hamilton-Jacobi equation characterizing the value function and then approximate the value function itself. To go further and possibly improve the obtained approximation, we can consider a perturbative approach around our quadratic approximation. This means that we regard the real Hamiltonian functions as small perturbations of the quadratic functions used to approximate them and consider then a first order approximation (the zero-th order approximation being that obtained in Section 4.3).

Formally, writing

$$H_\xi^{i,b}(p) = \check{H}^{i,b}(p) + \epsilon h^{i,b}(p), \quad H_\xi^{i,a}(p) = \check{H}^{i,a}(p) + \epsilon h^{i,a}(p), \quad \text{and} \quad \theta(t, q) = \check{\theta}(t, q) + \epsilon \eta(t, q),$$

and plugging these expressions in Eq. (4.5) in the limit case where $Q^i = +\infty$ for all $i \in \{1, \dots, d\}$, we obtain

$$0 = \partial_t \theta(t, q) - \frac{1}{2} \gamma q^\top \Sigma q + \sum_{i=1}^d z^i H_\xi^{i,b} \left(\frac{\theta(t, q) - \theta(t, q + z^i e^i)}{z^i} \right) + \sum_{i=1}^d z^i H_\xi^{i,a} \left(\frac{\theta(t, q) - \theta(t, q - z^i e^i)}{z^i} \right)$$

$$\begin{aligned}
&= \partial_t \check{\theta}(t, q) - \frac{1}{2} \gamma q^\top \Sigma q + \sum_{i=1}^d z^i \check{H}^{i,b} \left(\frac{\check{\theta}(t, q) - \check{\theta}(t, q + z^i e^i)}{z^i} \right) + \sum_{i=1}^d z^i \check{H}^{i,a} \left(\frac{\check{\theta}(t, q) - \check{\theta}(t, q - z^i e^i)}{z^i} \right) \\
&\quad + \epsilon \left(\partial_t \eta(t, q) + \sum_{i=1}^d z^i h^{i,b} \left(\frac{\check{\theta}(t, q) - \check{\theta}(t, q + z^i e^i)}{z^i} \right) + \sum_{i=1}^d z^i h^{i,a} \left(\frac{\check{\theta}(t, q) - \check{\theta}(t, q - z^i e^i)}{z^i} \right) \right. \\
&\quad \quad \quad + \sum_{i=1}^d \check{H}^{i,b'} \left(\frac{\check{\theta}(t, q) - \check{\theta}(t, q + z^i e^i)}{z^i} \right) (\eta(t, q) - \eta(t, q + z^i e^i)) \\
&\quad \quad \quad \left. + \sum_{i=1}^d \check{H}^{i,a'} \left(\frac{\check{\theta}(t, q) - \check{\theta}(t, q - z^i e^i)}{z^i} \right) (\eta(t, q) - \eta(t, q - z^i e^i)) \right) + o(\epsilon) \\
&= \epsilon \left(\partial_t \eta(t, q) + \sum_{i=1}^d z^i h^{i,b} \left(\frac{\check{\theta}(t, q) - \check{\theta}(t, q + z^i e^i)}{z^i} \right) + \sum_{i=1}^d z^i h^{i,a} \left(\frac{\check{\theta}(t, q) - \check{\theta}(t, q - z^i e^i)}{z^i} \right) \right. \\
&\quad \quad \quad + \sum_{i=1}^d \left(-\check{H}^{i,b'} \left(\frac{\check{\theta}(t, q) - \check{\theta}(t, q + z^i e^i)}{z^i} \right) \right) (\eta(t, q + z^i e^i) - \eta(t, q)) \\
&\quad \quad \quad \left. + \sum_{i=1}^d \left(-\check{H}^{i,a'} \left(\frac{\check{\theta}(t, q) - \check{\theta}(t, q - z^i e^i)}{z^i} \right) \right) (\eta(t, q - z^i e^i) - \eta(t, q)) \right) + o(\epsilon).
\end{aligned}$$

Therefore,

$$\begin{aligned}
0 &= \partial_t \eta(t, q) + \sum_{i=1}^d z^i h^{i,b} \left(\frac{\check{\theta}(t, q) - \check{\theta}(t, q + z^i e^i)}{z^i} \right) + \sum_{i=1}^d z^i h^{i,a} \left(\frac{\check{\theta}(t, q) - \check{\theta}(t, q - z^i e^i)}{z^i} \right) \\
&\quad + \sum_{i=1}^d \left(-\check{H}^{i,b'} \left(\frac{\check{\theta}(t, q) - \check{\theta}(t, q + z^i e^i)}{z^i} \right) \right) (\eta(t, q + z^i e^i) - \eta(t, q)) \\
&\quad + \sum_{i=1}^d \left(-\check{H}^{i,a'} \left(\frac{\check{\theta}(t, q) - \check{\theta}(t, q - z^i e^i)}{z^i} \right) \right) (\eta(t, q - z^i e^i) - \eta(t, q)),
\end{aligned}$$

and we have the terminal condition $\eta(T, q) = 0$.

By Feynman-Kac representation theorem, we have

$$\eta(t, q) = \mathbb{E}^{\check{\mathbb{P}}} \left[\int_t^T \left(\sum_{i=1}^d z^i h^{i,b} \left(\frac{\check{\theta}(s, q_{s-}^{t,q}) - \check{\theta}(s, q_{s-}^{t,q} + z^i e^i)}{z^i} \right) + \sum_{i=1}^d z^i h^{i,a} \left(\frac{\check{\theta}(s, q_{s-}^{t,q}) - \check{\theta}(s, q_{s-}^{t,q} - z^i e^i)}{z^i} \right) \right) ds \right],$$

where under $\check{\mathbb{P}}$ the process $(q_s^{t,q})_{s \in [t, T]}$ satisfies

$$dq_s^{t,q} = \sum_{i=1}^d z^i (d\check{N}_s^{i,b} - d\check{N}_s^{i,a}) e^i \quad \text{and} \quad q_t^{t,q} = q,$$

with, for each $i \in \{1, \dots, d\}$, $\check{N}^{i,b}$ and $\check{N}^{i,a}$ constructed like $N^{i,b}$ and $N^{i,a}$ but with respective

intensities given at time s by

$$-\check{H}^{i,b'} \left(\frac{\check{\theta}(s, q_{s-}^{t,q}) - \check{\theta}(s, q_{s-}^{t,q} + z^i e^i)}{z^i} \right) \quad \text{and} \quad -\check{H}^{i,a'} \left(\frac{\check{\theta}(s, q_{s-}^{t,q}) - \check{\theta}(s, q_{s-}^{t,q} - z^i e^i)}{z^i} \right).$$

In practice, it means that we can approximate $\theta(t, q)$ by

$$\check{\theta}(t, q) + \mathbb{E}^{\mathbb{P}} \left[\int_t^T \left(\sum_{i=1}^d z^i (H_\xi^{i,b} - \check{H}^{i,b}) \left(\frac{\check{\theta}(s, q_{s-}^{t,q}) - \check{\theta}(s, q_{s-}^{t,q} + z^i e^i)}{z^i} \right) + \sum_{i=1}^d z^i (H_\xi^{i,a} - \check{H}^{i,a}) \left(\frac{\check{\theta}(s, q_{s-}^{t,q}) - \check{\theta}(s, q_{s-}^{t,q} - z^i e^i)}{z^i} \right) \right) ds \right].$$

Of course this new approximation is not a closed-form one. However, it can be computed using a Monte-Carlo simulation for a specific (t, q) . In particular, it means that upon receiving a request for quote from a client and if time permits (which depends on asset class and market conditions), a market maker can perform a Monte-Carlo simulation to obtain an approximation of the value function at the relevant points to compute a quote that more accurately accounts for the liquidity of the requested asset than a quote computed using the closed forms of Section 4.3.3.

4.5 A multi-asset market making model with additional features

4.5.1 A more general model

In Section 4.2.1 we presented a multi-asset extension to the classical single-asset market making model of Avellaneda and Stoikov. This extension can itself be extended to encompass important features of OTC markets. In this section we extend our results to a more general multi-asset market making model with drift in prices to model the views of the market maker, client tiering, distributed requested sizes for each asset and each tier, and fixed transaction costs for each asset and each tier.

In terms of modeling, the addition of drifts to the price processes is straightforward. Formally, we assume that for each $i \in \{1, \dots, d\}$, the dynamics of the price process $(S_t^i)_{t \in \mathbb{R}_+}$ of asset i is now given by

$$dS_t^i = \mu^i dt + \sigma^i dW_t^i,$$

where σ^i and $(W_t^i)_{t \in \mathbb{R}_+}$ are defined as in Section 4.2.1 and where μ^i is a real constant. In what follows, we denote by μ the vector $\mu = (\mu^1, \dots, \mu^d)^\top$.

In OTC markets, market makers often divide their clients into groups, called tiers, for instance because they do not have the same commercial relationship with all clients or because the propensity to transact given a quote differs across clients. In particular, they can answer/stream different quotes to clients from different tiers.⁹ Let us denote here by $N \in \mathbb{N}^*$ the number of such tiers.

⁹There can also be tiers to proxy the existence of trading platforms with different clients and/or different costs.

In addition to introducing tiers, we can drop the assumption of constant request size per asset and consider instead that, for each asset and each tier, the size of the requests at the bid and at the ask are distributed according to known distributions.

Mathematically, the bid and ask quotes that the market maker propose are then modeled by the maps

$$S^{i,n,b} : (\omega, t, z) \in \Omega \times [0, T] \times \mathbb{R}_+^* \mapsto S_t^{i,n,b}(\omega, z) \in \mathbb{R} \quad \text{and} \quad S^{i,n,a} : (\omega, t, z) \in \Omega \times [0, T] \times \mathbb{R}_+^* \mapsto S_t^{i,n,a}(\omega, z) \in \mathbb{R},$$

where $i \in \{1, \dots, d\}$ is the index of the asset, $n \in \{1, \dots, N\}$ is the index of the tier, and $z \in \mathbb{R}_+^*$ is the size of the request (in number of assets). In the same vein as in Section 4.2.1, we introduce¹⁰

$$\delta_t^{i,n,b}(z) = S_t - S_t^{i,n,b}(z) \quad \text{and} \quad \delta_t^{i,n,a}(z) = S_t^{i,n,a}(z) - S_t,$$

and the maps $(\delta_t^{i,n,b}(\cdot))_{t \in \mathbb{R}_+}$ and $(\delta_t^{i,n,a}(\cdot))_{t \in \mathbb{R}_+}$ are assumed to be \mathbb{F} -predictable and bounded from below by a given constant $-\delta_\infty < 0$.¹¹

With these new features in mind, we introduce for each asset $i \in \{1, \dots, d\}$ and for each tier $n \in \{1, \dots, N\}$ the processes $(N_t^{i,n,b})_{t \in \mathbb{R}_+}$ and $(N_t^{i,n,a})_{t \in \mathbb{R}_+}$ modeling the number of transactions in asset i with clients from tier n at the bid and at the ask, respectively. They are \mathbb{R}_+^* -marked point processes, with respective intensity kernels $(\lambda_t^{i,n,b}(dz))_{t \in \mathbb{R}_+^*}$ and $(\lambda_t^{i,n,a}(dz))_{t \in \mathbb{R}_+^*}$ given by

$$\lambda_t^{i,n,b}(dz) = \Lambda^{i,n,b}(\delta_t^{i,n,b}(z)) \mathbb{1}_{\{q_{t-}^i + z \leq Q^i\}} \nu^{i,n,b}(dz) \quad \text{and} \quad \lambda_t^{i,n,a}(dz) = \Lambda^{i,n,a}(\delta_t^{i,n,a}(z)) \mathbb{1}_{\{q_{t-}^i - z \geq -Q^i\}} \nu^{i,n,a}(dz),$$

where $\nu^{i,n,b}$ and $\nu^{i,n,a}$ are the two probability measures representing the distribution of the requested sizes at the bid and at the ask respectively, for asset i and tier n , and where $\Lambda^{i,n,b}$ and $\Lambda^{i,n,a}$ satisfy the same assumptions as those satisfied by the intensity functions of Section 4.2.1.

For asset $i \in \{1, \dots, d\}$, the resulting inventory $(q_t^i)_{t \in \mathbb{R}_+}$ has dynamics

$$dq_t^i = \sum_{n=1}^N \int_{\mathbb{R}_+^*} z N^{i,n,b}(dt, dz) - \sum_{n=1}^N \int_{\mathbb{R}_+^*} z N^{i,n,a}(dt, dz),$$

where for each tier $n \in \{1, \dots, N\}$, $N^{i,n,b}(dt, dz)$ and $N^{i,n,a}(dt, dz)$ are the random measures associated with the processes $(N_t^{i,n,b})_{t \in \mathbb{R}_+}$ and $(N_t^{i,n,a})_{t \in \mathbb{R}_+}$, respectively.

Finally, we consider the addition of fixed transaction costs.¹² For that purpose, we introduce for each asset $i \in \{1, \dots, d\}$ and for each tier $n \in \{1, \dots, N\}$ two real numbers $c^{i,n,b}$ and $c^{i,n,a}$ modelling the fixed cost of a transaction in asset i with a client from tier n , at the bid and at the ask, respectively.

The resulting cash process $(X_t)_{t \in \mathbb{R}_+}$ has, consequently, the following dynamics:

$$dX_t = \sum_{i=1}^d \sum_{n=1}^N \int_{\mathbb{R}_+^*} \left[\left(\delta_t^{i,n,b}(z) z - c^{i,n,b} \right) N^{i,n,b}(dt, dz) + \left(\delta_t^{i,n,a}(z) z - c^{i,n,a} \right) N^{i,n,a}(dt, dz) \right] - \sum_{i=1}^d S_t^i dq_t^i.$$

¹⁰ ω is omitted in what follows.

¹¹This additional constraint of a fixed lower bound is just a technical one to be able to state theorems in the general case where request sizes are distributed (see [26]).

¹²Proportional transaction costs can be considered in the initial model through shifts in the intensity functions.

4.5.2 The Hamilton-Jacobi equation

In this new setting, one can again show that the two optimization problems introduced in Section 4.2.1 boil down to the resolution of a Hamilton-Jacobi equation of the form

$$\begin{aligned}
0 &= \partial_t \theta(t, q) + \mu^\top q - \frac{\gamma}{2} q^\top \Sigma q \\
&+ \sum_{i=1}^d \sum_{n=1}^N \int_{\mathbb{R}_+^*} \mathbb{1}_{\{q^i + z \leq Q^i\}} z H_\xi^{i,n,b} \left(z, \frac{\theta(t, q) - \theta(t, q + ze^i) + c^{i,n,b}}{z} \right) \nu^{i,n,b}(dz) \\
&+ \sum_{i=1}^d \sum_{n=1}^N \int_{\mathbb{R}_+^*} \mathbb{1}_{\{q^i - z \geq -Q^i\}} z H_\xi^{i,n,a} \left(z, \frac{\theta(t, q) - \theta(t, q - ze^i) + c^{i,n,a}}{z} \right) \nu^{i,n,a}(dz),
\end{aligned} \tag{4.21}$$

with terminal condition

$$\theta(t, q) = 0, \tag{4.22}$$

where $\xi = \gamma$ in the case of Model A and $\xi = 0$ in the case of Model B, and where the functions $H_\xi^{i,n,b}$ and $H_\xi^{i,n,a}$ are defined by

$$H_\xi^{i,n,b}(z, p) := \begin{cases} \sup_{\delta > -\delta_\infty} \frac{\Lambda^{i,n,b}(\delta)}{\xi z} (1 - \exp(-\xi z(\delta - p))) & \text{if } \xi > 0, \\ \sup_{\delta > -\delta_\infty} \Lambda^{i,n,b}(\delta)(\delta - p) & \text{if } \xi = 0 \end{cases}$$

and

$$H_\xi^{i,n,a}(z, p) := \begin{cases} \sup_{\delta > -\delta_\infty} \frac{\Lambda^{i,n,a}(\delta)}{\xi z} (1 - \exp(-\xi z(\delta - p))) & \text{if } \xi > 0, \\ \sup_{\delta > -\delta_\infty} \Lambda^{i,n,a}(\delta)(\delta - p) & \text{if } \xi = 0. \end{cases}$$

Remark 2. When $\xi = 0$, the dependence in z of the Hamiltonian functions vanishes. Nevertheless, we keep the variable z for the sake of consistency.

Following a method similar to that developed in [26], we can show that, for a given $\xi \geq 0$, there exists a unique bounded function $\theta : [0, T] \times \prod_{i=1}^d [-Q^i, Q^i] \rightarrow \mathbb{R}$, C^1 in time, solution of Eq. (4.21) with terminal condition (4.22).

Moreover, a classical verification argument enables to go from θ to optimal controls for both Model A and Model B. The optimal quotes as functions of θ are given by the two theorems that follow.

In the case of Model A, the result is the following:

Theorem 3. *Let us consider the solution θ of Eq. (4.21) with terminal condition (4.22), for $\xi = \gamma$.*

Then, for $i \in \{1, \dots, d\}$ and $n \in \{1, \dots, N\}$, the optimal bid and ask quotes as functions of the trade size z , $S_t^{i,n,b}(z) = S_t^i - \delta_t^{i,n,b}(z)$ and $S_t^{i,n,a}(z) = S_t^i + \delta_t^{i,n,a*}(z)$ in Model A are characterized*

by

$$\begin{aligned}\delta_t^{i,n,b*}(z) &= \tilde{\delta}_\gamma^{i,n,b*} \left(z, \frac{\theta(t, q_{t-}) - \theta(t, q_{t-} + ze^i) + c^{i,n,b}}{z} \right) \quad \text{for } q_{t-} + ze^i \in \prod_{j=1}^d [-Q^j, Q^j], \\ \delta_t^{i,n,a*}(z) &= \tilde{\delta}_\gamma^{i,n,a*} \left(z, \frac{\theta(t, q_{t-}) - \theta(t, q_{t-} - ze^i) + c^{i,n,a}}{z} \right) \quad \text{for } q_{t-} - ze^i \in \prod_{j=1}^d [-Q^j, Q^j],\end{aligned}$$

where the functions $\tilde{\delta}_\gamma^{i,n,b*}(\cdot, \cdot)$ and $\tilde{\delta}_\gamma^{i,n,a*}(\cdot, \cdot)$ are defined by

$$\begin{aligned}\tilde{\delta}_\gamma^{i,n,b*}(z, p) &= \Lambda^{i,n,b^{-1}} \left(\gamma z H_\gamma^{i,n,b}(z, p) - \partial_p H_\gamma^{i,n,b}(z, p) \right) \vee (-\delta_\infty), \\ \tilde{\delta}_\gamma^{i,n,a*}(z, p) &= \Lambda^{i,n,a^{-1}} \left(\gamma z H_\gamma^{i,n,a}(z, p) - \partial_p H_\gamma^{i,n,a}(z, p) \right) \vee (-\delta_\infty).\end{aligned}$$

For Model B, the result is the following:

Theorem 4. *Let us consider the solution θ of Eq. (4.21) with terminal condition (4.22), for $\xi = 0$.*

Then, for $i \in \{1, \dots, d\}$ and $n \in \{1, \dots, N\}$, the optimal bid and ask quotes as functions of the trade size z , $S_t^{i,n,b}(z) = S_t^i - \delta_t^{i,n,b}(z)$ and $S_t^{i,n,a}(z) = S_t^i + \delta_t^{i,n,a*}(z)$ in Model B are characterized by*

$$\begin{aligned}\delta_t^{i,n,b*}(z) &= \tilde{\delta}_0^{i,n,b*} \left(z, \frac{\theta(t, q_{t-}) - \theta(t, q_{t-} + ze^i) + c^{i,n,b}}{z} \right) \quad \text{for } q_{t-} + ze^i \in \prod_{j=1}^d [-Q^j, Q^j], \\ \delta_t^{i,n,a*}(z) &= \tilde{\delta}_0^{i,n,a*} \left(z, \frac{\theta(t, q_{t-}) - \theta(t, q_{t-} - ze^i) + c^{i,n,a}}{z} \right) \quad \text{for } q_{t-} - ze^i \in \prod_{j=1}^d [-Q^j, Q^j],\end{aligned}$$

where the functions $\tilde{\delta}_0^{i,n,b*}(\cdot, \cdot)$ and $\tilde{\delta}_0^{i,n,a*}(\cdot, \cdot)$ are defined by

$$\tilde{\delta}_0^{i,n,b*}(z, p) = \Lambda^{i,n,b^{-1}} \left(-\partial_p H_0^{i,n,b}(z, p) \right) \vee (-\delta_\infty) \quad \text{and} \quad \tilde{\delta}_0^{i,n,a*}(z, p) = \Lambda^{i,n,a^{-1}} \left(-\partial_p H_0^{i,n,a}(z, p) \right) \vee (-\delta_\infty).$$

4.5.3 Quadratic approximation

As before, let us replace for all $i \in \{1, \dots, d\}$ and $n \in \{1, \dots, N\}$, the Hamiltonian functions $H_\xi^{i,n,b}$ and $H_\xi^{i,n,a}$ by the functions

$$\check{H}^{i,n,b} : (z, p) \mapsto \alpha_0^{i,n,b}(z) + \alpha_1^{i,n,b}(z)p + \frac{1}{2}\alpha_2^{i,n,b}(z)p^2 \quad \text{and} \quad \check{H}^{i,n,a} : (z, p) \mapsto \alpha_0^{i,n,a}(z) + \alpha_1^{i,n,a}(z)p + \frac{1}{2}\alpha_2^{i,n,a}(z)p^2.$$

Remark 3. *Here, $\alpha_j^{i,n,b}$ and $\alpha_j^{i,n,a}$ (for $j \in \{0, 1, 2\}$) are functions of z . A natural choice for the functions $(\check{H}^{i,n,b})_{(i,n) \in \{1, \dots, d\} \times \{1, \dots, N\}}$ and $(\check{H}^{i,n,a})_{(i,n) \in \{1, \dots, d\} \times \{1, \dots, N\}}$ derives from Taylor expansions around $p = 0$. In that case,*

$$\forall i \in \{1, \dots, d\}, \forall n \in \{1, \dots, N\}, \forall j \in \{0, 1, 2\}, \quad \alpha_j^{i,n,b}(z) = \partial_p^j H_\xi^{i,n,b}(z, 0) \quad \text{and} \quad \alpha_j^{i,n,a} = \partial_p^j H_\xi^{i,n,a}(z, 0).$$

If we consider the limit case where $Q^i = +\infty$ for all $i \in \{1, \dots, d\}$, Eq. (4.21) then becomes

$$\begin{aligned}
0 = & \partial_t \check{\theta}(t, q) + \mu^\top q - \frac{\gamma}{2} q^\top \Sigma q + \sum_{i=1}^d \sum_{n=1}^N \left(\int_{\mathbb{R}_+^*} z \alpha_0^{i,n,b}(z) \nu^{i,n,b}(dz) + \int_{\mathbb{R}_+^*} z \alpha_0^{i,n,a}(z) \nu^{i,n,a}(dz) \right) \\
& + \sum_{i=1}^d \sum_{n=1}^N \left(\int_{\mathbb{R}_+^*} \alpha_1^{i,n,b}(z) (\check{\theta}(t, q) - \check{\theta}(t, q + ze^i) + c^{i,n,b}) \nu^{i,n,b}(dz) \right. \\
& \quad \left. + \int_{\mathbb{R}_+^*} \alpha_1^{i,n,a}(z) (\check{\theta}(t, q) - \check{\theta}(t, q - ze^i) + c^{i,n,a}) \nu^{i,n,a}(dz) \right) \\
& + \frac{1}{2} \sum_{i=1}^d \sum_{n=1}^N \left(\int_{\mathbb{R}_+^*} \frac{1}{z} \alpha_2^{i,n,b}(z) (\check{\theta}(t, q) - \check{\theta}(t, q + ze^i) + c^{i,n,b})^2 \nu^{i,n,b}(dz) \right. \\
& \quad \left. + \int_{\mathbb{R}_+^*} \frac{1}{z} \alpha_2^{i,n,a}(z) (\check{\theta}(t, q) - \check{\theta}(t, q - ze^i) + c^{i,n,a})^2 \nu^{i,n,a}(dz) \right), \quad (4.23)
\end{aligned}$$

with terminal condition

$$\check{\theta}(T, q) = 0. \quad (4.24)$$

Using the same ansatz as in Section 4.3, we obtain the following result (we omit the proof as it follows the same logic as for that of Proposition 1):

Proposition 4. *Let us introduce for all $i \in \{1, \dots, d\}, n \in \{1, \dots, N\}, j \in \{0, 1, 2\}, k \in \mathbb{N}$, the following constants:*

$$\begin{aligned}
\Delta_{j,k}^{i,n,b} &= \int_{\mathbb{R}_+^*} z^k \alpha_j^{i,n,b}(z) \nu^{i,n,b}(dz) \quad \text{and} \quad \Delta_{j,k}^{i,n,a} = \int_{\mathbb{R}_+^*} z^k \alpha_j^{i,n,a}(z) \nu^{i,n,a}(dz), \\
V_{j,k}^b &= \left(\sum_{n=1}^N \Delta_{j,k}^{1,n,b}, \dots, \sum_{n=1}^N \Delta_{j,k}^{d,n,b} \right)^\top \quad \text{and} \quad V_{j,k}^a = \left(\sum_{n=1}^N \Delta_{j,k}^{1,n,a}, \dots, \sum_{n=1}^N \Delta_{j,k}^{d,n,a} \right)^\top, \\
\tilde{V}_{j,k}^b &= \left(\sum_{n=1}^N c^{1,n,b} \Delta_{j,k}^{1,n,b}, \dots, \sum_{n=1}^N c^{d,n,b} \Delta_{j,k}^{d,n,b} \right)^\top \quad \text{and} \quad \tilde{V}_{j,k}^a = \left(\sum_{n=1}^N c^{1,n,a} \Delta_{j,k}^{1,n,a}, \dots, \sum_{n=1}^N c^{d,n,a} \Delta_{j,k}^{d,n,a} \right)^\top, \\
\chi_{j,k}^b &= \sum_{i=1}^d \sum_{n=1}^N \Delta_{j,k}^{i,n,b} \quad \text{and} \quad \chi_{j,k}^a = \sum_{i=1}^d \sum_{n=1}^N \Delta_{j,k}^{i,n,a}, \\
\tilde{\chi}_{j,k}^b &= \sum_{i=1}^d \sum_{n=1}^N c^{i,n,b} \Delta_{j,k}^{i,n,b} \quad \text{and} \quad \tilde{\chi}_{j,k}^a = \sum_{i=1}^d \sum_{n=1}^N c^{i,n,a} \Delta_{j,k}^{i,n,a}, \\
\hat{\chi}_{j,k}^b &= \sum_{i=1}^d \sum_{n=1}^N (c^{i,n,b})^2 \Delta_{j,k}^{i,n,b} \quad \text{and} \quad \hat{\chi}_{j,k}^a = \sum_{i=1}^d \sum_{n=1}^N (c^{i,n,a})^2 \Delta_{j,k}^{i,n,a},
\end{aligned}$$

and

$$D_{j,k}^b = \text{diag} \left(\sum_{n=1}^N \Delta_{j,k}^{1,n,b}, \dots, \sum_{n=1}^N \Delta_{j,k}^{d,n,b} \right) \quad \text{and} \quad D_{j,k}^a = \text{diag} \left(\sum_{n=1}^N \Delta_{j,k}^{1,n,a}, \dots, \sum_{n=1}^N \Delta_{j,k}^{d,n,a} \right).$$

Let us consider three differentiable functions $A : [0, T] \rightarrow S_d^{++}$, $B : [0, T] \rightarrow \mathbb{R}^d$, and $C : [0, T] \rightarrow \mathbb{R}$ solutions of the system of ordinary differential equations

$$\begin{cases} A'(t) = 2A(t) (D_{2,1}^b + D_{2,1}^a) A(t) - \frac{1}{2}\gamma\Sigma, \\ B'(t) = \mu + 2A(t) (V_{1,1}^b - V_{1,1}^a) + 2A(t) (D_{2,2}^b - D_{2,2}^a) \mathcal{D}(A(t)) \\ \quad + 2A(t) (D_{2,1}^b + D_{2,1}^a) B(t) + 2A(t) (\tilde{V}_{2,0}^b - \tilde{V}_{2,0}^a), \\ C'(t) = \text{Tr}(D_{0,1}^b + D_{0,1}^a) + \text{Tr}((D_{1,2}^b + D_{1,2}^a) A(t)) + (V_{1,1}^b - V_{1,1}^a)^\top B(t) \\ \quad + (\tilde{\chi}_{1,0}^b + \tilde{\chi}_{1,0}^a) + \frac{1}{2}\mathcal{D}(A(t))^\top (D_{2,3}^b + D_{2,3}^a) \mathcal{D}(A(t)) \\ \quad + B(t)^\top (D_{2,2}^b - D_{2,2}^a) \mathcal{D}(A(t)) + (\tilde{V}_{2,1}^b + \tilde{V}_{2,1}^a)^\top \mathcal{D}(A(t)) \\ \quad + \frac{1}{2}B(t)^\top (D_{2,1}^b + D_{2,1}^a) B(t) + (\tilde{V}_{2,0}^b - \tilde{V}_{2,0}^a)^\top B(t) \\ \quad + \frac{1}{2}(\hat{\chi}_{2,0}^b + \hat{\chi}_{2,0}^a). \end{cases} \quad (4.25)$$

with terminal conditions

$$A(T) = 0, B(T) = 0, \text{ and } C(T) = 0. \quad (4.26)$$

Then $\check{\theta} : (t, q) \in [0, T] \times \mathbb{R}^d \mapsto -q^\top A(t)q - q^\top B(t) - C(t)$ is solution of Eq. (4.23) with terminal condition (4.24).

We can now solve (4.25) with terminal conditions (4.26) in closed form. This is the purpose of the following proposition whose proof is omitted (see Proposition 2 for a similar proof).

Proposition 5. Assume $\sum_{n=1}^N \Delta_{2,1}^{i,n,b} + \Delta_{2,1}^{i,n,a} > 0$ for all $i \in \{1, \dots, d\}$. The system of ODEs (4.25) with terminal conditions (4.26) admits the unique solution

$$\begin{aligned} A(t) &= \frac{1}{2}D_+^{-\frac{1}{2}}\hat{A}\left(e^{\hat{A}(T-t)} - e^{-\hat{A}(T-t)}\right)\left(e^{\hat{A}(T-t)} + e^{-\hat{A}(T-t)}\right)^{-1}D_+^{-\frac{1}{2}}, \\ B(t) &= -2e^{-2\int_t^T A(u)D_+ du} \int_t^T e^{2\int_s^T A(u)D_+ du} \left(\frac{1}{2}\mu + A(s)\left(V_- + \tilde{V}_- + D_- \mathcal{D}(A(s))\right)\right) ds, \\ C(t) &= -\text{Tr}(D_{0,1}^b + D_{0,1}^a)(T-t) - \text{Tr}\left((D_{1,2}^b + D_{1,2}^a) \int_t^T A(s)ds\right) - V_-^\top \int_t^T B(s)ds \\ &\quad - \frac{1}{2} \int_t^T \mathcal{D}(A(s))^\top (D_{2,3}^b + D_{2,3}^a) \mathcal{D}(A(s))ds - \frac{1}{2} \int_t^T B(s)^\top D_+ B(s)ds \\ &\quad - \int_t^T B(s)^\top D_- \mathcal{D}(A(s))ds - (\tilde{\chi}_{1,0}^b + \tilde{\chi}_{1,0}^a)(T-t) - \frac{1}{2}(\hat{\chi}_{2,0}^b + \hat{\chi}_{2,0}^a)(T-t) \\ &\quad - (\tilde{V}_{2,1}^b + \tilde{V}_{2,1}^a)^\top \int_t^T \mathcal{D}(A(s))ds, \end{aligned}$$

where

$$D_+ = D_{2,1}^b + D_{2,1}^a, \quad D_- = D_{2,2}^b - D_{2,2}^a, \quad V_- = V_{1,1}^b - V_{1,1}^a, \quad \tilde{V}_- = \tilde{V}_{2,0}^b - \tilde{V}_{2,0}^a, \quad \text{and } \hat{A} = \sqrt{\gamma}\left(D_+^{\frac{1}{2}}\Sigma D_+^{\frac{1}{2}}\right)^{\frac{1}{2}}.$$

Now, using the same method as in Section 4.3, we get the following asymptotic results:

Proposition 6. Let (A, B, C) be the solution of the system of ODEs (4.25) with terminal conditions (4.26).

If $D_+^{\frac{1}{2}}\mu \in \text{Im}(\widehat{A})$, then,

$$\begin{aligned}
A(0) &\xrightarrow{T \rightarrow +\infty} \frac{1}{2}\sqrt{\gamma}\Gamma, \\
B(0) &\xrightarrow{T \rightarrow +\infty} -D_+^{-\frac{1}{2}}\widehat{A}^+D_+^{\frac{1}{2}}\mu - D_+^{-\frac{1}{2}}\widehat{A}\widehat{A}^+D_+^{-\frac{1}{2}}\left(V_- + \tilde{V}_- + \frac{1}{2}\sqrt{\gamma}D_- \mathcal{D}(\Gamma)\right), \\
\frac{C(0)}{T} &\xrightarrow{T \rightarrow +\infty} -\text{Tr}(D_{0,1}^b + D_{0,1}^a) - \frac{1}{2}\sqrt{\gamma}\text{Tr}((D_{1,2}^b + D_{1,2}^a)\Gamma) + V_-^\top D_+^{-\frac{1}{2}}\widehat{A}^+D_+^{\frac{1}{2}}\mu \\
&\quad + V_-^\top D_+^{-\frac{1}{2}}\widehat{A}\widehat{A}^+D_+^{-\frac{1}{2}}\left(V_- + \tilde{V}_- + \frac{1}{2}\sqrt{\gamma}D_- \mathcal{D}(\Gamma)\right) - \frac{1}{8}\gamma \mathcal{D}(\Gamma)^\top (D_{2,3}^b + D_{2,3}^a) \mathcal{D}(\Gamma) \\
&\quad - \frac{1}{2}\mu^\top D_+^{\frac{1}{2}}\widehat{A}^+\widehat{A}^+D_+^{\frac{1}{2}}\mu - \mu^\top D_+^{\frac{1}{2}}\widehat{A}^+D_+^{-\frac{1}{2}}\left(V_- + \tilde{V}_- + \frac{1}{2}\sqrt{\gamma}D_- \mathcal{D}(\Gamma)\right) \\
&\quad - \frac{1}{2}\left(V_- + \tilde{V}_- + \frac{1}{2}\sqrt{\gamma}D_- \mathcal{D}(\Gamma)\right)^\top D_+^{-\frac{1}{2}}\widehat{A}\widehat{A}^+D_+^{-\frac{1}{2}}\left(V_- + \tilde{V}_- + \frac{1}{2}\sqrt{\gamma}D_- \mathcal{D}(\Gamma)\right) \\
&\quad + \frac{1}{2}\sqrt{\gamma}\mu^\top D_+^{\frac{1}{2}}\widehat{A}^+D_+^{-\frac{1}{2}}D_- \mathcal{D}(\Gamma) + \frac{1}{2}\sqrt{\gamma}\left(V_- + \tilde{V}_- + \frac{1}{2}\sqrt{\gamma}D_- \mathcal{D}(\Gamma)\right)^\top D_+^{-\frac{1}{2}}\widehat{A}\widehat{A}^+D_+^{-\frac{1}{2}}D_- \mathcal{D}(\Gamma) \\
&\quad - (\hat{\chi}_{2,0}^b + \hat{\chi}_{2,0}^a) - \frac{1}{2}(\hat{\chi}_{2,0}^b + \hat{\chi}_{2,0}^a) - \frac{1}{2}\sqrt{\gamma}\left(\tilde{V}_{2,1}^b + \tilde{V}_{2,1}^a\right)^\top \mathcal{D}(\Gamma),
\end{aligned}$$

where $\Gamma = D_+^{-\frac{1}{2}}\left(D_+^{\frac{1}{2}}\Sigma D_+^{\frac{1}{2}}\right)^{\frac{1}{2}}D_+^{-\frac{1}{2}}$ and \widehat{A}^+ is the Moore-Penrose generalized inverse of \widehat{A} .

Remark 4. The assumption $D_+^{\frac{1}{2}}\mu \in \text{Im}(\widehat{A})$ is satisfied when $\mu = 0$ or when Σ is invertible. If this assumption is not satisfied, then it can be shown that $\frac{B(0)}{T} \xrightarrow{T \rightarrow +\infty} -D_+^{-\frac{1}{2}}\widehat{A}^+\widehat{A}D_+^{\frac{1}{2}}\mu$. In particular, there is no constant asymptotic approximation of the quotes. In fact, if the assumption $D_+^{\frac{1}{2}}\mu \in \text{Im}(\widehat{A})$ is not satisfied, the market maker may have an incentive to propose very good quotes to clients in order to build portfolios bearing positive return at no risk.

