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# Steinhaus triangles and graphs and contributions to numerical semigroups, Tower of Hanoi problems, Ramsey theory and Kneser transversals

Jonathan Chappelon

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# HABILITATION A DIRIGER DES RECHERCHES DE L'UNIVERSITÉ DE MONTPELLIER

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Institut Montpellierain Alexander Grothendieck

Steinhaus triangles and graphs and contributions to  
numerical semigroups, Tower of Hanoi problems, Ramsey  
theory and Kneser transversals

Présentée par Jonathan CHAPPELON  
Le 13 décembre 2021

Devant le jury composé de

Jean-Christophe NOVELLI, Professeur des universités, Université Gustave Eiffel  
Jean-Paul ALLOUCHE, Directeur de recherche émérite, Sorbonne Université - CNRS  
Andreas HINZ, Professeur, Ludwig-Maximilians-Universität München  
Shalom ELIAHOU, Professeur des universités, Université du Littoral Côte d'Opale  
Jorge Luis RAMIREZ ALFONSIN, Professeur des universités, Université de Montpellier

Président du Jury et Rapporteur  
Rapporteur  
Rapporteur  
Examineur  
Examineur



UNIVERSITÉ  
DE MONTPELLIER

"Je déclare avoir respecté, dans la conception et la rédaction de ce mémoire d'HDR, les valeurs et principes d'intégrité scientifique destinés à garantir le caractère honnête et scientifiquement rigoureux de tout travail de recherche, visés à l'article L.211-2 du Code de la recherche et énoncés par la Charte nationale de déontologie des métiers de la recherche et la Charte d'intégrité scientifique de l'Université de Montpellier. Je m'engage à les promouvoir dans le cadre de mes activités futures d'encadrement de recherche."

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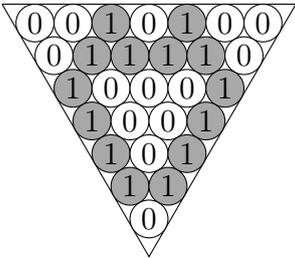
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# Introduction

This habilitation thesis is organized in two independent parts: a first part about my main results on Steinhaus triangles and graphs and a second part about other results obtained on numerical semigroups, Tower of Hanoi problems, Ramsey theory and Kneser transversals.

The topic of the first part, Steinhaus triangles and graphs, is my oldest research theme, starting with my doctoral thesis, that I defended in 2008. Since then, I have continued to work on and explore problems on these structures and on generalizations. This part is divided into five chapters: two chapters about binary Steinhaus triangles, the first chapter on balanced triangles and the second one on Steinhaus triangles having symmetric properties, one chapter on Steinhaus graphs and finally two chapters on generalizations of binary Steinhaus triangles, one for a generalization with modular numbers and one other for generalizations in higher dimensions with balanced simplices.

In Chapter 1, the structure of binary Steinhaus triangles is explored. This is one of the most elementary structures in mathematics, that even a schoolboy could understand. The figure below consists of 14 zeroes and 14 ones. They are arranged in such a way that under each pair of equal numbers there appears a zero and under different numbers there appears a one. If the first row has  $n$  terms, then in an analogous figure there would be



$\frac{n(n+1)}{2}$  terms, our example corresponds to the case  $n = 7$ . In his famous book entitled "One hundred problems in elementary mathematics", Hugo Steinhaus asked in 1958 whether it is possible to construct a figure analogous to the above one and beginning with  $n$  terms in the highest row. A binary Steinhaus triangle is a down-pointing binary triangle built with the same local rule and we say that a triangle is balanced if it contains as many zeroes as ones. The Steinhaus problem was positively solved for the first time in 1972 by Heiko Harborth. He proved that, for every positive integer  $n \equiv 0$  or  $3 \pmod{4}$ , there exists a binary sequence of length  $n$  which is the first row of a balanced Steinhaus triangle. His proof is constructive and based on pseudo-periodic binary sequences. Since then, many solutions of the Steinhaus problem have appeared in the literature. All known solutions are constructive and involve pseudo-periodic binary sequences. After explaining the reasons for choosing pseudo-periodic binary sequences, some of these solutions are presented in this chapter. In the last part of this chapter, a recent solution of the Steinhaus problem obtained from purely periodic binary sequences is detailed. If we apply the same

local rule from an infinite binary sequence, instead of a finite one, we obtain an infinite grid of zeroes and ones that is called its orbit. The main result here is that there exist periodic binary sequences, of length period 24, whose orbit contains balanced binary Steinhaus triangles for all the admissible sizes. Moreover, this result is also valid for a generalization of the Steinhaus problem in two directions: for binary Steinhaus triangles containing an odd number of terms whose numbers of zeroes and ones only differ by 1 (almost-balanced triangles) and also for up-pointing binary triangles built with the same local rule (binary generalized Pascal triangles). The strength of this result is that from only one periodic binary sequence, it is possible to explicitly obtain the existence of almost-balanced binary triangles, of these two kinds, for all possible sizes. This result has been published in [C7].

The study of binary Steinhaus triangles continues in Chapter 2 with binary triangles having symmetric properties. It is straightforward to see that the set  $\mathcal{ST}(n)$  of binary Steinhaus triangles of size  $n$ , i.e., whose first row is of length  $n$ , is a vector space over  $\mathbb{Z}/2\mathbb{Z}$  of dimension  $n$ . Moreover, the local rule implies that  $\mathcal{ST}(n)$  is closed under the 120 degrees rotation and under horizontal reflection. This implies a faithful representation of the dihedral group  $D_3$  on  $\mathcal{ST}(n)$ . A binary Steinhaus triangle is said to be rotationally symmetric, horizontally symmetric or dihedrally symmetric if it is invariant under the 120 degrees rotation, the horizontal reflection or both, respectively. The linear subspaces  $\mathcal{RST}(n)$ ,  $\mathcal{HST}(n)$  and  $\mathcal{DST}(n)$  of rotationnaly symmetric, horizontally symmetric and dihedrally symmetric binary Steinhaus triangles, respectively, have been studied by André Barbé in [26], where he determined the dimension of each of them. Bases of  $\mathcal{RST}(n)$ ,  $\mathcal{HST}(n)$  and  $\mathcal{DST}(n)$  are obtained by Josep Brunat and Montserrat Maureso in [39]. In Chapter 2, for each of these three linear subspaces, we give new bases which are simpler than those mentioned. They are obtained by considering elementary properties of generalized binomial coefficients. A journal version [C8] of these results is submitted for publication.

Chapter 3 concerns the structure of Steinhaus graphs. A Steinhaus graph of order  $n$  is a simple graph whose adjacency matrix has an upper-triangular part which is a binary Steinhaus triangle of size  $n-1$ . It is straightforward to see that the set  $\mathcal{SG}(n)$  of Steinhaus graphs of order  $n$  is a vector space over  $\mathbb{Z}/2\mathbb{Z}$  of dimension  $n-1$  that is isomorphic to  $\mathcal{ST}(n-1)$ . This family of simple graphs was introduced by John C. Molluzzo in [89]. Franz A. Delahan proved in [48] that any simple graph of order  $n$  is isomorphic to an induced subgraph of a Steinhaus graph of order  $\binom{n}{2} + 1$ . A natural problem on Steinhaus graphs is to characterize those, or their associated binary sequences, having a given graph property such as connectedness, planarity, bipartition, regularity, etc. In Chapter 3, we focus on regular Steinhaus graphs, that are Steinhaus graphs where every vertex has the same degree, and on the weaker case of parity-regular Steinhaus graphs, that are Steinhaus graphs for which all the vertex degrees are of the same parity. A parity-regular Steinhaus graph is said to be either even or odd according to the parity of its vertices. The first part of Chapter 3 is a complete study of the linear subspace  $\mathcal{ESG}(n)$  of even Steinhaus graphs of order  $n$ . The characterization of even Steinhaus graphs has already been done by Wayne M. Dymacek in [51], where the dimension of  $\mathcal{ESG}(n)$  is given by considering that the adjacency matrix of an even Steinhaus graph is symmetric with respect to its anti-diagonal. This result is retrieved here by showing that the anti-diagonal terms of the adjacency matrix of a Steinhaus graph, not necessarily even, can be expressed in function of its vertex degrees. This last result has been published in [C3]. After that, it is shown that there exists an isomorphism between  $\mathcal{ESG}(n)$  and a certain linear subspace of  $\mathcal{DST}(n)$ . Therefore, using results highlighted in Chapter 2, explicit vector bases of

$\mathcal{ESG}(n)$  and  $\mathcal{PRSG}(n)$  are given, for all positive integers  $n$ . This new result comes from [C8]. The last part of this chapter concerns regular Steinhaus graphs. The complete list of regular Steinhaus graphs was conjectured by Wayne M. Dymacek in [51]. Since then, this conjecture was verified up to 117 vertices by Shalom Eliahou and Maxime Augier in [24]. In the end of Chapter 3, we show how results obtained on  $\mathcal{DST}(n)$  permits us to complete the study of regular Steinhaus graphs and to push forward the verification of Dymacek's conjecture up to 1500 vertices for regular Steinhaus graphs of odd degree. This result has also been published in [C3].

In Chapter 4, the structure of binary Steinhaus triangles is extended to triangles of numbers modulo a positive integer  $m$  built with the same local rule as the standard Pascal triangle modulo  $m$ . A Steinhaus triangle modulo  $m$  is said to be balanced if it contains all the elements of  $\mathbb{Z}/m\mathbb{Z}$  with the same multiplicity. In 1976, John C. Molluzzo asked if there exists a balanced Steinhaus triangle of size  $n$  modulo  $m$ , for every positive integers  $m$  and  $n$  such that  $\binom{n+1}{2}$  is divisible by  $m$ . For  $m = 2$ , it corresponds to the Steinhaus problem introduced in Chapter 1. The Molluzzo problem constitutes the main topic of Chapter 4. Until my doctoral thesis, this problem was solved in the affirmative only for  $m = 3$  and  $5$  in [30]. Since then, this problem has been positively solved, by constructive approaches, for small values of  $m$ : for  $m \in \{3, 5, 7\}$  in [C1] and for  $m = 4$  in [C9]. First counter-examples appeared in [C1], where it is proved that there does not exist balanced Steinhaus triangles of size  $5$  in  $\mathbb{Z}/15\mathbb{Z}$  and of size  $6$  in  $\mathbb{Z}/21\mathbb{Z}$ . Nevertheless, this problem can be positively answered for an infinite number of values  $m$ . In the first part of this chapter, it is shown that there exist balanced Steinhaus triangles, for all the possible sizes, in the case where  $m$  is a power of  $3$ . This result is obtained by studying Steinhaus triangles associated with arithmetic progressions. The main result here is that, when  $m$  is an odd number, the Steinhaus triangle whose first row is an arithmetic progression with an invertible common difference and of length  $k\alpha m$  is balanced, for all non-negative integers  $k$  and where  $\alpha$  is the multiplicative order of  $2^m$  modulo  $m$ . This implies that there exists an infinite number of balanced Steinhaus triangles modulo  $m$ , for every odd number  $m$ . In particular, when  $m$  is a power of  $3$ , this permits to positively answer the Molluzzo problem in this case. This result has been published in [C2]. In the second part of Chapter 4, we are interested in a particular sequence of integers

$$\text{US} = (\dots, -3, -3, 5, -2, -2, 3, -1, -1, 1, 0, \mathbf{0}, -1, 1, 1, -3, 2, 2, -5, 3, 3, -7, \dots)$$

that is called the universal sequence. This sequence can be seen as an interlacing of three arithmetic progressions. After studying Steinhaus triangles, or more generally orbits, associated with interlaced arithmetic progressions, we obtain that the orbit of the projection of US in  $\mathbb{Z}/m\mathbb{Z}$  contains infinitely many balanced Steinhaus triangles, when  $m$  is odd. More precisely, for every odd number  $m$ , this orbit contains balanced Steinhaus triangles of size  $km$ , for all non-negative integers  $k$ . The proof of this result is based on arithmetic triangles, that are, triangles of numbers modulo  $m$ , not necessarily Steinhaus triangles, where all rows and diagonals are arithmetic progressions with the same common differences, respectively. This is a noticeable improvement of the previous result on arithmetic progressions because here we obtain balanced Steinhaus triangles for all sizes multiple of  $m$ , not only for multiples of  $\alpha m$ . This result does not completely solve the Molluzzo problem for other moduli but, in the case where  $m$  is an odd prime power, we know that there exist balanced Steinhaus triangles for  $2/3$  of the admissible sizes. These results on the universal sequence have been published in [C5]. In the last part of Chapter 4, a construction method that consists in attempting to lift in  $\mathbb{Z}/4\mathbb{Z}$  specific known solutions in

$\mathbb{Z}/2\mathbb{Z}$  is detailed. This permits to obtain a complete positive answer to Molluzzo problem for  $m = 4$ . Up to now, this is the only known solution to this problem for  $m > 2$  even. This work has been carried out with Shalom Eliahou and has been published in [C9].

In Chapter 5, the structure of Steinhaus triangles is considered for other local rules and for higher dimensions, not only in dimension 2. An additive cellular automaton is a linear map on the set of infinite multidimensional arrays of elements in  $\mathbb{Z}/m\mathbb{Z}$ . In this chapter, we consider simplices appearing in orbits generated from arithmetic arrays by additive cellular automata. We show that they are a source of balanced simplices, that are simplices containing all the elements of  $\mathbb{Z}/m\mathbb{Z}$  with the same multiplicity. For any additive cellular automaton of dimension 1 or higher, the existence of infinitely many balanced simplices of  $\mathbb{Z}/m\mathbb{Z}$  appearing in such orbits is obtained, and this, for an infinite number of values  $m$ . The special case of the Pascal cellular automata, the cellular automata generating the Pascal simplices, that are a generalization of the Pascal triangle into arbitrary dimension, is studied in detail. This constitutes a natural generalization of the result obtained on Steinhaus triangles associated with arithmetic progressions in Chapter 4. This generalization has been published in [C6].

The second part of this habilitation thesis is divided into four chapters. Each chapter is about a research topic on which I began to work after my doctoral thesis: numerical semigroups, Tower of Hanoi problems, Ramsey theory and Kneser transversals.

Chapter 6 concerns numerical semigroups. A semigroup is a set with an internal binary operation that is associative. A numerical semigroup  $S$  is a subsemigroup of  $\mathbb{N}$  such that  $0 \in S$  and  $\mathbb{N} \setminus S$  is finite. It is well-known that, for every numerical semigroup  $S$ , there exist a minimal nonempty set  $A = \{a_1, \dots, a_n\}$  of relatively prime positive integers such that  $S = \langle A \rangle$ , i.e., each element of  $S$  can be expressed as  $x_1 a_1 + \dots + x_n a_n$ , where  $x_1, \dots, x_n$  are non-negative integers. The set  $A$  is called the set of generators of  $S$ . For every numerical semigroup  $S$ , we consider the locally finite poset  $(\mathbb{Z}, \leq_S)$  induced by  $S$  on the set of integers  $\mathbb{Z}$  defined by  $x \leq_S y$  if and only if  $y - x \in S$ , for all integers  $x$  and  $y$ . The Möbius function is an important concept associated with locally finite posets introduced by Gian-Carlo Rota in [98]. In the first part of Chapter 6, we investigate the Möbius function  $\mu_S$  associated with the poset  $(\mathbb{Z}, \leq_S)$ , where  $S$  is a numerical semigroup. Up to 2013, the only known result concerning  $\mu_S$  was an old theorem due to James A. Deddens in 1979 [47] that determines the value of  $\mu_S$  when  $S = \langle a_1, a_2 \rangle$  has exactly two generators. Here, we introduce a new approach to investigate  $\mu_S$  when  $S$  is an arithmetic numerical semigroup, that is, when  $S = \langle a, a + d, \dots, a + kd \rangle$  for some positive integers  $a, d, k$  such that  $a$  and  $d$  are coprime and  $k < a$ . The main result obtained here is a recursive formula for  $\mu_S$  when  $S$  is an arithmetic numerical semigroup. In particular, this permits us to give a new proof of Deddens's result and an explicit formula in the case where  $k = 2$  and  $a$  even, i.e., when  $S = \langle 2q, 2q + d, 2q + 2d \rangle$  for relatively prime positive integers  $q$  and  $d$ . This is a joint work with Jorge Luis Ramírez Alfonsín that has been published in [C16]. After that, we continue the investigation of the Möbius function associated with locally finite posets  $(\mathbb{Z}^m, \leq_S)$  arising now from subsemigroups  $S$  of  $\mathbb{Z}^m$ . This extends the work introduced before. We develop another approach to study  $\mu_S$  by using the Hilbert series of  $S$ . The latter enables us to provide formulas for  $\mu_S$  when  $S$  belongs to certain families of semigroups such as semigroups with a unique Betti element or three generated complete intersection numerical semigroups. This second work on  $\mu_S$  has been carried out with Ignacio García-Marco, Luis Pedro Montejano and Jorge Luis Ramírez Alfonsín and has been published in [C10]. The last part of this chapter is about square Frobenius numbers. For every numerical semigroup  $S$ , the Frobenius number  $g(S)$  is the largest integer which

is not an element of  $S$ . It is well-known that  $g(\langle a_1, a_2 \rangle) = a_1 a_2 - a_1 - a_2$ . However to calculate  $g(S)$  is a difficult problem in general. Here, we consider the following variant of the Frobenius number of  $S$ : let  ${}^2r(S)$  be the largest square number not belonging to  $S$ . We give an upper bound for  ${}^2r(S)$  for all arithmetic numerical semigroups  $S$ . The latter turns out to be the exact value of  ${}^2r(\langle s_1, s_2 \rangle)$  under certain conditions. We present an exact formula for  ${}^2r(\langle s_1, s_1 + d \rangle)$  when  $d = 3, 4$  and  $5$ . We study  ${}^2r(\langle s_1, s_1 + 1 \rangle)$  and  ${}^2r(\langle s_1, s_1 + 2 \rangle)$  and put forward two relevant conjectures. This is a joint work with Jorge Luis Ramírez Alfonsín and a journal version [C17] of these results is submitted for publication.

In Chapter 7, Tower of Hanoi problems are discussed. The Tower of Hanoi problem was introduced by Edouard Lucas in 1883 [85] for the case of 3 pegs and  $n$  disks of different sizes. Initially,  $n$  disks are placed on one of the 3 pegs with the largest at the bottom. Then, at each time one of the topmost disks is moved to a peg with a larger disk on the top or to an empty peg. The goal of the problem is to transfer all the disks from the initial peg to the peg of destination with the minimum number of moves. A simple recursive argument shows that  $2^n - 1$  moves are necessary and sufficient to carry out this task. This Tower of Hanoi problem was then extended to the case of 4 pegs by Dudeney in 1907 [50] and to arbitrary  $k \geq 3$  pegs by Stewart in 1939 [108]. In 1941, Frame [66] and Stewart [109] independently proposed algorithms which achieve the same numbers of moves for the  $k$ -peg Tower of Hanoi problem with  $k \geq 4$  pegs. Thus, these numbers are called the Frame-Stewart numbers. It is known that these numbers are optimal for the 4 pegs problem [37] and they are supposed to be optimal for  $k > 4$  pegs. The Frame-Stewart numbers are defined by the following recurrence relation:

$$S_k(n) = \min_{1 \leq t \leq n} \{2 \cdot S_k(n-t) + S_{k-1}(t)\}, \quad S_3(n) = 2^n - 1.$$

In the first part of Chapter 7, we generalize this recurrence relation to

$$G_k(n) = \min_{1 \leq t \leq n} \{p_k \cdot G_k(n-t) + q_k \cdot G_{k-1}(t)\}, \quad G_3(n) = p_3 \cdot G_3(n-1) + q_3,$$

for two sequences of arbitrary positive integers  $(p_i)_{i \geq 3}$  and  $(q_i)_{i \geq 3}$  and we show that the sequence of differences  $(G_k(n) - G_k(n-1))_{n \geq 1}$  consists of numbers of the form  $(\prod_{i=3}^k q_i) \cdot (\prod_{i=3}^k p_i^{\alpha_i})$ , with  $\alpha_i \geq 0$  for all  $i$ , arranged in nondecreasing order. We also apply this result to analyze recurrence relations for the Tower of Hanoi problems on several graphs. This work has been carried out with Akihiro Matsuura and has been published in [C14]. In the last part of Chapter 7, results obtained about a two-player combinatorial game based on Tower of Hanoi are studied. First, it is well-known that in the 3 pegs problem, the minimal number of moves is  $2^n - 1$  to transfer a tower of  $n$  disks. But there are also other variations to the game, involving for example real number weights on the moves of the disks. This gives rise to a similar type of problem, but where the final score seeks to be optimized. We study extensions of the one-player setting to two players, invoking classical winning conditions in combinatorial game theory such as the player who moves last wins, or the highest score wins. Here we solve both these winning conditions on 3 pegs. This is a joint work with Urban Larsson and Akihiro Matsuura that has been published in [C11].

Problems in Ramsey theory typically ask a question of the form: how big must some substructure be to guarantee that a particular property holds? In the first part of Chapter 8, we are interested in a problem of Ramsey theory on numbers. For any positive integers  $l$  and  $m$ , a set of integers is said to be (weakly)  $l$ -sum-free modulo  $m$

if it contains no (pairwise distinct) elements  $x_1, x_2, \dots, x_l, y$  satisfying the congruence  $x_1 + \dots + x_l \equiv y \pmod{m}$ . It is known that, for any positive integers  $k$  and  $l$ , there exists a largest integer  $n$  for which the set of the first  $n$  positive integers  $\{1, 2, \dots, n\}$  admits a partition into  $k$  (weakly)  $l$ -sum-free sets modulo  $m$ . This number is called the generalized (weak) Schur number modulo  $m$ , associated with  $k$  and  $l$ . This is a modular version of the standard (weak) Schur numbers for which only few exact values are known. It is straightforward to see that modular Schur numbers always constitute lower bounds of standard Schur numbers. Here, for all positive integers  $k$  and  $l$ , the exact value of these modular Schur numbers are determined for the moduli  $m = 1, 2$  and  $3$ . This work has been carried out with María Pastora Revuelta Marchena and María Isabel Sanz Domínguez and has been published in [C18]. In the last part of Chapter 8, several results for Ramsey theory on graphs are exposed. For any simple graphs  $G$  and  $H$ , let  $R(G, H)$  be the smallest integer  $N$  such that any 2-coloring (say red and blue) of the edges of the complete graph  $K_n$ , where  $n \geq N$ , there is either a red copy of  $G$  or a blue copy of  $H$ . It is well-known, by Ramsey Theorem, that  $R(G, H)$  is always finite, for any graphs  $G$  and  $H$ . After giving different recursive formulas, we obtain new exact numbers or new bounds on Ramsey numbers for different families of graphs: essentially for complete graphs with a dropped edge or triangle and for complete graphs with dropped stars. This is a joint work with Luis Pedro Montejano and Jorge Luis Ramírez Alfonsín. These results have been published in [C15, C19].

The last chapter of this thesis concerns Kneser transversals. For any positive integers  $k, d, \lambda$  with both  $d, k \geq \lambda$ , let  $m(k, d, \lambda)$  be the maximum positive integer  $n$  such that every set of  $n$  points (not necessarily in general position) in  $\mathbb{R}^d$  has the property that the convex hulls of all  $k$ -sets have a common transversal  $(d - \lambda)$ -plane. In [23], the following inequalities were obtained

$$d - \lambda + k + \left\lceil \frac{k}{\lambda} \right\rceil - 1 \leq m(k, d, \lambda) < d + 2(k - \lambda) + 1.$$

It turns out that  $m(k, d, \lambda)$  is strongly connected with other interesting problems, for instance, the chromatic number of Kneser hypergraphs and a discrete version of Rado's centerpoint theorem. In Chapter 9, we introduce a natural discrete version  $m^*$  of  $m$  by considering the existence of complete Kneser transversals. A complete Kneser transversal of a finite subset  $X$  of  $\mathbb{R}^d$  is a  $(d - \lambda)$ -plane that is transversal to the convex hulls of all  $k$ -sets of  $X$  and that contains  $(d - \lambda) + 1$  points of  $X$ . We study the relation between  $m^*$  and  $m$  and give a number of lower and upper bounds of  $m^*$  as well as the exact value in some cases. The main ingredient for proofs are Radon's partition theorem as well as oriented matroids tools. By studying the alternating oriented matroid, we obtain the asymptotic behavior of the function  $m^*$  for the family of cyclic polytopes. This work has been carried out with Leonardo Martínez-Sandoval, Luis Montejano, Luis Pedro Montejano and Jorge Luis Ramírez Alfonsín and has been published in [C12]. In the last part of Chapter 9, we focus on the existence of (complete) Kneser transversals for  $\lambda = 2, 3$ . In order to do this, we introduce the notions of stability and instability for (complete) Kneser transversals. We first give a stability result for collections of  $d + 2(k - \lambda)$  points in  $\mathbb{R}^d$  with  $k - \lambda \geq 2$  and  $\lambda = 2, 3$ . We then present a description of Kneser transversals  $L$  of collections of  $d + 2(k - \lambda)$  points in  $\mathbb{R}^d$  with  $k - \lambda \geq 2$  for  $\lambda = 2, 3$ . We show that either  $L$  is a complete Kneser transversal or it contains  $d - 2(\lambda - 1)$  points and the remaining  $2(k - 1)$  points of  $X$  are matched in  $k - 1$  pairs in such a way that  $L$  intersects the corresponding closed segments determined by them. The latter leads to new upper

and lower bounds (in the case when  $\lambda = 2$  and  $3$ ) for  $m(k, d, \lambda)$ . Finally, by using oriented matroid machinery, we present some computational results (closely related to the stability and unstability notions). We determine the existence of (complete) Kneser transversals for each of the 246 different order types of configurations of 7 points in  $\mathbb{R}^3$ . This is also a joint work with Leonardo Martínez-Sandoval, Luis Montejano, Luis Pedro Montejano and Jorge Luis Ramírez Alfonsín and it has been published in [C13].



# Part I

## Steinhaus triangles and graphs



# Chapter 1

## Balanced binary triangles

### 1.1 Preliminary results on binary triangles

In this chapter, we mainly consider two kinds of binary triangles built with the same local rule as the Sierpiński triangle or the standard Pascal triangle modulo 2.

#### 1.1.1 Binary Steinhaus triangles

**Definition 1.1.1** (Binary Steinhaus triangle). Let  $n$  be a positive integer. A *binary Steinhaus triangle* of size  $n$  is a down-pointing triangle  $(a_{i,j})_{1 \leq i \leq j \leq n}$  of elements of the cyclic group  $\mathbb{Z}/2\mathbb{Z} = \{0, 1\}$  satisfying the same local rule as the standard Pascal triangle modulo 2, that is,

$$a_{i,j} = a_{i-1,j-1} + a_{i-1,j}, \quad (\text{LR})$$

for all integers  $i$  and  $j$  such that  $2 \leq i \leq j \leq n$ , where the sum is the sum in  $\mathbb{Z}/2\mathbb{Z}$ . Note that (0) and (1) are the binary Steinhaus triangles of size 1 and we say that  $\emptyset$  is the binary Steinhaus triangle of size 0.

An example of a binary Steinhaus triangle of size 7 is depicted in Figure 1.1.

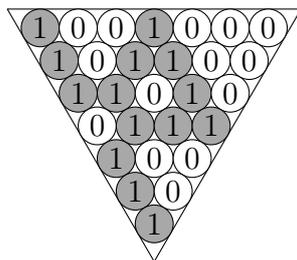


Figure 1.1: A binary Steinhaus triangle of size 7

**Notation 1.1.2.** For any non-negative integer  $n$ , the set of binary Steinhaus triangles of size  $n$  is denoted by  $\mathcal{ST}(n)$ .

It is straightforward to see that  $\mathcal{ST}(n)$  is a vector space over  $\mathbb{Z}/2\mathbb{Z}$ , by considering the sum

$$(a_{i,j})_{1 \leq i \leq j \leq n} + (b_{i,j})_{1 \leq i \leq j \leq n} = (a_{i,j} + b_{i,j})_{1 \leq i \leq j \leq n},$$

for all  $(a_{i,j})_{1 \leq i \leq j \leq n}$  and  $(b_{i,j})_{1 \leq i \leq j \leq n}$  in  $\mathcal{ST}(n)$ .

Moreover, it is clear that a binary Steinhaus triangle  $(a_{i,j})_{1 \leq i \leq j \leq n}$  is completely determined by its first row  $(a_{1,j})_{1 \leq j \leq n}$ . Indeed, by induction on  $i$  and using (LR), we obtain that

$$a_{i,j} = \sum_{k=0}^{i-1} \binom{i-1}{k} a_{1,j-k}, \quad (1.1)$$

for all integers  $i$  and  $j$  such that  $1 \leq i \leq j \leq n$ , where the binomial coefficient  $\binom{a}{b}$  is the coefficient of the monomial  $X^b$  in the expansion of  $(1+X)^a$ , for all non-negative integers  $a$  and  $b$ . Therefore, in the sequel, we use the following

**Notation 1.1.3.** The binary Steinhaus triangle whose first row is the binary sequence  $S$  is denoted by  $\nabla S$ .

The triangle in Figure 1.1 is then  $\nabla(1001000)$ .

For two binary sequences  $S_1 = (a_j)_{1 \leq j \leq n}$  and  $S_2 = (b_j)_{1 \leq j \leq n}$  of the same length  $n \geq 1$ , their *sum* is the sequence  $S_1 + S_2 = (a_j + b_j)_{1 \leq j \leq n}$  of length  $n$ . The set  $\mathbb{Z}/2\mathbb{Z}^{\{1, \dots, n\}}$  of binary sequences of length  $n$  can then be seen as a vector space over  $\mathbb{Z}/2\mathbb{Z}$  of dimension  $n$ . Since the linear map

$$\begin{aligned} \mathbb{Z}/2\mathbb{Z}^{\{1, \dots, n\}} &\longrightarrow \mathcal{ST}(n) \\ S &\longmapsto \nabla S \end{aligned}$$

is an isomorphism, we obtain the following

**Proposition 1.1.4.** *For any non-negative integer  $n$ , the set  $\mathcal{ST}(n)$  is a vector space over  $\mathbb{Z}/2\mathbb{Z}$  of dimension  $n$ .*

This kind of binary triangles has been introduced by Hugo Steinhaus in his problem book [106, 107], where he posed, as unsolved, the following

**Problem 1.1.5** (Steinhaus, 1958). *Does there exist, for every non-negative integer  $n$  such that  $n \equiv 0$  or  $3 \pmod{4}$ , a Steinhaus triangle of size  $n$  containing as many 0's as 1's?*

The triangle  $\nabla(1001000)$  depicted in Figure 1.1 solves this problem for  $n = 7$ , since it contains 14 zeroes and 14 ones. Note that, since a triangle of size  $n$  contains  $\binom{n+1}{2}$  elements, the condition  $n \equiv 0$  or  $3 \pmod{4}$  is a necessary and sufficient condition for having a triangle of size  $n$  with an even number of terms. The Steinhaus Problem was solved for the first time by Heiko Harborth in 1972 [71]. Since then, many solutions of this problem have appeared in the literature [60, 61, 57] and [C7]. All known solutions are constructive and some of these will be presented later in this chapter.

## 1.1.2 Binary generalized Pascal triangles

**Definition 1.1.6** (Binary generalized Pascal triangle). Let  $n$  be a positive integer. A *binary generalized Pascal triangle* of size  $n$  is an up-pointing triangle  $(a_{i,j})_{1 \leq j \leq i \leq n}$  of elements of  $\mathbb{Z}/2\mathbb{Z}$  verifying the local rule (LR), that is,

$$a_{i,j} = a_{i-1,j-1} + a_{i-1,j},$$

for all integers  $i, j$  such that  $2 \leq j < i \leq n$ . Note that (0) and (1) are the binary generalized Pascal triangles of size 1 and we say that  $\emptyset$  is the binary generalized Pascal triangle of size 0.

It is clear that a generalized Pascal triangle  $(a_{i,j})_{1 \leq j \leq i \leq n}$  is completely determined by its left side  $L = (a_{i,1})_{1 \leq i \leq n}$  and its right side  $R = (a_{i,i})_{1 \leq i \leq n}$ . Indeed, by induction on  $i$  and using (LR), we obtain that

$$a_{i,j} = \sum_{k=2}^{i-j+1} \binom{i-k-1}{j-2} a_{k,1} + \sum_{k=2}^j \binom{i-k-1}{j-k} a_{k,k}, \tag{1.2}$$

for all integers  $i$  and  $j$  such that  $2 \leq j \leq i-1 \leq n-1$ . Note that the first terms of  $L$  and  $R$  correspond.

**Notation 1.1.7.** We denote by  $\Delta(L, R)$  the binary generalized Pascal triangle whose left and right sides are the sequences  $L$  and  $R$ , respectively.

An example of binary generalized Pascal triangle of size 7 is depicted in Figure 1.2. Moreover, note that, for the constant binary sequences  $L = R = 1 \cdots 1$  of length  $n$ , the triangle  $\Delta(L, R)$  corresponds to the first  $n$  rows of the standard Pascal triangle modulo 2, the Sierpiński triangle.

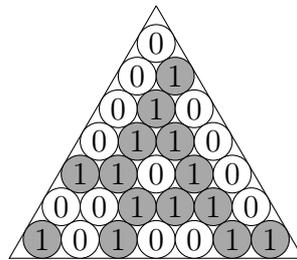


Figure 1.2: The binary generalized Pascal triangle  $\Delta(0000101, 0100001)$

**Notation 1.1.8.** For any non-negative integer  $n$ , the set of binary generalized Pascal triangles of size  $n$  is denoted by  $\mathcal{PT}(n)$ .

**Proposition 1.1.9.** For any positive integer  $n$ , the set  $\mathcal{PT}(n)$  is a vector space over  $\mathbb{Z}/2\mathbb{Z}$  of dimension  $2n - 1$ .

Moreover, there exists a natural isomorphism between  $\mathcal{PT}(n)$  and  $\mathcal{ST}(2n - 1)$ , for all positive integers  $n$ . Indeed, as depicted in Figure 1.3, a binary generalized Pascal triangle of size  $n$  can be seen as a subtriangle of a binary Steinhaus triangle of size  $2n - 1$ . More details on this result will be given later.

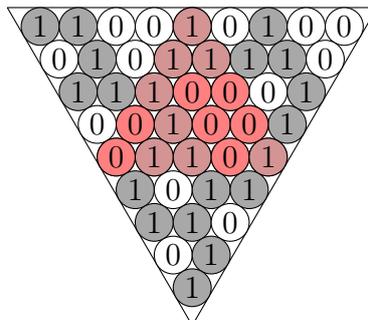


Figure 1.3: A binary generalized Pascal triangle inside a Steinhaus triangle

### 1.1.3 Derived and antiderived sequences of binary sequences

These two kinds of binary triangles are constituted by binary sequences. We continue by introducing the notions of derived and antiderived sequences of binary sequences.

**Definition 1.1.10** (Derived sequence). Let  $S = (a_j)_{1 \leq j \leq n}$  be a sequence of  $\mathbb{Z}/2\mathbb{Z}$ . The *derived sequence*  $\partial S$  of  $S$  is the sequence

$$\partial S = (a_j + a_{j+1})_{1 \leq j \leq n-1} \quad (1.3)$$

of length  $n - 1$ , when  $n \geq 2$ , and the empty sequence, when  $n \leq 1$ .

It is clear that the derivation map  $\partial$  is linear, i.e.,  $\partial(S_1 + S_2) = \partial S_1 + \partial S_2$  for all binary sequences  $S_1$  and  $S_2$  of same length.

**Definition 1.1.11** (Iterated derived sequences). The *iterated derived sequences*  $\partial^i S$  of a finite binary sequence  $S$  are recursively defined by  $\partial^i S = \partial(\partial^{i-1} S)$ , for all  $i \geq 1$ , with  $\partial^0 S = S$ .

For any finite binary sequence  $S$ , its Steinhaus triangle  $\nabla S$  can then be seen as the collection  $(\partial^i S)_{0 \leq i \leq n-1}$ , where, for every  $i \in \{1, \dots, n\}$ , the  $i$ th row of  $\nabla S$  corresponds to the derived sequence  $\partial^{i-1} S$ .

**Definition 1.1.12** (Antiderived sequences). Let  $S$  be a binary sequence of length  $n$ . An *antiderived sequence* of  $S$  is a sequence  $T$  of length  $n + 1$  such that  $\partial T = S$ .

**Notation 1.1.13** (Constant sequences). For any non-negative integer  $n$ , the constant sequence of length  $n$  equal to  $x$  is denoted by  $(x)_n$ . For  $n = 1$ , the sequence  $(x)_1$  is simply denoted by  $(x)$ .

**Proposition 1.1.14.** Let  $S = (a_j)_{1 \leq j \leq n}$  be a sequence of  $\mathbb{Z}/2\mathbb{Z}$  of length  $n$ . The sequence  $S$  admits two antiderived sequences  $T_1$  and  $T_2$  and these sequences are complementary, i.e.,  $T_1 + T_2 = (1)_{n+1}$ . For any  $i \in \{1, \dots, n + 1\}$  and any  $x \in \mathbb{Z}/2\mathbb{Z}$ , the antiderived sequence of  $S$  whose  $i$ th term is  $x$  is the sequence  $\int_{i,x} S = (b_j)_{1 \leq j \leq n+1}$  of length  $n + 1$  defined by

$$b_j = x + \sum_{k=1}^{i-1} a_k + \sum_{k=1}^{j-1} a_k, \quad (1.4)$$

for all  $j \in \{1, \dots, n + 1\}$ .

For example, the sequence  $S = 0100$  admits the two antiderived sequences  $00111$  and  $11000$ . Further, it is straightforward to obtain a fundamental theorem of calculus.

**Proposition 1.1.15.** Let  $S = (a_j)_{1 \leq j \leq n}$  be a binary sequence of length  $n$ . For any  $i \in \{1, \dots, n + 1\}$  and any  $x \in \mathbb{Z}/2\mathbb{Z}$ , we have that

- i)  $\partial \left( \int_{i,x} S \right) = S$ ,
- ii)  $\int_{i,x} (\partial S) = S + (a_i + x)_n$ .

The notions of derived sequences and antiderived sequences can easily be adapted for infinite binary sequences. A similar result as Proposition 1.1.15 has been obtained in this case in [90].

### 1.1.4 The multiplicity function of binary triangles

Binary triangles also can be seen as finite multisets of elements of  $\mathbb{Z}/2\mathbb{Z}$ .

**Notation 1.1.16.** For any finite multiset  $M$  of elements of  $\mathbb{Z}/2\mathbb{Z}$ , let  $\mathbf{m}_M$  denote its *multiplicity function*, that is, the function  $\mathbf{m}_M : \mathbb{Z}/2\mathbb{Z} \rightarrow \mathbb{N}$  that assigns to each element  $x \in \mathbb{Z}/2\mathbb{Z}$  its multiplicity in  $M$ . The *cardinality* of  $M$  is then the sum  $|M| = \mathbf{m}_M(0) + \mathbf{m}_M(1)$ .

For any binary sequence  $S$ , since every element of the triangle  $\nabla S$  can be expressed in function of the terms of the sequence  $S$  by (1.1), it is easy to see that the parity of  $\mathbf{m}_{\nabla S}(1)$  is only related to the terms of the sequence  $S$ .

**Proposition 1.1.17** (Harborth [71] - Chang [42]). *Let  $S = (a_j)_{1 \leq j \leq n}$  be a binary sequence of length  $n$ . Then,*

$$\mathbf{m}_{\nabla S}(1) \equiv \sum_{j=1}^n \left( \binom{n+1}{k} + 1 \right) a_j \pmod{2}.$$

Moreover,  $\mathbf{m}_{\nabla S}(1)$  is even for all binary sequences of length  $n$  if and only if  $n = 2^k - 2$ , for some  $k \geq 2$

The smallest and largest number of each element in a binary triangle are known.

**Notation 1.1.18.** For any  $n_1$ -tuple  $X_1$  and any  $n_2$ -tuple  $X_2$ ,

$$X_1 = (a_0, a_1, \dots, a_{n_1-1}) \quad \text{and} \quad X_2 = (b_0, b_1, \dots, b_{n_2-1}),$$

the *concatenation*  $X_1.X_2$  is the  $(n_1 + n_2)$ -tuple

$$X_1.X_2 = (a_0, a_1, \dots, a_{n_1-1}, b_0, b_1, \dots, b_{n_2-1}).$$

For any  $n$ -tuple  $X$ , the  $kn$ -tuple  $X^k$  is recursively defined by  $X^k = X.X^{k-1}$  for all integers  $k \geq 2$ , with  $X^1 = X$ . For any  $n$ -tuple  $X = (a_0, a_1, \dots, a_{n-1})$ , the infinite sequence  $X^\infty = (b_j)_{j \in \mathbb{N}}$  is defined by  $b_{kn+j} = a_j$  for all  $k \in \mathbb{N}$  and for all  $j \in \{0, 1, \dots, n-1\}$ . For any infinite sequence  $S = (a_j)_{j \in \mathbb{N}}$  and any positive integers  $n_1$  and  $n_2$  such that  $n_1 \leq n_2$ , we denote by  $S[n_1, n_2]$  the subsequence  $S[n_1, n_2] = (a_{n_1}, a_{n_1+1}, \dots, a_{n_2})$ . Moreover, for any positive integer  $n$ , we denote by  $S[n]$  the initial segment of length  $n$  of  $S$ , that is, the  $n$ -tuple  $S[n] = S[0, n-1] = (a_0, a_1, \dots, a_{n-1})$ .

**Proposition 1.1.19** (Harborth [71] - Chang [42]). *For any binary sequence  $S$  of length  $n$ , we have*

$$0 \leq \mathbf{m}_{\nabla S}(1) \leq \left\lceil \frac{2}{3} \binom{n+1}{2} \right\rceil.$$

Moreover,  $\mathbf{m}_{\nabla S}(1) = 0$  if and only if  $S = (0)_n$  and  $\mathbf{m}_{\nabla S}(1) = \left\lceil \frac{2}{3} \binom{n+1}{2} \right\rceil$  if and only if  $S$  is in

$$\begin{cases} \{(110)^\infty[n], (101)^\infty[n], (011)^\infty[n]\} & \text{for } n \equiv 0, 2 \pmod{3}, \\ \{(110)^\infty[n], (101)^\infty[n]\} & \text{for } n \equiv 1 \pmod{3}. \end{cases}$$

**Proposition 1.1.20** (Harborth-Hurlbert [72]). *For any binary generalized Pascal triangle  $\Delta$  of size  $n$ , we have*

$$0 \leq \mathbf{m}_\Delta(1) \leq \left\lceil \frac{2}{3} \binom{n+1}{2} \right\rceil + \varepsilon,$$

where

$$\varepsilon = \begin{cases} 2 & \text{if } n \equiv 1 \pmod{3}, n \neq 1, \\ 1 & \text{if } n \equiv 0, 2 \pmod{3}, n \neq 8, \\ 0 & \text{if } n = 1, \\ 3 & \text{if } n = 8. \end{cases}$$

Determining all possible values of the multiplicity function of a binary triangle is a very difficult problem. However, Chang managed to obtain in [42] the four smallest and three largest possible numbers of 1's in binary Steinhaus triangles.

**Proposition 1.1.21** (Chang [42]). *Let  $S$  be a binary sequence of length  $n \geq 1$ . If  $\mathbf{m}_{\nabla S}(1) > 0$ , then  $\mathbf{m}_{\nabla S}(1) \geq n$ . Moreover,  $\mathbf{m}_{\nabla S}(1) = n$  if and only if  $S$  is one of the following sequences:*

- $1 \cdots 1, 10 \cdots 0$  and  $0 \cdots 01$ ,
- $010$  for  $n = 3$ .

**Proposition 1.1.22** (Chang [42]). *Let  $S$  be a binary sequence of length  $n \geq 4$ . If  $\mathbf{m}_{\nabla S}(1) > n$ , then  $\mathbf{m}_{\nabla S}(1) \geq n - 1 + \lfloor \frac{n}{2} \rfloor$ . Moreover,  $\mathbf{m}_{\nabla S}(1) = n - 1 + \lfloor \frac{n}{2} \rfloor$  if and only if  $S$  is one of the following sequences:*

- $010 \cdots 0$  and  $0 \cdots 010$ ,
- $0 \cdots 011$  and  $110 \cdots 0$ ,
- $(01)^{\frac{n}{2}}$  and  $(10)^{\frac{n}{2}}$  if  $n$  is even and  $0(10)^{\frac{n-1}{2}}$  if  $n$  is odd,
- $001100, 001000$  and  $000110$  for  $n = 6$ ,
- $0001000$  for  $n = 7$ .

**Proposition 1.1.23** (Chang [42]). *Let  $S$  be a binary sequence of odd length  $n \geq 5$ . Then,  $\mathbf{m}_{\nabla S}(1) = n - 1 + \frac{n+1}{2}$  if and only if  $S$  is one of the following sequences:*

- $1(01)^{\frac{n-1}{2}}$ ,
- $0 \cdots 011$  and  $110 \cdots 0$ ,
- $00100, 01100$  and  $00110$  for  $n = 5$ .

**Proposition 1.1.24** (Chang [42]). *Let  $S$  be a binary sequence of length  $n \geq 7$ . If  $\mathbf{m}_{\nabla S}(1) > n - 1 + \lfloor \frac{n}{2} \rfloor$ , then*

$$\mathbf{m}_{\nabla S}(1) \geq \begin{cases} 2n - 4 & \text{if } n \equiv 2 \pmod{4} \text{ or } n = 11, \\ 2n - 3 & \text{otherwise.} \end{cases}$$

**Proposition 1.1.25** (Chang [42]). *Let  $S$  be a binary sequence of length  $n \geq 1$ . Then,  $\mathbf{m}_{\nabla S}(1) = \lfloor \frac{2}{3} \binom{n+1}{2} \rfloor - 1$  if and only if  $S$  is one of the following sequences:*

- $0$  for  $n = 1$ ,
- $111, 010, 100$  and  $001$  for  $n = 3$ ,
- $0110, 1001, 1110$  and  $0111$  for  $n = 4$ ,

- 01110, 01011, 11010, 11101, 10111, 01001 and 10010 for  $n = 5$ ,
- $0(110)^{\frac{n-1}{3}}$  for  $n \equiv 1 \pmod{3}$ .

**Proposition 1.1.26** (Chang [42]). *Let  $S$  be a binary sequence of length  $n \geq 6$ . If  $\mathbf{m}_{\nabla S}(1) < \lceil \frac{2}{3} \binom{n+1}{2} \rceil - 1$ , then  $\mathbf{m}_{\nabla S}(1) \leq \lfloor \frac{n^2+1}{3} \rfloor$ . Moreover, if  $\mathbf{m}_{\nabla S}(1) = \lfloor \frac{n^2+1}{3} \rfloor$ , then the sequence  $S$  cannot contain three consecutive 0's, except for the sequences 110001 and 100011 for  $n = 6$  and 110001110 and 011100011 for  $n = 9$ .*

### 1.1.5 The average number of each element in binary triangles

For any positive integer  $n$  and every  $x \in \mathbb{Z}/2\mathbb{Z}$ , it is possible to determine the average number of  $x$  in a binary triangle of size  $n$ , i.e.,

$$\frac{1}{2^n} \sum_{\nabla \in ST(n)} \mathbf{m}_{\nabla}(x) \quad \text{and} \quad \frac{1}{2^{2n-1}} \sum_{\Delta \in PT(n)} \mathbf{m}_{\Delta}(x).$$

**Proposition 1.1.27.** *For any non-negative integer  $n$ , the average number of 0's and 1's in a binary Steinhaus triangle or in a binary generalized Pascal triangle of size  $n$  is exactly  $\frac{1}{2} \binom{n+1}{2}$ .*

*Proof.* Since any binary sequence  $S$  of length  $n$  admits exactly two antiderived sequences  $\int_{i,0} S$  and  $\int_{i,1} S$  with  $\int_{i,0} S + \int_{i,1} S = (1)_{n+1}$  from Proposition 1.1.14.  $\square$

The Steinhaus Problem then corresponds to the determination of the existence of binary Steinhaus triangles with an average number of 0's and 1's, that are binary Steinhaus triangles  $\nabla$  with  $\mathbf{m}_{\nabla}(0) = \mathbf{m}_{\nabla}(1)$ .

**Definition 1.1.28** (Balanced multiset). A finite multiset  $M$  of elements of  $\mathbb{Z}/2\mathbb{Z}$  is said to be *balanced* if  $\mathbf{m}_M(0) = \mathbf{m}_M(1)$ . Note that the cardinality of a balanced multiset is an even number.

For instance, since they contain 14 zeroes and 14 ones, the triangles depicted in Figure 1.1 and Figure 1.2 are balanced binary triangles of size 7.

Since a binary triangle of size  $n$  is constituted by  $\binom{n+1}{2}$  elements of  $\mathbb{Z}/2\mathbb{Z}$ , it is clear that there exists a balanced binary triangle of size  $n$  only if the binomial coefficient  $\binom{n+1}{2}$  is even. Moreover,

$$\binom{n+1}{2} \equiv 0 \pmod{2} \iff n \equiv 0 \text{ or } 3 \pmod{4}.$$

Therefore, it is natural to consider the Steinhaus Problem. When  $n$  is a positive integer such that  $n \equiv 1$  or  $2 \pmod{4}$ , i.e., when  $\binom{n+1}{2}$  is an odd number, we can consider the similar problem of determining if there exist almost-balanced Steinhaus triangles of size  $n$ .

**Definition 1.1.29** (Almost-balanced multiset). A finite multiset  $M$  of elements of  $\mathbb{Z}/2\mathbb{Z}$  is said to be *almost-balanced* if  $|\mathbf{m}_M(0) - \mathbf{m}_M(1)| \leq 1$ .

**Proposition 1.1.30.** *Let  $M$  be a finite multiset of elements of  $\mathbb{Z}/2\mathbb{Z}$  that is almost-balanced. Then, the multiset  $M$  is balanced if and only if  $|M|$  is even.*

**Corollary 1.1.31.** *Let  $T$  be an almost-balanced binary triangle of size  $n$ . If  $n \equiv 0$  or  $3 \pmod{4}$ , then  $\mathbf{m}_T(0) = \mathbf{m}_T(1)$  and  $T$  is (perfectly) balanced. Otherwise, if  $n \equiv 1$  or  $2 \pmod{4}$ , then  $\mathbf{m}_T(0) = \mathbf{m}_T(1) \pm 1$ .*

The Steinhaus Problem can then be generalized in two directions: for binary generalized Pascal triangles and when the number of elements constituting a triangle is odd.

**Problem 1.1.32** (Chappelon [C7]). *Does there exist, for any positive integer  $n$ , an almost-balanced binary Steinhaus triangle and an almost-balanced binary generalized Pascal triangle of size  $n$ ?*

This chapter is organized as follows. In the next section, solutions of the Steinhaus Problem based on pseudo-periodic binary sequences are presented. After that, a recent solution of Problem 1.1.32 is studied in details in Section 3. This problem is positively solved in this chapter by considering fully periodic binary sequences generating triangles that have a periodic structure.

## 1.2 Pseudo-periodic balanced binary Steinhaus triangles

In this section, different solutions of the Steinhaus Problem are presented. All these solutions are constructive and balanced binary triangles given here are always associated with binary sequences that are pseudo-periodic.

### 1.2.1 Binary Steinhaus triangles of pseudo-periodic sequences

**Definition 1.2.1** (Pseudo-periodic sequences). Let  $u = (u_n)_{n \in N}$  be a (finite or infinite) sequence indexed on  $N = \mathbb{N}$  or  $N = \{0, 1, \dots, m\}$ , with  $m \in \mathbb{N}$ .

- i) The sequence  $u$  is said to be *periodic* if there exists a positive integer  $p$  such that  $u_{n+p} = u_n$ , for all  $n \in N$  such that  $n + p \in N$ . Then, we say that the sequence  $u$  is  *$p$ -periodic* of *period*  $P = \{u_0, \dots, u_{p-1}\}$ .
- ii) The sequence  $u$  is said to be *pseudo-periodic* if there exists positive integers  $i$  and  $p$  such that  $u_{n+p} = u_n$ , for all  $n \in N \setminus \{0, \dots, i-1\}$  such that  $n + p \in N$ . Then, we say that the sequence  $u$  is  *$(i, p)$ -pseudo-periodic* of *period*  $P = \{u_i, \dots, u_{i+p-1}\}$  from  $I = \{u_0, \dots, u_{i-1}\}$ .

The interest of pseudo-periodic sequences is that this structure is preserved under the derivation process.

**Proposition 1.2.2.** *Let  $I = \{x_1, \dots, x_i\}$  be an  $i$ -tuple and  $P = \{y_1, \dots, y_p\}$  be a  $p$ -tuple of elements in  $\mathbb{Z}/2\mathbb{Z}$ . Then, the derived sequence  $\partial S$  of the  $(i, p)$ -pseudo-periodic sequence  $S = I.P^\infty$  is also an  $(i, p)$ -pseudo-periodic sequence. More precisely, we have*

$$\partial(I.P^\infty) = I'.P'^\infty,$$

with  $I' = (x_1 + x_2, x_2 + x_3, \dots, x_i + y_1)$  and  $P' = (y_1 + y_2, y_2 + y_3, \dots, y_p + y_1)$ .

Moreover, the sequence of derived sequences of a pseudo-periodic sequence is also pseudo-periodic.

**Proposition 1.2.3.** *For any pseudo-periodic binary sequence  $S$ , the sequence of derived sequences  $(\partial^i S)_{i \geq 0}$  is pseudo-periodic.*

*Proof.* Suppose that  $S$  is an  $(i, p)$ -pseudo-periodic binary sequence. We know from Proposition 1.2.2 that the sequence  $\partial^k S$  is also an  $(i, p)$ -pseudo-periodic sequence, for all  $k \geq 0$ . Since the number of  $(i, p)$ -pseudo-periodic binary sequences is finite ( $2^{i+p}$ ), it follows that there exist integers  $k_1$  and  $k_2$  such that  $0 \leq k_1 < k_2$  and  $\partial^{k_1} S = \partial^{k_2} S$ . We conclude that the sequence  $(\partial^k S)_{k \geq 0}$  is  $(k_2 - k_1)$ -pseudo-periodic. □

Let  $S$  be an  $(i, p)$ -pseudo-periodic binary sequence. From Proposition 1.2.2, we know that all its iterated derived sequences  $\partial^k S$  are also  $(i, p)$ -pseudo-periodic, for all  $k \geq 0$ , and from Proposition 1.2.3, there exist non-negative integers  $i'$  and  $p'$  such that the sequence  $(\partial^k S)_{k \geq 0}$  is  $(i', p')$ -pseudo-periodic. Then, for  $\alpha = \text{lcm}(i, p, i', p')$ , we have that the sequence  $(\partial^k S)_{k \geq 0}$  and every derived sequence  $\partial^k S$ , for all  $k \geq 0$ , are  $(\alpha, \alpha)$ -periodic. It follows that binary Steinhaus triangles  $\nabla S[k\alpha]$  have an iterative structure, as depicted in Figure 1.4.

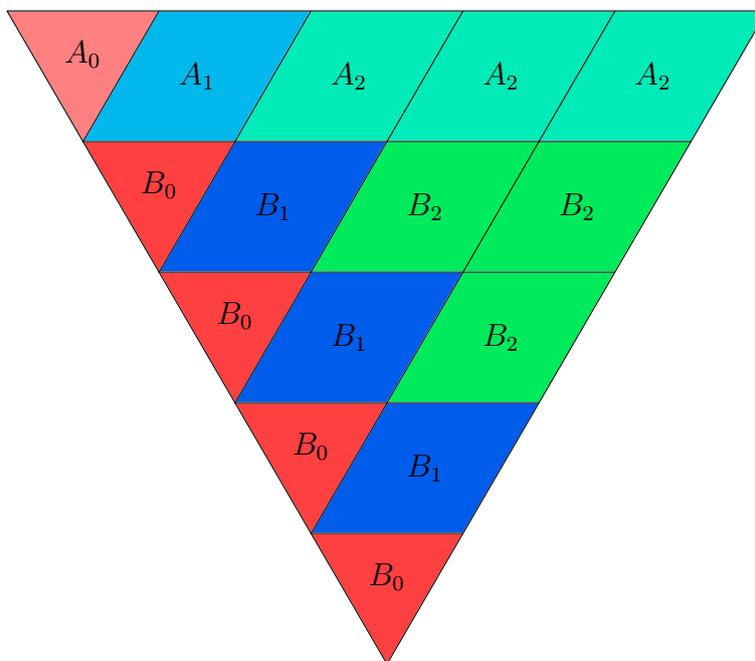


Figure 1.4: Structure of binary Steinhaus triangles of pseudo-periodic sequences

**Proposition 1.2.4.** *Let  $S$  be an  $(\alpha, \alpha)$ -pseudo-periodic binary sequence such that its sequence of iterated derived sequences  $(\partial^i S)_{i \geq 0}$  is also  $(\alpha, \alpha)$ -pseudo-periodic. Then, the triangles  $\nabla S[k\alpha]$  are balanced for all  $k \geq 0$  if and only if they are balanced for all  $k \in \{1, 2, 3, 4\}$ .*

### 1.2.2 Harborth’s solution for the Steinhaus Problem

The first solution for the Steinhaus Problem appeared in 1972 in [71], where Harborth has shown that there exist, for all the admissible sizes, balanced Steinhaus triangles whose first row is a pseudo-periodic sequence of period length 12 and for which the sequence of iterated derived sequences is fully periodic of period length 12.

**Theorem 1.2.5** (Harborth [71]). *For any  $l \in \{3, 4, 7, 8, 11, 12\}$ , there exist  $I \in \{0, 1\}^l$  and  $P \in \{0, 1\}^{12}$  such that the sequence  $(\partial^i(I.P^\infty))_{i \geq 0}$  of derivatives of  $I.P^\infty$  is 12-periodic, i.e.,*

$$\partial^{12}(I.P^\infty) = I.P^\infty,$$

and such that the Steinhaus triangle  $\nabla(I.P^k)$ , of size  $12k + l$ , is balanced, for all  $k \geq 0$ .

The method can easily be explained: after determining the set of tuples  $I$  and  $P$  such that  $\partial^{12}(I.P^\infty) = I.P^\infty$ , the periodic structure of  $\nabla(I.P^k)$ , as depicted in Figure 1.5, implies that the Steinhaus triangles  $\nabla(I.P^k)$  are balanced, for all  $k \geq 0$ , if and only if the elementary blocks

$$\begin{aligned} B_0 &= \nabla I \\ B_1 &= \nabla(I.P) \setminus (B_0 \sqcup B_0) \\ B_2 &= \nabla(I.P^2) \setminus (\nabla(I.P) \sqcup B_0 \sqcup B_1) \end{aligned}$$

are balanced.

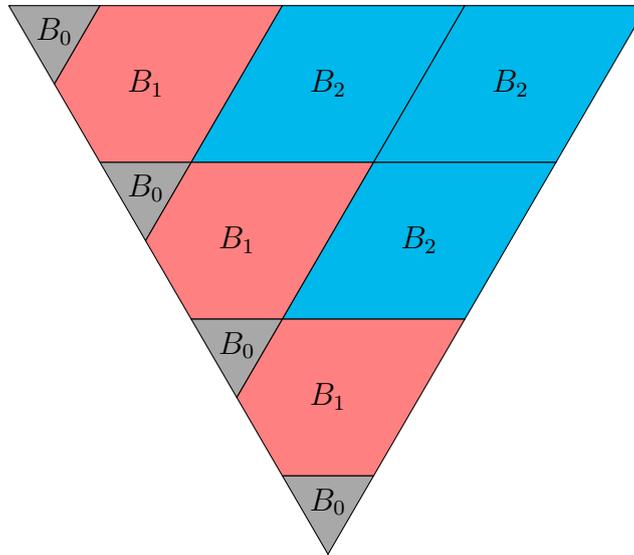


Figure 1.5: Structure of balanced triangles in Harborth's solution

### 1.2.3 Strongly balanced binary Steinhaus triangles

**Definition 1.2.6.** Let  $S$  be a sequence of  $\mathbb{Z}/2\mathbb{Z}$  of length  $n$ . The binary Steinhaus triangle  $\nabla S$  is said to be *strongly balanced* if the sub-triangles  $\nabla S[n - 4t]$  are balanced for all  $t \in \{0, 1, \dots, \lfloor \frac{n}{4} \rfloor\}$ .

In [60], Eliahou and Hachez have shown that there exist, for all non-negative integers  $n$  such that  $n \equiv 0$  or  $3 \pmod{4}$ , strongly balanced Steinhaus triangles of size  $n$ . More precisely, they completely determined the set of sequences generating strongly balanced Steinhaus triangles of size  $n$ , for all admissible sizes.

**Theorem 1.2.7** (Eliahou-Hachez [60]). *The generating function  $g(t) = \sum_{n=0}^{\infty} sb(n)t^n$  of the number  $sb(n)$  of strongly balanced Steinhaus triangles of size  $n$  is given by the following rational function:*

$$g(t) = \frac{4t^{92}}{1-t^4} + f_0(t) + \frac{(14 + 12t^4 + 14t^8)t^{127}}{1-t^{12}} + f_3(t),$$

where  $f_0(t)$  and  $f_3(t)$  are the following polynomials:

$$f_0(t) = 1 + 6t^4 + 18t^8 + 30t^{12} + 52t^{16} + 80t^{20} + 88t^{24} + 106t^{28} + 116t^{32} + 124t^{36} \\ + 106t^{40} + 92t^{44} + 92t^{48} + 90t^{52} + 64t^{56} + 44t^{60} + 38t^{64} + 32t^{68} + 20t^{72} \\ + 20t^{76} + 8t^{80} + 8t^{84} + 6t^{88},$$

$$f_3(t) = 4t^3 + 8t^7 + 16t^{11} + 26t^{15} + 36t^{19} + 48t^{23} + 48t^{27} + 66t^{31} + 88t^{35} + 108t^{39} \\ + 114t^{43} + 90t^{47} + 88t^{51} + 104t^{55} + 92t^{59} + 60t^{63} + 48t^{67} + 28t^{71} + 26t^{75} \\ + 26t^{79} + 20t^{83} + 16t^{87} + 18t^{91} + 14t^{95} + 14t^{99} + 14t^{103} + 14t^{107} + 16t^{111} \\ + 14t^{115} + 14t^{119} + 16t^{123}.$$

Considering the binary sequences  $Q_1, \dots, Q_4$  and  $R_1, \dots, R_{12}$  of Table 1.1, the set of strongly balanced Steinhaus triangles of sufficiently large size can then be described.

$Q_1$	$(0100) \cdot (001001011100)^\infty$	$R_1$	$(001) \cdot (010000100001)^\infty$
$Q_2$	$(010010000111)^\infty$	$R_2$	$(0011110) \cdot (001101010110)^\infty$
$Q_3$	$(0101) \cdot (011000011000)^\infty$	$R_3$	$(010) \cdot (000101000010)^\infty$
$Q_4$	$(0101) \cdot (101000101000)^\infty$	$R_4$	$(0100001) \cdot (010010111100001010111111)^\infty$
		$R_5$	$(0100001) \cdot (100100001001)^\infty$
		$R_6$	$(0101011) \cdot (010101100011)^\infty$
		$R_7$	$(0101011) \cdot (010111111101011010011101)^\infty$
		$R_8$	$(010) \cdot (101110110010)^\infty$
		$R_9$	$(100) \cdot (001000010100)^\infty$
		$R_{10}$	$(1000010) \cdot (110001101010)^\infty$
		$R_{11}$	$(1111101) \cdot (011000110101)^\infty$
		$R_{12}$	$(111) \cdot (110110000111)^\infty$

Table 1.1: Sequences generating strongly balanced Steinhaus triangles

**Theorem 1.2.8** (Eliahou-Hachez [60]). *Let  $n$  be a non-negative integer.*

- i) *If  $n \equiv 0 \pmod{4}$ , then the Steinhaus triangles  $\nabla(Q_i[n])$  are strongly balanced, for all  $i \in \{1, 2, 3, 4\}$ . Conversely, every strongly balanced Steinhaus triangle of size  $n$  with  $n \equiv 0 \pmod{4}$  and  $n \geq 92$  is one of the triangles  $\nabla(Q_i[n])$ , where  $i \in \{1, 2, 3, 4\}$ .*
- ii) *If  $n \equiv 3 \pmod{4}$ , then the Steinhaus triangles  $\nabla(R_i[n])$  are strongly balanced, for all  $i \in \{1, \dots, 12\}$ . Moreover, if  $n \geq 127$ , then every strongly balanced Steinhaus triangle of size  $n$  is one of the triangles  $\nabla(R_i[n])$ , where  $i \in \{1, \dots, 12\}$ , with the following exceptions:*
  - *if  $n \equiv 3 \pmod{12}$ , there are two more strongly balanced Steinhaus triangles of size  $n$ , namely  $\nabla(R_5[n-4] \cdot (0101))$  and  $\nabla(R_8[n-4] \cdot (0100))$ .*
  - *if  $n \equiv 7 \pmod{12}$ , there are two more strongly balanced Steinhaus triangles of size  $n$ , namely  $\nabla(R_8[n-8] \cdot (01001000))$ , and either  $\nabla(R_5[n-8] \cdot (01011111))$  if  $n \equiv 7 \pmod{24}$ , or  $\nabla(R_5[n-8] \cdot (01011010))$  if  $n \equiv 19 \pmod{24}$ .*

The method can easily be explained: assume that the binary sequence  $X$  of length  $n$  generates a strongly balanced Steinhaus triangle. An extension of  $X$  is any binary sequence  $Y$  such that  $Y[n] = X$ . Let  $Y$  be any one of the 16 possible extensions of  $X$  of

length  $n + 4$ . Then, the Steinhaus triangle  $\nabla Y$  is strongly balanced if and only if  $\nabla X$  is balanced. This holds because  $\nabla X$  is itself strongly balanced.

Note that the sequences in Theorem 1.2.8 are pseudo-periodic of period length 12 or 24, as in the Harborth’s proof. The proof directly follows from the pseudo-periodic structure of the triangles as depicted in Figure 1.6 where the blocks  $T, A_1, A_3, B_1 \sqcup B_3, C_1 \sqcup C_3, A_2, B_2$  and  $C_2$  are balanced.

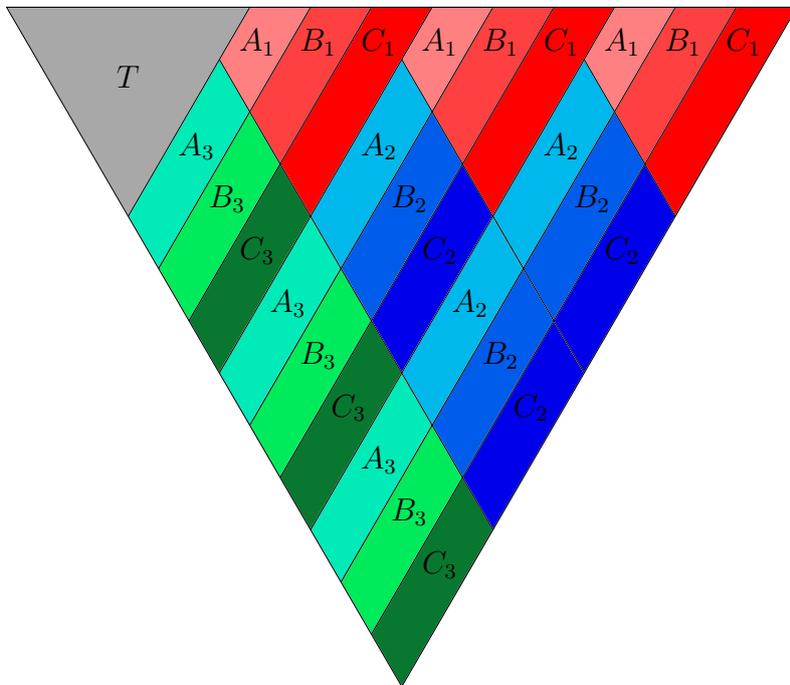


Figure 1.6: Structure of the Steinhaus triangle  $\nabla(Q_1[52])$

### 1.2.4 Other solutions for the Steinhaus Problem

There exist other interesting solutions to the Steinhaus Problem.

**Definition 1.2.9.** Let  $S = (a_j)_{1 \leq j \leq n}$  be a sequence of  $\mathbb{Z}/2\mathbb{Z}$  of length  $n$ . The *reversed sequence* of  $S$  is the sequence  $\bar{S} = (a_{n-j+1})_{1 \leq j \leq n}$ . The sequence  $S$  is said to be *symmetric* if  $\bar{S} = S$ , i.e., if  $a_{n-j+1} = a_j$ , for all  $j \in \{1, \dots, n\}$ . It is said to be *antisymmetric* if  $S + \bar{S} = (1)_n$ , i.e., if  $x_{n-j+1} \equiv x_j + 1 \pmod 2$ , for all  $j \in \{1, \dots, n\}$ .

A solution, based on binary Steinhaus triangles generated from symmetric and anti-symmetric sequences, appeared in [61].

**Theorem 1.2.10** (Eliahou-Hachez [61]). *Let  $n$  be a non-negative integer.*

- i) There exists a symmetric binary sequence of length  $n$  that generates a balanced Steinhaus triangle if and only if  $n \equiv 0, 3$  or  $7 \pmod 8$ .*
- ii) There exists an antisymmetric binary sequence of length  $n$  that generates a balanced Steinhaus triangle if and only if  $n \equiv 4 \pmod 8$ .*

Another solution, based on Steinhaus triangles generated from balanced binary sequences, that are binary sequences containing as many 0’s as 1’s, can be found in [57].

**Theorem 1.2.11** (Eliahou-Marín-Revuelta [57]). *For any integer  $n$  divisible by 4, there exists a balanced binary sequence of length  $n$  that generates a balanced Steinhaus triangle.*

This produces a complete solution for the Steinhaus Problem since if  $S$  is a balanced binary sequence of length  $n \equiv 0 \pmod{4}$  such that  $\nabla S$  is balanced, then it is clear that  $\nabla(\partial S)$  is a balanced Steinhaus triangle of size  $n - 1 \equiv 3 \pmod{4}$ .

Note that the solutions in [61, 57] admit a pseudo-periodic structure too. In the next section, a fully periodic solution of the Steinhaus Problem is given.

## 1.3 Periodic almost-balanced binary triangles

In this section, we present a recent solution (Chappelon 2017 [C7]) of Problem 1.1.32, that is a generalization of the Steinhaus Problem in two directions: for binary generalized Pascal triangles and when the number of elements constituting a triangle is odd. This problem is solved in the positive in this section. The solution presented here is constructive and based on periodic binary triangles, that are binary triangles where each row or column is a periodic sequence.

### 1.3.1 Orbits of doubly infinite binary sequences

Let us begin with some definitions and terminology. As for infinite or finite binary sequences, the derivation process can be considered on doubly infinite binary sequences.

**Definition 1.3.1** (Derived sequence). Let  $S = (a_j)_{j \in \mathbb{Z}}$  be a doubly infinite sequence of  $\mathbb{Z}/2\mathbb{Z}$ . The *derived sequence*  $\partial S$  is the sequence

$$\partial S = (a_{j-1} + a_j)_{j \in \mathbb{Z}}.$$

This derivation process can be iterated and, for every positive integer  $i$ , the  $i$ -th derived sequence  $\partial^i S$  is recursively defined by  $\partial^i S = \partial(\partial^{i-1} S)$ , with  $\partial^0 S = S$ .

**Definition 1.3.2** (Orbit). Let  $S = (a_j)_{j \in \mathbb{Z}}$  be a doubly infinite sequence of  $\mathbb{Z}/2\mathbb{Z}$ . The *orbit*  $\mathcal{O}_S$  is the sequence of all the iterated derived sequences of  $S$ , that is,

$$\mathcal{O}_S = (\partial^i S)_{i \in \mathbb{N}}.$$

The orbit of  $S$  can also be seen as the doubly indexed sequence  $\mathcal{O}_S = (a_{i,j})_{(i,j) \in \mathbb{N} \times \mathbb{Z}}$  recursively defined by

- i)  $a_{0,j} = a_j$ , for all  $j \in \mathbb{Z}$ , and
- ii)  $a_{i,j} = a_{i-1,j-1} + a_{i-1,j}$ , for all  $i \geq 1$  and for all  $j \in \mathbb{Z}$ .

Using the local rule (LR), it is straightforward to see that any element of the orbit  $\mathcal{O}_S$  can be expressed in function of the terms of the sequence  $S$ .

**Proposition 1.3.3.** *Let  $S = (a_j)_{j \in \mathbb{Z}}$  be a doubly infinite sequence of  $\mathbb{Z}/2\mathbb{Z}$ . In the orbit  $\mathcal{O}_S = (a_{i,j})_{(i,j) \in \mathbb{N} \times \mathbb{Z}}$ , we have*

$$a_{i,j} = \sum_{k=0}^i \binom{i}{k} a_{j-k},$$

for all  $(i,j) \in \mathbb{N} \times \mathbb{Z}$ .

An example of orbit  $\mathcal{O}_S$  associated with the sequence

$$S = \dots\dots 0010101100001100001001110 \dots\dots$$

is depicted in Figure 1.7.

0	0	1	0	1	0	1	1	0	0	0	0	1	1	0	0	0	0	1	0	0	1	1	1	0
1	0	1	1	1	1	1	0	1	0	0	0	1	0	1	0	0	0	1	1	0	1	0	0	1
1	1	1	0	0	0	0	1	1	1	0	0	1	1	1	1	0	0	1	0	1	1	1	0	1
1	0	0	1	0	0	0	1	0	0	1	0	1	0	0	0	1	0	1	1	1	0	0	1	1
0	1	0	1	1	0	0	1	1	0	1	1	1	1	0	0	1	1	1	0	0	1	0	1	0
1	1	1	1	0	1	0	1	0	1	1	0	0	0	1	0	1	0	0	1	0	1	1	1	1
0	0	0	0	1	1	1	1	1	1	0	1	0	0	1	1	1	1	1	0	1	1	1	0	0
0	0	0	0	1	0	0	0	0	0	1	1	1	0	1	0	0	0	1	1	0	0	1	0	0
0	0	0	0	1	1	0	0	0	0	1	0	0	1	1	1	0	0	1	0	1	0	1	1	0
1	0	0	0	1	0	1	0	0	0	1	1	0	1	0	0	1	0	1	1	1	0	1	1	0
1	1	0	0	1	1	1	1	0	0	1	0	1	1	1	0	1	1	1	0	0	0	0	1	1
0	0	1	0	1	0	0	0	1	0	1	1	1	0	0	1	0	0	1	0	0	0	1	0	0
1	0	1	1	1	1	0	0	1	1	1	0	0	1	0	1	0	1	1	0	0	1	1	0	1
0	1	1	0	0	0	1	0	1	0	0	1	0	1	1	1	1	1	1	1	0	1	0	1	0

Figure 1.7: Binary triangles appearing in an orbit  $\mathcal{O}_S$

Binary triangles can then be considered as appearing in orbits of binary sequences. Example of triangles appearing in an orbit  $\mathcal{O}_S$  is represented in Figure 1.7.

**Notation 1.3.4.**

- i) Let  $\nabla S(i_0, j_0, n)$  denote the down-pointing triangle whose *principal vertex* is at the position  $(i_0, j_0) \in \mathbb{N} \times \mathbb{Z}$  in the orbit  $\mathcal{O}_S = (a_{i,j})_{(i,j) \in \mathbb{N} \times \mathbb{Z}}$  and of *size*  $n$ , i.e., the binary Steinhaus triangle

$$\nabla S(i_0, j_0, n) = (a_{i_0+i, j_0+j})_{0 \leq i \leq j \leq n-1}.$$

- ii) Let  $\Delta S(i_0, j_0, n)$  denote the up-pointing triangle whose *principal vertex* is at the position  $(i_0, j_0) \in \mathbb{N} \times \mathbb{Z}$  in the orbit  $\mathcal{O}_S$  and of *size*  $n$ , i.e., the binary generalized Pascal triangle

$$\Delta S(i_0, j_0, n) = (a_{i_0+i, j_0+j})_{0 \leq j \leq i \leq n-1}.$$

The main result of this section is the following

**Theorem 1.3.5** (Chappelon [C7]). *There exists a binary doubly infinite sequence  $S$  such that its orbit  $\mathcal{O}_S$  contains almost-balanced Steinhaus triangles and almost-balanced generalized Pascal triangles of size  $n$ , for all non-negative integers  $n$ .*

This theorem positively solves Problem 1.1.32 for the two kinds of triangles, even when the triangles contain an odd number of terms. Note that the existence of almost-balanced Steinhaus triangles with odd cardinality was first announced, without proof, in [60]. For the generalized Pascal triangles, the result was known but has not been published before.

### 1.3.2 Periodic orbits

The definition and notations for (in)finite periodic sequences previously used can be extended to the case of doubly infinite periodic sequences.

**Notation 1.3.6.** For any  $n$ -tuple  $X = (a_0, a_1, \dots, a_{n-1})$ , the doubly infinite sequence  $X^\infty = (b_j)_{j \in \mathbb{Z}}$  is defined by  $b_{kn+j} = a_j$  for all  $k \in \mathbb{Z}$  and for all  $j \in \{0, 1, \dots, n-1\}$ . For any doubly infinite sequence  $S = (a_j)_{j \in \mathbb{Z}}$  and any positive integer  $n$ , we denote by  $S[n]$  the initial segment of length  $n$  of  $S$ , that is, the  $n$ -tuple  $S[n] = (a_0, a_1, \dots, a_{n-1})$ .

**Definition 1.3.7** (Periodic sequence). Let  $p$  be a positive integer and let  $S = (a_j)_{j \in \mathbb{Z}}$  be a doubly infinite sequence of elements in  $\mathbb{Z}/2\mathbb{Z}$ . The sequence  $S$  is said to be *periodic of period  $p$* , or  *$p$ -periodic*, if  $a_{j+p} = a_j$  for all  $j \in \mathbb{Z}$ . The  $p$ -periodicity of  $S$  is denoted by  $S = (a_0, a_1, \dots, a_{p-1})^\infty$ , where the  $p$ -tuple  $(a_0, a_1, \dots, a_{p-1})$  is a *period* of  $S$ .

As for (in)finite binary sequences, it is clear that the periodicity of a doubly infinite sequence is preserved under the derivation process.

**Proposition 1.3.8.** For any  $p$ -tuple  $(a_0, a_1, \dots, a_{p-1})$  of  $\mathbb{Z}/2\mathbb{Z}$ , we have

$$\partial(a_0, a_1, \dots, a_{p-1})^\infty = (a_{p-1} + a_0, a_0 + a_1, \dots, a_{p-2} + a_{p-1})^\infty$$

Moreover, its sequence of derived sequences is pseudo-periodic.

**Proposition 1.3.9** (Chappelon [C7]). The orbit of a periodic sequence is a pseudo-periodic sequence.

*Proof.* As in the proof of Proposition 1.2.3. □

Here, we will study the special case where the orbit is fully periodic.

**Definition 1.3.10** (Periodic orbit). Let  $S$  be a doubly infinite binary sequence. Its orbit  $\mathcal{O}_S = (a_{i,j})_{(i,j) \in \mathbb{N} \times \mathbb{Z}}$  is said to be  *$p$ -periodic* if every row and every column is a  $p$ -periodic sequence, i.e., if the equalities

$$a_{i,j+p} = a_{i,j} \quad \text{and} \quad a_{i+p,j} = a_{i,j}$$

hold for all  $i \in \mathbb{N}$  and all  $j \in \mathbb{Z}$ . In other words, the orbit  $(a_{i,j})_{(i,j) \in \mathbb{N} \times \mathbb{Z}}$  is  $p$ -periodic if the equality

$$a_{i,j} = a_{\bar{i}, \bar{j}}$$

holds, for all  $(i, j) \in \mathbb{N} \times \mathbb{Z}$ , where  $\bar{x}$  is the remainder in the euclidean division of  $x$  by  $p$ . Any square  $P_{i_0, j_0} = (a_{i_0+i, j_0+j})_{0 \leq i, j \leq p-1}$  of size  $p$  is said to be a *period* of the  $p$ -periodic orbit  $\mathcal{O}_S$ .

*Remark 1.3.11.* All the periods of a  $p$ -periodic orbit have the same multiplicity function, i.e., we have  $\mathbf{m}_{P_{i_0, j_0}} = \mathbf{m}_{P_{0,0}}$  for all  $(i_0, j_0) \in \mathbb{N} \times \mathbb{Z}$ .

For example, as depicted in Figure 1.8, the orbit  $\mathcal{O}_{X^\infty}$  associated with the 6-tuple  $X = 010100$  is 6-periodic. Note that a binary triangle appearing in a periodic orbit is simply a periodic binary triangle, as defined above.

The set of  $p$ -tuples of  $\mathbb{Z}/2\mathbb{Z}$  that generate  $p$ -periodic orbits is given in the following

0	1	0	1	0	0	0	1	0	1	0	0	0	1	0	1	0	0
0	1	1	1	1	0	0	1	1	1	1	0	0	1	1	1	1	0
0	1	0	0	0	1	0	1	0	0	0	1	0	1	0	0	0	1
1	1	1	0	0	1	1	1	1	0	0	1	1	1	1	0	0	1
0	0	0	1	0	1	0	0	0	1	0	1	0	0	0	1	0	1
1	0	0	1	1	1	1	0	0	1	1	1	1	1	0	0	1	1
0	1	0	1	0	0	0	1	0	1	0	0	0	1	0	1	0	0
0	1	1	1	1	0	0	1	1	1	1	0	0	1	1	1	1	0
0	1	0	0	0	1	0	1	0	0	0	1	0	1	0	0	0	1
1	1	1	0	0	1	1	1	1	0	0	1	1	1	1	0	0	1
0	0	0	1	0	1	0	0	0	1	0	1	0	0	0	1	0	1
1	0	0	1	1	1	1	0	0	1	1	1	1	0	0	1	1	1

Figure 1.8: The 6-periodic orbit  $\mathcal{O}_{010100^\infty}$

**Theorem 1.3.12** (Harborth [71] - Chappelon [C7]). *The orbit  $\mathcal{O}_{X^\infty}$  associated with the  $p$ -tuple  $X = (a_0, a_1, \dots, a_{p-1})$  is  $p$ -periodic if and only if the vector  $v_X = (a_0, a_1, \dots, a_{p-1})^t$  is in the kernel of the matrix  $W_p$  which is the Wendt matrix of size  $p$  modulo 2, i.e., the circulant matrix of the binomial coefficients modulo 2*

$$W_p = \begin{pmatrix} \binom{p}{p} & \binom{p}{p-1} & \binom{p}{p-2} & \dots & \binom{p}{1} \\ \binom{p}{1} & \binom{p}{p} & \binom{p}{p-1} & \dots & \binom{p}{2} \\ \vdots & \vdots & \vdots & & \vdots \\ \binom{p}{p-1} & \binom{p}{p-2} & \binom{p}{p-3} & \dots & \binom{p}{p} \end{pmatrix}.$$

**Notation 1.3.13.** The set of  $p$ -tuples  $X$  that generate  $p$ -periodic orbits  $\mathcal{O}_{X^\infty}$  is denoted by  $\mathcal{PO}_p$ .

The set  $\mathcal{PO}_p$  is then a  $\mathbb{Z}/2\mathbb{Z}$ -vector space isomorphic to the kernel of the Wendt matrix  $W_p$  of size  $p$  modulo 2. Table 1.2 gives  $\dim \ker(W_p)$  and  $|\mathcal{PO}_p| = 2^{\dim \ker(W_p)}$  for the first few values of  $p$ .

$p$	1	2	3	4	5	6	7	8	9	10	11	12
$\dim \ker(W_p)$	0	0	2	0	0	4	6	0	2	0	0	8
$p$	13	14	15	16	17	18	19	20	21	22	23	24
$\dim \ker(W_p)$	0	12	14	0	0	4	0	0	8	0	0	16

Table 1.2: The first few values of  $\dim \ker(W_p)$

For example, for  $p = 6$ , we have  $\dim \ker(W_6) = 4$  and  $|\mathcal{PO}_6| = 2^4 = 16$ . There are then 16 different 6-tuples that generate a 6-periodic orbit. More precisely, the set  $\mathcal{PO}_6$  is given by

$$\begin{aligned} \mathcal{PO}_6 &= \langle 000101, 001010, 010001, 100010 \rangle \\ &= \{000000, 000101, 001010, 001111, 010001, 010100, 011011, 011110, \\ &\quad 100010, 100111, 101000, 101101, 110011, 110110, 111001, 111100\}. \end{aligned}$$

We recall here that the 6-tuple  $X = 010100$  generates a 6-periodic orbit as depicted in Figure 1.8.

### 1.3.3 Symmetry group of $\mathcal{PO}_p$

In this subsection, a symmetry group on the set of  $p$ -tuples that generate  $p$ -periodic orbits is defined. First, the notion of translation and the action of the dihedral group  $D_3$  on periodic orbits are introduced.

#### Translation

**Definition 1.3.14.** Let  $\mathcal{O}_{X^\infty} = (a_{\bar{i}, \bar{j}})_{(i,j) \in \mathbb{N} \times \mathbb{Z}}$  be the  $p$ -periodic orbit associated with  $X = (a_0, a_1, \dots, a_{p-1}) \in \mathcal{PO}_p$ . The *translate* of  $X$  by the vector  $(u, v) \in \mathbb{Z}^2$  is the  $p$ -tuple  $\mathbf{t}_{u,v}(X) = (a_{\overline{-u}, \overline{j-v}})_{0 \leq j \leq p-1}$ .

From Proposition 1.3.3, we know that

$$\mathbf{t}_{u,v}(X) = \left( \sum_{k=0}^{\overline{-u}} \binom{\overline{-u}}{k} a_{\overline{j-v-k}} \right)_{0 \leq j \leq p-1}.$$

From the definition of  $\mathbf{t}_{u,v}(X)$ , it is clear that

$$\mathcal{O}_{\mathbf{t}_{u,v}(X)^\infty} = (a_{\overline{i-u}, \overline{j-v}})_{(i,j) \in \mathbb{N} \times \mathbb{Z}}.$$

Therefore  $\mathbf{t}_{u,v}$  is an automorphism of  $\mathcal{PO}_p$ . Moreover, the application

$$\begin{aligned} (\mathbb{Z}^2, +) &\longrightarrow (\text{Aut}(\mathcal{PO}_p), \circ) \\ (u, v) &\longmapsto \mathbf{t}_{u,v} \end{aligned}$$

is a group morphism.

