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THÈSE

Pour l'obtention du grade de docteur de

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École doctorale: Sciences et Techniques de l'Ingénieur de Tunis

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présentée par

Yosra BOUGHANJA

Inégalités de stabilité dans les problèmes inverses pour équations parabolique et de transport

Directeurs de thèse : Mourad BELLASSOUED et Éric SOCCORSI

Soutenue le 18 décembre 2020

Devant le jury composé de :

Mourad BELLASSOUED	Professeur, Université de Tunis El Manar ENIT -LAMSIN	Encadrant
Éric SOCCORSI	Maitre de conférences HDR, Aix Marseille Université-CPT	Encadrant
Thierry DAUDÉ	Maitre de conférences HDR, Université de Franche-Comté	Rapporteur
Moez KHENISSI	Professeur, Université de Sousse	Rapporteur
Amel BEN ABDA	Professeur, Université de Tunis El Manar	Présidente
Assia BENABDALLAH	Professeur, Aix Marseille Université	Examinatrice
Anna DOUBOVA	Professeur, Université de Séville	Invitée

THÈSE

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Abstract

This PhD is devoted to the determination of unknown coefficients appearing in the linear Boltzmann equation or the parabolic equation. Its main objective is to establish results of uniqueness and stability in the identification of these coefficients from ad-hoc data. The first part of the thesis is devoted to the study of the inverse problems associated with the linear Boltzmann equation (transport equation). The inverse problem considered concerns the identification of the diffusion and absorption coefficients from knowledge of the albedo operator, which models all possible boundary measurements. The second part of the thesis is devoted to treat inverse problems that concern parabolic equations. Precisely, we specify our study on determining a coefficient, arising in the parabolic equation, from full and partial measurement of the solution on the boundary in an arbitrary time interval.
Key words: Inverse problem, Albedo operator Stabilty estimate, Dirichlet to Neumann operator, Parabolic equation, Lineair Boltzmann equation.

Résumé

Cette thèse est consacrée à la détermination de coefficients inconnus apparaissant dans l'équation linéaire de Boltzmann ou l'équation de la chaleur. Son objectif principal est d'établir des résultats d'unicité et de stabilité dans l'identification de ces coefficients à partir de données ad-hoc.
La première partie de la thèse est consacrée à l'étude des problèmes inverses associés à l'équation linéaire de Boltzmann (équation de transport).
Le problème inverse considéré concerne l'identification des coefficients de diffusion et d'absorption à partir de la mesure latérale du flux sortant de la solution.
La deuxième partie de la thèse est consacrée à la résolution d'un problème inverse pour l'équation de la chaleur. Nous étudions la stabilité de la détermination du coefficient d'ordre zéro à partir de données de type Neumann complètes ou partielles, mesurées sur un intervalle de temps arbitraire.

Mots clés: Problème inverse, Opérateur d'albédo, Estimation de stabilité opérateur, Dirichlet-Neuman, Équation parabolique, Équation de Boltzmann linéaire.

Résumé détaillé des résultats obtenus

Les travaux menés au cours de cette thèse ont eu pour principal objet l'identification de coefficients apparaissant dans l'équation linéaire de Boltzmann ou l'équation de la chaleur. La première partie de la thèse est consacrée à l'étude des problèmes inverses en transport. Il s'agit d'identifier le coefficient d'absorption et le coefficient de diffusion intervenants dans l'équation de Boltzmann linéaire, à partir de l'opérateur d'albédo qui nous permet de mesurer le flux sortant à partir du flux entrant. La deuxième partie de la thèse est consacrée à la résolution d'un problème inverse pour l'équation de la chaleur. Nous étudions la stabilité de la détermination du coefficient d'ordre zéro à partir de données de type Neumann complètes ou partielles, mesurées sur un intervalle de temps arbitraire.

L'équation de Boltzmann linéaire

L'équation de Boltzmann linéaire décrit l'évolution d'une population de particules physiques (neutrons, photons,...), sous l'hypothèse qu'elles n'interagissent pas entre-elles (ce qui nous amène à écarter les termes non linéaires).
Soit Ω un domaine convexe et borné de $\mathbb{R}^n$ ($n \geq 2$) de classe C^1 et $\mathscr{S} = \mathbb{S}^{n-1}$ la sphère unité de $\mathbb{R}^n$. On note $Q = \mathscr{S} \times \Omega$ et $\Sigma = \mathscr{S} \times \partial\Omega$. On note $\nu(x)$ le vecteur unitaire normal en tout point x de la frontière $\partial\Omega$, et on considère les ensembles suivants :

$$\Gamma^{\pm} = \{(x,\theta) \in \Gamma \,:\, \pm\theta \cdot \nu(x) > 0\}, \quad \Sigma^{\pm} = [0,T] \times \Gamma^{\pm}, \quad \Gamma = \partial\Omega \times \mathscr{S}.$$

L'ensemble Σ^+(respectivement Σ^-) est l'ensemble des points de l'espace des phases correspondant à des particules entrantes (respectivement sortantes). Étant donnée $f \in L^p(0,T;L^p(\Gamma^-,d\xi))$, $p \in [1,+\infty[$, et une fonction $a(x,\theta) \in L^\infty(Q)$, le problème direct consiste à trouver $u = u(t,x,\theta)$, solution de ce système:

$$\begin{cases} \partial_t u + \theta \cdot \nabla u(t,x,\theta) + a(x,\theta)u = \mathscr{N}_k[u](t,x,\theta) & \text{in} \quad Q_T := (0,T) \times Q, \\ u(0,x,\theta) = u_0 & \text{in} \quad Q, \\ u(t,x,\theta) = f(t,x,\theta) & \text{on} \quad \Sigma^-, \end{cases} \tag{0.0.1}$$

où t et x sont respectivement la variable de temporelle et la variable spatiale. Ici, u_0 la condition initiale, f désigne la condition aux limites et T est la durée de l'expérience. L'opérateur intégral $\mathscr{N}_k$ est donné par :

$$\mathscr{N}_k[u](t,x,\theta) = \int_{\mathscr{S}} k(x,\theta,\theta')u(t,x,\theta')d\theta',$$

où $d\theta'$ désigne la mesure de surface sur $\mathscr{S}$ associée à la mesure de Lebesgue dans $\mathbb{R}^n$.

Le problème inverse auquel nous nous intéressons dans cette partie consiste à identifier a et k à partir de la mesure de flux sortant à travers la partie Σ^+ de la frontière du domaine de la solution u de l'équation (0.0.1). Cela nous amène à considérer l'opérateur d'albédo $\mathscr{A}_{a,k}$ qui nous exprime le flux sortant à partir du flux entrant.

Problème direct

Dans cette partie, nous allons introduire les notations spécifiques à ce contexte et nous donnerons un théorème d'existence et d'unicité de la solution au problème de Boltzmann linéaire.

On considère $\mathscr{L}_p^\pm(\Sigma^\pm) := L^p(0,T;L^p(\Gamma^\pm,d\xi))$, où $d\xi = |\omega \cdot \nu(x)|d\sigma d\theta$. On introduit aussi les espaces:

$$\mathscr{W}_p := \{u \in L^p(Q),\ \theta \cdot \nabla u \in L^p(Q);\ \int_\Gamma |\theta \cdot \nu(x)|u(x,\theta)|^p d\theta dx < \infty\}.$$

Étant donnés $M_0 > 0$, M_1et $M_2 > 0$, on considère les ensembles admissibles de coefficients a et k

$$\mathscr{A}(M_0) = \{a \in \mathscr{C}^1(\overline{Q});\ \|a\|_{L^\infty(Q)} \leq M_0\}.$$

et

$$\mathscr{K}(M_1, M_2) = \{k \in L^\infty(\Omega, L^2(\mathscr{S} \times \mathscr{S}));\ \int_{\mathscr{S}} |k(x,\theta,\theta')| d\theta' \leq M_1, \\ \int_{\mathscr{S}} |k(x,\theta,\theta')| d\theta \leq M_2\}.$$

Le problème direct associé à l'équation de Boltzmann linéaire (0.0.1) est résolu par le théorème d'existence et d'unicité suivant.

Théorème 1. *([43, Chap. 21, 3.1] et [P1, Appendice A]) Soient $T > 0$, $p \geq 1$, $(a,k) \in \mathscr{A}(M_0) \times \mathscr{K}(M_1, M_2)$ et $u_0 \in L^\infty(Q)$. Alors, pour tout $f \in \mathscr{L}_p^-(\Sigma^-)$, il existe une unique solution u de* (0.0.1)*, vérifiant*

$$u \in \mathscr{C}^1([0,T]; L^p(Q)) \cap \mathscr{C}([0,T]; \mathscr{W}_p).$$

De plus, il existe une constante $C > 0$, telle que

$$\|u(t,\cdot,\cdot)\|_{L^p(Q)} + \|u\|_{\mathscr{L}_p^+(\Sigma^+)} \leq C(\|u_0\|_{L^p(Q)} + \|f\|_{\mathscr{L}_p^-(\Sigma^-)}), \quad \forall\, t \in (0,T). \quad (0.0.2)$$

Classes de jauge équivalentes de l'équation de Boltzmann linéaire

Commençons par supposer que la condition initiale de (0.0.1) $u_0 = 0$, et que les données connues sont constituées par la mesures sur le bord Σ^+ de la solution u. Cela nous amène à considérer l'opérateur d'albédo $\mathscr{A}_{a,k}$, défini comme suit:

$$\begin{array}{rccl} \mathscr{A}_{a,k}: & \mathscr{L}_p^-(\Sigma^-) & \longrightarrow & \mathscr{L}_p^+(\Sigma^+) \\ & f & \longmapsto & \mathscr{A}_{a,k}[f] = u_{|\Sigma^+}. \end{array}$$

D'après (0.0.2), l'opérateur d'albédo $\mathscr{A}_{a,k}$ est continu de $\mathscr{L}_p^-(\Sigma^-)$ dans $\mathscr{L}_p^+(\Sigma^+)$. On note $\|\mathscr{A}_{a,k}\|_p$ sa norme.

Nous considérons le problème inverse de la détermination du coefficient d'absorption anisotrope (dépendant de la direction θ), pour lequel même l'unicité est un

problème ouvert. Dans [66] il est prouvé que l'opérateur d'albédo est invariant par transformation de jauge pour l'équation de transport stationnaire, ce qui constitue une objection à l'unicité de la détermination de a à partir de l'opérateur d'albédo. Pour l'équation de transport dépendant du temps, nous avons réussi à caractériser une famille de transformations de jauge des coefficients d'absorption et de diffusion, qui ne modifie pas l'opérateur d'albédo. Celui-ci ne permet donc au mieux de récupérer la classe de jauge correspondante. Soit $\mu \in \mathscr{C}^1(Q)$ tel que $\mu(x,\theta) > 0$ et $\mu(x,\theta) = 1$, pour $(x,\theta) \in \Gamma$. Alors pour $\tilde{u} = \mu(x,\theta)u$, on a

$$\mathscr{B}_{\tilde{a},\tilde{k}}[\tilde{u}](t,x,\theta) = \partial_t \tilde{u} + \theta \cdot \nabla \tilde{u}(t,x,\theta) + \tilde{a}(x,\theta)\tilde{u} - \mathscr{N}_{\tilde{k}}[\tilde{u}](t,x,\theta) = 0 \quad \text{dans } Q_T, \tag{0.0.3}$$

où $(\tilde{a},\tilde{k})$ est donné par

$$\tilde{a}(x,\theta) = a(x,\theta) - \theta \cdot \nabla_x \log \mu(x,\theta), \quad \tilde{k}(x,\theta,\theta') = \frac{\mu(x,\theta)}{\mu(x,\theta')} k(x,\theta,\theta'). \tag{0.0.4}$$

Puisque $\mu = 1$ sur Γ, les données au bord sont inchangées. Par conséquent, si $\mathscr{A}_{\tilde{a},\tilde{k}}$ est l'opérateur d'albédo correspondant à la paire $(\tilde{a},\tilde{k})$, on a $\mathscr{A}_{\tilde{a},\tilde{k}} = \mathscr{A}_{a,k}$. En fait, le coefficient d'absorption peut être déterminé de façon stable pour l'opérateur d'albédo modulo la transformation de jauge (0.0.4).
C'est le résultat principal de cette partie. Il s'énonce comme suit:

Théorème 2. *Si $T > Diam(\Omega)$, alors il existe $C > 0$ telle que pour $\theta \in \mathscr{S}$ et $x \in \mathbb{R}^n$, l'inégalité suivante*

$$\left| \int_{\mathbb{R}} a(x+s\theta,\theta)ds \right| \leq C \| \mathscr{A}_{a_2,k_2} - \mathscr{A}_{a_1,k_1} \|_1, \tag{0.0.5}$$

à lieu pour tout $(a_j,k_j) \in \mathscr{A}(M_0) \times \mathscr{K}(M_1,M_2)$, $j = 1,2$, vérifiant

$$\partial_x^\alpha a_1(x,\theta) = \partial_x^\alpha a_2(x,\theta),\ (x,\theta) \in \Gamma,\ \alpha \in (\mathbb{N} \cup \{0\})^n,\ |\alpha| \leq 1. \tag{0.0.6}$$

Comme conséquence importante de ce théorème, nous avons le corollaire suivant.

Corollaire 3. *Si $T > Diam(\Omega)$ et que $(a_j,k_j) \in \mathscr{A}(M_0) \times \mathscr{K}(M_1,M_2), j = 1,2$, sont tels que $\mathscr{A}_{a_1,k_1} = \mathscr{A}_{a_2,k_2}$, alors il existe une function positive $\mu(x,\theta) \in L^\infty(Q)$*

vérifiant $\theta \cdot \nabla_x \mu \in L^\infty(Q)$ *et* $\mu(x,\theta) = 1$ *pour* $(x,\theta) \in \Gamma$, *pour laquelle on a :*

$$a_1(x,\theta) = a_2(x,\theta) - \theta \cdot \nabla (\log \mu(x,\theta)), \quad \forall (x,\theta) \in Q. \tag{0.0.7}$$

Le problème de l'unicité lorsque les coefficients dépendent du temps

Nous considérons le problème mixte associé à l'équation de Boltzmann linéaire suivante:

$$\begin{cases} \partial_t u + \theta \cdot \nabla u + a(t,x)u = \mathscr{N}_k[u](t,x,\theta) & \text{in} \quad Q_T, \\ u(0,x,\theta) = u_0 & \text{in} \quad Q, \\ u(t,x,\theta) = f(t,x,\theta) & \text{on} \quad \Sigma^-, \end{cases} \tag{0.0.8}$$

où le noyau k de $\mathscr{N}_k$, dépent du temps cette fois:

$$\mathscr{N}_k[u](t,x,\theta) = \int_{\mathscr{S}} k(t,x,\theta,\theta')u(t,x,\theta')d\theta'.$$

Dans cette partie, nous prouvons que les coefficients d'absorption a et de diffusion k, dépendant du temps, sont déterminés de façons unique dans un sous-domaine strict de $(0,T) \times \Omega$, par la connaissance de l'opérateur d'albédo. Introduisons maintenant quelques notations. Soit $r > 0$ tel que $T > 2r$. Sans perte de généralité, on suppose que $\Omega \subseteq B(0,r/2) = \{x \in \mathbb{R}^n, |x| < r/2\}$ et on note $X_r = (0,T) \times B(0,r/2)$. Dans un première temps, nous avons montré dans le cas où la condition initiale u_0 est nulle, qu'il est impossible d'identifier le coefficient d'absorption dépendant du temps a, sur l'ensemble conique $\mathscr{C}_r$ suivant:

$$\mathscr{C}_r = \left\{(t,x) \in X_r,\ |x| \leq \frac{r}{2} - t,\ 0 \leq t \leq \frac{r}{2}\right\}, \tag{0.0.9}$$

à partir de l'opérateur d'albédo.

Étant donnée

$$\mathscr{B}_r = \left\{x \in \mathbb{R}^n,\ \frac{r}{2} < |x| < T - \frac{r}{2}\right\}.$$

On considère les cônes lumineux:

$$\mathscr{C}_r^+ = \left\{(t,x) \in X_r,\ |x| < t - \frac{r}{2},\ t > \frac{r}{2}\right\}, \tag{0.0.10}$$

$$\mathscr{C}_r^- = \left\{(t,x) \in X_r,\ |x| < T - \frac{r}{2} - t,\ T - \frac{r}{2} > t\right\}. \tag{0.0.11}$$

Posons ensuite

$$X_{r,\sharp} = \Omega_T \cap \mathscr{C}_r^+, \qquad \text{et} \quad X_{r,*} = \Omega_T \cap \mathscr{C}_r^+ \cap \mathscr{C}_r^-,$$

puis nous introduisons l'ensemble admissible des coefficients d'absorption a dans $X_{r,*}$ et $X_{r,\sharp}$. Étant donnée $a_0 \in \mathscr{C}^1(\overline{\Omega}_T)$, on pose :

$$\mathscr{A}^*(M_0) = \{a \in \mathscr{A}(M_0);\ a = a_0 \text{ in } \overline{\Omega}_T \setminus X_{r,*}\}.$$

et

$$\mathscr{A}^\sharp(M_0) = \{a \in \mathscr{A}(M_0);\ a = a_0 \text{ in } \overline{\Omega}_T \setminus X_{r,\sharp}\}.$$

L'ensemble admissible de coefficients de diffusion k est défini comme suit:

$$\mathscr{K}(M_1, M_2) = \{k \in L^\infty(\Omega_T, L^2(\mathscr{S} \times \mathscr{S}));\ \int_{\mathscr{S}} |k(t,x,\theta,\theta')| d\theta' \leq M_1, \\ \int_{\mathscr{S}} |k(t,x,\theta,\theta')| d\theta \leq M_2\}.$$

On a alors le résultat de non unicité suivant:

Théorème 4. *Soit $(a,k) \in \mathscr{A}(M_0) \times \mathscr{K}(M_1, M_2)$ tel que $Supp(a) \subset \mathscr{C}_r$ et $Supp(k(\cdot,\theta,\theta')) \subset \mathscr{C}_r$. Alors, on a $\mathscr{A}_{a,k} = \mathscr{A}_{0,0}$.*

Cependant nous avons montré que le coefficient d'absorption dépendant du temps peut être déterminé dans $X_{r,*}$ par l'opérateur d'albédo.

Théorème 5. *Soient $T > 2r$ et $(a_j, k_j) \in \mathscr{A}^*(M_0) \times \mathscr{K}(M_1, M_2)$, $j = 1, 2$. Alors, on a l'implication*

$$\mathscr{A}_{a_1,k_1} = \mathscr{A}_{a_2,k_2} \quad \Rightarrow \quad a_1 = a_2 \text{ sur } X_{r,*}.$$

Nous avons prouvé également un résultat d'identification pour le coefficient de diffusion k dans $X_{r,*}$.

Théorème 6. *Soient $(a_j, k_j) \in \mathscr{A}(M_0) \times \mathscr{K}(M_1, M_2)$, $j = 1, 2$, tel que $k_j(t,x,\theta,\theta') = g_j(t,x)\kappa(\theta,\theta')$, avec $g_j \in L^\infty(X_{r,*})$, $\kappa \in L^\infty(\mathscr{S} \times \mathscr{S})$ et $\kappa(\theta,\theta) \neq 0$, $\forall \theta \in \mathscr{S}$. Alors, si $T > 2r$ et $\mathscr{A}_{a_1,k_1} = \mathscr{A}_{a_2,k_2}$, on a $g_1 = g_2$ p.p. dans $X_{r,*}$.*

Détermination stable du coefficient d'absorption dépendant du temps

Lorsque le coefficient d'absorption ne dépend pas du temps t, Bal et Jollivet [7] ont prouvé un résultat de stabilité (et donc d'unicité) pour le coefficient d'absorption par rapport à l'opérateur d'albédo. Dans la section précédente, nous avons prouvé qu'il est impossible de déterminer le coefficient d'absorption a dépendant du temps dans tout le domaine, lorsque la condition initiale est égale zéro ($u_0 = 0$), à partir de la connaissance $\mathscr{A}_{a,k}(f)$, où f décrit un ensemble de conditions de type Dirichlet . Dans cette section, nous considérerons trois ensembles de données f différents et nous traitons le problème de stabilité du problème inverse considéré.

Dans le premier cas, nous prouverons une estimation de stabilité de type log pour le coefficient d'absorption a par rapport à l'opérateur d'albédo dans un sous-ensemble de domaine $X_{r,*}$.

Théorème 7. *Soit $T > 2r$. Alors il existe trois constantes $C > 0$, $\mu \in (0,1)$ et $m \in (0,1)$, telles que si $\|\mathscr{A}_{a_1,k_1} - \mathscr{A}_{a_2,k_2}\|_1 \leq m$, on a*

$$\|a_1 - a_2\|_{H^{-1}(X_{r,*})} \leq C\left(\|\mathscr{A}_{a_1,k_1} - \mathscr{A}_{a_2,k_2}\|_1^{\mu/2} + |\log\|\mathscr{A}_{a_1,k_1} - \mathscr{A}_{a_2,k_2}\|_1|^{-1}\right), \tag{0.0.12}$$

pour tout $(a_j, k_j) \in \mathscr{A}^*(M_0) \times \mathscr{K}(M_1, M_2)$, $j = 1, 2$.
De plus, si on suppose que $a_j \in H^{s+1}(\Omega_T)$, $s > \frac{n-1}{2}$, avec $\|a_j\|_{H^{s+1}(\Omega_T)} \leq M$, $j = 1, 2$, pour une constante $M > 0$, alors il existe $C > 0$ et $\mu_1 \in (0,1)$ tels que

$$\|a_1 - a_2\|_{L^\infty(X_{r,*})} \leq C\left(\|\mathscr{A}_{a_1,k_1} - \mathscr{A}_{a_2,k_2}\|_1^{\mu/2} + |\log\|\mathscr{A}_{a_1,k_1} - \mathscr{A}_{a_2,k_2}\|_1|^{-1}\right)^{\mu_1}, \tag{0.0.13}$$

où C ne dépend que de Ω, T, n et M_i, $i = 0, 1, 2$.

Ensuite, nous supposons $u(T, \cdot)$ connu, $u_0 = 0$, et étendons ce résultat à la région $X_{r,\sharp} \supset X_{r,*}$. On définit pour cela l'opérateur suivant

$$\begin{array}{rccl} \tilde{\mathscr{A}}_{a,k}: & \mathscr{L}_p^-(\Sigma^-) & \to & \mathscr{K}_p^+ := \mathscr{L}_p^+(\Sigma^+) \times L^p(Q) \\ & f & \mapsto & \tilde{\mathscr{A}}_{a,k}(f) = (u_{|\Sigma^+}, u(T, \cdot, \cdot)). \end{array}$$

Théorème 8. *Soit $T > 2r$. Alors il existe $C > 0$, $\mu \in (0,1)$ et $m \in (0,1)$, tels que pour $\|\tilde{\mathscr{A}}_{a_1,k_1} - \tilde{\mathscr{A}}_{a_2,k_2}\|_1 \leq m$, on a*

$$\|a_1 - a_2\|_{H^{-1}(X_{r,\sharp})} \leq C\left(\|\tilde{\mathscr{A}}_{a_1,k_1} - \tilde{\mathscr{A}}_{a_2,k_2}\|_1^{\mu/2} + |\log\|\tilde{\mathscr{A}}_{a_1,k_1} - \tilde{\mathscr{A}}_{a_2,k_2}\|_1|^{-1}\right), \tag{0.0.14}$$

pour tout $(a_j, k_j) \in \mathscr{A}^\sharp(M_0) \times \mathscr{K}(M_1, M_2)$, $j = 1, 2$.
De plus, si on suppose que $a_j \in H^{s+1}(\Omega_T)$, $s > \frac{n-1}{2}$, avec $\|a_j\|_{H^{s+1}(\Omega_T)} \leq M$, $j = 1, 2$, pour $M > 0$, alors il existe $C > 0$ et $\mu_1 \in (0,1)$ tels que

$$\|a_1 - a_2\|_{L^\infty(X_{r,\sharp})} \leq C\left(\|\tilde{\mathscr{A}}_{a_1,k_1} - \tilde{\mathscr{A}}_{a_2,k_2}\|_1^{\mu/2} + |\log\|\tilde{\mathscr{A}}_{a_1,k_1} - \tilde{\mathscr{A}}_{a_2,k_2}\|_1|^{-1}\right)^{\mu_1}, \tag{0.0.15}$$

où C ne dépend que de Ω, T, n et M_i, $i = 0, 1, 2$.

Enfin, nous avons prouvé une estimation de stabilité de type log pour ce problème sur tout le domaine, à partir des données précédentes obtenues en faisant varier la condition initiale. Nous définissons l'espace suivant $\mathscr{K}_p^- := \mathscr{L}_p^-(\Sigma^-) \times L^p(Q)$. L'ensemble des observations considérées dans ce cas est défini par l'opérateur linéaire borné $\mathscr{I}_{a,k}$

$$\begin{array}{rccl} \mathscr{I}_{a,k}: & \mathscr{K}_p^- & \longrightarrow & \mathscr{K}_p^+ \\ & g = (f, u_0) & \longmapsto & \mathscr{I}_{a,k}(g) = (u_{|\Sigma^+}, u(T,\cdot,\cdot)). \end{array}$$

On note

$$\mathscr{I}_{a,k}^1(g) = u_{|\Sigma^+}, \quad \mathscr{I}_{a,k}^2(g) = u(T,\cdot,\cdot).$$

où u désigne la solution du système (0.0.1).

Théorème 9. *Soit $T > 2r$. Alors il existe $C > 0$, $\mu \in (0,1)$ et $m \in (0,1)$, tels que si $\|\mathscr{I}_{a_1,k_1} - \mathscr{I}_{a_2,k_2}\|_1 \leq m$, on a*

$$\|a_1 - a_2\|_{H^{-1}(\Omega_T)} \leq C\left(\|\mathscr{I}_{a_1,k_1} - \mathscr{I}_{a_2,k_2}\|_1^{\mu/2} + |\log\|\mathscr{I}_{a_1,k_1} - \mathscr{I}_{a_2,k_2}\|_1|^{-1}\right), \tag{0.0.16}$$

pour tout $(a_j, k_j) \in \mathscr{A}(M_0) \times \mathscr{K}(M_1, M_2)$, $j = 1, 2$.
De plus, si on suppose que $a_j \in H^{s+1}(\Omega_T)$, $s > \frac{n-1}{2}$, avec $\|a_j\|_{H^{s+1}(\Omega_T)} \leq M$, $j = 1, 2$, pour $M > 0$, alors existe $C > 0$ et $\mu_1 \in (0,1)$, tels que

$$\|a_1 - a_2\|_{L^\infty(\Omega_T)} \leq C\left(\|\mathscr{I}_{a_1,k_1} - \mathscr{I}_{a_2,k_2}\|_1^{\mu/2} + |\log\|\mathscr{I}_{a_1,k_1} - \mathscr{I}_{a_2,k_2}\|_1|^{-1}\right)^{\mu_1}, \tag{0.0.17}$$

où C ne dépend que de Ω, T, n et M_i, $i = 0, 1, 2$.

La dérivation des résultats de stabilité et d'unicité des sections et s'inspire de Cipolati et Motta [39, 40] (pour l'équation de Boltzmann linéaire avec un coefficient d'absorption ne dépendant que de l'espace), ainsi que de Bellassoued et Ben Aicha [11] et Ben Aicha [12](pour l'équation hyperbolique).

Stabilité dans la détermination de l'absorption dépendant du temps à partir d'un nombre fini de mesures

Dans cette partie, nous prouvons une estimation de stabilité pour le coefficient d'absorption a, dépendant du temps, apparaissant dans l'équation linéaire de Boltzmann (0.0.1), par rapport un nombre fini de mesures de la solution sur le bord. Nous utilisons les mêmes notations que ci-dessus et examinons les trois situations de la section précédente.
Soit $a_0 \in \mathscr{C}^1(\overline{\Omega}_T)$ et $\{\rho_1^*, \ldots, \rho_k^*\}$ un ensemble linéairement indépendant dans $C^1(X_{r,*})$. On note

$$\chi_k^* = \mathrm{Vect}\{\rho_1^*, \cdots, \rho_k^*\}.$$

Étant donnée une constante $M > 0$, on fixe

$$\chi_k^*(M, a_0) = \{a \in \chi_k^*(M),\ a = a_0 \text{ dans } X_r \setminus X_{r,*};\ \|a\|_{L^\infty(\Omega_T)} \leq M\},$$

où

$$\chi_k^*(M) = \{a \in \chi_k^*;\ \text{telle que } a \in C^1(\overline{\Omega}_T)\}.$$

Nous avons prouvé une inégalité stabilité lipschitzienne dans le sous-ensemble spécifique $X_{r,*}$

Théorème 10. *Soit $T > 2r$. Alors il existe $\omega_i \in \mathscr{S}$ et $f_i \in L^1(\Sigma^+_{\omega_i})$, $i = 1 \cdots k$, tels que*

$$\|a_1 - a_2\|_{L^\infty(X_{r,*})} \leq C \sup_{1 \leq i \leq k} \|\left(\tilde{\mathscr{A}}_{a_2,k_2} - \tilde{\mathscr{A}}_{a_1,k_1}\right)(f^{(i)})(\cdot, \cdot, \omega_i)\|_{L^1(\Sigma^+_{\omega_i})}, \qquad (0.0.18)$$

pour tout $(a_j, k_j) \in \chi_k(M, a_0) \times \mathscr{K}(M_1, M_2)$, $j = 1, 2$ tel que $k_j = a_j k_0$, où $k_0 \in \mathscr{K}(M_1, M_2)$..

Remarque 11. *Nous pouvons étendre le résultat de stabilité précédent à $X_{r,\sharp}$ et sur tout le domaine.*

Les résultats de la première section ont été publiés dans [P2], alors que ceux de la deuxième et troisième sections sont décrits dans [P1]. L'approche utilisée ici repose essentiellement sur la construction d'un ensemble adapté de solutions de type "optique géométrique" pour l'équation de Boltzmann linéaire.

Détermination du potentiel parabolique par des données latérales sur un intervalle de temps arbitraire

En 2001, Canuto et Kavian ont prouvé dans [27] que la connaissance de l' opérateur Dirichlet Neumann mesurée à un instant donné détermine de manière unique le coefficient d'ordre zéro (ou potentiel) indépendant du temps, d'une équation parabolique. Mais la question de stabilité reste toujours ouverte. Plus récemment, Choulli et Kian [32] ont établi une estimation de stabilité (logarithmique) pour le potentiel par rapport à l'opérateur Dirichlet Neumann parabolique mesuré sur tout l'intervalle de temps. Leur stratégie est basée sur la construction de solutions de type optique géométrique (CGO) appropriées. Une question naturelle découlant des résultats de [27] et [32] est de savoir si le potentiel peut être déterminé de manière stable par l'opérateur DN parabolique mesuré sur un sous-intervalle arbitraire (éventuellement réduit à un instant) de l'espace temporel.

La deuxième partie de la thèse est ainsi consacrée à l'étude de problèmes inverses aux limites associés à l'équation parabolique suivante.

$$\begin{cases} \partial_t u - \Delta u + q(x)u = 0 & \text{in} \quad Q, \\ u(0,x) = u_0 & \text{in} \quad \Omega, \\ u(t,x) = f(t,x), & \text{on} \quad \Sigma, \end{cases} \tag{0.0.19}$$

où $q \in L^\infty(\Omega)$ est supposé réel. Ce système décrit l'évolution au cours du temps $t \in (0,T)$ de la chaleur se propageant à travers un milieu homogène.

Dans cette partie, nous traitons le problème inverse de la détermination du coefficient différentiel d'ordre zéro, à partir de la mesure complète partielle de la solution sur la frontière physique $\partial\Omega$, sur un intervalle de temps arbitrairement petit. Nous prouvons une inégalité de stabilité logarithmique par rapport à l'opérateur Dirichlet Neumann complet ou partiel avec deux approches différentes.

Notations et problème direct

Pour définir l'opérateur DN associé à (0.0.19), nous introduisons $H_0^{r,s}(\Sigma)$, $r,s>0$ l'espace fonctionnel

$$H_0^{r,s}(\Sigma) = L^2(0,T;H^r(\partial\Omega)) \cap H_0^s(0,T;L^2(\partial\Omega)).$$

Nous supposons que la condition initiale $u_0 = 0$. Les données dont on dispose sont les mesures de la dérivée normale de la solution u_q sur le bord sur un sous-intervalle temporel $(T_* - \epsilon, T_* + \epsilon)$ avec $T_* \in (0,T)$, $\epsilon \in (0, \min(T_*, T - T_*))$. Cela nous amène à considérer l'opérateur de DN parabolique associé au système (0.0.19), défini par

$$\begin{array}{rccl} \tilde{\Lambda}_q : & H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma_0) & \longrightarrow & H^{-\frac{3}{2},-\frac{3}{4}}(\Sigma_0), \\ & f & \longmapsto & \tilde{\Lambda}_q(f) = \partial_\nu u_q, \end{array} \tag{0.0.20}$$

où $\Sigma_0 = (T_* - \epsilon, T_* + \epsilon) \times \partial\Omega$ et $\nu(x)$ est le vecteur unitaire normal de $\partial\Omega$ en x. Ensuite, on introduit l'ensemble admissible des potentiels réels inconnus, q. Pour $M > 0$, on note:

$$\mathscr{M} = \{q \in L^\infty(\Omega); \quad \|q\|_{L^\infty(\Omega)} \leq M\}.$$

Considérons les espaces $\mathscr{K}_\pm(Q) = \{u \in L^2(Q);\ (\pm\partial_t - \Delta)u \in L^2(Q)\}$, associés à la norme

$$\|u\|_{\mathscr{K}_\pm(Q)} = \left(\|u\|_{L^2(Q)} + \|(\pm\partial_t - \Delta)u\|_{L^2(Q)}\right)^{1/2}.$$

Nous fixons $\omega \in \mathscr{S}^{n-1}$, la sphère unitaire de $\mathbb{R}^n$, introduisons

$$\Gamma^\pm = \{x \in \Gamma\ :\ \pm\omega \cdot \nu(x) > 0\}, \tag{0.0.21}$$

et posons $\Sigma^{\pm}(\omega) = \Gamma^{\pm} \times (0, T)$. Pour un choix arbitraire $\omega_0 \in \mathscr{S}^{n-1}$, fixons $\varrho > 0$ et introduisons

$$\Gamma_{+,\varrho}(\omega_0) = \{x \in \Gamma;\ \nu(x) \cdot \omega_0 > \varrho\}; \quad \Gamma_{-,\varrho}(\omega_0) = \Gamma \setminus \overline{\Gamma_{+,\varrho}(\omega_0)},$$

$$\Sigma_{+,\varrho}(\omega_0) = (T_* - \epsilon, T_* + \epsilon) \times \Gamma_{+,\varrho}(\omega_*), \quad \Sigma_{-,\varrho}(\omega_0) = \Sigma \setminus \overline{\Sigma_{+,\varrho}(\omega_0)}.$$

Alors, l'opérateur Dirichlet Neumann local est défini comme suit:

$$\begin{array}{rccl} \Lambda_q^{\sharp}: & H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma_0) & \longrightarrow & H^{-\frac{3}{2},-\frac{3}{4}}(\Sigma_{-,\varrho}(\omega_0)), \\ & f & \longmapsto & \Lambda_q^{\sharp}(f) = \partial_\nu u_{q|\Sigma_{-,\varrho}(\omega_0)}. \end{array} \tag{0.0.22}$$

Avant de passer à l'étude du problème inverse il convient d'énoncer le résultat d'existence et d'unicité suivant pour le système (0.0.22).

Lemme 12. *[32, Proposition 2.3 et Proposition 2.4] Soient $q \in L^\infty(\Omega)$ et $f \in H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma)$. Alors le système* (0.0.19) *admet une unique solution du transposition $u_q \in \mathscr{K}_+(Q)$ satisfaisant*

$$\|u_q\|_{\mathscr{K}_+(Q)} \leq C\|f\|_{H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma)}, \tag{0.0.23}$$

où la constante C ne dépend que de Q et de M. De plus, l'opérateur Dirichlet Neumann parabolique

$$\Lambda_q : f \mapsto \partial_\nu u_q,$$

est un opérateur borné de $H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma)$ dans $H^{-\frac{3}{4},-\frac{3}{2}}(\Sigma)$.

Identification stable du potentiel à partir des données limites complètes

Nous avons prouvé un résultat de stabilité logarithmique pour le coefficient q apparaissant dans l'équation parabolique de (0.0.19) à partir de l'opérateur Dirichlet-Neumann global. Soit

$$\phi_\epsilon(t) = \begin{cases} \epsilon^{-\frac{5}{8}(n-2)} e^{2|\Omega|\epsilon^{-\frac{1}{2}}} t & \text{in} \quad t \geq e^{-2|\Omega|\epsilon^{-\frac{1}{2}}}, \\ |\log t|^{-\frac{5}{16}(n-2)} & \text{in} \quad 0 < t < e^{-2|\Omega|\epsilon^{-\frac{1}{2}}}. \end{cases}$$

Théorème 13. *Soient $q_1, q_2 \in \mathscr{M}$. Alors il existe une constante $C > 0$, qui ne dépend que de M et Ω, telle que*

$$\|q_1 - q_2\|_{H^{-1}(\Omega)} \leq C\phi_\epsilon(\|\tilde{\Lambda}_2 - \tilde{\Lambda}_1\|),$$

où $\|\tilde{\Lambda}_2 - \tilde{\Lambda}_1\|$ représente la norme de $\tilde{\Lambda}_2 - \tilde{\Lambda}_1$ dans $\mathscr{B}\left(H^{-1/2,-1/4}(\Sigma_0); H^{1/2,1/4}(\Sigma_0)\right)$.

Détermination du potentiel à partir de mesures partielles

Dans cette partie, on s'intéresse à l'identification du potentiel q à partir de la connaissance de l'opérateur Dirichlet Neumann partiel.

Théorème 14. *Soient $q_j \in \mathscr{M}$, $j = 1, 2$. Alors il existe deux constantes $C > 0$ et $\mu \in (0, 1)$, ne dépendant que M, Ω et s telles que*

$$\|q_1 - q_2\|_{H^{-1}(\Omega)} \leq C\left(\|\Lambda_1^\sharp - \Lambda_2^\sharp\| + (\log|\log\|\Lambda_1^\sharp - \Lambda_2^\sharp\||)^{-2/\mu}\right)^{\mu/2}.$$

Où $\|\Lambda_1^\sharp - \Lambda_2^\sharp\|$ représente la norme de $\Lambda_1^\sharp - \Lambda_2^\sharp$ dans $\mathscr{B}\left(H^{-1/2,-1/4}(\Sigma_0); H^{1/2,1/4}(\Sigma_{-,\varrho}(\omega_0')\right)$.

Nous avons prouvé nos résultats (Théorèmes 13, 14) par deux approche différentes. Dans la première, nous avons construit des solutions de type "optique géométrique" complexes qui sont décalées dans le temps de façon à être "centrées" en T_*. Dans la deuxième, nous avons utilisé les fonctions "optique géométrique" complexes introduites dans [32] multipliées par une fonction de troncature temporelle. Dans les deux cas, nous avons obtenu les mêmes types d'estimations de stabilité, la seule différence apparaissant dans les constantes multiplicatives, dans la première méthode la constante qu'on obtient c'est $\epsilon^{-\frac{5}{8}(n-2)}e^{2|\Omega|\epsilon^{-\frac{1}{2}}}$ et dans la deuxième c'est $\epsilon^{-(\frac{n}{8}+1)}e^{2|\Omega|\epsilon^{-\frac{1}{2}}}$. Ces résultats établissent que le potentiel est déterminé de manière log-stable par l'opérateur Dirichlet-Neumann parabolique complet ou partiel mesuré sur l'intervalle $(T_* - \epsilon, T_* + \epsilon)$ (sous réserve qu'il soit suffisamment petit). Mais, comme le préfacteur dans les estimations de stabilité explose lorsque ϵ tend vers 0, il n'est pas possible de récupérer le potentiel parabolique de façon stable à partir de la connaissance de l'opérateur Dirichlet-Neumann parabolique à l'instant $T_* \in (0, T)$.

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General introduction

Inverse problems is a very interesting research topic considering their applications in several scientific fields. They are defined by opposition to direct problems, in the sense that the unknowns in an inverse problem are known quantities for the associated direct problems. A more operational definition is that an inverse problem involves is to determine causes from knowledge of the effects, witch is the opposite of direct problem, consisting of deduce the effects from the knowledge of the cause.

Predicting the future state of a physical system, knowing its current state, is an example type of the direct problem. We can consider various inverse problems: for example, reconstructing the past state of the system knowing its current state, or determining parameters of the system, knowing (part of) its evolution. This last problem is that of the identification of parameters, which will be our main concern in the following. Solving these inverse problems helps define good modelling mathematics associated with the physical case studied. Indeed, the inverse problem consists in determining a domain, an initial condition or a parameter intervening in the system which characterizes the physical environment. To resolve an inverse problem, we need to have additional data. These data are generally experimental data. From such data, the existence and uniqueness of a solution to the inverse problem is not often guaranteed. Worse, very often small perturbation on the measurements cause very big variations in the physical parameters sought. This is the reason why the inverse problems are typically ill-posed in the sense of Hadamard [47]. An ill-posed problem is one which doesn't meet the three Hadamard criteria for being well-posed. These criteria are:

- Having a solution,
- Having a unique solution,
- Having a solution that depends continuously on the parameters or input data.

My research interests during the preparation of my PhD were focused on the analysis of inverse problems of determining some coefficients appearing in Linear Boltzmann equation and parabolic equation. The main and common objective is to establish uniqueness and stability results for the identification of these coefficients.

Part 1: Transport inverse problems:

The First part of the thesis is devoted to the study of inverse problems associated to the linear Boltzmann equation.

Boltzmann linear equation (Transport equation) posed in the phase space (position, direction and time) provides an excellent approximation of particle behaviour. The transport equation offer accurate descriptions for the propagation of particles in scattering media such as e.g. photons in human tissues in the application of optical tomography [2], it also operating calculations and control of nuclear reactors or in problems arising from radiative hydrodynamics (see G. Bell, S. Glasstone [24], J.Bussac, P.Reuss [44], J.J. Duderstadt, W.R. Martin [20]. In the latter medical imaging modality, optical parameters are reconstructed from available photon density readings at detectors. Such detectors are typically optical fibers, which collect photons coming from different directions. The optical sources are also typically optical fibers and photons are thus emitted over a larger range of directions.

Inverse transport theory concerns the reconstruction of the scattering and absorption coefficients in a linear Boltzmann equation from knowledge of the albedo operator, which models all possible boundary measurements.

Inverse transport theory has many applications, in medical imaging and optical tomography [2, 58] and optical molecular imaging [30]. Applications in remote sensing in the atmosphere are considered in [54]. Inverse transport can also be

used efficiently for imaging using high frequency waves propagating in highly heterogeneous media; see e.g. [9, 24]. For general references and earlier review papers on inverse transport, we refer the reader to e.g.[5, 55, 62].
When scattering is weak that's mean the interaction between particles is negligible that is leads to discard non-linear term, a balance equation on the density of particles moving was given by Boltzmann. For the time-depend case this equation writes as

$$\partial_t u + \theta \cdot \nabla u(t,x,\theta) + a(x)u = \mathscr{N}_k[u](t,x,\theta) \quad \text{in} \quad Q_T := (0,T) \times Q,$$

Where $T > 0$, $\mathscr{S} = \mathbb{S}^{n-1}$ the unit sphere of $\mathbb{R}^n$, $Q := \mathscr{S} \times \Omega$ where Ω a bounded and convex domain of $\mathbb{R}^n$, $n \geq 2$ and $\mathscr{N}_k$ is the integral operator with kernel $k(x,\theta,\theta')$ defined by

$$\mathscr{N}_k[u](t,x,\theta) = \int_{\mathscr{S}} k(x,\theta,\theta')u(t,x,\theta')d\theta',$$

where $d\theta'$ denotes the surface measure on $\mathscr{S}$ associated to the Lebesgue measure in $\mathbb{R}^n$.
The function u represent the density of particles at x travelling in the direction θ, $a(x)$ is the absorption coefficient at x and $k(x,\theta,\theta')$ is the scattering coefficient (or the collision kernel). Let $\Gamma^{\pm}$ denote the incoming and outgoing bundles

$$\Gamma^{\pm} = \{(x,\theta) \in \Gamma \,:\, \pm\theta \cdot \nu(x) > 0\}, \quad \Sigma^{\pm} = [0,T] \times \Gamma^{\pm}, \quad \Gamma = \partial\Omega \times \mathscr{S},$$

where $\nu = \nu(x)$ denotes the unit outward normal to $\partial\Omega$ at x. The medium is probed with the given radiation

$$u_{|\Sigma^-} = f^-.$$

The exiting radiation $u_{|\Sigma^+}$ is detected thus defining the albedo operator $\mathscr{A}_{a,k}$ that takes the incoming flux f^- to the outgoing flux $u_{|\Sigma^+}$

$$\mathscr{A}_{a,k}[f^-] = u_{|\Sigma^+}.$$

Our focus here is the inverse problem of recovery the coefficients in the linear Boltzmann equation via boundary knowledge of the albedo operator. More precisely, we are interested to recover a and k by giving the incoming flux of particles on the boundary and measuring the outgoing one.
There is a wide bibliography devoted to the problem of recovering coefficients appearing in the linear Boltzmann equation. But they are mostly concerned with time-independent scattering coefficient and the stationary case when the absorption coefficient is an isotropic (θ-dependent).

The uniqueness of the reconstruction of the absorption and scattering coefficients from the albedo operator for the stationary transport equation, when the scattering coefficient k does not depends on time t, both in the case that a is assumed to be a function of the position x alone and the case that a depends on direction as well as position x, was proved by Choulli and Stefanov in [35] and Tamasan [67]. Under certain smallness assumptions on the absorption and the scattering parameters a stability results was obtained in the stationary transport equation in two or three dimensional case by Romanov [61, 62] and in two-dimensional case stability result in [65] imposes a smallness condition on the scattering coefficient k. In three or higher dimensions, stability of the reconstruction results of general scattering and absorption coefficients were established by Bal and Jollivet [6].

Also, for the time-dependent Boltzmann equation, when the scattering coefficient does not depend of time t, Bal and Jollivet [7] proved a stability results on the reconstruction of the absorption and scattering parameters for a given error in the measured albedo operator, where the theoretical approach is based on the singular decomposition of Schwartz kernel of the albedo operator. In the case where the unknown absorption coefficient a is depending only on the spatial variable, Cipolatti, Motta and Roberty [40] and Cipolatti [39] proved a uniqueness and stability result by means of geometric optics solutions.
This parts deals with study the stability and uniqueness in the inverse problem of determining the coefficients appearing in the linear Boltzmann equation equation from boundary knowledge of the albedo operator $\mathscr{A}_{a,k}$.

First of all, we introduce, in Chapter 1, some important results needed to study these inverse problems.

In Chapter 2, we deal with the inverse transport problem through anisotropic absorption coefficient, where the challenge here that it was not clear whether the uniqueness failed. In [66] they proved that for the stationary transport equation, the albedo operator is invariant under the gauge transformation and a stability estimate for gauge equivalent classes was proved for the stationary transport equation in [56]. Here we can see the same objection to the uniqueness for the time-dependent transport equation, then we managed to prove that a certain family of gauge transformation of the absorption and scattering coefficients does not change the albedo operator and that the albedo operator can recover uniquely the class of gauge equivalent of the absorption coefficient, we also prove a stability estimate for the absorption coefficient up to a gauge transformation.

In Chapter 4, we inspired from the work of Bellassoued for the the dissipative wave equation, we prove stability estimates in the recovery of the time-dependent absorption coefficient a appearing in the dynamical Boltzmann equation via different types of measurements and over different sub-domain of Q_T. It was actually a challenging problem as unique determination of the coefficient, in the case where the initial condition is zero, is not even guaranteed on the whole domain Q_T. Because of this obstruction to uniqueness, we were able to prove a uniqueness theorem for this problem, only in a precise subset of the domain Q_T. A better description of this obstruction to uniqueness and a rigorous proof of the uniqueness result are given in Chapter 3 in witch we prove that the time-depend absorption and scattering coefficients can be uniquely determined in a precise subset of the domain from the knowledge of the albedo operator.

We finish this part by Chapter 5, we prove a Lipschitz stability estimate in the determination of the absorption coefficient appearing in the linear Boltzmann equation from a finite number of data measured on a suitable sub-boundary. Moreover, we show that we can extend this result to the determination of the absorption coefficient in a larger region, and then in the whole domain provided

that we have much more data.

Part 2: Parabolic inverse problems:

The second part of the thesis is devoted to treat inverse problems that concern parabolic equations. Precisely, in this second part, we specify our study on determining a coefficient, arising in the parabolic equation, from full and partial measurement of the solution on the boundary in an arbitrary time interval. Physically our inverse problem in this part, is to determine the coefficient, containing certain properties of the body, by applying a heat source over part of the body boundary and measure the temperature over another part of the body boundary.

The problem of identifying coefficient appearing in a parabolic equation has been treated, very well extensively. There are many works related to this topic. We reefer to [31, 49] for an overview of such problems. In [49], Isakov proved a uniqueness result for the potential appearing in the parabolic equation from the knowledge of the solution on the lateral boundary over the entire time-span together with the solution in the whole domain at the final time. In [31] Choulli proved a logarithmic stability result for this problem with the same data. Later on, in [32] Choulli and Kian, improved this stability estimate, they do not remaining the data at the final time. Cannon and Esteva [25] proved a logarithmic stability estimate for the determination of the support of a source term in a one-dimensional parabolic equation from a boundary measurement. This last result has been extended in [26] to the three-dimensional heat equation. Choulli and Yamamoto established in [38] a stability estimate for the inverse problem of determining a source term appearing in a heat equation from Neumann boundary measurements.

In 2001, Canuto and Kavian proved in [27] that knowledge of the parabolic Dirichlet-to-Neumann (D-to-N) map measured at a given time uniquely determines the (time-independent) potential of the parabolic equation. But the stability issue, that is the problem to know whether the potential depends continuously on these data (the parabolic D-to-N map measured at a single time) is still open. More recently, Choulli and Kian [32] established in a (logarithmic)

stability estimate for the parabolic potential with respect to the parabolic D-to-N map measured at the entire time-span. Their strategy is based on the construction of suitable optics geometrical solution of the parabolic equation. A natural question arising from [27] and [32] is to know whether the potential can be stably determined by the parabolic D-to-N map measured on an arbitrarily small subinterval of the time space.

In Chapter 6, we shall prove by an appropriate modification of the optics geometrical functions introduced in [32] that answer is positive.