4.5.4 From value functions to heuristics and quotes

Quotes: the general case

The greedy quoting strategy associated with our closed-form proxy of the value function leads to the following quotes for all $i \in \{1, \dots, d\}$ and $n \in \{1, \dots, N\}$:

$$\begin{aligned}
\check{\delta}_t^{i,n,b}(z) &= \tilde{\delta}_\xi^{i,n,b*} \left(z, \frac{\check{\theta}(t, q_{t-}) - \check{\theta}(t, q_{t-} + ze^i) + c^{i,n,b}}{z} \right) \\
&= \tilde{\delta}_\xi^{i,n,b*} \left(z, 2q_{t-}^\top A(t)e^i + ze^{i^\top} A(t)e^i + e^{i^\top} B(t) + \frac{c^{i,n,b}}{z} \right), \\
\check{\delta}_t^{i,n,a}(z) &= \tilde{\delta}_\xi^{i,n,a*} \left(z, \frac{\check{\theta}(t, q_{t-}) - \check{\theta}(t, q_{t-} - ze^i) + c^{i,n,a}}{z} \right) \\
&= \tilde{\delta}_\xi^{i,n,a*} \left(z, -2q_{t-}^\top A(t)e^i + ze^{i^\top} A(t)e^i - e^{i^\top} B(t) + \frac{c^{i,n,a}}{z} \right),
\end{aligned}$$

where $\tilde{\delta}_\xi^{i,n,b*}$ and $\tilde{\delta}_\xi^{i,n,a*}$ are given in Theorems 3 and 4 for $\xi = \gamma$ and $\xi = 0$ respectively (depending on whether one considers Model A or Model B).

The asymptotic regime exhibited in the above paragraphs can then serve to obtain the following simple closed-form approximations:

$$\begin{aligned}
\check{\delta}_t^{i,n,b}(z) &= \tilde{\delta}_\xi^{i,n,b*} \left(z, \sqrt{\gamma} q_{t-}^\top \Gamma e^i + \frac{1}{2} \sqrt{\gamma} z e^{i^\top} \Gamma e^i - e^{i^\top} D_+^{-\frac{1}{2}} \hat{A}^+ D_+^{\frac{1}{2}} \mu \right. \\
&\quad \left. - e^{i^\top} D_+^{-\frac{1}{2}} \hat{A} \hat{A}^+ D_+^{-\frac{1}{2}} \left(V_- + \tilde{V}_- + \frac{1}{2} \sqrt{\gamma} D_- \mathcal{D}(\Gamma) \right) + \frac{c^{i,n,b}}{z} \right), \tag{4.27}
\end{aligned}$$

$$\begin{aligned}
\check{\delta}_t^{i,n,a}(z) &= \tilde{\delta}_\xi^{i,n,a*} \left(z, -\sqrt{\gamma} q_{t-}^\top \Gamma e^i + \frac{1}{2} \sqrt{\gamma} z e^{i^\top} \Gamma e^i + e^{i^\top} D_+^{-\frac{1}{2}} \hat{A}^+ D_+^{\frac{1}{2}} \mu \right. \\
&\quad \left. + e^{i^\top} D_+^{-\frac{1}{2}} \hat{A} \hat{A}^+ D_+^{-\frac{1}{2}} \left(V_- + \tilde{V}_- + \frac{1}{2} \sqrt{\gamma} D_- \mathcal{D}(\Gamma) \right) + \frac{c^{i,n,a}}{z} \right). \tag{4.28}
\end{aligned}$$

If we assume that for all $i \in \{1, \dots, d\}$ and for all $n \in \{1, \dots, N\}$ we have $\nu^{i,n,b} = \nu^{i,n,a}$ and $\Lambda^{i,n,b} = \Lambda^{i,n,a}$, then $\forall i \in \{1, \dots, d\}, \forall n \in \{1, \dots, N\}, H^{i,n,b} = H^{i,n,a}$, and it is thus natural to chose symmetric approximations of the Hamiltonian functions, i.e. $\forall i \in \{1, \dots, d\}, \forall n \in \{1, \dots, N\}, \check{H}^{i,n,b} = \check{H}^{i,n,a}$. In that case, (4.27) and (4.28) simplify into

$$\check{\delta}_t^{i,n,b}(z) = \tilde{\delta}_\xi^{i,n,b*} \left(z, \sqrt{\gamma} q_{t-}^\top \Gamma e^i + \frac{1}{2} \sqrt{\gamma} z e^{i^\top} \Gamma e^i - e^{i^\top} D_+^{-\frac{1}{2}} \hat{A}^+ D_+^{\frac{1}{2}} \mu + \frac{c^{i,n,b}}{z} \right), \tag{4.29}$$

$$\check{\delta}_t^{i,n,a}(z) = \tilde{\delta}_\xi^{i,n,a*} \left(z, -\sqrt{\gamma} q_{t-}^\top \Gamma e^i + \frac{1}{2} \sqrt{\gamma} z e^{i^\top} \Gamma e^i + e^{i^\top} D_+^{-\frac{1}{2}} \hat{A}^+ D_+^{\frac{1}{2}} \mu + \frac{c^{i,n,a}}{z} \right). \tag{4.30}$$

All these approximations of the quotes can be used directly or as a starting point in iterative methods designed to compute the optimal quotes (policy iteration algorithms, actor-critic algorithms, etc.).

Quotes: the case of symmetric exponential intensities

If we assume that for all $i \in \{1, \dots, d\}$ and for all $n \in \{1, \dots, N\}$ we have $\nu^{i,n,b} = \nu^{i,n,a} =: \nu^{i,n}$ and intensity functions given by

$$\Lambda^{i,n,b}(\delta) = \Lambda^{i,n,a}(\delta) = A^{i,n} e^{-k^{i,n} \delta}, \quad A^{i,n}, k^{i,n} > 0,$$

then (see [62]), in the limit case where $\delta_\infty = +\infty$ the Hamiltonian functions are given, for all $i \in \{1, \dots, d\}$ and $n \in \{1, \dots, N\}$, by

$$H_\xi^{i,n,b}(z,p) = H_\xi^{i,n,a}(z,p) = \frac{A^{i,n}}{k^{i,n}} C_\xi^{i,n}(z) \exp(-k^{i,n}p),$$

where

$$C_\xi^{i,n}(z) = \begin{cases} \left(1 + \frac{\xi z}{k^{i,n}}\right)^{-\left(1 + \frac{k^{i,n}}{\xi z}\right)} & \text{if } \xi > 0 \\ e^{-1} & \text{if } \xi = 0, \end{cases}$$

and the functions $\tilde{\delta}_\xi^{i,n,b*}$ and $\tilde{\delta}_\xi^{i,n,a*}$ are given, for all $i \in \{1, \dots, d\}$ and $n \in \{1, \dots, N\}$, by

$$\tilde{\delta}_\xi^{i,n,b*}(z,p) = \tilde{\delta}_\xi^{i,n,a*}(z,p) = \begin{cases} p + \frac{1}{\xi z} \log\left(1 + \frac{\xi z}{k^{i,n}}\right) & \text{if } \xi > 0 \\ p + \frac{1}{k^{i,n}} & \text{if } \xi = 0. \end{cases}$$

Therefore, if we consider the quadratic approximation of the Hamiltonian functions based on their Taylor expansion around $p = 0$ (see Remark 1), then (4.29) and (4.30) become

$$\begin{aligned} \check{\delta}_t^{i,n,b}(z) &= \begin{cases} \sqrt{\gamma} \left(q_t^\top \Gamma e^i + \frac{1}{2} z^i e^{i^\top} \Gamma e^i - \frac{1}{\gamma} e^{i^\top} D_+^{-\frac{1}{2}} \widehat{A}^+ D_+^{\frac{1}{2}} \mu \right) + \frac{c^{i,n,b}}{z} + \frac{1}{\gamma z} \log\left(1 + \frac{\gamma z}{k^{i,n}}\right) & \text{in Model A,} \\ \sqrt{\gamma} \left(q_t^\top \Gamma e^i + \frac{1}{2} z^i e^{i^\top} \Gamma e^i - \frac{1}{\gamma} e^{i^\top} D_+^{-\frac{1}{2}} \widehat{A}^+ D_+^{\frac{1}{2}} \mu \right) + \frac{c^{i,n,b}}{z} + \frac{1}{k^{i,n}} & \text{in Model B.} \end{cases} \\ \check{\delta}_t^{i,n,a}(z) &= \begin{cases} \sqrt{\gamma} \left(q_t^\top \Gamma e^i - \frac{1}{2} z^i e^{i^\top} \Gamma e^i + \frac{1}{\gamma} e^{i^\top} D_+^{-\frac{1}{2}} \widehat{A}^+ D_+^{\frac{1}{2}} \mu \right) + \frac{c^{i,n,b}}{z} + \frac{1}{\gamma z} \log\left(1 + \frac{\gamma z}{k^{i,n}}\right) & \text{in Model A,} \\ \sqrt{\gamma} \left(q_t^\top \Gamma e^i - \frac{1}{2} z^i e^{i^\top} \Gamma e^i + \frac{1}{\gamma} e^{i^\top} D_+^{-\frac{1}{2}} \widehat{A}^+ D_+^{\frac{1}{2}} \mu \right) + \frac{c^{i,n,b}}{z} + \frac{1}{k^{i,n}} & \text{in Model B.} \end{cases} \end{aligned}$$

where $\Gamma = D_+^{-\frac{1}{2}} (D_+^{\frac{1}{2}} \Sigma D_+^{\frac{1}{2}})^{\frac{1}{2}} D_+^{-\frac{1}{2}}$ and

$$D_+ = \text{diag} \left(2 \sum_{n=1}^N A^{1,n} k^{1,n} \int_{\mathbb{R}_+^*} C_\xi^{1,n}(z) z \nu^{1,n}(dz), \dots, 2 \sum_{n=1}^N A^{d,n} k^{d,n} \int_{\mathbb{R}_+^*} C_\xi^{d,n}(z) z \nu^{d,n}(dz) \right).$$

4.6 Numerical results

To evaluate the performance of our approximations, we test them in the two assets case, for which we can compute the true optimal quotes in Model B by using a Euler scheme to solve the HJB equation.

We consider the following characteristics for the two assets:

- Asset prices: $S_0^1 = S_0^2 = 100 \text{ €}$.
- Drifts: $\mu^1 = 0.1 \text{ €} \cdot \text{day}^{-1}$, $\mu^2 = -0.1 \text{ €} \cdot \text{day}^{-1}$.
- Volatilities: $\sigma^1 = 1.2 \text{ €} \cdot \text{day}^{-\frac{1}{2}}$, $\sigma^2 = 0.6 \text{ €} \cdot \text{day}^{-\frac{1}{2}}$.
- Correlation: $\rho = 0.5$.

This corresponds to a covariance matrix Σ given by

$$\Sigma = \begin{pmatrix} (\sigma^1)^2 & \rho\sigma^1\sigma^2 \\ \rho\sigma^1\sigma^2 & (\sigma^2)^2 \end{pmatrix} = \begin{pmatrix} 1.44 & 0.36 \\ 0.36 & 0.36 \end{pmatrix}.$$

The intensity functions are given for all $i \in \{1, 2\}$ by:

$$\Lambda^{i,b}(\delta) = \Lambda^{i,a}(\delta) = \lambda_{RFQ} \frac{1}{1 + e^{\alpha_\Lambda + \beta_\Lambda \delta}},$$

with $\lambda_{RFQ} = 30 \text{ day}^{-1}$, $\alpha_\Lambda = 0.7$, and $\beta_\Lambda = 30 \text{ €}^{-1}$. This corresponds to 30 requests per day, a probability of $\frac{1}{1+e^{0.7}} \simeq 33\%$ to trade when the answered quote is the reference price and a probability of $\frac{1}{1+e^{0.4}} \simeq 40\%$ to trade when the answered quote is the reference price improved by 1 cent.

Request sizes are distributed according to a Gamma distribution $\Gamma(\alpha_\mu, \beta_\mu)$ with $\alpha_\mu = 4$ and $\beta_\mu = 4 \cdot 10^{-4}$. This corresponds to an average request size of 10000 assets (i.e. approximately 1000000 €) and a standard deviation equal to half the average.

We consider only one tier and no fixed transaction costs, and take the following parameters for the optimization problem:

- Time horizon given by $T = 7$ days. This horizon ensures convergence towards stationary quotes at time $t = 0$.
- Risk aversion parameter: $\gamma = 8 \cdot 10^{-6} \text{ €}^{-1}$.

4.6.1 Value function and optimal quotes

We first approximate the solution θ to (4.21) using a monotone implicit Euler scheme on a grid with 25×97 points for the inventory and a discretization of the request size distribution with 4 sizes: $z^1 = 6250$, $z^2 = 12500$, $z^3 = 18750$, and $z^4 = 25000$ assets – thereafter ordered by very small, small, large, and very large size – with respective probability $p^1 = 0.534$, $p^2 = 0.350$, $p^3 = 0.097$ and $p^4 = 0.019$. We impose risk limits $Q^1 = 75000$ and $Q^2 = 300000$, i.e. no trade that would result in an inventory q^1 for asset 1 such that $|q^1| > 75000$ is admitted, and similarly no trade that would result in an inventory q^2 for asset 2 such that $|q^2| > 300000$ is admitted.

The value function (at time $t = 0$) is plotted in Figure 4.1.

As discussed above, we choose $T = 7$ days to ensure convergence of the optimal quotes to their stationary values. This is illustrated in Figure 4.2.

We plot in Figures 4.3 and 4.4 the optimal bid quotes of asset 1 and asset 2 respectively, for different values of the inventory.

To observe the impact of the request size on the optimal quotes, we plot in Figures 4.5, 4.6, 4.7, and 4.8 respectively the functions

$$q^1 \mapsto \bar{\delta}^{1,b}(0, q^1, 0, z^k), \quad k \in \{1, \dots, 4\},$$

$$q^2 \mapsto \bar{\delta}^{1,b}(0, 0, q^2, z^k), \quad k \in \{1, \dots, 4\},$$

$$q^1 \mapsto \bar{\delta}^{2,b}(0, q^1, 0, z^k), \quad k \in \{1, \dots, 4\},$$

and

$$q^2 \mapsto \bar{\delta}^{2,b}(0, 0, q^2, z^k), \quad k \in \{1, \dots, 4\},$$

where $\bar{\delta}^{i,b}$ represents the optimal bid quote for asset i as a function of time, inventory, and size of request. We see that accounting for the size of requests impacts the optimal bid quotes, especially for a large inventory. The monotonicity of the quotes is of course unsurprising.

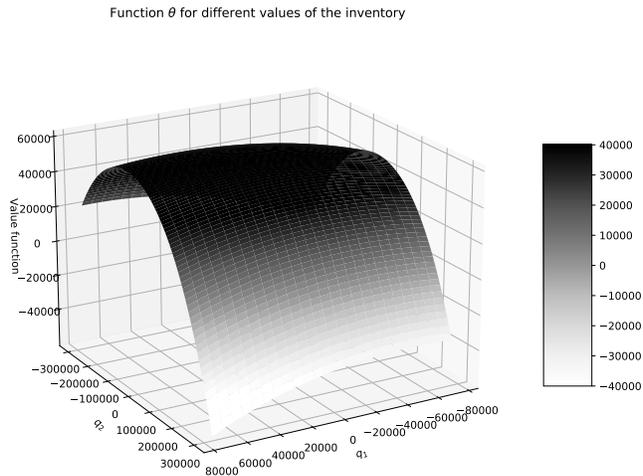


Figure 4.1: Function θ at time $t = 0$ for different values of the inventory.

Study of the convergence towards stationary quotes

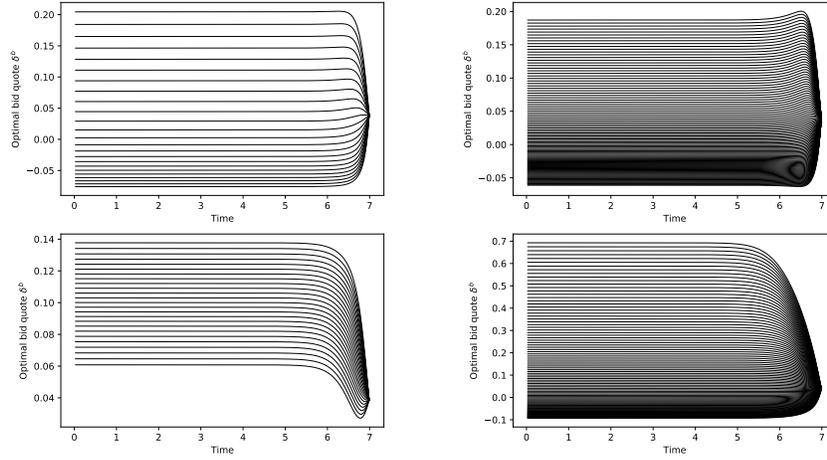


Figure 4.2: Optimal bid quotes as a function of time for different values of the inventory (very small trades) – top left: quotes of asset 1 for different values of inventory q^2 ($q^1 = 0$), top right: quotes of asset 1 for different values of inventory q^1 ($q^2 = 0$), bottom left: quotes of asset 2 for different values of inventory q^2 ($q^1 = 0$), bottom right: quotes of asset 2 for different values of inventory q^1 ($q^2 = 0$).

Optimal bid quote for asset 1 for different values of the inventory (very small trades)

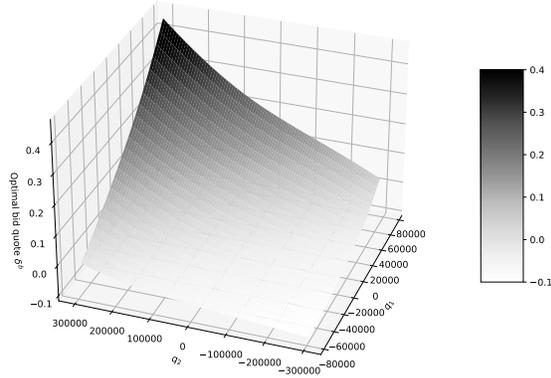


Figure 4.3: Optimal bid quote at $t = 0$ for asset 1 as a function of the inventory (very small trades).

Optimal bid quote for asset 2 for different values of the inventory (very small trades)

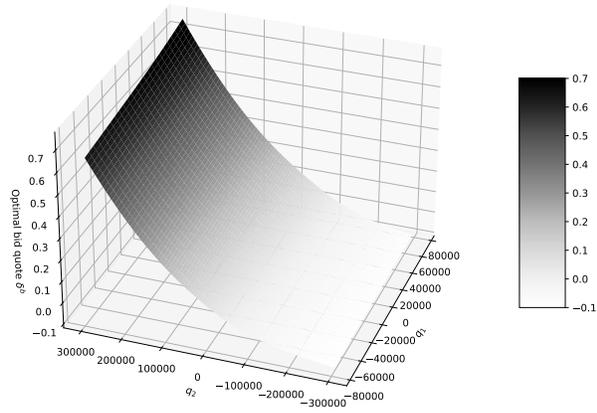


Figure 4.4: Optimal bid quote at $t = 0$ for asset 2 as a function of the inventory (very small trades).

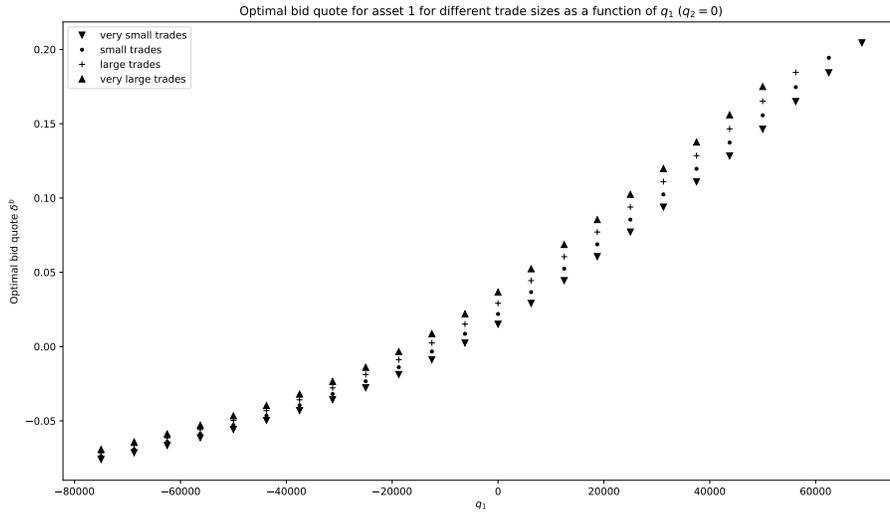


Figure 4.5: Optimal bid quote at $t = 0$ for asset 1 for different trade sizes as a function of q^1 ($q^2 = 0$).

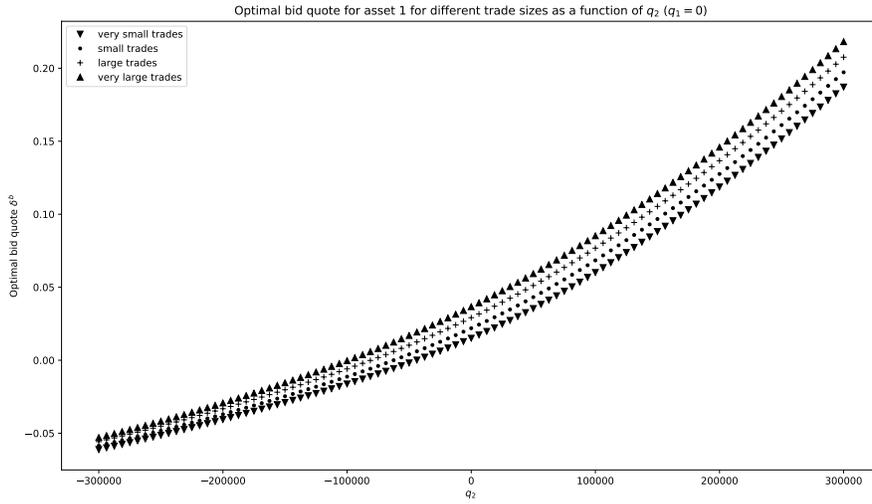


Figure 4.6: Optimal bid quote at $t = 0$ for asset 1 for different trade sizes as a function of q^2 ($q^1 = 0$).

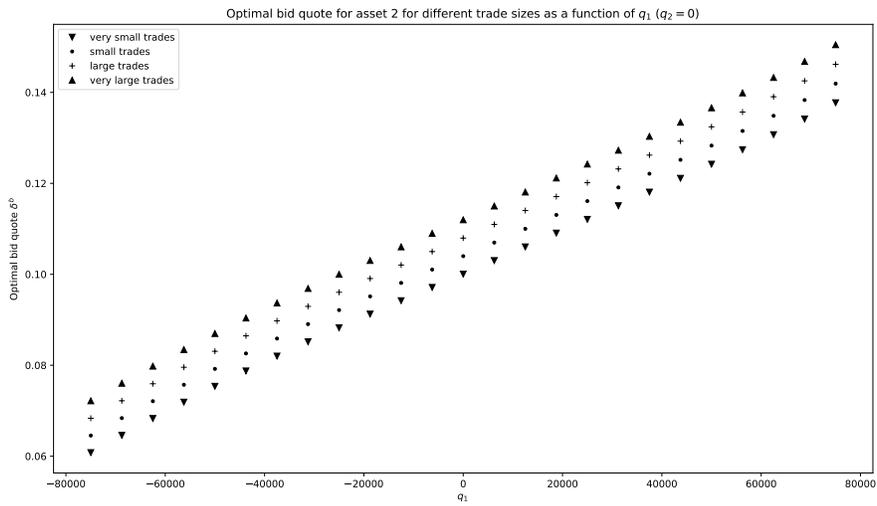


Figure 4.7: Optimal bid quote at $t = 0$ for asset 2 for different trade sizes as a function of q^1 ($q^2 = 0$).

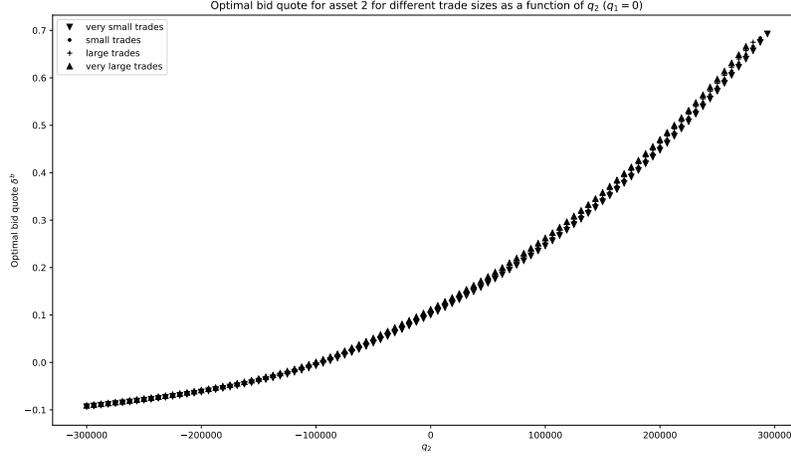


Figure 4.8: Optimal bid quote at $t = 0$ for asset 2 for different trade sizes as a function of q^2 ($q^1 = 0$).

4.6.2 Closed-form approximations

We now move on to the study of the results given by our closed-form approximations. We first plot in Figure 4.9 the approximate function θ given by Proposition 4. As a first remark, we can see that although there seems to be quite a difference in terms of level between the true value function and the approximate one, the general shape is the same, and thus the finite differences should be similar, which is what matters to compute quotes. This is confirmed in the next subsection, when we compare the strategies.

As before, we plot in Figures 4.10 and 4.11 the optimal bid quotes of asset 1 and asset 2 respectively, for different values of the inventory.

We also plot in Figures 4.12, 4.13, 4.14, and 4.15 respectively the functions

$$q^1 \mapsto \hat{\delta}^{1,b}(q^1, 0, z^k), \quad k \in \{1, \dots, 4\},$$

$$q^2 \mapsto \hat{\delta}^{1,b}(0, q^2, z^k), \quad k \in \{1, \dots, 4\},$$

$$q^1 \mapsto \hat{\delta}^{2,b}(q^1, 0, z^k), \quad k \in \{1, \dots, 4\},$$

and

$$q^2 \mapsto \hat{\delta}^{2,b}(0, q^2, z^k), \quad k \in \{1, \dots, 4\},$$

where $\hat{\delta}^{i,b}$ represents the closed-form approximation of the optimal bid quote for asset i as a function of inventory and size of request, and we compare those quotes with the optimal bid quotes given by the resolution of Eq. 4.21.

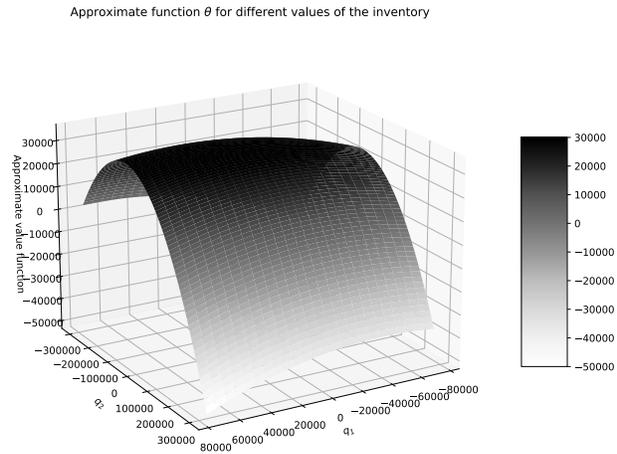


Figure 4.9: Approximate function θ at $t = 0$ of the problem for different values of the inventory.

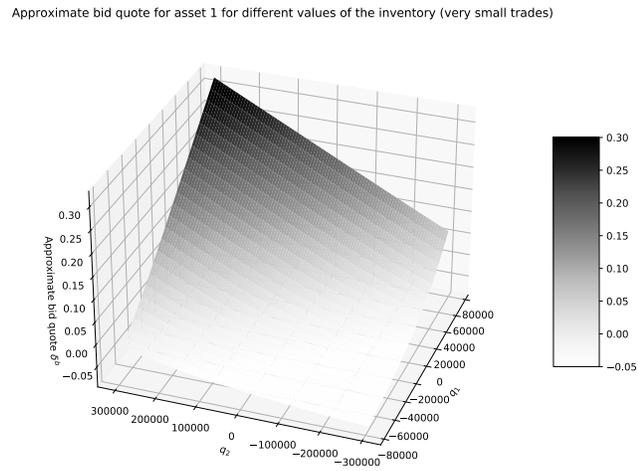


Figure 4.10: Approximate optimal bid quote at $t = 0$ for asset 1 as a function of the inventory (very small trades).

Approximate bid quote for asset 2 for different values of the inventory (very small trades)

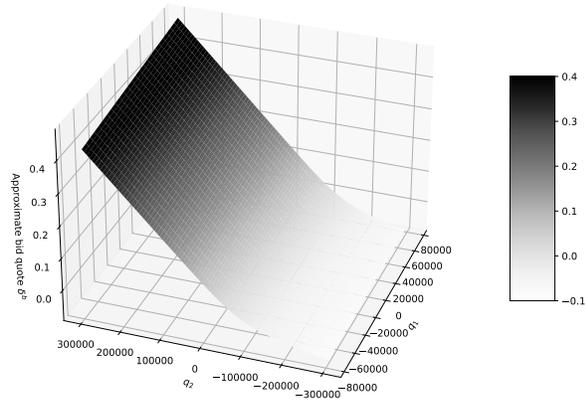


Figure 4.11: Approximate optimal bid quote at $t = 0$ for asset 2 as a function of the inventory (very small trades).

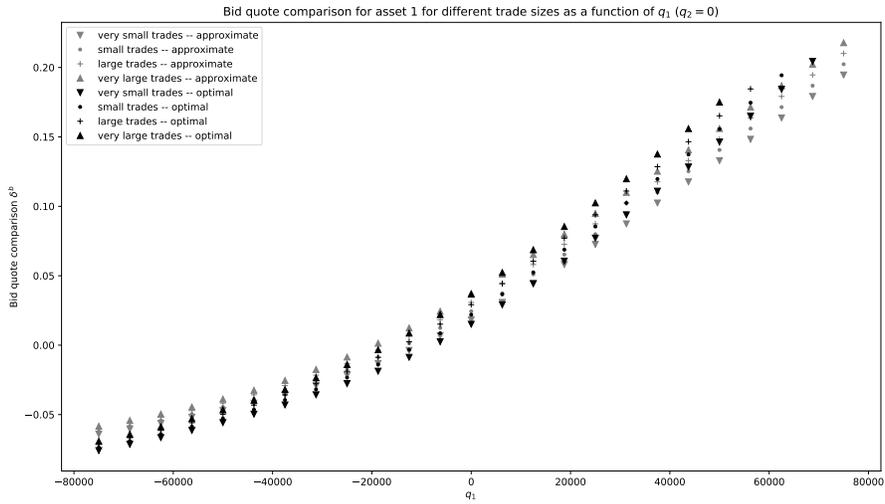


Figure 4.12: Approximate optimal bid quote for asset 1 for different trade sizes as a function of q^1 ($q^2 = 0$).

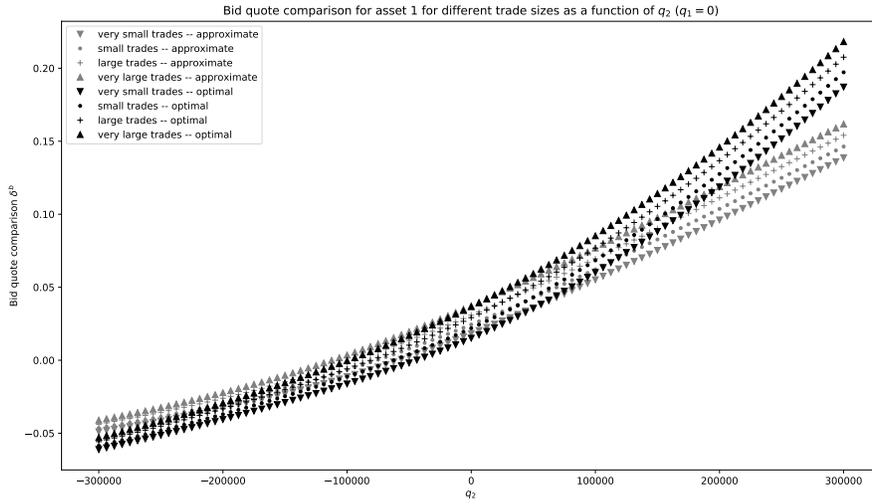


Figure 4.13: Approximate optimal bid quote for asset 1 for different trade sizes as a function of q^2 ($q^1 = 0$).

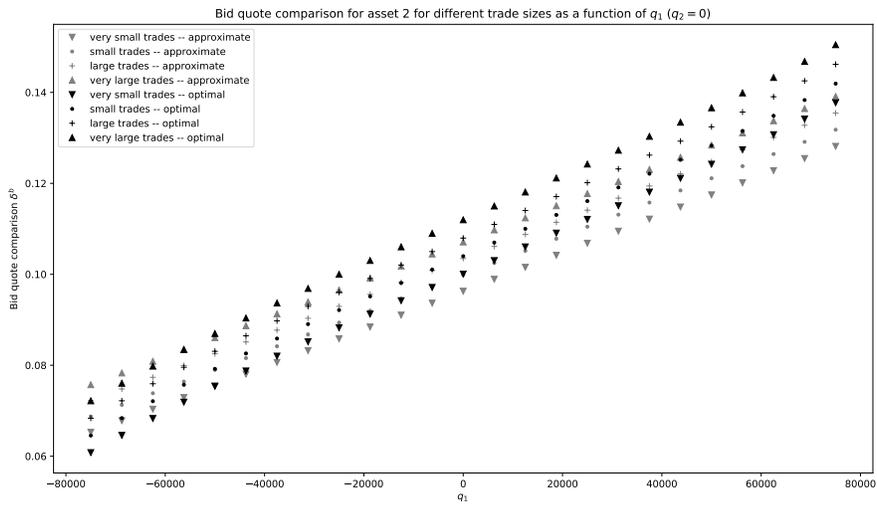


Figure 4.14: Approximate optimal bid quote for asset 2 for different trade sizes as a function of q^1 ($q^2 = 0$).

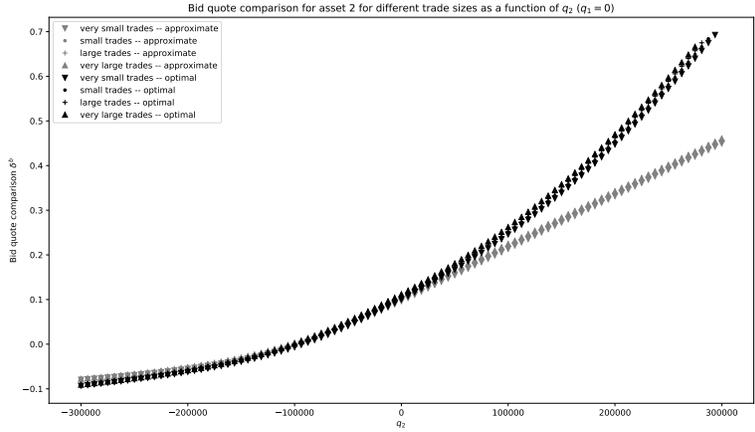


Figure 4.15: Approximate optimal bid quote for asset 2 for different trade sizes as a function of q^2 ($q^1 = 0$).

As a first observation, we can see that the closed-form approximations get really close to the true optimal quotes, except maybe for very large inventories. We finally study the performance of the approximations, by using 4000 Monte-Carlo simulations. We plot in Figures 4.16 and 4.17 the distribution of the PnL of a market maker, when quoting the true optimal quotes in the first case, and when quoting the closed-form approximations in the second case.