For example, the translate of the 6-tuple  $010100 \in \mathcal{PO}_6$  (Figure 1.8) by the vector  $(2, 3)$  is  $\mathbf{t}_{2,3}(010100) = 101000$ , as we can see in its orbit  $\mathcal{O}_{\mathbf{t}_{2,3}(010100)^\infty} = \mathcal{O}_{101000^\infty}$  depicted in Figure 1.9.

1	0	1	0	0	0	1	0	1	0	0	0	1	0	1	0	0	0
1	1	1	1	0	0	1	1	1	1	0	0	1	1	1	1	0	0
1	0	0	0	1	0	1	0	0	0	0	1	0	1	0	0	0	1
1	1	0	0	1	1	1	1	0	0	1	1	1	1	0	0	1	1
0	0	1	0	1	0	0	0	1	0	1	0	0	0	1	0	1	0
0	0	1	1	1	1	0	0	1	1	1	1	0	0	1	1	1	1
1	0	1	0	0	0	1	0	1	0	0	0	1	0	1	0	0	0
1	1	1	1	0	0	1	1	1	1	0	0	1	1	1	1	0	0
1	0	0	0	1	0	1	0	0	0	1	0	1	0	0	0	0	1
1	1	0	0	1	1	1	1	0	0	1	1	1	1	0	0	1	1
0	0	1	0	1	0	0	0	1	0	1	0	0	0	1	0	1	0
0	0	1	1	1	1	0	0	1	1	1	1	0	0	1	1	1	1

Figure 1.9: The translate  $\mathbf{t}_{2,3}(010100) = 101000$

#### The dihedral group $D_3$

First, consider the binary Steinhaus triangles  $\nabla S = (a_{i,j})_{0 \leq i \leq j \leq n-1}$  of size  $n$ . The *left and right sides* of  $\nabla S$  are the sequences  $l(S) = (a_{n-1-i, n-1-i})_{0 \leq i \leq n-1}$  and  $r(S) = (a_{i, n-1})_{0 \leq i \leq n-1}$ ,

respectively. From (1.1), we know that  $l(S)$  and  $r(S)$  can be expressed as functions of the elements of  $S = (a_j)_{0 \leq j \leq n-1}$

$$l(S) = \left( \sum_{k=0}^{n-1-i} \binom{n-1-i}{k} a_{n-1-i-k} \right)_{0 \leq i \leq n-1} \quad \text{and} \quad r(S) = \left( \sum_{k=0}^i \binom{i}{k} a_{n-1-k} \right)_{0 \leq i \leq n-1}.$$

The *reversed sequence* of  $S$  is the sequence read from the right to the left, that is  $i(S) = (a_{n-1-j})_{0 \leq j \leq n-1}$ .

Due to the symmetries involved in the local rule that generates  $\nabla S$ , the Pascal local rule modulo 2, it is known that the Steinhaus triangles  $\nabla l(S)$ ,  $\nabla r(S)$  and  $\nabla i(S)$  correspond to the rotations of  $\mp 120$  degrees around the center of the triangle  $\nabla S$  and the reflection across the vertical line through the center of  $\nabla S$ , respectively. More precisely, for all integers  $i$  and  $j$  such that  $1 \leq i \leq j \leq n-1$ , we have

$$a_{i-1,j-1} + a_{i-1,j} = a_{i,j} \iff a_{i-1,j} + a_{i,j} = a_{i-1,j-1} \iff a_{i-1,j-1} + a_{i,j} = a_{i-1,j}. \quad (1.5)$$

Therefore

$$\nabla l(S) = \nabla (a_{n-1-j,n-1-j})_{0 \leq j \leq n-1} = (a_{n-1-j,n-1-j+i})_{0 \leq i \leq j \leq n-1},$$

$$\nabla r(S) = \nabla (a_{j,n-1})_{0 \leq j \leq n-1} = (a_{j-i,n-1-i})_{0 \leq i \leq j \leq n-1},$$

$$\nabla i(S) = \nabla (a_{0,n-1-j})_{0 \leq j \leq n-1} = (a_{i,n-1+i-j})_{0 \leq i \leq j \leq n-1}.$$

Since

$$r^3 = i^2 = (ir)^2 = id_{(\mathbb{Z}/2\mathbb{Z})^n},$$

the subgroup of  $(\text{Aut}((\mathbb{Z}/2\mathbb{Z})^n), \circ)$ , the group of automorphisms of the vector space of  $n$ -tuples over  $\mathbb{Z}/2\mathbb{Z}$ , generated by  $r$  and  $i$  is isomorphic to the dihedral group  $D_3$

$$\langle r, i \mid r^3 = i^2 = (ir)^2 = id_{(\mathbb{Z}/2\mathbb{Z})^n} \rangle = D_3.$$

As depicted in Figure 1.10, it is easy to see that the multiplicity function of a Steinhaus triangle is invariant under the action of the dihedral group  $D_3$ . Indeed, for any finite sequence  $S$ , we have

$$\mathbf{m}_{\nabla S} = \mathbf{m}_{\nabla r(S)} = \mathbf{m}_{\nabla i(S)}.$$

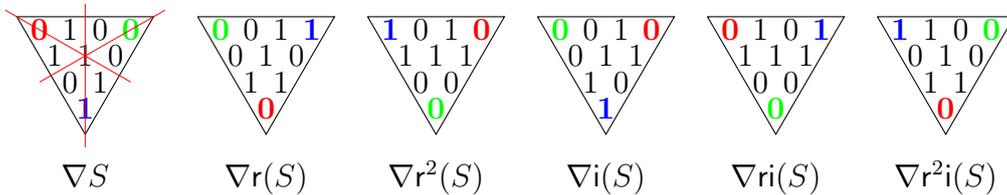


Figure 1.10: Action of  $D_3$  on  $\nabla(0100)$

The study of rotationally symmetric triangles and dihedrally symmetric triangles, that are triangles  $\nabla S$  such that  $S = r(S)$  and  $S = r(S) = i(S)$ , respectively, can be found in [26, 39] and [C8] and in next chapter.

Now, we consider the restrictions of  $r$  and  $i$  to the vector space  $\mathcal{PO}_p$  of  $p$ -tuples that generate  $p$ -periodic orbits. Since we only consider these restrictions, they are also denoted by  $r$  and  $i$  in the sequel.

**Proposition 1.3.15** (Chappelton [C7]). *For all positive integers  $p$ , we have*

$$r(\mathcal{PO}_p) = i(\mathcal{PO}_p) = \mathcal{PO}_p.$$

It follows that  $r$  and  $i$  are automorphisms of the vector space  $\mathcal{PO}_p$  and the subgroup of  $(\text{Aut}(\mathcal{PO}_p), \circ)$  generated by  $r$  and  $i$  is also isomorphic to the dihedral group  $D_3$

$$D_3 = \langle r, i \rangle = \{id_{\mathcal{PO}_p}, r, r^2, i, ri, r^2i\}.$$

More precisely, for any  $p$ -tuple  $X$ , we have

$$\begin{aligned} \mathcal{O}_{X^\infty} &= (a_{\bar{i}, \bar{j}})_{(i,j) \in \mathbb{N} \times \mathbb{Z}} & \mathcal{O}_{i(X)^\infty} &= (a_{\bar{i}, \overline{i-j-1}})_{(i,j) \in \mathbb{N} \times \mathbb{Z}} \\ \mathcal{O}_{r(X)^\infty} &= (a_{\overline{j-i}, \overline{-i-1}})_{(i,j) \in \mathbb{N} \times \mathbb{Z}} & \mathcal{O}_{ri(X)^\infty} &= (a_{\overline{-j-1}, \overline{-i-1}})_{(i,j) \in \mathbb{N} \times \mathbb{Z}} \\ \mathcal{O}_{r^2(X)^\infty} &= (a_{\overline{-j-1}, \overline{i-j-1}})_{(i,j) \in \mathbb{N} \times \mathbb{Z}} & \mathcal{O}_{r^2i(X)^\infty} &= (a_{\overline{j-i}, \bar{j}})_{(i,j) \in \mathbb{N} \times \mathbb{Z}} \end{aligned}$$

For instance, a representation of  $\mathcal{O}_{g(010100)^\infty}$  for all  $g \in D_3$  is given in Figure 1.11.

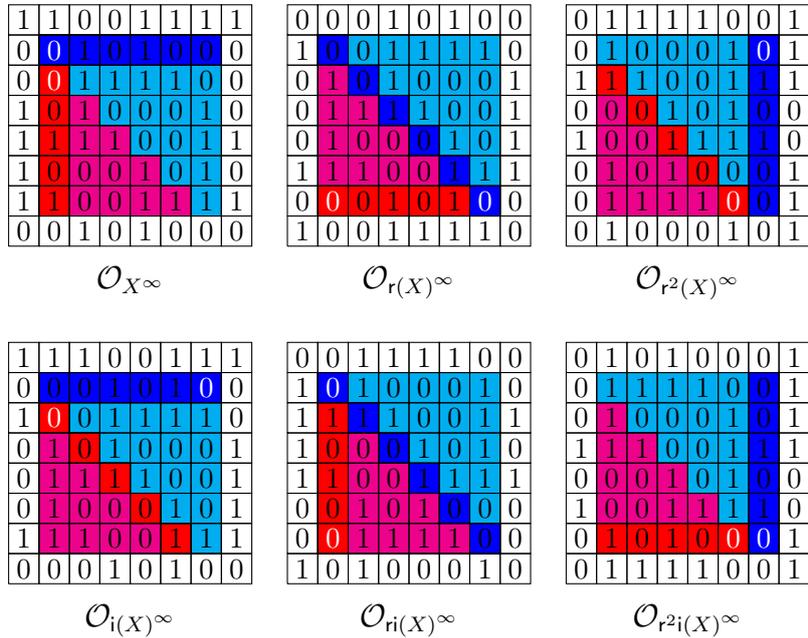


Figure 1.11: Action of  $D_3$  on  $\mathcal{O}_{010100}^\infty$

### The symmetry group of $\mathcal{PO}_p$

Let  $G$  be the subgroup of  $(\text{Aut}(\mathcal{PO}_p), \circ)$  generated by  $r$ ,  $i$ ,  $t_{1,0}$  and  $t_{0,1}$ , that is,

$$G := \langle r, i, t_{1,0}, t_{0,1} \rangle.$$

As in  $D_3$ , the equality  $ir = r^2i$  holds in  $G$ . The equalities involving the translations are listed below.

**Proposition 1.3.16.** *For all  $(u, v) \in \mathbb{Z}^2$ , the equalities  $rt_{u,v} = t_{v-u, -u}r$  and  $it_{u,v} = t_{u, u-v}i$  hold.*

From these equalities, it is clear that each element  $g \in G$  can be uniquely written as

$$g = t_{u,v} r^\alpha i^\beta$$

with  $u, v \in \{0, 1, \dots, p-1\}$ ,  $\alpha \in \{0, 1, 2\}$  and  $\beta \in \{0, 1\}$ . Therefore  $G$  is a group of order  $|G| = 6p^2$ .

### Equivalence classes of $\mathcal{PO}_p$

Now, we consider the binary relation  $\sim_G$  on the set  $\mathcal{PO}_p$  defined by  $X_1 \sim_G X_2$  if and only if there exists  $g \in G$  such that  $X_2 = g(X_1)$ . Since  $G$  is a subgroup of  $(\text{Aut}(\mathcal{PO}_p), \circ)$ , it is clear that  $\sim_G$  is an equivalence relation on  $\mathcal{PO}_p$ . Therefore, to search almost-balanced triangles, it is sufficient to examine only one representative of each equivalence class in the set  $\overline{\mathcal{PO}_p} := \mathcal{PO}_p / \sim_G$ . In the sequel, the equivalence class of the tuple  $X$  is denoted by  $\overline{X}$  and the lexicographically smallest tuple  $X$  is used as the representative of each equivalence class  $\overline{X}$ .

For example, for  $p = 6$ ,  $\overline{\mathcal{PO}_6}$  consists of 3 equivalence classes that contain the 16 tuples of  $\mathcal{PO}_6$  generating 6-periodic orbits. More precisely,

$$\begin{aligned} \overline{\mathcal{PO}_6} &= \{\{000000\}, \{000101, 001010, 001111, 010001, 010100, 011110, 100010, \\ &\quad 100111, 101000, 110011, 111001, 111100\}, \{011011, 101101, 110110\}\} \\ &= \{\overline{000000}, \overline{000101}, \overline{011011}\} \end{aligned}$$

since

$$\begin{aligned} 001010 &= t_{0,5}(000101) & 100010 &= t_{0,1}(000101) & 101101 &= t_{0,1}(011011) \\ 001111 &= t_{1,3}(000101) & 100111 &= t_{1,4}(000101) & 110110 &= t_{0,2}(011011) \\ 010001 &= t_{0,2}(000101) & 101000 &= t_{0,3}(000101) & & \\ 010100 &= t_{0,4}(000101) & 110011 &= t_{1,5}(000101) & & \\ 011110 &= t_{1,2}(000101) & 111001 &= t_{1,0}(000101) & & \\ & & 111100 &= t_{1,1}(000101) & & \end{aligned}$$

The 6-periodic orbits associated with these 3 equivalence classes are depicted in Figure 1.12.

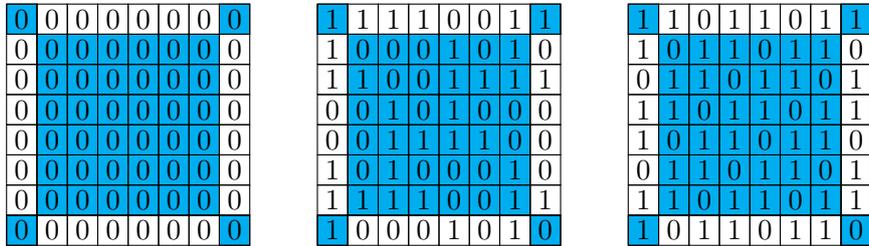


Figure 1.12: The set  $\overline{\mathcal{PO}_6} = \{\overline{000000}, \overline{000101}, \overline{011011}\}$

Table 1.3 gives  $|\overline{\mathcal{PO}_p}|$  for the first few values of  $p$ .

$p$	1	2	3	4	5	6	7	8	9	10	11	12
$ \mathcal{PO}_p $	1	1	$2^2$	1	1	$2^4$	$2^6$	1	$2^2$	1	1	$2^8$
$ \overline{\mathcal{PO}_p} $	1	1	2	1	1	3	3	1	2	1	1	7
$p$	13	14	15	16	17	18	19	20	21	22	23	24
$ \mathcal{PO}_p $	1	$2^{12}$	$2^{14}$	1	1	$2^4$	1	1	$2^8$	1	1	$2^{16}$
$ \overline{\mathcal{PO}_p} $	1	13	30	1	1	3	1	1	6	1	1	92

Table 1.3: The first few values of  $|\overline{\mathcal{PO}_p}|$

### 1.3.4 Family of periodic almost-balanced Steinhaus triangles with the same principal vertex

In this subsection, we determine necessary and sufficient conditions for obtaining, in a  $p$ -periodic orbit, an infinite family of almost-balanced Steinhaus triangles with the same principal vertex.

**Proposition 1.3.17** (Chappelon [C7]). *Let  $S = X^\infty$  with  $X \in \mathcal{PO}_p$ ,  $(i_0, j_0) \in \mathbb{N} \times \mathbb{Z}$  and  $r \in \{0, 1, \dots, p-1\}$ . The Steinhaus triangles*

$$T_k := \nabla S(i_0, j_0, kp + r)$$

*are almost-balanced for all non-negative integers  $k$  if and only if the triangle  $T_0$ , the multiset difference  $T_1 \setminus T_0$  and the period  $P$  are almost-balanced, with  $p$  divisible by 4.*

This is the reason why, in the sequel of this section, we only consider  $p$ -periodic orbits with a balanced period and where  $p$  is divisible by 4.

Note that the period of the orbit generated from every element of a same equivalence class of  $\overline{\mathcal{PO}_p}$  has the same multiplicity function. Let us denote by  $\overline{\mathcal{BPO}_p}$  the set of all the equivalence classes of  $\overline{\mathcal{PO}_p}$  having a balanced period. Table 1.4 gives  $|\overline{\mathcal{BPO}_p}|$  for the first few values of  $p$  divisible by 4.

$p$	4	8	12	16	20	24
$ \mathcal{PO}_p $	1	1	256	1	1	65536
$ \overline{\mathcal{PO}_p} $	1	1	7	1	1	92
$ \overline{\mathcal{BPO}_p} $	0	0	2	0	0	17

Table 1.4: The first few values of  $|\overline{\mathcal{BPO}_p}|$

More precisely, we obtain

$$\overline{\mathcal{BPO}_{12}} = \{\overline{Y_1}, \overline{Y_2}\} = \{\overline{0000011110111}, \overline{000101000101}\}$$

and  $\overline{\mathcal{BPO}_{24}} = \{\overline{X_1}, \overline{X_2}, \dots, \overline{X_{17}}\}$ , where the 17 representatives  $X_i$  are given in Table 1.5. Note that  $X_{16} = Y_1^2$  and  $X_{17} = Y_2^2$ . Therefore the orbits  $\mathcal{O}_{X_{16}^\infty}$  and  $\mathcal{O}_{X_{17}^\infty}$  correspond to  $\mathcal{O}_{Y_1^\infty}$  and  $\mathcal{O}_{Y_2^\infty}$ , respectively. A representation of the orbits generated from the elements of  $\overline{\mathcal{BPO}_{12}}$  and  $\overline{\mathcal{BPO}_{24}}$  can be found in Appendix A.

### 1.3.5 Periodic almost-balanced binary triangles

In this subsection we will prove Theorem 1.3.5, the main result of this section.

Let  $X$  be a  $p$ -tuple of  $\mathbb{Z}/2\mathbb{Z}$ , with  $p$  divisible by 4, such that  $\overline{X}$  is in  $\overline{\mathcal{BPO}_p}$  and let  $S := X^\infty$ . Now, for each remainder  $r \in \{0, 1, \dots, p-1\}$  and for each position  $(i_0, j_0) \in \{0, 1, \dots, p-1\}^2$ , we test if the blocks  $\nabla S(i_0, j_0, r)$  and  $\nabla S(i_0, j_0, p+r) \setminus \nabla S(i_0, j_0, r)$  are almost-balanced. If this is the case, we know from Proposition 1.3.17 that the Steinhaus triangles  $\nabla S(i_0, j_0, kp + r)$  are almost-balanced for all non-negative integers  $k$ .

Let  $\mathcal{R}_{\overline{X}}$  denote the set of remainders  $r \in \{0, 1, \dots, p-1\}$  for which there exists a position  $(i_0, j_0) \in \{0, 1, \dots, p-1\}^2$  such that the Steinhaus triangles  $\nabla S(i_0, j_0, kp + r)$  are almost-balanced for all non-negative integers  $k$ .

$i$	$X_i$	$i$	$X_i$
1	000000000110101101101011	10	000000101001110110011111
2	000000001001011110010111	11	000001000010010100100001
3	000000010010011000100111	12	000001000101101101011111
4	000000010110111001101111	13	000001001000001110000111
5	000000010111110001111101	14	000001001111110111111001
6	000000100001111100011101	15	000001100001100100011111
7	000000100010100100101011	16	000001110111000001110111
8	000000100011101100111001	17	000101000101000101000101
9	000000101000111110001101		

Table 1.5: The representatives  $X_i$  of  $\overline{\mathcal{BPO}}_{24} = \{\overline{X_1}, \overline{X_2}, \dots, \overline{X_{17}}\}$ 

From Table 1.4, the first values of  $p$ , divisible by 4, for which  $\overline{\mathcal{BPO}}_p \neq \emptyset$  are 12 and 24. For  $p = 12$ , we find that  $\mathcal{R}_{\overline{Y_i}} = \emptyset$  for each of the two equivalence classes  $\overline{\mathcal{BPO}}_{12} = \{\overline{Y_1}, \overline{Y_2}\}$ . For  $p = 24$ , we find that  $\mathcal{R}_{\overline{X_i}} \neq \emptyset$  for the first 15 of the 17 equivalence classes of  $\overline{\mathcal{BPO}}_{24} = \{\overline{X_1}, \overline{X_2}, \dots, \overline{X_{17}}\}$ . Note that the two equivalence classes  $\overline{X}$  of  $\overline{\mathcal{BPO}}_{24}$  such that  $\mathcal{R}_{\overline{X}} = \emptyset$  are exactly of the form  $X = Y^2$  with  $\overline{Y} \in \overline{\mathcal{BPO}}_{12}$  (as already seen  $X_{16} = Y_1^2$  and  $X_{17} = Y_2^2$ ). More precisely, Table 1.6 gives the exact number of remainders constituting  $\mathcal{R}_{\overline{X}}$  for each  $\overline{X} \in \overline{\mathcal{BPO}}_{24}$ .

$i$	$X_i$	$ \mathcal{R}_{\overline{X_i}} $
1	000000000110101101101011	18
2	000000001001011110010111	16
3	000000010010011000100111	23
4	<b>000000010110111001101111</b>	<b>24</b>
5	000000010111110001111101	17
6	<b>000000100001111100011101</b>	<b>24</b>
7	<b>000000100010100100101011</b>	<b>24</b>
8	<b>000000100011101100111001</b>	<b>24</b>
9	<b>000000101000111110001101</b>	<b>24</b>
10	000000101001110110011111	23
11	<b>000001000010010100100001</b>	<b>24</b>
12	000001000101101101011111	23
13	000001001000001110000111	20
14	000001001111110111111001	20
15	000001100001100100011111	23
16	000001110111000001110111	0
17	000101000101000101000101	0

Table 1.6:  $|\mathcal{R}_{\overline{X}}|$  for all  $\overline{X} \in \overline{\mathcal{BPO}}_{24}$ 

For six equivalence classes  $\overline{X}$  of  $\overline{\mathcal{BPO}}_{24}$ , we find that  $|\mathcal{R}_{\overline{X}}| = 24$  and thus, from these 24-tuples, we obtain the proof of Theorem 1.3.5 for Steinhaus triangles, i.e., there exist periodic orbits containing almost-balanced Steinhaus triangles of size  $n$  for all  $n \geq 1$ .

For instance, in the orbit  $\mathcal{O}_{X_9^\infty}$  associated with the 24-tuple

$$X_9 = 000000101000111110001101,$$

the existence of almost-balanced Steinhaus triangles for all the possible sizes can be obtained from at least 4 positions. Table 1.7 gives positions  $(i_0, j_0)$  in the orbit  $\mathcal{O}_{X_9^\infty}$  for which the Steinhaus triangles  $\nabla X_9^\infty(i_0, j_0, 24k + r)$  are almost-balanced for all non-negative integers  $k$  and the corresponding 24-tuples  $Z$  such that  $\nabla Z^\infty[24k + r] = \nabla X_9^\infty(i_0, j_0, 24k + r)$ .

$r$	$(i_0, j_0)$	$Z$
0, 4, 7, 8, 12, 13, 15, 16, 21, 22, 23	(1, 11)	010000100101110000011110
1, 2, 3, 5, 10, 17, 18, 19, 20, 21	(1, 6)	111100100001001011100000
0, 1, 6, 9, 14, 22, 23	(6, 9)	000111010101000101001100
11	(3, 3)	000110011101001011001011

Table 1.7: Almost-balanced Steinhaus triangles  $\nabla X_9^\infty(i_0, j_0, 24k + r) = \nabla Z^\infty[24k + r]$

The family of almost-balanced Steinhaus triangles  $\nabla X_9^\infty(6, 9, 24k + 6)$ , appearing in the orbit  $\mathcal{O}_{X_9^\infty}$ , is depicted in Figure 1.13, where empty and full squares correspond to 0 and 1 respectively. Indeed, we can verify that the blocks  $T_0 := \nabla X_9^\infty(6, 9, 6)$ ,  $T_1 \setminus T_0 := \nabla X_9^\infty(6, 9, 30) \setminus \nabla X_9^\infty(6, 9, 6)$  and the period  $P$  are almost-balanced, since their multiplicity functions, given in Table 1.8, are constant or almost constant.

$x$	$\mathbf{m}_{T_0}(x)$	$\mathbf{m}_{T_1 \setminus T_0}(x)$	$\mathbf{m}_P(x)$
0	11	222	288
1	10	222	288

Table 1.8: The multiplicity functions of  $T_0$ ,  $T_1 \setminus T_0$  and  $P$

The following proposition concludes the proof of Theorem 1.3.5 by showing that in an orbit  $\mathcal{O}_{X^\infty}$  generated from a  $p$ -tuple  $X$  such that  $\overline{X} \in \overline{\mathcal{BPO}}_p$ , the existence of almost-balanced Steinhaus triangles implies that of almost-balanced generalized Pascal triangles.

**Proposition 1.3.18** (Chappelton [C7]). *Let  $S = X^\infty$  with  $\overline{X} \in \overline{\mathcal{BPO}}_p$ ,  $(i_0, j_0) \in \mathbb{N} \times \mathbb{Z}$ ,  $r \in \{0, 1, \dots, p-1\}$  and  $p$  divisible by 4. Then, the Steinhaus triangles  $\nabla S(i_0, j_0, kp + r)$  are almost-balanced for all non-negative integers  $k$  if and only if the generalized Pascal triangles  $\Delta S(i_0 + r + 1, j_0 + r, kp + (p - 1 - r))$  are almost-balanced for all non-negative integers  $k$ .*

Using Proposition 1.3.18 and the families of almost-balanced Steinhaus triangles appearing in the orbit  $\mathcal{O}_{X_9^\infty}$  associated with the 24-tuple  $X_9 = 000000101000111110001101$  given in Table 1.7, we obtain the existence of almost-balanced generalized Pascal triangles for all the possible sizes.

Moreover, in the orbit  $\mathcal{O}_{X_9^\infty}$ , the existence of almost-balanced generalized Pascal triangles for all the possible sizes can also be obtained from only 6 positions. This result is not obtained by using Proposition 1.3.18 but by testing, at each position  $(i_0, j_0)$  and for each remainder  $r$ , if the elementary blocks  $V_1$  and  $V_0 \cup V_1$  are almost-balanced, where  $V_1 = \Delta X_9^\infty(i_0, j_0, r)$  and  $V_0 = \Delta X_9^\infty(i_0, j_0, p+r) \setminus (\Delta X_9^\infty(i_0, j_0, r) \cup \Delta X_9^\infty(i_0 + p, j_0 + p, r))$ . The corresponding values appear in Table 1.9.

Figure 1.13: The balanced Steinhaus triangles  $\nabla X_9^\infty(6, 9, 24k + 6)$ 

$r$	$(i_0, j_0)$	$Z_t$	$Z_r$
1, 7, 15, 23	(0, 9)	010001000111110100111001	000110010000010000011101
4, 5, 12, 13, 21	(7, 22)	011111110100110100110010	001010010111111101010110
3, 6, 14, 19, 22	(7, 4)	011110011111101110000010	010001001010001011100110
2, 10, 18, 20	(4, 15)	100111011100010101011000	110100010100011010010111
0, 8, 16, 22	(1, 2)	001011100111100001010110	000010000011101000110010
1, 3, 9, 11, 17	(6, 7)	011000100111011100010101	000001110100011001000001

Table 1.9: Almost-balanced triangles  $\Delta X_9^\infty(i_0, j_0, 24k+r) = \Delta(Z_t^\infty[24k+r], Z_r^\infty[24k+r])$

# Chapter 2

## Symmetric binary triangles

In this chapter, we study binary triangles that are invariant under the action of subgroups of the dihedral group  $D_3$ .

### 2.1 Symmetric binary Steinhaus triangles

Let  $n$  be a positive integer. We consider the set  $\mathcal{ST}(n)$  of binary Steinhaus triangles of size  $n$ . We already know that  $\mathcal{ST}(n)$  is a vector space over  $\mathbb{Z}/2\mathbb{Z}$  of dimension  $n$ . We begin by recalling that the dihedral group  $D_3$  acts on  $\mathcal{ST}(n)$ .

Let  $(a_{i,j})_{1 \leq i \leq j \leq n}$  be a binary Steinhaus triangle of size  $n$ . The local rule (LR)

$$a_{i,j} = a_{i-1,j-1} + a_{i-1,j}$$

can also be written as

$$a_{i-1,j-1} = a_{i-1,j} + a_{i,j} \quad \text{or} \quad a_{i,j} = a_{i-1,j} + a_{i-1,j-1},$$

for all integers  $i$  and  $j$  such that  $2 \leq i \leq j \leq n$ . This is the reason why the 120 degrees rotation and the horizontal reflection of a Steinhaus triangle are also Steinhaus triangles, of the same size.

**Definition 2.1.1** (Rotation and horizontal reflection). Let  $r$  and  $h$  be the 120 degrees rotation and the horizontal reflection of binary Steinhaus triangles, that are the automorphisms of  $\mathcal{ST}(n)$  defined by

$$\begin{aligned} r : \quad \mathcal{ST}(n) &\longrightarrow \mathcal{ST}(n) \\ (a_{i,j})_{1 \leq i \leq j \leq n} &\longmapsto (a_{j-i+1, n-i+1})_{1 \leq i \leq j \leq n} \end{aligned}$$

and

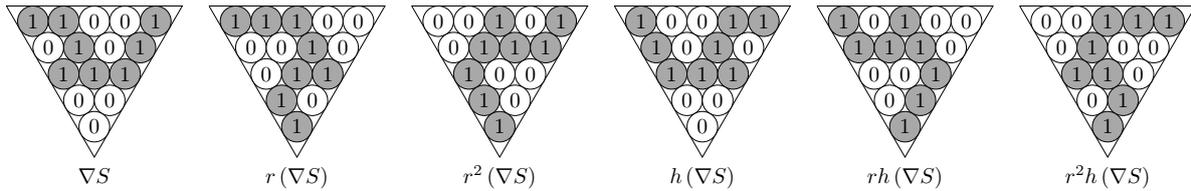
$$\begin{aligned} h : \quad \mathcal{ST}(n) &\longrightarrow \mathcal{ST}(n) \\ (a_{i,j})_{1 \leq i \leq j \leq n} &\longmapsto (a_{i, n-j+i})_{1 \leq i \leq j \leq n} \end{aligned}$$

for all non-negative integers  $n$ .

These automorphisms verify the following identities

$$r^3 = h^2 = hrhr = id_{\mathcal{ST}(n)},$$

where  $id_{\mathcal{ST}(n)}$  is the identity map on  $\mathcal{ST}(n)$ . Therefore, the subgroup  $\langle r, h \rangle$  generated by  $r$  and  $h$ , of the automorphism group of  $\mathcal{ST}(n)$ , is isomorphic to the dihedral group  $D_3$ .

Figure 2.1: Action of  $D_3$  on  $\nabla(11001)$ 

This induces a faithful representation of  $D_3$  on  $\mathcal{ST}(n)$ , for all non-negative integers  $n$ . In the sequel, the automorphism subgroup  $\langle r, h \rangle$  is simply denoted by  $D_3$ . For instance, for  $S = (11001)$  and for all  $g \in D_3$ , the Steinhaus triangles  $g(\nabla S)$  are depicted in Figure 2.1.

For any subgroup  $G$  of  $D_3$  and any non-negative integer  $n$ , we consider the linear subspace of invariant triangles of  $\mathcal{ST}(n)$  under  $G$ , that is,

$$\mathcal{ST}(n)^G = \{\nabla \in \mathcal{ST}(n) \mid \forall g \in G, g(\nabla) = \nabla\}.$$

It is well known that there are exactly 6 subgroups of  $D_3$ , that are  $\{id_{\mathcal{ST}(n)}\}$ ,  $\langle h \rangle$ ,  $\langle rh \rangle$ ,  $\langle r^2h \rangle$ ,  $\langle r \rangle$  and  $D_3$ . Obviously, we have  $\mathcal{ST}(n)^G = \mathcal{ST}(n)$  for the trivial subgroup  $G = \{id_{\mathcal{ST}(n)}\}$ . Moreover, by the linear maps

$$\begin{array}{ccc} \mathcal{ST}(n)^{\langle h \rangle} & \longrightarrow & \mathcal{ST}(n)^{\langle rh \rangle} \\ \nabla & \longmapsto & r^2(\nabla) \end{array}$$

and

$$\begin{array}{ccc} \mathcal{ST}(n)^{\langle h \rangle} & \longrightarrow & \mathcal{ST}(n)^{\langle r^2h \rangle} \\ \nabla & \longmapsto & r(\nabla) \end{array}$$

it is clear that the three linear subspaces  $\mathcal{ST}(n)^{\langle h \rangle}$ ,  $\mathcal{ST}(n)^{\langle rh \rangle}$  and  $\mathcal{ST}(n)^{\langle r^2h \rangle}$  are isomorphic to each other. Therefore, for all non-negative integers  $n$ , we only consider the linear subspaces  $\mathcal{ST}(n)^{\langle h \rangle}$ ,  $\mathcal{ST}(n)^{\langle r \rangle}$  and  $\mathcal{ST}(n)^{D_3}$ .

**Notation 2.1.2.** For all non-negative integers  $n$ , the linear subspaces  $\mathcal{ST}(n)^{\langle h \rangle}$ ,  $\mathcal{ST}(n)^{\langle r \rangle}$  and  $\mathcal{ST}(n)^{D_3}$  are denoted by  $\mathcal{HST}(n)$ ,  $\mathcal{RST}(n)$  and  $\mathcal{DST}(n)$ , respectively. Obviously, these vector spaces simply correspond to  $\ker(h - id_{\mathcal{ST}(n)})$ ,  $\ker(r - id_{\mathcal{ST}(n)})$  and their intersection, respectively.

**Definition 2.1.3** (Symmetric binary Steinhaus triangles). A binary Steinhaus triangle  $\nabla$  of  $\mathcal{HST}(n)$ ,  $\mathcal{RST}(n)$  or  $\mathcal{DST}(n)$  is said to be *horizontally symmetric*, *rotationally symmetric* or *dihedrally symmetric*, respectively, and verifies  $h(\nabla) = \nabla$ ,  $r(\nabla) = \nabla$  or  $r(\nabla) = h(\nabla) = \nabla$ , respectively.

Examples of such symmetric Steinhaus triangles appear in Figure 2.2.

In [26], it was proved that

- $\dim \mathcal{HST}(n) = \lfloor \frac{n}{2} \rfloor$ ,
- $\dim \mathcal{RST}(n) = \lfloor \frac{n}{3} \rfloor + \delta_{1, (n \bmod 3)}$ ,
- $\dim \mathcal{DST}(n) = \lfloor \frac{n+3}{6} \rfloor + \delta_{1, (n \bmod 6)}$ ,

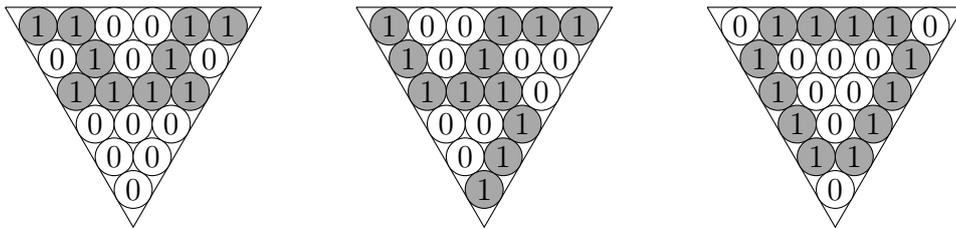


Figure 2.2: Triangles of  $\mathcal{HST}(6)$ ,  $\mathcal{RST}(6)$  and  $\mathcal{DST}(6)$ .

for all non-negative integers  $n$ , where  $\delta_{i,(n \bmod j)}$  is equal to 1, if  $n \equiv i \pmod j$ , and 0 otherwise. Bases of  $\mathcal{HST}(n)$ ,  $\mathcal{RST}(n)$  and  $\mathcal{DST}(n)$ , for all non-negative integers  $n$ , are obtained in [39]. In this chapter, we give new bases, for each of these three linear subspaces, which are simpler than those mentioned. They are obtained by considering elementary properties of generalized binomial coefficients and the notion of generating index sets of Steinhaus triangles. These results can be found in [C8] for the most part.

## 2.2 Generating index sets

Let  $n$  be a positive integer. We denote by  $\nabla(n)$  the index set of Steinhaus triangles of size  $n$ , that is,

$$\nabla(n) = \{(i, j) \in \mathbb{N}^2 \mid 1 \leq i \leq j \leq n\}.$$

**Definition 2.2.1** (Generating index set of  $\mathcal{ST}(n)$ ). A subset  $G$  of  $\nabla(n)$  is said to be a *generating index set* of  $\mathcal{ST}(n)$  if the knowledge of the values  $a_{i,j}$ , for all  $(i, j) \in G$ , uniquely determines the whole Steinhaus triangle  $(a_{i,j})_{1 \leq i \leq j \leq n}$ , i.e., if the linear map

$$\begin{aligned} \pi_G : \quad \mathcal{ST}(n) &\longrightarrow \{0, 1\}^G \\ (a_{i,j})_{1 \leq i \leq j \leq n} &\longmapsto (a_{i,j})_{(i,j) \in G} \end{aligned}$$

is an isomorphism.

Since  $\dim \mathcal{ST}(n) = n$ , we deduce that the cardinality of a generating index set of  $\mathcal{ST}(n)$  is always  $n$ . From (1.1), it is clear that the set of top row indices of a Steinhaus triangle of size  $n$ , that is,

$$G_1 = \{(1, 1), (1, 2), \dots, (1, n)\},$$

is a generating index set of  $\mathcal{ST}(n)$ . Note that  $\pi_{G_1}^{-1}(S) = \nabla S$ , for all  $S \in \{0, 1\}^{G_1}$ . It follows that the set  $G$  is a generating index set of  $\mathcal{ST}(n)$  if and only if the linear map  $\pi_G \circ \pi_{G_1}^{-1} : \{0, 1\}^{G_1} \rightarrow \{0, 1\}^G$  is an isomorphism. For instance, the 16 generating index sets of  $\mathcal{ST}(3)$  (4 up to the action of the dihedral group  $D_3$ ) are depicted in Figure 2.3, where a disk is either black if its position is in the generating index set or white otherwise.

Since the sets of right side indices,

$$G_r = \{(1, n), (2, n), \dots, (n, n)\},$$

and left side indices,

$$G_l = \{(1, 1), (2, 2), \dots, (n, n)\},$$

of a Steinhaus triangle  $\nabla$  of size  $n$  can be seen as the sets of top row indices of the Steinhaus triangles  $r(\nabla)$  and  $r^2(\nabla)$ , respectively, it follows that  $G_r$  and  $G_l$  are generating index

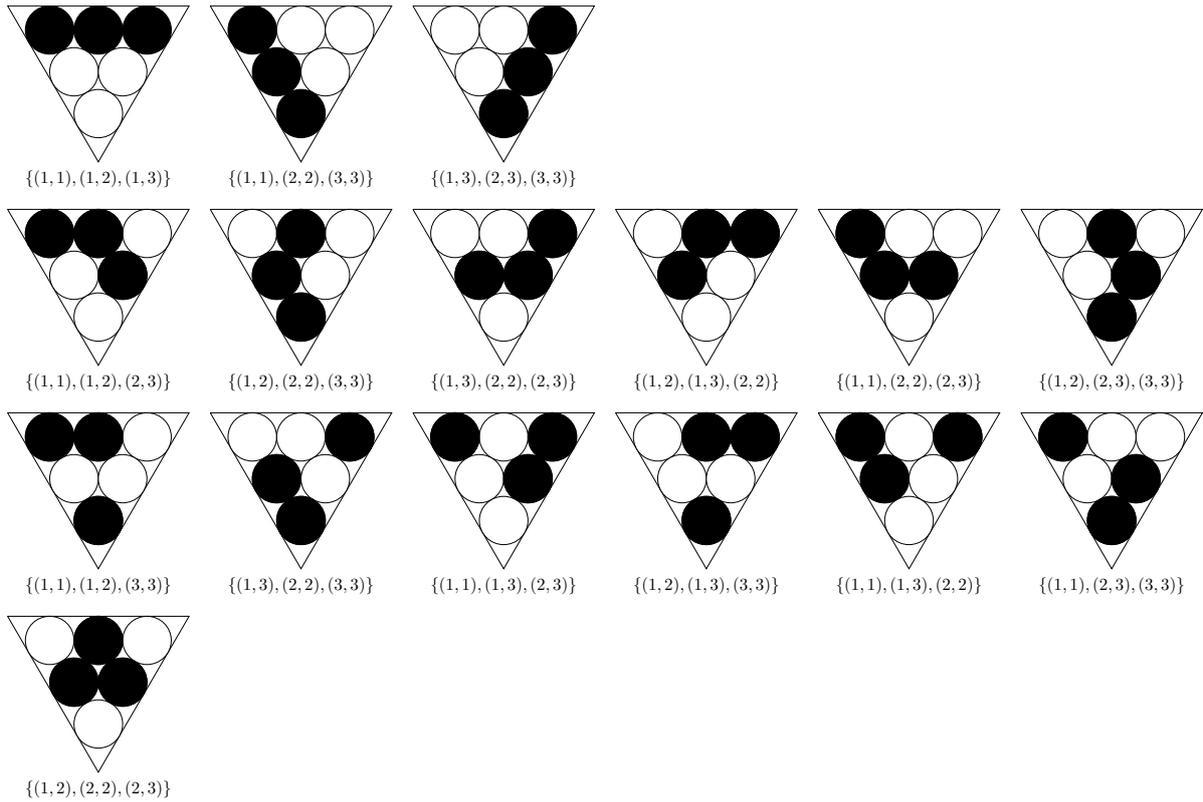


Figure 2.3: Generating index sets of  $\mathcal{ST}(3)$

sets of  $\mathcal{ST}(n)$  too. Therefore, each element of a Steinhaus triangle can be expressed in function of the terms of its first row, of its right side or of its left side.

For any non-negative integers  $a$  and  $b$  such that  $b \leq a$ , the binomial coefficient  $\binom{a}{b}$  is the coefficient of the monomial  $X^b$  in the polynomial expansion of the binomial power  $(1+X)^a$ . It corresponds to the number of ways to choose  $b$  elements in a set of  $a$  elements. Here, we extend this notation by supposing that  $\binom{a}{b} = 0$ , for all integers  $b$  such that  $b < 0$  or  $b > a$ . For this generalization, the Pascal identity

$$\binom{a}{b} = \binom{a-1}{b-1} + \binom{a-1}{b}$$

holds, for all positive integers  $a$  and all integers  $b$ .

**Lemma 2.2.2.** *Let  $(a_{i,j})_{1 \leq i \leq j \leq n}$  be a binary Steinhaus triangle of size  $n$ . Then, we have*

$$a_{i,j} \equiv \sum_{k=1}^n \binom{i-1}{j-k} a_{1,k} \equiv \sum_{k=1}^n \binom{n-j}{k-i} a_{k,n} \equiv \sum_{k=1}^n \binom{j-i}{k-i} a_{k,k} \pmod{2},$$

for all integers  $i, j$  such that  $1 \leq i \leq j \leq n$ .

**Proposition 2.2.3.** *Let  $G = \{(i_1, j_1), (i_2, j_2), \dots, (i_n, j_n)\}$  be a subset of  $\nabla(n)$  whose cardinality is  $|G| = n$ . Then, the set  $G$  is a generating index set of  $\mathcal{ST}(n)$  if and only if  $\det(M_G) \equiv 1 \pmod{2}$ , where*

$$M_G = \left( \binom{i_k - 1}{j_k - l} \right)_{1 \leq k, l \leq n}.$$

The notion of generating index sets and the result of Proposition 2.2.3 appear in a more general context in [28, 29], where it is also proved that the set of generating index sets of  $\mathcal{ST}(n)$  define a matroid called the Pascal matroid modulo 2. Note that a generating index set is simply called a generating set in [28, 29].

The notion of generating index set can be extended to any linear subspace of  $\mathcal{ST}(n)$ .

**Definition 2.2.4** (Generating index set of linear subspaces). Let  $\mathcal{V}$  be a linear subspace of  $\mathcal{ST}(n)$ . A subset  $G$  of  $\nabla(n)$  is said to be a *generating index set* of  $\mathcal{V}$  if the linear map

$$\begin{aligned} \pi_G : \quad \mathcal{V} &\longrightarrow \{0, 1\}^G \\ (a_{i,j})_{1 \leq i \leq j \leq n} &\longmapsto (a_{i,j})_{(i,j) \in G} \end{aligned}$$

is an isomorphism. Note that  $|G| = \dim \mathcal{V}$ , for any generating index set  $G$  of  $\mathcal{V}$ .

In this chapter, we consider generating index sets of the linear subspaces  $\mathcal{RST}(n)$ ,  $\mathcal{HST}(n)$  and  $\mathcal{DST}(n)$ , for all non-negative integers  $n$ .

## 2.3 Rotationally symmetric Steinhaus triangles

In this section, after characterizing rotationally symmetric Steinhaus triangles, we determine, for all non-negative integers  $n$ , generating index sets and bases of  $\mathcal{RST}(n)$ .

### 2.3.1 Characterizations of $\mathcal{RST}(n)$

First, by definition of the automorphism  $r$ , we have

$$r \left( (a_{i,j})_{1 \leq i \leq j \leq n} \right) = (a_{j-i+1, n-i+1})_{1 \leq i \leq j \leq n} = \nabla(a_{j,n})_{1 \leq j \leq n},$$

for any Steinhaus triangle  $(a_{i,j})_{1 \leq i \leq j \leq n} = \nabla(a_{1,j})_{1 \leq j \leq n}$ . Therefore, a Steinhaus triangle  $(a_{i,j})_{1 \leq i \leq j \leq n}$  is rotationally symmetric if and only if its first row  $(a_{1,j})_{1 \leq j \leq n}$  and its right side  $(a_{j,n})_{1 \leq j \leq n}$  correspond.

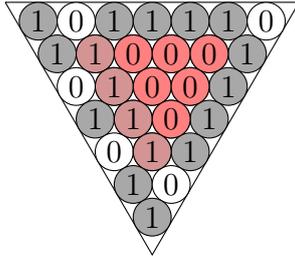
**Proposition 2.3.1.** *The Steinhaus triangle  $(a_{i,j})_{1 \leq i \leq j \leq n}$  is rotationally symmetric if and only if  $(a_{1,j})_{1 \leq j \leq n} = (a_{j,n})_{1 \leq j \leq n}$ .*

**Definition 2.3.2** (Operator H). Let H be the linear map that assigns, to each Steinhaus triangle of order  $n \geq 3$ , its subtriangle of order  $n - 3$  obtained by removing its first row and its left and right sides, that is,

$$\begin{aligned} \text{H} : \quad \mathcal{ST}(n) &\longrightarrow \mathcal{ST}(n - 3) \\ (a_{i,j})_{1 \leq i \leq j \leq n} &\longmapsto (a_{1+i, 2+j})_{1 \leq i \leq j \leq n-3} \end{aligned}$$

Note that the linear map H is surjective. Indeed, for any  $\nabla S' \in \mathcal{ST}(n - 3)$ , it is easy to verify that  $\nabla S' = \text{H}(\nabla S)$  if and only if  $S$  is one of the eight sequences of the form  $S = (x_1) \cdot \int_{i,x} S' \cdot (x_2)$ , where  $x_1, x_2 \in \{0, 1\}$  and  $\int_{i,x} S'$  is one of the two antiderived sequences of  $S'$ . Examples of a Steinhaus triangle  $\nabla S$  and its subtriangle  $\text{H}(\nabla S)$  are depicted in Figure 2.4.

**Notation 2.3.3.** For any binary sequence  $S = (a_j)_{1 \leq j \leq n}$ , we denote by  $\sigma(S)$  its sum  $\sigma(S) = \sum_{j=1}^n a_j$ , i.e., the number of ones in  $S$ , and by  $\sigma_2(S)$  its sum modulo 2.

Figure 2.4:  $H(\nabla(1011110)) = \nabla(1000)$ 

For any positive integer  $n \geq 3$ , by definition of  $\mathcal{RST}(n)$  and  $H$ , it is clear that  $H(\mathcal{RST}(n)) \subset \mathcal{RST}(n-3)$ . The precise relationship between a rotationally symmetric Steinhaus triangle  $\nabla S$  and its subtriangle  $H(\nabla S)$  is given in the following

**Proposition 2.3.4.** *Let  $S$  be a finite binary sequence of length  $n \geq 3$ . The Steinhaus triangle  $\nabla S$  is rotationally symmetric if and only if  $H(\nabla S) = \nabla S'$  is rotationally symmetric and  $S = (\sigma_2(S')) \cdot \int_{i,x} S' \cdot (\sigma_2(S'))$ , for some  $i \in \{1, \dots, n-2\}$  and some  $x \in \{0, 1\}$ .*

Proposition 2.3.4 appears in [27] in a more general context.

### 2.3.2 Generating index sets of $\mathcal{RST}(n)$

Using Proposition 2.3.4, we are now ready to determine generating index sets of the linear subspace of rotationally symmetric Steinhaus triangles, for every non-negative integer  $n$ .

**Theorem 2.3.5** (Chappelon [C8]). *Let  $n$  be a non-negative integer. The set*

$$G_R = \left\{ (i, j_i) \mid i \in \left\{ 1, \dots, \left\lfloor \frac{n}{3} \right\rfloor + \delta_{1, (n \bmod 3)} \right\} \right\},$$

where  $j_i \in \{2i, \dots, n-i\}$  for all  $i \in \{1, \dots, \lfloor \frac{n}{3} \rfloor\}$  and  $j_{\frac{n+2}{3}} = \frac{2n+1}{3}$  when  $n \equiv 1 \pmod{3}$ , is a generating index set of  $\mathcal{RST}(n)$ .

**Corollary 2.3.6** (Chappelon [C8]). *Let  $n$  be a non-negative integer. The set*

$$G_R := \left\{ \left( i, n - \left\lfloor \frac{n}{3} \right\rfloor \right) \mid i \in \left\{ 1, \dots, \left\lfloor \frac{n}{3} \right\rfloor + \delta_{1, (n \bmod 3)} \right\} \right\}$$

is a generating index set of  $\mathcal{RST}(n)$ .

Since the dimension of  $\mathcal{RST}(n)$  corresponds to the cardinality of a generating index set  $G_R$ , it is straightforward to obtain the following

**Corollary 2.3.7.**  $\dim \mathcal{RST}(n) = \left\lfloor \frac{n}{3} \right\rfloor + \delta_{1, (n \bmod 3)}$ , for all non-negative integers  $n$ .

### 2.3.3 Bases of $\mathcal{RST}(n)$

Now, using the generating index sets  $G_R$  introduced before, we determine bases of the linear subspace of rotationally symmetric Steinhaus triangles.

First, we consider the linear map  $\rho : \mathcal{ST}(n) \rightarrow \mathcal{RST}(n)$  defined by  $\rho = r^2 + r + id_{\mathcal{ST}(n)}$ , for all non-negative integers  $n$ . Obviously, this map is surjective since  $\rho(\nabla) = \nabla$ , for all  $\nabla \in \mathcal{RST}(n)$ . Moreover, as detailed below, all the terms of  $\rho(\nabla)$  can be expressed in function of these  $\nabla$ .

**Proposition 2.3.8** (Chappelon [C8]). *For all  $(a_{i,j})_{1 \leq i \leq j \leq n} \in \mathcal{ST}(n)$ , we have*

$$\rho \left( (a_{i,j})_{1 \leq i \leq j \leq n} \right) = (a_{i,j} + a_{j-i+1, n-i+1} + a_{n-j+1, n+i-j} \bmod 2)_{1 \leq i \leq j \leq n}.$$

**Notation 2.3.9** (Triangles  $U_n$ ). For any non-negative integer  $n$ , let  $U_n$  be the Steinhaus triangle of size  $n$  defined by

$$U_n = \rho(\nabla(1)_n).$$

It is clear that  $U_0 = \emptyset$ ,  $U_1 = \nabla(1)$ ,  $U_2 = \nabla(00)$  and  $U_n = \nabla(011 \cdots 110)$  for  $n \geq 3$ , since

$$U_n = \rho(\nabla(1)_n) = \nabla(1 \cdots 1) + \nabla(10 \cdots 0) + \nabla(0 \cdots 01) = \nabla((1 \cdots 1) + (10 \cdots 0) + (0 \cdots 01)),$$

for all positive integers  $n$ .

The Steinhaus triangles  $U_n$  are depicted in Figure 2.5, for the first few values of  $n$ . Moreover, an explicit formula for the terms of  $U_n$  is given in the following

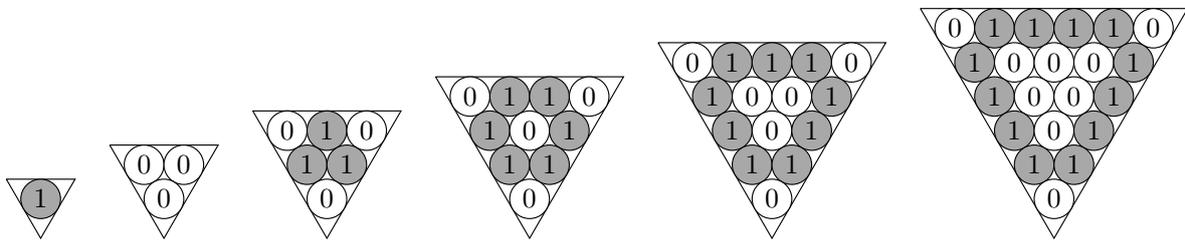


Figure 2.5:  $U_n$  for  $n \in \{1, \dots, 6\}$

**Proposition 2.3.10** (Chappelon [C8]). *For any non-negative integer  $n$ , we have*

$$U_n = (\delta_{i,1} + \delta_{i,j} + \delta_{j,n} \bmod 2)_{1 \leq i \leq j \leq n}.$$

**Corollary 2.3.11.**  $H(U_n) = \nabla(0)_{n-3}$ , for all positive integers  $n \geq 3$ .

For any non-negative integer  $k$  such that  $3k \leq n$ , we consider the iterated operator

$$H^k = \underbrace{HH \cdots H}_{k \text{ times}} : \begin{array}{ccc} \mathcal{ST}(n) & \longrightarrow & \mathcal{ST}(n-3k) \\ (a_{i,j})_{1 \leq i \leq j \leq n} & \longmapsto & (a_{k+i, 2k+j})_{1 \leq i \leq j \leq n-3k} \end{array}$$

Using the operators  $H^k$  and the generating index set  $G_R$ , we obtain a family of bases of  $\mathcal{RST}(n)$ , for all non-negative integers  $n$ .

**Theorem 2.3.12** (Chappelon [C8]). *Let  $n$  and  $m$  be non-negative integers such that  $m = \lfloor \frac{n}{3} \rfloor + \delta_{1, (n \bmod 3)}$ . For every  $k \in \{0, \dots, m-1\}$ , let  $\nabla_k \in \mathcal{RST}(n)$  such that  $H^k(\nabla_k) = U_{n-3k}$ . Then, the set  $\{\nabla_0, \dots, \nabla_{m-1}\}$  is a basis of  $\mathcal{RST}(n)$ .*

Since  $U_{n-3k} = \rho((1)_{n-3k})$  by definition, for all non-negative integers  $n$  and  $k$  such that  $3k \leq n$ , this leads to the following

**Corollary 2.3.13** (Chappelon [C8]). *Let  $n$  and  $m$  be non-negative integers such that  $m = \lfloor \frac{n}{3} \rfloor + \delta_{1, (n \bmod 3)}$ . For every  $k \in \{0, \dots, m-1\}$ , let  $S_k$  be a binary sequence of length  $n$  such that  $\partial^k S_k = (1)_{n-3k}$ . Then, the set  $\{\rho(\nabla S_0), \dots, \rho(\nabla S_{m-1})\}$  is a basis of  $\mathcal{RST}(n)$ .*

Binomial coefficients  $\binom{a}{b}$  have been defined before for any non-negative integer  $a$  and any integer  $b$ . Now, we extend this definition for negative integers  $a$ . For any integers  $a$  and  $b$ , let  $\binom{a}{b}$  denote the integers recursively defined by

- $\binom{a}{0} = 1$ , for all  $a \in \mathbb{Z}$ ,
- $\binom{0}{b} = 0$ , for all  $b \in \mathbb{N}^*$ ,
- $\binom{a}{b} = \binom{a-1}{b-1} + \binom{a-1}{b}$ , for all  $a, b \in \mathbb{Z}$ .

When  $a$  is non-negative, it corresponds with the previous definition. Moreover, for any negative integer  $a$ , the following equality holds

$$\binom{a}{b} = \begin{cases} 0 & \text{for } b < 0, \\ (-1)^b \binom{b-a-1}{b} & \text{for } b \geq 0. \end{cases}$$

Here, we mainly consider the *infinite Pascal matrix modulo 2*, that is, the doubly indexed sequence  $\left(\binom{a}{b}_2\right)_{(a,b) \in \mathbb{Z}^2}$ , where  $\binom{a}{b}_2$  is the value of  $\binom{a}{b} \bmod 2$ . The first few values of this doubly infinite sequence are shown in Figure 2.6, where the terms  $\binom{a}{0}_2$  are in blue, for all integers  $a$ , and the terms  $\binom{0}{b}_2$  are in red, for all positive integers  $b$ .

**Notation 2.3.14.** For any integers  $k$  and  $l$ , let  $S_{k,l}^{(n)}$  be the subsequence of length  $n$  of the  $k$ th column of the infinite Pascal matrix modulo 2 defined by

$$S_{k,l}^{(n)} = \left( \binom{l+j-1}{k}_2 \right)_{1 \leq j \leq n} = \left( \binom{l}{k}_2, \binom{l+1}{k}_2, \dots, \binom{l+n-1}{k}_2 \right).$$

For instance, the sequence  $S_{5,2}^{(7)} = \left( \binom{j+1}{5}_2 \right)_{1 \leq j \leq 7} = (0001010)$  appears in yellow in Figure 2.6.

Since we retrieve the local rule (LR) in the infinite Pascal matrix modulo 2, it is straightforward to obtain the following

**Proposition 2.3.15.** *Let  $k$  and  $l$  be two integers and let  $n$  be a positive integer. Then,*

$$\partial^i S_{k,l}^{(n)} = S_{k-i,l}^{(n-i)} = \left( \binom{l+j-1}{k-i}_2 \right)_{1 \leq j \leq n-i},$$

for all  $i \in \{0, \dots, n-1\}$  and

$$\nabla S_{k,l}^{(n)} = \left( \binom{l+j-i}{k+1-i}_2 \right)_{1 \leq i \leq j \leq n}.$$

For instance, the Steinhaus triangle  $\nabla S_{10,-10}^{(7)} = \nabla(0000110)$  appears in green in Figure 2.6.

We are now ready for giving explicit bases of  $\mathcal{RST}(n)$ , for every non-negative integer  $n$ , using Corollary 2.3.13 with binary sequences  $S_{k,l}^{(n)}$ .

**Theorem 2.3.16** (Chappelon [C8]). *Let  $n$  and  $m$  be non-negative integers such that  $m = \lfloor \frac{n}{3} \rfloor + \delta_{1,(n \bmod 3)}$ . For any integers  $l_0, \dots, l_{m-1}$ , the set  $\left\{ \rho \left( \nabla S_{0,l_0}^{(n)} \right), \dots, \rho \left( \nabla S_{m-1,l_{m-1}}^{(n)} \right) \right\}$  is a basis of  $\mathcal{RST}(n)$ . Moreover, we have*

$$\rho \left( \nabla S_{k,l_k}^{(n)} \right) = \left( \binom{l_k+j-i}{k+1-i} + \binom{l_k+n-j}{k+i-j} + \binom{l_k+i-1}{k+j-n} \bmod 2 \right)_{1 \leq i \leq j \leq n}, \quad (2.1)$$

for all  $k \in \{0, \dots, m-1\}$ .

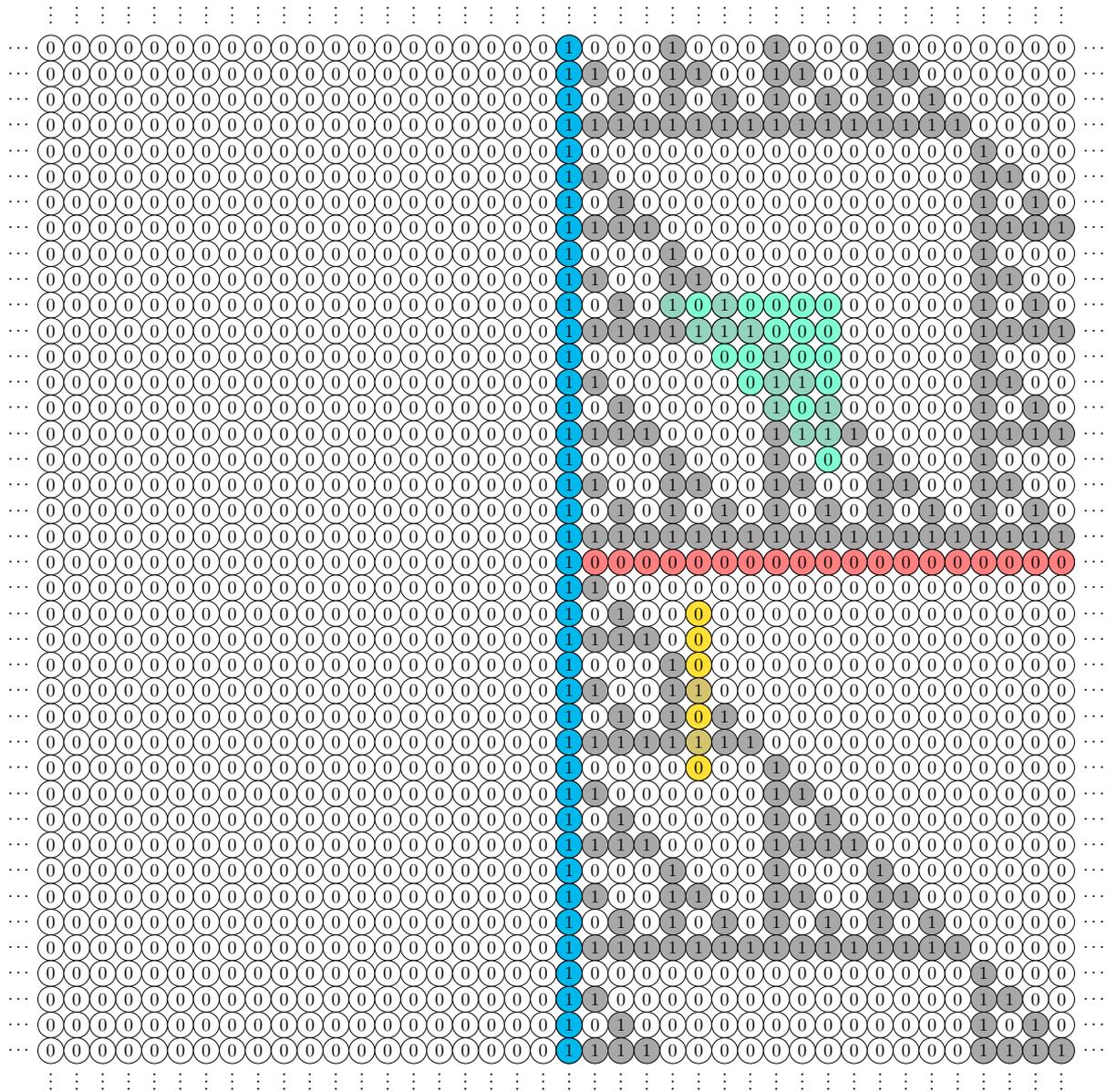


Figure 2.6: The infinite Pascal matrix modulo 2 with  $\binom{a}{0}_2$  in blue,  $\binom{0}{b}_2$  in red,  $S_{5,2}^{(7)}$  in yellow and  $\nabla S_{10,-10}^{(7)}$  in green

*Remark 2.3.17.* For any integer  $l_0$ , we have  $\rho(\nabla S_{0,l_0}^{(n)}) = \rho(\nabla(1)_n) = U_n$ .

For instance, for  $n = 10$  and  $l_0 = l_1 = l_2 = l_3 = 0$ , we obtain

$k$	$S_{k,0}^{(10)}$	$\rho(\nabla S_{k,0}^{(10)})$
0	(1111111111)	$\nabla_0 = \nabla(0111111110)$
1	(0101010101)	$\nabla_1 = \nabla(1001010111)$
2	(0011001100)	$\nabla_2 = \nabla(0001001000)$
3	(0001000100)	$\nabla_3 = \nabla(0010001100)$

All the rotationally symmetric Steinhaus triangles of size 10 are depicted in Figure 2.7, where the elements of the basis  $\{\nabla_0, \nabla_1, \nabla_2, \nabla_3\}$  are in red and, for every  $\nabla \in \mathcal{RST}(10)$ , the coordinate vector  $(x_0, x_1, x_2, x_3)$  of  $\nabla = x_0\nabla_0 + x_1\nabla_1 + x_2\nabla_2 + x_3\nabla_3$  is given.

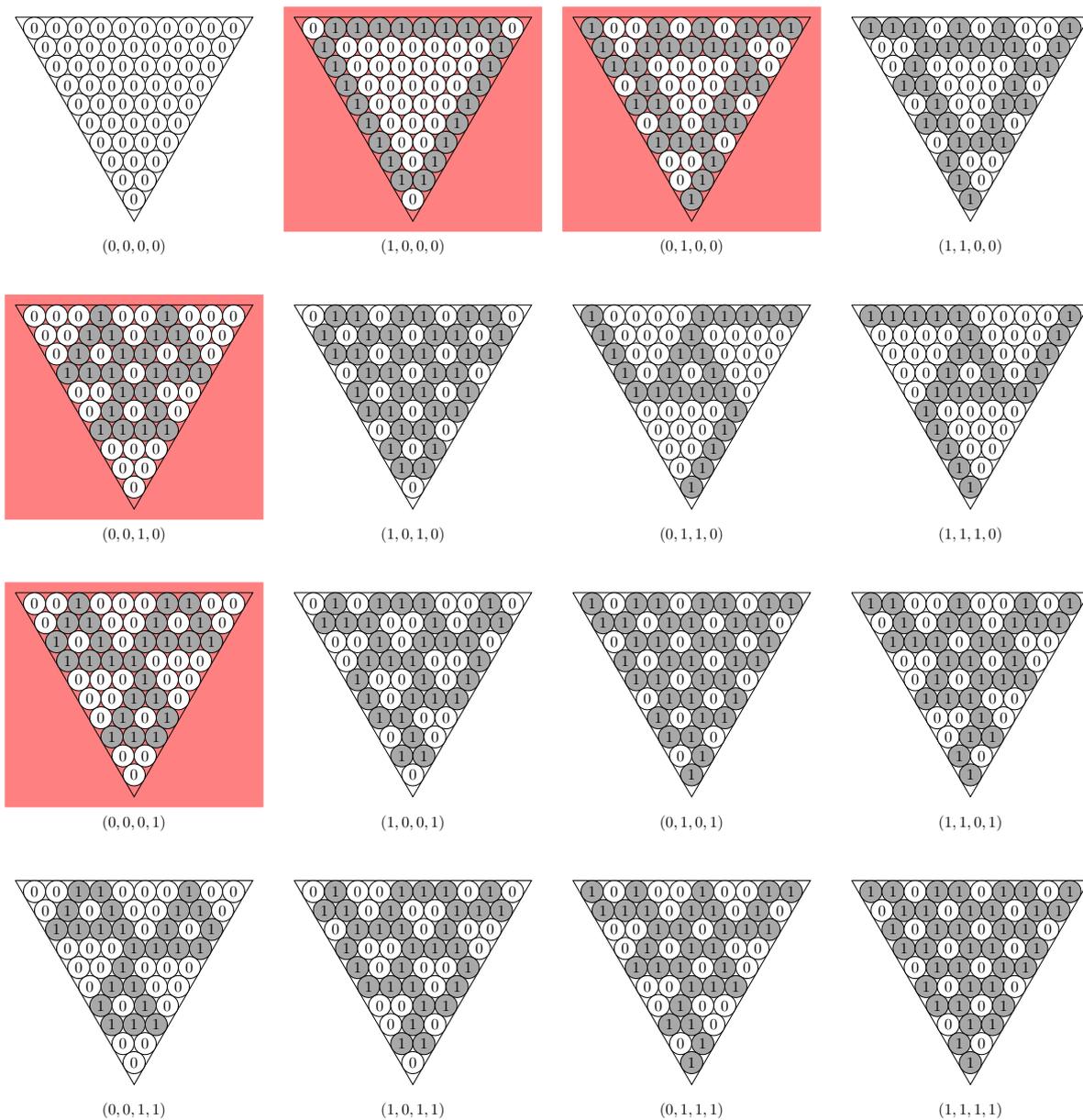


Figure 2.7: The 16 triangles of  $\mathcal{RST}(10)$  where the 4 red triangles form a basis

## 2.4 Horizontally symmetric Steinhaus triangles

In this section, we characterize the horizontally symmetric Steinhaus triangles and we give a generating index set of  $\mathcal{HST}(n)$ . This permits us to obtain bases of  $\mathcal{HST}(n)$ , for all non-negative integers  $n$ .

### 2.4.1 Characterizations of $\mathcal{HST}(n)$

**Definition 2.4.1** (Symmetric sequence). A binary sequence  $S = (a_j)_{1 \leq j \leq n}$  is said to be *symmetric* if  $a_{n-j+1} = a_j$ , for all  $j \in \{1, \dots, n\}$ .

For instance, the sequence (010010010) is symmetric. As shown in the following result, the symmetry is preserved under the derivation process.

**Proposition 2.4.2** (Chappelon [C8]). *The binary sequence  $S$  is symmetric if and only if  $\partial S$  is symmetric and  $\sigma_2(\partial S) = 0$ .*

It follows that the horizontal symmetry of a Steinhaus triangle is only related to the symmetry of its first row.

**Proposition 2.4.3** (Chappelon [C8]). *The Steinhaus triangle  $\nabla S$  is horizontally symmetric if and only if the sequence  $S$  is symmetric.*

Now, we show that the horizontal symmetry of a Steinhaus triangle only depends on the values of middle terms of its rows of odd lengths.

**Proposition 2.4.4** (Chappelon [C8]). *The Steinhaus triangle  $(a_{i,j})_{1 \leq i \leq j \leq n}$ , of size  $n$ , is horizontally symmetric if and only if  $a_{n-2i, n-i} = 0$ , for all  $i \in \{0, \dots, \lfloor \frac{n}{2} \rfloor - 1\}$ .*

### 2.4.2 Generating index set of $\mathcal{HST}(n)$

**Proposition 2.4.5** (Chappelon [C8]). *Let  $n$  be a non-negative integer. The set*

$$G_H := \left\{ (1, j) \mid j \in \left\{ 1, \dots, \left\lceil \frac{n}{2} \right\rceil \right\} \right\}$$

*is a generating index set of  $\mathcal{HST}(n)$ .*

Since the dimension of  $\mathcal{HST}(n)$  corresponds to the cardinality of the generating index set  $G_H$ , it is straightforward to obtain the following

**Corollary 2.4.6.**  $\dim \mathcal{HST}(n) = \left\lceil \frac{n}{2} \right\rceil$ , for all non-negative integers  $n$ .

### 2.4.3 Bases of $\mathcal{HST}(n)$

**Notation 2.4.7.** Let  $n$  be a positive integer. For any positive integer  $k \in \{1, \dots, n\}$ , we denote by  $E_k^{(n)}$  the binary sequence of length  $n$  consisting only of zeroes, except at position  $k$ . In other words, we have

$$E_k^{(n)} = \left( \binom{0}{j-k} \right)_{1 \leq j \leq n},$$

for all  $k \in \{1, \dots, n\}$ .

Since we retrieve the local rule (LR) of the infinite Pascal matrix modulo 2, we obtain that

$$\nabla E_k^{(n)} = \left( \binom{i-1}{j-k}_2 \right)_{1 \leq i \leq j \leq n},$$

for all integers  $k \in \{1, \dots, n\}$ .

**Proposition 2.4.8** (Chappelton [C8]). *Let  $n$  be a positive integer. The set  $\{\nabla_1, \dots, \nabla_{\lfloor \frac{n}{2} \rfloor}\}$  is a basis of  $\mathcal{HST}(n)$ , where*

$$\nabla_k = \nabla \left( E_k^{(n)} + E_{n-k+1}^{(n)} \right) = \left( \binom{i-1}{j-k} + \binom{i-1}{j-n+k-1} \pmod{2} \right)_{1 \leq i \leq j \leq n},$$

for all  $k \in \{1, \dots, \lfloor \frac{n}{2} \rfloor\}$ , and

$$\nabla_{\frac{n+1}{2}} = \nabla E_{\frac{n+1}{2}}^{(n)} = \left( \binom{i-1}{j-\frac{n+1}{2}}_2 \right)_{1 \leq i \leq j \leq n},$$

when  $n$  is odd.

Now, we show that the generating index set  $G_H$  also permits us to obtain a basis from the sequences  $S_{k,l}^{(n)} = \left( \binom{l+j-1}{k}_2 \right)_{1 \leq j \leq n}$  introduced before.

**Theorem 2.4.9** (Chappelton [C8]). *Let  $n$  be a positive integer. The set  $\{\nabla_1, \dots, \nabla_{\lfloor \frac{n}{2} \rfloor}\}$  is a basis of  $\mathcal{HST}(n)$ , where*

$$\nabla_k = \nabla S_{n-2k, -k}^{(n)} = \left( \binom{-k+j-i}{n-2k+1-i}_2 \right)_{1 \leq i \leq j \leq n},$$

for all  $k \in \{1, \dots, \lfloor \frac{n}{2} \rfloor\}$ , and  $\nabla_{\frac{n+1}{2}} = \nabla(1)_n$ , when  $n$  is odd.

*Remark 2.4.10.* When  $n$  is even, we have  $\nabla_{\frac{n}{2}} = \nabla S_{0, -\frac{n}{2}}^{(n)} = \nabla(1)_n$ . Therefore,  $\nabla_{\lfloor \frac{n}{2} \rfloor} = \nabla(1)_n$ , for all integers  $n$ .

For instance, for  $n = 7$ , we obtain

$$\begin{aligned} \nabla_1 &= \nabla S_{5, -1}^{(7)} = \nabla(1000001), & \nabla_2 &= \nabla S_{3, -2}^{(7)} = \nabla(0100010), \\ \nabla_3 &= \nabla S_{1, -3}^{(7)} = \nabla(1010101), & \nabla_4 &= \nabla(1111111). \end{aligned}$$

All the horizontally symmetric Steinhaus triangles of size 7 are depicted in Figure 2.8, where the elements of the basis  $\{\nabla_1, \nabla_2, \nabla_3, \nabla_4\}$  are in red and, for every  $\nabla \in \mathcal{HST}(7)$ , the coordinate vector  $(x_1, x_2, x_3, x_4)$  of  $\nabla = x_1 \nabla_1 + x_2 \nabla_2 + x_3 \nabla_3 + x_4 \nabla_4$  is given.

## 2.5 Dihedrally symmetric Steinhaus triangles

In this section, after characterizing dihedrally symmetric Steinhaus triangles, we determine, for all non-negative integers  $n$ , generating index sets and a basis of  $\mathcal{DST}(n)$ .

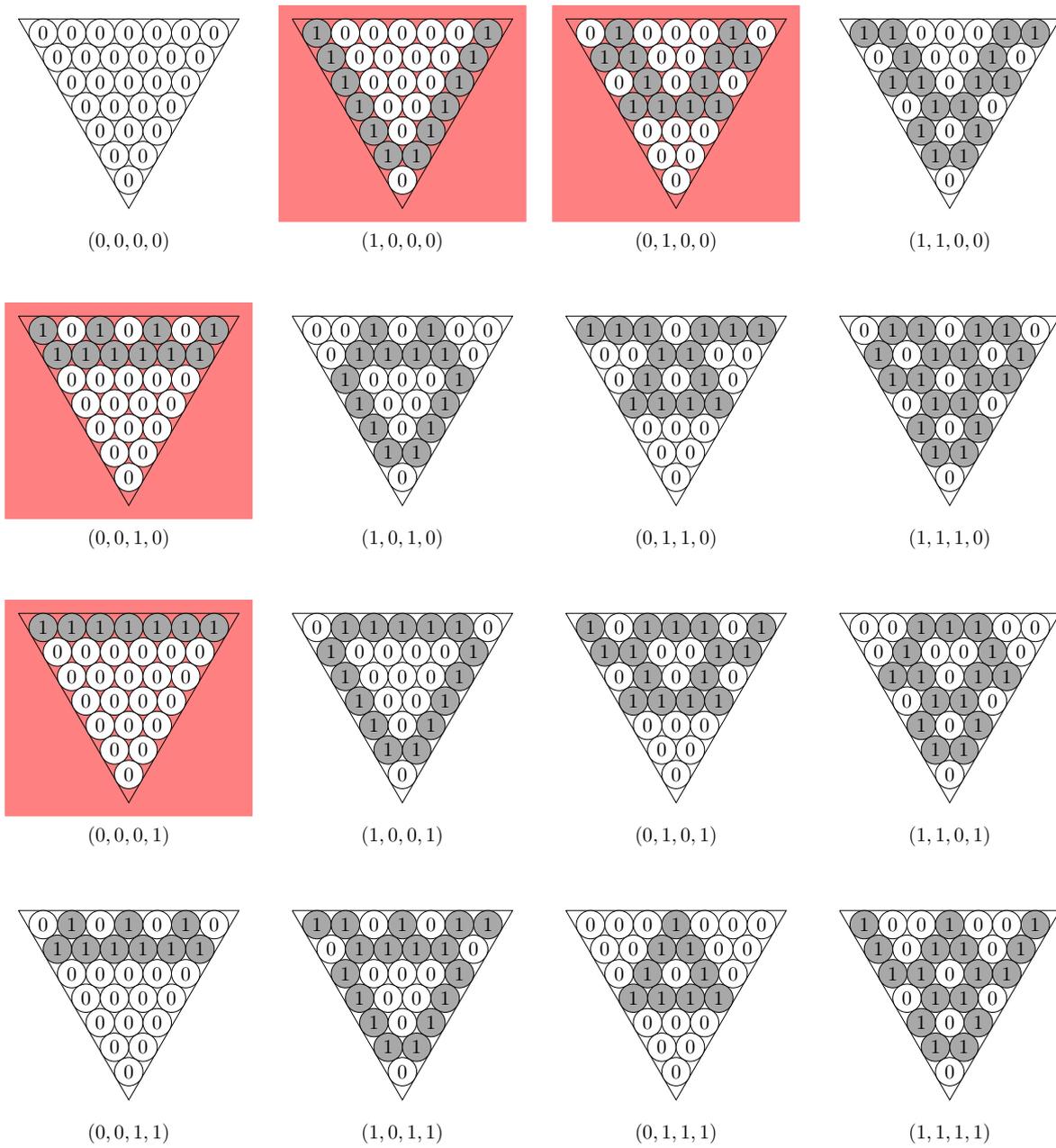


Figure 2.8: The 16 triangles of  $\mathcal{HST}(7)$  where the 4 red triangles form a basis

### 2.5.1 Characterizations of $\mathcal{DST}(n)$

We begin by showing that the dihedral symmetry of a Steinhaus triangle is only related to the symmetry of its first row and of its right and left sides.

**Proposition 2.5.1.** *The Steinhaus triangle  $\nabla$  is dihedrally symmetric if and only if two of the three Steinhaus triangles  $\nabla$ ,  $r(\nabla)$  and  $r^2(\nabla)$  are horizontally symmetric.*

**Corollary 2.5.2.** *The Steinhaus triangle  $\nabla S = (a_{i,j})_{1 \leq i \leq j \leq n}$  is dihedrally symmetric if and only if two of the three sequences, its first row  $(a_{1,j})_{1 \leq j \leq n}$ , its right side  $(a_{j,n})_{1 \leq j \leq n}$  or its left side  $(a_{j,j})_{1 \leq j \leq n}$ , are symmetric.*

Proposition 2.5.1 also permits us to show that the dihedral symmetry of a Steinhaus triangle only depends on the values of middle terms of its rows, its columns or its diagonals of odd lengths.

**Corollary 2.5.3.** *The Steinhaus triangle  $\nabla S = (a_{i,j})_{1 \leq i \leq j \leq n}$  is dihedrally symmetric if and only if two of the sets  $\{a_{n-2i,n-i} \mid i \in \{0, \dots, \lfloor \frac{n}{2} \rfloor - 1\}\}$ ,  $\{a_{i,2i-1} \mid i \in \{1, \dots, \lfloor \frac{n}{2} \rfloor\}\}$  and  $\{a_{i,n-i+1} \mid i \in \{1, \dots, \lfloor \frac{n}{2} \rfloor\}\}$  are sets of zeroes.*

For any positive integer  $n \geq 3$ , by definition of  $\mathcal{DST}(n)$  and  $H$ , it is clear that  $H(\mathcal{DST}(n)) \subset \mathcal{DST}(n-3)$ . The precise relationship between a dihedrally symmetric Steinhaus triangle  $\nabla S$  and its subtriangle  $H(\nabla S)$  is given in the following

**Proposition 2.5.4.** *Let  $S$  be a binary sequence of length  $n \geq 3$ . The Steinhaus triangle  $\nabla S$  is dihedrally symmetric if and only if  $H(\nabla S) = \nabla S'$  is dihedrally symmetric,  $\sigma_2(S') = 0$  and  $S = (0) \cdot \int_{i,x} S' \cdot (0)$ , for some  $i \in \{1, \dots, n-2\}$  and some  $x \in \{0, 1\}$ .*

Proposition 2.5.4 appears in [27] in a more general context.

**Notation 2.5.5.** For any non-negative integer  $n$ , the set of dihedrally symmetric Steinhaus triangles  $\nabla S$  of size  $n$  with  $\sigma(S)$  even is denoted by  $\mathcal{DST}_0(n)$ .

It is clear that  $\mathcal{DST}_0(n)$  is a linear subspace of  $\mathcal{DST}(n)$ . Moreover, the vector space  $\mathcal{DST}(n)$  can be expressed in function of its linear subspace  $\mathcal{DST}_0(n)$ .

**Proposition 2.5.6** (Chappelon [C8]). *Let  $n$  be a non-negative integer. Then, we have*

$$\mathcal{DST}(n) = \begin{cases} \mathcal{DST}_0(n) & \text{for } n \text{ even,} \\ \mathcal{DST}_0(n) \sqcup (\mathcal{DST}_0(n) + U_n) & \text{for } n \text{ odd,} \end{cases}$$

where  $\sqcup$  is the disjoint union of two sets.

**Corollary 2.5.7** (Chappelon [C8]). *Let  $S$  be a binary sequence of length  $n \geq 3$ . For  $n$  even, the Steinhaus triangle  $\nabla S$  is in  $\mathcal{DST}_0(n)$  if and only if  $H(\nabla S) = \nabla S'$  is in  $\mathcal{DST}_0(n-3)$  and  $S = (0) \cdot \int_{i,x} S' \cdot (0)$ , for some  $i \in \{1, \dots, n-2\}$  and some  $x \in \{0, 1\}$ . For  $n$  odd, the Steinhaus triangle  $\nabla S$  is in  $\mathcal{DST}_0(n)$  if and only if  $H(\nabla S) = \nabla S'$  is in  $\mathcal{DST}_0(n-3)$  and  $S = (0) \cdot \int_{\frac{n-1}{2},0} S' \cdot (0)$ .*

### 2.5.2 Generating index sets of $\mathcal{DST}(n)$

We begin this subsection by giving generating index sets of  $\mathcal{DST}_0(n)$ , for all non-negative integers  $n$ . This comes from Corollary 2.5.7.

**Theorem 2.5.8** (Chappelton [C8]). *Let  $n$  and  $m$  be non-negative integers such that  $m = \lfloor \frac{n}{6} \rfloor + \delta_{4,(n \bmod 6)}$ . For every integer  $i \in \{1, \dots, \lfloor \frac{n}{3} \rfloor\}$ , let  $j_i \in \{2i, \dots, n - i\}$ . Then, the set*

$$G_{D_0} = \{(2i + 1, j_{2i+1}) \mid i \in \{0, \dots, m - 1\}\},$$

when  $n$  is even, or

$$G_{D_0} = \{(2i, j_{2i}) \mid i \in \{1, \dots, m\}\},$$

when  $n$  is odd, is a generating index set of  $\mathcal{DST}_0(n)$ .

Since the dimension of  $\mathcal{DST}_0(n)$  corresponds to the cardinality of the generating index set  $G_{D_0}$ , it is easy to obtain the following

**Corollary 2.5.9.**  $\dim \mathcal{DST}_0(n) = \lfloor \frac{n}{6} \rfloor + \delta_{4,(n \bmod 6)}$ , for all non-negative integers  $n$ .

Using Proposition 2.5.6 and Theorem 2.5.8, we are now ready to give a generating index set of  $\mathcal{DST}(n)$ , for all non-negative integers  $n$ .

**Theorem 2.5.10** (Chappelton [C8]). *Let  $n$  and  $m$  be non-negative integers such that  $m = \lfloor \frac{n+3}{6} \rfloor + \delta_{1,(n \bmod 6)}$ . For every integer  $i \in \{1, \dots, \lfloor \frac{n}{3} \rfloor\}$ , let  $j_i \in \{2i, \dots, n - i\}$ . Then, the set*

$$G_D = \{(2i + 1, j_{2i+1}) \mid i \in \{0, \dots, m - 1\}\},$$

when  $n$  is even, or

$$G_D = \{(1, j_1)\} \cup \{(2i, j_{2i}) \mid i \in \{1, \dots, m - 1\}\},$$

when  $n$  is odd, is a generating index set of  $\mathcal{DST}(n)$ .

**Corollary 2.5.11** (Chappelton [C8]). *Let  $n$  and  $m$  be non-negative integers such that  $m = \lfloor \frac{n+3}{6} \rfloor + \delta_{1,(n \bmod 6)}$ . The set*

$$G_D = \left\{ \left( 2i + 1, n - \left\lfloor \frac{n}{3} \right\rfloor \right) \mid i \in \{0, \dots, m - 1\} \right\},$$

when  $n$  is even, or

$$G_D = \left\{ \left( 1, n - \left\lfloor \frac{n}{3} \right\rfloor \right) \right\} \cup \left\{ \left( 2i, n - \left\lfloor \frac{n}{3} \right\rfloor \right) \mid i \in \{1, \dots, m - 1\} \right\},$$

when  $n$  is odd, is a generating index set of  $\mathcal{DST}(n)$ .