Part I

Linear Boltzmann inverse problems

Chapter 1

Linear Boltzmann equation

1.1 Introduction

In this chapter, we will present in a general way the linear Boltzmann equation. We will introduce the definitions and notations specific to this context and we will give the theorem of existence and uniqueness of solution to the linear Boltzmann equation. Let Ω be a bounded and convex domain of $\mathbb{R}^n$, $n \geq 2$. Here and henceforth, the space of velocities $\mathscr{S} = \mathbb{S}^{n-1}$ is chosen to be the unit sphere of $\mathbb{R}^n$ to simplify the presentation, we denote $Q := \Omega \times \mathscr{S}$ and for $T > 0$, we denote $Q_T = (0,T) \times Q$. Let $\nu(x)$ denotes the unit outward normal to $\partial\Omega$ at x. Let $\Gamma = \partial\Omega \times \mathscr{S}$, $\Gamma^{\pm} = \{(x,\theta) \in \Gamma \,:\, \pm\theta \cdot \nu(x) > 0\}$ and $\Sigma^{\pm} = (0,T) \times \Gamma^{\pm}$. We consider the the following linear Boltzmann transport equation in Q_T:

$$\begin{cases} \partial_t u + \theta \cdot \nabla u(t,x,\theta) + a(x,\theta)u = \mathscr{N}_k[u](t,x,\theta) & \text{in} \quad Q_T, \\ u(0,x,\theta) = u_0(x,\theta) & \text{in} \quad Q, \\ u(t,x,\theta) = f(t,x,\theta), & \text{on} \quad \Sigma^-. \end{cases} \tag{1.1.1}$$

where $\mathscr{N}_k$ is the integral operator with kernel $k(t,x,\theta,\theta')$ defined by

$$\mathscr{N}_k[u](t,x,\theta) = \int_{\mathscr{S}} k(t,x,\theta,\theta')u(t,x,\theta')d\theta'. \tag{1.1.2}$$

In order to well define the albedo operator as a trace operator on the outcoming boundary, we consider the spaces $\mathscr{L}_p^{\pm}(\Sigma^{\pm}) := L^p(0,T;L^p(\Gamma^{\pm},d\xi))$, where $d\xi = |\theta \cdot \nu(x)|dxd\theta$, equipped with the norm

$$\|f\|_{\mathscr{L}_p^{\pm}(\Sigma^{\pm})} = \|f\|_{L^p(0,T;L^p(\Gamma^{\pm},d\xi))}.$$

We introduce the spaces

$$\mathscr{W}_p := \{u \in L^p(Q),\ \theta \cdot \nabla u \in L^p(Q);\ \int_\Gamma |\theta \cdot \nu(x)| u(x,\theta)|^p d\theta dx < \infty\}.$$

We also consider the operator $A : D(A) \to L^p(Q)$, defined by $(Au)(x,\theta) = \theta \cdot \nabla u(x,\theta)$, with

$$D(A) = \{u \in \mathscr{W}_p;\ u_{|\Gamma^+} = 0\}.$$

Given $M_0, M_1, M_2 > 0$, we consider the set of admissible coefficients a and k

$$\mathscr{A}(M_0) = \{a \in \mathscr{C}^1(\overline{Q});\ \|a\|_{L^\infty(Q)} \leq M_0\}.$$

$$\mathscr{K}(M_1, M_2) = \{k \in L^\infty(\Omega_T, L^2(\mathscr{S} \times \mathscr{S}));\ \int_{\mathscr{S}} |k(t,x,\theta,\theta')| d\theta' \leq M_1,$$
$$\int_{\mathscr{S}} |k(t,x,\theta,\theta')| d\theta \leq M_2\}.$$

1.2 Well-posedness result

In this section, we aim to prove the existence, uniqueness of the solution u of the initial boundary value problem (1.1.1). We refer the reader to (Chapter 21 paragraph 3.1 [43]) for the complete demonstration

Theorem 1.2.1. *Let $T > 0$, $p \geq 1$ are given, $(a,k) \in \mathscr{A}(M_0) \times \mathscr{K}(M_1, M_2)$ and $u_0 \in L^\infty(Q)$. For $f \in \mathscr{L}_p^-(\Sigma^-)$ the unique solution u of (1.1.1) satisfies*

$$u \in \mathscr{C}^1([0,T]; L^p(Q)) \cap \mathscr{C}([0,T]; \mathscr{W}_p).$$

Moreover, there is a constant $C > 0$ such that

$$\|u(t,\cdot,\cdot)\|_{L^p(Q)} + \|u\|_{\mathscr{L}_p^+(\Sigma^+)} \leq C(\|u_0\|_{L^p(Q)} + \|f\|_{\mathscr{L}_p^-(\Sigma^-)}), \quad \forall\, t \in (0,T). \quad (1.2.3)$$

Proof. The Theorem and its demonstration are presented in a more general form in [43] in chapter 21, paragraph 3.1. Using the fact that $(a,k) \in \mathscr{A}(M_0) \times \mathscr{K}(M_1, M_2)$, one can see from [43], that there exists a unique solution u of (1.1.1) satisfies

$$u \in \mathscr{C}^1([0,T]; L^p(Q)) \cap \mathscr{C}([0,T]; \mathscr{W}_p).$$

Multiplying the first equation of (1.1.1) by the complex conjugate of $|u|^{p-2}\overline{u}$, integrating it over Q and taking its real part, we get:

$$\begin{aligned}\mathrm{Re}\int_Q (\partial_t u(t)|u(t)|^{p-2}\overline{u}(t)+\theta\cdot\nabla u(t)|u(t)|^{p-2}\overline{u}(t)+a(x,\theta)u(t)|u(t)|^{p-2}\overline{u}(t))d\theta dx\\ = \mathrm{Re}\int_Q \mathcal{N}_k[u](t)|u(t)|^{p-2}\overline{u}(t)d\theta dx + \mathrm{Re}\int_Q \nu(t)|u(t)|^{p-2}\overline{u}(t)d\theta dx.\end{aligned}$$

Then, we have

$$\begin{aligned}\frac{1}{p}\frac{d}{dt}\int_Q |u(t)|^p d\theta dx + \frac{1}{p}\int_Q \mathrm{div}(\theta|u(t)|^p)d\theta dx + \int_Q a(x,\theta)|u(t)|^p d\theta dx\\ = \mathrm{Re}\int_Q \mathcal{N}_k[u](t)|u(t)|^{p-2}\overline{u}(t)d\theta dx + \mathrm{Re}\int_Q \nu(t)|u(t)|^{p-2}\overline{u}(t)d\theta dx.\end{aligned}$$

Then by Gauss identity, we get

$$\begin{aligned}\frac{1}{p}\frac{d}{dt}\int_Q |u(t)|^p d\theta dx + \frac{1}{p}\int_{\Gamma^+} \theta\cdot\nu(x)|u(t)|^p d\theta dx + \frac{1}{p}\int_{\Gamma^-} \theta\cdot\nu(x)|u(t)|^p d\theta dx\\ + \int_Q a(x,\theta)|u(t)|^p d\theta dx\\ = \mathrm{Re}\int_Q (\mathcal{N}_k[u](t)|u(t)|^{p-2}\overline{u}(t)d\theta dx + \mathrm{Re}\int_Q \nu(t)|u(t)|^{p-2}\overline{u}(t)d\theta dx. \quad (1.2.4)\end{aligned}$$

By (1.3.9) and Hölder's inequality, we get

$$\begin{aligned}\int_Q |\mathcal{N}_k[u(t)]||u(t)|^{p-1}dxd\theta &\le \left(\int_Q |\mathcal{N}_k[u(t)]|^p dxd\theta\right)^{\frac{1}{p}}\left(\int_Q |u(t)|^{(p-1)q}dxd\theta\right)^{\frac{1}{q}}\\ &\le C_p\|u(t)\|_{L^p(Q)}\left(\int_Q |u(t)|^p dxd\theta\right)^{1-\frac{1}{p}}\\ &\le C_p\|u(t)\|_{L^p(Q)}\|u(t)\|_{L^p(Q)}^{p(1-\frac{1}{p})}\\ &\le C_p\|u(t)\|_{L^p(Q)}^p,\end{aligned} \quad (1.2.5)$$

where $C_p = M_1^{1/q}M_2^{1/p}$ with $\frac{1}{p}+\frac{1}{q}=1$. On the other hand Young inequality implies

$$\int_Q |\nu(t)||u(t)|^{p-1}d\theta dx \le \frac{1}{p}\|\nu(t)\|_{L^p(Q)}^p + \frac{1}{q}\|u(t)\|_{L^p(Q)}^p. \quad (1.2.6)$$

Then, we obtain

$$\begin{aligned}\frac{1}{p}\frac{d}{dt}\int_Q |u(t)|^p d\theta dx + \frac{1}{p}\int_{\Gamma^+} |\theta\cdot\nu(x)||u(t)|^p d\theta dx \\ \leq \frac{1}{p}\int_{\Gamma^-} |\theta\cdot\nu(x)||f(t)|^p d\theta dx + C_p\|u(t)\|^p_{L^p(Q)} \\ + \|a\|_{L^\infty(Q)}\|u(t)\|^p_{L^p(Q)} + \frac{1}{p}\|v(t)\|^p_{L^p(Q)} + \frac{1}{q}\|u(t)\|^p_{L^p(Q)},\end{aligned} \tag{1.2.7}$$

we conclude

$$\frac{d}{dt}\|u(t)\|^p_{L^p(Q)} \leq \int_{\Gamma^-} |\theta\cdot\nu(x)||f(t)|^p d\theta dx + C_1\|u(t)\|^p_{L^p(Q)} + \|v(t)\|^p_{L^p(Q)}.$$

Now, integrating this last inequality over $(0,t)$, we get

$$\|u(t)\|^p_{L^p(Q)} \leq \|f\|^p_{\mathscr{L}^-_p(\Sigma^-)} + \|v\|^p_{L^p(Q_T)} + \|u_0\|^p_{L^p(Q)} + C_1\int_0^t \|u(s)\|^p_{L^p(Q)} ds. \tag{1.2.8}$$

By using Grönwall's inequality, we get

$$\|u(t)\|^p_{L^p(Q)} \leq C_T\left(\|f\|^p_{\mathscr{L}^-_p(\Sigma^-)} + \|v\|^p_{L^p(Q_T)} + \|u_0\|^p_{L^p(Q)}\right). \tag{1.2.9}$$

By integrating (1.2.9) over $(0,T)$, we get

$$\begin{aligned}\frac{1}{p}\|u(T)\|^p_{L^p(Q)} - \frac{1}{p}\|u_0\|^p_{L^p(Q)} + \frac{1}{p}\int_0^T \|u(s)\|^p_{L^p(\Gamma^+;d\xi)} ds \\ \leq \frac{1}{p}\int_0^T \|f(s)\|^p_{L^p(\Gamma^-;d\xi)} ds + C_1\int_0^T \|u(s)\|^p_{L^p(Q)} ds.\end{aligned} \tag{1.2.10}$$

Therefore by using (1.2.9), we get

$$\|u\|^p_{\mathscr{L}^+_p(\Sigma^+)} \leq C\left(\|f\|^p_{\mathscr{L}^-_p(\Sigma^-)} + \|v\|^p_{L^p(Q_T)} + \|u_0\|^p_{L^p(Q)}\right). \tag{1.2.11}$$

In light of (1.2.9) and (1.2.11), the proof of Theorem 1.2.1 is completed. □

By assuming that the initial condition $u_0 = 0$ and our known data will be given only by boundary measurements. That will lead us to consider the albedo operator $\mathscr{A}_{a,k}$, that maps the incoming flux on the boundary Σ^- into the outgoing one, defined as follows

$$\begin{array}{rccl}\mathscr{A}_{a,k}: & \mathscr{L}^-_p(\Sigma^-) & \longrightarrow & \mathscr{L}^+_p(\Sigma^+) \\ & f & \longmapsto & \mathscr{A}_{a,k}[f] = u_{|\Sigma^+}.\end{array}$$

From (1.2.3) the albedo operator map $\mathscr{A}_{a,k}$ is continuous from $\mathscr{L}_p^-(\Sigma^-)$ to $\mathscr{L}_p^+(\Sigma^+)$. We denote by $\|\mathscr{A}_{a,k}\|_p$ its norm.

Next, we consider now the following initial boundary value problem for the linear Boltzmann equation with source term $v \in L^p(Q_T)$:

$$\begin{cases} \partial_t \psi + \theta \cdot \nabla \psi + a(x,\theta)\psi = \mathscr{N}_k[\psi] + v(t,x,\theta) & \text{in } Q_T, \\ \psi(0,x,\theta) = 0 & \text{in } Q, \\ \psi(t,x,\theta) = 0 & \text{on } \Sigma^-. \end{cases} \tag{1.2.12}$$

Lemma 1.2.2. *Let $T > 0$, $(a,k) \in \mathscr{A}(M_0) \times \mathscr{K}(M_1, M_2)$. Assuming $v \in L^p(Q_T)$, then there exists a unique solution ψ to* (1.2.12) *such that*

$$\psi \in \mathscr{C}^1([0,T]; L^p(Q)) \cap \mathscr{C}([0,T]; D(A)).$$

Furthermore, there is a constant $C > 0$ such that

$$\|\psi(t,\cdot,\cdot)\|_{L^p(Q)} \leq C \|v\|_{L^p(Q_T)}, \quad \forall\, t \in (0,T). \tag{1.2.13}$$

1.3 Geometric optics solution for linear Boltzmann equation

In this section, we build appropriate geometric optics solutions to the linear Boltzmann equation which are used in the derivation of the stability estimates presented in next chapters. By selecting suitably small $\varepsilon > 0$, we assume that

$$T > \operatorname{Diam}(\Omega) + 4\varepsilon. \tag{1.3.1}$$

Denote

$$\Omega_\varepsilon = \left\{ x \in \mathbb{R}^n \backslash \overline{\Omega};\ \operatorname{dist}(x,\Omega) < \varepsilon \right\}.$$

Let $\phi^\pm \in \mathscr{C}_0^\infty(\Omega_\varepsilon, \mathscr{C}(\mathscr{S}))$. Then, for any $\theta \in \mathscr{S}$, we have

$$\operatorname{Supp} \phi^+(\cdot,\theta) \cap \Omega = \emptyset, \tag{1.3.2}$$

and

$$(\operatorname{Supp} \phi^-(\cdot,\theta) \pm T\theta) \cap \Omega = \emptyset.$$

Let

$$\varphi^{\pm}(t,x,\theta)=\phi^{\pm}(x-t\theta,\theta),\quad (t,x,\theta)\in\mathbb{R}\times\mathbb{R}^n\times\mathscr{S}.$$

Then $\varphi^{\pm}(t,x,\theta)$ solves the following transport equation

$$(\partial_t+\theta\cdot\nabla)\,\varphi(t,x,\theta)=0,\quad (t,x,\theta)\in\mathbb{R}\times\mathbb{R}^n\times\mathscr{S}. \tag{1.3.3}$$

For $\lambda>0$, we define

$$\varphi^{\pm}_{\lambda}(t,x,\theta)=\varphi^{\pm}(t,x,\theta)e^{\pm i\lambda(t-x\cdot\theta)},\quad (t,x,\theta)\in\mathbb{R}\times\mathbb{R}^n\times\mathscr{S}. \tag{1.3.4}$$

Finally, for $a\in\mathscr{C}^1(Q)$, we set

$$b_a(t,x,\theta)=\exp\left(-\int_0^t a(x-s\theta,\theta)ds\right),\quad (t,x,\theta)\in\mathbb{R}\times\mathbb{R}^n\times\mathscr{S},$$

where we have extended $a(\cdot,\theta)$ by 0 outside Ω. We have

$$\begin{aligned}\theta\cdot\nabla\quad b_a(t,x,\theta)&=-b_a\textstyle\int_0^t\theta\cdot\nabla a(x-s\theta,\theta)ds=b_a(t,x,\theta)\int_0^t\left(\frac{d}{ds}a(x-s\theta,\theta)\right)ds\\&=[a(x-t\theta,\theta)-a(x,\theta)]\,b_a(t,x,\theta)\\&=-\partial_t b_a(t,x,\theta)-a(x,\theta)b_a(t,x,\theta).\end{aligned}$$

Therefore b_a satisfies the following transport equation

$$(\partial_t+\theta\cdot\nabla+a(x,\theta))\,b_a(t,x,\theta)=0,\quad (t,x,\theta)\in\mathbb{R}\times\mathbb{R}^n\times\mathscr{S}. \tag{1.3.5}$$

At several points we will need extension of the absorption coefficient a_j to $\mathbb{R}^n\times\mathscr{S}$. The following basic extension result will be used for this.

Lemma 1.3.1. *Let $a_j\in\mathscr{C}^1(\overline{Q})$, $j=1,2$. Let $\tilde{\Omega}$ be a smooth open bounded domain of $\mathbb{R}^n$, containing $\overline{\Omega}$. Then, there exist two function $\tilde{a}_1$ and $\tilde{a}_2$ in $\mathscr{C}^1(\mathbb{R}^n\times\mathscr{S})$, satisfy Supp$(\tilde{a}_j(\cdot,\theta))\subset\overline{\Omega}$, for any $\theta\in\mathscr{S}$, such that we have*

$$\tilde{a}_j(\cdot,\theta)=a_j(\cdot,\theta)\ \textit{in}\ \Omega,\ j=1,2,\ \textit{and}\ \tilde{a}_1(\cdot,\theta)=\tilde{a}_2(\cdot,\theta)\ \textit{in}\ \tilde{\Omega}\setminus\Omega.$$

Moreover, we have

$$\|\tilde{a}_j\|_{C^1(\mathbb{R}^n\times\mathscr{S})}\le C\left(\|a_1\|_{C^1(\overline{Q})}+\|a_2\|_{C^1(\overline{Q})}\right), \tag{1.3.6}$$

for some $C>0$, depending only $\tilde{\Omega}$ and Ω and for any a_j $j=1,2$, satisfy the condition

$$\partial_x^{\alpha}a_1(x,\theta)=\partial_x^{\alpha}a_2(x,\theta),\ (x,\theta)\in\Gamma,\ \alpha\in(\mathbb{N}\cup\{0\})^n,\ |\alpha|\le 1. \tag{1.3.7}$$

Proof. By using [42] Theorem 16.11, we can find $\tilde{a}_1$ and $\tilde{a}_1$ in $\mathscr{C}^1(\mathbb{R}^n \times \mathscr{S})$, such that $\mathrm{Supp}(\tilde{a}_1(\cdot,\theta)) \subset \overline{\Omega}$, $\tilde{a}_j(\cdot,\theta) = a_j(\cdot,\theta)$ in Ω and (1.3.6) holds true for $j = 1$. We denote

$$\tilde{a}_2(x,\theta) = \begin{cases} a_2(x,\theta), & (x,\theta) \in \Omega \times \mathscr{S} \\ \tilde{a}_1(x,\theta), & (x,\theta) \in \mathbb{R}^n \times \mathscr{S}, \end{cases} \tag{1.3.8}$$

Then, from (2.2.24), we have $\tilde{a}_2$ in $\mathscr{C}^1(\mathbb{R}^n \times \mathscr{S})$, and it satisfies (1.3.6) for $j = 2$. □

Here and henceforth, we will denote the extension $\tilde{a}_j$ by a_j, $j = 1.2$. Therefore, b_a satisfies the transport equation (1.3.5).

Lemma 1.3.2. *There exists $C > 0$ such that for any $p \geq 1$, we have*

$$\left\| \mathscr{N}_k[\varphi_\lambda^\pm b_a](t,\cdot,\cdot) \right\|_{L^p(Q)} \leq C \left\| \phi^\pm \right\|_{L^p(\mathbb{R}^n \times \mathscr{S})}, \quad \forall t \in (0,T), \tag{1.3.9}$$

for any $(a,k) \in \mathscr{A}(M_0) \times \mathscr{K}(M_1, M_2)$ Moreover, we have

$$\lim_{\lambda \to \infty} \left\| \mathscr{N}_k[\varphi_\lambda^\pm b_a] \right\|_{L^2(Q_T)} = 0, \tag{1.3.10}$$

where C depends only on T, Ω, M, M_1 and M_2. Here $\varphi_\lambda^\pm$ as they are defined in (1.3.4) and $\mathscr{N}_k$ is defined by (1.1.2).

Proof. We can write

$$\mathscr{N}_k[\varphi_\lambda^\pm b_a](t,x,\theta) = e^{\pm i\lambda t} \Phi_\lambda^\pm(t,x,\theta), \tag{1.3.11}$$

where we set

$$\Phi_\lambda^\pm(t,x,\theta) = \int_{\mathscr{S}} k(t,x,\theta,\theta') \varphi^\pm(t,x,\theta') e^{\pm i\lambda x \cdot \theta'} b_a(t,x,\theta') d\theta'. \tag{1.3.12}$$

By using Hölder's inequality, we have

$$\left| \Phi_\lambda^\pm(t,x,\theta) \right|^p \leq M_1^{p/p'} e^{pTM} \int_{\mathscr{S}} \left| k(t,x,\theta,\theta') \right| \left| \varphi^\pm(t,x,\theta') \right|^p d\theta', \tag{1.3.13}$$

with $\frac{1}{p} + \frac{1}{p'} = 1$. Applying Fubini's Theorem, we deduce that

$$\left\| \Phi_\lambda^\pm(t,\cdot) \right\|_{L^p(Q)}^p \leq e^{pTM} M_1^{p/p'} M_2 \int_{\mathscr{S}} \int_\Omega \left| \varphi^\pm(t,x,\theta') \right|^p dx d\theta'$$

$$\leq C^p \int_{\mathscr{S}} \int_{\mathbb{R}^n} \left|\phi^{\pm}(x,\theta')\right|^p dx d\theta', \tag{1.3.14}$$

where $C = e^{TM}\max(1+M_1, M_2)$. This proves (1.3.9). Now we rewrite $\Phi_\lambda^{\pm}$ as

$$\Phi_\lambda^{\pm}(t,x,\theta) = \int_{\mathscr{S}} \tilde{k}_{t,x}(\theta,\theta') e^{\mp i\lambda x\cdot\theta'} d\theta', \tag{1.3.15}$$

where the kernel $\tilde{k}_{t,x}$ is given by

$$\tilde{k}_{t,x}(\theta,\theta') = k(t,x,\theta,\theta')\varphi^{\pm}(t,x,\theta')b_a(t,x,\theta'), \tag{1.3.16}$$

using the fact that $\tilde{k}_{t,x} \in L^2(\mathscr{S}\times\mathscr{S})$, a.e., $(t,x)\in\Omega_T$, thus the integral operator with kernel $\tilde{k}_{t,x}$ is compact in $L^2(\mathscr{S})$. Using the fact that $\theta \mapsto e^{\mp i\lambda x\cdot\theta}$ converge weakly to zero in $L^2(\mathscr{S})$ when $\lambda\to\infty$, we get

$$\lim_{\lambda\to\infty} \left\|\Phi_\lambda^{\pm}(t,x,\cdot)\right\|_{L^2(\mathscr{S})} = 0, \quad \text{a.e., } (t,x)\in\Omega_T. \tag{1.3.17}$$

The Lebesgue Theorem implies that

$$\lim_{\lambda\to\infty} \left\|\Phi_\lambda^{\pm}\right\|_{L^2(Q_T)} = 0 = \lim_{\lambda\to\infty} \left\|\mathscr{N}_k[\varphi_\lambda^{\pm} b_a]\right\|_{L^2(Q_T)}. \tag{1.3.18}$$

This completes the proof. □

Lemma 1.3.3. *Let $p\geq 1$, $(a,k)\in\mathscr{A}(M_0)\times\mathscr{K}(M_1,M_2)$. For any $\phi^+\in\mathscr{C}_0^\infty(\Omega_\varepsilon,\mathscr{C}(\mathscr{S}))$ and any $\lambda>0$ the following system*

$$\begin{cases} \partial_t u + \theta\cdot\nabla u + a(x,\theta)u = \mathscr{N}_k[u](t,x,\theta) & (t,x,\theta)\in Q_T, \\ u(0,x,\theta) = 0 & (x,\theta)\in Q, \end{cases} \tag{1.3.19}$$

admits a solution $u_\lambda^+ \in \mathscr{C}^1([0,T];L^p(Q))\cap\mathscr{C}([0,T];\mathscr{W}_p)$, of the form

$$u_\lambda^+(t,x,\theta) = \varphi_\lambda^+(t,x,\theta)b_a(t,x,\theta) + r_\lambda^+(t,x,\theta),$$

where r_λ^+ satisfies

$$\left\|r_\lambda^+\right\|_{\mathscr{C}([0,T];L^p(Q))} \leq C\left\|\phi^+\right\|_{L^p(\mathbb{R}^n\times\mathscr{S})}, \text{ and } \quad \lim_{\lambda\to\infty}\left\|r_\lambda^+\right\|_{L^2(Q_T)} = 0, \tag{1.3.20}$$

for some positive C is a constant depending on M, M_1, M_2 and T.

Proof. Recall that we seek u_λ^+ in the form:

$$u_\lambda^+(t,x,\theta) = \varphi_\lambda^+(t,x,\theta)b_a(t,x,\theta) + r_\lambda^+(t,x,\theta).$$

By a simple computation and using (1.3.3) and (1.3.5), we get

$$\begin{aligned}
&\partial_t u_\lambda^+ + \theta\cdot\nabla u_\lambda^+ + a(x,\theta)u_\lambda^+ - \mathcal{N}_k[u_\lambda^+] \\
&\qquad = \partial_t r_\lambda^+(t,x,\theta) + \theta\cdot\nabla r_\lambda^+(t,x,\theta) + a(x,\theta)r - \mathcal{N}_k[r_\lambda^+(t,x,\theta)] \\
&\qquad + (\partial_t + \theta\cdot\nabla + a(x,\theta))\big(\varphi_\lambda^+(t,x,\theta)b_a(t,x,\theta)\big) + \mathcal{N}_k[\varphi_\lambda^+ b_a](t,x,\theta).
\end{aligned}$$

We choose $r_\lambda^+(t,x,\theta)$ solution of the following homogeneous boundary value problem

$$\begin{cases}
\partial_t r + \theta\cdot\nabla r + a(x,\theta)r = \mathcal{N}_k[r] + v_\lambda(t,x,\theta) & \text{in } Q_T, \\
r(0,x,\theta) = 0 & \text{in } Q, \\
r(t,x,\theta) = 0 & \text{on } \Sigma^-,
\end{cases} \tag{1.3.21}$$

where the source term v_λ is given by

$$v_\lambda(t,x,\theta) = \mathcal{N}_k[\varphi_\lambda^+ b_a](t,x,\theta). \tag{1.3.22}$$

Now, to complete our proof it is sufficient to show that r_λ^+ satisfies the estimates (1.3.20) .

Furthermore, thanks to Lemma 1.3.4 there is a constant $C > 0$, such that

$$\left\| r_\lambda^+(t,\cdot,\cdot)\right\|_{L^p(Q)} \le C\,\|v_\lambda\|_{L^p(Q_T)}, \quad \forall\, t\in(0,T), \tag{1.3.23}$$

by (1.3.10) and (1.3.10), we get (1.3.20). This completes the proof. □

In a similar way we get the following Lemma.

Lemma 1.3.4. *Let $q \ge 1$, $(a,k) \in \mathcal{A}(M_0)\times\mathcal{K}(M_1,M_2)$. For any $\phi^- \in \mathcal{C}_0^\infty(\Omega_\varepsilon, \mathcal{C}(\mathcal{S}))$ and any $\lambda > 0$ the following system*

$$\begin{cases}
\partial_t u + \theta\cdot\nabla u - a(x,\theta)u = -\mathcal{N}_k^*[u](t,x,\theta), & (t,x,\theta)\in Q_T, \\
u(T,x,\theta) = 0, & (x,\theta)\in Q,
\end{cases} \tag{1.3.24}$$

where

$$\mathcal{N}_k^*[u](t,x,\theta) = \int_{\mathcal{S}} k(t,x,\theta,\theta')u(t,x,\theta)d\theta,$$

admit solution $u_\lambda^- \in \mathscr{C}([0,T];\mathscr{W}_q(Q)) \cap \mathscr{C}^1([0,T];L^q(Q))$, *of the form*

$$u_\lambda^-(t,x,\theta) = \varphi_\lambda^-(t,x,\theta)b_{-a}(t,x,\theta) + r_\lambda^-(t,x,\theta),$$

where r_λ^- *satisfies*

$$\begin{cases} \partial_t r + \theta\cdot\nabla r - a(x,\theta)r = -\mathcal{N}_k^*[r] - \mathcal{N}_k^*[\varphi_\lambda^- b_a](t,x,\theta), & \text{in } Q_T, \\ r(T,x,\theta) = 0, & \text{in } Q, \\ r(t,x,\theta) = 0, & \text{on } \Sigma^+, \end{cases} \tag{1.3.25}$$

and, there is a constant $C > 0$ *such that*

$$\left\|r_\lambda^-\right\|_{\mathscr{C}([0,T];L^q(Q))} \le C\left\|\phi^-\right\|_{L^q(\mathbb{R}^n\times\mathcal{S})}, \text{ and } \lim_{\lambda\to\infty}\left\|r_\lambda^-\right\|_{L^2(Q_T)} = 0, \tag{1.3.26}$$

for some positive C *is a constant depending on* M, M_1, M_2 *and* T.

1.4 Preliminaries

In order to study our inverse problem in the next chapters, we need also to introduce some crucial tool to prove the main results, there are the the Poisson kernel and the light-ray transform.

1.4.1 Poisson kernel

We start by introducing the Poisson kernel in $B(0,1) \subset \mathbb{R}^n$, defined by

$$P(\omega,\theta) = \frac{1-|\omega|^2}{\alpha_n|\omega-\theta|^n}, \quad \omega \in B(0,1),\ \theta \in \mathcal{S}.$$

Let $0 < h < 1$, we define $\vartheta_h : \mathcal{S}\times\mathcal{S} \to \mathbb{R}$ as

$$\vartheta_h(\omega,\theta) = P(h\omega,\theta). \tag{1.4.27}$$

Let us recall the following Lemma, which follows from (Theorem 2.48 of [46]) and Lemma 4.3 of [18]).

Lemma 1.4.1. *Let ϑ_h given by (1.4.27), $h \in (0,1)$. Then we have the following properties:*

$$0 \leq \vartheta_h(\omega,\theta) \leq \frac{2}{\alpha_n(1-h)^{n-1}}, \quad \forall\, h \in (0,1),\ \forall\, \omega,\ \theta \in \mathscr{S}. \tag{1.4.28}$$

$$\int_{\mathscr{S}} \vartheta_h(\omega,\theta)d\theta = 1, \quad \forall h \in (0,1),\ \forall\, \omega \in \mathscr{S}. \tag{1.4.29}$$

$$\int_{\mathscr{S}} \vartheta_h(\omega,\theta)\,|\omega-\theta|\,d\theta \leq C(1-h)^{1/2n}, \quad \forall\, h \in (0,1),\ \forall\, \omega \in \mathscr{S}. \tag{1.4.30}$$

$$\lim_{h\to 1}\int_{\mathscr{S}} \vartheta_h(\omega,\theta)g(\theta)d\theta = g(\omega),\ \forall\, \omega \in \mathscr{S}, \tag{1.4.31}$$

here the limit is taken in the topology of $L^p(\mathscr{S})$, $p \in [1,+\infty[$ and uniformly on $\mathscr{S}$ if $g \in \mathscr{C}(\mathscr{S})$.

1.4.2 The X-ray Transform

For f function in $\mathbb{R}^n$, $\omega \in \mathscr{S}$, $x \in \mathbb{R}^n$ we define the X-ray transform $\mathscr{P}$ as:

$$\mathscr{P}(\omega,x) = \int_{\mathbb{R}} f(x+s\omega)ds,$$

as has been pointed out that $\mathscr{P}(\omega,x)$ represents the integral of f over the straight line passing through x in the direction ω. It is easy to see that $\mathscr{P}(\omega,x)$ does not change if x moved in the direction ω. Therefore we restrict x to $\omega^{\perp} = \{x \in \mathbb{R}^n; x\cdot\omega = 0\}$ and we can consider $\mathscr{P}$ as a function on the tangent bundle $\mathscr{T} = \{(\omega,x) : \omega \in \mathscr{S},\ x \in \omega^{\perp}\}$.

Moreover, (see [57]) we can prove that for $f \in L^1(\mathbb{R}^n)$ and $\omega \in \mathscr{S}$. Then $\mathscr{P}(\omega,\cdot) \in L^1(\omega^{\perp})$ and

$$\mathscr{F}(\mathscr{P}(\omega,\cdot))(\xi) = (2\pi)^{-\left(\frac{n-1}{2}\right)}\int_{\omega^{\perp}} e^{-ix\cdot\xi}\mathscr{P}(f)(\omega,x)d\mu_x = \sqrt{2\pi}\widehat{f}(\xi),$$

where $d\mu_x$ is the $(n-1)$—dimensional standard volume on $\omega^{\perp}$ and $\widehat{f}(\xi) = (2\pi)^{-\frac{n}{2}}\displaystyle\int_{\mathbb{R}^n} f(y)e^{-y\cdot\xi}dy$.

Chapter 2

Stable determination of the Gauge equivalent classes in the linear Boltzmann equation

2.1 Introduction

In this chapter, we treat the inverse problem of determining the scattering and the absorption coefficients appearing in the linear Boltzmann equation via boundary measurements. We prove that gauge-equivalent of the coefficients yield the same albedo operator. Here we show an obstruction to non-uniqueness in the time-dependent transport equation. Namely, given $\mu \in \mathscr{C}^1(Q)$ be such that $\mu(x,\theta) > 0$ and $\mu(x,\theta) = 1$, for $(x,\theta) \in \Gamma$, let $\tilde{u} = \mu(x,\theta)u$, one has

$$\mathscr{B}_{\tilde{a},\tilde{k}}[\tilde{u}](t,x,\theta) = \partial_t \tilde{u} + \theta \cdot \nabla \tilde{u}(t,x,\theta) + \tilde{a}(x,\theta)\tilde{u} - \mathscr{N}_{\tilde{k}}[\tilde{u}](t,x,\theta) = 0 \quad \text{in } Q_T, \tag{2.1.1}$$

where $(\tilde{a},\tilde{k})$ is given by

$$\tilde{a}(x,\theta) = a(x,\theta) - \theta \cdot \nabla_x \log \mu(x,\theta), \quad \tilde{k}(x,\theta,\theta') = \frac{\mu(x,\theta)}{\mu(x,\theta')} k(x,\theta,\theta'). \tag{2.1.2}$$

Since $\mu = 1$ on Γ, the boundary data do not change. Therefore, if $\mathscr{A}_{\tilde{a},\tilde{k}}$ is the albedo operator corresponding to the pair $(\tilde{a},\tilde{k})$, then $\mathscr{A}_{\tilde{a},\tilde{k}} = \mathscr{A}_{a,k}$. This relation called gauge equivalent. In this chapter we will study the question of stability

of the determination of the gauge equivalent classes.

2.2 Stability and uniqueness of the absorption coefficient modulo the gauge transformation

This section is dedicated to prove our main stability results. The idea of the proof is usually based on the Exponentially Growing Solutions already built in Chapter 1. Here and henceforth, we extend a in $\mathbb{R}^n \times \mathscr{S}$ by $a = a_2 - a_1$ in $\overline{Q}$ and $a = 0$ on $(\mathbb{R}^n \setminus \overline{\Omega}) \times \mathscr{S}$. We denote

$$a(x,\theta) = (a_2 - a_1)(x,\theta), \quad b_a(t,x,\theta) = b_{a_2} b_{-a_1}(t,x,\theta) = \exp\Big(-\int_0^t a(x-s\theta,\theta)\,ds\Big).$$

Here, we recall the definition of b_{-a_1} and b_{a_2}

$$b_{-a_1}(x,t) = \exp\Big(\int_0^t a_1(x-s\theta,\theta)\,ds\Big), \quad b_{a_2}(t,x,\theta) = \exp\Big(-\int_0^t a_2(x-s\theta,\theta)\,ds\Big).$$

Lemma 2.2.1. *Let $(a_j,k_j) \in \mathscr{A}(M_0) \times \mathscr{K}(M_1,M_2)$, $j = 1, 2$. There exists $C > 0$ such that for any $\phi_j \in \mathscr{C}_0^\infty(\Omega_\varepsilon)$, $j = 1,2$, the following estimate holds true*

$$\Big|\int_{\Omega_\varepsilon \times \mathscr{S}} \vartheta_h(\omega,\theta)\phi_1(y)\phi_2(y)\Big[b_a(T,y+T\theta,\theta) - 1\Big]\,d\theta\,dy\Big|$$
$$\leq C\|\mathscr{A}_{a_2,k_2} - \mathscr{A}_{a_1,k_1}\|_1 \|\phi_1\|_{L^1(\mathbb{R}^n)} \|\phi_2\|_{L^\infty(\mathbb{R}^n)}. \quad (2.2.3)$$

Here C depends only on Ω, T, M, M_1 and M_2.

Proof. Let $\vartheta_h(\omega,\cdot)$ as it was defined in (1.4.27) and let $p,q \geq 1$, with $\frac{1}{p} + \frac{1}{q} = 1$. Choosing

$$\phi^+(y,\theta) = \vartheta_h(\omega,\theta)\phi_1(y), \quad \phi^-(y,\theta) = \phi_2(y).$$

Then we have Supp $\phi^+(\cdot,\theta) \cap \Omega = \emptyset$, Lemma 1.3.3 guarantees that there exists a geometrical optics solution $u_\lambda^+ \in \mathscr{C}^1([0,T]; L^p(Q)) \cap \mathscr{C}([0,T]; \mathscr{W}_p)$ to the equation

$$\begin{cases} \mathscr{B}_{a_2,k_2}[u](t,x,\theta) = 0 & (t,x,\theta) \in Q_T, \\ u(0,x,\theta) = 0 & (x,\theta) \in Q, \end{cases}$$

in the following form

$$u_\lambda^+(t,x,\theta) = \varphi_\lambda^+(t,x,\theta) b_{a_2}(t,x,\theta) + r_\lambda^+(t,x,\theta), \tag{2.2.4}$$

corresponding to a_2 and k_2, where $r_\lambda^+(t,x,\theta)$ obeys (1.3.20). We denote by f_λ the function

$$f_\lambda(t,x,\theta) = u_\lambda^+(t,x,\theta)_{|\Sigma^-} = \varphi_\lambda^+(t,x,\theta) b_{a_2}(t,x,\theta)_{|\Sigma^-},$$

and we consider a solution u_1 to the following system

$$\begin{cases} \mathscr{B}_{a_1,k_1}[u_1](t,x,\theta) = 0 & (t,x,\theta) \in Q_T, \\ u_1(0,x,\theta) = 0 & (x,\theta) \in Q, \\ u_1(t,x,\theta) = f_\lambda(t,x,\theta) & (t,x,\theta) \in \Sigma^-. \end{cases}$$

Now if we set $u = u_1 - u_\lambda^+$. Then, u solves

$$\begin{cases} \mathscr{B}_{a_1,k_1}[u](t,x,\theta) = a(x,\theta) u_\lambda^+ - \mathscr{N}_k[u_\lambda^+] & (t,x,\theta) \in Q_T, \\ u(0,x,\theta) = 0 & (x,\theta) \in Q, \\ u(t,x,\theta) = 0 & (t,x,\theta) \in \Sigma^-, \end{cases} \tag{2.2.5}$$

where $a = a_2 - a_1$ and $k = k_2 - k_1$. Meanwhile, Lemma 1.3.4 and the fact that $(\text{Supp } \phi^-(\cdot,\theta) \pm T\theta) \cap \Omega = \emptyset$, guarantee the existence of a geometrical optic solution $u_\lambda^- \in \mathscr{C}^1([0,T]; L^q(Q)) \cap \mathscr{C}([0,T]; \mathscr{W}_q)$ to the backward problem

$$\begin{cases} \mathscr{B}^*_{a_1,k_1}[u](t,x,\theta) = 0, & (t,x,\theta) \in Q_T, \\ u(T,x,\theta) = 0, & (x,\theta) \in Q, \end{cases}$$

corresponding to a_1 and k_1, in the form

$$u_\lambda^-(t,x,\theta) = \varphi_\lambda^-(t,x,\theta) b_{-a_1}(t,x,\theta) + r_\lambda^-(t,x,\theta), \tag{2.2.6}$$

where $r_\lambda^-(t,x,\theta)$ obeys (1.3.26) with $p = q$. We multiply the first equation of the system (2.2.5) by u_λ^-, then by integrating by part and using Green's formula, we obtain

$$\begin{aligned} \int_{Q_T} a(x,\theta) u_\lambda^+ u_\lambda^- \, d\theta \, dx \, dt - \int_{Q_T} \mathscr{N}_k[u_\lambda^+](t,x,\theta) u_\lambda^-(t,x,\theta) d\theta \, dx \, dt \\ = \int_{\Sigma^+} \theta \cdot \nu(x) (\mathscr{A}_{a_2,k_2} - \mathscr{A}_{a_1,k_1})(f_\lambda) u_\lambda^- \, d\theta \, dx \, dt. \end{aligned} \tag{2.2.7}$$

This last estimate, follow from the fact that we have

$$\int_{Q_T} \mathcal{N}_{k_1}[u](t,x,\theta)u_\lambda^-(t,x,\theta)dxd\theta dt = \int_{Q_T} u(t,x,\theta)\mathcal{N}_{k_1}^*[u_\lambda^-](t,x,\theta)dxd\theta dt.$$

Now, we replace the solutions u_λ^+ and u_λ^- by their forms, we get

$$\begin{aligned}\int_{Q_T} & a(x,\theta)u_\lambda^+ u_\lambda^- d\theta\, dx\, dt \\ &= \int_{Q_T} a(x,\theta)\varphi_\lambda^+(t,x,\theta)\varphi_\lambda^-(t,x,\theta)b_a(t,x,\theta)d\theta\, dx\, dt + \mathcal{E}_{1,\lambda}, \quad (2.2.8)\end{aligned}$$

where $\mathcal{E}_{1,\lambda}$ represents the sum of the integrals that contain terms with r_λ^+ and r_λ^-.

Bearing in mind that there exists $C > 0$ such that for any $\lambda \geq 0$, we have

$$|\mathcal{E}_{1,\lambda}| \leq C\left(\left\|r_\lambda^+\right\|_{L^2(Q_T)} + \left\|r_\lambda^-\right\|_{L^2(Q_T)}\right). \quad (2.2.9)$$

Then by (1.3.20) and (1.3.26), we have

$$\lim_{\lambda\to\infty} \mathcal{E}_{1,\lambda} = 0. \quad (2.2.10)$$

Meanwhile, we have

$$\begin{aligned}\int_{Q_T} & \mathcal{N}_k[u_\lambda^+]u_\lambda^- d\theta\, dx\, dt \\ &= \int_{Q_T} \mathcal{N}_k[\varphi_\lambda^+ b_{a_2}](t,x,\theta)\varphi_\lambda^-(t,x,\theta)b_{-a_1}(t,x,\theta)d\theta\, dx\, dt + \mathcal{E}_{2,\lambda}, \quad (2.2.11)\end{aligned}$$

where

$$|\mathcal{E}_{2,\lambda}| \leq C\left(\left\|r_\lambda^+\right\|_{L^2(Q_T)} + \left\|r_\lambda^-\right\|_{L^2(Q_T)}\right). \quad (2.2.12)$$

Thus from (1.3.20) and (1.3.26), we get

$$\lim_{\lambda\to\infty} \mathcal{E}_{2,\lambda} = 0. \quad (2.2.13)$$

From Cauchy-Schwartz inequality and (1.3.18), we find

$$\lim_{\lambda\to\infty} \int_{Q_T} \mathcal{N}_k[\varphi_\lambda^+ b_{a_2}](t,x,\theta)\varphi_\lambda^-(t,x,\theta)b_{-a_1}(t,x,\theta)d\theta\, dx\, dt = 0. \quad (2.2.14)$$

We rewrite (2.2.7), as

$$\int_{Q_T} a(x,\theta)\varphi_\lambda^+(t,x,\theta)\varphi_\lambda^-(t,x,\theta)b_a(t,x,\theta)d\theta\,dx\,dt$$
$$= \int_{\Sigma^+} \theta\cdot\nu(x)(\mathscr{A}_{a_2,k_2}-\mathscr{A}_{a_1,k_1})(f_\lambda)u_\lambda^-\,d\theta\,dx\,dt - \mathscr{E}_\lambda, \quad (2.2.15)$$

where $\mathscr{E}_\lambda$ is defined by

$$\mathscr{E}_\lambda = \mathscr{E}_{1,\lambda} - \mathscr{E}_{2,\lambda} - \int_{Q_T} \mathscr{N}_k[\varphi_\lambda^+ b_{a_2}](t,x,\theta)\varphi_\lambda^-(t,x,\theta)b_{-a_1}(t,x,\theta)d\theta\,dx\,dt. \quad (2.2.16)$$

Next, taking the limit as $\lambda\to+\infty$, then from (2.2.10), (2.2.13) and (2.2.14), we get

$$\lim_{\lambda\to\infty}\mathscr{E}_\lambda = 0. \quad (2.2.17)$$

Next we have $u_\lambda^+ = f_\lambda$ on Σ^-, then for $q\geq 1$ such that $\frac{1}{p}+\frac{1}{q}=1$, we get the following estimate

$$\Big|\int_{\Sigma^+}(\mathscr{A}_{a_2,k_2}-\mathscr{A}_{a_1,k_1})(f_\lambda)u^-\,dx\,d\theta\,dt\Big|$$
$$\leq C\|\mathscr{A}_{a_2,k_2}-\mathscr{A}_{a_1,k_1}\|_p\,\|f_\lambda\|_{\mathscr{L}_p^-(\Sigma^-)}\|u_\lambda^-\|_{\mathscr{L}_q^+(\Sigma^+)}.$$

Next, as the following estimates hold

$$\|u_\lambda^-\|_{\mathscr{L}_q^+(\Sigma^+)}\leq C\|\phi^-\|_{L^\infty(\mathbb{R}^n\times\mathscr{S})},\quad \lim_{p\to1}\|f_\lambda\|_{\mathscr{L}_p^-(\Sigma^-)} = \|f_\lambda\|_{\mathscr{L}_1^-(\Sigma^-)}\leq C\|\phi^+\|_{L^1(\mathbb{R}^n\times\mathscr{S})},$$

and note that, since $\mathscr{A}_{a_j,k_j}$, $j=1,2$ are continuous from $\mathscr{L}_{p'}^-(\Sigma^-)$ to $\mathscr{L}_{p'}^+(\Sigma^+)$ for any $p'\geq 1$, we get for any p, $1<p<2$, by Riesz-Thorin's Theorem, the following interpolation inequality

$$\|\mathscr{A}_{a_2,k_2}-\mathscr{A}_{a_1,k_1}\|_p\leq\|\mathscr{A}_{a_2,k_2}-\mathscr{A}_{a_1,k_1}\|_2^{1-\kappa_p}\|\mathscr{A}_{a_2,k_2}-\mathscr{A}_{a_1,k_1}\|_1^{\kappa_p}.$$

with $\kappa_p=(2-p)/p$. So that

$$\limsup_{p\to1}\|\mathscr{A}_{a_2,k_2}-\mathscr{A}_{a_1,k_1}\|_p\leq\|\mathscr{A}_{a_2,k_2}-\mathscr{A}_{a_1,k_1}\|_1. \quad (2.2.18)$$

therefore, by (2.2.15), (2.2.17) and (2.2.18), we get

$$\Big|\int_{Q_T} a(x,\theta)\varphi^+\varphi^-(t,x,\theta)b_a(t,x,\theta)d\theta\,dx\,dt\Big|$$
$$\leq C\|\mathscr{A}_{a_2,k_2}-\mathscr{A}_{a_1,k_1}\|_1\|\phi^+\|_{L^1(\mathbb{R}^n\times\mathscr{S})}\|\phi^-\|_{L^\infty(\mathbb{R}^n\times\mathscr{S})}. \quad (2.2.19)$$

Then, from the fact that $a(\cdot,\theta)=0$ outside Ω and performing the changes of variables $y=x-t\theta$, we get

$$\Big|\int_0^T\int_{\mathbb{R}^n\times\mathscr{S}} a(y+t\theta,\theta)\phi^+(y,\theta)\phi^-(y,\theta)b_a(t,y+t\theta,\theta)\,d\theta dy\,dt\Big| \\ \leq C\|\mathscr{A}_{a_2,k_2}-\mathscr{A}_{a_1,k_1}\|_1\|\phi^+\|_{L^1(\mathbb{R}^n\times\mathscr{S})}\|\phi^-\|_{L^\infty(\mathbb{R}^n\times\mathscr{S})}. \quad (2.2.20)$$

Moreover, we have

$$\int_0^T\int_{\mathbb{R}^n\times\mathscr{S}} a(y+t\theta,\theta)\phi^+(y,\theta)\phi^-(y,\theta)b_a(t,y+t\theta,\theta)\,d\theta\,dy\,dt \\ =-\int_{\mathbb{R}^n\times\mathscr{S}}\phi^+(y,\theta)\phi^-(y,\theta)\int_0^T\frac{d}{dt}\Big[\exp\Big(-\int_0^t a(y+s\theta,\theta)\,ds\Big)\Big]dt\,d\theta\,dy \\ =-\int_{\mathbb{R}^n\times\mathscr{S}}(\phi^+\phi^-)(y,\theta)\Big[b_a(T,y+T\theta,\theta)-1\Big]d\theta\,dy, \quad (2.2.21)$$

then, we conclude the desired estimate given by

$$\Big|\int_{\mathbb{R}^n\times\mathscr{S}}(\phi^+\phi^-)(y,\theta)\Big[b_a(T,y+T\theta,\theta)-1\Big]d\theta\,dy\Big| \\ \leq C\|\mathscr{A}_{a_2,k_2}-\mathscr{A}_{a_1,k_1}\|_1\|\phi^+\|_{L^1(\mathbb{R}^n\times\mathscr{S})}\|\phi^-\|_{L^\infty(\mathbb{R}^n\times\mathscr{S})}. \quad (2.2.22)$$

This completes the proof of the lemma. □

Theorem 2.2.2. *Let $T>Diam(\Omega)$, then there exists $C>0$ such that for each $\theta\in\mathscr{S}$ and $x\in\mathbb{R}^n$, we have*

$$\Big|\int_{\mathbb{R}} a(x+s\theta,\theta)ds\Big|\leq C\|\mathscr{A}_{a_2,k_2}-\mathscr{A}_{a_1,k_1}\|_1, \quad (2.2.23)$$

for any $(a_j,k_j)\in\mathscr{A}(M_0)\times\mathscr{K}(M_1,M_2)$, $j=1,2$, satisfy the condition

$$\partial_x^\alpha a_1(x,\theta)=\partial_x^\alpha a_2(x,\theta),\ (x,\theta)\in\Gamma,\ \alpha\in(\mathbb{N}\cup\{0\})^n,\ |\alpha|\leq 1. \quad (2.2.24)$$

Proof. Let $\omega\in\mathscr{S}$ and $\phi_j\in\mathscr{C}_0^\infty(\Omega_\varepsilon)$, $j=1,2$. By (1.4.31) we have

$$\lim_{h\to 1}\int_{\mathscr{S}}\vartheta_h(\omega,\theta)\Big[b_a(T,y+T\theta,\theta)-1\Big]d\theta=\Big[b_a(T,y+T\omega,\omega)-1\Big], \quad (2.2.25)$$

then, we obtain the following estimate

$$\Big|\int_{\Omega_\varepsilon}\phi_1(y)\phi_2(y)\Big[b_a(T,y+T\omega,\omega)-1\Big]dy\Big|\leq C\|\mathscr{A}_{a_2,k_2}-\mathscr{A}_{a_1,k_1}\|_1\|\phi_1\|_{L^1(\mathbb{R}^n)}\|\phi_2\|_{L^\infty(\mathbb{R}^n)}. \tag{2.2.26}$$

We conclude that

$$\Big|b_a(T,y+T\omega,\omega)-1\Big|\leq C\|\mathscr{A}_{a_2,k_2}-\mathscr{A}_{a_1,k_1}\|_1,\quad \forall\, y\in\Omega_\varepsilon,\ \forall\,\omega\in\mathscr{S}. \tag{2.2.27}$$

Bearing in mind that $|X|\leq e^M\,|e^X-1|$ for any X real satisfying $|X|\leq M$, one gets

$$\Big|\int_0^T a(y+s\omega,\omega)\,ds\Big|\leq e^{MT}\Big|b_a(T,y+T\omega,\omega)-1\Big|,$$

here $X=\displaystyle\int_0^T a(y+s\omega,\omega)ds$. Thus using (2.2.27) we get the following estimate

$$\Big|\int_0^T a(y+s\omega,\omega)\,ds\Big|\leq C\|\mathscr{A}_{a_2,k_2}-\mathscr{A}_{a_1,k_1}\|_1,\quad \forall\, y\in\Omega_\varepsilon,\quad \forall\omega\in\mathscr{S}. \tag{2.2.28}$$

Moreover, since $\mathrm{supp}(a(\cdot,\omega))\subset\Omega$ and $T>\mathrm{Diam}(\Omega)$ we have

$$\int_0^T a(y+s\omega,\omega)ds=\int_0^{+\infty}a(y+s\omega,\omega)ds,$$

we deduce that

$$\Big|\int_0^\infty a(y+s\omega,\omega)\,ds\Big|\leq C\|\mathscr{A}_{a_2,k_2}-\mathscr{A}_{a_1,k_1}\|_1,\quad \forall y\in\Omega_\varepsilon,\ \forall\omega\in\mathscr{S}. \tag{2.2.29}$$

Now let $x\in\Omega_\varepsilon$. Since Ω is convex we distinguish two cases:

Case 1. $x-s\omega\notin\Omega$ for all $s\geq 0$. Then

$$\int_0^\infty a(x-s\omega,\omega)ds=0.$$

Therefore we have by (2.2.29)

$$\Big|\int_{\mathbb{R}} a(y+s\omega,\omega)\,ds\Big|\leq C\|\mathscr{A}_{a_2,k_2}-\mathscr{A}_{a_1,k_1}\|_1. \tag{2.2.30}$$

Case 2. There exists $s\geq 0$ such that $x-s\omega\in\Omega$. Let

$$s^*=\sup\left\{s\geq 0;\,(x-s\omega)\in\overline{\Omega}\right\}.$$

We choose $\eta \in (0,\varepsilon)$ such that we have $y = x-(s^*+\eta)\omega \in \Omega_\varepsilon$. Then, since $x - s\omega \notin \Omega$ for $s > s^*+\eta$ by the definition of s^*, we have

$$\int_0^\infty a(x-s\omega,\omega)ds = \int_0^{s^*+\eta} a(x-s\omega,\omega)ds = \int_0^{s^*+\eta} a(y+s\omega,\omega)ds.$$

At the last integral, we changed the variable: s by $s^*+\eta-s$.