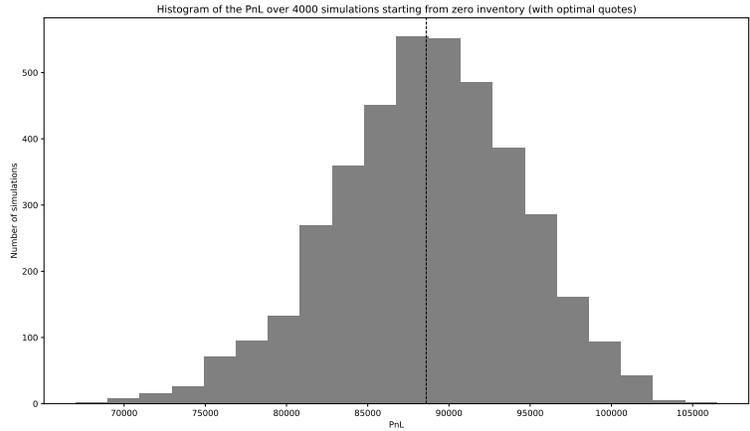


Figure 4.16: Distribution of the PnL of a market maker using the optimal quotes.

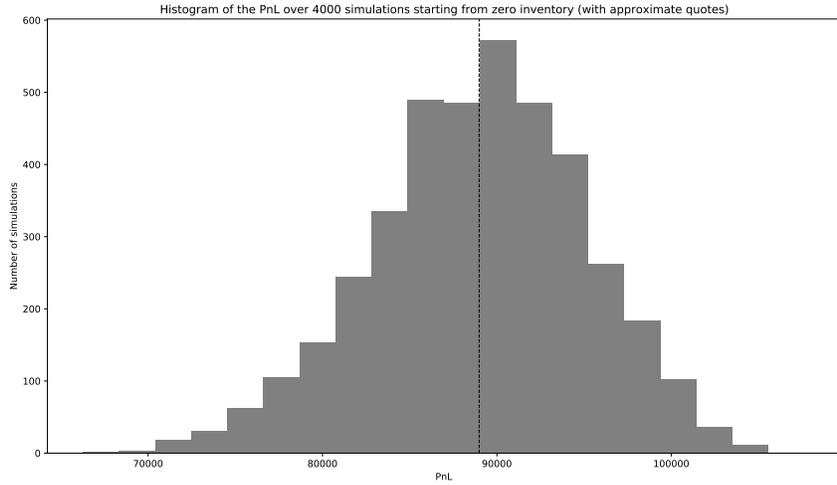


Figure 4.17: Distribution of the PnL of a market maker using the closed-form approximations.

When using the optimal quotes, the market maker gets an average PnL of 88600€ with a standard deviation of 86900€. When using the closed-form approximation, the performance is very similar, as she gets an average PnL of 89000€ with a standard deviation of 87500€.

These results are really satisfying in terms of performance. We see that, although the approximate optimal quotes can sometimes get quite far from the true optimal quotes, they do so only for values of the inventory that we should almost never reach, and therefore this gap does not really impact the average PnL nor its standard deviation. We believe that what we observed here in dimension 2 is general, and that the results in higher dimensions would be just as good.

Conclusion

We proposed closed-form approximations for the value functions associated with many multi-asset extensions of the market making models available in the literature. These closed-form approximations have been obtained through the “approximation” of a Hamilton-Jacobi equation by another Hamilton-Jacobi equation that can be simplified into a Riccati equation and two linear ordinary differential equations, all solvable in closed form. These closed-form approximations can be used for various purposes, in particular to design quoting strategies through a greedy approach. The resulting closed-form approximations of the optimal quotes generalize those obtained by Guéant, Lehalle, and Fernandez-Tapia in [65] to a general framework suitable for practical use.

Part II

Other problems in mathematical finance

Chapter 5

On bid and ask side-specific tick sizes

Abstract

The tick size, which is the smallest increment between two consecutive prices for a given asset, is a key parameter of market microstructure. In particular, the behavior of high frequency market makers is highly related to its value. We take the point of view of an exchange and investigate the relevance of having different tick sizes on the bid and ask sides of the order book. Using an approach based on the model with uncertainty zones, we show that when side-specific tick sizes are suitably chosen, it enables the exchange to improve the quality of liquidity provision.

5.1 Introduction

The tick size is the smallest increment between two consecutive prices on a trading instrument. It is fixed by the exchange or regulator and typically depends on both the price of the asset and the traded volume, see [73, 80]. It is a crucial parameter of market microstructure and its value is often subject of debates: a too small tick size leads to very frequent price changes whereas a too large tick size prevents the price from moving freely according to the investor's views. In this article, we focus on so-called large tick assets, that is assets for which the spread is most of the time equal to one tick. Such assets represent a large number of financial products, especially in Europe since MIFID II regulation, see [80].

The tick size has a major influence on the ecosystem of financial markets, in particular on the activity of high frequency traders. Being usually considered as market makers, these agents are the main liquidity providers for most heavily traded financial assets. This means that they propose prices at which they are ready to buy (bid price) and sell (ask price) units of financial products. In [54], the authors investigate the behavior of high frequency traders with respect to the relative tick size, which is defined as the ratio between the tick size and the price level. One of their findings is that everything else equal, stocks with a lower relative tick size attract a greater proportion of high frequency traders, see also [44, 86]. This is because they can rapidly marginally adjust their quotes to seize price priority. In the case of a large tick asset, speed is still an important feature as market participants have to compete for queue priority in the order book, see [74, 87].

Market makers (typically high frequency traders) face a complex optimization problem: making money out of the bid-ask spread (the difference between the bid and ask prices) while mitigating the inventory risk associated to price changes. This problem is usually addressed via stochastic control theory tools, see for example [13, 39, 40, 61, 65]. In classical market making models, the so-called efficient price, which represents the market consensus on the value of the asset at a given time, around which the market maker posts his quotes, is a continuous semi-martingale. The quotes of the market maker are continuous in terms of price values and not necessarily multiple of the tick size. However, in actual financial markets, transaction prices are obviously lying on the discrete tick grid. This discreteness of prices is a key feature which cannot be neglected at the high frequency scale since it plays a fundamental role in the design of market making strategies in practice. To get a more realistic market making model, one therefore needs to build a relevant continuous-time price dynamic with discrete state space to take into account this very important microstructural property of the asset.

To this end, we borrow the framework of the model with uncertainty zones introduced in [93, 94]. In this model, transaction prices are discrete and the current transaction price is modified only when the underlying continuous efficient price process crosses some predetermined zones. In our approach, we also consider that there exists an efficient price that market participants have in mind when making their trading decisions. Based on this efficient price, market participants build “fair” bid and “fair” ask prices. These two prices are lying on the tick grid and represent the views of market participants on reasonable and tradable values for buying and selling, regardless of any inventory constraint. In our setting, depending on his views and his inventory constraint, the market maker chooses whether or not to quote a constant volume at these fair bid and ask prices. This is a stylized viewpoint as in practice the market maker will probably quote a larger spread rather than not quoting at all. The market takers increase (resp. decrease) their current “fair” bid price if the efficient price becomes “sufficiently” higher (resp. lower) than their current fair bid price and similarly for the ask side. The mechanism to determine whether the efficient price is sufficiently higher (resp. lower) than the current price is that of the model with uncertainty zones, described in Section 5.2.

Usual market making models include a symmetric running penalty for the inventory process, often defined as $\phi \int_0^T Q_t^2 dt$ where Q_t is the inventory of the market maker at time $t \in [0, T]$, $\phi > 0$ is a risk aversion parameter and T is the end of the trading period. It is well-known, see for example [3], that for regulatory and operational reasons, market participants and especially market makers are reluctant to have a short inventory at the end of the trading day. This is mainly due to constraints imposed by the exchange/regulator and to the overnight repo rate that they have to pay. This asymmetry between long and short terminal inventory of the market maker gives the intuition of the potential relevance of some kind of asymmetry in the market design between buy and sell orders.

If some kind of asymmetry is implemented at the microstructure level, it can have important consequences on the profit of exchanges, as it notably depends on the number of processed orders. Typical ways to optimize the number of orders on platforms are the choice of relevant tick sizes and suitable fee schedules (which subsidize liquidity provision and tax liquidity consumption). In [52], the authors highlight the importance of differentiating maker and taker fees in order to increase the trading rate. In the more recent studies [17, 48], optimal make-take fees schedules are designed

based on contract theory. In this work, the asymmetry we consider is not between liquidity consumers and liquidity providers but between buyers and sellers.

The goal of the paper [14] is to show the possible benefits for an exchange in terms of liquidity provision of side-specific tick sizes. To this end, we build an agent-based model where a high frequency market maker acts on a large tick asset. The exchange is mitigating the activity on its platform by choosing suitable tick sizes on the bid and ask sides. This means we have a different tick grid for buy and sell orders. For given the tick sizes chosen by the exchange, we formulate the stochastic control problem faced by the market maker who needs to maximize his Profit and Loss (PnL for short) while controlling his inventory risk, taking into account asymmetry between short and long inventory. We show existence and uniqueness of a viscosity solution to the Hamilton-Jacobi-Bellman (HJB for short) equation associated to this problem. Then, we derive a quasi-closed form for the optimal controls of the market maker (up to the value function). In particular, the role of the tick size in the decision of whether or not to quote is explicit: essentially, a large tick size implies a large profit per trade for the market maker but less market orders coming from market takers, and conversely.

Next, we solve the optimization problem of the exchange which can select optimal tick sizes knowing the associated trading response of the market maker. In our model, the exchange earns a fixed fee when a transaction occurs. Therefore, its remuneration is related to the quality of the liquidity provided by the market maker on its platform. Numerical results show that side-specific tick sizes are more suitable than symmetric ones both for the market maker and the exchange. The former is able to trigger more alternations in the sign of market orders, which is beneficial both for spread pocketing and inventory management (in contrast with the case where sequences of buy orders are followed by sequences of sell orders). The latter increases the number of transactions on its platform. We also show that a tick size asymmetry can offset short inventory constraints, therefore increasing the gains of both the market maker and the exchange.

The paper is organized as follows. In Section 5.2, we give a reminder on the model with uncertainty zones and explain how we revisit it for market making purposes. The market maker and exchange's problems are described in Section 5.3. We also state here our results about existence and uniqueness of a viscosity solution associated to the control problem of the market maker and derive its optimal controls. Finally, Section 5.4 is devoted to numerical results and their interpretations. Proofs are relegated to an appendix.

5.2 The model with uncertainty zones

In this section, we provide a reminder on the model with uncertainty zones introduced in [93, 94], and we adapt it to the framework of a market making problem with side-specific tick values. It is commonly admitted that low frequency financial price data behave like a continuous Brownian semi-martingale. However this is clearly not the case for high frequency data. The model with uncertainty zones reproduces sparingly and accurately the behavior of ultra high frequency transaction data of a large tick asset. It is based on a continuous-time semi-martingale efficient price and a one dimensional parameter $\eta \in [0, \frac{1}{2}]$. The key idea of the model is that when a transaction occurs at some value on the tick grid, the efficient price is close enough to this value at the transaction

time. This proximity is measured through the parameter η .

We define the efficient price $(S_t)_{t \in [0, T]}$ on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$ where T is the trading horizon. The logarithm of the efficient price $(Y_t)_{t \in [0, T]}$ is an \mathcal{F}_t -adapted continuous Brownian semi-martingale of the form

$$Y_t = \log(S_t) = \log(S_0) + \int_0^t a_s ds + \int_0^t \sigma_s^- dW_s,$$

where W is an \mathcal{F} -Brownian motion, and $(\sigma_t)_{t \in [0, T]}$ is an \mathcal{F} -adapted process with càdlàg paths and $(a_t)_{t \in [0, T]}$ is \mathcal{F} -progressively measurable. Transaction prices lie on two fixed tick grids, defined by $\{k\alpha^a, k\alpha^b\}$ where α^a (resp. α^b) is the tick size on the ask (resp. bid) side and $k \in \mathbb{Z}$. For $0 \leq \eta^i \leq \frac{1}{2}$ and $i \in \{a, b\}$, we define the zone $U_k^i = [0, \infty) \times (d_k^i, u_k^i)$ with

$$d_k^i = (k + \frac{1}{2} - \eta^i)\alpha^i, u_k^i = (k + \frac{1}{2} + \eta^i)\alpha^i. \quad (5.1)$$

Therefore U_k^a is a band of size $2\eta^a\alpha^a$ around the ask mid-tick grid value $(k + \frac{1}{2})\alpha^a$ and U_k^b is a band of size $2\eta^b\alpha^b$ around the bid mid-tick grid value $(k + \frac{1}{2})\alpha^b$. We call these bands the uncertainty zones. The zones on the bid and ask sides are characterized by the parameters η^b, η^a which control the width of the uncertainty zones. We will see in the next section how the fair bid and ask prices are deduced from the efficient price dynamics across the uncertainty zones. In particular, the larger η^i , the farther from the last traded price (on the bid or ask side) the efficient price has to be so that a price change occurs. The idea behind the model with uncertainty zones is that, in some sense, market participants feel more comfortable when the asset price is constant than when it is constantly moving. However, there are times when the transaction price has to change because they consider that the last traded price value is not reasonable anymore.

For sake of simplicity, we assume that transaction prices cannot jump by more than one tick. We also define the time series of bid and ask transaction times leading to a price change as $(\tau_j^b, \tau_j^a)_{j \geq 0}$. The last traded bid or ask price process is characterized by the couples of transaction times and transaction prices with price changes $(\tau_j; P_{\tau_j^i}^i)_{j \geq 0}$ where $P_{\tau_j^i}^i = S_{\tau_j^i}^{(\alpha^i)}$, the superscript (α^i) denoting the rounding to the nearest α^i .

Remark 5. *The trade price goes up by one tick when the efficient price process S crosses the uncertainty zone in the upward direction, and conversely it goes down by one tick when the efficient price process S crosses the uncertainty zone in the downward direction.*

One can actually show that the efficient price can be retrieved from transaction data using the equation

$$S_{\tau_j^i} = S_{\tau_j^i}^{(\alpha^i)} - \alpha^i \left(\frac{1}{2} - \eta^i \right) \text{sgn} \left(S_{\tau_j^i}^{(\alpha^i)} - S_{\tau_{j-1}^i}^{(\alpha^i)} \right), \quad i \in \{a, b\}, j \in \mathbb{N}.$$

This formula is particularly useful in order to derive ultra high frequency estimators of volatility and covariation (see [94]). The parameters η^i can be estimated very easily. Let $N_{\alpha^i, t}^{(a)}$ and $N_{\alpha^i, t}^{(c)}$ be respectively the number of alternations and continuations¹ of one tick over the period $[0, t]$. Then,

¹An alternation/continuation corresponds to two consecutive price changes in the opposite/same direction.

an estimator of η^i over $[0, t]$ is given by

$$\hat{\eta}_{\alpha^i, t} = \frac{N_{\alpha^i, t}^{(c)}}{2N_{\alpha^i, t}^{(a)}}.$$

We refer to [93, 94] for further details on these estimation procedures. In this paper, we use the model with uncertainty zones for market making purposes rather than for statistical estimation.

5.3 High frequency market making under side-specific tick values and interaction with the exchange

5.3.1 The market maker's problem

We consider a high frequency market maker acting on an asset whose efficient price S_t has the dynamics

$$dS_t = \sigma dW_t,$$

where $\sigma > 0$ denotes the volatility of the asset. He uses the model with uncertainty zones described earlier to materialize his views on the fair bid and ask prices. He increases (resp. decreases) his bid price if the efficient price is “sufficiently” higher (resp. lower) than his current fair bid price. The notion of “sufficiently” higher or lower is determined by the uncertainty zones parameters η^a, η^b , and the tick sizes α^a, α^b . If η^a is small (resp. large), the market maker changes more (resp. less) frequently his ask price, and similarly for the bid price with η^b . This leads to the following definition of fair bid and ask prices of the market maker S^a, S^b .²

$$\begin{aligned} S_t^a &= S_{t-}^a + \alpha^a \mathbf{1}_{\{S_t - S_{t-}^a > (\frac{1}{2} + \eta^a)\alpha^a\}} - \alpha^a \mathbf{1}_{\{S_t - S_{t-}^a < -(\frac{1}{2} + \eta^a)\alpha^a\}}, \\ S_t^b &= S_{t-}^b + \alpha^b \mathbf{1}_{\{S_t - S_{t-}^b > (\frac{1}{2} + \eta^b)\alpha^b\}} - \alpha^b \mathbf{1}_{\{S_t - S_{t-}^b < -(\frac{1}{2} + \eta^b)\alpha^b\}}. \end{aligned}$$

Thus the fair bid (resp. ask) is modified when the efficient price is close enough to a new tradable price on the tick grid with mesh α^b (resp. α^a).

Remark 6. *Note that in the case $\alpha^a = \alpha^b, \eta^a = \eta^b$, the fair best bid is equal to the fair best ask. This means that at a given time, a buy or sell order would be at the same price. In this situation, in our stylized view, the market maker would probably quote only on one side (bid or ask). It is consistent with the standard form of the model with uncertainty zones, where, at a given time, transactions can only happen only on one side of the market, depending on the location of the efficient price. Still, the market maker collects the spread from transactions occurring at different times as it is the case in practice.*

We assume a constant volume of transaction equal to one. The market maker can choose to be present or not for a transaction at the bid (with a price S^b) or at the ask (with a price S^a). The

²Note that we can have situations where the bid price is above the ask price. However, recall that S^a and S^b are only views about the fair bid and ask prices under the constraint that they have to lie on the tick grids. This does not imply an arbitrage opportunity, as in this configuration the intelligent market maker would not quote on both sides.

corresponding cash process at terminal time T is given by

$$X_T = \int_0^T \left(S_t^a dN_t^a - S_t^b dN_t^b \right),$$

where the N_t^i represent the number of transactions on the bid or ask side between 0 and t . In this framework, the inventory of the market maker is given by $Q_t = N_t^b - N_t^a \in \mathcal{Q} = [-\tilde{q}, \tilde{q}]$ where \tilde{q} is the risk limit of the market maker. For $i \in \{a, b\}$, the dynamics of N_t^i is that of a point process with intensity

$$\lambda(\ell_t^i, Q_t) := \frac{\lambda \ell_t^i}{1 + (\kappa \alpha^i)^2} \mathbf{1}_{\{\phi(i)Q_t > -\tilde{q}\}}, \quad \phi(i) = \mathbf{1}_{\{i=a\}} - \mathbf{1}_{\{i=b\}}.$$

The process $\ell_t^i \in \{0, 1\}$ is the market maker's control which lies in the set of \mathcal{F} – predictable processes with values in $\{0, 1\}$ denoted by \mathcal{L} . The parameter $\kappa > 0$ controls the sensitivity of the intensities to α^i , and $\lambda > 0$ is a scale parameter. When the market maker does not want to be present on the bid (resp. ask side) at the price S^b (resp. S^a) he sets $\ell^b = 0$ (resp. $\ell^a = 0$) and conversely. In our large tick asset setting, the situation where the market maker is not present is a simplified way to model the case where the market maker's quote is higher than the best possible limit. At a given time $t \in [0, T]$, when $\ell_t^b = 0$ (resp. $\ell_t^a = 0$), the intensity of the point process N_t^b (resp. N_t^a) is equal to zero so that there are no incoming transactions. In addition to this, market takers are more confident to send market orders when the tick size is small, as the market maker has more flexibility to adjust his bid and ask prices.³ This explains the decreasing shape of the intensities of market order arrivals from market takers with respect to the tick size. The chosen parametric form for the intensities ensures no degenerate behavior when the tick size gets close to zero. The parameter κ and the quadratic shape of the intensity have no influence on the qualitative results obtained in Section 5.4, but, on the quantitative side, they fix the position of the optimal tick sizes and their dependence on the penalization parameters.

The marked-to-market value of the market maker's portfolio at time t is defined as $Q_t S_t$. His optimization problem writes

$$\sup_{\ell \in \mathcal{L}} \mathbb{E} \left[X_T + Q_T (S_T - A Q_T) - \phi \int_0^T Q_s^2 ds - \phi_- \int_0^T |Q_s|^2 \mathbf{1}_{Q_s < 0} ds \right], \quad (5.2)$$

where $\phi > 0$ represents the risk-aversion parameter of the market maker, $\phi_- > 0$ is the additional risk aversion of the market maker toward short position on $[0, T]$ and $A Q_T^2$, with $A > 0$, is a penalty term for the terminal inventory position regardless of its sign. In this setting, the market maker wishes to hold a terminal inventory close to zero because of the quadratic penalty $A Q_T^2$. The term $\phi \int_0^T Q_s^2 ds$ penalizes long or short positions over the trading period. In the following, we will choose T large enough so that the final penalization has little impact on the results. Problem (5.2) can of course be rewritten as

$$\sup_{\ell \in \mathcal{L}} \mathbb{E} \left[Q_T (S_T - A Q_T) + \int_0^T \left\{ S_s^a \lambda(\ell_s^a) - S_s^b \lambda(\ell_s^b) - \phi Q_s^2 - \phi_- Q_s^2 \mathbf{1}_{Q_s < 0} \right\} ds \right].$$

³When the tick size is smaller, the market takers are more willing to trade. This does not necessarily lead to a higher number of orders as it depends on the market maker's presence.

We define the corresponding value function h defined on the open set

$$\mathcal{D} = \left\{ (S^a, S^b, S) \in \alpha^a \mathbb{Z} \times \alpha^b \mathbb{Z} \times \mathbb{R} \text{ such that } -\left(\frac{1}{2} + \eta^a\right) \alpha^a < S - S^a < \left(\frac{1}{2} + \eta^a\right) \alpha^a \right. \\ \left. \text{and } -\left(\frac{1}{2} + \eta^b\right) \alpha^b < S - S^b < \left(\frac{1}{2} + \eta^b\right) \alpha^b \right\}$$

by

$$h(t, S^a, S^b, S, q) = \sup_{\ell \in \mathcal{L}_t} \mathbb{E}_{t, S^a, S^b, S, q} \left[Q_T (S_T - A Q_T) + \int_t^T \left\{ S_s^a \lambda(\ell_s^a) - S_s^b \lambda(\ell_s^b) \right. \right. \\ \left. \left. - \phi Q_s^2 - \phi_- Q_s^2 \mathbf{1}_{Q_s < 0} \right\} ds \right], \quad (5.3)$$

where \mathcal{L}_t denotes the restriction of admissible controls to $[t, T]$. We define the boundary $\partial \mathcal{D}$ of \mathcal{D} as

$$\partial \mathcal{D} = \left\{ (S^a, S^b, S) \in \alpha^a \mathbb{Z} \times \alpha^b \mathbb{Z} \times \mathbb{R} \text{ such that } S - S^a = \pm \left(\frac{1}{2} + \eta^a\right) \alpha^a \text{ and/or } S - S^b = \pm \left(\frac{1}{2} + \eta^b\right) \alpha^b \right\},$$

and write $\bar{\mathcal{D}} = \mathcal{D} \cup \partial \mathcal{D}$. For given (S^a, S^b) , if $(S^a, S^b, S) \in \partial \mathcal{D}$, it means that S corresponds to an efficient price value that triggers a modification of the fair bid or ask price.

The Hamilton-Jacobi-Bellman equation associated to this stochastic control problem is given by

$$0 = \partial_t h(t, S^a, S^b, S, q) - \phi q^2 - \phi_- q^2 \mathbf{1}_{q < 0} + \frac{1}{2} \sigma^2 \partial_{SS} h(t, S^a, S^b, S, q) \\ + \frac{\lambda}{1 + (\kappa \alpha^a)^2} \max_{\ell^a \in \{0, 1\}} \left\{ \ell^a \left(S^a + h(t, S^a, S^b, S, q - \ell^a) - h(t, S^a, S^b, S, q) \right) \right\} \\ + \frac{\lambda}{1 + (\kappa \alpha^b)^2} \max_{\ell^b \in \{0, 1\}} \left\{ \ell^b \left((-S^b) + h(t, S^a, S^b, S, q + \ell^b) - h(t, S^a, S^b, S, q) \right) \right\}, \quad (5.4)$$

for $(t, S^a, S^b, S, q) \in [0, T] \times \mathcal{D} \times \mathcal{Q}$, with terminal condition

$$h(T, S^a, S^b, S, q) = q(S - Aq). \quad (5.5)$$

Let us consider the function h defined in 5.3. For $(t, S^a, S^b, S, q) \in [0, T] \times \partial \mathcal{D} \times \mathcal{Q}$, and $(t_n, S_n^a, S_n^b, S_n, q_n)_{n \in \mathbb{N}}$ a sequence in $[0, T] \times \mathcal{D} \times \mathcal{Q}$ which converges to (t, S^a, S^b, S, q) , we will show that $h(t_n, S_n^a, S_n^b, S_n, q_n)$ converges independently of the sequence and we denote by $h(t, S^a, S^b, S, q)$ its limit. On $[0, T] \times \partial \mathcal{D} \times \mathcal{Q}$, we will show the following boundary conditions (which we will naturally impose for the solution of 5.4):

$$0 = \mathbf{1}_{\{S - S^a = (\frac{1}{2} + \eta^a) \alpha^a, S - S^b < (\frac{1}{2} + \eta^b) \alpha^b\}} \left(h(t, S^a + \alpha^a, S^b, S, q) - h(t, S^a, S^b, S, q) \right) \\ + \mathbf{1}_{\{S - S^a < (\frac{1}{2} + \eta^a) \alpha^a, S - S^b = (\frac{1}{2} + \eta^b) \alpha^b\}} \left(h(t, S^a, S^b + \alpha^b, S, q) - h(t, S^a, S^b, S, q) \right) \\ + \mathbf{1}_{\{S - S^a = (\frac{1}{2} + \eta^a) \alpha^a, S - S^b = (\frac{1}{2} + \eta^b) \alpha^b\}} \left(h(t, S^a + \alpha^a, S^b + \alpha^b, S, q) - h(t, S^a, S^b, S, q) \right) \\ + \mathbf{1}_{\{S - S^a = -(\frac{1}{2} + \eta^a) \alpha^a, S - S^b > -(\frac{1}{2} + \eta^b) \alpha^b\}} \left(h(t, S^a - \alpha^a, S^b, S, q) - h(t, S^a, S^b, S, q) \right) \\ + \mathbf{1}_{\{S - S^a > -(\frac{1}{2} + \eta^a) \alpha^a, S - S^b = -(\frac{1}{2} + \eta^b) \alpha^b\}} \left(h(t, S^a, S^b - \alpha^b, S, q) - h(t, S^a, S^b, S, q) \right) \\ + \mathbf{1}_{\{S - S^a = -(\frac{1}{2} + \eta^a) \alpha^a, S - S^b = -(\frac{1}{2} + \eta^b) \alpha^b\}} \left(h(t, S^a - \alpha^a, S^b - \alpha^b, S, q) - h(t, S^a, S^b, S, q) \right). \quad (5.6)$$

In other words, the value function varies continuously when the efficient price leaves an uncertainty zone and the prices S^a and S^b are modified.⁴ In the following, we say that a function defined on $[0, T] \times \mathcal{D} \times \mathcal{Q}$ satisfies the continuity conditions if it satisfies (5.6).

The following proposition is of particular importance for the existence and uniqueness of a viscosity solution associated to the control problem of the market maker.

Proposition 7. *The function h defined in Equation (5.3) is continuous on \mathcal{D} and satisfies the continuity conditions (5.6).*

The proof is given in Appendix 5.6.1 and relies on the specific structure of our model based on hitting times of a Brownian motion. We now state the main theorem of this article, whose proof is relegated to Appendix 5.6.2.

Theorem 5. *The value function h is the unique continuous viscosity solution to Equation (5.4) on $[0, T] \times \mathcal{D} \times \mathcal{Q}$ with terminal condition (5.5) and satisfying the continuity conditions.*

The value function depends on five variables. However, as (S^a, S^b) takes value in $\alpha^a \mathbb{Z} \times \alpha^b \mathbb{Z}$, it can essentially be reduced to three variables as we now explain. For any $(i, j) \in \mathbb{Z}^2$, we introduce the function $h^{i,j}$ defined on

$$[0, T] \times \underbrace{\left(\alpha^a i - \left(\frac{1}{2} + \eta^a\right) \alpha^a, \alpha^a i + \left(\frac{1}{2} + \eta^a\right) \alpha^a \right) \cap \left(\alpha^b j - \left(\frac{1}{2} + \eta^b\right) \alpha^b, \alpha^b j + \left(\frac{1}{2} + \eta^b\right) \alpha^b \right)}_{=\mathcal{D}_{i,j}} \times \mathcal{Q}$$

by $h^{i,j}(t, S, q) = h(t, \alpha^a i, \alpha^b j, S, q)$. Then $h^{i,j}$ is the solution of the following HJB equation:

$$\begin{aligned} 0 = & \partial_t h^{i,j}(t, S, q) - \phi q^2 - \phi_-(q)_- \mathbf{1}_{q < 0} + \frac{1}{2} \sigma^2 \partial_{SS} h^{i,j}(t, S, q) \\ & + \frac{\lambda}{1 + (\kappa \alpha^a)^2} \max_{\ell^a \in \{0, 1\}} \left\{ \ell^a \left(\alpha^a i + h^{i,j}(t, S, q - \ell^a) - h^{i,j}(t, S, q) \right) \right\} \\ & + \frac{\lambda}{1 + (\kappa \alpha^b)^2} \max_{\ell^b \in \{0, 1\}} \left\{ \ell^b \left(-\alpha^b j + h^{i,j}(t, S, q + \ell^b) - h^{i,j}(t, S, q) \right) \right\}, \end{aligned}$$

with terminal condition $h^{i,j}(T, S, q) = q(S - A_Q)$ and natural Dirichlet boundary conditions for $S \in \partial \mathcal{D}_{i,j}$:

$$\begin{aligned} h^{i,j}(t, S, q) = & h^{i+1,j}(t, S, q) \mathbf{1}_{\{S - \alpha^a i = (\frac{1}{2} + \eta^a) \alpha^a, S - \alpha^b j < (\frac{1}{2} + \eta^b) \alpha^b\}} \\ & + h^{i,j+1}(t, S, q) \mathbf{1}_{\{S - \alpha^a i < (\frac{1}{2} + \eta^a) \alpha^a, S - \alpha^b j = (\frac{1}{2} + \eta^b) \alpha^b\}} \\ & + h^{i+1,j+1}(t, S, q) \mathbf{1}_{\{S - \alpha^a i = (\frac{1}{2} + \eta^a) \alpha^a, S - \alpha^b j = (\frac{1}{2} + \eta^b) \alpha^b\}} \\ & + h^{i-1,j}(t, S, q) \mathbf{1}_{\{S - \alpha^a i = -(\frac{1}{2} + \eta^a) \alpha^a, S - \alpha^b j > -(\frac{1}{2} + \eta^b) \alpha^b\}} \\ & + h^{i,j-1}(t, S, q) \mathbf{1}_{\{S - \alpha^a i > -(\frac{1}{2} + \eta^a) \alpha^a, S - \alpha^b j = -(\frac{1}{2} + \eta^b) \alpha^b\}} \\ & + h^{i-1,j-1}(t, S, q) \mathbf{1}_{\{S - \alpha^a i = -(\frac{1}{2} + \eta^a) \alpha^a, S - \alpha^b j = -(\frac{1}{2} + \eta^b) \alpha^b\}}. \end{aligned}$$

⁴Note that, as the terminal condition does not depend on S^a and S^b , it also satisfies this boundary condition on $\partial \mathcal{D}$.

From this, we derive the optimal controls of the market maker as

$$\begin{aligned}\ell^{*a}(t, i, j, S, q) &= \mathbf{1}_{\{\alpha^a i + h^{i,j}(t, S, q-1) - h^{i,j}(t, S, q) > 0\}}, \\ \ell^{*b}(t, i, j, S, q) &= \mathbf{1}_{\{-\alpha^b j + h^{i,j}(t, S, q+1) - h^{i,j}(t, S, q) > 0\}}.\end{aligned}$$

The practical interest of Theorem 5 is that it allows us to compute the value function and optimal controls based on a finite difference scheme. Examples of computations of the value function are given in Section 5.4 and Appendix 5.6.3. Having described the problem of the market maker, we now turn to the optimization problem of the platform.

5.3.2 The platform's problem

The market maker acts on a platform whose goal is to maximize the number of market orders on $[0, T]$. The intensities of arrival of market orders are functions of ℓ^a, ℓ^b , which are themselves functions of α^a, α^b . We assume that the platform is risk-neutral and earns a fixed taker cost $c > 0$ for each market order.⁵ Therefore its optimization problem is defined as

$$\sup_{(\alpha^a, \alpha^b) \in \mathbb{R}_+^2} \mathbb{E}^{\ell^{*a}, \ell^{*b}} [X_T^p],$$

given the optimal controls (ℓ^{*a}, ℓ^{*b}) of the market maker and $X_t^p = c(N_t^a + N_t^b)$.

It is easy to observe that this problem boils down to maximizing the function v defined below over \mathbb{R}_+^2 :

$$v(\alpha^a, \alpha^b) := \mathbb{E} \left[\int_0^T c\lambda \left\{ \frac{\ell^{*a}(t, S_t^a, S_t^b, S_t, q_t)}{1 + (\kappa\alpha^a)^2} + \frac{\ell^{*b}(t, S_t^a, S_t^b, S_t, q_t)}{1 + (\kappa\alpha^b)^2} \right\} dt \right].$$

Here we clearly see the tradeoff of the platform. A small tick size α^a increase the term $(1 + (\kappa\alpha^a)^2)^{-1}$. This is because it attracts more buy market orders. However, the optimal control $\ell^{*,a}$ is more often equal to zero: the gain of the market maker may be too small if he quotes at the price S^a , therefore he regularly sets $\ell^{*,a} = 0$. The problem is similar on the bid side. On the other hand, a large tick size increases the gain of the market maker if a transaction occurs, but decreases the number of market orders sent by market takers, hence decreasing the trading volume.

We study numerically this problem in the next section by computing the value of v on a two dimensional grid and finding its maximum.

5.4 Numerical results

In this section, we show from numerical experiments the benefits of side-specific tick values in terms of increase of their value function for both the market maker and the platform. Also, we fix reference values η_0 and α_0 . From them, to choose the parameter η^i associated to a given tick size

⁵More complex fee schedules can be handled in this framework. We can for example add a component which is proportional to the amount of cash traded.

α^i we use a result from [44] which gives the new value of the parameter η^i in case of a change of tick size from α_0 to α^i . This formula writes

$$\eta^i = \eta_0 \sqrt{\frac{\alpha_0}{\alpha^i}}. \quad (5.7)$$

In the following, we only consider values of α^a and α^b such that the underlying remains a large tick asset both on the bid and ask sides, that is $\eta^a \leq \frac{1}{2}, \eta^b \leq \frac{1}{2}$.

For the first experiments, we set $T = 40s, \bar{q} = 5, \sigma = 0.01s^{-\frac{1}{2}}, A = 0.1, \kappa = 10, \phi = 0.005, \lambda = 4, \eta_0 = 0.3$ and $\alpha_0 = 0.01$ which correspond to reasonable values to model a liquid asset. In particular, taking a larger T has no impact on the dependence of our results on the tick sizes. To remain in the large tick regime, we investigate values of α^i satisfying $0.0045 \leq \alpha^i \leq 0.05$ for $i = a, b$.

5.4.1 Similar tick values on both sides

In this section we investigate the case where $\alpha^a = \alpha^b$. We plot in Figure 5.1 the value functions of the market maker and the exchange, respectively h and v , for various values of $\alpha = \alpha^a = \alpha^b$. We fix the efficient price $S = 10.5$, the inventory $q = 0$ and we only consider values of α so that $0.5/\alpha \in \mathbb{N}$.

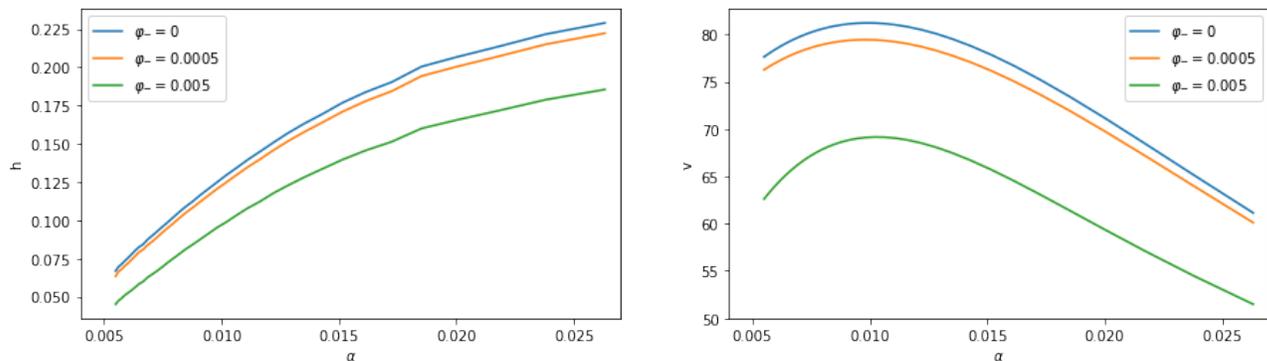


Figure 5.1: Value function h (on the left) and v (on the right) for $\phi_- = 0$ in blue, $\phi_- = 0.0005$ in orange, $\phi_- = 0.005$ in green, as a function of $\alpha = \alpha^a = \alpha^b$.