Since the dimension of  $\mathcal{DST}(n)$  corresponds to the cardinality of the generating index set  $G_D$ , it is straightforward to obtain the following

**Corollary 2.5.12.**  $\dim \mathcal{DST}(n) = \lfloor \frac{n+3}{6} \rfloor + \delta_{1,(n \bmod 6)}$ , for all non-negative integers  $n$ .

### 2.5.3 Basis of $\mathcal{DST}(n)$

First, using the operators  $H^k$  and the generating index sets  $G_D$  introduced before, we obtain a family of bases of  $\mathcal{DST}(n)$ , for all non-negative integers  $n$ .

**Theorem 2.5.13** (Chappelton [C8]). *Let  $n$  and  $m$  be non-negative integers such that  $m = \lfloor \frac{n+3}{6} \rfloor + \delta_{1,(n \bmod 6)}$ . For every  $k \in \{0, \dots, \lfloor \frac{n}{3} \rfloor - 1\}$ , let  $\nabla_k \in \mathcal{DST}(n)$  such that  $H^k(\nabla_k) = U_{n-3k}$ . Then, the set*

$$\{\nabla_{2k} \mid k \in \{0, \dots, m-1\}\},$$

when  $n$  is even, or

$$\{\nabla_0\} \cup \{\nabla_{2k+1} \mid k \in \{0, \dots, m-2\}\},$$

when  $n$  is odd, is a basis of  $\mathcal{DST}(n)$ .

*Remark 2.5.14.*  $\nabla_0 = U_n$  in the previous result.

Now, we consider the restriction of the linear map  $\rho$  on the linear subspace  $\mathcal{HST}(n)$ , i.e., the linear map  $\rho|_{\mathcal{HST}(n)} : \mathcal{HST}(n) \rightarrow \mathcal{DST}(n)$  defined by  $\rho = r^2 + r + id_n$ , for all non-negative integers  $n$ . Obviously, this map is surjective since  $\rho(\nabla) = \nabla$ , for all  $\nabla \in \mathcal{DST}(n)$ . Since  $U_{n-3k} = \rho((1)_{n-3k})$  by definition, for all non-negative integers  $n$  and  $k$  such that  $3k \leq n$ , this leads to the following

**Corollary 2.5.15** (Chappelton [C8]). *Let  $n$  and  $m$  be non-negative integers such that  $m = \lfloor \frac{n+3}{6} \rfloor + \delta_{1,(n \bmod 6)}$ . For every  $k \in \{0, \dots, \lfloor \frac{n}{3} \rfloor - 1\}$ , let  $S_k$  be a symmetric binary sequence of length  $n$  such that  $\partial^k S_k = (1)_{n-k}$ . Then, the set*

$$\{\rho(\nabla S_{2k}) \mid k \in \{0, \dots, m-1\}\},$$

when  $n$  is even, or

$$\{\rho(\nabla S_0)\} \cup \{\rho(\nabla S_{2k+1}) \mid k \in \{0, \dots, m-2\}\},$$

when  $n$  is odd, is a basis of  $\mathcal{DST}(n)$ .

*Remark 2.5.16.*  $\rho(\nabla S_0) = \rho(\nabla(1)_n) = U_n$  in the previous result.

We end this section by giving an explicit basis of  $\mathcal{DST}(n)$  in terms of the  $n$ -length binary sequences

$$S_{k,l}^{(n)} = \left( \binom{l+j-1}{k} \right)_2 \Big|_{1 \leq j \leq n},$$

for all integers  $k$  and  $l$ .

**Theorem 2.5.17** (Chappelton [C8]). *Let  $n$  and  $m$  be non-negative integers such that  $m = \lfloor \frac{n+3}{6} \rfloor + \delta_{1,(n \bmod 6)}$ . For every  $k \in \{0, \dots, \lfloor \frac{n}{3} \rfloor - 1\}$  of same parity as  $n$ , let*

$$\nabla_k = \rho \left( \nabla S_{k, \frac{k-n}{2}}^{(n)} \right) = \left( \binom{\frac{k-n}{2} + j - i}{k+1-i} + \binom{\frac{k+n}{2} - j}{k+i-j} + \binom{\frac{k-n}{2} + i - 1}{k+j-n} \bmod 2 \right) \Big|_{1 \leq i \leq j \leq n}.$$

Then, the set

$$\{\nabla_{2k} \mid k \in \{0, \dots, m-1\}\},$$

when  $n$  is even, or

$$\{U_n\} \cup \{\nabla_{2k+1} \mid k \in \{0, \dots, m-2\}\},$$

when  $n$  is odd, is a basis of  $\mathcal{DST}(n)$ .

**Corollary 2.5.18** (Chappelton [C8]). *Let  $n$  and  $m$  be non-negative integers such that  $m = \lfloor \frac{n}{6} \rfloor + \delta_{4,(n \bmod 6)}$ . For every  $k \in \{0, \dots, \lfloor \frac{n}{3} \rfloor - 1\}$  of same parity as  $n$ , let*

$$\nabla_k = \rho \left( \nabla S_{k, \frac{k-n}{2}}^{(n)} \right) = \left( \binom{\frac{k-n}{2} + j - i}{k + 1 - i} + \binom{\frac{k+n}{2} - j}{k + i - j} + \binom{\frac{k-n}{2} + i - 1}{k + j - n} \bmod 2 \right)_{1 \leq i \leq j \leq n}.$$

Then, the set

$$\{\nabla_{2k} \mid k \in \{0, \dots, m - 1\}\},$$

when  $n$  is even, or

$$\{\nabla_{2k+1} \mid k \in \{0, \dots, m - 1\}\},$$

when  $n$  is odd, is a basis of  $\mathcal{DST}_0(n)$ .

For instance, for  $n = 22$ , we obtain

$k$	$S_{2k, k-11}^{(22)}$	$\rho \left( \nabla S_{2k, k-11}^{(22)} \right)$
0	(11111111111111111111)	$\nabla_0 = \nabla(011111111111111111110)$
1	(1100110011001100110011)	$\nabla_2 = \nabla(0110110011001100110110)$
2	(1000011110000111100001)	$\nabla_4 = \nabla(0111111110000111111110)$
3	(0000001100000011000000)	$\nabla_6 = \nabla(000000100000010000000)$

All the dihedrally symmetric Steinhaus triangles of size 22 are depicted in Figure 2.9, where the elements of the basis  $\{\nabla_0, \nabla_2, \nabla_4, \nabla_6\}$  are in red and, for every  $\nabla \in \mathcal{DST}(22)$ , the coordinate vector  $(x_0, x_2, x_4, x_6)$  of  $\nabla = x_0\nabla_0 + x_2\nabla_2 + x_4\nabla_4 + x_6\nabla_6$  is given.

## 2.6 Symmetric generalized Pascal triangles

In this section, we consider binary generalized Pascal triangles. The set of binary generalized Pascal triangles of size  $n$  is denoted by  $\mathcal{PT}(n)$ . Since the set of generalized Pascal triangles is closed under addition modulo 2, it follows that  $\mathcal{PT}(n)$  is a vector space over  $\mathbb{Z}/2\mathbb{Z}$ . Since a generalized Pascal triangle is uniquely determined by its left and right sides, which have the same first term, the dimension of  $\mathcal{PT}(n)$  is  $2n - 1$ , for all positive integers  $n$ . Moreover, there exists a natural isomorphism between  $\mathcal{PT}(n)$  and  $\mathcal{ST}(2n - 1)$ , for all positive integers  $n$ . Indeed, as depicted in Figure 2.10, a generalized Pascal triangle of size  $n$  can be seen as a subtriangle of a Steinhaus triangle of size  $2n - 1$ .

**Definition 2.6.1** (Isomorphism  $\gamma$ ). For any positive integer  $n$ , let  $\gamma$  be the linear map defined by

$$\begin{aligned} \gamma : \quad \mathcal{ST}(2n - 1) &\longrightarrow \mathcal{PT}(n) \\ (a_{i,j})_{1 \leq i \leq j \leq 2n-1} &\longmapsto (a_{i, n-1+j})_{1 \leq j \leq i \leq n} \end{aligned}$$

The linear map  $\gamma$  is well defined since the generalized Pascal triangles and the Steinhaus triangles share the same local rule (LR).

**Proposition 2.6.2.** *The linear map  $\gamma : \mathcal{ST}(2n - 1) \longrightarrow \mathcal{PT}(n)$  is an isomorphism.*

As for Steinhaus triangles, the action of the dihedral group  $D_3 = \langle r', h' \rangle$  on  $\mathcal{PT}(n)$  can be considered, where the automorphisms  $r'$  and  $h'$  of  $\mathcal{PT}(n)$  are defined by

$$\begin{aligned} r' : \quad \mathcal{PT}(n) &\longrightarrow \mathcal{PT}(n) \\ (a_{i,j})_{1 \leq j \leq i \leq n} &\longmapsto (a_{n+j-i, n+1-i})_{1 \leq j \leq i \leq n} \end{aligned}$$

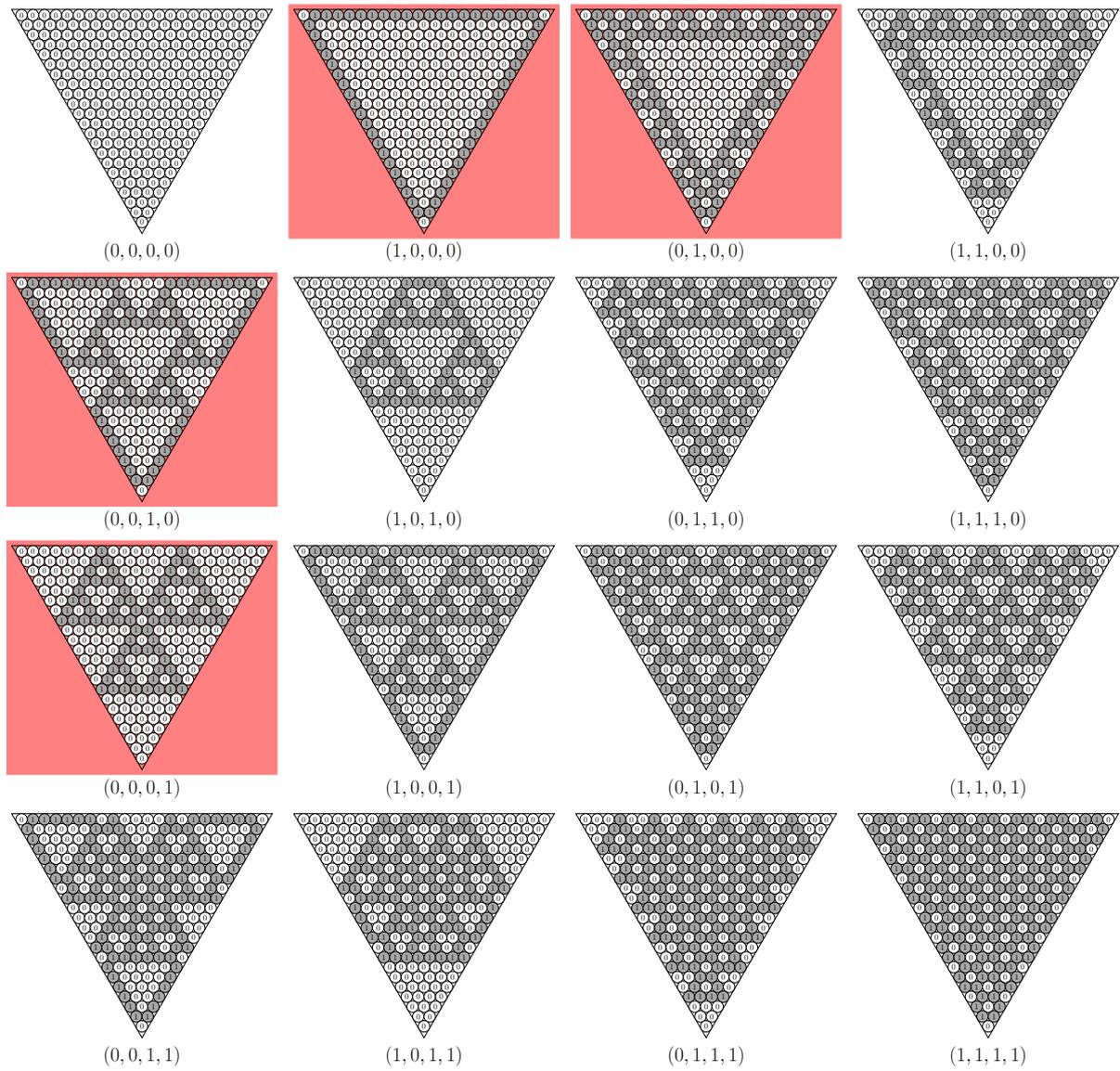


Figure 2.9: The 16 triangles of  $DST(22)$  where the 4 red triangles form a basis

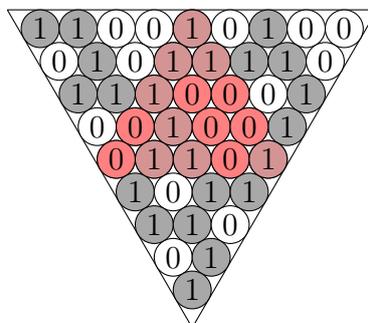


Figure 2.10:  $\gamma(\nabla(110010100)) = \Delta((11100), (11001))$

and

$$\begin{aligned}
 h' : \mathcal{PT}(n) &\longrightarrow \mathcal{PT}(n) \\
 (a_{i,j})_{1 \leq j \leq i \leq n} &\longmapsto (a_{i,1-j+i})_{1 \leq j \leq i \leq n}
 \end{aligned}$$

For instance, for  $L = (11100)$  and  $R = (11001)$  and for all  $g \in D_3$ , the generalized Pascal triangles  $g(\Delta(L, R))$  are depicted in Figure 2.11.

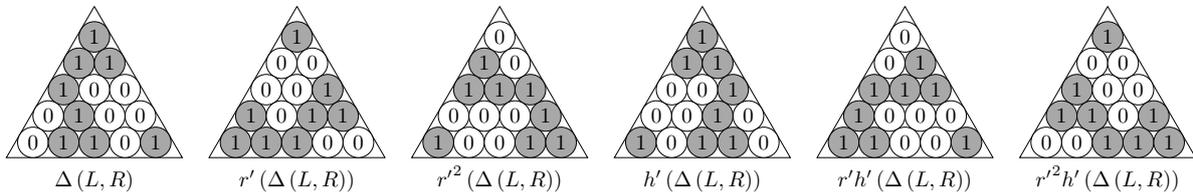


Figure 2.11: Action of  $D_3$  on  $\Delta((11100), (11001))$

**Proposition 2.6.3.** *For any positive integer  $n$ , we have*

$$\gamma r = r' \gamma \quad \text{and} \quad \gamma h = h' \gamma.$$

**Definition 2.6.4** (Symmetric binary generalized Pascal triangles). A generalized Pascal triangle  $\Delta$  of size  $n$  is said to be

- *rotationally symmetric* if  $r'(\Delta) = \Delta$ ,
- *horizontally symmetric* if  $h'(\Delta) = \Delta$ ,
- *dihedrally symmetric* if  $r'(\Delta) = h'(\Delta) = \Delta$ .

The sets of horizontally symmetric, rotationally symmetric and dihedrally symmetric generalized Pascal triangles of size  $n$  are denoted by  $\mathcal{HP}\mathcal{T}(n)$ ,  $\mathcal{RP}\mathcal{T}(n)$  and  $\mathcal{DP}\mathcal{T}(n)$ , respectively, for all non-negative integers  $n$ . In other words, the sets  $\mathcal{HP}\mathcal{T}(n)$ ,  $\mathcal{RP}\mathcal{T}(n)$  and  $\mathcal{DP}\mathcal{T}(n)$  are simply the linear subspaces  $\ker(h' - id_{\mathcal{PT}(n)})$ ,  $\ker(r' - id_{\mathcal{PT}(n)})$  and  $\ker(h' - id_{\mathcal{PT}(n)}) \cap \ker(r' - id_{\mathcal{PT}(n)})$ , respectively, where  $id_{\mathcal{PT}(n)}$  is the identity map on  $\mathcal{PT}(n)$ , for all non-negative integers  $n$ .

Examples of symmetric binary generalized Pascal triangles appear in Figure 2.12.

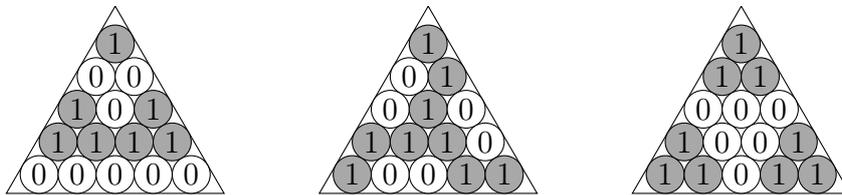


Figure 2.12: Triangles of  $\mathcal{HP}\mathcal{T}(5)$ ,  $\mathcal{RP}\mathcal{T}(5)$  and  $\mathcal{DP}\mathcal{T}(5)$ .

It is now easy to see that a symmetric generalized Pascal triangles of size  $n$  corresponds to a symmetric Steinhaus triangle of size  $2n - 1$ , for all positive integers  $n$ .

**Proposition 2.6.5** (Chappelon [C8]). *For any positive integer  $n$ , a Steinhaus triangle  $\nabla$ , of size  $2n - 1$ , is horizontally, rotationally, or dihedrally symmetric if and only if the generalized Pascal triangle  $\gamma(\nabla)$ , of size  $n$ , is horizontally, rotationally, or dihedrally symmetric, respectively.*

**Corollary 2.6.6** (Chappelton [C8]). *The linear map  $\gamma$  induces isomorphisms of  $\mathcal{HST}(2n-1)$  upon  $\mathcal{HPT}(n)$ ,  $\mathcal{RST}(2n-1)$  upon  $\mathcal{RPT}(n)$  and  $\mathcal{DST}(2n-1)$  upon  $\mathcal{DPT}(n)$ , respectively, for all positive integers  $n$ .*

Using the isomorphism  $\gamma$  and the results of the previous sections, we obtain the dimension and a basis for each linear subspace of symmetric generalized Pascal triangles of size  $n$ , for all positive integers  $n$ .

**Proposition 2.6.7.** *For any positive integer  $n$ , we have*

- $\dim \mathcal{HPT}(n) = n$ ,
- $\dim \mathcal{RPT}(n) = 2 \lfloor \frac{n-1}{3} \rfloor + 1$ ,
- $\dim \mathcal{DPT}(n) = \lceil \frac{n}{3} \rceil$ .

**Theorem 2.6.8** (Chappelton [C8]). *Let  $n$  and  $m$  be positive integers such that  $m = 2 \lfloor \frac{n-1}{3} \rfloor + 1$ . For any integers  $l_0, \dots, l_{m-1}$ , the set*

$$\left\{ \gamma \rho \left( \nabla S_{k, l_k}^{(2n-1)} \right) \mid k \in \{0, \dots, m-1\} \right\}$$

*is a basis of  $\mathcal{RPT}(n)$ , where*

$$\gamma \rho \left( \nabla S_{k, l_k}^{(2n-1)} \right) = \left( \binom{l_k + j - i + n - 1}{k + 1 - i} + \binom{l_k + n - j}{k + i - j - n + 1} + \binom{l_k + i - 1}{k + j - n} \bmod 2 \right)_{1 \leq j \leq i \leq n},$$

*for all  $k \in \{0, \dots, m-1\}$ .*

**Theorem 2.6.9** (Chappelton [C8]). *Let  $n$  be a positive integer. The set*

$$\left\{ \gamma \left( S_{2(n-k)-1, -k}^{(2n-1)} \right) \mid k \in \{1, \dots, n-1\} \right\} \cup \left\{ \gamma \left( \nabla(1)_{2n-1} \right) \right\}$$

*is a basis of  $\mathcal{HPT}(n)$ , where*

$$\gamma \left( \nabla S_{2(n-k)-1, -k}^{(2n-1)} \right) = \left( \binom{-k + j - i + n - 1}{2(n-k) - i} \right)_{1 \leq j \leq i \leq n},$$

*for all  $k \in \{1, \dots, \lfloor \frac{n}{2} \rfloor\}$ , and  $\gamma \left( \nabla(1)_{2n-1} \right) = \Delta \left( (1) \cdot (0)_{n-1}, (1) \cdot (0)_{n-1} \right)$ .*

**Theorem 2.6.10** (Chappelton [C8]). *Let  $n$  and  $m$  be positive integers such that  $m = \lceil \frac{n}{3} \rceil$ . Then, the set*

$$\left\{ \gamma \left( U_{2n-1} \right) \right\} \cup \left\{ \gamma \rho \left( S_{2k+1, k-n+1}^{(2n-1)} \right) \mid k \in \{0, \dots, m-2\} \right\}$$

*is a basis of  $\mathcal{DST}(n)$ , where  $\gamma \left( U_{2n-1} \right) = \Delta \left( (1) \cdot (0)_{n-2} \cdot (1), (1) \cdot (0)_{n-2} \cdot (1) \right)$  and*

$$\gamma \rho \left( S_{2k+1, k-n+1}^{(2n-1)} \right) = \left( \binom{k + j - i}{2k - i + 2} + \binom{k - j + 1}{i - j + 2k - n + 2} + \binom{k - n + i}{2k + j - n + 1} \bmod 2 \right)_{1 \leq j \leq i \leq n},$$

*for all  $k \in \{0, \dots, m-2\}$ .*

We end this section by giving bases obtained from Theorem 2.6.8 for  $\mathcal{RPT}(6)$ , from Theorem 2.6.9 for  $\mathcal{HPT}(4)$  and from Theorem 2.6.10 for  $\mathcal{DPT}(11)$ .

For  $n = 7$  and  $k_0 = k_1 = k_2 = k_3 = k_4 = 0$ , we obtain the following basis

$$\left\{ \gamma\rho \left( \nabla S_{k,0}^{(13)} \right) \mid k \in \{0, 1, 2, 3, 4\} \right\}$$

of  $\mathcal{RPT}(7)$ , where

$k$	$S_{k,0}^{(13)}$	$\rho \left( \nabla S_{k,0}^{(13)} \right)$	$\gamma\rho \left( \nabla S_{k,0}^{(13)} \right)$
0	(1111111111111)	$\nabla(0111111111110)$	$\Delta_0 = \Delta((1000001), (1000001))$
1	(0101010101010)	$\nabla(0001010101000)$	$\Delta_1 = \Delta((0100010), (0100010))$
2	(0011001100110)	$\nabla(0101001100010)$	$\Delta_2 = \Delta((1110101), (1010111))$
3	(0001000100010)	$\nabla(0100000101010)$	$\Delta_3 = \Delta((0000010), (0100000))$
4	(0000111100001)	$\nabla(1111011110001)$	$\Delta_4 = \Delta((1011001), (1001101))$

All the rotationnaly symmetric generalized Pascal triangles of size 7 are depicted in Figure 2.13, where the elements of the basis  $\{\Delta_0, \Delta_1, \Delta_2, \Delta_3, \Delta_4\}$  are in red and, for every  $\Delta \in \mathcal{RPT}(7)$ , the coordinate vector  $(x_0, x_1, x_2, x_3, x_4)$  of  $\Delta = x_0\Delta_0 + x_1\Delta_1 + x_2\Delta_2 + x_3\Delta_3 + x_4\Delta_4$  is given.

For  $n = 4$ , we obtain the following basis

$$\left\{ \gamma \left( S_{7-2k,-k}^{(7)} \right) \mid k \in \{1, 2, 3\} \right\} \cup \left\{ \gamma(\nabla(1)_7) \right\}$$

of  $\mathcal{HPT}(4)$ , where

$k$	$S_{7-2k,-k}^{(7)}$	$\gamma\rho \left( \nabla S_{7-2k,-k}^{(7)} \right)$
1	(1000001)	$\Delta_1 = \Delta((0001), (0001))$
2	(0100010)	$\Delta_2 = \Delta((0011), (0011))$
3	(1010101)	$\Delta_3 = \Delta((0100), (0100))$

and  $\Delta_4 = \gamma(\nabla(1)_7) = \Delta((1000), (1000))$ . All the horizontally symmetric generalized Pascal triangles of size 4 are depicted in Figure 2.14, where the elements of the basis  $\{\Delta_1, \Delta_2, \Delta_3, \Delta_4\}$  are in red and, for every  $T \in \mathcal{HPT}(4)$ , the coordinate vector  $(x_1, x_2, x_3, x_4)$  of  $\Delta = x_1\Delta_1 + x_2\Delta_2 + x_3\Delta_3 + x_4\Delta_4$  is given.

For  $n = 11$ , we obtain the following basis

$$\left\{ \gamma(U_{21}) \right\} \cup \left\{ \gamma\rho \left( \nabla S_{2k+1,k-10}^{(21)} \right) \mid k \in \{0, 1, 2\} \right\}$$

of  $\mathcal{DPT}(11)$ , where  $\Delta_0 = \gamma(U_{21}) = \Delta((10000000001), (10000000001))$  and

$k$	$S_{2k+1,k-10}^{(21)}$	$\rho \left( \nabla S_{2k+1,k-10}^{(21)} \right)$	$\gamma\rho \left( \nabla S_{2k+1,k-10}^{(21)} \right)$
0	(0101010101010101010)	$\nabla(0001010101010101000)$	$\Delta_1 = \Delta((01000000010), (01000000010))$
1	(100010001000100010001)	$\nabla(011110001000100011110)$	$\Delta_2 = \Delta((00110001100), (00110001100))$
2	(000001010000010100000)	$\nabla(000000010000010000000)$	$\Delta_3 = \Delta((00010001000), (00010001000))$

All the dihedrally symmetric generalized Pascal triangles of size 11 are depicted in Figure 2.15, where the elements of the basis  $\{\Delta_0, \Delta_1, \Delta_2, \Delta_3\}$  are in red and, for every  $\Delta \in \mathcal{DPT}(11)$ , the coordinate vector  $(x_0, x_1, x_2, x_3)$  of  $\Delta = x_0\Delta_0 + x_1\Delta_1 + x_2\Delta_2 + x_3\Delta_3$  is given.

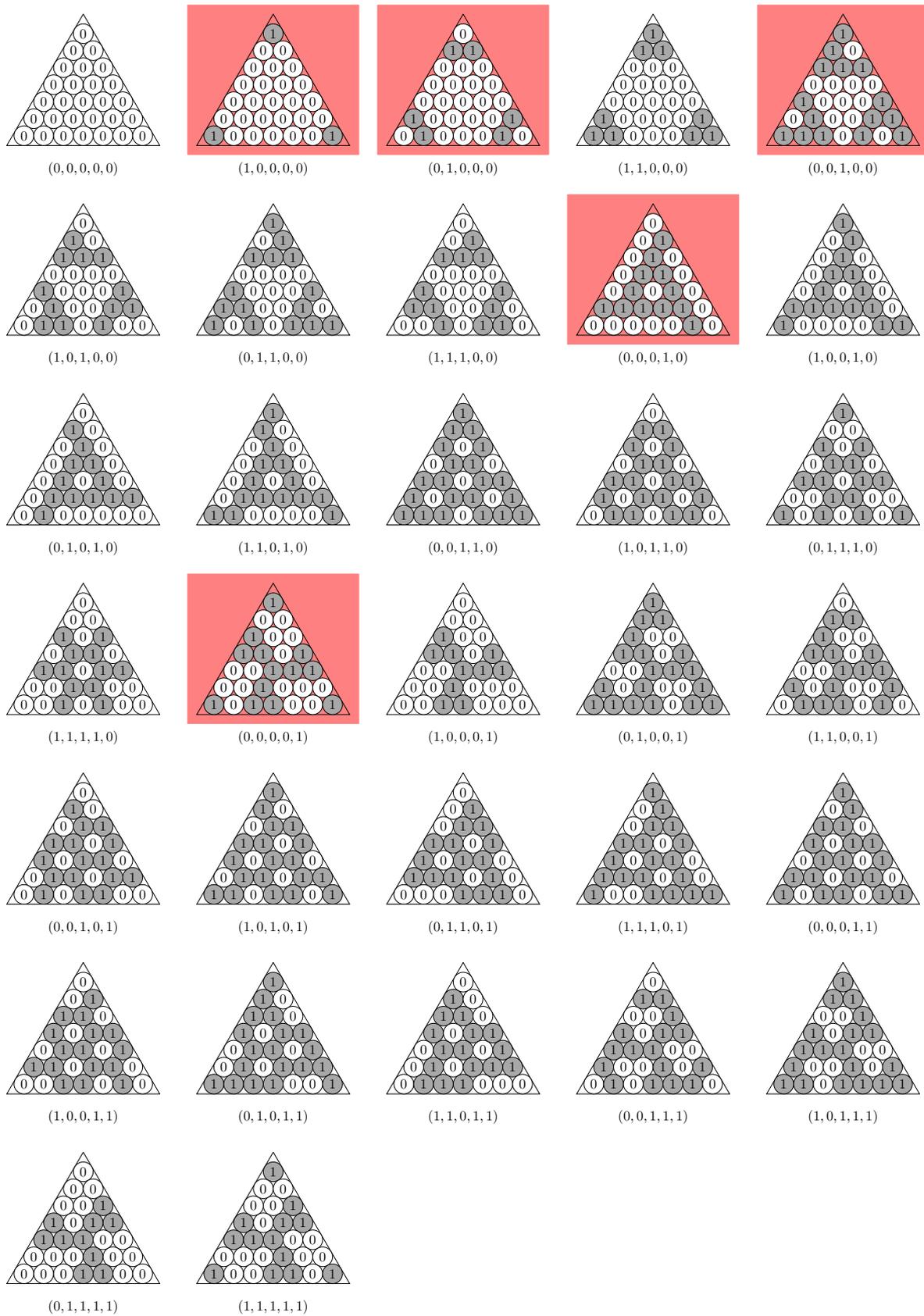


Figure 2.13: The 32 triangles of  $\mathcal{RPT}(7)$  where the 5 red triangles form a basis

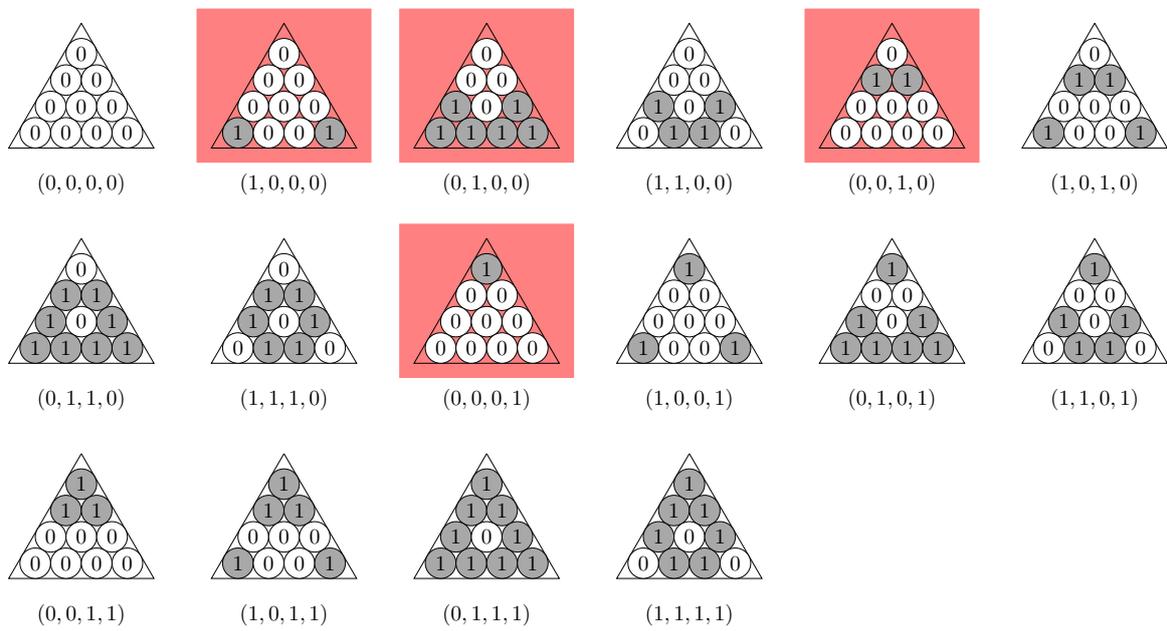


Figure 2.14: The 16 triangles of  $\mathcal{HPT}(4)$  where the 4 red triangles form a basis

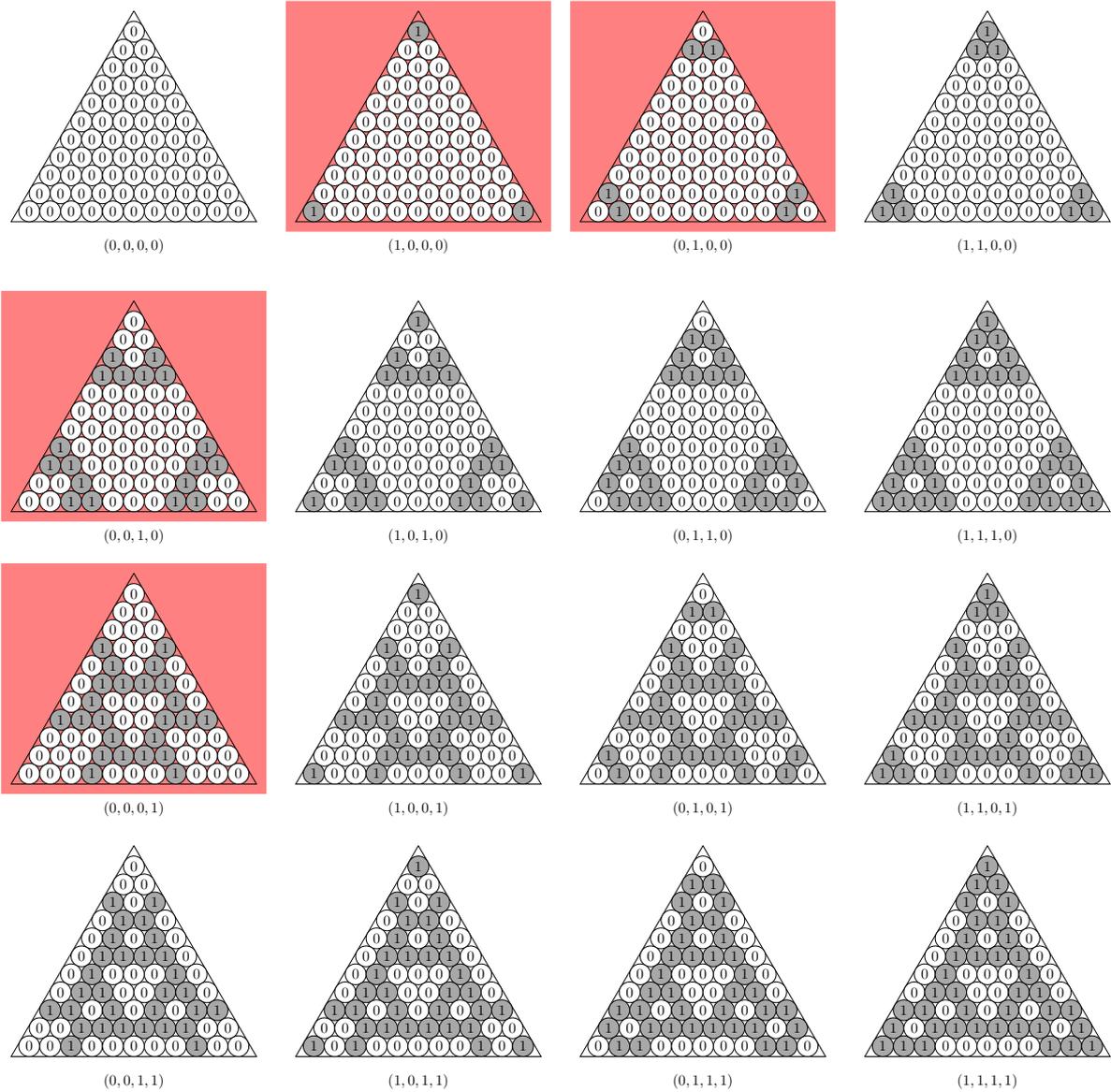


Figure 2.15: The 16 triangles of  $\mathcal{DPT}(11)$  where the 4 red triangles form a basis

# Chapter 3

## Regular Steinhaus graphs

In this chapter, we introduce the Steinhaus graphs that are a family of simple graphs related to binary Steinhaus triangles.

### 3.1 The family of Steinhaus graphs

**Definition 3.1.1** (Steinhaus graph). A *Steinhaus graph* of order  $n \geq 1$  is a simple graph whose adjacency matrix has an upper-triangular part which is a binary Steinhaus triangle of size  $n - 1$ . For any sequence  $S = (a_1, a_2, \dots, a_{n-1})$  of  $\mathbb{Z}/2\mathbb{Z}$  of length  $n - 1$ , its associated Steinhaus graph  $G(S)$  is the simple graph of order  $n$  whose adjacency matrix  $M(S) = (a_{i,j})_{1 \leq i, j \leq n}$  verifies

- i)  $a_{i,j} = a_{j,i}$ , for all  $i, j \in \{1, \dots, n\}$ , (symmetry)
- ii)  $a_{i,i} = 0$ , for all  $i \in \{1, \dots, n\}$ , (diagonal of zeroes)
- iii)  $a_{1,j} = a_{j-1}$ , for all  $j \in \{2, \dots, n - 1\}$ , (sequence  $S$ )
- iv)  $a_{i,j} = a_{i-1,j-1} + a_{i-1,j}$ , for all integers  $i, j$  such that  $2 \leq i < j \leq n$ , (local rule of  $\nabla S$ ).

The adjacency matrix  $M(S)$  is said to be the *Steinhaus matrix* of  $S$ .

For example, for  $S = (0010100)$ , the Steinhaus graph  $G(S)$  and its adjacency matrix  $M(S)$  are depicted in Figure 3.1.

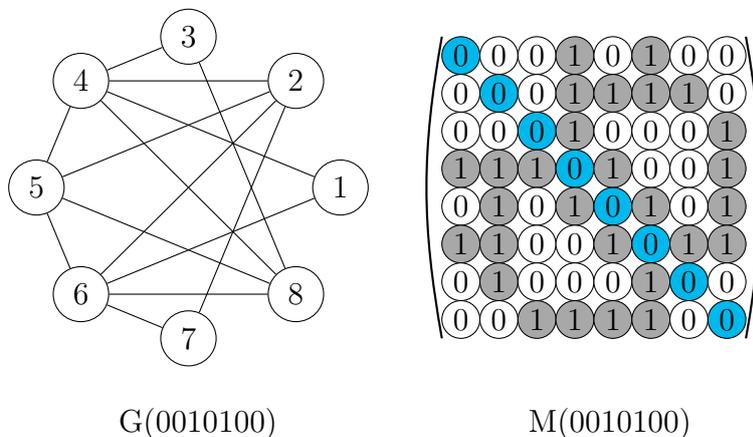


Figure 3.1: The Steinhaus graph  $G(0010100)$  and its adjacency matrix  $M(0010100)$

**Notation 3.1.2.** For all positive integers  $n$ , the set of Steinhaus graphs of order  $n$  is denoted by  $\mathcal{SG}(n)$  and the set of Steinhaus matrices of size  $n$  by  $\mathcal{SM}(n)$ .

It is clear that there is a natural correspondence between  $\mathcal{SG}(n)$  (or  $\mathcal{SM}(n)$ ) and  $\mathcal{ST}(n-1)$ . Therefore, for all positive integers  $n$ , the set  $\mathcal{SG}(n)$  (or  $\mathcal{SM}(n)$ ) is a vector space over  $\mathbb{Z}/2\mathbb{Z}$  of dimension  $n-1$ .

The family of Steinhaus graphs has been introduced in [89]. In [48], it was proved that any simple graph of order  $n$  is isomorphic to an induced subgraph of a Steinhaus graph of order  $\binom{n}{2} + 1$ . A general problem on Steinhaus graphs is to characterize those, or their associated binary sequences, having a given graph property such as connectedness, planarity, bipartition, regularity, etc. It is easy to see that a Steinhaus graph is either connected or totally disconnected (the edgeless graph). The bipartite Steinhaus graphs are characterized in [52, 53, 43] and the planar ones in [55].

In this chapter, we are interested in Steinhaus graphs that are regular, i.e., Steinhaus graphs whose vertices are of the same degree. In 1979, the two following conjectures on regular Steinhaus graphs were given by W. Dymacek.

**Conjecture 3.1.3** (Dymacek [51]). *The regular Steinhaus graphs of even degree are the zero-edge graph on  $n$  vertices, for all positive integers  $n$ , and the Steinhaus graph  $G(S)$  on  $n = 3m + 1$  vertices generated by the periodic sequence  $S = (110)^m$  of length  $3m$ , for all positive integers  $m$ .*

**Conjecture 3.1.4** (Dymacek [51]). *The complete graph on two vertices  $K_2$  is the only regular Steinhaus graph of odd degree.*

These conjectures were verified up to 117 vertices in [24] and up to 1500 vertices in [C3] for the odd case.

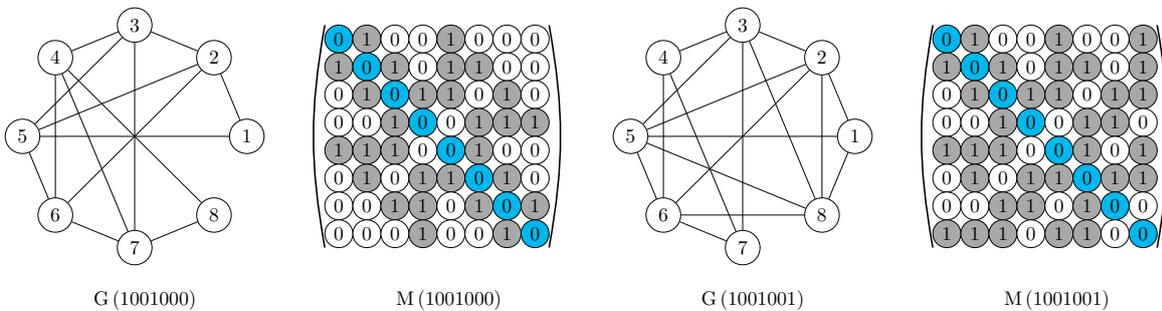


Figure 3.2: Even and odd Steinhaus graphs

## 3.2 Parity-regular Steinhaus graphs

In this section, we study the weaker case of parity-regular Steinhaus graphs, that are Steinhaus graphs for which all the vertex degrees are of the same parity.

**Definition 3.2.1** (Even and odd Steinhaus graphs). A Steinhaus graph is said to be *even* (resp. *odd*) if every vertex has even degree (resp. odd degree). A *parity-regular* Steinhaus graph is a Steinhaus graph that is even or odd.

Examples of even and odd Steinhaus graphs are given in Figure 3.2.

**Notation 3.2.2.** For all positive integers  $n$ , the sets of even Steinhaus graphs and of odd Steinhaus graphs of order  $n$  are denoted by  $\mathcal{ESG}(n)$  and  $\mathcal{OSG}(n)$ , respectively. For all positive integers  $n$ , the set of parity-regular Steinhaus graphs of order  $n$  is denoted by  $\mathcal{PRSG}(n)$ , that is,  $\mathcal{PRSG}(n) = \mathcal{ESG}(n) \cup \mathcal{OSG}(n)$ .

In this section, we prove that  $\mathcal{ESG}(n)$  is a linear subspace of  $\mathcal{SG}(n)$  of dimension  $\lfloor \frac{n-1}{3} \rfloor$ , for all positive integers  $n$ , and  $\mathcal{OSG}(n)$  is an affine subspace of direction  $\mathcal{ESG}(n)$ , for all even numbers  $n$ . Obviously, since the number of vertices of odd degrees is always even,  $\mathcal{OSG}(n) = \emptyset$  when  $n$  is odd.

### 3.2.1 Doubly-symmetric Steinhaus matrices

**Definition 3.2.3** (Doubly-symmetric Steinhaus matrices). A Steinhaus matrix  $M(S) = (a_{i,j})_{1 \leq i,j \leq n}$  is said to be *doubly-symmetric* if all its diagonals are symmetric, i.e., if

$$a_{n-j+1,n-i+1} = a_{i,j}$$

for all  $i, j \in \{1, \dots, n\}$ . In other words, the Steinhaus matrix  $M(S)$  is doubly-symmetric if the Steinhaus triangle  $r^2(\nabla S)$  is horizontally symmetric.

An example of doubly-symmetric Steinhaus matrix of size 8 is given in Figure 3.3. First, we characterize doubly-symmetric Steinhaus matrices.

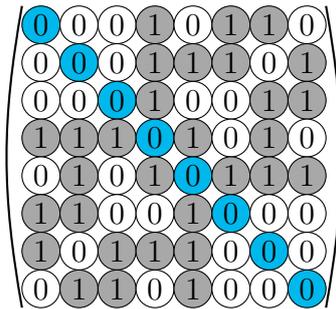


Figure 3.3: A doubly-symmetric Steinhaus matrix of size 8

**Proposition 3.2.4.** Let  $M = (a_{i,j})_{1 \leq i,j \leq n}$  be a Steinhaus matrix of size  $n \geq 3$ . Then, the following assertions are equivalent:

- i) the matrix  $M$  is doubly-symmetric,
- ii) the over-diagonal  $(a_{i,i+1})_{1 \leq i \leq n-1}$  of  $M$  is a symmetric sequence,
- iii) the entries  $a_{i,n-i+1}$  of the anti-diagonal of  $M$  vanish for all  $i \in \{1, \dots, \lfloor \frac{n-1}{2} \rfloor\}$ .

*Proof.* We know that the Steinhaus matrix  $M(S)$  is doubly-symmetric if and only if the Steinhaus triangle  $r^2(\nabla S)$  is horizontally symmetric. Therefore, the equivalence  $i) \Leftrightarrow ii)$  directly comes from Proposition 2.4.3 and the equivalence  $i) \Leftrightarrow iii)$  from Proposition 2.4.4.  $\square$

The main result of this subsection is the following

**Theorem 3.2.5** (Dymacek [51]). *The adjacency matrix of an even Steinhaus graph is a doubly-symmetric Steinhaus matrix whose diagonals are zero-sum.*

*Remark 3.2.6.* If a Steinhaus matrix is doubly-symmetric, then all its diagonals are symmetric and thus we already know that all its diagonals, except the over-diagonal  $(a_{i,i+1})_{1 \leq i \leq n-1}$ , have to be zero-sum. The additional information here is that  $a_{\frac{n}{2}, \frac{n}{2}+1} = 0$  in a Steinhaus matrix  $(a_{i,j})_{1 \leq i,j \leq n}$  of even size  $n$ .

The proof given here is based on the following result which shows that the anti-diagonal entries of a Steinhaus matrix are determined by the vertex degrees of its associated Steinhaus graph.

**Theorem 3.2.7** (Chappelon [C3]). *Let  $G$  be a Steinhaus graph on  $n \geq 2$  vertices and  $M = (a_{i,j})$  its associated Steinhaus matrix. Then every anti-diagonal entry of  $M$  can be expressed by means of the vertex degrees of  $G$ . If we denote by  $\deg(V_i)$  the degree of the vertex  $V_i$  in  $G$ , then we have*

$$a_{i,n-i+1} \equiv \sum_{k=0}^{i-1} \binom{i-1}{k} \deg(V_{i+k+1}) \equiv \sum_{k=0}^{i-1} \binom{i-1}{k} \deg(V_{n-i-k}) \pmod{2},$$

for all  $i \in \{1, \dots, \lfloor \frac{n}{2} \rfloor\}$ .

*Remark 3.2.8.* We deduce from Theorem 3.2.7 a necessary condition on the vertex degrees of a given labelled graph to be a Steinhaus graph. Indeed, vertex degrees of a Steinhaus graph on  $n$  vertices must satisfy the following binary equations:

$$\sum_{k=0}^{i-1} \binom{i-1}{k} \deg(V_{i+k+1}) \equiv \sum_{k=0}^{i-1} \binom{i-1}{k} \deg(V_{n-i-k}) \pmod{2}, \quad \text{for all } 1 \leq i \leq \lfloor \frac{n}{2} \rfloor.$$

More generally, an open problem, corresponding to Question 3 in [54], is to determine if an arbitrary graph, not necessary labelled, is isomorphic to a Steinhaus graph.

It is clear that Dymacek's theorem comes from Theorem 3.2.7 using Proposition 3.2.4.

### 3.2.2 Even Steinhaus graphs

Recall that, for any positive integer  $n \geq 3$ , the linear map  $H : \mathcal{ST}(n) \rightarrow \mathcal{ST}(n-3)$ , introduced in Chapter 2, is the linear map that assigns to each binary Steinhaus triangle of size  $n$  its subtriangle of size  $n-3$  obtained by removing its first row and its left and right sides. Now, we consider the linear map  $H$  on graphs.

**Definition 3.2.9** (Operator  $H$ ). Let  $H : \mathcal{SG}(n) \rightarrow \mathcal{SG}(n-3)$  be the linear map that assigns to each Steinhaus graph  $G(S)$  of order  $n$  the Steinhaus graph of order  $n-3$  whose upper-triangular part of its adjacency matrix is the Steinhaus triangle  $H(\nabla S)$ . More precisely,

$$\begin{aligned} H : \mathcal{SG}(n) &\longrightarrow \mathcal{SG}(n-3) \\ G(S) &\longmapsto G(\partial S[2, n-3]) \end{aligned}$$

Note that the linear map  $H$  is surjective. Indeed, for any  $G(S') \in \mathcal{SG}(n-3)$ , it is easy to verify that  $G(S') = H(G(S))$  if and only if  $S$  is one of the eight sequences of the form  $S = (x_1) \cdot \int_{i,x} S' \cdot (x_2)$ , where  $x_1, x_2 \in \mathbb{Z}/2\mathbb{Z}$  and  $\int_{i,x} S'$  is one of the two antiderived sequences of  $S'$ . Examples of Steinhaus graphs  $G(S)$  and  $H(G(S))$  and their adjacency matrices are depicted in Figure 3.4.

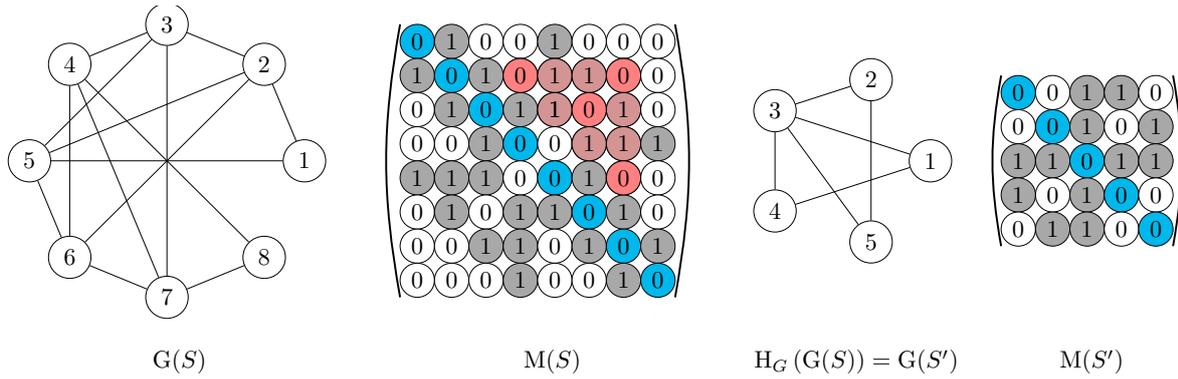


Figure 3.4:  $G(1001000)$  and  $H(G(1001000)) = G(0110)$

**Theorem 3.2.10** (Dymacek [51]). *Let  $S$  be a finite binary sequence of length  $n \geq 3$ . The Steinhaus graph  $G(S)$  is even if and only if  $H(G(S)) = G(S')$  is even and  $S = \left(\sigma_2 \left(\int_{i,x} S'\right)\right) \cdot \int_{i,x} S' \cdot (0)$ , for some  $i \in \{1, \dots, n-3\}$  and some  $x \in \mathbb{Z}/2\mathbb{Z}$ , where  $\sigma_2(T)$  is the sum of the elements of  $T$  modulo 2.*

**Corollary 3.2.11.**  $\dim \mathcal{ESG}(n) = \lfloor \frac{n-1}{3} \rfloor$ , for all positive integers  $n$ .

### 3.2.3 Parity-regular Steinhaus graphs

We end this section by giving the dimension of the linear subspace  $\mathcal{PRSG}(n)$  of parity-regular Steinhaus graphs, for all positive integers  $n$ . By definition, we know that

$$\mathcal{PRSG}(n) = \mathcal{ESG}(n) \sqcup \mathcal{OSG}(n),$$

where  $\mathcal{OSG}(n)$  is the set of odd Steinhaus graphs of order  $n$ , for all positive integers  $n$ . As already remarked, it is clear that  $\mathcal{OSG}(n) = \emptyset$ , when  $n$  is odd. For  $n$  even, we obtain the following

**Proposition 3.2.12.** *For any positive integer  $n$ ,*

$$\mathcal{OSG}(2n) = \mathcal{ESG}(2n) + G((0)_{2n-2} \cdot (1)).$$

The proof is based on the following lemma, where the linear map  $\iota$  is defined by

$$\begin{aligned} \iota : \mathcal{SG}(2n) &\longrightarrow \mathcal{SG}(2n) \\ G(S) &\longmapsto G(S + (0)_{2n-2} \cdot (1)) \end{aligned}$$

for all positive integers  $n$ .

**Lemma 3.2.13.** *For any positive integer  $n$ , the Steinhaus graph  $G(S)$  of order  $2n$  is even if and only if  $\iota(G(S))$  is odd.*

It immediately follows that, for any positive integer  $n$ , we have

$$\mathcal{PRSG}(2n-1) = \mathcal{ESG}(2n-1) \tag{3.1}$$

and

$$\mathcal{PRSG}(2n) = \mathcal{ESG}(2n) \sqcup (\mathcal{ESG}(2n) + G((0)_{2n-2} \cdot (1))). \tag{3.2}$$

Therefore, we retrieve the following

**Proposition 3.2.14.**  $\dim \mathcal{PRSG}(n) = \lfloor \frac{n-1}{3} \rfloor + \delta_{0,(n \bmod 2)}$ , for all positive integers  $n$ .

### 3.3 Vector bases of parity-regular Steinhaus graphs

Bases of  $\mathcal{PRSG}(n)$  have been computed, for  $n \leq 30$ , in [24]. In this section, we determine bases of  $\mathcal{ESG}(n)$  and  $\mathcal{PRSG}(n)$ , for all positive integers  $n$ . This is achieved by showing that the vector space  $\mathcal{ESG}(n)$  is isomorphic to a particular linear subspace of  $\mathcal{DST}(2n-1)$ , for all positive integers  $n$ .

#### 3.3.1 Isomorphism between $\mathcal{ESG}(n)$ and $\mathcal{DST}_0(2n-1)$

**Definition 3.3.1** (Interlacing sequence). For any binary sequence  $S = (a_j)_{1 \leq j \leq n}$  of length  $n$ , we denote by  $ir(S)$  the *interlacing of the sequence  $S$  and its reversed sequence*, that is, the sequence  $ir(S) = (b_j)_{1 \leq j \leq 2n}$  of length  $2n$  defined by

$$b_{2j-1} = a_j \quad \text{and} \quad b_{2j} = a_{n-j+1},$$

for all  $j \in \{1, \dots, n\}$ .

For instance, for  $S = (101000)$ , we have  $ir(S) = (100010010001)$ .

**Definition 3.3.2** (Operator  $\theta$ ). For any positive integer  $n$ , let  $\theta$  be the linear map

$$\begin{aligned} \theta : \mathcal{SG}(n) &\longrightarrow \mathcal{ST}(2n-1) \\ G(S) &\longmapsto \nabla \int_{n,0} ir(S) \end{aligned}$$

Note that the Steinhaus triangle  $\nabla S \in \mathcal{ST}(n-1)$  is then a subtriangle of  $\theta(G(S)) \in \mathcal{ST}(2n-1)$ . Indeed, for the sequence  $S = (a_j)_{1 \leq j \leq n-1}$  and the Steinhaus triangle  $\theta(G(S)) = \nabla \int_{n,0} ir(S) = (a_{i,j})_{1 \leq i \leq j \leq 2n-1}$ , the Steinhaus triangle  $\nabla S$  is simply the subtriangle  $(a_{2i,2j})_{1 \leq i \leq j \leq n-1}$ , since  $a_{2,2j} = a_j$ , for all  $j \in \{1, \dots, n-1\}$ , by definition of  $\theta$ , and, using the local rule (LR), we have

$$a_{2i,2j} \equiv a_{2i-1,2j-1} + a_{2i-1,2j} \equiv a_{2i-2,2j-2} + 2a_{2i-2,2j-1} + a_{2i-2,2j} \equiv a_{2i-2,2j-2} + a_{2i-2,2j} \pmod{2},$$

for all integers  $i$  and  $j$  such that  $2 \leq i \leq j \leq n-1$ . For instance, for the sequence  $S = (101000)$ , the Steinhaus triangle  $\theta(G(S))$  is depicted in Figure 3.5, where the subtriangle  $\nabla S$  appears in red.

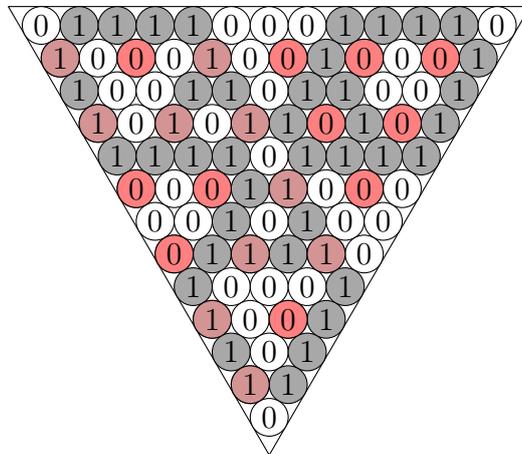


Figure 3.5: The Steinhaus triangle  $\theta(G(101000))$  where  $\nabla(101000)$  appears in red

By definition of the linear map  $ir$ , we know that the sequence  $ir(S)$  is symmetric and  $\sigma_2(ir(S)) = 0$ . It follows from Proposition 2.4.2 that the sequence  $\int_{n,0} ir(S)$  is symmetric too. Therefore, using Proposition 2.4.3, we have

$$\theta(G(S)) = \nabla \int_{n,0} ir(S) \in \mathcal{HST}(2n - 1),$$

for all  $G(S) \in \mathcal{SG}(n)$ . Moreover, since the middle term of the sequence  $\int_{n,0} ir(S)$  is 0 by definition, we obtain that

$$\sigma_2 \left( \int_{n,0} ir(S) \right) = 0,$$

for any sequence  $S$  of length  $n - 1$ .

The main result of this section is to show that the restriction of  $\theta$  to the linear subspace of even Steinhaus graphs  $\mathcal{ESG}(n)$  induces an isomorphism between  $\mathcal{ESG}(n)$  and  $\mathcal{DST}_0(2n - 1)$ , for all positive integers  $n$ .

**Theorem 3.3.3** (Chappelon [C8]). *Let  $S$  be a binary sequence of length  $n - 1 \geq 0$ . Then, the Steinhaus graph  $G(S)$  is even if and only if the Steinhaus triangle  $\theta(G(S))$  is dihedrally symmetric.*

**Corollary 3.3.4** (Chappelon [C8]). *For any positive integer  $n$ , the restriction*

$$\theta|_{\mathcal{ESG}(n)} : \mathcal{ESG}(n) \longrightarrow \mathcal{DST}_0(2n - 1)$$

*is an isomorphism.*

Using Corollary 3.3.4 with Corollary 2.5.9 permits us to obtain a new proof that

$$\dim \mathcal{ESG}(n) = \dim \mathcal{DST}_0(2n - 1) = \left\lfloor \frac{2n - 1}{6} \right\rfloor + \delta_{4, (2n-1 \bmod 6)} = \left\lfloor \frac{2n - 1}{6} \right\rfloor = \left\lfloor \frac{n - 1}{3} \right\rfloor,$$

for all positive integers  $n$ .

### 3.3.2 Basis of $\mathcal{ESG}(n)$

Using Theorem 3.3.3 and the results of Chapter 2, we are now ready for giving a basis of  $\mathcal{ESG}(n)$ , for all positive integers  $n$ .

**Theorem 3.3.5** (Chappelon [C8]). *Let  $n$  be a positive integer. The set*

$$\left\{ \psi \rho \left( \nabla S_{2k+1, k-n+1}^{(2n-1)} \right) \mid k \in \left\{ 0, \dots, \left\lfloor \frac{n-1}{3} \right\rfloor - 1 \right\} \right\}$$

*is a basis of  $\mathcal{ESG}(n)$ , where  $\psi \rho \left( \nabla S_{2k+1, k-n+1}^{(2n-1)} \right) = G(S_k)$  with*

$$S_k = \left( \binom{k - n + 2j - 1}{2k} + \binom{k + n - 2j}{2k - 2j + 3} + \binom{k - n + 2}{2k - 2n + 2j + 2} \bmod 2 \right)_{1 \leq j \leq n-1},$$

*for all  $k \in \{0, \dots, \lfloor \frac{n-1}{3} \rfloor - 1\}$ .*

For instance, for  $n = 12$ , we obtain

$k$	$S_{2k+1, k-11}^{(23)}$	$\rho \left( \nabla S_{2k+1, k-11}^{(23)} \right)$	$\psi \left( \rho \left( \nabla S_{2k+1, k-11}^{(23)} \right) \right)$
0	(1010101010101010101010101010101)	$\nabla(011010101010101010101010110)$	$G_1 = G(111111111110)$
1	(010001000100010001000100010)	$\nabla(00010100010001000101000)$	$G_2 = G(01101010110)$
2	(10000010100000101000001)	$\nabla(01111110100000101111110)$	$G_3 = G(10011001000)$

All the even Steinhaus graphs of order 12 are depicted in Figure 3.6, where the elements of the basis  $\{G_1, G_2, G_3\}$  are in red and, for every  $G \in \mathcal{ESG}(12)$ , the coordinate vector  $(x_1, x_2, x_3)$  of  $G = x_1G_1 + x_2G_2 + x_3G_3$  is given.

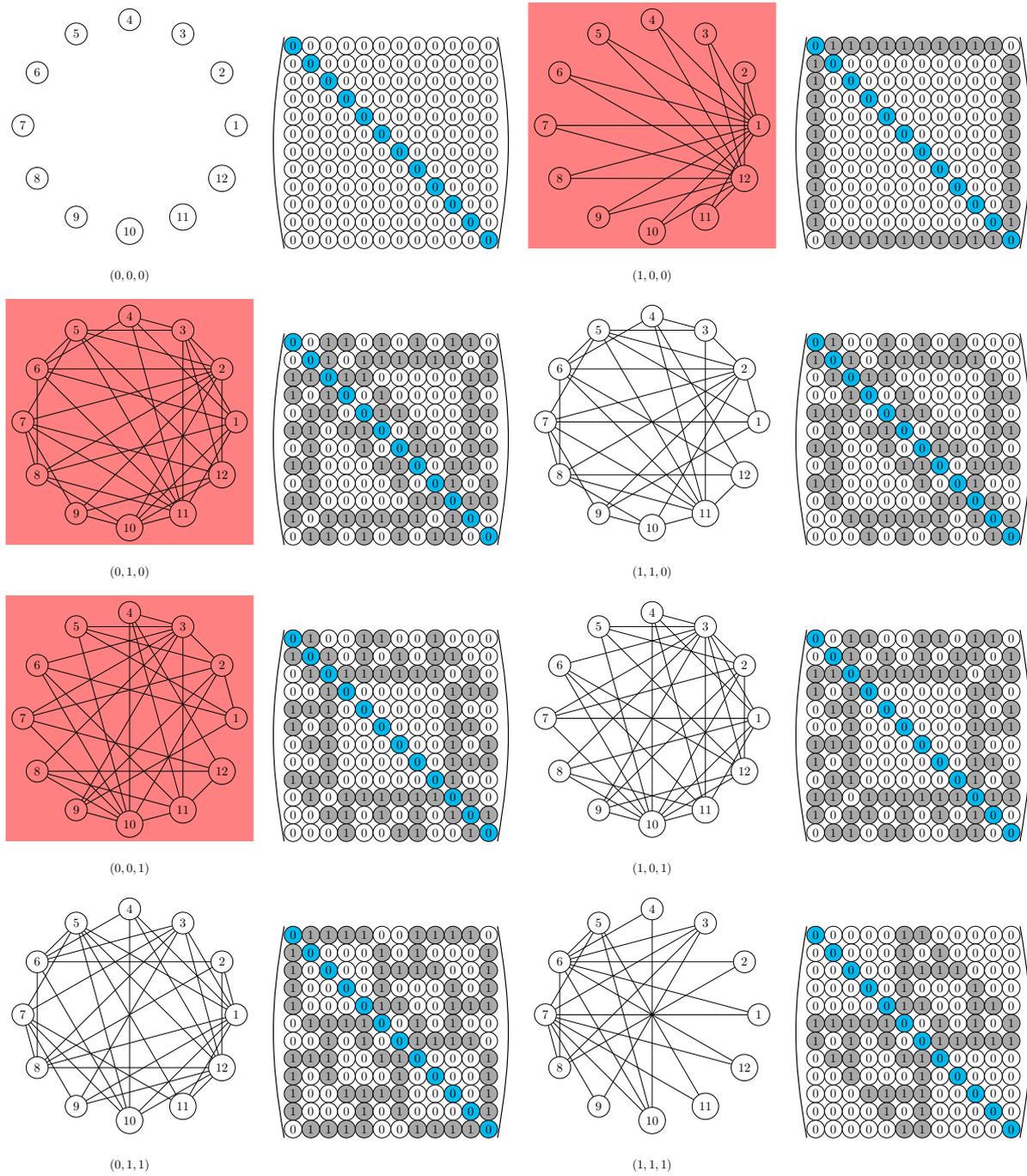


Figure 3.6: The 8 graphs of  $\mathcal{ESG}(12)$  where the 3 red graphs form a basis

### 3.3.3 Basis of $\mathcal{PRSG}(n)$

We end this section by giving a basis of the linear subspace  $\mathcal{PRSG}(n)$  of parity-regular Steinhaus graphs, for all positive integers  $n$ . By definition, we know that  $\mathcal{PRSG}(n) = \mathcal{ESG}(n) \sqcup \mathcal{OSG}(n)$ , where  $\mathcal{OSG}(n)$  is the set of odd Steinhaus graphs of order  $n$ , for all positive integers  $n$ . As already remarked, it is clear that  $\mathcal{OSG}(n) = \emptyset$ , when  $n$  is odd, and

$$\mathcal{OSG}(n) = \mathcal{ESG}(n) + G((0)_{n-2} \cdot (1)),$$

when  $n$  is even. It immediately follows that, for any positive integer  $n$ , we have

$$\mathcal{PRSG}(2n-1) = \mathcal{ESG}(2n-1) \tag{3.3}$$

and

$$\mathcal{PRSG}(2n) = \mathcal{ESG}(2n) \sqcup (\mathcal{ESG}(2n) + G((0)_{2n-2} \cdot (1))). \tag{3.4}$$

Combining the identities (3.3) and (3.4) with Theorem 3.3.5, we obtain the following

**Theorem 3.3.6** (Chappelon [C8]). *Let  $n$  be a positive integer. The set*

$$\left\{ \psi\rho \left( \nabla S_{2k+1, k-n+1}^{(2n-1)} \right) \mid k \in \left\{ 0, \dots, \left\lfloor \frac{n-1}{3} \right\rfloor - 1 \right\} \right\},$$

when  $n$  is odd, or the set

$$\{G((0)_{n-2} \cdot (1))\} \cup \left\{ \psi\rho \left( \nabla S_{2k+1, k-n+1}^{(2n-1)} \right) \mid k \in \left\{ 0, \dots, \left\lfloor \frac{n-1}{3} \right\rfloor - 1 \right\} \right\},$$

when  $n$  is even, is a basis of  $\mathcal{PRSG}(n)$ , where  $\psi\rho \left( \nabla S_{2k+1, k-n+1}^{(2n-1)} \right) = G(S_k)$  with

$$S_k = \left( \binom{k-n+2j-1}{2k} + \binom{k+n-2j}{2k-2j+3} + \binom{k-n+2}{2k-2n+2j+2} \pmod{2} \right)_{1 \leq j \leq n-1},$$

for all  $k \in \{0, \dots, \lfloor \frac{n-1}{3} \rfloor - 1\}$ .

For instance, for  $n = 12$ , we obtain

$$G_0 = G(000000000001), \quad G_1 = G(111111111110), \quad G_2 = G(01101010110), \quad G_3 = G(10011001000).$$

All the parity-regular Steinhaus graphs of order 12 are depicted in Figure 3.6 for even graphs and in Figure 3.7 for odd ones, where the elements of the basis  $\{G_0, G_1, G_2, G_3\}$  are in red and, for every  $G \in \mathcal{PRSG}(12)$ , the coordinate vector  $(x_0, x_1, x_2, x_3)$  of  $G = x_0G_0 + x_1G_1 + x_2G_2 + x_3G_3$  is given  $((x_1, x_2, x_3)$  when  $x_0 = 0$  in Figure 3.6).

## 3.4 Regular Steinhaus graphs of odd degree

In this section, we will study in detail the structure of Steinhaus matrices associated with regular Steinhaus graphs of odd degree.

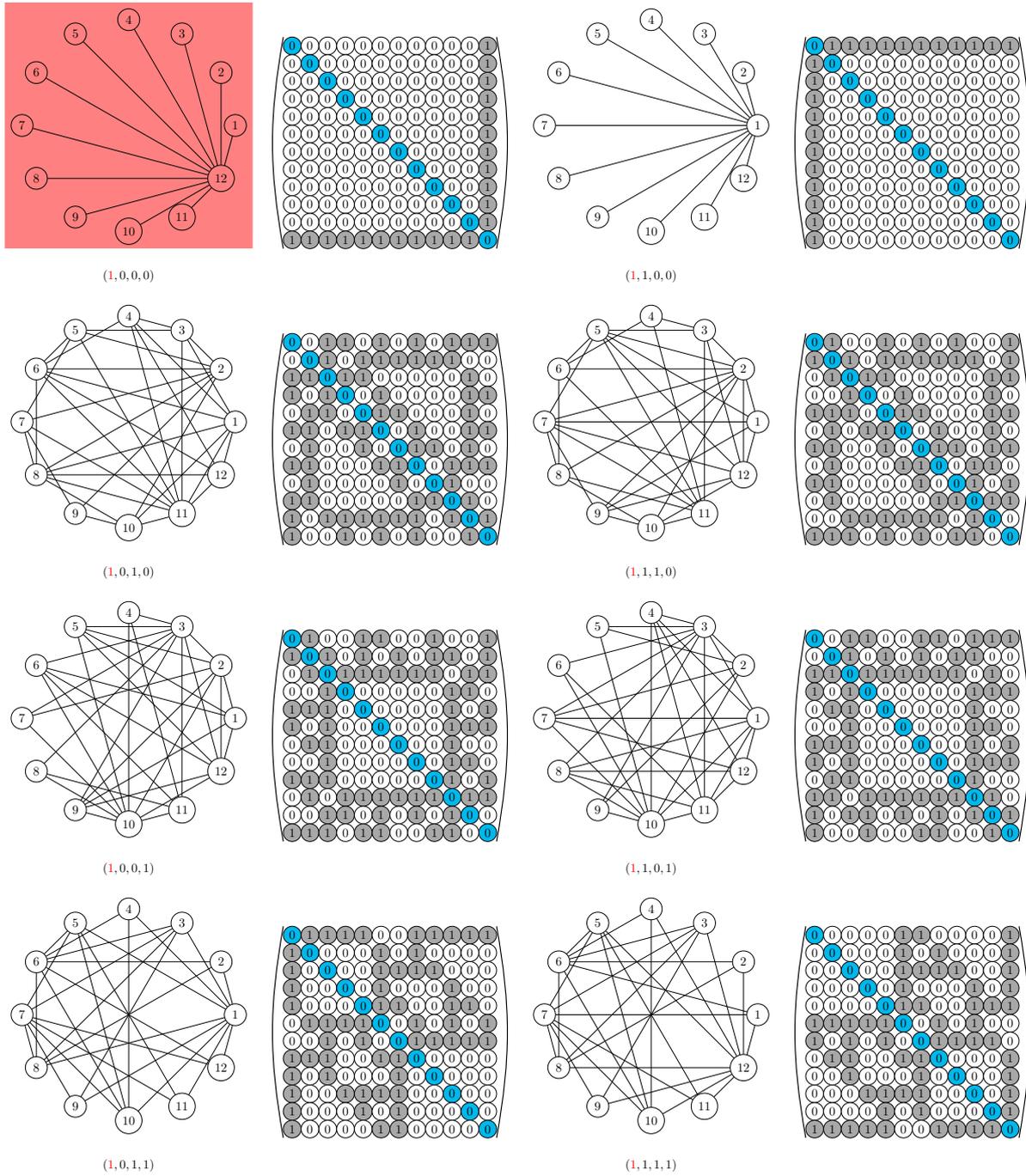


Figure 3.7: The 8 graphs of  $\mathcal{OS}(12)$

### 3.4.1 Multi-symmetric Steinhaus matrices

Let  $G$  be a Steinhaus graph on  $n \geq 1$  vertices. Then, for every integer  $1 \leq i \leq n$ , we denote by  $G \setminus \{V_i\}$  the graph obtained from  $G$  by deleting its  $i$ th vertex  $V_i$  and its incident edges in  $G$ . Since the adjacency matrix of the graph  $G \setminus \{V_1\}$  (resp.  $G \setminus \{V_n\}$ ) is the Steinhaus matrix obtained by removing the first row (resp. the last column) in the adjacency matrix of  $G$ , it follows that the graph  $G \setminus \{V_1\}$  (resp.  $G \setminus \{V_n\}$ ) is a Steinhaus graph on  $n - 1$  vertices.

Bailey and Dymacek studied the regular Steinhaus graphs of odd degree in [25], where the following theorem is stated, using Dymacek's theorem.

**Theorem 3.4.1** (Bailey-Dymacek [25]). *Let  $G$  be a regular Steinhaus graph of odd degree  $d$  on  $2n \geq 4$  vertices. Then  $d = n$ , the Steinhaus graph  $G \setminus \{V_1, V_{2n}\}$  is regular of even degree  $n - 1$ , and  $a_{1,j} = a_{1,2n-j+1}$  for all  $2 \leq j \leq 2n - 1$ .*

*Remark 3.4.2.* In every simple graph, there are an even number of vertices of odd degree. Therefore parity-regular Steinhaus graphs of odd type and thus regular Steinhaus graphs of odd degree have an even number of vertices.

In their theorem, the authors studied the form of the sequence associated with  $G$ . We are more interested in the Steinhaus matrix of  $G \setminus \{V_1, V_{2n}\}$  in the sequel.

**Definition 3.4.3** (Multi-symmetric matrix). A square matrix of size  $n \geq 1$  is said to be *multi-symmetric* if  $M$  is doubly-symmetric and each row of the upper-triangular part of  $M$  is a symmetric sequence, that is

$$a_{i,j} = a_{i,n-j+i+1}, \quad \text{for all } 1 \leq i < j \leq n.$$

It is clear that a Steinhaus matrix  $M(S)$  is multi-symmetric if and only if the binary Steinhaus triangle  $\nabla S$  is dihedrally symmetric.

First, it is easy to see that each column of the upper-triangular part of a multi-symmetric matrix is also a symmetric sequence.

**Proposition 3.4.4.** *Let  $M = (a_{i,j})$  be a multi-symmetric matrix of size  $n$ . Then, each column of the upper-triangular part of  $M$  is a symmetric sequence, that is  $a_{i,j} = a_{j-i,j}$  for all  $1 \leq i < j \leq n$ .*

As for doubly-symmetric Steinhaus matrices, multi-symmetric Steinhaus matrices can be characterized as follows.

**Proposition 3.4.5** (Chappelton [C3]). *Let  $M = (a_{i,j})$  be a Steinhaus matrix of size  $n \geq 3$ . Then the following assertions are equivalent:*

- (i) *the matrix  $M$  is multi-symmetric,*
- (ii) *the first row, the last column and the over-diagonal of  $M$  are symmetric sequences,*
- (iii) *the entries  $a_{i,n-i+1}$ ,  $a_{n-2i+1,n-i+1}$  and  $a_{i,2i}$  vanish for all  $1 \leq i \leq \lfloor \frac{n-1}{2} \rfloor$ .*

We now refine Theorem 3.4.1.

**Theorem 3.4.6.** *Let  $G$  be a regular Steinhaus graph of odd degree  $n$  on  $2n \geq 4$  vertices. Then  $G \setminus \{V_1, V_{2n}\}$  is a regular Steinhaus graph of even degree  $n - 1$  whose associated Steinhaus matrix is multi-symmetric.*

*Remark 3.4.7.* By Theorem 3.4.6, it is easy to show that Conjecture 3.1.3 implies Conjecture 3.1.4. Indeed, if Conjecture 3.1.3 is true, then the zero-edge graph on  $n$  vertices is the only regular Steinhaus graph of even degree whose Steinhaus matrix is multi-symmetric. It follows, by Theorem 3.4.6, that if  $G(s)$  is a regular Steinhaus graph of odd degree on  $n + 2$  vertices then  $s = (0 \cdots 01)$  or  $s = (1 \cdots 1)$ . Therefore the Steinhaus graph  $G(s)$  is the star graph on  $n + 2$  vertices which is not a regular Steinhaus graph.

In the sequel of this subsection we will study in detail the multi-symmetric Steinhaus matrices. First, in order to determine a parametrization of these matrices, we use the operator  $H$  which assigns to each matrix  $M = (a_{i,j})$  in  $\mathcal{SM}(n)$  the Steinhaus matrix  $H(M) = (b_{i,j})$  in  $\mathcal{SM}(n-3)$  defined by  $b_{i,j} = a_{i-1,j-2}$ , for all  $1 \leq i < j \leq n-3$ .

**Proposition 3.4.8** (Chappelon [C3]). *Let  $M = (a_{i,j})$  be a Steinhaus matrix of size  $n \geq 4$ . Then the extension  $M$  of  $H(M)$  only depends on the parameters  $a_{1,2}$ ,  $a_{1,j_0}$  and  $a_{1,n}$ , with  $j_0$  in  $\{3, \dots, n-1\}$ .*

Therefore, for every Steinhaus matrix  $N$  of size  $n-3$ , there exist 8 distinct Steinhaus matrices  $M$  of size  $n$  such that  $H(M) = N$ . We can also use this operator to determine parametrizations of multi-symmetric Steinhaus matrices.

**Proposition 3.4.9** (Chappelon [C3]). *Let  $M = (a_{i,j})$  be a multi-symmetric Steinhaus matrix of size  $n$ . Let  $j_i$  be an element of the set  $\{2i+1, \dots, n-i\}$  for all  $1 \leq i \leq \lfloor \frac{n-1}{3} \rfloor$ . Then the matrix  $M$  depends on the following parameters:*

- $a_{1,j_1}$  and  $\{a_{2i,j_{2i}} \mid 1 \leq i \leq \lceil \frac{n}{6} \rceil - 1\}$ , for  $n$  even,
- $\{a_{2i+1,j_{2i+1}} \mid 0 \leq i \leq \lceil \frac{n-3}{6} \rceil - 1\}$ , for  $n$  odd.

For all positive integers  $n$ , the number of multi-symmetric Steinhaus matrices of size  $n$  immediately follows.

**Theorem 3.4.10** (Chappelon [C3]). *Let  $n$  be a positive integer. If we denote by  $MS(n)$  the number of multi-symmetric Steinhaus matrices of size  $n$ , then we have*

$$MS(n) = \begin{cases} 2^{\lceil \frac{n}{6} \rceil} & , \text{ for } n \text{ even,} \\ 2^{\lceil \frac{n-3}{6} \rceil} & , \text{ for } n \text{ odd.} \end{cases}$$

### 3.4.2 Vertex degrees of Steinhaus graphs associated with multi-symmetric Steinhaus matrices

In this section, we analyse the vertex degrees of a Steinhaus graph associated with a multi-symmetric Steinhaus matrix of size  $n$ . We begin with the case of doubly-symmetric Steinhaus matrices.

**Proposition 3.4.11.** *Let  $n$  be a positive integer and  $G$  be a Steinhaus graph on  $n$  vertices whose Steinhaus matrix is doubly-symmetric. Then, for all  $1 \leq i \leq n$ , we have*

$$\deg(V_i) = \deg(V_{n-i+1}).$$

We shall now see that, for a Steinhaus graph associated with a multi-symmetric Steinhaus matrix, the knowledge of the vertex degrees modulo 4 imposes strong conditions on the entries of its Steinhaus matrix. In order to prove this result, we distinguish different cases depending on the parity of  $n$ .

**Proposition 3.4.12** (Chappelon [C3]). *Let  $n$  be an even number and  $G$  be a Steinhau graph on  $n$  vertices whose Steinhau matrix  $M = (a_{i,j})$  is multi-symmetric. Then, we have*

$$\begin{aligned} \deg(V_1) &= \deg(V_n) \equiv a_{1, \frac{n}{2}+1} \pmod{2}, \\ \deg(V_2) &= \deg(V_{n-1}) \equiv 2a_{1, \frac{n}{2}+1} \pmod{4}, \\ \deg(V_3) &= \deg(V_{n-2}) \equiv 2a_{2, \frac{n}{2}+1} \pmod{4}, \\ \deg(V_{2i}) &= \deg(V_{n-2i+1}) \equiv 2a_{2, 2i+1} + 2a_{i, 2i+1} \pmod{4}, \quad \text{for all } 2 \leq i \leq \frac{n}{2} - 2. \end{aligned}$$

*Remark 3.4.13.* Let  $n$  be an even number. In every Steinhau graph on  $n$  vertices whose Steinhau matrix is multi-symmetric the fourth vertex  $V_4$  has a degree divisible by 4.

**Proposition 3.4.14** (Chappelon [C3]). *Let  $n$  be an odd number and  $G$  be a Steinhau graph on  $n$  vertices whose Steinhau matrix  $M = (a_{i,j})$  is multi-symmetric. Then, we have*

$$\begin{aligned} \deg(V_1) &= \deg(V_n) \equiv 0 \pmod{2}, \\ \deg(V_2) &= \deg(V_{n-1}) \equiv 2a_{1, \frac{n+1}{2}} \pmod{4}, \\ \deg(V_{2i}) &\equiv 2a_{i+1, 2i+1} + 2a_{2i-1, 2i+1} + 2a_{2i-1, \frac{n-1}{2}+i} \pmod{4}, \quad \text{for all } 2 \leq i \leq \frac{n-3}{2}, \\ \deg(V_{2i+1}) &\equiv 2a_{2, 2i+2} \pmod{4}, \quad \text{for all } 1 \leq i \leq \frac{n-3}{2}. \end{aligned}$$

*Remark 3.4.15.* Let  $n$  be an odd number. In every Steinhau graph on  $n$  vertices whose Steinhau matrix is multi-symmetric the third vertex  $V_3$  has a degree divisible by 4.

### 3.4.3 Regular Steinhau graphs modulo 4 with multi-symmetric Steinhau matrices

In this subsection, we consider the multi-symmetric Steinhau matrices associated with Steinhau graphs which are regular modulo 4, i.e. where all vertex degrees are equal modulo 4. First, we determine an upper bound of the number of these matrices. Two cases are distinguished, according to the parity of  $n$ .

**Theorem 3.4.16** (Chappelon [C3]). *For all odd numbers  $n$ , there are at most  $2^{\lceil \frac{n}{30} \rceil}$  multi-symmetric Steinhau matrices of size  $n$  whose associated Steinhau graphs are regular modulo 4.*

**Theorem 3.4.17** (Chappelon [C3]). *For all even numbers  $n$ , there are at most  $2^{\lceil \frac{n}{24} \rceil}$  multi-symmetric Steinhau matrices of size  $n$  whose associated Steinhau graphs are regular modulo 4.*

Using these explicit parametrizations of the multi-symmetric Steinhau matrices whose Steinhau graphs are regular modulo 4, we obtain the following result by computer search:

**Computational Result 3.4.18** (Chappelon [C3]). *For all positive integers  $n \leq 1500$ , the zero-edge graph on  $n$  vertices is the only Steinhau graph on  $n$  vertices with a multi-symmetric Steinhau matrix and which is regular modulo 4.*

This result can easily be proved for all odd numbers in the special case of regular Steinhau graphs on  $n$  vertices whose Steinhau matrices are multi-symmetric.

**Theorem 3.4.19** (Chappelon [C3]). *For all odd numbers  $n$ , there is no regular Steinhau graph on  $n$  vertices whose Steinhau matrix is multi-symmetric, except the zero-edge graph on  $n$  vertices.*

Finally, the above computational result permits us to extend the verification of Conjecture 3.1.4 up to  $n \leq 1500$  vertices. Indeed, as proved in the remark following Theorem 3.4.6, for a Steinhaus graph  $G$  on  $2n$  vertices, if  $G \setminus \{V_1, V_{2n}\}$  is the zero-edge graph on  $2n - 2$  vertices, then  $G$  is the star graph on  $2n$  vertices which is not a regular graph. Therefore, by Theorem 3.4.6, we obtain

**Theorem 3.4.20** (Chappelon [C3]). *There is no regular Steinhaus graph of odd degree on  $2 < n \leq 1500$  vertices.*

# Chapter 4

## Balanced triangles modulo $m$

In this chapter, we are interested in a generalization of the Steinhaus Problem for triangles of elements of  $\mathbb{Z}/m\mathbb{Z}$  built with the same local rule as the standard Pascal triangle modulo a positive integer  $m$ .

### 4.1 The Molluzzo Problem

**Definition 4.1.1** (Steinhaus triangle modulo  $m$ ). A *Steinhaus triangle modulo  $m$*  of size  $n \geq 1$  is a down-pointing triangle  $(a_{i,j})_{1 \leq i \leq j \leq n}$  of elements of  $\mathbb{Z}/m\mathbb{Z}$  satisfying the same local rule as the standard Pascal triangle modulo  $m$ , that is,

$$a_{i,j} = a_{i-1,j-1} + a_{i-1,j}, \quad (\text{LR})$$

for all integers  $i$  and  $j$  such that  $2 \leq i \leq j \leq n$ , where the sum is the sum in  $\mathbb{Z}/m\mathbb{Z}$ . Note that  $(x)$ , for all  $x \in \mathbb{Z}/m\mathbb{Z}$ , are Steinhaus triangles modulo  $m$  of size 1 and we say that  $\emptyset$  is the Steinhaus triangle modulo  $m$  of size 0.