Here we claim that $y+s\omega \notin \Omega$ for all $s > s^*+\eta$. In fact, let $y+s_0\omega = x+(s_0-s^*-\eta)\omega \in \Omega$ for some $s_0 > s^*+\eta$. Then, since Ω is convex and x is on the segment connecting $x-s^*\omega \in \overline{\Omega}$ and $x+(s_0-s^*-\eta)\omega$, we see by $s_0-s^*-\eta > 0$ that $x \in \overline{\Omega}$, which is impossible by $x \in \Omega_\varepsilon \subset \mathbb{R}^n \setminus \overline{\Omega}$. Therefore

$$\int_0^\infty a(x-s\omega,\omega)ds = \int_0^\infty a(y+s\omega,\omega)ds, \quad \forall y \in \Omega_\varepsilon,\ \forall \omega \in \mathscr{S}.$$

Hence, by (2.2.29) we have

$$\left|\int_{-\infty}^0 a(x+s\omega,\omega)ds\right| = \left|\int_0^\infty a(y+s\omega,\omega)ds\right| \le C\|\mathscr{A}_{a_2,k_2} - \mathscr{A}_{a_1,k_1}\|_1.$$

Thus we conclude that

$$\left|\int_{\mathbb{R}} a(y+s\omega,\omega)ds\right| \le C\|\mathscr{A}_{a_2,k_2} - \mathscr{A}_{a_1,k_1}\|_1, \quad \forall y \in \Omega_\varepsilon,\ \forall \omega \in \mathscr{S}. \tag{2.2.31}$$

Let $x \in \mathbb{R}^n$ and $x+s\omega \notin \overline{\Omega}$ for any $s \in \mathbb{R}$. Then $\displaystyle\int_{\mathbb{R}} a(x+s\omega,\omega)ds = 0$ because $\operatorname{Supp} a(\cdot,\omega) \subset \overline{\Omega}$. Next let $x \in \mathbb{R}^n$ and $x+s_0\omega \in \overline{\Omega}$ for some $s_0 \in \mathbb{R}$. Then we can choose some $s_1 \in \mathbb{R}$ such that $x+s_1\omega \in \Omega_\varepsilon$, we have

$$\int_{\mathbb{R}} a(x+s\omega,\omega)ds = \int_{\mathbb{R}} a(x+s_1\omega+(s-s_1)\omega,\omega)ds.$$

Performing the changes of variables $t = s-s_1$ and $y = x+s_1\omega$, we get by (2.2.31)

$$\left|\int_{\mathbb{R}} a(y+t\omega,\omega)dt\right| \le C\|\mathscr{A}_{a_2,k_2} - \mathscr{A}_{a_1,k_1}\|_1, \quad \forall y \in \mathbb{R}^n,\ \forall \omega \in \mathscr{S}.$$

This completes the proof of Theorem 2.2.2. □

As a consequence of Theorem 2.2.2, we have the following result

Corollary 2.2.3. *Let $T > Diam(\Omega)$ and $(a_j, k_j) \in \mathscr{A}(M_0) \times \mathscr{K}(M_1, M_2), j = 1, 2,$ such that $\mathscr{A}_{a_1,k_1} = \mathscr{A}_{a_2,k_2}$. Then there exist a positive function $\mu(x,\theta) \in L^\infty(Q)$ with $\theta \cdot \nabla_x \mu \in L^\infty(Q)$ and $\mu(x,\theta) = 1$ for $(x,\theta) \in \Gamma$, such that we have*

$$a_1(x,\theta) = a_2(x,\theta) - \theta \cdot \nabla(\log \mu(x,\theta)), \quad \forall (x,\theta) \in Q. \tag{2.2.32}$$

Proof. Since we are assuming that $\mathscr{A}_{a_2,k_2} = \mathscr{A}_{a_1,k_1}$, then from the previous Theorem, we have

$$\int_{\mathbb{R}} a(y + t\omega, \omega)\, dt = 0, \quad \forall y \in \mathbb{R}^n, \quad \forall \omega \in \mathscr{S}. \tag{2.2.33}$$

Let consider

$$\Phi(y,\omega) = \int_{-\infty}^{0} a(y + t\omega, \omega)\, dt, \quad (y,\omega) \in \mathbb{R}^n \times \mathscr{S}. \tag{2.2.34}$$

Then $\Phi \in L^\infty(Q)$ with

$$\begin{aligned} &\omega \cdot \nabla \Phi(y,\omega) = a(y,\omega) && \text{in} \quad Q, \\ &\Phi(y,\omega) = 0, && \text{on} \quad \Gamma^-. \end{aligned} \tag{2.2.35}$$

From (2.2.33) we also get $\Phi(y,\omega) = 0$ on Γ^+, therefore $\Phi(y,\omega) = 0$ for $(y,\omega) \in \Gamma$.

Then if we pick $\mu(y,\omega) = e^{\Phi(y,\omega)}$, we get

$$a_1(y,\omega) = a_2(y,\omega) + \omega \cdot \nabla \log \mu(y,\omega).$$

This completes the proof. □

2.3 Application

In this section, we prove that if we have an additional hypothesis on the absorption coefficient, we can do even better. For example, suppose that $a(x,\theta) = \theta \cdot B(x)$ where, $B = (b_j)_{1 \le j \le n} \in C^2(\bar{\Omega}; \mathbb{R}^n)$.

$$b_k(x) = (B_1 - B_2)(x) \cdot e_k = B(x) \cdot e_k,$$

where $(e_k)_{1\leq k\leq n}$ is the canonical basis of $\mathbb{R}^n$. Then the non-uniqueness manifested in (2.1.2) says that the best we could hope to reconstruct from the albedo operator is the 2-form given by

$$\operatorname{curl}(B)=\sum_{i,j=1}^{n}\left(\frac{\partial b_i}{\partial x_j}-\frac{\partial b_j}{\partial x_i}\right)dx_j\wedge dx_i.$$

We denote by

$$\sigma_{j,k}(x)=\frac{\partial b_k}{\partial x_j}(x)-\frac{\partial b_j}{\partial x_k}(x),\quad j,k=1,\dots,n. \tag{2.3.36}$$

Inspired by the work of [14] and [17], we estimate the Fourier transform of $\sigma_{j,k}$ as follows.

Lemma 2.3.1. *There exists constant $C>0$, such that for any $\xi\in\mathbb{R}^n$, we have*

$$<\xi>^{-1}|\widehat{\sigma}_{j,k}(\xi)|\leq C\|\mathscr{A}_{a_1,k_1}-\mathscr{A}_{a_2,k_2}\|_1, \tag{2.3.37}$$

where $\langle\xi\rangle=\sqrt{1+|\xi|^2}$.

Proof. Since we are assuming that $a(x,\omega)=\omega\cdot B(x)$, the identity in Theorem 2.2.2 is reduce to

$$\left|\int_{\mathbb{R}}\omega\cdot B(x+t\omega)dt\right|\leq C\|\mathscr{A}_{a_1,k_1}-\mathscr{A}_{a_2,k_2}\|_1,\quad \forall\omega\in\mathscr{S},\ \forall x\in\mathbb{R}^n. \tag{2.3.38}$$

Let $\xi\in\omega^{\perp}$. Performing the changes of variables $z=x+t\omega\in\omega^{\perp}\oplus\mathbb{R}\omega=\mathbb{R}^n$, we have the following identity

$$z\cdot\xi=x\cdot\xi+t\omega\cdot\xi=x\cdot\xi,$$

with $dz=dt\,d\mu_x$. Thus, we get

$$\begin{aligned}\int_{\omega^{\perp}}e^{-ix\cdot\xi}\int_{\mathbb{R}}\omega\cdot B(x+t\omega)\,dt\,d\mu_x &= \int_{\omega^{\perp}}\int_{\mathbb{R}}e^{-ix\cdot\xi-t\xi\cdot\omega}\omega\cdot B(x+t\omega)\,dt\,d\mu_x\\ &= \int_{\mathbb{R}^n}e^{-iz\cdot\xi}\omega\cdot B(z)\,dz.\end{aligned}$$

Assume that $\Omega\subset B_R$, with $R>0$. Using the fact that Supp $B\subset\Omega$, we get from (2.3.38).

$$\left|\int_{\mathbb{R}^n}e^{-iz\cdot\xi}\omega\cdot B(z)\,dz\right| \leq \int_{\omega^{\perp}\cap B_R}\left|e^{-ix\cdot\xi}\int_{\mathbb{R}}\omega\cdot B(x+t\omega)\,dt\right|dz$$

$$\leq \quad C\|\mathscr{A}_{a_2,k_2} - \mathscr{A}_{a_1,k_1}\|_1. \tag{2.3.39}$$

For $\xi \in \mathbb{R}^n$, we define $\omega = \frac{\xi_j e_k - \xi_k e_j}{|\xi_j e_k - \xi_k e_j|}$. Multiplying (2.3.39) by $|\xi_j e_k - \xi_k e_j|$, we obtain

$$\Big|\int_{\mathbb{R}^n} e^{-iz\cdot\xi}\Big(\xi_j b_k(z) - \xi_k b_j(z)\Big)\,dz\Big| \leq C\langle\xi\rangle\|\mathscr{A}_{a_2,k_2} - \mathscr{A}_{a_1,k_1}\|_1.$$

This together with (2.3.36) yield

$$|\widehat{\sigma}_{j,k}(\xi)| \leq C\langle\xi\rangle\|\mathscr{A}_{a_2,k_2} - \mathscr{A}_{a_1,k_1}\|_1.$$

This ends the proof. □

We now state the main result of this section.

Theorem 2.3.2. *Let $B_1,\ B_2$ be two real $C^2(\bar{\Omega};\mathbb{R}^n)$ vector fields on Ω, $\|B_i\|_{C^2(\bar{\Omega};\mathbb{R}^n)} \leq M,\ i = 1,2$ and $T > Diam(\Omega)$. Then there exist a constant $C > 0$ and $\eta \in (0,1)$ such that*

$$\|curl\, B_1 - curl\, B_2\|_{H^{-1}(\Omega)} \leq C\|\mathscr{A}_{a_2,k_2} - \mathscr{A}_{a_1,k_1}\|_1^{\eta},$$

where C depends on Ω, M, T and d.

Proof. We start by decompose the $H^{-1}(\mathbb{R}^n)$ norm as follows

$$\|\sigma_{j,k}\|^2_{H^{-1}(\mathbb{R}^n)} = \int_{|\xi|\leq\gamma} |\widehat{\sigma}_{j,k}(\xi)|^2\langle\xi\rangle^{-2}\,d\xi + \int_{|\xi|>\gamma} |\widehat{\sigma}_{j,k}(\xi)|^2\langle\xi\rangle^{-2}\,d\xi.$$

Then, we have

$$\|\sigma_{j,k}\|^2_{H^{-1}(\mathbb{R}^n)} \leq C\Big(R^n\|\langle\xi\rangle^{-1}\widehat{\sigma}_{j,k}\|^2_{L^\infty(B_\gamma)} + \frac{1}{\gamma^2}\|\sigma_{j,k}\|^2_{L^2(\mathbb{R}^n)}\Big).$$

This entails that

$$\|\sigma_{j,k}\|^2_{H^{-1}(\mathbb{R}^n)} \leq C\Big(\gamma^n\|\mathscr{A}_{a_2,k_2} - \mathscr{A}_{a_1,k_1}\|_1^2 + \frac{1}{\gamma^2}\Big).$$

In particular,

$$\|\sigma_{j,k}\|^2_{H^{-1}(\mathbb{R}^n)} \leq C \min_{\gamma\in[0,+\infty[}(\gamma^n\|\mathscr{A}_{a_2,k_2} - \mathscr{A}_{a_1,k_1}\|_1^2 + \frac{1}{\gamma^2}), \tag{2.3.40}$$

which yields

$$\|\sigma_{j,k}\|^2_{H^{-1}(\mathbb{R}^n)} \leq C(\|\mathscr{A}_{a_2,k_2} - \mathscr{A}_{a_1,k_1}\|_1^{4/n+2}).$$

Then we have

$$\|\text{curl B}_1 - \text{curl B}_2\|_{H^{-1}(\Omega)} \leq \sum_{j,k} \|\sigma_{j,k}\|_{H^{-1}(\mathbb{R}^n)} \leq C\|\text{curl B}_1 - \text{curl B}_2\|_1^{\eta},$$

where $\eta = 2/n + 2$ This completes the proof of our Theorem. □

As an easy consequence of this Theorem. By interpolation inequality, we can readily derive the following stability estimate.

Corollary 2.3.3. *Let $B_1,\ B_2$ be two real $C^2(\bar{\Omega};\mathbb{R}^n)$ vector fields on Ω, $T > Diam(\Omega)$, such that $\|B_i\|_{H^{s+1}(\Omega)} \leq M,\ i = 1,2$, where $s > \frac{n-1}{2}$. Then there exist a constant $C > 0$ and $\eta' \in (0,1)$ such that*

$$\|curl\ B_1 - curl\ B_2\|_{L^\infty(\Omega)} \leq C\|\mathscr{A}_{a_2,k_2} - \mathscr{A}_{a_1,k_1}\|_1^{\eta'},$$

where C depends on Ω, M, T and d.

Proof. Let us now consider δ such that $s > \frac{n-1}{2} + \delta$. By Sobolev's embedding theorem, since $B_j \in C^2(\bar{\Omega},\mathbb{R}^n)$, we obtain

$$\begin{aligned}
\|\text{curl B}_1 - \text{curl B}_2\|_{L^\infty(\Omega)} &\leq C\|\text{curl B}_1 - \text{curl B}_2\|_{H^{\frac{n+1}{2}+\delta}(\mathbb{R}^n)} \\
&\leq C\|\text{curl B}_1 - \text{curl B}_2\|^{1-\beta}_{H^{-1}(\mathbb{R}^n)}\|\text{curl B}_1 - \text{curl B}_2\|^{\beta}_{H^{s+1}(\mathbb{R}^n)} \\
&\leq C\|\text{curl B}_1 - \text{curl B}_2\|_1^{\eta'},
\end{aligned}$$

which yields the desired estimate. □

Chapter 3

Uniqueness and non uniqueness in determining the time dependent coefficients the linear Boltzmann equation

3.1 Introduction

In this chapter, we will discuss the problem of the uniqueness of inverse coefficients problem. Throughout this chapter we suppose that $n \geq 2$. Here we recall the notions of the previous chapters. Let Ω be a bounded and convex domain of $\mathbb{R}^n$. Here and henceforth, the space of velocities $\mathscr{S} = \mathbb{S}^{n-1}$ is chosen to be the unit sphere of $\mathbb{R}^n$ to simplify the presentation, we denote $Q := \mathscr{S} \times \Omega$ and for $T > 0$, we denote $Q_T = (0, T) \times Q$. Let $\nu(x)$ denotes the unit outward normal to $\partial\Omega$ at x. The initial-boundary value problem for the the linear Boltzmann equation under study is as follows :

$$\partial_t u + \theta \cdot \nabla u(t, x, \theta) + a(t, x)u = \mathscr{N}_k[u](t, x, \theta) \quad \text{in} \quad Q_T, \tag{3.1.1}$$

where $a \in L^\infty((0, T)\times\Omega)$ and $\mathscr{N}_k$ is the integral operator with kernel $k(t, x, \theta, \theta')$ defined by

$$\mathscr{N}_k[u](t, x, \theta) = \int_{\mathscr{S}} k(t, x, \theta, \theta')u(t, x, \theta')d\theta'. \tag{3.1.2}$$

3.2 Non-uniqueness in determining time-dependent coefficients in the linear Boltzmann equation

In this section we aim to show that it is hopeless to recover the time-dependent absorption coefficient a in the cloaking region. We consider initial-boundary value problem for the linear Boltzmann equation (3.1.1)

$$\begin{cases} \partial_t u + \theta \cdot \nabla u(t,x,\theta) + a(t,x)u = \mathscr{N}_k[u](t,x,\theta) & \text{in} \quad Q_T, \\ u(0,x,\theta) = 0 & \text{in} \quad Q, \\ u(t,x,\theta) = f(t,x,\theta) & \text{on} \quad \Sigma^-. \end{cases} \tag{3.2.1}$$

We focus on showing a non-uniqueness result in the cloaking region $\mathscr{C}_r$, for the identification of the time-dependent coefficients a and k appearing in the linear Boltzmann equation (3.2.1), from measurements given by the albedo operator that maps the incoming flux on the boundary Σ^- into the outgoing one, defined as follows

$$\begin{array}{rccc} \mathscr{A}_{a,k}: & \mathscr{L}_p^-(\Sigma^-) & \longrightarrow & \mathscr{L}_p^+(\Sigma^+) \\ & f & \longmapsto & \mathscr{A}_{a,k}[f] = u_{|\Sigma^+}. \end{array}$$

From (1.2.3) the albedo operator map $\mathscr{A}_{a,k}$ is continuous from $\mathscr{L}_p^-(\Sigma^-)$ to $\mathscr{L}_p^+(\Sigma^+)$. We denote by $\|\mathscr{A}_{a,k}\|_p$ its norm. Let us first introduce the following notations. Let $r > 0$ be such that $T > 2r$. Without loss of generality, we assume that $\Omega \subseteq B(0,r/2) = \{x \in \mathbb{R}^n, |x| < r/2\}$. We set $X_r = (0,T) \times B(0,r/2)$. We define

$$V = \bigcup_{0 \le \tau \le t'} D(\tau) = \bigcup_{0 \le \tau \le t'} (\mathscr{C}_r \cap \{t = \tau\}),$$

where $0 < t' < r/2$ and $\mathscr{C}_r$ is the *cloaking* region :

$$\mathscr{C}_r = \left\{(t,x) \in X_r,\ |x| \le \frac{r}{2} - t,\ 0 \le t \le \frac{r}{2}\right\}. \tag{3.2.2}$$

Moreover, we denote by

$$\mathscr{S} = \partial \mathscr{C}_r \cap (\Omega \times]0,t'[), \quad \text{and} \quad \partial V = \mathscr{S} \cup D(t') \cup D(0).$$

We denote $d\sigma$ the surface element of $\mathscr{S}$ and the vector $(\eta, \mu_1, \mu_2, \cdots, \mu_n) \in \mathbb{R}^{n+1}$ is the outward unit normal vector at $(x,t) \in \mathscr{S}$ such that

$$\eta = \Big(\sum_{j=1}^{n} \mu_j^2\Big)^{1/2}. \tag{3.2.3}$$

Indeed, $\partial\mathscr{C}_r$ is implicitly defined by

$$\partial\mathscr{C}_r = \{(t,x),\ \varrho(t,x) := |x|^2 - (\frac{r}{2} - t)^2 = 0,\ 0 \le t \le \frac{r}{2}\},$$

the outward unit vector $(\eta, \mu_1, \dots, \mu_n)$ at $(t,x) \in \partial\mathscr{C}_r$ is given by

$$\begin{aligned}(\eta, \mu_1, \dots, \mu_n) &= \left(\frac{\partial_t \varrho}{\sqrt{|\partial_t\varrho|^2 + |\nabla\varrho|^2}}, \frac{\nabla\varrho}{\sqrt{|\partial_t\varrho|^2 + |\nabla\varrho|^2}}\right) \\ &= \frac{1}{\sqrt{2}|x|}\left(\frac{r}{2} - t, x\right), \qquad (t,x) \in \partial\mathscr{C}_r.\end{aligned}$$

This imply (3.2.3).

Let $\Omega_T = (0,T) \times \Omega$. Given $M_0, M_1, M_2 > 0$, we consider the set of admissible coefficients a and k

$$\mathscr{A}(M_0) = \{a \in \mathscr{C}^1(\overline{\Omega}_T);\ \|a\|_{L^\infty(\Omega_T)} \le M_0\}.$$

$$\begin{aligned}\mathscr{K}(M_1, M_2) = \{k \in L^\infty(\Omega_T, L^2(\mathscr{S} \times \mathscr{S}));\ &\int_{\mathscr{S}} |k(t,x,\theta,\theta')| d\theta' \le M_1, \\ &\int_{\mathscr{S}} |k(t,x,\theta,\theta')| d\theta \le M_2\}.\end{aligned}$$

Lemma 3.2.1. *Let $(a,k) \in \mathscr{A}(M_0) \times \mathscr{K}(M_1, M_2)$. For $f \in \mathscr{L}_p^-(\Sigma^-)$ we denote by u the solution of the linear Boltzmann equation*

$$\begin{cases} \partial_t u + \theta \cdot \nabla u + a(t,x)u = \mathscr{N}_k[u](t,x,\theta) & \text{in} \quad Q_T, \\ u(0,x,\theta) = 0 & \text{in} \quad Q, \\ u(t,x,\theta) = f(t,x,\theta) & \text{on} \quad \Sigma^-. \end{cases}$$

Then, for any $(t,x) \in \mathscr{C}_r$ and $\theta \in \mathscr{S}$, we have $u(t,x,\theta) = 0$.

Proof. We denote by $Pu = \partial_t u + \theta \cdot \nabla u + a(t,x)u - \mathscr{N}_k[u]$. By direct calculations, we easily verify that

$$\begin{aligned}&\int_{\mathscr{S}} \int_V 2Pu(t,x,\theta)u(t,x,\theta) dx dt d\theta \\ &\quad = \int_{\mathscr{S}} \int_V 2\partial_t u(t,x,\theta)u(t,x,\theta) dx dt d\theta \\ &\quad + \int_{\mathscr{S}} \int_V 2\theta \cdot \nabla u(t,x,\theta)u(t,x,\theta) dx dt d\theta \\ &\quad + \int_{\mathscr{S}} \int_V 2a(t,x)|u(t,x,\theta)|^2 dx dt d\theta\end{aligned}$$

$$
\begin{aligned}
&-\int_{\mathscr{S}}\int_V 2\mathscr{N}_k[u](t,x,\theta)u(t,x,\theta)dxdtd\theta\\
&=\int_{\mathscr{S}}\int_V \frac{d}{dt}|u(t,x,\theta)|^2dxdtd\theta+\int_{\mathscr{S}}\int_V \operatorname{div}(\theta\,|u(t,x,\theta)|^2)dxdtd\theta\\
&+\int_{\mathscr{S}}\int_V 2a(t,x)|u(t,x,\theta)|^2dxdtd\theta\\
&-\int_{\mathscr{S}}\int_V 2\mathscr{N}_\kappa[u](t,x,\theta)u(t,x,\theta)dxdtd\theta.
\end{aligned}
$$

Applying the divergence theorem, one gets

$$
\begin{aligned}
&\int_{\mathscr{S}}\int_V 2Pu(t,x,\theta)u(t,x,\theta)dxdtd\theta\\
&\quad=\int_{\mathscr{S}}\int_{\mathscr{S}}|u(t,x,\theta)|^2\eta d\sigma d\theta+\int_{\mathscr{S}}\int_{\mathscr{S}}(\theta\cdot\mu)|u(t,x,\theta)|^2d\sigma d\theta\\
&\quad+\int_{\mathscr{S}}\int_{D(t')}|u(t',x,\theta)|^2dxd\theta-\int_{\mathscr{S}}\int_{D(0)}|u(0,x,\theta)|^2dxd\theta\\
&+\int_{\mathscr{S}}\int_V 2a(t,x)|u(t,x,\theta)|^2dxdtd\theta-\int_{\mathscr{S}}\int_V 2\mathscr{N}_\kappa[u](t,x,\theta)u(t,x,\theta)dxdtd\theta.
\end{aligned}
$$

By using (3.2.3), we can see that

$$
\begin{aligned}
\int_{\mathscr{S}}\int_{\mathscr{S}}|u(t,x,\theta)|^2\eta d\sigma d\theta+&\int_{\mathscr{S}}\int_{\mathscr{S}}(\theta\cdot\mu)|u(t,x,\theta)|^2d\sigma d\theta\\
&\geq\int_{\mathscr{S}}\int_{\mathscr{S}}|u(t,x,\theta)|^2(\eta-|\mu|)d\sigma d\theta=0.
\end{aligned}
$$

Then, since $u(0,x,\theta)=0$, we get

$$
\begin{aligned}
\int_{\mathscr{S}}\int_{D(t')}|u(t',x,\theta)|^2dxd\theta\leq&\int_{\mathscr{S}}\int_V 2Pu(t,x,\theta)u(t,x,\theta)dxdtd\theta\\
&-\int_{\mathscr{S}}\int_V 2|a(t,x)||u(t,x,\theta)|^2dxdtd\theta\\
&+\int_{\mathscr{S}}\int_V 2|\mathscr{N}_k[u](t,x,\theta)||u(t,x,\theta)|dxdtd\theta.
\end{aligned}
$$

On the other hand, by Cauchy-Schwartz inequality and Fubini's theorem, we can see that

$$
\int_{\mathscr{S}}\int_V 2\mathscr{N}_k[u](t,x,\theta)u(t,x,\theta)dxdtd\theta\leq C\int_{\mathscr{S}}\int_V|u(t,x,\theta)|^2dxdtd\theta, \tag{3.2.4}
$$

where, the positive constant C is depending on M_1 and M_2. We can also see that

$$\int_{\mathscr{S}}\int_V 2|a(t,x)||u(t,x,\theta)|^2 dxdtd\theta \leq 2M_0 \int_{\mathscr{S}}\int_V |u(t,x,\theta)|^2 dxdtd\theta. \tag{3.2.5}$$

Now, using the fact that $Pu(t,x,\theta)=0$ for any $(t,x)\in V$ and $\theta\in\mathscr{S}$, yet from (3.2.4) and (3.2.5) we get

$$\int_{\mathscr{S}}\int_{D(t')} |u(t',x,\theta)|^2 dxd\theta \leq C\int_0^{t'}\int_{\mathscr{S}}\int_{D(t)} |u(t,x,\theta)|^2 dxd\theta dt.$$

In view of Grönwall's Lemma we end up deducing that $u(t',x,\theta)=0$ for any $(t',x)\in D(t')$, $t'\in(0,r/2)$ and $\theta\in\mathscr{S}$. This completes the proof of the lemma. □

3.3 The non uniqueness result

In this section, we prove by using Lemma 3.2.1, that it is hopeless to recover the time-dependent absorption and scattering coefficients in the cloaking region $\mathscr{C}_r$ in the case where the initial condition is zero.

Theorem 3.3.1. *For any $(a,k)\in\mathscr{A}(M_0)\times\mathscr{K}(M_1,M_2)$, such that Supp$(a)\subset\mathscr{C}_r$ and Supp$(k(\cdot,\theta,\theta'))\subset\mathscr{C}_r$, we have $\mathscr{A}_{a,k}=\mathscr{A}_{0,0}$.*

Proof. Let $(a,k)\in\mathscr{A}(M_0)\times\mathscr{K}(M_1,M_2)$ such that Supp$(a)\subset\mathscr{C}_r$. Let $f\in\mathscr{L}_p^-(\Sigma^-)$ and u satisfy

$$\begin{cases} \partial_t u+\theta\cdot\nabla u+a(t,x)u=\mathscr{N}_\kappa[u](t,\theta,x) & \text{in} \quad Q_T,\\ u(0,x,\theta)=0 & \text{in} \quad Q,\\ u(t,x,\theta)=f(t,x,\theta) & \text{on} \quad \Sigma^-. \end{cases}$$

Since from Lemma 3.2.1, we have $u(t,x,\theta)=0$ for $(t,x)\in\mathscr{C}_r$ and $\theta\in\mathscr{S}$ and using the fact that
Supp$(a)\subset\mathscr{C}_r$, we deduce that u solves also the following transport boundary-value problem

$$\begin{cases} \partial_t v + \theta \cdot \nabla v = 0 & \text{in} \quad Q_T, \\ v(0,x,\theta) = 0 & \text{in} \quad Q, \\ v(t,x,\theta) = f(t,x,\theta) & \text{on} \quad \Sigma^-. \end{cases}$$

Then, we conclude that $\mathscr{A}_{a,k}(f) = \mathscr{A}_{0,0}(f)$ for all $f \in \mathscr{L}_p^-(\Sigma^-)$. Then the proof of Theorem is completed. □

3.4 Geometrical optics solutions for the Boltzmann equation

The present section we proceed as in Chapter 1, to construct a geometrical optics solutions for the linear Boltzmann equation (3.2.1) with time-dependent coefficients, which we will use them to proof of our main results. Firstly we need to introduce some notation. Let $r > 0$ be such that $T > 2r$. We assume that $\Omega \subseteq B(0, r/2)$. We consider the annular region around the domain Ω,

$$\mathscr{B}_r = \left\{ x \in \mathbb{R}^n,\ \frac{r}{2} < |x| < T - \frac{r}{2} \right\}.$$

Setting the *forward* and *backward* light cone:

$$\mathscr{C}_r^+ = \left\{ (t,x) \in X_r,\ |x| < t - \frac{r}{2},\ t > \frac{r}{2} \right\}, \tag{3.4.1}$$

$$\mathscr{C}_r^- = \left\{ (t,x) \in X_r,\ |x| < T - \frac{r}{2} - t,\ T - \frac{r}{2} > t \right\}. \tag{3.4.2}$$

Finally, we denote

$$X_{r,*} = \Omega_T \cap \mathscr{C}_r^+ \cap \mathscr{C}_r^-.$$

It is clear that, $X_{r,*} \subset \Omega_T$.

Now, let $\phi^\pm \in \mathscr{C}_0^\infty(\mathscr{B}_r, \mathscr{C}(\mathscr{S}))$. Then, for any $\theta \in \mathscr{S}$, we have

$$\operatorname{Supp} \phi^+(\cdot, \theta) \cap \Omega = \emptyset. \tag{3.4.3}$$

Since $T > 2r$, we have also

$$(\operatorname{Supp} \phi^-(\cdot, \theta) \pm T\theta) \cap \Omega = \emptyset. \tag{3.4.4}$$

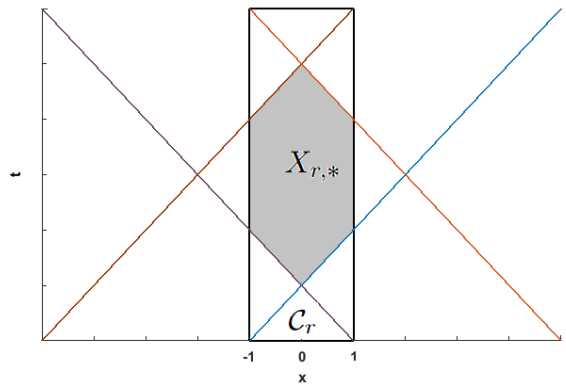

Figure 3.1: Particular case $\Omega = [-1,1]$

We set

$$\varphi^{\pm}(t,x,\theta) = \phi^{\pm}(x - t\theta, \theta), \quad (t,x,\theta) \in \mathbb{R} \times \mathbb{R}^n \times \mathscr{S}.$$

Then $\varphi^{\pm}(t,x,\theta)$ solves the following transport equation

$$(\partial_t + \theta \cdot \nabla)\varphi(t,x,\theta) = 0, \quad (t,x,\theta) \in \mathbb{R} \times \mathbb{R}^n \times \mathscr{S}. \tag{3.4.5}$$

Moreover, for $\lambda > 0$, setting

$$\varphi_{\lambda}^{\pm}(t,x,\theta) = \varphi^{\pm}(t,x,\theta)e^{\pm i\lambda(t - x\cdot\theta)}, \quad (t,x,\theta) \in \mathbb{R} \times \mathbb{R}^n \times \mathscr{S}. \tag{3.4.6}$$

Finally, for $a \in \mathscr{A}(M_0)$, we set

$$b_a(t,x,\theta) = \exp\left(-\int_0^t a(s, x - (t-s)\theta)ds\right), \quad (t,x,\theta) \in \mathbb{R} \times \mathbb{R}^n \times \mathscr{S},$$

where we have extended a by 0 outside $\overline{\Omega}_T$. We have

$$\theta \cdot \nabla b_a(t,x,\theta) \;=\; -b_a \int_0^t \theta \cdot \nabla a(s, x - (t-s)\theta)\, ds.$$

This implies in particular that

$$\begin{aligned}
\partial_t b_a(t,x,\theta) &= -a(t,x)b_a(t,x,\theta) - \left(\int_0^t \frac{d}{dt} a(s, x - (t-s)\theta)ds\right) b_a(t,x,\theta) \\
&= -a(t,x)b_a(t,x,\theta) + \left(\int_0^t \theta \cdot \nabla a(s, x - (t-s)\theta)ds\right) b_a(t,x,\theta).
\end{aligned}$$

Therefore b_a satisfies the following transport equation

$$(\partial_t + \theta \cdot \nabla + a(t,x))b_a(t,x,\theta) = 0, \quad (t,x,\theta) \in \mathbb{R} \times \mathbb{R}^n \times \mathscr{S}. \tag{3.4.7}$$

Lemma 3.4.1. *Let $p \geq 1$ and $(a,k) \in \mathscr{A}(M_0) \times \mathscr{K}(M_1, M_2)$. For any $\phi^+ \in \mathscr{C}_0^\infty(\mathscr{B}_r, \mathscr{C}(\mathscr{S}))$ and any $\lambda > 0$ the following system*

$$\begin{cases} \partial_t u + \theta \cdot \nabla u + a(t,x)u = \mathscr{N}_k[u](t,x,\theta) & (t,x,\theta) \in Q_T, \\ u(0,x,\theta) = 0 & (x,\theta) \in Q, \end{cases} \tag{3.4.8}$$

has a solution of the form

$$u_\lambda^+(t,x,\theta) = \varphi_\lambda^+(t,x,\theta) b_a(t,x,\theta) + \psi_\lambda^+(t,x,\theta), \tag{3.4.9}$$

satisfying

$$u_\lambda^+ \in \mathscr{C}([0,T]; W_p(Q)) \cap \mathscr{C}^1([0,T]; L^p(Q)),$$

where ψ_λ^+ satisfies

$$\begin{cases} \partial_t \psi + \theta \cdot \nabla \psi + a(t,x)\psi = \mathscr{N}_k[\psi] + \mathscr{N}_k[\varphi_\lambda^+ b_a](t,x,\theta) & \text{in } Q_T, \\ \psi(0,x,\theta) = 0 & \text{in } Q, \\ \psi(t,x,\theta) = 0 & \text{on } \Sigma^-, \end{cases} \tag{3.4.10}$$

and, there is a constant $C > 0$ such that

$$\left\| \psi_\lambda^+ \right\|_{\mathscr{C}([0,T]; L^p(Q))} \leq C \left\| \phi^+ \right\|_{L^p(\mathbb{R}^n \times \mathscr{S})}. \tag{3.4.11}$$

Here C is a constant depending on Ω, T and M_i, $i = 0,1,2$.

Moreover, we have

$$\lim_{\lambda \to \infty} \left\| \psi_\lambda^+ \right\|_{L^2(Q_T)} = 0. \tag{3.4.12}$$

Proof. Let $\psi_\lambda^+(t,x,\theta)$ solves the following homogeneous boundary value problem

$$\begin{cases} \partial_t \psi + \theta \cdot \nabla \psi + a(t,x)\psi = \mathscr{N}_k[\psi] + v_\lambda(t,x,\theta) & \text{in } Q_T, \\ \psi(0,x,\theta) = 0 & \text{in } Q, \\ \psi(t,x,\theta) = 0 & \text{on } \Sigma^-, \end{cases} \tag{3.4.13}$$

where the source term v_λ is given by

$$v_\lambda(t,x,\theta) = -(\partial_t + \theta \cdot \nabla + a(t,x))\big(\varphi_\lambda^+(t,x,\theta) b_a(t,x,\theta)\big) + \mathscr{N}_k[\varphi_\lambda^+ b_a](t,x,\theta). \tag{3.4.14}$$

To prove the Lemma it would be enough to show that ψ_λ^+ satisfies the estimates (3.4.11) and (3.4.12).

By a simple computation and using (3.4.5) and (3.4.7), we have

$$v_\lambda(t,x,\theta) = \mathscr{N}_k[\varphi_\lambda^+ b_a](t,x,\theta). \tag{3.4.15}$$

By Lemma 1.3.4 there is a constant $C > 0$, such that

$$\left\|\psi_\lambda^+(t,\cdot,\cdot)\right\|_{L^p(Q)} \leq C \left\|v_\lambda\right\|_{L^p(Q_T)}, \quad \forall\, t \in (0,T). \tag{3.4.16}$$

From (1.3.10) and (3.4.16) we obtain (3.4.11) and (3.4.12). This completes the proof of the Lemma. □

We prove by proceeding as in the proof of Lemma 3.4.1, the following Lemma.

Lemma 3.4.2. *Let $q \geq 1$ and $(a,k) \in \mathscr{A}(M_0) \times \mathscr{K}(M_1,M_2)$. For any $\phi^- \in \mathscr{C}_0^\infty(\mathscr{B}_r;\mathscr{C}(\mathscr{S}))$ and any $\lambda > 0$ the following system*

$$\begin{cases} \partial_t u + \theta\cdot\nabla u - a(t,x)u = -\mathscr{N}_k^*[u](t,x,\theta) & (t,x,\theta) \in Q_T, \\ u(T,x,\theta) = 0 & (x,\theta) \in Q, \end{cases} \tag{3.4.17}$$

where

$$\mathscr{N}_k^*[u](t,x,\theta) = \int_{\mathscr{S}} k(t,x,\theta',\theta)u(t,x,\theta')d\theta',$$

has a solution of the form

$$u_\lambda^-(t,x,\theta) = \varphi_\lambda^-(t,x,\theta)b_{-a}(t,x,\theta) + \psi_\lambda^-(t,x,\theta),$$

satisfying

$$u_\lambda^- \in \mathscr{C}([0,T];W_q(Q)) \cap \mathscr{C}^1([0,T];L^q(Q)),$$

where ψ_λ^- satisfies

$$\begin{cases} \partial_t\psi + \theta\cdot\nabla\psi - a(t,x)\psi = -\mathscr{N}_k^*[\psi] - \mathscr{N}_k^*[\varphi_\lambda^- b_a](t,x,\theta) & \text{in } Q_T, \\ \psi(T,x,\theta) = 0 & \text{in } Q, \\ \psi(t,x,\theta) = 0 & \text{on } \Sigma^+, \end{cases} \tag{3.4.18}$$

and, there is a constant $C > 0$ such that

$$\left\|\psi_\lambda^-\right\|_{\mathscr{C}([0,T];L^q(Q))} \leq C \left\|\phi^-\right\|_{L^q(\mathbb{R}^n\times\mathscr{S})}. \tag{3.4.19}$$

Here C is a constant depending only on Ω, T and M_i, $i = 0,1,2$.

Moreover, we have

$$\lim_{\lambda\to\infty} \left\|\psi_\lambda^-\right\|_{L^2(Q_T)} = 0. \tag{3.4.20}$$

3.5 Uniqueness in determining time-dependent absorption coefficient in the linear Boltzmann equation

In this section, we aim to treat the inverse problem of determining a time-dependent absorption and scattering coefficients in the linear Boltzmann equation. We prove that the absorption and scattering coefficients can be uniquely determined in a precise subset of the domain. Here we need to introduce the admissible set of the absorption coefficient a in $X_{r,*}$. Given $a_0 \in \mathscr{C}^1(\overline{\Omega}_T)$, we set

$$\mathscr{A}^*(M_0) = \{a \in \mathscr{A}(M_0);\ a = a_0 \text{ in } \overline{\Omega}_T \setminus X_{r,*}\}.$$

Let $(a_j, k_j) \in \mathscr{A}^*(M_0) \times \mathscr{K}(M_1, M_2)$, $j = 1,2$. Here, we recall the definition of b_{-a_1} and b_{a_2}

$$b_{-a_1}(x,t) = \exp\Big(\int_0^t a_1(s, x-(t-s)\theta)\,ds\Big),$$

$$b_{a_2}(t,x,\theta) = \exp\Big(-\int_0^t a_2(s, x-(t-s)\theta)\,ds\Big),$$

and we define

$$a = a_2 - a_1, \text{ and } b_a(t,x,\theta) = b_{a_2} b_{-a_1}(t,x,\theta) = \exp\Big(-\int_0^t a(s, x-(t-s)\theta)\,ds\Big),$$

where we have extend a by 0 outside $\overline{\Omega}_T$.

Let us also, consider the conic set

$$E = \{(\tau,\xi) \in \mathbb{R}\times\mathbb{R}^n,\ |\tau| \leq |\xi|\}.$$

We denote by $\widehat{\rho}$ the Fourier transform of a function $\rho \in L^1(\mathbb{R}^{n+1})$ with respect to (t,x):

$$\widehat{\rho}(\tau,\xi) = \int_{\mathbb{R}} \int_{\mathbb{R}^n} \rho(t,x) e^{-ix\cdot\xi} e^{-it\tau} dx dt.$$

3.6 Uniqueness result of the absorption coefficient

In this section, we prove that the time-dependent absorption coefficient can be uniquely determined with respect to the albedo operator in a specific subset of the domain $X_{r,*}$.

We start by a preliminary identity for the absorption coefficient a appearing in the linear Boltzmann equation, by means of the geometric optics solutions.

Lemma 3.6.1. *Let $(a_j, k_j) \in \mathscr{A}^*(M_0) \times \mathscr{K}(M_1, M_2)$, $j = 1, 2$. Assume that $\mathscr{A}_{a_2,k_2} - \mathscr{A}_{a_1,k_1}$, then for any $\phi^\pm \in \mathscr{C}_0^\infty(\mathscr{B}_r, \mathscr{C}(\mathscr{S}))$, the following identity holds true*

$$\Big| \int_{\mathbb{R}^n \times \mathscr{S}} (\phi^+\phi^-)(y,\theta) \Big[\exp\Big(-\int_0^T a(s, y+s\theta)\, ds\Big) - 1 \Big] d\theta\, dy \Big| = 0 \ \forall \omega \in \mathscr{S},\ a.e, x \in \mathbb{R}^n. \tag{3.6.1}$$

Proof. In view of Lemma 3.4.1, and using the fact that Supp $\phi^+(\cdot,\theta) \cap \Omega = \emptyset$, there exists a G.O-solution u_λ^+ to the equation

$$\begin{cases} \partial_t u + \theta \cdot \nabla u + a_2(t,x) u = \mathscr{N}_{k_2}[u](t,x,\theta) & (t,x,\theta) \in Q_T, \\ u(0,x,\theta) = 0 & (t,x) \in Q, \end{cases}$$

in the following form

$$u_\lambda^+(t,x,\theta) = \varphi_\lambda^+(t,x,\theta) b_{a_2}(t,x,\theta) + \psi_\lambda^+(t,x,\theta),$$

corresponding to the coefficients a_2 and k_2, where $\psi_\lambda^+(t,x,\theta)$ satisfies (3.4.11) and (3.4.12). Next, let us denote by f_λ the function

$$f_\lambda(t,x,\theta) := u_\lambda^+(t,x,\theta)_{|\Sigma^-} = \varphi_\lambda^+(t,x,\theta) b_{a_2}(t,x,\theta)_{|\Sigma^-}.$$

We denote by u_1 the solution of

$$\begin{cases} \partial_t u_1 + \theta \cdot \nabla u_1 + a_1(t,x)u_1 = \mathcal{N}_{k_1}[u_1] & (t,x,\theta) \in Q_T, \\ u_1(0,x,\theta) = 0 & (x,\theta) \in Q, \\ u_1(t,x,\theta) = f_\lambda(t,x,\theta) & (t,x,\theta) \in \Sigma^-. \end{cases}$$

Putting $u = u_1 - u_\lambda^+$. Then, u is a solution to the following system

$$\begin{cases} \partial_t u + \theta \cdot \nabla u + a_1(t,x)u = \mathcal{N}_{k_1}[u] + a(t,x)u_\lambda^+ - \mathcal{N}_k[u_\lambda^+] & (t,x,\theta) \in Q_T, \\ u(0,x,\theta) = 0 & (x,\theta) \in Q, \\ u(t,x,\theta) = 0 & (t,x,\theta) \in \Sigma^-, \end{cases} \tag{3.6.2}$$

where $a = a_2 - a_1$ and $k = k_2 - k_1$. On the other hand by Lemma 3.4.2 and the fact that $(\text{Supp } \phi_1(\cdot,\theta) \pm T\theta) \cap \Omega = \emptyset$, we can find of a solution u_λ^- to the backward problem

$$\begin{cases} \partial_t u + \theta \cdot \nabla u - a_1(t,x)u = -\mathcal{N}_{k_1}^*[u](t,x,\theta) & \text{in } Q_T, \\ u(T,x,\theta) = 0 & \text{in } Q, \end{cases}$$

corresponding to the coefficients a_1 and k_1, in the form

$$u_\lambda^-(t,x,\theta) = \varphi_\lambda^-(t,x,\theta) b_{-a_1}(t,x,\theta) + \psi_\lambda^-(t,x,\theta), \tag{3.6.3}$$

here the remainder term $\psi_\lambda^-(t,x,\theta)$ satisfies the estimate (3.4.19) and (3.4.20). Multiplying the first equation of (3.6.2) by u_λ^-, integrating by parts and using Green's formula, we obtain

$$\int_{Q_T} a(t,x)u_\lambda^+ u_\lambda^- \, d\theta \, dx \, dt - \int_{Q_T} \mathcal{N}_k[u_\lambda^+](t,x,\theta) u_\lambda^-(t,x,\theta) d\theta \, dx \, dt = 0 \tag{3.6.4}$$

By replacing u_λ^+ and u_λ^- by their expressions, we obtain

$$\int_{Q_T} a(t,x)u_\lambda^+ u_\lambda^- d\theta \, dx \, dt = \int_{Q_T} a(t,x)\varphi_\lambda^+(t,x,\theta)\varphi_\lambda^-(t,x,\theta) b_a(t,x,\theta) d\theta \, dx \, dt$$
$$+ \int_{Q_T} a(t,x)\varphi_\lambda^-(t,x,\theta) b_{-a_1}(t,x,\theta)\psi_\lambda^+(t,x,\theta) d\theta \, dx \, dt$$

$$
\begin{aligned}
&+\int_{Q_T} a(t,x)\varphi_\lambda^+(t,x,\theta)b_{a_2}(t,x,\theta)\psi_\lambda^-(t,x,\theta)d\theta\,dx\,dt\\
&+\int_{Q_T} a(t,x)\psi_\lambda^-(t,x,\theta)\psi_\lambda^+(t,x,\theta)d\theta\,dx\,dt\\
&=\int_{Q_T} a(t,x)\varphi_\lambda^+(t,x,\theta)\varphi_\lambda^-(t,x,\theta)b_a(t,x,\theta)d\theta\,dx\,dt+\mathscr{I}_{1,\lambda},
\end{aligned}
\tag{3.6.5}
$$

where $\mathscr{I}_{1,\lambda}$ represents the sum of the integrals that contain terms with ψ_λ^+ and ψ_λ^-.