When $\phi_- = 0$, the value of the exchange reaches its maximum at $\alpha \simeq 0.012$. An increase of ϕ_- leads to a reduction of the number of transactions. However the optimal tick value for the exchange is not significantly modified.

The optimal tick value for the market maker is larger than that of the exchange. This is because the exchange is only interested in attracting orders while the market maker's gain per trade (not taking into account the inventory risk) is linear with respect to the tick value. The trade-off of the exchange is the following: on the one hand, he would like to implement a quite small tick value (to attract market orders) but on the other hand, he must ensure a reasonable presence of market maker.

When ϕ_- increases, the value function of the market maker decreases, for all tick values. This is no surprise since ϕ_- corresponds to an inventory penalization, hence reducing the market maker's PnL.

In Figure 5.2, we subtract the value function when $\phi_- = 0$ to the other value functions displayed in Figure 5.1. We remark that for the market maker, the larger the tick the more significant the penalization of short inventory in terms of value function. We observe the opposite phenomenon for the exchange: the difference is essentially slightly increasing with respect to α . In particular, we see a quite strong impact of the penalization on the value function of the exchange when the tick size is small.

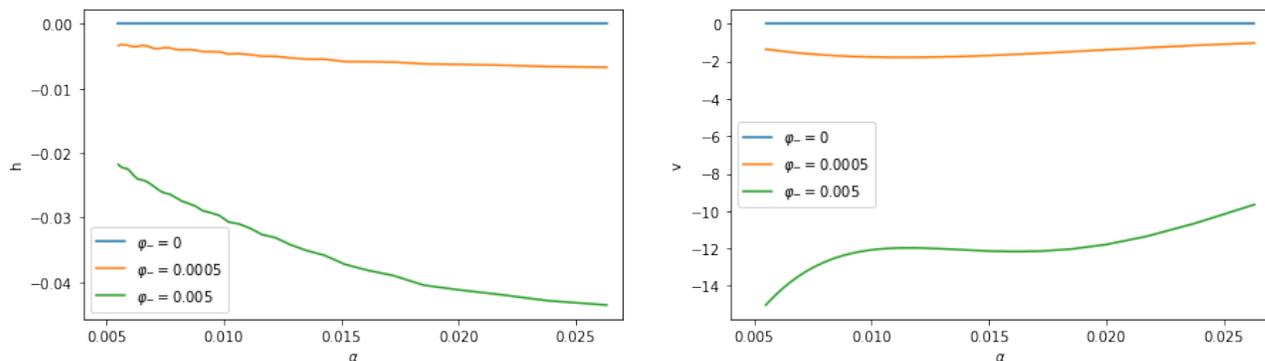


Figure 5.2: Variation of the functions h and v (difference between $\phi_- = 0$ in blue, $\phi_- = 0.0005$ in orange, $\phi_- = 0.005$ in green, and $\phi_- = 0$ as a function of $\alpha = \alpha^a = \alpha^b$).

We now study the case of side-specific tick values.

5.4.2 Side-specific tick values: additional opportunities for the market maker

We set $\alpha^b = 0.0124$ (optimal tick size in the non side-specific case) and let α^a vary. We plot the value functions of the market maker and the exchange in Figure 5.3. Again we observe that both value functions are decreasing with respect to ϕ_- . From the point of view of the market maker, having non side-specific tick values is sub-optimal, even in the case $\phi_- = 0$. This is because when the two tick values are different, it is possible for S^a to be greater than S^b and orders to arrive with the same intensities on both sides: the market maker can collect the spread. It is not possible in the non side-specific case, where the market maker can only pocket the spread from buy and sell orders at two different times. Side-specific tick values are also clearly beneficial for the exchange. The transaction flow increases for $\alpha^a > \alpha^b$ because of the good liquidity provided by the market maker, and for $\alpha^a < \alpha^b$ because of the high number of incoming market orders.

Remark 7. Remark that with shifted grids (same tick values on both sides but with a grid shifted compared to the other), those additional opportunities for the market maker would remain. In section 5.4.3, we will see however, that, from the point of view of the exchange, side-specific tick values are much more interesting.

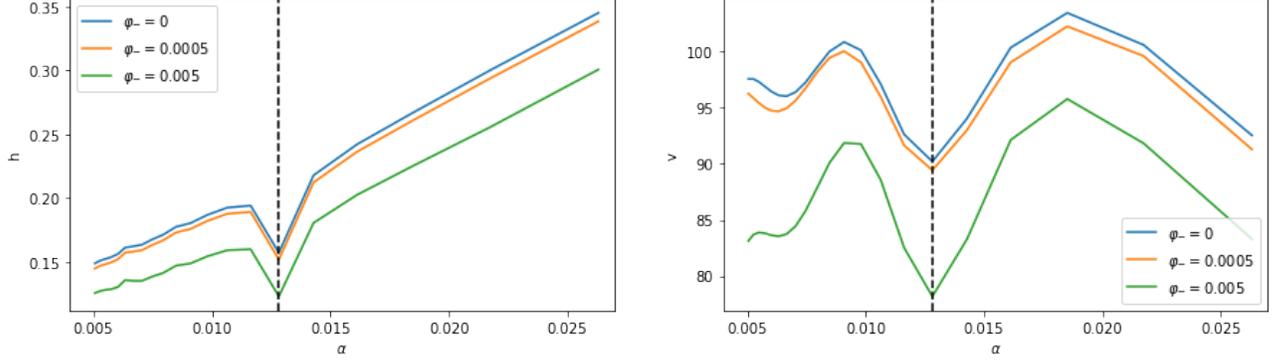


Figure 5.3: Value function h (on the left) and v (on the right) as functions of α^a , for $\alpha^b = 0.0124$, for $\phi_- = 0$ in blue, $\phi_- = 0.0005$ in orange, $\phi_- = 0.005$ in green.

5.4.3 Side-specific tick values: effect of ϕ_-

We plot the two-dimensional value functions of the market maker and the exchange for side-specific tick values.

First we take $\phi_- = 0$ in Figure 5.4. We note that the opportunity for the market maker mentioned above remains present for all tick values and that the value functions are symmetric around the axis $\alpha^b = \alpha^a$ (side-specific tick values are preferred). Furthermore, we see that the exchange prefers smaller tick values than the market maker. The optimal values for the exchange lie on an anti-diagonal which goes from $(\alpha^a = 0.0045, \alpha^b = 0.025)$ to $(\alpha^a = 0.025, \alpha^b = 0.0045)$. On this line the number of transactions varies little. It seems however that the optimum is on the edges of the zone in which the asset remains large tick: the two couples (α^a, α^b) mentioned above.

If the tick values are too large the intensities of the market orders become too small and the number of transactions diminishes. If both ticks are too small, the market maker does not trade much because the gain per trade becomes too little compared to the inventory cost (recall that the intensity of market orders is upper bounded). However, the case where one tick is quite small and the other is large is suitable for the market maker: for example, if $\alpha^a < \alpha^b$ his strategy is to be long and liquidate his long position fast if needed thanks to the small value of α^a which ensures a large number of incoming market orders. This explains why the optimal tick values given by the exchange are side-specific and symmetric with respect to the axis $\alpha^a = \alpha^b$. More precisely, the choice of ticks $(\alpha^a = 0.0045, \alpha^b = 0.025)$ or $(\alpha^a = 0.025, \alpha^b = 0.0045)$ seems optimal.

We now plot in Figure 5.5 the value function for $\phi_- = 0.005$. This non-zero parameter implies a clear decrease of the value function of the market maker and the reduction of the number of transactions. An important remark is that the value functions are no-longer symmetric around the axis $\alpha^b = \alpha^a$.

For clarity we plot in Figure 5.6 the difference of the value functions when $\phi_- = 0.005$ and when $\phi_- = 0$ as a function of α^a, α^b . We see that the added component is not symmetric regarding to the axis $\alpha^b = \alpha^a$ and both the market maker and the exchange tend to prefer the case $\alpha^b > \alpha^a$.

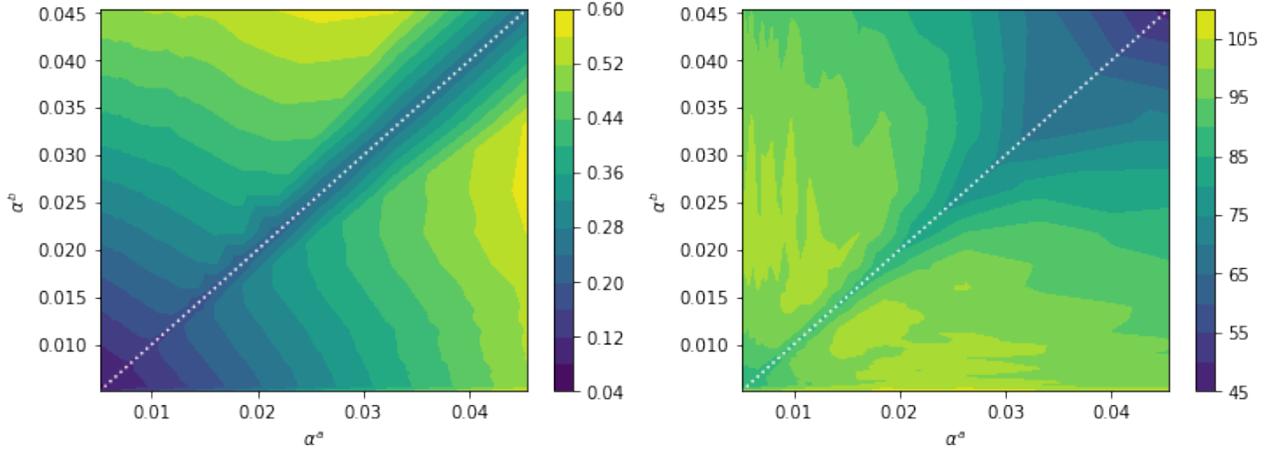


Figure 5.4: Value function h (on the left) and v (on the right) as functions of α^a and α^b , for $\phi_- = 0$.

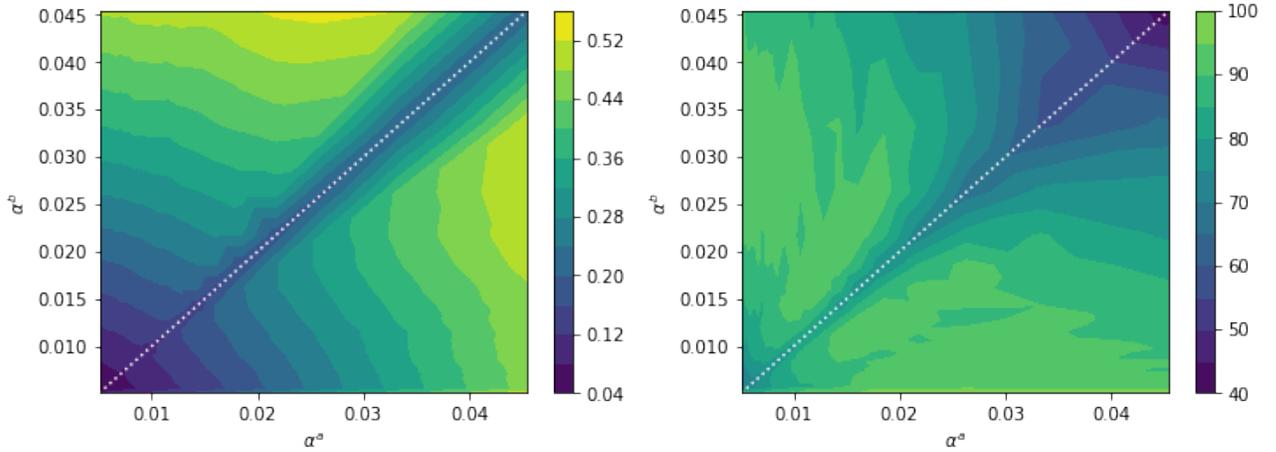


Figure 5.5: Value function h (on the left) and v (on the right) as functions of α^a and α^b , for $\phi_- = 0.005$.

It is particularly clear for the market maker's problem where the difference between the values at $(\alpha^a = 0.0045, \alpha^b = 0.025)$ and $(\alpha^a = 0.025, \alpha^b = 0.0045)$ is approximately 0.03 which is roughly 10% of the value function. Indeed, as explained above, having α^b quite large and α^a rather small essentially ensures that the market maker can maintain a positive inventory all along the trading trajectory: attractive PnL for incoming buy orders and possibility to quickly reduce a positive inventory. The right-hand side of Figure 5.6 is harder to interpret, as the changes are comparatively smaller and influenced more by the kink described in 5.4.2 and its analogues at $\alpha^b = 2\alpha^a$ and $\alpha^b = \frac{1}{2}\alpha^a$.

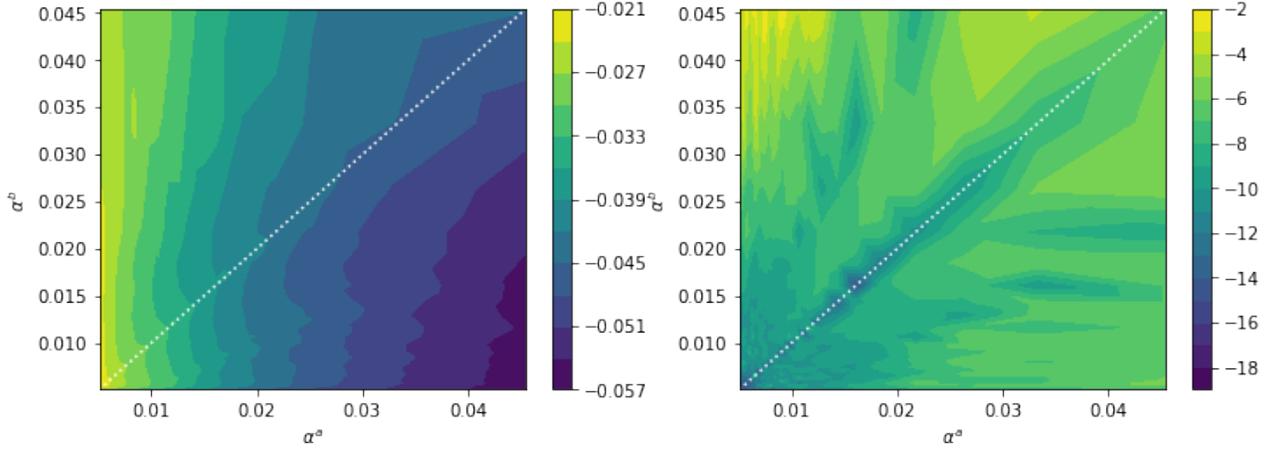


Figure 5.6: Difference between the value function h (on the left) and v (on the right) as functions of α^a and α^b , between the case $\phi_- = 0.005$ and the case $\phi_- = 0$.

The exchange leans more towards a larger α^b compared to α^a . To see that more clearly, we fix $\alpha^a = 0.0045$ and plot in Figure 5.7 the value functions h and v , as functions of α^b , for different values of ϕ .

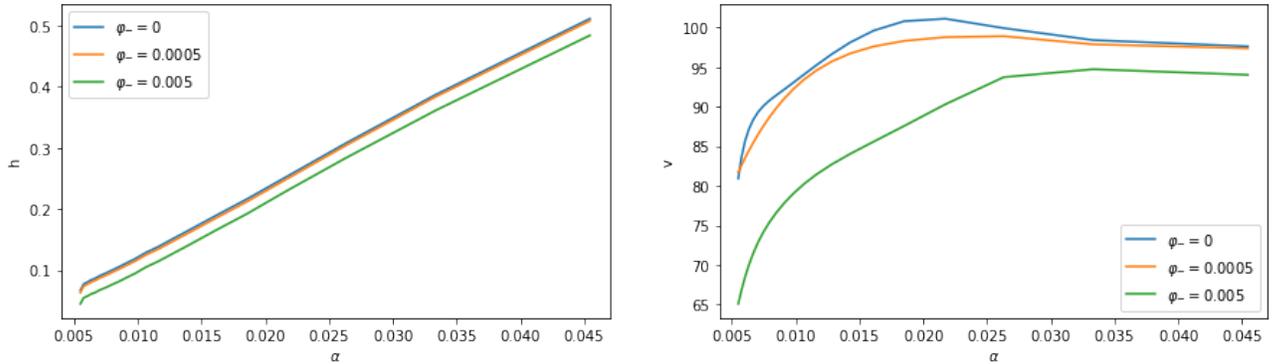


Figure 5.7: Value functions h and v for $\alpha^a = 0.0045$, as functions of α^b , for different values of ϕ .

The value function of the market maker is increasing in α^b . This is the same phenomenon as already observed in Figure 5.5. The value of the exchange has a maximum which is increasing in ϕ_- : as the penalization gets more side-specific, the optimal tick values displayed by the exchange become more asymmetric. Indeed, for $\phi_- = 0$, the optimum is reached for $\alpha^b \simeq 0.024$, while for $\phi_- = 0.005$, it is obtained for $\alpha^b \simeq 0.034$. Note that a relevant tick value set by the exchange can compensate for his loss of value function due to an increase of ϕ_- . By choosing a new tick size optimally when going from $\phi_- = 0$ to $\phi_- = 0.005$, the loss in value function is of 7% only. Keeping $\alpha^b = 0.024$ would lead to a loss of 15%. Note that the compensation can be total for the market

maker (and even exceeds the loss) but is only partial for the exchange.

5.5 Conclusion

A suitable choice of tick values by the exchange is a subtle equilibrium. If the platform imposes the same tick value on the bid and ask sides, it has to be sufficiently large to ensure significant PnL per trade for the market maker and therefore good liquidity provision, and sufficiently small to attract market orders from market takers. When allowing for side-specific tick values with no constraint on short inventory, the optimal tick values for the exchange are of the form (α_1^*, α_2^*) or symmetrically (α_2^*, α_1^*) with $\alpha_1^* < \alpha_2^*$. In this case, the market maker can take advantage from additional trading opportunities and increase his activity. The exchange benefits from this situation because of the higher number of trades on his platform. Moreover, when there is a penalty for short inventory positions of the market maker, there is only one optimal couple of tick values. In this case, the market maker and subsequently the exchange prefer $\alpha^b > \alpha^a$ and the difference between α^a and α^b at the optimum becomes larger. Finally, note that side-specific tick values could have subtle consequences in a multi-platform setting. This issue is left for further study, as well as the situation where market takers are more strategic in their execution.

5.6 Appendix

5.6.1 Proof of Proposition 7

First we prove the continuity of h on $\mathcal{D} \times [0, T)$.

Let $q \in \mathcal{Q}$, $t_1 \in [0, T)$, $(s^a, s^b, s_1) \in \mathcal{D}$. Note that $\{s \in \mathbb{R}, (s^a, s^b, s) \in \mathcal{D}\}$ is an open interval containing s_1 , which we denote by $(s^{\leftarrow}, s^{\rightarrow})$. If the process S_t starts from a point $s \in (s^{\leftarrow}, s^{\rightarrow})$ with $S_t^a = s^a$ and $S_t^b = s^b$, S_t^a and S_t^b will not jump as long as S_t stays in $(s^{\leftarrow}, s^{\rightarrow})$. We will prove that the function $(t, s) \in [0, T) \times (s^{\leftarrow}, s^{\rightarrow}) \mapsto h(t, s^a, s^b, s, q)$ is continuous at (t_1, s_1) .

We fix $\eta > 0$. There is a ball with positive diameter \mathcal{B} in $[0, T) \times (s^{\leftarrow}, s^{\rightarrow})$ centered on (t_1, s_1) and some $\epsilon > 0$ such that, if $(t_2, s_2) \in \mathcal{B}$, then

$$\mathbb{E}[\tau^1 - t_1 | S_{t_1} = s_1] < \eta, \quad \mathbb{E}[\tau^2 - t_2 | S_{t_2} = s_2] < \eta, \quad (5.8)$$

$$\mathbb{P}[\tau^1 < T | S_{t_1} = s_1] > 1 - \eta, \quad \mathbb{P}[\tau^2 < T | S_{t_2} = s_2] > 1 - \eta, \quad (5.9)$$

and⁶

$$\begin{aligned} \inf_{\ell \in \mathcal{L}} \mathbb{P}[\inf_{t_1 \leq s \leq \tau^1} Q_s = \sup_{t_1 \leq s \leq \tau^1} Q_s = q | S_{t_1} = s_1, Q_{t_1} = q] &> 1 - \eta, \\ \inf_{\ell \in \mathcal{L}} \mathbb{P}[\inf_{t_2 \leq s \leq \tau^2} Q_s = \sup_{t_2 \leq s \leq \tau^2} Q_s = q | S_{t_2} = s_2, Q_{t_2} = q] &> 1 - \eta, \end{aligned} \quad (5.10)$$

where we write

$$\tau^1 = T \wedge \inf\{t \geq t_1, S_t^{t_1, s_1} = (s_1 \vee s_2) + \epsilon \text{ or } S_t^{t_1, s_1} = (s_1 \wedge s_2) - \epsilon\},$$

$$\tau^2 = T \wedge \inf\{t \geq t_2, S_t^{t_2, s_2} = (s_1 \vee s_2) + \epsilon \text{ or } S_t^{t_2, s_2} = (s_1 \wedge s_2) - \epsilon\}.$$

⁶These inequalities can be attained independently of the control ℓ as S is independent from Q .

The quantities τ^1 and τ^2 are stopping times such that $t_1 \leq \tau^1 \leq T$ a.s. and $t_2 \leq \tau^2 \leq T$ a.s. We impose

$$s^{\leftarrow} < (s_1 \wedge s_2) - \epsilon < s_1 < (s_1 \vee s_2) + \epsilon < s^{\rightarrow}, \quad s^{\leftarrow} < (s_1 \wedge s_2) - \epsilon < s_2 < (s_1 \vee s_2) + \epsilon < s^{\rightarrow}$$

for any $(t_2, s_2) \in \mathcal{B}$ by taking a smaller ball \mathcal{B} and a smaller ϵ if necessary. In particular, this tells us that if $(S_{t_1}, S_{t_1}^a, S_{t_1}^b) = (s_1, s^a, s^b)$, S_t^a does not jump between t_1 and τ^1 . Similarly, if $(S_{t_2}, S_{t_2}^a, S_{t_2}^b) = (s_2, s^a, s^b)$, S_t^b does not jump between t_2 and τ^2 .

Let some arbitrary $(t_2, s_2) \in \mathcal{B}$ and τ^1 and τ^2 the associated stopping times. Using the dynamic programming principle, we obtain

$$h(t_1, s^a, s^b, s_1, q) = \sup_{\ell \in \mathcal{L}} \mathbb{E} \left[h(\tau^1, S_{\tau^1}^a, S_{\tau^1}^b, S_{\tau^1}, Q_{\tau^1}) + \int_{t_1}^{\tau^1} \{-\phi Q_t^2 - \phi_- Q_t^2 \mathbf{1}_{Q_t < 0}\} dt \mid S_{t_1} = s_1, S_{t_1}^a = s^a, S_{t_1}^b = s^b, Q_{t_1} = q \right].$$

This can be rewritten as

$$h(t_1, s^a, s^b, s_1, q) = \sup_{\ell \in \mathcal{L}} \mathbb{E} \left[\sum_{\bar{q} \in \mathcal{Q}} (h(\tau^1, s^a, s^b, S_{\tau^1}, \bar{q}) \mathbf{1}_{\{Q_{\tau^1} = \bar{q}\}} + \int_{t_1}^{\tau^1} \{-\phi Q_t^2 - \phi_- Q_t^2 \mathbf{1}_{Q_t < 0}\} \mathbf{1}_{\{Q_t = \bar{q}\}}) dt \mid S_{t_1} = s_1, Q_{t_1} = q \right].$$

Noticing from 5.3 that h is bounded on $[0, T] \times \{s^a\} \times \{s^b\} \times [s^{\leftarrow}, s^{\rightarrow}] \times \mathcal{Q}$, we deduce by 5.10 that

$$\left| h(t_1, s^a, s^b, s_1, q) - \sup_{\ell \in \mathcal{L}} \mathbb{E} \left[h(\tau^1, s^a, s^b, S_{\tau^1}, q) + \int_{t_1}^{\tau^1} \{-\phi q^2 - \phi_-(q)_-^2\} dt \mid S_{t_1} = s_1, Q_{t_1} = q \right] \right| \leq C\eta$$

for a constant C which depends only on s^a, s^b , and $(q)_- = q^2 \mathbf{1}_{q < 0}$. The expectation above does not depend on the control ℓ , hence we drop the supremum and fix an arbitrary control $\ell = 0$. We denote by \mathbb{E}^0 the expectation under the probability measure given by this control. The expectation neither depends on the process Q_t , so we drop the conditioning with respect to Q_{t_1} .

This leads to

$$\left| h(t_1, s^a, s^b, s_1, q) - \mathbb{E}^0 \left[h(\tau^1, s^a, s^b, S_{\tau^1}, q) + \int_{t_1}^{\tau^1} \{-\phi q^2 - \phi_-(q)_-^2\} dt \mid S_{t_1} = s_1 \right] \right| \leq C\eta.$$

Similarly, starting from (t_2, s^a, s^b, s_2, q) with $(t_2, s_2) \in \mathcal{B}$, we get

$$\left| h(t_2, s^a, s^b, s_2, q) - \mathbb{E}^0 \left[h(\tau^2, s^a, s^b, S_{\tau^2}, q) + \int_{t_2}^{\tau^2} \{-\phi q^2 - \phi_-(q)_-^2\} dt \mid S_{t_2} = s_2 \right] \right| \leq C\eta,$$

and we deduce that

$$\begin{aligned} |h(t_1, s^a, s^b, s_1, q) - h(t_2, s^a, s^b, s_2, q)| &\leq \left| \mathbb{E}^0 \left[h(\tau^1, s^a, s^b, S_{\tau^1}, q) + \int_{t_1}^{\tau^1} \{-\phi q^2 - \phi_-(q)_-^2\} dt \mid S_{t_1} = s_1 \right] \right. \\ &\quad \left. - \mathbb{E}^0 \left[h(\tau^2, s^a, s^b, S_{\tau^2}, q) + \int_{t_2}^{\tau^2} \{-\phi q^2 - \phi_-(q)_-^2\} dt \mid S_{t_2} = s_2 \right] \right| + 2C\eta \\ &\leq \left| \mathbb{E}^0 \left[h(\tau^1, s^a, s^b, S_{\tau^1}, q) \mid S_{t_1} = s_1 \right] - \mathbb{E}^0 \left[h(\tau^2, s^a, s^b, S_{\tau^2}, q) \mid S_{t_2} = s_2 \right] \right| \\ &\quad + \left| \phi q^2 - \phi_-(q)_-^2 \right| \left(\mathbb{E}^0 [\tau^1 - t_1 \mid S_{t_1} = s_1] + \mathbb{E}^0 [\tau^2 - t_2 \mid S_{t_2} = s_2] \right) + 2C\eta. \end{aligned} \tag{5.11}$$

Using (5.8), we get

$$\left| \mathbb{E}^0[\tau^1 - t_1 | S_{t_1} = s_1] + E^0[\tau^2 - t_2 | S_{t_2} = s_2] \right| < 2\eta. \quad (5.12)$$

Also, the conditional laws

$$\begin{aligned} (\tau^1 | S_{t_1} = s_1, S_{\tau^1} = (s_1 \vee s_2) + \epsilon, \tau^1 < T), & \quad (\tau^1 | S_{t_1} = s_1, S_{\tau^1} = (s_1 \wedge s_2) - \epsilon, \tau^1 < T), \\ (\tau^2 | S_{t_2} = s_2, S_{\tau^2} = (s_1 \vee s_2) + \epsilon, \tau^2 < T), & \quad (\tau^2 | S_{t_2} = s_2, S_{\tau^2} = (s_1 \wedge s_2) - \epsilon, \tau^2 < T), \end{aligned}$$

have bounded continuous densities, which we denote by $f^{1,+}$, $f^{1,-}$, $f^{2,+}$ and $f^{2,-}$ respectively (see for example [32], Formula 3.0.6). So, by decomposing the first term in (5.11) with respect to the values of S_{τ^1} and S_{τ^2} , we can write

$$\begin{aligned} & \left| \mathbb{E}^0[h(\tau^1, s^a, s^b, S_{\tau^1}, q) | S_{t_1} = s_1] - \mathbb{E}^0[h(\tau^2, s^a, s^b, S_{\tau^2}, q) | S_{t_2} = s_2] \right| \\ \leq & \left| \sum_{j \in \{+, -\}} \int_0^T h(t, s^a, s^b, s_j, q) (f^{1,j}(t) \mathbb{P}^0[S_{\tau^1} = s_j, \tau^1 < T | S_{t_1} = s_1] - f^{2,j}(t) \mathbb{P}^0[S_{\tau^2} = s_j, \tau^2 < T | S_{t_2} = s_2]) dt \right| \\ & + \left| \mathbb{E}^0[h(\tau^1, s^a, s^b, S_{\tau^1}, q) \mathbf{1}_{\{S_{\tau^1} \neq s_+, S_{\tau^1} \neq s_-\} \cup \{\tau^1 = T\}} | S_{t_1} = s_1] \right| \\ & + \left| \mathbb{E}^0[h(\tau^2, s^a, s^b, S_{\tau^2}, q) \mathbf{1}_{\{S_{\tau^2} \neq s_+, S_{\tau^2} \neq s_-\} \cup \{\tau^2 = T\}} | S_{t_2} = s_2] \right| \end{aligned} \quad (5.13)$$

where $s_+ = s_1 \vee s_2 + \epsilon$ and $s_- = s_1 \wedge s_2 - \epsilon$. Remark that the event $S_{\tau^1} \neq s_+, S_{\tau^1} \neq s_-, S_{t_1} = s_1$ happens only if $\tau^1 = T$ so that $\mathbb{P}^0[\{S_{\tau^1} \neq s_+, S_{\tau^1} \neq s_-\} \cup \{\tau^1 = T\} | S_{t_1} = s_1] < \eta$ by (5.9). Similarly $\mathbb{P}^0[\{S_{\tau^2} \neq s_+, S_{\tau^2} \neq s_-\} \cup \{\tau^2 = T\} | S_{t_2} = s_2] < \eta$ by (5.9). As a consequence, using again (5.11), (5.12) and the fact that h is bounded on $[0, T] \times \{s^a\} \times \{s^b\} \times [s^{\leftarrow}, s^{\rightarrow}] \times \mathcal{Q}$, we get

$$\begin{aligned} & |h(t_1, s^a, s^b, s_1, q) - h(t_2, s^a, s^b, s_2, q)| \\ \leq & \left| \sum_{j \in \{+, -\}} \int h(t, s^a, s^b, s_j, q) (f^{1,j}(t) \mathbb{P}^0[S_{\tau^1} = s_j, \tau^1 < T | S_{t_1} = s_1] - f^{2,j}(t) \mathbb{P}^0[S_{\tau^2} = s_j, \tau^2 < T | S_{t_2} = s_2]) dt \right| \\ & + (2|\phi q^2 - \phi_-(q)_-^2| + 4C)\eta. \end{aligned}$$

Recall that the $f^{1,+}$, $f^{1,-}$, $f^{2,+}$ and $f^{2,-}$ depend on s_2 and t_2 . We have

$$\begin{aligned} & |\mathbb{P}^0[S_{\tau^2} = s_+, \tau^2 < T | S_{t_2} = s_2] f^{2,+} - \mathbb{P}^0[S_{\tau^1} = s_+, \tau^1 < T | S_{t_1} = s_1] f^{1,+}| \xrightarrow{(t_2, s_2) \rightarrow (t_1, s_1)} 0, \\ & |\mathbb{P}^0[S_{\tau^2} = s_-, \tau^2 < T | S_{t_2} = s_2] f^{2,-} - \mathbb{P}^0[S_{\tau^1} = s_-, \tau^1 < T | S_{t_1} = s_1] f^{1,-}| \xrightarrow{(t_2, s_2) \rightarrow (t_1, s_1)} 0 \end{aligned}$$

point-wise on $[0, T]$ directly by [32] Formula 3.0.6 and Appendix 11. Having fixed ϵ and using again [32] Formula 3.0.6 and Appendix 11, we see that the above functions are uniformly bounded with respect to $(s_2, t_2) \in \mathcal{B}$. So, using that h is bounded on $[0, T] \times \{s^a\} \times \{s^b\} \times [s^{\leftarrow}, s^{\rightarrow}] \times \mathcal{Q}$, we can apply the dominated convergence theorem to deduce that

$$\begin{aligned} & \left| \sum_{j \in \{+, -\}} \int_0^T h(t, s^a, s^b, s_j, q) (f^{1,j}(t) \mathbb{P}^0[S_{\tau^1} = s_j, \tau^1 < T | S_{t_1} = s_1] - f^{2,j}(t) \mathbb{P}^0[S_{\tau^2} = s_j, \tau^2 < T | S_{t_2} = s_2]) dt \right| \\ & \xrightarrow{(t_2, s_2) \rightarrow (t_1, s_1)} 0. \end{aligned}$$

Thus we have shown that h is continuous at the point (t_1, S^a, S^b, s_1, q) . The case $t_1 = T$ is treated the same way.

The continuity conditions can be proved using the same lines: fixing $q \in \mathcal{Q}$, $t_1 \in [0, T]$ and $(S^a, S^b, s_1) \in \partial\mathcal{D}$, choosing (t_2, s_2) close enough to (t_1, s_1) and applying the dynamic programming principle between t_1 and τ^1 , and t_2 and τ^2 , for τ^1 and τ^2 two well-chosen stopping times (for example

$$\tau^1 = T \wedge \inf \left\{ t > t_1, S_t = s_1 + \epsilon \text{ or } S_t = \frac{s^{\leftarrow} \wedge s_1}{2} \right\}, \quad \tau^2 = T \wedge \inf \left\{ t > t_2, S_t = s_1 + \epsilon \text{ or } S_t = \frac{s^{\leftarrow} \wedge s_1}{2} \right\}.$$

with $\epsilon > 0$ small enough, for a boundary inducing an upward jump).

5.6.2 Proof of Theorem 5

We first prove that the value function of the market maker's problem is indeed a viscosity solution of (5.4).

Proposition 8. *The value function h is a continuous viscosity solution on $[0, T] \times \mathcal{D} \times \mathcal{Q}$ of (5.4). Furthermore, $h(T, S^a, S^b, S, q) = q(S - Aq)$ for all $(S^a, S^b, S, q) \in \mathcal{D} \times \mathcal{Q}$, and*

$$\begin{aligned} h(t, S^a, S^b, S, q) = & \mathbf{1}_{\{S - S^a = (\frac{1}{2} + \eta^a)\alpha^a, S - S^b < (\frac{1}{2} + \eta^b)\alpha^b\}} h(t, S^a + \alpha^a, S^b, S, q) \\ & + \mathbf{1}_{\{S - S^a < (\frac{1}{2} + \eta^a)\alpha^a, S - S^b = (\frac{1}{2} + \eta^b)\alpha^b\}} h(t, S^a, S^b + \alpha^b, S, q) \\ & + \mathbf{1}_{\{S - S^a = (\frac{1}{2} + \eta^a)\alpha^a, S - S^b = (\frac{1}{2} + \eta^b)\alpha^b\}} h(t, S^a + \alpha^a, S^b + \alpha^b, S, q) \\ & + \mathbf{1}_{\{S - S^a = -(\frac{1}{2} + \eta^a)\alpha^a, S - S^b > -(\frac{1}{2} + \eta^b)\alpha^b\}} h(t, S^a - \alpha^a, S^b, S, q) \\ & + \mathbf{1}_{\{S - S^a > -(\frac{1}{2} + \eta^a)\alpha^a, S - S^b = -(\frac{1}{2} + \eta^b)\alpha^b\}} h(t, S^a, S^b - \alpha^b, S, q) \\ & + \mathbf{1}_{\{S - S^a = -(\frac{1}{2} + \eta^a)\alpha^a, S - S^b = -(\frac{1}{2} + \eta^b)\alpha^b\}} h(t, S^a - \alpha^a, S^b - \alpha^b, S, q), \end{aligned}$$

for all $(t, S^a, S^b, S, q) \in [0, T] \times \partial\mathcal{D} \times \mathcal{Q}$.