**Definition 4.1.2** (Generalized Pascal triangle modulo  $m$ ). A *generalized Pascal triangle modulo  $m$*  of size  $n \geq 1$  is an up-pointing triangle  $(a_{i,j})_{1 \leq j \leq i \leq n}$  of elements of  $\mathbb{Z}/m\mathbb{Z}$  verifying the local rule (LR), that is,

$$a_{i,j} = a_{i-1,j-1} + a_{i-1,j},$$

for all integers  $i, j$  such that  $2 \leq j < i \leq n$ . Note that  $(x)$ , for all  $x \in \mathbb{Z}/m\mathbb{Z}$ , are generalized Pascal triangles modulo  $m$  of size 1 and we say that  $\emptyset$  is the generalized Pascal triangle modulo  $m$  of size 0.

Examples of a Steinhaus triangle and a generalized Pascal triangle modulo 7 of size 7 are given in Figure 4.1.

As for the binary case  $m = 2$  introduced in Chapter 1, all the elements of a Steinhaus triangle are determined by its first row and all the elements of a generalized Pascal triangle by its left and right sides. More precisely, in any Steinhaus triangle  $(a_{i,j})_{1 \leq i \leq j \leq n}$ , we have

$$a_{i,j} = \sum_{k=0}^{i-1} \binom{i-1}{k} a_{1,j-k},$$

for all integers  $i$  and  $j$  such that  $1 \leq i \leq j \leq n$ , and in any generalized Pascal triangle  $(a_{i,j})_{1 \leq j \leq i \leq n}$ , we have

$$a_{i,j} = \sum_{k=2}^{i-j+1} \binom{i-k-1}{j-2} a_{k,1} + \sum_{k=2}^j \binom{i-k-1}{j-k} a_{k,k},$$

for all integers  $i$  and  $j$  such that  $1 \leq j \leq i \leq n$ .

**Notation 4.1.3.** Let  $\nabla S$  denote the Steinhaus triangle whose first row is the sequence  $S$  of  $\mathbb{Z}/m\mathbb{Z}$  and let  $\Delta(L, R)$  denote the generalized Pascal triangle whose left and right sides are the sequences  $L$  and  $R$  of  $\mathbb{Z}/m\mathbb{Z}$ , respectively. Note that the first terms of  $L$  and  $R$  must correspond.

**Notation 4.1.4.** For any positive integer  $m$  and any non-negative integer  $n$ , the sets of Steinhaus triangles and generalized Pascal triangles modulo  $m$  of size  $n$  are denoted by  $\mathcal{ST}_m(n)$  and  $\mathcal{PT}_m(n)$ , respectively.

For any positive integer  $m$  and any non-negative integer  $n$ , the sets  $\mathcal{ST}_m(n)$  and  $\mathcal{PT}_m(n)$  are  $\mathbb{Z}/m\mathbb{Z}$ -modules. Moreover, for any positive integer  $n$ , there exists a natural isomorphism between  $\mathcal{ST}_m(2n-1)$  and  $\mathcal{PT}_m(n)$  that is

$$\begin{aligned} \gamma : \quad \mathcal{ST}_m(2n-1) &\longrightarrow \mathcal{PT}_m(n) \\ (a_{i,j})_{1 \leq i \leq j \leq 2n-1} &\longmapsto (a_{i,n-1+j})_{1 \leq j \leq i \leq n} \end{aligned}$$

Steinhaus triangles and generalized Pascal triangles can be viewed as finite multisets of  $\mathbb{Z}/m\mathbb{Z}$ .

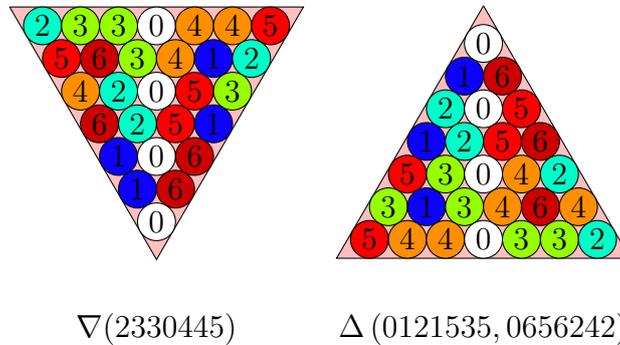


Figure 4.1: Examples of triangles of  $\mathbb{Z}/7\mathbb{Z}$  of size 7

**Definition 4.1.5** (Multiset). A finite *multiset* of elements of  $\mathbb{Z}/m\mathbb{Z}$  is a set of  $\mathbb{Z}/m\mathbb{Z}$  for which repeated elements are allowed. A finite multiset  $M$  of  $\mathbb{Z}/m\mathbb{Z}$  corresponds to a function  $\mathbf{m}_M : \mathbb{Z}/m\mathbb{Z} \rightarrow \mathbb{N}$ , the *multiplicity function* associated to  $M$ , which assigns its multiplicity in  $M$  to each element of  $\mathbb{Z}/m\mathbb{Z}$ . The *cardinality* of  $M$ , denoted by  $|M|$ , is the number of elements of  $M$  counted with multiplicity, that is the non-negative integer  $|M| = \sum_{x \in \mathbb{Z}/m\mathbb{Z}} \mathbf{m}_M(x)$ .

**Definition 4.1.6** (Balanced multiset). A finite multiset  $M$  of  $\mathbb{Z}/m\mathbb{Z}$  is said to be *balanced* if its multiplicity function is constant, i.e., if  $\mathbf{m}_M(x) = \mathbf{m}_M(y)$  for all  $x, y \in \mathbb{Z}/m\mathbb{Z}$ .

In 1976, John C. Molluzzo proposed the following problem on balanced Steinhaus triangles modulo  $m$ .

**Problem 4.1.7** (Molluzzo [89]). *Let  $m$  be a positive integer. For every non-negative integer  $n$  such that the binomial coefficient  $\binom{n+1}{2}$  is divisible by  $m$ , does there exist a balanced Steinhaus triangle of size  $n$  in  $\mathbb{Z}/m\mathbb{Z}$ ?*

For  $m = 2$ , we retrieve the Steinhaus Problem of Chapter 1. Since then, this problem has been positively solved, by constructive approaches, for small values of  $m$ : for  $m \in \{3, 5\}$  in [30], for  $m \in \{3, 5, 7\}$  in [C1], for  $m = 4$  in [C9]. First counter-examples appeared in [C1], where it is proved that there does not exist balanced Steinhaus triangles of size 5 in  $\mathbb{Z}/15\mathbb{Z}$  and of size 6 in  $\mathbb{Z}/21\mathbb{Z}$ . Nevertheless, this problem can be positively answered for an infinite number of values  $m$ . Indeed, as showed in [C1, C2], there exist balanced Steinhaus triangles, for all the possible sizes, in the case where  $m$  is a power of 3. This result was obtained by studying Steinhaus triangles associated with arithmetic progressions. Even if the Molluzzo Problem is not completely solved for the other odd values of  $m$ , we know that there exist infinitely many balanced Steinhaus triangles in every  $\mathbb{Z}/m\mathbb{Z}$  with  $m$  odd. This weak version of the Molluzzo Problem was posed in [C9].

**Problem 4.1.8** (Weak Molluzzo's Problem). *Let  $m$  be a positive integer. Do there exist infinitely many balanced Steinhaus triangles of  $\mathbb{Z}/m\mathbb{Z}$ ?*

Problem 4.1.8 is thus solved for the odd numbers  $m$ . For the even values, the cases  $m = 2$  and  $m = 4$  come from the solutions of Problem 4.1.7 and a solution is known from [59] for  $m \in \{6, 8, 10\}$ . This problem is completely open for the even numbers  $m \geq 12$ .

This chapter is organized as follows. After giving some preliminary results about Steinhaus triangles and the Molluzzo Problem in Section 2, we study in details the Steinhaus triangles generated by arithmetic progressions in Section 3. This permits us to obtain a positive answer to Problem 4.1.7 for  $m = 3^k$ , for all  $k \geq 0$ , and a positive answer to Problem 4.1.8 for any odd value of  $m$ . In Section 4, we show that there exists a particular sequence of integers whose projection of its orbit in  $\mathbb{Z}/m\mathbb{Z}$  contains balanced Steinhaus triangles of size  $km$  modulo  $m$ , for all  $k \geq 0$  and for all odd values of  $m$ . Finally, in Section 5, we are interested in the Molluzzo Problem for even values of  $m$ . We manage to lift to  $\mathbb{Z}/4\mathbb{Z}$  some solutions for  $\mathbb{Z}/2\mathbb{Z}$  obtained in Chapter 1.

## 4.2 Preliminary results

### 4.2.1 Derived and antiderived sequences of sequences modulo $m$

As for binary sequences in Chapter 1, we introduce the notions of derived and antiderived sequences of sequences of  $\mathbb{Z}/m\mathbb{Z}$ .

**Definition 4.2.1** (Derived sequence). Let  $S = (a_j)_{1 \leq j \leq n}$  be a sequence of  $\mathbb{Z}/m\mathbb{Z}$ . The *derived sequence*  $\partial S$  of  $S$  is the sequence

$$\partial S = (a_j + a_{j+1})_{1 \leq j \leq n-1} \quad (4.1)$$

of length  $n - 1$ , when  $n \geq 2$ , and the empty sequence, when  $n \leq 1$ .

It is clear that the derivation map  $\partial$  is linear, i.e.,  $\partial(S_1 + S_2) = \partial S_1 + \partial S_2$  for all sequences  $S_1$  and  $S_2$  of same length.

**Definition 4.2.2** (Iterated derived sequence). The *iterated derived sequences*  $\partial^i S$  of a finite sequence  $S$  of  $\mathbb{Z}/m\mathbb{Z}$  are recursively defined by  $\partial^i S = \partial(\partial^{i-1} S)$ , for all  $i \geq 1$ , with  $\partial^0 S = S$ .

For any finite sequence  $S$  of  $\mathbb{Z}/m\mathbb{Z}$ , its Steinhaus triangle  $\nabla S$  can then be seen as the collection  $(\partial^i S)_{0 \leq i \leq n-1}$ , where, for every  $i \in \{1, \dots, n\}$ , the  $i$ th row of  $\nabla S$  corresponds to the derived sequence  $\partial^{i-1} S$ .

**Definition 4.2.3** (Antiderived sequence). Let  $S$  be a sequence of  $\mathbb{Z}/m\mathbb{Z}$  of length  $n$ . An *antiderived sequence* of  $S$  is a sequence  $T$  of length  $n + 1$  such that  $\partial T = S$ .

**Proposition 4.2.4.** Let  $S = (a_j)_{1 \leq j \leq n}$  be a sequence of  $\mathbb{Z}/m\mathbb{Z}$  of length  $n$ . The sequence  $S$  admits  $m$  antiderived sequences  $T_1, \dots, T_m$  and these sequences verify

$$\{T_{k,i} \mid k \in \{1, \dots, m\}\} = \mathbb{Z}/m\mathbb{Z},$$

for all  $i \in \{1, \dots, n + 1\}$ , where  $T_{k,i}$  is the  $i$ th term of the sequence  $T_k$ . For any  $i \in \{1, \dots, n + 1\}$  and any  $x \in \mathbb{Z}/m\mathbb{Z}$ , the antiderived sequence of  $S$  whose  $i$ th term is  $x$  is the sequence  $\int_{i,x} S = (b_j)_{1 \leq j \leq n+1}$  of length  $n + 1$  defined by

$$b_j = (-1)^{i+j} x + \sum_{k=1}^{i-1} (-1)^{j+k} a_k - \sum_{k=1}^{j-1} (-1)^{j+k} a_k, \quad (4.2)$$

for all  $j \in \{1, \dots, n + 1\}$ .

For example, the sequence  $S = 0143$  of  $\mathbb{Z}/5\mathbb{Z}$  admits the five antiderived sequences 00130, 14221, 23312, 32403 and 41044. Further, it is straightforward to obtain a fundamental theorem of calculus.

**Proposition 4.2.5.** Let  $S = (a_j)_{1 \leq j \leq n}$  be a binary sequence of length  $n$ . For any  $i \in \{1, \dots, n + 1\}$  and any  $x \in \mathbb{Z}/m\mathbb{Z}$ , we have that

$$i) \quad \partial \left( \int_{i,x} S \right) = S,$$

$$ii) \quad \int_{i,x} (\partial S) = S + \left( (-1)^{i+j} (x - a_i) \right)_{1 \leq j \leq n}.$$

## 4.2.2 The average number of each element in triangles modulo $m$

Using the derivative and antiderivative operations on sequences of  $\mathbb{Z}/m\mathbb{Z}$  introduced before, it is now easy to determine, for any positive integer  $n$  and every  $x \in \mathbb{Z}/m\mathbb{Z}$ , the average number of  $x$  in a Steinhaus triangle or a generalized Pascal triangle modulo  $m$  of size  $n$ , i.e.,

$$\frac{1}{m^n} \sum_{\nabla \in ST_m(n)} \mathbf{m}_{\nabla}(x) \quad \text{and} \quad \frac{1}{m^{2n-1}} \sum_{\Delta \in PT_m(n)} \mathbf{m}_{\Delta}(x).$$

**Proposition 4.2.6.** For any non-negative integer  $n$ , the average number of  $x$  in a Steinhaus triangle or in a generalized Pascal triangle modulo  $m$  of size  $n$  is exactly  $\frac{1}{m} \binom{n+1}{2}$ , for all  $x \in \mathbb{Z}/m\mathbb{Z}$ .

The Molluzzo Problem then corresponds to the determination of the existence of Steinhaus triangles modulo  $m$  with an average number of each element of  $\mathbb{Z}/m\mathbb{Z}$ , that are Steinhaus triangles  $\nabla$  with  $\mathbf{m}_{\nabla}(x) = \mathbf{m}_{\nabla}(y)$ , for all  $x, y \in \mathbb{Z}/m\mathbb{Z}$ .

### 4.2.3 Balanced multisets under projection maps

**Definition 4.2.7.** For every factor  $q$  of the positive integer  $m$ , we denote by  $\pi_q$  the canonical surjective morphism  $\pi_q : \mathbb{Z}/m\mathbb{Z} \longrightarrow \mathbb{Z}/q\mathbb{Z}$ . For a finite multiset  $M$  of elements in  $\mathbb{Z}/m\mathbb{Z}$ , we define, and denote by

$$\pi_q(M) = \{\pi_q(x) \mid x \in M\},$$

its projected multiset in  $\mathbb{Z}/q\mathbb{Z}$ .

We now study the behaviour of balanced multisets in  $\mathbb{Z}/m\mathbb{Z}$  under the projection morphism  $\pi_q : \mathbb{Z}/m\mathbb{Z} \longrightarrow \mathbb{Z}/q\mathbb{Z}$ .

**Theorem 4.2.8** (Chappelon [C2]). *Let  $q$  be a divisor of  $m$  and  $M$  be a multiset of elements in  $\mathbb{Z}/m\mathbb{Z}$ . Then, the multiset  $M$  is balanced if, and only if, its projected multiset  $\pi_q(M)$  is balanced and the multiplicity function  $\mathbf{m}_M : \mathbb{Z}/m\mathbb{Z} \longrightarrow \mathbb{N}$  is constant on each coset of the subgroup  $q\mathbb{Z}/m\mathbb{Z}$ .*

*Proof.* For every  $x$  in  $\mathbb{Z}/m\mathbb{Z}$ , it is clear that the multiplicity of  $\pi_q(x)$  in  $\pi_q(M)$  is the sum of the multiplicities in  $M$  of all the elements of the coset  $x + q\mathbb{Z}/m\mathbb{Z}$ , that is,

$$\mathbf{m}_{\pi_q(M)}(\pi_q(x)) = \sum_{k=0}^{\frac{m}{q}-1} \mathbf{m}_M(x + kq), \quad \forall x \in \mathbb{Z}/m\mathbb{Z}.$$

This completes the proof. □

*Remark 4.2.9.* Theorem 4.2.8 is true only when  $|M|$  is divisible by  $m$ . For instance, as depicted in Figure 4.2, the Steinhaus triangle  $\nabla(153)$  is balanced modulo 10 while its projection  $\pi_5(\nabla(153)) = \nabla(103)$  is not in  $\mathbb{Z}/5\mathbb{Z}$ .

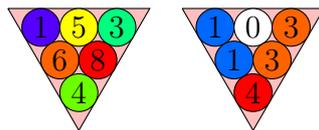


Figure 4.2: The Steinhaus triangle  $\nabla(153)$  of  $\mathbb{Z}/10\mathbb{Z}$  and its projection  $\pi_5(\nabla(153))$  in  $\mathbb{Z}/5\mathbb{Z}$

Therefore, in the sequel of this chapter, we only consider triangles of size  $n$  such that  $\binom{n+1}{2}$  is divisible by  $m$ .

**Notation 4.2.10.** The set of all prime numbers is denoted by  $\mathcal{P}$ . For every prime number  $p$ , we denote by  $v_p(m)$  the  $p$ -adic valuation of  $m$ , i.e. the greatest exponent  $e \geq 0$  for which  $p^e$  divides  $m$ . The prime factorization of  $m$  may then be written as

$$m = \prod_{p \in \mathcal{P}} p^{v_p(m)}.$$

We denote by  $\omega(m)$  the number of distinct prime factors of  $m$ , i.e. the number of primes  $p$  for which  $v_p(m) \geq 1$ .

The set of all positive integers  $n$  such that  $m$  divides  $\binom{n+1}{2}$  is described in the following

**Theorem 4.2.11** (Chappelon [C2]). *Let  $m$  be a positive integer. The set of all non-negative integers  $n$  such that the binomial coefficient  $\binom{n+1}{2}$  is a multiple of  $m$  is a disjoint union of  $2^{\omega(m)}$  distinct classes modulo  $2m$  if  $m$  is even, and of the same number of distinct classes modulo  $m$  if  $m$  is odd. This set comprises the classes  $2m\mathbb{N}$  and  $(2m-1) + 2m\mathbb{N}$  if  $m$  is even, and the classes  $m\mathbb{N}$  and  $(m-1) + m\mathbb{N}$  if  $m$  is odd.*

**Corollary 4.2.12.** *Let  $p$  be an odd prime number and  $k$  be a positive integer. For every non-negative integer  $n$ , we have*

$$\binom{n+1}{2} \equiv 0 \pmod{p^k} \iff n \equiv 0 \text{ or } -1 \pmod{p^k}.$$

Similarly, for every non-negative integer  $n$ , we have

$$\binom{n+1}{2} \equiv 0 \pmod{2^k} \iff n \equiv 0 \text{ or } -1 \pmod{2^{k+1}}.$$

For instance, for  $m = 825 = 3 \cdot 5^2 \cdot 11$ , the set of non-negative integers  $n$  such that the binomial coefficient  $\binom{n+1}{2}$  is divisible by 825 is the disjoint union of the 8 classes  $a + 825\mathbb{N}$  with  $a \in \{0, 99, 275, 374, 450, 549, 725, 824\}$ .

## 4.3 Balanced Steinhaus triangles from arithmetic progressions

### 4.3.1 Steinhaus triangles of arithmetic progressions

In this subsection, we describe the structure of Steinhaus triangles associated with arithmetic progressions of  $\mathbb{Z}/m\mathbb{Z}$ .

**Notation 4.3.1.** Let  $\text{AP}(a, d, n)$  denote the *arithmetic progression* beginning with  $a$ , with common difference  $d$  and of length  $n$ , that is,

$$\text{AP}(a, d, n) = (a, a + d, a + 2d, \dots, a + (n-1)d),$$

for any  $a, d \in \mathbb{Z}/m\mathbb{Z}$  and any non-negative integer  $n$ .

First, it is straightforward to see that the structure of arithmetic progressions is preserved under the derivation process.

**Proposition 4.3.2.** *Let  $a, d \in \mathbb{Z}/m\mathbb{Z}$ . Then,*

$$\partial \text{AP}(a, d, n) = \text{AP}(2a + d, 2d, n-1),$$

for all positive integers  $n$ .

**Corollary 4.3.3.** *Let  $a, d \in \mathbb{Z}/m\mathbb{Z}$ . Then,*

$$\partial^i \text{AP}(a, d, n) = \text{AP}(2^i a + 2^{i-1} id, 2^i d, n-i),$$

for all integers  $n$  and  $i$  such that  $1 \leq i \leq n$ .

**Corollary 4.3.4.** *Let  $a, d \in \mathbb{Z}/m\mathbb{Z}$ . If  $\nabla\text{AP}(a, d, n) = (a_{i,j})_{1 \leq i \leq j \leq n}$ , then*

$$a_{i,j} = \begin{cases} a + (j-1)d & \text{for } i = 1 \text{ and for all } j \in \{1, \dots, n\}, \\ 2^{i-1}a + 2^{i-2}(2j-i)d & \text{for all integers } i, j \text{ s.t. } 2 \leq i \leq j \leq n. \end{cases}$$

For any even number  $m$ , we deduce from Corollary 4.3.4 that in the Steinhaus triangle  $\nabla\text{AP}(a, d, n) = (a_{i,j})_{1 \leq i \leq j \leq n}$ , we have

$$a_{i,j} \in \{2k \mid k \in \mathbb{Z}/m\mathbb{Z}\}$$

for all integers  $i$  and  $j$  such that  $3 \leq i \leq j \leq n$ . Therefore, if  $\nabla\text{AP}(a, d, n)$  is balanced, then all the elements of  $\{2k+1 \mid k \in \mathbb{Z}/m\mathbb{Z}\}$  are in the first two rows of  $\nabla\text{AP}(a, d, n)$ . It follows that

$$n + (n-1) \geq \frac{1}{2} \binom{n+1}{2}$$

and thus  $n \leq 6$ . Moreover, since  $m$  is even, we have that  $\binom{n+1}{2}$  must be divisible by 2. This implies that  $n \in \{0, 3, 4\}$ .

**Proposition 4.3.5.** *Let  $m$  be an even integer and let  $n$  be a non-negative integer. If  $\nabla\text{AP}(a, d, n)$  is balanced, then  $n \in \{0, 3, 4\}$ .*

The Steinhaus triangle of size 0 is always balanced in  $\mathbb{Z}/m\mathbb{Z}$ , for any positive integer  $m$ . For any even number  $m$  and any positive integer  $n$ , the complete determination of balanced triangles  $\nabla\text{AP}(a, d, n)$  is given below.

**Theorem 4.3.6** (Chappelon [C2]). *Let  $m$  be an even number. Let  $a, d \in \mathbb{Z}/m\mathbb{Z}$  and let  $n$  be a positive integer. Then the Steinhaus triangle generated from  $X = \text{AP}(a, d, n)$  is balanced if, and only if, we have*

$$\begin{cases} m = 2 & \text{and } X \in \{010, 111, 0101, 1010\}, \\ \text{or} \\ m = 6 & \text{and } X \in \{135, 234, 432, 531\}. \end{cases}$$

In the sequel of this section, we suppose that  $m$  is an odd number. In this case, it is easy to see that among the  $m$  antiderived sequences of an arithmetic progression, only one is also an arithmetic progression.

**Proposition 4.3.7.** *Let  $m$  be an odd number and let  $a, d \in \mathbb{Z}/m\mathbb{Z}$ . Then, the sequence  $\text{AP}(2^{-1}a - 2^{-2}d, 2^{-1}d, n+1)$  is the only arithmetic progression whose derived sequence is the arithmetic progression  $\text{AP}(a, d, n)$ .*

In contrast, for  $n$  even, there is no such unicity statement. For example, in  $\mathbb{Z}/8\mathbb{Z}$ , the arithmetic progressions 37373 and 11111 are distinct but have the same derived sequence 2222.

### 4.3.2 Balanced Steinhaus triangles from arithmetic progressions modulo $m$ odd

We begin by showing that the common difference of a balanced Steinhaus triangle of an arithmetic progression in  $\mathbb{Z}/m\mathbb{Z}$  must be invertible.

**Theorem 4.3.8** (Chappelon [C2]). *Let  $m$  be an odd number and let  $a, d \in \mathbb{Z}/m\mathbb{Z}$ . If  $d$  is non-invertible, then the Steinhaus triangle  $\nabla\text{AP}(a, d, n)$  is not balanced, for all positive integers  $n$ .*

We continue by studying Steinhaus triangles of arithmetic progressions with invertible common differences.

**Definition 4.3.9.** For every coprime integers  $m$  and  $a$ , we denote by  $\text{ord}_m(a)$  the multiplicative order of  $a$  modulo  $m$ , i.e. the smallest positive integer  $e$  such that  $a^e \equiv 1 \pmod m$ , namely

$$\text{ord}_m(a) = \min \{e \in \mathbb{N}^* \mid a^e \equiv 1 \pmod m\}.$$

Note that  $\text{ord}_m(a)$  is always a divisor of  $\varphi(m)$ , the totient of  $m$ .

The following theorem is the main result of this section.

**Theorem 4.3.10** (Chappelon [C2]). *Let  $m$  be an odd number. Let  $a, d \in \mathbb{Z}/m\mathbb{Z}$ , with  $d$  invertible. Then, the Steinhaus triangle  $\nabla\text{AP}(a, d, n)$  is balanced, for all positive integers  $n \equiv 0$  or  $-1 \pmod{\text{ord}_m(2^m)m}$ .*

For example, in  $\mathbb{Z}/7\mathbb{Z}$ , the Steinhaus triangle  $\nabla\text{AP}(1, 3, 20)$  is balanced since 3 is an invertible element of  $\mathbb{Z}/7\mathbb{Z}$  and  $\text{ord}_7(2^7) = 3$ . Indeed, each element of  $\mathbb{Z}/7\mathbb{Z}$  occurs 30 times in this Steinhaus triangle.

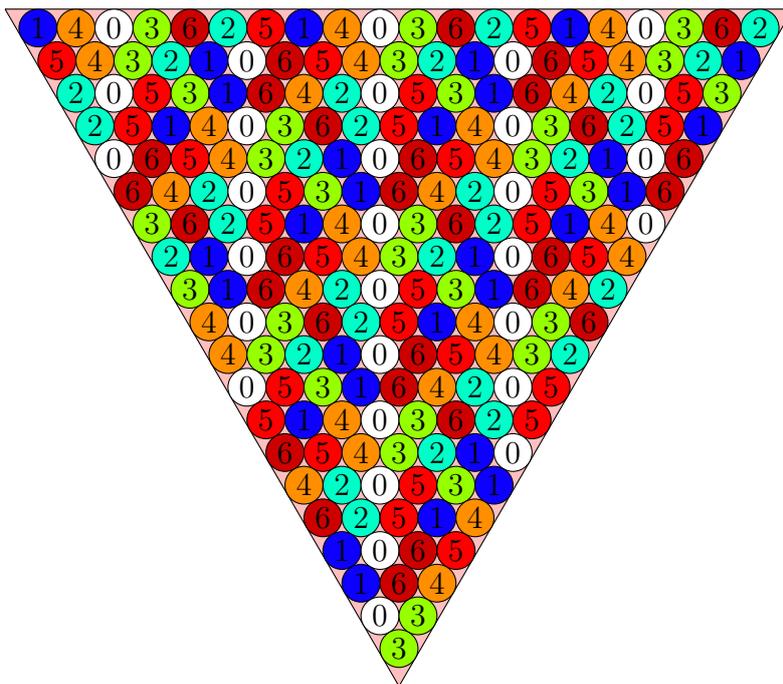


Figure 4.3: The Steinhaus triangle  $\nabla\text{AP}(1, 3, 20)$

Since there are  $n$  distinct elements  $a$  in  $\mathbb{Z}/m\mathbb{Z}$  and  $\varphi(m)$  distinct invertible elements  $d$  in  $\mathbb{Z}/m\mathbb{Z}$ , it follows that, for each positive integer  $n$ , there exist exactly  $m\varphi(m)$  distinct arithmetic progressions  $\text{AP}(a, d, n)$  with invertible common difference in  $\mathbb{Z}/m\mathbb{Z}$  and of length  $n$ . Therefore, for  $m$  odd, Theorem 4.3.10 implies that there exist at least  $m\varphi(m)$  balanced Steinhaus triangles of size  $n$  for every positive integer  $n \equiv 0$  or  $-1 \pmod{\text{ord}_m(2^m)m}$ . However, this is not sufficient to completely settle the Molluzzo Problem, as shown by the following proposition. This shortcoming will be partly overcome in the next subsection.

**Proposition 4.3.11.** *Let  $m > 1$  be an odd number. Then*

$$\text{ord}_m(2^m) \geq 2.$$

### 4.3.3 The antisymmetric case

Here, we refine Theorem 4.3.10 by considering antisymmetric sequences of  $\mathbb{Z}/m\mathbb{Z}$ .

**Definition 4.3.12** (Antisymmetric sequence). Let  $X = (x_1, x_2, \dots, x_n)$  be a finite sequence of length  $n \geq 1$  in  $\mathbb{Z}/m\mathbb{Z}$ . The sequence  $X$  is said to be *antisymmetric* if  $x_{m-i+1} = -x_i$ , for all  $i \in \{1, \dots, n\}$ .

We first show that the antisymmetry is preserved by the derivation process and we study the condition to have an antisymmetric antiderived sequence of an antisymmetric sequence.

**Proposition 4.3.13** (Chappelon [C2]). *Let  $X = (x_1, x_2, \dots, x_n)$  be a finite sequence of length  $n \geq 1$  in  $\mathbb{Z}/m\mathbb{Z}$ . Then the sequence  $X$  is antisymmetric if, and only if, its derived sequence  $\partial X$  is also antisymmetric and  $x_{\lceil \frac{n}{2} \rceil} + x_{n-\lceil \frac{n}{2} \rceil+1} = 0$ , where  $\lceil \frac{n}{2} \rceil$  is the ceiling of  $\frac{n}{2}$ .*

**Proposition 4.3.14** (Chappelon [C2]). *Let  $m$  be an odd number. Let  $a, d \in \mathbb{Z}/m\mathbb{Z}$ . Then, the arithmetic progression  $\text{AP}(a, d, n)$  of length  $n \geq 2$  is antisymmetric if, and only if, its derived sequence  $\text{AP}(2a + d, 2d, n - 1)$  is also antisymmetric.*

In contrast, for  $m$  even, this proposition is not true. For instance, for  $n = 8$ , the arithmetic progression  $X = 01234$  is not antisymmetric in  $\mathbb{Z}/8\mathbb{Z}$  but its derived sequence  $\partial X = 1357$  is.

We now determine arithmetic progressions which are antisymmetric in  $\mathbb{Z}/m\mathbb{Z}$  for  $m$  odd.

**Proposition 4.3.15** (Chappelon [C2]). *Let  $m$  be an odd number. Let  $d$  be in  $\mathbb{Z}/m\mathbb{Z}$  and  $n$  be a positive integer. Then, there exists a unique antisymmetric arithmetic progression of length  $n$  and with common difference  $d$ . Moreover, if  $n$  is a multiple of  $m$ , then the unique antisymmetric arithmetic progression with common difference  $d$  and of length  $n$  is the sequence  $\text{AP}(2^{-1}d, d, n)$ . If  $n \equiv -1 \pmod{m}$ , then the unique antisymmetric arithmetic progression with common difference  $d$  and of length  $n$  is the sequence  $\text{AP}(d, d, n)$ .*

If  $m$  is even, the above unicity does not hold in general. For example, in  $\mathbb{Z}/8\mathbb{Z}$ , the antisymmetric sequences 02460 and 46024 are both arithmetic progressions of length  $n = 5$  and of common difference  $d = 2$ .

**Definition 4.3.16.** For every coprime integers  $m$  and  $a$ , we denote by  $\text{pord}_m(a)$  the *projective multiplicative order* of  $a$  modulo  $m$ , i.e., the smallest positive integer  $e$  such that  $a^e \equiv \pm 1 \pmod{m}$ , namely

$$\text{pord}_m(a) = \min \{e \in \mathbb{N}^* \mid a^e \equiv \pm 1 \pmod{m}\}.$$

Note that we have the alternative  $\text{ord}_m(a) = \text{pord}_m(a)$  or  $\text{ord}_m(a) = 2\text{pord}_m(a)$ . Moreover,  $\text{ord}_m(a) = 2\text{pord}_m(a)$  if and only if there exists a power  $e$  of  $a$  such that  $a^e \equiv -1 \pmod{m}$ .

We now improve Theorem 4.3.10 by considering the antisymmetric arithmetic progressions with invertible common difference. There are exactly  $\varphi(m)$  such sequences, for every length, by Proposition 4.3.15.

**Theorem 4.3.17** (Chappelon [C2]). *Let  $m$  be an odd number and  $d$  be an invertible element in  $\mathbb{Z}/m\mathbb{Z}$ . Then*

- for every  $n \equiv 0 \pmod{\text{pord}_m(2^m)}$ , the triangle  $\nabla\text{AP}(2^{-1}d, d, n)$  is balanced,
- for every  $n \equiv -1 \pmod{\text{pord}_m(2^m)}$ , the triangle  $\nabla\text{AP}(d, d, n)$  is balanced.

We shall now see that this theorem answers in the affirmative the Molluzzo Problem in  $\mathbb{Z}/3^k\mathbb{Z}$  for all positive integers  $k$  and gives a partial answer in the general odd case.

**Corollary 4.3.18** (Chappelon [C2]). *The Molluzzo Problem is completely solved in  $\mathbb{Z}/3^k\mathbb{Z}$  for all positive integers  $k$ . In other words, there exists a balanced Steinhau triangle of size  $n$  of  $\mathbb{Z}/3^k\mathbb{Z}$  for all positive integers  $n$  such that  $\binom{n+1}{2}$  is divisible by  $3^k$ .*

*Proof.* Let  $k$  be a positive integer. We have

$$\text{pord}_{3^k}(2^{3^k}) = \text{pord}_3(2^3) = 1.$$

Let  $d$  be an invertible element in  $\mathbb{Z}/3^k\mathbb{Z}$ . Then, Theorem 4.3.17 implies that

- $\text{AP}(2^{-1}d, d, n)$  is balanced for every positive integer  $n \equiv 0 \pmod{3^k}$ ,
- $\text{AP}(d, d, n)$  is balanced for every positive integer  $n \equiv -1 \pmod{3^k}$ .

Finally, from Corollary 4.2.12, we know that  $3^k$  divides the binomial coefficient  $\binom{n+1}{2}$  if, and only if, the positive integer  $n$  is congruent to 0 or  $-1$  modulo  $3^k$ .  $\square$

For every odd number  $m$ , Theorem 4.2.11 and Theorem 4.3.17 partly solve the Molluzzo Problem in  $\mathbb{Z}/m\mathbb{Z}$  in the exact proportion of  $\frac{1}{2^{\omega(m)-1}\text{pord}_m(2^m)}$ , where  $\omega(m)$  is the number of distinct prime factors of  $m$ . Indeed, if we consider the sets

$$N(m) = \left\{ n \in \mathbb{N} \mid \binom{n+1}{2} \equiv 0 \pmod{m} \right\},$$

and

$$B(m) = \{n \in \mathbb{N} \mid \exists \text{ a balanced sequence in } \mathbb{Z}/m\mathbb{Z} \text{ of length } n\},$$

then clearly  $B(m) \subset N(m)$ . Moreover, the Molluzzo Problem can be reformulated as the question whether  $B(m) = N(m)$  for all  $m > 1$ .

It follows from Theorem 4.2.11 and Theorem 4.3.17 that

$$\frac{|B(m) \cap \{0, 1, \dots, k\}|}{|N(m) \cap \{0, 1, \dots, k\}|} \geq \frac{1}{2^{\omega(m)-1}\text{pord}_m(2^m)},$$

for all  $k \geq \text{pord}_m(2^m)$ . Since  $2^{\omega(m)-1}\text{pord}_m(2^m) \geq 2$  for every odd number  $m \neq 3^k$ , it follows that our method gives a complete solution to the Molluzzo Problem for the powers of three only. For example, for  $m = 5^k$ , we have  $2^{\omega(m)-1}\text{pord}_m(2^m) = 2$ , whence our results in this case produce balanced sequences for half of the admissible lengths.

## 4.4 A universal integer sequence generating balanced triangles modulo $m$ odd

### 4.4.1 Orbits and triangles modulo $m$

In this section, we are also interested in the Molluzzo Problem and its weak version for odd values of  $m$ , for Steinhaus triangles and for generalized Pascal triangles modulo  $m$ . These triangles can be seen as finite subparts of the orbit associated with a doubly infinite sequence of elements of  $\mathbb{Z}/m\mathbb{Z}$ , where the derivation process is the same as defined above for finite sequences, with the same local rule as the standard Pascal triangle modulo  $m$ .

**Definition 4.4.1** (Derived sequences). Let  $m$  be a positive integer and let  $S = (a_j)_{j \in \mathbb{Z}}$  be a doubly infinite sequence of elements of  $\mathbb{Z}/m\mathbb{Z}$ . The *derived sequence* of  $S$  is the sequence

$$\partial S = (a_j + a_{j+1})_{j \in \mathbb{Z}}.$$

The  $i$ th *derived sequence*  $\partial^i S$  is recursively defined by  $\partial^i S = \partial(\partial^{i-1} S)$ , for all integers  $i \geq 1$ , with  $\partial^0 S = S$ .

**Definition 4.4.2** (Orbit). Let  $m$  be a positive integer and let  $S = (a_j)_{j \in \mathbb{Z}}$  be a doubly infinite sequence of elements of  $\mathbb{Z}/m\mathbb{Z}$ . The *orbit* of  $S$  is the sequence

$$\mathcal{O}_S = (\partial^i S)_{i \in \mathbb{N}}.$$

The orbit of  $S$  can also be seen as the  $(\mathbb{N} \times \mathbb{Z})$ -indexed sequence

$$\mathcal{O}_S = (a_{i,j})_{(i,j) \in \mathbb{N} \times \mathbb{Z}}$$

where  $a_{i,j}$  is the  $j$ th term of  $\partial^i S$ , for all  $i \in \mathbb{N}$  and all  $j \in \mathbb{Z}$ . Then, we have

$$a_{i+1,j} = a_{i,j} + a_{i,j+1},$$

for all integers  $i \geq 1$  and  $j$ , by the local rule of this cellular automaton. Moreover, we know that

$$a_{i,j} = \sum_{k=0}^i \binom{i}{k} a_{j+k}$$

for all integers  $i \geq 0$  and  $j$ .

**Notation 4.4.3.** Let  $S$  be a doubly infinite sequence of  $\mathbb{Z}/m\mathbb{Z}$  and let  $\mathcal{O}_S = (a_{i,j})_{(i,j) \in \mathbb{N} \times \mathbb{Z}}$  its orbit. For every  $i \in \mathbb{N}$ , the  $i$ th *row* of  $\mathcal{O}_S$  is the sequence  $R_i = \partial^i S = (a_{i,j})_{j \in \mathbb{Z}}$  and, for every  $j \in \mathbb{Z}$ , the  $j$ th *diagonal* and the  $j$ th *anti-diagonal* of  $\mathcal{O}_S$  are the sequences  $D_j = (a_{i,j})_{i \in \mathbb{N}}$  and  $AD_j = (a_{i,j-i})_{i \in \mathbb{N}}$  respectively.

Examples of Steinhaus triangles and generalized Pascal triangles appearing in an orbit of a doubly infinite sequence of  $\mathbb{Z}/5\mathbb{Z}$  are depicted in Figure 4.4.

**Definition 4.4.4** (Interlaced arithmetic progression). For all positive integers  $n$  and  $k$  and for all  $k$ -tuples of elements  $A = (a_0, \dots, a_{k-1})$  and  $D = (d_0, \dots, d_{k-1})$  in  $\mathbb{Z}/m\mathbb{Z}$ , or in  $\mathbb{Z}$ , the  $k$ -*interlaced arithmetic progression*  $IAP(A, D)$  is the sequence with first terms  $(a_0, \dots, a_{k-1})$  and with common differences  $(d_0, \dots, d_{k-1})$ , that is the doubly infinite sequence  $IAP(A, D) = (a_j)_{j \in \mathbb{Z}}$  defined by  $a_{j_0+jk} = a_{j_0} + jd_{j_0}$ , for all  $j \in \mathbb{Z}$  and for every  $j_0 \in \{0, 1, \dots, k-1\}$ . For  $k = 1$ , we denote by  $AP(a_0, d_0)$  the arithmetic progression with first element  $a_0$  and with common difference  $d_0$ .

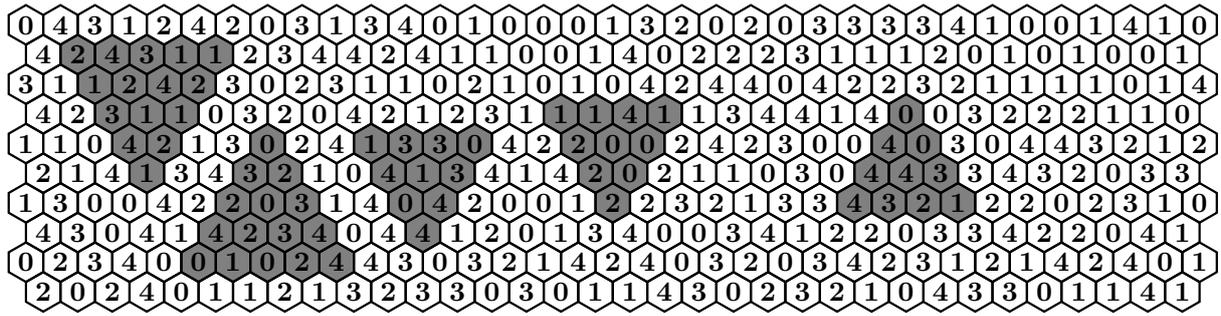


Figure 4.4: Examples of triangles in  $\mathbb{Z}/5\mathbb{Z}$ :  $\nabla(24311)$ ,  $\Delta(03240, 02344)$ ,  $\nabla(1330)$ ,  $\nabla(1141)$  and  $\Delta(0444, 0031)$

Let  $S = \text{IAP}((0, -1, 1), (1, -2, 1))$ . We shall show that this sequence has the remarkable property that the orbit of its projection  $\pi_m(S)$  contains infinitely many balanced Steinhaus figures for every odd number  $m$ . For this reason, we shall call this sequence the *universal sequence* and denote it by US. The first few terms of US, where  $\mathbf{0}$  is the term of index 0, are given below:

$$\text{US} = (\dots, -3, -3, 5, -2, -2, 3, -1, -1, 1, 0, \mathbf{0}, -1, 1, 1, -3, 2, 2, -5, 3, 3, -7, \dots).$$

The following theorem is the main result of this section.

**Theorem 4.4.5** (Chappelon [C5]). *Let  $m$  be an odd number. Then, the orbit of the projection  $\pi_m(\text{US})$  of the universal sequence in  $\mathbb{Z}/m\mathbb{Z}$  contains:*

- i) *balanced Steinhaus triangles of size  $n$  for every  $n \equiv 0 \pmod m$  or  $n \equiv -1 \pmod{3m}$ . This partially solves the Molluzzo Problem for  $2/3$  of the admissible sizes  $n$ , in the case where  $m$  is an odd prime power.*
- ii) *balanced generalized Pascal triangles of size  $n$  for every  $n \equiv -1 \pmod m$  or  $n \equiv 0 \pmod{3m}$ . This also gives a partial solution of the analogous Molluzzo Problem for generalized Pascal triangles, for  $2/3$  of the admissible sizes  $n$ , in the case where  $m$  is an odd prime power.*

It would be highly desirable to have a similar result for  $m$  even, but this is widely open. This section is organized as follows. In Subsection 2, we study doubly arithmetic triangles (DAT for short) in  $\mathbb{Z}/m\mathbb{Z}$ . These are triangles whose rows are arithmetic progressions with the same common difference and whose diagonals are also arithmetic progressions with the same common difference. We show that these triangles constitute a source of balanced multisets in  $\mathbb{Z}/m\mathbb{Z}$ , for  $m$  odd, while they are never balanced in  $\mathbb{Z}/m\mathbb{Z}$ , for  $m$  even. In Subsection 3, interlaced doubly arithmetic orbits, i.e., orbits that are an interlacing of doubly arithmetic structures, are considered. We determine all the interlaced doubly arithmetic orbits in  $\mathbb{Z}$  and, in Subsection 4, we show that the projection of these particular orbits in  $\mathbb{Z}/m\mathbb{Z}$ , for  $m$  odd, contains infinitely many balanced triangles. This result is refined in Subsection 5, by considering antisymmetric sequences. In Subsection 6, a particular case of this antisymmetric refinement leads to the universal sequence US and Theorem 4.4.5.

### 4.4.2 DAT: a source of balanced multisets

**Definition 4.4.6** (DAT). For all positive integers  $m$  and  $n$  and for all elements  $a, d_1$  and  $d_2$  in  $\mathbb{Z}/m\mathbb{Z}$ , the *doubly arithmetic triangle*  $\text{DAT}(a, d_1, d_2, n)$  is the triangle of size  $n$  in  $\mathbb{Z}/m\mathbb{Z}$ , with first element  $a$  and where each diagonal and each row are arithmetic progressions with respective common differences  $d_1$  and  $d_2$ , that is the multiset in  $\mathbb{Z}/m\mathbb{Z}$  defined by

$$\text{DAT}(a, d_1, d_2, n) = \{a + id_1 + jd_2 \mid 0 \leq i \leq n - 1, 0 \leq j \leq n - 1 - i\}.$$

Obviously, the anti-diagonals of a DAT are arithmetic progressions with common difference  $d_1 - d_2$ .

Here, we show that doubly arithmetic triangles constitute a source of balanced multisets in  $\mathbb{Z}/m\mathbb{Z}$ , for  $m$  odd. We begin by determining a necessary condition, on the common differences  $d_1$  and  $d_2$ , to obtain a balanced DAT in  $\mathbb{Z}/m\mathbb{Z}$ .

**Proposition 4.4.7** (Chappelon [C5]). *Let  $m$  and  $n$  be two positive integers and let  $a, d_1, d_2 \in \mathbb{Z}/m\mathbb{Z}$ . If the doubly arithmetic triangle  $\text{DAT}(a, d_1, d_2, n)$  is balanced, then its common differences  $d_1, d_2$  and  $d_1 - d_2$  are invertible in  $\mathbb{Z}/m\mathbb{Z}$ .*

*Remark 4.4.8.* For  $m$  even, there is no balanced DAT in  $\mathbb{Z}/m\mathbb{Z}$  since at least one element of  $\{d_1, d_2, d_1 - d_2\}$  is not invertible in  $\mathbb{Z}/m\mathbb{Z}$ , by the parity of  $m$ .

*Remark 4.4.9.* The condition on  $n$  to be a positive integer such that  $\binom{n+1}{2}$  is divisible by  $m$  is not sufficient: as depicted in Figure 4.5, the triangle  $\text{DAT}(0, 8, 1, 5)$  is not balanced in  $\mathbb{Z}/15\mathbb{Z}$ , although its cardinality  $\binom{6}{2} = 15$  is divisible by  $m = 15$  and its common differences 8, 1 and 7 are invertible in  $\mathbb{Z}/15\mathbb{Z}$ .

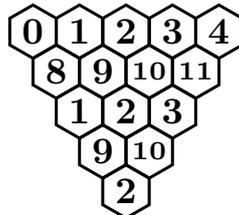


Figure 4.5: The doubly arithmetic triangle  $\text{DAT}(0, 8, 1, 5)$  in  $\mathbb{Z}/15\mathbb{Z}$

The following theorem is the main result of this subsection.

**Theorem 4.4.10** (Chappelon [C5]). *Let  $m$  be an odd number and let  $d_1, d_2 \in \mathbb{Z}/m\mathbb{Z}$  be invertible such that  $d_1 - d_2$  is also invertible. Then, the doubly arithmetic triangle  $\text{DAT}(a, d_1, d_2, n)$  is balanced in  $\mathbb{Z}/m\mathbb{Z}$  for all  $n \equiv 0$  or  $-1 \pmod m$ .*

*Remark 4.4.11.* For  $m$  odd and for every  $d \in \mathbb{Z}/m\mathbb{Z}$  invertible, the doubly arithmetic triangles  $\text{DAT}(a, d, -d, n)$ ,  $\text{DAT}(a, d, 2d, n)$  and  $\text{DAT}(a, 2d, d, n)$  are balanced in  $\mathbb{Z}/m\mathbb{Z}$ , for all  $n \equiv 0$  or  $-1 \pmod m$ .

Let  $m$  be a positive integer and let  $d_1$  and  $d_2$  be two elements of  $\mathbb{Z}/m\mathbb{Z}$ . The orbit  $\mathcal{O}_S$ , associated with a doubly infinite sequence  $S$  in  $\mathbb{Z}/m\mathbb{Z}$ , is said to be  $(d_1, d_2)$ -*doubly arithmetic* if each subtriangle appearing in it is a DAT with common differences  $(d_1, d_2)$ , that is if  $\mathcal{O}_S$  is an orbit where all the diagonals are arithmetic progressions with the same

common difference  $d_1$  and where all the rows are arithmetic progressions with the same common difference  $d_2$ .

Now, we prove that, for every positive integer  $m$ , there does not exist a doubly arithmetic orbit in  $\mathbb{Z}/m\mathbb{Z}$ , except the trivial orbit generated by the sequence of zeros in  $\mathbb{Z}/m\mathbb{Z}$ .

**Proposition 4.4.12.** *Let  $m$  be a positive integer. The orbit associated with the sequence of zeros is the only doubly arithmetic orbit in  $\mathbb{Z}/m\mathbb{Z}$ .*

Even if there does not exist a non-trivial doubly arithmetic orbit, the results of this subsection will be useful in next subsections, where orbits with an interlaced doubly arithmetic structure are studied.

### 4.4.3 Interlaced doubly arithmetic orbits of integers

**Definition 4.4.13** (Interlaced doubly arithmetic orbit). For all positive integers  $m$ ,  $k_1$  and  $k_2$  and for every doubly infinite sequence  $S$  in  $\mathbb{Z}/m\mathbb{Z}$ , or in  $\mathbb{Z}$ , the orbit  $\mathcal{O}_S = (a_{i,j})_{(i,j) \in \mathbb{N} \times \mathbb{Z}}$  is said to be  $(k_1, k_2)$ -interlaced doubly arithmetic if, for every  $i_0 \in \{0, 1, \dots, k_1 - 1\}$  and every  $j_0 \in \{0, 1, \dots, k_2 - 1\}$ , the subsequence  $(a_{i_0 + ik_1, j_0 + jk_2})_{(i,j) \in \mathbb{N} \times \mathbb{Z}}$  is doubly arithmetic, i.e., if we have

$$a_{i_0 + ik_1, j_0 + jk_2} = a_{i_0, j_0} + i(a_{i_0 + k_1, j_0} - a_{i_0, j_0}) + j(a_{i_0, j_0 + k_2} - a_{i_0, j_0}),$$

for all  $i \in \mathbb{N}$  and all  $j \in \mathbb{Z}$ .

Determining all interlaced doubly arithmetic orbits (IDAO for short) in  $\mathbb{Z}/m\mathbb{Z}$  seems to be very difficult. Nevertheless, IDAO in  $\mathbb{Z}$  are determined in this subsection and their projection in  $\mathbb{Z}/m\mathbb{Z}$  will be considered in the sequel. First, it is clear that the sequence  $S$  associated with a  $(k_1, k_2)$ -interlaced doubly arithmetic orbit  $\mathcal{O}_S$  is a  $k_2$ -interlaced arithmetic progression. We begin by showing that the interlaced arithmetic structure of a sequence is preserved under the derivation process.

**Proposition 4.4.14.** *Let  $m$  be a positive integer. Let  $(a_0, \dots, a_{k-1})$  and  $(d_0, \dots, d_{k-1})$  be two  $k$ -tuples of elements in  $\mathbb{Z}/m\mathbb{Z}$ , or in  $\mathbb{Z}$ . Then, we have*

$$\begin{aligned} \partial \text{IAP}((a_0, \dots, a_{k-1}), (d_0, \dots, d_{k-1})) \\ = \text{IAP}((a_0 + a_1, \dots, a_{k-2} + a_{k-1}, a_{k-1} + a_0 + d_0), (d_0 + d_1, \dots, d_{k-2} + d_{k-1}, d_{k-1} + d_0)). \end{aligned}$$

We can now explicitly determine all the iterated derived sequences of an interlaced arithmetic progression.

**Proposition 4.4.15.** *Let  $m$  be a positive integer. Let  $A$  and  $D$  be two  $k$ -tuples of elements in  $\mathbb{Z}/m\mathbb{Z}$ , or in  $\mathbb{Z}$ . Then, for every non-negative integer  $i$ , we have*

$$\partial^i \text{IAP}(A, D) = \text{IAP}(AC_{\mathbf{i}} + D\mathbf{T}_{\mathbf{i}}, DC_{\mathbf{i}}),$$

where  $\mathbf{C}_{\mathbf{i}}$  is the circulant matrix of size  $k$  defined by

$$\mathbf{C}_{\mathbf{i}} = \text{Circ} \left( \sum_{l \geq 0} \binom{i}{lk}, \sum_{l \geq 0} \binom{i}{lk-1}, \dots, \sum_{l \geq 0} \binom{i}{lk+1} \right),$$

and where  $\mathbf{T}_{\mathbf{i}}$  is the Toeplitz matrix of size  $k$  where the  $(r, s)$ -entry of  $\mathbf{T}_{\mathbf{i}}$  is

$$(\mathbf{T}_{\mathbf{i}})_{r,s} = \sum_{l \geq 0} l \binom{i}{r-s+lk},$$

for all  $r, s \in \{1, \dots, k\}$ .

The main result of this subsection is the complete characterization of IDAO in  $\mathbb{Z}$ .

**Theorem 4.4.16** (Chappelon [C5]). *Every interlaced doubly arithmetic orbit  $\mathcal{O}_S$  in  $\mathbb{Z}$  is generated by an interlaced arithmetic progression of the form*

$$S = \text{IAP}((a_0, a_1, a_2), (d, -2d - 3\Sigma, d + 3\Sigma)),$$

where  $a_0, a_1, a_2$  and  $d$  are integers, and  $\Sigma := a_0 + a_1 + a_2$ .

We begin by showing that the interlaced arithmetic progressions listed in Theorem 4.4.16 will generate interlaced doubly arithmetic orbits of integers.

**Proposition 4.4.17.** *Let  $a_0, a_1, a_2, d \in \mathbb{Z}$  and let  $\Sigma = a_0 + a_1 + a_2$ . Then, the orbit  $\mathcal{O}_S$  associated with  $S = \text{IAP}((a_0, a_1, a_2)(d, -2d - 3\Sigma, d + 3\Sigma))$  is  $(6, 3)$ -interlaced doubly arithmetic.*

Now, we show that there is no other sequence generating IDAO in  $\mathbb{Z}$ . Since any  $(k_1, k_2)$ -IDAO is also a  $(k_1 k_2, k_1 k_2)$ -IDAO, we suppose that we have  $k_1 = k_2 = k$  in the sequel. The problem of determining all  $(k, k)$ -IDAO can then be converted into a system of linear equations.

**Proposition 4.4.18** (Chappelon [C5]). *Let  $m$  be a positive integer. Let  $A$  and  $D$  be two  $k$ -tuples of elements in  $\mathbb{Z}/m\mathbb{Z}$ , or in  $\mathbb{Z}$ , and let  $S = \text{IAP}(A, D)$  be a  $k$ -interlaced arithmetic progression. Then, the orbit  $\mathcal{O}_S$  is  $(k, k)$ -interlaced doubly arithmetic if and only if  $A$  and  $D$  satisfy*

$$\begin{pmatrix} \mathbf{W}_k^2 & \mathbf{W}_k \mathbf{T}_k^T \\ \mathbf{0}_k & \mathbf{W}_k \end{pmatrix} \begin{pmatrix} A^T \\ D^T \end{pmatrix} = 0,$$

where  $\mathbf{W}_k = \mathbf{C}_k - \mathbf{I}_k = \text{Circ} \left( \binom{k}{0}, \binom{k}{1}, \dots, \binom{k}{k-1} \right)$ , that is the Wendt matrix of size  $k$ .

In [112], E. Wendt investigated the resultant of  $X^k - 1$  and  $(X + 1)^k - 1$ , which corresponds to the determinant of  $\mathbf{W}_k$ . E. Lehmer was the first to prove that the determinant of  $\mathbf{W}_k$  vanishes if and only if  $k$  is divisible by 6 [83]. It is also easy to deduce from her proof that the Wendt matrix  $\mathbf{W}_k$  is of rank  $k$  if  $k$  is not divisible by 6 and of rank  $k - 2$  otherwise.

**Proposition 4.4.19.**

$$\text{rank}(\mathbf{W}_k) = \begin{cases} k & \text{if } k \not\equiv 0 \pmod{6}, \\ k - 2 & \text{if } k \equiv 0 \pmod{6}. \end{cases}$$

We are now able to prove the main theorem of this section.

*Proof of Theorem 4.4.16.* If  $k$  is not divisible by 6, then the Wendt matrix  $\mathbf{W}_k$  is of rank  $k$  by Proposition 4.4.19. This implies that  $A = D = (0, \dots, 0)$  and thus  $S$  is the sequence of zeros. Otherwise, if  $k$  is divisible by 6, then Proposition 4.4.17 implies that the vector space of  $(k, k)$ -interlaced doubly arithmetic orbits is of dimension greater than or equal to 4. Moreover, since  $\text{rank}(\mathbf{W}_k^2) = \text{rank}(\mathbf{W}_k) = k - 2$  by Proposition 4.4.19, it follows that the matrix

$$\begin{pmatrix} \mathbf{W}_k^2 & \mathbf{W}_k \mathbf{T}_k^T \\ \mathbf{0}_k & \mathbf{W}_k \end{pmatrix}$$

is of rank greater than or equal to  $2k - 4$ . Therefore, there is no other  $(k, k)$ -IDAO than those listed in Theorem 4.4.16. This completes the proof. □

#### 4.4.4 Balanced triangles modulo $m$ odd

For any odd number  $m$ , the projection in  $\mathbb{Z}/m\mathbb{Z}$  of an IDAO in  $\mathbb{Z}$ , obtained in the previous subsection, contains infinitely many balanced triangles modulo  $m$ .

**Theorem 4.4.20** (Chappelon [C5]). *Let  $m$  be an odd number and let  $a_0, a_1, a_2, d \in \mathbb{Z}/m\mathbb{Z}$ . Define  $\Sigma := a_0 + a_1 + a_2$ . If  $d, d + 3\Sigma$ , and  $2d + 3\Sigma$  are invertible, then, the following triangles, contained in the orbit of  $S = \text{IAP}((a_0, a_1, a_2), (d, -2d - 3\Sigma, d + 3\Sigma))$ , are balanced:*

- i) every Steinhaus triangle of size  $n$  in  $\mathcal{O}_S$ , for every  $n \equiv 0$  or  $-1 \pmod{6m}$ ,
- ii) every generalized Pascal triangle of size  $n$  in  $\mathcal{O}_S$ , for every  $n \equiv 0$  or  $-1 \pmod{6n}$ .

*Proof.* Let  $\mathcal{O}_S = (a_{i,j})_{(i,j) \in \mathbb{N} \times \mathbb{Z}}$  be the orbit associated with  $S$ . Consider the subsequences  $S_{i_0, j_0} = (a_{i_0+6i, j_0+6j})_{(i,j) \in \mathbb{N} \times \mathbb{Z}}$ , for  $i_0$  and  $j_0$  in  $\{0, 1, 2, 3, 4, 5\}$ . Each of these 36 subsequences is doubly arithmetic since the orbit  $\mathcal{O}_S$  is  $(6, 3)$ -interlaced doubly arithmetic by Proposition 4.4.17. Table 4.1 gives their common differences  $d_1, d_2, d_1 - d_2$ . Thus, each

$S_{i_0, j_0}$	$d_1$	$d_2$	$d_1 - d_2$
$S_{1,2}, S_{1,5}, S_{3,1}, S_{3,4}, S_{5,0}, S_{5,3}$	$2d$	$2(2d + 3\Sigma)$	$-2(d + 3\Sigma)$
$S_{0,1}, S_{0,4}, S_{2,0}, S_{2,3}, S_{4,2}, S_{4,5}$	$-2d$	$-2(2d + 3\Sigma)$	$2(d + 3\Sigma)$
$S_{1,1}, S_{1,4}, S_{3,0}, S_{3,3}, S_{5,2}, S_{5,5}$	$2(d + 3\Sigma)$	$-2d$	$2(2d + 3\Sigma)$
$S_{0,0}, S_{0,3}, S_{2,2}, S_{2,5}, S_{4,1}, S_{4,4}$	$-2(d + 3\Sigma)$	$2d$	$-2(2d + 3\Sigma)$
$S_{0,2}, S_{0,5}, S_{2,1}, S_{2,4}, S_{4,0}, S_{4,3}$	$2(2d + 3\Sigma)$	$2(d + 3\Sigma)$	$2d$
$S_{1,0}, S_{1,3}, S_{3,2}, S_{3,5}, S_{5,1}, S_{5,4}$	$-2(2d + 3\Sigma)$	$-2(d + 3\Sigma)$	$-2d$

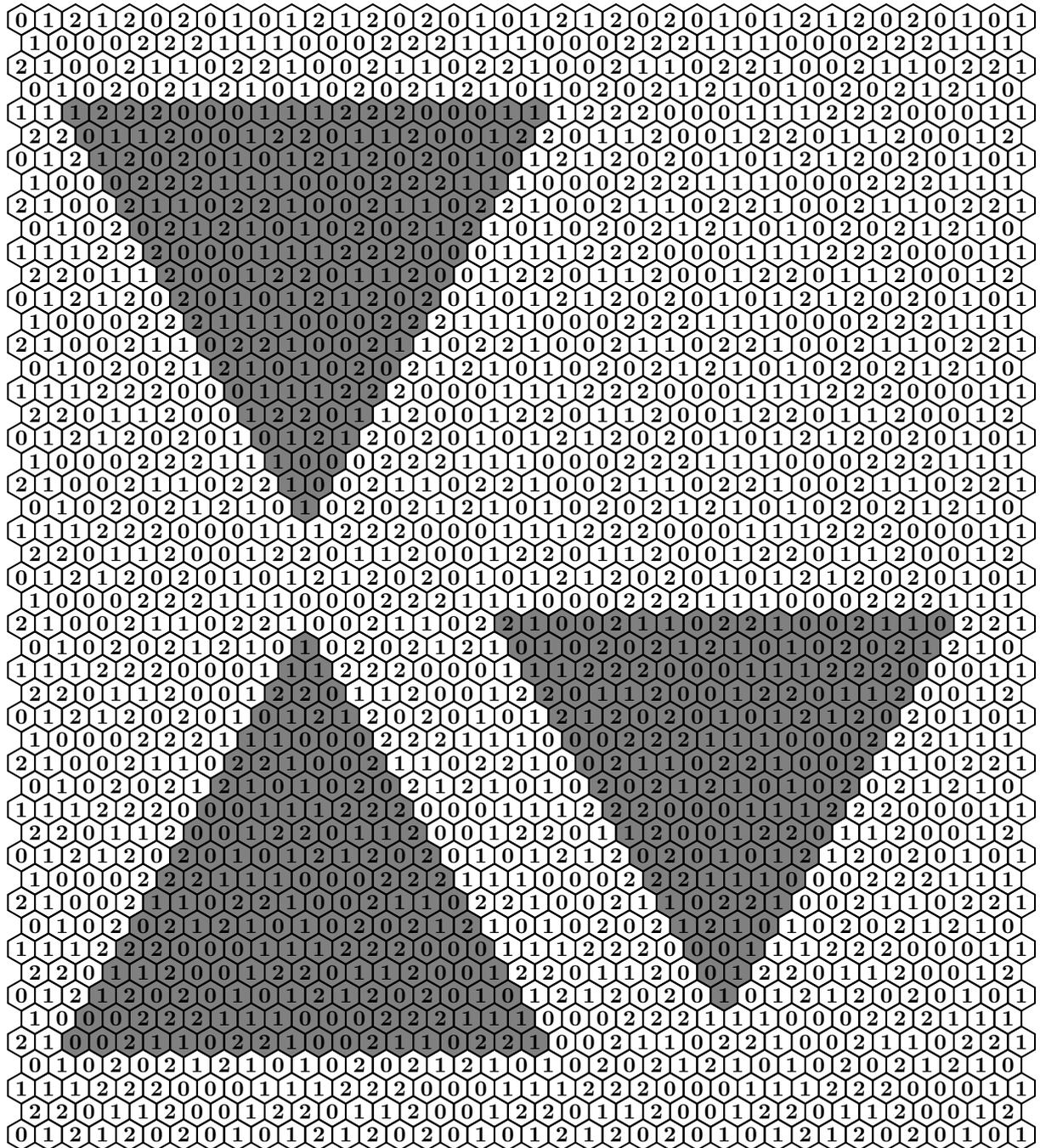
Table 4.1:  $S_{i_0, j_0} = (a_{i_0, j_0} + id_1 + jd_2)_{(i,j) \in \mathbb{N} \times \mathbb{Z}}$

subsequence  $S_{i_0, j_0}$  is doubly arithmetic, with invertible common differences  $d_1, d_2$  and  $d_1 - d_2$ . Let  $\lambda \geq 1$  and let  $\nabla$  be a Steinhaus triangle of size  $n = 6\lambda m$  or  $n = 6\lambda m - 1$ , that appears in  $\mathcal{O}_S$ . Since  $\nabla \cap S_{i_0, j_0}$ , for  $i_0$  and  $j_0$  in  $\{0, 1, 2, 3, 4, 5\}$ , is a doubly arithmetic triangle of order  $\lambda m$  or  $\lambda m - 1$  and with invertible common differences  $d_1, d_2$  and  $d_1 - d_2$ , it follows from Theorem 4.4.10 that the 36 subtriangles are balanced. Therefore their union, the Steinhaus triangle  $\nabla$ , is also balanced in  $\mathbb{Z}/m\mathbb{Z}$ . Similarly, every generalized Pascal triangle of size  $n$  in  $\mathcal{O}_S$  is balanced, for all  $n \equiv 0$  or  $-1 \pmod{6m}$ , since it can be decomposed into 36 subtriangles, which are balanced doubly arithmetic triangles by Theorem 4.4.10 again.  $\square$

The case where  $a_0 = 0, a_1 = 1, a_2 = 2$  and  $d = 1$  in  $\mathbb{Z}/3\mathbb{Z}$ , i.e., the orbit associated with the sequence  $\text{IAP}((0, 1, 2), (1, 1, 1))$ , is illustrated in Figure 4.6. In this example, balanced triangles are depicted in gray: there are a balanced Steinhaus triangle of size 18, a balanced generalized Pascal triangle of size 18 and a balanced Steinhaus triangle of size 17.

#### 4.4.5 The antisymmetric case

Here, we refine Theorem 4.4.20 by considering antisymmetric sequences in  $\mathbb{Z}/m\mathbb{Z}$ . The main interest of the antisymmetric sequences in  $\mathbb{Z}/m\mathbb{Z}$  is that their multiplicity function admits a certain symmetry. Indeed, it is clear that, if  $S$  is an antisymmetric sequence in  $\mathbb{Z}/m\mathbb{Z}$ , then its multiplicity function  $\mathbf{m}_S$  satisfies  $\mathbf{m}_S(x) = \mathbf{m}_S(-x)$ , for all  $x$  in  $\mathbb{Z}/m\mathbb{Z}$ . The same equality appears for the multiplicity function of triangles generated from antisymmetric sequences.

Figure 4.6: Balanced triangles in the orbit of  $\text{IAP}((0, 1, 2), (1, 1, 1))$  in  $\mathbb{Z}/3\mathbb{Z}$ .

**Proposition 4.4.21** (Chappelon [C5]). *Let  $m$  be a positive integer and let  $S$  be an anti-symmetric sequence of length  $n \geq 1$  in  $\mathbb{Z}/m\mathbb{Z}$ . Then, we have  $\mathbf{m}_{\nabla S}(x) = \mathbf{m}_{\nabla S}(-x)$  for all  $x \in \mathbb{Z}/m\mathbb{Z}$ .*

**Proposition 4.4.22** (Chappelon [C5]). *Let  $m$  be a positive integer and let  $S$  be an anti-symmetric sequence of length  $2n-1 \geq 1$  in  $\mathbb{Z}/m\mathbb{Z}$ . Then, we have  $\mathbf{m}_{\gamma(\nabla S)}(x) = \mathbf{m}_{\gamma(\nabla S)}(-x)$  for all  $x \in \mathbb{Z}/m\mathbb{Z}$ .*

*Remark 4.4.23.* The triangle  $\gamma(\nabla S)$  is the generalized Pascal triangle of size  $n$  associated with the Steinhaus triangle  $\nabla S$  of size  $2n-1$  by the isomorphism  $\gamma : \mathcal{ST}_m(2n-1) \rightarrow \mathcal{PT}_m(n)$  defined at the beginning of this section.

Now, for  $m$  odd, we determine all the sequences generating IDAO in  $\mathbb{Z}$  and such that the first  $3m$  terms of their projection in  $\mathbb{Z}/m\mathbb{Z}$  are antisymmetric.

**Proposition 4.4.24** (Chappelon [C5]). *Let  $m$  be an odd number. Let  $a_0, a_1, a_2, d \in \mathbb{Z}/m\mathbb{Z}$  and let  $\Sigma = a_0 + a_1 + a_2$ . Then, the subsequence  $S_n = \text{IAP}((a_0, a_1, a_2), (d, -2d - 3\Sigma, d + 3\Sigma))[0, n-1]$ , of length  $n \equiv 0 \pmod{3m}$  in  $\mathbb{Z}/m\mathbb{Z}$ , is antisymmetric if and only if  $\Sigma = 0$  and  $a_1 = -d$ , i.e., if we have  $S_n = \text{IAP}((a, -d, d-a), (d, -2d, d))[0, n-1]$ .*

Let  $m$  be an odd number and let  $a$  and  $d$  be two elements in  $\mathbb{Z}/m\mathbb{Z}$  with  $d$  invertible. We refine Theorem 4.4.20 by considering the orbit  $\mathcal{O}_S = (a_{i,j})_{(i,j) \in \mathbb{N} \times \mathbb{Z}}$  of the sequence  $S = \text{IAP}((a, -d, d-a), (d, -2d, d))$ . Let  $\nabla_0$  be the Steinhaus triangle, of size  $3m$ , generated by the first  $3m$  terms of  $S$  and let  $\Delta_0$  be the generalized Pascal triangle, of size  $3m-1$ , adjacent with  $\nabla_0$  as depicted in Figure 4.7, that are  $\nabla_0 = \nabla S[0, 3m-1]$  and  $\Delta_0 = \gamma(\nabla \partial S[1, 6m-3])$ .

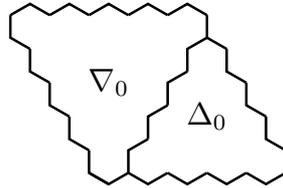


Figure 4.7:  $\nabla_0 = \nabla S[0, 3m-1]$  and  $\Delta_0 = \gamma(\nabla \partial S[1, 6m-3])$

We begin by showing that these triangles are balanced in  $\mathbb{Z}/m\mathbb{Z}$ .

**Proposition 4.4.25** (Chappelon [C5]). *Let  $m$  be an odd number and let  $a, d \in \mathbb{Z}/m\mathbb{Z}$  with  $d$  invertible. Consider the 3-interlaced arithmetic progression  $S = \text{IAP}((a, -d, d-a)(d, -2d, d))$ . Then, the triangles  $\nabla_0 = \nabla S[0, 3m-1]$  and  $\Delta_0 = \gamma(\nabla \partial S[1, 6m-3])$  are balanced in  $\mathbb{Z}/m\mathbb{Z}$ .*

Finally, we obtain the refinement of Theorem 4.4.20 announced above.

**Theorem 4.4.26** (Chappelon [C5]). *Let  $m$  be an odd number and let  $a, d \in \mathbb{Z}/m\mathbb{Z}$  with  $d$  invertible. Then, the following triangles, contained in the orbit of  $S = \text{IAP}((a, -d, d-a), (d, -2d, d))$ , are balanced:*

- i) *the Steinhaus triangles  $\nabla S[0, 3\lambda m-1]$  of size  $3\lambda m$ , and  $\nabla \partial S[0, 3\lambda m-2]$  of size  $3\lambda m-1$ , for every positive integer  $\lambda$ ,*
- ii) *the generalized Pascal triangles  $\gamma(\nabla \partial S[-n, n-2])$  of size  $n$ , for every positive integer  $n \equiv 0$  or  $-1 \pmod{3m}$ .*

*Proof.* For every positive integer  $\lambda$ , the Steinhaus triangle  $\nabla S[0, 3\lambda m - 1]$  and the Pascal triangle  $\gamma(\nabla\partial S[-3\lambda m, 3\lambda m - 2])$  are balanced because they are multiset unions of the elementary triangles  $\nabla_0, -\nabla_0, \Delta_0$  and  $-\Delta_0$ , which are balanced in  $\mathbb{Z}/m\mathbb{Z}$  by Proposition 4.4.25. The Steinhaus triangle  $\nabla\partial S[0, 3\lambda m - 2]$  is balanced, since it is obtained from  $\nabla S[0, 3\lambda m - 1]$  by rejecting its first row, which is a 3-interlaced arithmetic progression with invertible common differences and of length  $3\lambda m$  and thus contains  $3\lambda$  times each element of  $\mathbb{Z}/m\mathbb{Z}$ . Similarly, the Pascal triangle  $\gamma(\nabla\partial S[-3\lambda m + 1, 3\lambda m - 3])$  is balanced, since it is obtained from  $\gamma(\nabla\partial S[-3\lambda m, 3\lambda m - 2])$  by rejecting its last row, which is also balanced. □

### 4.4.6 The universal sequence modulo $m$ odd

Let  $US = \text{IAP}((0, -1, 1), (1, -2, 1))$  be the universal sequence of integers introduced before. In this subsection, we refine Theorem 4.4.26 by studying this universal sequence modulo an odd number  $m$ , namely the sequence

$$S = d\pi_n(US) = \text{IAP}((0, -d, d), (d, -2d, d)),$$

where  $d$  is invertible in  $\mathbb{Z}/m\mathbb{Z}$ . It corresponds to the sequence  $S$  of Theorem 4.4.26 with  $a = 0$ . First, each element of its orbit  $\mathcal{O}_S = (a_{i,j})_{(i,j) \in \mathbb{N} \times \mathbb{Z}}$  can be expressed as a function of  $d$ .

**Proposition 4.4.27** (Chappelon [C5]). *Let  $m$  be an odd number and let  $d \in \mathbb{Z}/m\mathbb{Z}$  be invertible. Consider the orbit  $\mathcal{O}_S = (a_{i,j})_{(i,j) \in \mathbb{N} \times \mathbb{Z}}$  of  $S = \text{IAP}((0, -d, d), (d, -2d, d))$  in  $\mathbb{Z}/m\mathbb{Z}$ . Then, we have*

$$a_{i,j} = (-1)^i \sum_{k>0} \binom{k}{j + 2i - k} (-1)^k (k - i)d.$$

for all non-negative integers  $i$  and  $j$ .

In the sequel of this subsection, we suppose that  $m$  is an odd number and that  $S$  is the universal sequence modulo  $m$ , that is  $S = \text{IAP}((0, -d, d), (d, -2d, d))$ , where  $d$  is an invertible element in  $\mathbb{Z}/m\mathbb{Z}$ . Let  $\nabla_1, \nabla_2$  and  $\nabla_3$  be the Steinhaus triangles of size  $m$  associated with the sequences  $S[0, m - 1], S[m, 2m - 1]$  and  $S[2m, 3m - 1]$ , respectively, and let  $\Delta_1, \Delta_2$  and  $\Delta_3$  be their adjacent generalized Pascal triangles of size  $m - 1$ , as depicted in Figure 4.8, that are:  $\nabla_1 = \nabla S[0, m - 1], \nabla_2 = \nabla S[m, 2m - 1], \nabla_3 = \nabla S[2m, 3m - 1], \Delta_1 = \gamma(\nabla\partial S[1, 2m - 3]), \Delta_2 = \gamma(\nabla\partial S[m + 1, 3m - 3])$  and  $\Delta_3 = \gamma(\nabla\partial S[2m + 1, 4m - 3])$ .

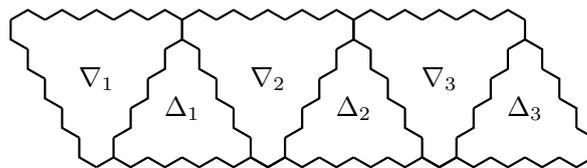


Figure 4.8: The elementary triangles  $\nabla_1, \nabla_2, \nabla_3, \Delta_1, \Delta_2$  and  $\Delta_3$

We begin by showing that these triangles, or unions of them, are balanced in  $\mathbb{Z}/m\mathbb{Z}$ .

**Proposition 4.4.28** (Chappelon [C5]). *Let  $m$  be an odd number and let  $d \in \mathbb{Z}/m\mathbb{Z}$  be invertible. Consider the universal sequence  $S = \text{IAP}((0, -d, d), (d, -2d, d))$  modulo  $m$  and the elementary triangles  $\nabla_1 = \nabla S[0, m-1]$ ,  $\nabla_2 = \nabla S[m, 2m-1]$ ,  $\nabla_3 = \nabla S[2m, 3m-1]$ ,  $\Delta_1 = \gamma(\nabla\partial S[1, 2m-3])$ ,  $\Delta_2 = \gamma(\nabla\partial S[m+1, 3m-3])$  and  $\Delta_3 = \gamma(\nabla\partial S[2m+1, 4m-3])$ . Then, the multisets  $\nabla_2$ ,  $\nabla_1 \cup \nabla_3$ ,  $\Delta_3$  and  $\Delta_1 \cup \Delta_2$  are balanced in  $\mathbb{Z}/m\mathbb{Z}$ .*

We are now ready to give the main result of this section.

**Theorem 4.4.29** (Chappelon [C5]). *Let  $m$  be an odd number and let  $d \in \mathbb{Z}/m\mathbb{Z}$  be invertible. Then, the following triangles, contained in the orbit associated with the universal sequence  $S = \text{IAP}((0, -d, d), (d, -2d, d))$  in  $\mathbb{Z}/m\mathbb{Z}$ , are balanced:*

- i) *the Steinhaus triangles  $\nabla S[n, 2n-1]$ , for every  $n \equiv 0 \pmod{m}$ , and  $\nabla\partial S[0, n-1]$ , for every  $n \equiv -1 \pmod{3m}$ ,*
- ii) *the generalized Pascal triangle  $\gamma(\nabla\partial S[-n, n-2])$ , for every  $n \equiv -1 \pmod{m}$  or  $n \equiv 0 \pmod{3m}$ .*

## 4.5 The Molluzzo Problem modulo $m$ even

Despite its apparent simplicity, the problem of Molluzzo is very challenging, as testified by the scarcity of available results. We have seen that this problem is positively solved for  $m = 2$  in Chapter 1 (the Steinhaus Problem) and for  $m = 3^k$ , for every  $k \geq 1$ , in Section 3 of this chapter. In this section, we positively solve this problem for the case  $m = 4$ . The construction method, presented here and in [C9], consists in attempting to lift to  $\mathbb{Z}/4\mathbb{Z}$  specific known solutions in  $\mathbb{Z}/2\mathbb{Z}$ .

### 4.5.1 A solution for $m = 4$

First, from Corollary 4.2.12, we know that the set of non-negative integers  $n$  for which the binomial coefficient  $\binom{n+1}{2}$  is divisible by  $m = 4$  is exactly the set of non-negative integers congruent to 0 or  $-1$  modulo 8, i.e.,

$$N(4) = \left\{ n \in \mathbb{N} \mid \binom{n+1}{2} \equiv 0 \pmod{4} \right\} = 8\mathbb{N} \cup (8\mathbb{N} + 7).$$

As in [60] for the case  $m = 2$ , the solution presented here involves the concept of *strongly balanced triangles*.

**Definition 4.5.1** (Strongly balanced triangle). Let  $S$  be a finite sequence of length  $n \geq 0$  in  $\mathbb{Z}/4\mathbb{Z}$ . The Steinhaus triangle  $\nabla S$  is said to be *strongly balanced* if, for every  $t \in \{0, \dots, \lfloor \frac{n}{8} \rfloor\}$ , the Steinhaus triangle  $\nabla S[n-8t]$  is balanced.

Here is the main result of this section.

**Theorem 4.5.2** (Chappelon-Eliahou [C9]). *There exists a balanced Steinhaus triangle of size  $n$  in  $\mathbb{Z}/4\mathbb{Z}$  if and only if  $\binom{n+1}{2} \equiv 0 \pmod{4}$ . More precisely, consider the following*

infinite pseudo-periodic sequences in  $\mathbb{Z}/4\mathbb{Z}$ :

$$\begin{aligned} S_1 &= 01220232(212113220030232311200232)^\infty, \\ S_2 &= 21210130(200132022112002110220130)^\infty, \\ T_1 &= 0120021(212202102023032200322021)^\infty, \\ T_2 &= 1000212(312223301210312003103232)^\infty, \\ T_3 &= 1200210(220101222032222103000210)^\infty, \\ T_4 &= 2102203(232002102021230022302203)^\infty. \end{aligned}$$

Then, for all integers  $i, j, k$  such that  $1 \leq i \leq 2$ ,  $1 \leq j \leq 4$  and  $k \geq 0$ , the Steinhaus triangles of the initial segments  $S_i[8k]$  and  $T_j[8k + 7]$  are strongly balanced.

## 4.5.2 The construction method

We now explain how our solution was constructed. Let  $m_1, m_2 \geq 2$  be integers, with  $m_2$  a multiple of  $m_1$ . Consider the canonical quotient map

$$\pi : \mathbb{Z}/m_2\mathbb{Z} \longrightarrow \mathbb{Z}/m_1\mathbb{Z}.$$

If  $\nabla$  is a Steinhaus triangle in  $\mathbb{Z}/m_2\mathbb{Z}$ , then  $\pi(\nabla)$  is a Steinhaus triangle in  $\mathbb{Z}/m_1\mathbb{Z}$ . Moreover, if  $\nabla$  is balanced of size  $n$  such that  $\binom{n+1}{2}$  is divisible by  $m_2$ , we know from Theorem 4.2.8 that  $\pi(\nabla)$  is balanced in  $\mathbb{Z}/m_1\mathbb{Z}$ . Thus, an obvious strategy for constructing balanced Steinhaus triangles in  $\mathbb{Z}/m_2\mathbb{Z}$  consists in trying to lift to  $\mathbb{Z}/m_2\mathbb{Z}$  known balanced Steinhaus triangles in  $\mathbb{Z}/m_1\mathbb{Z}$ . This route is tricky, as illustrated by Theorems 4.5.6 and 4.5.7 below. It allowed us to solve the case  $m = 4$  of the Molluzzo Problem, but neither the case  $m = 6$  nor the case  $m = 8$  so far.

We shall restrict our attention to *strongly balanced* Steinhaus triangles. These were defined earlier in  $\mathbb{Z}/4\mathbb{Z}$  only. We now generalize them to  $\mathbb{Z}/m\mathbb{Z}$  for all even moduli.

**Definition 4.5.3.** Let  $m \geq 2$  be an even modulus. Let  $S$  be a finite sequence of length  $n \geq 0$  in  $\mathbb{Z}/m\mathbb{Z}$ . The Steinhaus triangle  $\nabla S$  is said to be *strongly balanced* if, for every  $t \in \{0, \dots, \lfloor \frac{n}{2m} \rfloor\}$ , the Steinhaus triangle  $\nabla S[n - 2mt]$  is balanced.

Note that this definition coincides with Definition 4.5.1 for  $m = 4$ . From now on, we assume that  $m_1 = m$  is an even number, and that  $m_2 = 2m_1$ . The following notation helps to measure, roughly speaking, to what extent strong solutions in  $\mathbb{Z}/m\mathbb{Z}$  can be lifted to strong solutions in  $\mathbb{Z}/2m\mathbb{Z}$ .

**Notation 4.5.4.** Let  $S$  be an infinite sequence in  $\mathbb{Z}/m\mathbb{Z}$ . For  $n \geq 0$ , let  $a_n(S)$  denote the number of sequences  $T$  in  $\mathbb{Z}/2m\mathbb{Z}$ , of length  $n$ , such that

- $\nabla T$  is a strongly balanced Steinhaus triangle in  $\mathbb{Z}/2m\mathbb{Z}$ ;
- $\pi(T) = S[n]$ , the initial segment of length  $n$  in  $S$ .

We denote by  $\mathcal{G}_S(t) = \sum_{n=0}^{\infty} a_n(S)t^n$  the generating function of the numbers  $a_n(S)$ .

We shall use this notation as a convenient device for exhibiting the value of the  $a_n(S)$  for all  $n$  at once. For our present purposes, the favorable case occurs when  $\mathcal{G}_S(t)$  is an infinite series, not just a polynomial. Indeed,  $\mathcal{G}_S(t)$  is an infinite series if and only if there exists infinitely many strongly balanced Steinhaus triangles in  $\mathbb{Z}/2m\mathbb{Z}$ , which lift those in  $\mathbb{Z}/m\mathbb{Z}$  generated by initial segments of  $S$ .

**From  $\mathbb{Z}/2\mathbb{Z}$  to  $\mathbb{Z}/4\mathbb{Z}$** 

Here we set  $m = 2$ . Several types of balanced Steinhaus triangles of length  $4k$  or  $4k + 3$  in  $\mathbb{Z}/2\mathbb{Z}$  are known. We focus here on the ones given in [60], which have the added property of being strongly balanced.