Note that there exists $C>0$ such that for any $\lambda\geq 0$, we have

$$
|\mathscr{I}_{1,\lambda}|\leq C\left(\left\|\psi_\lambda^+\right\|_{L^2(Q_T)}+\left\|\psi_\lambda^-\right\|_{L^2(Q_T)}\right). \tag{3.6.6}
$$

Then by (3.4.12) and (3.4.20), we get

$$
\lim_{\lambda\to\infty}\mathscr{I}_{1,\lambda}=0. \tag{3.6.7}
$$

Moreover, we have

$$
\begin{aligned}
\int_{Q_T}\mathscr{N}_k[u_\lambda^+]u_\lambda^- d\theta\,dx\,dt&=\int_{Q_T}\mathscr{N}_k[\varphi_\lambda^+ b_{a_2}](t,x,\theta)\varphi_\lambda^-(t,x,\theta)b_{-a_1}(t,x,\theta)d\theta\,dx\,dt\\
&+\int_{Q_T}\mathscr{N}_k[\psi_\lambda^+](t,x,\theta)\varphi_\lambda^-(t,x,\theta)b_{-a_1}(t,x,\theta)d\theta\,dx\,dt\\
&+\int_{Q_T}\mathscr{N}_k[\varphi_\lambda^+ b_{a_2}](t,x,\theta)\psi_\lambda^-(t,x,\theta)d\theta\,dx\,dt\\
&+\int_{Q_T}\mathscr{N}_k[\psi_\lambda^+](t,x,\theta)\psi_\lambda^-(t,x,\theta)d\theta\,dx\,dt\\
&=\int_{Q_T}\mathscr{N}_k[\varphi_\lambda^+ b_{a_2}](t,x,\theta)\varphi_\lambda^-(t,x,\theta)b_{-a_1}(t,x,\theta)d\theta\,dx\,dt+\mathscr{I}_{2,\lambda},
\end{aligned}
$$

where

$$
\begin{aligned}
\mathscr{I}_{2,\lambda}&=\int_{Q_T}\mathscr{N}_k[\psi_\lambda^+](t,x,\theta)\varphi_\lambda^-(t,x,\theta)b_{-a_1}(t,x,\theta)d\theta\,dx\,dt\\
&+\int_{Q_T}\mathscr{N}_k[\varphi_\lambda^+ b_{a_2}](t,x,\theta)\psi_\lambda^-(t,x,\theta)d\theta\,dx\,dt+\int_{Q_T}\mathscr{N}_k[\psi_\lambda^+](t,x,\theta)\psi_\lambda^-(t,x,\theta)d\theta\,dx\,dt.
\end{aligned}
$$

By (1.3.9), we get

$$
|\mathscr{I}_{2,\lambda}|\leq C\left(\left\|\psi_\lambda^+\right\|_{L^2(Q_T)}+\left\|\psi_\lambda^-\right\|_{L^2(Q_T)}\right). \tag{3.6.8}
$$

Then by (3.4.12) and (3.4.20), we have

$$\lim_{\lambda\to\infty} \mathscr{I}_{2,\lambda} = 0, \tag{3.6.9}$$

and by the Cauchy-Schwartz inequality, we obtain

$$\left|\int_{Q_T} \mathscr{N}_k[\varphi_\lambda^+ b_{a_2}](t,x,\theta)\varphi_\lambda^-(t,x,\theta)b_{-a_1}(t,x,\theta)d\theta\, dx\, dt\right| \leq C\left\|\mathscr{N}_k[\varphi_\lambda^+ b_{a_2}]\right\|_{L^2(Q_T)}. \tag{3.6.10}$$

Then by (1.3.10), we have

$$\lim_{\lambda\to\infty} \int_{Q_T} \mathscr{N}_k[\varphi_\lambda^+ b_{a_2}](t,x,\theta)\varphi_\lambda^-(t,x,\theta)b_{-a_1}(t,x,\theta)d\theta\, dx\, dt = 0. \tag{3.6.11}$$

In light of (3.6.4), we have

$$\int_{Q_T} a(t,x)\varphi^+(t,x,\theta)\varphi^-(t,x,\theta)b_a(t,x,\theta)d\theta\, dx\, dt - \mathscr{I}_\lambda = 0, \tag{3.6.12}$$

where the remainder term $\mathscr{I}_\lambda$ is given by

$$\mathscr{I}_\lambda = \mathscr{I}_{1,\lambda} - \mathscr{I}_{2,\lambda} - \int_{Q_T} \mathscr{N}_k[\varphi_\lambda^+ b_{a_2}](t,x,\theta)\varphi_\lambda^-(t,x,\theta)b_{-a_1}(t,x,\theta)d\theta\, dx\, dt.$$

Taking the limit as $\lambda \to +\infty$, we get from (3.6.7), (3.6.9) and (3.6.11)

$$\lim_{\lambda\to\infty} \mathscr{I}_\lambda = 0. \tag{3.6.13}$$

We get, by (3.6.12) and (3.6.13),

$$\left|\int_{Q_T} a(t,x)\varphi^+\varphi^-(t,x,\theta)b_a(t,x,\theta)d\theta\, dx\, dt\right| = 0.$$

Then, using the fact $a(t,\cdot) = 0$ outside Ω and by changing of variables $y = x - t\theta$, one gets the following estimation

$$\left|\int_0^T \int_{\mathbb{R}^n\times\mathscr{S}} a(t,y+t\theta)\phi^+(y,\theta)\phi^-(y,\theta)\exp\Big(-\int_0^t a(s,y+s\theta)\,ds\Big)d\theta dy\, dt\right| = 0.$$

Bearing in mind that

$$\int_0^T \int_{\mathbb{R}^n\times\mathscr{S}} a(t,y+t\theta)\phi^+(y,\theta)\phi^-(y,\theta)\exp\Big(-\int_0^t a(s,y+s\theta)\,ds\Big)d\theta\, dy\, dt$$

$$= -\int_0^T \int_{\mathbb{R}^n\times\mathscr{S}} \phi^+(y,\theta)\phi^-(y,\theta)\frac{d}{dt}\Big[\exp\Big(-\int_0^t a(s,y+s\theta)\,ds\Big)\Big]\,d\theta\,dy\,dt$$
$$= -\int_{\mathbb{R}^n\times\mathscr{S}} (\phi^+\phi^-)(y,\theta)\Big[\exp\Big(-\int_0^T a(s,y+s\theta)\,ds\Big)-1\Big]\,d\theta\,dy,$$

we conclude the desired estimate given by

$$\Big|\int_{\mathbb{R}^n\times\mathscr{S}} (\phi^+\phi^-)(y,\theta)\Big[\exp\Big(-\int_0^T a(s,y+s\theta)\,ds\Big)-1\Big]\,d\theta\,dy\Big| = 0.$$

□

We move now to give and prove the main statement of this section.

Theorem 3.6.2. *Let $T > 2r$ and $(a_j,k_j) \in \mathscr{A}^*(M_0)\times\mathscr{K}(M_1,M_2)$, $j=1,2$. Then, we have*

$$\mathscr{A}_{a_1,k_1} = \mathscr{A}_{a_2,k_2} \quad \textit{implies} \quad a_1 = a_2 \textit{ on } X_{r,*}.$$

Proof. Let $\varrho_h(\omega,\theta)$ defined by (1.4.27). Selecting

$$\phi^+(y,\theta) = \varrho_h(\omega,\theta)\phi_1(y), \quad \phi^-(y,\theta) = \phi_2(y),$$

we obtain from (3.6.1), that for any $h \in (0,1)$,

$$\Big|\int_{\mathscr{B}_r\times\mathscr{S}} \varrho_h(\omega,\theta)\phi_1(y)\phi_2(y)\Big[\exp\Big(-\int_0^T a(s,y+s\theta)\,ds\Big)-1\Big]\,d\theta\,dy\Big| = 0.$$

Since $\|\varrho_h(\omega,\cdot)\|_{L^1(\mathscr{S})} = 1$, and by (1.4.31) we have

$$\lim_{h\to 1}\int_{\mathscr{S}} \varrho_h(\omega,\theta)\Big[\exp\Big(-\int_0^T a(s,y+s\theta)\,ds\Big)-1\Big]\,d\theta = \Big[\exp\Big(-\int_0^T a(s,y+s\omega)\,ds\Big)-1\Big].$$

Then, we get

$$\Big|\int_{\mathscr{B}_r} \phi_1(y)\phi_2(y)\Big[\exp\Big(-\int_0^T a(s,y+s\omega)\,ds\Big)-1\Big]\,dy\Big| = 0.$$

Let the bilinear form $\mathscr{B} : L^1(\mathscr{B}_r)\times L^\infty(\mathscr{B}_r) \longrightarrow \mathbb{R}$ given by

$$\mathscr{B}(\phi_1,\phi_2) = \int_{\mathscr{B}_r} \phi_1(y)\phi_2(y)R(y,\omega)\,dy,$$

where

$$R(y,\omega) = \exp\Big(-\int_0^T a(s,y+s\omega)\,ds\Big)-1.$$

Then we have

$$|\mathscr{B}(\phi_1,\phi_2)| \leq C\|R(\cdot,\omega)\|_{L^\infty(\mathscr{B}_r)}\|\phi_1\|_{L^1(\mathbb{R}^n)}\|\phi_2\|_{L^\infty(\mathbb{R}^n)}.$$

We conclude that

$$\|\mathscr{B}\| = \|R(\cdot,\omega)\|_{L^\infty(\mathscr{B}_r)} = 0,$$

where $\|\mathscr{B}\|$ stands for the norm of the bilinear form $\mathscr{B}$. That is

$$\Big|\exp\Big(-\int_0^T a(s,y+s\omega)\,ds\Big)-1\Big| = 0, \quad \forall\, y\in\mathscr{B}_r,\ \forall\,\omega\in\mathscr{S}. \tag{3.6.14}$$

Using the fact that $|X|\leq e^M\,|e^X-1|$ for any X real satisfying $|X|\leq M$ we found out that

$$\Big|\int_0^T a(s,y+s\omega)\,ds\Big| \leq e^{MT}\Big|\exp\Big(-\int_0^T a(s,y+s\omega)\,ds\Big)-1\Big|,$$

where $X = \displaystyle\int_0^T a(s,y+s\omega)ds$. We conclude in light of (3.6.14) the following estimate

$$\Big|\int_0^T a(s,y+s\omega)\,ds\Big| = 0, \quad \forall\, y\in\mathscr{B}_r, \quad \omega\in\mathscr{S}. \tag{3.6.15}$$

Using the fact that $a = a_2 - a_1 = 0$ outside $X_{r,*}$, this entails that for all $y\in\mathscr{B}_r$ and $\omega\in\mathscr{S}$, we have

$$\Big|\int_{\mathbb{R}} a(s,y+s\omega)\,ds\Big| = 0. \tag{3.6.16}$$

Now, if we assume $y\in B(0,r/2)$, we get $|y+t\omega|\geqslant|t|-|y|\geq|t|-r/2$. Hence, one can see that

$(t,y+t\omega)\notin\mathscr{C}_r^+$, if $t>r/2$. On the other hand, we have $(t,y+t\omega)\notin\mathscr{C}_r^+$, if $t\leqslant r/2$. Thus, we conclude that $(t,y+t\omega)\notin\mathscr{C}_r^+\supset X_{r,*}$ for $t\in\mathbb{R}$. This and the fact that $a=a_2-a_1=0$ outside $X_{r,*}$, entails that for all $y\in B(0,r/2)$ and $\omega\in\mathscr{S}$, we have

$$a(t,y+t\omega) = 0,\ \forall t\in\mathbb{R}.$$

By a similar way, we prove for $|y|\geqslant T-r/2$, that $(t,y+t\omega)\notin\mathscr{C}_r^-\supset X_{r,*}$ for $t\in\mathbb{R}$ and then $a(t,y+t\omega)=0$. Hence, we conclude that

$$\Big|\int_{\mathbb{R}} a(s,y+s\omega)\,ds\Big| = 0, \quad \forall\,\omega\in\mathscr{S},\ y\notin\mathscr{B}_r. \tag{3.6.17}$$

Thus, by (3.6.15) and (3.7.40) we finish the proof of the lemma by getting

$$\left|\int_{\mathbb{R}} a(s, y+s\omega)\, ds\right| = 0, \quad \forall\, \omega \in \mathscr{S},\ y \in \mathbb{R}^n.$$

Next, we turn our attention to the Fourier transform of a. let $(\tau, \xi) \in E$ and $\zeta \in \mathscr{S}$ be such that $\xi \cdot \zeta = 0$. Setting

$$\omega = \frac{\tau}{|\xi|^2} \cdot \xi + \sqrt{1 - \frac{\tau^2}{|\xi|^2}} \cdot \zeta.$$

Then, one can see that $\omega \in \mathscr{S}$ and $\omega \cdot \xi = \tau$. On the other hand by the change of variable $x = y + t\omega$, we have for all $\xi \in \mathbb{R}^n$ and $\omega \in \mathscr{S}$, the following identity

$$\begin{aligned}\int_{\mathbb{R}^n} \mathscr{R}(a)(y,\omega) e^{-iy\cdot\xi} dy &= \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}} a(s, y+s\omega)\, ds\right) e^{-iy\cdot\xi} dy \\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}} a(t,x) e^{-ix\cdot\xi} e^{-it\omega\cdot\xi} dx dt \\ &= \widehat{a}(\omega\cdot\xi, \xi) = \widehat{a}(\tau, \xi),\end{aligned}$$

where we have set $(\omega \cdot \xi, \xi) = (\tau, \xi) \in E$. Bearing in mind that for any $t \in \mathbb{R}$, $\operatorname{Supp} a(t, \cdot) \subset \Omega \subset B(0, r/2)$, we obtain

$$\widehat{a}(\tau, \xi) = 0.$$

Hence, by the injectivity of the Fourier transform we get the diserd result. This completes the proof of Theorem 3.6.2. □

3.7 Uniqueness of the scattering coefficient

This section is concerned to the proof of a partial result on the identification of the scattering coefficient in $X_{r,*}$. Let $(a_j, k_j) \in \mathscr{A}^*(M_0) \times \mathscr{K}(M_1, M_2)$, $j = 1, 2$ such that

$$\mathscr{A}_{a_1,k_1} = \mathscr{A}_{a_2,k_2}.$$

Let $\omega \in \mathscr{S}$ and $\phi_j \in \mathscr{C}_0^\infty(\mathscr{K}_r)$, $j = 1, 2$. Moreover, let $\vartheta_h(\omega, \cdot)$ defined by (1.4.27). Selecting

$$\phi^+(y,\theta) = \vartheta_h(\omega,\theta)\phi_1(y), \quad \phi^-(y,\theta) = \vartheta_{h'}(\omega,\theta)\phi_2(y),\ h, h' \in (0,1).$$

Moreover, we denote

$$\varphi^+_{h,\lambda}(t,x,\theta)=\varphi_{1,h}(t,x,\theta)e^{i\lambda(t-x\cdot\theta)},\quad \varphi^-_{h',\lambda}(t,x,\theta)=\varphi_{2,h'}(t,x,\theta)e^{-i\lambda(t-x\cdot\theta)}, \tag{3.7.18}$$

where

$$\varphi_{1,h}(t,x,\theta)=\vartheta_h(\omega,\theta)\phi_1(x-t\theta),\quad \varphi_{2,h'}(t,x,\theta)=\vartheta_{h'}(\omega,\theta)\phi_2(x-t\theta).$$

By Lemma 3.4.1, the following system

$$\begin{aligned}&\partial_t u+\theta\cdot\nabla u+a_2(x,\theta)u=\mathscr{N}_{k_2}[u](t,x,\theta), && (t,x,\theta)\in Q_T,\\ &u(0,x,\theta)=0, && (x,\theta)\in Q,\end{aligned} \tag{3.7.19}$$

admit solution $u^+_{h,\lambda}\in\mathscr{C}([0,T];\mathscr{W}_p(Q))\cap\mathscr{C}^1([0,T];L^p(Q))$, in the form

$$u^+_{h,\lambda}(t,x,\theta)=\varphi^+_{h,\lambda}(t,x,\theta)b_{a_2}(t,x,\theta)+r^+_{h,\lambda}(t,x,\theta),$$

where $r^+_{h,\lambda}$ satisfies

$$\begin{cases}\partial_t r^+_{h,\lambda}+\theta\cdot\nabla r^+_{h,\lambda}+a_2(x,\theta)r^+_{h,\lambda}=\mathscr{N}_{k_2}[r^+_{h,\lambda}]+\mathscr{N}_{k_2}[\varphi^+_{h,\lambda}b_{a_2}](t,x,\theta) & \text{in } Q_T,\\ r^+_{h,\lambda}(0,x,\theta)=0 & \text{in } Q,\\ r^+_{h,\lambda}(t,x,\theta)=0 & \text{on } \Sigma^-.\end{cases} \tag{3.7.20}$$

We consider a solution $u_{1,h}$ to the following system

$$\begin{cases}\partial_t u_{1,h}+\theta\cdot\nabla u_{1,h}+a_1(x,\theta)u_{1,h}=\mathscr{N}_{k_1}[u_{1,h}] & \text{in } Q_T,\\ u_{1,h}(0,x,\theta)=0 & \text{in } Q,\\ u_{1,h}(t,x,\theta)=f_{h,\lambda}(t,x,\theta):=u^+_{h,\lambda}(t,x,\theta) & \text{on } \Sigma^-.\end{cases}$$

Setting $u=u_{1,h}-u^+_{h,\lambda}$. Then, u is a solution to the following system

$$\begin{cases}\partial_t u+\theta\cdot\nabla u+a_1(x,\theta)u=\mathscr{N}_{k_1}[u]+a(x,\theta)u^+_\lambda-\mathscr{N}_k[u^+_{h,\lambda}] & \text{in } Q_T,\\ u(0,x,\theta)=0 & \text{in } Q,\\ u(t,x,\theta)=0 & \text{on } \Sigma^-.\end{cases} \tag{3.7.21}$$

here $a=a_2-a_1$ and $k=k_2-k_1$. Likewise by Lemma 3.4.2, the following system

$$\begin{aligned}&\partial_t u+\theta\cdot\nabla u-a_1(x,\theta)u=-\mathscr{N}^*_{k_1}[u](t,x,\theta) && \text{in } Q_T,\\ &u(T,x,\theta)=0 && \text{in } Q,\end{aligned}$$

admit solution $u^-_{h',\lambda} \in \mathscr{C}([0,T];\mathscr{W}_q(Q)) \cap \mathscr{C}^1([0,T];L^q(Q))$, of the form

$$u^-_{h',\lambda}(t,x,\theta) = \varphi^-_{h',\lambda}(t,x,\theta)b_{-a_1}(t,x,\theta) + r^-_{h',\lambda}(t,x,\theta),$$

where $r^-_{h',\lambda}$ satisfies

$$\begin{cases} \partial_t r^-_{h',\lambda} + \theta\cdot\nabla r^-_{h',\lambda} - a_1(x,\theta)r^-_{h',\lambda} = -\mathscr{N}^*_{k_1}[r^-_{h',\lambda}] - \mathscr{N}^*_{k_1}[\varphi^-_{h',\lambda}b_{a_1}](t,x,\theta) & \text{in } Q_T, \\ r^-_{h',\lambda}(T,x,\theta) = 0, & \text{in } Q, \\ r^-_{h',\lambda}(t,x,\theta) = 0 & \text{on } \Sigma^+. \end{cases} \tag{3.7.22}$$

Multiplying the first equation of the system (3.7.21) by $u^-_{h',\lambda}$, integrating by parts and using Green's formula, we obtain

$$\int_{Q_T} a(x,\theta)u^+_{h,\lambda}\,u^-_{h',\lambda}\,d\theta dx dt - \int_{Q_T} \mathscr{N}_{k'}[u^+_{h,\lambda}](t,x,\theta)u^-_{h',\lambda}(t,x,\theta)d\theta\ dx\,dt$$
$$= \int_{\Sigma^+} \theta\cdot\nu(x)(\mathscr{A}_{a'_2,k'_2} - \mathscr{A}_{a_1,k_1})(f_{h,\lambda})u^-_{h,\lambda}\,d\theta\,dx\,dt. \tag{3.7.23}$$

Since $\mathscr{A}_{a_1,k_1} = \mathscr{A}_{a_2,k_2}$, so $a = 0$ follows from Theorem 3.6.2. This allows us to rewrite the identity (3.7.23) as

$$\int_{Q_T} \mathscr{N}_k[u^+_{h,\lambda}](t,x,\theta)u^-_{h',\lambda}(t,x,\theta)d\theta\ dx\,dt = 0.$$

Now, we replace the solutions $u^+_{h,\lambda}$ and $u^-_{h,\lambda}$ by their forms, we have

$$\begin{aligned} -\int_{Q_T} &\mathscr{N}_k[\varphi^+_{h,\lambda}b_{a_2}](t,x,\theta)\varphi^-_{h',\lambda}(t,x,\theta)b_{-a_1}(t,x,\theta)d\theta\,dx\,dt \\ &= \int_{Q_T} \mathscr{N}_k[r^+_{h,\lambda}](t,x,\theta)\varphi^-_{h',\lambda}(t,x,\theta)b_{-a_1}(t,x,\theta)d\theta\,dx\,dt \\ &\quad + \int_{Q_T} \mathscr{N}_k[\varphi^+_{h,\lambda}b_{a_2}](t,x,\theta)r^-_{h',\lambda}(t,x,\theta)d\theta\,dx\,dt \\ &\quad + \int_{Q_T} \mathscr{N}_k[r^+_{h,\lambda}](t,x,\theta)r^-_{h',\lambda}(t,x,\theta)d\theta\,dx\,dt. \end{aligned} \tag{3.7.24}$$

Now, we denote the left-hand side of (3.7.24) by

$$\mathscr{L}_{\lambda,h,h'}(\omega) = \int_{Q_T} \mathscr{N}_k[\varphi^+_{h,\lambda}b_{a_2}](t,x,\theta)\varphi^-_{h',\lambda}(t,x,\theta)b_{-a_1}(t,x,\theta)d\theta\,dx\,dt, \tag{3.7.25}$$

and the right-hand side of (3.7.24) by

$$\mathscr{E}_{\lambda,h,h'}(\omega) = \int_{Q_T} \mathscr{N}_k[r^+_{h,\lambda}](t,x,\theta)\varphi^-_{h',\lambda}(t,x,\theta)b_{-a_1}(t,x,\theta)d\theta\, dx\, dt$$
$$+ \int_{Q_T} \mathscr{N}_k[\varphi^+_{h,\lambda} b_{a_2}](t,x,\theta)r^-_{h',\lambda}(t,x,\theta)d\theta\, dx\, dt$$
$$+ \int_{Q_T} \mathscr{N}_k[r^+_{h,\lambda}](t,x,\theta)r^-_{h',\lambda}(t,x,\theta)d\theta\, dx\, dt. \quad (3.7.26)$$

Lemma 3.7.1. *Let $\omega \in \mathscr{S}$ and $r^+_{h,\lambda}$ the unique solution to* (3.7.20). *We denote by $\zeta^+_{\lambda,\omega}$ be the solution of the following problem*

$$\begin{cases} \partial_t \zeta^+_{\lambda,\omega} + \theta\cdot\nabla\zeta^+_{\lambda,\omega} + a_2(x,\theta)\zeta^+_{\lambda,\omega} = \mathscr{N}_{k_2}[\zeta^+_{\lambda,\omega}] + k_2(t,x,\theta,\omega)\varphi^+_\lambda(t,x,\omega)b_{a_2}(t,x,\omega) \\ \zeta^+_{\lambda,\omega}(0,x,\theta) = 0, \\ \zeta^+_{\lambda,\omega}(t,x,\theta) = 0 \end{cases} \quad (3.7.27)$$

where φ^+_λ is given by (3.4.6). Then we have

$$\lim_{h\to 1}\left\| r^+_{h,\lambda} - \zeta^+_{\lambda,\omega}\right\|_{L^2(Q_T)} = 0. \quad (3.7.28)$$

Moreover, we get

$$\lim_{h\to 1}\left\| \mathscr{N}_{k_2}[r^+_{h,\lambda}] - \mathscr{N}_{k_2}[\zeta^+_{\lambda,\omega}]\right\|_{L^2(Q_T)} = 0. \quad (3.7.29)$$

Proof. We denote $w_{h,\lambda}(t,x,\theta) = r^+_{h,\lambda} - \zeta^+_{\lambda,\omega}$, then we get

$$\begin{cases} \partial_t w_{h,\lambda} + \theta\cdot\nabla w_{h,\lambda} + a_2(x,\theta)w_{h,\lambda} = \mathscr{N}_k[w_{h,\lambda}] + \mathscr{N}_{k_2}[\varphi^+_{h,\lambda} b_{a_2}] - S_{\lambda,\omega}(t,x,\theta) & \text{in } Q_T, \\ w_{h,\lambda}(0,x,\theta) = 0, & \text{in } Q, \\ w_{h,\lambda}(t,x,\theta) = 0 & \text{on } \Sigma^-, \end{cases} \quad (3.7.30)$$

where we set

$$S_{\lambda,\omega}(t,x,\theta) = k_2(t,x,\theta,\omega)\varphi^+_\lambda(t,x,\omega)b_{a_2'}(t,x,\omega).$$

Then, from Lemma 1.3.4 we get

$$\left\| w_{h,\lambda}\right\|_{L^2(Q_T)} \le C\left\| \mathscr{N}_{k_2}[\varphi^+_{h,\lambda} b_{a_2}] - S_{\lambda,\omega}\right\|_{L^2(Q_T)},$$

where C is a constant depending only on Ω, T. Furthermore, by (1.4.31) we get

$$\lim_{h\to 1}\mathscr{N}_{k_2}[(\varphi^+_{h,\lambda})b_{a_2}](t,x,\theta) = S_{\lambda,\omega}(t,x,\theta), \quad \text{in } L^2(\mathscr{S}).$$

Thus, we obtain

$$\lim_{h\to 1}\left\|\mathcal{N}_{k_2}[(\varphi^+_{h,\lambda}b_{a_2}]-S_{\lambda,\omega}\right\|_{L^2(Q_T)}=0,$$

which implies that

$$\lim_{h\to 1}\left\|r^+_{h,\lambda}-\zeta^+_{\lambda,\omega}\right\|_{L^2(Q_T)}=0.$$

This completes the proof of Lemma.

We prove by proceeding as the proof of Lemma 3.7.1 the following Lemma. □

Lemma 3.7.2. *Let $\omega\in\mathcal{S}$ and $r^-_{h,\lambda}$ the unique solution to* (3.7.22). *We denote by $\zeta^-_{\lambda,\omega}$ be the solution of the following problem*

$$\begin{cases}\partial_t\zeta^-+\theta\cdot\nabla\zeta^--a_1(x,\theta)\zeta^-=-\mathcal{N}^*_{k_1}[\zeta^-]-k_1(t,x,\theta,\omega)\varphi^-_\lambda(t,x,\omega)b_{a_1}(t,x,\omega),\\ \zeta^-(T,x,\theta)=0,\\ \zeta^-(t,x,\theta)=0,\end{cases}\tag{3.7.31}$$

where φ^-_λ is given by (3.4.6). Then we have

$$\lim_{h'\to 1}\left\|r^-_{h',\lambda}-\zeta^-_{\lambda,\omega}\right\|_{L^2(Q_T)}=0.\tag{3.7.32}$$

Moreover, we get

$$\lim_{h'\to 1}\left\|\mathcal{N}^*_{k_1}[r^-_{h,\lambda}]-\mathcal{N}^*_{k_1}[\zeta^-_{\lambda,\omega}]\right\|_{L^2(Q_T)}=0.\tag{3.7.33}$$

Lemma 3.7.3. *Let $\mathcal{L}_{\lambda,h,h'}$ and $\mathcal{E}_{\lambda,h,h'}$, as they are defined respectively in* (3.7.25) *and* (3.7.26). *Then, we have*

$$\lim_{\substack{h\to 1\\ h'\to 1}}\mathcal{L}_{\lambda,h,h'}(\omega)=\mathcal{L}(\omega),\quad\forall\omega\in\mathcal{S},$$

and

$$\lim_{\substack{h\to 1\\ h'\to 1}}\mathcal{E}_{\lambda,h,h'}(\omega)=\mathcal{E}_\lambda(\omega),\quad\forall\omega\in\mathcal{S},$$

where $\mathcal{L}$ and $\mathcal{E}_\lambda$ are respectively given by

$$\mathcal{L}(\omega)=\int_0^T\int_\Omega k(t,x,\omega,\omega)\phi_1(x-t\omega)\phi_2(x-t\omega)dxdt.$$

$$\mathscr{E}_\lambda(\omega)$$
$$= \int_0^T \int_\Omega \Big(\mathscr{N}_k[\zeta^+_{\lambda,\omega}]\phi_2(x-t\omega)b_{-a_1}e^{-i\lambda(t-\omega\cdot x)} + \mathscr{N}^*_k[\zeta^-_{\lambda,\omega}]\phi_1(x-t\omega)b_{a_2}e^{i\lambda(t-\omega\cdot x)}\Big)\,dx\,dt$$
$$+ \int_{Q_T} \mathscr{N}_k[\zeta^+_{\lambda,\omega}]\zeta^-_{\lambda,\omega}\,d\theta\,dx\,dt. \quad (3.7.34)$$

Proof. By replacing $\varphi^+_{h,\lambda}$ and $\varphi^-_{h',\lambda}$ by their expressions and by using (1.4.28) and (1.4.29), we obtain

$$|\mathscr{N}_k[\varphi^+_{h,\lambda}b_{a_2}](t,x,\theta)\varphi^-_{h',\lambda}(t,x,\theta)b_{-a_1}(t,x,\theta)| \le C_n(1-h')^{1-n},$$

where C depend on d, M, M_1, M_2 and T. Therefore, by using (1.4.29) and Lebesgue's Theorem, one gets

$$\lim_{h\to 1}\int_{Q_T} \mathscr{N}_k[\varphi^+_{h,\lambda}b_{a_2}](t,x,\theta)\vartheta_{h'}(\omega,\theta)\phi_2(x-t\theta)e^{-i\lambda(t-x\cdot\theta)}b_{-a_1}(t,x,\theta)d\theta\,dx\,dt$$
$$= \mathscr{L}_{\lambda,h'}(\omega).$$

where

$$\mathscr{L}_{\lambda,h'}(\omega)$$
$$= \int_{Q_T} k(t,x,\theta,\omega)\phi_1(x-t\omega)e^{-i\lambda x\cdot\omega}b_{a'_2}(t,x,\omega)\vartheta_{h'}(\omega,\theta)\phi_2(x-t\theta)e^{i\lambda x\cdot\theta}b_{-a_1}(t,x,\theta)d\theta\,dx\,dt.$$

Similarly, we obtain that

$$\lim_{h'\to 1}\mathscr{L}_{\lambda,h'}(\omega) = \mathscr{L}(\omega), \qquad \forall\omega\in\mathscr{S},$$

where we have used $a'_2 = a_1$.

We rewrite $\mathscr{E}_{\lambda,h,h'}(\omega)$ as $\mathscr{E}_{\lambda,h,h'}(\omega) = \mathscr{E}^1_{\lambda,h,h'}(\omega) + \mathscr{E}^2_{\lambda,h,h'}(\omega) + \mathscr{E}^3_{\lambda,h,h'}(\omega)$, where

$$\mathscr{E}^1_{\lambda,h,h'}(\omega) = \int_{Q_T} \mathscr{N}_k[r^+_{h,\lambda}](t,x,\theta)\varphi^-_{h',\lambda}(t,x,\theta)b_{-a_1}(t,x,\theta)d\theta\,dx\,dt$$
$$= \int_{Q_T} \mathscr{N}_k[r^+_{h,\lambda}](t,x,\theta)\vartheta_{h'}(\omega,\theta)\phi_2(x-t\theta)e^{-i\lambda(t-x\cdot\theta)}b_{-a_1}(t,x,\theta)d\theta\,dx\,dt,$$
$$(3.7.35)$$

$$\begin{aligned}\mathscr{E}^2_{\lambda,h,h'}(\omega) &= \int_{Q_T} \mathscr{N}_k[\varphi^+_{h,\lambda} b_{a_2}](t,x,\theta) r^-_{h',\lambda}(t,x,\theta) d\theta\, dx\, dt \\ &= \int_{Q_T} \mathscr{N}_{k'}[\vartheta_h(\omega,\theta)\phi_1(x-t\theta)e^{i\lambda(t-x\cdot\theta)} b_{a_2}](t,x,\theta) r^-_{h',\lambda}(t,x,\theta) d\theta\, dx\, dt,\end{aligned} \tag{3.7.36}$$

$$\mathscr{E}^3_{\lambda,h,h'}(\omega) = \int_{Q_T} \mathscr{N}_k[r^+_{h,\lambda}](t,x,\theta) r^-_{h',\lambda}(t,x,\theta) d\theta\, dx\, dt. \tag{3.7.37}$$

Taking the limit as $h \to 1$ in the above expressions, we get from Lemma 3.7.1 and (1.4.31)

$$\lim_{h\to 1} \mathscr{E}^1_{\lambda,h,h'}(\omega) = \int_{Q_T} \mathscr{N}_k[\zeta^+_{\lambda,\omega}](t,x,\theta)\vartheta_{h'}(\omega,\theta)\phi_2(x-t\theta)e^{-i\lambda(t-x\cdot\theta)} b_{-a_1}(t,x,\theta) d\theta\, dx\, dt,$$

$$\lim_{h\to 1} \mathscr{E}^2_{\lambda,h,h'}(\omega) = \int_{Q_T} k'(t,x,\theta,\omega)\phi_1(x-t\omega)e^{i\lambda(t-x\cdot\omega)} b_{a_2}(t,x,\omega) r^-_{h',\lambda}(t,x,\theta) d\theta\, dx\, dt,$$

$$\lim_{h\to 1} \mathscr{E}^3_{\lambda,h,h'}(\omega) = \int_{Q_T} \mathscr{N}_k[\zeta^+_{\lambda,\omega}](t,x,\theta) r^-_{h',\lambda}(t,x,\theta) d\theta\, dx\, dt.$$

Now, taking the limit as $h' \to 1$ by a similar reasoning we obtain $\lim\limits_{\substack{h\to 1\\ h'\to 1}} \mathscr{E}_{\lambda,h,h'}(\omega) = \mathscr{E}_\lambda(\omega)$, for any $\omega \in \mathscr{S}$. This completes the proof of the lemma. □

Lemma 3.7.4. *Let $\lambda > 0$, and $\mathscr{E}_\lambda$ given by* (3.7.34). *Then, we have*

$$\lim_{\lambda\to\infty} \mathscr{E}_\lambda(\omega) = 0, \quad \forall \omega \in \mathscr{S}.$$

Proof. By using the fact that $\zeta^\pm_{\lambda,\omega} \rightharpoonup 0$ in $L^2(\Omega)$ and that the integral operator $\mathscr{N}_k$ is compact, we get

$$\lim_{\lambda\to\infty} \left\| \mathscr{N}_k[\zeta^+_{\lambda,\omega}](t,\cdot,\theta) \right\|_{L^2(\Omega)} = \lim_{\lambda\to\infty} \left\| \mathscr{N}^*_k[\zeta^-_{\lambda,\omega}](t,\cdot,\theta) \right\|_{L^2(\Omega)} = 0, \quad \forall t \in (0,T),\ \theta \in \mathscr{S}.$$

The Lebesgue Theorem implies that

$$\lim_{\lambda\to\infty} \left\| \mathscr{N}_k[\zeta^+_{\lambda,\omega}] \right\|_{L^2(Q_T)} = \lim_{\lambda\to\infty} \left\| \mathscr{N}^*_k[\zeta^-_{\lambda,\omega}] \right\|_{L^2(Q_T)} = 0.$$

This complete the proof. □

We are now in position to state and prove the main result of this section.

Theorem 3.7.5. *Let $(a_j, k_j) \in \mathscr{A}(M_0)\times\mathscr{K}(M_1, M_2)$,, $j = 1, 2$, such that $k_j(t, x, \theta, \theta') = g_j(t, x)\kappa(\theta, \theta')$, with $g_j \in L^\infty(X_{r,*})$, $\kappa \in L^\infty(\mathscr{S} \times \mathscr{S})$ and $\kappa(\theta, \theta) \neq 0$, $\forall\theta \in \mathscr{S}$. If $T > 2r$ and $\mathscr{A}_{a_1,k_1} = \mathscr{A}_{a_2,k_2}$, then $g_1 = g_2$ a.e, in $X_{r,*}$.*

Proof. Since we have by (3.7.24), that $\mathscr{L}_{\lambda,h,h'}(\omega) = \mathscr{E}_{\lambda,h,h'}(\omega)$ for all $\omega \in \mathscr{S}$, we take the limit in h, h' and λ and we get, from Lemmas 3.7.3 and 3.7.4, that

$$\int_0^T \int_{\mathbb{R}^n} k(t, x, \omega, \omega)\phi_1(x - t\omega)\phi_2(x - t\omega)dxdt = 0.$$

where k is extended by 0 outside Ω. Since $k_j(t, x, \omega, \omega) = g_j(t, x)\kappa(\omega, \omega)$, $j = 1, 2$, where $\kappa \in L^\infty(\mathscr{S} \times \mathscr{S})$ and $\kappa(\omega, \omega) \neq 0$ a.e on $\mathscr{S}$. Performing the changes of variables, $y = x - t\omega$, for $g = g_2 - g_1$, we get

$$\int_0^T \int_{\mathbb{R}^n} g(t, y+t\omega)\kappa(\omega, \omega)\phi_1(y)\phi_2(y)dydt = 0,\ \forall\, \phi_1, \phi_2 \in \mathscr{C}_0^\infty(\mathscr{K}_r),\ \forall\, \omega \in \mathscr{S}.$$

Put $\phi_1(x) = \phi_2(x)$, we obtain

$$\int_0^T \int_{\mathbb{R}^n} g(t, y + t\omega)\kappa(\omega, \omega)\phi_1^2(y)dydt = 0,\ \forall\, \phi_1 \in \mathscr{C}_0^\infty(\mathscr{K}_r),\ \forall\, \omega \in \mathscr{S}. \tag{3.7.38}$$

Now, we consider a non negative function $\phi_0 \in \mathscr{C}_0^\infty(B(0, 1))$ such that $\|\phi_0\|_{L^2(\mathbb{R}^n)} = 1$. We define

$$\phi_h(x) = h^{-n/2}\phi_0\left(\frac{x - y}{h}\right),$$

where $y \in \mathscr{K}_r$. Then, for $h > 0$ sufficiently small we obtain

$$\operatorname{Supp}\phi_h \cap \Omega = \emptyset. \quad (\operatorname{Supp}\phi_h \pm T\theta) \cap \Omega = \emptyset.$$

We rewrite (3.7.38) with $\phi_1 = \phi_h$, we deduce that

$$\lim_{h\mapsto 0}\int_{\mathbb{R}^n} \phi_h^2(y)\kappa(\omega, \omega)\left(\int_0^T g(y + t\omega)\,dt\right)dx$$
$$= \kappa(\omega, \omega)\int_0^T g(t, y + t\omega)\,dt = 0,\ \ \forall\, \omega \in \mathscr{S},\ \forall y \in \mathscr{K}_r.$$

Moreover, since $\kappa(\omega, \omega) \neq 0$, we have

$$\int_0^T g(t, y + t\omega)\,dt = 0,\ \ \forall\, \omega \in \mathscr{S},\ \forall y \in \mathscr{K}_r.$$

Since $\mathrm{Supp}(g) \subset X_{r,*}$, we conclude that

$$\int_{\mathbb{R}} g(t, y + t\omega)\, dt = 0, \tag{3.7.39}$$

for each $(\omega, y) \in \mathscr{S} \times \mathscr{K}_r$. Furthermore, if $y \in B(0, r/2)$, we have $|y + t\omega| \geqslant |t| - |y| \geq |t| - r/2$. Hence, one can see that $(t, y + t\omega) \notin \mathscr{C}_r^+$, if $t > r/2$. On the other hand, we have $(t, y + t\omega) \notin \mathscr{C}_r^+$, if $t \leqslant r/2$. Thus, we conclude that $(t, y + t\omega) \notin \mathscr{C}_r^+ \supset X_{r,*}$ for $t \in \mathbb{R}$. This and the fact that $\mathrm{Supp}(g) \subset X_{r,*}$, entails that for all $y \in B(0, r/2)$ and $\omega \in \mathscr{S}$, we have

$$g(t, y + t\omega) = 0, \ \forall t \in \mathbb{R}.$$

By a similar way, we prove for $|y| \geqslant T - r/2$, that $(t, y + t\omega) \notin \mathscr{C}_r^- \supset X_{r,*}$ for $t \in \mathbb{R}$ and then $g(t, y + t\omega) = 0$. Hence, we conclude that

$$\int_{\mathbb{R}} g(t, y + t\omega)\, dt = 0, \quad \forall \omega \in \mathscr{S}, \ \forall y \notin \mathscr{K}_r. \tag{3.7.40}$$

Thus, by (3.7.39) and (3.7.40), we get

$$\int_{\mathbb{R}} g(t, y + t\omega)\, dt = 0, \quad \forall \omega \in \mathscr{S}, \ \forall y \in \mathbb{R}^n. \tag{3.7.41}$$

We now turn our attention to the Fourier transform of g. Let $\xi \in \mathbb{R}^n$. In light of (3.7.41) and by the use of Fubini's Theorem, we get

$$\int_{\mathbb{R}} \int_{\mathbb{R}^n} g(t, y + t\omega) e^{-iy\cdot\xi}\, dy dt = 0, \quad \forall \omega \in \mathscr{S}, \ \forall y \in \mathbb{R}^n. \tag{3.7.42}$$

Performing the changes of variables $x = y + t\omega$, one gets

$$\int_{\mathbb{R}} \int_{\mathbb{R}^n} g(t, x) e^{-ix\cdot\xi} e^{it(\omega\cdot\xi)}\, dx dt = 0, \quad \forall \omega \in \mathscr{S}, \ \forall x \in \mathbb{R}^n. \tag{3.7.43}$$

Let us now consider $(\xi, \tau) \in E$ and $\zeta' \in \mathscr{S}$ be such that $\xi \cdot \zeta' = 0$. Setting

$$\omega = \frac{\tau}{|\xi|^2} \cdot \xi + \sqrt{1 - \frac{\tau^2}{|\xi|^2}} \cdot \zeta.$$

then $(\xi, \omega \cdot \xi) = (\xi, \tau) \in E$. We then deduce that $\widehat{g}(\xi, \tau) = 0$ in the set E. By an argument of analyticity, we extend this result to $\mathbb{R}^{n+1}$. Hence, by the injectivity of the Fourier we get the desired result. This ends the proof. □

Chapter 4

Stable determination of a time-depend coefficient for the linear Boltzmann equation

4.1 Introduction

This chapter we aim to the study the stability issues in of an inverse from the observations made at the boundary. We consider initial-boundary value problem for the linear Boltzmann equation

$$\begin{cases} \partial_t u + \theta \cdot \nabla u(t,x,\theta) + a(t,x)u = \mathscr{N}_k[u](t,x,\theta) & \text{in} \quad Q_T := (0,T) \times Q, \\ u(0,x,\theta) = u_0(x,\theta) & \text{in} \quad Q := \mathscr{S} \times \Omega, \\ u(t,x,\theta) = f(t,x,\theta) & \text{on} \quad \Sigma^-. \end{cases} \tag{4.1.1}$$

where $a \in L^\infty((0,T)\times\Omega)$ and $\mathscr{N}_k$ is the integral operator with kernel $k(t,x,\theta,\theta')$ defined by

$$\mathscr{N}_k[u](t,x,\theta) = \int_{\mathscr{S}} k(x,\theta,\theta')u(t,x,\theta')d\theta'. \tag{4.1.2}$$

In Chapter 3, we have proved that the unique determination of the time-dependent coefficient a in the whole domain, is fails if we assume that the medium is quiet initially ($u_0 = 0$) and our data are only the response $\mathscr{A}_{a,k}(f)$ where f denotes the disturbance used to probe the medium. In this chapter, we will consider

three different sets of data and we will treat the stability issue for the inverse problem under consideration in three different cases. In the first case, we we will prove a log-type stability estimate for the time-dependent absorption coefficient a with respect to the albedo operator in a subset of domain. Next we assume that we know $u(T,\cdot)$, with $u_0 = 0$, we will extend this result to a larger region and finally, we will prove a log-type stability estimate for this problem over the whole domain when our data is the response of the medium for all possible initial states.

Let us introduce some notations. Let $r > 0$ be such that $T > 2r$. Without loss of generality, we assume that $\Omega \subseteq B(0,r/2) = \{x \in \mathbb{R}^n, |x| < r/2\}$. We set $X_r = (0,T) \times B(0,r/2)$, $\Omega_T = (0,T) \times \Omega$ and we consider the annular region around the domain Ω,

$$\mathscr{B}_r = \left\{x \in \mathbb{R}^n,\ \frac{r}{2} < |x| < T - \frac{r}{2}\right\}.$$

Setting the *forward* and *backward* light cone:

$$\mathscr{C}_r^+ = \left\{(t,x) \in X_r,\ |x| < t - \frac{r}{2},\ t > \frac{r}{2}\right\}, \tag{4.1.3}$$

$$\mathscr{C}_r^- = \left\{(t,x) \in X_r,\ |x| < T - \frac{r}{2} - t,\ T - \frac{r}{2} > t\right\}. \tag{4.1.4}$$

Let the *cloaking* region

$$\mathscr{C}_r = \left\{(t,x) \in X_r,\ |x| \le \frac{r}{2} - t,\ 0 \le t \le \frac{r}{2}\right\}. \tag{4.1.5}$$

Finally, we denote

$$X_{r,\sharp} = \Omega_T \cap \mathscr{C}_r^+, \qquad \text{and} \quad X_{r,*} = \Omega_T \cap \mathscr{C}_r^+ \cap \mathscr{C}_r^-.$$

It is clear that, $X_{r,*} \subset X_{r,\sharp} \subset \Omega_T$.

In order to give the main statements of this chapter, we need first to introduce the set of the admissible coefficients a and k. Given $M_0, M_1, M_2 > 0$, we consider

$$\mathscr{A}(M_0) = \{a \in \mathscr{C}^1(\overline{\Omega}_T);\ \|a\|_{L^\infty(\Omega_T)} \le M_0\}.$$

Given $a_0 \in \mathscr{C}^1(\overline{\Omega}_T)$, we set

$$\mathscr{A}^*(M_0) = \{a \in \mathscr{A}(M_0);\ a = a_0 \text{ in } \overline{\Omega}_T \setminus X_{r,*}\}.$$

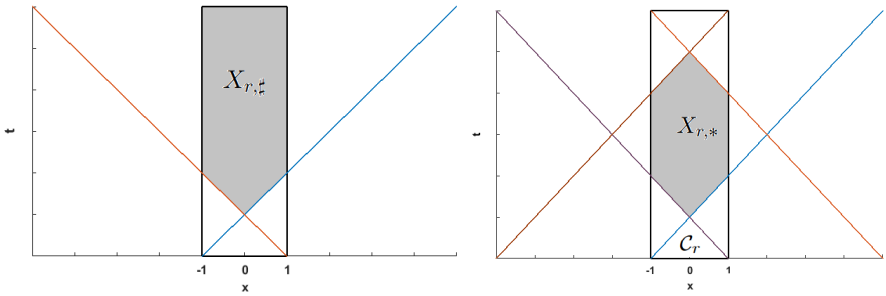

Figure 4.1: Particular case $\Omega = [-1, 1]$

$$\mathscr{A}^{\sharp}(M_0) = \{a \in \mathscr{A}(M_0);\ a = a_0 \text{ in } \overline{\Omega}_T \setminus X_{r,\sharp}\}.$$

and

$$\mathscr{K}(M_1, M_2) = \{k \in L^{\infty}(\Omega, L^2(\mathscr{S} \times \mathscr{S}));\ \int_{\mathscr{S}} |k(x,\theta,\theta')| d\theta' \leq M_1, \\ \int_{\mathscr{S}} |k(x,\theta,\theta')| d\theta \leq M_2\}.$$

4.2 Stable determination of absorption coefficient from boundary measurements

In this section we prove a log-type stability estimate for a appearing in the initial boundary value problem (4.1.1) with $u_0 = 0$. Our goal here is to show that the time dependent coefficient a depends stably on the albedo operator $\mathscr{A}_{a,k}$. This is by means of the geometric optics solutions introduced in Chapter 3 and the light-ray transform. Let's first recall the CGO solutions constructed in the previous chapter .

$$u_{\lambda}^{+}(t,x,\theta) = \varphi_{\lambda}^{+}(t,x,\theta) b_a(t,x,\theta) + \psi_{\lambda}^{+}(t,x,\theta),$$

$$u_{\lambda}^{-}(t,x,\theta) = \varphi_{\lambda}^{-}(t,x,\theta) b_{-a}(t,x,\theta) + \psi_{\lambda}^{-}(t,x,\theta),$$

where b_{-a_1} and b_{a_2} defined as follow

$$b_{-a_1}(x,t) = \exp\Big(\int_0^t a_1(s, x-(t-s)\theta)\, ds\Big),$$

$$b_{a_2}(t,x,\theta) = \exp\Big(-\int_0^t a_2(s, x-(t-s)\theta)\,ds\Big),$$

and $\psi_\lambda^\pm$ satisfies

$$\left\|\psi_\lambda^+\right\|_{\mathscr{C}([0,T];L^p(Q))} \leq C\left\|\phi^+\right\|_{L^p(\mathbb{R}^n\times\mathscr{S})}, \quad \lim_{\lambda\to\infty}\left\|\psi_\lambda^+\right\|_{L^2(Q_T)} = 0. \tag{4.2.6}$$

and

$$\left\|\psi_\lambda^-\right\|_{\mathscr{C}([0,T];L^q(Q))} \leq C\left\|\phi^-\right\|_{L^q(\mathbb{R}^n\times\mathscr{S})} \quad \lim_{\lambda\to\infty}\left\|\psi_\lambda^-\right\|_{L^2(Q_T)} = 0.. \tag{4.2.7}$$

Here C is a constant depending only on Ω, T and M_i, $i = 0,1,2$.

4.2.1 Preliminary estimate

The main purpose of this section is to give a preliminary estimate, which relates the difference of the absorption coefficients to albedo operator. Let $(a_j, k_j) \in \mathscr{A}^*(M_0) \times \mathscr{K}(M_1, M_2)$, $j = 1,2,$. We set

$a = a_2 - a_1$, and $b_a(t,x,\theta) = b_{a_2} b_{-a_1}(t,x,\theta) = \exp\Big(-\int_0^t a(s, x-(t-s)\theta)\,ds\Big)$,

where we have extend a by 0 outside $\overline{\Omega}_T$.

We only assume that Supp $\phi^\pm(\cdot,\theta) \subset \mathscr{B}_r$, in such a way we have

$$\mathrm{Supp}(\phi^+)(\cdot,\theta) \cap \Omega = \emptyset \quad \forall\,\theta \in \tag{4.2.8}$$

and

$$\mathrm{Supp}(\phi^-)(\cdot,\theta) \pm T\theta) \cap \Omega = \emptyset, \quad \forall\,\theta \in \mathscr{S}. \tag{4.2.9}$$

We start by collecting a preliminary estimate which is needed to prove the main statement of this section.