Proof. Let $(\bar{S}^a, \bar{S}^b, \bar{q}) \in \alpha^a\mathbb{Z} \times \alpha^b\mathbb{Z} \times \mathcal{Q}$, and $(t_n, S_n)_{n \in \mathbb{N}} \in [0, T] \times \mathbb{R}$ be a sequence such that

$$t_n \xrightarrow{n \rightarrow +\infty} \hat{t} \in [0, T),$$

$$S_n \xrightarrow{n \rightarrow +\infty} \hat{S} \in \mathbb{R},$$

$$h(t_n, \bar{S}^a, \bar{S}^b, S_n, \bar{q}) \xrightarrow{n \rightarrow +\infty} h(\hat{t}, \bar{S}^a, \bar{S}^b, \hat{S}, \bar{q}),$$

with $(\bar{S}^a, \bar{S}^b, \hat{S}) \in \mathcal{D}$. Without loss of generality we can assume that $(\bar{S}^a, \bar{S}^b, S_n) \in \mathcal{D}$ for all $n \in \mathbb{N}$.

Let us first consider the case $\hat{t} = T$. Let us take two arbitrary controls $\ell_s^a = \ell_s^b = 0$, for all $s \in [0, T)$, then for all $n \in \mathbb{N}$ we have

$$h(t_n, \bar{S}^a, \bar{S}^b, S_n, \bar{q}) \geq \mathbb{E}_{t_n, \bar{S}^a, \bar{S}^b, S_n, \bar{q}} \left[Q_T(S_T - A Q_T) - \phi \int_{t_n}^T Q_s^2 ds - \phi_- \int_{t_n}^T Q_s^2 \mathbf{1}_{Q_s < 0} ds \right],$$

and by dominated convergence we can obtain

$$h(T, \bar{S}^a, \bar{S}^b, \hat{S}, \bar{q}) \geq \bar{q}(\hat{S} - A\bar{q}).$$

Now let us consider the case $\hat{t} < T$. Let $\varphi : [0, T) \times \mathcal{D} \times \mathcal{Q} \rightarrow \mathbb{R}$ be a continuous function, \mathcal{C}^1 in t , \mathcal{C}^2 in S and such that $0 = \min_{[0, T) \times \mathcal{D}} (h - \varphi) = (h - \varphi)(\hat{t}, \bar{S}^a, \bar{S}^b, \hat{S}, \bar{q})$. We also assume that $h = \varphi$ only at the point $(\hat{t}, \bar{S}^a, \bar{S}^b, \hat{S}, \bar{q})$. Let us assume that there exists $\eta > 0$ such that

$$\begin{aligned} 2\eta \leq & \partial_t \varphi(\hat{t}, \bar{S}^a, \bar{S}^b, \hat{S}, \bar{q}) - \phi \bar{q}^2 - \phi_- \bar{q}^2 \mathbf{1}_{\bar{q} < 0} + \frac{1}{2} \sigma^2 \partial_{SS} \varphi(\hat{t}, \bar{S}^a, \bar{S}^b, \hat{S}, \bar{q}) \\ & + \frac{\lambda}{1 + (\kappa \alpha^a)^2} \max_{\ell^a \in \{0, 1\}} \left\{ \ell^a \left(\bar{S}^a + \varphi(\hat{t}, \bar{S}^a, \bar{S}^b, \hat{S}, \bar{q} - \ell^a) - \varphi(\hat{t}, \bar{S}^a, \bar{S}^b, \hat{S}, \bar{q}) \right) \right\} \\ & + \frac{\lambda}{1 + (\kappa \alpha^b)^2} \max_{\ell^b \in \{0, 1\}} \left\{ \ell^b \left(-\bar{S}^b + \varphi(\hat{t}, \bar{S}^a, \bar{S}^b, \hat{S}, \bar{q} + \ell^b) - \varphi(\hat{t}, \bar{S}^a, \bar{S}^b, \hat{S}, \bar{q}) \right) \right\}. \end{aligned}$$

Then we must have

$$\begin{aligned} 0 \leq & \partial_t \varphi(t, \bar{S}^a, \bar{S}^b, S, \bar{q}) - \phi \bar{q}^2 - \phi_- \bar{q}^2 \mathbf{1}_{\bar{q} < 0} + \frac{1}{2} \sigma^2 \partial_{SS} \varphi(t, \bar{S}^a, \bar{S}^b, S, \bar{q}) \\ & + \frac{\lambda}{1 + (\kappa \alpha^a)^2} \max_{\ell^a \in \{0, 1\}} \left\{ \ell^a \left(\bar{S}^a + \varphi(t, \bar{S}^a, \bar{S}^b, S, \bar{q} - \ell^a) - \varphi(t, \bar{S}^a, \bar{S}^b, S, \bar{q}) \right) \right\} \\ & + \frac{\lambda}{1 + (\kappa \alpha^b)^2} \max_{\ell^b \in \{0, 1\}} \left\{ \ell^b \left(-\bar{S}^b + \varphi(t, \bar{S}^a, \bar{S}^b, S, \bar{q} + \ell^b) - \varphi(t, \bar{S}^a, \bar{S}^b, S, \bar{q}) \right) \right\}, \end{aligned}$$

for all $(t, S) \in B = ((\hat{t} - r, \hat{t} + r) \cap [0, T)) \times (\hat{S} - r, \hat{S} + r)$ for some $r > 0$. Without loss of generality, we can assume that B contains the sequence $(t_n, S_n)_n$ and that for all $(t, S) \in B$, we have $(\bar{S}^a, \bar{S}^b, S) \in \mathcal{D}$. We can choose the value of η such that

$$\varphi(t, \bar{S}^a, \bar{S}^b, S, \bar{q}) + \eta \leq h(t, \bar{S}^a, \bar{S}^b, S, \bar{q})$$

on $\partial_p B := \left(((\hat{t} - r, \hat{t} + r) \cap [0, T)) \times \left(\{\hat{S} - r\} \cup \{\hat{S} + r\} \right) \right) \cup \left(\{\hat{t} + r\} \times [\hat{S} - r, \hat{S} + r] \right)$. We can also assume that

$$\varphi(t, S^a, S^b, S, q) + \eta \leq h(t, S^a, S^b, S, q),$$

for $(t, S^a, S^b, S, q) \in \tilde{B}$ with

$$\tilde{B} = \left\{ (t, \bar{S}^a, \bar{S}^b, S, q) \mid (t, S) \in B, q \in \{\bar{q} - 1, \bar{q} + 1\} \cap \mathcal{Q} \right\}.$$

We introduce the set

$$B_{\mathcal{D}} = \{(t, \bar{S}^a, \bar{S}^b, S, \bar{q}) \mid (t, S) \in B\}$$

and set $\pi_n = \inf\{t \geq t_n \mid (t, S_t^a, S_t^b, S_t, q_t) \notin B_{\mathcal{D}}\}$ with $S_{t_n}^i = \bar{S}^i$, $q_{t_n} = \bar{q}$, $S_{t_n} = S_n$, where the processes are controlled by

$$\ell_t^a = \mathbf{1}_{\{S_t^a + \varphi(t, S_t^a, S_t^b, S_t, q_{t-}) - \varphi(t, S_t^a, S_t^b, S_t, q_{t-}) > 0\}},$$

$$\ell_t^b = \mathbf{1}_{\{-S_t^b + \varphi(t, S_t^a, S_t^b, S_t, q_{t-} + 1) - \varphi(t, S_t^a, S_t^b, S_t, q_{t-}) > 0\}}.$$

Using Itô's formula and noting that S_t^a, S_t^b do not jump between t_n and π_n , we derive

$$\begin{aligned} \varphi(\pi_n, S_{\pi_n}^a, S_{\pi_n}^b, S_{\pi_n}, q_{\pi_n}) &= \varphi(t_n, \bar{S}^a, \bar{S}^b, S_n, \bar{q}) + \int_{t_n}^{\pi_n} \left\{ \partial_t \varphi(t, S_t^a, S_t^b, S_t, q_t) + \frac{1}{2} \sigma^2 \partial_{SS} \varphi(t, S_t^a, S_t^b, S_t, q_t) \right\} dt \\ &\quad + \int_{t_n}^{\pi_n} \lambda(\ell_t^a) \{ \varphi(t, S_t^a, S_t^b, S_t, q_{t-} - \ell_t^a) - \varphi(t, S_t^a, S_t^b, S_t, q_{t-}) \} dt \\ &\quad + \int_{t_n}^{\pi_n} \lambda(\ell_t^b) \{ \varphi(t, S_t^a, S_t^b, S_t, q_{t-} + \ell_t^b) - \varphi(t, S_t^a, S_t^b, S_t, q_{t-}) \} dt \\ &\quad + \int_{t_n}^{\pi_n} \sigma \partial_S \varphi(t, S_t^a, S_t^b, S_t, q_t) dW_t \\ &\quad + \int_{t_n}^{\pi_n} \{ \varphi(t, S_t^a, S_t^b, S_t, q_{t-} - \ell_t^a) - \varphi(t, S_t^a, S_t^b, S_t, q_{t-}) \} d\tilde{N}_t^a \\ &\quad + \int_{t_n}^{\pi_n} \{ \varphi(t, S_t^a, S_t^b, S_t, q_{t-} + \ell_t^b) - \varphi(t, S_t^a, S_t^b, S_t, q_{t-}) \} d\tilde{N}_t^b \\ &\geq \varphi(t_n, \bar{S}^a, \bar{S}^b, S_n, \bar{q}) \\ &\quad - \int_{t_n}^{\pi_n} \{ S_t^a \lambda(\ell_t^a) - S_t^b \lambda(\ell_t^b) - \phi q_t^2 - \phi_- q_t^2 \mathbf{1}_{q_t < 0} \} dt \\ &\quad + \int_{t_n}^{\pi_n} \sigma \partial_S \varphi(t, S_t^a, S_t^b, S_t, q_t) dW_t \\ &\quad + \int_{t_n}^{\pi_n} \{ \varphi(t, S_t^a, S_t^b, S_t, q_{t-} - \ell_t^a) - \varphi(t, S_t^a, S_t^b, S_t, q_{t-}) \} d\tilde{N}_t^a \\ &\quad + \int_{t_n}^{\pi_n} \{ \varphi(t, S_t^a, S_t^b, S_t, q_{t-} + \ell_t^b) - \varphi(t, S_t^a, S_t^b, S_t, q_{t-}) \} d\tilde{N}_t^b. \end{aligned}$$

Then by taking the expectation we get

$$\varphi(t_n, \bar{S}^a, \bar{S}^b, S_n, \bar{q}) \leq \mathbb{E} \left[\varphi(\pi_n, S_{\pi_n}^a, S_{\pi_n}^b, S_{\pi_n}, q_{\pi_n}) + \int_{t_n}^{\pi_n} \{ S_t^a \lambda(\ell_t^a) - S_t^b \lambda(\ell_t^b) - \phi q_t^2 - \phi_- q_t^2 \mathbf{1}_{q_t < 0} \} dt \right].$$

Thus

$$\varphi(t_n, \bar{S}^a, \bar{S}^b, S_n, \bar{q}) \leq -\eta + \mathbb{E} \left[h(\pi_n, S_{\pi_n}^a, S_{\pi_n}^b, S_{\pi_n}, q_{\pi_n}) + \int_{t_n}^{\pi_n} \{ S_t^a \lambda(\ell_t^a) - S_t^b \lambda(\ell_t^b) - \phi q_t^2 - \phi_- q_t^2 \mathbf{1}_{q_t < 0} \} dt \right].$$

As

$$\begin{aligned} \varphi(t_n, \bar{S}^a, \bar{S}^b, S_n, \bar{q}) &\xrightarrow{n \rightarrow +\infty} \varphi(\hat{t}, \bar{S}^a, \bar{S}^b, \hat{S}, \bar{q}) = h(\hat{t}, \bar{S}^a, \bar{S}^b, \hat{S}, \bar{q}), \\ h(t_n, \bar{S}^a, \bar{S}^b, S_n, \bar{q}) &\xrightarrow{n \rightarrow +\infty} h(\hat{t}, \bar{S}^a, \bar{S}^b, \hat{S}, \bar{q}), \end{aligned}$$

there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $h(t_n, \bar{S}^a, \bar{S}^b, S_n, \bar{q}) - \frac{\eta}{2} \leq \varphi(t_n, \bar{S}^a, \bar{S}^b, S_n, \bar{q})$ and we deduce

$$h(t_n, \bar{S}^a, \bar{S}^b, S_n, \bar{q}) \leq -\frac{\eta}{2} + \mathbb{E} \left[h(\pi_n, S_{\pi_n}^a, S_{\pi_n}^b, S_{\pi_n}, q_{\pi_n}) + \int_{t_n}^{\pi_n} \{ S_t^a \lambda(\ell_t^a) - S_t^b \lambda(\ell_t^b) - \phi q_t^2 - \phi_- q_t^2 \mathbf{1}_{q_t < 0} \} dt \right],$$

which contradicts the dynamic programming principle. Therefore,

$$\begin{aligned}
0 &\geq \partial_t \varphi(\hat{t}, \bar{S}^a, \bar{S}^b, \hat{S}, \bar{q}) - \phi \bar{q}^2 - \phi_- \bar{q}^2 \mathbf{1}_{\bar{q} < 0} + \frac{1}{2} \sigma^2 \partial_{SS} \varphi(\hat{t}, \bar{S}^a, \bar{S}^b, \hat{S}, \bar{q}) \\
&\quad + \frac{\lambda}{1 + (\kappa \alpha^a)^2} \max_{\ell^a \in \{0,1\}} \left\{ \ell^a \left(\bar{S}^a + \varphi(\hat{t}, \bar{S}^a, \bar{S}^b, \hat{S}, \bar{q} - \ell^a) - \varphi(\hat{t}, \bar{S}^a, \bar{S}^b, \hat{S}, \bar{q}) \right) \right\} \\
&\quad + \frac{\lambda}{1 + (\kappa \alpha^b)^2} \max_{\ell^b \in \{0,1\}} \left\{ \ell^b \left(-\bar{S}^b + \varphi(\hat{t}, \bar{S}^a, \bar{S}^b, \hat{S}, \bar{q} + \ell^b) - \varphi(\hat{t}, \bar{S}^a, \bar{S}^b, \hat{S}, \bar{q}) \right) \right\},
\end{aligned}$$

and h is a viscosity supersolution of the HJB equation on $[0, T] \times \mathcal{D} \times \mathcal{Q}$.

The proof for the subsolution part is identical. \square

For a given $\rho > 0$, we introduce the function \tilde{h} such that

$$\tilde{h}(t, S^a, S^b, S, q) = e^{\rho t} h(t, S^a, S^b, S, q) \quad \forall (t, S^a, S^b, S, q) \in [0, T] \times \mathcal{D} \times \mathcal{Q}.$$

Then \tilde{h} is a viscosity solution of the following HJB equation:

$$\begin{aligned}
0 &= -\rho \tilde{h}(t, S^a, S^b, S, q) + \partial_t \tilde{h}(t, S^a, S^b, S, q) - \phi q^2 - \phi_- q^2 \mathbf{1}_{q < 0} + \frac{1}{2} \sigma^2 \partial_{SS} \tilde{h} \\
&\quad + \frac{\lambda}{1 + (\kappa \alpha^a)^2} \max_{\ell^a \in \{0,1\}} \left\{ \ell^a \left(e^{\rho t} S^a + \tilde{h}(t, S^a, S^b, S, q - \ell^a) - \tilde{h}(t, S^a, S^b, S, q) \right) \right\} \\
&\quad + \frac{\lambda}{1 + (\kappa \alpha^b)^2} \max_{\ell^b \in \{0,1\}} \left\{ \ell^b \left(e^{\rho t} (-S^b) + \tilde{h}(t, S^a, S^b, S, q + \ell^b) - \tilde{h}(t, S^a, S^b, S, q) \right) \right\}, \quad (5.14)
\end{aligned}$$

and we see that proving a maximum principle for (5.14) is equivalent to proving one for (5.4).

Definition 1. Let $U : [0, T] \times \mathcal{D} \times \mathcal{Q} \rightarrow \mathbb{R}$ be continuous with respect to (t, S) . For $(\hat{t}, \bar{S}^a, \bar{S}^b, \hat{S}, \bar{q}) \in [0, T] \times \mathcal{D} \times \mathcal{Q}$, we say that $(y, p, A) \in \mathbb{R}^3$ is in the subset $\mathcal{P}^-U(\hat{t}, \bar{S}^a, \bar{S}^b, \hat{S}, \bar{q})$ (resp. the superjet $\mathcal{P}^+U(\hat{t}, \bar{S}^a, \bar{S}^b, \hat{S}, \bar{q})$) if

$$\begin{aligned}
U(t, \bar{S}^a, \bar{S}^b, S, \bar{q}) &\geq U(\hat{t}, \bar{S}^a, \bar{S}^b, \hat{S}, \bar{q}) + y(t - \hat{t}) + p(S - \hat{S}) + \frac{1}{2} A(S - \hat{S})^2 + o\left(|t - \hat{t}| + |S - \hat{S}|^2\right), \\
\left(\text{resp. } U(t, \bar{S}^a, \bar{S}^b, S, \bar{q}) &\leq U(\hat{t}, \bar{S}^a, \bar{S}^b, \hat{S}, \bar{q}) + y(t - \hat{t}) + p(S - \hat{S}) + \frac{1}{2} A(S - \hat{S})^2 + o\left(|t - \hat{t}| + |S - \hat{S}|^2\right)\right),
\end{aligned}$$

for all (t, S) such that $(t, \bar{S}^a, \bar{S}^b, S, \bar{q}) \in [0, T] \times \mathcal{D} \times \mathcal{Q}$.

We also define $\bar{\mathcal{P}}^-U(\hat{t}, \bar{S}^a, \bar{S}^b, \hat{S}, \bar{q})$ as the set of points $(y, p, A) \in \mathbb{R}^3$ such that there exists a sequence $(t_n, \bar{S}^a, \bar{S}^b, S_n, \bar{q}, y_n, p_n, A_n) \in [0, T] \times \mathcal{D} \times \mathcal{Q} \times \mathcal{P}^-U(t_n, \bar{S}^a, \bar{S}^b, S_n, \bar{q})$ satisfying

$$(t_n, \bar{S}^a, \bar{S}^b, S_n, \bar{q}, y_n, p_n, A_n) \xrightarrow{n \rightarrow +\infty} (\hat{t}, \bar{S}^a, \bar{S}^b, \hat{S}, \bar{q}, y, p, A).$$

The set $\bar{\mathcal{P}}^+U(\hat{t}, \bar{S}^a, \bar{S}^b, \hat{S}, \bar{q})$ is defined similarly.

Let us recall one of the definitions of viscosity solutions which we are going to use for the proof of the uniqueness.

Lemma 1. *A continuous function \tilde{U} is a viscosity supersolution (resp. subsolution) to (5.14) on $[0, T] \times \mathcal{D} \times \mathcal{Q}$ if and only if for all $(\hat{t}, \bar{S}^a, \bar{S}^b, \hat{S}, \bar{q}) \in [0, T] \times \mathcal{D} \times \mathcal{Q}$ and all $(\hat{y}, \hat{p}, \hat{A}) \in \tilde{\mathcal{P}}^- U(\hat{t}, \bar{S}^a, \bar{S}^b, \hat{S}, \bar{q})$ (resp. $\tilde{\mathcal{P}}^+ U(\hat{t}, \bar{S}^a, \bar{S}^b, \hat{S}, \bar{q})$), we have*

$$\begin{aligned} & -\rho \tilde{U}(\hat{t}, \bar{S}^a, \bar{S}^b, \hat{S}, \bar{q}) + \hat{y} - \phi \bar{q}^2 - \phi_- \bar{q}^2 \mathbf{1}_{q < 0} + \frac{1}{2} \sigma^2 A \\ & + \frac{\lambda}{1 + (\kappa \alpha^a)^2} \max_{\ell^a \in \{0, 1\}} \left\{ \ell^a \left(e^{\rho t} \bar{S}^a + U(\hat{t}, \bar{S}^a, \bar{S}^b, \hat{S}, \bar{q} - \ell^a) - U(\hat{t}, \bar{S}^a, \bar{S}^b, \hat{S}, \bar{q}) \right) \right\} \\ & + \frac{\lambda}{1 + (\kappa \alpha^b)^2} \max_{\ell^b \in \{0, 1\}} \left\{ \ell^b \left(e^{\rho t} (-\bar{S}^b) + U(\hat{t}, \bar{S}^a, \bar{S}^b, \hat{S}, \bar{q} + \ell^b) - U(\hat{t}, \bar{S}^a, \bar{S}^b, \hat{S}, \bar{q}) \right) \right\} \leq 0 \end{aligned}$$

(resp. ≥ 0).

We refer to [33] for a proof of this result. We can now state a maximum principle from which the uniqueness can be easily deduced:

Proposition 9. *Let U (resp. V) be a continuous viscosity supersolution (resp. subsolution) of (5.4) with polynomial growth on $[0, T] \times \mathcal{D} \times \mathcal{Q}$ and satisfying the continuity conditions (5.6). If $U \geq V$ on $\{T\} \times \mathcal{D} \times \mathcal{Q}$, then $U \geq V$ on $[0, T] \times \mathcal{D} \times \mathcal{Q}$.*

Proof. As before, we introduce the functions \tilde{U} and \tilde{V} such that

$$\tilde{U}(t, S^a, S^b, S, q) = e^{\rho t} U(t, S^a, S^b, S, q) \quad \text{and} \quad \tilde{V}(t, S^a, S^b, S, q) = e^{\rho t} V(t, S^a, S^b, S, q).$$

Then \tilde{U} and \tilde{V} are respectively viscosity supersolution and subsolution of Equation (5.14) on $[0, T] \times \mathcal{D} \times \mathcal{Q}$ with $\tilde{U} \geq \tilde{V}$ on $\{T\} \times \mathcal{D} \times \mathcal{Q}$. To prove the proposition, it is enough to prove that $\tilde{U} \geq \tilde{V}$ on $[0, T] \times \mathcal{D} \times \mathcal{Q}$. We proceed by contradiction. Let us assume that $\sup_{[0, T] \times \mathcal{D} \times \mathcal{Q}} \tilde{V} - \tilde{U} > 0$. Let $p \in \mathbb{N}^*$

such that

$$\lim_{\|S\|_2 \rightarrow +\infty} \sup_{\substack{t \in [0, T], q \in \mathcal{Q} \\ (S, S^a, S^b) \in \mathcal{D}}} \frac{|\tilde{U}(t, S^a, S^b, S, q)| + |\tilde{V}(t, S^a, S^b, S, q)|}{1 + \|S\|_2^{2p}} = 0,$$

where $\|\cdot\|_2$ is the Euclidian norm. Then there exists $(\hat{t}, \bar{S}^a, \bar{S}^b, \hat{S}, \bar{q}) \in [0, T] \times \mathcal{D} \times \mathcal{Q}$ such that

$$\begin{aligned} 0 & < \tilde{V}(\hat{t}, \bar{S}^a, \bar{S}^b, \hat{S}, \bar{q}) - \tilde{U}(\hat{t}, \bar{S}^a, \bar{S}^b, \hat{S}, \bar{q}) - \phi(\hat{t}, \hat{S}, \hat{S}, \bar{q}) \\ & = \sup_{(t, S^a, S^b, S, q)} \tilde{V}(t, S^a, S^b, S, q) - \tilde{U}(t, S^a, S^b, S, q) - \phi(t, S, S, q), \end{aligned}$$

where

$$\phi(t, S, R, q) := \varepsilon e^{-\mu t} (1 + \|S\|_2^{2p} + \|R\|_2^{2p}),$$

with $\varepsilon > 0$, $\mu > 0$. The choice of the function ϕ allows us to look for a supremum in a bounded set with respect to (S, S^a, S^b) . Then the supremum is either reached for a point in $[0, T] \times \mathcal{D} \times \mathcal{Q}$ or on $[0, T] \times \partial \mathcal{D} \times \mathcal{Q}$ (recall that \mathcal{D} is open). But the continuity conditions tell us that if the supremum is reached on $[0, T] \times \partial \mathcal{D} \times \mathcal{Q}$, it is also reached in $[0, T] \times \mathcal{D} \times \mathcal{Q}$. Since $\tilde{U} \geq \tilde{V}$ on $\{T\} \times \mathcal{D} \times \mathcal{Q}$, it is clear that $\hat{t} < T$.

Then, for all $n \in \mathbb{N}^*$, we can find $(t_n, S_n, R_n) \in [0, T] \times \mathbb{R}^2$ such that $(\bar{S}^a, \bar{S}^b, S_n), (\bar{S}^a, \bar{S}^b, R_n) \in \mathcal{D}$ and

$$\begin{aligned} 0 &< \tilde{V}(t_n, \bar{S}^a, \bar{S}^b, S_n, \bar{q}) - \tilde{U}(t_n, \bar{S}^a, \bar{S}^b, R_n, \bar{q}) \\ &\quad - \phi(t_n, S_n, R_n, \bar{q}) - n|S_n - R_n|^2 - \left(|t_n - \hat{t}|^2 + |S_n - \hat{S}|^4 \right) \\ &= \sup_{(t, S, R)} \tilde{V}(t, \bar{S}^a, \bar{S}^b, S, \bar{q}) - \tilde{U}(t, \bar{S}^a, \bar{S}^b, R, \bar{q}) \\ &\quad - \phi(t, S, R, \bar{q}) - n|S - R|^2 - \left(|t - \hat{t}|^2 + |S - \hat{S}|^4 \right). \end{aligned}$$

Then, we have

$$(t_n, S_n, R_n) \xrightarrow{n \rightarrow +\infty} (\hat{t}, \hat{S}, \hat{S}),$$

and

$$\begin{aligned} &\tilde{V}(t_n, \bar{S}^a, \bar{S}^b, S_n, \bar{q}) - \tilde{U}(t_n, \bar{S}^a, \bar{S}^b, R_n, \bar{q}) \\ &\quad - \phi(t_n, S_n, R_n) - n|S_n - R_n|^2 - \left(|t_n - \hat{t}|^2 + |S_n - \hat{S}|^4 \right) \\ &\xrightarrow{n \rightarrow +\infty} \tilde{V}(\hat{t}, \bar{S}^a, \bar{S}^b, \hat{S}, \bar{q}) - \tilde{U}(\hat{t}, \bar{S}^a, \bar{S}^b, \hat{S}, \bar{q}) - \phi(\hat{t}, \hat{S}, \hat{S}). \end{aligned}$$

For $n \in \mathbb{N}^*$, let us write for $(t, S, R) \in [0, T] \times \mathbb{R}^2$

$$\varphi_n(t, S, R) := \phi(t, S, R) + n|S - R|^2 + |t - \hat{t}|^2 + |S - \hat{S}|^4.$$

Then Ishii's Lemma (see [20, 43]) guarantees that for any $\eta > 0$, we can find $(y_n^1, p_n^1, A_n^1) \in \bar{\mathcal{P}}^+ \tilde{V}(t_n, \bar{S}^a, \bar{S}^b, S_n, \bar{q})$ and $(y_n^2, p_n^2, A_n^2) \in \bar{\mathcal{P}}^- \tilde{U}(t_n, \bar{S}^a, \bar{S}^b, R_n, \bar{q})$ such that

$$y_n^1 - y_n^2 = \partial_t \varphi_n(t_n, S_n, R_n), \quad (p_n^1, p_n^2) = (\partial_S \varphi_n, -\partial_R \varphi_n)(t_n, S_n, R_n)$$

and

$$\begin{pmatrix} A_n^1 & 0 \\ 0 & -A_n^2 \end{pmatrix} \leq H_{SR} \varphi_n(t_n, S_n, R_n) + \eta (H_{SR} \varphi_n(t_n, S_n, R_n))^2,$$

where $H_{SR} \varphi_n(t_n, \dots)$ denotes the Hessian of $\varphi_n(t_n, \dots)$. Applying Lemma 1, we get

$$\begin{aligned} \rho \left(\tilde{V}(t_n, \bar{S}^a, \bar{S}^b, S_n, \bar{q}) - \tilde{U}(t_n, \bar{S}^a, \bar{S}^b, R_n, \bar{q}) \right) &\leq y_n^1 - y_n^2 + \frac{1}{2} \sigma^2 (A_n^1 - A_n^2) \\ &+ \frac{\lambda}{1 + (\kappa \alpha^a)^2} \max_{\ell^a \in \{0, 1\}} \left\{ \ell^a \left(e^{\rho t_n} \bar{S}^a + \tilde{V}(t_n, \bar{S}^a, \bar{S}^b, S_n, \bar{q} - \ell^a) - \tilde{V}(t_n, \bar{S}^a, \bar{S}^b, S_n, \bar{q}) \right) \right\} \\ &+ \frac{\lambda}{1 + (\kappa \alpha^b)^2} \max_{\ell^b \in \{0, 1\}} \left\{ \ell^b \left(e^{\rho t_n} (-\bar{S}^b) + \tilde{V}(t_n, \bar{S}^a, \bar{S}^b, S_n, \bar{q} + \ell^b) - \tilde{V}(t_n, \bar{S}^a, \bar{S}^b, S_n, \bar{q}) \right) \right\} \\ &- \frac{\lambda}{1 + (\kappa \alpha^a)^2} \max_{\ell^a \in \{0, 1\}} \left\{ \ell^a \left(e^{\rho t_n} \bar{S}^a + \tilde{U}(t_n, \bar{S}^a, \bar{S}^b, R_n, \bar{q} - \ell^a) - \tilde{U}(t_n, \bar{S}^a, \bar{S}^b, R_n, \bar{q}) \right) \right\} \\ &- \frac{\lambda}{1 + (\kappa \alpha^b)^2} \max_{\ell^b \in \{0, 1\}} \left\{ \ell^b \left(e^{\rho t_n} (-\bar{S}^b) + \tilde{U}(t_n, \bar{S}^a, \bar{S}^b, R_n, \bar{q} + \ell^b) - \tilde{U}(t_n, \bar{S}^a, \bar{S}^b, R_n, \bar{q}) \right) \right\}. \end{aligned}$$

Moreover, we have

$$H_{SR}\varphi_n(t_n, S_n, R_n) = \begin{pmatrix} \partial_{SS}^2\phi(t_n, S_n, R_n) + 2n + 12(S_n - \hat{S})^2 & \partial_{SR}^2\phi(t_n, S_n, R_n) - 2n \\ \partial_{SR}^2\phi(t_n, S_n, R_n) - 2n & \partial_{SR}^2\phi(t_n, S_n, R_n) + 2n \end{pmatrix}.$$

It follows that

$$\begin{aligned} & \rho \left(\tilde{V}(t_n, \bar{S}^a, \bar{S}^b, S_n, \bar{q}) - \tilde{U}(t_n, \bar{S}^a, \bar{S}^b, R_n, \bar{q}) \right) \leq \partial_t \phi(t_n, S_n, R_n) + 2(t_n - \hat{t}) \\ & + \frac{1}{2} \sigma^2 \left(\partial_{SS}^2\phi(t_n, S_n, R_n) + \partial_{RR}^2\phi(t_n, S_n, R_n) + 2\partial_{SR}^2\phi(t_n, S_n, R_n) + 12(S_n - \hat{S}) \right) + \eta C_n \\ & + \frac{\lambda}{1 + (\kappa\alpha^a)^2} \max_{\ell^a \in \{0,1\}} \left\{ \ell^a \left(e^{\rho t_n} \bar{S}^a + \tilde{V}(t_n, \bar{S}^a, \bar{S}^b, S_n, \bar{q} - \ell^a) - \tilde{V}(t_n, \bar{S}^a, \bar{S}^b, S_n, \bar{q}) \right) \right\} \\ & + \frac{\lambda}{1 + (\kappa\alpha^b)^2} \max_{\ell^b \in \{0,1\}} \left\{ \ell^b \left(e^{\rho t_n} (-\bar{S}^b) + \tilde{V}(t_n, \bar{S}^a, \bar{S}^b, S_n, \bar{q} + \ell^b) - \tilde{V}(t_n, \bar{S}^a, \bar{S}^b, S_n, \bar{q}) \right) \right\} \\ & - \frac{\lambda}{1 + (\kappa\alpha^a)^2} \max_{\ell^a \in \{0,1\}} \left\{ \ell^a \left(e^{\rho t_n} \bar{S}^a + \tilde{U}(t_n, \bar{S}^a, \bar{S}^b, R_n, \bar{q} - \ell^a) - \tilde{U}(t_n, \bar{S}^a, \bar{S}^b, R_n, \bar{q}) \right) \right\} \\ & - \frac{\lambda}{1 + (\kappa\alpha^b)^2} \max_{\ell^b \in \{0,1\}} \left\{ \ell^b \left(e^{\rho t_n} (-\bar{S}^b) + \tilde{U}(t_n, \bar{S}^a, \bar{S}^b, R_n, \bar{q} + \ell^b) - \tilde{U}(t_n, \bar{S}^a, \bar{S}^b, R_n, \bar{q}) \right) \right\}, \end{aligned}$$

where C_n does not depend on η . Therefore, as the maximums on the right-hand side are always positive, we deduce that for all $n \in \mathbb{N}^*$,

$$\begin{aligned} & \rho \left(\tilde{V}(t_n, \bar{S}^a, \bar{S}^b, S_n, \bar{q}) - \tilde{U}(t_n, \bar{S}^a, \bar{S}^b, R_n, \bar{q}) \right) \leq \partial_t \phi(t_n, S_n, R_n) + 2(t_n - \hat{t}) \\ & + \frac{1}{2} \sigma^2 \left(\partial_{SS}^2\phi(t_n, S_n, R_n) + \partial_{RR}^2\phi(t_n, S_n, R_n) + 2\partial_{SR}^2\phi(t_n, S_n, R_n) + 12(S_n - \hat{S}) \right) \\ & + \frac{\lambda}{1 + (\kappa\alpha^a)^2} \max_{\ell^a \in \{0,1\}} \left\{ \ell^a \left(e^{\rho t_n} \bar{S}^a + \tilde{V}(t_n, \bar{S}^a, \bar{S}^b, S_n, \bar{q} - \ell^a) - \tilde{V}(t_n, \bar{S}^a, \bar{S}^b, S_n, \bar{q}) \right) \right\} \\ & + \frac{\lambda}{1 + (\kappa\alpha^b)^2} \max_{\ell^b \in \{0,1\}} \left\{ \ell^b \left(e^{\rho t_n} (-\bar{S}^b) + \tilde{V}(t_n, \bar{S}^a, \bar{S}^b, S_n, \bar{q} + \ell^b) - \tilde{V}(t_n, \bar{S}^a, \bar{S}^b, S_n, \bar{q}) \right) \right\}. \end{aligned}$$

As \tilde{V} is continuous and $(t_n, S_n)_n$ converges to (\hat{t}, \hat{S}) , the last two terms are bounded from above by some constant M . Then by sending n to infinity, we get

$$\begin{aligned} & \rho \left(\tilde{V}(\hat{t}, \bar{S}^a, \bar{S}^b, \hat{S}, \bar{q}) - \tilde{U}(\hat{t}, \bar{S}^a, \bar{S}^b, \hat{S}, \bar{q}) \right) \leq \partial_t \phi(\hat{t}, \hat{S}, \hat{S}) \\ & + \frac{1}{2} \sigma^2 \left(\partial_{SS}^2\phi(\hat{t}, \hat{S}, \hat{S}) + \partial_{RR}^2\phi(\hat{t}, \hat{S}, \hat{S}) + 2\partial_{SR}^2\phi(\hat{t}, \hat{S}, \hat{S}) \right) + M. \end{aligned}$$

For $\mu > 0$ large enough, the right-hand side is strictly negative, and as $\rho > 0$ we get

$$\tilde{V}(\hat{t}, \bar{S}^a, \bar{S}^b, \hat{S}, \bar{q}) - \tilde{U}(\hat{t}, \bar{S}^a, \bar{S}^b, \hat{S}, \bar{q}) < 0,$$

hence the contradiction. □

5.6.3 Effects of the uncertainty zones on h

We keep the same parameters as in Section 5.4 and take $\alpha^a = 0.01$ and $\alpha^b = 0.00625$. We plot the value function of the market maker's problem (the function h) on some small range of values of S . Note that $S = 10.5$ is on both discrete grids. We distinguish 4 possible cases, depending on

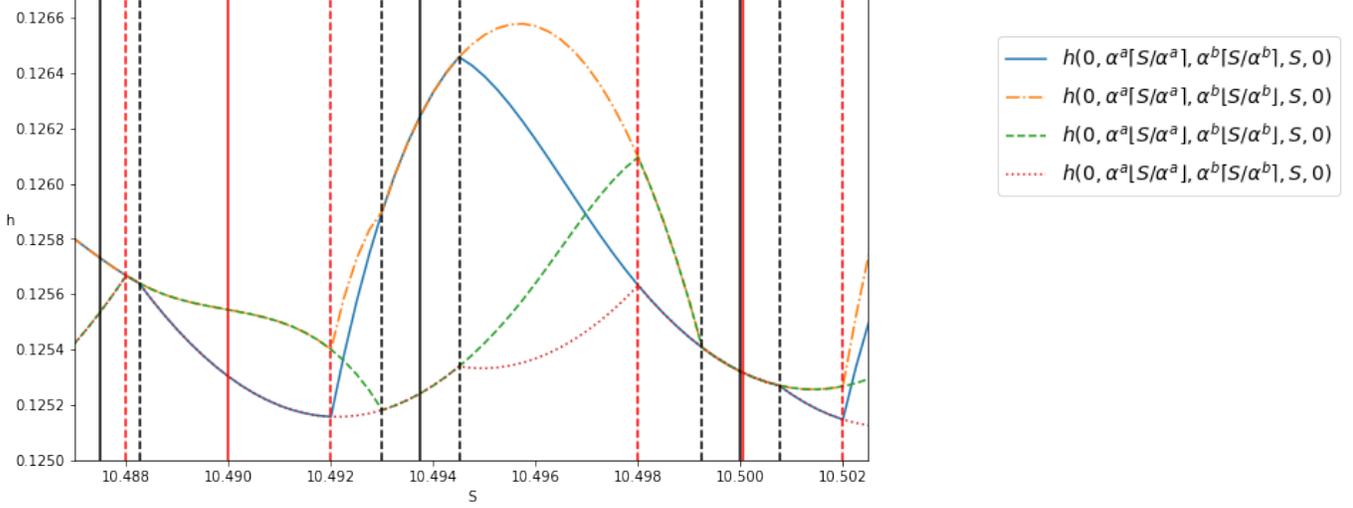


Figure 5.8: Value function h of the market maker for $q = 0$, as a function of S .

whether

- $S^a = \alpha^a \lfloor S/\alpha^a \rfloor$ and $S^b = \alpha^b \lfloor S/\alpha^b \rfloor$ (green dots),
- $S^a = \alpha^a \lfloor S/\alpha^a \rfloor$ and $S^b = \alpha^b \lceil S/\alpha^b \rceil$ (red dash-dots),
- $S^a = \alpha^a \lceil S/\alpha^a \rceil$ and $S^b = \alpha^b \lfloor S/\alpha^b \rfloor$ (orange dash),
- $S^a = \alpha^a \lceil S/\alpha^a \rceil$ and $S^b = \alpha^b \lceil S/\alpha^b \rceil$ (blue solid).