**Theorem 4.5.5** (Eliahou-Hachez [60]). *Let  $Q_1, \dots, Q_4$  and  $R_1, \dots, R_{12}$  be the following eventually periodic sequences of  $\mathbb{Z}/2\mathbb{Z}$ :*

$$\begin{aligned} Q_1 &= 0100(001001011100)^\infty, & R_4 &= 0100001(010010111100001010111111)^\infty, \\ Q_2 &= (010010000111)^\infty, & R_5 &= 0100001(100100001001)^\infty, \\ Q_3 &= 0101(011000011000)^\infty, & R_6 &= 0101011(010101100011)^\infty, \\ Q_4 &= 0101(101000101000)^\infty, & R_7 &= 0101011(010111111101011010011101)^\infty, \\ & & R_8 &= 010(101110110010)^\infty, \\ R_1 &= 001(010000100001)^\infty, & R_9 &= 100(001000010100)^\infty, \\ R_2 &= 0011110(001101010110)^\infty, & R_{10} &= 1000010(110001101010)^\infty, \\ R_3 &= 010(000101000010)^\infty, & R_{11} &= 1111101(011000110101)^\infty, \\ & & R_{12} &= 111(110110000111)^\infty. \end{aligned}$$

For all integers  $i, j, k$  such that  $1 \leq i \leq 4$ ,  $1 \leq j \leq 12$  and  $k \geq 0$ , the Steinhaus triangles  $\nabla Q_i[4k]$  and  $\nabla R_j[4k + 3]$  are strongly balanced in  $\mathbb{Z}/2\mathbb{Z}$ .

Can we lift some initial segments of these sequences to sequences in  $\mathbb{Z}/4\mathbb{Z}$  which generate strongly balanced Steinhaus triangles? To answer this question, we have determined by computer the numbers  $a_n(S)$  for all 16 sequences  $S$  in Theorem 4.5.5 and all  $n \geq 1$ . In 11 out of the 16 cases, the numbers  $a_n(S)$  turn out to vanish for all sufficiently large  $n$ , i.e. the series  $\mathcal{G}_S(t)$  is just a polynomial. But remarkably, in the remaining 5 cases, the  $a_n(S)$  turn out to be *ultimately periodic* and non-vanishing, so that the infinite series  $\mathcal{G}_S(t)$  is actually a *rational function*. These 16 series are displayed below; the 5 infinite ones occur for the sequences  $Q_1, Q_3, R_3, R_9, R_{10}$ .

**Theorem 4.5.6** (Chappelton-Eliahou [C9]). *The generating functions  $\mathcal{G}_S(t)$  of  $Q_1, \dots, Q_4$  and  $R_1, \dots, R_{12}$  are:*

$$\begin{aligned} \mathcal{G}_{Q_1}(t) &= 1 + 8t^8 + 34t^{16} + 58t^{24} + 84t^{32} + 88t^{40} + 86t^{48} + 82t^{56} + 60t^{64} + 36t^{72} + \\ &\quad + 34t^{80} + 28t^{88} + 16t^{96} + \frac{2t^{104}}{1 - t^8}, \end{aligned}$$

$$\begin{aligned} \mathcal{G}_{Q_2}(t) &= 1 + 4t^8 + 14t^{16} + 32t^{24} + 36t^{32} + 48t^{40} + 44t^{48} + 26t^{56} + 22t^{64} + 8t^{72} + \\ &\quad + 6t^{80} + 4t^{88} + 2t^{96}, \end{aligned}$$

$$\begin{aligned} \mathcal{G}_{Q_3}(t) &= 1 + 8t^8 + 28t^{16} + 46t^{24} + 78t^{32} + 124t^{40} + 118t^{48} + 96t^{56} + 78t^{64} + 60t^{72} + \\ &\quad + 28t^{80} + 20t^{88} + 14t^{96} + 10t^{104} + 4t^{112} + 6t^{120} + 4t^{128} + 6t^{136} + 4t^{144} + 2t^{152} + \\ &\quad + 2t^{160} + 2t^{168} + 2t^{176} + 2t^{184} + 2t^{192} + 2t^{200} + 4t^{208} + \frac{2t^{216}}{1 - t^8}, \end{aligned}$$

$$\mathcal{G}_{Q_4}(t) = 1 + 8t^8 + 26t^{16} + 42t^{24} + 66t^{32} + 62t^{40} + 52t^{48} + 36t^{56} + 26t^{64} + 12t^{72} + 6t^{80},$$

$$\mathcal{G}_{R_1}(t) = 0, \quad \mathcal{G}_{R_2}(t) = 0,$$

$$\begin{aligned} \mathcal{G}_{R_3}(t) = & 10t^7 + 38t^{15} + 70t^{23} + 88t^{31} + 76t^{39} + 54t^{47} + 44t^{55} + 28t^{63} + 16t^{71} + 8t^{79} + \\ & + 4t^{87} + 4t^{95} + 4t^{103} + 4t^{111} + 4t^{119} + 6t^{127} + 4t^{135} + 6t^{143} + \frac{4t^{151}}{1-t^8}, \end{aligned}$$

$$\begin{aligned} \mathcal{G}_{R_4}(t) = & 10t^7 + 52t^{15} + 102t^{23} + 136t^{31} + 152t^{39} + 118t^{47} + 108t^{55} + 80t^{63} + 60t^{71} + \\ & + 32t^{79} + 20t^{87} + 8t^{95} + 2t^{103}, \end{aligned}$$

$$\mathcal{G}_{R_5}(t) = 10t^7,$$

$$\begin{aligned} \mathcal{G}_{R_6}(t) = & 10t^7 + 30t^{15} + 66t^{23} + 96t^{31} + 96t^{39} + 94t^{47} + 66t^{55} + 42t^{63} + 24t^{71} + 8t^{79} + \\ & + 2t^{87} + 2t^{95}, \end{aligned}$$

$$\begin{aligned} \mathcal{G}_{R_7}(t) = & 10t^7 + 60t^{15} + 138t^{23} + 204t^{31} + 304t^{39} + 266t^{47} + 246t^{55} + 148t^{63} + 64t^{71} + \\ & + 36t^{79} + 14t^{87} + 10t^{95} + 8t^{103}, \end{aligned}$$

$$\mathcal{G}_{R_8}(t) = 10t^7,$$

$$\begin{aligned} \mathcal{G}_{R_9}(t) = & 10t^7 + 42t^{15} + 80t^{23} + 130t^{31} + 164t^{39} + 174t^{47} + 126t^{55} + 68t^{63} + 38t^{71} + \\ & + 20t^{79} + 22t^{87} + 12t^{95} + 2t^{103} + 2t^{111} + 2t^{119} + 2t^{127} + 2t^{135} + 2t^{143} + 2t^{151} + \\ & + 2t^{159} + 2t^{167} + 2t^{175} + 2t^{183} + 2t^{191} + 2t^{199} + 4t^{207} + \frac{2t^{215}}{1-t^8}, \end{aligned}$$

$$\begin{aligned} \mathcal{G}_{R_{10}}(t) = & 10t^7 + 58t^{15} + 98t^{23} + 130t^{31} + 160t^{39} + 138t^{47} + 132t^{55} + 84t^{63} + 64t^{71} + \\ & + 34t^{79} + 14t^{87} + 8t^{95} + 6t^{103} + 2t^{111} + 2t^{119} + 4t^{127} + \frac{2t^{135}}{1-t^8}, \end{aligned}$$

$$\mathcal{G}_{R_{11}}(t) = 4t^7 + 16t^{15} + 26t^{23} + 32t^{31} + 30t^{39} + 30t^{47} + 26t^{55} + 12t^{63} + 8t^{71} + 2t^{79},$$

$$\mathcal{G}_{R_{12}}(t) = 4t^7.$$

The origin of our sequences  $S_1, S_2, T_1, T_2, T_3, T_4$ , solving the Molluzzo Problem in  $\mathbb{Z}/4\mathbb{Z}$ , is now clear. Indeed, they are lifts to  $\mathbb{Z}/4\mathbb{Z}$  of the 5 sequences  $Q_1, Q_3, R_3, R_9, R_{10}$  in  $\mathbb{Z}/2\mathbb{Z}$  with  $\mathcal{G}_S(t)$  infinite. More precisely, we have

$$\pi(S_1) = Q_1, \quad \pi(S_2) = Q_3, \quad \pi(T_1) = \pi(T_4) = R_3, \quad \pi(T_2) = R_{10}, \quad \pi(T_3) = R_9,$$

as the reader may readily check.

### From $\mathbb{Z}/4\mathbb{Z}$ to $\mathbb{Z}/8\mathbb{Z}$

Having solved the problem in  $\mathbb{Z}/4\mathbb{Z}$  with Theorem 4.5.2, can we lift our solutions  $S_1, S_2, T_1, T_2, T_3, T_4$  to sequences in  $\mathbb{Z}/8\mathbb{Z}$  giving rise to infinitely many strongly balanced Steinhaus triangles? Unfortunately, the answer is no, as shown by the following computational result.

**Theorem 4.5.7** (Chappelon-Eliahou [C9]). *The generating functions  $\mathcal{G}_S(t)$  of sequences*

$S_1, S_2, T_1, T_2, T_3, T_4$  are polynomials only:

$$\mathcal{G}_{S_1}(t) = 1 + 16t^{16} + 46t^{32} + 32t^{48} + 14t^{64},$$

$$\mathcal{G}_{S_2}(t) = 1 + 22t^{16} + 60t^{32} + 56t^{48} + 28t^{64} + 6t^{80},$$

$$\mathcal{G}_{T_1}(t) = 14t^{15} + 40t^{31} + 40t^{47} + 24t^{63} + 8t^{79} + 2t^{95} + 2t^{111},$$

$$\mathcal{G}_{T_2}(t) = 30t^{15} + 66t^{31} + 76t^{47} + 32t^{63} + 12t^{79},$$

$$\mathcal{G}_{T_3}(t) = 14t^{15} + 54t^{31} + 42t^{47} + 34t^{63} + 12t^{79} + 2t^{95},$$

$$\mathcal{G}_{T_4}(t) = 14t^{15} + 54t^{31} + 64t^{47} + 40t^{63} + 10t^{79} + 2t^{95}.$$

Summarizing, at this stage, it is not even known whether there exist infinitely many balanced Steinhaus triangles in  $\mathbb{Z}/8\mathbb{Z}$ .

# Chapter 5

## Balanced simplices

### 5.1 Simplices generated by additive cellular automata

Let us define the generalization of Steinhaus triangles and generalized Pascal triangles that we consider here and in [C6]. Let  $n$  and  $m$  be positive integers. Throughout this chapter,  $n$  will denote the dimension of the objects studied and  $m$  the order of the finite cyclic group  $\mathbb{Z}/m\mathbb{Z}$ . For any integers  $a$  and  $b$  such that  $a < b$ , we let  $[a, b]$  denote the set of the integers between  $a$  and  $b$ , that is,  $[a, b] := \{a, a + 1, \dots, b\}$  and  $[a, b]^n$  the Cartesian product of  $n$  copies of  $[a, b]$ . For any  $n$ -tuple of elements  $u$ , we let  $u_i$  denote its  $i$ th component for all  $i \in [1, n]$ , that is,  $u = (u_1, \dots, u_n)$ . For two  $n$ -tuples  $u$  and  $v$  and an integer  $\lambda$ , we consider the sum  $u + v := (u_1 + v_1, \dots, u_n + v_n)$ , the product  $u \cdot v := (u_1 v_1, \dots, u_n v_n)$  and the scalar product  $\lambda u := (\lambda u_1, \dots, \lambda u_n)$ .

**Definition 5.1.1** (ACA). Let  $r$  be a non-negative integer and let  $W = (w_j)_{j \in [-r, r]^n}$  be an  $n$ -dimensional array of integers of size  $(2r+1)^n$ . The *additive cellular automaton* (ACA for short) over  $\mathbb{Z}/m\mathbb{Z}$  associated with  $W$  is the map  $\partial$  which assigns, to every  $n$ -dimensional infinite array of  $\mathbb{Z}/m\mathbb{Z}$ , a new array by a linear transformation whose coefficients are those of  $W$ . More precisely, the map  $\partial$  is defined by

$$\partial((a_i)_{i \in \mathbb{Z}^n}) = \left( \sum_{j \in [-r, r]^n} w_j a_{i+j} \right)_{i \in \mathbb{Z}^n},$$

for all arrays  $(a_i)_{i \in \mathbb{Z}^n}$  of elements in  $\mathbb{Z}/m\mathbb{Z}$ . We say that  $\partial$  is of *dimension*  $n \geq 1$  and of *weight*  $W$  with *radius*  $r \geq 0$ .

**Definition 5.1.2** (Orbit). Let  $A = (a_i)_{i \in \mathbb{Z}^n}$  be an infinite array of  $\mathbb{Z}/m\mathbb{Z}$  of dimension  $n$ . The *orbit*  $\mathcal{O}_A$  generated from  $A$  by the ACA  $\partial$  is the collection of all the  $n$ -dimensional arrays obtained from  $A$  by successive applications of  $\partial$ , that is,

$$\mathcal{O}_A := \{\partial^j(A) \mid j \in \mathbb{N}\},$$

where  $\partial^j$  is recursively defined by  $\partial^j(A) = \partial(\partial^{j-1}(A))$  for all  $j \geq 1$  and  $\partial^0(A) = A$ . The orbit  $\mathcal{O}_A$  can also be seen as the  $(n+1)$ -dimensional array  $(a_{i,j})_{(i,j) \in \mathbb{Z}^n \times \mathbb{N}}$  of  $\mathbb{Z}/m\mathbb{Z}$  whose  $j$ th row  $R_j := (a_{i,j})_{i \in \mathbb{Z}^n}$  corresponds to  $\partial^j(A)$ , for all  $j \in \mathbb{N}$ .

**Definition 5.1.3** (Simplices). Let  $A = (a_i)_{i \in \mathbb{Z}^n}$  be an infinite array of  $\mathbb{Z}/m\mathbb{Z}$  of dimension  $n$ . Let  $\varepsilon \in \{-1, 1\}^n$  and let  $s$  be a positive integer. The *simplex* of size  $s$ , with orientation

$\varepsilon$  and whose principal vertex is at the coordinates  $j \in \mathbb{Z}^n$  in  $A$ , is the multiset of  $\mathbb{Z}/m\mathbb{Z}$  defined and denoted by

$$\Delta(j, \varepsilon, s) := \{a_{j+\varepsilon \cdot k} \mid k \in \mathbb{N}^n \text{ such that } k_1 + \dots + k_n \leq s - 1\}.$$

For  $n = 2$  and  $n = 3$ , it is called a triangle and a tetrahedron, respectively.

$a_{0,0}$	$a_{1,0}$	2	3	4	0	1	2	3	4	0	1	2	3	4	0
$a_{0,1}$	$a_{1,1}$	2	1	0	4	3	2	1	0	4	3	2	1	0	4
2	3	4	0	1	2	3	4	0	1	2	3	4	0	1	2
2	1	0	4	3	2	1	0	4	3	2	1	0	4	3	2
4	0	1	2	3	4	0	1	2	3	4	0	1	2	3	4
0	4	3	2	1	0	4	3	2	1	0	4	3	2	1	0
1	2	3	4	0	1	2	3	4	0	1	2	3	4	0	1
4	3	2	1	0	4	3	2	1	0	4	3	2	1	0	4
2	3	4	0	1	2	3	4	0	1	2	3	4	0	1	2
2	1	0	4	3	2	1	0	4	3	2	1	0	4	3	2
4	0	1	2	3	4	0	1	2	3	4	0	1	2	3	4
0	4	3	2	1	0	4	3	2	1	0	4	3	2	1	0
1	2	3	4	0	1	2	3	4	0	1	2	3	4	0	1
4	3	2	1	0	4	3	2	1	0	4	3	2	1	0	4
2	3	4	0	1	2	3	4	0	1	2	3	4	0	1	2
2	1	0	4	3	2	1	0	4	3	2	1	0	4	3	2

Figure 5.1: Example of triangles  $\Delta((2, 2), ++, 5)$ ,  $\Delta((2, 13), +-, 5)$ ,  $\Delta((13, 2), -+, 5)$  and  $\Delta((13, 13), --, 5)$  appearing in an orbit  $\mathcal{O}_A = (a_{i,j})_{(i,j) \in \mathbb{Z} \times \mathbb{N}}$  of  $\mathbb{Z}/5\mathbb{Z}$  generated by the ACA of weight  $W = (2, 1, 1)$

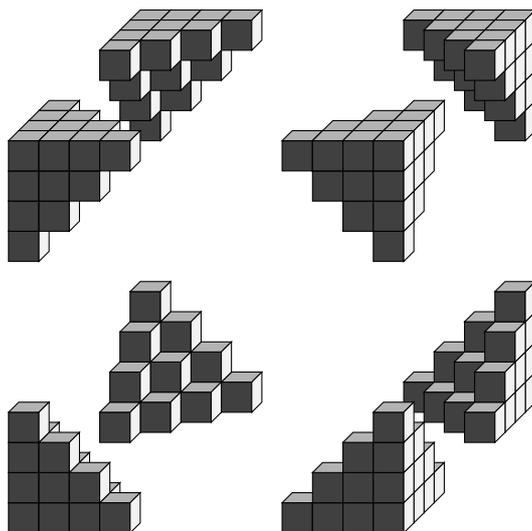


Figure 5.2: The eight possible orientations of a tetrahedron

In this chapter, we mainly consider simplices of dimension  $n$  appearing in the orbit generated from an infinite array of  $\mathbb{Z}/m\mathbb{Z}$  by an ACA of dimension  $n - 1$ . Examples of triangles, for the four possible orientations in dimension  $n = 2$ , are depicted in Figure 5.1.

In Figure 5.2, for dimension  $n = 3$ , the eight possible orientations of a tetrahedron are represented.

The goal in this chapter is to prove the existence of balanced simplices appearing in certain orbits generated by ACA. Sufficient conditions for obtaining this result will be detailed throughout this chapter. As seen in the previous chapters, this notion of balanced simplices generated by ACA essentially appears in the literature in the case of the Pascal cellular automaton of dimension 1.

**Definition 5.1.4** ( $\text{PCA}_n$ ). The *Pascal cellular automaton* of dimension  $n$  is the ACA of radius  $r = 1$  and whose weight array  $W = (w_i)_{i \in [-1,1]^n}$  is defined by

$$w_i = \begin{cases} 1 & \text{if } i \in \{0_{\mathbb{Z}^n}, -e_1, -e_2, \dots, -e_n\}, \\ 0 & \text{otherwise,} \end{cases}$$

where  $(e_1, e_2, \dots, e_n)$  is the canonical basis of the vector space  $\mathbb{Z}^n$ . It is denoted by  $\text{PCA}_n$ .

For instance,  $W = (1, 1, 0)$  for  $\text{PCA}_1$  and  $W = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}$  for  $\text{PCA}_2$ .

*Remark 5.1.5.* Let  $A = (a_i)_{i \in \mathbb{Z}^{n-1}}$  be the  $(n-1)$ -dimensional array of  $\mathbb{Z}/m\mathbb{Z}$  defined by  $a_i = 1$  for  $i = 0_{\mathbb{Z}^n}$  and  $a_i = 0$  otherwise. If  $(a_{i,j})_{(i,j) \in \mathbb{Z}^{n-1} \times \mathbb{N}}$  is the orbit  $\mathcal{O}_A$  generated by the  $\text{PCA}_{n-1}$ , then  $a_{i,j}$  is the multinomial coefficient

$$a_{i,j} = \binom{j}{i_1, \dots, i_{n-1}, j - \sum_{k=1}^{n-1} i_k} = \frac{j!}{i_1! \cdots i_{n-1}! (j - \sum_{k=1}^{n-1} i_k)!}$$

for all  $i \in \mathbb{N}^{n-1}$  such that  $i_1 + \cdots + i_{n-1} \leq j$ , and  $a_{i,j} = 0$  otherwise. Thus, we retrieve the coefficients of the Pascal  $n$ -simplex modulo  $m$ . This is the reason why this specific ACA is called the Pascal cellular automaton.

Let  $A = (a_i)_{i \in \mathbb{Z}}$  be a doubly infinite sequence of  $\mathbb{Z}/m\mathbb{Z}$ . We consider the orbit generated from  $A$  by  $\text{PCA}_1$ , i.e., the infinite array  $\mathcal{O}_A = (a_{i,j})_{(i,j) \in \mathbb{Z} \times \mathbb{N}}$  defined by  $a_{i,0} = a_i$  for all  $i \in \mathbb{Z}$  and

$$a_{i,j} = a_{i-1,j-1} + a_{i,j-1}$$

for all  $(i,j) \in \mathbb{Z} \times \mathbb{N}^*$ . Then, the  $(-+)$ -triangles and the  $(+-)$ -triangles appearing in a such orbit correspond with the Steinhaus triangles and the generalized Pascal triangles modulo  $m$ , respectively.

As already seen before, the proof of Theorem 4.4.5 is based on the elementary object that is the arithmetic triangle and its main interest is that it is very often balanced. In this paper, we consider a generalization in higher dimensions of arithmetic triangles.

**Definition 5.1.6** (Arithmetic arrays and simplices). Let  $n$  and  $m$  be positive integers. Let  $A = (a_i)_{i \in \mathbb{Z}^n}$  be an array of  $\mathbb{Z}/m\mathbb{Z}$ . The array  $A$  is said to be *arithmetic* with first element  $a$  and with common difference  $d = (d_1, \dots, d_n) \in (\mathbb{Z}/m\mathbb{Z})^n$  if

$$a_i = a + i_1 d_1 + \cdots + i_n d_n,$$

for all  $i = (i_1, \dots, i_n) \in \mathbb{Z}^n$ . The arithmetic array with first element  $a \in \mathbb{Z}/m\mathbb{Z}$  and with common difference  $d \in (\mathbb{Z}/m\mathbb{Z})^n$  is denoted by  $\text{AA}(a, d)$ . The *arithmetic simplex* of size  $s$ , with first element  $a \in \mathbb{Z}/m\mathbb{Z}$  and with common difference  $d = (d_1, \dots, d_n) \in (\mathbb{Z}/m\mathbb{Z})^n$ , is

the simplex  $\Delta(0_{\mathbb{Z}^n}, + \cdots +, s)$  appearing in the array  $\text{AA}(a, d) = (a_i)_{i \in \mathbb{Z}^n}$  and is denoted by  $\text{AS}(a, d, s)$ , that is,

$$\text{AS}(a, d, s) = \{a + i_1 d_1 + \cdots + i_n d_n \mid i \in \mathbb{N}^n \text{ such that } i_1 + \cdots + i_n \leq s - 1\}.$$

For  $n = 1$ , the arithmetic progression  $\text{AS}(a, d, s)$  is also denoted by  $\text{AP}(a, d, s)$ .

*Remark 5.1.7.* The following multiset identities hold for arithmetic simplices:

$$\begin{aligned} \text{AS}(a, (d_1, \dots, d_n), s) &= \text{AS}(a + (s - 1)d_1, (-d_1, d_2 - d_1, \dots, d_n - d_1), s) \\ &= \text{AS}(a + (s - 1)d_2, (d_1 - d_2, -d_2, d_3 - d_2, \dots, d_n - d_2), s) \\ &\dots\dots \\ &= \text{AS}(a + (s - 1)d_n, (d_1 - d_n, \dots, d_{n-1} - d_n, -d_n), s), \end{aligned}$$

and  $\text{AS}(a, (d_1, \dots, d_n), s) = \text{AS}(a, (d_{\pi(1)}, \dots, d_{\pi(n)}), s)$ , where  $\pi$  is a permutation of  $[1, n]$ .

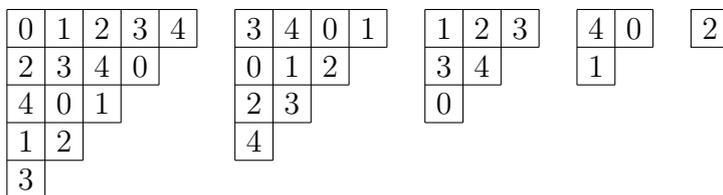


Figure 5.3: The arithmetic tetrahedron  $\text{AS}(0, (1, 2, 3), 5)$  in  $\mathbb{Z}/5\mathbb{Z}$

For example, the arithmetic tetrahedron  $\text{AS}(0, (1, 2, 3), 5)$  of  $\mathbb{Z}/5\mathbb{Z}$  is depicted in Figure 5.3. The successive rows of this tetrahedron are the arithmetic triangles  $\text{AS}(0, (1, 2), 5)$ ,  $\text{AS}(3, (1, 2), 4)$ ,  $\text{AS}(1, (1, 2), 3)$ ,  $\text{AS}(4, (1, 2), 2)$  and  $\text{AS}(2, (1, 2), 1)$  of  $\mathbb{Z}/5\mathbb{Z}$ .

Return now to the general case of an ACA of dimension  $n - 1$ , with a weight array  $W = (w_j)_{j \in [-r, r]^{n-1}}$  of radius  $r \in \mathbb{N}$ . Let us define

$$\sigma := \sum_{j \in [-r, r]^{n-1}} w_j \quad \text{and} \quad \sigma_k := \sum_{j \in [-r, r]^{n-1}} j_k w_j, \quad \text{for all } k \in [1, n - 1].$$

For any integers  $a$  and  $b$ , we let  $\text{gcd}(a, b)$  and  $\text{lcm}(a, b)$  denote the greatest common divisor and the least common multiplier of  $a$  and  $b$ , respectively. Let  $x \in \mathbb{Z}/m\mathbb{Z}$ . We also let  $\text{gcd}(x, m)$  denote the greatest common divisor of  $m$  and any representant of the residue class  $x$ .

Using properties of arithmetic simplices, the following theorem, which is the main result of this chapter, will be obtained.

**Theorem 5.1.8** (Chappelton [C6]). *Let  $n \geq 2$  and  $m$  be two positive integers such that  $\text{gcd}(m, n!) = 1$ . Suppose that  $\sigma$  is invertible modulo  $m$ . Let  $a \in \mathbb{Z}/m\mathbb{Z}$ ,  $d \in (\mathbb{Z}/m\mathbb{Z})^{n-1}$  and  $\varepsilon \in \{-1, 1\}^n$  such that  $d_i$ , for all  $i \in [1, n]$ , and  $\varepsilon_j d_j - \varepsilon_i d_i$ , for all distinct integers  $i, j \in [1, n]$ , are invertible, where  $d_n := \sigma^{-1} \sum_{k=1}^{n-1} \sigma_k d_k$ . Then, in the orbit  $\mathcal{O}_{\text{AA}(a, d)}$ , every  $n$ -simplex with orientation  $\varepsilon$  and of size  $s$  is balanced modulo  $m$ , for all  $s \equiv -t \pmod{\text{lcm}(\text{ord}_m(\sigma), m)}$ , where  $t \in [0, n - 1]$ .*

*Remark 5.1.9.* For any integer  $\sigma$  which is invertible modulo  $m$ , the identity

$$\text{lcm}(\text{ord}_m(\sigma), m) = \text{ord}_m(\sigma^m) m$$

holds. A complete study of this arithmetic function can be found in [C4].

For  $n = 2$ ,  $m$  odd and  $W = (1, 1, 0)$ , the weight sequence of  $\text{PCA}_1$ , we retrieve Theorem 4.3.10. Indeed, in this case, we have  $\sigma = 2$ ,  $\sigma_1 = -1$ ,  $d_1 = d$ ,  $d_2 = \sigma^{-1}\sigma_1 d_1 = -2^{-1}d$  and  $\varepsilon_1 d_1 - \varepsilon_2 d_2 = \pm 2^{-1}d$  for  $\varepsilon = (\pm 1, \mp 1)$ , which are invertibles of  $\mathbb{Z}/m\mathbb{Z}$ .

In the special case of  $\text{PCA}_{n-1}$ , Theorem 5.1.8 gives a positive answer to the equivalent problem of the weak Molluzzo Problem, in higher dimensions, for an infinite number of values  $m$ .

**Corollary 5.1.10** (Chappelon [C6]). *Let  $n \geq 2$  be a positive integer. For every positive integer  $m$  such that  $\gcd(m, (3(n-1))!) = 1$ , there exist infinitely many balanced  $n$ -simplices of  $\mathbb{Z}/m\mathbb{Z}$  generated by  $\text{PCA}_{n-1}$ , for all possible orientations  $\varepsilon \in \{-1, 1\}^n$ . In the special case of the two orientations  $\varepsilon = +\cdots + -$  or  $\varepsilon = -\cdots - +$ , the existence of an infinite number of such balanced simplices is verified for every  $m$  such that  $\gcd(m, n!) = 1$  for  $n$  even and for every  $m$  such that  $\gcd(m, (\frac{3n+1}{2})!) = 1$  for  $n$  odd.*

This chapter is organized as follows. After giving some basic results on balanced simplices and orbits of arithmetic arrays generated by ACA in Section 2, we study, in Section 3, arithmetic simplices and we give some sufficient conditions on them to be balanced, for any dimension  $n \geq 2$ . Moreover, in dimension 2 and 3, we also provide necessary conditions on arithmetic triangles and arithmetic tetrahedra for being balanced. This leads to Theorem 5.1.8 in Section 4. Moreover, using the specificities on balanced arithmetic tetrahedra in dimension 2, highlighted in Section 3, we complete Theorem 5.1.8 for balanced tetrahedra. In Section 5, we consider the special case where simplices have the additional geometric property of being composed of antisymmetric sequences. This permits us to obtain more results for ACA of dimension 1 generating balanced triangles. Finally, the problem of determining the existence of balanced triangles and tetrahedra generated by  $\text{PCA}_1$  and  $\text{PCA}_2$ , for the remaining open cases, is posed in the last section.

## 5.2 Preliminaries

We begin this section with the terminology on simplices that we will use in the sequel.

**Definition 5.2.1** (Vertices, edges, facets and rows of simplices). Let  $A = (a_i)_{i \in \mathbb{Z}^n}$  be an infinite array of dimension  $n$  of  $\mathbb{Z}/m\mathbb{Z}$ . Let  $\Delta = \Delta(j, \varepsilon, s)$  be the  $n$ -simplex of size  $s$  of  $\mathbb{Z}/m\mathbb{Z}$  with principal vertex at position  $j \in \mathbb{Z}^n$  in  $A$  and with orientation  $\varepsilon \in \{-1, 1\}^n$ . Let  $(e_1, e_2, \dots, e_n)$  denote the canonical basis of the vector space  $\mathbb{Z}^n$  and let  $e_0 := 0_{\mathbb{Z}^n}$ . The  $n+1$  vertices  $V_0, \dots, V_n$  of  $\Delta$  are defined by  $V_k(\Delta) := a_{j+(s-1)\varepsilon \cdot e_k}$  for all  $k \in [0, n]$ , where  $V_0(\Delta) := a_j$  (principal vertex) and

$$V_k(\Delta) := a_{j+\varepsilon \cdot (s-1)e_k} = a_{j_1, \dots, j_{k-1}, j_k+\varepsilon_k(s-1), j_{k+1}, \dots, j_n},$$

for all  $k \in [1, n]$ . The  $\binom{n+1}{2}$  edges  $E_{k,l}$  of  $\Delta$  are sequences of length  $s$  defined by

$$\begin{aligned} E_{k,l}(\Delta) &:= \{a_{j+\varepsilon \cdot ((s-1-x)e_k + xe_l)} \mid x \in [0, s-1]\} \\ &= \{a_{j+\varepsilon \cdot (s-1)e_k}, a_{j+\varepsilon \cdot ((s-2)e_k + e_l)}, a_{j+\varepsilon \cdot ((s-3)e_k + 2e_l)}, \dots, a_{j+\varepsilon \cdot (s-1)e_l}\}, \end{aligned}$$

for all distinct integers  $k, l \in [0, n]$ . The  $n+1$  facets  $F_0, \dots, F_n$  of  $\Delta$  are the  $(n-1)$ -simplices of size  $s$  defined by

$$F_k(\Delta) := \{a_{j+\varepsilon \cdot l} \mid l \in \mathbb{N}^n \text{ such that } l_k = 0 \text{ and } l_1 + \dots + l_n \leq s-1\},$$

for all  $l \in [1, n]$  and

$$F_0(\Delta) := \{a_{j+\varepsilon \cdot l} \mid l \in \mathbb{N}^n \text{ such that } l_1 + \cdots + l_n = s - 1\}.$$

For every  $k \in [0, s - 1]$ , the  $k$ th row of  $\Delta$  is the  $(n - 1)$ -simplex of size  $s - k$  defined by

$$R_k(\Delta) := \{a_{j+\varepsilon \cdot l} \mid l \in \mathbb{N}^n \text{ such that } l_n = k \text{ and } l_1 + \cdots + l_{n-1} \leq s - k - 1\}.$$

### 5.2.1 Sizes of balanced simplices

In this subsection, the admissible sizes of balanced simplices are studied. First, the cardinality of an  $n$ -simplex of size  $s$  is determined.

**Proposition 5.2.2.** *Let  $\Delta$  be an  $n$ -simplex of size  $s$  appearing in an  $n$ -dimensional array. Then, the multiset cardinality of  $\Delta$  is  $|\Delta| = \binom{s+n-1}{n}$ .*

The divisibility of  $\binom{s+n-1}{n}$  by  $m$  is obviously a necessary condition for having a balanced  $n$ -simplex of  $\mathbb{Z}/m\mathbb{Z}$  of size  $s$ . When  $m$  is a composite number, to give all the sizes  $s$  for which the binomial  $\binom{s+n-1}{n}$  is divisible by  $m$  is tedious and not really important here because the results that we obtain in this chapter are only for some of them, not for all the admissible sizes. It would correspond to a generalization of Theorem 4.2.11. Nevertheless, we can see that the sizes involved in Theorem 5.1.8 are admissible for this problem.

**Proposition 5.2.3.** *Let  $n, p, k, s$  be positive integers such that  $p$  is prime and  $p > n \geq 2$ . Then, the binomial coefficient  $\binom{s+n-1}{n}$  is divisible by  $p^k$  if and only if  $s \equiv -t \pmod{p^k}$  for  $t \in [0, n - 1]$ .*

**Proposition 5.2.4.** *Let  $m$  and  $n$  be two positive integers such that  $\gcd(m, n!) = 1$  and let  $s$  be a positive integer such that  $s \equiv -t \pmod{m}$ , where  $t \in [0, n - 1]$ . Then, the binomial coefficient  $\binom{s+n-1}{n}$  is divisible by  $m$ .*

### 5.2.2 Orbits of arithmetic arrays

In this subsection, the orbits of arithmetic arrays are studied in detail. Let  $n \geq 2$  be a positive integer. First, we show that the arithmetic structure is preserved under the action of  $\partial$  for any weight array  $W = (w_i)_{i \in [-r, r]^{n-1}}$ , of radius  $r \in \mathbb{N}$ .

**Proposition 5.2.5.** *Let  $a \in \mathbb{Z}/m\mathbb{Z}$  and let  $d = (d_1, \dots, d_{n-1}) \in (\mathbb{Z}/m\mathbb{Z})^{n-1}$ . Then,*

$$\partial \text{AA}(a, d) = \text{AA} \left( \sigma a + \sum_{k=1}^{n-1} \sigma_k d_k, \sigma d \right),$$

where  $\sigma$  and  $\sigma_k$  are the coefficients

$$\sigma := \sum_{j \in [-r, r]^{n-1}} w_j, \quad \sigma_k := \sum_{j \in [-r, r]^{n-1}} j_k w_j, \quad \text{for all } k \in [1, n - 1].$$

**Proposition 5.2.6.** *Let  $a \in \mathbb{Z}/m\mathbb{Z}$  and let  $d = (d_1, \dots, d_{n-1}) \in (\mathbb{Z}/m\mathbb{Z})^{n-1}$ . Then,*

$$\partial^i \text{AA}(a, d) = \text{AA} \left( \sigma^i a + i \sigma^{i-1} \sum_{k=1}^{n-1} \sigma_k d_k, \sigma^i d \right),$$

for all  $i \in \mathbb{N}$ .

Thus, the elements of the orbit of an arithmetic array  $\text{AA}(a, d)$  are entirely determined in function of  $a$ ,  $d$ ,  $\sigma$  and  $\sigma_k$  for all  $k \in [1, n-1]$ .

**Proposition 5.2.7.** *Let  $a \in \mathbb{Z}/m\mathbb{Z}$  and let  $d \in (\mathbb{Z}/m\mathbb{Z})^{n-1}$ . Let  $\mathcal{O}_{\text{AA}(a,d)} = (a_i)_{i \in \mathbb{Z}^{n-1} \times \mathbb{N}}$  be the orbit of the arithmetic array  $\text{AA}(a, d)$ . Then,*

$$a_i = (\partial^{i_n} \text{AA}(a, d))_{i_1, \dots, i_{n-1}} = \sigma^{i_n} \left( a + \sum_{k=1}^n i_k d_k \right),$$

for all  $i \in \mathbb{Z}^{n-1} \times \mathbb{N}$ , where  $d_n := \sigma^{-1} \sum_{k=1}^{n-1} \sigma_k d_k$ .

*Remark 5.2.8.* For  $\mathcal{O}_{\text{AA}(a,d)} = (a_i)_{i \in \mathbb{Z}^{n-1} \times \mathbb{N}}$  and for every  $(i_1, \dots, i_{n-1}) \in \mathbb{Z}^{n-1}$ , the sequence  $(a_{i_1, \dots, i_{n-1}, i_n})_{i_n \in \mathbb{N}}$  is the arithmetico-geometric sequence with first element  $a + i_1 d_1 + \dots + i_{n-1} d_{n-1}$ , with common difference  $d_n := \sigma^{-1} \sum_{k=1}^{n-1} \sigma_k d_k$  and common ratio  $\sigma$ .

We deduce from Proposition 5.2.7 that two distinct ACA can generate the same orbit from an arithmetic array. For instance, for any ACA of weight array  $W = (w_i)_{i \in [-r, r]^{n-1}}$  of radius  $r$ , we can consider the ACA of weight array  $\overline{W} = (\overline{w}_i)_{i \in [-1, 1]^{n-1}}$  of radius 1 defined by

$$\overline{w}_i = \begin{cases} \sigma(W) - \sum_{k=1}^{n-1} \sigma_k(W) & , \text{ if } i = 0_{\mathbb{Z}^n}, \\ \sigma_k(W) & , \text{ if } i = e_k, \text{ for all } k \in [1, n-1], \\ 0 & , \text{ otherwise.} \end{cases}$$

Then, it is clear that we have

$$\sigma(\overline{W}) = \sum_{i \in [-1, 1]^{n-1}} \overline{w}_i = \sigma(W),$$

and

$$\sigma_k(\overline{W}) = \sum_{i \in [-1, 1]^{n-1}} i_k \overline{w}_i = \sigma_k(W),$$

for all  $k \in [1, n-1]$ . Therefore, in the sequel of this chapter, the coefficients  $\sigma$  and  $\sigma_k$  will be more important than the elements of the weight array  $W$  themselves.

Now, we prove that, in the orbit of an arithmetic array of  $\mathbb{Z}/m\mathbb{Z}$ , if there exists a balanced simplex of sufficiently large size, then  $\sigma$  is invertible modulo  $m$ .

**Proposition 5.2.9** (Chappelon [C6]). *Let  $a \in \mathbb{Z}/m\mathbb{Z}$  and let  $d \in (\mathbb{Z}/m\mathbb{Z})^{n-1}$ . In the orbit  $\mathcal{O}_{\text{AA}(a,d)}$ , if an  $n$ -simplex of size  $s > \frac{n}{m-1} + \frac{3}{2} + \sqrt{\frac{mn^2}{(m-1)^2} + \frac{1}{4}}$  is balanced in  $\mathbb{Z}/m\mathbb{Z}$ , then  $\sigma$  is invertible modulo  $m$ .*

*Remark 5.2.10.* For all integers  $m \geq 2$ , we have  $\frac{n}{m-1} + \frac{3}{2} + \sqrt{\frac{mn^2}{(m-1)^2} + \frac{1}{4}} < \frac{5n+3}{2}$ .

This is the reason why we suppose, in the sequel of this paper, that  $\sigma$  is invertible modulo  $m$ . We end this section by showing that a simplex in the orbit of an arithmetic array can be decomposed into arithmetic subsimplices.

**Proposition 5.2.11** (Chappelon [C6]). *Let  $a \in \mathbb{Z}/m\mathbb{Z}$  and let  $d = (d_1, \dots, d_{n-1}) \in (\mathbb{Z}/m\mathbb{Z})^{n-1}$ . Let  $\alpha$  and  $s$  be two positive integers such that  $\alpha$  is divisible by  $\text{ord}_m(\sigma)$*

and  $s \equiv -t \pmod{\alpha}$ , where  $t \in [0, n-1]$ , and let  $\varepsilon \in \{-1, 1\}^n$ . Let  $\Delta(j, \varepsilon, s)$  be the  $n$ -simplex appearing in the orbit  $\mathcal{O}_{AA(a,d)} = (a_i)_{i \in \mathbb{Z}^{n-1} \times \mathbb{N}}$ . Then, for every  $k \in [0, \alpha-1]^n$ , the subsimplex

$$\text{SS}_k := \{a_{j+\varepsilon \cdot (k+\alpha l)} \mid l \in \mathbb{N}^n \text{ such that } (k_1 + \alpha l_1) + \cdots + (k_n + \alpha l_n) \leq s-1\},$$

obtained from  $\Delta(j, \varepsilon, s)$  by extracting one term every  $\alpha$  in each component, is the arithmetic simplex

$$\text{SS}_k = \text{AS} \left( a_{j+\varepsilon \cdot k}, \alpha \sigma^{j_n + \varepsilon_n k_n} \varepsilon \cdot \tilde{d}, \left\lfloor \frac{s}{\alpha} \right\rfloor - \left\lfloor \frac{\sum_{u=1}^n k_u + t}{\alpha} \right\rfloor \right),$$

where  $\tilde{d} = (d_1, \dots, d_{n-1}, \sigma^{-1} \sum_{u=1}^{n-1} \sigma_u d_u)$ .

From the previous proposition, we know that every  $n$ -simplex  $\Delta$  of size  $\lambda\alpha - t$ , where  $\alpha$  is a multiple of  $\text{ord}_m(\sigma)$  and  $t \in [0, n-1]$ , appearing in the orbit of an arithmetic array can be decomposed into  $\alpha^n$  arithmetic  $n$ -simplices of sizes in  $[\lambda - (n-1), \lambda]$ . Therefore, in next section, the arithmetic simplices will be studied in detail.

### 5.3 Balanced arithmetic simplices

In this section, we will see that arithmetic simplices are a source of balanced multisets of  $\mathbb{Z}/m\mathbb{Z}$ . First, we show, in the general case  $n \geq 1$ , that there exists sufficient conditions on arithmetic simplices for being balanced. After that, in dimension  $n = 2$  and  $n = 3$ , i.e., for arithmetic triangles and arithmetic tetrahedra, necessary conditions for being balanced are also given.

#### 5.3.1 The general case: in dimension $n \geq 1$

We begin this subsection by showing that, when  $n \geq 2$ , the edges, the facets and the rows of an arithmetic simplex are also arithmetic.

**Proposition 5.3.1.** *Let  $a \in \mathbb{Z}/m\mathbb{Z}$ ,  $d = (d_1, \dots, d_n) \in (\mathbb{Z}/m\mathbb{Z})^n$  and let  $s$  be a positive integer. Let  $\Delta := \text{AS}(a, d, s)$  and let  $d_0 := 0$ . Then, we have*

$$V_i(\Delta) = a + (s-1)d_i,$$

for all  $i \in [0, n]$ ,

$$E_{i,j}(\Delta) = \text{AP}(V_i(\Delta), d_j - d_i, s),$$

for all distinct integers  $i, j \in [0, n]$ ,

$$R_i(\Delta) = \text{AS}(a + id_n, (d_1, \dots, d_{n-1}), s-i),$$

for all  $i \in [0, n]$ ,

$$F_i(\Delta) = \text{AS}(a, (d_1, \dots, d_{i-1}, d_{i+1}, \dots, d_n), s),$$

for all  $i \in [1, n]$ , and

$$F_0(\Delta) = \text{AS}(a + (s-1)d_1, (d_2 - d_1, \dots, d_n - d_1), s).$$

Moreover, for all  $i \in [0, n]$ , we have

$$\Delta \setminus F_i(\Delta) = \text{AS}(a + d_i, d, s-1).$$

The following theorem, which gives sufficient conditions on arithmetic simplices for being balanced, is the main result of this section.

**Theorem 5.3.2** (Chappelon [C6]). *Let  $n$  and  $m$  be two positive integers with  $\gcd(m, n!) = 1$ . Let  $a \in \mathbb{Z}/m\mathbb{Z}$  and let  $d = (d_1, \dots, d_n) \in (\mathbb{Z}/m\mathbb{Z})^n$  such that  $d_i$ , for all  $i \in [1, n]$ , and  $d_j - d_i$ , for all distinct integers  $i, j \in [1, n]$ , are invertible. Then, the arithmetic simplex  $\text{AS}(a, d, s)$  is balanced for all  $s \equiv -t \pmod{m}$ , with  $t \in [0, n - 1]$ .*

### 5.3.2 In dimension 2

In this subsection, we only consider arithmetic triangles over  $\mathbb{Z}/m\mathbb{Z}$ . Necessary conditions on the common differences  $d_1$ ,  $d_2$  and  $d_2 - d_1$  of  $\text{AS}(a, (d_1, d_2), s)$ , depicted in Figure 5.4, for being balanced in  $\mathbb{Z}/m\mathbb{Z}$  are determined.

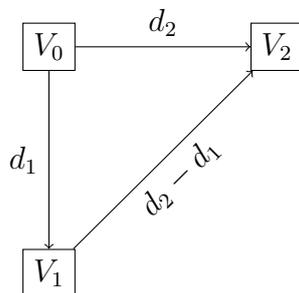


Figure 5.4: Common differences of an arithmetic triangle

**Theorem 5.3.3** (Chappelon [C6]). *Let  $m$  and  $s$  be two positive integers and let  $a, d_1, d_2 \in \mathbb{Z}/m\mathbb{Z}$ . If the arithmetic triangle  $\text{AS}(a, (d_1, d_2), s)$  is balanced, then the common differences  $d_1$ ,  $d_2$  and  $d_2 - d_1$  are all invertible.*

It follows from this theorem that there does not exist balanced arithmetic triangles in  $\mathbb{Z}/m\mathbb{Z}$  for  $m$  even. Nevertheless, in the case where  $m$  is an even number, the multiplicity function of an arithmetic triangle of  $\mathbb{Z}/m\mathbb{Z}$  can be completely determined when exactly two of the three common differences  $d_1, d_2, d_1 - d_2$  are invertible and the size  $s$  is such that  $s \equiv 0$  or  $-1 \pmod{m}$ .

**Proposition 5.3.4** (Chappelon [C6]). *Let  $m$  and  $s$  be two positive integers such that  $s \equiv 0$  or  $-1 \pmod{m}$ . Let  $a, d_1, d_2 \in \mathbb{Z}/m\mathbb{Z}$  and let  $\Delta = \text{AS}(a, (d_1, d_2), s)$ . If  $d_2$  and  $d_2 - d_1$  are invertible, then*

$$\mathbf{m}_\Delta(x) = \mathbf{m}_\Delta(x + \gcd(d_1, m)),$$

for all  $x \in \mathbb{Z}/m\mathbb{Z}$ , and

$$\mathbf{m}_\Delta(a + id_2) = \frac{1}{m} \binom{s+1}{2} + \left\lceil \frac{s}{m} \right\rceil \left( \frac{\gcd(d_1, m) - 1}{2} - i \right),$$

for all integers  $i \in [0, \gcd(d_1, m) - 1]$ .

For example, for  $m = 12$ ,  $a = 0$ ,  $d_1 = 1$  and  $d_2 = 5$ , we obtain that  $d_2 - d_1 = 4$ ,  $\gcd(d_1 - d_2, m) = 4$  and the multiplicity function of  $\Delta = \text{AS}(a, (d_1, d_2), m)$  is given in Table 5.1.

$x$	0	1	2	3	4	5	6	7	8	9	10	11
$m_{\Delta}(x)$	5	6	7	8	5	6	7	8	5	6	7	8

Table 5.1: Multiplicity function of  $\Delta = \text{AS}(0, (1, 5), 12)$  in  $\mathbb{Z}/12\mathbb{Z}$

### 5.3.3 In dimension 3

In this subsection, we only consider the arithmetic tetrahedron  $\text{AS}(a, (d_1, d_2, d_3), s)$  in  $\mathbb{Z}/m\mathbb{Z}$ . We determine necessary and sufficient conditions on the common differences  $d_1, d_2, d_3, d_2 - d_1, d_3 - d_2$  and  $d_1 - d_3$  of  $\text{AS}(a, (d_1, d_2, d_3), s)$ , depicted in Figure 5.5, for being balanced in  $\mathbb{Z}/m\mathbb{Z}$ .

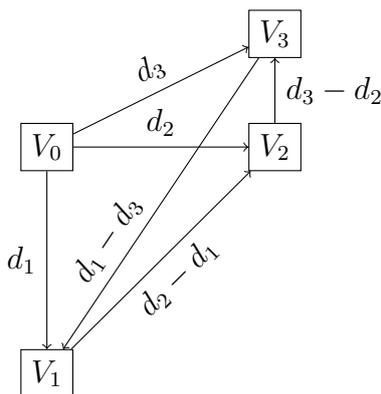


Figure 5.5: Common differences of an arithmetic tetrahedron

**Definition 5.3.5** (Adjacent common differences). Among the six common differences  $d_1, d_2, d_3, d_2 - d_1, d_3 - d_2$  and  $d_1 - d_3$  of  $\text{AS}(a, (d_1, d_2, d_3), s)$ , two of them are said to be adjacent if they have a vertex in common. The couples of non-adjacent common differences of  $\text{AS}(a, (d_1, d_2, d_3), s)$  are  $(d_1, d_3 - d_2), (d_2, d_1 - d_3)$  and  $(d_3, d_2 - d_1)$ . The twelve other couples of common differences are said to be adjacent (See Figure 5.6).

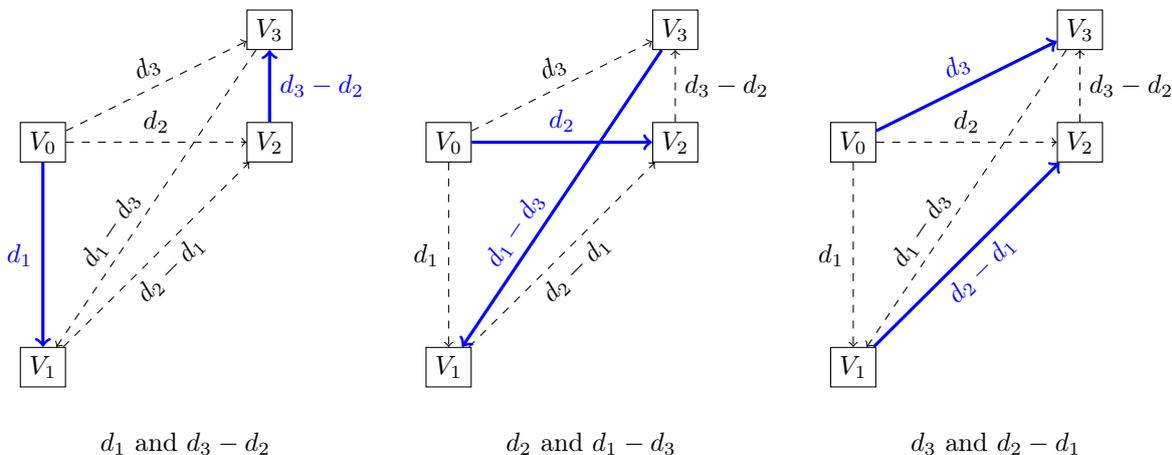


Figure 5.6: Non-adjacent common differences of  $\text{AS}(a, (d_1, d_2, d_3), s)$

**Theorem 5.3.6** (Chappelon [C6]). *Let  $m$  and  $s$  be two positive integers and let  $a, d_1, d_2, d_3$  be in  $\mathbb{Z}/m\mathbb{Z}$ . Let  $D := \{d_1, d_2, d_3, d_2 - d_1, d_3 - d_2, d_1 - d_3\}$  be the set of common differences of the arithmetic tetrahedron  $\Delta = \text{AS}(a, (d_1, d_2, d_3), s)$ . If  $\Delta$  is balanced in  $\mathbb{Z}/m\mathbb{Z}$  and*

- i)  $m$  is odd, then all the elements of  $D$  are invertible.*
- ii)  $m$  is even, then all the elements of  $D$  are invertible, except two of them, say  $\delta_1$  and  $\delta_2$ , which are non-adjacent and such that  $\gcd(\delta_1, m) = \gcd(\delta_2, m) = 2$ .*

We continue by showing that there is no balanced arithmetic tetrahedron in  $\mathbb{Z}/m\mathbb{Z}$  when  $m$  is divisible by 3.

**Theorem 5.3.7** (Chappelon [C6]). *Let  $m$  and  $s$  be two positive integers such that  $m$  is a multiple of 3. There is no balanced arithmetic tetrahedron of size  $s$  in  $\mathbb{Z}/m\mathbb{Z}$ .*

In the end of this subsection, we prove that the necessary conditions on the common differences of balanced arithmetic tetrahedra highlighted in Theorem 5.3.6 are also sufficient for certain sizes.

**Theorem 5.3.8** (Chappelon [C6]). *Let  $m$  be an odd number not divisible by 3 and let  $a, d_1, d_2, d_3 \in \mathbb{Z}/m\mathbb{Z}$  such that  $d_1, d_2, d_3, d_2 - d_1, d_3 - d_2$  and  $d_1 - d_3$  are invertible. Then, the arithmetic tetrahedron  $\text{AS}(a, (d_1, d_2, d_3), s)$  is balanced for all  $s \equiv 0, -1, \text{ or } -2 \pmod{m}$ .*

*Proof.* Theorem 5.3.2 for  $n = 3$ . □

**Theorem 5.3.9** (Chappelon [C6]). *Let  $m$  be an even number not divisible by 3 and let  $a, d_1, d_2, d_3 \in \mathbb{Z}/m\mathbb{Z}$  such that  $\gcd(d_1, m) = \gcd(d_3 - d_2, m) = 2$  and  $d_2, d_3, d_2 - d_1$  and  $d_1 - d_3$  are invertible. Then, the arithmetic tetrahedron  $\text{AS}(a, (d_1, d_2, d_3), s)$  is balanced for all  $s \equiv 0 \text{ or } -2 \pmod{m}$ .*

*Remark 5.3.10.* When  $m$  is even and not divisible by 3, if we suppose that  $\gcd(d_1, m) = \gcd(d_3 - d_2, m) = 2$  and  $d_2, d_3, d_2 - d_1$  and  $d_1 - d_3$  are invertible, then the arithmetic tetrahedron  $\text{AS}(a, (d_1, d_2, d_3), s)$  is not balanced for all  $s \equiv -1 \pmod{m}$ . Indeed, it can be seen as the multiset difference of the arithmetic tetrahedron  $\text{AS}(a - d_3, (d_1, d_2, d_3), s + 1)$ , which is balanced by Theorem 5.3.9, and the arithmetic triangle  $\text{AS}(a - d_3, (d_1, d_2), s + 1)$ , which is not balanced by Theorem 5.3.3.

## 5.4 Balanced simplices generated from arithmetic arrays

We are now ready to show that the orbits generated from arithmetic arrays by additive cellular automata are a source of balanced simplices.

### 5.4.1 The general case: in dimension $n \geq 2$

**For any ACA**

Using properties of arithmetic simplices, Theorem 5.1.8, which is the main result of this chapter, is obtained.

**Theorem 5.4.1** (Chappelon [C6]). *Let  $n \geq 2$  and  $m$  be two positive integers such that  $\gcd(m, n!) = 1$ . Suppose that  $\sigma$  is invertible modulo  $m$ . Let  $a \in \mathbb{Z}/m\mathbb{Z}$ ,  $d \in (\mathbb{Z}/m\mathbb{Z})^{n-1}$  and  $\varepsilon \in \{-1, 1\}^n$  such that  $d_i$ , for all  $i \in [1, n]$ , and  $\varepsilon_j d_j - \varepsilon_i d_i$ , for all distinct integers  $i, j \in [1, n]$ , are invertible, where  $d_n := \sigma^{-1} \sum_{k=1}^{n-1} \sigma_k d_k$ . Then, in the orbit  $\mathcal{O}_{AA(a,d)}$ , every  $n$ -simplex with orientation  $\varepsilon$  and of size  $s$  is balanced modulo  $m$ , for all  $s \equiv -t \pmod{\text{lcm}(\text{ord}_m(\sigma), m)}$ , where  $t \in [0, n-1]$ .*

*Remark 5.4.2.* If  $\sigma \equiv 1 \pmod{m}$ , then the  $n$ -simplex  $\Delta(j, \varepsilon, s)$  appearing in  $\mathcal{O}_{AA(a,d)}$  is an arithmetic simplex and Theorem 5.1.8 is simply Theorem 5.3.2 on balanced arithmetic simplices.

### For the Pascal cellular automata

Here, we investigate the consequences of Theorem 5.1.8 on the existence of balanced  $n$ -simplices, in the case where the ACA considered is  $\text{PCA}_{n-1}$  and we obtain Corollary 5.1.10.

**Corollary 5.4.3** (Chappelon [C6]). *Let  $n \geq 2$  be a positive integers. For every positive integer  $m$  such that  $\gcd(m, (3(n-1))!) = 1$ , there exist infinitely many balanced  $n$ -simplices of  $\mathbb{Z}/m\mathbb{Z}$  generated by  $\text{PCA}_{n-1}$ , for all possible orientations  $\varepsilon \in \{-1, 1\}^n$ . In the special case of the two orientations  $\varepsilon = + \cdots + -$  or  $\varepsilon = - \cdots - +$ , the existence of an infinite number of such balanced simplices is verified for every  $\mathbb{Z}/m\mathbb{Z}$  such that  $\gcd(m, n!) = 1$ , if  $n$  is even, and for every  $\mathbb{Z}/m\mathbb{Z}$  such that  $\gcd(m, (\frac{3n+1}{2})!) = 1$ , if  $n$  is odd.*

## 5.4.2 In dimension 3

In this subsection, we show that, in dimension 3, a similar result as Theorem 5.1.8 can be obtained for certain even values of  $m$  by using Theorem 5.3.9. As a corollary, the special case of the Pascal cellular automaton  $\text{PCA}_2$  is studied.

### For any ACA

**Theorem 5.4.4** (Chappelon [C6]). *Let  $m$  be an even number not divisible by 3 such that  $\sigma \in \mathbb{Z}/m\mathbb{Z}$  is invertible and  $\sigma \equiv 1 \pmod{2^{v_2(m)}}$ , where  $v_2(m)$  is the highest exponent  $u$  such that  $2^u$  divides  $m$ . Let  $a \in \mathbb{Z}/m\mathbb{Z}$ ,  $d = (d_1, d_2) \in (\mathbb{Z}/m\mathbb{Z})^2$  and  $\varepsilon \in \{-1, 1\}^3$  such that  $\varepsilon_2 d_2$ ,  $\varepsilon_3 d_3$ ,  $\varepsilon_2 d_2 - \varepsilon_1 d_1$ ,  $\varepsilon_1 d_1 - \varepsilon_3 d_3$  are invertible in  $\mathbb{Z}/m\mathbb{Z}$  and  $\gcd(\varepsilon_1 d_1, m) = \gcd(\varepsilon_3 d_3 - \varepsilon_2 d_2, m) = 2$ , where  $d_3 := \sigma^{-1}(\sigma_1 d_1 + \sigma_2 d_2)$ . Then, in the orbit  $\mathcal{O}_{AA(a,d)}$ , every tetrahedron with orientation  $\varepsilon$  and of size  $s$  is balanced, for all  $s \equiv 0$  or  $-2 \pmod{\text{lcm}(\text{ord}_m(\sigma), m)}$ .*

### For the Pascal cellular automaton

Here, we investigate the consequences of Theorem 5.4.4 on the existence of balanced tetrahedra, in the case where the ACA considered is  $\text{PCA}_2$ .

**Corollary 5.4.5** (Chappelon [C6]). *For even numbers  $m$  not divisible by 3 such that  $v_2(m) = 1$ , there exist infinitely many balanced tetrahedra of  $\mathbb{Z}/m\mathbb{Z}$  generated by  $\text{PCA}_2$ , for all orientations  $\varepsilon = +++$ ,  $+ - +$ ,  $+ - -$ ,  $- + +$ ,  $- + -$  and  $- - -$ . In the remaining case of the two orientations  $\varepsilon = ++ -$  or  $\varepsilon = - - +$ , the existence of an infinite number of such balanced tetrahedra is verified for every  $\mathbb{Z}/m\mathbb{Z}$  of even order  $m$  such that  $v_2(m) = 1$  and  $\gcd(m, 3.5) = 1$ .*

## 5.5 The antisymmetric case

We begin this section by defining the antisymmetric sequences and the antisymmetric simplices.

**Definition 5.5.1** (Antisymmetric sequences). A finite sequence  $S = (a_1, \dots, a_s)$  of length  $s \geq 1$  in  $\mathbb{Z}/m\mathbb{Z}$  is said to be *antisymmetric* if  $a_i + a_{s-i+1} = 0$  for all  $i \in [1, s]$ .

For instance, the sequence  $S = 2210433$  is antisymmetric in  $\mathbb{Z}/5\mathbb{Z}$ .

**Definition 5.5.2** (Antisymmetric simplices). Let  $A = (a_i)_{i \in \mathbb{Z}^n}$  be an infinite array of elements in  $\mathbb{Z}/m\mathbb{Z}$  and let  $\Delta(j, \varepsilon, s)$  be the  $n$ -simplex of size  $s$ , with orientation  $\varepsilon \in \{-1, 1\}^n$  and whose principal vertex is  $a_j$  in  $A$ , that is,

$$\Delta(j, \varepsilon, s) = \{a_{j+\varepsilon \cdot k} \mid k \in \mathbb{N}^n \text{ such that } k_1 + \dots + k_n \leq s - 1\}.$$

Let  $u$  and  $v$  be two distinct integers in  $[0, n]$ . The simplex  $\Delta(j, \varepsilon, s)$  is said to be  $(u, v)$ -*antisymmetric* if all its subsequences in the same direction of the edge  $E_{u,v}$  between the vertices  $V_u$  and  $V_v$  are antisymmetric. More precisely,  $\Delta(j, \varepsilon, s)$  is  $(0, v)$ -antisymmetric if we have

$$a_{j+\varepsilon \cdot k} + a_{j+\varepsilon \cdot s(k)} = 0, \text{ where } s(k) = \left( k_1, \dots, k_{v-1}, s - 1 - \sum_{l=1}^n k_l, k_{v+1}, \dots, k_n \right),$$

for all  $k \in \mathbb{N}^n$  such that  $k_1 + \dots + k_n \leq s - 1$  and, for  $u, v \geq 1$ ,  $\Delta(j, \varepsilon, s)$  is  $(u, v)$ -antisymmetric if we have

$$a_{j+\varepsilon \cdot k} + a_{j+\varepsilon \cdot t(k)} = 0, \text{ where } (t(k))_l = k_{\tau(l)} \text{ for all } l \in [1, n],$$

where  $\tau$  is the transposition  $(u, v)$ , for all  $k \in \mathbb{N}^n$  such that  $k_1 + \dots + k_n \leq s - 1$ .

For instance, the tetrahedron depicted in Figure 5.7 is  $(1, 2)$ -antisymmetric. Moreover, each row of this tetrahedron is an  $(1, 2)$ -antisymmetric triangle.

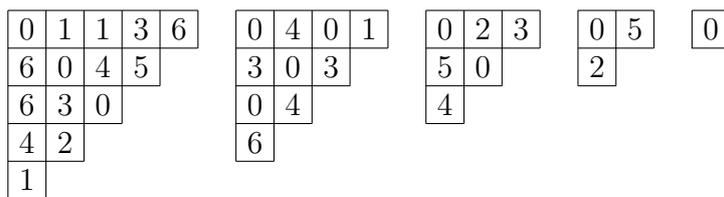


Figure 5.7: An  $(1, 2)$ -antisymmetric tetrahedron in  $\mathbb{Z}/7\mathbb{Z}$

### 5.5.1 Antisymmetric simplices

Let  $m$  and  $n$  be two positive integers such that  $n \geq 2$  and  $\gcd(m, n!) = 1$ . In the sequel of this section, we consider  $n$ -simplices  $\Delta(j, \varepsilon, s)$  appearing in the orbit of the arithmetic array  $AA(a, d)$ , where  $a \in \mathbb{Z}/m\mathbb{Z}$  and  $d = (d_1, d_2, \dots, d_{n-1}) \in (\mathbb{Z}/m\mathbb{Z})^{n-1}$ , generated by an ACA of weight array  $W = (w_l)_{l \in [-r, r]^{n-1}}$ . The elements of this orbit are denoted by  $\mathcal{O}_{AA(a, d)} = (a_i)_{i \in \mathbb{Z}^{n-1} \times \mathbb{N}}$ . As already defined before,

$$\sigma := \sum_{l \in [-r, r]^{n-1}} w_l \quad \text{and} \quad \sigma_k = \sum_{l \in [-r, r]^{n-1}} l_k w_l, \text{ for all } k \in [1, n - 1].$$

Moreover, suppose that  $\sigma$  is invertible modulo  $m$  and let

$$d_n := \sigma^{-1} \sum_{k=1}^{n-1} \sigma_k d_k.$$

In this subsection, necessary conditions on simplices for being antisymmetric are determined.

**Proposition 5.5.3** (Chappelon [C6]). *Let  $u$  and  $v$  be two distinct integers in  $[1, n]$ . If  $\Delta(j, \varepsilon, s)$  is  $(u, v)$ -antisymmetric, then  $d_w = 0$  for all  $w \in [1, n] \setminus \{u, v\}$  and  $\varepsilon_u d_u + \varepsilon_v d_v = 0$ .*

**Proposition 5.5.4** (Chappelon [C6]). *Let  $v$  be an integer in  $[1, n]$ . If  $\Delta(j, \varepsilon, s)$  is  $(0, v)$ -antisymmetric, then  $2\varepsilon_w d_w = \varepsilon_v d_v$  for all  $w \in [1, n] \setminus \{v\}$ .*

For  $n \geq 3$ , it is easy to see from Proposition 5.5.3 and Proposition 5.5.4 that if the simplex  $\Delta(j, \varepsilon, s)$  is  $(u, v)$ -antisymmetric, then there is at least one element among the elements  $\varepsilon_i d_i$ , for all  $i \in [1, n]$ , and  $\varepsilon_j d_j - \varepsilon_i d_i$ , for all distinct integers  $i, j \in [1, n]$ , which is non-invertible and equal to zero in  $\mathbb{Z}/m\mathbb{Z}$ . In this case, the hypotheses of Theorem 5.3.2 are not satisfied. Therefore, in the next subsection, we only consider the case of dimension  $n = 2$ .

## 5.5.2 In dimension 2

An arithmetic array of dimension 1 is simply called an *arithmetic progression* and is denoted by  $\text{AP}(a, d)$  or  $\text{AP}(a, d, s)$ , for an arithmetic progression of length  $s$ , that is,

$$\text{AP}(a, d, s) = (a, a + d, a + 2d, \dots, a + (s - 1)d).$$

Let  $W = (w_{-r}, \dots, w_r) \in \mathbb{Z}^{2r+1}$  be the weight sequence of the ACA of dimension 1 that we consider here. First, we know that the derived sequence of an arithmetic progression is also an arithmetic progression. Indeed, we have

$$\partial \text{AP}(a, d) = \text{AP}(\sigma a + \sigma' d, \sigma d),$$

where

$$\sigma = \sum_{i=-r}^r w_i \quad \text{and} \quad \sigma' = \sum_{i=-r}^r i w_i.$$

As already remarked, for  $\overline{W} = (0, \sigma - \sigma', \sigma')$ , we obtain that  $\sigma(\overline{W}) = \sigma(W)$  and  $\sigma'(\overline{W}) = \sigma'(W)$ . Thus, the orbits of  $\text{AP}(a, d)$  are the same if we consider  $W$  or  $\overline{W}$ . Therefore, in the sequel of this subsection, we only consider the case where  $r = 1$  and  $W = (0, \sigma - \sigma', \sigma')$ .

### (1,2)-antisymmetric triangles

First, we know from Proposition 5.5.3 that if the triangle  $\Delta(j, \varepsilon, s)$ , appearing in the orbit  $\mathcal{O}_{\text{AP}(a,d)}$  with  $d$  invertible, is  $(1, 2)$ -antisymmetric, then

$$\varepsilon_1 d + \varepsilon_2 \sigma^{-1} \sigma' d = 0 \iff \varepsilon_1 \sigma + \varepsilon_2 \sigma' = 0 \iff \sigma' = -\varepsilon_1 \varepsilon_2 \sigma.$$

So, we deduce that

$$W = (0, (1 + \varepsilon_1 \varepsilon_2) \sigma, -\varepsilon_1 \varepsilon_2 \sigma).$$

Since  $\Delta(j, \varepsilon, s)$  is  $(1, 2)$ -antisymmetric, we know that  $a_j = 0$  and  $a_{j+\varepsilon \cdot e_1} + a_{j+\varepsilon \cdot e_2} = 0$ . It follows that

$$a_j = 0 \iff a + j_1 d_1 + j_2 d_2 = 0,$$

and

$$\begin{aligned} a_{j+\varepsilon \cdot e_1} + a_{j+\varepsilon \cdot e_2} = 0 &\iff \sigma^{j_2}(a + (j_1 + \varepsilon_1)d_1 + j_2 d_2) + \sigma^{j_2+\varepsilon_2}(a + j_1 d_1 + (j_2 + \varepsilon_2)d_2) = 0 \\ &\implies \varepsilon_1 d_1 + \sigma^{\varepsilon_2} \varepsilon_2 d_2 = 0 \\ &\iff \varepsilon_1 + \sigma^{\varepsilon_2} \varepsilon_2 \sigma^{-1} \sigma' = 0. \end{aligned}$$

This implies that  $\sigma^{\varepsilon_2} = 1$  and thus  $\sigma = 1$ . Therefore,

$$W = (0, 1 + \varepsilon_1 \varepsilon_2, -\varepsilon_1 \varepsilon_2).$$

Finally, since  $\sigma = 1$ , we know that  $\Delta(j, \varepsilon, s)$  is an arithmetic triangle which is already balanced for all  $s \equiv 0$  or  $-1 \pmod m$  by Theorem 5.3.2.

### **(0,2)-antisymmetric triangles**

First, we know from Proposition 5.5.4 that if the triangle  $\Delta(j, \varepsilon, s)$ , appearing in the orbit  $\mathcal{O}_{\text{AP}(a,d)}$  with  $d$  invertible, is  $(0, 2)$ -antisymmetric, then

$$\varepsilon_2 \sigma^{-1} \sigma' = 2\varepsilon_1 \iff \sigma' = 2\varepsilon_1 \varepsilon_2 \sigma.$$

So, we deduce that

$$W = (0, (1 - 2\varepsilon_1 \varepsilon_2)\sigma, 2\varepsilon_1 \varepsilon_2 \sigma).$$

Since  $\Delta(j, \varepsilon, s)$  is  $(0, 2)$ -antisymmetric, we know that  $a_{j+(s-1)\varepsilon \cdot e_1} = 0$  and  $a_{j+(s-2)\varepsilon \cdot e_1} + a_{j+\varepsilon \cdot ((s-2)e_1 + e_2)} = 0$ . It follows that

$$a_{j+(s-1)\varepsilon \cdot e_1} = 0 \iff a + (j_1 + \varepsilon_1(s-1))d_1 + j_2 d_2 = 0,$$

and

$$\begin{aligned} a_{j+(s-2)\varepsilon \cdot e_1} + a_{j+\varepsilon \cdot ((s-2)e_1 + e_2)} = 0 &\iff \sigma^{j_2}(a + (j_1 + \varepsilon_1(s-2))d_1 + j_2 d_2) \\ &\quad + \sigma^{j_2+\varepsilon_2}(a + (j_1 + \varepsilon_1(s-2))d_1 + (j_2 + \varepsilon_2)d_2) = 0 \\ &\implies -\varepsilon_1 d_1 + \sigma^{\varepsilon_2}(-\varepsilon_1 d_1 + \varepsilon_2 d_2) = 0 \\ &\iff -\varepsilon_1 + \sigma^{\varepsilon_2}(-\varepsilon_1 + \varepsilon_2 \sigma^{-1} \sigma') = 0 \\ &\iff \sigma' = \frac{1 + \sigma^{\varepsilon_2}}{\sigma^{\varepsilon_2}} \varepsilon_1 \varepsilon_2 \sigma \end{aligned}$$

This implies that  $\sigma^{\varepsilon_2} = 1$  and thus  $\sigma = 1$ . Therefore,

$$W = (0, 1 - 2\varepsilon_1 \varepsilon_2, 2\varepsilon_1 \varepsilon_2).$$

Finally, since  $\sigma = 1$ , we know that  $\Delta(j, \varepsilon, s)$  is an arithmetic triangle which is already balanced for all  $s \equiv 0$  or  $-1 \pmod m$  by Theorem 5.3.2.

### (0,1)-antisymmetric triangles

First, we know from Proposition 5.5.4 that if the simplex  $\Delta(j, \varepsilon, s)$  is (0, 1)-antisymmetric, then

$$\varepsilon_1 = 2\varepsilon_2\sigma^{-1}\sigma' \iff \sigma = 2\varepsilon_1\varepsilon_2\sigma'.$$

So, we deduce that

$$W = (0, (2\varepsilon_1\varepsilon_2 - 1)\sigma', \sigma').$$

Now, we refine Theorem 5.1.8 in this case by considering triangles that have the additional property to be (0, 1)-antisymmetric.

**Theorem 5.5.5** (Chappelton [C6]). *Let  $m$  be an odd positive integer and let  $W \in \mathbb{Z}^{2r+1}$  such that  $\sigma = 2\sigma'$  and  $\sigma$  is invertible modulo  $m$ . Let  $a, d \in \mathbb{Z}/m\mathbb{Z}$  such that  $d$  is invertible. Then, in the orbit  $\mathcal{O}_{\text{AP}(a,d)}$ , every (0, 1)-antisymmetric triangle of orientation  $(++)$  or  $(--)$  and of size  $s$  is balanced, for all positive integers  $s \equiv 0$  or  $-1 \pmod{\text{lcm}(\text{pord}_m(\sigma), m)}$ , where  $\text{pord}_m(\sigma)$  is the multiplicative order of  $\sigma$  in  $(\mathbb{Z}/m\mathbb{Z})^* / \{-1, 1\}$ .*

**Theorem 5.5.6** (Chappelton [C6]). *Let  $m$  be an odd positive integer and let  $W \in \mathbb{Z}^{2r+1}$  such that  $\sigma = -2\sigma'$  and  $\sigma$  is invertible modulo  $m$ . Let  $a, d \in \mathbb{Z}/m\mathbb{Z}$  such that  $d$  is invertible. Then, in the orbit  $\mathcal{O}_{\text{AP}(a,d)}$ , every (0, 1)-antisymmetric triangle of orientation  $(-+)$  or  $(+-)$  and of size  $s$  is balanced, for all positive integers  $s \equiv 0$  or  $-1 \pmod{\text{lcm}(\text{pord}_m(\sigma), m)}$ , where  $\text{pord}_m(\sigma)$  is the multiplicative order of  $\sigma$  in  $(\mathbb{Z}/m\mathbb{Z})^* / \{-1, 1\}$ .*

## 5.6 Open problems

For the Pascal cellular automaton of dimension 1, the following problems remain open.

**Problem 5.6.1.** *For  $m$  even, do there exist infinitely many balanced triangles of  $\mathbb{Z}/m\mathbb{Z}$ , with any orientation, generated by  $\text{PCA}_1$ ?*

**Problem 5.6.2.** *For  $m$  odd divisible by 3, do there exist infinitely many balanced triangles of  $\mathbb{Z}/m\mathbb{Z}$ , with orientations  $++$  and  $--$ , generated by  $\text{PCA}_1$ ?*

For the Pascal cellular automaton of dimension 2, the following problems remain open.

**Problem 5.6.3.** *For  $m$  divisible by 3, do there exist infinitely many balanced tetrahedra of  $\mathbb{Z}/m\mathbb{Z}$ , with any orientation, generated by  $\text{PCA}_2$ ?*

**Problem 5.6.4.** *For  $m$  even such that  $v_2(m) \geq 2$ , do there exist infinitely many balanced tetrahedra of  $\mathbb{Z}/m\mathbb{Z}$ , with any orientation, generated by  $\text{PCA}_2$ ?*

**Problem 5.6.5.** *For  $m$  odd divisible by 5, do there exist infinitely many balanced tetrahedra of  $\mathbb{Z}/m\mathbb{Z}$ , with orientations  $++-$ ,  $+ - +$ ,  $+ - -$ ,  $- + +$ ,  $- + -$  and  $- - -$ , generated by  $\text{PCA}_2$ ?*

**Problem 5.6.6.** *For  $m$  divisible by 5 such that  $v_2(m) \leq 1$ , do there exist infinitely many balanced tetrahedra of  $\mathbb{Z}/m\mathbb{Z}$ , with orientations  $++-$  and  $- - +$ , generated by  $\text{PCA}_2$ ?*

## Part II

Other results on numerical semigroups,  
Tower of Hanoi, Ramsey theory and  
Kneser transversals



# Chapter 6

## Numerical semigroups

The main topic of this chapter is the numerical semigroups. The results presented here can be found in the publications [C16, C10] and the manuscript [C17].

### 6.1 The Möbius function of numerical semigroup posets

#### 6.1.1 Numerical semigroups

**Definition 6.1.1** (Semigroup). A *semigroup* is a set  $S$  with an internal binary operation  $\cdot : S \times S \rightarrow S$  that is associative, i.e.,

$$a.(b.c) = (a.b).c,$$

for all  $a, b, c \in S$ .

**Definition 6.1.2** (Numerical semigroup). A *numerical semigroup* is a subsemigroup  $S$  of  $\mathbb{N}$  such that  $0 \in S$  and  $\mathbb{N} \setminus S$  is finite.

**Notation 6.1.3.** Let  $A = \{a_1, \dots, a_n\}$  be a finite set of non-negative integers. The subsemigroup of  $\mathbb{N}$  generated by  $A$  is denoted by  $\langle A \rangle$ , i.e.,

$$\langle A \rangle = \{x_1 a_1 + \dots + x_n a_n \mid (x_1, \dots, x_n) \in \mathbb{N}^n\}.$$

For any non-negative integer  $x$ , we say that  $x$  is *representable* as a non-negative integer combination of  $a_1, \dots, a_n$  if and only if  $x \in \langle A \rangle$ .

**Proposition 6.1.4.** *The subsemigroup  $S$  of  $\mathbb{N}$  is a numerical semigroup if and only if there exists a finite subset  $A$  of  $\mathbb{N}$  such that  $S = \langle A \rangle$  with  $\gcd(A) = 1$ .*

**Corollary 6.1.5.** *Let  $A = \{a_1, \dots, a_n\}$  be a finite set of non-negative integers such that  $\gcd(A) = 1$ . Then, there exists an integer  $N$  such that any integer  $x \geq N$  is representable as a non-negative integer combination of  $a_1, \dots, a_n$ .*

**Definition 6.1.6** (Gaps, Frobenius number, genus and multiplicity). Let  $S$  be a numerical semigroup. The positive integers not in  $S$  are called the *gaps* of  $S$ . The largest gap of  $S$  is called the *Frobenius number* of  $S$  and is denoted by  $g(S)$ . The number of gaps of  $S$ , denoted by  $N(S)$  (that is,  $N(S) = |\mathbb{N} \setminus S|$ ) is called the *genus* of  $S$ . The *multiplicity* of  $S$  is the smallest positive element  $m(S)$  belonging to  $S$ .

It is well-known that

$$g(\langle\{a_1, a_2\}\rangle) = a_1 a_2 - a_1 - a_2,$$

for any coprime positive integers  $a_1$  and  $a_2$ . However, calculating  $g(S)$  is a difficult problem in general. In [95] was shown that computing  $g(S)$  is *NP-hard*. An extensive literature on the Frobenius number and numerical semigroups can be found in [96, 97].

### 6.1.2 The Möbius function of locally finite posets

The *Möbius function* is an important concept associated with *locally finite posets*.

**Definition 6.1.7** (Poset). A *partial order* is a binary relation  $\leq_P$  over a set  $P$  that is reflexive, antisymmetric and transitive, i.e.,

- i)  $a \leq_P a$ , for all  $a \in P$ , (reflexivity)
- ii) if  $a \leq_P b$  and  $b \leq_P a$ , then  $a = b$ , for all  $a, b \in P$ , (antisymmetry)
- iii) if  $a \leq_P b$  and  $b \leq_P c$ , then  $a \leq_P c$ , for all  $a, b, c \in P$ . (transitivity)

A set with a partial order is called a *partially ordered set*, or *poset* for short.

**Definition 6.1.8** (Locally finite poset). Let  $(P, \leq_P)$  be a poset. The *strict partial order*  $<_P$  is the reduction of  $\leq_P$  given by,  $a <_P b$  if and only if  $a \leq_P b$  and  $a \neq b$ . For any  $a$  and  $b$  in the poset  $P$ , the *segments* between  $a$  and  $b$  are defined by

$$\begin{aligned} [a, b]_P &= \{c \in P \mid a \leq_P c \leq_P b\}, & ]a, b]_P &= \{c \in P \mid a <_P c \leq_P b\}, \\ [a, b[_P &= \{c \in P \mid a \leq_P c <_P b\}, & ]a, b[_P &= \{c \in P \mid a <_P c <_P b\}. \end{aligned}$$

A poset is said to be *locally finite* if every segment has finite cardinality.

**Definition 6.1.9** (Chain). Let  $a$  and  $b$  be elements of the locally finite poset  $(P, \leq_P)$ . A *chain* of length  $l \geq 0$  between  $a$  and  $b$  is a subset of  $[a, b]_P$  containing  $a$  and  $b$ , with cardinality  $l + 1$  and totally ordered by  $\leq_P$ , that is  $\{a_0, a_1, \dots, a_l\} \subset [a, b]_P$  such that

$$a = a_0 <_P a_1 <_P a_2 <_P \dots <_P a_{l-1} <_P a_l = b.$$

For any nonnegative integer  $l$ , we denote by  $C_l(a, b)$  the set of all chains of length  $l$  between  $a$  and  $b$ . The cardinality of  $C_l(a, b)$  is denoted by  $c_l(a, b)$ . This number is always finite because the poset  $P$  is supposed to be locally finite.

For instance, the number of chains  $c_3(1, 12)$ , where the poset is the set  $\mathbb{N}$  partially ordered by divisibility, is equal to 3. Indeed, there are exactly 3 chains of length 3 between 1 and 12 in  $[1, 12]_{\mathbb{N}} = \{1, 2, 3, 4, 6, 12\}$ , which are  $\{1, 2, 4, 12\}$ ,  $\{1, 2, 6, 12\}$  and  $\{1, 3, 6, 12\}$ .

**Definition 6.1.10** (Möbius function). For any locally finite poset  $P$ , the *Möbius function*  $\mu_P$  is the integer-valued function on  $P \times P$  defined by

$$\mu_P(a, b) = \sum_{l \geq 0} (-1)^l c_l(a, b), \tag{6.1}$$

for all elements  $a$  and  $b$  of the poset  $P$ .

*Remark 6.1.11.*

- i) The sum involved in the definition of the Möbius function is always finite because, for  $a$  and  $b$  given, there exists a maximal length of a possible chain between  $a$  and  $b$  since the segment  $[a, b]_P$  has finite cardinality.
- ii) If we consider the Möbius function of the poset obtained from the positive integers partially ordered by the divisibility, we retrieve the classical Möbius arithmetic function on the integers.

The Möbius function has been extremely useful to investigate many different problems. For instance, the *inclusion-exclusion principle* can be retrieved by considering the set of all subsets of a finite set partially ordered by inclusion.

The concept of Möbius function for a locally finite poset  $(P, \leq_P)$  was introduced by Rota in [98] as the inverse of the zeta function in the incidence algebra of a locally finite poset. Let us see this with more detail. Consider the set  $\mathcal{I}(P)$  of all real-valued functions  $f : P \times P \rightarrow \mathbb{R}$  for which  $f(a, b) = 0$  if  $a \not\leq_P b$ . The sum  $+$  and the multiplication by scalars are defined as usual in  $\mathcal{I}(P)$ . The product of two functions  $f$  and  $g$  in  $\mathcal{I}(P)$  is defined by

$$(f \times g)(a, b) = \sum_{c \in [a, b]_P} f(a, c)g(c, b),$$

for all  $(a, b) \in P \times P$ . Then  $(\mathcal{I}(P), +, \cdot, \times)$  appears as an associative algebra over  $\mathbb{R}$ . This is the *incidence algebra* of  $P$ . The *Kronecker delta function*  $\delta \in \mathcal{I}(P)$ , defined by

$$\delta(a, b) = \begin{cases} 1 & \text{if } a = b, \\ 0 & \text{otherwise,} \end{cases}$$

for all  $(a, b) \in P \times P$ , is the identity element of  $\mathcal{I}(P)$ . The *zeta function*  $\zeta_P \in \mathcal{I}(P)$  is defined by

$$\zeta_P(a, b) = \begin{cases} 1 & \text{if } a \leq_P b, \\ 0 & \text{otherwise,} \end{cases}$$

for all  $(a, b) \in P \times P$ .

Rota [98] proved that the zeta function  $\zeta_P$  (called the *inverse function*) is invertible in  $\mathcal{I}(P)$  and showed that  $\mu_P$  is recursively defined as follows: for all  $(a, b) \in P \times P$ , by

$$\mu_P(a, a) = 1 \quad \text{and} \quad \mu_P(a, b) = - \sum_{c \in [a, b]_P} \mu_P(a, c) \quad \text{if } a <_P b. \quad (6.2)$$

Let us see that both definitions of  $\mu_S$  given by (6.1) and by (6.2) are equivalent. For, let  $a$  and  $b$  be two elements of the locally finite poset  $P$  such that  $a <_P b$ . Then,

$$c_l(a, b) = \sum_{c \in [a, b]_P} c_{l-1}(a, c) = \sum_{c \in ]a, b]_P} c_{l-1}(c, b), \quad (6.3)$$

for all positive integers  $l$ . Indeed, every chain  $\{a_0, a_1, \dots, a_l\} \in C_l(a, b)$  can be seen as an extension of a chain of  $C_{l-1}(a, a_{l-1})$  or of  $C_{l-1}(a_1, b)$ .