Lemma 4.2.1. *Let $(a_j, k_j) \in \mathscr{A}^*(M_0) \times \mathscr{K}(M_1, M_2)$, $j = 1, 2$. There exists $C > 0$ such that for any $\phi^\pm \in \mathscr{C}_0^\infty(\mathscr{B}_r, \mathscr{C}(\mathscr{S}))$, the following estimate holds true*

$$\Big|\int_{\mathbb{R}^n\times\mathscr{S}}(\phi^+\phi^-)(y,\theta)\Big[\exp\Big(-\int_0^T a(s,y+s\theta)\,ds\Big)-1\Big]d\theta\,dy\Big|$$
$$\leq C\|\mathscr{A}_{a_2,k_2}-\mathscr{A}_{a_1,k_1}\|_1\|\phi^+\|_{L^1(\mathbb{R}^n\times\mathscr{S})}\|\phi^-\|_{L^\infty(\mathbb{R}^n\times\mathscr{S})}. \quad (4.2.10)$$

Here C depends only on Ω, T and M_i, $i=0,1,2$.

Proof. In view of Lemma 3.4.1, and using the fact that Supp $\phi^+(\cdot,\theta)\cap\Omega=\emptyset$, there exists a G.O-solution u_λ^+ to the equation

$$\begin{cases}\partial_t u+\theta\cdot\nabla u+a_2(t,x)u=\mathscr{N}_{k_2}[u](t,x,\theta) & (t,x,\theta)\in Q_T,\\ u(0,x,\theta)=0 & (t,x)\in Q,\end{cases}$$

in the following form

$$u_\lambda^+(t,x,\theta)=\varphi_\lambda^+(t,x,\theta)b_{a_2}(t,x,\theta)+\psi_\lambda^+(t,x,\theta),$$

corresponding to the coefficients a_2 and k_2, where $\psi_\lambda^+(t,x,\theta)$ satisfies (4.2.6). Next, let us denote by f_λ the function

$$f_\lambda(t,x,\theta):=u_\lambda^+(t,x,\theta)_{|\Sigma^-}=\varphi_\lambda^+(t,x,\theta)b_{a_2}(t,x,\theta)_{|\Sigma^-}.$$

We denote by u_1 the solution of

$$\begin{cases}\partial_t u_1+\theta\cdot\nabla u_1+a_1(t,x)u_1=\mathscr{N}_{k_1}[u_1] & (t,x,\theta)\in Q_T,\\ u_1(0,x,\theta)=0 & (x,\theta)\in Q,\\ u_1(t,x,\theta)=f_\lambda(t,x,\theta) & (t,x,\theta)\in\Sigma^-.\end{cases}$$

Putting $u=u_1-u_\lambda^+$. Then, u is a solution to the following system

$$\begin{cases}\partial_t u+\theta\cdot\nabla u+a_1(t,x)u=\mathscr{N}_{k_1}[u]+a(t,x)u_\lambda^+-\mathscr{N}_k[u_\lambda^+] & (t,x,\theta)\in Q_T,\\ u(0,x,\theta)=0 & (x,\theta)\in Q,\\ u(t,x,\theta)=0 & (t,x,\theta)\in\Sigma^-,\end{cases} \quad (4.2.11)$$

where $a=a_2-a_1$ and $k=k_2-k_1$. On the other hand by Lemma 3.4.2 and the fact that (Supp $\phi_1(\cdot,\theta)\pm T\theta)\cap\Omega=\emptyset$, we can find of a solution u_λ^- to the backward problem

$$\begin{cases}\partial_t u+\theta\cdot\nabla u-a_1(t,x)u=-\mathscr{N}_{k_1}^*[u](t,x,\theta) & \text{in } Q_T,\\ u(T,x,\theta)=0 & \text{in } Q,\end{cases}$$

corresponding to the coefficients a_1 and k_1, in the form

$$u_\lambda^-(t,x,\theta) = \varphi_\lambda^-(t,x,\theta) b_{-a_1}(t,x,\theta) + \psi_\lambda^-(t,x,\theta), \tag{4.2.12}$$

here the remainder term $\psi_\lambda^-(t,x,\theta)$ satisfies the estimate (4.2.7). Multiplying the first equation of (4.2.11) by u_λ^-, integrating by parts and using Green's formula, we obtain

$$\begin{aligned}\int_{Q_T} a(t,x) u_\lambda^+ u_\lambda^- \, d\theta \, dx \, dt - \int_{Q_T} \mathscr{N}_k[u_\lambda^+](t,x,\theta) u_\lambda^-(t,x,\theta) d\theta \, dx \, dt \\ = \int_{\Sigma^+} \theta \cdot \nu(x) (\mathscr{A}_{a_2,k_2} - \mathscr{A}_{a_1,k_1})(f_\lambda) u_\lambda^- \, d\theta \, dx \, dt. \end{aligned} \tag{4.2.13}$$

Where we have used

$$\int_{Q_T} \mathscr{N}_{k_1}[u](t,x,\theta) u_\lambda^-(t,x,\theta) d\theta \, dx \, dt = \int_{Q_T} u(t,x,\theta) \mathscr{N}_{k_1}^*[u_\lambda^-](t,x,\theta) d\theta \, dx \, dt.$$

By replacing u_λ^+ and u_λ^- by their expressions, we obtain

$$\begin{aligned}&\int_{Q_T} a(t,x) u_\lambda^+ u_\lambda^- d\theta \, dx \, dt \\ &= \int_{Q_T} a(t,x) \varphi_\lambda^+(t,x,\theta) \varphi_\lambda^-(t,x,\theta) b_a(t,x,\theta) d\theta \, dx \, dt + \mathscr{I}_{1,\lambda}, \end{aligned} \tag{4.2.14}$$

where $\mathscr{I}_{1,\lambda}$ represents the sum of the integrals that contain terms with ψ_λ^+ and ψ_λ^-.

Note that there exists $C > 0$ such that for any $\lambda \geq 0$, we have

$$|\mathscr{I}_{1,\lambda}| \leq C \left(\left\| \psi_\lambda^+ \right\|_{L^2(Q_T)} + \left\| \psi_\lambda^- \right\|_{L^2(Q_T)} \right). \tag{4.2.15}$$

Then by (4.2.6) and (4.2.7), we get

$$\lim_{\lambda \to \infty} \mathscr{I}_{1,\lambda} = 0. \tag{4.2.16}$$

Moreover, we have

$$\int_{Q_T} \mathscr{N}_k[u_\lambda^+] u_\lambda^- d\theta \, dx \, dt = \int_{Q_T} \mathscr{N}_k[\varphi_\lambda^+ b_{a_2}](t,x,\theta) \varphi_\lambda^-(t,x,\theta) b_{-a_1}(t,x,\theta) d\theta \, dx \, dt + \mathscr{I}_{2,\lambda},$$

where

$$\mathscr{I}_{2,\lambda} = \int_{Q_T} \mathscr{N}_k[\psi_\lambda^+](t,x,\theta)\varphi_\lambda^-(t,x,\theta)b_{-a_1}(t,x,\theta)d\theta\,dx\,dt$$
$$+\int_{Q_T} \mathscr{N}_k[\varphi_\lambda^+ b_{a_2}](t,x,\theta)\psi_\lambda^-(t,x,\theta)d\theta\,dx\,dt+\int_{Q_T} \mathscr{N}_k[\psi_\lambda^+](t,x,\theta)\psi_\lambda^-(t,x,\theta)d\theta\,dx\,dt.$$

By (1.3.9), we get

$$|\mathscr{I}_{2,\lambda}| \le C\left(\left\|\psi_\lambda^+\right\|_{L^2(Q_T)} + \left\|\psi_\lambda^-\right\|_{L^2(Q_T)}\right). \tag{4.2.17}$$

Then by (4.2.6) and (4.2.7), we have

$$\lim_{\lambda\to\infty} \mathscr{I}_{2,\lambda} = 0, \tag{4.2.18}$$

and by the Cauchy-Schwartz inequality, we obtain

$$\left|\int_{Q_T} \mathscr{N}_k[\varphi_\lambda^+ b_{a_2}](t,x,\theta)\varphi_\lambda^-(t,x,\theta)b_{-a_1}(t,x,\theta)d\theta\,dx\,dt\right| \le C\left\|\mathscr{N}_k[\varphi_\lambda^+ b_{a_2}]\right\|_{L^2(Q_T)}. \tag{4.2.19}$$

Then by (1.3.10), we have

$$\lim_{\lambda\to\infty}\int_{Q_T} \mathscr{N}_k[\varphi_\lambda^+ b_{a_2}](t,x,\theta)\varphi_\lambda^-(t,x,\theta)b_{-a_1}(t,x,\theta)d\theta\,dx\,dt = 0. \tag{4.2.20}$$

In light of (4.2.13), we have

$$\int_{Q_T} a(t,x)\varphi^+(t,x,\theta)\varphi^-(t,x,\theta)b_a(t,x,\theta)d\theta\,dx\,dt$$
$$= \int_{\Sigma^+} \theta\cdot\nu(x)(\mathscr{A}_{a_2,k_2} - \mathscr{A}_{a_1,k_1})(f_\lambda)u_\lambda^-\,d\theta\,dx\,dt - \mathscr{I}_\lambda, \tag{4.2.21}$$

where the remainder term $\mathscr{I}_\lambda$ is given by

$$\mathscr{I}_\lambda = \mathscr{I}_{1,\lambda} - \mathscr{I}_{2,\lambda} - \int_{Q_T} \mathscr{N}_k[\varphi_\lambda^+ b_{a_2}](t,x,\theta)\varphi_\lambda^-(t,x,\theta)b_{-a_1}(t,x,\theta)d\theta\,dx\,dt.$$

Taking the limit as $\lambda\to+\infty$, we get from (4.2.16), (4.2.18) and (4.2.20)

$$\lim_{\lambda\to\infty}\mathscr{I}_\lambda = 0. \tag{4.2.22}$$

Furthermore on Σ^-, we have $u_\lambda^+ = f_\lambda$, then we get the following estimate

$$\left|\int_{\Sigma^+} (\mathscr{A}_{a_2,k_2} - \mathscr{A}_{a_1,k_1})(f_\lambda)\,u_\lambda^-\,dx\,d\theta\,dt\right| \le C\|\mathscr{A}_{a_2,k_2} - \mathscr{A}_{a_1,k_1}\|_p\,\|f_\lambda\|_{\mathscr{L}_p^-(\Sigma^-)}\|u_\lambda^-\|_{\mathscr{L}_q^+(\Sigma^+)}. \tag{4.2.23}$$

Using the fact that

$$\|u_\lambda^-\|_{\mathscr{L}_q^+(\Sigma^+)} \le C\|\phi^-\|_{L^\infty(\mathbb{R}^n\times\mathscr{S})}, \quad \lim_{p\to 1}\|f_\lambda\|_{\mathscr{L}_p^-(\Sigma^-)} = \|f_\lambda\|_{\mathscr{L}_1^-(\Sigma^-)} \le C\|\phi^+\|_{L^1(\mathbb{R}^n\times\mathscr{S})}, \tag{4.2.24}$$

and

$$\limsup_{p\to 1}\|\mathscr{A}_{a_2,k_2} - \mathscr{A}_{a_1,k_1}\|_p \le \|\mathscr{A}_{a_2,k_2} - \mathscr{A}_{a_1,k_1}\|_1. \tag{4.2.25}$$

We get, by (4.2.21), (4.2.22) and (4.2.23),

$$\Big|\int_{Q_T} a(t,x)\varphi^+\varphi^-(t,x,\theta)b_a(t,x,\theta)d\theta\,dx\,dt\Big| \le C\|\mathscr{A}_{a_2,k_2} - \mathscr{A}_{a_1,k_1}\|_1\|\phi^+\|_{L^1(\mathbb{R}^n\times\mathscr{S})}\|\phi^-\|_{L^\infty(\mathbb{R}^n\times\mathscr{S})}.$$

Then, using the fact $a(t,\cdot) = 0$ outside Ω and by changing of variables $y = x - t\theta$, one gets the following estimation

$$\Big|\int_0^T\int_{\mathbb{R}^n\times\mathscr{S}} a(t,y+t\theta)\phi^+(y,\theta)\phi^-(y,\theta)\exp\Big(-\int_0^t a(s,y+s\theta)\,ds\Big)d\theta dy\,dt\Big| \le C\|\mathscr{A}_{a_2,k_2} - \mathscr{A}_{a_1,k_1}\|_1\|\phi^+\|_{L^1(\mathbb{R}^n\times\mathscr{S})}\|\phi^-\|_{L^\infty(\mathbb{R}^n\times\mathscr{S})}.$$

Bearing in mind that

$$\int_0^T\int_{\mathbb{R}^n\times\mathscr{S}} a(t,y+t\theta)\phi^+(y,\theta)\phi^-(y,\theta)\exp\Big(-\int_0^t a(s,y+s\theta)\,ds\Big)d\theta\,dy\,dt = -\int_{\mathbb{R}^n\times\mathscr{S}}(\phi^+\phi^-)(y,\theta)\Big[\exp\Big(-\int_0^T a(s,y+s\theta)\,ds\Big)-1\Big]d\theta\,dy,$$

we conclude the desired estimate given by

$$\Big|\int_{\mathbb{R}^n\times\mathscr{S}}(\phi^+\phi^-)(y,\theta)\Big[\exp\Big(-\int_0^T a(s,y+s\theta)\,ds\Big)-1\Big]d\theta\,dy\Big| \le C\|\mathscr{A}_{a_2,k_2} - \mathscr{A}_{a_1,k_1}\|_1\|\phi^+\|_{L^1(\mathbb{R}^n\times\mathscr{S})}\|\phi^-\|_{L^\infty(\mathbb{R}^n\times\mathscr{S})}.$$

This completes the proof of the lemma. □

4.2.2 Stability for the light-ray transform

The light-ray transform $\mathscr{R}$ of a function $f \in L^1(\mathbb{R}^{n+1})$ is defined by

$$\mathscr{R}(f)(\theta,x) := \int_{\mathbb{R}} f(t,x+t\theta)dt.$$

Using the above Lemma 4.2.1, we can estimate the light-ray transform of the difference of the two absorption coefficients as follows:

Lemma 4.2.2. *Let* $(a_j, k_j) \in \mathscr{A}^*(M_0) \times \mathscr{K}(M_1, M_2)$, $j = 1, 2$, *then there exists* $C > 0$ *such that*

$$|\mathscr{R}(a)(\omega, x)| \leq C\|\mathscr{A}_{a_1,k_1} - \mathscr{A}_{a_2,k_2}\|_1, \quad \forall\, \omega \in \mathscr{S},\ a.e,\ x \in \mathbb{R}^n. \tag{4.2.26}$$

Here C *is a constant depending only on* Ω, T *and* M_i, $i = 0, 1, 2$.

Proof. Let $\omega \in \mathscr{S}$ and $\phi_j \in \mathscr{C}_0^\infty(\mathscr{B}_r)$, $j = 1, 2$. Moreover, let $\varrho_h(\omega, \theta)$ defined by (1.4.27). Selecting

$$\phi^+(y, \theta) = \varrho_h(\omega, \theta)\phi_1(y), \quad \phi^-(y, \theta) = \phi_2(y),$$

we obtain from (4.2.10), that for any $h \in (0, 1)$,

$$\Big|\int_{\mathscr{B}_r \times \mathscr{S}} \varrho_h(\omega, \theta)\phi_1(y)\phi_2(y)\Big[\exp\Big(-\int_0^T a(s, y + s\theta)\, ds\Big) - 1\Big] d\theta\, dy\Big|$$
$$\leq C\|\mathscr{A}_{a_2,k_2} - \mathscr{A}_{a_1,k_1}\|_1 \|\phi_1\|_{L^1(\mathbb{R}^n)} \|\phi_2\|_{L^\infty(\mathbb{R}^n)} \|\varrho_h(\omega, \cdot)\|_{L^1(\mathscr{S})}.$$

Since $\|\varrho_h(\omega, \cdot)\|_{L^1(\mathscr{S})} = 1$, and by (1.4.31) we have

$$\lim_{h \to 1} \int_{\mathscr{S}} \varrho_h(\omega, \theta)\Big[\exp\Big(-\int_0^T a(s, y + s\theta)\, ds\Big) - 1\Big] d\theta = \Big[\exp\Big(-\int_0^T a(s, y + s\omega)\, ds\Big) - 1\Big].$$

Then, we get

$$\Big|\int_{\mathscr{B}_r} \phi_1(y)\phi_2(y)\Big[\exp\Big(-\int_0^T a(s, y + s\omega)\, ds\Big) - 1\Big] dy\Big|$$
$$\leq C\|\mathscr{A}_{a_2,k_2} - \mathscr{A}_{a_1,k_1}\|_1 \|\phi_1\|_{L^1(\mathbb{R}^n)} \|\phi_2\|_{L^\infty(\mathbb{R}^n)}.$$

We conclude that

$$\Big|\exp\Big(-\int_0^T a(s, y + s\omega)\, ds\Big) - 1\Big| \leq C\|\mathscr{A}_{a_2,k_2} - \mathscr{A}_{a_1,k_1}\|_1, \quad \forall\, y \in \mathscr{B}_r,\ \forall\, \omega \in \mathscr{S}. \tag{4.2.27}$$

Using the fact that $|X| \leq e^M |e^X - 1|$ for any X real satisfying $|X| \leq M$ we found out that

$$\Big|\int_0^T a(s, y + s\omega)\, ds\Big| \leq e^{MT}\Big|\exp\Big(-\int_0^T a(s, y + s\omega)\, ds\Big) - 1\Big|,$$

where $X = \int_0^T a(s, y + s\omega)ds$. We conclude in light of (4.2.27) the following estimate

$$\left| \int_0^T a(s, y + s\omega)\, ds \right| \leq C \| \mathscr{A}_{a_2,k_2} - \mathscr{A}_{a_1,k_1} \|_1, \quad \forall\, y \in \mathscr{B}_r, \quad \omega \in \mathscr{S}. \quad (4.2.28)$$

Using the fact that $a = a_2 - a_1 = 0$ outside $X_{r,*}$, this entails that for all $y \in \mathscr{B}_r$ and $\omega \in \mathscr{S}$, we have

$$\left| \int_{\mathbb{R}} a(s, y + s\omega)\, ds \right| \leq C \| \mathscr{A}_{a_2,k_2} - \mathscr{A}_{a_1,k_1} \|_1. \quad (4.2.29)$$

Now, if we assume $y \in B(0, r/2)$, we get $|y + t\omega| \geqslant |t| - |y| \geq |t| - r/2$. Hence, one can see that
$(t, y + t\omega) \notin \mathscr{C}_r^+$, if $t > r/2$. On the other hand, we have $(t, y + t\omega) \notin \mathscr{C}_r^+$, if $t \leqslant r/2$. Thus, we conclude that $(t, y + t\omega) \notin \mathscr{C}_r^+ \supset X_{r,*}$ for $t \in \mathbb{R}$. This and the fact that $a = a_2 - a_1 = 0$ outside $X_{r,*}$, entails that for all $y \in B(0, r/2)$ and $\omega \in \mathscr{S}$, we have

$$a(t, y + t\omega) = 0, \ \forall t \in \mathbb{R}.$$

By a similar way, we prove for $|y| \geqslant T - r/2$, that $(t, y + t\omega) \notin \mathscr{C}_r^- \supset X_{r,*}$ for $t \in \mathbb{R}$ and then $a(t, y + t\omega) = 0$. Hence, we conclude that

$$\left| \int_{\mathbb{R}} a(s, y + s\omega)\, ds \right| \leq C \| \mathscr{A}_{a_1,k_1} - \mathscr{A}_{a_2,k_2} \|_1, \quad \forall\, \omega \in \mathscr{S},\ y \notin \mathscr{B}_r. \quad (4.2.30)$$

Thus, by (4.2.28) and (4.2.30) we finish the proof of the lemma by getting

$$\left| \int_{\mathbb{R}} a(s, y + s\omega)\, ds \right| \leq C \| \mathscr{A}_{a_1,k_1} - \mathscr{A}_{a_2,k_2} \|_1, \quad \forall\, \omega \in \mathscr{S},\ y \in \mathbb{R}^n.$$

The proof of Lemma 4.2.1 is complete. □

Our goal is to obtain an estimate linking the Fourier transform with respect to (t, x) of the absorption coefficient $a = a_1 - a_2$ to the measurement $(\mathscr{A}_{a_1,k_1} - \mathscr{A}_{a_2,k_2})$ in the conic set $E = \{(\tau, \xi) \in \mathbb{R} \times \mathbb{R}^n,\ |\tau| \leq |\xi|\}$.

Lemma 4.2.3. *Let $(a_j, k_j) \in \mathscr{A}^*(M_0) \times \mathscr{K}(M_1, M_2)$, $j = 1, 2$, then there exists $C > 0$, such that the following estimate*

$$|\widehat{a}(\tau, \xi)| \leq C \|\mathscr{A}_{a_1,k_1} - \mathscr{A}_{a_2,k_2}\|_1, \tag{4.2.31}$$

holds for any $(\tau, \xi) \in E$.

Proof. Let $(\tau, \xi) \in E$ and $\zeta \in \mathscr{S}$ be such that $\xi \cdot \zeta = 0$. Setting

$$\omega = \frac{\tau}{|\xi|^2} \cdot \xi + \sqrt{1 - \frac{\tau^2}{|\xi|^2}} \cdot \zeta.$$

Then, one can see that $\omega \in \mathscr{S}$ and $\omega \cdot \xi = \tau$. On the other hand by the change of variable $x = y + t\omega$, we have for all $\xi \in \mathbb{R}^n$ and $\omega \in \mathscr{S}$, the following identity

$$\begin{aligned}
\int_{\mathbb{R}^n} \mathscr{R}(a)(y, \omega) e^{-iy\cdot\xi} dy &= \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}} a(s, y + s\omega)\, ds \right) e^{-iy\cdot\xi} dy \\
&= \int_{\mathbb{R}^n} \int_{\mathbb{R}} a(t, x) e^{-ix\cdot\xi} e^{-it\omega\cdot\xi} dx dt \\
&= \widehat{a}(\omega \cdot \xi, \xi) = \widehat{a}(\tau, \xi),
\end{aligned}$$

where we have set $(\omega \cdot \xi, \xi) = (\tau, \xi) \in E$. Bearing in mind that for any $t \in \mathbb{R}$, $\operatorname{Supp} a(t, \cdot) \subset \Omega \subset B(0, r/2)$, we obtain

$$\int_{\mathbb{R}^n \cap B(0, \frac{r}{2} + T)} \mathscr{R}(a)(y, \omega) e^{-iy\cdot\xi} dy = \widehat{a}(\tau, \xi).$$

Then, in view of Lemma 4.2.2, we complets the proof of the Lemma 4.2.3. □

We now give the following result which is proved in [11].

Lemma 4.2.4. *Let $\mathscr{O}$ be an open set of $B(0, 1) \subset \mathbb{R}^{n+1}$, and F an analytic function in $B(0, 2)$, such that: there exist constant M, $\eta > 0$ such that*

$$\|\partial^\gamma F\|_{L^\infty(B(0,2))} \leq \frac{M|\gamma|!}{\eta^{|\gamma|}}, \quad \gamma \in (\mathbb{N} \cup \{0\})^{n+1}.$$

Then,

$$\|F\|_{L^\infty(B(0,1))} \leq (2M)^{1-\mu} \|F\|^{\mu}_{L^\infty(\mathscr{O})}.$$

where $\mu \in (0, 1)$ depends on n, η and $|\mathscr{O}|$. Here $|\gamma| = \gamma_1 + \cdots + \gamma_{n+1}$ and $B(0, \rho) = \{y \in \mathbb{R}^{n+1},\ |y| < \rho\}$.

We are now in position to state our main result of this section

Theorem 4.2.5. *Let $T > 2r$. There exist $C > 0$ and $\mu, m \in (0,1)$, such that if $\|\mathscr{A}_{a_1,k_1} - \mathscr{A}_{a_2,k_2}\|_1 \leq m$, we have*

$$\|a_1 - a_2\|_{H^{-1}(X_{r,*})} \leq C\left(\|\mathscr{A}_{a_1,k_1} - \mathscr{A}_{a_2,k_2}\|_1^{\mu/2} + |\log\|\mathscr{A}_{a_1,k_1} - \mathscr{A}_{a_2,k_2}\|_1|^{-1}\right), \tag{4.2.32}$$

for any $(a_j, k_j) \in \mathscr{A}^*(M_0) \times \mathscr{K}(M_1, M_2)$, $j = 1, 2$.
Moreover, if we assume that $a_j \in H^{s+1}(\Omega_T)$, $s > \frac{n-1}{2}$, with $\|a_j\|_{H^{s+1}(\Omega_T)} \leq M$, $j = 1, 2$, for some constant $M > 0$, then there exist $C > 0$ and $\mu_1 \in (0,1)$ such that

$$\|a_1 - a_2\|_{L^\infty(X_{r,*})} \leq C\left(\|\mathscr{A}_{a_1,k_1} - \mathscr{A}_{a_2,k_2}\|_1^{\mu/2} + |\log\|\mathscr{A}_{a_1,k_1} - \mathscr{A}_{a_2,k_2}\|_1|^{-1}\right)^{\mu_1}, \tag{4.2.33}$$

where C depends only on Ω, T, n and M_i, $i = 0, 1, 2$.

Proof. For a fixed $\alpha > 0$, we set $F_\alpha(\tau, \xi) = \widehat{a}(\alpha(\tau, \xi))$, for all $(\tau, \xi) \in \mathbb{R}^{n+1}$. It is easy to see that F_α is analytic and we have

$$\begin{aligned}|\partial^\gamma F_\alpha(\tau, \xi)| &= |\partial^\gamma \int_{\mathbb{R}^{n+1}} a(t,x) e^{-ix\cdot\xi} e^{-it\omega\cdot\xi} dx dt| \\ &= \left|\int_{\mathbb{R}^{n+1}} a(t,x)(-i)^{|\gamma|}\alpha^{|\gamma|}(t,x)^\gamma e^{-i\alpha(t,x)\cdot(\tau,\xi)} dx dt\right|.\end{aligned}$$

This entails that

$$|\partial^\gamma F_\alpha(\tau, \xi)| \leq \int_{\mathbb{R}^{n+1}} |a(t,x)|\alpha^{|\gamma|}(|x|^2+t^2)^{\frac{|\gamma|}{2}} dx dt \leq \|a\|_{L^1(X_{r,*})}\alpha^{|\gamma|}(2T^2)^{\frac{|\gamma|}{2}} \leq C\frac{|\gamma|!}{(T^{-1})^{|\gamma|}}e^\alpha.$$

Then, applying Lemma 4.2.4 in the set $\mathscr{O} = \overset{\circ}{E} \cap B(0,1)$ with $M = Ce^\alpha$ and $\eta = T^{-1}$, where

$$\overset{\circ}{E} = \{(\tau, \xi) \in \mathbb{R} \times \mathbb{R}^n,\ |\tau| < |\xi|\}.$$

So, we can find a constant $\mu \in (0,1)$, such that we have for all $(\tau, \xi) \in B(0,1)$, the following estimation

$$|F_\alpha(\tau,\xi)| = |\widehat{a}(\alpha(\tau,\xi))| \le Ce^{\alpha(1-\mu)}\|F_\alpha\|^{\mu}_{L^\infty(\mathscr{O})}, \quad (\tau,\xi)\in B(0,1).$$

Now, we will find an estimate for the Fourier transform of the absorption coefficient a in a suitable ball. Using the fact that $\alpha\overset{\circ}{E} = \{\alpha(\tau,\xi),\ (\tau,\xi)\in\overset{\circ}{E}\} = \overset{\circ}{E}$, we obtain for all $(\tau,\xi)\in B(0,\alpha)$

$$\begin{aligned}|\widehat{a}(\tau,\xi)| = |F_\alpha(\alpha^{-1}(\tau,\xi))| &\le Ce^{\alpha(1-\mu)}\|F_\alpha\|^{\mu}_{L^\infty(\mathscr{O})}\\ &\le Ce^{\alpha(1-\mu)}\|\widehat{a}\|^{\mu}_{L^\infty(B(0,\alpha)\cap\overset{\circ}{E})}\\ &\le Ce^{\alpha(1-\mu)}\|\widehat{a}\|^{\mu}_{L^\infty(\overset{\circ}{E})}.\end{aligned} \tag{4.2.34}$$

Our goal now is to deduce an estimate that links the unknown coefficient a to the measurement $\mathscr{A}_{a_1,k_1} - \mathscr{A}_{a_2,k_2}$. To obtain such estimate, we need first to decompose the $H^{-1}(\mathbb{R}^{n+1})$ norm of a into the following way

$$\begin{aligned}&\|a\|^{2/\mu}_{H^{-1}(\mathbb{R}^{n+1})}\\ &= \left(\int_{|(\tau,\xi)|<\alpha}(1+|(\tau,\xi)|^2)^{-1}|\widehat{a}(\tau,\xi)|^2 d\xi d\tau + \int_{|(\tau,\xi)|\ge\alpha}(1+|(\tau,\xi)|^2)^{-1}|\widehat{a}(\tau,\xi)|^2 d\xi d\tau\right)^{1/\mu}\\ &\le C\left(\alpha^{n+1}\|\widehat{a}\|_{L^\infty(B(0,\alpha))} + \alpha^{-2}\|a\|^2_{L^2(\mathbb{R}^{n+1})}\right)^{1/\mu}.\end{aligned}$$

Therefore, by using (6.5.88) and Lemma 4.2.4, we get

$$\begin{aligned}\|a\|^{2/\mu}_{H^{-1}(\mathbb{R}^{n+1})} &\le C\left(\alpha^{n+1}e^{2\alpha(1-\mu)}\|\mathscr{A}_{a_1,k_1} - \mathscr{A}_{a_2,k_2}\|_1^{2\mu} + \alpha^{-2}\right)^{1/\mu}\\ &\le C\left(\alpha^{\frac{n+1}{\mu}}e^{2\frac{\alpha(1-\mu)}{\mu}}\|\mathscr{A}_{a_1,k_1} - \mathscr{A}_{a_2,k_2}\|_1^2 + \alpha^{-2/\mu}\right)\\ &\le C\left(e^{N\alpha}\|\mathscr{A}_{a_1,k_1} - \mathscr{A}_{a_2,k_2}\|_1^2 + \alpha^{-2/\mu}\right),\end{aligned}$$

where N depends on n and μ. Next, we minimize the right hand-side of the above inequality with respect to α. We need to take α sufficiently large. So, there exists a constant $c>0$ such that if $0<\|\mathscr{A}_{a_1,k_1} - \mathscr{A}_{a_2,k_2}\|_1 < c$, and

$$\alpha = \frac{1}{N}|\log\|\mathscr{A}_{a_1,k_1} - \mathscr{A}_{a_2,k_2}\|_1|,$$

then, we have the following estimation

$$\begin{aligned}\|a\|_{H^{-1}(X_{r,*})} \le \|a\|_{H^{-1}(\mathbb{R}^{n+1})} &\le C\left(\|\mathscr{A}_{a_1,k_1} - \mathscr{A}_{a_2,k_2}\|_1 + |\log\|\mathscr{A}_{a_1,k_1} - \mathscr{A}_{a_2,k_2}\|_1|^{-2/\mu}\right)^{\mu/2}\\ &\le C\left(\|\mathscr{A}_{a_1,k_1} - \mathscr{A}_{a_2,k_2}\|_1^{\mu/2} + |\log\|\mathscr{A}_{a_1,k_1} - \mathscr{A}_{a_2,k_2}\|_1|^{-1}\right).\end{aligned}$$

Let us now consider δ such that $s > \frac{n-1}{2} + \delta$. By using Sobolev's interpolation inequality, we get

$$\begin{aligned}\|a\|_{L^\infty(X_{r,*})} &\leq C\|a\|_{H^{\frac{n+1}{2}+\delta}(X_{r,*})}\\ &\leq C\|a\|^{1-\beta}_{H^{-1}(X_{r,*})}\|a\|^{\beta}_{H^{s+1}(X_{r,*})}\\ &\leq C\|a\|^{1-\beta}_{H^{-1}(X_{r,*})},\end{aligned}$$

for some $\beta \in (0,1)$. This completes the proof. □

4.3 Determination of the absorption coefficients from boundary measurements and final observations

In this section, we will extend the stability estimates obtained in the first case to a larger region $X_{r,\sharp} \supset X_{r,*}$. We shall consider the geometric optics solutions constructed in Chapter 3, associated with a function ϕ^+ obeying $\operatorname{Supp}\phi^+(\cdot,\theta)\cap\Omega = \emptyset$. Note that this time, we have more flexibility on the choice of the support of the function ϕ^- and we don't need to assume that $\operatorname{Supp}\phi^-(\cdot,\theta)\pm T\theta\cap\Omega = \emptyset$ any more. We define the measurements in this case are given by the following response operator

$$\begin{aligned}\tilde{\mathscr{A}}_{a,k}:\quad \mathscr{L}^-_p(\Sigma^-) &\to \mathscr{K}^+_p := \mathscr{L}^+_p(\Sigma^+)\times L^p(Q)\\ f &\mapsto \tilde{\mathscr{A}}_{a,k}(f) = (u_{|\Sigma^+}, u(T,\cdot,\cdot)),\end{aligned}$$

associated to the problem (4.1.1) with $u_0 = 0$. We denote by

$$\tilde{\mathscr{A}}^1_{a,k}(f) = u_{|\Sigma^+},\quad \tilde{\mathscr{A}}^2_{a,k}(f) = u(T,\cdot,\cdot).$$

Here we will find that the absorption coefficient a can be stably recovered in a larger region if we further know the final data of the solution u of the linear Boltzmann equation (4.1.1). In the rest of this section, we define $a = a_2 - a_1$ in $\overline{\Omega}_T$ and $a = 0$ on $\mathbb{R}^{n+1}\setminus\overline{\Omega}_T$. We shall first prove the following statement:

Lemma 4.3.1. *Let $(a_j,k_j)\in\mathscr{A}^\sharp(M_0)\times\mathscr{K}(M_1,M_2)$, $j=1,2$. There exists $C>0$ such that for any $\phi^\pm\in\mathscr{C}^\infty_0(\mathbb{R}^n,\mathscr{C}(\mathscr{S}))$, with Supp $\phi^\pm(\cdot,\theta)\cap B(0,r/2)=\emptyset$, for*

all $\theta \in \mathscr{S}$, the following estimate holds true

$$\Big|\int_{\mathbb{R}^n\times\mathscr{S}} (\phi^+\phi^-)(y,\theta)\Big[\exp\Big(-\int_0^T a(s,y+s\theta)\,ds\Big)-1\Big]d\theta\,dy\Big|$$
$$\leq C\|\tilde{\mathscr{A}}_{a_1,k_1}-\tilde{\mathscr{A}}_{a_2,k_2}\|_1\|\phi^+\|_{L^1(\mathbb{R}^n\times\mathscr{S})}\|\phi^-\|_{L^\infty(\mathbb{R}^n\times\mathscr{S})}.$$

Here C depends only on Ω, T, and M_i, $i=0,1,2$.

Proof. In view of Lemma 3.4.1, and using the fact that Supp $\phi^+(\cdot,\theta)\cap\Omega=\emptyset$, there exists a G.o solution u_λ^+ to the equation

$$\begin{cases} \partial_t u+\theta\cdot\nabla u+a_2(t,x)u=\mathscr{N}_{k_2}[u](t,x,\theta) & \text{in } Q_T,\\ u(0,x,\theta)=0 & \text{in } Q,\end{cases}$$

in the following form

$$u_\lambda^+(t,x,\theta)=\varphi_\lambda^+(t,x,\theta)b_{a_2}(t,x,\theta)+\psi_\lambda^+(t,x,\theta), \tag{4.3.1}$$

corresponding to the absorption coefficients a_2 and the scattering coefficients k_2, where the remainder term $\psi_\lambda^+(t,x,\theta)$ satisfies (4.2.6). Next, we set the function

$$f_\lambda(t,x,\theta):=u_\lambda^+(t,x,\theta)_{|\Sigma^-}=\varphi_\lambda^+(t,x,\theta)b_{a_2}(t,x,\theta).$$

We denote by u_1 the solution of

$$\begin{cases} \partial_t u_1+\theta\cdot\nabla u_1+a_1(t,x)u_1=\mathscr{N}_{k_1}[u_1] & \text{in } Q_T,\\ u_1(0,x,\theta)=0 & \text{in } Q,\\ u_1(t,x,\theta)=f_\lambda(t,x,\theta) & \text{on } \Sigma^-.\end{cases}$$

Putting $u=u_1-u_\lambda^+$. Then, u is a solution to the following system

$$\begin{cases} \partial_t u+\theta\cdot\nabla u+a_1(t,x)u=\mathscr{N}_{k_1}[u]+a(t,x)u_\lambda^+-\mathscr{N}_k[u_\lambda^+] & \text{in } Q_T,\\ u(0,x,\theta)=0 & \text{in } Q,\\ u(t,x,\theta)=0 & \text{on } \Sigma^-.\end{cases} \tag{4.3.2}$$

On the other hand, Lemma 3.4.2 guarantees the existence of a G.O-solution u_λ^- to the adjoint problem

$$\partial_t u+\theta\cdot\nabla u-a_1(t,x)u=-\mathscr{N}_{k_1}^*[u](t,x,\theta) \quad \text{in } Q_T,$$

corresponding to the coefficients a_1 and k_1, in the form

$$u_\lambda^-(t,x,\theta) = \varphi_\lambda^-(t,x,\theta) b_{-a_1}(t,x,\theta) + \psi_\lambda^-(t,x,\theta), \tag{4.3.3}$$

where $\psi_\lambda^-(t,x,\theta)$ satisfies (4.2.7). We multiply the first equation of (4.3.2) by u_λ^-, and we integrate by parts, we obtain

$$\begin{aligned}\int_{Q_T} a(t,x) u_\lambda^+ u_\lambda^- \, d\theta dx dt - \int_{Q_T} \mathscr{N}_k[u_\lambda^+](t,x,\theta) u_\lambda^-(t,x,\theta) d\theta \, dx \, dt \\ = \int_{\Sigma^+} \theta \cdot \nu(x) (\tilde{\mathscr{A}}^1_{a_2,k_2} - \tilde{\mathscr{A}}^1_{a_1,k_1})(f_\lambda) u_\lambda^- \, d\theta \, dx \, dt \\ + \int_{\mathscr{S}\times\Omega} (\tilde{\mathscr{A}}^2_{a_2,k_2} - \tilde{\mathscr{A}}^2_{a_1,k_1})(f_\lambda) u_\lambda^-(T,x,\theta) \, d\theta \, dx.\end{aligned} \tag{4.3.4}$$

By replacing u_λ^+ and u_λ^- by their expressions, we obtain

$$\begin{aligned}&\Big| \int_{\mathbb{R}^n\times\mathscr{S}} (\phi^+\phi^-)(y,\theta) \Big[\exp\Big(-\int_0^T a(s, y+s\theta) ds \Big) - 1 \Big] d\theta dy \Big| \\ &\le C \Big[\|(\tilde{\mathscr{A}}^1_{a_2,k_2} - \tilde{\mathscr{A}}^1_{a_1,k_1}) f_\lambda\|_{\mathscr{L}^+_p(\Sigma^+)} + \|(\tilde{\mathscr{A}}^2_{a_2,k_2} - \tilde{\mathscr{A}}^2_{a_1,k_1}) f_\lambda\|_{L^p(Q)}\| \Big] \Big[\|u_\lambda^-\|_{\mathscr{L}^+_q(\Sigma^+)} + \|u_\lambda^-(T)\|_{L^q(Q)} \Big] \\ &\qquad \le C \|\tilde{\mathscr{A}}_{a_2,k_2} - \tilde{\mathscr{A}}_{a_1,k_1}\|_p \, \|f_\lambda\|_{\mathscr{L}^-_p(\Sigma^-)} \|z_\lambda^-\|_{\mathscr{K}^+_q},\end{aligned}$$

where

$$z_\lambda^- = (u^-_{\lambda|\Sigma^+}, u_\lambda^-(T,\cdot,\cdot)).$$

Then, by using (4.2.24) and (4.2.25) and the fact that

$$\begin{aligned}\|u_\lambda^-(T,\cdot,\cdot)\|_{L^q(Q)} &\le C \left(\|\varphi_\lambda^-(T,\cdot,\cdot)\|_{L^q(Q)} + \|\psi_\lambda^-(T,\cdot,\cdot)\|_{L^q(Q)} \right) \\ &\le C \|\phi^\pm\|_{L^\infty(\mathbb{R}^n\times\mathscr{S})}.\end{aligned} \tag{4.3.5}$$

We complete the proof of Lemma 4.3.1. □

Lemma 4.3.2. *Let $(a_j, k_j) \in \mathscr{A}^\sharp(M_0) \times \mathscr{K}(M_1, M_2)$, $j = 1, 2$, then there exists $C > 0$ such that*

$$|\mathscr{R}(a)(\theta, x)| \le C \|\tilde{\mathscr{A}}_{a_1,k_1} - \tilde{\mathscr{A}}_{a_2,k_2}\|_1, \quad \forall\, \theta \in \mathscr{S},\ x \in \mathbb{R}^n. \tag{4.3.6}$$

Proof. We consider $\phi^\pm$, such that $\operatorname{Supp} \phi^\pm(\cdot,\theta) \cap B\left(0, \frac{r}{2}\right)$. Using Lemma 4.3.1 and repeating the arguments used in Lemma 4.2.2, we obtain this estimate

$$\left|\int_{\mathbb{R}} a(s, y+s\theta)ds\right| \leq C\|\tilde{\mathscr{A}}_{a_1,k_1} - \tilde{\mathscr{A}}_{a_2,k_2}\|_1, \quad \forall\, \theta \in \mathscr{S},\ y \notin B\left(0, \frac{r}{2}\right). \quad (4.3.7)$$

Let now $y \in B\left(0, \frac{r}{2}\right)$, let us show that $a(t, y+t\theta) = 0$, for any $t \in \mathbb{R}$. Indeed, we have

$$|y+t\theta| \geq |t| - |y|.$$

So, we deduce that for all $t > \frac{r}{2}$ we have $(t, y+t\theta) \notin \mathscr{C}_r^+$. And if $t \leq \frac{r}{2}$, we have also that $(t, y+t\theta) \notin \mathscr{C}_r^+$. Since $X_{r,\sharp} = X \cap \mathscr{C}_r^+$. We have $(t, y+t\theta) \notin X_{r,\sharp}$, for any $t \in \mathbb{R}$. Then using the fact that $a = a_2 - a_1 = 0$ outside $X_{r,\sharp}$ we obtain,

$$\int_{\mathbb{R}} a(t, y+t\theta)dt = 0, \quad \forall\, y \in B\left(0, \frac{r}{2}\right). \quad (4.3.8)$$

In light of (4.3.7) and (4.3.8), the proof of Lemma 4.3.2 is completed. □

Next, using the above result and in the same way as in Section 2, we drive our second result.

Theorem 4.3.3. *Let $T > 2r$. There exist $C > 0$ and $\mu, m \in (0,1)$, such that if $\|\tilde{\mathscr{A}}_{a_1,k_1} - \tilde{\mathscr{A}}_{a_2,k_2}\|_1 \leq m$, we have*

$$\|a_1 - a_2\|_{H^{-1}(X_{r,\sharp})} \leq C\left(\|\tilde{\mathscr{A}}_{a_1,k_1} - \tilde{\mathscr{A}}_{a_2,k_2}\|_1^{\mu/2} + |\log\|\tilde{\mathscr{A}}_{a_1,k_1} - \tilde{\mathscr{A}}_{a_2,k_2}\|_1|^{-1}\right), \quad (4.3.9)$$

for any $(a_j, k_j) \in \mathscr{A}^\sharp(M_0) \times \mathscr{K}(M_1, M_2)$, $j = 1, 2$.
Moreover, if we assume that $a_j \in H^{s+1}(\Omega_T)$, $s > \frac{n-1}{2}$, with $\|a_j\|_{H^{s+1}(\Omega_T)} \leq M$, $j = 1, 2$, for some constant $M > 0$, then there exist $C > 0$ and $\mu_1 \in (0,1)$ such that

$$\|a_1 - a_2\|_{L^\infty(X_{r,\sharp})} \leq C\left(\|\tilde{\mathscr{A}}_{a_1,k_1} - \tilde{\mathscr{A}}_{a_2,k_2}\|_1^{\mu/2} + |\log\|\tilde{\mathscr{A}}_{a_1,k_1} - \tilde{\mathscr{A}}_{a_2,k_2}\|_1|^{-1}\right)^{\mu_1}, \quad (4.3.10)$$

where C depends only on Ω, T, n and M_i, $i = 0, 1, 2$.

4.4 Determination of the absorption coefficient from boundary measurements and final observation by varying the initial data

In this section we deal with the same problem treated in Section 2 and 3, except our data will be the response of the medium for all possible initial data. Here we will determine the time-dependent coefficient a over the whole domain. We define the following space $\mathscr{K}_p^- := \mathscr{L}_p^-(\Sigma^-)\times L^p(Q)$. Let's first define the operator $\mathscr{I}_{a,k}$

$$\begin{array}{rccc} \mathscr{I}_{a,k}: & \mathscr{K}_p^- & \longrightarrow & \mathscr{K}_p^+ \\ & g=(f,u_0) & \longmapsto & \mathscr{I}_{a,k}(g)=(u_{|\Sigma^+}, u(T,\cdot,\cdot)). \end{array}$$

We denote by

$$\mathscr{I}^1_{a,k}(g)=u_{|\Sigma^+}, \quad \mathscr{I}^2_{a,k}(g)=u(T,\cdot,\cdot).$$

We start by introducing the following Lemma in which we get an estimate linking the Light-ray transform of the absorption coefficient a and the associated response maps $\mathscr{I}_{a,k}$.

Lemma 4.4.1. *Let $(a_j,k_j)\in\mathscr{A}(M_0)\times\mathscr{K}(M_1,M_2)$, $j=1,2$, then there exists $C>0$ such that*

$$|\mathscr{R}(a)(\theta,x)|\le C\|\mathscr{I}_{a_1,k_1}-\mathscr{I}_{a_2,k_2}\|_1, \qquad \forall\ \theta\in\mathscr{S},\ x\in\mathbb{R}^n. \tag{4.4.1}$$

Proof. Let $\phi^\pm\in\mathscr{C}_0^\infty(\mathbb{R}^n,\mathscr{C}(\mathscr{S}))$. For λ sufficiently large, Lemma 3.4.2 guarantees the existence of G.O-solution u_λ^+ to

$$\partial_t u+\theta\cdot\nabla u+a_2(t,x)u=\mathscr{N}_{k_2}[u](t,x,\theta)$$

in the following form

$$u_\lambda^+(t,x,\theta)=\varphi_\lambda^+(t,x,\theta)b_{a_2}(t,x,\theta)+\psi_\lambda^+(t,x,\theta), \tag{4.4.2}$$

corresponding to the coefficients a_2 and k_2, here the remainder term $\psi_\lambda^+(t,x,\theta)$ satisfies (3.4.11), (3.4.12). Now, we denote by f_λ the function

$$f_\lambda(t,x,\theta) := u_\lambda^+(t,x,\theta)_{|\Sigma^-} = \varphi_\lambda^+(t,x,\theta) b_{a_2}(t,x,\theta) \quad \text{and} \quad u_0(x,\theta) = u_\lambda^+(0,x,\theta).$$

We remark here that u_0 not necessary zero since (4.2.8) not necessary satisfied. We denote by u_1 the solution of

$$\begin{cases} \partial_t u_1 + \theta \cdot \nabla u_1 + a_1(t,x) u_1 = \mathscr{N}_{k_1}[u_1] & \text{in } Q_T, \\ u_1(0,x,\theta) = u_0(x,\theta) & \text{in } Q, \\ u_1(t,x,\theta) = f_\lambda(t,x,\theta) & \text{on } \Sigma^-. \end{cases}$$

Putting $u = u_1 - u_\lambda^+$. Then, u is a solution to the following system

$$\begin{cases} \partial_t u + \theta \cdot \nabla u + a_1(t,x) u = \mathscr{N}_{k_1}[u] + a(t,x) u_\lambda^+ - \mathscr{N}_k[u_\lambda^+] & \text{in } Q_T, \\ u(0,x,\theta) = 0 & \text{in } Q, \\ u(t,x,\theta) = 0 & \text{on } \Sigma^-, \end{cases} \tag{4.4.3}$$

where $a = a_2 - a_1$ and $k = k_2 - k_1$. Applying Lemma 3.4.2, once more for λ large enough, we may find a G.O-solution u_λ^- to the backward problem of (4.1.1)

$$\partial_t u + \theta \cdot \nabla u - a_1(t,x) u = -\mathscr{N}_{k_1}^*[u](t,x,\theta),$$

corresponding to the coefficients a_1 and k_1, in the form

$$u_\lambda^-(t,x,\theta) = \varphi_\lambda^-(t,x,\theta) b_{-a_1}(t,x,\theta) + \psi_\lambda^-(t,x,\theta), \tag{4.4.4}$$

where $\psi_\lambda^-(t,x,\theta)$ satisfies (4.2.7). Multiplying the equation (4.4.3) by u_λ^-, integrating by parts and using Green's formula, we get

$$\begin{aligned} \int_{Q_T} a(t,x) u_\lambda^+ u_\lambda^- \, d\theta dx dt - \int_{Q_T} \mathscr{N}_k[u_\lambda^+](t,x,\theta) u_\lambda^-(t,x,\theta) d\theta \, dx \, dt \\ = \int_{\Sigma^+} \theta \cdot \nu(x) (\mathscr{I}^1_{a_2,k_2} - \mathscr{I}^1_{a_1,k_1})(g_\lambda) u_\lambda^- \, d\theta \, dx \, dt \\ + \int_{\mathscr{S} \times \mathbb{R}^n} (\mathscr{I}^2_{a_2,k_2} - \mathscr{I}^2_{a_1,k_1})(g_\lambda) u_\lambda^-(T,x,\theta) \, d\theta \, dx, \end{aligned} \tag{4.4.5}$$

where

$$g_\lambda = (u^+_{\lambda|\Sigma^-}, u_\lambda^+(0,\cdot,\cdot)).$$

Now, replacing u_λ^+ and u_λ^- by their expressions and proceeding as in the proof of Lemma 4.3.1, we get

$$\Big|\int_{\mathbb{R}^n\times\mathscr{S}}(\phi^+\phi^-)(y,\theta)\Big[\exp\Big(-\int_0^T a(s,y+s\theta)\,ds\Big)-1\Big]d\theta\,dy\Big|$$
$$\leq C\|\mathscr{I}_{a_2,k_2}-\mathscr{I}_{a_1,k_1}\|_1\|\phi^+\|_{L^1(\mathbb{R}^n\times\mathscr{S})}\|\phi^-\|_{L^\infty(\mathbb{R}^n\times\mathscr{S})}.$$

Now, in order to complete the proof of Lemma 4.4.1, it will be enough to fix $y\in\mathbb{R}^n$, consider ϕ^+ defined as before, and proceed as in the proof of Lemma 4.3.2. □

Next, by repeating the arguments used in the previous sections, we drive the result of this part as following:

Theorem 4.4.2. *Let $T>2r$. There exist $C>0$ and $\mu,m\in(0,1)$, such that if $\|\mathscr{I}_{a_1,k_1}-\mathscr{I}_{a_2,k_2}\|_1\leq m$, we have*

$$\|a_1-a_2\|_{H^{-1}(\Omega_T)}\leq C\Big(\|\mathscr{I}_{a_1,k_1}-\mathscr{I}_{a_2,k_2}\|_1^{\mu/2}+|\log\|\mathscr{I}_{a_1,k_1}-\mathscr{I}_{a_2,k_2}\|_1|^{-1}\Big),\quad (4.4.6)$$

for any $(a_j,k_j)\in\mathscr{A}(M_0)\times\mathscr{K}(M_1,M_2)$, $j=1,2$.
Moreover, if we assume that $a_j\in H^{s+1}(\Omega_T)$, $s>\frac{n-1}{2}$, with $\|a_j\|_{H^{s+1}(\Omega_T)}\leq M$, $j=1,2$, for some constant $M>0$, then there exist $C>0$ and $\mu_1\in(0,1)$ such that

$$\|a_1-a_2\|_{L^\infty(\Omega_T)}\leq C\Big(\|\mathscr{I}_{a_1,k_1}-\mathscr{I}_{a_2,k_2}\|_1^{\mu/2}+|\log\|\mathscr{I}_{a_1,k_1}-\mathscr{I}_{a_2,k_2}\|_1|^{-1}\Big)^{\mu_1},\quad (4.4.7)$$

where C depends only on Ω,T,n and M_i, $i=0,1,2$.