Note that depending on the value of S , some of those cases can be excluded. The solid vertical red and black lines represent respectively the values on the ask ($\alpha^a\mathbb{Z}$) and the bid grid ($\alpha^b\mathbb{Z}$). The dotted vertical lines represent the limits of the uncertainty zones on each side.

In the uncertainty zones, the value function h depends non-trivially on S^a and S^b . Thanks to the continuity conditions at the boundaries of the uncertainty zones, we get a smooth behavior of h when S exits a zone. Remark that when $S \in [10 \pm ((\frac{1}{2} - \eta^a)\alpha^a) \wedge ((\frac{1}{2} - \eta^b)\alpha^b)]$, necessarily $S^a = S^b = 10$.

In our example, $\alpha^a > \alpha^b$ and $(\frac{1}{2} - \eta_a)\alpha^a > (\frac{1}{2} - \eta_b)\alpha^b$. So, if S is in $(10 + (\frac{1}{2} - \eta_b)\alpha^b, (\frac{1}{2} - \eta_a)\alpha^a)$, necessarily $S^a = 10$, but S^b can take either the value 10 or $10 + \alpha^b$ depending on whether S comes from higher prices or lower prices. This is why there are two curves in the interval $(10 + (\frac{1}{2} - \eta_b)\alpha^b, (\frac{1}{2} - \eta_a)\alpha^a)$. At $(\frac{1}{2} - \eta_a)\alpha^a$, two additional curves appear as S^a can also be two different values.

Chapter 6

Optimal execution and statistical arbitrage under Ornstein-Uhlenbeck dynamics

Abstract

In recent years, academics, regulators, and market practitioners have increasingly addressed liquidity issues. Amongst the numerous problems addressed, the optimal execution of large orders is probably the one that has attracted the most research works, mainly in the case of single-asset portfolios. In practice, however, optimal execution problems often involve large portfolios comprising numerous assets, and models should consequently account for risks at the portfolio level. In this paper, we address multi-asset optimal execution in a model where prices have multivariate Ornstein-Uhlenbeck dynamics and where the agent maximizes the expected (exponential) utility of her P&L. We use the tools of stochastic optimal control and simplify the initial multidimensional Hamilton-Jacobi-Bellman equation into a system of ordinary differential equations (ODEs) involving a Matrix Riccati ODE for which classical existence theorems do not apply. By using *a priori* estimates obtained thanks to optimal control tools, we nevertheless prove an existence and uniqueness result for the latter ODE, and then deduce a verification theorem that provides a rigorous solution to the execution problem. Using numerical methods we eventually illustrate our results and discuss their implications. In particular, we show how our model can be used to build statistical arbitrage strategies.

Key words: Optimal execution, Statistical arbitrage, Stochastic optimal control, Riccati equations.

6.1 Introduction

When executing large blocks of assets, financial agents need to control their overall trading costs by finding the optimal balance between trading rapidly to minimize the market price risk and trading

slowly to minimize execution costs and market impact. Building on the first rigorous approaches introduced by Bertsimas and Lo in [30] and Almgren and Chriss in [10] and [11], many models for the optimal execution of large orders have been proposed in the last two decades. Subsequently, almost all practitioners today slice their large orders into small (child) orders according to optimized trading schedules inspired by the academic literature.

The basic Almgren-Chriss model is a discrete-time model where the agent posts market orders (MOs) to maximize a mean-variance objective function. Many extensions of this seminal model have been proposed. Regarding the framework, (Forsyth and Kennedy, [51]) examines the use of quadratic variation rather than variance in the objective function, (Schied and Schöneborn, [96]) uses stochastic control tools to characterize and find optimal strategies for a Von Neumann-Morgenstern investor, and (Guéant, [60]) provides results for optimal liquidation within a Von Neumann-Morgenstern expected utility framework with general market impact functions and derives subsequent results for block trade pricing. As for the model parameters, (Almgren, [7]) studies the case of random execution costs, (Almgren, [8, 9]) addresses stochastic liquidity and volatility, (Lehalle, [81]) discusses how to take into account statistical aspects of the variability of estimators of the main exogenous variables such as volumes or volatilities in the optimization phase, and (Cartea and Jaimungal, [38]) provides a closed-form strategy incorporating order flows from all agents. Furthermore, numerous market impact and limit order book (LOB) models have also been studied. For instance, (Obizhaeva and Wang, [89]), later generalized in (Alfonsi, Fruth and Schied, [5]), proposes a single-asset market impact model where price dynamics are derived from a dynamic LOB model with resilience, (Alfonsi and Schied, [6]) derives explicit optimal execution strategies in a discrete-time LOB model with general shape functions and an exponentially decaying price impact, (Gatheral, [55]) uses the no-dynamic-arbitrage principle to address the viability of market impact models, and (Gatheral, Schied and Slynko, [57]) obtains explicit optimal strategies with a transient market impact in an expected cost minimization setup. As for order and execution strategy types, the Almgren-Chriss framework focuses on orders of the Implementation Shortfall (IS) type with MOs only. Other execution strategies have been studied in the literature, like Volume-Weighted Average Price (VWAP) orders in (Konishi, [76]), (Frei and Westray, [53]) and (Guéant and Royer, [67]), but also Target Close (TC) orders and Percentage of Volume (POV) orders, in (Guéant, [61]). Besides, several models focusing on optimal execution with limit orders have been proposed, as in (Bayraktar and Ludkovski, [22]), but also in (Guéant, Lehalle, and Fernandez-Tapia, [64]) and (Guéant and Lehalle, [63]). Regarding the existence of several venues, the case of optimal splitting of orders across different liquidity pools has been addressed in (Laruelle, Lehalle, and Pages, [79]), in (Cartea, Jaimungal, and Penalva, [39]), and more recently in (Baldacci and Manziuk, [16]).

Another recent and important stream of the optimal execution literature deals with adding predictive signals of future price changes.¹ Typical examples of these signals include order book imbalances, forecasts of the future order flow of market participants, and other price-based technical indicators. The usual formalism in the literature with predictive signals is to consider Brownian or Black-Scholes dynamics, along with independent mean-reverting Markov signals. The case of Ornstein-Uhlenbeck-type signals is of special interest as it usually leads to closed-form formulas. For the interested reader, we refer to (Belak and Muhle-Karbe, [23]) where the authors consider

¹We consider this stream of the literature to be closely related to our topic of multi-asset optimal execution. Indeed, when trading an asset, the dynamics of another asset within or outside the portfolio can be regarded as a predictive signal that can enhance the execution process.

optimal execution with general Markov signals and an application to “target zone models”, and to (Lehalle and Neuman, [83]) and (Neuman and Voß, [88]) in which the authors provide an optimal trading framework incorporating Markov signals and a transient market impact.

In practice, operators routinely face the problem of having to execute simultaneously large orders regarding various assets, such as in block trading for funds facing large subscriptions or withdrawals, or when considering multi-asset trades in statistical arbitrage trading strategies. More generally, banks and market makers manage their (il)liquidity and market risk, when it comes to executing trades, in the context of a central risk book; hence the need for multi-asset models. However, in contrast to the single-asset case, the existing literature on the joint execution scheduling of large orders in multiple assets, or a single asset inside a multi-asset portfolio, is rather limited. Besides, most papers simply consider correlated Brownian motions when modelling the joint dynamics of prices. The problem of using single-asset models or unrealistic multivariate models for portfolio trading is that they do not balance execution and market impact with price risk at the portfolio or strategy level, and the resulting trading curves of individual assets usually turn out to be suboptimal.

The first paper presenting a way to build multi-asset trading curves in an optimized way is (Almgren and Chriss, [11]). Almgren and Chriss consider indeed, in an appendix of their seminal paper, a multi-asset extension of their discrete-time model – see Appendix 6.4 for a solution of the classical portfolio execution problem in a continuous-time setting with correlated Brownian dynamics for prices. Several extensions to this model have been proposed since then. (Lehalle, [82]) considers adding an inventory constraint to balance the different portfolio lines during the portfolio execution process. (Schied and Schöneborn, [97]) shows, under general continuous-time multidimensional price and market impact dynamics and for an exponential utility objective function, that deterministic strategies are optimal. In (Cartea, Jaimungal, and Penalva, [39]), the authors use stochastic control tools to derive optimal execution strategies for basic multi-asset trading algorithms such as optimal entry/exit times and cointegration-based statistical arbitrage. (Bismuth, Guéant, and Pu, [31]) addresses optimal portfolio liquidation (along with other portfolio related problems) by coupling Bayesian learning and stochastic control to derive optimal strategies under uncertainty on model parameters in the Almgren-Chriss framework. Regarding the literature around the addition of predictive signals, (Emschwiller, Petit, and Bouchaud, [47]) extends optimal trading with Markovian predictors to the multi-asset case, with linear trading costs, using a mean-field approach that reduces the problem to a single-asset one.

A notable model for the multivariate dynamics of financial variables that goes beyond the simple one where prices diffuse like correlated Brownian motions is the multivariate Ornstein-Uhlenbeck (multi-OU) model. It is especially attractive because it is parsimonious, and yet general enough to cover a wide spectrum of multi-dimensional dynamics. Multi-OU dynamics offer a large coverage since particular cases include correlated Brownian motions but also cointegrated dynamics which are heavily used in statistical arbitrage. (Cartea, Gan, and Jaimungal, [37]) is, to our knowledge, the pioneering paper in the use of the multi-OU model for the price dynamics in a multi-asset optimal execution problem. Indeed, the authors proposed an interesting model where the asset prices have multi-OU dynamics and the agent maximizes an objective function given by the expectation of the P&L minus a running penalty related to the instantaneous variance of the portfolio. In their approach, the problem boils down to a system of ODEs involving a Matrix Riccati ODE for which the classical existence theorems related to linear-quadratic control theory apply.

In the paper [24], we propose a model similar to the one in [37], but where the objective function is of the Von Neumann-Morgenstern type: an expected exponential utility of the P&L.² By using classical stochastic optimal control tools we show that the problem boils down to solving a system of ODEs involving a Matrix Riccati ODE. However, unlike what happens in [37], the use of an expected exponential utility framework to account for the risk leads to a Matrix Riccati ODE for which classical existence theorems do not apply. By using *a priori* estimates obtained thanks to optimal control tools, we nevertheless prove an existence and uniqueness result for the latter ODE, and then deduce a verification theorem that provides a rigorous solution to the execution problem.

The main contribution of this paper is therefore to propose a model for multi-asset portfolio execution under multi-OU price dynamics in an expected utility framework that accounts for the overall risk associated with the execution process. We focus on the problem where an agent is in charge of unwinding a large portfolio, but also illustrate the use of our results for multi-asset statistical arbitrage purposes.

The remainder of this paper is organized as follows. In Section 2 we present the optimal execution problem in the form of a stochastic optimal control problem and show that solving the associated Hamilton-Jacobi-Bellman (HJB) equation boils down to solving a system of ODEs involving a Matrix Riccati ODE. We then prove a global existence result for that ODE and eventually provide a solution to the initial stochastic optimal control problem thanks to a verification argument. In Section 3, we then illustrate our results with numerical approximations of the optimal strategies and numerical simulations of prices. Our examples focus on optimal liquidation but we also illustrate and discuss the use of our results for building statistical arbitrage strategies. The core of the paper is followed by two appendices: one dedicated to the special case where the multi-OU dynamics reduces to a simple correlated Brownian dynamics and another dedicated to some form of limit case where execution costs are ignored – the latter case being useful to obtain *a priori* estimates for our general problem.

6.2 The optimal liquidation problem

6.2.1 Modelling framework and notations

In this paper, we consider a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{F} = (\mathcal{F}_t)_{t \in [0, T]})$ satisfying the usual conditions. We assume this probability space to be large enough to support all the processes we introduce.

We consider a market with $d \in \mathbb{N}^*$ assets, and a trader wishing to liquidate her portfolio over a period of time $[0, T]$, with $T > 0$. Her inventory process³ $(q_t)_{t \in [0, T]} = (q_t^1, \dots, q_t^d)_{t \in [0, T]}^\top$ evolves as

$$dq_t = v_t dt, \tag{6.1}$$

²Our model accounts therefore for the risk in a different manner than the model presented in [39]. Comparisons are difficult to carry out as risk aversion parameters in the two models have different meanings.

³The superscript \top designates the transpose operator. It transforms here a line vector into a column vector.

with $q_0 \in \mathbb{R}^d$ given, where $(v_t)_{t \in [0, T]} = (v_t^1, \dots, v_t^d)_{t \in [0, T]}^\top$ represents the trading rate of the trader for each asset.

The prices of the d assets are modelled as a d -dimensional Ornstein-Uhlenbeck process $(S_t)_{t \in [0, T]} = (S_t^1, \dots, S_t^d)_{t \in [0, T]}^\top$.⁴

$$dS_t = R(\bar{S} - S_t)dt + VdW_t, \quad (6.2)$$

with $S_0 \in \mathbb{R}^d$ given, where $\bar{S} \in \mathbb{R}^d$, $R \in \mathcal{M}_d(\mathbb{R})$, $V \in \mathcal{M}_{d, k}(\mathbb{R})$, and $(W_t)_{t \in [0, T]} = (W_t^1, \dots, W_t^k)_{t \in [0, T]}^\top$ is a k -dimensional standard Brownian motion (with independent coordinates) for some $k \in \mathbb{N}^*$. For what follows, we introduce $\Sigma = VV^\top$.

Finally, the process $(X_t)_{t \in [0, T]}$ modelling the trader's cash account has the dynamics

$$dX_t = -v_t^\top S_t dt - L(v_t)dt, \quad (6.3)$$

with $X_0 \in \mathbb{R}$ given, where $L : \mathbb{R}^d \rightarrow \mathbb{R}_+$ is a function representing the temporary market impact or execution costs of the trader. In this paper, we mainly consider the case where L is a positive-definite quadratic form (see below).

The trader aims at maximizing the expected utility of her wealth at the end of the trading window $[0, T]$. This wealth is the sum of the amount X_T on the cash account at time T and the value of the remaining inventory evaluated here as $q_T^\top S_T - \ell(q_T)$, where the term $\ell(q_T)$ is a discount applied to the Mark-to-Market (MtM) value that proxies liquidity and market price risk and penalizes any terminal non-zero position. In what follows, we mainly consider the case where the penalty function $\ell : \mathbb{R}^d \rightarrow \mathbb{R}_+$ is a positive-semidefinite quadratic form (see below).

Mathematically, the trader therefore wants to solve the dynamic optimization problem

$$\sup_{v \in \mathcal{A}} \mathbb{E} \left[-e^{-\gamma(X_T + q_T^\top S_T - \ell(q_T))} \right], \quad (6.4)$$

where $\gamma > 0$ is the absolute risk aversion parameter of the trader, and \mathcal{A} is the set of admissible controls, to be defined below.

To define the set of admissible controls \mathcal{A} , we first introduce a notion of "linear growth" relevant in our context.

Definition 2. *Let $t \in [0, T]$. An \mathbb{R}^d -valued, \mathbb{F} -adapted process $(\zeta_s)_{s \in [t, T]}$ is said to satisfy a linear growth condition on $[t, T]$ with respect to $(S_s)_{s \in [t, T]}$ if there exists a constant $C_T > 0$ such that for all $s \in [t, T]$,*

$$\|\zeta_s\| \leq C_T \left(1 + \sup_{\tau \in [t, s]} \|S_\tau\| \right)$$

almost surely.

⁴The generalization with a permanent impact component is straightforward.

We then define for all $t \in [0, T]$:

$$\mathcal{A}_t = \{(v_s)_{s \in [t, T]}, \mathbb{R}^d\text{-valued, } \mathbb{F}\text{-adapted, satisfying a linear growth condition with respect to } (S_s)_{s \in [t, T]}\}, \quad (6.5)$$

and take $\mathcal{A} := \mathcal{A}_0$.

It is natural to use the tools of stochastic optimal control to solve the above dynamic optimization problem. Let us define the value function of the problem $u : [0, T] \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$ as

$$u(t, x, q, S) = \sup_{v \in \mathcal{A}_t} \mathbb{E} \left[-e^{-\gamma(X_T^{t,x,S,v} + (q_T^{t,q,v})^\top S_T^{t,S} - \ell(q_T^{t,q,v}))} \right], \quad (6.6)$$

where for $(t, x, q, S) \in [0, T] \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d$ and $v \in \mathcal{A}_t$, the processes $(q_s^{t,q,v})_{s \in [t, T]}$, $(S_s^{t,S})_{s \in [t, T]}$, and $(X_s^{t,x,S,v})_{s \in [t, T]}$ have respective dynamics

$$\begin{aligned} dq_s^{t,q,v} &= v_s ds, \\ dS_s^{t,S} &= R(\bar{S} - S_s^{t,S}) ds + V dW_s, \end{aligned}$$

and

$$dX_s^{t,x,S,v} = -v_s^\top S_s^{t,S} ds - L(v_s) ds,$$

with $S_t^{t,S} = S$, $q_t^{t,q,v} = q$, and $X_t^{t,x,S,v} = x$.

6.2.2 Hamilton-Jacobi-Bellman equation

The HJB equation associated with the problem (6.4) is given by

$$\begin{aligned} 0 &= \partial_t w(t, x, q, S) + \sup_{v \in \mathbb{R}^d} (-(v^\top S + L(v)) \partial_x w(t, x, q, S) + v^\top \nabla_q w(t, x, q, S)) \\ &\quad + (\bar{S} - S)^\top R^\top \nabla_S w(t, x, q, S) + \frac{1}{2} \text{Tr}(\Sigma D_S^2 w(t, x, q, S)), \end{aligned} \quad (6.7)$$

for all $(t, x, q, S) \in [0, T] \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d$ with the terminal condition

$$w(T, x, q, S) = -e^{-\gamma(x + q^\top S - \ell(q))} \quad \forall (x, q, S) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d. \quad (6.8)$$

In order to study (6.7), we are going to use the following ansatz:

$$w(t, x, q, S) = -e^{-\gamma(x + q^\top S + \theta(t, q, S))} \quad \forall (t, x, q, S) \in [0, T] \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d. \quad (6.9)$$

The interest of this ansatz is based on the following proposition:

Proposition 8. *Let $\tau < T$. If there exists $\theta : [\tau, T] \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$ solution to*

$$\begin{aligned} 0 &= \partial_t \theta(t, q, S) + \sup_{v \in \mathbb{R}^d} (v^\top \nabla_q \theta(t, q, S) - L(v)) + \frac{1}{2} \text{Tr}(\Sigma D_S^2 \theta(t, q, S)) \\ &\quad - \frac{\gamma}{2} (q + \nabla_S \theta(t, q, S))^\top \Sigma (q + \nabla_S \theta(t, q, S)) + (\bar{S} - S)^\top R^\top (\nabla_S \theta(t, q, S) + q) \end{aligned} \quad (6.10)$$

on $[\tau, T] \times \mathbb{R}^d \times \mathbb{R}^d$, with terminal condition

$$\theta(T, q, S) = -\ell(q) \quad \forall (q, S) \in \mathbb{R}^d \times \mathbb{R}^d, \quad (6.11)$$

then the function $w : [\tau, T] \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$ defined by

$$w(t, x, q, S) = -e^{-\gamma(x+q^\top S+\theta(t,q,S))} \quad \forall (t, x, q, S) \in [\tau, T] \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d$$

is a solution to (6.7) on $[\tau, T] \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d$ with terminal condition (6.8).

Proof. Let $\theta : [\tau, T] \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$ be a solution to (6.10) on $[\tau, T] \times \mathbb{R}^d \times \mathbb{R}^d$ with terminal condition (6.11), then we have for all $(t, x, q, S) \in [\tau, T] \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d$:

$$\begin{aligned} & \partial_t w(t, x, q, S) + \sup_{v \in \mathbb{R}^d} (-(v^\top S + L(v))\partial_x w(t, x, q, S) + v^\top \nabla_q w(t, x, q, S)) \\ & + (\bar{S} - S)^\top R^\top \nabla_S w(t, x, q, S) + \frac{1}{2} \text{Tr} (\Sigma D_{SS}^2 w(t, x, q, S)) \\ = & -\gamma \partial_t \theta(t, q, S) w(t, x, q, S) + \sup_{v \in \mathbb{R}^d} (\gamma(v^\top S + L(v))w(t, x, q, S) - \gamma v^\top (\nabla_q \theta(t, q, S) + S)w(t, x, q, S)) \\ & + \frac{\gamma^2}{2} \text{Tr} (\Sigma(q + \nabla_S \theta(t, q, S))(q + \nabla_S \theta(t, q, S))^\top w(t, x, q, S)) \\ & - \gamma(\bar{S} - S)^\top R^\top (\nabla_S \theta(t, q, S) + q)w(t, x, q, S) - \frac{1}{2} \text{Tr} (\gamma \Sigma D_{SS}^2 \theta(t, q, S)w(t, x, q, S)) \\ = & -\gamma w(t, x, q, S) \left(\partial_t \theta(t, q, S) + \sup_{v \in \mathbb{R}^d} (v^\top \nabla_q \theta(t, q, S) - L(v)) + \frac{1}{2} \text{Tr} (\Sigma D_{SS}^2 \theta(t, q, S)) \right. \\ & \left. - \frac{\gamma}{2} (q + \nabla_S \theta(t, q, S))^\top \Sigma (q + \nabla_S \theta(t, q, S)) + (\bar{S} - S)^\top R^\top (\nabla_S \theta(t, q, S) + q) \right) \\ = & 0. \end{aligned}$$

As it is straightforward to verify that w satisfies the terminal condition (6.8), the result is proved. \square

Assumption 1. From now on, we assume that the functions L and ℓ are of the form $L(v) = v^\top \eta v$ and $\ell(q) = q^\top \Gamma q$, for some $\eta \in \mathcal{S}_d^{++}(\mathbb{R})$ and $\Gamma \in \mathcal{S}_d^+(\mathbb{R})$.

With the above assumption, the Legendre-Fenchel transform of L writes

$$H : p \in \mathbb{R}^d \mapsto \sup_{v \in \mathbb{R}^d} v^\top p - v^\top \eta v = \frac{1}{4} p^\top \eta^{-1} p, \quad (6.12)$$

as the supremum is reached at $v^* = \frac{1}{2} \eta^{-1} p$.

Consequently, we get the following HJB equation for θ :

$$\begin{aligned} 0 = & \partial_t \theta(t, q, S) + \frac{1}{4} \nabla_q \theta(t, q, S)^\top \eta^{-1} \nabla_q \theta(t, q, S) + \frac{1}{2} \text{Tr} (\Sigma D_{SS}^2 \theta(t, q, S)) \\ & - \frac{\gamma}{2} (q + \nabla_S \theta(t, q, S))^\top \Sigma (q + \nabla_S \theta(t, q, S)) + (\bar{S} - S)^\top R^\top (\nabla_S \theta(t, q, S) + q), \end{aligned} \quad (6.13)$$

with terminal condition

$$\theta(T, q, S) = -q^\top \Gamma q \quad \forall (q, S) \in \mathbb{R}^d \times \mathbb{R}^d. \quad (6.14)$$

To further study (6.13), we introduce a second ansatz and look for a solution θ of the following form:

$$\theta(t, q, S) = q^\top A(t)q + q^\top B(t)S + S^\top C(t)S + D(t)^\top q + E(t)^\top S + F(t) \quad \forall (t, q, S) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^d. \quad (6.15)$$

The interest of this ansatz is stated in the following proposition:

Proposition 9. *Let $\tau < T$. Assume there exist $A \in C^1([\tau, T], \mathcal{S}_d(\mathbb{R}))$, $B \in C^1([\tau, T], \mathcal{M}_d(\mathbb{R}))$, $C \in C^1([\tau, T], \mathcal{S}_d(\mathbb{R}))$, $D \in C^1([\tau, T], \mathbb{R}^d)$, $E \in C^1([\tau, T], \mathbb{R}^d)$, $F \in C^1([\tau, T], \mathbb{R})$ satisfying the system of ODEs*

$$\begin{cases} A'(t) = \frac{\gamma}{2}(B(t) + I_d)\Sigma(B(t)^\top + I_d) - A(t)\eta^{-1}A(t) \\ B'(t) = (B(t) + I_d)R + 2\gamma(B(t) + I_d)\Sigma C(t) - A(t)\eta^{-1}B(t) \\ C'(t) = R^\top C(t) + C(t)R + 2\gamma C(t)\Sigma C(t) - \frac{1}{4}B(t)^\top \eta^{-1}B(t) \\ D'(t) = -(B(t) + I_d)R\bar{S} + \gamma(B(t) + I_d)\Sigma E(t) - A(t)\eta^{-1}D(t) \\ E'(t) = -2C(t)R\bar{S} + R^\top E(t) + 2\gamma C(t)\Sigma E(t) - \frac{1}{2}B(t)^\top \eta^{-1}D(t) \\ F'(t) = -\bar{S}^\top R^\top E(t) - \text{Tr}(\Sigma C(t)) + \frac{\gamma}{2}E(t)^\top \Sigma E(t) - \frac{1}{4}D(t)^\top \eta^{-1}D(t), \end{cases} \quad (6.16)$$

where I_d denotes the identity matrix in $\mathcal{M}_d(\mathbb{R})$, with terminal conditions

$$A(T) = -\Gamma, \quad B(T) = C(T) = D(T) = E(T) = F(T) = 0. \quad (6.17)$$

Then the function θ defined by (6.15) satisfies (6.13) on $[\tau, T] \times \mathbb{R}^d \times \mathbb{R}^d$ with terminal condition (6.14).

Proof. Let us consider $A \in C^1([\tau, T], \mathcal{S}_d(\mathbb{R}))$, $B \in C^1([\tau, T], \mathcal{M}_d(\mathbb{R}))$, $C \in C^1([\tau, T], \mathcal{S}_d(\mathbb{R}))$, $D \in C^1([\tau, T], \mathbb{R}^d)$, $E \in C^1([\tau, T], \mathbb{R}^d)$, $F \in C^1([\tau, T], \mathbb{R})$ verifying (6.16) on $[\tau, T]$ with terminal condition (6.17). Let us consider $\theta : [\tau, T] \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$ defined by (6.15). Then we obtain for all $(t, q, S) \in [\tau, T] \times \mathbb{R}^d \times \mathbb{R}^d$:

$$\begin{aligned} & \partial_t \theta(t, q, S) + \frac{1}{4} \nabla_q \theta(t, q, S)^\top \eta^{-1} \nabla_q \theta(t, q, S) + \frac{1}{2} \text{Tr}(\Sigma D_{SS}^2 \theta(t, q, S)) \\ & - \frac{\gamma}{2} (q + \nabla_S \theta(t, q, S))^\top \Sigma (q + \nabla_S \theta(t, q, S)) + (\bar{S} - S)^\top R^\top (\nabla_S \theta(t, q, S) + q), \\ = & q^\top A'(t)q + q^\top B'(t)S + S^\top C'(t)S + D'(t)^\top q + E'(t)^\top S + F'(t) \\ & + q^\top A(t)\eta^{-1}A(t)q + q^\top A(t)\eta^{-1}B(t)S + \frac{1}{4} S^\top B(t)^\top \eta^{-1}B(t)S \\ & + D(t)^\top \eta^{-1}A(t)q + \frac{1}{2} (D(t)^\top \eta^{-1}(B(t))S + \frac{1}{4} D(t)^\top \eta^{-1}D(t) \\ & + \text{Tr}(\Sigma C(t)) - \frac{\gamma}{2} (q + B(t)^\top q + 2C(t)S + E(t))^\top \Sigma (q + B(t)^\top q + 2C(t)S + E(t)) \\ & + \bar{S}^\top R^\top q + \bar{S}^\top R^\top (B(t)^\top q + 2C(t)S + E(t)) - S^\top R^\top q - S^\top R^\top (B(t)^\top q + 2C(t)S + E(t)) \end{aligned}$$

$$\begin{aligned}
&= q^\top \left(A'(t) - \frac{\gamma}{2}(B(t) + I_d)\Sigma(B(t)^\top + I_d) + \frac{1}{4}(2A(t))\eta^{-1}(2A(t)) \right) q \\
&\quad + q^\top \left(B'(t) - (I_d + B(t))R - 2\gamma(B(t) + I_d)\Sigma C(t) + A(t)\eta^{-1}B(t) \right) S \\
&\quad + S^\top \left(C'(t) - R^\top C(t) - C(t)R - 2\gamma C(t)\Sigma C(t) + \frac{1}{4}B(t)^\top \eta^{-1}B(t) \right) S \\
&\quad + \left(D'(t) + (B(t) + I_d)R\bar{S} - \gamma(B(t) + I_d)\Sigma E(t) + A(t)\eta^{-1}D(t) \right)^\top q \\
&\quad + \left(E'(t) + 2C(t)R\bar{S} - R^\top E(t) - 2\gamma C(t)\Sigma E(t) + \frac{1}{2}B(t)^\top \eta^{-1}D(t) \right)^\top S \\
&\quad + \left(F'(t) + \bar{S}^\top R^\top E(t) + \text{Tr}(\Sigma C(t)) - \frac{\gamma}{2}E(t)^\top \Sigma E(t) + \frac{1}{4}D(t)^\top \eta^{-1}D(t) \right) \\
&= 0.
\end{aligned}$$

As it is straightforward to verify that θ satisfies the terminal condition (6.14), the result is proved. \square

Remark 14. *Two remarks can be made on the system of ODEs (6.16):*

- *This system of ODEs can clearly be decomposed into three groups of equations: the first three ODEs for A , B , and C are independent of the others and can be solved as a first step; once we know A , B , and C we can solve the linear ODEs for D and E , and finally F can be obtained with a simple integration;*
- *When $R = 0$ (i.e. in the case where the prices S of the d assets are correlated arithmetic Brownian motions), there is a trivial solution to the last five equations which is $B = C = D = E = F = 0$. A can then be found as shown in Appendix 6.4.*

It is noteworthy that the first system, i.e.

$$\begin{cases} A'(t) = \frac{\gamma}{2}(B(t) + I_d)\Sigma(B(t)^\top + I_d) - A(t)\eta^{-1}A(t) \\ B'(t) = (B(t) + I_d)R + 2\gamma(B(t) + I_d)\Sigma C(t) - A(t)\eta^{-1}B(t) \\ C'(t) = R^\top C(t) + C(t)R + 2\gamma C(t)\Sigma C(t) - \frac{1}{4}B(t)^\top \eta^{-1}B(t) \end{cases} \quad (6.18)$$

boils down to a Matrix Riccati ODE. Indeed, defining $P : [0, T] \rightarrow \mathcal{S}_{2d}(\mathbb{R})$ as

$$P(t) = \begin{pmatrix} A(t) & \frac{1}{2}B(t) \\ \frac{1}{2}B(t)^\top & C(t) \end{pmatrix}, \quad (6.19)$$

we see that (6.18) with terminal condition $A(T) = -\Gamma$ and $B(T) = C(T) = 0$ is equivalent to

$$P'(t) = Q + Y^\top P(t) + P(t)Y + P(t)UP(t), \quad (6.20)$$

with terminal condition

$$P(T) = \begin{pmatrix} -\Gamma & 0 \\ 0 & 0 \end{pmatrix} \in \mathcal{S}_{2d}(\mathbb{R}), \quad (6.21)$$

where

$$Q = \frac{1}{2} \begin{pmatrix} \gamma\Sigma & R \\ R^\top & 0 \end{pmatrix} \in \mathcal{S}_{2d}(\mathbb{R}), \quad Y = \begin{pmatrix} 0 & 0 \\ \gamma\Sigma & R \end{pmatrix} \in \mathcal{M}_{2d}(\mathbb{R}), \quad U = \begin{pmatrix} -\eta^{-1} & 0 \\ 0 & 2\gamma\Sigma \end{pmatrix} \in \mathcal{S}_{2d}(\mathbb{R}).$$

When compared to the Matrix Riccati ODEs arising in the linear-quadratic optimal control literature, the distinctive aspect of our equation is that the matrix U characterizing the quadratic term in the Riccati equation has both positive and negative eigenvalues. In particular, we cannot rely on existing results coming from linear-quadratic control theory to prove that there exists a solution to (6.20) with terminal condition (6.21). In this paper, we address the existence of a solution by using *a priori* estimates for the value function.

Regarding the set of equations (6.18), there exists a unique local solution by Cauchy-Lipschitz theorem. In the following section, we therefore first state a verification theorem that solves the problem when on an interval $[\tau, T]$, and use that very result to address global existence and uniqueness of a solution on $[0, T]$.

6.2.3 Main mathematical results

Theorem 5. *Let $\tau < T$. Let $A \in C^1([\tau, T], \mathcal{S}_d(\mathbb{R}))$, $B \in C^1([\tau, T], \mathcal{M}_d(\mathbb{R}))$, $C \in C^1([\tau, T], \mathcal{S}_d(\mathbb{R}))$, $D \in C^1([\tau, T], \mathbb{R}^d)$, $E \in C^1([\tau, T], \mathbb{R}^d)$, $F \in C^1([\tau, T], \mathbb{R})$ be a solution to the system (6.16) on $[\tau, T)$ with terminal condition (6.17), and consider the function θ defined by (6.15) and the associated function w defined by (6.9).*

Then for all $(t, x, q, S) \in [\tau, T] \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d$ and $v = (v_s)_{s \in [t, T]} \in \mathcal{A}_t$, we have

$$\mathbb{E} \left[-e^{-\gamma(X_T^{t,x,S,v} + (q_T^{t,q,v})^\top S_T^{t,S} - \ell(q_T^{t,q,v}))} \right] \leq w(t, x, q, S). \quad (6.22)$$

Moreover, equality is obtained in (6.22) by taking the optimal control $(v_s^)_{s \in [t, T]} \in \mathcal{A}_t$ given by the closed-loop feedback formula*

$$v_s^* = \frac{1}{2} \eta^{-1} (2A(s)q_s^{t,q,v} + B(s)S_s^{t,S} + D(s)). \quad (6.23)$$

In particular, $w = u$ on $[\tau, T] \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d$.