Obviously, the identity  $\mu_S(a, a) = 1$  directly comes from (6.1) since  $c_0(a, a) = 1$  and  $c_l(a, a) = 0$  for all  $l \geq 1$ . By combining (6.3) and (6.1), for all  $a <_P b$ , we obtain that

$$\mu_P(a, b) = \sum_{l \geq 0} (-1)^l c_l(a, b) = c_0(a, b) + \sum_{l \geq 1} (-1)^l \sum_{c \in [a, b]_P} c_{l-1}(a, c).$$

Finally, since  $a \neq b$ , it follows that  $c_0(a, b) = 0$  and thus

$$\mu_P(a, b) = \sum_{c \in [a, b]_P} \sum_{l \geq 0} (-1)^{l+1} c_l(a, c) = - \sum_{c \in [a, b]_P} \mu_P(a, c).$$

Similarly, using the second identity of (6.3), we can also prove that, whenever  $a <_P b$ , we have

$$\mu_P(a, b) = - \sum_{c \in [a, b]_P} \mu_P(c, b).$$

Therefore the two definitions of the Möbius function (for a locally finite poset) are the same. All the results presented in next section are derived from the recursive formula presented in (6.2).

### 6.1.3 Numerical semigroup posets

For any numerical semigroup  $S$ , we consider the binary relation  $\leq_S$  on the set of integers  $\mathbb{Z}$  defined by

$$x \leq_S y \iff y - x \in S,$$

for all integers  $x$  and  $y$ .

**Proposition 6.1.12.** *For any numerical semigroup  $S$ ,  $(\mathbb{Z}, \leq_S)$  is a locally finite poset.*

*Proof.* The binary relation  $\leq_S$  is reflexive since  $0 \in S$ , antisymmetric since  $S \cap (-S) = \{0\}$  and reflexive by associativity of the addition in  $S$ . Therefore, the relation  $\leq_S$  is a partial order over  $\mathbb{Z}$ . Moreover,  $(\mathbb{Z}, \leq_S)$  is a locally finite poset since

$$\left| [x, y]_{(\mathbb{Z}, \leq_S)} \right| \leq y - x + 1,$$

for all  $x, y \in \mathbb{Z}$ . □

**Notation 6.1.13.** For any numerical semigroup  $S$ , the Möbius function of the locally finite poset  $(\mathbb{Z}, \leq_S)$  is denoted by  $\mu_S$ .

It is easy to see that  $\mu_S$  can be considered as a univariate function on  $\mathbb{Z}$ .

**Proposition 6.1.14.** *Let  $S$  be a numerical semigroup. Then, in the locally finite poset  $(\mathbb{Z}, \leq_S)$ , we have*

$$c_l(x, y) = c_l(0, y - x), \tag{6.4}$$

for all  $x, y \in \mathbb{Z}$  and for all  $l \geq 0$ .

**Corollary 6.1.15.** *Let  $S$  be a numerical semigroup. Then, in the locally finite poset  $(\mathbb{Z}, \leq_S)$ , we have*

$$\mu_S(x, y) = \mu_S(0, y - x),$$

for all  $x, y \in \mathbb{Z}$ .

In the sequel of this section, we shall only consider the reduced Möbius function  $\mu_S : \mathbb{Z} \rightarrow \mathbb{Z}$  defined by

$$\mu_S(x) = \mu_S(0, x),$$

for all  $x \in \mathbb{Z}$ . The recursive formula given by (6.2) can be more easily presented when the locally finite poset is  $(\mathbb{Z}, \leq_S)$ .

**Proposition 6.1.16** (Chappelton-Ramírez [C16]). *Let  $S$  be a numerical semigroup and let  $x \in \mathbb{Z} \setminus \{0\}$ . Then,*

$$\mu_S(x) = - \sum_{y \in S \setminus \{0\}} \mu_S(x - y) \iff \sum_{y \in S} \mu_S(x - y) = 0.$$

### 6.1.4 The Möbius function of arithmetic semigroup posets

Using the recursive formula of Proposition 6.1.16, the Möbius function of numerical semigroup posets have been explicitly determined, or semi-explicitly determined, for some families of numerical semigroups in [C16].

For  $S = \langle a, b \rangle$

For a numerical semigroup  $S$  generated from two coprime positive integers, the following theorem, due to Deddens in 1979, gives an explicit formula for  $\mu_S$ .

**Theorem 6.1.17** (Deddens [47]). *Let  $a$  and  $b$  be two relatively positive integers and let  $S = \langle a, b \rangle$ . Then, we have*

$$\mu_S(x) = \begin{cases} 1 & \text{if } x \geq 0 \text{ and } x \equiv 0 \text{ or } a + b \pmod{ab}, \\ -1 & \text{if } x \geq 0 \text{ and } x \equiv a \text{ or } b \pmod{ab}, \\ 0 & \text{otherwise,} \end{cases}$$

for all  $x \in \mathbb{Z}$ .

A proof based on the recursive formula of Proposition 6.1.16 can be found in [C16].

#### For arithmetic semigroups

**Definition 6.1.18** (Arithmetic semigroup). Let  $a$  and  $d$  be two coprime positive integers and let  $k$  be a positive integer such that  $k \leq a - 1$ . The numerical semigroup

$$S = \langle a, a + d, \dots, a + kd \rangle$$

generated by the arithmetic progression  $\text{AP}(a, d, k + 1)$  is called an *arithmetic semigroup*.

A recursive formula for  $\mu_S$  is obtained in [C16] when  $S = \langle a, a + d, \dots, a + kd \rangle$  is an arithmetic semigroup. The following key remark led us to guess such recursion. If  $x = m_a a + m_d d$  such that  $m_a \geq 0$  and  $0 \leq m_d \leq a - 1$  then,

$$x \in S \iff m_a \geq \left\lceil \frac{m_d}{k} \right\rceil.$$

**Theorem 6.1.19** (Chappelon-Ramirez [C16]). *Let  $S = \langle a, a + d, \dots, a + kd \rangle$  such that  $\gcd(a, d) = 1$  and let  $a = qk + r$  with  $0 \leq r < k$ . Let  $x \in \mathbb{Z} \setminus \{0, a, a + kd, a + (a + kd)\}$ , then*

$$\mu_S(x) = \begin{cases} \mu_S(x - q(a + kd)) + \sum_{i=1}^{k-1} \mu_S(x - (a + id) - q(a + kd)) \\ - \mu_S(x - (a + id)) & \text{if } r = 0, \\ \mu_S(x - (q + 1)(a + kd)) + \sum_{i=r}^{k-1} \mu_S(x - (a + id) - q(a + kd)) \\ - \sum_{i=1}^{k-1} \mu_S(x - (a + id)) & \text{if } r = 1, \\ \mu_S(x - (q + 1)(a + kd)) + \sum_{i=1}^{r-1} \mu_S(x - (a + id) - (q + 1)(a + kd)) \\ + \sum_{i=r}^{k-1} \mu_S(x - (a + id) - q(a + kd)) - \sum_{i=1}^{k-1} \mu_S(x - (a + id)) & \text{if } r \geq 2. \end{cases}$$

The proof is based on Proposition 6.1.16 and on the following lemma where it is proved that in arithmetic semigroups there exists a unique representation of elements.

**Lemma 6.1.20.** *Let  $x \in S = \langle a, a + d, \dots, a + kd \rangle$  with  $2 \leq k \leq a - 1$ . Then, there exists a unique triplet  $(x_0, x_i, x_k) \in \mathbb{N} \times \{0, 1\} \times \{0, \dots, \lfloor \frac{a}{k} \rfloor\}$  such that*

$$x = x_0a + x_i(a + id) + x_k(a + kd)$$

for some  $1 \leq i \leq k - 1$  with  $ix_i + kx_k < a$ .

**Notation 6.1.21.** Let  $(x_0, x_i, x_k) \in \mathbb{N} \times \{0, 1\} \times \{0, \dots, \lfloor a/k \rfloor\}$  with  $1 \leq i \leq k - 1$  and  $ix_i + kx_k < a$ . We denote by  $[x_0, x_i, x_k]$  the element in  $S = \langle a, a + d, \dots, a + kd \rangle$  given by the representation of Lemma 6.1.20.

**For semigroups**  $\langle 2a, 2a + d, 2a + 2d \rangle$

Recall that the *multiplicity function* of a multiset  $A$  of  $\mathbb{N}$  is the function

$$\mathbf{m}_A : \mathbb{N} \longrightarrow \mathbb{N}$$

which assigns to each element  $x \in \mathbb{N}$  its multiplicity, that is, the number of times that  $x$  appears in the multiset  $A$ .

Let  $a = 2q$  and  $d \in \mathbb{N}^*$  such that  $\gcd(a, d) = \gcd(q, d) = 1$ . For each  $i \in \{-1, 0, 1\}$ , we consider the following multisets.

$$A_i = \{m(q + d) + i \mid m \in \mathbb{N}\},$$

$$B_i = \{m(q + d) - nd + i \mid m \in \mathbb{N} \setminus \{0, 1\}, n \in \{1, 2, \dots, \lfloor m/2 \rfloor\}\},$$

$$C_i = A_i \cup B_i.$$

As we mentioned above, given a triple  $(x_0, x_1, x_2) \in \mathbb{N} \times \{0, 1\} \times \{0, \dots, q - 1\}$ , we denote by  $[x_0, x_1, x_2]$  the element in  $S$  given by the representation in Lemma 6.1.20. We shall consider this representation for all  $x_0 \in \mathbb{Z}$ , i.e.,

$$[x_0, x_1, x_2] = x_0a + x_1(a + d) + x_2(a + 2d)$$

for all  $(x_0, x_1, x_2) \in \mathbb{Z} \times \{0, 1\} \times \{0, \dots, q - 1\}$ . In this case, it is clear that

$$[x_0, x_1, x_2] \in S \iff x_0 \in \mathbb{N},$$

for all  $(x_0, x_1, x_2) \in \mathbb{Z} \times \{0, 1\} \times \{0, \dots, q - 1\}$ .

**Theorem 6.1.22** (Chappelon-Ramirez [C16]). *Let  $q$  and  $d$  be two coprime positive integers and let*

$$S = \langle 2q, 2q + d, 2q + 2d \rangle.$$

Let  $(x_0, x_1, x_2) \in \mathbb{Z} \times \{0, 1\} \times \{0, \dots, q - 1\}$ . Then,

$$\mu_S([x_0, x_1, x_2]) = \begin{cases} (-1)^{x_1} (\mathbf{m}_{A_0} - \mathbf{m}_{A_1} + 2\mathbf{m}_{B_0} - \mathbf{m}_{B_{-1}} - \mathbf{m}_{B_1}) (x_0) & \text{if } x_2 = 0, \\ (-1)^{x_1} (2\mathbf{m}_{C_0} - \mathbf{m}_{C_{-1}} - \mathbf{m}_{C_1}) (x_0 - x_2) & \text{if } x_2 \geq 1. \end{cases}$$

We notice that if  $x_0 - x_2$  is a constant then we should have the same value for  $\mu_S([x_0, 0, x_2])$ . The latter is illustrated by the first values of  $\mu_S([x_0, 0, x_2])$ , listed in Table 6.1, for the case when  $a = 22$  and  $d = 5$ . Indeed, we can see appearing diagonals (corresponding to  $x_0 - x_2$  constant) with the same value.

Table 6.1: First values of  $\mu_S([x_0, 0, x_2])$  for  $q = 11$  and  $d = 5$ .

$x_0 \backslash x_2$	0	1	2	3	4	5	6	7	8	9	10
0	<b>1</b>	<b>-1</b>	0	0	0	0	0	0	0	0	0
1	<b>-1</b>	<b>2</b>	<b>-1</b>	0	0	0	0	0	0	0	0
2	0	<b>-1</b>	<b>2</b>	<b>-1</b>	0	0	0	0	0	0	0
	0	0	<b>-1</b>	<b>2</b>	<b>-1</b>	0	0	0	0	0	0
	0	0	0	<b>-1</b>	<b>2</b>	<b>-1</b>	0	0	0	0	0
	0	0	0	0	<b>-1</b>	<b>2</b>	<b>-1</b>	0	0	0	0
	0	0	0	0	0	<b>-1</b>	<b>2</b>	<b>-1</b>	0	0	0
	0	0	0	0	0	0	<b>-1</b>	<b>2</b>	<b>-1</b>	0	0
	0	0	0	0	0	0	0	<b>-1</b>	<b>2</b>	<b>-1</b>	0
	0	0	0	0	0	0	0	0	<b>-1</b>	<b>2</b>	<b>-1</b>
$q - 1$	0	0	0	0	0	0	0	0	0	<b>-1</b>	<b>2</b>
$q$	0	0	0	0	0	0	0	0	0	0	<b>-1</b>
$q + 1$	0	0	0	0	0	0	0	0	0	0	0
	0	0	0	0	0	0	0	0	0	0	0
	0	0	0	0	0	0	0	0	0	0	0
$q + d - 1$	<b>0</b>	0	0	0	0	0	0	0	0	0	0
$q + d$	<b>1</b>	<b>-1</b>	0	0	0	0	0	0	0	0	0
$q + d + 1$	<b>-1</b>	<b>2</b>	<b>-1</b>	0	0	0	0	0	0	0	0
$q + d + 2$	0	<b>-1</b>	<b>2</b>	<b>-1</b>	0	0	0	0	0	0	0
	0	0	<b>-1</b>	<b>2</b>	<b>-1</b>	0	0	0	0	0	0
	0	0	0	<b>-1</b>	<b>2</b>	<b>-1</b>	0	0	0	0	0
	0	0	0	0	<b>-1</b>	<b>2</b>	<b>-1</b>	0	0	0	0
	0	0	0	0	0	<b>-1</b>	<b>2</b>	<b>-1</b>	0	0	0
	0	0	0	0	0	0	<b>-1</b>	<b>2</b>	<b>-1</b>	0	0
	0	0	0	0	0	0	0	<b>-1</b>	<b>2</b>	<b>-1</b>	0
$2q + d - 1$	<b>-1</b>	0	0	0	0	0	0	0	0	<b>-1</b>	<b>2</b>
$2q + d$	<b>2</b>	<b>-1</b>	0	0	0	0	0	0	0	0	<b>-1</b>
$2q + d + 1$	<b>-1</b>	<b>2</b>	<b>-1</b>	0	0	0	0	0	0	0	0
	0	<b>-1</b>	<b>2</b>	<b>-1</b>	0	0	0	0	0	0	0
	0	0	<b>-1</b>	<b>2</b>	<b>-1</b>	0	0	0	0	0	0
$2q + 2d - 1$	<b>0</b>	0	0	<b>-1</b>	<b>2</b>	<b>-1</b>	0	0	0	0	0
$2q + 2d$	<b>1</b>	<b>-1</b>	0	0	<b>-1</b>	<b>2</b>	<b>-1</b>	0	0	0	0
$2q + 2d + 1$	<b>-1</b>	<b>2</b>	<b>-1</b>	0	0	<b>-1</b>	<b>2</b>	<b>-1</b>	0	0	0
$2q + 2d + 2$	0	<b>-1</b>	<b>2</b>	<b>-1</b>	0	0	<b>-1</b>	<b>2</b>	<b>-1</b>	0	0
	0	0	<b>-1</b>	<b>2</b>	<b>-1</b>	0	0	<b>-1</b>	<b>2</b>	<b>-1</b>	0
	0	0	0	<b>-1</b>	<b>2</b>	<b>-1</b>	0	0	<b>-1</b>	<b>2</b>	<b>-1</b>
	0	0	0	0	<b>-1</b>	<b>2</b>	<b>-1</b>	0	0	<b>-1</b>	<b>2</b>
	0	0	0	0	0	<b>-1</b>	<b>2</b>	<b>-1</b>	0	0	<b>-1</b>
	0	0	0	0	0	0	<b>-1</b>	<b>2</b>	<b>-1</b>	0	0
	0	0	0	0	0	0	0	<b>-1</b>	<b>2</b>	<b>-1</b>	0
$3q + 2d - 1$	<b>-1</b>	0	0	0	0	0	0	0	0	<b>-1</b>	<b>2</b>
$3q + 2d$	<b>2</b>	<b>-1</b>	0	0	0	0	0	0	0	0	<b>-1</b>
$3q + 2d + 1$	<b>-1</b>	<b>2</b>	<b>-1</b>	0	0	0	0	0	0	0	0
	0	<b>-1</b>	<b>2</b>	<b>-1</b>	0	0	0	0	0	0	0

Continued on next page

Table 6.1 – continued from previous page

$x_0 \backslash x_2$	0	1	2	3	4	5	6	7	8	9	10
	0	0	-1	2	-1	0	0	0	0	0	0
$3q + 3d - 1$	0	0	0	-1	2	-1	0	0	0	0	0
$3q + 3d$	1	-1	0	0	-1	2	-1	0	0	0	0
$3q + 3d + 1$	-1	2	-1	0	0	-1	2	-1	0	0	0
$3q + 3d + 2$	0	-1	2	-1	0	0	-1	2	-1	0	0
	0	0	-1	2	-1	0	0	-1	2	-1	0
	0	0	0	-1	2	-1	0	0	-1	2	-1

## 6.2 The Möbius function of semigroup posets through Hilbert series

In this section, posets associated with subsemigroups of  $\mathbb{Z}^m$  are considered.

### 6.2.1 Semigroup posets

**Notation 6.2.1.** Let  $A = \{a_1, \dots, a_n\}$  be a subset of  $\mathbb{Z}^m$ . The subsemigroup of  $\mathbb{Z}^m$  generated by  $A$  is denoted by  $\langle A \rangle$ , i.e.,

$$\langle A \rangle = \langle a_1, \dots, a_n \rangle = \{x_1 a_1 + \dots + x_n a_n \mid x_1, \dots, x_n \in \mathbb{N}\}.$$

**Definition 6.2.2.** For any subsemigroup  $S$  of  $\mathbb{Z}^m$ , we consider the binary relation  $\leq_S$  on  $\mathbb{Z}^m$  defined by

$$x \leq_S y \iff y - x \in S,$$

for all  $x, y \in \mathbb{Z}^m$ .

**Proposition 6.2.3.** Let  $S$  be a subsemigroup of  $\mathbb{Z}^m$ . Then,  $(\mathbb{Z}^m, \leq_S)$  is a locally finite poset if and only if  $S \cap (-S) = \{0\}$ .

*Proof.* The reflexivity and the transitivity of  $\leq_S$  come from  $0 \in S$  and the associativity of the addition in  $S$ , respectively. Moreover, the relation  $\leq_S$  is antisymmetric if and only if  $S \cap (-S) = \{0\}$ . Finally, if  $S \cap (-S) = \{0\}$ , then  $(\mathbb{Z}^m, \leq_S)$  is locally finite.  $\square$

**Definition 6.2.4** (Pointed semigroup). A subsemigroup  $S$  of  $\mathbb{Z}^m$  such that  $S \cap (-S) = \{0\}$  is said to be *pointed*.

Let  $\mu_S$  denote the Möbius function associated with  $(\mathbb{Z}^m, \leq_S)$ . It is easy to see that  $\mu_S$  can be considered as a univariate function on  $\mathbb{Z}^m$ . Indeed, for all  $x, y \in \mathbb{Z}^m$  and for all  $l \geq 0$ , one can observe that  $c_l(x, y) = c_l(0, y - x)$ . Thus, we obtain

$$\mu_S(x, y) = \mu_S(0, y - x)$$

for all  $x, y \in \mathbb{Z}^m$ . Therefore, in the sequel of this section, we shall only consider the reduced Möbius function  $\mu_S : \mathbb{Z}^m \rightarrow \mathbb{Z}$  defined by

$$\mu_S(x) := \mu_S(0, x),$$

for all  $x \in \mathbb{Z}^m$ . Thus, as for the numerical semigroup posets, the formula given by (6.2) may now be simplified when the locally finite poset is  $(\mathbb{Z}^m, \leq_S)$ .

**Proposition 6.2.5** (CGMR [C10]). *Let  $S$  be a pointed semigroup and let  $x \in \mathbb{Z}^m$ . Then,*

$$\sum_{b \in S} \mu_S(x - b) = \begin{cases} 1 & \text{if } x = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 6.2.5 will be very useful to obtain most of the following results.

### 6.2.2 The Hilbert and Möbius series

Here, results relating the Hilbert series of the semigroup  $S$  with the Möbius function of the poset  $(\mathbb{Z}^m, \leq_S)$  are given. First, some basic notions on multivariate Hilbert series are quickly recalled. A complete study of multivariate Hilbert series can be found in [81].

Let  $k$  be any field and let  $S = \langle a_1, \dots, a_n \rangle$  be a subsemigroup of  $\mathbb{Z}^m$ . The semigroup  $S$  induces a grading on the ring of polynomials  $R := k[x_1, \dots, x_n]$  by assigning  $\deg_S(x_i) := a_i \in \mathbb{Z}^m$ , for all  $i \in \{1, \dots, n\}$ . Then, the  $S$ -degree of the monomial  $m := x_1^{\alpha_1} \cdots x_n^{\alpha_n}$  is  $\deg_S(m) := \sum_{i=1}^n \alpha_i a_i \in \mathbb{Z}^m$ . A polynomial is said to be  $S$ -homogeneous if all of its monomials have the same  $S$ -degree and an ideal is  $S$ -homogeneous if it is generated by  $S$ -homogeneous polynomials. For all  $b \in \mathbb{Z}^m$ , we denote by  $R_b$  the  $k$ -vector space generated by all  $S$ -homogeneous polynomials of  $S$ -degree  $b$ .

Whenever  $S$  is pointed, the  $k$ -vector space  $R_b$  has finite dimension, for all  $b \in \mathbb{Z}^m$  (see [81, Proposition 4.1.19]). Let  $I \subset R$  be an  $S$ -homogeneous ideal. The *multigraded Hilbert function* of  $M := R/I$  is

$$HF_M : \mathbb{Z}^m \longrightarrow \mathbb{N},$$

defined by  $HF_M(b) := \dim_k(R_b) - \dim_k(R_b \cap I)$ , for all  $b \in \mathbb{Z}^m$ .

For every  $b = (b_1, \dots, b_m) \in \mathbb{Z}^m$ , we denote by  $\mathbf{t}^b$  the monomial  $t_1^{b_1} \cdots t_m^{b_m}$  in the Laurent polynomial ring  $\mathbb{Z}[t_1, \dots, t_m, t_1^{-1}, \dots, t_m^{-1}]$ . The *multivariate Hilbert series of  $M$*  is the following formal power series in  $\mathbb{Z}[[t_1, \dots, t_m, t_1^{-1}, \dots, t_m^{-1}]]$ :

$$\mathcal{H}_M(\mathbf{t}) := \sum_{b \in \mathbb{Z}^m} HF_M(b) \mathbf{t}^b.$$

We denote by  $I_S$  the *toric ideal of  $S$* , i.e., the kernel of the homomorphism of  $k$ -algebras

$$\varphi : R \longrightarrow k[t_1, \dots, t_m, t_1^{-1}, \dots, t_m^{-1}]$$

induced by  $\varphi(x_i) = \mathbf{t}^{a_i}$ , for all  $i \in \{1, \dots, n\}$ . It is well known that  $I_S$  is  $S$ -homogeneous (see [111, Corollary 4.3]). Moreover, the multivariate Hilbert series of  $M = R/I_S$  with respect to the grading induced by  $S$  is

$$\mathcal{H}_M(\mathbf{t}) = \sum_{b \in S} \mathbf{t}^b. \tag{6.5}$$

Indeed,  $R_b = \{0\}$  and  $HF_M(b) = 0$  whenever  $b \notin S$ . In addition, if  $b \in S$ ,  $\varphi$  induces an isomorphism of  $k$ -vector spaces between  $R_b/(R_b \cap I)$  and  $\{\alpha \mathbf{t}^b \mid \alpha \in k\}$ , for all  $b \in S$ . Hence,  $HF_M(b) = 1$  in this case.

**Definition 6.2.6** (Hilbert series of a semigroup). The multivariate Hilbert series of  $R/I_S$  is called the *Hilbert series of  $S$*  and is denoted by  $\mathcal{H}_S(\mathbf{t})$ .

**Theorem 6.2.7** (CGMR [C10]). *Let  $S$  be a pointed semigroup and let  $c_1, \dots, c_k$  be nonzero vectors in  $\mathbb{Z}^m$ . If we set*

$$(1 - \mathbf{t}^{c_1}) \cdots (1 - \mathbf{t}^{c_k}) \mathcal{H}_S(\mathbf{t}) = \sum_{b \in \mathbb{Z}^m} f_b \mathbf{t}^b \in \mathbb{Z}[[t_1, \dots, t_m, t_1^{-1}, \dots, t_m^{-1}]],$$

then,

$$\sum_{b \in \mathbb{Z}^m} f_b \mu_S(x - b) = 0$$

for all  $x \notin \{\sum_{i \in A} c_i \mid A \subset \{1, \dots, k\}\}$ .

*Remark 6.2.8.* Notice that the formula  $(1 - \mathbf{t}^{c_1}) \cdots (1 - \mathbf{t}^{c_k}) \mathcal{H}_S(\mathbf{t}) = \sum_{b \in \mathbb{Z}^m} f_b \mathbf{t}^b$  might have an infinite number of terms. Nevertheless, for every  $x \in \mathbb{Z}^m$ , the formula given here  $\sum_{b \in \mathbb{Z}^m} f_b \mu_S(x - b) = 0$  only involves a finite number of nonzero summands, since  $S$  is pointed.

The following example illustrates how to apply Theorem 6.2.7 to compute  $\mu_S$ .

**Example 6.2.9.** Consider the semigroup  $S = \langle 2, 3 \rangle \subset \mathbb{N}$ . We observe that  $S = \mathbb{N} \setminus \{1\}$ . Hence,  $\mathcal{H}_S(t) = 1 + \sum_{b \geq 2} t^b \in \mathbb{Z}[[t]]$  and  $t^2 \mathcal{H}_S(t) = t^2 + \sum_{b \geq 4} t^b$ . It follows that

$$(1 - t^2) \mathcal{H}_S(t) = 1 + t^3.$$

Applying Theorem 6.2.7, we get that

$$\mu_S(x) + \mu_S(x - 3) = 0,$$

for all  $x \in \mathbb{Z} \setminus \{0, 2\}$ . Furthermore, by direct computation, we have  $\mu_S(0) = 1$ ,  $\mu_S(2) = -1$  and  $\mu_S(x) = 0$  for all  $x < 0$ . This leads to the formula

$$\mu_S(x) = \begin{cases} 1 & \text{if } x \geq 0 \text{ and } x \equiv 0 \text{ or } 5 \pmod{6}, \\ -1 & \text{if } x \geq 0 \text{ and } x \equiv 2 \text{ or } 3 \pmod{6}, \\ 0 & \text{otherwise.} \end{cases}$$

**Definition 6.2.10** (Möbius series of a semigroup). The *Möbius series* of  $S$  is the generating function  $\mathcal{G}_S$  of the Möbius function

$$\mathcal{G}_S(\mathbf{t}) := \sum_{b \in \mathbb{Z}^m} \mu_S(b) \mathbf{t}^b \in \mathbb{Z}[[t_1, \dots, t_m, t_1^{-1}, \dots, t_m^{-1}]].$$

**Theorem 6.2.11** (CGMR [C10]). *Let  $S$  be a pointed semigroup. Then,*

$$\mathcal{H}_S(\mathbf{t}) \cdot \mathcal{G}_S(\mathbf{t}) = 1.$$

Theorem 6.2.11 states that, whenever we can explicitly compute the inverse of  $\mathcal{H}_S(\mathbf{t})$ , we will be able to obtain  $\mu_S$ . This idea is illustrated in next example.

**Example 6.2.12.** Let  $\{e_1, \dots, e_m\}$  denote the canonical basis of  $\mathbb{N}^m$  and let  $S$  be the semigroup defined by  $S = \langle e_1, \dots, e_m \rangle = \mathbb{N}^m$ . Clearly, we have that

$$\mathcal{H}_{\mathbb{N}^m}(\mathbf{t}) = \sum_{b \in \mathbb{N}^m} \mathbf{t}^b = \frac{1}{(1 - t_1) \cdots (1 - t_m)}.$$

Therefore, by Theorem 6.2.11, we obtain

$$\mathcal{G}_{\mathbb{N}^m}(\mathbf{t}) = (1 - t_1) \cdots (1 - t_m) = \sum_{A \subset \{1, \dots, m\}} (-1)^{|A|} \prod_{i \in A} t_i = \sum_{A \subset \{1, \dots, m\}} (-1)^{|A|} \mathbf{t}^{\sum_{i \in A} e_i}.$$

So we derive the following formula for  $\mu_{\mathbb{N}^m}$ :

$$\mu_{\mathbb{N}^m}(x) = \begin{cases} (-1)^{|A|} & \text{if } x = \sum_{i \in A} e_i \text{ for some } A \subset \{1, \dots, m\}, \\ 0 & \text{otherwise.} \end{cases}$$

**Definition 6.2.13** (Complete intersection pointed semigroup). A pointed semigroup  $S = \langle a_1, \dots, a_n \rangle$  is called a *complete intersection semigroup* if its corresponding toric ideal  $I_S$  is a *complete intersection ideal*, i.e., if  $I_S$  is generated by  $n - d$   $S$ -homogeneous polynomials, where  $d$  is the dimension of the  $\mathbb{Q}$ -vector space spanned by  $a_1, \dots, a_n$ .

Characterizations of complete intersection toric ideals can be found in [65].

**Notation 6.2.14.** Let  $B = (b_1, b_2, \dots, b_k)$  be a  $k$ -tuple of nonzero vectors in  $\mathbb{Z}^m$  such that the semigroup  $\mathcal{T} := \langle b_1, \dots, b_k \rangle$  is pointed and let  $b \in \mathbb{Z}^m$ . We denote by  $d_B(b)$  the number of non-negative integer representations of  $b$  by  $b_1, \dots, b_k$ , that is, the number of solutions of  $b = \sum_{i=1}^k x_i b_i$ , where  $x_i$  is a nonnegative integer for all  $i$ .

*Remark 6.2.15.* Since  $\mathcal{T}$  is pointed, we know that  $d_B(b)$  is finite, for all  $b \in \mathbb{Z}^m$ . Moreover,  $d_B(0) = 1$ .

**Proposition 6.2.16** (Theorem 5.8.15 [81]). *Let  $B = (b_1, b_2, \dots, b_k)$  be a  $k$ -tuple of nonzero vectors in  $\mathbb{Z}^m$  such that the semigroup  $\mathcal{T} := \langle b_1, \dots, b_k \rangle$  is pointed. The generating function of  $d_B(b)$  is given by*

$$\sum_{b \in \mathbb{Z}^m} d_B(b) \mathbf{t}^b = \frac{1}{(1 - \mathbf{t}^{b_1})(1 - \mathbf{t}^{b_2}) \cdots (1 - \mathbf{t}^{b_k})}.$$

**Corollary 6.2.17** (CGMR [C10]). *Let  $S$  be a complete intersection pointed semigroup and assume that  $I_S$  is generated by  $n - d$   $S$ -homogeneous polynomials of  $S$ -degrees  $b_1, \dots, b_{n-d} \in \mathbb{Z}^m$ . Then,*

$$\mu_S(x) = \sum_{A \subset \{1, \dots, n\}} (-1)^{|A|} d_B \left( x - \sum_{i \in A} b_i \right),$$

for all  $x \in \mathbb{Z}^m$ , where  $B = \{b_1, \dots, b_{n-d}\}$ .

### 6.2.3 Explicit formulas for the Möbius function

In this subsection, we exploit the results of the previous subsection to obtain explicit formulas for  $\mu_S$  when  $S$  is a semigroup with a unique Betti element and when  $S$  is a complete intersection numerical semigroup generated by three elements. These results are consequences of Corollary 6.2.17. However, they can also be obtained with a different proof by using Theorem 6.2.7.

### Semigroups with a unique Betti element

**Definition 6.2.18** (Semigroup with a unique Betti element). A semigroup  $S \subset \mathbb{N}^m$  is said to have a *unique Betti element*  $b \in \mathbb{N}^m$  if its corresponding toric ideal is generated by a set of  $S$ -homogeneous polynomials of common  $S$ -degree  $b$ .

**Proposition 6.2.19** (Corollary 10 [68]). *Subsemigroups of  $\mathbb{N}^m$  with a unique Betti element are always complete intersection.*

**Theorem 6.2.20** (CGMR [C10]). *Let  $S = \langle a_1, \dots, a_n \rangle \subset \mathbb{N}^m$  be a semigroup with a unique Betti element  $b \in \mathbb{N}^m$ . If we denote by  $d$  the dimension of the  $\mathbb{Q}$ -vector space generated by  $a_1, \dots, a_n$ , then we have*

$$\mu_S(x) = \sum_{j=1}^t (-1)^{|A_j|} \binom{k_{A_j} + n - d - 1}{k_{A_j}},$$

where

$$\{A_1, \dots, A_t\} = \left\{ A \subset \{1, \dots, n\} \mid \text{there exists } k_A \in \mathbb{N} \text{ such that } x - \sum_{i \in A} a_i = k_A b \right\}.$$

When  $m = 1$ , i.e., when  $S = \langle a_1, \dots, a_n \rangle \subset \mathbb{N}$ ,  $S$  is a numerical semigroup with a unique Betti element  $b \in \mathbb{N}$  if and only if there exist pairwise relatively prime integers  $b_1, \dots, b_n \geq 2$  such that  $a_i := \prod_{j \neq i} b_j$ , for all  $i \in \{1, \dots, n\}$ , and  $b = \prod_{i=1}^n b_i$  (see [68]). In this setting, Theorem 6.2.20 can be refined as follows.

**Corollary 6.2.21** (CGMR [C10]). *Let  $S = \langle a_1, \dots, a_n \rangle \subset \mathbb{N}$  be a numerical semigroup with a unique Betti element  $b \in \mathbb{N}$ . Then,*

$$\mu_S(x) = \begin{cases} (-1)^{|A|} \binom{k + n - 2}{k} & \text{if } x = \sum_{i \in A} a_i + kb \text{ for some } A \subset \{1, \dots, n\}, k \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$

As a direct consequence of this result, we recover Theorem 6.1.17.

### Three generated complete intersection numerical semigroups

We provide a semi-explicit formula for  $\mu_S$ , when  $S$  is a complete intersection numerical semigroup minimally generated by the set  $\{a_1, a_2, a_3\}$ . When  $S = \langle a_1, a_2, a_3 \rangle \subset \mathbb{N}$ , Herzog proves in [75] that  $S$  is a complete intersection if and only if  $\gcd(a_i, a_j) a_k \in \langle a_i, a_j \rangle$  with  $\{i, j, k\} = \{1, 2, 3\}$ . Suppose that  $da_1 \in \langle a_2, a_3 \rangle$ , where  $d := \gcd(a_2, a_3)$ .

For every  $x \in \mathbb{Z}$ , there exists a unique  $\alpha(x) \in \{0, \dots, d-1\}$  such that  $\alpha(x)a_1 \equiv x \pmod{d}$ . It is easy to check that, for every  $x, y \in \mathbb{Z}$ ,

$$\alpha(x - y) = \begin{cases} \alpha(x) - \alpha(y) & \text{if } \alpha(x) \geq \alpha(y), \\ d + \alpha(x) - \alpha(y) & \text{otherwise.} \end{cases} \quad (6.6)$$

**Theorem 6.2.22** (CGMR [C10]). *Let  $S = \langle a_1, a_2, a_3 \rangle$  be a numerical semigroup such that  $da_1 \in \langle a_2, a_3 \rangle$ , where  $d := \gcd(a_2, a_3)$ . For all  $x \in \mathbb{Z}$ , we have that  $\mu_S(x) = 0$ , if  $\alpha(x) \geq 2$ , and*

$$\mu_S(x) = (-1)^\alpha (d_B(x') - d_B(x' - a_2) - d_B(x' - a_3) + d_B(x' - a_2 - a_3))$$

otherwise, where  $x' := x - \alpha(x)a_1$  and  $B := (da_1, a_2 a_3/d)$ .

*Remark 6.2.23.*

- i) Theorem 6.2.22 yields an algorithm for computing  $\mu_S(x)$ , for all  $x \in \mathbb{Z}$ , which relies on the computation of four values of  $d_B(y)$ , where  $B = (da_1, a_2a_3/d)$ . It is worth mentioning that in [96, Section 4.4] there are several results and methods to compute these values.
- ii) Theorem 6.2.22 generalizes [C16, Theorem 3], where the authors provide a semi-explicit formula for  $S = \langle 2q, 2q + e, 2q + 2e \rangle$  where  $q, e \in \mathbb{Z}^+$  and  $\gcd(2q, 2q + e, 2q + 2e) = 1$ . Indeed, if  $S = \langle a, a + e, \dots, a + ke \rangle$  with  $\gcd(a, e) = 1$  and  $k \geq 2$ , then  $S$  is a complete intersection if and only if  $k = 2$  and  $a$  is even (see [32]).

### 6.2.4 When is a poset equivalent to a semigroup poset?

A natural question is whether a poset  $\mathcal{P}$  is *isomorphic* to a poset associated with a semigroup  $S$  since, in such a case, one might be able to calculate  $\mu_{\mathcal{P}}$  by computing  $\mu_S$  instead.

Let  $(\mathcal{P}, \leq_{\mathcal{P}})$  be a locally finite poset. For every  $x \in \mathcal{P}$ , we set  $\mathcal{P}_x := \{y \in \mathcal{P} \mid x \leq_{\mathcal{P}} y\}$  and we consider the restricted Möbius function  $\mu_{\mathcal{P}}(-, x) : \mathcal{P}_x \rightarrow \mathbb{Z}$ . It is clear that, if there exists a pointed semigroup  $S$  and an order isomorphism  $\psi : (\mathcal{P}_x, \leq_{\mathcal{P}}) \rightarrow (S, \leq_S)$ , then  $\mu_{\mathcal{P}}(-, x)$  can be computed by means of the Möbius function of  $(S, \leq_S)$ , since  $\mu_{\mathcal{P}}(y, x) = \mu_S(\psi(y))$  for all  $y \in \mathcal{P}_x$ .

The poset  $\mathcal{P}_x$  is said to be *autoequivalent* if and only if, for all  $y \in \mathcal{P}_x$ , there exists an order isomorphism  $g_y : \mathcal{P}_x \rightarrow \mathcal{P}_y$  such that  $g_y \circ g_z = g_z \circ g_y$ , for all  $y, z \in \mathcal{P}_x$ , and  $g_x$  is the identity. For all  $y \in \mathcal{P}_x$ , we set  $l_1(y) := \{z \in \mathcal{P} \mid \text{there is no } u \in \mathcal{P} \text{ such that } y \leq u \leq z\}$ . Whenever  $\mathcal{P}_x$  is autoequivalent with isomorphisms  $\{g_y\}_{x \leq y}$  and  $l_1(x)$  is a finite set of  $n$  elements, we associate to  $\mathcal{P}$  a subgroup  $L_{\mathcal{P}} \subset \mathbb{Z}^n$  in the following way.

Let  $l_1(x) = \{x_1, \dots, x_n\} \subset \mathcal{P}$  and consider the map

$$f : \mathbb{N}^n \longrightarrow \mathcal{P}$$

defined as  $f(0, \dots, 0) = x$ , and for all  $\alpha \in \mathbb{N}^n$  and all  $i \in \{1, \dots, n\}$ ,  $f(\alpha + e_i) = g_{x_i}(f(\alpha))$ , where  $\{e_1, \dots, e_n\}$  is the canonical basis of  $\mathbb{N}^n$ . In particular,  $f(e_i) = g_{x_i}(f(0)) = g_{x_i}(x) = x_i$ , for all  $i \in \{1, \dots, n\}$ .

**Lemma 6.2.24.**  *$f$  is well defined and is surjective.*

Now, we set  $L_{\mathcal{P}} := \{\alpha - \beta \in \mathbb{Z}^n \mid f(\alpha) = f(\beta)\}$ .

**Lemma 6.2.25.**  *$L_{\mathcal{P}}$  is a subgroup of  $\mathbb{Z}^n$ .*

If  $L$  is a subgroup of  $\mathbb{Z}^n$ , then its *saturation* is the group defined by

$$\text{Sat}(L) := \{\gamma \in \mathbb{Z}^n \mid \text{there exists } d \in \mathbb{Z}^+ \text{ such that } d\gamma \in L\}.$$

**Theorem 6.2.26** (CGMR [C10]). *Let  $\mathcal{P}$  be a locally finite poset and let  $x \in \mathcal{P}$ . Then,  $(\mathcal{P}_x, \leq)$  is isomorphic to  $(S, \leq_S)$  for some (pointed) semigroup  $S \subset \mathbb{Z}^m$  if and only if  $\mathcal{P}_x$  is autoequivalent,  $l_1(x)$  is finite and  $L_{\mathcal{P}} = \text{Sat}(L_{\mathcal{P}})$ .*

The necessity direction of Theorem 6.2.26 can be stated in algebraic terms as : whenever  $\mathcal{P}_x$  is autoequivalent and  $l_1(x)$  is finite, the subgroup  $L_{\mathcal{P}}$  defines a lattice ideal  $I := (\{\mathbf{x}^\alpha - \mathbf{x}^\beta \mid \alpha - \beta \in L_{\mathcal{P}}\})$ . Moreover,  $\mathcal{P}_x$  is isomorphic to a semigroup poset  $(S, \leq_S)$  if and only if the ideal  $I$  itself is the toric ideal of a semigroup  $S$ . The latter holds if and only if  $I$  is prime or, equivalently, if  $L_{\mathcal{P}} = \text{Sat}(L_{\mathcal{P}})$  (see [56]).

### 6.3 The square Frobenius number

**Definition 6.3.1** (*P*-type functions). Given a particular (arithmetical, number theoretical, etc.) *Property P*, one might consider the following two *P*-type functions of a numerical semigroup  $S$ :

$${}^P r(S) := \text{the largest integer having property } P \text{ not belonging to } S$$

and

$${}_{P} r(S) := \text{the smallest integer having property } P \text{ belonging to } S.$$

Notice that the multiplicity and the Frobenius number are *P*-type functions where *P* is the property of being a positive integer. *P*-type functions were introduced by J.L. Ramírez Alfonsín (often mentioned during his lectures) with the hope to understand better certain properties *P* in terms of linear forms. In this spirit, we consider the property of being *perfect k*-power integer (that is, integers of the form  $m^k$  for some integers  $m, k > 1$ ).

**Definition 6.3.2** (Power Frobenius number). Let  $k \geq 2$  be an integer, we define

$${}^{k\text{-power}} r(S) := \text{the largest perfect } k\text{-power integer not belonging to } S.$$

This *k*-power variant of  $g(S)$  is called the *k*-power Frobenius number of  $S$ , we may write  ${}^{k} r(S)$  for short.

In this section we investigate the 2-power Frobenius number, we call it the *square Frobenius number*.

#### 6.3.1 Arithmetic semigroups

Let  $a, d$  and  $k$  be positive integers such that  $a$  and  $d$  are relatively prime. Throughout this subsection, we denote by  $S_A$  the semigroup generated by the *arithmetic progression* whose first element is  $a$ , with common difference  $d$  and of length  $k + 1$ , that is,

$$S_A = \langle a, a + d, a + 2d, \dots, a + kd \rangle.$$

Note that the integers  $a, a + d, \dots, a + kd$  are relatively prime if and only if  $\gcd(a, d) = 1$ .

We shall start by giving a necessary and sufficient condition for a square to belong to  $S_A$ .

**Definition 6.3.3** (Modular multiplicative inverse). For any integer  $x$  coprime to  $d$ , a *multiplicative inverse modulo d* of  $x$  is an integer  $y$  such that  $xy \equiv 1 \pmod{d}$ .

**Proposition 6.3.4** (Chappelon-Ramírez [C17]). *Let  $i$  be an integer and let  $\lambda_i$  be the unique integer in  $\{0, 1, \dots, d - 1\}$  such that  $\lambda_i a + i^2 \equiv 0 \pmod{d}$ . In other words, the integer  $\lambda_i$  is the rest in the Euclidean division of  $-a^{-1}i^2$  by  $d$ , where  $a^{-1}$  is a multiplicative inverse of  $a$  modulo  $d$ . Then,*

$$(a - i)^2 \in S_A \text{ if and only if } (i + kd)^2 \leq \left( \left( \left\lfloor \frac{i^2 + \lambda_i a}{ad} \right\rfloor + k \right) d - \lambda_i \right) (a + kd).$$

*Remark 6.3.5.* We have that  $\lambda_0 = 0$  and  $\lambda_i > 0$  for all integers  $i$  such that  $\gcd(i, d) = 1$  with  $d \geq 2$ . Moreover,  $\lambda_i = \lambda_{d-i}$  for all  $i \in \{1, 2, \dots, d - 1\}$ .

The above characterization permits us to obtain an upper-bound of  ${}^2r(S_A)$  when  $a$  is larger enough compared to  $d \geq 3$ .

**Definition 6.3.6.** Let  $\lambda^*$  be the integer defined by

$$\lambda^* = \max_{0 \leq i \leq d-1} \{ \lambda_i \in \{0, 1, \dots, d-1\} \mid \lambda_i a + i^2 \equiv 0 \pmod{d} \}.$$

Let  $\{\alpha_1 < \dots < \alpha_n\} \subseteq \{0, 1, \dots, d-1\}$  such that  $\lambda_{\alpha_j} = \lambda^*$  and take  $\alpha_{n+1} = d + \alpha_1$ . Let  $j \in \{1, \dots, n\}$  be the index such that

$$(\mu d + \alpha_j)^2 \leq (kd - \lambda^*)(a + kd) < (\mu d + \alpha_{j+1})^2, \quad (6.7)$$

for some integer  $\mu \geq 0$ .

*Remark 6.3.7.*

(a) The above index  $j$  exists and it is unique. Indeed, we clearly have that there is an integer  $\mu$  such that

$$\mu d \leq \sqrt{(kd - \lambda^*)(a + kd)} < (\mu + 1)d.$$

Since  $0 \leq \alpha_1 < \dots < \alpha_n \leq d-1$ , then the interval  $[\mu d, (\mu + 1)d]$  can be refined into intervals of the form  $[\mu d + \alpha_i, \mu d + \alpha_{i+1}[$  for each  $i = 1, \dots, n-1$ . Therefore, there is a unique index  $j$  verifying equation (6.7).

(b) We have that  $\mu d + \alpha_{n+1} = (\mu + 1)d + \alpha_1$ .

The following two propositions give us useful information on the sequence of indexes  $\alpha_1, \dots, \alpha_n$ .

**Proposition 6.3.8.** *We have that  $\alpha_i + \alpha_{n+1-i} = d$ , for all  $i \in \{1, \dots, n\}$ .*

**Proposition 6.3.9.** *If  $d \geq 3$  then  $n \geq 2$  and  $1 \leq \alpha_1 < \frac{d}{2} < \alpha_n \leq d-1$ .*

**Definition 6.3.10.** Let us now consider the integer function  $h(a, d, k)$  defined as

$$h(a, d, k) := (a - ((\mu - k)d + \alpha_{j+1}))^2.$$

*Remark 6.3.11.* We notice that the function  $h(a, d, k)$  can always be computed for any relatively prime integers  $a$  and  $d$  and any positive integer  $k$ . It is enough to calculate  $\lambda_i$  for each  $i = 0, \dots, d-1$ , from which  $\lambda^*$  and the set of  $\alpha_i$ 's can be obtained and thus the desired  $\mu$  and  $\alpha_{j+1}$  can be computed.

**Theorem 6.3.12** (Chappelon-Ramírez [C17]). *Let  $d \geq 3$  and  $a + kd \geq 4kd^3$ . Then,*

$${}^2r(S_A) \leq h(a, d, k).$$

*Remark 6.3.13.* The above proof can be adapted if we consider the weaker condition  $a + kd > 4(kd - \lambda^*)d^2 + d^2$  instead of  $a + kd \geq 4kd^3$ .

We believe that the upper bound  $h(a, d, k)$  of  ${}^2r(S_A)$  given in Theorem 6.3.12 is actually an equality. We are able to establish the latter in the case when  $k = 1$  for any  $d \geq 3$ .

**Corollary 6.3.14** (Chappelon-Ramírez [C17]). *Let  $d \geq 3$  and  $a + d \geq 4d^3$ . Then,*

$${}^2r(\langle a, a + d \rangle) = h(a, d, 1).$$

Unfortunately, the value of  ${}^2r(\langle a, a + d \rangle)$  given in the above corollary does not hold in general (if the condition  $a + d \geq 4d^3$  is not satisfied). However, as we will see below, the number of values of  $a$  not satisfying the equality  ${}^2r(\langle a, a + d \rangle) = h(a, d, 1)$  is finite for each fixed  $d$ .

### 6.3.2 Formulas for $\langle a, a + d \rangle$ with small $d \geq 3$

Here, we investigate the value of  ${}^2r(\langle a, a + d \rangle)$  when  $d$  is small.

For any positive integer  $d \geq 3$ , we may define the set  $E(d)$  to be the set of integers  $a$  coprime to  $d$  not satisfying the equality of Corollary 6.3.14, that is,

$$E(d) := \{a \in \mathbb{N} \setminus \{0, 1\} \mid \gcd(a, d) = 1 \text{ and } {}^2r(\langle a, a + d \rangle) \neq h(a, d, 1)\}.$$

Since  $\lambda^* \leq d - 1$  then, from Corollary 6.3.14, we obtain that  $E(d) \subset [2, 4d^3 - 1] \cap \mathbb{N}$ . We completely determine the set  $E(d)$  for a few values of  $d \geq 3$  by computer calculations, see Table 6.2.

$d$	$ E(d) $	$E(d)$
3	0	$\emptyset$
4	0	$\emptyset$
5	5	$\{2, 4, 13, 27, 32\}$
6	0	$\emptyset$
7	10	$\{2, 3, 4, 9, 16, 18, 19, 23, 30, 114\}$
8	5	$\{5, 9, 21, 45, 77\}$
9	5	$\{2, 4, 7, 8, 16\}$
10	14	$\{3, 9, 13, 23, 27, 33, 43, 123, 133, 143, 153, 163, 333, 343\}$
11	14	$\{2, 3, 4, 5, 7, 8, 9, 14, 16, 18, 25, 36, 38, 47\}$
12	9	$\{13, 19, 25, 31, 67, 79, 139, 151, 235\}$

Table 6.2:  $E(d)$  for the first values of  $d \geq 3$

The exact values of  ${}^2r(\langle a, a + d \rangle)$  when  $a \in E(d)$ , for  $d \in \{3, \dots, 12\}$ , are given in Table 6.3. For each value  $d \in \{3, \dots, 12\}$ , an explicit formula can be presented for  ${}^2r(\langle a, a + d \rangle)$  excluding the values given in Table 6.2. The latter can be done by using (essentially) the same arguments as those applied in the proofs of Theorem 6.3.12 and Corollary 6.3.14.

**Theorem 6.3.15** (Chappelon-Ramírez [C17]). *Let  $a \geq 3$  be an integer not divisible by 3 and let  $S = \langle a, a + 3 \rangle$ . Then,*

$${}^2r(S) = \begin{cases} (a - (3b - 1))^2 & \text{if either } (3b + 1)^2 \leq a + 3 < (3b + 2)^2 \text{ and } a \equiv 1 \pmod{3} \\ & \text{or } (3b + 1)^2 \leq 2(a + 3) < (3b + 2)^2 \text{ and } a \equiv 2 \pmod{3}, \\ (a - (3b + 1))^2 & \text{if either } (3b + 2)^2 \leq a + 3 < (3b + 4)^2 \text{ and } a \equiv 1 \pmod{3} \\ & \text{or } (3b + 2)^2 \leq 2(a + 3) < (3b + 4)^2 \text{ and } a \equiv 2 \pmod{3}. \end{cases}$$

**Theorem 6.3.16** (Chappelon-Ramírez [C17]). *Let  $a \geq 3$  be an odd integer and let  $S = \langle a, a + 4 \rangle$ . Then,*

$${}^2r(S) = \begin{cases} (a - (4b - 1))^2 & \text{if either } (4b + 1)^2 \leq a + 4 < (4b + 3)^2 \text{ and } a \equiv 1 \pmod{4} \\ & \text{or } (4b + 1)^2 \leq 3(a + 4) < (4b + 3)^2 \text{ and } a \equiv 3 \pmod{4}, \\ (a - (4b + 1))^2 & \text{if either } (4b + 3)^2 \leq a + 4 < (4b + 5)^2 \text{ and } a \equiv 1 \pmod{4} \\ & \text{or } (4b + 3)^2 \leq 3(a + 4) < (4b + 5)^2 \text{ and } a \equiv 3 \pmod{4}. \end{cases}$$

$d$	$a$	${}^2r(\langle a, a+d \rangle)$	$h(a, d, 1)$	$d$	$a$	$\mathcal{SG}(\langle a, a+d \rangle)$	$h(a, d, 1)$
5	2	1	0	10	43	$40^2$	$39^2$
5	4	1	$2^2$	10	123	$110^2$	$109^2$
5	13	$10^2$	$9^2$	10	133	$120^2$	$119^2$
5	27	$21^2$	$20^2$	10	143	$130^2$	$129^2$
5	32	$26^2$	$25^2$	10	153	$140^2$	$139^2$
7	2	1	$5^2$	10	163	$150^2$	$149^2$
7	3	$2^2$	0	10	333	$310^2$	$309^2$
7	4	$5^2$	$6^2$	10	343	$320^2$	$319^2$
7	9	$7^2$	$6^2$	11	2	$3^2$	$4^2$
7	16	$14^2$	$13^2$	11	3	$5^2$	$9^2$
7	18	$17^2$	$16^2$	11	4	$5^2$	$6^2$
7	19	$14^2$	$13^2$	11	5	$7^2$	$9^2$
7	23	$21^2$	$20^2$	11	7	$4^2$	$2^2$
7	30	$28^2$	$27^2$	11	8	$7^2$	$12^2$
7	114	$105^2$	$104^2$	11	9	$8^2$	$12^2$
8	5	$4^2$	$3^2$	11	14	$13^2$	$19^2$
8	9	$10^2$	$12^2$	11	16	$14^2$	$20^2$
8	21	$16^2$	$15^2$	11	18	$15^2$	$13^2$
8	45	$36^2$	$35^2$	11	25	$22^2$	$20^2$
8	77	$64^2$	$63^2$	11	36	$33^2$	$31^2$
9	2	$3^2$	$4^2$	11	38	$36^2$	$34^2$
9	4	$3^2$	$6^2$	11	47	$44^2$	$42^2$
9	7	$6^2$	$11^2$	12	13	$14^2$	$18^2$
9	8	$6^2$	$4^2$	12	19	$17^2$	$16^2$
9	16	$9^2$	$12^2$	12	25	$26^2$	$30^2$
10	3	$2^2$	$7^2$	12	31	$29^2$	$28^2$
10	9	$7^2$	$12^2$	12	67	$59^2$	$58^2$
10	13	$10^2$	$9^2$	12	79	$71^2$	$70^2$
10	23	$20^2$	$19^2$	12	139	$125^2$	$124^2$
10	27	$26^2$	$25^2$	12	151	$137^2$	$136^2$
10	33	$30^2$	$29^2$	12	235	$215^2$	$214^2$

Table 6.3:  ${}^2r(\langle a, a+d \rangle)$  and  $h(a, d, 1)$  when  $a \in E(d)$  for  $d \in \{3, \dots, 12\}$

**Theorem 6.3.17** (Chappelon-Ramírez [C17]). *Let  $a \geq 2$  be an integer not divisible by 5 and let  $S = \langle a, a + 5 \rangle$ . Then,*

$${}^2r(S) = \begin{cases} 1 & \text{if } a = 2 \text{ or } 4, \\ 10^2 & \text{if } a = 13, \\ (a - 6)^2 & \text{if } a = 27 \text{ or } 32, \\ (a - (5b - 2))^2 & \text{if either } (5b + 2)^2 \leq a + 5 < (5b + 3)^2 \text{ and } a \equiv 4 \pmod{5} \\ & \text{or } (5b + 2)^2 \leq 2(a + 5) < (5b + 3)^2 \text{ and } a \equiv 2 \pmod{5}, \\ (a - (5b - 1))^2 & \text{if either } (5b + 1)^2 \leq a + 5 < (5b + 4)^2 \text{ and } a \equiv 1 \pmod{5} \\ & \text{or } (5b + 1)^2 \leq 2(a + 5) < (5b + 4)^2 \text{ and } a \equiv 3 \pmod{5}, a \neq 13, \\ (a - (5b + 1))^2 & \text{if either } (5b + 4)^2 \leq a + 5 < (5b + 6)^2 \text{ and } a \equiv 1 \pmod{5} \\ & \text{or } (5b + 4)^2 \leq 2(a + 5) < (5b + 6)^2 \text{ and } a \equiv 3 \pmod{5}, \\ (a - (5b + 2))^2 & \text{if either } (5b + 3)^2 \leq a + 5 < (5b + 7)^2 \text{ and } a \equiv 4 \pmod{5}, a \neq 4 \\ & \text{or } (5b + 3)^2 \leq 2(a + 5) < (5b + 7)^2 \text{ and } a \equiv 2 \pmod{5}, a \neq 2, 27, 32. \end{cases}$$

### 6.3.3 Study of $\langle a, a + 1 \rangle$

We investigate the square Frobenius number of  $\langle a, a + 1 \rangle$  with  $a \geq 2$ . We first study the case when neither  $a$  nor  $a + 1$  is a square integer. Using similar arguments as those applied in the proofs of Theorem 6.3.12 and Corollary 6.3.14, we obtain the following

**Proposition 6.3.18** (Chappelon-Ramírez [C17]). *Let  $a$  be a positive integer such that  $b^2 < a < a + 1 < (b + 1)^2$  for some integer  $b \geq 1$ . Then,*

$${}^2r(\langle a, a + 1 \rangle) = (a - b)^2.$$

Let  $(u_n)_{n \geq 1}$  be the recursive sequence defined by

$$u_1 = 1, u_2 = 2, u_3 = 3, u_{2n} = u_{2n-1} + u_{2n-2} \text{ and } u_{2n+1} = u_{2n} + u_{2n-2} \text{ for all } n \geq 2. \quad (6.8)$$

The first few values of  $(u_n)_{n \geq 1}$  are

$$1, 2, 3, 5, 7, 12, 17, 29, 41, 70, 99, 169, 239, 408, 577, 985, \dots \quad (\text{A002965})$$

This sequence appears in a number of other contexts. For instance, it corresponds to the denominators of Farey fraction approximations to  $\sqrt{2}$ , where the fractions are  $\frac{1}{1}, \frac{2}{1}, \frac{3}{2}, \frac{4}{3}, \frac{7}{5}, \frac{10}{7}, \frac{17}{12}, \frac{24}{17} \dots$ , see [104].

We pose the following conjecture in the case when either  $a$  or  $a + 1$  is a square integer.

**Conjecture 6.3.19** (Chappelon-Ramírez [C17]). *Let  $(u_n)_{n \geq 1}$  be the recursive sequence given in (6.8). If  $a = b^2$  for some integer  $b \geq 1$  then*

$${}^2r(\langle a, a + 1 \rangle) = \begin{cases} (a - \lfloor b\sqrt{2} \rfloor)^2 & \text{if } b \notin \bigcup_{n \geq 0} \{u_{4n+1}, u_{4n+2}\}, \\ (a - \lfloor b\sqrt{3} \rfloor)^2 & \text{if } b \in \bigcup_{n \geq 0} \{u_{4n+1}, u_{4n+2}\}. \end{cases}$$

If  $a + 1 = b^2$  for some integer  $b \geq 1$  then

$${}^2r(\langle a, a + 1 \rangle) = \begin{cases} (a - \lfloor b\sqrt{2} \rfloor)^2 & \text{if } b \notin \bigcup_{n \geq 1} \{u_{4n-1}, u_{4n}\}, \\ (a - \lfloor b\sqrt{3} \rfloor)^2 & \text{if } b \in \bigcup_{n \geq 1} \{u_{4n}, u_{4n+3}\}, \\ 2^2 & \text{if } b = u_3 = 3. \end{cases}$$

The formulas of Conjecture 6.3.19 have been verified by computer for all integers  $a \geq 2$  up to  $10^6$ .

### 6.3.4 Study of $\langle a, a + 2 \rangle$

We investigate the square Frobenius number of  $\langle a, a + 2 \rangle$  with  $a \geq 3$  odd. We first study the case when neither  $a$  nor  $a + 2$  is a square integer. Using similar arguments as those applied in the proofs of Theorem 6.3.12 and Corollary 6.3.14, we obtain the following

**Proposition 6.3.20** (Chappelon-Ramírez [C17]). *Let  $a \geq 3$  be an odd integer such that  $(2b + 1)^2 < a < a + 2 < (2b + 3)^2$  for some integer  $b \geq 1$ . Then,*

$${}^2r(\langle a, a + 2 \rangle) = (a - (2b + 1))^2.$$

We pose the following conjecture in the case when either  $a$  or  $a + 2$  is a square integer.

**Conjecture 6.3.21** (Chappelon-Ramírez [C17]). *Let  $(u_n)_{n \geq 1}$  be the recursive sequence given in (6.8). If  $a = (2b + 1)^2$  for some integer  $b \geq 1$  then*

$${}^2r(\langle a, a + 2 \rangle) = \begin{cases} \left( a - 2 \left\lfloor \frac{(2b+1)\sqrt{2}}{2} \right\rfloor \right)^2 & \text{if } (2b + 1) \notin \bigcup_{n \geq 1} \{u_{4n+1}\}, \\ (a - \lfloor (2b + 1)\sqrt{3} \rfloor)^2 & \text{if } (2b + 1) \in \bigcup_{n \geq 2} \{u_{4n+1}\}, \\ 38^2 & \text{if } 2b + 1 = u_5 = 7. \end{cases}$$

If  $a + 2 = (2b + 1)^2$  for some integer  $b \geq 1$  then

$${}^2r(\langle a, a + 2 \rangle) = \begin{cases} \left( a - 2 \left\lfloor \frac{(2b+1)\sqrt{2}}{2} \right\rfloor \right)^2 & \text{if } (2b + 1) \notin \bigcup_{n \geq 0} \{u_{4n+3}\}, \\ (a - \lfloor (2b + 1)\sqrt{3} \rfloor)^2 & \text{if } (2b + 1) \in \bigcup_{n \geq 0} \{u_{4n+3}\}. \end{cases}$$

The formulas of Conjecture 6.3.21 have been verified by computer for all odd integers  $a \geq 3$  up to  $10^6$ .

### 6.3.5 Remarks and open problems

In the process of investigating square Frobenius numbers different problems arose. We naturally consider the  $P$ -type function  ${}_k\text{-power}r(S) = {}_k r(S)$  defined as,

$${}_k r(S) := \text{the smallest perfect } k\text{-power integer belonging to } S.$$

It is clear that

$$s \leqslant {}_k r(S) \leqslant s^k \tag{6.9}$$

where  $s$  is the multiplicity of  $S$ .

**Theorem 6.3.22** (Chappelon-Ramírez [C17]). *Let  $S_A = \langle a, a + d, \dots, a + kd \rangle$  where  $a, d, k$  are positive integers with  $\gcd(a, d) = 1$ . If  $\lceil \sqrt{a} \rceil \leqslant d \leqslant \frac{ak}{1+2k}$  then*

$${}_2 r(S_A) \leqslant (a - d)^2.$$

**Problem 6.3.23.** *Let  $k \geqslant 2$  be an integer and let  $S$  be a numerical semigroup. Investigate the computational complexity to determine  ${}_k r(S)$  and/or  ${}_k r(S)$ .*

Or more ambitious,

**Question 6.3.24.** *Let  $k \geqslant 2$  be an integer. Is there a closed formula for  ${}_k r(S)$  and/or  ${}_k r(S)$  for any semigroup  $S$ ?*

Perhaps a first step on this direction might be the following.

**Problem 6.3.25.** *Give a formula for  ${}_2 r(\langle F_i, F_j \rangle)$  and/or  ${}_2 r(\langle F_i, F_j \rangle)$  with  $\gcd(F_i, F_j) = 1$  where  $F_k$  denotes the  $k^{\text{th}}$  Fibonacci number. What about  ${}_2 r(\langle a^2, b^2 \rangle)$  where  $a$  and  $b$  are relatively prime integers? We clearly have that  ${}_2 r(\langle a^2, b^2 \rangle) = a^2$  for  $1 \leqslant a < b$ .*

# Chapter 7

## Tower of Hanoi

In this chapter, we are interested in problems related to the Tower of Hanoi. The first section is about results obtained in [C14] on generalized Frame-Stewart numbers, that are optimal numbers of moves in the Tower of Hanoi problem on 4 pegs and supposed to be optimal for  $k \geq 5$  pegs. In the second section, results obtained in [C11] about a two-player combinatorial game based on the Tower of Hanoi are studied. An extensive literature on Tower of Hanoi problems can be found in [76].

### 7.1 Generalized Frame-Stewart numbers

The Tower of Hanoi problem was introduced by Édouard Lucas in 1883 [85] for the case of 3 pegs and  $n$  disks of different sizes. Initially,  $n$  disks are placed on one of the 3 pegs with the largest at the bottom. Then, at each time one of the topmost disks is moved to a peg with a larger disk on the top or to an empty peg. The goal of the problem is to transfer all the disks from the initial peg to the peg of destination with the minimum number of moves. A simple recursive argument shows that  $2^n - 1$  moves are necessary and sufficient to carry out this task. This Tower of Hanoi problem was then extended to the case of 4 pegs by Dudeney in 1907 [50] and to arbitrary  $k \geq 3$  pegs by Stewart in 1939 [108]. In 1941, Frame [66] and Stewart [109] independently proposed algorithms which achieve the same numbers of moves for the  $k$ -peg Tower of Hanoi problem with  $k \geq 4$  pegs. Klavžar et al.[79] showed that seven different approaches to the  $k$ -peg Tower of Hanoi problem, including those by Frame and Stewart, are all equivalent, that is, achieve the same numbers of moves. Thus, these numbers are called the *Frame-Stewart numbers* [78].

In 2014, T. Bousch showed in [37] that the Frame-Stewart numbers are the optimal numbers of moves for the case of 4 pegs. Somewhat surprisingly, the optimal solution for the multi-peg Tower of Hanoi problem with  $k \geq 5$  pegs is not known yet. So far, the best upper bounds are achieved by the Frame-Stewart numbers and the best lower bounds are obtained by Chen et al. [44]. Since the upper bounds are believed to be optimal, they are called the *presumed optimal* solution.

Stewart's recursive algorithm for the  $k$ -peg Tower of Hanoi problem is summarized as follows. For a positive integer  $t \leq n$ ,

- i) recursively transfer a pile of  $n - t$  smallest disks from the first peg to a temporary peg, using  $k$  pegs;

- ii) transfer the remaining pile of  $t$  largest disks from the first peg to the final peg using  $k - 1$  pegs, ignoring the peg occupied by the  $n - t$  smallest disks;
- iii) recursively transfer the pile of  $n - t$  smallest disks from the temporary peg to the final peg, using  $k$  pegs.

The algorithm chooses the integer  $t$  such that the number of moves is minimized.

**Definition 7.1.1** (Frame-Stewart numbers). The Frame-Stewart numbers  $S_k(n)$  satisfy the following recurrence relations:

$$S_k(n) = \min_{1 \leq t \leq n} \{2 \cdot S_k(n - t) + S_{k-1}(t)\}, \text{ for } n \geq 1, k \geq 4,$$

$$S_3(n) = 2^n - 1, \text{ for } n \geq 1, \text{ and } S_k(0) = 0, \text{ for } k \geq 3.$$

When  $k = 4$  for instance,  $S_4(n)$  is obtained by the following simple formula:

$$S_4(n) - S_4(n - 1) = 2^{i-1}, \text{ for } \binom{i}{2} < n \leq \binom{i+1}{2},$$

where  $\binom{i}{2}$  is the binomial coefficient equal to  $i(i - 1)/2$ . In the general case  $k \geq 4$ , the number  $S_k(n)$  is obtained by several different approaches, e.g., [66, 78, 79, 87, 109].

In [88], the following general recurrence relation was considered to clarify the combinatorial structure latent in the recurrence relation for  $S_k(n)$  and to cope with the recurrence relations for the Tower of Hanoi *on graphs* in which pegs are placed on vertices of a given graph and disks are only moved along the edges:

$$T(n) = \min_{1 \leq t \leq n} \{\alpha \cdot T(n - t) + \beta \cdot (2^t - 1)\}, \text{ for } n \geq 1, \text{ and } T(0) = 0,$$

where  $\alpha$  and  $\beta$  are arbitrary positive integers. It was shown that the sequence of differences  $(T(n) - T(n - 1))_{n \geq 1}$  consists of numbers of the form  $\beta \cdot 2^i \cdot \alpha^j$ , with  $i, j \geq 0$ , arranged in nondecreasing order. When  $\alpha = 3$ ,  $2^i \cdot \alpha^j$  increases as  $1, 2, 3, 2^2, 2 \cdot 3, 2^3, 3^2, 2^2 \cdot 3, 2^4, 2 \cdot 3^2, \dots$ . These numbers are called “3-smooth numbers” [105] and have been studied extensively in number theory, in relation to the distribution of prime numbers [73] and to new number representations [36, 62]. The formulation and analysis of  $T(n)$ , however, has some defects such that (i) it is only focused on the 4-peg case with no consideration for the general case  $k \geq 3$ ; and (ii) even in the 4-peg case, term  $2^i \cdot \alpha^j$  consists of constant 2 and parameter  $\alpha$ , which might admit further generalization.

In this section, we fully generalize the recurrence relations for the previous  $S_k(n)$  and  $T(n)$  and obtain the exact formulas.

**Definition 7.1.2** (Generalized Frame-Stewart numbers). We define the following recurrence relations for two sequences of arbitrary positive integers  $(p_i)_{i \geq 3}$  and  $(q_i)_{i \geq 3}$ :

$$G_k(n) = \min_{1 \leq t \leq n} \{p_k \cdot G_k(n - t) + q_k \cdot G_{k-1}(t)\}, \text{ for } n \geq 1, k \geq 4,$$

$$G_3(n) = p_3 \cdot G_3(n - 1) + q_3, \text{ for } n \geq 1, \text{ and } G_k(0) = 0, \text{ for } k \geq 3.$$

The numbers  $G_k(n)$  are called the *generalized Frame-Stewart numbers*.

Then, we show that the sequence of differences  $(G_k(n) - G_k(n - 1))_{n \geq 1}$  consists of numbers of the form  $(\prod_{i=3}^k q_i) \cdot (\prod_{i=3}^k p_i^{\alpha_i})$ , with  $\alpha_i \geq 0$  for all  $i$ , arranged in nondecreasing order.

**Theorem 7.1.3** (Chappelon-Matsuura [C14]). *For every positive integer  $n$  and for two sequences of arbitrary positive integers  $(p_i)_{i \geq 3}$  and  $(q_i)_{i \geq 3}$ , we have*

$$G_k(n) = q \cdot \sum_{j=1}^n u_j^k$$

where  $q = \prod_{i=3}^k q_i$  and  $u_j^k$  is the  $j$ th term of the sequence  $(u_j^k)_{j \geq 1}$  of integers  $\prod_{i=3}^k p_i^{\alpha_i}$ , with  $\alpha_i \geq 0$  for all  $i$ , arranged in nondecreasing order.

*Remark 7.1.4.* Note that  $G_k(n)$  is equal to  $S_k(n)$  when  $(p_i, q_i) = (2, 1)$  for all  $i \geq 3$  and  $G_4(n)$  is equal to  $T(n)$  when  $(p_3, q_3) = (2, 1)$  and  $(p_4, q_4) = (\alpha, \beta)$ .

### 7.1.1 Basic results on smooth number sequences

Let  $(p_i)_{i \geq 3}$  be a sequence of positive integers. We consider the sequence  $(u_j^k)_{j \geq 1}$  of all the integers of the form  $\prod_{i=3}^k p_i^{\alpha_i}$ , where  $\alpha_i \geq 0$  for all  $i$ , arranged in nondecreasing order. For instance, for  $(p_3, p_4) = (2, 2)$  and  $(p_3, p_4) = (2, 3)$ , the first few terms of  $(u_j^4)_{j \geq 1}$  are  $(1, 2, 2, 2^2, 2^2, 2^2, 2^3, \dots)$  and  $(1, 2, 3, 2^2, 2 \cdot 3, 2^3, 3^2, \dots)$ , respectively. When there is some  $i_0$  such that  $p_{i_0}$  is equal to 1, then by definition  $(u_j^k)_{j \geq 1}$  is the constant sequence of 1's, for every  $k \geq i_0$ . We note that  $(u_j^k)_{j \geq 1}$  is closely related to *smooth numbers* which have been explored extensively in number theory. A positive integer is called *B-smooth* if none of its prime factors are greater than a positive integer  $B$ . The sequence  $(u_j^k)_{j \geq 1}$  then consists of  $B$ -smooth numbers for  $B = \max_{3 \leq i \leq k} \{p_i\}$ .

Here, we restrict to the case where all the  $p_i$ 's are greater than 1 and obtain a simple lemma on a certain *recursive* structure of the smooth number sequence  $(u_j^k)_{j \geq 1}$ , which is useful for the proof of Theorem 7.1.3.

**Lemma 7.1.5** (Chappelon-Matsuura [C14]). *Let  $k \geq 4$  and let  $(f_k(j))_{j \geq 1}$  be the sequence of positive integers defined by  $f_k(1) = 1$  and  $f_k(j) = \min \{l > f_k(j-1) \mid u_l^k = u_{j-1}^{k-1}\}$  for  $j \geq 2$ . Then, for every integer  $n$  such that  $f_k(j) < n < f_k(j+1)$ , we have  $u_n^k = p_k \cdot u_{n-j}^k$ .*

### 7.1.2 Sketch of proof of the main result

Let  $G_k^1(n)$  denote the special case of  $G_k(n)$  associated with arbitrary sequence  $(p_i)_{i \geq 3}$  and with the constant sequence  $(q_i)_{i \geq 3}$  with  $q_i = 1$  for  $i \geq 3$ . There exists a simple relationship between numbers  $G_k(n)$  and  $G_k^1(n)$ .

**Proposition 7.1.6** (Chappelon-Matsuura [C14]). *For every nonnegative integer  $n$  and for every sequence of integers  $(q_i)_{i \geq 3}$ , we have*

$$G_k(n) = q \cdot G_k^1(n),$$

where  $q = \prod_{i=3}^k q_i$ .

By Proposition 7.1.6, it is sufficient to prove Theorem 7.1.3 for  $G_k^1(n)$  instead of  $G_k(n)$ . Now, we show at which argument  $G_k^1(n) = \min_{1 \leq t \leq n} \{p_k \cdot G_k^1(n-t) + G_{k-1}^1(t)\}$  takes its minimum.

**Lemma 7.1.7** (Chappelon-Matsuura [C14]). *Let  $n$  be a positive integer. Suppose that  $p_i > 1$  for all  $3 \leq i \leq k$ . Suppose also that  $\Delta G_i^1(l) = G_i^1(l) - G_i^1(l-1) = u_l^i$  for  $3 \leq i \leq k-1$  and  $l \geq 1$  and that  $\Delta G_k^1(l) = u_l^k$  for  $1 \leq l \leq n-1$ . Let  $j$  be the integer such that  $f_k(j) \leq n < f_k(j+1)$ . Then, for  $1 \leq t \leq n$ ,  $G_{k,n}^1(t) = p_k \cdot G_k^1(n-t) + G_{k-1}^1(t)$  takes its minimum at  $t = j$ .*

Using Proposition 7.1.6 and Lemma 7.1.7, we are now ready to obtain Theorem 7.1.3, the main result of this section.

**Theorem 7.1.8** (Chappelon-Matsuura [C14]). *For every positive integer  $n$  and for two sequences of arbitrary positive integers  $(p_i)_{i \geq 3}$  and  $(q_i)_{i \geq 3}$ , we have*

$$G_k(n) = q \cdot \sum_{j=1}^n u_j^k$$

where  $q = \prod_{i=3}^k q_i$  and  $u_j^k$  is the  $j$ th term of the sequence  $(u_j^k)_{j \geq 1}$  of integers  $\prod_{i=3}^k p_i^{\alpha_i}$ , with  $\alpha_i \geq 0$  for all  $i$ , arranged in nondecreasing order.

**Corollary 7.1.9** (Chappelon-Matsuura [C14]). *Let  $k \geq 4$  and  $j \geq 1$ . For every integer  $n$  such that  $f_k(j) \leq n < f_k(j+1)$ ,*

$$G_k(n) = p_k \cdot G_k(n-j) + q_k \cdot G_{k-1}(j).$$

We end this subsection by considering the special case where  $p_i = p \geq 1$  for all  $i$ .

**Proposition 7.1.10** (Chappelon-Matsuura [C14]). *Let  $p_i = p \geq 1$  for all  $3 \leq i \leq k$ . Then, for all integers  $j \geq 0$  and  $n \geq 1$  such that*

$$\binom{k+j-3}{k-2} < n \leq \binom{k+j-2}{k-2},$$

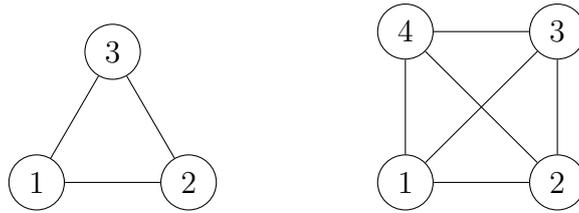
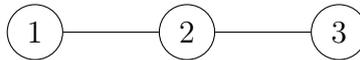
$u_n^k = p^j$  and  $G_k^1(n)$  can be computed as follows:

$$G_k^1(n) = \sum_{m=0}^{j-1} \binom{k+m-3}{k-3} p^m + \left( n - \binom{k+j-3}{k-2} \right) p^j.$$

### 7.1.3 Application: the Tower of Hanoi on graphs

Let  $G = (V, E)$  be a simple graph with the set of vertices  $V = \{v_1, \dots, v_k\}$  and the set of edges  $E$ . A  $k$ -peg Tower of Hanoi problem can be considered on  $G$ : the  $k$  pegs are placed on the vertices  $v_1, \dots, v_k$  and transfer of disks is allowed between the pegs  $v_i$  and  $v_j$  only if there is an edge between  $v_i$  and  $v_j$ . The original  $k$ -peg Tower of Hanoi problem then corresponds to the Tower of Hanoi problem on the complete graph  $K_k$ . The cases of  $k = 3$  and  $k = 4$  are illustrated in Figure 7.1.

The main application of the generalized Frame-Stewart numbers is in giving upper bounds of the number of moves for the Tower of Hanoi problem on some simple graphs. For the Tower of Hanoi problem on the complete graph with  $k \geq 3$  vertices and  $n \geq 0$  disks, we retrieve the Frame-Stewart numbers  $S_k(n)$ . In the sequel, we consider other special cases where  $G$  is the path graph  $P_3$  or the star graph  $S_k$ .

Figure 7.1: The original Tower of Hanoi problem with 3 pegs ( $K_3$ ) and 4 pegs ( $K_4$ )Figure 7.2: The path graph  $P_3$ 

### On the path graph $P_3$

The following theorem shows that the optimal number of moves for the Tower of Hanoi problem on the path graph  $P_3$  is given by the generalized Frame-Stewart numbers.

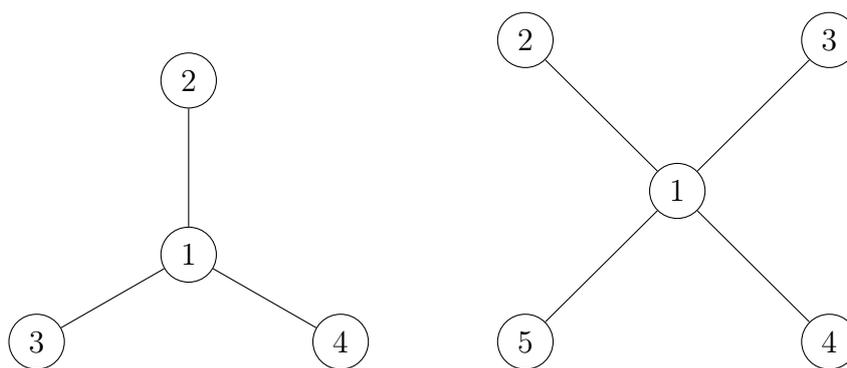
**Theorem 7.1.11.** *Consider the Tower of Hanoi problem on  $P_3$ , as depicted in Figure 7.2. The minimum number of moves to transfer  $n \geq 1$  disks*

- from peg 1 to peg 3 is  $G_3(n) = 2 \cdot \sum_{i=0}^{n-1} 3^i$ , where  $(p_3, q_3) = (3, 2)$ ;
- from peg 1 to peg 2 is  $G_3^1(n) = \sum_{i=0}^{n-1} 3^i$ , where  $(p_3, q_3) = (3, 1)$ .

This theorem is rather well-known (e.g., see [100]).

### On the star graph $S_k$

We end this subsection by considering the Tower of Hanoi problem on the star graph  $S_k$  with  $k$  vertices and  $k - 1$  edges. For  $k = 3$ , the graph  $S_3$  corresponds to the path graph  $P_3$ . The star graphs for  $k = 4$  and  $k = 5$  are depicted in Figure 7.3.

Figure 7.3: The star graphs  $S_4$  and  $S_5$ 

Stockmeyer [110] considered the Tower of Hanoi problem on the star graph  $S_4$ , where all the  $n$  disks are transferred from one leaf of the graph to another leaf (for instance, the problem of transferring disks in the minimal number of moves from peg 2 to peg 3 in Figure 7.3). He described a recursive algorithm which achieved a good (seemingly the best) upper bound; thus, called it the *presumed optimal* algorithm. T. Bousch proved in [38] that this number is optimal for  $S_4$ . Here, we generalize this algorithm to the star

graph  $S_k$  for arbitrary  $k \geq 3$  and show that disks can be transferred from one leaf to another in  $G_k(n)$  moves.

**Theorem 7.1.12** (Chappelon-Matsuura [C14]). *Let  $k \geq 3$  be an integer. Consider the Tower of Hanoi problem on the star graph  $S_k$  in which  $n \geq 1$  disks are transferred from one leaf of the graph to another leaf. Then, an upper bound on the minimal number of moves to solve this problem is given by the generalized Frame-Stewart number  $G_k(n)$ , where  $(p_3, q_3) = (3, 2)$  and  $(p_i, q_i) = (2, 1)$  for  $4 \leq i \leq k$ .*

## 7.2 Two-player Tower of Hanoi

Tower of Hanoi (TH) is traditionally a *one player game* [74, 49]. Here we let two players, Anh (first player) and Bao (second player), alternate turns and play a game on three or more pegs with various numbers of disks. We will begin by analyzing games under the following *impartial* rules (the move options do not depend on who is to move) [22, 31, 45].

Let  $n \geq 1$  and  $k \geq 3$  be positive integers. Two players alternate in transferring precisely one out of  $n$  disks (of different sizes) on  $k$  pegs, Peg 1, ..., Peg  $k$ . The *starting position* is as usual for the Tower of Hanoi; *the tower* (i.e., all the disks in decreasing size) is placed on the *starting peg* (Peg 1), and at each stage of the game, a larger disk cannot be placed on top of a smaller. In addition, the current player cannot move the disk that the previous player just moved. The game ends when the tower has been transferred to some predetermined *final peg*, and some predetermined condition determines *who wins*. It is not allowed to transfer the tower to a non-final peg. We detail five ending conditions in the sequel. A *position* is any configuration of the  $n$  disks on the  $k$  pegs such that no larger disk is on top of a smaller disk, and in case of a non-starting position, a note on which disk was just moved.

Thus, a position  $(S_1, \dots, S_k; \alpha)$  is an ordered partitioning of the set  $\{1, \dots, n\}$  in  $k$  sub-sets

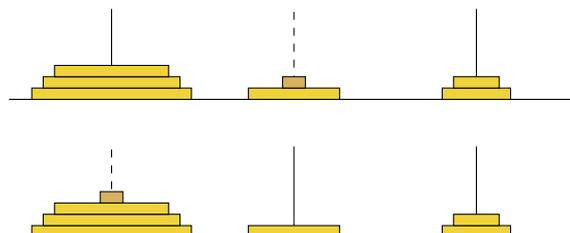
$$\bigcup_{i \in \{1, \dots, k\}} S_i = \{1, \dots, n\}$$

such that  $S_i \cap S_j = \emptyset$  for any  $i \neq j$ , and an integer  $\alpha \in \{1, \dots, k\}$ .

Let us exemplify our game with a position on seven disks and three pegs, and where the disk at the dashed peg has just been moved by the previous player and hence cannot be moved by the current player; below it we find its two legal options.



Move options:



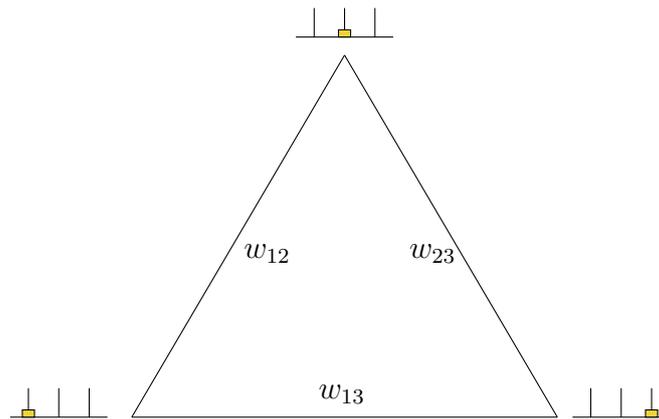


Figure 7.4: A graph representation of Tower of Hanoi, with weighted move edges, for  $n = 1$

Henceforth it will be clear by the context which disk was moved last, so we omit the dashed emphasis. Note that, by this rule, there are exactly  $3(2 \cdot 3^{n-1} - 1) = 2(3^n - 3) + 3$  positions of two-player TH on 3 pegs and  $n$  disks; 2 positions for each configuration of the non-Tower configurations (depending on which disk was just moved) and the 3 Tower configurations. A game is *cyclic* (or *loopy* [46]) if there is at least one position which can be revisited during (not necessarily optimal) play. Clearly TH is cyclic.

**Normal play** One winning condition is adapted from the one-player game: the player who plays the last disk (on top of the rest of the tower) wins. This corresponds to a classical convention for two-player games: a player who cannot move loses, which is called *normal play* [22, 31, 45, 102]. If no player can force a win in this setting, then the game is declared drawn.

One of first observations is that in spite of the cyclic nature of the game, if we play on just three pegs, Anh wins the normal play convention. Bao's moves will be forced throughout the game and the proof is an adaptation of the well known one-player result. In fact, Anh's moves also have a restriction; she always has to move the smallest disk, but this turns out to be an advantage. In that section, we also note that the game is drawn on four or more pegs.