Chapter 5

Stability in the determination of a time-dependent coefficients for the linear Boltzmann equation with a finite number of measurements

5.1 Introduction

In this chapter, we study the stability in the inverse problem of determining the time dependent coefficients appearing in the linear Boltzmann equation (3.2.1), via a finite number of measurements on the boundary. We prove in dimension $n \geq 2$ a lipshitz stability estimate in the determination of the absorption coefficient appearing in the linear Boltzmann equation from a finite number of data measured on a boundary. Here, analogously to the previous chapter, we will use the same notation and the same three cases as in Chapter 4. For this, we need to recall some notion given in Chapter 4. Let $r > 0$ be such that $T > 2r$, such that $\Omega \subseteq B(0, r/2) = \{x \in \mathbb{R}^n, |x| < r/2\}$. We set $X_r = (0, T) \times B(0, r/2)$ and we consider the annular region around the domain Ω,

$$\mathscr{B}_r = \left\{ x \in \mathbb{R}^n,\ \frac{r}{2} < |x| < T - \frac{r}{2} \right\}.$$

We define the *forward* and *backward* light cone:

$$\mathscr{C}_r^+ = \left\{(t,x) \in X_r,\ |x| < t - \frac{r}{2},\ t > \frac{r}{2}\right\}, \tag{5.1.1}$$

$$\mathscr{C}_r^- = \left\{(t,x) \in X_r,\ |x| < T - \frac{r}{2} - t,\ T - \frac{r}{2} > t\right\}. \tag{5.1.2}$$

We denote

$$X_{r,\sharp} = \Omega_T \cap \mathscr{C}_r^+, \qquad \text{and} \quad X_{r,*} = \Omega_T \cap \mathscr{C}_r^+ \cap \mathscr{C}_r^-.$$

5.2 Stability result of the absorption coefficient in a specific subset of the whole domain

In this section, we will prove a stability result of the time-dependent absorption coefficient a in a specific subset $X_{r,*}$ of the whole domain, without any assumptions about the initial or final data of the solution. In the rest of this section, we extend a in $\mathbb{R}^{n+1}$ by $a = a_2 - a_1$ in $\overline{\Omega}_T$ and $a = 0$ on $\mathbb{R}^{n+1} \setminus \overline{\Omega}_T$. This is by means of the geometric optics solutions introduced in Chapter 3, associated with a function $\phi^{\pm}$, where assume that Supp $\phi^{\pm}(\cdot,\theta) \subset \mathscr{B}_r$, in such a way we have

$$\text{Supp}(\phi^+)(\cdot,\theta) \cap \Omega = \emptyset \quad \forall\, \theta \quad \text{Supp}(\phi^-)(\cdot,\theta) \pm T\theta) \cap \Omega = \emptyset, \quad \forall\, \theta \in \mathscr{S}.$$

We consider the admissible set of the absorption coefficient a in $X_{r,*}$. Given $a_0 \in \mathscr{C}^1(\overline{\Omega}_T)$, let $\left\{\rho_1^*, \ldots, \rho_k^*\right\}$ the linearly independent set in $C^1(X_{r,*})$. We denote

$$\chi_k^* = \text{Span}\left\{\rho_1^*, \cdots, \rho_k^*\right\}.$$

We set

$$\chi_k^*(M, a_0) = \{a \in \chi_k^*(M),\ a = a_0 \text{ in } X_r \setminus X_{r,*};\ \|a\|_{L^\infty(\Omega_T)} \le M\},$$

where

$$\chi_k^*(M) = \{a \in \chi_k^*;\ \text{such that } a \in C^1(\overline{\Omega}_T)\}.$$

We start with the following lemma, which is given in [41], will be essential for the proof of our main result.

Lemma 5.2.1. *There exist $\omega_j \in \mathscr{S}$ and $\phi_j \in \mathscr{C}_0^\infty(\mathscr{B}_r)$, $j = 1, \cdots, k$ such that the matrix $A^* = (a^*_{ij})_{i,j}$, with entries defined by*

$$a^*_{ij} := \int_{\mathbb{R}^n} \mathscr{R}(\rho^*_i)(\omega_j, x)\phi_j^2(x)dx,$$

is invertible. Here

$$\mathscr{R}(\rho^*_i)(\omega_j, x) := \int_{\mathbb{R}} \rho^*_i(t, x - t\omega_j)dt.$$

Next, we show an estimate which relates the differential of the two absorption coefficients to the albedo operator.

Lemma 5.2.2. *Let $(a_j, k_j) \in \chi^*(M, a_0) \times \mathscr{K}(M_1, M_2)$, $j = 1, 2$ such that $k_j = a_j k_0$, with $k_0 \in \mathscr{K}(M_1, M_2)$ and $\omega \in \mathscr{S}$. There exists $C_1 > 0$, such that for any $\phi_1 \in \mathscr{C}_0^\infty(\mathscr{B}_r)$ and any large $\lambda \geq 0$, there exist $f_\lambda \in L^1(\Sigma^-)$ such that the following estimate holds true*

$$\Big| \int_{\mathbb{R}^n} \int_{\mathbb{R}} \phi_1^2(y) a(s, y + s\omega)\, ds \Big| \leq C(\lambda)\|a\|_{L^\infty(\Omega_T)} + C_1 \| \big(\tilde{\mathscr{A}}_{a_2,k_2} - \tilde{\mathscr{A}}_{a_1,k_1}\big)(f_\lambda)(\cdot, \cdot, \omega)) \|_{L^1(\Sigma^+_\omega)}.$$

Here $C(\lambda) \to 0$, $\lambda \to \infty$ and C_1 depends only on Ω, T, M and M_i, $i = 1, 2$ and $\tilde{\mathscr{A}}_{a_j,k_j}(f_\lambda)$ denotes the zero extension of $\mathscr{A}_{a_j,k_j}(f_\lambda)$ on $\partial\Omega$ and Σ^+_ω is given by

$$\Sigma^+_\omega = \{(t, x) \in (0, T) \times \partial\Omega \,:\, \omega \cdot \nu(x) > 0\}.$$

Proof. In view of Lemma 3.4.1, and using the fact that Supp $\phi^+(\cdot, \theta) \cap \Omega = \emptyset$, there exists a G.O-solution u^+_λ to the equation

$$\begin{cases} \partial_t u + \theta \cdot \nabla u + a_2(t, x)u = \mathscr{N}_{k_2}[u](t, x, \theta) & (t, x, \theta) \in Q_T, \\ u(0, x, \theta) = 0 & (t, x) \in Q, \end{cases}$$

in the following form

$$u^+_\lambda(t, x, \theta) = \varphi^+_\lambda(t, x, \theta) b_{a_2}(t, x, \theta) + \psi^+_\lambda(t, x, \theta),$$

corresponding to the coefficients a_2 and k_2, where $\psi^+_\lambda(t, x, \theta)$ satisfies (3.4.11) and (3.4.12). Next, let us denote by f_λ the function

$$f_\lambda(t, x, \theta) := u^+_\lambda(t, x, \theta)_{|\Sigma^-} = \varphi^+_\lambda(t, x, \theta) b_{a_2}(t, x, \theta)_{|\Sigma^-}.$$

We denote by u_1 the solution of

$$\begin{cases} \partial_t u_1 + \theta \cdot \nabla u_1 + a_1(t,x)u_1 = \mathcal{N}_{k_1}[u_1] & (t,x,\theta) \in Q_T, \\ u_1(0,x,\theta) = 0 & (x,\theta) \in Q, \\ u_1(t,x,\theta) = f_\lambda(t,x,\theta) & (t,x,\theta) \in \Sigma^-. \end{cases}$$

Putting $u = u_1 - u_\lambda^+$. Then, u is a solution to the following system

$$\begin{cases} \partial_t u + \theta \cdot \nabla u + a_1(t,x)u = \mathcal{N}_{k_1}[u] + a(t,x)u_\lambda^+ - \mathcal{N}_k[u_\lambda^+] & (t,x,\theta) \in Q_T, \\ u(0,x,\theta) = 0 & (x,\theta) \in Q, \\ u(t,x,\theta) = 0 & (t,x,\theta) \in \Sigma^-, \end{cases} \tag{5.2.1}$$

where $a = a_2 - a_1$ and $k = k_2 - k_1 = ak_0$. On the other hand by Lemma 3.4.2 and the fact that $(\text{Supp } \phi_1(\cdot,\theta) \pm T\theta) \cap \Omega = \emptyset$, we can find of a solution u_λ^- to the backward problem

$$\begin{cases} \partial_t u + \theta \cdot \nabla u - a_1(t,x)u = -\mathcal{N}_{k_1}^*[u](t,x,\theta) & \text{in } Q_T, \\ u(T,x,\theta) = 0 & \text{in } Q, \end{cases}$$

corresponding to the coefficients a_1 and k_1, in the form

$$u_\lambda^-(t,x,\theta) = \varphi_\lambda^-(t,x,\theta) b_{-a_1}(t,x,\theta) + \psi_\lambda^-(t,x,\theta), \tag{5.2.2}$$

here the remainder term $\psi_\lambda^-(t,x,\theta)$ satisfies the estimates (3.4.19) and (3.4.20). Multiplying the first equation of (5.2.1) by u_λ^-, integrating by parts and using Green's formula, we obtain

$$\begin{aligned} \int_{Q_T} a(t,x)u_\lambda^+ u_\lambda^- \, d\theta \, dx \, dt - \int_{Q_T} \mathcal{N}_k[u_\lambda^+](t,x,\theta) u_\lambda^-(t,x,\theta) d\theta \, dx \, dt \\ = \int_{\Sigma^+} \theta \cdot \nu(x)(\mathcal{A}_{a_2,k_2} - \mathcal{A}_{a_1,k_1})(f_\lambda) u_\lambda^- \, d\theta \, dx \, dt. \end{aligned} \tag{5.2.3}$$

By replacing u_λ^+ and u_λ^- by their expressions, we obtain

$$\begin{aligned} \int_{Q_T} a(t,x)u_\lambda^+ u_\lambda^- d\theta \, dx \, dt = \int_{Q_T} a(t,x)\varphi_\lambda^+(t,x,\theta)\varphi_\lambda^-(t,x,\theta) b_a(t,x,\theta) d\theta \, dx \, dt \\ + \int_{Q_T} a(t,x)\varphi_\lambda^-(t,x,\theta) b_{-a_1}(t,x,\theta)\psi_\lambda^+(t,x,\theta) d\theta \, dx \, dt \end{aligned}$$

$$
\begin{aligned}
&+\int_{Q_T} a(t,x)\varphi_\lambda^+(t,x,\theta)b_{a_2}(t,x,\theta)\psi_\lambda^-(t,x,\theta)d\theta\,dx\,dt\\
&+\int_{Q_T} a(t,x)\psi_\lambda^-(t,x,\theta)\psi_\lambda^+(t,x,\theta)d\theta\,dx\,dt\\
&=\int_{Q_T} a(t,x)\varphi_\lambda^+(t,x,\theta)\varphi_\lambda^-(t,x,\theta)b_a(t,x,\theta)d\theta\,dx\,dt+\mathscr{I}_{1,\lambda},
\end{aligned}
\tag{5.2.4}
$$

where $\mathscr{I}_{1,\lambda}$ represents the sum of the integrals that contain terms with ψ_λ^+ and ψ_λ^-.

Firstly, we will prove that there exists $C(\lambda)$, where $C(\lambda) > 0$, $C(\lambda) \to 0$ as $\lambda \to \infty$, such that

$$
|\mathscr{I}_{1,\lambda}| + \left|\int_{Q_T} \mathscr{N}_k[u_\lambda^+]u_\lambda^- d\theta\,dx\,dt\right| \le C(\lambda)\|a\|_{L^\infty(\Omega_T)}. \tag{5.2.5}
$$

By (3.4.12) and (3.4.20), we get $\lambda \ge 0$,

$$
\begin{aligned}
|\mathscr{I}_{1,\lambda}| &\le C\|a\|_{L^\infty(\Omega_T)}\left(\left\|\psi_\lambda^+\right\|_{L^2(Q_T)} + \left\|\psi_\lambda^-\right\|_{L^2(Q_T)}\right)\\
&\le C_1(\lambda)\|a\|_{L^\infty(\Omega_T)}, \quad C_1(\lambda)\to 0,\ \lambda\to\infty.
\end{aligned}
\tag{5.2.6}
$$

Moreover, we have

$$
\begin{aligned}
\int_{Q_T} \mathscr{N}_k[u_\lambda^+]u_\lambda^- d\theta\,dx\,dt &= \int_{Q_T} \mathscr{N}_k[\varphi_\lambda^+ b_{a_2}](t,x,\theta)\varphi_\lambda^-(t,x,\theta)b_{-a_1}(t,x,\theta)d\theta\,dx\,dt\\
&+\int_{Q_T} \mathscr{N}_k[\psi_\lambda^+](t,x,\theta)\varphi_\lambda^-(t,x,\theta)b_{-a_1}(t,x,\theta)d\theta\,dx\,dt\\
&+\int_{Q_T} \mathscr{N}_k[\varphi_\lambda^+ b_{a_2}](t,x,\theta)\psi_\lambda^-(t,x,\theta)d\theta\,dx\,dt\\
&+\int_{Q_T} \mathscr{N}_k[\psi_\lambda^+](t,x,\theta)\psi_\lambda^-(t,x,\theta)d\theta\,dx\,dt\\
&=\int_{Q_T} a(t,x)\mathscr{N}_{k_0}[\varphi_\lambda^+ b_{a_2}](t,x,\theta)\varphi_\lambda^-(t,x,\theta)b_{-a_1}(t,x,\theta)d\theta\,dx\,dt+\mathscr{I}_{2,\lambda},
\end{aligned}
$$

where

$$
\begin{aligned}
\mathscr{I}_{2,\lambda} &= \int_{Q_T} a(t,x)\mathscr{N}_{k_0}[\psi_\lambda^+](t,x,\theta)\varphi_\lambda^-(t,x,\theta)b_{-a_1}(t,x,\theta)d\theta\,dx\,dt\\
&\quad+\int_{Q_T} a(t,x)\mathscr{N}_{k_0}[\varphi_\lambda^+ b_{a_2}](t,x,\theta)\psi_\lambda^-(t,x,\theta)d\theta\,dx\,dt
\end{aligned}
$$

$$+\int_{Q_T} a(t,x)\mathcal{N}_{k_0}[\psi_\lambda^+](t,x,\theta)\psi_\lambda^-(t,x,\theta)d\theta\,dx\,dt.$$

By (1.3.9), we get

$$|\mathscr{I}_{2,\lambda}| \le C\|a\|_{L^\infty(\Omega_T)}\Big(\left\|\psi_\lambda^+\right\|_{L^2(Q_T)} + \left\|\psi_\lambda^-\right\|_{L^2(Q_T)}\Big). \tag{5.2.7}$$

Then by (3.4.12) and (3.4.20), we have

$$|\mathscr{I}_{2,\lambda}| \le C_1(\lambda)\|a\|_{L^\infty(\Omega_T)}, \quad C_1(\lambda)\to 0,\ \lambda\to\infty. \tag{5.2.8}$$

On the other hand, by the Cauchy-Schwartz inequality, we obtain

$$\left|\int_{Q_T} a(t,x)\mathcal{N}_{k_0}[\varphi_\lambda^+ b_{a_2}](t,x,\theta)\varphi_\lambda^-(t,x,\theta)b_{-a_1}(t,x,\theta)d\theta\,dx\,dt\right| \le C\|a\|_{L^\infty(\Omega_T)}\left\|\mathcal{N}_{k_0}[\varphi_\lambda^+ b_{a_2}]\right\|_{L^2(Q_T)}, \tag{5.2.9}$$

and by (1.3.10), we have

$$\int_{Q_T} a(t,x)\mathcal{N}_{k_0}[\varphi_\lambda^+ b_{a_2}](t,x,\theta)\varphi_\lambda^-(t,x,\theta)b_{-a_1}(t,x,\theta)d\theta\,dx\,dt \le C_2(\lambda)\|a\|_{L^\infty(\Omega_T)},$$
$$C_2(\lambda)\to 0,\ \lambda\to\infty. \tag{5.2.10}$$

In light of (5.2.4), we have

$$\int_{Q_T} a(t,x)\varphi^+(t,x,\theta)\varphi^-(t,x,\theta)b_a(t,x,\theta)d\theta\,dx\,dt = \int_{\Sigma^+}\theta\cdot\nu(x)(\mathscr{A}_{a_2,k_2}-\mathscr{A}_{a_1,k_1})(f_\lambda)u_\lambda^-\,d\theta\,dx\,dt - \mathscr{I}_\lambda, \tag{5.2.11}$$

where the remainder term $\mathscr{I}_\lambda$ is given by

$$\mathscr{I}_\lambda = \mathscr{I}_{1,\lambda} - \mathscr{I}_{2,\lambda} - \int_{Q_T}\mathcal{N}_k[\varphi_\lambda^+ b_{a_2}](t,x,\theta)\varphi_\lambda^-(t,x,\theta)b_{-a_1}(t,x,\theta)d\theta\,dx\,dt.$$

We get from (5.2.6), (5.2.8) and (5.2.10)

$$|\mathscr{I}_\lambda| \le C(\lambda)\|a\|_{L^\infty(\Omega_T)}, \quad C(\lambda)\to 0,\ \lambda\to\infty. \tag{5.2.12}$$

Then, we conclude that there exists $C(\lambda)$ verifies (5.2.5).

Thus, by (5.2.11) and (5.2.5), we get

$$\Big|\int_{Q_T} a(t,x)\varphi^+\varphi^-(t,x,\theta)b_a(t,x,\theta)d\theta\,dx\,dt\Big| \leq C(\lambda)\|a\|_{L^\infty(\Omega_T)} + \int_{\Sigma^+} |(\mathscr{A}_{a_2,k_2} - \mathscr{A}_{a_1,k_1})(f_\lambda)u_\lambda^-|\,d\theta\,dx\,dt.$$

Next, using the fact $a(t,\cdot) = 0$ outside Ω and by changing of variables $y = x - t\theta$, we get

$$\Big|\int_{Q_T} a(t,x)\varphi^+\varphi^-(t,x,\theta)b_a(t,x,\theta)d\theta\,dx\,dt\Big| = \Big|\int_0^T\int_{\mathbb{R}^n\times\mathscr{S}} a(t,y+t\theta)\phi^+(y,\theta)\phi^-(y,\theta)\exp\Big(-\int_0^t a(s,y+s\theta)\,ds\Big)d\theta dy\,dt\Big|,$$

and bearing in mind that

$$\begin{aligned}&\int_0^T\int_{\mathbb{R}^n\times\mathscr{S}} a(t,y+t\theta)\phi^+(y,\theta)\phi^-(y,\theta)\exp\Big(-\int_0^t a(s,y+s\theta)\,ds\Big)d\theta\,dy\,dt\\ &= -\int_0^T\int_{\mathbb{R}^n\times\mathscr{S}} \phi^+(y,\theta)\phi^-(y,\theta)\frac{d}{dt}\Big[\exp\Big(-\int_0^t a(s,y+s\theta)\,ds\Big)\Big]d\theta\,dy\,dt\\ &= -\int_{\mathbb{R}^n\times\mathscr{S}} (\phi^+\phi^-)(y,\theta)\Big[\exp\Big(-\int_0^T a(s,y+s\theta)\,ds\Big)-1\Big]d\theta\,dy,\end{aligned}$$

then, we get the following estimate

$$\Big|\int_{\mathbb{R}^n\times\mathscr{S}} (\phi^+\phi^-)(y,\theta)\Big[\exp\Big(-\int_0^T a(s,y+s\theta)\,ds\Big)-1\Big]d\theta\,dy\Big| \leq C(\lambda)\|a\|_{L^\infty(\Omega_T)} + \int_{\Sigma^+} |(\mathscr{A}_{a_2,k_2} - \mathscr{A}_{a_1,k_1})(f_\lambda)u_\lambda^-|\,d\theta\,dx\,dt.$$

Now, let $\omega \in \mathscr{S}$ and $\phi_j \in \mathscr{C}_0^\infty(\mathscr{B}_r)$, $j = 1,2$. Moreover, let $\varrho_h(\omega,\cdot)$ defined by (1.4.27). Selecting

$$\phi^+(y,\theta) = \varrho_h(\omega,\theta)\phi_1(y), \quad \phi^-(y,\theta) = \phi_1(y).$$

Then, we have

$$\Big|\int_{\mathbb{R}^n\times\mathscr{S}} \varrho_h(\omega,\theta)\phi_1^2(y)\Big[\exp\Big(-\int_0^T a(s,y+s\theta)\,ds\Big)-1\Big]d\theta\,dy\Big|$$

$$\leq C(\lambda)\|a\|_{L^\infty(\Omega_T)} + \int_{\Sigma^+} |(\mathscr{A}_{a_2,k_2} - \mathscr{A}_{a_1,k_1})(f_\lambda)(t,x,\theta)u_\lambda^-(t,x,\theta)|\, d\theta\, dx\, dt.$$

where

$$u_\lambda^-(t,x,\omega) = \phi_1(x - t\theta)e^{\int_0^t a_1(s,x-s\theta)ds - i\lambda(t - x\cdot\theta)}.$$

Next, by (1.4.31) we get

$$\lim_{h\to 1}\int_{\mathbb{R}^n\times\mathscr{S}} \varrho_h(\omega,\theta)\phi_1^2(y)\Big[\exp\Big(-\int_0^T a(s,y+s\theta)\,ds\Big) - 1\Big]d\theta\, dy$$
$$= \int_{\mathbb{R}^n} \phi_1^2(y)\Big[\exp\Big(-\int_0^T a(s,y+s\theta)\,ds\Big) - 1\Big]dy, \quad (5.2.13)$$

and

$$\lim_{h\to 1}\int_{\Sigma^+} |(\mathscr{A}_{a_2,k_2} - \mathscr{A}_{a_1,k_1})(f_\lambda)(t,x,\theta)u_\lambda^-(t,x,\theta)|\, d\theta\, dx\, dt$$
$$= \int_{\Sigma_\omega^+} |(\mathscr{A}_{a_2,k_2} - \mathscr{A}_{a_1,k_1})(f_\lambda)(t,x,\omega)\Psi_\lambda^-(t,x,\omega)|\, dx\, dt, \quad (5.2.14)$$

where

$$\Psi_\lambda^-(t,x,\omega) = \phi_1(x - t\omega)e^{\int_0^t a_1(s,x-s\omega)ds - i\lambda(t - x\cdot\omega)}.$$

Then, we get the following estimate

$$\int_{\mathbb{R}^n} \phi_1^2(y)\Big[\exp\Big(-\int_0^T a(s,y+s\theta)\,ds\Big) - 1\Big]dy$$
$$\leq C(\lambda)\|a\|_{L^\infty(\Omega_T)} + \int_{(0,T)\times\partial\Omega} |(\tilde{\mathscr{A}}_{a_2,k_2} - \tilde{\mathscr{A}}_{a_1,k_1})(f_\lambda)(t,x,\omega)\Psi_\lambda^-(t,x,\omega)|\, dx\, dt, \quad (5.2.15)$$

where $\tilde{\mathscr{A}}_{a_j,k_j}(f_\lambda)$ denotes the zero extension of $\mathscr{A}_{a_j,k_j}(f_\lambda)$ on $\partial\Omega$.

That is

$$\Big|\int_{\mathbb{R}^n} \phi_1^2(y)\exp\Big(-\int_0^T a(s,y+s\omega)\,ds\Big) - 1\Big| \quad (5.2.16)$$
$$\leq C(\lambda)\|a\|_{L^\infty(\Omega_T)} + C\|\big(\tilde{\mathscr{A}}_{a_2,k_2} - \tilde{\mathscr{A}}_{a_1,k_1}\big)(f_\lambda)(\cdot,\cdot,\omega)\|_{L^1(\Sigma_\omega^+)}, \quad \forall\, y\in\mathscr{B}_r,\ \forall\,\omega\in\mathscr{S}. \quad (5.2.17)$$

Next, using the fact that $|X| \leq e^M\,|e^X - 1|$ for any X real satisfying $|X| \leq M$ we found out that

$$\Big|\int_0^T a(s,y+s\omega)\,ds\Big| \leq e^{MT}\Big|\exp\Big(-\int_0^T a(s,y+s\omega)\,ds\Big) - 1\Big|,$$

where $X = \int_0^T a(s, y + s\omega)ds$.
We conclude in light of (5.2.16) the following estimate

$$\Big|\int_{\mathbb{R}^n}\int_0^T \phi_1^2(y)a(s, y + s\omega)\,ds\Big| \tag{5.2.18}$$

$$\leq C(\lambda)\|a\|_{L^\infty(\Omega_T)} + C\|\big(\tilde{\mathscr{A}}_{a_2,k_2} - \tilde{\mathscr{A}}_{a_1,k_1}\big)(f_\lambda)(\cdot,\cdot,\omega)\|_{L^1(\Sigma_\omega^+)}, \quad \forall\, y \in \mathscr{B}_r,\ \forall\, \omega \in \mathscr{S}. \tag{5.2.19}$$

Using the fact that $a = a_2 - a_1 = 0$ outside $X_{r,*}$, this entails that for all $y \in \mathscr{B}_r$ and $\omega \in \mathscr{S}$, we have

$$\Big|\int_{\mathbb{R}^n}\int_{\mathbb{R}} \phi_1^2(y)a(s, y + s\omega)\,ds\Big| \tag{5.2.20}$$

$$\leq C(\lambda)\|a\|_{L^\infty(\Omega_T)} + C\|\big(\tilde{\mathscr{A}}_{a_2,k_2} - \tilde{\mathscr{A}}_{a_1,k_1}\big)(f_\lambda)(\cdot,\cdot,\omega)\|_{L^1(\Sigma_\omega^+)}, \quad \forall\, y \in \mathscr{B}_r,\ \forall\, \omega \in \mathscr{S}. \tag{5.2.21}$$

Now, if we assume $y \in B(0, r/2)$, we get $|y + t\omega| \geqslant |t| - |y| \geq |t| - r/2$. Hence, one can see that
$(s, y + s\omega) \notin \mathscr{C}_r^+$, if $s > r/2$. On the other hand, we have $(s, y + s\omega) \notin \mathscr{C}_r^+$, if $s \leqslant r/2$. Thus, we conclude that $(s, y + s\omega) \notin \mathscr{C}_r^+ \supset X_{r,*}$ for $s \in \mathbb{R}$. This and the fact that $a = a_2 - a_1 = 0$ outside $X_{r,*}$, entails that for all $y \in B(0, r/2)$ and $\omega \in \mathscr{S}$, we have

$$a(t, y + s\omega) = 0,\ \forall s \in \mathbb{R}.$$

By a similar way, we prove for $|y| \geqslant T - r/2$, that $(s, y + s\omega) \notin \mathscr{C}_r^- \supset X_{r,*}$ for $s \in \mathbb{R}$ and then $a(s, y + s\omega) = 0$. Hence, we conclude that

$$\Big|\int_{\mathbb{R}^n}\int_{\mathbb{R}} \phi_1^2(y)a(s, y + s\omega)\,ds\Big| \tag{5.2.22}$$

$$\leq C(\lambda)\|a\|_{L^\infty(\Omega_T)} + C\|\big(\tilde{\mathscr{A}}_{a_2,k_2} - \tilde{\mathscr{A}}_{a_1,k_1}\big)(f_\lambda)(\cdot,\cdot,\omega)\|_{L^1(\Sigma_\omega^+)}, \quad \forall\, \omega \in \mathscr{S},\ y \notin \mathscr{B}_r. \tag{5.2.23}$$

Thus, by (5.2.20) and (5.2.22), we finish the proof of the lemma by getting

$$\Big|\int_{\mathbb{R}^n}\int_{\mathbb{R}} \phi_1^2(y)a(s, y + s\omega)\,ds\Big|$$

$$\leq C(\lambda)\|a\|_{L^\infty(\Omega_T)}+C\|\left(\tilde{\mathscr{A}}_{a_2,k_2}-\tilde{\mathscr{A}}_{a_1,k_1}\right)(f_\lambda)(\cdot,\cdot,\omega)\|_{L^1(\Sigma^+_\omega)}, \quad \forall\, \omega\in\mathscr{S},\ y\in\mathbb{R}^n. \tag{5.2.24}$$

This completes the proof of the lemma. □

We are now in position to state our result.

Theorem 5.2.3. *Let $T > 2r$, then there exist $\omega_i \in \mathscr{S}$ and $f_i \in L^1(\Sigma^+_{\omega_i})$, $i = 1\cdots k$, such that*

$$\|a_1-a_2\|_{L^\infty(X_{r,*})} \leq C \sup_{1\leq i\leq k} \|\left(\tilde{\mathscr{A}}_{a_2,k_2}-\tilde{\mathscr{A}}_{a_1,k_1}\right)(f^{(i)})(\cdot,\cdot,\omega_i)\|_{L^1(\Sigma^+_{\omega_i})}, \tag{5.2.25}$$

for any $(a_j,k_j)\in\chi^*_k(M,a_0)\times\mathscr{K}(M_1,M_2)$, $j=1,2$, such that $k_j = a_jk_0$, where $k_0\in\mathscr{K}(M_1,M_2)$.

Proof. We consider in (5.2.16) ω as the direction $\omega_1,\cdots,\omega_k$, and ϕ_1 as the $\phi_1^1,\cdots,\phi^k$ given by Lemma (5.2.1). Define maps B_i for $i=1,\cdots,k$ by

$$B_i^*(a) = \int_0^T\int_\Omega \left(\phi_1^{(i)}(x+t\omega_i)\right)^2 a(t,x)dxdt, \quad a\in\chi^*_k.$$

Let $f_\lambda^{(i)}\in L^1(\Sigma^-)$ satisfy the assumptions in Lemma 5.2.2 with respect to $\phi_1^{(i)}$ and ω_i, $i=1,\cdots,k$. Then, for any $a_1,a_2\in\chi^*_k(M,a_0)$, we have

$$|B_i^*(a)| \leq C(\lambda)\|a\|_{L^\infty(\Omega_T)} + C\|\left(\tilde{\mathscr{A}}_{a_2,k_2}-\tilde{\mathscr{A}}_{a_1,k_1}\right)(f_\lambda^{(i)})(\cdot,\cdot,\omega_i)\|_{L^1(\Sigma^+_{\omega_i})}. \tag{5.2.26}$$

Let $e_1=(1,0,\cdots,0)$, $e_2=(0,1,0,\cdots,0)$, $\cdots$, $e_n=(0,0,\cdots,0,1)$ and let us define a linear map $B^*:\chi^*_k\to\mathbb{R}^k$ on the finite dimensional space χ^*_k as follows

$$B^*(a) = \sum_{j=1}^k B_j^*(a)e_j, \quad \rho\in\chi^*_k.$$

Then, we get

$$\begin{aligned}|B^*(a)| &\leq \sup_{1\leq i\leq k}|B_i^*(a)|\\ &\leq C(\lambda)\|a\|_{L^\infty(\Omega_T)} + C\sup_{1\leq i\leq k}\|\left(\tilde{\mathscr{A}}_{a_2,k_2}-\tilde{\mathscr{A}}_{a_1,k_1}\right)(f_\lambda^{(i)})(\cdot,\cdot,\omega_i)\|_{L^1(\Sigma^+_{\omega_i})}.\end{aligned}$$

Seeing that B^* is independent of λ so will be $(B^*)^{-1}$, and we have:

$$
\begin{aligned}
|a| &\leq |(B^*)^{-1}B^*(a)| \\
&\leq C(\lambda)\|a\|_{L^\infty(\Omega_T)} + C \sup_{1\leq i\leq k} \|\left(\tilde{\mathscr{A}}_{a_2,k_2} - \tilde{\mathscr{A}}_{a_1,k_1}\right)(f_\lambda^{(i)})(\cdot,\cdot,\omega_i)\|_{L^1(\Sigma^+_{\omega_i})}, \quad (5.2.27)
\end{aligned}
$$

So, we have

$$
\|a\|_{L^\infty(\Omega_T)} \leq C(\lambda)\|a\|_{L^\infty(\Omega_T)} + C \sup_{1\leq i\leq k} \|\left(\tilde{\mathscr{A}}_{a_2,k_2} - \tilde{\mathscr{A}}_{a_1,k_1}\right)(f_\lambda^{(i)})(\cdot,\cdot,\omega_i)\|_{L^1(\Sigma^+_{\omega_i})}. \tag{5.2.28}
$$

We now choose λ such that $C(\lambda) \leq 1/2$, we complete the proof of the Theorem.

□

Remark 5.2.4. *We proceed as in previous section we can extend the previous stability result in the large regions $X_{r,\sharp}$ and on the whole domain.*

Part II

Parabolic inverse problems

Chapter 6

Stable determination of coefficient in a parabolic equation from a Dirichlet-to-Neumann map on an arbitrary small time

6.1 Introduction

In this chapter, we analyse the inverse problem of determining the zero-th order coefficient in a parabolic equation, from full or partial observation of the solution on the boundary, when the measurement is performed on in an arbitrarily small time interval. Namely, we prove log stable determination of this coefficient with respect to the full or partial Dirichlet-to-Neumann map measured on an arbitrary time subinterval of $(0, T)$. We tackle this problem with two different approaches. In the first one, we use appropriate time-shifted complex geometrical optics functions. In the second approach, we use the CGO functions introduced in [32] that we multiply by a suitable time varying truncation function.

6.2 Analysis of the forward problem

In this section we introduce some notations and we recall some known facts about the well-posedness of the forward parabolic problem.

Let Ω be a C^2-bounded domain of $\mathbb{R}^n$, $n \geq 2$. For $T > 0$, we set $Q = \Omega \times (0,T)$ and $\Sigma = \partial\Omega \times (0,T)$. We consider the following diffusion equation with boundary conditions

$$\begin{cases} \partial_t u - \Delta u + q(x)u = 0 & \text{in} \quad Q, \\ u(0,x) = 0 & \text{in} \quad \Omega, \\ u(t,x) = f(t,x), & \text{on} \quad \Sigma, \end{cases} \tag{6.2.1}$$

where $q \in L^\infty(\Omega)$. This system describes the time evolution of the heat propagation through a homogeneous body.

We consider the parabolic Dirichlet-to-Neumann map (abbreviated to D-to-N map in the following) associated to the system (6.2.1), given by

$$\Lambda_q : f \mapsto \partial_\nu u_{q|\Sigma},$$

where u_q is the solution of (6.2.1) and ν denotes the outward unit normal vector on Σ.

In order to well define the D-to-N associated to (6.2.1), we introduce $H_0^{r,s}$, $r,s > 0$ the functional space

$$H_0^{r,s}(\Sigma) = L^2(0,T;H^r(\partial\Omega)) \cap H_0^s(0,T;L^2(\partial\Omega)).$$

We also define the spaces $\mathscr{K}_\pm(Q) = \{u \in L^2(Q);\ (\pm\partial_t - \Delta)u \in L^2(Q)\}$, with norm

$$\|u\|_{\mathscr{K}_\pm(Q)} = \left(\|u\|_{L^2(Q)} + \|(\pm\partial_t - \Delta)u\|_{L^2(Q)}\right)^{1/2}.$$

Following Lions and Magenes [53], the dual space of $H_0^{r,s}(\Sigma)$ is $H^{-r,-s}(\Sigma)$, and Choulli and Kian [32] proved that the dual space of $H^{\frac{1}{2},\frac{1}{4}}(\Sigma)$ is $H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma)$.

We recall the following Lemma on the unique existence of a weak solution to the problem (6.2.1). It is know that this problem is well-posed, and we have the following result which is borrowed from (Proposition 2.3 and Proposition 2.4 [32]).

Lemma 6.2.1. *Let $q \in L^\infty(\Omega)$, $f \in H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma)$. Then the IBVP* (6.2.1) *admits a unique transposition solution $u_q \in \mathscr{K}_+(Q)$ satisfying*

$$\|u_q\|_{\mathscr{K}_+(Q)} \leq C\|f\|_{H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma)}, \tag{6.2.2}$$

where the constant C depends only on Q and M. Therefore the parabolic DtN map Λ_q is a bounded operator from $H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma)$ to $H^{-\frac{3}{4},-\frac{3}{2}}(\Sigma)$.

We also have this important Lemma shows that $\Lambda_2 - \Lambda_1$, with $q_1, q_2 \in L^\infty(\Omega)$, has a better regularity than Λ_1 and Λ_2 individually.

Lemma 6.2.2. *Let $q_1, q_2 \in L^\infty(\Omega)$. Then $\Lambda_2 - \Lambda_1$ is a bounded operator from $H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma)$ to $H^{\frac{1}{2},\frac{1}{4}}(\Sigma)$.*

Pick $T_* \in (0,T)$, $\epsilon \in (0, \min(T_*, T - T_*))$ and set $\Sigma_0 = \partial\Omega \times (T_* - \epsilon, T_* + \epsilon)$. We introduce the time localized parabolic Dirichlet-to-Neumann map associated to the system (6.2.1), given as

$$\begin{array}{rccl} \tilde{\Lambda}_q : & H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma_0) & \longrightarrow & H^{-\frac{3}{2},-\frac{3}{4}}(\Sigma_0), \\ & f & \longmapsto & \tilde{\Lambda}_q(f) = \partial_\nu u_{q|\Sigma_0}, \end{array} \tag{6.2.3}$$

where $\Sigma_0 = (T_* - \epsilon, T_* + \epsilon) \times \Gamma$, $\Gamma = \partial\Omega$, $T_* \in (0,T)$, $\epsilon \in (0, min(T_*, T - T_*))$ and u_q is the unique solution of (6.2.1) and $\nu(x)$ denotes the unit outward normal to $\partial\Gamma$ at x. It follow from Lemma 6.2.1 that $\tilde{\Lambda}_q$ is a bounded operator from $H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma_0)$ to $H^{-\frac{3}{4},-\frac{3}{2}}(\Sigma_0)$.

For $\omega \in \mathscr{S}^{n-1}$, the unit sphere of $\mathbb{R}^n$, and we introduce

$$\Gamma^\pm = \{x \in \Gamma \,:\, \pm\omega \cdot \nu(x) > 0\}, \tag{6.2.4}$$

and we denote by $\Sigma^\pm(\omega) = \Gamma^\pm \times (0,T)$. Next, we choose $\omega_0 \in \mathscr{S}^{n-1}$, and fix $\varrho > 0$, we introduce

$$\Gamma_{+,\varrho}(\omega_0) = \{x \in \Gamma;\ \nu(x) \cdot \omega_0 > \varrho\}; \quad \Gamma_{-,\varrho}(\omega_0) = \Gamma \setminus \overline{\Gamma_{+,\varrho}(\omega_0)},$$

$$\Sigma_{+,\varrho}(\omega_0) = (T_* - \epsilon, T_* + \epsilon) \times \Gamma_{+,\varrho}(\omega_0), \quad \Sigma_{-,\varrho}(\omega_0) = \Sigma \setminus \overline{\Sigma_{+,\varrho}(\omega_0)}.$$

We introduce now, the local time localized D-to-N map by

$$\begin{array}{rccl} \Lambda_q^\sharp : & H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma_0) & \longrightarrow & H^{-\frac{3}{2},-\frac{3}{4}}(\Sigma_{-,\varrho}(\omega_0)), \\ & f & \longmapsto & \Lambda_q^\sharp(f) = \partial_\nu u_{q|\Sigma_{-,\varrho}(\omega_0)}. \end{array} \tag{6.2.5}$$

It follow from Lemma 6.2.1 that $\tilde{\Lambda}_q$ is a bounded operator from $H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma_0)$ to $H^{-\frac{3}{4},-\frac{3}{2}}(\Sigma_{-,\varrho}(\omega_0))$.

6.3 Carleman type estimates

In this section, inspired by [16], we drive a parabolic version for an elliptic Carleman inequality, which is a key ingredient for building suitable CGO solutions. In order to formulate our Carleman estimate, we introduce the following weight function φ_λ, for $\omega \in \mathscr{S}^{n-1}$ and $\lambda > 0$,

$$\varphi_\lambda(t,x;\omega) = \lambda\omega \cdot x + \lambda^2 t, \quad (t,x) \in Q.$$

For the sake of simplicity, we use in the sequel the following notations

$$\mathscr{L}^2_\pm = L^2(Q, e^{\pm\varphi_\lambda} dxdt).$$

Set

$$\mathscr{M} = \{q \in L^\infty(\Omega); \quad \|q\|_{L^\infty(\Omega)} \leq M\}.$$

Then we have the following Carleman type estimate.

Lemma 6.3.1. *Let $q \in \mathscr{M}$. Then, there exist two constants $\lambda_0 > 0$ and $C > 0$, both of them depending only on M and Q, such that the following estimate:*

$$\lambda^2 \int_Q e^{-2\varphi_\lambda}|u|^2 dxdt + \int_\Omega e^{-2\varphi_\lambda}|u(T,x)|^2 dx + \lambda \int_{\Sigma^+(\omega)} (\omega \cdot \nu) e^{-2\varphi_\lambda}|\partial_\nu u|^2 d\sigma dt$$
$$\leq C \int_Q e^{-2\varphi_\lambda}|(\partial_t - \Delta + q)u|^2 dxdt - \lambda \int_{\Sigma^-(\omega)} (\omega \cdot \nu) e^{-2\varphi_\lambda}|\partial_\nu u|^2 d\sigma dt,$$

holds for every $u \in H^{2,1}(Q)$ satisfying $u_{|\Sigma} = 0$ and $u_{|t=0} = 0$, and for all $\lambda \geq \lambda_0$.

Proof. Firstly, we conjugate the operator $P = (\partial_t - \Delta)$ by the exponential weight function e^{φ_λ}

$$P_{\lambda,\varphi} v = e^{-\varphi_\lambda} P e^{\varphi_\lambda} v,$$

where v is given by

$$v(t,x) = e^{-\varphi_\lambda} u(t,x), \quad (t,x) \in Q.$$

By simple computation, we may write $P_{\lambda,\varphi}$ as

$$P_{\lambda,\varphi}v = -\Delta v + (\partial_t - 2\lambda\omega\cdot\nabla)v = P_2 v + P_1 v.$$

Next, let us denote

$$\mathscr{I}_\lambda = \int_Q e^{-2\varphi_\lambda}|(\partial_t - \Delta)u(t,x)|^2 dxdt = \int_Q |P_{\lambda,\varphi}v(t,x)|^2 dxdt.$$

Thus, we have

$$\mathscr{I}_\lambda = \int_Q \Big(|P_2 v(t,x)|^2 + |P_1 v(t,x)|^2 + 2P_2 v(t,x)P_1 v(t,x)\Big)dxdt.$$

On the other hand, we have

$$\begin{aligned}
2\int_Q P_2 v(t,x)P_1 v(t,x)dxdt &= 2\int_Q -\Delta v(\partial_t v - 2\lambda\omega\cdot\nabla v)dxdt \\
&= 2\int_Q -\Delta v\partial_t v dxdt + 4\lambda\int_Q \Delta v(\omega\cdot\nabla v)dxdt \\
&= \mathscr{I}_{1,\lambda} + \mathscr{I}_{2,\lambda}.
\end{aligned}$$

Now, by using Green's formula and the fact that $v_{|t=0} = 0$, we get

$$\begin{aligned}
\mathscr{I}_{1,\lambda} &= -2\int_Q \Delta v\partial_t v dxdt \\
&= -2\int_Q \nabla v\cdot\nabla(\partial_t v)dxdt \\
&= -\int_Q \frac{d}{dt}|\nabla v|^2 dxdt \\
&= -\int_\Omega |\nabla v(T,x)|^2 dx.
\end{aligned}$$

Using Poincaré inequality, we get

$$\int_\Omega |\nabla v(T,x)|^2 dx \geq C\int_\Omega |v(T,x)|^2 dx, \quad \text{where } C = C(\Omega).$$

Next, by a simple calculation, we get

$$
\begin{aligned}
\mathscr{I}_{2,\lambda} &= 2\lambda \int_Q \Delta v(\omega \cdot \nabla v) dxdt \\
&= 2\lambda \int_Q \sum_{k=1}^n \partial_k^2 v(t,x) \sum_{j=1}^n \omega^j \partial_j v(t,x) dxdt \\
&= 2\lambda \sum_{j,k=1}^n \int_Q \partial_k^2 v(t,x) \omega^j \partial_j v(t,x) dxdt \\
&= 2\lambda \sum_{k=1}^n \int_Q \partial_k \Big(\sum_{j=1}^n \partial_k v \omega^j \partial_j v \Big) dxdt - \lambda \sum_{k=1}^n \int_Q \sum_{j=1}^n \omega^j \partial_j |\partial_k v|^2 dxdt \\
&= 2\lambda \sum_{k=1}^n \int_Q \partial_k \big((\partial_k v) \omega \cdot \nabla v(t,x) \big) dxdt - \lambda \int_Q \operatorname{div}(\omega |\nabla v|^2) dxdt \\
&= 2\lambda \int_Q \operatorname{div}(\nabla v \omega \cdot \nabla v) dxdt - \lambda \int_Q \operatorname{div}(\omega |\nabla v|^2) dxdt, \\
&= 2\lambda \int_\Sigma (\nabla v \omega \cdot \nabla v) \cdot \nu d\sigma dt - \lambda \int_\Sigma (\omega \cdot \nu) |\nabla v|^2 d\sigma dt.
\end{aligned}
$$

Here, we have set $\omega = (\omega^1, \cdots, \omega^n)$. Since $\partial_\nu v = \nabla v \cdot \nu$ on Σ, we see that

$$
|\nabla v|^2 = |\partial_\nu v|^2, \quad \omega \cdot \nabla v = (\omega \cdot \nu) \partial_\nu v, \quad \text{on} \quad \Sigma,
$$

and by the divergence theorem, $\mathscr{I}_{2,\lambda}$ reads

$$
\begin{aligned}
\mathscr{I}_{2,\lambda} &= 2\lambda \int_\Sigma (\omega \cdot \nu) |\partial_\nu v|^2 d\sigma dt - \lambda \int_\Sigma (\omega \cdot \nu) |\partial_\nu v|^2 d\sigma dt \\
&= \lambda \int_\Sigma (\omega \cdot \nu) |\partial_\nu v|^2 d\sigma dt.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\mathscr{I}_\lambda &\geq \int_Q \Big(|P_2 v(t,x)|^2 + |P_1 v(t,x)|^2 \Big) dxdt + \int_\Omega |\nabla v(T,x)|^2 dx \\
&\geq \int_\Omega |v(T,x)|^2 dx + \int_Q |P_1 v|^2 dxdt.
\end{aligned} \tag{6.3.6}
$$

On the other hand, multiplying $P_1 v(s,x)$ by $v(s,x)$ and integrating over $(0,t) \times \Omega$, for $t \in (0,T)$ we obtain,

$$
\begin{aligned}
\int_0^t \int_\Omega \Big(P_1 v(s,x) \Big) v(s,x) dxds &= \int_0^t \int_\Omega \partial_t v(s,x) v(s,x) dxds \\
&\quad - 2\lambda \int_0^t \int_\Omega v(s,x) (\omega \cdot \nabla v(s,x)) dxds.
\end{aligned} \tag{6.3.7}
$$

Now, we set $\xi = \left(\frac{1}{\sqrt{1+4\lambda}}, \frac{-2\lambda\omega}{\sqrt{1+4\lambda}}\right) = (\xi_1, \xi') \in \mathbb{R}^{n+1}$, and we consider $v \in C_0^\infty((0,+\infty)\times\Omega)$, we have

$$v(t,x) = \int_{-\infty}^{0} \partial_s (v((t,x)+s\xi))\, ds, \quad (t,x) \in Q.$$

We denote (t,x) by y. We have

$$\begin{aligned} v(t,x) &= \int_{-\infty}^{0} \partial_s (v(y+s\xi)\, ds = \int_{-\infty}^{0} \xi \cdot \nabla_y (v(y+s\xi)\, ds \\ &= \int_{-\infty}^{0} \left(\xi_1 \partial_t v(y+s\xi) + \xi' \cdot \nabla v(y+s\xi\right) ds \\ &= \frac{1}{\sqrt{1+4\lambda}} \int_{-\infty}^{0} (\partial_t v(y+s\xi) - 2\lambda\omega \cdot \nabla v(y+s\xi))\, ds \qquad (6.3.8) \\ &= \frac{1}{\sqrt{1+4\lambda}} \int_{-\infty}^{0} (\partial_t v((t,x)+s\xi) - 2\lambda\omega \cdot \nabla v((t,x)+s\xi))\, ds \\ &= \frac{1}{\sqrt{1+4\lambda}} \int_{-\infty}^{0} \Big(P_1 v((t,x)+s\xi)\Big) ds. \end{aligned}$$

Pick, $R > 0$, so large that $\Omega \subset \{|x| \le R; x \in \mathbb{R}^n\}$. For $s < -4R$, $x \in \Omega$ and $\lambda > 1$, we see that

$$|x + s\xi'| \ge \frac{-s}{2} - |x| > R.$$

Then, for $s < -4R$ and $x \in \Omega$, we have $x + s\xi' \notin \Omega$. Therefore

$$\begin{aligned} |v(t,x)| = \Big| \int_{-\infty}^{0} \partial_s (v(y+s\xi)\, ds \Big|^2 &= \Big| \int_{-4R}^{0} \partial_s (v(t,x)+s\xi)\, ds \Big|^2 \\ &\le \int_{-4R}^{0} |\partial_s (v(t,x)+s\xi)|^2 ds \\ &\le \frac{1}{1+4\lambda} \int_{-\infty}^{0} |P_1 v((t,x)+s\xi)|^2 ds, \quad (t,x) \in Q. \qquad (6.3.9) \end{aligned}$$

Using the Cauchy Schwarz inequality, we obtain

$$\begin{aligned} \int_Q |v(t,x)| dx dt &\le 4R \int_Q \int_{-4R}^{0} |\partial_s (v(t,x)+s\xi)|^2 ds dx dt \\ &= \frac{4R}{1+4\lambda} \int_0^t \int_\Omega \int_{-4R}^{0} \Big| P_1 v((t,x)+s\xi) \Big| ds dx dt \\ &= \frac{4R}{1+4\lambda} \int_0^T \int_\Omega \int_{-4R}^{0} \Big| P_1 v((t+s\xi_1, x+s\xi') \Big| ds dx dt \end{aligned}$$

$$= \frac{4R}{1+4\lambda}\int_{\mathbb{R}}\int_{\mathbb{R}^n}\int_{-4R}^{0}\Big|P_1 v((t+s\xi_1, x+s\xi')\Big| ds dx dt.$$

Using the change of variables, $\tau = t+s\xi_1$, $\mu = x+s\xi'$, and we applying Fubini's theorem, we get

$$\int_Q |v(t,x)| \leq \frac{4R}{1+4\lambda}\int_{-4R}^{0}\int_{\mathbb{R}}\int_{\mathbb{R}^n}\Big|P_1 v((\tau,\mu)\Big| d\tau d\mu ds \leq \frac{16R^2}{1+4\lambda}\int_Q \Big|P_1 v((t,x)\Big| dx dt. \tag{6.3.10}$$

Now, by inserting (6.3.10) into (6.3.6), we get that

$$\mathscr{I}_\lambda \geq C\int_Q \lambda^2 |v(t,x)|^2 dx dt + \int_\Omega |v(T,x)|^2 dx + \lambda\int_\Sigma (\omega\cdot\nu)|\partial_\nu v|^2 d\sigma dt.$$

Bearing in mind that we have $v(t,x) = e^{-\lambda\varphi}u(t,x)$, then we obtain all for $q \in \mathscr{M}$,

$$\mathscr{I}_\lambda = \int_Q e^{-2\varphi_\lambda}|(\partial_t - \Delta)u(t,x)|^2 dx dt \leq \int_Q e^{-2\varphi_\lambda}|(\partial_t - \Delta + q(x))u(t,x)|^2 dx dt + M\int_Q e^{-2\varphi_\lambda}|u(t,x)|^2 dx dt. \tag{6.3.11}$$

Thus, we have

$$\begin{aligned} C\int_Q e^{-2\varphi_\lambda}\lambda^2|u(t,x)|^2 dx dt + \int_\Omega e^{-2\varphi_\lambda}|u(T,x)| dx + \lambda\int_{\Sigma^+(\omega)}(\omega\cdot\nu)e^{-2\varphi_\lambda}|\partial_\nu u|^2 d\sigma dt \\ \leq \int_Q e^{-2\varphi_\lambda}|(\partial_t^2 - \Delta + q(x))u(t,x)|^2 dx dt \\ + M\int_Q e^{-2\varphi_\lambda}|u(t,x)|^2 dx dt - \lambda\int_{\Sigma^-(\omega)}(\omega\cdot\nu)e^{-2\varphi_\lambda}|\partial_\nu u|^2 d\sigma dt. \end{aligned} \tag{6.3.12}$$

Therefore, for λ sufficiently large, it holds true that

$$\begin{aligned} &\int_Q e^{-2\varphi_\lambda}\lambda^2|u(t,x)|^2 dx dt + \int_\Omega e^{-2\varphi_\lambda}|u(T,x)| dx + \lambda\int_{\Sigma^+(\omega)}(\omega\cdot\nu)e^{-2\varphi_\lambda}|\partial_\nu u|^2 d\sigma dt \\ &\leq \int_Q e^{-2\varphi_\lambda}|(\partial_t^2 - \Delta + q(x))u(t,x)|^2 dx dt - \lambda\int_{\Sigma^-(\omega)}(\omega\cdot\nu)e^{-2\varphi_\lambda}|\partial_\nu u|^2 d\sigma dt. \end{aligned} \tag{6.3.13}$$

This completes the proof of Lemma 6.3.1. □

In a similar way, we can prove the following Carleman estimate.