Proof. Let $t \in [\tau, T)$, we first prove that $(v_s^*)_{s \in [t, T]} \in \mathcal{A}_t$ (i.e., $(v_s^*)_{s \in [t, T]}$ is well-defined and admissible). Let us consider the Cauchy initial value problem

$$\forall s \in [t, T], \quad \frac{d\tilde{q}_s}{ds} = \frac{1}{2} \eta^{-1} (2A(s)\tilde{q}_s + B(s)S_s^{t,S} + D(s)), \quad \tilde{q}_t = q.$$

The unique solution of that Cauchy problem writes

$$\tilde{q}_s = \exp \left(\int_t^s \phi(\varrho) d\varrho \right) \left(q + \int_t^s \psi(\varrho, S_\varrho^{t,S}) \exp \left(- \int_t^\varrho \phi(\varsigma) d\varsigma \right) d\varrho \right), \quad (6.24)$$

where ϕ and ψ are defined by

$$\begin{aligned} \phi : s \in [t, T] &\mapsto \eta^{-1} A(s), \\ \psi : (s, S) \in [t, T] \times \mathbb{R}^d &\mapsto \frac{1}{2} \eta^{-1} (B(s)S + D(s)). \end{aligned}$$

Then v^* can be written as

$$v_s^* = \frac{d\tilde{q}_s}{ds} = \phi(s) \exp\left(\int_t^s \phi(\varrho) d\varrho\right) \left(q + \int_t^s \psi(\varrho, S_\varrho^{t,S}) \exp\left(-\int_t^\varrho \phi(\zeta) d\zeta\right) d\varrho\right) + \psi(s, S_s^{t,S}). \quad (6.25)$$

We see from the definition of ϕ and the affine form of ψ in S that v^* satisfies a linear growth condition, and is therefore in \mathcal{A}_t .

Let us consider $(t, x, q, S) \in [\tau, T] \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d$ and $v = (v_s)_{s \in [t, T]} \in \mathcal{A}_t$. We now prove that

$$\mathbb{E} \left[w(T, X_T^{t,x,S,v}, q_T^{t,q,v}, S_T^{t,S}) \right] \leq w(t, x, q, S). \quad (6.26)$$

We use the following notations for readability:

$$\begin{aligned} \forall s \in [t, T], \quad w(s, X_s^{t,x,S,v}, q_s^{t,q,v}, S_s^{t,S}) &= w_s^{t,x,q,S,v}, \\ \forall s \in [t, T], \quad \theta(s, q_s^{t,q,v}, S_s^{t,S}) &= \theta_s^{t,q,S,v}. \end{aligned}$$

By Itô's formula, we have $\forall s \in [\tau, T]$

$$dw_s^{t,x,q,S,v} = \mathcal{L}^v w_s^{t,x,q,S,v} ds + (\nabla_S w_s^{t,x,q,S,v})^\top V dW_s, \quad (6.27)$$

where

$$\begin{aligned} \mathcal{L}^v w_s^{t,x,q,S,v} &= \partial_t w_s^{t,x,q,S,v} - (v^\top S + v^\top \eta v) \partial_x w_s^{t,x,q,S,v} + v^\top \nabla_q w_s^{t,x,q,S,v} \\ &\quad + (\bar{S} - S)^\top R^\top \nabla_S w_s^{t,x,q,S,v} + \frac{1}{2} \text{Tr}(\Sigma D_{SS}^2 w_s^{t,x,q,S,v}). \end{aligned} \quad (6.28)$$

From (6.9) and (6.15) we have

$$\begin{aligned} \nabla_S w_s^{t,x,q,S,v} &= -\gamma w_s^{t,x,q,S,v} (q_s^{t,q,v} + \nabla_S \theta_s^{t,q,S,v}) \\ &= -\gamma w_s^{t,x,q,S,v} (q_s^{t,q,v} + B(s)^\top q_s^{t,q,v} + 2C(s) S_s^{t,S} + E(s)). \end{aligned} \quad (6.29)$$

We define $\forall s \in [t, T]$,

$$\kappa_s^{q,S,v} = -\gamma (q_s^{t,q,v} + B(s)^\top q_s^{t,q,v} + 2C(s) S_s^{t,S} + E(s)), \quad (6.30)$$

$$\xi_{t,s}^{q,S,v} = \exp\left(\int_t^s \kappa_\varrho^{q,S,v} V dW_\varrho - \frac{1}{2} \int_t^s \kappa_\varrho^{q,S,v} \Sigma \kappa_\varrho^{q,S,v} d\varrho\right). \quad (6.31)$$

We then have

$$d\left(w_s^{t,x,q,S,v} \left(\xi_{t,s}^{q,S,v}\right)^{-1}\right) = \left(\xi_{t,s}^{q,S,v}\right)^{-1} \mathcal{L}^v w_s^{t,x,q,S,v} ds. \quad (6.32)$$

By definition of w , $\mathcal{L}^v w_s^{t,x,q,S,v} \leq 0$. Moreover, equality holds for the control reaching the sup in (6.12). The sup is reached for the unique value

$$v_s = \frac{1}{2} \eta^{-1} \nabla_q \theta_s^{t,q,S,v} \quad (6.33)$$

$$= \frac{1}{2} \eta^{-1} (2A(s) q_s^{t,q,v} + B(s) S_s^{t,S} + D(s)), \quad (6.34)$$

which corresponds to the case $(v_s)_{s \in [t, T]} = (v_s^*)_{s \in [t, T]}$.

As a consequence, $\left(w_s^{t, x, q, S, v} \left(\xi_{t, s}^{q, S, v} \right)^{-1} \right)_{s \in [t, T]}$ is nonincreasing and therefore

$$w(T, X_T^{t, x, S, v}, q_T^{t, q, v}, S_T^{t, S}) \leq w(t, x, q, S) \xi_{t, T}^{q, S, v}, \quad (6.35)$$

with equality when $(v_s)_{s \in [t, T]} = (v_s^*)_{s \in [t, T]}$.

Taking expectations we get

$$\mathbb{E} \left[w \left(T, X_T^{t, x, S, v}, q_T^{t, q, v}, S_T^{t, S} \right) \right] \leq w(t, x, q, S) \mathbb{E} \left[\xi_{t, T}^{q, S, v} \right]. \quad (6.36)$$

We proceed to prove that $\mathbb{E} \left[\xi_{t, T}^{q, S, v} \right]$ is equal to 1. To do so, we use that $\xi_{t, t}^{q, S, v} = 1$ and prove that $(\xi_{t, s}^{q, S, v})_{s \in [t, T]}$ is a martingale under $(\mathbb{P}; \mathbb{F} = (\mathcal{F}_s)_{s \in [t, T]})$.

We know that $(q_s^{t, q, v})_{s \in [t, T]}$ satisfies a linear growth condition with respect to $(S_s^{t, S})_{s \in [t, T]}$ since v is an admissible control. Given the form of κ , there exists a constant C such that, almost surely,

$$\sup_{s \in [t, T]} \|\kappa_s^{q, S, v}\|^2 \leq C \left(1 + \sup_{s \in [t, T]} \|W_s - W_t\|^2 \right). \quad (6.37)$$

By using classical properties of the Brownian motion, we prove that

$$\exists \epsilon > 0, \forall s \in [t, T], \quad \mathbb{E} \left[\exp \left(\frac{1}{2} \int_s^{(s+\epsilon) \wedge T} (\kappa_\rho^{q, S, v})^\top \Sigma \kappa_\rho^{q, S, v} d\rho \right) \right] < +\infty. \quad (6.38)$$

From Novikov condition, we see that $(\xi_{t, s}^{q, S, v})_{s \in [t, T]}$ is a martingale under $(\mathbb{P}; \mathbb{F} = (\mathcal{F}_s)_{s \in [t, T]})$.

We obtain

$$\mathbb{E} \left[w(T, X_T^{t, x, S, v}, q_T^{t, q, v}, S_T^{t, S}) \right] \leq w(t, x, q, S), \quad (6.39)$$

with equality when $(v_s)_{s \in [t, T]} = (v_s^*)_{s \in [t, T]}$.

We conclude that

$$u(t, x, q, S) = \sup_{(v_s)_{s \in [t, T]} \in \mathcal{A}_t} \mathbb{E} \left[-\exp \left(-\gamma \left(X_T^{t, x, S, v} + (q_T^{t, q, v})^\top S_T^{t, S} - \ell(q_T^{t, q, v}) \right) \right) \right] \quad (6.40)$$

$$= \mathbb{E} \left[-\exp \left(-\gamma \left(X_T^{t, x, S, v^*} + (q_T^{t, q, v^*})^\top S_T^{t, S} - \ell(q_T^{t, q, v^*}) \right) \right) \right] \quad (6.41)$$

$$= w(t, x, q, S). \quad (6.42)$$

□

We will next proceed to prove existence and uniqueness of a solution to the system of ODEs (6.16) on $[0, T]$ with terminal condition (6.17), or equivalently to (6.20) with terminal condition (6.21).⁵

Theorem 6. *There exists a unique solution $A \in C^1([0, T], \mathcal{S}_d(\mathbb{R}))$, $B \in C^1([0, T], \mathcal{M}_d(\mathbb{R}))$, $C \in C^1([0, T], \mathcal{S}_d(\mathbb{R}))$, $D \in C^1([0, T], \mathbb{R}^d)$, $E \in C^1([0, T], \mathbb{R}^d)$, $F \in C^1([0, T], \mathbb{R})$ to the system of ODEs (6.16) on $[0, T]$ with terminal condition (6.17).*

Proof. To prove Theorem 6, it is enough, as explained in Remark 14, to show existence and uniqueness for $A \in C^1([0, T], \mathcal{S}_d(\mathbb{R}))$, $B \in C^1([0, T], \mathcal{M}_d(\mathbb{R}))$, and $C \in C^1([0, T], \mathcal{S}_d(\mathbb{R}))$, or equivalently, existence and uniqueness on $[0, T]$ of a solution $P \in C^1([0, T], \mathcal{S}_{2d}(\mathbb{R}))$ to (6.20) with terminal condition (6.21).

First, by Cauchy-Lipschitz theorem, there exists a unique maximal solution⁶ (A, B, C) to the system of ODEs (6.18) with terminal condition (6.17) defined on an open interval $(t_{\min}, t_{\max}) \ni T$.

We now show that $t_{\min} = -\infty$, which implies our theorem.

By contradiction, let us assume that $t_{\min} \in (-\infty, T)$ and let $\tau \in (t_{\min}, T)$.

Starting from values $(t, x, q, S) \in [\tau, T] \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d$, let us consider the suboptimal strategy $v = (0)_{s \in [t, T]} \in \mathcal{A}_t$ for which $\forall s \in [t, T], q_s^{t, q, v} = q$ and

$$\mathbb{E} \left[-\exp \left(-\gamma \left(X_T^{t, x, S, v} + (q_T^{t, q, v})^\top S_T^{t, S} - \ell(q_T^{t, q, v}) \right) \right) \right] = \mathbb{E} \left[-\exp \left(-\gamma \left(x + q^\top S + q^\top \left(S_T^{t, S} - S \right) - q^\top \Gamma q \right) \right) \right]. \quad (6.43)$$

Since $(S_s^{t, S})_{s \in [t, T]}$ follows multivariate Ornstein-Uhlenbeck dynamics, we know that

$$S_T^{t, S} - S = \left(I - e^{-R(T-t)} \right) (\bar{S} - S) + \int_t^T e^{-R(T-s)} V dW_s.$$

Then $S_T^{t, S} - S \sim \mathcal{N} \left((I - e^{-R(T-t)}) (\bar{S} - S), \Sigma_t \right)$, where the covariance matrix is defined by

$$\Sigma_t = \int_t^T e^{-R(T-s)} \Sigma e^{-R^\top(T-s)} ds.$$

Then,

$$\begin{aligned} & \mathbb{E} \left[-\exp \left(-\gamma \left(X_T^{t, x, S, v} + (q_T^{t, q, v})^\top S_T^{t, S} - \ell(q_T^{t, q, v}) \right) \right) \right] \\ &= -\exp \left(-\gamma (x + q^\top S) \right) \exp \left(-\gamma \left(q^\top \left(I - e^{-R(T-t)} \right) (\bar{S} - S) - q^\top \Gamma q - \frac{1}{2} \gamma q^\top \Sigma_t q \right) \right). \end{aligned} \quad (6.44)$$

Since the strategy is sub-optimal, if we consider θ defined as in (6.15), we have by Theorem 5

$$-\exp \left(-\gamma (x + q^\top S + \theta(t, q, S)) \right) \geq -\exp \left(-\gamma (x + q^\top S) \right) \exp \left(-\gamma \left(q^\top \left(I - e^{-R(T-t)} \right) (\bar{S} - S) - q^\top \Gamma q - \frac{1}{2} \gamma q^\top \Sigma_t q \right) \right). \quad (6.45)$$

⁵The result in fact holds on $(-\infty, T]$ as the initial time plays no role.

⁶The fact that A and C are symmetric is itself a consequence of Cauchy-Lipschitz theorem since (A, B, C) and (A^\top, B, C^\top) are solution of the same Cauchy problem.

We conclude that for all $(t, q, S) \in [\tau, T] \times \mathbb{R}^d \times \mathbb{R}^d$,

$$\begin{aligned} \theta(t, q, S) &= \begin{pmatrix} q \\ S \end{pmatrix}^\top P(t) \begin{pmatrix} q \\ S \end{pmatrix} + D(t)^\top q + E(t)^\top S + F(t) \\ &\geq \begin{pmatrix} q \\ S \end{pmatrix}^\top \begin{pmatrix} -\frac{\gamma}{2}\Sigma_t - \Gamma & -\frac{1}{2}(I - e^{-R(T-t)}) \\ -\frac{1}{2}(I - e^{-R^\top(T-t)}) & 0 \end{pmatrix} \begin{pmatrix} q \\ S \end{pmatrix} + \bar{S}^\top (I - e^{-R^\top(T-t)}) q, \end{aligned}$$

where $P(t) = \begin{pmatrix} A(t) & \frac{1}{2}B(t) \\ \frac{1}{2}B(t)^\top & C(t) \end{pmatrix}$.

We therefore necessarily have, for the natural order on symmetric matrices,⁷

$$\forall t \in [\tau, T], \quad P(t) \geq \begin{pmatrix} -\frac{\gamma}{2}\Sigma_t - \Gamma & -\frac{1}{2}(I - e^{-R(T-t)}) \\ -\frac{1}{2}(I - e^{-R^\top(T-t)}) & 0 \end{pmatrix}.$$

Now, for $(t, x, q, S) \in [\tau, T] \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d$, we have

$$\sup_{v \in \mathcal{A}_t} \mathbb{E} \left[-\exp \left(-\gamma \left(X_T^{t,x,S,v} + (q_T^{t,q,v})^\top S_T^{t,S} - (q_T^{t,q,v})^\top \Gamma q_T^{t,q,v} \right) \right) \right] \quad (6.46)$$

$$\begin{aligned} &= \sup_{v \in \mathcal{A}_t} \mathbb{E} \left[-\exp \left(-\gamma \left(x + q^\top S + \int_t^T (q_s^{t,q,v})^\top dS_s - \int_t^T L(v_s) ds - (q_T^{t,q,v})^\top \Gamma q_T^{t,q,v} \right) \right) \right] \\ &\leq \exp(-\gamma(x + q^\top S)) \sup_{v \in \mathcal{A}_t} \mathbb{E} \left[-\exp \left(-\gamma \left(\int_t^T (q_s^{t,q,v})^\top dS_s \right) \right) \right], \end{aligned} \quad (6.47)$$

If $(v_s)_{s \in [t, T]} \in \mathcal{A}_t$, it is straightforward to see that the process $(q_s^{t,q,v})_{s \in [t, T]}$ is in the space of admissible controls $\mathcal{A}_t^{\text{Merton}}$ defined in (6.53) in Appendix 6.5 (in which we study a Merton problem that can be regarded as a limit case of ours when the execution costs and terminal costs vanish). Therefore,

$$\begin{aligned} &\sup_{v \in \mathcal{A}_t} \mathbb{E} \left[-\exp \left(-\gamma \left(X_T^{t,x,S,v} + (q_T^{t,q,v})^\top S_T^{t,S} - (q_T^{t,q,v})^\top \Gamma q_T^{t,q,v} \right) \right) \right] \\ &\leq \exp(-\gamma(x + q^\top S)) \sup_{q \in \mathcal{A}_t^{\text{Merton}}} \mathbb{E} \left[-\exp \left(-\gamma \left(\int_t^T q_s^\top dS_s \right) \right) \right]. \end{aligned} \quad (6.48)$$

As shown in Appendix 6.5, inequality (6.48) writes

$$-\exp(-\gamma(x + q^\top S + \theta(t, q, S))) \leq -\exp\left(-\gamma\left(x + q^\top S + \hat{\theta}(t, S)\right)\right),$$

where $\hat{\theta}(t, S) = S^\top \hat{C}(t)S + \hat{E}(t)^\top S + \hat{F}(t)$ with $\hat{C} \in C^1([\tau, T], \mathcal{S}_d(\mathbb{R}))$, $\hat{E} \in C^1([\tau, T], \mathbb{R}^d)$, $\hat{F} \in C^1([\tau, T], \mathbb{R})$ defined by

$$\begin{cases} \hat{C}(t) = (T-t) \frac{1}{2\gamma} R^\top \Sigma^{-1} R, \\ \hat{E}(t) = (T-t) \frac{1}{\gamma} R^\top \Sigma^{-1} R \bar{S}, \\ \hat{F}(t) = \frac{1}{4\gamma} (T-t)^2 \text{Tr}(R^\top \Sigma^{-1} R \Sigma) + (T-t) \frac{1}{2\gamma} \bar{S}^\top R^\top \Sigma^{-1} R. \end{cases}$$

⁷For $\underline{M}, \bar{M} \in \mathcal{S}_d(\mathbb{R})$, $\underline{M} \leq \bar{M}$ if and only if $\bar{M} - \underline{M} \in \mathcal{S}_d^+(\mathbb{R})$.

We conclude that for all $(t, q, S) \in [\tau, T] \times \mathbb{R}^d \times \mathbb{R}^d$,

$$\begin{aligned} \theta(t, q, S) &= \begin{pmatrix} q \\ S \end{pmatrix}^\top P(t) \begin{pmatrix} q \\ S \end{pmatrix} + D(t)^\top q + E(t)^\top S + F(t) \\ &\leq \begin{pmatrix} q \\ S \end{pmatrix}^\top \begin{pmatrix} 0 & 0 \\ 0 & \hat{C}(t) \end{pmatrix} \begin{pmatrix} q \\ S \end{pmatrix} + \hat{E}(t)^\top S + \hat{F}(t). \end{aligned}$$

Therefore,

$$\forall t \in [\tau, T], \quad P(t) \leq \begin{pmatrix} 0 & 0 \\ 0 & \hat{C}(t) \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & (T-t) \frac{1}{2\gamma} R^\top \Sigma^{-1} R \end{pmatrix}.$$

We have therefore $\forall \tau \in (t_{\min}, T)$, $\forall t \in [\tau, T]$:

$$\begin{pmatrix} -\frac{\gamma}{2} \Sigma_t - \Gamma & -\frac{1}{2} (I - e^{-R(T-t)}) \\ -\frac{1}{2} (I - e^{-R^\top(T-t)}) & 0 \end{pmatrix} \leq P(t) \leq \begin{pmatrix} 0 & 0 \\ 0 & (T-t) \frac{1}{2\gamma} R^\top \Sigma^{-1} R \end{pmatrix}.$$

As t_{\min} is supposed to be finite, there exists $\underline{M}, \overline{M} \in \mathcal{S}_d(\mathbb{R})$ with $\underline{M} \leq \overline{M}$ such that $\forall t \in [t_{\min}, T]$, $P(t)$ stays in the compact set $\{M \in \mathcal{S}_d(\mathbb{R}) \mid \underline{M} \leq M \leq \overline{M}\}$. This contradicts the maximality of the solution, hence $t_{\min} = -\infty$. □

Theorem 6 implies that Theorem 5 can be applied with $\tau = 0$. In particular, our optimal execution problem is solved and the optimal strategy is given by the closed-loop feedback control (6.23). In the next section, we illustrate our results with simulations of prices and numerical approximations of the optimal strategies.

6.3 Numerical results

6.3.1 Single-asset case

In this section, we study the case of a trader dealing with a single asset S with the following parameters:

- Initial price: $S_0 = \$100$,
- Mean-reversion parameter: $R = 0 \text{ day}^{-1}$, $R = 1 \text{ day}^{-1}$ or $R = 10 \text{ day}^{-1}$ (see below in the examples),
- Long-term average: $\bar{S} = \$100$,
- Volatility: $\sigma = 5 \text{ \$} \cdot \text{day}^{-\frac{1}{2}}$,
- Temporary impact: $L(v) = \eta v^2$, with $\eta = 1 \cdot 10^{-3} \text{ \$} \cdot \text{day}$.

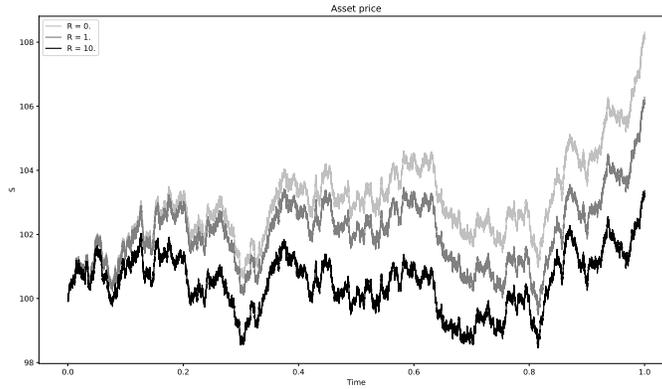


Figure 6.1: Trajectory of the asset price for different values of R .

Figure 6.1 represents trajectories of the price process $(S_t)_{t \in [0, T]}$ for different values of R , using the same Brownian paths.

We consider a trader wishing to unwind a portfolio with $q_0 = 1000$ assets over the time interval $[0, T]$ where $T = 1$ day. In order to enforce almost complete liquidation, we set $\Gamma = 12$ \$.

We consider the case where the absolute risk aversion parameter is $\gamma = 1 \cdot 10^{-3} \text{ \$}^{-1}$. For the three price trajectories of Figure 6.1, we plot the optimal execution strategy and the corresponding inventory process in Figure 6.2 and Figure 6.3 respectively.

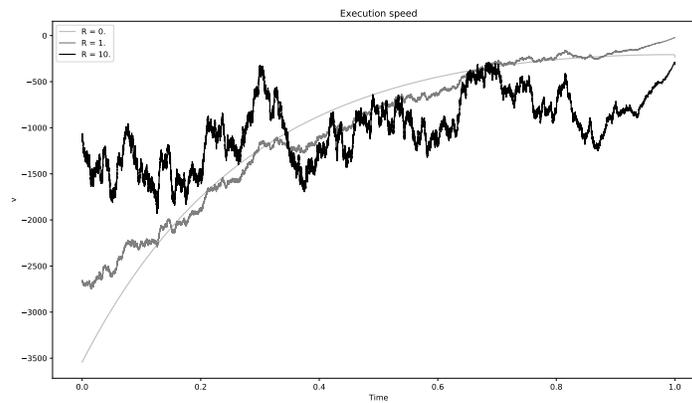


Figure 6.2: Execution speed $(v_t)_{t \in [0, T]}$ for different values of R ($\gamma = 1 \cdot 10^{-3} \text{ \$}^{-1}$).

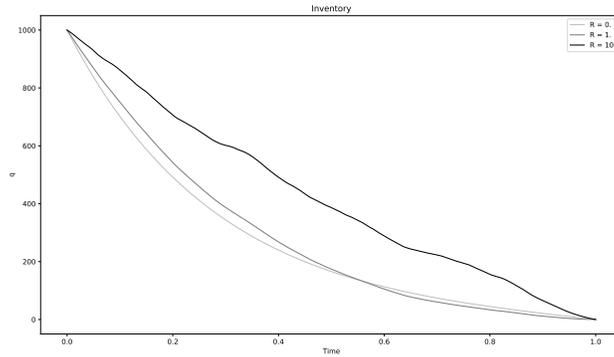


Figure 6.3: Trajectory of the inventory $(q_t)_{t \in [0, T]}$ for different values of R ($\gamma = 1 \cdot 10^{-3} \text{ \$}^{-1}$).

An interesting observation can be made here: the higher the mean-reversion parameter, the lower the influence of price risk on the execution strategy. In particular, when R is large, the trader acts almost as if she was performing a VWAP/TWAP execution plus a simple mean-reverting statistical arbitrage strategy: the average level of $(v_t)_{t \in [0, T]}$ in the case where $R = 10 \text{ day}^{-1}$ is indeed driven by the total number of assets to sell and its oscillations are highly correlated to those of $(S_t)_{t \in [0, T]}$: the trader sells faster when the price is above \bar{S} and slower when it is below \bar{S} .

Given the above observation, it is natural to illustrate how our model can be used to build a statistical arbitrage strategy by setting $q_0 = 0$ and $\Gamma = 0$: the trader has no initial inventory and just wants to maximize the expected utility of the MtM value of her portfolio at time T . We chose $R = 10 \text{ day}^{-1}$ to focus on mean reversion and extend the trading window by setting $T = 9$ days. A trajectory of the price process $(S_t)_{t \in [0, T]}$ is plotted in Figure 6.4.

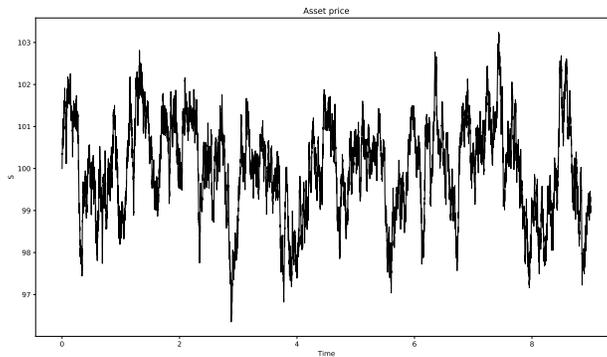


Figure 6.4: Asset price $(S_t)_{t \in [0, T]}$ for $R = 10 \text{ day}^{-1}$.

We then plot in Figure 6.5 the optimal execution strategy $(v_t)_{t \in [0, T]}$ for different values of the risk aversion parameter γ : $\gamma = 1 \cdot 10^{-1} \text{ \$}^{-1}$, $\gamma = 1 \cdot 10^{-2} \text{ \$}^{-1}$ and $\gamma = 1 \cdot 10^{-7} \text{ \$}^{-1}$.

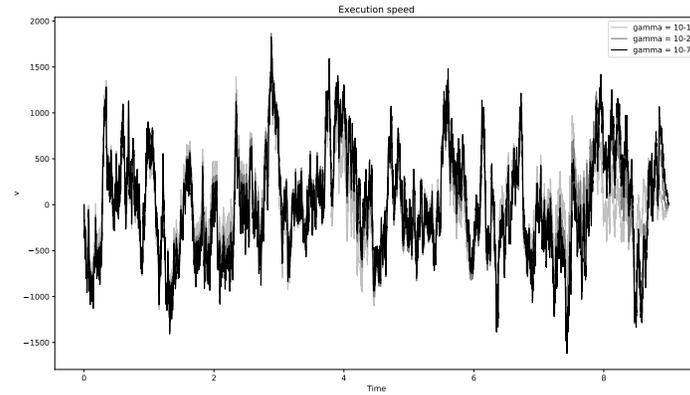


Figure 6.5: Execution speed $(v_t)_{t \in [0, T]}$ for different values of γ ($R = 10 \text{ day}^{-1}$).

We observe that, as expected, the optimal strategies look highly correlated to the price trajectory. In Figure 6.6, we plot the corresponding inventory of the trader as a function of time for the different values of γ .

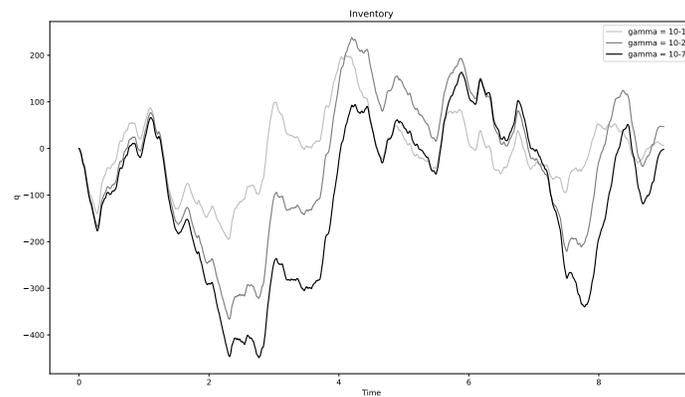


Figure 6.6: Trajectory of the inventory $(q_t)_{t \in [0, T]}$ for different values of γ ($R = 10 \text{ day}^{-1}$).

In all three cases, as expected, the trader sells the asset when the price is above \bar{S} , and starts buying when it goes below \bar{S} : her inventory is mean-reverting toward 0. Of course, the higher the risk aversion, the closer to 0 her inventory remains.

We finally perform 1500 Monte-Carlo simulations and plot, in Figures 6.7, 6.8, and 6.9 the distributions of the MtM value at time T for $\gamma = 1 \cdot 10^{-1} \text{ \$}^{-1}$, $\gamma = 1 \cdot 10^{-2} \text{ \$}^{-1}$ and $\gamma = 1 \cdot 10^{-7} \text{ \$}^{-1}$ respectively.

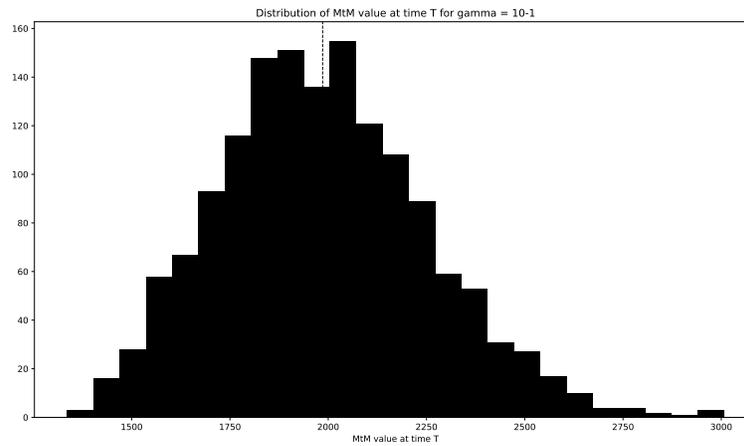


Figure 6.7: Distribution of MtM value at time T for $\gamma = 1 \cdot 10^{-1} \text{ \$}^{-1}$ ($R = 10 \text{ day}^{-1}$).

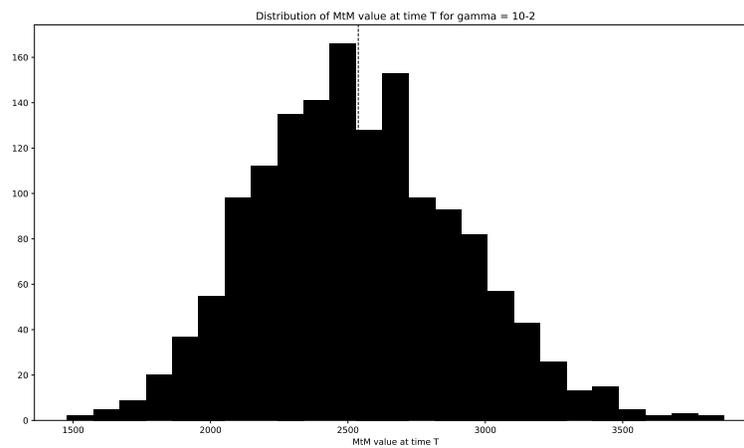


Figure 6.8: Distribution of MtM value at time T for $\gamma = 1 \cdot 10^{-2} \text{ \$}^{-1}$ ($R = 10 \text{ day}^{-1}$).

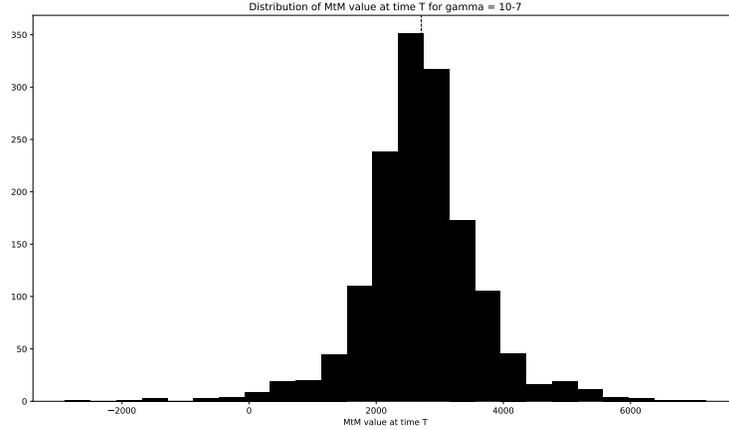


Figure 6.9: Distribution of MtM value at time T for $\gamma = 1 \cdot 10^{-7} \text{ \$}^{-1}$ ($R = 10 \text{ day}^{-1}$).

We see that our strategy allows to make money on average by taking advantage of the mean reversion. In the first case (Figure 6.7), we get an average MtM value of \$1986 and a standard deviation of \$266. In the second case (Figure 6.8), we get an average MtM value of \$2538 and a standard deviation of \$373. In the third case (Figure 6.9), we get an average MtM value of \$2708 and a standard deviation of \$920.

6.3.2 Multi-asset case

We now study the case of a trader in charge of 2 assets S^1, S^2 with the following parameters:

- Initial price: $S_0^1 = S_0^2 = \$100$,
- Mean-reversion matrix: $R = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$ (no-cointegration case) or $R = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$ (cointegration case),
- Long-term average: $\bar{S} := (\bar{S}^1, \bar{S}^2) = (\$100, \$100)$,
- Quadratic covariation matrix: $\Sigma = \begin{pmatrix} 25 & 7.5 \\ 7.5 & 25 \end{pmatrix}$ (which corresponds to an arithmetic volatility of $5 \text{ \$} \cdot \text{day}^{-\frac{1}{2}}$ for the two assets, and a correlation of 0.3),
- Temporary impact: $L(v) = v^T \eta v$, with $\eta = (1 \cdot 10^{-3} \text{ \$} \cdot \text{day}) \times I_2$.

We assume that the trader has an initial inventory $q_0 = (1000, 1000)$ and that she wants to liquidate within $T = 2$ days. Her risk aversion is given by $\gamma = 2 \cdot 10^{-3} \$^{-1}$. We penalize the remaining inventory with the matrix $\Gamma = 12 \times I_2$.

Let us first consider that the matrix R is given by $\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$ (the no-cointegration case). We simulate in Figure 6.10 a corresponding trajectory for the prices of the two assets. We then plot in Figures 6.11 and 6.12 the optimal strategy and the associated inventories, respectively.

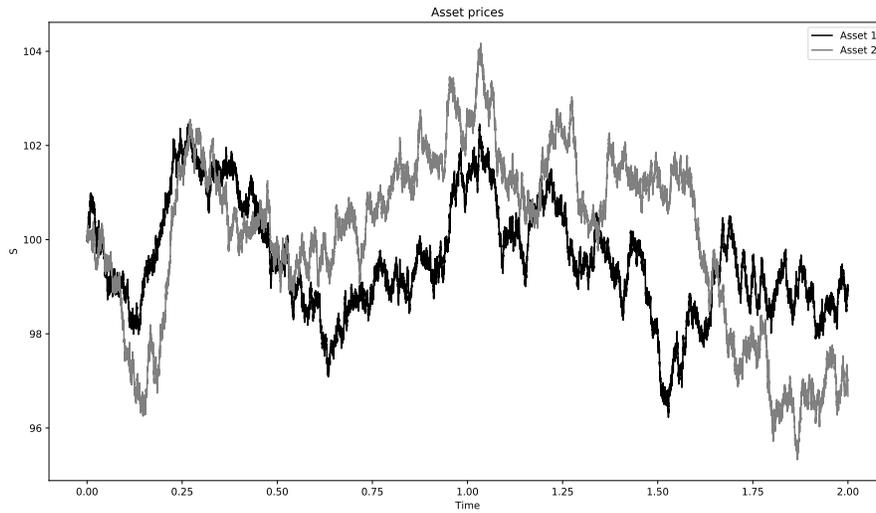


Figure 6.10: Asset prices $(S_t^1)_{t \in [0, T]}$ and $(S_t^2)_{t \in [0, T]}$ in the no-cointegration case.