**Scoring play** Scoring two-player TH is the following game. The players move as in the normal play variation, but at each stage of game the current player gains a given real weight  $w_{ij} = w_{ji}$ , for a move from Peg  $i$  to Peg  $j$  (and the game starts with score 0 for each player). Thus, with three pegs there are three *move edges*,  $\{i, j\} \in \{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$  and three real weights  $w_{12}, w_{13}$  and  $w_{23}$ ; see Figure 7.4 for an illustration. For example, if Anh's current score is  $s$  and she moves along edge  $\{i, j\}$ , then her updated score is  $s + w_{ij}$ . The player who obtains the largest score when the game ends wins.

If the game terminates but none of the players can claim a victory (because their terminal scores are equal) then the game is a tie. We also adapt the convention of drawn games from normal play, so a game is declared a draw if no player can force a win, by terminating the game.

The main result of this section is that Anh wins nearly all scoring games on three pegs. The reason is partly the same as in normal play, but in this setting the optimal move sequence varies depending on the given weights, and the proof is non-trivial. The

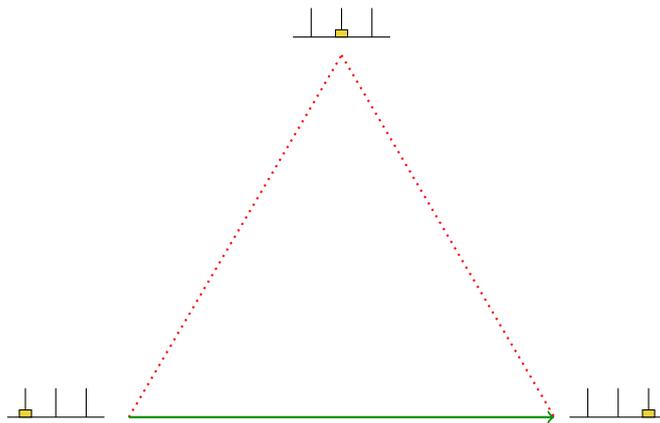


Figure 7.5: A play of two-player TH for  $n = 1$ , (EC1)

only case when Anh cannot force a win is when all weights are equal and non-positive; otherwise she can attain an arbitrary high score by adhering to certain intermediate repetitive patterns. Questions of termination are often very hard (e.g., Turing machines), but in our three-peg setting it will be easy to distinguish drawn from winning. The first player, namely Anh, controls all the moves under optimal play; if she cannot win, it will be easy to play drawn. The question of minimizing the number of moves in the two-player setting is also studied.

### 7.2.1 Ending conditions for the two-player Tower of Hanoi

Consider the following variations of the two-player Tower of Hanoi; the game ends when the tower has been transferred to a *final peg*, which is

- (EC1) a given peg, distinct from the starting peg;
- (EC2) the starting peg, but the largest disk has to be moved at least once;
- (EC3) the starting peg, but the smallest disk has to be moved at least once;
- (EC4) any peg, but the largest disk has to be moved at least once; and
- (EC5) any peg, but the smallest disk has to be moved at least once.

Ending conditions (EC2) and (EC3) are not applicable when  $n = 1$ . In Figure 7.4 to Figure 7.7, we illustrate the idea of going from the one-player setting to the two-player setting using the standard graph representation. Each edge will now be directed, and the direction depends on the previous move. In Figures 7.5 and 7.7 we show which moves are possible in two-player Tower of Hanoi for  $n = 1$  and 2 respectively (red dotted edges are illegal; green move edges are directed with direction depending on a given initial move).

### 7.2.2 General play

In this section we regard *an odd or even number of moves* in the one-player game in the same sense as for the two-player games; no disk will be moved twice unless some other disk has been moved in between. Each odd numbered move thus moves the smallest disk, and in fact we could equivalently have chosen to let Anh lead the game. Recall that the

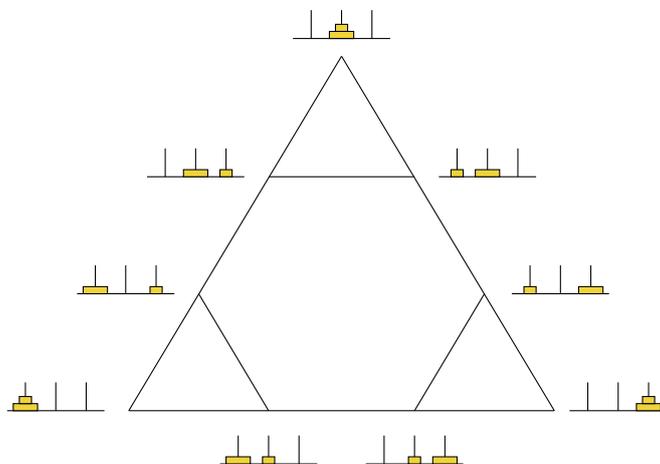


Figure 7.6: A graph representation of one-player TH for  $n = 2$

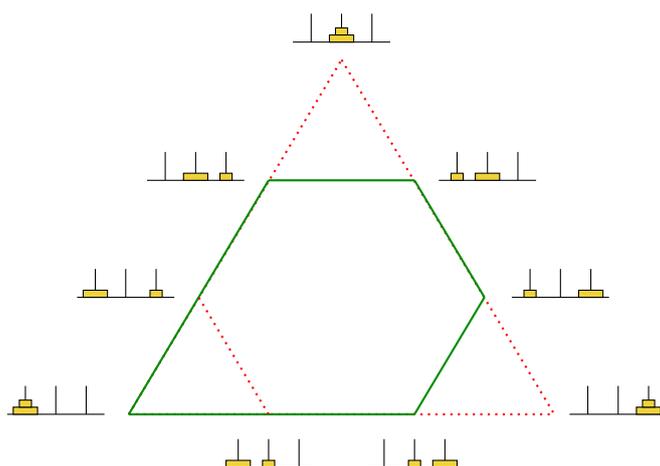


Figure 7.7: A play sequence of two-player TH for  $n = 2$ , (EC2); depicting the first row in Table 7.1

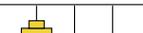
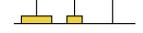
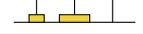
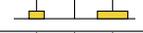
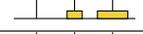
Position	Sequence of moves	Number of moves
	13 – 12 – 13 – 23 – 12 – 13 – 12	7
	12	1
	13	1
	13 – 12 – 13	3
	13 – 12 – 23	3
	12 – 13 – 12 – 23 – 13	5
	12 – 13 – 12	3
	13 – 12 – 13 – 23 – 12	5
	12 – 13 – 23	3

Table 7.1: Any Tower of Hanoi position on two disks and three pegs can be reached in an odd number of moves

starting position throughout the paper is all disks on Peg 1. The following two results are easy well-known consequences of the graph representation of Tower of Hanoi.

**Theorem 7.2.1** (CLM [C11]). *For three pegs,  $n \geq 2$  disks can be transferred from the starting position to any position using an odd number of moves. That is, in the two-player version, the first player can force play to any position in an odd number of moves.*

**Corollary 7.2.2** (CLM [C11]). *For three pegs,  $n \geq 2$  disks can be transferred from the starting position to any intermediate position using an even number of moves.*

### 7.2.3 Normal play: two-player Tower of Hanoi

In the normal play variation of the two-player Tower of Hanoi, Anh can avoid drawn simply by adhering to the well known *minimal algorithm* for the one-player Tower of Hanoi (Bao's moves will be forced all through the game), using precisely  $2^n - 1$  moves. However, she can also choose freely among all odd-length move paths.

**Corollary 7.2.3** (CLM [C11]). *For three pegs and  $n \geq 1$  disks, the two-player Tower of Hanoi game terminates and the first player wins. This is true for any ending condition and also from any position, provided that the previous player did not move the smallest disk.*

Games on four pegs are mostly loopy.

**Theorem 7.2.4** (CLM [C11]). *The two-player Tower of Hanoi on four or more pegs is a draw game if the number of disks is three or more, given any ending condition.*

For completeness, let us also give the rest of the  $k$ -peg observations with  $k \geq 4$ . For (EC1-3), if there are two disks, Bao never has to move the largest disk to a final peg and hence the game is drawn. For (EC1), if there is only one disk, then Anh wins in the first move. For (EC2,3), if there is only one disk, the special rule is invoked and Bao wins in his first move. For (EC4,5), if there are two disks, Bao has to move the largest disk to a final peg and hence loses. If there is only one disk, then Anh wins in the first move.

### 7.2.4 Scoring play: two-player Tower of Hanoi with weights

As stated in the introduction, for the scoring variation of the normal play setting, we provide real weights to the *move edges*, in the three-peg case,  $w_{12}$ ,  $w_{13}$  and  $w_{23}$  respectively. As usual, the two players alternate in moving, and a player gets  $w_{ij}$  points for a move along edge  $\{i, j\}$ . The player who has most points when the game ends wins. We begin by giving the solution of the game with less than three disks.

We will use  $A_{ij}(n)$  and  $B_{ij}(n)$ , for the total points for Anh and Bao respectively, of the two-player Tower of Hanoi game, for transferring  $n$  disks from Peg  $i$  to Peg  $j$  by a given algorithm, for example the minimal algorithm, and we let the *total score* be  $\Delta_{ij}(n) = A_{ij}(n) - B_{ij}(n)$ , or just  $\Delta(n)$ . Hence, if  $\Delta(n) > 0$  then Anh wins, if  $\Delta(n) = 0$ , then the game is tie, and otherwise Bao wins.

**Theorem 7.2.5** (CLM [C11]). *Consider the two-player Tower of Hanoi game on two disks, three pegs and three weights of real numbers  $w_{12}$ ,  $w_{13}$ , and  $w_{23}$ . In case of (EC4) or (EC5), the first player wins if either of the following inequalities holds:*

$$w_{12} + w_{23} - w_{13} > 0 \quad (7.1)$$

$$3w_{13} - w_{12} - w_{23} > 0 \quad (7.2)$$

$$w_{13} + w_{23} - w_{12} > 0 \quad (7.3)$$

$$3w_{12} - w_{13} - w_{23} > 0 \quad (7.4)$$

$$w_{12} + w_{13} - w_{23} > 0 \quad (7.5)$$

*In case of (EC2) or (EC3), she wins if (7.5) holds. In case of (EC1), she wins if (7.1) or (7.2) holds. Otherwise the game is a draw.*

*If the game is played on only one disk, then (EC2,3) are not applicable. For (EC1) the first player wins if  $w_{13} > 0$ ; loses if  $w_{13} < 0$ ; and the game is a tie otherwise. The second player wins (EC4,5) if  $w_{12} < 0$  and  $w_{13} < 0$ . The game is a tie if at least one of these weights is 0 and the others are non-positive. Otherwise the first player wins.*

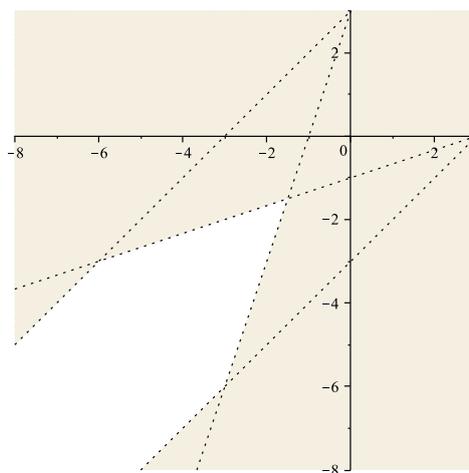


Figure 7.8: Consider (EC1) for  $n = 2$ . Here  $w_{23} = -3$  and the other weights represent the  $x$ - and  $y$ -axes. The game is drawn in the white area. Compare this picture with the result for  $n \geq 3$  in Theorem 7.2.6, where the class of drawn games would have been represented by a single white dot at  $(-3, -3)$ .

It turns out that the case  $n \geq 3$  has fewer drawn games, allowing a simpler description; it relies on the general ideas in Subsection 7.2.1. By relabeling the pegs, it suffices to analyze the case of transferring the disks from Peg 1 to Peg 3 and the case from Peg 1 to Peg 1 .

**Theorem 7.2.6** (CLM [C11]). *Given  $n \geq 3$  disks, three pegs and three weights of real numbers  $w_{13}$ ,  $w_{12}$ , and  $w_{23}$ . Then, for the two-player Tower of Hanoi game, the first player wins every game, except in the case  $w_{12} = w_{13} = w_{23} \leq 0$  for which the game is a draw.*

We can adapt this proof to scoring play from an arbitrary position of disks for the three-peg case. But we have to remember that in the two-player setting a position carries also a memory of the last move.

**Corollary 7.2.7.** *Consider an arbitrary Tower of Hanoi position on  $n \geq 3$  disks, three pegs and three weights of real numbers  $w_{13}$ ,  $w_{12}$ , and  $w_{23}$ . Then the first player wins, unless  $w_{12} = w_{13} = w_{23} \leq 0$ , and provided the previous player did not move the smallest disk.*

## 7.2.5 The minimal number of moves for winning

### The minimal number of moves for winning normal play

Traditionally, in the one-player setting, the interest has often been focused on the minimal number of moves for transferring the tower. Here, we analyze our variations of the two-player game in this sense. It is not a big surprise that the minimal number of moves to win normal play is the same as the number of moves in the one-player minimal algorithm, but let us sum up the state of the art before we move on to the more challenging analysis of minimum number of moves for winning scoring play.

**Theorem 7.2.8** (CLM [C11]). *The minimum number of moves for transferring  $n \geq 1$  disks from one peg to another peg is  $2^n - 1$ . The minimum number of moves for transferring  $n \geq 2$  disks from one peg to the same peg is  $2^{n+1} - 1$ , if the largest disk has to be moved; and it is seven if only the smallest disk has to be moved.*

Now, we estimate the minimal number of moves for winning the normal play two-player Tower of Hanoi. In case the game is drawn, then we say that the minimal number of moves is infinite.

**Theorem 7.2.9** (CLM [C11]). *Let  $M_l(n)$  denote the minimal number of moves needed for winning a normal play game on  $l \geq 3$  pegs and  $n \geq 1$  disks. Then,*

$$M_3(n) = \begin{cases} 2^n - 1, & \text{for } n \geq 1, \text{ for (EC1,4),} \\ \begin{cases} 2^{n+1} - 1, & \text{for } n \geq 2, \text{ for (EC2),} \\ 7, & \text{for } n \geq 2, \text{ for (EC3),} \end{cases} \\ \begin{cases} 2^n - 1, & \text{for } n \leq 2, \text{ for (EC5),} \\ 7, & \text{for } n \geq 3, \text{ for (EC5).} \end{cases} \end{cases}$$

$$M_l(n) = \begin{cases} 1, & \text{for } n = 1, l \geq 4 \text{ and for (EC1,4,5),} \\ 3, & \text{for } n = 2, l \geq 4 \text{ and for (EC4,5),} \\ \infty, & \text{otherwise for } l \geq 4. \end{cases}$$

### The minimal number of moves for winning scoring play

When the players move blindly (ignoring winning) and just follow the classical minimal algorithm, we obtain the total scores according to the following two lemmas.

**Lemma 7.2.10.** *Given  $n \geq 1$  disks, three pegs and three weights of real numbers  $w_{12}$ ,  $w_{13}$ , and  $w_{23}$ . Then, for the two-player weighted Tower of Hanoi game of transferring  $n$  disks from Peg 1 to Peg 3 by the minimal algorithm, the total score is*

- $\Delta_{13}(n) = w_{13}$  if  $n$  is odd;
- $\Delta_{13}(n) = w_{12} + w_{23} - w_{13}$  if  $n$  is even.

**Lemma 7.2.11.** *Given  $n \geq 2$  disks, three pegs and three weights of real numbers  $w_{12}$ ,  $w_{13}$ , and  $w_{23}$ . Then, for the two-player weighted Tower of Hanoi game of transferring  $n$  disks from Peg 1 to Peg 1 by the minimal algorithm, if we suppose that the largest disk be moved, the total score is*

- $\Delta_{11}(n) = 3w_{23} - w_{12} - w_{13}$  if  $n$  is odd;
- $\Delta_{11}(n) = w_{12} + w_{13} - w_{23}$  if  $n$  is even.

We estimate the minimal number of moves for winning scoring Tower of Hanoi under the five ending conditions (EC1-5). We recall that the starting peg is Peg 1 and for (EC1) we suppose the tower is transferred to Peg 3.

**Theorem 7.2.12** (CLM [C11]). *Let  $M_3(n)$  denote the minimal number of moves needed for a winning game on three pegs and  $n \geq 1$  disks. Then, for  $n \leq 2$ ,*

- for (EC1):

$$M_3(1) = \begin{cases} 1 & \text{if } w_{13} \neq 0, \\ \infty & \text{otherwise,} \end{cases}$$

$$3 \leq M_3(2) \leq \begin{cases} 3 & \text{if } w_{12} + w_{23} > w_{13}, \\ 5 & \text{if } 3w_{13} > w_{12} + w_{23}, \\ \infty & \text{otherwise,} \end{cases}$$

- for (EC2,3):

$$M_3(1) = \infty,$$

$$7 \leq M_3(2) \leq \begin{cases} 7 & \text{if } w_{12} + w_{13} > w_{23}, \\ \infty & \text{otherwise,} \end{cases}$$

- for (EC4,5):

$$M_3(1) = \begin{cases} 1 & \text{if } \max\{w_{12}, w_{13}\} \neq 0, \\ \infty & \text{otherwise,} \end{cases}$$

$$3 \leq M_3(2) \leq \begin{cases} 3 & \text{if } w_{12} + w_{23} > w_{13} \text{ or } w_{13} + w_{23} > w_{12}, \\ 5 & \text{if } 3w_{13} > w_{12} + w_{23} \text{ or } 3w_{12} > w_{13} + w_{23}, \\ 7 & \text{if } w_{12} + w_{13} > w_{23}, \\ \infty & \text{otherwise.} \end{cases}$$

For  $n \geq 3$ , if  $w_{12} = w_{13} = w_{23} = \alpha$ , we have

$$M_3(n) = \begin{cases} 2^n - 1 & \text{if } \alpha > 0, \\ \infty & \text{if } \alpha \leq 0. \end{cases}$$

Otherwise, suppose that

$$\beta_1 = \begin{cases} w_{13} & \text{if } n \text{ odd,} \\ w_{12} + w_{23} - w_{13} & \text{if } n \text{ even,} \end{cases}$$

$$\beta_2 = \begin{cases} 3w_{23} - w_{12} - w_{13} & \text{if } n \text{ odd,} \\ w_{12} + w_{13} - w_{23} & \text{if } n \text{ even,} \end{cases}$$

$$\beta_3 = \begin{cases} \max\{w_{13}, w_{12}\} & \text{if } n \text{ odd,} \\ \max\{w_{12} + w_{23} - w_{13}, w_{13} + w_{23} - w_{12}\} & \text{if } n \text{ even,} \end{cases}$$

and  $\gamma = \max\{w_{ij} + w_{ik} - 2w_{jk} \mid \{i, j, k\} = \{1, 2, 3\}\}$ .

Then,

- for (EC1):

$$M_3(n) = 2^n - 1 \text{ if } \beta_1 > 0,$$

otherwise,

$$2^n \leq M_3(n) \leq 2^n + 15 + 16 \left\lfloor \frac{-\beta_1}{2\gamma} \right\rfloor,$$

except for  $n = 3$  with (i)  $\gamma = w_{13} + w_{23} - 2w_{12}$  and (ii)  $\gamma = w_{12} + w_{13} - 2w_{23}$ , where

$$8 \leq M_3(n) \leq \min \left\{ 27 + 16 \left\lfloor \frac{2(w_{12} + w_{23}) - 3w_{13}}{2\gamma} \right\rfloor, 29 + 16 \left\lfloor \frac{-w_{13}}{2\gamma} \right\rfloor \right\},$$

- for (EC2):

$$M_3(n) = 2^{n+1} - 1 \text{ if } \beta_2 > 0,$$

$$2^{n+1} \leq M_3(n) \leq 2^{n+1} + 15 + 16 \left\lfloor \frac{-\beta_2}{2\gamma} \right\rfloor \text{ otherwise,}$$

- for (EC3):

$$M_3(n) = \begin{cases} 7 & \text{if } w_{12} + w_{13} > w_{23}, \\ 15 & \text{if } w_{12} + w_{13} \leq w_{23} \text{ and } \beta_2 > 0, \end{cases}$$

$$16 \leq M_3(n) \leq 31 + 16 \left\lfloor \frac{-\beta_2}{2\gamma} \right\rfloor \text{ otherwise,}$$

- for (EC4):

$$M_3(n) = 2^n - 1 \text{ if } \beta_3 > 0,$$

otherwise,

$$2^n \leq M_3(n) \leq \min \left\{ 2^n + 15 + 16 \left\lfloor \frac{-\beta_3}{2\gamma} \right\rfloor, 2^{n+1} + 15 + 16 \left\lfloor \frac{-\beta_2}{2\gamma} \right\rfloor \right\},$$

except for (i)  $n = 3$  and  $\gamma = w_{13} + w_{23} - 2w_{12}$ , where

$$8 \leq M_3(3) \leq \min \left\{ 23 + 16 \left\lfloor \frac{-w_{12}}{2\gamma} \right\rfloor, 27 + 16 \left\lfloor \frac{2(w_{12} + w_{23}) - 3w_{13}}{2\gamma} \right\rfloor, 29 + 16 \left\lfloor \frac{-w_{13}}{2\gamma} \right\rfloor \right\},$$

$$31 + 16 \left\lfloor \frac{-\beta_2}{2\gamma} \right\rfloor \Bigg\},$$

for (ii)  $n = 3$  and  $\gamma = w_{12} + w_{23} - 2w_{13}$ , where

$$8 \leq M_3(3) \leq \min \left\{ 23 + 16 \left\lfloor \frac{-w_{13}}{2\gamma} \right\rfloor, 27 + 16 \left\lfloor \frac{2(w_{13} + w_{23}) - 3w_{12}}{2\gamma} \right\rfloor, 29 + 16 \left\lfloor \frac{-w_{12}}{2\gamma} \right\rfloor, \right. \\ \left. 31 + 16 \left\lfloor \frac{-\beta_2}{2\gamma} \right\rfloor \right\},$$

and for (iii)  $n = 3$  and  $\gamma = w_{12} + w_{13} - 2w_{23}$ , where

$$8 \leq M_3(3) \leq \min \left\{ 27 + 16 \left\lfloor \frac{2(w_{12} + w_{23}) - 3w_{13}}{2\gamma} \right\rfloor, 29 + 16 \left\lfloor \frac{-w_{13}}{2\gamma} \right\rfloor, \right. \\ \left. 27 + 16 \left\lfloor \frac{2(w_{13} + w_{23}) - 3w_{12}}{2\gamma} \right\rfloor, 29 + 16 \left\lfloor \frac{-w_{12}}{2\gamma} \right\rfloor, 31 + 16 \left\lfloor \frac{-\beta_2}{2\gamma} \right\rfloor \right\},$$

• for (EC5):

$$7 \leq M_3(n) \leq \begin{cases} 7 & \text{if } w_{12} + w_{13} > w_{23} \text{ or } (n = 3 \text{ and } \beta_3 > 0), \\ 15 & \text{if } (w_{12} + w_{13} \leq w_{23} \text{ and } \beta_2 > 0) \text{ or } (n = 4 \text{ and } \beta_3 > 0), \\ 2^n - 1 & \text{if } \beta_3 > 0, \end{cases}$$

otherwise,

$$8 \leq M_3(n) \leq \min \left\{ 31 + 16 \left\lfloor \frac{-\beta_2}{2\gamma} \right\rfloor, 2^n + 15 + 16 \left\lfloor \frac{-\beta_3}{2\gamma} \right\rfloor \right\},$$

except for the cases (i), (ii), and (iii) that are the same with (EC4).



# Chapter 8

## Ramsey theory

Problems in Ramsey theory typically ask a question of the form: how big must some substructure be to guarantee that a particular property holds? In this chapter, we are interested in a problem of Ramsey theory on numbers in the first section. It corresponds to the study of modular Schur numbers obtained in [C18]. In the second section, results obtained in [C15, C19] about Ramsey theory on graphs are exposed.

### 8.1 Modular Schur numbers

In [70], Guy proposed two unsolved problems in elementary number theory. The first one is the Schur's problem of partitioning integers into sum-free classes (Problem E12). Schur proved in [101] that if the set of the first  $\lceil k!e \rceil$  positive integers is partitioned into  $k$  parts any way, then  $x + y = z$  can be solved in integers within one part. Let  $S(k)$  be the largest integer  $n$  such that there exists a partition of the first  $n$  positive integers  $\{1, 2, \dots, n\}$  into  $k$  parts with no solution to the equation

$$x + y = z,$$

in any part. The exact value of  $S(k)$  is known only for  $k \in \{1, 2, 3, 4\}$  and the Problem E12 is to determine it for  $k \geq 5$ . The second unsolved problem proposed by Guy is a modular version of this Schur's problem (Problem E13). It was posed by Abbott and Wang in [21]. Let  $T(k)$  be the largest integer  $n$  such that there exists a partition of  $\{1, 2, \dots, n\}$  into  $k$  parts, with no solution to the congruence

$$x + y \equiv z \pmod{n + 1},$$

in any part. Abbott and Wang determined that  $T(k) = S(k)$  for  $k \in \{1, 2, 3, 4\}$ , and they conjectured that the equality is true for all positive integers  $k$ . The main purpose here is to study and explicitly determine modular generalizations of Schur numbers.

#### 8.1.1 Modular generalized Schur numbers

**Definition 8.1.1** (Partition). Let  $k$  be a positive integer and let  $S$  be a set of integers. A  $k$ -partition of  $S$  is a set  $P = \{S_1, \dots, S_k\}$  of subsets of  $S$  such that any element of  $S$  is contained in exactly one element of  $P$ .

**Definition 8.1.2** (Sum-free set). Let  $l$  be a positive integer. A set of integers is said to be  $l$ -sum-free if it contains no elements  $x_1, \dots, x_l, y$  satisfying

$$x_1 + \dots + x_l = y.$$

**Definition 8.1.3** (Generalized Schur number). For every positive integer  $k$ , the *generalized Schur number*  $S(k, l)$  is the largest integer  $n$  for which the set of the first  $n$  positive integers  $\{1, 2, \dots, n\}$  admits a  $k$ -partition into  $l$ -sum-free sets.

For  $l = 2$ , the numbers  $S(k, 2) = S(k)$  are known as *Schur numbers*. They have been introduced by Schur himself in 1916 [101], in order to study a modular version of Fermat's Last Theorem. He proved that those numbers are always finite, for every positive integer  $k$ . The first few Schur numbers are given in Table 8.1.

$k$	1	2	3	4	5	6	7
$S(k, 2)$	1	4	13	44	$160 \leq \dots \leq 305$	$\geq 536$	$\geq 1680$

Table 8.1: The first few Schur numbers  $S(k, 2)$

The exact value of  $S(4, 2)$  was obtained by Baumert [20]. The lower and upper bounds of  $S(5, 2)$  are due to Exoo [63] and Sanz [99], respectively. Finally, the lower bounds of  $S(6, 2)$  and  $S(7, 2)$  were obtained by Fredricksen and Sweet [67] by considering symmetric sum-free partitions.

Many generalizations of Schur numbers have appeared since their introduction. Here, the generalized Schur numbers that we consider are similarly defined in [33, 82]. These numbers are always finite (see [93, 82] for instance).

**Definition 8.1.4** (Modular sum-free set). Let  $m$  be a positive integer. A set of integers is said to be  *$l$ -sum-free modulo  $m$*  if it contains no elements  $x_1, \dots, x_l, y$  satisfying

$$x_1 + \dots + x_l \equiv y \pmod{m}.$$

**Definition 8.1.5** (Modular generalized Schur number). For every positive integer  $k$ , the *generalized Schur number modulo  $m$* , denoted by  $S_m(k, l)$ , is the largest integer  $n$  for which the set of the first  $n$  positive integers  $\{1, 2, \dots, n\}$  admits a  $k$ -partition into  $l$ -sum-free sets modulo  $m$ .

Obviously, for every modulus  $m$ , the inequality

$$S_m(k, l) \leq S(k, l) \tag{8.1}$$

holds because a  $l$ -sum-free set modulo  $m$  of integers is also  $l$ -sum-free. Moreover, since  $m + \dots + m \equiv m \pmod{m}$ , a  $l$ -sum-free set of integers modulo  $m$  does not contain the integer  $m$ . Therefore, we have

$$S_m(k, l) \leq m - 1. \tag{8.2}$$

For  $l = 2$ , Abbott and Wang investigated in [21] the numbers

$$T(k) = \max \{n \in \mathbb{N} \mid S_{n+1}(k, 2) = n\},$$

where  $k$  is a positive integer. They obtained that  $T(k) = S(k, 2)$  for  $k \in \{1, 2, 3, 4\}$  and they conjectured that the equality is true for all positive integers  $k$ .

Here, we explicitly determine the modular generalized Schur numbers  $S_m(k, l)$  for small values of  $m$ : for all moduli  $m \in \{1, 2, 3\}$ . For  $m = 1$ , the result is clear. Indeed,

$$S_1(k, l) = 0,$$

for all  $k \geq 1$  and  $l \geq 1$ , since every positive integer  $x$  verifies  $x + \dots + x \equiv x \pmod{1}$  and thus, there does not exist non-empty  $l$ -sum-free set modulo 1. For  $m = 2$  and  $m = 3$ , the exact values of  $S_m(k, l)$  are given by the following theorems.

**Theorem 8.1.6** (CRS [C18]). *Let  $k$  and  $l$  be two positive integers. Then,*

$$S_2(k, l) = \begin{cases} 0 & \text{for } l \text{ odd,} \\ 1 & \text{for } l \text{ even.} \end{cases}$$

**Theorem 8.1.7** (CRS [C18]). *Let  $k$  and  $l$  be two positive integers. Then,*

$$S_3(k, l) = \begin{cases} 0 & \text{for } k \geq 1 \text{ and } l \equiv 1 \pmod{3}, \\ 1 & \text{for } k = 1 \text{ and } l \equiv 0, 2 \pmod{3}, \\ 2 & \text{for } k \geq 2 \text{ and } l \equiv 0, 2 \pmod{3}. \end{cases}$$

A simple proof of these theorems can be found in [C18].

### 8.1.2 Modular generalized weak Schur numbers

**Definition 8.1.8** (Weakly sum-free set). A set of integers is said to be *weakly  $l$ -sum-free* if it contains no pairwise distinct elements  $x_1, \dots, x_l, y$  satisfying

$$x_1 + \dots + x_l = y.$$

**Definition 8.1.9** (Modular generalized weak Schur number). For every positive integer  $k$ , the *generalized weak Schur number*  $WS(k, l)$  is the largest integer  $n$  for which the set of the first  $n$  positive integers  $\{1, 2, \dots, n\}$  admits a  $k$ -partition into weakly  $l$ -sum-free sets.

For  $l = 2$ , the numbers  $WS(k, 2)$  are called *weak Schur numbers*. The first few weak Schur numbers are given in Table 8.2.

$k$	1	2	3	4	5	6
$WS(k, 2)$	2	8	23	66	$\geq 196$	$\geq 575$

Table 8.2: The first few weak Schur numbers  $WS(k, 2)$

The exact value of  $WS(4, 2)$  was obtained by Blanchard, Harary and Reis [35]. The lower bounds of  $WS(5, 2)$  and  $WS(6, 2)$  are due to Eliahou, Marín, Revuelta and Sanz [58].

More generally, the generalized weak Schur numbers are always finite (see [103, 77, 82] for instance). Moreover, the generalized weak Schur numbers appear as a good upper bound for the generalized Schur numbers, since a weakly  $l$ -sum-free set of integers is also  $l$ -sum-free. Therefore, we have

$$S(k, l) \leq WS(k, l), \quad (8.3)$$

for all positive integers  $k$  and  $l$ . A trivial lower bound for the weak Schur numbers is

$$kl \leq WS(k, l), \quad (8.4)$$

because each of the  $k$  weakly sum-free sets can contain  $l$  distinct integers without solution of the equation  $x_1 + \dots + x_l = y$ . Better lower bounds for  $WS(k, l)$  can be found in [99].

**Definition 8.1.10** (Modular weakly sum-free set). A set of integers is said to be *weakly  $l$ -sum-free modulo  $m$*  if it contains no pairwise distinct elements  $x_1, \dots, x_l, y$  satisfying

$$x_1 + \dots + x_l \equiv y \pmod{m}.$$

**Definition 8.1.11** (Modular generalized weak Schur number). For every positive integer  $k$ , the *generalized weak Schur number modulo  $m$* , denoted by  $WS_m(k, l)$ , is the largest integer  $n$  for which the set of the first  $n$  positive integers  $\{1, 2, \dots, n\}$  admits a  $k$ -partition into weakly  $l$ -sum-free sets modulo  $m$ .

For every modulus  $m$ , the inequality

$$WS_m(k, l) \leq WS(k, l) \tag{8.5}$$

holds because a weakly  $l$ -sum-free set modulo  $m$  of integers is also weakly  $l$ -sum-free.

Abbott and Wang conjectured that  $T(k)$  is equal to  $S(k, 2)$ , for all positive integers  $k$ . Here, in the weak case, it appears that considering similar numbers as  $T(k)$  is without great interest. Indeed, as we can see in Table 8.3 for  $m \in \{1, \dots, 15\}$ , the values of modular generalized weak Schur numbers, for  $k = 2$  and  $l = 2$ , seem to be very difficult to predict. For  $m \geq 16$ , we have  $WS_m(2, 2) = WS(2, 2)$  because  $WS_m(2, 2) \leq WS(2, 2) = 8$  and because, for two distinct integers  $x, y \in \{1, \dots, 8\}$ , we always have  $x + y \leq 15 < m$ .

$m$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$WS_m(2, 2)$	4	5	4	5	6	6	7	6	7	7	7	8	8	8	8

Table 8.3: The modular generalized weak Schur numbers  $WS_m(2, 2)$

Here, we explicitly determine the modular generalized weak Schur numbers  $WS_m(k, l)$  for small values of  $m$ : for all moduli  $m \in \{1, 2, 3\}$ . For  $m = 1$ , we obtain that

$$WS_1(k, l) = kl,$$

for all  $k \geq 1$  and  $l \geq 1$ , since a weakly  $l$ -sum-free set modulo 1 has cardinality of at most  $l$  because every  $l + 1$  distinct positive integers  $x_1, \dots, x_l, y$  verify  $x_1 + \dots + x_l \equiv y \pmod 1$ . For  $m = 2$  and  $m = 3$ , the following theorems are proved in [C18].

**Theorem 8.1.12** (CRS [C18]). *Let  $k$  and  $l$  be two positive integers. Then,*

$$WS_2(k, l) = \begin{cases} l + 1 & \text{for } k = 1 \text{ and } l \equiv 0, 1 \pmod 4, \\ l & \text{for } k = 1 \text{ and } l \equiv 2, 3 \pmod 4, \\ 2(k - 1)l + 1 & \text{for } k \geq 2 \text{ and } l \text{ even,} \\ k(l + 1) & \text{for } \begin{cases} k \geq 2 \text{ and } l \equiv 1 \pmod 4, \\ k \geq 2 \text{ even and } l \equiv 3 \pmod 4, \end{cases} \\ k(l + 1) - 1 & \text{for } k \geq 3 \text{ odd and } l \equiv 3 \pmod 4. \end{cases}$$

**Theorem 8.1.13** (CRS [C18]). *Let  $k$  and  $l$  be two positive integers. Then,*

$$WS_3(k, l) = \begin{cases} 3k & \text{for } k \geq 1 \text{ and } l = 1, \\ l & \text{for } k = 1 \text{ and } l \geq 2, \\ 2l + 2 & \text{for } k = 2 \text{ and } l \geq 2, l \equiv 0, 1, 5 \pmod 9, \\ 2l + 1 & \text{for } k = 2 \text{ and } \begin{cases} l = 3, \\ l \geq 5, l \equiv 2, 3, 4, 6, 7, 8 \pmod 9, \end{cases} \\ 2l & \text{for } k = 2 \text{ and } l \in \{2, 4\}, \\ 3(k - 2)l + 2 & \text{for } k \geq 3 \text{ and } l \equiv 0, 2 \pmod 3, \\ k(l + 1) & \text{for } \begin{cases} k = 3, k \geq 5 \text{ and } l \geq 2, l \equiv 1 \pmod 3, \\ k = 4 \text{ and } l \geq 2, l \equiv 1, 7 \pmod 9, \end{cases} \\ 4l + 3 & \text{for } k = 4 \text{ and } l \equiv 4 \pmod 9. \end{cases}$$

## 8.2 Ramsey numbers of certain families of graphs

**Definition 8.2.1** (Ramsey number). Let  $G$  and  $H$  be two graphs. Let  $R(G, H)$  be the smallest integer  $N$  such that for any 2-coloring (say, red and blue) of the edges of the complete graph  $K_n$ , with  $n \geq N$ , there is either a red copy of  $G$  or a blue copy of  $H$ .

It is known, by Ramsey Theorem, that  $R(G, H)$  is always finite, for any graphs  $G$  and  $H$ . For instance, the first few Ramsey numbers  $R(K_a, K_b)$  are given in Table 8.4.

$a \backslash b$	1	2	3	4	5	6	7
1	1	1	1	1	1	1	1
2		2	3	4	5	6	7
3			6	9	14	18	23
4				18	25	$36 \leq \dots \leq 41$	$49 \leq \dots \leq 61$
5					$43 \leq \dots \leq 48$	$58 \leq \dots \leq 87$	$80 \leq \dots \leq 143$
6						$102 \leq \dots \leq 165$	$115 \leq \dots \leq 298$
7							$205 \leq \dots \leq 540$

Table 8.4: The first few Ramsey numbers  $R(K_a, K_b)$

In this section, we present Ramsey numbers for complete graphs with dropped cliques and for complete graphs with dropped stars that appears in [C19] and [C15], respectively. An excellent survey on Ramsey numbers for small values can be found in [94].

### 8.2.1 Ramsey for complete graphs with dropped edge or triangle

**Definition 8.2.2.** Let  $k \geq t \geq 1$  be positive integers. We denote by  $K_{[k,t]}$  the complete graph on  $k$  vertices from which a set of edges, induced by a clique of order  $t$ , has been dropped.

For instance, the graphs  $K_{[5,3]}$  and  $K_{[4,2]}$  are depicted in Figure 8.1.

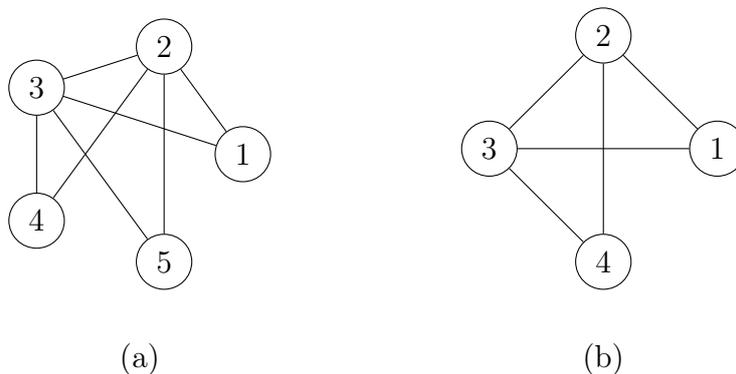


Figure 8.1: (a)  $K_{[5,3]}$  and (b)  $K_{[4,2]}$

#### A recursive formula

First, a generalization of the Ramsey number for  $r$  colors can be considered.

**Definition 8.2.3** (Generalized Ramsey number). Let  $k_1, \dots, k_r$  and  $t_1, \dots, t_r$  be positive integers with  $k_i \geq t_i$  for all  $i \in \{1, \dots, r\}$ . Let  $R([k_1, t_1], \dots, [k_r, t_r])$  be the smallest integer  $n$  such that for any  $r$ -edge coloring of  $K_n$  there always occurs a monochromatic  $K_{[k_i, t_i]}$  for some  $i$ .

In the case when  $k_i = t_i$  for some  $i$ , we set

$$R([k_1, t_1], \dots, [k_{i-1}, t_{i-1}], [t_i, t_i], [k_{i+1}, t_{i+1}], \dots, [k_r, t_r]) \leq t_i.$$

We note that equality is reached at  $\min_{1 \leq i \leq r} \{t_i \mid t_i = k_i\}$ , since the set of all the edges of  $K_{[t_i, t_i]}$  (which is empty) can always be colored with color  $i$ . We also notice that the case  $R([k_1, 1], \dots, [k_r, 1])$  is exactly the classical Ramsey number  $R(k_1, \dots, k_r)$  (the smallest integer  $n$  such that for any  $r$ -edge coloring of  $K_n$  there always occurs a monochromatic  $K_{k_i}$  for some  $i$ ).

The following recursive inequality is classical in Ramsey theory

$$R(k_1, k_2, \dots, k_r) \leq R(k_1 - 1, k_2, \dots, k_r) + R(k_1, k_2 - 1, \dots, k_r) + \dots + R(k_1, k_2, \dots, k_r - 1) - (r - 2) \tag{8.6}$$

In the same spirit, we have the following generalization.

**Lemma 8.2.4** (CMR [C19]). Let  $r \geq 2$  and let  $k_1, \dots, k_r$  and  $t_1, \dots, t_r$  be positive integers with  $k_i \geq t_i + 1 \geq 2$  for all  $i$ . Then,

$$R([k_1, t_1], \dots, [k_r, t_r]) \leq R([k_1 - 1, t_1], [k_2, t_2], \dots, [k_r, t_r]) + R([k_1, t_1], [k_2 - 1, t_2], \dots, [k_r, t_r]) + \dots + R([k_1, t_1], [k_2, t_2], \dots, [k_r - 1, t_r]) - (r - 2).$$

Lemma 8.2.4 allows us to give the following upper bound for  $R([k_1, t_1], \dots, [k_r, t_r])$ .

**Theorem 8.2.5** (CMR [C19]). Let  $r \geq 2$  be a positive integer and let  $k_1, \dots, k_r$  and  $t_1, \dots, t_r$  be positive integers such that  $k_i \geq t_i$  for all  $i \in \{1, \dots, r\}$ . Then,

$$R([k_1, t_1], \dots, [k_r, t_r]) \leq \max_{1 \leq i \leq r} \{t_i\} \binom{k_1 + \dots + k_r - (t_1 + \dots + t_r)}{k_1 - t_1, k_2 - t_2, \dots, k_r - t_r}$$

where  $\binom{n_1 + n_2 + \dots + n_r}{n_1, n_2, \dots, n_r}$  is the multinomial coefficient defined by  $\binom{n_1 + n_2 + \dots + n_r}{n_1, n_2, \dots, n_r} = \frac{(n_1 + \dots + n_r)!}{n_1! n_2! \dots n_r!}$ , for all nonnegative integers  $n_1, \dots, n_r$ .

Theorem 8.2.5 can be considered as a natural generalization of the well-known explicit upper bound for classical Ramsey numbers. Indeed, an immediate consequence of Theorem 8.2.5 (by taking  $t_i = 1$  for all  $i$ ) is the following classical upper bound due to Graham and Rödl [69, (2.48)].

$$R([k_1, 1], \dots, [k_r, 1]) \leq \binom{k_1 + \dots + k_r - r}{k_1 - 1, \dots, k_r - 1}. \tag{8.7}$$

Let  $k \geq t \geq 2$  and  $r \geq 2$  be integers and let  $R_r([k, t]) = R(\underbrace{[k, t], \dots, [k, t]}_r)$ .

Another straight consequence of Theorem 8.2.5 (by taking  $k = k_1 = \dots = k_r$  and  $t = t_1 = \dots = t_r$ ) is the following inequality

$$R_r([k, t]) \leq t \binom{r(k-t)}{k-t, \dots, k-t} \tag{8.8}$$

Moreover, if  $t = 1$  then

$$R_r([k, 1]) \leq \frac{(rk-r)!}{((k-1)!)^r}. \tag{8.9}$$

**New Ramsey numbers obtained**

Using the recursive formula of Lemma 8.2.4 and other Ramsey numbers already known, the following theorems are proved in [C19].

**Theorem 8.2.6** (CMR [C19]). *Let  $n \geq 2$  be an integer. Then,*

$$R([n, 2], [4, 3]) = \begin{cases} 2 & \text{for } n = 2, \\ 5 & \text{for } n = 3, \\ 3n - 5 & \text{for } n \geq 4. \end{cases}$$

**Theorem 8.2.7** (CMR [C19]). *Let  $n \geq 2$  be an integer. Then,*

$$R([n, 3], [4, 3]) = \begin{cases} 3 & \text{for } n = 3, \\ 6 & \text{for } n = 4, \\ 8 & \text{for } n = 5, \\ 11 & \text{for } n = 6, \\ 3n - 8 & \text{for } n \geq 7. \end{cases}$$

**Theorem 8.2.8** (CMR [C19]). *Let  $n \geq 2$  be an integer. Then,*

$$R([n, 2], [5, 3]) \begin{cases} = 2 & \text{for } n = 2, \\ = 7 & \text{for } n = 3, \\ \leq 3 \binom{n+1}{2} - 5n + 4 & \text{for } n \geq 4. \end{cases}$$

**8.2.2 Ramsey for complete graphs with dropped stars**

**Definition 8.2.9** (Complete graph with dropped star). Let  $K_n - K_{1,s}$  be the complete graph on  $n$  vertices from which the edges of the star graph  $K_{1,s}$  are dropped. Notice that  $K_n - K_{1,1} = K_n - e$  is the complete graph on  $n$  vertices from which an edge is dropped and  $K_n - K_{1,2} = K_n - P_3$  is the complete graph on  $n$  vertices from which a path on three vertices is dropped.

Here, we investigate  $R(K_m - e, K_n - K_{1,s})$  and  $R(K_m, K_n - K_{1,s})$  for a variety of integers  $m, n$  and  $s$ .

**Main result**

Let  $G$  be a graph and denote by  $G^v$  the graph obtained from  $G$  to which a new vertex  $v$ , incident to all the vertices of  $G$ , is added. The main result of this subsection is the following

**Theorem 8.2.10** (CMR [C15]). *Let  $n$  and  $s$  be positive integers. Let  $G_1$  be any graph and let  $N$  be an integer such that  $N \geq R(G_1^v, K_n)$ . If  $\left\lceil \frac{(s+1)(N-n)}{n} \right\rceil \geq R(G_1, K_{n+1} - K_{1,s})$ , then  $R(G_1^v, K_{n+1} - K_{1,s}) \leq N$ .*

Using Theorem 8.2.10, new exact values or bounds for Ramsey numbers of certain complete graphs with dropped stars are obtained.

**Some exact values for  $R(K_m - e, K_n - K_{1,s})$**

**Corollary 8.2.11** (CMR [C15]).  $R(K_3 - e, K_{m+1} - K_{1,s}) = 2m - 1$ , for all  $s \in \{1, \dots, m - 1\}$ .

**Corollary 8.2.12** (CMR [C15]).

i)  $R(K_4 - e, K_5 - K_{1,3}) = 11$ .

ii)  $R(K_4 - e, K_6 - K_{1,s}) = 16$  for any  $3 \leq s \leq 4$ .

iii)  $R(K_4 - e, K_7 - K_{1,s}) = 21$  for any  $4 \leq s \leq 5$ .

**Corollary 8.2.13** (CMR [C15]).  $21 \leq R(K_4 - e, K_7 - K_{1,3}) \leq 22$ .

**Corollary 8.2.14** (CMR [C15]).  $R(K_5 - e, K_5 - K_{1,3}) = 19$ .

**Corollary 8.2.15** (CMR [C15]).  $R(K_5 - e, K_6 - K_{1,s}) = R(K_5 - e, K_5)$  for  $s = 3, 4$ .

**New upper bounds for  $R(K_m, K_n - P_3)$**

Let us first notice that, by taking  $G_1 = K_m$  in Theorem 8.2.10, we obtain

**Corollary 8.2.16** (CMR [C15]). Let  $N$  be an integer such that  $N \geq R(K_{m+1}, K_n)$ . If  $\left\lceil \frac{(s+1)(N-n)}{n} \right\rceil \geq R(K_m, K_{n+1} - K_{1,s})$ , then  $R(K_{m+1}, K_{n+1} - K_{1,s}) \leq N$ .

Using this result, we obtain

**Corollary 8.2.17** (CMR [C15]).

i)  $R(K_6, K_4 - P_3) = R(K_6, K_3) = 18$  with  $m = 5$  and  $n = 3$ ,

ii)  $R(K_7, K_4 - P_3) = R(K_7, K_3) = 23$  with  $m = 6$  and  $n = 3$ ,

iii)  $R(K_8, K_4 - P_3) = R(K_8, K_3) = 28$  with  $m = 7$  and  $n = 3$ ,

iv)  $R(K_9, K_4 - P_3) = R(K_9, K_3) = 36$  with  $m = 8$  and  $n = 3$ ,

v)  $R(K_{10}, K_4 - P_3) = R(K_{10}, K_3) \leq 43$  with  $m = 9$  and  $n = 3$ .

**Corollary 8.2.18** (CMR [C15]).

i)  $R(K_6, K_5 - P_3) \leq 41$ .

ii)  $R(K_7, K_5 - P_3) \leq 61$ .

iii)  $R(K_8, K_5 - P_3) \leq 85$ .

iv)  $R(K_9, K_5 - P_3) \leq 117$ .

v)  $R(K_{10}, K_5 - P_3) \leq 159$ .

**Corollary 8.2.19** (CMR [C15]). If  $37 \leq R(K_6, K_4)$ , then  $R(K_6, K_5 - P_3) = R(K_6, K_4)$ .

**Corollary 8.2.20** (CMR [C15]).

i)  $25 \leq R(K_4, K_6 - P_3) \leq 27$ .

ii)  $R(K_5, K_6 - P_3) \leq 49$ .

iii)  $R(K_6, K_6 - P_3) \leq 87$ .

**Corollary 8.2.21** (CMR [C15]). *For each  $3 \leq m \leq 5$  and each  $7 \leq n \leq 16$ , we have that  $R(K_m, K_n - P_3) \leq u(m, n)$ , where the value of  $u(m, n)$  is given in the  $(m, n)$  entry of the below table (the value between parentheses is the best previously known upper bound).*

$m \setminus n$	7	8	9	10	11	12	13	14	15	16
3					44(47)	52(59)	61(72)	70(86)	80(101)	91(117)
4	41(49)	61(72)		115(136)	154(183)	199(242)	253(319)	313(405)	383(506)	466(623)
5	87(105)	143(177)	222(277)							

**Some bounds for  $R(K_m, K_n - K_{1,s})$  when  $s \geq 3$**

**Corollary 8.2.22** (CMR [C15]). *For each  $6 \leq m \leq 15$ , we have that  $R(K_m, K_6 - K_{1,3}) \leq u(m)$ , where the value of  $u(m)$  is given in the below table (the value between parentheses is the best previously known upper bound).*

$m$	6	7	8	9	10	11	12	13	14	15
$b_u$	87(90)	143(151)	216(235)	316(350)	442(499)	633(690)	848(928)	1139(1219)	1461(1568)	1878(1568)

**Corollary 8.2.23** (CMR [C15]). *For each  $4 \leq m \leq 11$ , we have that  $R(K_m, K_7 - K_{1,3}) \leq u(m)$ , where the value of  $u(m)$  is given in the below table (the value between parentheses is the best previously known upper bound).*

$m$	4	5	6	7	8	9	10	11
$b_u$	41(43)	87(90)	165(180)	298(331)	495(566)	780(916)	1175(1415)	1804(2105)

**More equalities**

**Corollary 8.2.24** (CMR [C15]).

- i)  $R(K_4, K_7 - K_{1,s}) = R(K_4, K_6)$  for  $s \geq 3$ .
- ii)  $R(K_4, K_8 - K_{1,s}) = R(K_4, K_7)$  for  $s \geq 3$ .
- iii)  $R(K_4, K_9 - K_{1,s}) = R(K_4, K_8)$  for  $s \geq 4$ .
- iv)  $R(K_4, K_{10} - K_{1,s}) = R(K_4, K_9)$  for  $s \geq 4$ .
- v)  $R(K_4, K_{11} - K_{1,s}) = R(K_4, K_{10})$  for  $s \geq 5$ .
- vi)  $R(K_4, K_{12} - K_{1,s}) = R(K_4, K_{11})$  for  $s \geq 6$ .
- vii)  $R(K_4, K_{13} - K_{1,s}) = R(K_4, K_{12})$  for  $s \geq 6$ .
- viii)  $R(K_4, K_{14} - K_{1,s}) = R(K_4, K_{13})$  for  $s \geq 7$ .
- ix)  $R(K_4, K_{15} - K_{1,s}) = R(K_4, K_{14})$  for  $s \geq 8$ .
- x)  $R(K_4, K_{16} - K_{1,s}) = R(K_4, K_{15})$  for  $s \geq 9$ .

**Wheels versus  $K_n - K_{1,s}$**

Here, we obtain further relating results by applying Theorem 8.2.10 to other graphs. Indeed, we may consider  $G_1$  as the cycle on  $n - 1$  vertices  $C_{n-1}$ , and thus  $G_1^v$  will be the wheel  $W_n$  by taking the new vertex  $v$  incident to all the vertices of  $C_{n-1}$ .

**Corollary 8.2.25** (CMR [C15]).

- i)  $R(W_5, K_6 - K_{1,s}) = 27$  for  $s = 3, 4, 5$ .
- ii)  $R(W_5, K_7 - K_{1,s}) = R(W_5, K_6)$  for  $s = 4, 5, 6$ .
- iii)  $R(W_5, K_8 - K_{1,s}) = R(W_5, K_7)$  for  $s = 4, 5, 6, 7$ .



# Chapter 9

## Kneser transversals

### 9.1 Complete Kneser transversals

**Definition 9.1.1** (Function  $m$ ). Let  $k, d, \lambda$  be integers with  $d, k \geq \lambda \geq 1$ . Consider the function  $m(k, d, \lambda)$ , defined to be the maximum positive integer  $n$  such that every set of  $n$  points (not necessarily in general position) in  $\mathbb{R}^d$  has the property that the convex hulls of all  $k$ -sets have a common transversal  $(d - \lambda)$ -plane.

In [23], the following inequalities were obtained

$$d - \lambda + k + \left\lceil \frac{k}{\lambda} \right\rceil - 1 \leq m(k, d, \lambda) < d + 2(k - \lambda) + 1. \quad (9.1)$$

An interesting feature of the value of  $m(k, d, \lambda)$  is its strong connection with the chromatic number of Kneser hypergraphs [80, 84] as well as with the Rado's centerpoint theorem [92]. Indeed, for the former it is proved in [23] that

$$\text{if } m(k, d, \lambda) < n, \text{ then } d - \lambda + 1 < \chi(KG^{\lambda+1}(n, k)).$$

For the latter, recall that the well-known Rado's centerpoint theorem [92] states that if  $X$  is a bounded measurable set in  $\mathbb{R}^d$  then there exists a point  $x \in \mathbb{R}^d$  such that

$$\text{measure}(P \cap X) \geq \frac{\text{measure}(X)}{d + 1}$$

for each half-space  $P$  that contains  $x$  (see also [91] for the case  $d = 2$ ).

Independently Bukh and Matousek [40, Section 6] and Arocha, Bracho, Montejano and Ramírez-Alfonsín in [23] consider the following generalization of a discrete version of Rado's centerpoint theorem. Let  $n, d, \lambda \geq 1$  be integers with  $d \geq \lambda$  and let

$\tau(n, d, \lambda) \stackrel{\text{def}}{=} \text{the maximum positive integer } \tau \text{ such that for any collection } X \text{ of } n \text{ points in } \mathbb{R}^d, \text{ there is a } (d - \lambda)\text{-plane } L_X \text{ such that any closed half-space } H \text{ through } L_X \text{ contains at least } \tau \text{ points.}$

By the hyperplane separation theorem we have that  $n - \tau(n, d, \lambda) + 1$  is equal to the minimum positive integer  $k$  such that for any collection  $X$  of  $n$  points in  $\mathbb{R}^d$  there is a common transversal  $(d - \lambda)$ -plane to the convex hulls of all  $k$ -sets, which is essentially  $m(k, d, \lambda)$ . Therefore, any improvement to the lower or upper bounds for  $m(k, d, \lambda)$  will give important insight on the above interesting problem.

The purpose here is to introduce and study a discrete version of the function  $m(k, d, \lambda)$  which allows us to improve lower or upper bounds for  $m(k, d, \lambda)$ .

**Definition 9.1.2** (Kneser transversal). Let  $k, d, \lambda$  be integers with  $k, d \geq \lambda \geq 1$  and let  $X \subset \mathbb{R}^d$  be a finite set. We call  $L$  a *Kneser transversal* of  $X$  if it is a  $(d - \lambda)$ -plane transversal to the convex hulls of all  $k$ -sets of  $X$ . If in addition  $L$  contains  $(d - \lambda) + 1$  points of  $X$ , then  $L$  is a *complete Kneser  $(d - \lambda)$ -transversal*.

**Definition 9.1.3** (Function  $m^*$ ). Let  $k, d, \lambda$  be integers with  $k, d \geq \lambda \geq 1$ . Let  $m^*(k, d, \lambda)$  be the maximum positive integer  $n$  such that every set of  $n$  points (not necessarily in general position) in  $\mathbb{R}^d$  has a complete Kneser  $(d - \lambda)$ -transversal to the convex hulls of its  $k$ -sets.

This is a natural discrete version of the original function  $m$ . Indeed, consider a set of points  $X$  in  $\mathbb{R}^d$ . The existence of an arbitrary  $(d - \lambda)$ -plane transversal to the convex hull of the  $k$ -sets of  $X$  is not an invariant of the order type. For example, if  $d = 2$  and  $X$  is the vertex set of a regular hexagon then the center is a 0-plane transversal to the convex hull of the 4-sets. But by suitably perturbing these 6 points slightly we lose this property, see Figure 9.1.

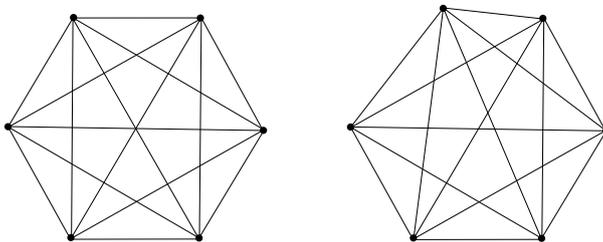


Figure 9.1: A 0-plane transversal that is not invariant of the order type

On the other hand, the existence of a complete Kneser  $(d - \lambda)$ -transversal to the convex hull of the  $k$ -sets is an invariant of the order type. This allows us to study  $m^*$  using oriented matroid theory. Since the parameter  $m^*$  requires additional conditions on the transversals, we clearly have

$$m^*(k, d, \lambda) \leq m(k, d, \lambda).$$

The case  $k = \lambda$  is easy to deal with.

**Proposition 9.1.4** (CMMMR [C12]). *The value of  $m^*(k, d, k)$  is  $d$ .*

*Proof.* If we have  $d$  points or less in  $\mathbb{R}^d$ , then we can choose any subset  $T$  with  $d - k + 1$  elements, and it will have non-empty intersection with any  $k$ -set. Therefore,  $\text{aff}(T)$  will be a Kneser transversal. On the other hand, if we choose  $d + 1$  affinely independent points in  $\mathbb{R}^d$ , then any  $(d - k + 1)$ -set  $T$  will leave  $k$  points in its complement, and therefore by affine independence  $\text{aff}(T)$  cannot be a Kneser transversal.  $\square$

From here on, we will assume that  $k \geq \lambda + 1$ . It turns out that the function  $m^*$  has two different behaviors. The arguments for the case  $\lambda - 1 \geq \lceil \frac{d}{2} \rceil$ , are usually simpler than those for the case  $\lambda - 1 < \lceil \frac{d}{2} \rceil$ . For this reason, we define

$$\alpha(d, \lambda) = \frac{\lambda - 1}{\lceil \frac{d}{2} \rceil}$$

and we call  $\alpha \geq 1$  the trivial range and  $\alpha < 1$  the non-trivial range. In this section, we investigate  $m^*$ . We present the exact value of  $m^*$  in the trivial range and give bounds and some exact values of  $m^*$  in the non-trivial range.

### 9.1.1 Kneser transversals from Radon partitions

Here, we provide tools from convex geometry to detect complete Kneser transversals using Radon partitions.

Let  $d$  be a positive integer. Consider  $d + 2$  points  $v_1, v_2, \dots, v_{d+2}$  in general position in  $\mathbb{R}^d$ . Radon's theorem states that there exists a unique partition  $\{1, 2, \dots, d + 2\} = A \cup B$  such that

$$\text{Conv} \left( \bigcup_{i \in A} v_i \right) \cap \text{Conv} \left( \bigcup_{i \in B} v_i \right) \neq \emptyset.$$

Moreover, Radon's theorem states that this intersection is a unique point in the interior of each convex hull.

The following proposition is a generalization of the well-known Carathéodory's theorem that states that if a point  $p$  lies in the convex hull of a set  $S$  in  $\mathbb{R}^d$ , then there is a subset  $S'$  of  $S$  consisting of at most  $d + 1$  points such that  $p$  lies in the convex hull of  $S'$ . It is not difficult to prove that the set  $S'$  has exactly  $d + 1$  points if the set  $S$  is in general position in  $\mathbb{R}^d$ .

**Proposition 9.1.5** (CMMMR [C12]). *Let  $d$  and  $\lambda$  be positive integers with  $d \geq \lambda$  and let  $S$  and  $T$  be two disjoint sets of points in general position in  $\mathbb{R}^d$  such that  $|S| \geq \lambda + 1$  and  $|T| = d - \lambda + 1$ . Then the following two statements are equivalent:*

- $\text{Conv}(S) \cap \text{aff}(T) \neq \emptyset$ ,
- $\text{Conv}(S') \cap \text{aff}(T) \neq \emptyset$  for a subset  $S' \subseteq S$  such that  $|S'| = \lambda + 1$ .

The following result will be very useful in the sequel.

**Proposition 9.1.6** (CMMMR [C12]). *Let  $S$  and  $T$  be two disjoint and non-empty sets of points in  $\mathbb{R}^d$  such that  $|S| + |T| = d + 2$  and  $S \cup T$  is in general position. Then  $\text{Conv}(S) \cap \text{aff}(T) \neq \emptyset$  if and only if all the points of  $S$  are in the same set in the Radon partition of  $S \cup T$ .*

*Remark 9.1.7.* In the case in which  $S \cup T$  is not in general position then we still have a Radon partition, but it might not be unique. If in one of those Radon partitions we have that all the points from  $S$  belong to the same set, then the proof above shows that we also conclude that  $\text{Conv}(S) \cap \text{aff}(T) \neq \emptyset$ .

As a consequence of the above proposition and remark we get the following lemma.

**Lemma 9.1.8** (CMMMR [C12]). *Let  $X$  be any set of  $d + 2$  distinct points in  $\mathbb{R}^d$  and let  $\lfloor \frac{d+2}{2} \rfloor \leq t \leq d + 1$ . Then  $X$  can be partitioned into two disjoint sets  $S$  and  $T$  such that  $|T| = t$  and  $\text{Conv}(S) \cap \text{aff}(T) \neq \emptyset$ .*

Using this lemma, we obtain the following lower bound in the non-trivial range.

**Theorem 9.1.9** (CMMMR [C12]). *In the non-trivial range, when  $\alpha(d, \lambda) < 1$ , we have that*

$$(d - \lambda + 1) + k \leq m^*(k, d, \lambda).$$

*Proof.* Let  $X$  be a collection of  $(d - \lambda + 1) + k$  points in  $\mathbb{R}^d$ . Since  $k \geq \lambda + 1$ , then  $|X| \geq d + 2$ . Let  $Y$  be a  $(d + 2)$ -subset of  $X$ .

Since we are in the case  $\alpha(d, \lambda) < 1$ , we have that  $\lfloor \frac{d+2}{2} \rfloor \leq d - \lambda + 1 \leq d + 1$ . Therefore, by Lemma 9.1.8 we can give a partition of  $Y$  into two disjoint sets  $S$  and  $T$  such that  $|T| = d - \lambda + 1$  and  $\text{Conv}(S) \cap \text{aff}(T) \neq \emptyset$ .

We claim that  $\text{aff}(T)$  is a Kneser transversal for  $X$ . If a  $T$  intersects a  $k$ -set, then  $\text{aff}(T)$  is clearly a transversal to its convex hull. Since  $X$  has  $(d - \lambda + 1) + k$  points, there is exactly one  $k$ -subset that does not intersect  $T$ : the complement of  $T$  in  $X$ . But this  $k$ -set contains  $S$ , for which we know  $\text{Conv}(S) \cap \text{aff}(T) \neq \emptyset$ . This shows that  $\text{aff}(T)$  is a transversal to the convex hull of all  $k$ -sets.  $\square$

## 9.1.2 Matroids and cyclic polytopes

**Definition 9.1.10** (Moment curve and cyclic polytope). The *moment curve* in  $\mathbb{R}^d$  is defined parametrically as the map  $\gamma : \mathbb{R} \rightarrow \mathbb{R}^d$ ,  $t \mapsto (t, t^2, \dots, t^d)$ . A *cyclic polytope* is the convex hull of a finite set of points on the moment curve.

Here, we study the function  $m^*$  for sets of vertices of cyclic polytopes. Basic notions of oriented matroids can be found in [34].

**Definition 9.1.11** (Alternating oriented matroid). The oriented matroids associated with cyclic polytopes on  $n$  vertices of dimension  $d$  are called *alternating oriented matroids* and they are denoted by  $A(r, n)$  with  $r = d + 1$ .

A well-known fact in oriented matroid theory is that the circuits of oriented matroid theory arising from a configuration of points can be interpreted as minimal Radon-partitions induced by the signs of the elements. For example, if we have the set of points  $V = \{v_1, v_2, v_3, v_4, v_5, v_6\}$  in  $\mathbb{R}^3$  and if  $C = \{v_1, v_2, v_4, v_5, v_6\}$  is a signed circuit with  $++--+$ , this means that the sets  $A = \{v_1, v_2, v_6\}$  and  $B = \{v_4, v_5\}$  form a Radon partition, that is  $\text{Conv}(A) \cap \text{Conv}(B) \neq \emptyset$ .

Suppose that the ground set of  $A(r, n)$  is  $[n]$  and let  $C$  be one of its circuits. A well-known fact [34, Section 9.4] is that

$$|C| = r + 1 \text{ and if its elements are increasingly ordered, then they are alternatively signed } +-+-\dots \quad (9.2)$$

Therefore, minimal Radon partitions of cyclic polytopes are well understood.

**Definition 9.1.12** (Function  $m^*$ ). Let  $k, d, \lambda$  be integers with  $k, d \geq \lambda \geq 1$ . Let  $\zeta(k, d, \lambda)$  be the maximum number of vertices that the cyclic polytope in  $\mathbb{R}^d$  can have, so that it has a complete Kneser  $(d - \lambda)$ -transversal to the convex hulls of its  $k$ -sets of vertices.

We clearly have

$$m^*(k, d, \lambda) \leq \zeta(k, d, \lambda).$$

We will give upper and lower bounds for  $\zeta(k, d, \lambda)$ . First we deal with some easy special cases. If  $\lambda = 0$ , then any  $d - 0$  transversal is the whole space, and then we can have as many points as we want. Also, in the trivial range, the following theorem states that the precise value of  $\zeta(k, d, \lambda)$  and  $m^*(k, d, \lambda)$  is  $d - \lambda + k$ .

**Theorem 9.1.13** (CMMMR [C12]). *When  $\alpha(d, \lambda) \geq 1$ , we have that*

$$m^*(k, d, \lambda) = \zeta(k, d, \lambda) = d - \lambda + k.$$

*In particular, when  $\alpha(d, \lambda) \geq 1$ , the vertex set of the cyclic polytope with at least  $(d - \lambda + 1) + k$  points does not have a complete Kneser  $(d - \lambda)$ -transversal to the convex hulls of its  $k$ -sets.*

*Proof.* Clearly  $d - \lambda + k \leq m^*(k, d, \lambda)$  since every  $(d - \lambda + 1)$ -set intersects all the  $k$ -sets for any set of  $d - \lambda + k$  points in  $\mathbb{R}^d$ . Let  $S$  be the cyclic polytope with  $d - \lambda + 1 + k$  points in  $\mathbb{R}^d$ . Let  $T \subseteq S$  be any set with  $d - \lambda + 1$  points,  $K$  a subset of  $S \setminus T$  with  $k$  points and  $K'$  a subset of  $K$  with  $\lambda + 1$  points. By (9.2) the Radon partition of  $T \cup K'$  can have at most  $\lceil \frac{d+2}{2} \rceil$  elements with the same sign. Since by hypothesis  $|K'| = \lambda + 1 > \lceil \frac{d+2}{2} \rceil$ , then  $K'$  has at least two elements with different signs and hence by Proposition 9.1.6 we have  $\text{Conv}(K') \cap \text{aff}(T) = \emptyset$ . Therefore  $\text{Conv}(K) \cap \text{aff}(T) = \emptyset$  by Proposition 9.1.5.  $\square$

**Definition 9.1.14.** Let  $k, d, \lambda$  be integers with  $k, d \geq \lambda \geq 1$ . Let us define  $\beta(\lambda, j) = \frac{j+\lambda-1}{2}$  for each integer  $j$  such that  $j + \lambda$  is an odd number. Let

$$z(k, d, \lambda) \stackrel{\text{def}}{=} (d - \lambda + 1) + \max_{\substack{j \in \{\lambda+1, \dots, d-\lambda+2\} \\ j+\lambda \text{ is odd}}} \left( \left\lfloor \frac{k-1}{\beta(\lambda, j)} \right\rfloor \cdot j + (k-1)_{\text{mod} \beta(\lambda, j)} \right)$$

$$Z(k, d, \lambda) \stackrel{\text{def}}{=} (d - \lambda + 1) + \lfloor (2 - \alpha(d, \lambda))(k - 1) \rfloor$$

Using simple combinatorial arguments and (9.2) the structure of minimal Radon partitions of cyclic polytopes, it is proved in [C12] that  $z(k, d, \lambda)$  and  $Z(k, d, \lambda)$  are lower and upper bounds, respectively, of  $\zeta(k, d, \lambda)$  in the non-trivial range.

**Theorem 9.1.15** (CMMMR [C12]). *In the non-trivial range, when  $\alpha(d, \lambda) < 1$ ,*

$$z(k, d, \lambda) \leq \zeta(k, d, \lambda) \leq Z(k, d, \lambda).$$

**Corollary 9.1.16.** *In the non-trivial range, when  $\alpha(d, \lambda) < 1$ , we have that*

$$(d - \lambda + 1) + k \leq m^*(k, d, \lambda) \leq (d - \lambda + 1) + \lfloor (2 - \alpha(d, \lambda))(k - 1) \rfloor.$$

The bounds found by Theorem 9.1.15 are asymptotically correct in terms of  $k$ .

**Theorem 9.1.17** (CMMMR [C12]). *In the non trivial-range, we have that*

$$\lim_{k \rightarrow \infty} \frac{\zeta(k, d, \lambda)}{k} = 2 - \alpha(d, \lambda).$$

## 9.2 Codimension two and three Kneser transversals

In this section, we present various results on the existence of (complete) Kneser transversals for  $\lambda = 2, 3$ . In order to do this, we introduce the notions of *stability* and *instability* for (complete) Kneser transversals. After that, by using oriented matroid machinery, we present some computational results (closely related to the stability and unstability notions). We determine the existence of (complete) Kneser transversals for each of the 246 different order types of configurations of 7 points in  $\mathbb{R}^3$ . The results presented here can be found in [C13].

### 9.2.1 Stability

As already seen before, the existence of a Kneser transversal is not necessarily an invariant of the *order type*. For example, for  $d = 2$ , let  $X$  be the vertex set of a regular hexagon. Then, the center of such hexagon is a 0-plane transversal to the convex hull of the 4-sets. But, by a suitable 'slight' perturbation of the 6 vertices of the hexagon we lose this property, see Figure 9.1.

The situation for complete Kneser transversals is different. Indeed, in the previous section, we have seen how to detect complete Kneser transversal by using only Radon partitions implying that the existence of such transversals is an invariant of the order type. This naturally lead us to consider the notions of *stability* and *instability*.

**Definition 9.2.1** (Stable Kneser transversal). A Kneser transversal is said to be *stable* (resp. *unstable*) if the given set of points can be *slightly perturbed* (move each point to, not more than  $\epsilon > 0$  distance of their original position) such that the new configuration of points admits (if there is any) only complete Kneser transversals (resp. the new configuration of points does not admit a Kneser transversal).

First, (complete) Kneser transversals for codimensions 2 and 3 can be determined.

**Theorem 9.2.2** (CMMMR [C13]). *Let  $X = \{x_1, x_2, \dots, x_n\}$  be a collection of  $n = d + 2(k - \lambda)$  points in general position in  $\mathbb{R}^d$ . Suppose that  $L$  is a  $(d - \lambda)$ -plane transversal to the convex hulls of all  $k$ -sets of  $X$  with  $\lambda = 2, 3$  and  $k \geq \lambda + 2$  and  $d \geq \lambda$ . Then, either*

- (1)  $L$  is a complete Kneser transversal (i.e., it contains  $d - \lambda + 1$  points of  $X$ ) or
- (2)  $|X \cap L| = d - 2(\lambda - 1)$  and the other  $2(k - 1)$  points of  $X$  are matched in  $k - 1$  pairs in such a way that  $L$  intersects the corresponding closed segments determined by them.

It is not difficult to generalize Theorem 9.2.2 for  $\lambda > 3$  if we ask  $d - 2(\lambda - 1)$  points of  $X$  to be contained in a  $(d - \lambda)$ -plane transversal  $L$  and a collection of simplices (formed with the rest of points) of different dimensions (not necessarily intervals) to be all intersected by  $L$ . However, we have not been able to prove that also for  $\lambda > 3$  every simplex can be chosen to be an interval.

**Theorem 9.2.3** (CMMMR [C13]). *Let  $X = \{x_1, \dots, x_n\}$  be a finite collection of  $n = d + 2(k - \lambda)$  points in  $\mathbb{R}^d$ , with  $k - \lambda \geq 2$ ,  $\lambda = 2, 3$  and  $d \geq \lambda$ . For every  $\epsilon > 0$  there exists  $X' = \{x'_1, \dots, x'_n\}$ , a collection of points in  $\mathbb{R}^d$  in general position such that  $|x_i - x'_i| < \epsilon$ , for every  $i = 1, \dots, n$ , and with the property that every transversal  $(d - \lambda)$ -plane to the convex hull of the  $k$ -sets of  $X'$  is complete (i.e., it contains  $d - \lambda + 1$  points of  $X'$ ).*

## 9.2.2 Bounds for $m(k, d, \lambda)$ when $\lambda = 2, 3$

We start by proving the following upper bound

**Theorem 9.2.4** (CMMMR [C13]). *Let  $\lambda = 2, 3$ ,  $k - \lambda \geq 2$  and  $2(\lambda - 1) \leq d \leq 2(k - 2)$ . Then,*

$$m(k, d, \lambda) < d + 2(k - \lambda).$$

*Proof.* Let  $X = \{x_1, \dots, x_n\}$  be a finite collection of points in  $\mathbb{R}^d$  embedded in the moment curve on  $n = d + 2(k - \lambda)$  vertices. On one hand, by Theorem 9.2.3, there exists a collection of points  $X' = \{x'_1, \dots, x'_n\}$  in  $\mathbb{R}^d$  in general position with the order type as the cyclic polytope and with the property that every  $(d - \lambda)$ -plane transversal to the convex hull of the  $k$ -sets of  $X'$  is complete (i.e., it contains  $d - \lambda + 1$  points of  $X'$ ). On the other hand, by Corollary 9.1.16, we have that

$$m^*(k, d, \lambda) \leq (d - \lambda + 1) + \left\lceil \left( 2 - \frac{\lambda - 1}{\lfloor \frac{d}{2} \rfloor} \right) (k - 1) \right\rceil$$

which is strictly smaller than  $d + 2(k - \lambda)$  when  $\lceil \frac{d}{2} \rceil < k - 1$  and thus implying that  $X'$  does not admit a complete transversal. Therefore, the collection of points  $X'$  in  $\mathbb{R}^d$  does not have a  $(d - \lambda)$ -plane transversal to the convex hull of the  $k$ -sets implying that  $m(k, d, \lambda) < d + 2(k - \lambda)$ .  $\square$

The following result due to Bukh, Matoušek and Nivasch [40] was proved by using equivariant topology (see also [41] and [86] for some related results).

Let  $X = \{x_1, x_2, \dots, x_n\}$  be a collection of  $n$  points in  $\mathbb{R}^d$ . Then, there exists a codimension two affine plane  $L$  and  $2d - 1$  hyperplanes passing through  $L$  that divide  $\mathbb{R}^d$  into  $4d - 2$  parts, each containing at most  $\frac{n}{4d-2} + O(1)$  points of  $X$ .

Every hyperplane  $H$  through  $L$  leaves at least  $2d - 2$  of these parts on each side of  $H$ . Therefore, on each side of  $H$  we must have at least

$$n - 2d \left( \frac{n}{4d - 2} + O(1) \right) = \frac{(2d - 2)n}{4d - 2} - 2dO(1)$$

points of  $X$  and thus

$$\left\lfloor \frac{(2d - 2)n}{4d - 2} - 2dO(1) \right\rfloor \leq \tau(n, d, 2).$$

Furthermore, if  $k \geq \frac{2dn}{4d-2} + 2dO(1)$ , the codimension two affine plane  $L$  intersects the convex hull of every  $k$ -set of  $X$ , implying that

$$\left\lceil \frac{(4d - 2)}{2d} (k - 2dO(1)) \right\rceil \leq m(k, d, 2). \tag{9.3}$$

From Equation (9.3) and Theorem 9.2.4, we obtain

$$2 - \frac{1}{d} \leq \limsup_{k \rightarrow \infty} \frac{m(k, d, 2)}{k} \leq 2.$$

Therefore, for  $d \geq 3, \lambda = 2$  and  $k$  large enough the conjectured value [23, Conjecture 1]  $m(k, d, \lambda) = d - \lambda + k + \lceil \frac{k}{\lambda} \rceil - 1$  does not hold.

### 9.2.3 Computational results

General background on oriented matroid theory can be found in [34].

**Definition 9.2.5** (Abstract order type). An *abstract order type* is the relabeling class of an acyclic oriented matroid. The abstract order types of realizable oriented matroids are called *order types* corresponding to *isomorphism types* of configurations of points in the Euclidean space.

In [23], it is proved that  $m(3, 2, 4) = 6$  (i.e., there always exists a transversal line to all tetrahedra formed by any configuration of 6 points in  $\mathbb{R}^3$ ) and that there is never a transversal line to all tetrahedra formed by any configuration of 8 points in general position in  $\mathbb{R}^3$ .

What about transversal lines to all tetrahedra in configurations of 7 points in  $\mathbb{R}^3$ ?

We will answer this question by classifying the configurations of 7 points in  $\mathbb{R}^3$  having a (complete) Kneser line (if any). It is known that there are 5083 abstract order types of rank  $r = 4$  ( $d = 3$ ) of cardinality  $n = 7$  [64]. Among these 5083 abstract order types, 246 of them are the order type of some configuration of points in general position.

### Complete Kneser transversal line

We investigate whether there exists a complete transversal line to the tetrahedra of  $E = \{x_1, \dots, x_7\}$ . For this, we first detect when the line joining  $x_{i_1}$  and  $x_{i_2}$  intersects the interior of the triangle  $(x_{i_3}, x_{i_4}, x_{i_5})$ . The following proposition is a direct consequence of Proposition 9.1.6.

**Proposition 9.2.6** (CMMMR [C13]). *Let  $E := \{x_1, \dots, x_7\}$  be a set of 7 points in general position in  $\mathbb{R}^3$  and let  $\mathcal{M} = (E, B)$  its associated oriented matroid. Then, the line  $(x_{i_1}, x_{i_2})$  intersects the interior of the triangle  $(x_{i_3}, x_{i_4}, x_{i_5})$  if and only if  $sg(x_{i_3}) = sg(x_{i_4}) = sg(x_{i_5})$  in the circuit  $\{x_{i_1}, x_{i_2}, x_{i_3}, x_{i_4}, x_{i_5}\}$  of  $\mathcal{M}$ , where  $sg(x)$  stands for the sign of  $x$  in  $\mathcal{M}$ .*

*Proof.* Since  $\mathcal{M}$  is acyclic, it is not possible that the elements of the circuit  $C := \{x_{i_1}, x_{i_2}, x_{i_3}, x_{i_4}, x_{i_5}\}$  have all the same sign. Moreover, as depicted in Figure 9.2, the line  $(x_{i_1}, x_{i_2})$  intersects the interior of the triangle  $(x_{i_3}, x_{i_4}, x_{i_5})$  if and only if the Radon partition associated with  $C$  is one of  $\{\{x_{i_1}, x_{i_2}\}, \{x_{i_3}, x_{i_4}, x_{i_5}\}\}$ ,  $\{\{x_{i_1}\}, \{x_{i_2}, x_{i_3}, x_{i_4}, x_{i_5}\}\}$  or  $\{\{x_{i_2}\}, \{x_{i_1}, x_{i_3}, x_{i_4}, x_{i_5}\}\}$ .  $\square$

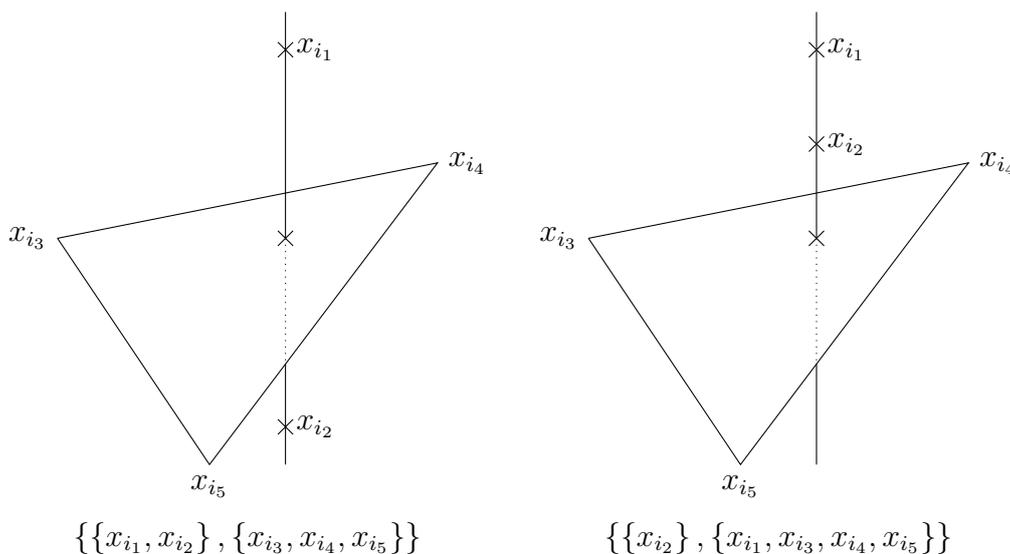


Figure 9.2: Radon partitions where the line  $(x_{i_1}, x_{i_2})$  intersects the triangle  $(x_{i_3}, x_{i_4}, x_{i_5})$

We notice that since the points of  $E$  are in general position, then the line  $(x_{i_1}, x_{i_2})$  cannot intersect the triangle  $(x_{i_3}, x_{i_4}, x_{i_5})$  on a vertex or an edge.

For each of the  $\binom{7}{2} = 21$  pairs  $(x_{i_1}, x_{i_2})$ , we determine, by using Proposition 9.2.6, if the line  $(x_{i_1}, x_{i_2})$  intersects the  $\binom{5}{3} = 10$  triangles of  $E \setminus \{x_{i_1}, x_{i_2}\}$ . Since the points are in general position, it is easy to see that if  $(x_{i_1}, x_{i_2})$  intersects a tetrahedron  $T$  whose vertices are in  $E$ , then  $(x_{i_1}, x_{i_2})$  intersects at least two faces (two triangles) of  $T$  (c.f. Proposition 9.1.5). Therefore, if  $(x_{i_1}, x_{i_2})$  intersects the triangle  $(x_{i_3}, x_{i_4}, x_{i_5})$ , it intersects both tetrahedra  $(x_{i_3}, x_{i_4}, x_{i_5}, x_{i_6})$  and  $(x_{i_3}, x_{i_4}, x_{i_5}, x_{i_7})$ . Finally, if the line  $(x_{i_1}, x_{i_2})$  intersects the  $\binom{5}{4} = 5$  tetrahedra generated from  $E \setminus \{x_{i_1}, x_{i_2}\}$ , it immediately follows that  $(x_{i_1}, x_{i_2})$  is transversal to all the tetrahedra of  $E$ .

For instance, for  $\mathcal{M} = OT(7, 4, 2)$  in the classification given in [64], that is the abstract order type representing a point configuration having the chirotope  $\chi_{\mathcal{M}} : \mathcal{B} \rightarrow \{0, -, +\}$

given by

$$\begin{array}{cccccccccccccccc} & 1 & 1 & 1 & 1 & 2 & 1 & 1 & 1 & 2 & 1 & 1 & 2 & 1 & 2 & 3 & 1 & 1 & 1 \\ & 2 & 2 & 2 & 3 & 3 & 2 & 2 & 3 & 3 & 2 & 3 & 3 & 4 & 4 & 4 & 2 & 2 & 3 \\ & 3 & 3 & 4 & 4 & 4 & 3 & 4 & 4 & 4 & 5 & 5 & 5 & 5 & 5 & 5 & 3 & 4 & 4 \\ & 4 & 5 & 5 & 5 & 5 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 7 & 7 & 7 \\ \chi_{\mathcal{M}} = & + & - & + & - & - & - & + & - & - & - & + & + & - & - & + & + & - & + \\ & 2 & 1 & 1 & 2 & 1 & 2 & 3 & 1 & 1 & 2 & 1 & 2 & 3 & 1 & 2 & 3 & 4 & & \\ & 3 & 2 & 3 & 3 & 4 & 4 & 4 & 2 & 3 & 3 & 4 & 4 & 4 & 5 & 5 & 5 & 5 & & \\ & 4 & 5 & 5 & 5 & 5 & 5 & 5 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & & \\ & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & & \\ \chi_{\mathcal{M}} = & + & + & - & - & + & + & + & + & - & - & + & - & + & + & + & - & + & + \end{array}$$

the line  $L$  going through 1 and 5 is a complete Kneser transversal line. Indeed, by Proposition 9.2.6, we know that  $L$  intersects the triangles  $(2, 3, 4)$ ,  $(2, 3, 6)$ ,  $(2, 6, 7)$  and  $(3, 4, 7)$  since the corresponding circuits are

$$\{\overline{12345}\}, \{\overline{12\overline{356}}\}, \{\overline{12\overline{567}}\}, \{\overline{13\overline{457}}\},$$

implying that  $L$  intersects the 5 tetrahedra  $(2, 3, 4, 6)$ ,  $(2, 3, 4, 7)$ ,  $(2, 3, 6, 7)$ ,  $(2, 4, 6, 7)$  and  $(3, 4, 6, 7)$ .