Lemma 6.3.2. *Let $q \in \mathscr{M}$. Then, there exist two constants $\lambda_0 > 0$ and $C > 0$ depending only on M and Q, such that, the following estimate*

$$\lambda^2 \int_Q e^{2\varphi_\lambda}|u|^2 dxdt + \int_\Omega e^{2\varphi_\lambda}|u(0,x)|^2 dx - \lambda \int_{\Sigma^-(\omega)} (\omega\cdot\nu)e^{-2\varphi_\lambda}|\partial_\nu u|^2 d\sigma dt$$
$$\leq C\int_Q e^{2\varphi_\lambda}|(-\partial_t - \Delta + q)u|^2 dxdt + \lambda \int_{\Sigma^+(\omega)} (\omega\cdot\nu)e^{-2\varphi_\lambda}|\partial_\nu u|^2 d\sigma dt,$$

holds for every $u \in H^{2,1}(Q)$ such that $u_{|\Sigma} = 0$ and $u_{|t=T} = 0$, and for all $\lambda \geq \lambda_0$.

6.4 First approach: time shifted CGO functions

6.4.1 Construction of CGO solutions

In this section, we construct suitable time-shifted CGO solutions to the parabolic equation, which are key ingredients to proof of the stability inequality. We first introduce the following notations, for $T_0, T_1 > 0$, we set $Q_0 = (T_0, T_1)\times\Omega$, $\Sigma_* := \times(T_0, T_1)\times\Gamma$, $\Sigma_*^\pm = (T_0, T_1)\times\Gamma^\pm$ and $\mathscr{L}^2_{*,\pm,\varphi_\lambda\gamma} = L^2(\Sigma_*^\pm, e^{\pm\varphi_\lambda}\gamma dxdt)$, $\gamma = |\omega\cdot\nu|$. We start by the following lemmas.

Lemma 6.4.1. *Let $q \in \mathscr{M}$ and let $\lambda \geq \lambda_0$. Then, for all $(g,h) \in \mathscr{L}^2_-(Q_0) \times \mathscr{L}^2_{*,-,-\varphi_\lambda\gamma^{-1}}$, there exists $z \in \mathscr{K}_+(Q_0)$, such that*

$$\begin{cases} (\partial_t - \Delta + q)z = g, & in \quad Q_0, \\ z(t,x) = h, & on \quad \Sigma_*^-, \\ z(T_0,x) = 0, & in \quad \Omega. \end{cases} \tag{6.4.14}$$

Moreover, we have

$$\|z\|_{\mathscr{L}^2_-(Q_0)} \leq C\left(\lambda^{-1}\|g\|_{\mathscr{L}^2_-(Q_0)} + \lambda^{-\frac{1}{2}}\|h\|_{\mathscr{L}^2_{*,-,-\varphi_\lambda\gamma^{-1}}}\right). \tag{6.4.15}$$

for some constant $C > 0$ depending only on Ω and M.

Proof. Firstly, we define the following space

$$\mathscr{F} = \{v \in H^{2,1}(Q_0),\ v = 0 \text{ on } \Sigma,\ v(T_1,\cdot) = 0 \text{ in } \Omega\}.$$

Next, we consider the linear subspace S of $\mathscr{L}^2_+(Q_0)\oplus\mathscr{L}^2_{*,+,\varphi_\lambda\gamma\lambda}$ defined as follows

$$S=\{(-\partial_t-\Delta+q)\,v,\partial_\nu v_{|\Sigma^-_*};\ v\in\mathscr{F}^+_T\}.$$

We consider the linear functional $\mathscr{L}$ defined on the linear subspace S of $\mathscr{L}^2_+\oplus\mathscr{L}^2_{*,-,\varphi_\lambda\gamma\lambda}$ as follows:

$$\mathscr{L}((-\partial_t-\Delta+q)v,\partial_\nu v_{|\Sigma^+_*})=\langle v,g\rangle-\langle\partial_\nu v,h\rangle_{L^2(\Sigma^-_*)}.$$

Using Lemma 6.3.2, we get

$$\begin{aligned}
&|\mathscr{L}((-\partial_t-\Delta+q)v,\partial_\nu v_{|\Sigma^+_*})|\le\|v\|_{\mathscr{L}^2_+(Q_0)}\|g\|_{\mathscr{L}^2_-(Q_0)}+\|\partial_\nu v\|_{\mathscr{L}^2_{*,-,\varphi_\lambda\gamma}}\|h\|_{\mathscr{L}^2_{*,-,-\varphi_\lambda\gamma^-}}\\
&\le\left(\lambda^{-1}\|g\|_{\mathscr{L}^2_-(Q_0)}+\lambda^{-\frac12}\|h\|_{\mathscr{L}^2_{*,+,-\varphi_\lambda\gamma^-}}\right)\left(\lambda\|v\|_{\mathscr{L}^2_+(Q_0)}+\lambda^{\frac12}\|\partial_\nu v\|_{\mathscr{L}^2_{*,+,\varphi_\lambda\gamma}}\right)\\
&\le C\left(\lambda^{-1}\|g\|_{\mathscr{L}^2_-(Q_0)}+\lambda^{-\frac12}\|h\|_{\mathscr{L}^2_{*,-,-\varphi_\lambda\gamma^-}}\right)\left(\|(-\partial_t-\Delta+q)v\|_{\mathscr{L}^2_+(Q_0)}+\|\partial_\nu v\|_{\mathscr{L}^2_{*,-,\varphi_\lambda\gamma\lambda}}\right)\\
&\le C\left(\lambda^{-1}\|g\|_{\mathscr{L}^2_-(Q_0)}+\lambda^{-\frac12}\|h\|_{\mathscr{L}^2_{*,-,-\varphi_\lambda\gamma^-}}\right)\|(-\partial_t-\Delta+q)v,\partial_\nu v_{|\Sigma^-})\|_{\mathscr{L}^2_+(Q_0)\oplus\mathscr{L}^2_{*,-,\varphi_\lambda\gamma\lambda}}.
\end{aligned}$$

Thus

$$\|\mathscr{L}\|\le C\left(\lambda^{-1}\|g\|_{\mathscr{L}^2_-(Q_0)}+\lambda^{-\frac12}\|h\|_{\mathscr{L}^2_{*,-,-\varphi_\lambda\gamma^-}}\right).$$

Therefore $\mathscr{L}$ is continuous in the subspace S of $\mathscr{L}^2_+(Q_0)\oplus\mathscr{L}^2_{*,-,\lambda\varphi_\lambda\gamma\lambda}$. So, using Hahn-Banach theorem we can extend $\mathscr{L}$ to a linear functional which is also denoted $\mathscr{L}$, with the same norm on the space $\mathscr{L}^2_+(Q_0)\oplus\mathscr{L}^2_{*,-,\varphi_\lambda\gamma\lambda}$. Furthermore, By using Riesz theorem, there exists $(z,w)\in\mathscr{L}^2_-(Q_0)\oplus\mathscr{L}^2_{*,+,\varphi_\lambda\gamma\lambda}$, such that

$$\begin{aligned}
\mathscr{L}\left((-\partial_t-\Delta+q)v,\partial_\nu v_{|\Sigma^+_*}\right)&=\langle(-\partial_t-\Delta+q)v,z\rangle_{L^2(Q_0)}+\langle\partial_\nu v,w\rangle_{L^2(\Sigma^+)}\\
&=\langle v,g\rangle_{L^2(Q_0)}-\langle\partial_\nu v,h\rangle_{L^2(\Sigma^-_*)},
\end{aligned}$$

and z satisfying

$$\|\mathscr{L}\|=\|z\|_{\mathscr{L}^2_-(Q_0)}\le C\left(\lambda^{-1}\|g\|_{\mathscr{L}^2_-(Q_0)}+\lambda^{-\frac12}\|h\|_{\mathscr{L}^2_{*,-,-\varphi_\lambda\gamma^{-1}}}\right).$$

Here $v=0$ on Σ^* is enforced in the $H^{3/2,3/4}(\Sigma^*)$ (resp., $H^{1/2,1/4}(\Sigma^*)$)-sense, and $v(\cdot,T_1)=0$ in Ω, should be understood in the $H^1(\Omega)$-sense. We have proved that there exists $(z,w)\in\mathscr{L}^2_-(Q_0)\oplus\mathscr{L}^2_{*,+,\varphi_\lambda\gamma\lambda}$ such that we have $\langle(-\partial_t-\Delta+q)v,z\rangle_{L^2(Q_0)}+\langle\partial_\nu v,w\rangle_{L^2(\Sigma^+_*)}=\langle v,g\rangle_{L^2(Q_0)}-\langle\partial_\nu v,h\rangle_{L^2(\Sigma^-)}$ for all $v\in\mathscr{F}^+_T$. Therefore, the above equality holds for all $v\in C^\infty_0(Q_0)$, and we have

$$\langle(-\partial_t-\Delta+q)v,z\rangle_{L^2(Q_0)}+\langle\partial_\nu v,w\rangle_{L^2(\Sigma^+_*)}=\langle(-\partial_t-\Delta)v,z\rangle_{L^2(Q_0)}+\langle v,qz\rangle_{L^2(Q_0)}+\langle\partial_\nu v,w\rangle_{L^2(\Sigma^+_*)}$$

$$= \langle v, g\rangle_{L^2(Q_0)}, -\langle \partial_\nu v, h\rangle_{L^2(\Sigma_*^-)},$$

hence

$$\langle(-\partial_t - \Delta)v, z\rangle_{C_0^\infty(Q_0),(C_0^\infty(Q_0))'} + \langle v, qz\rangle_{C_0^\infty(Q_0),(C_0^\infty(Q_0))'} = \langle v, g\rangle_{C_0^\infty(Q_0),(C_0^\infty(Q_0))'},$$

or equivalently

$$\langle v, (\partial_t - \Delta + q)z\rangle_{C_0^\infty(Q_0),(C_0^\infty(Q_0))'} = \langle v, g\rangle_{C_0^\infty(Q_0),(C_0^\infty(Q_0))'}.$$

As a consequence we have $(\partial_t - \Delta + q)z = g$ in $(C_0^\infty(Q_0))'$. But, since $g \in \mathscr{L}_-^2(Q_0)$, this identity holds in $L^2(\Omega)$ and thus

$$z \in \mathscr{K}_+(Q_0) = \{u \in L^2(Q_0),\ (\partial_t - \Delta + q)u \in L^2(Q_0)\} = \{u \in L^2(Q_0),\ (\partial_t - \Delta)u \in L^2(Q_0)\}.$$

The next step is to use the following generalized Green formula, which is true for all $z \in \mathscr{K}_+(Q_0)$ and all $v \in \mathscr{F}_T^+$:

$$\begin{aligned} & \langle(\partial_t - \Delta + q)z, v\rangle_{L^2(Q_0)} - \langle z, (-\partial_t - \Delta + q)v\rangle_{L^2(Q_0)} \\ = \ & -\langle z, \partial_\nu v\rangle_{L^2(\Sigma_*)} - \langle z(\cdot, T_0), v(\cdot, T_0)\rangle_{L^2(\Omega)}. \end{aligned}$$

Now, bearing in mind that $\langle(-\partial_t - \Delta + q)v, z\rangle_{L^2(Q_0)} = \langle v, g\rangle_{L^2(Q_0)}$ for all $v \in \mathscr{F}_T^+$ and that $(\partial_t - \Delta + q)z = g$ in $L^2(Q_0)$, we obtain that

$$\begin{aligned} & \langle(\partial_t - \Delta + q)z, v\rangle_{L^2(Q_0)} - \langle z, (-\partial_t - \Delta + q)v\rangle_{L^2(Q_0)} \\ = \ & \langle w, \partial_\nu v\rangle_{L^2(\Sigma_*^+)} - \langle \partial_\nu v, h\rangle_{L^2(\Sigma_*^-)}. \end{aligned}$$

Thus

$$\langle z, \partial_\nu v\rangle_{L^2(\Sigma_*)} + \langle w, \partial_\nu v\rangle_{L^2(\Sigma_*^+)} = -\langle z(T_0, \cdot), v(T_0, \cdot)\rangle_{L^2(\Omega)} - \langle \partial_\nu v, h\rangle_{L^2(\Sigma_*^-)},\ v \in \mathscr{F}_T^+. \tag{6.4.16}$$

Therefore, taking $v(t, x) = (T - t)w(x)$ for all $w \in C_0^\infty(\Omega)$, we get

$$\langle z(T_0, \cdot), w\rangle_{L^2(\Omega)} = 0,\ w \in C_0^\infty(\Omega).$$

which implies that $z(T_0,\cdot) = 0$. As a consequence we have $\langle z, \partial_\nu v\rangle_{L^2(\Sigma_*)} + \langle w, \partial_\nu v\rangle_{L^2(\Sigma_*^+)} = -\langle \partial_\nu v, h\rangle_{L^2(\Sigma_*^-)} 0$ for all $v \in \mathscr{F}_T^+$, by (6.4.16). Moreover, since

$$\partial_\nu \mathscr{F}_T^+ = \{\partial_\nu v,\ v \in \mathscr{F}_T^+\},$$

from [53, Chap. 4, Theorem 2.3], then we necessarily have $z = h$ in Σ_*^-.

This completes the proof of the lemma.

□

In a similar way, we can prove the following lemma.

Lemma 6.4.2. *Let $q \in \mathscr{M}$ and let $\lambda \geq \lambda_0$, for all $(g,h) \in \mathscr{L}_+^2(Q_0) \times \mathscr{L}^2_{*,+,\varphi_\lambda,\gamma^{-1}}$ there exist $C > 0$ and $z \in \mathscr{K}_-(Q_0)$, such that*

$$\begin{cases} (-\partial_t - \Delta + q)z = g, & in \quad Q_0, \\ z(t,x) = 0, & on \quad \Sigma_*^+, \\ z(T_1,x) = 0, & in \quad \Omega. \end{cases} \tag{6.4.17}$$

Moreover, we have

$$\|z\|_{\mathscr{L}_-^2(Q_0)} \leq C\left(\lambda^{-1}\|g\|_{\mathscr{L}_+^2(Q_0)} + \lambda^{-\frac{1}{2}}\|h\|_{\mathscr{L}^2_{*,+,\varphi_\lambda\gamma^{-1}}}\right). \tag{6.4.18}$$

for some constant $C > 0$ depending only on Ω and M.

Next, we set $Q' := (T_* - \epsilon, T) \times \Omega$, $Q'' := (0, T_* + \epsilon) \times \Omega$, $\Sigma'_\pm := (T_* - \epsilon, T) \times \Gamma^\pm$ and $\Sigma''_\pm := (0, T_* + \epsilon) \times \Gamma^\pm$.

Lemma 6.4.3. *Let $q \in \mathscr{M}$, $\omega \in \mathscr{S}^{n-1}$, $\lambda \geq \lambda_0$ and let $\xi \in \mathbb{R}^n$ satisfy $\xi \cdot \omega = 0$. Then there exist $\Psi_\lambda^+ \in \mathscr{K}_+(Q')$ satisfying*

$$\begin{cases} (\partial_t - \Delta + q(x))(\Psi_\lambda^+ e^{\varphi_\lambda}) = -(\partial_t - \Delta + q(x))(e^{\varphi_\lambda}(1 - e^{-\lambda^{\frac{3}{4}}(t - T_* + \epsilon)})e^{-ix\cdot\xi}), & in\ Q', \\ \Psi_\lambda^+(t,x) = 0, & on \quad \Sigma'_-, \\ \Psi_\lambda^+(T_* - \epsilon, x) = 0, & in \quad \Omega. \end{cases} \tag{6.4.19}$$

Moreover, there exists a constant $C > 0$, depending only on Ω and M, such that

$$\|\Psi_\lambda^+\|_{L^2(Q')} \leq C(\lambda^{-\frac{1}{4}} + \lambda^{-1}\langle\xi\rangle^2), \tag{6.4.20}$$

where $\langle\xi\rangle = \sqrt{1 + |\xi|^2}$.

Proof. By a simple calculation, we get that

$$-(\partial_t - \Delta + q(x))(e^{\varphi_\lambda}(1 - e^{-\lambda^{\frac{3}{4}}(t-T_*+\epsilon)})e^{-ix\cdot\xi}) = \mathscr{R},$$

where $\mathscr{R} = e^{\varphi_\lambda}e^{-ix\cdot\xi}[(|\xi|^2 + q)(1 - e^{-\lambda^{\frac{3}{4}}(t-T_*+\epsilon)}) - \lambda^{\frac{3}{4}}e^{-\lambda^{\frac{3}{4}}(t-T_*+\epsilon)}]$.
Evidently, $\mathscr{R} \in \mathscr{L}^2_-(Q')$ and

$$\|\mathscr{R}\|_{\mathscr{L}^2_-(Q')} \leq C(\langle\xi\rangle^2 + \lambda^{\frac{3}{4}}).$$

From Lemma 6.4.1, there exist a solution $w \in \mathscr{K}_-(Q')$, satisfying

$$\begin{cases} (\partial_t - \Delta + q(x))w = \mathscr{R}, & \text{in} \quad Q', \\ w(t,x) = 0, & \text{on} \quad \Sigma'_-, \\ w(T_* - \epsilon, x) = 0, & \text{in} \quad \Omega, \end{cases} \tag{6.4.21}$$

where $\Psi^+_\lambda = e^{-\varphi_\lambda}w$ satisfies (6.4.19). Moreover, we get

$$\|\Psi^+_\lambda\|_{L^2(Q')} = \|w\|_{\mathscr{L}^2_-(Q')} \leq C\lambda^{-1}\|\mathscr{R}\|_{\mathscr{L}^2_-(Q')} \leq C(\langle\xi\rangle^2\lambda^{-1} + \lambda^{-\frac{1}{4}}).$$

which entails (6.4.20).

□

In the same way as above we have the following lemma.

Lemma 6.4.4. *Let $q \in \mathscr{M}$ and let $\lambda \geq \lambda_0$, where λ_0 is as in Lemma 6.4.1. Then, there exists $\Psi^-_\lambda \in \mathscr{K}_-(Q'')$ satisfying is*

$$\begin{cases} (-\partial_t - \Delta + q(x))(\Psi^-_\lambda e^{-\varphi_\lambda}) = -(-\partial_t - \Delta + q(x))(e^{-\varphi_\lambda}(1 - e^{-\lambda^{\frac{3}{4}}(T_*+\epsilon-t)})), & \textit{in} \quad Q'', \\ \Psi^-_\lambda(t,x) = 0, & \textit{on} \quad \Sigma''_+, \\ \Psi^-_\lambda(T_* + \epsilon, x) = 0, & \textit{in} \quad \Omega. \end{cases} \tag{6.4.22}$$

Moreover, there exists a constant $C > 0$, depending only on Ω and M, such that

$$\|\Psi^-_\lambda\|_{L^2(Q'')} \leq C\lambda^{-\frac{1}{4}}. \tag{6.4.23}$$

Having build the remainder terms Ψ^+_λ and Ψ^-_λ, we conclude this section by the following lemmas.

Lemma 6.4.5. *Let $q \in \mathscr{M}$, let $\omega \in \mathscr{S}^{n-1}$, and let $\xi \in \mathbb{R}^n$ satisfy $\xi \cdot \omega = 0$. Then the following system*

$$\begin{cases} \partial_t u - \Delta u + q(x)u = 0 & in \quad Q, \\ u(T_* - \epsilon, x) = 0 & in \quad \Omega, \end{cases} \tag{6.4.24}$$

admits a solution $u_\lambda^+ \in \mathscr{K}_+(Q)$ such that

$$\begin{cases} u_\lambda^+ = 0 & in\, Q \setminus Q', \\ u_\lambda^+ = e^{\varphi_\lambda}((1 - e^{-\lambda^{\frac{3}{4}}(t-T_*+\epsilon)})e^{-ix\cdot\xi}) + \Psi_\lambda^+) & in \quad Q', \end{cases} \tag{6.4.25}$$

where $\Psi_\lambda^+ \in \mathscr{K}_+(Q')$ satisfies $\Psi_\lambda^+ = 0$ on Σ'_- and

$$\|\Psi_\lambda^+\|_{L^2(Q')} \leq C(\langle\xi\rangle^2\lambda^{-1} + \lambda^{-\frac{1}{4}}), \tag{6.4.26}$$

For some positive constant C depending only on M and Ω.

Proof. Firstly, we consider a solution u_λ^+ of the following system

$$\begin{cases} \partial_t u_\lambda^+ - \Delta u_\lambda^+ + q(x)u_\lambda^+ = 0 & \text{in} \quad Q', \\ u_\lambda^+(T_* - \epsilon, x) = 0 & \text{in} \quad \Omega. \end{cases} \tag{6.4.27}$$

As we want $u_\lambda^+ = e^{\varphi_\lambda}((1 - e^{-\lambda^{\frac{3}{4}}(t-T_*+\epsilon)})e^{-ix\cdot\xi}) + \Psi_\lambda^+)$, in Q', we seek Ψ_λ^+ to be solution of

$$\begin{cases} (\partial_t - \Delta + q(x))(\Psi_\lambda^+ e^{\varphi_\lambda}) = -(\partial_t - \Delta + q(x))(e^{\varphi_\lambda}((1 - e^{-\lambda^{\frac{3}{4}}(t-T_*+\epsilon)})e^{-ix\cdot\xi}) & \text{in} \quad Q', \\ \Psi_\lambda^+(t,x) = 0, & \text{on} \quad \Sigma'_- \\ \Psi_\lambda^+(T_* - \epsilon, x) = 0, & \text{in} \quad \Omega. \end{cases} \tag{6.4.28}$$

Such a solution Ψ_λ^+ lies in $\mathscr{K}_+(Q')$.

Set now

$$\begin{cases} u_\lambda^+ = 0 & \text{in}\, Q \setminus Q', \\ u_\lambda^+ = e^{\varphi_\lambda}((1 - e^{-\lambda^{\frac{3}{4}}(t-T_*+\epsilon)})e^{-ix\cdot\xi}) + \Psi_\lambda^+) & \text{in} \quad Q. \end{cases} \tag{6.4.29}$$

Then, for all $\varphi \in C_0^\infty(Q)$, using the fact that $u_\lambda^+ \in L^2(Q)$ and that $u_\lambda^+ = 0$ in $Q \setminus Q'$, we obtain

$$\langle(\partial_t - \Delta)u_\lambda^+, \varphi\rangle_{(C_0^\infty(Q))', C_0^\infty(Q)} = \langle u_\lambda^+, (-\partial_t - \Delta)\varphi\rangle_{(C_0^\infty(Q))', C_0^\infty(Q)}$$

$$= \int_Q u_\lambda^+(-\partial_t - \Delta)\varphi dxdt$$
$$= \int_{Q'} u_\lambda^+(-\partial_t - \Delta)\varphi dxdt.$$

Since $u_\lambda^+ \in \mathscr{K}_+(Q')$, by integrating by parts and using the fact that $\varphi(T,\cdot) = 0$ and $u_\lambda^+(T_* - \epsilon, x) = 0$, we obtain

$$\int_{Q'} u_\lambda^+(-\partial_t - \Delta)\varphi dxdt = \int_{Q'} (\partial_t - \Delta)u_\lambda^+ \varphi dxdt.$$

Therefore, we have

$$\begin{aligned}\langle(\partial_t - \Delta + q)u_\lambda^+, \varphi\rangle_{(C_0^\infty(Q))', C_0^\infty(Q)} &= \langle(\partial_t - \Delta)u_\lambda^+, \varphi\rangle_{(C_0^\infty(Q))', C_0^\infty(Q)} + \langle qu_\lambda^+, \varphi\rangle_{(C_0^\infty(Q))', C_0^\infty(Q)} \\ &= \int_{Q'} (\partial_t - \Delta + q)u_\lambda^+ \varphi dxdt = 0.\end{aligned}$$

Thus, we have $(\partial_t - \Delta + q)u_\lambda^+ = 0$ in $L^2(Q)$ and hence $u_\lambda^+ \in \mathscr{K}_+(Q)$. □

In the same way as in the proof of Lemma 6.4.5 we have the following lemma.

Lemma 6.4.6. *Let $q \in \mathscr{M}$. Then the following system*

$$\begin{cases} -\partial_t u - \Delta u + q(x)u = 0 & in \quad Q, \\ u(T_* + \epsilon, x) = 0 & in \quad \Omega \end{cases} \tag{6.4.30}$$

admits a solution $u_\lambda^- \in \mathscr{K}_-(Q)$ such that

$$\begin{cases} u_\lambda^- = 0 & in\, Q \setminus Q'', \\ u_\lambda^- = e^{-\varphi_\lambda}((1 - e^{-\lambda^{\frac{3}{4}}(T_* + \epsilon - t)}) + \Psi_\lambda^-) & in \quad Q'', \end{cases} \tag{6.4.31}$$

where $\Psi_\lambda^- \in \mathscr{K}_-(Q'')$ satisfies $\Psi_\lambda^- = 0$ on Σ_+'' and

$$\|\Psi_\lambda^-\|_{L^2(Q'')} \leq C\lambda^{-\frac{1}{4}}. \tag{6.4.32}$$

Here C a positive constant depending only on M and Ω.

6.4.2 Stable determination of the potential from the full boundary data

The purpose of the present section is to prove a logarithmic stability result in the determination of the coefficient q appearing in the parabolic equation of (6.2.1) from the full D-to-N map. Given q_1 and q_2 in $\mathscr{M}$, we set $q = q_1 - q_2$ and we extend q by 0 in $\mathbb{R}^n \setminus \Omega$. We start by the following estimate which relates q to the difference of D-to-N maps associated with q_1 and q_2, respectively. In this section, we denote $e^{\pm\varphi_\lambda}\gamma dxdt)$ by $\mathscr{L}^2_{\prime,\pm,\varphi_\lambda\gamma} = L^2(\Sigma'_\pm$.

Lemma 6.4.7. *Let $q_j \in \mathscr{M}$, $j = 1,2$, let $R > 0$, and let $\xi \in \mathbb{R}^n$ satisfy $\xi \cdot \omega = 0$. Then, there exists two constants $\lambda_0 > 0$ and $C > 0$, depending only on M and Ω, and there exists $\epsilon_0 \in (0,1)$, such that*

$$|\widehat{q}(\xi)| \leq C(\epsilon^{\frac{3}{8}} + R^2\epsilon^{\frac{31}{16}} + \|\tilde{\Lambda}_1 - \tilde{\Lambda}_2\|(\epsilon^{\frac{5}{8}} + R^2\epsilon^{\frac{13}{8}})e^{2|\Omega|\epsilon^{-\frac{1}{2}}}), \tag{6.4.33}$$

for any $|\xi| \leq R$ and $0 < \epsilon < \epsilon_0$. Here, $\|\tilde{\Lambda}_2 - \tilde{\Lambda}_1\|$ stands for the norm of $\tilde{\Lambda}_2 - \tilde{\Lambda}_1$ in $\mathscr{B}\left(H^{-1/2,-1/4}(\Sigma_0); H^{1/2,1/4}(\Sigma_0)\right)$.

Proof. In view of Lemma 6.4.5, there exists a solution u^+_λ to the equation

$$\begin{cases} \partial_t u - \Delta u + q(x)u = 0 & \text{in} \quad Q', \\ u(T_* - \epsilon, x) = 0 & \text{in} \quad \Omega, \end{cases} \tag{6.4.34}$$

of the form

$$\begin{cases} u^+_\lambda = 0 & \text{in } Q \setminus Q', \\ u^+_\lambda = e^{\varphi_\lambda}((1 - e^{-\lambda^{\frac{3}{4}}(t - T_* + \epsilon)})e^{-ix\cdot\xi}) + \Psi^+_\lambda) & \text{in} \quad Q', \end{cases} \tag{6.4.35}$$

where $\Psi^+_\lambda \in \mathscr{K}_+(Q')$ satisfies $\Psi^+_\lambda = 0$ on Σ'_-. Next, let us denote by $\mu_+(t,x)$ the function

$$\mu_+(t,x) = u^+_\lambda(t,x)_{|\Sigma'_-} = e^{\varphi_\lambda}((1 - e^{\lambda^{\frac{3}{4}}(t - T_* + \epsilon)})e^{-ix\cdot\xi}), \quad (t,x) \in \Sigma'_-.$$

Evidently, $\mu_+(t,x) \in \mathscr{L}^2_{\prime,-,-\varphi_\lambda\gamma}$, so there exists $u_1 \in \mathscr{K}_+(Q')$ which is solution of

$$\begin{cases} \partial_t u_1 - \Delta u_1 + q_1(x)u_1 = 0 & \text{in} \quad Q', \\ u_1(T_* - \epsilon, x) = 0 & \text{in} \quad \Omega, \\ u_1(t,x) = \mu_+(t,x), & \text{on} \quad \Sigma'_-. \end{cases} \tag{6.4.36}$$

Thus, setting $u_1 = 0$ in $Q \setminus Q'$, we get by arguing as in the derivation of Lemma 6.4.5 that $u_1 \in \mathscr{K}_+(Q)$ and that $(\partial_t - \Delta + q)u_1 = 0$ in Q. Next, put $u = u_1 - u_\lambda^+$. Then, u is a solution to the following system

$$\begin{cases} \partial_t u - \Delta u + q_1(x)u = q(x)u_\lambda^+ & \text{in} \quad Q, \\ u(T_* - \epsilon, x) = 0 & \text{in} \quad \Omega, \\ u(t,x) = 0, & \text{on} \quad \Sigma'_-, \end{cases} \tag{6.4.37}$$

where we recall that $q(x) = q_1(x) - q_2(x)$. We point out that $u = 0$ in $Q \setminus Q'$. On the other hand, Lemma 6.4.6 guarantees the existence of the exponentially growing solutions u_λ^- to the backward problem

$$\begin{cases} -\partial_t u - \Delta u + q(x)u = 0 & \text{in} \quad Q, \\ u(T_* + \epsilon, x) = 0 & \text{in} \quad \Omega \end{cases} \tag{6.4.38}$$

such that

$$\begin{cases} u_\lambda^- = 0 & \text{in } Q \setminus Q'', \\ u_\lambda^- = e^{-\varphi_\lambda}((1 - e^{-\lambda^{\frac{3}{4}}(T_* + \epsilon - t)}) + \Psi_\lambda^-) & \text{in} \quad Q'', \end{cases} \tag{6.4.39}$$

where $\Psi_\lambda^- \in \mathscr{K}_-(Q'')$ satisfies $\Psi_\lambda^- = 0$ on Σ''_+. Let us denote by $\mu_-(t,x)$ the function

$$\mu_-(t,x) = u_\lambda^+(t,x)_{|\Sigma'_+} = e^{-\varphi_\lambda}(1 - e^{\lambda^{\frac{3}{4}}(T_* + \epsilon - t)}), \quad (t,x) \in \Sigma''_+.$$

Multiplying both sides of the first line of (6.4.37) by u_λ^- and integrating over Q, we get that

$$\langle (\partial_t - \Delta + q_1)u, u_\lambda^- \rangle_{L^2(Q_*)} = \langle q u_\lambda^+, u_\lambda^- \rangle_{L^2(Q_*)}, \tag{6.4.40}$$

where we recall that $Q_* = (T_* - \epsilon, T_* + \epsilon) \times \Omega$. This leads to

$$\langle (\tilde{\Lambda}_2 - \tilde{\Lambda}_1)\mu_+, \mu_- \rangle_{H^{1/2,1/4}(\Sigma_0), H^{-1/2,-1/4}(\Sigma_0)} = \langle q u_\lambda^+, u_\lambda^- \rangle_{L^2(Q_*)}, \tag{6.4.41}$$

with the aid of the Green formula, where $\Sigma_0^\pm = (T_* - \epsilon, T_* + \epsilon) \times \Gamma^\pm$.
By replacing u_λ^+ and u_λ^- by their expression, (6.4.35) and (6.4.39), respectively, we get:

$$2\epsilon \left| \int_\Omega q(x) e^{-ix\cdot\xi} dx \right| \leq \left| \int_{Q_*} Z(t,x) dx dt \right| + |\langle (\tilde{\Lambda}_2 - \tilde{\Lambda}_1)(\mu_+), \mu_- \rangle_{H^{1/2,1/4}(\Sigma_0), H^{-1/2,-1/4}(\Sigma_0)}|, \tag{6.4.42}$$

where

$$Z(t,x) = q(x)[(1-e^{-\lambda^{\frac{3}{4}}(t-T_*+\epsilon)})e^{-ix\cdot\xi}\Psi_\lambda^- + \Psi_\lambda^+(1-e^{-\lambda^{\frac{3}{4}}(T_*+\epsilon-t)})$$
$$+\Psi_\lambda^+\Psi_\lambda^- + e^{-ix\cdot\xi}(e^{-2\lambda^{\frac{3}{4}}\epsilon} - e^{-\lambda^{\frac{3}{4}}(t-T_*+\epsilon)} - e^{-\lambda^{\frac{3}{4}}(T_*+\epsilon-t)}]. \quad (6.4.43)$$

Notice for all $\tau \in (0, 2\epsilon)$ that

$$(1-e^{-\lambda^{\frac{3}{4}}\tau})^2 = \int_0^{\lambda^{\frac{3}{4}}\tau} \frac{d}{dt}(1-e^{-t})^2 dt = 2\int_0^{\lambda^{\frac{3}{4}}\tau} e^{-t}(1-e^{-t})dt \le \min(1, 2\tau\lambda^{\frac{3}{4}}),$$

so we get

$$\|1-e^{-\lambda^{\frac{3}{4}}(t-T_*+\epsilon)}\|_{L^2(T_*-\epsilon,T_*+\epsilon)} \le 2\min(1, \lambda^{\frac{3}{8}}\epsilon^{\frac{1}{2}})\epsilon^{\frac{1}{2}}.$$

On the other hand, $\Psi_\lambda^+ \in \mathscr{K}_+(Q_*)$ is solution of

$$\begin{cases} (\partial_t - \Delta + q(x))(\Psi_\lambda^+ e^{\varphi_\lambda}) = \mathscr{R}, & \text{in } Q_*, \\ \Psi_\lambda^+(t,x) = 0, & \text{on } \Sigma_0^+, \\ \Psi_\lambda^+(T_*-\epsilon, x) = 0, & \text{in } \Omega, \end{cases} \quad (6.4.44)$$

where $\mathscr{R} = e^{\varphi_\lambda}e^{-ix\cdot\xi}[(|\xi|^2+q)(1-e^{-\lambda^{\frac{3}{4}}(t-T_*+\epsilon)}) - \lambda^{\frac{3}{4}}e^{-\lambda^{\frac{3}{4}}(t-T_*+\epsilon)}]$.
Evidently, $\mathscr{R} \in \mathscr{L}_-^2(Q_*)$ and

$$\|\mathscr{R}\|_{\mathscr{L}_-^2(Q_*)} \le C(\langle\xi\rangle^2\min(1, \lambda^{\frac{3}{8}}\epsilon^{\frac{1}{2}})\epsilon^{\frac{1}{2}} + \lambda^{\frac{3}{4}}\epsilon^{\frac{1}{2}}),$$

where C is a positive constant depending only on M and Ω.
From Lemma 6.4.1, there exists a solution $w \in \mathscr{K}_-(Q_*)$ satisfying

$$\begin{cases} (\partial_t - \Delta + q(x))w = \mathscr{R}, & \text{in } \quad Q', \\ w(t,x) = 0, & \text{on } \quad \Sigma', \\ w(T_*-\epsilon, x) = 0, & \text{in } \quad \Omega, \end{cases} \quad (6.4.45)$$

where $\Psi_\lambda^+ = e^{-\varphi_\lambda}w$ satisfies (6.4.44). Moreover, by (6.4.15), we get

$$\|\Psi_\lambda^+\|_{L^2(Q_*)} = \|w\|_{\mathscr{L}_-^2(Q_*)} \le C\lambda^{-1}\|\mathscr{R}\|_{\mathscr{L}_-^2(Q_*)} \le C(\langle\xi\rangle^2\min(1, \lambda^{\frac{3}{8}}\epsilon^{\frac{1}{2}})\epsilon^{\frac{1}{2}}\lambda^{-1} + \lambda^{-\frac{1}{4}}\epsilon^{\frac{1}{2}}), \quad (6.4.46)$$

where $C > 0$ depending only on M and Ω. In the same way, we obtain that

$$\|\Psi_\lambda^-\|_{L^2(Q_*)} \le C\lambda^{-\frac{1}{4}}\epsilon^{\frac{1}{2}}, \quad (6.4.47)$$

where $C > 0$ depending only on M and Ω.

Using the fact that $\tilde{\Lambda}_2 - \tilde{\Lambda}_1 \in \mathscr{B}\left(H^{-1/2,-1/4}(\Sigma_0); H^{1/2,1/4}(\Sigma_0)\right)$, we find that

$$\begin{aligned}
&|\langle(\tilde{\Lambda}_2 - \tilde{\Lambda}_1)\mu_+, \mu_-\rangle_{H^{1/2}(\Sigma_0),H^{-1/2,-1/4}(\Sigma_0)}| \\
&\quad \leq \tilde{\Lambda}_2 - \tilde{\Lambda}_1\|_{\mathscr{B}\left(H^{-1/2,-1/4}(\Sigma_0);H^{1/2,1/4}(\Sigma_0)\right)}\|\mu_+\|_{H^{-1/2,-1/4}(\Sigma_0^-)}\|\mu_-\|_{H^{-1/2,-1/4}(\Sigma_0^+)},
\end{aligned}$$

where

$$\begin{aligned}
\|\mu_+\|_{H^{-1/2,-1/4}(\Sigma_0^-)} &\leq \|u_\lambda^+\|_{\mathscr{K}^+(Q_*)} \leq C\|e^{\varphi_\lambda}((1 - e^{-\lambda^{\frac{3}{4}}(t-T_*+\epsilon)})e^{-ix\cdot\xi} + \Psi_\lambda^+)\|_{L^2(Q_*)} \\
&\leq Ce^{\lambda|\Omega|+\lambda^2(T_*+\epsilon)}[|\Omega|\min(1, \lambda^{\frac{3}{8}}\epsilon^{\frac{1}{2}})\epsilon^{\frac{1}{2}} + \epsilon^{\frac{1}{2}}\lambda^{-\frac{1}{4}} + \min(1, \lambda^{\frac{3}{8}}\epsilon^{\frac{1}{2}})\epsilon^{\frac{1}{2}}\lambda^{-1}R^2],
\end{aligned}$$

and

$$\begin{aligned}
\|\mu_-\|_{H^{-1/2,-1/4}(\Sigma_0^+)} \leq \|u_\lambda^-\|_{\mathscr{K}^-(Q_*)} &\leq C\|e^{-\varphi_\lambda}((1 - e^{-\lambda^{\frac{3}{4}}(T_*+\epsilon-t)}) + \Psi_\lambda^-)\|_{L^2(Q_*)} \\
&\leq Ce^{\lambda|\Omega|-\lambda^2(T_*-\epsilon)}[|\Omega|\min(1, \lambda^{\frac{3}{8}}\epsilon^{\frac{1}{2}})\epsilon^{\frac{1}{2}} + \epsilon^{\frac{1}{2}}\lambda^{-\frac{1}{4}}],
\end{aligned}$$

where $C > 0$ depends only on M and Ω.

Thus,

$$\begin{aligned}
&|\langle(\tilde{\Lambda}_2 - \tilde{\Lambda}_1)\mu_+, \mu_-\rangle_{H^{1/2,1/4}(\Sigma_0),H^{-1/2,-1/4}(\Sigma_0)}| \\
&\quad \leq C\|\tilde{\Lambda}_2 - \tilde{\Lambda}_1\|e^{2(\lambda|\Omega|+\lambda^2\epsilon)}\epsilon[|\Omega|\min(1, \epsilon\lambda^{\frac{3}{8}}\epsilon^{\frac{1}{2}}) + \min(1, \epsilon\lambda^{\frac{3}{8}}\epsilon^{\frac{1}{2}})\lambda^{-1}R^2 + \lambda^{-\frac{1}{4}}].
\end{aligned} \tag{6.4.48}$$

In view of (6.4.48) and in order to avoid the super exponential increase with respect to λ, of the prefactor of $\|\tilde{\Lambda}_1 - \tilde{\Lambda}_2\|$ in (6.4.48), we choose $\lambda^2\epsilon = 1$ i.e. $\lambda = \epsilon^{-\frac{1}{2}}$. Therefore, we have $\|\Psi_\lambda^+\|_{L^2(Q_*)} \leq C(R^2\epsilon^{\frac{21}{16}} + \epsilon^{\frac{5}{8}})$, $\|\Psi_\lambda^-\|_{L^2(Q_*)} \leq C\epsilon^{\frac{5}{8}}$ and

$$\|Z\|_{L^2(Q_*)} \leq C(\epsilon^{\frac{3}{8}} + R^2\epsilon^{\frac{31}{16}}). \tag{6.4.49}$$

Then, by (6.4.42), (6.4.48) and (6.4.49), we get

$$\left|\int_{\mathbb{R}^n} q(x)e^{-ix\cdot\xi}dx\right| \leq C\left(R^2\epsilon^{\frac{15}{16}} + \epsilon^{-\frac{5}{8}} + \|\tilde{\Lambda}_1 - \tilde{\Lambda}_2\|e^{2\epsilon^{-\frac{1}{2}}|\Omega|}(\epsilon^{-\frac{3}{8}} + R^2\epsilon^{\frac{5}{8}})\right),$$

where $C > 0$ depends only on M and Ω. This completes the proof of the lemma.

□

We now introduce the main result of this subsection. Set

$$\phi_\epsilon(t) = \begin{cases} \epsilon^{-\frac{5}{8}(n-2)} e^{2|\Omega|\epsilon^{-\frac{1}{2}}} t & \text{in} \quad t \geq e^{-2|\Omega|\epsilon^{-\frac{1}{2}}}, \\ |\log t|^{-\frac{5}{16}(n-2)} & \text{in} \quad 0 < t < e^{-2|\Omega|\epsilon^{-\frac{1}{2}}}. \end{cases}$$

Theorem 6.4.8. *Let $q_1, q_2 \in \mathscr{M}$. Then there exists $\epsilon_0 = \epsilon_0(T, \Omega, M) > 0$, and $C = C(T, \Omega, M) > 0$ such that,*

$$\|q_1 - q_2\|_{H^{-1}(\Omega)} \leq C\phi_\epsilon(\|\tilde{\Lambda}_2 - \tilde{\Lambda}_1\|), \quad \forall \epsilon \in (0, \epsilon_0).$$

Here $\|\tilde{\Lambda}_2 - \tilde{\Lambda}_1\|$ stands for the norm of $\tilde{\Lambda}_2 - \tilde{\Lambda}_1$ in $\mathscr{B}\left(H^{-1/2,-1/4}(\Sigma_0); H^{1/2,1/4}(\Sigma_0)\right)$.

Proof. In this proof, C denote a generic constant that depends only on M and Ω. Firstly, we need to decompose the $H^{-1}(\mathbb{R}^n)$ norm of q into the following way

$$\begin{aligned} \|q\|^2_{H^{-1}(\mathbb{R}^n)} &= \int_{|\xi|<R} (1+|\xi|^2)^{-1} |\widehat{q}(\xi)|^2 d\xi + \int_{|\xi|\geq R} (1+|\xi|^2)^{-1} |\widehat{q}(\xi)|^2 d\xi \\ &\leq R^n \|\widehat{q}\|^2_{L^\infty(B(0,R))} + R^{-2} \|q\|^2_{L^2(\mathbb{R}^n)}. \end{aligned}$$

Hence, in light of (6.4.33), we get

$$\begin{aligned} \|q\|^2_{H^{-1}(\mathbb{R}^n)} &\leq C\left(R^n\left[R^4\epsilon^{\frac{15}{8}} + \epsilon^{-\frac{5}{4}} + \|\tilde{\Lambda}_2 - \tilde{\Lambda}_1\|^2 e^{4|\Omega|\epsilon^{-\frac{1}{2}}}(\epsilon^{-\frac{3}{4}} + R^2\epsilon^{\frac{5}{4}})\right] + R^{-2}\right) \\ &\leq C\left(\epsilon^{\frac{15}{8}} R^{n+4} + \epsilon^{-\frac{5}{4}} R^n + (\epsilon^{\frac{5}{4}} R^2 + \epsilon^{-\frac{3}{4}}) e^{4|\Omega|\epsilon^{-\frac{1}{2}}} R^n \|\tilde{\Lambda}_2 - \tilde{\Lambda}_1\|^2 + R^{-2}\right). \end{aligned}$$

In this inequality, we take $R = \epsilon^{-s}$ and $0 < s < 1$, and obtain

$$\begin{aligned} \|q\|^2_{H^{-1}(\mathbb{R}^n)} &\leq C\left(\epsilon^{-s(n+4)+\frac{15}{8}} + \epsilon^{-\frac{5}{4}-sn} + e^{4|\Omega|\epsilon^{-\frac{1}{2}}} \epsilon^{-sn}(\epsilon^{\frac{5}{4}} R^2 + \epsilon^{-\frac{3}{4}}) \|\tilde{\Lambda}_2 - \tilde{\Lambda}_1\|^2 + \epsilon^{2s}\right) \\ &\leq C\left(\epsilon^{-s(n+4)+\frac{15}{8}} + e^{4|\Omega|\epsilon^{-\frac{1}{2}}} \epsilon^{-s(n+2)+\frac{5}{4}} \|\tilde{\Lambda}_2 - \tilde{\Lambda}_1\|^2\right). \end{aligned}$$

Then, we choose s such that $-s(n+4) + \frac{15}{8} = -s(n+2) + \frac{5}{4}$ i.e. $s = \frac{5}{16}$, then we get

$$\|q\|^2_{H^{-1}(\mathbb{R}^n)} \leq C\left(\epsilon^{-\frac{5}{16}(n-2)}(1 + e^{4|\Omega|\epsilon^{-\frac{1}{2}}} \|\tilde{\Lambda}_2 - \tilde{\Lambda}_1\|^2)\right).$$

If $\|\tilde{\Lambda}_2 - \tilde{\Lambda}_1\| \geq e^{-2|\Omega|\epsilon^{-\frac{1}{2}}}$, then we have

$$\|q\|_{H^{-1}(\mathbb{R}^n)} \leq C\epsilon^{-\frac{5}{8}(n-2)} e^{2|\Omega|\epsilon^{-\frac{1}{2}}} \|\tilde{\Lambda}_2 - \tilde{\Lambda}_1\|,$$

and if $\|\tilde{\Lambda}_2 - \tilde{\Lambda}_1\| < e^{-2|\Omega|\epsilon^{-\frac{1}{2}}}$, that is if $\epsilon > \frac{4|\Omega|^2}{(\log\|\tilde{\Lambda}_2 - \tilde{\Lambda}_1\|)^2}$, we obtain

$$\|q\|_{H^{-1}(\mathbb{R}^n)} \leq C\epsilon^{-\frac{5}{16}(n-2)} \leq C|\log\|\tilde{\Lambda}_2 - \tilde{\Lambda}_1\||^{-\frac{5}{16}(n-2)}.$$

□

6.4.3 Determining the potential from partial boundary measurements

In this section, inspired by [16], we prove a logarithmic stability estimate for the potential q from knowledge of the partial D-to-N map. As in Subsection 6.4.2, given two suitable potentials $q_1,\ q_2$ in $\mathscr{M}$, we set $q = q_1 - q_2$ and we extend it by $q = 0$ on $\mathbb{R}^n \setminus \Omega$. Let's first recall from (6.4.35) and (6.4.39) that

$$u_\lambda^+ = e^{\varphi_\lambda}((1 - e^{-\lambda^{\frac{3}{4}}(t-T_*+\epsilon)})e^{-ix\cdot\xi}) + \Psi_\lambda^+) \quad \text{in } Q',$$

and

$$u_\lambda^- = e^{-\varphi_\lambda}((1 - e^{-\lambda^{\frac{3}{4}}(T_*+\epsilon-t)}) + \Psi_\lambda^-) \quad \text{in } Q''.$$

We shall use the following notations. For $\varrho > 0$ and $\omega_0 \in \mathscr{S}^{n-1}$, we denote by

$$\mathscr{V}_\varrho(\omega_0) = \{\omega \in \mathscr{S}^{n-1};\ |\omega - \omega_0| < \frac{\varrho}{2}\},$$

a neighborhood around ω_0 in $\mathscr{S}^{n-1}$. Then, for each $\omega \in \mathscr{V}_\varrho(\omega_0)$, we have

$$\Sigma_{-,\frac{\varrho}{2}}(\omega) \subset \Sigma_{-,\varrho}(\omega_0).$$

Set

$$K_\varrho = \bigcup_{\omega \in \mathscr{V}_\varrho(\omega_0)} \omega^\perp,$$

and put

$$K_{\varrho,R} = \left\{\xi \in K_\varrho,\ |\xi| < R\right\}.$$

We start by the following lemma, estimating the Fourier transform of q in the conic set $K_{\varrho,R}$.