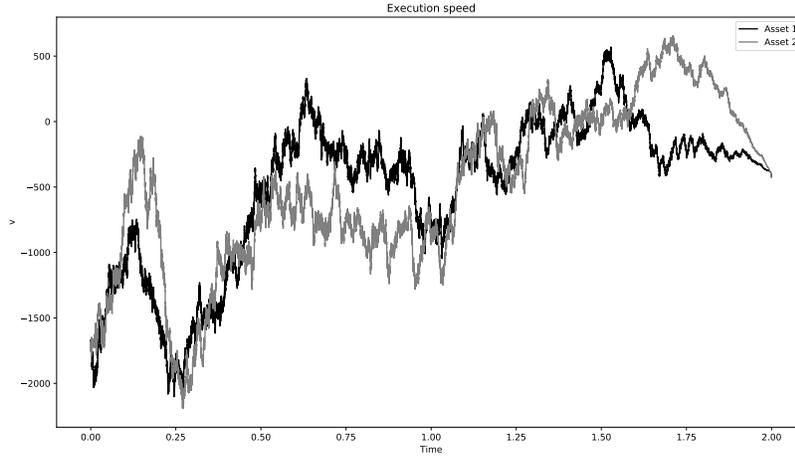


Figure 6.11: Execution speeds $(v_t^1)_{t \in [0, T]}$ and $(v_t^2)_{t \in [0, T]}$ in the no-cointegration case.

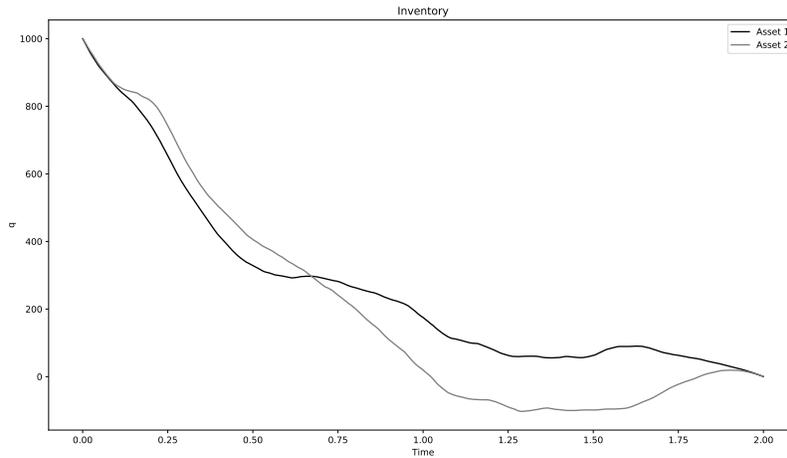


Figure 6.12: Trajectory of the inventories $(q_t^1)_{t \in [0, T]}$ and $(q_t^2)_{t \in [0, T]}$ in the no-cointegration case.

We now qualitatively compare these results with those obtained in the cointegration case where the matrix R is given by $\begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$. We simulate in Figure 6.13 a corresponding trajectory for the prices of the two assets. As before, we plot in Figures 6.14 and 6.15 the optimal strategy and the associated inventories, respectively.

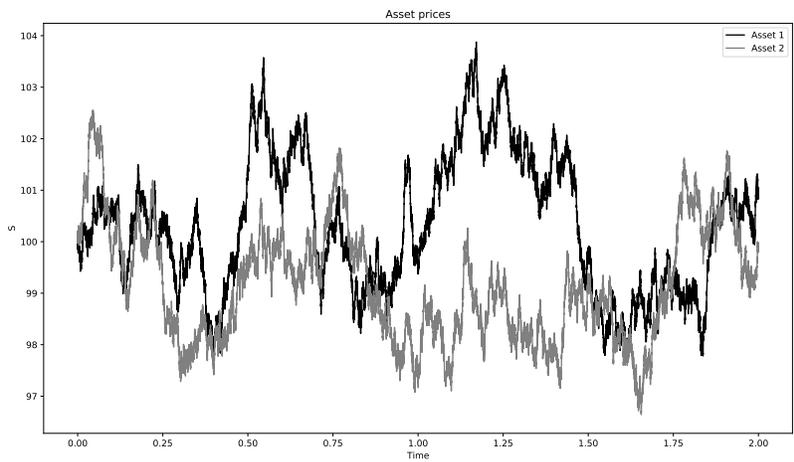


Figure 6.13: Asset prices $(S_t^1)_{t \in [0, T]}$ and $(S_t^2)_{t \in [0, T]}$ in the case of cointegration.

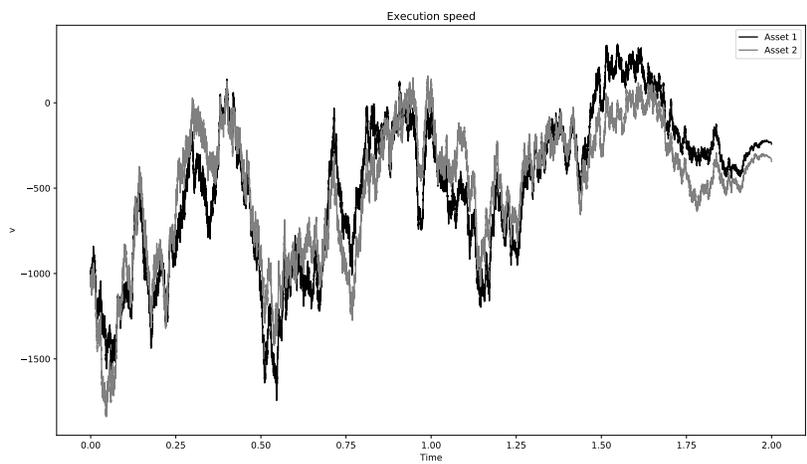


Figure 6.14: Execution speeds $(v_t^1)_{t \in [0, T]}$ and $(v_t^2)_{t \in [0, T]}$ in the case of cointegration.

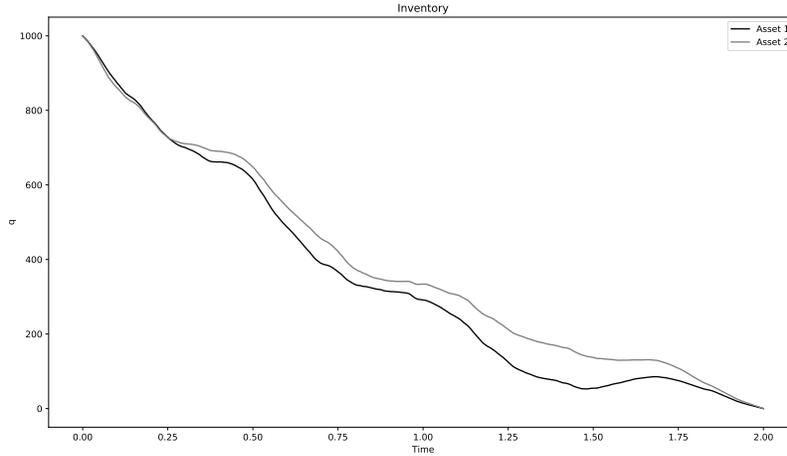


Figure 6.15: Trajectory of the inventories $(q_t^1)_{t \in [0, T]}$ and $(q_t^2)_{t \in [0, T]}$ in the case of cointegration.

As expected, the comparison of Figures 6.12 and 6.15 yields that, in the presence of cointegration, the trader tends to execute slower.

Conclusion

In this paper, we have shown how to account for cross-asset co-movements when executing trades in multiple assets. In our model, the agent has an exponential utility and the prices have multivariate Ornstein-Uhlenbeck dynamics, capturing the complex cross-asset dynamics of prices better than Brownian motions only. The advantage of our approach is twofold: (i) it accurately accounts for risk at the portfolio level, and (ii) it is versatile and can be used for basket execution, exogenous signal incorporation, and statistical arbitrage.

Our simulations show that considering cross-asset relations leads to different execution strategies. In particular, the presence of cointegration is exploited by the optimal strategy and usually leads to a reduction in the execution speed since the global variance of the portfolio is reduced.

The advantages for practitioners are numerous. Considering asset execution within a portfolio allows to manage risk across a wider basket of assets rather than considering only the risk of a single trade. Agents can hold securities on their balance sheets for longer, reducing market impact and execution costs. Moreover, from a regulation point of view, multivariate optimal execution models that naturally offset risks in a portfolio are of great interest. In fact, the new FRTB (Fundamental Review of the Trading Book) regulation will lead practitioners to assess liquidity risks within a centralized risk book for capital requirements. In this context, our model can reduce the liquidity risk of the execution process by taking into account the joint dynamics of the assets.

6.4 Appendix - Multi-asset optimal execution with correlated Brownian motions and execution costs

We consider in this appendix the problem of multi-asset optimal execution in the case where prices are correlated arithmetic Brownian motions. This problem is a special case of that presented in this paper, corresponding to $R = 0$ in the dynamics (6.2) of the asset prices. Therefore, the results presented in the paper apply. However, when $R = 0$, as mentioned in Remark 14, the system of ODEs (6.16) simplify since a trivial solution to the last five equations is $B = C = D = E = F = 0$. Therefore, the problem boils down to finding $A \in C^1([0, T], \mathcal{S}_d(\mathbb{R}))$ solution of the following terminal value problem:

$$\begin{cases} A'(t) &= \frac{\gamma}{2}\Sigma - A(t)\eta^{-1}A(t) \\ A(T) &= -\Gamma. \end{cases} \quad (6.49)$$

In this appendix we show that, when $\Sigma \in \mathcal{S}_d^{++}(\mathbb{R})$, A can be found in closed form.

For that purpose, we introduce the change of variables

$$a(t) = \eta^{-\frac{1}{2}}A(t)\eta^{-\frac{1}{2}} \quad \forall t \in [0, T].$$

and notice that (6.49) is equivalent to the terminal value problem

$$\begin{cases} a'(t) = \hat{A}^2 - a(t)^2 \\ a(T) = -C, \end{cases} \quad (6.50)$$

where $\hat{A} = \sqrt{\frac{\gamma}{2}} \left(\eta^{-\frac{1}{2}}\Sigma\eta^{-\frac{1}{2}} \right)^{\frac{1}{2}} \in \mathcal{S}_d^{++}(\mathbb{R})$ and $C = \eta^{-\frac{1}{2}}\Gamma\eta^{-\frac{1}{2}} \in \mathcal{S}_d^+(\mathbb{R})$.

To solve (6.50) we use a classical trick for Riccati equations, shown in the following Proposition:

Proposition 10. *Let $\xi : [0, T] \rightarrow \mathcal{S}_d(\mathbb{R})$ defined as*

$$\xi(t) = -\frac{\hat{A}^{-1}}{2} \left(I - e^{-2\hat{A}(T-t)} \right) - e^{-\hat{A}(T-t)} \left(C + \hat{A} \right)^{-1} e^{-\hat{A}(T-t)} \quad (6.51)$$

be the unique solution of the linear ODE

$$\begin{cases} \xi'(t) = \hat{A}\xi(t) + \xi(t)\hat{A} + I_d \\ \xi(T) = -\left(C + \hat{A} \right)^{-1}. \end{cases} \quad (6.52)$$

Then $\forall t \in [0, T]$, $\xi(t)$ is invertible and $a : t \in [0, T] \rightarrow \hat{A} + \xi(t)^{-1} \in \mathcal{S}_d(\mathbb{R})$ is the unique solution of (6.50).

Proof. First, we easily verify that ξ , defined in (6.51), is solution of the linear ODE (6.52). We see that, for all t in $[0, T]$, $\xi(t)$ is the sum of $-\frac{\hat{A}^{-1}}{2} \left(I - e^{-2\hat{A}(T-t)} \right) \in \mathcal{S}_d^{--}(\mathbb{R})$ and $-e^{-\hat{A}(T-t)} \left(C + \hat{A} \right)^{-1} e^{-\hat{A}(T-t)} \in \mathcal{S}_d^{--}(\mathbb{R})$, so $\xi(t) \in \mathcal{S}_d^{--}(\mathbb{R})$ and is invertible.

We also note that

$$a'(t) = -\xi(t)^{-1} \xi'(t) \xi(t)^{-1} = -\xi(t)^{-1} \hat{A} - \hat{A} \xi(t)^{-1} - \xi(t)^{-2} = \hat{A}^2 - \left(\hat{A} + \xi(t)^{-1} \right)^2 = \hat{A}^2 - a(t)^2$$

and $a(T) = -C$, hence the result. \square

We deduce the following corollary:

Corollary 1.

$$\forall t \in [0, T], \quad A(t) = \eta^{\frac{1}{2}} \left(\hat{A} - \left(\frac{\hat{A}^{-1}}{2} \left(I - e^{-2\hat{A}(T-t)} \right) + e^{-\hat{A}(T-t)} \left(C + \hat{A} \right)^{-1} e^{-\hat{A}(T-t)} \right)^{-1} \right) \eta^{\frac{1}{2}}.$$

6.5 Appendix - Merton portfolio optimization problem under Ornstein-Uhlenbeck dynamics and exponential utility

6.5.1 Modelling framework

We study in this appendix a Merton model where prices have multivariate Ornstein-Uhlenbeck dynamics. It is closely related to our model and can be seen as some form of limit case corresponding to no execution costs (i.e. $L = 0$) and no terminal penalty (i.e. $\ell = 0$).

The results obtained in this appendix are essential in our proof of existence of a solution to the system of ODEs (6.16) on $[0, T]$ with terminal condition (6.17) (see Theorem 6).

As in the paper, we consider a model with d assets, whose prices are modelled by a d -dimensional stochastic process $(S_t)_{t \in [0, T]} = (S_t^1, \dots, S_t^d)_{t \in [0, T]}^\top$ with dynamics

$$dS_t = R(\bar{S} - S_t)dt + VdW_t,$$

where $\bar{S} \in \mathbb{R}^d$, $R \in \mathcal{M}_d(\mathbb{R})$, $V \in \mathcal{M}_{d,k}(\mathbb{R})$, and $(W_t)_{t \in [0, T]} = (W_t^1, \dots, W_t^k)_{t \in [0, T]}^\top$ is a k -dimensional standard Brownian motion (with independent coordinates), for some $k \in \mathbb{N}^*$. As before, we write $\Sigma = VV^\top$.

We consider a trader optimizing her portfolio over the period $[0, T]$ by controlling at each time the number of each asset in her portfolio, i.e. she controls the d -dimensional process $(q_t)_{t \in [0, T]} = (q_t^1, \dots, q_t^d)_{t \in [0, T]}^\top$, where q_t^i denotes the number of assets i in the portfolio at time t , for each $i \in \{1, \dots, d\}$ ($t \in [0, T]$). The process $(q_t)_{t \in [0, T]}$ lies in the space of admissible controls \mathcal{A}_0^{Merton} , where for $t \in [0, T]$, the set \mathcal{A}_t^{Merton} is defined as

$$\mathcal{A}_t^{Merton} := \left\{ (q_s)_{s \in [t, T]}, \mathbb{R}^d\text{-valued, } \mathbb{F}\text{-adapted, satisfying a linear growth condition with respect to } (S_s)_{s \in [t, T]} \right\}. \quad (6.53)$$

We introduce the process $(\mathcal{V}_t)_{t \in [0, T]}$ modelling the MtM value of the trader's portfolio, i.e.

$$\forall t \in [0, T], \quad \mathcal{V}_t = \mathcal{V}_0 + \int_0^t q_s^\top dS_s, \quad \mathcal{V}_0 \in \mathbb{R} \text{ given}$$

For a given $\gamma > 0$, the trader aims at maximizing the following objective function:

$$\mathbb{E} \left[-e^{-\gamma \mathcal{V}_T} \right], \quad (6.54)$$

over the set of admissible controls $(q_t)_{t \in [0, T]} \in \mathcal{A}_0^{Merton}$. We define her value function $\hat{u} : [0, T] \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$ as

$$\hat{u}(t, \mathcal{V}, S) = \sup_{q \in \mathcal{A}_t^{Merton}} \mathbb{E} \left[-e^{-\gamma \mathcal{V}_T^{t, \mathcal{V}, S, q}} \right] \quad \forall (t, \mathcal{V}, S) \in [0, T] \times \mathbb{R} \times \mathbb{R}^d,$$

where $(\mathcal{V}_s^{t, \mathcal{V}, S, q})_{s \in [t, T]}$ denotes the process defined by

$$d\mathcal{V}_s^{t, \mathcal{V}, S, q} = q_s^\top dS_s^{t, S}, \quad \mathcal{V}_t^{t, \mathcal{V}, S, q} = \mathcal{V}$$

with

$$dS_s^{t, S} = R(\bar{S} - S_s^{t, S})ds + VdW_s, \quad S_t^{t, S} = S.$$

6.5.2 HJB equation

The HJB equation associated with Problem (6.54) is given by

$$\begin{aligned} 0 = & \partial_t \hat{u}(t, \mathcal{V}, S) + \nabla_S \hat{u}(t, \mathcal{V}, S)^\top R(\bar{S} - S) + \frac{1}{2} \text{Tr} \left(\Sigma D_{SS}^2 \hat{u}(t, \mathcal{V}, S) \right) \\ & + \sup_{q \in \mathbb{R}^d} \left\{ \partial_{\mathcal{V}} \hat{u}(t, \mathcal{V}, S) q^\top R(\bar{S} - S) + \frac{1}{2} \partial_{\mathcal{V}\mathcal{V}}^2 \hat{u}(t, \mathcal{V}, S) q^\top \Sigma q + \partial_{\mathcal{V}} \nabla_S \hat{u}(t, \mathcal{V}, S)^\top \Sigma q \right\} \end{aligned} \quad (6.55)$$

for all $(t, \mathcal{V}, S) \in [0, T] \times \mathbb{R} \times \mathbb{R}^d$, with terminal condition

$$\hat{u}(T, \mathcal{V}, S) = -e^{-\gamma \mathcal{V}} \quad \forall (\mathcal{V}, S) \in \mathbb{R} \times \mathbb{R}^d. \quad (6.56)$$

To solve the above HJB equation, we use the ansatz

$$\hat{u}(t, \mathcal{V}, S) = -e^{-\gamma(\mathcal{V} + \hat{\theta}(t, S))}. \quad (6.57)$$

Indeed, we have the following proposition:

Proposition 11. *If there exists $\hat{\theta} : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$ solution to*

$$0 = \partial_t \hat{\theta}(t, S) + \frac{1}{2} \text{Tr} \left(\Sigma D_{SS}^2 \hat{\theta}(t, S) \right) + \frac{1}{2\gamma} (\bar{S} - S)^\top R^\top \Sigma^{-1} R (\bar{S} - S) \quad (6.58)$$

on $[0, T] \times \mathbb{R}^d$, with terminal condition

$$\hat{\theta}(T, S) = 0 \quad \forall S \in \mathbb{R}^d, \quad (6.59)$$

then the function $\hat{w} : [0, T] \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$ defined by

$$\hat{w}(t, \mathcal{V}, S) = -e^{-\gamma(\mathcal{V} + \hat{\theta}(t, S))} \quad \forall (t, \mathcal{V}, S) \in [0, T] \times \mathbb{R} \times \mathbb{R}^d$$

is a solution to (6.55) on $[0, T] \times \mathbb{R} \times \mathbb{R}^d$ with terminal condition (6.56).

Proof. Let $\hat{\theta} : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$ be a solution to (6.58) on $[0, T] \times \mathbb{R}^d$ with terminal condition (6.59), then we have for all $(t, \mathcal{V}, S) \in [0, T] \times \mathbb{R} \times \mathbb{R}^d$:

$$\begin{aligned} & \partial_t \hat{w}(t, \mathcal{V}, S) + \nabla_S \hat{w}(t, \mathcal{V}, S)^\top R(\bar{S} - S) + \frac{1}{2} \text{Tr}(\Sigma D_{SS}^2 \hat{w}(t, \mathcal{V}, S)) \\ & + \sup_{q \in \mathbb{R}^d} \left\{ \partial_{\mathcal{V}} \hat{w}(t, \mathcal{V}, S) q^\top R(\bar{S} - S) + \frac{1}{2} \partial_{\mathcal{V}\mathcal{V}}^2 \hat{w}(t, \mathcal{V}, S) q^\top \Sigma q + \partial_{\mathcal{V}} \nabla_S \hat{w}(t, \mathcal{V}, S)^\top \Sigma q \right\} \\ = & -\gamma \partial_t \hat{\theta}(t, S) \hat{w}(t, \mathcal{V}, S) - \gamma \nabla_S \hat{\theta}(t, S) R(\bar{S} - S) \hat{w}(t, \mathcal{V}, S) - \gamma \hat{w}(t, \mathcal{V}, S) \frac{1}{2} \text{Tr}(\Sigma D_{SS}^2 \hat{\theta}(t, S)) \\ & + \frac{\gamma^2}{2} \hat{w}(t, \mathcal{V}, S) \nabla_S \hat{\theta}(t, S)^\top \Sigma \nabla_S \hat{\theta}(t, S) \\ & + \sup_{q \in \mathbb{R}^d} \left\{ -\gamma \hat{w}(t, \mathcal{V}, S) q^\top R(\bar{S} - S) + \frac{\gamma^2}{2} \hat{w}(t, \mathcal{V}, S) q^\top \Sigma q + \gamma^2 \hat{w}(t, \mathcal{V}, S) \nabla_S \hat{\theta}(t, \mathcal{V}, S)^\top \Sigma q \right\} \\ = & -\gamma \hat{w}(t, \mathcal{V}, S) \left(\partial_t \hat{\theta}(t, S) + \nabla_S \hat{\theta}(t, S) R(\bar{S} - S) + \frac{1}{2} \text{Tr}(\Sigma D_{SS}^2 \hat{\theta}(t, S)) - \frac{\gamma}{2} \nabla_S \hat{\theta}(t, S)^\top \Sigma \nabla_S \hat{\theta}(t, S) \right. \\ & \left. + \sup_{q \in \mathbb{R}^d} \left\{ q^\top \left(R(\bar{S} - S) - \gamma \Sigma \nabla_S \hat{\theta}(t, S) \right) - \frac{\gamma}{2} q^\top \Sigma q \right\} \right). \end{aligned}$$

The supremum in the last line is reached at

$$q^*(t, S) = \frac{1}{\gamma} \Sigma^{-1} R(\bar{S} - S) - \nabla_S \hat{\theta}(t, S),$$

and we obtain after simplifications:

$$\begin{aligned} & \partial_t \hat{w}(t, \mathcal{V}, S) + \nabla_S \hat{w}(t, \mathcal{V}, S)^\top R(\bar{S} - S) + \frac{1}{2} \text{Tr}(\Sigma D_{SS}^2 \hat{w}(t, \mathcal{V}, S)) \\ & + \sup_{q \in \mathbb{R}^d} \left\{ \partial_{\mathcal{V}} \hat{w}(t, \mathcal{V}, S) q^\top R(\bar{S} - S) + \frac{1}{2} \partial_{\mathcal{V}\mathcal{V}}^2 \hat{w}(t, \mathcal{V}, S) q^\top \Sigma q + \partial_{\mathcal{V}} \nabla_S \hat{w}(t, \mathcal{V}, S)^\top \Sigma q \right\} \\ = & -\gamma \hat{w}(t, \mathcal{V}, S) \left(\partial_t \hat{\theta}(t, S) + \frac{1}{2} \text{Tr}(\Sigma D_{SS}^2 \hat{\theta}(t, S)) + \frac{1}{2\gamma} (\bar{S} - S)^\top R^\top \Sigma^{-1} R (\bar{S} - S) \right) \\ = & 0. \end{aligned}$$

As \hat{w} satisfies the terminal condition (6.56), the result is proved. \square

We now use a second ansatz and look for a function $\hat{\theta}$ solution to (6.58) on $[0, T] \times \mathbb{R}^d$ with terminal condition (6.59) of the following form:

$$\hat{\theta}(t, S) = S^\top \hat{C}(t) S + \hat{E}(t)^\top S + \hat{F}(t), \quad (6.60)$$

We have indeed the following proposition:

Proposition 12. Assume there exists $\hat{C} \in C^1([0, T], \mathcal{S}_d(\mathbb{R}))$, $\hat{E} \in C^1([0, T], \mathbb{R}^d)$, $\hat{F} \in C^1([0, T], \mathbb{R})$ satisfying the system of ODEs

$$\begin{cases} \hat{C}'(t) &= -\frac{1}{2\gamma} R^\top \Sigma^{-1} R \\ \hat{E}'(t) &= \frac{1}{\gamma} R^\top \Sigma^{-1} R \bar{S} \\ \hat{F}'(t) &= -\text{Tr}(\hat{C}(t) \Sigma) - \frac{1}{2\gamma} \bar{S}^\top R^\top \Sigma^{-1} R \bar{S}, \end{cases} \quad (6.61)$$

with terminal condition

$$\hat{C}(T) = \hat{E}(T) = \hat{F}(T) = 0. \quad (6.62)$$

Then the function $\hat{\theta}$ defined by (6.60) satisfies (6.58) on $[0, T] \times \mathbb{R}^d$ with terminal condition (6.59).

Proof. Let us consider $\hat{C} \in C^1([0, T], \mathcal{S}_d(\mathbb{R}))$, $\hat{E} \in C^1([0, T], \mathbb{R}^d)$, $\hat{F} \in C^1([0, T], \mathbb{R})$ verifying (6.61) on $[0, T]$ with terminal condition (6.62). Let us consider $\hat{\theta} : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$ defined by (6.60). Then we obtain for all $(t, S) \in [0, T] \times \mathbb{R}^d$:

$$\begin{aligned} & \partial_t \hat{\theta}(t, S) + \frac{1}{2} \text{Tr}(\Sigma D_{SS}^2 \hat{\theta}(t, S)) + \frac{1}{2\gamma} (\bar{S} - S)^\top R^\top \Sigma^{-1} R (\bar{S} - S) \\ &= S^\top \hat{C}'(t) S + \hat{E}'(t)^\top S + \hat{F}'(t) + \text{Tr}(\hat{C}(t) \Sigma) + \frac{1}{2\gamma} (\bar{S} - S)^\top R^\top \Sigma^{-1} R (\bar{S} - S) \\ &= 0. \end{aligned}$$

As it is straightforward to verify that $\hat{\theta}$ satisfies the terminal condition (6.59), the result is proved. \square

It is straightforward to see that there exists a unique solution $\hat{C} \in C^1([0, T], \mathcal{S}_d(\mathbb{R}))$, $\hat{E} \in C^1([0, T], \mathbb{R}^d)$, $\hat{F} \in C^1([0, T], \mathbb{R})$ to (6.61) with terminal condition (6.62). We can then prove the following verification theorem.

Theorem 7. We consider the functions $\hat{C} \in C^1([0, T], \mathcal{S}_d(\mathbb{R}))$, $\hat{E} \in C^1([0, T], \mathbb{R}^d)$, $\hat{F} \in C^1([0, T], \mathbb{R})$ solutions to (6.61) with terminal condition

$$\hat{C}(T) = \hat{E}(T) = \hat{F}(T) = 0,$$

i.e. for all $t \in [0, T]$,

$$\begin{cases} \hat{C}(t) = (T - t) \frac{1}{2\gamma} R^\top \Sigma^{-1} R, \\ \hat{E}(t) = (T - t) \frac{1}{\gamma} R^\top \Sigma^{-1} R \bar{S}, \\ \hat{F}(t) = \frac{1}{4\gamma} (T - t)^2 \text{Tr}(R^\top \Sigma^{-1} R \Sigma) + (T - t) \frac{1}{2\gamma} \bar{S}^\top R^\top \Sigma^{-1} R. \end{cases}$$

We consider the function $\hat{\theta}$ defined by

$$\hat{\theta}(t, S) = S^\top \hat{C}(t) S + \hat{E}(t)^\top S + \hat{F}(t),$$

and the associated function \hat{w} defined by

$$\hat{w}(t, \mathcal{V}, S) = -e^{-\gamma(\mathcal{V} + \hat{\theta}(t, S))}.$$

For all $(t, \mathcal{V}, S) \in [0, T] \times \mathbb{R} \times \mathbb{R}^d$ and $q = (q_s)_{s \in [t, T]} \in \mathcal{A}_t^{Merton}$, we have

$$\mathbb{E} \left[-e^{-\gamma(\mathcal{V}_T^{\mathcal{V}, S, q})} \right] \leq \hat{w}(t, \mathcal{V}, S). \quad (6.63)$$

Moreover, equality is obtained in (6.63) by taking the optimal control $(q_s^*)_{s \in [t, T]} \in \mathcal{A}_t^{Merton}$ given by the closed-loop feedback formula

$$q_s^* = \frac{1}{\gamma} \Sigma^{-1} R(\bar{S} - S_s^{t, S}) - \hat{C}(s) S_s^{t, S} - \hat{E}(s). \quad (6.64)$$

In particular, $\hat{w} = \hat{u}$.

Proof. It is obvious that $(q_s^*)_{s \in [t, T]} \in \mathcal{A}_t^{Merton}$ (i.e., $(q_s^*)_{s \in [t, T]}$ is well-defined and admissible):

$$\exists C_T > 0, \forall s \in [t, T], \quad \|q_s^*\| \leq C_T \left(1 + \sup_{\tau \in [t, s]} \|S_\tau\| \right). \quad (6.65)$$

Let us consider $(t, \mathcal{V}, S) \in [0, T] \times \mathbb{R} \times \mathbb{R}^d$ and $q = (q_s)_{s \in [t, T]} \in \mathcal{A}_t^{Merton}$. We now prove that

$$\mathbb{E} \left[\hat{w}(T, \mathcal{V}_T^{\mathcal{V}, S}, S_T^{t, S}) \right] \leq \hat{w}(t, \mathcal{V}, S). \quad (6.66)$$

We use the following notations for readability

$$\forall s \in [t, T], \quad \hat{w}(s, \mathcal{V}_s^{t, \mathcal{V}, S, q}, S_s^{t, S}) = \hat{w}_s^{t, \mathcal{V}, S, q}, \quad (6.67)$$

$$\forall s \in [t, T], \quad \hat{\theta}(s, S_s^{t, S}) = \hat{\theta}_s^{t, S}. \quad (6.68)$$

By Itô's formula, we have $\forall s \in [0, T]$

$$d\hat{w}_s^{t, \mathcal{V}, S, q} = \mathcal{L}^q \hat{w}_s^{t, \mathcal{V}, S, q} ds + (\partial_{\mathcal{V}} \hat{w}_s^{t, \mathcal{V}, S, q} q_s + \nabla_S \hat{w}_s^{t, \mathcal{V}, S, q})^\top V dW_s, \quad (6.69)$$

where

$$\begin{aligned} \mathcal{L}^q \hat{w}_s^{t, \mathcal{V}, S, q} &= \partial_t \hat{w}_s^{t, \mathcal{V}, S, q} + (\nabla_S \hat{w}_s^{t, \mathcal{V}, S, q})^\top R(\bar{S} - S) + \partial_{\mathcal{V}} \hat{w}_s^{t, \mathcal{V}, S, q} q_s^\top R(\bar{S} - S) + \frac{1}{2} \text{Tr}(\Sigma D_S^2 \hat{w}_s^{t, \mathcal{V}, S, q}) \\ &+ \frac{1}{2} \partial_{\mathcal{V}}^2 \hat{w}_s^{t, \mathcal{V}, S, q} q_s^\top \Sigma q_s + (\partial_{\mathcal{V}} \nabla_S \hat{w}_s^{t, \mathcal{V}, S, q})^\top \Sigma q_s. \end{aligned} \quad (6.70)$$

We have

$$\begin{aligned} \nabla_S \hat{w}_s^{t, \mathcal{V}, S, q} &= -\gamma \hat{w}_s^{t, \mathcal{V}, S, q} \nabla_S \hat{\theta}_s^{t, S} \\ &= -\gamma \hat{w}_s^{t, \mathcal{V}, S, q} \left(2\hat{C}(s) S_s^{t, S} + \hat{E}(s) \right), \end{aligned} \quad (6.71)$$

and

$$\partial_{\mathcal{V}} \hat{w}_s^{t, \mathcal{V}, S, q} = -\gamma \hat{w}_s^{t, \mathcal{V}, S, q}. \quad (6.72)$$

We define $\forall s \in [t, T]$

$$\kappa_s^q = -\gamma \left(q_s + 2\hat{C}(s)S_s^{t,S} + \hat{E}(s) \right), \quad (6.73)$$

$$\xi_{t,s}^q = \exp \left(\int_t^s \kappa_\varrho^{q\top} V dW_\varrho - \frac{1}{2} \int_t^s \kappa_\varrho^{q\top} \Sigma \kappa_\varrho^q d\varrho \right). \quad (6.74)$$

We then have

$$d \left(\hat{w}_s^{t,\mathcal{V},S,q} (\xi_{t,s}^q)^{-1} \right) = (\xi_{t,s}^q)^{-1} \mathcal{L}^q \hat{w}_s^{t,\mathcal{V},S,q} ds. \quad (6.75)$$

By definition of \hat{w} , $\mathcal{L}^q \hat{w}_s^{t,\mathcal{V},S,q} \leq 0$.

Moreover, equality holds for the control reaching the sup in (6.55). It is easy to see that the sup is reached for the unique value

$$q_s = \frac{1}{\gamma} \Sigma^{-1} R (\bar{S} - S_s^{t,S}) - \nabla_S \hat{\theta}(t, S_s^{t,S}) \quad (6.76)$$

$$= \frac{1}{\gamma} \Sigma^{-1} R (\bar{S} - S_s^{t,S}) - 2\hat{C}(s)S_s^{t,S} - \hat{E}(s), \quad (6.77)$$

which corresponds to $(q_s)_{s \in [t, T]} = (q_s^*)_{s \in [t, T]}$.

As a consequence, $\left(\hat{w}_s^{t,\mathcal{V},S,q} (\xi_{t,s}^q)^{-1} \right)_{s \in [t, T]}$ is nonincreasing and therefore

$$\hat{w}(T, \mathcal{V}_T^{t,\mathcal{V},S,q}, S_T^{t,S}) \leq \hat{w}(t, \mathcal{V}, S) \xi_{t,T}^q, \quad (6.78)$$

with equality when $(q_s)_{s \in [t, T]} = (q_s^*)_{s \in [t, T]}$.

Taking expectation we get

$$\mathbb{E} \left[\hat{w}(T, \mathcal{V}_T^{t,\mathcal{V},S,q}, S_T^{t,S}) \right] \leq \hat{w}(t, \mathcal{V}, S) \mathbb{E} \left[\xi_{t,T}^q \right]. \quad (6.79)$$

We proceed to prove that $E \left[\xi_{t,T}^q \right]$ is equal to 1. To do so, we use that $\xi_{t,t}^q = 1$ and prove that $(\xi_{t,s}^q)_{s \in [t, T]}$ is a martingale under $(\mathbb{P}; \mathbb{F} = (\mathcal{F}_s)_{s \in [t, T]})$.

We know that $(q_s^{t,q})_{s \in [t, T]}$ satisfies a linear growth condition with respect to $(S_s^{t,S})_{s \in [t, T]}$. Given the form of κ one can easily show that there exists a constant C such that

$$\sup_{s \in [t, T]} \|\kappa_s^q\|^2 \leq C \left(1 + \sup_{s \in [t, T]} \|W_s - W_t\|^2 \right). \quad (6.80)$$

By using classical properties of the Brownian motion, we prove that

$$\exists \epsilon > 0, \forall s \in [t, T], \quad \mathbb{E} \left[\exp \left(\frac{1}{2} \int_s^{(s+\epsilon) \wedge T} \kappa_\varrho^{q\top} \Sigma \kappa_\varrho^q d\varrho \right) \right] < +\infty. \quad (6.81)$$

From Novikov condition, we see that $(\xi_{t,s}^q)_{s \in [t,T]}$ is a martingale under $(\mathbb{P}; \mathbb{F} = (\mathcal{F}_s)_{s \in [t,T]})$.

We obtain

$$\mathbb{E} \left[\hat{w} \left(T, \mathcal{V}_T^{t,\mathcal{V},S,q}, S_T^{t,S} \right) \right] \leq \hat{w}(t, \mathcal{V}, S), \quad (6.82)$$

with equality when $(q_s)_{s \in [t,T]} = (q_s^*)_{s \in [t,T]}$.

We conclude that

$$\hat{u}(t, \mathcal{V}, S) = \sup_{(q_s)_{s \in [t,T]} \in \mathcal{A}_t^{Merton}} \mathbb{E} \left[-\exp \left(-\gamma V_T^{t,\mathcal{V},S,q} \right) \right] \quad (6.83)$$

$$= \mathbb{E} \left[-\exp \left(-\gamma V_T^{t,\mathcal{V},S,q^*} \right) \right] \quad (6.84)$$

$$= \hat{w}(t, \mathcal{V}, S). \quad (6.85)$$

□

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