By applying the above method, we obtain the following result.

**Theorem 9.2.7** (CMMMR [C13]). *Among the 246 configurations of 7 points in general position in  $\mathbb{R}^3$  there are 124 admitting a complete Kneser transversal to the tetrahedra. These configurations correspond to the 124 realizable rank 4 oriented matroids on 7 elements given by the following set according to the classification in [64]*

$$A := \{2, 3, 5, 6, 8, 9, 10, 15, 16, 18, 20, 21, 25, 27, 28, 29, 33, 34, 35, 38, 40, 41, 43, 44, 45, 46, 47, 48, 50, 51, 52, 55, 56, 60, 62, 63, 64, 67, 68, 69, 70, 71, 72, 74, 76, 79, 85, 88, 92, 93, 94, 95, 96, 97, 98, 99, 100, 101, 102, 106, 107, 109, 110, 111, 112, 113, 118, 120, 123, 124, 125, 127, 132, 134, 135, 136, 140, 141, 142, 144, 145, 150, 151, 154, 155, 156, 157, 159, 160, 166, 167, 171, 172, 177, 178, 182, 183, 184, 185, 186, 187, 189, 191, 192, 195, 199, 200, 201, 206, 207, 208, 211, 212, 219, 220, 221, 224, 225, 228, 229, 234, 237, 243, 244\}.$$

**Kneser transversal line**

Let  $E := \{x_1, \dots, x_7\}$  be a set of 7 points in general position in  $\mathbb{R}^3$  and let  $\mathcal{M} = (E, \mathcal{B})$  be its associated oriented matroid. By Theorem 9.2.2, if there exist a non-complete Kneser transversal line to the convex hull of its 4-subsets, then the 7 points of  $\mathcal{M}$  must look as depicted in Figure 9.3. This implies that  $\mathcal{M}$  admits the following circuits

$$\{\overline{x_1 x_2 x_3 x_4 x_7}\}, \{\overline{x_1 x_2 x_5 x_6 x_7}\}, \{\overline{x_3 x_4 x_5 x_6 x_7}\}, \tag{9.4}$$

and cocircuits

$$\{\overline{x_3 x_4 x_5 x_6}\}, \{\overline{x_1 x_2 x_5 x_6}\}, \{\overline{x_1 x_2 x_3 x_4}\}. \tag{9.5}$$

However it is possible that certain configurations of 7 points  $\mathcal{M}$  having circuits as given in (9.4) and cocircuits as in (9.5) do not admit a transversal line to the convex hull

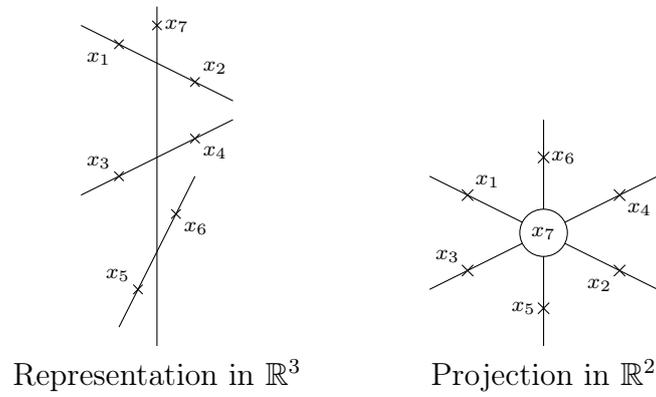


Figure 9.3: 7 points in  $\mathbb{R}^3$  with circuits and cocircuits satisfying (9.4) and (9.5) with a Kneser transversal line to all 4-sets

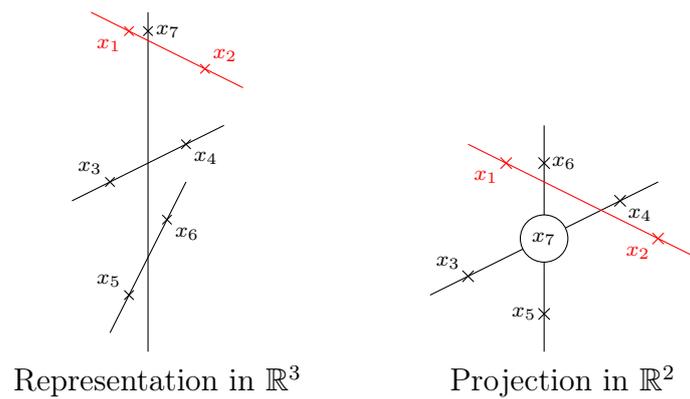


Figure 9.4: 7 points in  $\mathbb{R}^3$  with circuits and cocircuits satisfying (9.4) and (9.5) but without transversal line to all 4-sets

of its 4-subsets, see Figure 9.4 where this situation is illustrated. This example shows once more that the existence of Kneser Transversals is not an invariant of the order type in general.

Nevertheless, we can still identify whether a configuration of 7 points admits a Kneser transversal line. To this end, we consider the oriented matroid  $\mathcal{M}'$  associated with the configuration of 8 points  $E' := \{x_1, \dots, x_8\}$  in  $\mathbb{R}^3$ , not necessarily in general position, illustrated in Figure 9.5. The deletion of either point  $x_7$  or point  $x_8$  from  $\mathcal{M}$  yields a configuration on 7 points as represented in Figure 9.3 admitting thus a Kneser transversal line (containing either  $x_7$  or  $x_8$ ) to the all tetrahedra. We thus have that the line going through  $x_7$  and  $x_8$  would be a complete Kneser transversal line of  $E'$ . Moreover, any configuration on 7 points as represented in Figure 9.3 arises on this way.

We may thus detect all such configurations  $\mathcal{M}'$ . We do this by observing that an oriented matroid  $\mathcal{M}$  on 8 elements corresponds to such a configuration if and only if  $\mathcal{M}$  admits the following cocircuits

$$\{\overline{x_3 x_4 x_5 x_6}\}, \quad \{\overline{x_1 x_2 x_5 x_6}\}, \quad \{\overline{x_1 x_2 x_3 x_4}\}. \tag{9.6}$$

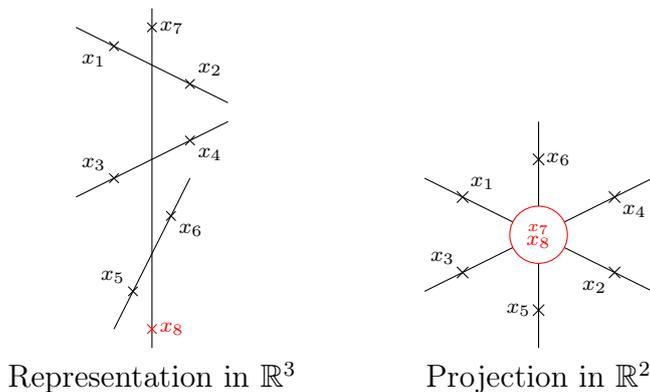


Figure 9.5: 8 points in  $\mathbb{R}^3$  with a complete transversal line to all 4-sets

For each of the 10775236 order types of 8 points in  $\mathbb{R}^3$ , we consider its associated oriented matroid  $\mathcal{M}$ . If  $\mathcal{M}$  admits cocircuits of the form (9.6), we delete  $x_7$  or  $x_8$  obtaining a configuration of 7 points in general position as in Figure 9.3 admitting a non-complete Kneser transversal line to all tetrahedra.

**Theorem 9.2.8** (CMMMR [C13]). *Among the 246 different order types of 7 points in general position in  $\mathbb{R}^3$  there are 124 admitting a representation for which there is a non-complete Kneser transversal line to all tetrahedra. These configurations correspond to the 124 realizable rank 4 oriented matroids on 7 elements given in by the following set according to the classification in [64]*

$$B := \{1, 2, 4, 6, 7, 10, 11, 12, 13, 14, 16, 19, 24, 26, 29, 30, 31, 32, 36, 38, 39, 40, 41, 42, 55, 58, 59, 60, 61, 62, 63, 65, 70, 71, 72, 74, 76, 77, 78, 79, 80, 81, 82, 83, 85, 86, 87, 88, 89, 90, 91, 93, 96, 97, 98, 99, 101, 102, 103, 104, 105, 111, 114, 117, 121, 122, 124, 126, 128, 129, 130, 138, 139, 140, 145, 146, 147, 148, 149, 151, 152, 153, 158, 165, 170, 171, 172, 173, 174, 175, 176, 177, 180, 185, 194, 196, 197, 198, 199, 201, 204, 206, 207, 209, 211, 212, 213, 214, 215, 217, 218, 219, 220, 230, 235, 236, 237, 238, 239, 240, 241, 242, 244, 246\}.$$

**Corollary 9.2.9** (CMMMR [C13]). *Among the 246 configurations of 7 points in general position in  $\mathbb{R}^3$ , 124 of them admit a Kneser transversal line to all the tetrahedra and 122 configurations do not admit a Kneser transversal line to the all the tetrahedra.*

*Proof.* By Theorems 9.2.7 and 9.2.8, we have

$$|A| = 124, |B| = 124, |A \cap B| = 46, |A \setminus B| = 78, |B \setminus A| = 78, |\overline{A \cup B}| = 44.$$

The 44 order types of  $\overline{A \cup B}$  do not admit Kneser transversal lines. By Theorem 9.2.3, for each of the 78 order types of  $B \setminus A$ , there exists a representation for which there is no Kneser transversal line.  $\square$

It turns out that all abstract order types of rank 4 with at most 7 elements are realizable [34, Corollary 8.3.3], that is, they correspond to 7 points in the space, but not necessarily in general position.

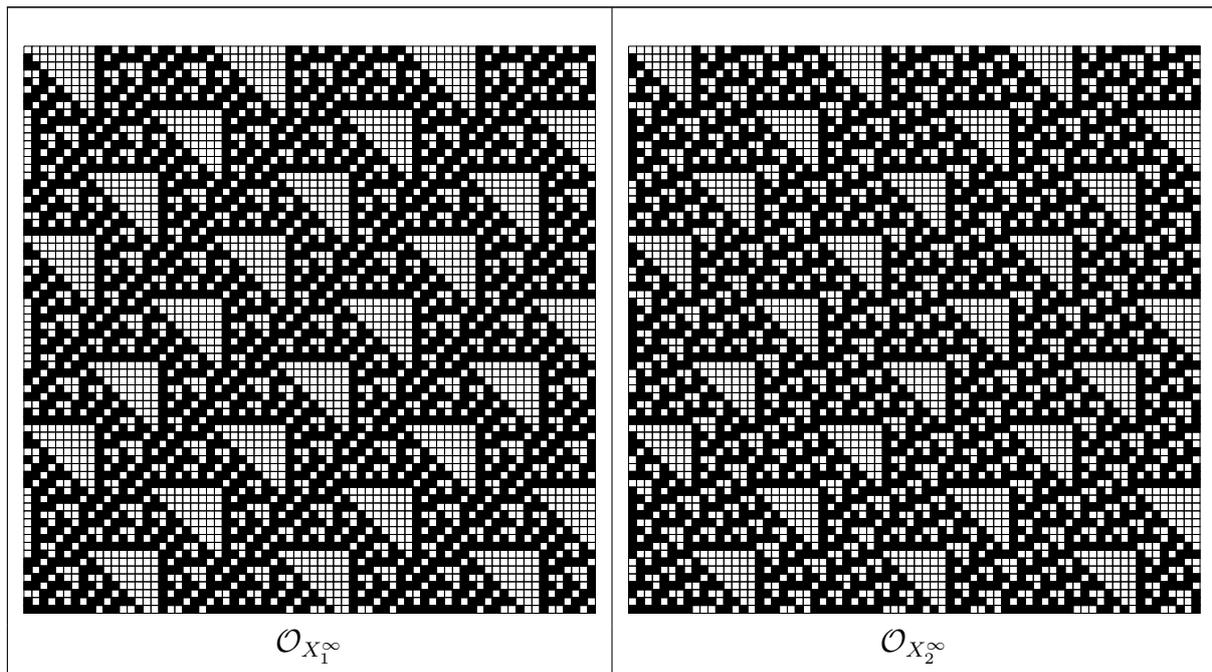
**Theorem 9.2.10** (CMMMR [C13]). *Among the 5083 abstract order types of rank  $r = 4$  ( $d = 3$ ) with  $n = 7$  there are 1158 having a complete Kneser transversal line to all the tetrahedra.*

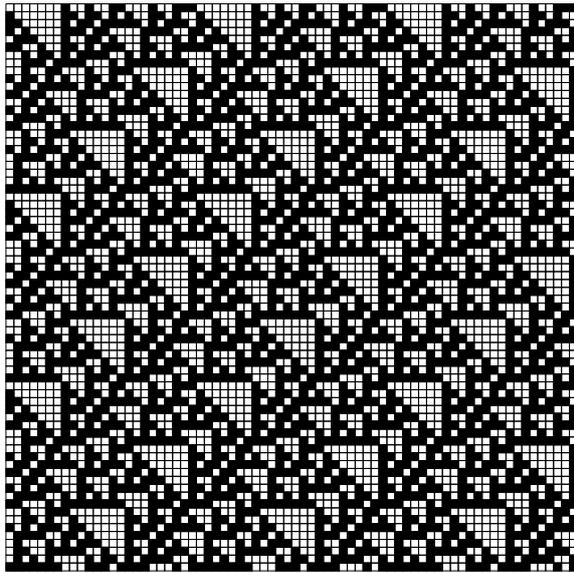
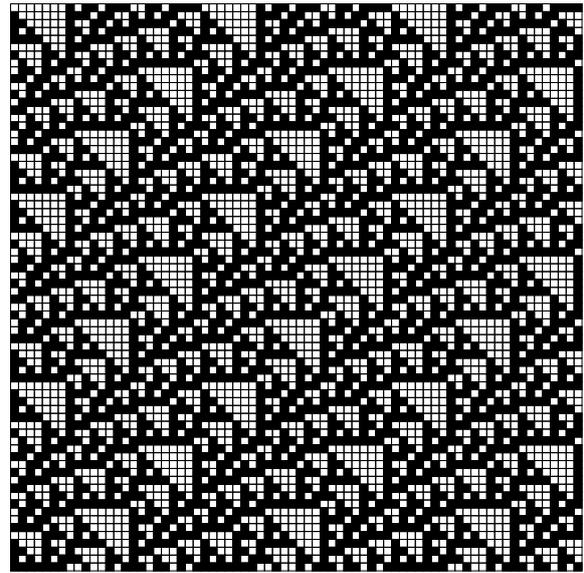
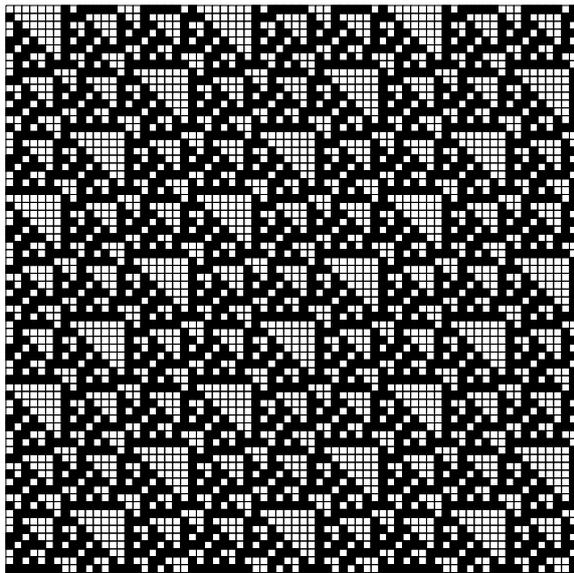
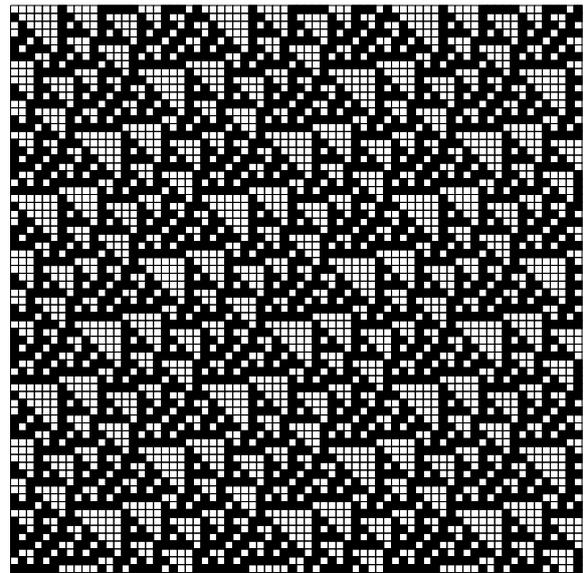
This calculation was obtained by applying the same arguments as those used for Theorem 9.2.7 (via Radon partitions) and by taking care of the cases when three or more points are collinear and when four or more points are coplanar (via the circuits of the the oriented matroid).

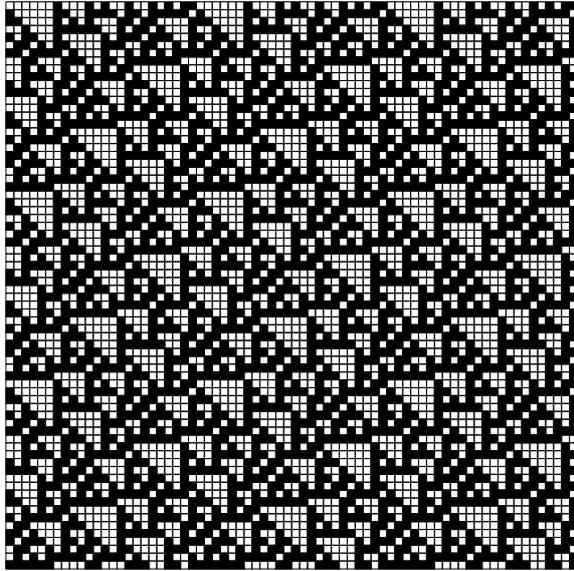
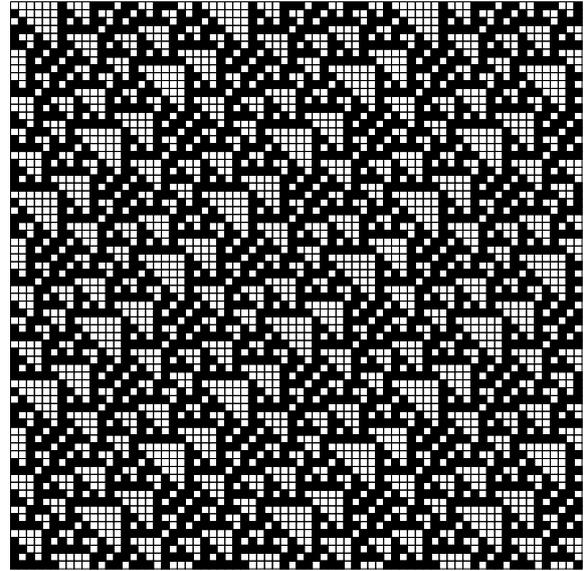
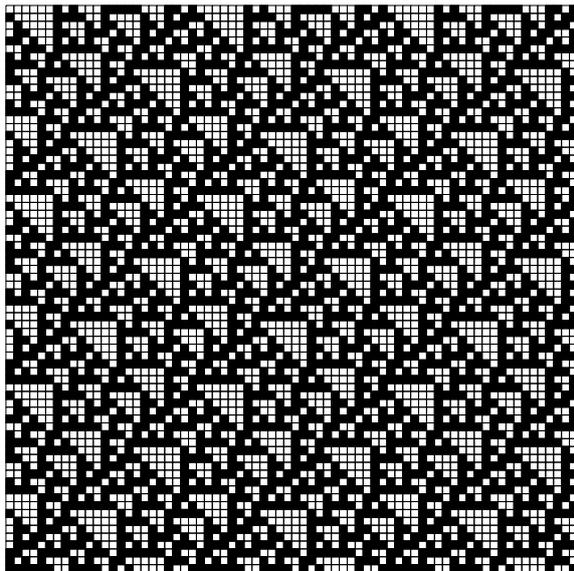
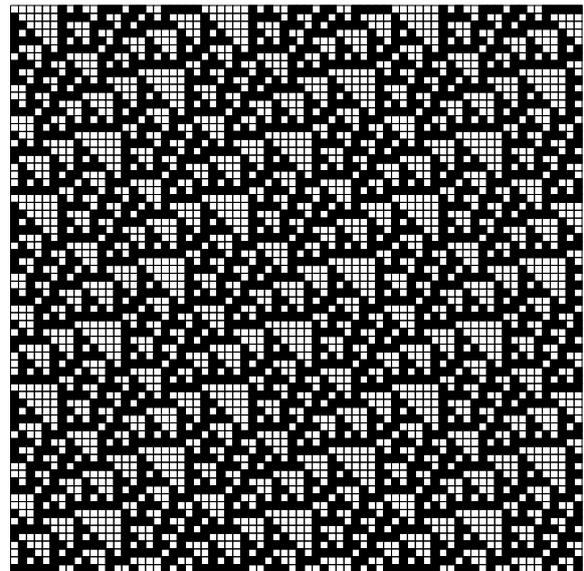
# Appendix A

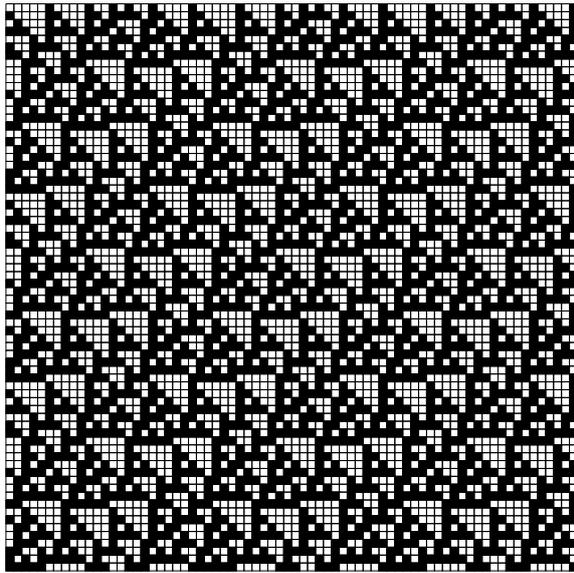
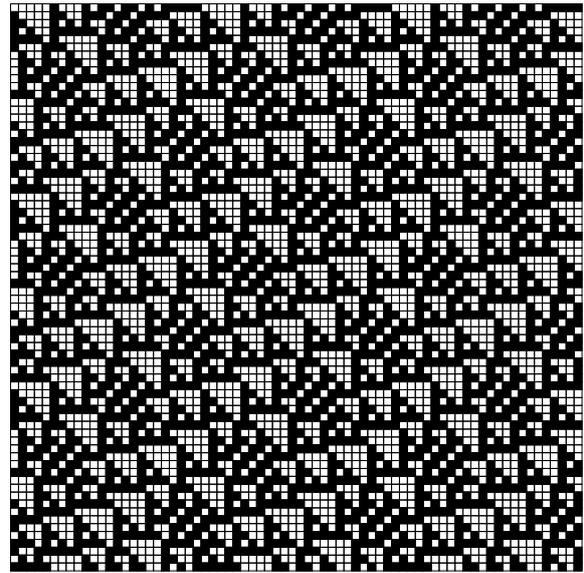
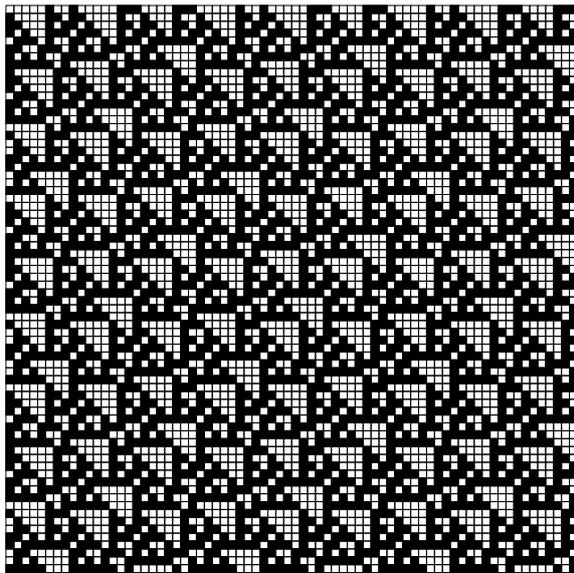
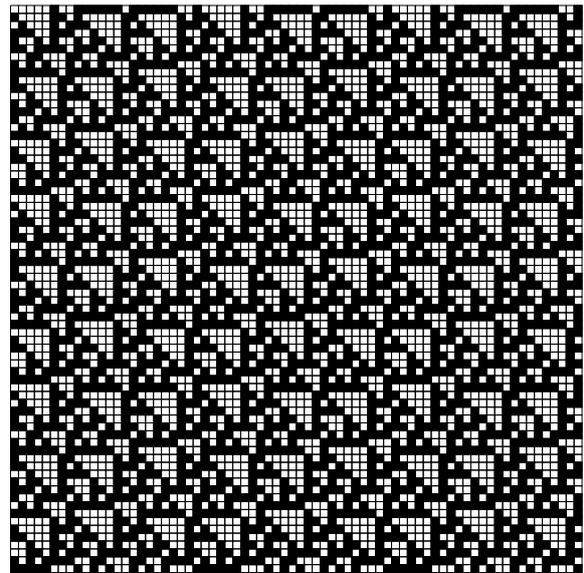
## The 24-periodic orbits with balanced periods

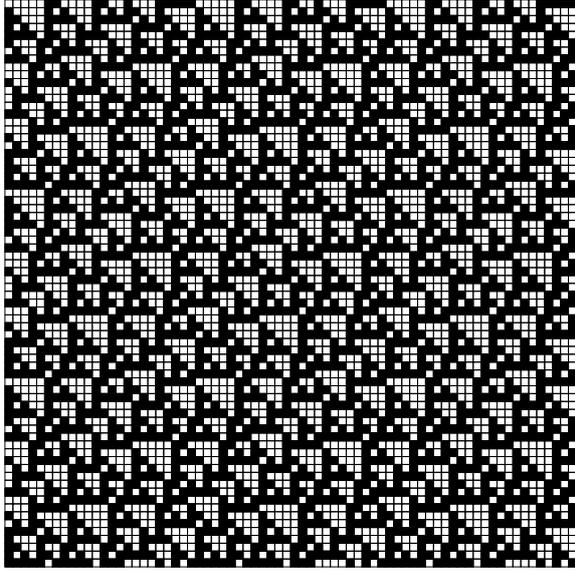
In this appendix, the orbits of representatives  $X_i$ , for all the 17 elements of  $\overline{\mathcal{BPO}}_{24} = \{\overline{X}_1, \overline{X}_2, \dots, \overline{X}_{17}\}$  of Chapter 1, are given. Moreover, we have also obtained the orbits of the elements of  $\overline{\mathcal{BPO}}_{12} = \{\overline{Y}_1, \overline{Y}_2\}$ . Indeed, as already remarked, we have  $\mathcal{O}_{Y_1^\infty} = \mathcal{O}_{X_{16}^\infty}$  and  $\mathcal{O}_{Y_2^\infty} = \mathcal{O}_{X_{17}^\infty}$  since  $X_{16} = Y_1^2$  and  $X_{17} = Y_2^2$ .



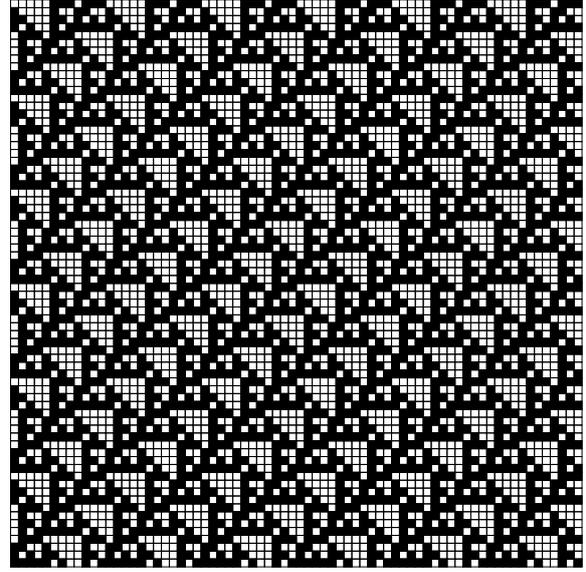
 $O_{X_3^\infty}$  $O_{X_4^\infty}$  $O_{X_5^\infty}$  $O_{X_6^\infty}$

 $\mathcal{O}_{X_7^\infty}$  $\mathcal{O}_{X_8^\infty}$  $\mathcal{O}_{X_9^\infty}$  $\mathcal{O}_{X_{10}^\infty}$

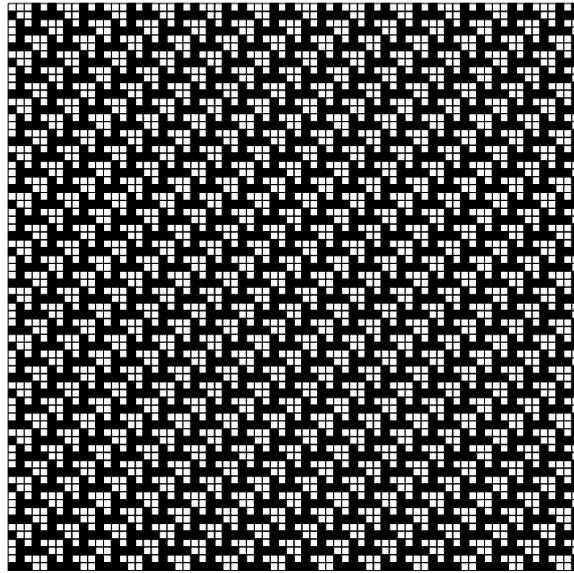
 $\mathcal{O}_{X_{11}^\infty}$  $\mathcal{O}_{X_{12}^\infty}$  $\mathcal{O}_{X_{13}^\infty}$  $\mathcal{O}_{X_{14}^\infty}$



$$\mathcal{O}_{X_{15}^{\infty}}$$



$$\mathcal{O}_{X_{16}^{\infty}} = \mathcal{O}_{Y_1^{\infty}}$$



$$\mathcal{O}_{X_{17}^{\infty}} = \mathcal{O}_{Y_2^{\infty}}$$



# Conclusion and future work

In this chapter, we conclude by giving an outlook to future work based on results obtained in this thesis.

## Triangles of average weight in elementary cellular automata

In Chapter 1, binary Steinhaus triangles have been introduced. The problem of Steinhaus can be seen as the determination whether there exist binary Steinhaus triangles of size  $n$  with an average number of zeroes and one, with  $\frac{1}{2}\binom{n+1}{2}$  zeroes and  $\frac{1}{2}\binom{n+1}{2}$  ones, for every non-negative integer  $n$ . The local rule in binary Steinhaus triangles corresponds to the local rule in the Sierpiński triangle or the standard Pascal triangle modulo 2. The Steinhaus Problem has been generalized for other local rules as for Steinhaus triangles modulo any positive integer  $m$  in Chapter 4 or for simplices generated by additive cellular automata in Chapter 5. Here, we consider a generalization for elementary cellular automata of dimension 1.

**Definition** (Elementary cellular automaton). The *elementary cellular automaton* associated with  $w : (\mathbb{Z}/2\mathbb{Z})^3 \rightarrow \mathbb{Z}/2\mathbb{Z}$  is the map  $\partial_w$  defined by

$$\partial_w((a_i)_{i \in \mathbb{Z}}) = (w(a_{i-1}, a_i, a_{i+1}))_{i \in \mathbb{Z}},$$

for all doubly infinite sequences  $(a_i)_{i \in \mathbb{Z}}$  of  $\mathbb{Z}/2\mathbb{Z}$ . There exist 256 elementary cellular automata. Among them, some are linear and we retrieve the Pascal cellular automaton which corresponds to the map associated with  $w(x, y, z) = y + z$ , for all  $(x, y, z) \in (\mathbb{Z}/2\mathbb{Z})^3$ .

For any doubly infinite sequence  $S$ , this derivation process can be repeated and we can consider the orbit  $\mathcal{O}_S = (\partial_w^i S)_{i \in \mathbb{N}}$ , where  $\partial_w^i$  is recursively defined by  $\partial_w^i S = \partial_w(\partial_w^{i-1} S)$  for all  $i \geq 1$  with  $\partial_w^0 S = S$ . Binary triangles appearing in a such orbit can then be considered as for binary Steinhaus triangles.

It is known that in binary Steinhaus triangles of size  $n$ , the average number of zeroes and ones is exactly  $\frac{1}{2}\binom{n+1}{2}$ , for all non-negative integers  $n$ . Another example, when  $w$  is the constant map equal to 1, it is easy to show that the average number of zeroes and ones in binary triangles of size  $n$  generated from  $\partial_w$  are

$$\frac{n}{2} \quad \text{and} \quad \frac{n}{2} + \binom{n}{2},$$

respectively, for all non-negative integers  $n$ .

In Chapter 1 and in [C7], a strategy for obtaining balanced binary triangles from periodic binary sequences has been explored. The objective here is to adapt this method in order to obtain triangles of average weight in any elementary cellular automaton.

**Problem P1.** For each of the 256 elementary cellular automata  $\partial_w$ , do there exist binary triangles of size  $n$  generated from  $\partial_w$  and with an average number of zeroes and ones, for all positive integers  $n$ ?

## Generalizations of symmetric binary triangles

In Chapter 2, symmetric binary triangles have been studied. Generating index sets and vector bases of linear subspaces of rotationally, horizontally and dihedrally symmetric binary triangles have been obtained. Here, we propose to extend this study in two directions: in higher dimensions with binary Steinhaus tetrahedra and for triangles of numbers modulo  $m$  built with a specific local rule for which the set of these triangles is invariant under the action of the dihedral group  $D_3$ .

### Symmetric binary Steinhaus tetrahedra

A binary Steinhaus tetrahedron of size  $n$  is a tetrahedron  $(a_{i,j,k})_{1 \leq i \leq j \leq k \leq n}$  of zeroes and ones with the (Pascal) local rule

$$a_{i,j,k} \equiv a_{i-1,j-1,k-1} + a_{i-1,j-1,k} + a_{i-1,j,k} \pmod{2}, \tag{LR}$$

for all integers  $i, j$  and  $k$  such that  $2 \leq i \leq j \leq k \leq n$ . The set of binary Steinhaus tetrahedra of size  $n$  is denoted by  $\mathcal{ST}^4(n)$ . Since a binary Steinhaus tetrahedron  $(a_{i,j,k})_{1 \leq i \leq j \leq k \leq n}$  is completely determined by its first row  $(a_{1,j,k})_{1 \leq j \leq k \leq n}$ , it follows that  $\mathcal{ST}^4(n)$  is a vector space over  $\mathbb{Z}/2\mathbb{Z}$  of dimension  $\binom{n+1}{2}$ . An example of binary Steinhaus tetrahedron of size 5 is depicted in Figure C1.

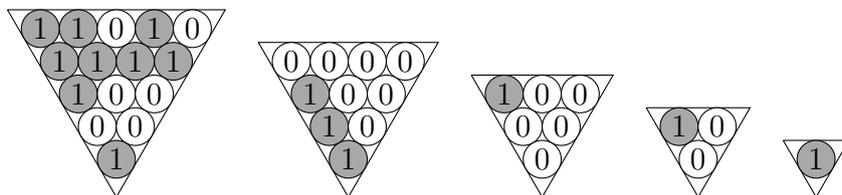


Figure C1: A binary Steinhaus tetrahedron of size 5

The symmetry group of regular tetrahedra is constituted by the identity map, 11 rotations (8 rotations about an axis through a vertex and the center of the opposite face by an angle of  $\pm \frac{2\pi}{3}$  and 3 rotations about an axis through centers of opposite edges by an angle of  $\frac{\pi}{2}$ ), 6 reflections and 6 rotofections. This symmetry group is isomorphic to  $\mathcal{S}_4$  and the subset of rotations is a subgroup of order 12 isomorphic to  $\mathcal{A}_4$ .

From the local rule (LR), for any binary Steinhaus tetrahedron  $T$  and for all  $g \in \mathcal{S}_4$ , it is easy to see that  $g(T)$  is also a binary Steinhaus tetrahedron, of the same size as  $T$ .

**Problem P2.** For every subgroup  $G$  of  $\mathcal{S}_4$  and every non-negative integer  $n$ , we consider the linear subspace of binary Steinhaus tetrahedra of size  $n$  defined by

$$\mathcal{ST}^4(n)^G = \{T \in \mathcal{ST}^4(n) \mid \forall g \in G, g(T) = T\}.$$

The question here is to characterize  $\mathcal{ST}^4(n)^G$ , to determine its dimension and a vector basis, for all non-negative integers  $n$  and for every subgroup  $G$  of  $\mathcal{S}_4$ .

### Symmetric triangles modulo $m$

Triangles similar to binary Steinhaus triangles can be considered for other kinds of numbers. Triangles defined from quasigroups have been studied in [27]. A *quasigroup*  $(G, \star)$  is a finite set  $G$  with a binary operation  $\star$  for which, for every  $a, b \in G$ , there exist unique elements  $x, y \in G$  such that  $a \star x = b$  and  $y \star a = b$ . A  $\nabla_\star$ -*configuration* of size  $n$  is a triangle  $(a_{i,j})_{1 \leq i \leq j \leq n}$  of elements in  $(G, \star)$  verifying the local rule

$$a_{i,j} = a_{i-1,j-1} \star a_{i-1,j},$$

for all integers  $i$  and  $j$  such that  $2 \leq i \leq j \leq n$ . For  $(G, \star) = (\mathbb{Z}/2\mathbb{Z}, +)$ , a  $\nabla_\star$ -configuration simply corresponds to a binary Steinhaus triangle.

A quasigroup  $(G, \star)$  is said to be *semisymmetric* if  $y \star (x \star y) = x$ , for all  $x, y \in G$ . It is easy to see that, for a semisymmetric quasigroup  $(G, \star)$ , the set of  $\nabla_\star$ -configurations is closed under 120 degrees rotation. Moreover, if  $(G, \star)$  is a commutative quasigroup, the set of  $\nabla_\star$ -configurations is also closed under the horizontal reflection.

A  $\nabla_\star$ -configuration is said to be *rotationally symmetric* if it is invariant by rotation and it is said to be *dihedrally symmetric* if it is invariant by rotation and by horizontal reflection (under the action of  $D_3$ ).

Rotationally symmetric  $\nabla_\star$ -configurations, for semisymmetric quasigroups, and dihedrally symmetric  $\nabla_\star$ -configurations, for commutative semisymmetric quasigroups, have been studied in [27]. The authors have obtained the cardinality of sets of rotationally symmetric and dihedrally symmetric  $\nabla_\star$ -configurations.

**Theorem** (Theorem 3.4 in [27]). *Let  $(G, \star)$  be a semisymmetric quasigroup. The number  $\text{conf}_{\mathcal{R}}(n)$  of rotationally symmetric  $\nabla_\star$ -configurations of size  $n$  is given by*

$$\text{conf}_{\mathcal{R}}(n) = \begin{cases} |G|^k & \text{if } n = 3k, \\ |G|^{k+1} & \text{if } n = 3k + 1, \\ |\text{Fix}(\star)||G|^k & \text{if } n = 3k + 2, \end{cases}$$

where  $\text{Fix}(\star) = \{x \in G \mid x \star x = x\}$ .

**Theorem** (Theorem 3.16 in [27]). *Let  $(G, \star)$  be a commutative semisymmetric quasigroup. The number  $\text{conf}_{\mathcal{D}}(n)$  of dihedrally symmetric  $\nabla_\star$ -configurations of size  $n$  is given by*

$$\text{conf}_{\mathcal{D}}(n) = \begin{cases} |G|^k & \text{if } n = 6k, \\ |G|^{k+1} & \text{if } n = 6k + 1, \\ |\text{Fix}(\star)||G|^k & \text{if } n = 6k + 2, \\ |G|^{k+1} & \text{if } n = 6k + 3, \\ |G|^{k+1} & \text{if } n = 6k + 4, \\ |\text{Fix}(\star)||G|^{k+1} & \text{if } n = 6k + 5, \end{cases}$$

where  $\text{Fix}(\star) = \{x \in G \mid x \star x = x\}$ .

Suppose now that  $G = \mathbb{Z}/m\mathbb{Z}$ , with  $m \geq 2$ , and let  $\star$  be the binary operation defined by  $x \star y = -(x + y)$ , for all  $x, y \in \mathbb{Z}/m\mathbb{Z}$ . Then, the quasigroup  $(\mathbb{Z}/m\mathbb{Z}, \star)$  is commutative and semisymmetric. The objective here is to study the sets of rotationally symmetric and dihedrally symmetric  $\nabla_\star$ -configurations of size  $n$  of  $(\mathbb{Z}/m\mathbb{Z}, \star)$ , that are denoted by  $\mathcal{RSC}_{(\mathbb{Z}/m\mathbb{Z}, \star)}(n)$  and  $\mathcal{DSC}_{(\mathbb{Z}/m\mathbb{Z}, \star)}(n)$ , respectively. Examples of rotationally symmetric and dihedrally symmetric  $\nabla_\star$ -configurations of  $\mathbb{Z}/6\mathbb{Z}$  are depicted in Figure C2.

It is clear that  $\mathcal{RSC}_{(\mathbb{Z}/m\mathbb{Z}, \star)}(n)$  and  $\mathcal{DSC}_{(\mathbb{Z}/m\mathbb{Z}, \star)}(n)$  are submodules of the free  $\mathbb{Z}/m\mathbb{Z}$ -module of  $\nabla_\star$ -configurations.

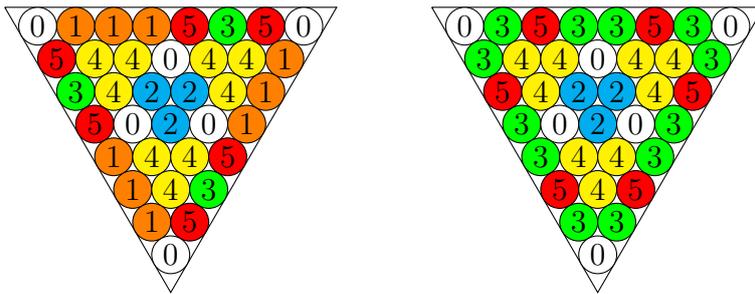


Figure C2: Triangles of  $\mathcal{RSC}_{(\mathbb{Z}/6\mathbb{Z}, \star)}(8)$  and  $\mathcal{DSC}_{(\mathbb{Z}/6\mathbb{Z}, \star)}(8)$

**Problem P3.** Let  $m$  be a positive integer. For every non-negative integer  $n$  and every submodule  $\mathcal{RSC}_{(\mathbb{Z}/m\mathbb{Z}, \star)}(n)$  and  $\mathcal{DSC}_{(\mathbb{Z}/m\mathbb{Z}, \star)}(n)$ , the question is to determine its length and a generating set.

### The Pascal Matroid

In Chapter 2, the notion of generating index set of binary Steinhaus triangles has been introduced in order to study linear subspaces of symmetric binary triangles. A subset  $G$  of the index set

$$\nabla(n) = \{(i, j) \in \mathbb{N}^2 \mid 1 \leq i \leq j \leq n\}$$

is said to be a *generating index set* of the vector space  $\mathcal{ST}(n)$  of binary Steinhaus triangles of size  $n$  if the knowledge of the values  $a_{i,j}$ , for all  $(i, j) \in G$ , uniquely determines the whole Steinhaus triangle  $(a_{i,j})_{1 \leq i \leq j \leq n}$ , i.e., if the linear map

$$\begin{aligned} \pi_G : \mathcal{ST}(n) &\longrightarrow (\mathbb{Z}/2\mathbb{Z})^G \\ (a_{i,j})_{1 \leq i \leq j \leq n} &\longmapsto (a_{i,j})_{(i,j) \in G} \end{aligned}$$

is an isomorphism. For instance, the index set in gray in Figure C3 is a generating index set for binary Steinhaus triangles of size 7.

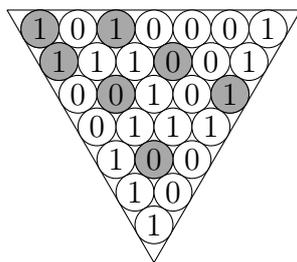


Figure C3: A generating index set of  $\mathcal{ST}(7)$

It is known that, for every non-negative integer  $n$ , the collection of generating index sets of  $\mathcal{ST}(n)$  has a matroid structure called *Pascal matroid*. This matroid has been introduced in [28] but without detailed analysis.

**Problem P4.** The main objective here is to study the Pascal matroid as well as other variations from a matroid theory perspective.

## Universal sequence of balanced Steinhaus triangles modulo $m$ even

In Chapter 4, the Molluzzo Problem on the existence of balanced Steinhaus triangles modulo  $m$  has been studied. Several solutions are presented, essentially in the case where  $m$  is odd. The main results obtained in Chapter 4 come from the analysis of Steinhaus triangles associated with arithmetic progressions and with interlaced arithmetic progressions. More precisely, the following two theorems are given.

**Theorem** (Chappelon [C2]). *Let  $m$  be an odd number. Let  $a, d \in \mathbb{Z}/m\mathbb{Z}$ , with  $d$  invertible, and let  $\text{AP}(a, d, n) = (a, a+d, \dots, a+(n-1)d)$  be the arithmetic progression beginning with  $a$ , with common difference  $d$  and of length  $n$ . Then, the Steinhaus triangle  $\nabla \text{AP}(a, d, n)$  is balanced, for all positive integers  $n \equiv 0$  or  $-1 \pmod{\text{ord}_m(2^m)m}$ , where  $\text{ord}_m(2^m)$  is the multiplicative order of  $2^m$  modulo  $m$ .*

This result gives a positive answer to the Molluzzo Problem when  $m$  is a power of three, i.e., for every non-negative integer  $k$ , there exists a balanced Steinhaus triangle modulo  $3^k$  of size  $n$  if and only if  $\binom{n+1}{2}$  is divisible by  $3^k$ . Moreover, for the other odd values of  $m$ , the existence of infinitely many balanced Steinhaus triangles modulo  $m$  is obtained. This positively answers the weak Molluzzo Problem for all odd values of  $m$ .

The second main result of Chapter 4 concerns a particular sequence of integers, called the *universal sequence*, that is the interlacing of three arithmetic progressions defined by

$$\begin{aligned} \text{US} &= \text{IAP}((0, -1, 1), (1, -2, 1)) \\ &= (\dots, -3, -3, 5, -2, -2, 3, -1, -1, 1, 0, \mathbf{0}, -1, 1, 1, -3, 2, 2, -5, 3, 3, -7, \dots). \end{aligned}$$

**Theorem** (Chappelon [C5]). *Let  $m$  be an odd number and let  $S = \pi_m(\text{US})$  be the projection of the universal sequence in  $\mathbb{Z}/m\mathbb{Z}$ . Then, the following Steinhaus triangles of size  $n$  are balanced:*

- $\nabla S[n, 2n - 1]$ , for all  $n \equiv 0 \pmod{m}$ ,
- $\nabla \partial S[0, n - 1]$ , for all  $n \equiv -1 \pmod{3m}$ .

In the last part of Chapter 4, a positive solution to the Molluzzo Problem for  $m = 4$  is obtained by lifting to  $\mathbb{Z}/4\mathbb{Z}$  specific known solutions in  $\mathbb{Z}/2\mathbb{Z}$ . The main objective here is to try to obtain the existence of balanced Steinhaus triangles modulo even numbers by lifting to  $\mathbb{Z}/2^k m\mathbb{Z}$  the universal sequence modulo  $m$  odd. The main idea is to continue the study of interlaced arithmetic progressions. Recent conjectures based on computer experiments are given below.

**Conjecture C1.** *Let  $S$  be the 12-interlaced arithmetic progression defined by*

$$S = \text{IAP}((1, 22, 13, 12, 26, 7, 0, 4, 5, 10, 12, 16), (17, 2, 23, 17, 30, 7, 7, 30, 17, 23, 2, 17)).$$

*For every  $m \in \{2, 4, 8, 16, 32\}$ , we consider the projection  $\pi_m(S)$  in  $\mathbb{Z}/m\mathbb{Z}$ . Then, the Steinhaus triangle  $\nabla \pi_m(S)[n, 2n - 1]$  of size  $n$  is balanced, for all  $n \equiv 0$  or  $-1 \pmod{12m}$ .*

**Conjecture C2.** *Let  $m$  be a positive integer not divisible by 64. Let  $k$  and  $m_0$  be the integers such that  $m = 2^k \cdot m_0$  with  $m_0$  odd and  $k \in \{0, 1, 2, 3, 4, 5\}$ . Let  $S$  be the 12-interlaced arithmetic progression of  $\mathbb{Z}/m\mathbb{Z}$  defined by*

$$\pi_{2^k}(S) = \text{IAP}((1, 22, 13, 12, 26, 7, 0, 4, 5, 10, 12, 16), (17, 2, 23, 17, 30, 7, 7, 30, 17, 23, 2, 17))$$

and

$$\pi_{m_0}(S) = \pi_{m_0}(\text{US}).$$

Then, the Steinhaus triangle  $\nabla S[n, 2n - 1]$  of size  $n$  is balanced, for all  $n \equiv 0$  or  $-1 \pmod{12m}$ .

The proof of these conjectures would give a positive answer to the weak Molluzzo Problem for all positive integers  $m$  not divisible by 64.

**Problem P5.** For every even number  $m$ , is it possible to obtain infinitely many balanced Steinhaus triangles modulo  $m$  by lifting the universal sequence US to  $\mathbb{Z}/m\mathbb{Z}$ ?

## $P$ -type functions for numerical semigroups

Let  $s_1, \dots, s_n$  be coprime positive integers. Let

$$S = \langle s_1, \dots, s_n \rangle = \left\{ \sum_{i=1}^n x_i s_i \mid \forall i \in \{1, \dots, n\}, x_i \in \mathbb{N} \right\}$$

denote the numerical semigroup generated by  $s_1, \dots, s_n$ .

As seen in Chapter 6, given a particular (arithmetical, number theoretical, etc.) *Property P*, one might consider the following two  $P$ -type functions of a numerical semigroup  $S$ :

$${}^P r(S) := \text{the largest integer having property } P \text{ not belonging to } S$$

and

$${}_P r(S) := \text{the smallest integer having property } P \text{ belonging to } S.$$

Notice that the multiplicity and the Frobenius number of a numerical semigroup are  $P$ -type functions where  $P$  is the property of being a positive integer. The case where  $P$  is the property of being a square number is studied Chapter 6 and in [C17].

**Problem P6.** Study these two  $P$ -type functions for other properties  $P$  such as being a triangular number, being a prime number or even being a cube, and this for different families of numerical semigroups.

## Generalizations of Frame-Stewart numbers

In Chapter 7, generalized Frame-Stewart numbers have been studied. These numbers are related to the Tower of Hanoi problem on graphs.

Stewart's recursive algorithm for the  $k$ -peg Tower of Hanoi problem is summarized as follows. For a positive integer  $t \leq n$ ,

- i) recursively transfer a pile of  $n - t$  smallest disks from the first peg to a temporary peg, using  $k$  pegs;
- ii) transfer the remaining pile of  $t$  largest disks from the first peg to the final peg using  $k - 1$  pegs, ignoring the peg occupied by the  $n - t$  smallest disks;
- iii) recursively transfer the pile of  $n - t$  smallest disks from the temporary peg to the final peg, using  $k$  pegs.

The algorithm chooses the integer  $t$  such that the number of moves is minimized.

**Definition** (Frame-Stewart numbers). The Frame-Stewart numbers  $S_k(n)$  satisfy the following recurrence relations:

$$S_k(n) = \min_{1 \leq t \leq n} \{2 \cdot S_k(n-t) + S_{k-1}(t)\}, \text{ for } n \geq 1, k \geq 4,$$

$$S_3(n) = 2^n - 1, \text{ for } n \geq 1, \text{ and } S_k(0) = 0, \text{ for } k \geq 3.$$

In Chapter 7 and in [C14], the Frame-Stewart numbers are generalized as follows and exact formulas are obtained.

**Definition** (Generalized Frame-Stewart numbers). Let  $(p_i)_{i \geq 3}$  and  $(q_i)_{i \geq 3}$  be two sequences of arbitrary positive integers. Let  $G_k(n)$  be the integers recursively defined by

$$G_k(n) = \min_{1 \leq t \leq n} \{p_k \cdot G_k(n-t) + q_k \cdot G_{k-1}(t)\}, \text{ for } n \geq 1, k \geq 4,$$

$$G_3(n) = p_3 \cdot G_3(n-1) + q_3, \text{ for } n \geq 1, \text{ and } G_k(0) = 0, \text{ for } k \geq 3.$$

The numbers  $G_k(n)$  are called the *generalized Frame-Stewart numbers*.

The main result is that the sequence of differences  $(G_k(n) - G_k(n-1))_{n \geq 1}$  consists of numbers of the form  $(\prod_{i=3}^k q_i) \cdot (\prod_{i=3}^k p_i^{\alpha_i})$ , with  $\alpha_i \geq 0$  for all  $i$ , arranged in nondecreasing order.

**Theorem** (Chappelon-Matsuura [C14]). *For every positive integer  $n$  and for two sequences of arbitrary positive integers  $(p_i)_{i \geq 3}$  and  $(q_i)_{i \geq 3}$ , we have*

$$G_k(n) = q \cdot \sum_{j=1}^n u_j^k$$

where  $q = \prod_{i=3}^k q_i$  and  $u_j^k$  is the  $j$ th term of the sequence  $(u_j^k)_{j \geq 1}$  of integers  $\prod_{i=3}^k p_i^{\alpha_i}$ , with  $\alpha_i \geq 0$  for all  $i$ , arranged in nondecreasing order.

The main objective here is to consider a more general version of generalized Frame-Stewart numbers and to obtain a similar result as in [C14]. More precisely, the following numbers are considered.

**Definition.** Let  $(F(n))_{n \in \mathbb{N}}$  be the family of non-negative integers defined by

$$F(0) = 0,$$

$$F(n) = \min \left\{ \sum_{i=1}^k F_i(m_i) \mid (m_1, \dots, m_k) \in \mathbb{N}^k, \sum_{i=1}^k m_i = n \right\} \text{ for } n \geq 1,$$

where, for every  $i \in \{1, 2, \dots, k\}$ ,  $(F_i(n))_{n \in \mathbb{N}}$  is a family of non-negative integers with  $F_i(0) = 0$ . For all positive integers  $n$ , the differences are denoted by  $\Delta F_i(n) = F_i(n) - F_i(n-1)$ , for all  $i \in \{1, \dots, k\}$ , and  $\Delta F(n) = F(n) - F(n-1)$ .

In a joint work in progress with Akihiro Matsuura, the following result was obtained.

**Theorem.** Let  $n$  be a positive integer. Suppose that, for all  $i \in \{1, \dots, k\}$ , the subsequence  $(\Delta F_i(j))_{1 \leq j \leq n}$  is monotonically increasing and consider the sequence  $(u_j)_{1 \leq j \leq kn}$  of these  $kn$  integers sorted in increasing order. Then,

$$F(l) = \sum_{j=1}^l u_j$$

for all  $l \in \{1, 2, \dots, n\}$ , i.e.,  $u_l = \Delta F(l)$  for all  $l \in \{1, 2, \dots, n\}$ .

*Remark.* The sequence of differences  $(\Delta F(l))_{1 \leq l \leq n}$  is monotonically increasing too.

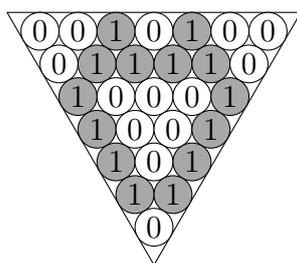
**Problem P7.** The question here is to study the generalization of Frame-Stewart numbers introduced before and to obtain explicit formulas in the case of the Tower of Hanoi problem on graphs.

# Résumé en français

Ce mémoire d'habilitation est organisé en deux parties indépendantes : une première partie qui traite de mes principaux résultats sur les triangles et les graphes de Steinhaus et une seconde partie à propos d'autres résultats obtenus sur les semigroupes numériques, des problèmes du type des tours de Hanoï, la théorie de Ramsey et sur les transversaux de Kneser.

Le thème de la première partie, les triangles et les graphes de Steinhaus, est mon plus ancien sujet de recherche, sur lequel j'ai commencé à travailler au cours de ma thèse de doctorat, soutenue en 2008. Depuis, j'ai continué à travailler et explorer divers problèmes sur ces structures ou sur des généralisations de celles-ci. Cette partie est divisée en cinq chapitres : deux chapitres à propos des triangles de Steinhaus binaires, le premier chapitre sur les triangles équilibrés et le second sur les triangles de Steinhaus possédant des propriétés de symétrie, un chapitre sur les graphes de Steinhaus et finalement deux chapitres sur de possibles généralisations des triangles de Steinhaus binaires, un sur une généralisation sur les nombres modulaires et un autre chapitre pour les généralisations en dimensions supérieures avec les simplexes équilibrés.

Dans le Chapitre 1, la structure de triangles de Steinhaus binaires est abordée. Il s'agit ici de l'une des structures mathématiques les plus élémentaires, que même un écolier pourrait comprendre. La figure ci-dessous est constituée de 14 zéros et de 14 uns. Ils sont arrangés de telle sorte que l'on a le chiffre zéro sous chaque paire d'éléments identiques et le chiffre un sous chaque paire d'éléments distincts. Si la première ligne était constituée



par  $n$  termes, alors dans une figure analogue on retrouverait  $\frac{n(n+1)}{2}$  éléments, l'exemple précédent correspondant au cas  $n = 7$ . Dans son livre intitulé "One hundred problems in elementary mathematics", Hugo Steinhaus posa la question en 1958 de savoir s'il est possible de construire une figure, analogue à la précédente, contenant  $n$  éléments dans sa première ligne. Un triangle de Steinhaus binaire est un triangle binaire, pointant vers le bas, construit avec la même règle locale. Un triangle est dit équilibré s'il contient autant de zéros que de uns. Le problème de Steinhaus a été résolu positivement pour la première fois en 1972 par Heiko Harborth. Il a montré que, pour tout entier positif  $n \equiv 0$  ou  $3 \pmod{4}$ , il existe une suite binaire de longueur  $n$  qui est la première ligne d'un

triangle de Steinhaus équilibré. Sa preuve est constructive et se base sur l'étude des suites binaires pseudo-périodiques. Depuis, de nombreuses solutions du problème de Steinhaus sont apparues dans la littérature. Toutes ces solutions sont constructives et mettent en jeu des suites binaires pseudo-périodiques. Quelques unes de ces solutions sont présentées dans ce chapitre, après avoir expliquer en quoi le choix de suites binaires pseudo-périodiques est pertinent.

Dans la dernière partie de ce chapitre, une solution récente du problème de Steinhaus obtenue à partir de suites binaires totalement périodiques est présentée. Si l'on applique la même règle locale à une suite binaire infinie, au lieu d'une suite finie, on obtient une grille infinie de zéros et de uns que nous appelons l'orbite de la suite. Le résultat principal de ce chapitre est qu'il existe des suites binaires périodiques, de période 24, dont l'orbite contient des triangles de Steinhaus binaires équilibrés pour toutes les tailles admissibles. De plus, ce résultat est également valide pour une généralisation du problème de Steinhaus dans deux directions : pour les triangles de Steinhaus binaires contenant un nombre impair d'éléments et dont le nombre de zéros et de uns ne diffèrent que d'au plus 1 (les triangles presque-équilibrés) mais aussi pour les triangles binaires, pointant vers le haut, construit avec la même règle locale (les triangles de Pascal généralisés). La force de ce résultat est qu'à partir d'une même suite binaire périodique, il est possible d'obtenir de manière explicite l'existence de triangles binaires presque-équilibrés, pour les deux types de triangles binaires, pour toutes les tailles possibles. Ce résultat a été publié dans [C7].

L'étude des triangles de Steinhaus binaires continue au Chapitre 2 avec les triangles binaires ayant des propriétés symétriques. Il est facile de voir que l'ensemble  $\mathcal{ST}(n)$  des triangles de Steinhaus binaires de taille  $n$ , c'est-à-dire, dont la première ligne est de longueur  $n$ , est un espace vectoriel sur  $\mathbb{Z}/2\mathbb{Z}$  de dimension  $n$ . De plus, la règle locale implique que  $\mathcal{ST}(n)$  est stable par la rotation d'angle 120 degrés ainsi que pour la réflexion horizontale. Cela implique une représentation fidèle du groupe diédral  $D_3$  sur  $\mathcal{ST}(n)$ . Un triangle de Steinhaus binaire est dit être symétrique par rotation, horizontalement symétrique ou diédralement symétrique s'il est invariant par la rotation d'angle 120 degrés, la réflexion horizontale ou les deux, respectivement. Les sous-espaces vectoriels  $\mathcal{RST}(n)$ ,  $\mathcal{HST}(n)$  et  $\mathcal{DST}(n)$  des triangles de Steinhaus symétriques par rotation, horizontalement symétriques et diédralement symétriques ont été étudiés par André Barbé dans [26], où il a déterminé pour chacun d'eux leur dimension. Des bases de  $\mathcal{RST}(n)$ ,  $\mathcal{HST}(n)$  et  $\mathcal{DST}(n)$  ont été obtenues par Josep Brunat et Montserrat Maureso dans [39]. Dans le Chapitre 2, pour chacun de ces trois sous-espaces vectoriels, de nouvelles bases, plus simples que celles précédemment citées, sont données. Elles sont obtenues en considérant des propriétés élémentaires des coefficients binomiaux généralisés. Un manuscrit [C8] sur ces résultats a été soumis à publication.

Le Chapitre 3 concerne la structure de graphes de Steinhaus. Un graphe de Steinhaus d'ordre  $n$  est un graphe simple dont la partie triangulaire supérieure de la matrice d'adjacence est un triangle de Steinhaus binaire de taille  $n - 1$ . Il est facile de voir que l'ensemble  $\mathcal{SG}(n)$  des graphes de Steinhaus d'ordre  $n$  est un espace vectoriel sur  $\mathbb{Z}/2\mathbb{Z}$  de dimension  $n - 1$  qui est isomorphe à  $\mathcal{ST}(n - 1)$ . Cette famille de graphes simples a été introduite par John C. Molluzzo dans [89]. Franz A. Delahan a montré dans [48] que tout graphe simple d'ordre  $n$  est isomorphe à un sous-graphe induit d'un graphe de Steinhaus d'ordre  $\binom{n}{2} + 1$ . Un problème classique sur les graphes de Steinhaus est de caractériser ceux, ou leurs suites binaires associées, ayant une propriété graphique donnée telle que la connectivité, la planarité, la propriété d'être biparti, la régularité, etc. Dans le Chapitre 3, nous nous

concentrons sur les graphes de Steinhaus réguliers, c'est-à-dire, les graphes de Steinhaus où tous les sommets sont de même degré, et sur le cas plus faible des graphes de Steinhaus réguliers de parité, c'est-à-dire, les graphes de Steinhaus où les degrés des sommets sont tous de même parité. Un graphe de Steinhaus régulier de parité est soit pair, soit impair, suivant la parité de ses sommets. La première partie du Chapitre 3 est une étude complète du sous-espace vectoriel  $\mathcal{ESG}(n)$  des graphes de Steinhaus pairs d'ordre  $n$ . La caractérisation des graphes de Steinhaus pairs a déjà été faite par Wayne M. Dymacek dans [51], où il détermine la dimension de  $\mathcal{ESG}(n)$  en prouvant que la matrice d'adjacence d'un graphe de Steinhaus pair est symétrique par rapporte à son antidiagonale. On retrouve ce résultat ici en montrant que les éléments de l'antidiagonale de la matrice d'adjacence d'un graphe de Steinhaus, non nécessairement pair, s'expriment en fonction des degrés de ses sommets. Ce dernier résultat a été publié dans [C3]. Après cela, il est montré qu'il existe un isomorphisme entre  $\mathcal{ESG}(n)$  et un certain sous-espace vectoriel de  $\mathcal{DST}(n)$ . Ainsi, en utilisant les résultats mis en lumière au Chapitre 2, des bases explicites de  $\mathcal{ESG}(n)$  et  $\mathcal{PRSG}(n)$  sont données, pour tout entier positif  $n$ . Ce nouveau résultat vient de [C8].

La dernière partie de ce chapitre concerne les graphes de Steinhaus réguliers. La liste complète des graphes de Steinhaus réguliers a été conjecturée par Wayne M. Dymacek dans [51]. Depuis, cette conjecture a été vérifiée jusque 117 sommets par Shalom Eliahou et Maxime Augier dans [24]. En fin de Chapitre 3, nous montrons comment les résultats obtenus sur  $\mathcal{DST}(n)$  permettent de compléter l'étude des graphes de Steinhaus réguliers et permettent de pousser la vérification de la conjecture de Dymacek jusqu'à 1500 sommets pour les graphes de Steinhaus réguliers de degré impair. Ce résultat a été publié dans [C3].

Au Chapitre 4, la structure de triangles de Steinhaus binaires est étendue au cas des triangles de nombres modulo un entier positif  $m$  construits avec la même règle locale que le triangle de Pascal classique modulo  $m$ . Un triangle de Steinhaus modulo  $m$  est dit équilibré s'il contient tous les éléments de  $\mathbb{Z}/m\mathbb{Z}$  avec la même multiplicité. En 1976, John C. Molluzzo demanda s'il existe un triangle de Steinhaus équilibré de taille  $n$  modulo  $m$ , pour tous entiers positifs  $m$  et  $n$  tels que  $\binom{n+1}{2}$  soit divisible par  $m$ . Pour  $m = 2$ , cela correspond au problème de Steinhaus introduit au Chapitre 1. Le problème de Molluzzo constitue le sujet principal du Chapitre 4. Jusqu'à ma thèse de doctorat, ce problème était résolu positivement uniquement pour  $m = 3$  et  $5$  dans [30]. Depuis, ce problème a été résolu positivement par une approche constructive pour de petites valeurs de  $m$  : pour  $m \in \{3, 5, 7\}$  dans [C1] et pour  $m = 4$  dans [C9]. Les premiers contre-exemples sont apparus dans [C1], où il est montré qu'il n'existe pas de triangles de Steinhaus équilibrés de taille 5 dans  $\mathbb{Z}/15\mathbb{Z}$  et de taille 6 dans  $\mathbb{Z}/21\mathbb{Z}$ . Néanmoins, ce problème admet une réponse positive pour une infinité de valeur de  $m$ . Dans la première partie de ce chapitre, on montre qu'il existe des triangles de Steinhaus équilibrés pour toutes les tailles possibles, dans le cas où  $m$  est une puissance de 3. Ce résultat est obtenu en étudiant les triangles de Steinhaus associés aux suites arithmétiques. Le résultat principal ici est que, lorsque  $m$  est un nombre impair, le triangle de Steinhaus dont la première ligne est une suite arithmétique de raison inversible et de longueur  $k\alpha m$  est équilibré, pour tout entier strictement positif  $k$  et où  $\alpha$  est l'ordre multiplicatif de  $2^m$  modulo  $m$ . Cela implique qu'il existe une infinité de triangles de Steinhaus équilibrés modulo  $m$  dès que le nombre  $m$  est impair. En particulier, lorsque  $m$  est une puissance de 3, cela permet de répondre positivement au problème de Molluzzo dans ce cas. Ce résultat a été publié dans [C2].

Dans la seconde partie du Chapitre 4, nous nous intéressons à une suite d'entiers

particulière

$$US = (\dots, -3, -3, 5, -2, -2, 3, -1, -1, 1, 0, \mathbf{0}, -1, 1, 1, -3, 2, 2, -5, 3, 3, -7, \dots)$$

qui est appelée la suite universelle. Cette suite peut être vue comme un entrelacement de trois suites arithmétiques. On étudie alors les triangles de Steinhaus, ou plus généralement les orbites, associés aux suites arithmétiques entrelacées et l'on obtient que l'orbite de la projection de  $US$  dans  $\mathbb{Z}/m\mathbb{Z}$  contient une infinité de triangles de Steinhaus équilibrés, lorsque  $m$  est un nombre impair. Plus précisément, pour tout nombre impair  $m$ , cette orbite contient des triangles de Steinhaus équilibrés de taille  $km$ , pour tout entier strictement positif  $k$ . La preuve de ce résultat est basée sur les triangles arithmétiques qui sont des triangles de nombres modulo  $m$ , non-nécessairement des triangles de Steinhaus, où toutes les lignes et toutes les diagonales sont des suites arithmétiques de mêmes raisons, respectivement. Il s'agit ici d'un progrès notable par rapport au résultat précédent sur les suites arithmétiques car l'on obtient maintenant l'existence de triangles de Steinhaus équilibrés pour toutes les tailles multiples de  $m$ , et pas seulement multiples de  $\alpha m$ . Ce résultat ne résout pas complètement le problème de Molluzzo pour d'autres moduli mais, dans le cas où  $m$  est une puissance de premier impair, nous savons qu'il existe des triangles de Steinhaus équilibrés pour  $2/3$  des tailles admissibles. Ces résultats sur la suite universelle ont été publiés dans [C5].

Dans la dernière partie du Chapitre 4, une méthode constructive qui consiste à essayer de relever dans  $\mathbb{Z}/4\mathbb{Z}$  des solutions déjà connues de  $\mathbb{Z}/2\mathbb{Z}$  est proposée. Cela permet d'obtenir une réponse complète et positive au problème de Molluzzo pour  $m = 4$ . A ce jour, il s'agit ici de la seule solution à ce problème pour  $m > 2$  pair. Ce travail a été réalisé en collaboration avec Shalom Eliahou et a été publié dans [C9].

Dans le Chapitre 5, on considère la structure de triangles de Steinhaus pour d'autres règles locales et en dimensions supérieures, pas seulement en dimension 2. Un automate cellulaire additif est une application linéaire sur l'ensemble des tableaux infinis multidimensionnels d'éléments dans  $\mathbb{Z}/m\mathbb{Z}$ . Dans ce chapitre, nous considérons les simplexes apparaissant dans les orbites engendrées à partir de tableaux arithmétiques par des automates cellulaires additifs. Il se trouve que ce sont des sources de simplexes équilibrés, c'est-à-dire, de simplexes contenant tous les éléments de  $\mathbb{Z}/m\mathbb{Z}$  avec la même multiplicité. Pour tout automate cellulaire additif de dimension 1 ou plus, l'existence d'une infinité de simplexes équilibrés dans  $\mathbb{Z}/m\mathbb{Z}$  apparaissant dans de telles orbites est obtenu, et ce pour une infinité de valeurs de  $m$ . Le cas particulier des automates cellulaires de Pascal, les automates cellulaires engendrant les simplexes de Pascal, c'est-à-dire une généralisation du triangle de Pascal en dimension quelconque, est étudié en détails. Cela constitue une généralisation naturelle du résultat sur les triangles de Steinhaus associés aux suites arithmétiques du Chapitre 4. Cette généralisation a été publiée dans [C6].

La seconde partie de ce mémoire d'habilitation est divisée en quatre chapitres. Chacun de ces chapitres est à propos d'un thème de recherche sur lequel j'ai commencé à travailler après mon doctorat : les semigroupes numériques, les problèmes de type tours de Hanoï, la théorie de Ramsey et les transversaux de Kneser.

Le Chapitre 6 concerne les semigroupes numériques. Un semigroupe est un ensemble avec une loi de composition interne qui est associative. Un semigroupe numérique  $S$  est un sous-semigroupe de  $\mathbb{N}$  tel que  $0 \in S$  et  $\mathbb{N} \setminus S$  soit fini. Il est connu que, pour tout semigroupe numérique  $S$ , il existe un sous-ensemble non-vide minimal  $A = \{a_1, \dots, a_n\}$

d'entiers strictement positifs et premiers entre eux tel que  $S = \langle A \rangle$ , c'est-à-dire, chaque élément de  $S$  peut s'exprimer comme  $x_1 a_1 + \dots + x_n a_n$ , où  $x_1, \dots, x_n$  sont des entiers naturels. L'ensemble  $A$  est appelé l'ensemble des générateurs de  $S$ . Pour tout semigroupe numérique  $S$ , on considère le poset localement fini  $(\mathbb{Z}, \leq_S)$  induit par  $S$  sur l'ensemble des entiers  $\mathbb{Z}$  défini par  $x \leq_S y$  si et seulement si  $y - x \in S$ , pour tous entiers  $x$  et  $y$ . La fonction de Möbius est un concept important associé aux posets localement finis qui a été introduit par Gian-Carlo Rota dans [98]. Dans la première partie du Chapitre 6, nous nous intéressons à la fonction de Möbius  $\mu_S$  associée au poset  $(\mathbb{Z}, \leq_S)$ , où  $S$  est un semigroupe numérique. Jusque 2013, le seul résultat connu concernant  $\mu_S$  était un vieux théorème dû à James A. Deddens en 1979 [47] qui détermine la valeur de  $\mu_S$  pour  $S = \langle a_1, a_2 \rangle$  le semigroupe numérique à deux générateurs. Ici, nous introduisons une nouvelle approche pour étudier  $\mu_S$  lorsque  $S$  est un semigroupe numérique arithmétique, c'est-à-dire, lorsque  $S = \langle a, a + d, \dots, a + kd \rangle$  pour des entiers positifs  $a, d, k$  tels que  $a$  et  $d$  soient relativement premiers et  $k < a$ . Le résultat principal obtenu ici est une formule récursive pour  $\mu_S$  lorsque  $S$  est un semigroupe numérique arithmétique. En particulier, cela permet d'obtenir une nouvelle preuve du résultat de Dedden ainsi qu'une formule explicite dans le cas où  $k = 2$  et  $a$  est pair, c'est-à-dire, lorsque  $S = \langle 2q, 2q + d, 2q + 2d \rangle$  pour des entiers positifs  $q$  et  $d$  premiers entre eux. Ce travail a été réalisé en collaboration avec Jorge Luis Ramírez Alfonsín et a été publié dans [C16].

Après cela, nous continuons l'étude de la fonction de Möbius associée aux posets localement finis  $(\mathbb{Z}^m, \leq_S)$  provenant dorénavant de sous-semigroupes  $S$  de  $\mathbb{Z}^m$ . Cela étend le travail précédemment introduit. Une nouvelle approche est développée afin d'étudier  $\mu_S$  en utilisant la série de Hilbert de  $S$ . Cette méthode nous a permis d'obtenir des formules explicites de  $\mu_S$  lorsque  $S$  est dans certaines familles de semigroupes tels que les semigroupes ayant un unique élément de Betti ou les semigroupes numériques à trois générateurs et d'intersection complète. Ce second travail sur  $\mu_S$  a été réalisé en collaboration avec Ignacio García-Marco, Luis Pedro Montejano et Jorge Luis Ramírez Alfonsín. Il a été publié dans [C10].

La dernière partie de ce chapitre est sur le nombre de Frobenius carré. Pour tout semigroupe numérique  $S$ , le nombre de Frobenius  $g(S)$  est le plus grand entier qui n'est pas un élément de  $S$ . Il est bien connu que  $g(\langle a_1, a_2 \rangle) = a_1 a_2 - a_1 - a_2$ . Cependant, déterminer de manière explicite  $g(S)$  est un problème difficile en général. Ici, nous nous intéressons à la variante suivante du nombre de Frobenius de  $S$  : soit  ${}^2r(S)$  le plus grand entier carré n'appartenant pas à  $S$ . Une borne supérieure de  ${}^2r(S)$  pour les semigroupes numériques arithmétiques  $S$  est donnée. Cette dernière s'avère être la valeur exacte de  ${}^2r(\langle s_1, s_2 \rangle)$  sous certaines conditions. Nous donnons une formule exacte pour  ${}^2r(\langle s_1, s_1 + d \rangle)$  lorsque  $d = 3, 4$  and  $5$ . Nous étudions  ${}^2r(\langle s_1, s_1 + 1 \rangle)$  et  ${}^2r(\langle s_1, s_1 + 2 \rangle)$  et mettons en avant deux conjectures. Il s'agit d'un travail en commun avec Jorge Luis Ramírez Alfonsín dont un manuscrit [C17] contenant ces résultats a été soumis à publication.

Dans le Chapitre 7, des problèmes de type tours de Hanoï sont présentés. Le problème de la tour de Hanoï a été introduit par Edouard Lucas en 1883 [85] dans le cas de 3 piquets et  $n$  disques de différentes tailles. Initialement,  $n$  disques sont placés sur un des 3 piquets avec le plus grand disque tout en bas. Alors, à chaque mouvement un des disque situé au dessus est déplacé sur un autre piquet contenant un disque plus grand sur le dessus ou sur un piquet vide. Le but de ce problème est de transférer tous les disques d'un piquet initial au piquet final avec un nombre minimal de mouvements. Un argument simple et récursif montre que  $2^n - 1$  mouvements sont nécessaires et suffisant pour mener à bien cette tâche. Ce problème de la tour de Hanoï a été étendu au cas de 4 piquets par

Dudeney en 1907 [50] et pour  $k \geq 3$  piquets par Stewart en 1939 [108]. En 1941, Frame [66] et Stewart [109] ont proposé de manière indépendante des algorithmes qui résolvent avec les mêmes nombres de mouvements ce problème de la tour de Hanoï à  $k$  piquets avec  $k \geq 4$  piquets. Ces nombres sont alors appelés les nombres de Frame-Stewart. Il est connu que ces nombres sont optimaux pour le problème à 4 piquets [37] et ils sont supposés optimaux pour  $k > 4$  piquets. Les nombres de Frame-Stewart sont définis par la formule de récursive suivante :

$$S_k(n) = \min_{1 \leq t \leq n} \{2 \cdot S_k(n-t) + S_{k-1}(t)\}, \quad S_3(n) = 2^n - 1.$$

Dans la première partie du Chapitre 7, nous considérons la généralisation suivante de cette formule de récursive

$$G_k(n) = \min_{1 \leq t \leq n} \{p_k \cdot G_k(n-t) + q_k \cdot G_{k-1}(t)\}, \quad G_3(n) = p_3 \cdot G_3(n-1) + q_3,$$

pour deux suites quelconques d'entiers positifs  $(p_i)_{i \geq 3}$  et  $(q_i)_{i \geq 3}$  et nous obtenons que la suite des différences  $(G_k(n) - G_k(n-1))_{n \geq 1}$  est constituée des nombres de la forme  $(\prod_{i=3}^k q_i) \cdot (\prod_{i=3}^k p_i^{\alpha_i})$ , où  $\alpha_i \geq 0$  pour tout  $i$ , ordonnée par ordre croissant. Nous appliquons également ce résultat afin d'analyser les relations de récurrence apparaissant dans les problèmes de tours de Hanoï sur certaines familles de graphes. Ce travail a été réalisé en collaboration avec Akihiro Matsuura et a été publié dans [C14].

Dans la dernière partie du Chapitre 7, on étudie des résultats obtenus sur un jeu combinatoire à deux joueurs basé sur les tours de Hanoï. Tout d'abord, il est bien connu que pour le problème à 3 piquets, le nombre minimal de mouvements afin de transférer une tour de  $n$  disques est  $2^n - 1$ . Mais il y a également d'autres variations de ce jeu, ajoutant par exemple des poids réels sur le mouvement des disques. Cela donne lieu à un type de problème similaire, mais où le score final cherche à être optimisé. Nous étudions les extensions du jeu à un joueur à deux joueurs, en considérant des conditions de victoire classiques dans la théorie des jeux combinatoires telles que le joueur qui se déplace en dernier gagne ou bien celui qui obtient le score le plus élevé gagne. Ici, nous résolvons le problème à 3 piquets pour ces deux conditions de victoire. Il s'agit d'un travail en commun avec Urban Larsson et Akihiro Matsuura qui a été publié dans [C11].

La théorie de Ramsey tente typiquement de répondre à des questions de la forme : combien d'éléments d'une certaine structure doivent être considérés pour qu'une propriété particulière se vérifie? Dans la première partie du Chapitre 8, nous nous intéressons à un problème de théorie de Ramsey sur les nombres. Pour tous entiers positifs  $l$  et  $m$ , un ensemble d'entiers est dit être (faiblement) libre de  $l$ -somme modulo  $m$  s'il ne contient pas d'éléments  $x_1, x_2, \dots, x_l, y$  (deux à deux distincts) satisfaisant la congruence  $x_1 + \dots + x_l \equiv y \pmod{m}$ . Il est connu que, pour tous entiers  $k$  et  $l$ , il existe un plus grand entier  $n$  pour lequel l'ensemble des  $n$  premiers entiers positifs  $\{1, 2, \dots, n\}$  admette une partition en  $k$  parties qui sont (faiblement) libre de  $l$ -somme modulo  $m$ . Ce nombre est appelé le nombre de Schur généralisé (faible) modulo  $m$ , associé à  $k$  et  $l$ . Il s'agit d'une version modulaire du nombre de Schur (faible) classique dont on ne connaît que très peu de valeurs exactes. Il est facile de voir que les nombres de Schur modulaires constituent une borne inférieure des nombres de Schur classiques. Ici, pour tous entiers positifs  $k$  et  $l$ , les valeurs exactes de ces nombres de Schur modulaires sont déterminées pour les moduli  $m = 1, 2$  and  $3$ . Ce travail a été réalisé en collaboration avec María Pastora Revuelta Marchena et María Isabel Sanz Domínguez et a été publié dans [C18].

Dans la dernière partie du Chapitre 8, plusieurs résultats de la théorie de Ramsey sur les graphes sont présentés. Pour tous graphes simples  $G$  et  $H$ , soit  $R(G, H)$  le plus petit entier  $N$  tel que pour toute 2-coloration (disons rouge et bleu) des arêtes du graphe complet  $K_n$ , où  $n \geq N$ , il y a soit une copie rouge de  $G$ , soit une copie bleue de  $H$ . Il est bien connu, par le théorème de Ramsey, que  $R(G, H)$  est toujours fini, pour tous graphes  $G$  et  $H$ . Après l'obtention de différentes formules récursives, nous obtenons de nouvelles valeurs exactes ou de nouvelles bornes de nombres de Ramsey pour différentes familles de graphes : en particulier pour les graphes complets privés d'un arête ou d'un triangle et pour les graphes complets privés d'une étoile. Il s'agit d'un travail en commun avec Luis Pedro Montejano and Jorge Luis Ramírez Alfonsín. Ces résultats ont été publiés dans [C15, C19].

Le dernier chapitre de ce mémoire concerne les transversaux de Kneser. Pour tous entiers positifs  $k, d, \lambda$  avec  $d, k \geq \lambda$ , soit  $m(k, d, \lambda)$  le plus grand entier  $n$  tel que tout ensemble de  $n$  points (non-nécessairement en position générale) dans  $\mathbb{R}^d$  a la propriété que les enveloppes convexes des sous-ensembles à  $k$  éléments admettent un  $(d - \lambda)$ -plan transversal commun. Les inégalités suivantes ont été obtenues dans [23]

$$d - \lambda + k + \left\lceil \frac{k}{\lambda} \right\rceil - 1 \leq m(k, d, \lambda) < d + 2(k - \lambda) + 1.$$

Il s'avère que  $m(k, d, \lambda)$  est fortement connecté avec d'autres problèmes intéressants, par exemple, le nombre chromatique des hypergraphes de Kneser et une version discrète du théorème du point central de Rado. Dans le Chapitre 9, nous introduisons de manière naturelle une version discrète  $m^*$  de  $m$  en considérant l'existence de transversaux complets de Kneser. Un transversal complet de Kneser d'un sous-ensemble fini  $X$  de  $\mathbb{R}^d$  est un  $(d - \lambda)$ -plan qui est transversal aux enveloppes convexes des  $k$ -sous-ensembles de  $X$  et qui contient  $(d - \lambda) + 1$  points de  $X$ . Nous étudions la relation entre  $m^*$  et  $m$  et donnons quelques bornes inférieures et supérieures de  $m^*$  ainsi que certaines valeurs exactes dans certains cas. L'ingrédient principal des preuves est aussi bien le théorème de partition de Randon que des outils de la théorie des matroïdes orientés. En étudiant le matroïde orienté alterné, nous obtenons le comportement asymptotique de la fonction  $m^*$  pour la famille des polytopes cycliques. Ce travail a été réalisé en collaboration avec Leonardo Martínez-Sandoval, Luis Montejano, Luis Pedro Montejano and Jorge Luis Ramírez Alfonsín et a été publié dans [C12].

Dans la dernière partie du Chapitre 9, nous nous concentrons sur l'existence de transversaux de Kneser (complets) pour  $\lambda = 2, 3$ . Dans ce but, nous introduisons les notions de stabilité et d'instabilité pour les transversaux de Kneser (complets). Tout d'abord, nous donnons un résultat de stabilité pour les collections de  $d + 2(k - \lambda)$  points dans  $\mathbb{R}^d$  avec  $k - \lambda \geq 2$  et  $\lambda = 2, 3$ . Nous présentons alors une description des transversaux de Kneser  $L$  des collections de  $d + 2(k - \lambda)$  points dans  $\mathbb{R}^d$  avec  $k - \lambda \geq 2$  pour  $\lambda = 2, 3$ . Nous montrons que soit  $L$  est un transversal de Kneser complet, soit il contient  $d - 2(\lambda - 1)$  points et les  $2(k - 1)$  points restants de  $X$  sont mis en correspondance dans  $k - 1$  paires de telle sorte que  $L$  intersecte les segments fermés correspondants déterminés par eux. Cela conduit à de nouvelles bornes supérieures et inférieures (dans le cas où  $\lambda = 2$  et  $3$ ) pour  $m(k, d, \lambda)$ . Finalement, en utilisant la théorie des matroïdes orientés, nous présentons des résultats calculatoires étroitement liés aux notions de stabilité et d'instabilité. Nous déterminons l'existence des transversaux de Kneser (complets) pour chacun des 246 types d'ordre différents de configurations de 7 points dans  $\mathbb{R}^3$ . Il s'agit également d'un travail en commun

avec Leonardo Martínez-Sandoval, Luis Montejano, Luis Pedro Montejano and Jorge Luis Ramírez Alfonsín qui a été publié dans [C13].

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