Lemma 6.4.9. *Let $q_j \in \mathscr{M}$, $j = 1, 2$, let $R > 0$, and let $\xi \in \mathbb{R}^n$ with $\xi \cdot \omega = 0$. Then there exists $C > 0$, depending only on M and Ω and there exists $\epsilon_0 \in (0, 1)$, such that we have*

$$|\widehat{q}(\xi)| \leq C(R^2\epsilon^{\frac{13}{16}} + \epsilon^{-\frac{1}{2}} + e^{2\epsilon^{-\frac{1}{2}}|\Omega|}\|\Lambda_1^\sharp - \Lambda_2^\sharp\|(\epsilon^{\frac{13}{16}}R^2 + \epsilon^{\frac{7}{16}})), \tag{6.4.50}$$

for any $|\xi| \leq R$ and $0 < \epsilon < \epsilon_0$. Here, $\|\Lambda_1^\sharp - \Lambda_2^\sharp\|$ stands for the norm of $\Lambda_1^\sharp - \Lambda_2^\sharp$ in $\mathscr{B}\left(H^{-1/2,-1/4}(\Sigma_0); H^{1/2,1/4}(\Sigma_{-,\varrho}(\omega_0))\right)$.

Proof. Using the same notation and proceeding as in the proof of Lemma 6.4.7, we obtain

$$\langle(\partial_t - \Delta + q_1)u, u_\lambda^-\rangle_{L^2(Q_*)} = \langle qu_\lambda^+, u_\lambda^-\rangle_{L^2(Q_*)}, \tag{6.4.51}$$

By replacing u_λ^+ and u_λ^- by their expressions, we obtain

$$2\epsilon\Big|\int_\Omega q(x)e^{-ix\cdot\xi}dx\Big| \le \Big|\int_{Q_*} Z(t,x)dxdt\Big| + \Big|\int_{\Sigma_{+,(\frac{\varrho}{2}}} \partial_\nu u)e^{-\varphi_\lambda}\Big(1 - e^{-\lambda^{\frac{1}{4}}(T_*+\epsilon-t)}\Big)dxdt\Big| + \Big|\int_{\Sigma_{-,\frac{\varrho}{2}}} (\partial_\nu u u_\lambda^-)dxdt\Big|, \tag{6.4.52}$$

where $Z(t,x)$ was defined in (6.4.43).

Next, by using (6.4.46) and (6.4.47) we get

$$\begin{aligned}
\Big|\int_{\Sigma_{-,\frac{\varrho}{2}}} \partial_\nu u u_\lambda^- dxdt\Big| &\le \|\partial_\nu u\|_{H^{\frac{1}{2},\frac{1}{4}}(\Sigma_{-,\frac{\varrho}{2}})}\|u_\lambda^-\|_{H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma_{-,\frac{\varrho}{2}})} \\
&\le \|\Lambda_1^\sharp(\mu_+) - \Lambda_2^\sharp(\mu_+)\|_{H^{\frac{1}{2},\frac{1}{4}}(\Sigma_{-,\frac{\varrho}{2}})}\epsilon^{\frac{1}{2}}e^{\lambda|\Omega|-\lambda^2(T_*-\epsilon)}[|\Omega|\min(1,\lambda^{\frac{3}{8}}\epsilon^{\frac{1}{2}}) + \lambda^{-\frac{1}{4}}] \\
&\le e^{\lambda|\Omega|-\lambda^2(T_*-\epsilon)}\epsilon^{\frac{1}{2}}[|\Omega|\min(1,\lambda^{\frac{3}{8}}\epsilon^{\frac{1}{2}}) + \lambda^{-\frac{1}{4}}]\|\Lambda_1^\sharp - \Lambda_2^\sharp\|\|\mu_+\|_{H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma_{-,\frac{\varrho}{2}})} \\
&\le C\|\Lambda_1^\sharp - \Lambda_2^\sharp\|e^{2(\lambda|\Omega|+\lambda^2\epsilon)}\epsilon[|\Omega|\min(1,\lambda^{\frac{3}{8}}\epsilon^{\frac{1}{2}}) + \min(1,\lambda^{\frac{3}{8}}\epsilon^{\frac{1}{2}})\lambda^{-1}R^2 + \lambda^{-1}].
\end{aligned} \tag{6.4.53}$$

Similarly, by using (6.4.47), we get

$$\begin{aligned}
&\Big|\int_{\Sigma_{+,\frac{\varrho}{2}}} \partial_\nu u(e^{-\varphi_\lambda}\Big(1 - e^{-\lambda^{\frac{1}{4}}(T_*+\epsilon-t}} + \Psi_\lambda^-\Big)dxdt\Big| \\
&\le \|\partial_\nu u e^{-\varphi_\lambda}\|_{L^2(\Sigma_{+,\frac{\varrho}{2}})}\|1 - e^{-\lambda^{\frac{1}{4}}(T_*+\epsilon-t)}\|_{L^2(\Sigma_{+,\frac{\varrho}{2}})} \\
&\le C\epsilon^{\frac{1}{2}}\|\partial_\nu u e^{-\varphi_\lambda}\|_{L^2(\Sigma_{+,\frac{\varrho}{2}})}(\min(1,\lambda^{\frac{3}{8}}\epsilon^{\frac{1}{2}})).
\end{aligned} \tag{6.4.54}$$

On the other hand, by Lemma 6.3.1 we find that

$$\begin{aligned}
\varrho\lambda\int_{\Sigma_{+,\frac{\varrho}{2}}} |\partial_\nu u|^2 e^{2-\varphi_\lambda}dxdt &\le \lambda\int_{\Sigma_0^+(\omega)} (\omega\cdot\nu)|\partial_\nu u|^2 e^{-2\varphi_\lambda}dxdt \\
&\le C\int_{Q_*} |\partial_t u - \Delta u + q(x)u|^2 e^{-2\varphi_\lambda}dxdt + \int_{\Sigma_0^-(\omega)} |\partial_\nu u|^2 e^{-2\varphi_\lambda}dxdt \\
&\le C\int_{Q_*} |q(x)u_\lambda^+|^2 e^{-2\varphi_\lambda}dxdt + e^{2(\lambda|\Omega|+\lambda^2(T_0+\epsilon))}\|\partial_\nu u\|^2_{L^2(\Sigma_0^-(\omega))}
\end{aligned}$$

$$
\begin{aligned}
&\leq C\int_{Q_*}|q(x)((1-e^{-\lambda^{\frac{3}{4}}(t-T_*+\epsilon)})e^{-ix\cdot\xi}+\Psi_\lambda^+))|^2dxdt+e^{2(\lambda|\Omega|+\lambda^2(T_0+\epsilon))}\|\partial_\nu u\|^2_{H^{\frac{1}{2},\frac{1}{4}}(\Sigma_0^-(\omega))}\\
&\leq C(\min(1,\lambda^{\frac{3}{8}}\epsilon^{\frac{1}{2}})\epsilon^{\frac{1}{2}}+\|\Psi_\lambda^+\|^2_{L^2(Q)}+e^{2(\lambda|\Omega|-\lambda^2(T-T_0-\epsilon))}\|\Lambda_1^\sharp-\Lambda_2^\sharp\|^2)\\
&\leq C(\min(1,\lambda^{\frac{3}{8}}\epsilon^{\frac{1}{2}})\epsilon^{\frac{1}{2}}+\min(1,\lambda^{\frac{3}{8}}\epsilon^{\frac{1}{2}})\epsilon^{\frac{1}{2}}R^4\lambda^{-2}+\epsilon^{\frac{1}{2}}\lambda^{-\frac{1}{2}}\\
&+e^{2(\lambda|\Omega|-\lambda^2(T-T_0-\epsilon))}(\min(1,\epsilon\lambda^{\frac{3}{8}}\epsilon^{\frac{1}{2}})\lambda^{-1}R^2+\lambda^{-\frac{1}{4}})\|\Lambda_1^\sharp-\Lambda_2^\sharp\|^2).
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
&\|\partial_\nu ue^{-\varphi_\lambda}\|_{L^2(\Sigma_{+,\frac{\rho}{2}})}\\
&\qquad\leq C(\min(1,\lambda^{\frac{3}{8}}\epsilon^{\frac{1}{2}})\epsilon(\lambda^{-\frac{1}{4}}+|\Omega|+R^2\lambda^{-1})\\
&\qquad+e^{2(\lambda|\Omega|-\lambda^2(T-T_0-\epsilon)}\|\Lambda_1^\sharp-\Lambda_2^\sharp\|(\min(1,\epsilon\lambda^{\frac{3}{8}}\epsilon^{\frac{1}{2}})\lambda^{-1}R^2+\lambda^{-\frac{1}{4}})). \quad (6.4.55)
\end{aligned}
$$

Then from (6.4.54) and (6.4.55), we obtain

$$
\begin{aligned}
&\Big|\int_{\Sigma_{+,\frac{\rho}{2}}}\partial_\nu ue^{-\varphi_\lambda}\Big(1-e^{-\lambda^{\frac{1}{4}}(T_*+\epsilon-t)})\Big)dxdt\Big|\\
&\qquad\leq C\epsilon^{\frac{1}{2}}(\min(1,\lambda^{\frac{3}{8}}\epsilon^{\frac{1}{2}})(|\Omega|+R^2\lambda^{-1}\\
&\qquad+e^{2(\lambda|\Omega|-\lambda^2(T-T_0-\epsilon)}(\min(1,\epsilon\lambda^{\frac{3}{8}}\epsilon^{\frac{1}{2}})\lambda^{-1}R^2+\lambda^{-\frac{1}{4}}))\|\Lambda_1^\sharp-\Lambda_2^\sharp\|). \quad (6.4.56)
\end{aligned}
$$

As in the proof of Lemma 6.4.7, we choose $\lambda=\epsilon^{-\frac{1}{2}}$. Then by (6.4.49), (6.4.52),(6.4.53) and (6.4.56), we get

$$
|\widehat{q}(\xi)|\leq C(R^2\epsilon^{\frac{13}{16}}+\epsilon^{-\frac{1}{2}}+e^{2\epsilon^{-\frac{1}{2}}|\Omega|}\|\Lambda_1^\sharp-\Lambda_2^\sharp\|(\epsilon^{\frac{13}{16}}R^2+\epsilon^{\frac{7}{16}})).
$$

This completes the proof of lemma. □

We are now ready to state the second main result of this chapter.

Theorem 6.4.10. *Let $q_j\in\mathscr{M}$, $j=1,2$. Then, there exist a $C>0$, that depends only on M,Ω, T, and $\mu\in(0,1)$, such that*

$$
\|q\|_{H^{-1}(\Omega)}\leq C\left(\|\Lambda_1^\sharp-\Lambda_2^\sharp\|+(\log|\log\|\Lambda_1^\sharp-\Lambda_2^\sharp\||)^{-2/\mu}\right)^{\mu/2}.
$$

Proof. By Lemma 6.4.9, we have

$$
|\widehat{q}(\xi)|\leq C(\epsilon^{-\frac{1}{2}}+\epsilon^{\frac{13}{16}}R^2+e^{2\epsilon^{-\frac{1}{2}}|\Omega|}\|\Lambda_1^\sharp-\Lambda_2^\sharp\|(\epsilon^{\frac{13}{16}}R^2+\epsilon^{\frac{7}{16}})).
$$

In this inequality, we take $R = \epsilon^{-s}$, $s > 0$, and obtain

$$|\widehat{q}(\xi)| \leq C(\epsilon^{-\frac{1}{2}} + \epsilon^{\frac{13}{16}-2s} + e^{2\epsilon^{-\frac{1}{2}}|\Omega|}\|\Lambda_1^\sharp - \Lambda_2^\sharp\|). \tag{6.4.57}$$

For a fixed $\tau > 0$, we set $F_\tau(\xi) = \widehat{q}(\tau\xi)$, for all $\xi \in \mathbb{R}^n$. It is easy to see that F_τ is analytic and that

$$\begin{aligned}|\partial^\gamma F_\tau(\xi)| &= |\partial^\gamma \int_{\mathbb{R}^n} q(x)e^{-ix\tau\cdot\xi}dx| \\ &= \Big|\int_{\mathbb{R}^n} q(x)(-i)^{|\gamma|}\tau^{|\gamma|}|x|^\gamma e^{-i\tau x\cdot\xi}dx\Big|.\end{aligned}$$

This entails that

$$|\partial^\gamma F_\tau(\tau)| \leq \int_{\mathbb{R}^n} |q(x)|\tau^{|\gamma|}(|x|)^{|\gamma|}dx \leq \|q\|_{L^1(\Omega)}\tau^{|\gamma|}(\mathrm{Diam}(\Omega))^{|\gamma|} \leq C|\gamma|!((\mathrm{Diam}(\Omega))^{|\gamma|}e^\tau,$$

for some constant $\mu \in (0,1)$, where C depends only on M. Then, applying Lemma 4.2.4 to the set $\mathscr{O} = K_{\varrho,R} \cap B(0,1)$ with $M = Ce^\tau$ and $\eta = (\mathrm{Diam}(\Omega))^{-1}$, we get that

$$|F_\tau(\xi)| = |\widehat{q}(\tau\xi))| \leq Ce^{\tau(1-\mu)}\|F_\tau\|^\mu_{L^\infty(\mathscr{O})}, \quad \xi \in B(0,1).$$

Now, we will find an estimate for the Fourier transform of q in a suitable ball. Hence by the fact that $\tau K_{\varrho,R} = \{\tau\xi,\ \xi \in K_{\varrho,R}\} = K_{\varrho,R}$, we obtain for all $\xi \in B(0,\tau)$ that

$$\begin{aligned}|\widehat{q}(\xi)| = |F_\tau(\tau^{-1}\xi)| &\leq Ce^{\tau(1-\mu)}\|F_\tau\|^\mu_{L^\infty(\mathscr{O})} \\ &\leq Ce^{\tau(1-\mu)}\|\widehat{q}\|^\mu_{L^\infty(B(0,\tau)\cap K_{\varrho,R})} \\ &\leq Ce^{\tau(1-\mu)}\|\widehat{q}\|^\mu_{L^\infty(K_{\varrho,R})}.\end{aligned} \tag{6.4.58}$$

Next we decompose the $H^{-1}(\mathbb{R}^n)$ norm of q into the sum

$$\begin{aligned}\|q\|^{2/\mu}_{H^{-1}(\mathbb{R}^n)} &= \left(\int_{|\xi|<\tau}(1+|\xi|^2)^{-1}|\widehat{q}(\xi|^2d\xi + \int_{|\xi|\geq\tau}(1+|\xi|^2)^{-1}|\widehat{q}(\xi)|^2d\xi\right)^{1/\mu} \\ &\leq C\left(\tau^n\|\widehat{q}\|^2_{L^\infty(B(0,\tau))} + \tau^{-2}\|q\|^2_{L^2(\mathbb{R}^n)}\right)^{1/\mu}.\end{aligned}$$

Therefore, by using (6.4.58), we get

$$\begin{aligned}\|q\|^{2/\mu}_{H^{-1}(\mathbb{R}^n)} &\leq C\left(\tau^n e^{2\tau(1-\mu)}(\epsilon^{-\mu} + \epsilon^{\frac{13}{4}\mu - s\mu} + e^{2\mu\epsilon^{-\frac{1}{2}}|\Omega|}\|\Lambda_1^\sharp - \Lambda_2^\sharp\|^{2\mu} + \tau^{-2}\right)^{1/\mu} \\ &\leq C\left(\tau^{\frac{n}{\mu}}e^{2\frac{\tau(1-\mu)}{\mu}}\epsilon^{-1} + \tau^{\frac{n}{\mu}}e^{2\frac{\tau(1-\mu)}{\mu}}e^{2\epsilon^{-\frac{1}{2}}}\|\Lambda_1^\sharp - \Lambda_2^\sharp\|^2 + \tau^{-2/\mu}\right).\end{aligned} \tag{6.4.59}$$

Let $\tau_0 \in (0,1)$ and for $0 < \tau < \tau_0$, set

$$\epsilon = \tau^{4\frac{n+2}{\mu}} e^{c\frac{\tau(1-\mu)}{\mu}}.$$

From $\tau < \tau_0$, we can assume that $0 < \epsilon < \epsilon_0$. Then $\tau^{\frac{n}{\mu}} e^{2\frac{\tau(1-\mu)}{\mu}} \epsilon^{-1} = \tau^{-2/\mu}$ and (6.4.59) yield

$$\|q\|_{H^{-1}(\mathbb{R}^n)} \leq C\left(\tau^{\frac{n}{\mu}} e^{\phi(\tau)} \|\Lambda_1^\sharp - \Lambda_2^\sharp\|^2 + \tau^{-2/\mu}\right)^{\mu/2},$$

where

$$\phi(\tau) = 2\frac{\tau(1-\mu)}{\mu} + \tau^{-\frac{4}{\mu}} e^{-2\frac{\tau(1-\mu)}{\mu}}.$$

Since $\tau^{\frac{n}{\mu}} e^{\phi(\tau)} \leq e^{e^{A\tau}}$, where A depends only on n, μ and Ω, we have

$$\|q\|_{H^{-1}(\Omega)} \leq \|q\|_{H^{-1}(\mathbb{R}^n)} \leq C\left(e^{e^{A\tau}} \|\Lambda_1^\sharp - \Lambda_2^\sharp\|^2 + \tau^{-2/\mu}\right)^{\mu/2},$$

where C is a positive constant depending only on M, Ω and T.

Now, in order to minimize the right-hand side with respect to τ, we set

$$\tau = \frac{1}{A} \log|\log\|\Lambda_1^\sharp - \Lambda_2^\sharp\||.$$

Thus, we have the following estimation :

$$\|q\|_{H^{-1}(\Omega)} \leq C\left(\|\Lambda_1^\sharp - \Lambda_2^\sharp\| + (\log|\log\|\Lambda_1^\sharp - \Lambda_2^\sharp\||)^{-2/\mu}\right)^{\mu/2}.$$

□

6.5 Second approach: time-localized CGO solution

6.5.1 Construction of CGO solutions

In this section, we construct CGO solutions for the parabolic equation, whose main part is supported on a suitable subinterval of $(0,T)$. These functions are obtained by multiplying the CGO functions introduced in [32] by a truncation function $h \in C_0^\infty(0,T)$.

Lemma 6.5.1. *Let $h \in C_0^\infty(0,T)$, let $\lambda \geq \lambda_0$ and let $\xi \in \mathbb{R}^n$ satisfy $\xi \cdot \omega = 0$. Then there exists $\Psi_\lambda^+ \in \mathscr{K}_+(Q)$ satisfying*

$$\begin{cases} (\partial_t - \Delta + q(x))(\Psi_\lambda^+ e^{\varphi_\lambda}) = -(\partial_t - \Delta + q(x))(e^{\varphi_\lambda} h(t) e^{-ix\cdot\xi}), & \text{in } Q, \\ \Psi_\lambda^+(t,x) = 0, & \text{on } \Sigma^-, \\ \Psi_\lambda^+(0,x) = 0, & \text{in } \Omega. \end{cases} \tag{6.5.60}$$

Moreover, there exists a constant $C > 0$ depending only M, T and Ω, such that

$$\|\Psi_\lambda^+\|_{L^2(Q)} \leq \frac{C}{\lambda} \langle \xi \rangle^2 \|h\|_{W^{1,2}(0,T)}, \tag{6.5.61}$$

where $\langle \xi \rangle = \sqrt{1 + |\xi|^2}$.

Proof. By a simple calculation, we get

$$-(\partial_t - \Delta + q(x))(\Psi_\lambda^+ e^{\varphi_\lambda}) = R,$$

where $R = e^{\varphi_\lambda} e^{-ix\cdot\xi}[(|\xi|^2 + q)h(t) - h'(t)]$.

Since

$$\|R\|_{\mathscr{L}_-^2(Q)} \leq C((1 + |\xi|^2)\|h\|_{L^2(0,T)} + \|h'\|_{L^2(0,T)}) \leq C\langle \xi \rangle^2 \|h\|_{W^{1,2}(0,T)}, \tag{6.5.62}$$

by Lemma 6.4.1, there exists $w \in \mathscr{K}_+(Q)$, satisfying

$$\begin{cases} (\partial_t - \Delta + q(x)w = R, & \text{in } Q, \\ w(t,x) = 0, & \text{on } \Sigma^-, \\ w(0,x) = 0, & \text{in } \Omega, \end{cases} \tag{6.5.63}$$

and

$$\|w\|_{\mathscr{L}_-^2(Q)} \leq C\lambda^{-1} \|R\|_{\mathscr{L}_-^2(Q)}, \tag{6.5.64}$$

for some constant C, which depends only on M and Ω.

Evidently, $\Psi_\lambda^+ = e^{-\varphi_\lambda} w$ satisfies (6.5.60). To prove that Ψ_λ^+ satisfies the estimate (6.5.61), it is enough to see that we have

$$\|\Psi_\lambda^+\|_{L^2(Q)} = \|w\|_{\mathscr{L}_-^2(Q)} \leq C\lambda^{-1} \|R\|_{\mathscr{L}_-^2(Q)} \leq C\|h\|_{W^{1,2}(0,T)} \lambda^{-1} \langle \xi \rangle^2.$$

From (6.5.62) and (6.5.64). This completes the proof.

□

In the same way, we have:

Lemma 6.5.2. *Let $h \in C_0^\infty(0,T)$ and let $\lambda \geq \lambda_0$. Then, there exists $\Psi_\lambda^- \in \mathscr{K}_-(Q)$ such that*

$$\begin{cases} (-\partial_t - \Delta + q(x))(\Psi_\lambda^- e^{-\varphi_\lambda}) = -(-\partial_t - \Delta + q(x))(e^{-\varphi_\lambda} h(t)), & in \quad Q, \\ \Psi_\lambda^-(t,x) = 0, & on \quad \Sigma^+, \\ \Psi_\lambda^-(T,x) = 0, & in \quad \Omega. \end{cases} \tag{6.5.65}$$

Moreover, there exists a constant $C > 0$, depending only on M, T and Ω, such that

$$\|\Psi_\lambda^-\|_{L^2(Q)} \leq \frac{C}{\lambda} \|h\|_{W^{1,2}(0,T)}. \tag{6.5.66}$$

Remark 6.5.3. *Notice that if supp $h = (T_* - \epsilon, T_* + \epsilon)$, for some $T_* \in (0,T)$ and $\epsilon \in min(T_*, T - T_*)$, then $\Psi_\lambda^+(t,\cdot) = 0$ for $t \in (0, T_* - \epsilon)$ and $\Psi_\lambda^-(t,\cdot) = 0$ for $t \in (T_* + \epsilon, T)$.*

Having defined the remainder terms Ψ_λ^+ and Ψ_λ^- with Lemmas 6.5.1 and 6.5.2, we turn now to building the principal part of the CGO functions needed by the proof of the stability result.

Lemma 6.5.4. *Let $q \in \mathscr{M}$. Then, for any $h \in C_0^\infty(0,T)$ and all $\lambda \geq \lambda_0$, the following system*

$$\begin{cases} \partial_t u - \Delta u + q(x) u = 0 & in \quad Q, \\ u(0,x) = 0 & in \quad \Omega, \end{cases} \tag{6.5.67}$$

admits a solution of the form

$$u_\lambda^+ = e^{\varphi_\lambda}(h(t) e^{-ix\cdot\xi} + \Psi_\lambda^+),$$

where Ψ_λ^+ is such that

$$\|\Psi_\lambda^+\|_{L^2(Q)} \leq \frac{C}{\lambda} \langle \xi \rangle^2 \|h\|_{W^{1,2}(0,T)}, \tag{6.5.68}$$

for some constant $C > 0$, which depends only on M, T and Ω, and verifies $\Psi_\lambda^+ = 0$ on Σ^-.

Proof. As we want u_λ^+ to be in the form $u_\lambda^+ = e^{\varphi_\lambda}(h(t)e^{-ix\cdot\xi} + \Psi_\lambda^+)$, Ψ_λ^+ is the solution of

$$\begin{cases} (\partial_t - \Delta + q(x))(\Psi_\lambda^+ e^{\varphi_\lambda}) = -(\partial_t - \Delta + q(x))(e^{\varphi_\lambda} h(t) e^{-ix\cdot\xi}) & \text{in} \quad Q, \\ \Psi_\lambda^+(t,x) = 0, & \text{on} \quad \Sigma^-, \\ \Psi_\lambda^+(0,x) = 0, & \text{in} \quad \Omega, \end{cases} \tag{6.5.69}$$

Thus, Lemma 6.5.1 provides the desired result. □

Similarly, by invoking Lemma 6.5.2 instead of Lemma 6.5.1 we prove the following lemma.

Lemma 6.5.5. *Let $q \in \mathscr{M}$. Then for any $h \in C_0^\infty(0,T)$ and all $\lambda \geq \lambda_0$, the following system*

$$\begin{cases} -\partial_t u - \Delta u + q(x)u = 0 & in \quad Q, \\ u(T,x) = 0 & in \quad \Omega, \end{cases} \tag{6.5.70}$$

has a solution of the form

$$u_\lambda^- = e^{-\varphi_\lambda}(h(t) + \Psi_\lambda^-),$$

where Ψ_λ^- satisfies $\Psi_\lambda^- = 0$ on Σ^+ and

$$\|\Psi_\lambda^-\|_{L^2(Q)} \leq \frac{C}{\lambda}\|h\|_{W^{1,2}(0,T)}, \tag{6.5.71}$$

for some constant $C > 0$ which depends only on M, T and Ω.

Remark 6.5.6. *Notice that if supp $h = (T_* - \epsilon, T_* + \epsilon)$ for some $T_* \in (0,T)$ and $\epsilon \in min(T_*, T - T_*)$, then we have $u_\lambda^+(t,\cdot) = 0$ for $t \in (0, T_* - \epsilon)$ and $u_\lambda^-(t,\cdot) = 0$ for $t \in (T_* + \epsilon, T)$.*

6.5.2 Stable determination from the full data

In this subsection we proceed as in Subsection 6.4.2 in order to prove our main stability result, but we use here the CGO functions built in Subsection 6.5.1. We start by the following preliminary estimate which relates the difference of two potentials q_1 and q_2 to the difference of their D-to-N maps. In this section we

write $\mathscr{L}^2_{\Sigma,\pm,\varphi_\lambda\gamma} = L^2(\Sigma^\pm, e^{\pm\varphi_\lambda}\gamma dxdt)$

Lemma 6.5.7. *Let $h \in C_0^\infty(0,T)$ be supported in $(T_* - \epsilon, T_* + \epsilon)$, let $q_j \in \mathscr{M}$, $j = 1,2$, let $R > 0$, and let $\xi \in \mathbb{R}^n$ satisfy $\xi \cdot \omega = 0$. Then, there exists two constants $\lambda_0 > 0$ and $C > 0$, depending only on M and Ω, and $\epsilon_0 \in (0,1)$, such that*

$$|\widehat{q}(\xi)| \leq C(\epsilon^{-\frac{1}{2}}R^2 + \|\tilde{\Lambda}_2 - \tilde{\Lambda}_1\| e^{2\epsilon^{-\frac{1}{2}}|\Omega|}[|\Omega| + \epsilon^{-\frac{1}{2}} + \epsilon^{-\frac{1}{2}}R^2]), \tag{6.5.72}$$

for any $|\xi| \leq R$ and $0 < \epsilon < \epsilon_0$. Here, $\|\tilde{\Lambda}_2 - \tilde{\Lambda}_1\|$ stands for the norm of $\tilde{\Lambda}_2 - \tilde{\Lambda}_1$ in $\mathscr{B}\left(H^{-1/2,-1/4}(\Sigma_0); H^{1/2,1/4}(\Sigma_0)\right)$.

Proof. Let $h \in C_0^\infty(0,T)$ be supported in $(T_* - \epsilon, T_* + \epsilon)$ for some $\epsilon \in (0, \min(T_*, T - T_*))$. Namely, we take

$$h(t) = \epsilon^{-\frac{1}{2}}\chi(\frac{t - T_*}{\epsilon}),$$

where $\chi \in C_0^\infty(\mathbb{R})$ is a positive function supported in the unit ball and such that $\|\chi\|_{L^2(\mathbb{R})} = 1$. In view of Lemma 6.5.4, there exists a solution u_λ^+ to the equation

$$\begin{cases} \partial_t u - \Delta u + q_2(x)u = 0 & \text{in} \quad Q, \\ u(0,x) = 0 & \text{in} \quad \Omega, \end{cases} \tag{6.5.73}$$

in the form

$$u_\lambda^+ = e^{\varphi_\lambda}(h(t)e^{-ix\cdot\xi} + \Psi_\lambda^+),$$

where Ψ_λ^+ satisfies (6.5.68). Next, let us denote by $\mu_+(t,x)$ the function

$$\mu_+(t,x) = e^{\varphi_\lambda}h(t)e^{-ix\cdot\xi}, \quad (t,x) \in \Sigma^-.$$

Evidently, $\mu_+(t,x) \in \mathscr{L}^2_{\Sigma,-,-\varphi_\lambda\gamma^{-1}}$.

We denote by u_1 the $\mathscr{K}_+(Q)$-solution of

$$\begin{cases} \partial_t u_1 - \Delta u_1 + q_1(x)u_1 = 0 & \text{in} \quad Q, \\ u_1(0,x) = 0 & \text{in} \quad \Omega, \\ u_1(t,x) = \mu_+(t,x), & \text{on} \quad \Sigma^-, \end{cases} \tag{6.5.74}$$

given by Lemma 6.2.1.

Notice that $\mu_+(t,x) = 0$ for a.e $t \in (0, T_* - \epsilon)$, hence $u_1(t,\cdot) = 0$ for a.e $t \in$

$(0, T_* - \epsilon)$. Put $u = u_1 - u_\lambda^+$. Then, $u \in \mathscr{K}_+(Q)$ is a solution to the following system

$$\begin{cases} \partial_t u - \Delta u + q_1(x)u = q(x)u_\lambda^+ & \text{in} \quad Q, \\ u(0,x) = 0 & \text{in} \quad \Omega, \\ u(t,x) = 0, & \text{on} \quad \Sigma^-, \end{cases} \tag{6.5.75}$$

where $q(x) = q_1(x) - q_2(x)$. On the other hand by Lemma 6.5.5, there exists a solution $u_\lambda^- \in \mathscr{K}_-(Q)$ to the backward problem

$$\begin{cases} -\partial_t u - \Delta u + q_1(x)u = 0 & \text{in} \quad Q, \\ u^-(T,x) = 0 & \text{in} \quad \Omega, \end{cases} \tag{6.5.76}$$

in the form

$$u_\lambda^- = e^{-\varphi_\lambda}(h(t) + \Psi_\lambda^-),$$

where Ψ_λ^- satisfies (6.5.71).

Bearing in mind that $u(t,\cdot) = u_\lambda^+(t,\cdot) = 0$ for a.e. $t \in (0, T_* - \epsilon)$ and that $u_\lambda^-(t,\cdot) = 0$ for a.e. $t \in (T_* + \epsilon, T)$, multiplying both sides of the first line of (6.5.75) by u_λ^- and integrating over $Q_* = (T_* - \epsilon, T_* + \epsilon) \times \Omega$, we get that

$$\langle (\partial_t - \Delta + q_1)u, u_\lambda^- \rangle_{L^2(Q_*)} = \langle q u_\lambda^+, u_\lambda^- \rangle_{L^2(Q_*)}. \tag{6.5.77}$$

Thus, putting $\Sigma_0 = (T_* - \epsilon, T_* + \epsilon) \times \partial\Omega$, remembering that $u_\lambda^+(T_* - \epsilon, \cdot) = u_\lambda^-(T_* + \epsilon, \cdot) = 0$ and integrating by parts in (6.5.77), we get that

$$\langle (\tilde{\Lambda}_2 - \tilde{\Lambda}_1)\mu_+, u_\lambda^- \rangle_{H^{1/2,1/4}(\Sigma_0), H^{-1/2,-1/4}(\Sigma_0)} = \langle q u_\lambda^+, u_\lambda^- \rangle_{L^2(Q_*)}. \tag{6.5.78}$$

By replacing u_λ^+ and u_λ^- by their expressions, applying Cauchy Schwarz inequality, we get

$$\begin{aligned} \Big| \int_{T_*-\epsilon}^{T_*+\epsilon} h^2(t)dt \int_\Omega q(x)e^{-ix\cdot\xi}dx \Big| \leq \Big| \int_{Q_*} q(x)h(t)e^{-ix\cdot\xi}\Psi_\lambda^- dxdt \Big| + \Big| \int_{Q_*} q(x)h(t)\Psi_\lambda^+ dxdt \Big| \\ + \Big| \int_{Q_*} q(x)\Psi_\lambda^+\Psi_\lambda^- dxdt \Big| + |\langle (\tilde{\Lambda}_1 - \tilde{\Lambda}_2)\mu_+, \mu_- \rangle_{H^{-1/2,-1/4}(\Sigma_0), H^{1/2,1/4}(\Sigma_0)}. \end{aligned} \tag{6.5.79}$$

Thus, by (6.5.68) and (6.5.71), we get

$$\Big| \int_{T_*-\epsilon}^{T_*+\epsilon} h^2(t)dt \int_\Omega q(x)e^{-ix\cdot\xi}dx \Big|$$

$$\leq C\|h\|^2_{W^{1,2}}(\frac{1}{\lambda}+\frac{1}{\lambda}\langle\xi\rangle^2+\frac{1}{\lambda^2}\langle\xi\rangle^2)+|\langle\Lambda_1'-\Lambda_2')(\mu_+),\mu_-\rangle_{H^{-1/2,-1/4}(\Sigma_0),H^{1/2,1/4}(\Sigma_0)}|$$
$$\leq \frac{C}{\lambda}\|h\|^2_{W^{1,2}(0,T)}(1+\langle\xi\rangle^2)+|\langle\tilde{\Lambda}_1-\tilde{\Lambda}_2)\mu_+,\mu_-\rangle_{H^{-1/2,-1/4}(\Sigma_0),H^{1/2,1/4}(\Sigma_0)}|,$$

where $C>0$ is a positive constant which depends only on M and Ω. Using the fact that $\tilde{\Lambda}_2-\tilde{\Lambda}_1\in\mathscr{B}\left(H^{-1/2,-1/4}(\Sigma_0);H^{1/2,1/4}(\Sigma_0)\right)$, we obtain

$$|\langle(\tilde{\Lambda}_2-\tilde{\Lambda}_1)\mu_+,\mu_-\rangle_{H^{1/2,1/4}(\Sigma_0),H^{-1/2,-1/4}(\Sigma_0)}|$$
$$\leq\|\tilde{\Lambda}_1-\tilde{\Lambda}_2\|_{\mathscr{B}\left(H^{-1/2,-1/4}(\Sigma_0);H^{1/2,1/4}(\Sigma_0)\right)}\|\mu_+\|_{H^{-1/2,-1/4}(\Sigma_0^-)}\|\mu_-\|_{H^{-1/2,-1/4}(\Sigma_0^+)}$$

Moreover, by (6.5.68), we get

$$\|\mu_+\|_{H^{-1/2,-1/4}(\Sigma_0)}\leq C\|u_\lambda^+\|_{\mathscr{K}^+(Q_*)}\leq C\|e^{\varphi_\lambda}(h(t)e^{-ix\cdot\xi}+\Psi_\lambda^+)\|_{L^2(Q_*)}$$
$$\leq C\|h\|_{W^{1,2}(0,T)}e^{\lambda|\Omega|+\lambda^2(T_*+\epsilon)}[|\Omega|+\lambda^{-1}R^2], \tag{6.5.80}$$

and by (6.5.71), we find that

$$\|\mu_-\|_{H^{-1/2,-1/4}(\Sigma_0)}\leq C\|u_\lambda^-\|_{\mathscr{K}^-(Q_*)}\leq C\|e^{-\varphi_\lambda}h(t)+\Psi_\lambda^-)\|_{L^2(Q_*)}$$
$$\leq C\|h\|_{W^{1,2}(0,T)}e^{\lambda|\Omega|-\lambda^2(T_*-\epsilon)}[|\Omega|+\lambda^{-1}], \tag{6.5.81}$$

where $C>0$ depends only on M, Ω and T. Thus,

$$|\langle(\tilde{\Lambda}_2-\tilde{\Lambda}_1)\mu_+,\mu_-\rangle_{H^{1/2,1/4}(\Sigma_0),H^{-1/2,-1/4}(\Sigma_0)}|$$
$$\leq C\|h\|^2_{W^{1,2}(0,T)}\|\tilde{\Lambda}_1-\tilde{\Lambda}_2\|e^{2(\lambda|\Omega|+\lambda^2\epsilon)}[|\Omega|+\lambda^{-1}+\lambda^{-1}R^2].$$

Using the fact that $\|h\|_{W^{1,2}(0,T)}\leq C\epsilon^{-\frac{1}{2}}$, for some positive constant C which depends only on χ, we get

$$\left|\int_{\mathbb{R}^n}q(x)e^{-ix\cdot\xi}dx\right|\leq C\|h\|^2_{W^{1,2}(0,T)}(\lambda^{-1}R^2+\|\tilde{\Lambda}_1-\tilde{\Lambda}_2\|e^{2(\lambda|\Omega|+\lambda^2\epsilon)}[|\Omega|+\lambda^{-1}+\lambda^{-1}R^2]$$
$$\leq C\epsilon^{-1}(\lambda^{-1}R^2+\lambda^{-1}+\|\tilde{\Lambda}_1-\tilde{\Lambda}_2\|e^{2(\lambda|\Omega|+\lambda^2\epsilon)}[|\Omega|+\lambda^{-1}+\lambda^{-1}R^2]).$$

We choose $\lambda=\epsilon^{-\frac{1}{2}}$, which amounts to saying that $\epsilon_0=\min(1,\lambda_0^{-2})$, and we find that

$$|\widehat{q}(\xi)| \leq C(\epsilon^{-\frac{1}{2}}R^2 + \|\tilde{\Lambda}_1 - \tilde{\Lambda}_2\| e^{2\epsilon^{-\frac{1}{2}}|\Omega|}[\epsilon^{-1}|\Omega| + \epsilon^{-\frac{1}{2}} + \epsilon^{-\frac{1}{2}}R^2]).$$

This completes the proof of the lemma.

□

We state now the main result of this subsection. For this purpose we set for all $\epsilon > 0$:

$$\phi_\epsilon(t) = \begin{cases} \epsilon^{-(\frac{n}{8}+1)} e^{2|\Omega|\epsilon^{-\frac{1}{2}}} t, & \text{in} \quad t \geq e^{-2|\Omega|\epsilon^{-\frac{1}{2}}}, \\ |\log t|^{-(n+2)}, & \text{in} \quad 0 < t < e^{-2|\Omega|\epsilon^{-\frac{1}{2}}}. \end{cases}$$

Theorem 6.5.8. *Let $q_1, q_2 \in \mathscr{M}$. Then there exists $\epsilon_0 = \epsilon_0(T, \Omega, M) > 0$, and $C = C(T, \Omega, M) > 0$, $\forall \epsilon \in (0, \epsilon_0)$, we have*

$$\|q_1 - q_2\|_{H^{-1}(\Omega)} \leq C\phi_\epsilon(\|\tilde{\Lambda}_2 - \tilde{\Lambda}_1\|).$$

Proof. Firstly, we need to decompose the $H^{-1}(\mathbb{R}^n)$ norm of q into the following sum

$$\begin{aligned} \|q\|^2_{H^{-1}(\mathbb{R}^n)} &= \int_{|\xi|<R} (1+|\xi|^2)^{-1} |\widehat{q}(\xi)|^2 d\xi + \int_{|\xi|\geq R} (1+|\xi|^2)^{-1} |\widehat{q}(\xi)|^2 d\xi \\ &\leq CR^n \|\widehat{q}\|^2_{L^\infty(B(0,R))} + R^{-2}\|q\|^2_{L^2(\mathbb{R}^n)}. \end{aligned}$$

In light of Lemma 6.5.7, we get

$$\|q\|^2_{H^{-1}(\mathbb{R}^n)} \leq C\left(R^n((\epsilon^{-1}R^4 + \|\tilde{\Lambda}_1 - \tilde{\Lambda}_2\|^2 e^{4\epsilon^{-\frac{1}{2}}|\Omega|}[\epsilon^{-2}|\Omega|^2 + \epsilon^{-1} + \epsilon^{-1}R^4]) + R^{-2}\right)$$

In this inequality, we take $R = \epsilon^{-s}$, $s \geqslant \frac{1}{4}$, and obtain

$$\begin{aligned} \|q\|^2_{H^{-1}(\mathbb{R}^n)} &\leq C\left(\epsilon^{-(n+4)s-1} + \epsilon^{-ns}\|\tilde{\Lambda}_1 - \tilde{\Lambda}_2\|^2 e^{4\epsilon^{-\frac{1}{2}}|\Omega|}(\epsilon^{-2}|\Omega|^2 + \epsilon^{-1} + \epsilon^{-4s-1}) + \epsilon^{2s}\right) \\ &\leq C\left(\epsilon^{-(n+4)s-1} + \epsilon^{-(n+4)s-1}\|\tilde{\Lambda}_1 - \tilde{\Lambda}_2\|^2 e^{4\epsilon^{-\frac{1}{2}}|\Omega|}\right) \\ &\leq C\epsilon^{-(n+4)s-1}\left(1 + \|\tilde{\Lambda}_1 - \tilde{\Lambda}_2\|^2 e^{4\epsilon^{-\frac{1}{2}}|\Omega|}\right). \end{aligned}$$

If $\|\tilde{\Lambda}_2 - \tilde{\Lambda}_1\| \geq e^{-2|\Omega|\epsilon^{-\frac{1}{2}}}$, then we have,

$$\|q\|_{H^{-1}(\mathbb{R}^n)} \leq C\epsilon^{\frac{-(n+4)}{2}s - \frac{1}{2}} e^{2|\Omega|\epsilon^{-\frac{1}{2}}} \|\tilde{\Lambda}_2 - \tilde{\Lambda}_1\|,$$

and if $\|\tilde{\Lambda}_2 - \tilde{\Lambda}_1\| < e^{-2|\Omega|\epsilon^{-\frac{1}{2}}}$, that is if $\epsilon > \frac{4|\Omega|^2}{(\log\|\tilde{\Lambda}_2-\tilde{\Lambda}_1\|)^2}$, then we obtain

$$\|q\|_{H^{-1}(\mathbb{R}^n)} \leq C\epsilon^{-(n+4)s-1} \leq C|\log\|\tilde{\Lambda}_2 - \tilde{\Lambda}_1\||^{-(n+4)s-1}.$$

Set, $s = \frac{1}{4}$, we obtain

$$\|q\|_{H^{-1}(\mathbb{R}^n)} \leq C|\log\|\tilde{\Lambda}_2 - \tilde{\Lambda}_1\||^{-(n+2)}.$$

□

6.5.3 Stability inequality for partial boundary measurements

The notation here are the same as Subsections 6.4.3 and 6.5.2. We proceed as in Subsection 6.4.3 and derive a stability result for the potential q, from the knowledge of the partial D-to-N map. We assume in the sequel that the function h is the same as in Lemma 6.5.7.

We start with the following lemma, which gives an estimate of the Fourier transform of the difference of q_1 and q_2 in $\mathscr{M}$, in the conic set $K_{\varrho,R}$.

Lemma 6.5.9. *Let $h \in C_0^\infty(0,T)$ be supported in $(T_* - \epsilon, T_* + \epsilon)$, let $q_j \in \mathscr{M}$, $j = 1,2$, let $R > 0$, and let $\xi \in \mathbb{R}^n$ with $\xi \cdot \omega = 0$. Then, there exist two constants $\lambda_0 > 0$ and $C > 0$, depending only on M and Ω, and there exists $\epsilon_0 \in (0,1)$, such that*

$$|\widehat{q}(\xi)| \leq C(\epsilon^{-\frac{1}{4}}R^2 + e^{2\epsilon^{-\frac{1}{2}}|\Omega|}\|\Lambda_1^\sharp - \Lambda_2^\sharp\|(\epsilon^{-\frac{1}{2}} + \epsilon^{-\frac{1}{2}}R^2), \tag{6.5.82}$$

for any $|\xi| \leq R$ and $0 < \epsilon < \epsilon_0$. Here, $\|\Lambda_1^\sharp - \Lambda_2^\sharp\|$ stands for the norm of $\Lambda_1^\sharp - \Lambda_2^\sharp$ in $\mathscr{B}\left(H^{-1/2,-1/4}(\Sigma_0); H^{1/2,1/4}(\Sigma_{-,\varrho}(\omega_0')\right)$.

Proof. Use the same notations and we proceed in the same way as in the proof of Lemma 6.5.7. We obtain that

$$\int_\Sigma (\partial_\nu u)u_\lambda^- dxdt = \int_Q q(x)u_\lambda^+ u_\lambda^- dxdt.$$

By replacing u_λ^+ and u_λ^- by their expressions, we obtain

$$\left|\int_Q h(t)q(x)e^{-ix\cdot\xi}dxdt\right| \leq \left|\int_Q A(t,x)dxdt\right| + \left|\int_{\Sigma_{+,\frac{\varrho}{2}}} (\partial_\nu u)e^{-\varphi_\lambda}h(t)dxdt\right|$$

$$+\Big|\int_{\Sigma_{-,\frac{\varrho}{2}}}(\partial_\nu u)u_\lambda^- dxdt\Big|, \quad (6.5.83)$$

where

$$A(t,x)=q(x)[h(t)e^{-ix\cdot\xi}\Psi_\lambda^- + h(t)\Psi_\lambda^+ + \Psi_\lambda^+\Psi_\lambda^-].$$

By using (6.5.68), (6.5.71) and the fact that $\|h\|_{W^{1,2}(0,T)} \le C\epsilon^{-\frac{1}{2}}$, for some constant $C>0$, which depends only on χ, we get that

$$\Big|\int_Q A(t,x)dxdt\Big| \le C\frac{1}{\epsilon}(\frac{1}{\lambda}+\frac{1}{\lambda}R^2+\frac{1}{\lambda^2}R^2), \quad (6.5.84)$$

where C a positive constant depending only on M, T and Ω. Again, by using (6.5.68), (6.5.71), we get

$$\begin{aligned}\Big|\int_{\Sigma_{-,\frac{\varrho}{2}}}(\partial_\nu u)u_\lambda^- dxdt\Big| &\le \|\partial_\nu u\|_{H^{\frac{1}{2},\frac{1}{4}}(\Sigma_{-,\frac{\varrho}{2}})}\|u_\lambda^-\|_{H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma_{-,\frac{\varrho}{2}})}\\ &\le \|\Lambda_1^\sharp(\mu_+)-\Lambda_2^\sharp(\mu_+)\|_{H^{\frac{1}{2},\frac{1}{4}}(\Sigma_{-,\frac{\varrho}{2}})}\|h\|_{W^{1,2}(0,T)}e^{\lambda|\Omega|-\lambda^2(T_*-\epsilon)}[|\Omega|+\lambda^{-1}]\\ &\le C\epsilon^{-1}\|\Lambda_1^\sharp-\Lambda_2^\sharp\|e^{2(\lambda|\Omega|+\lambda^2\epsilon)}[|\Omega|+\lambda^{-1}+\lambda^{-1}R^2].\end{aligned} \quad (6.5.85)$$

Furthermore, we have

$$\Big|\int_{\Sigma_{+,\frac{\varrho}{2}}}(\partial_\nu u)(e^{-\varphi_\lambda}h(t)dxdt\Big| \le \|(\partial_\nu u)e^{-\varphi_\lambda}\|_{L^2(\Sigma_{+,\frac{\varrho}{2}})}\|h\|_{L^2(\Sigma_{+,\frac{\varrho}{2}})} \le C\|(\partial_\nu u)e^{-\varphi_\lambda}\|_{L^2(\Sigma_{+,\frac{\varrho}{2}})}.$$

On the other hand, by Lemma 6.3.1 we get that

$$\begin{aligned}\varrho\lambda\int_{\Sigma_{+,\frac{\varrho}{2}}}|\partial_\nu u|^2e^{-2\varphi_\lambda}dxdt &\le \lambda\int_{\Sigma^+(\omega)}(\omega\cdot\nu)|\partial_\nu u|^2e^{-2\varphi_\lambda}dxdt\\ &\le C\int_Q|\partial_t u-\Delta u+q(x)u|^2e^{-2\varphi_\lambda}dxdt+\int_{\Sigma^-(\omega)}|\partial_\nu u|^2e^{-2\varphi_\lambda}dxdt\\ &\le C\int_Q|q(x)u_\lambda^+|^2e^{-2\varphi_\lambda}dxdt+e^{\lambda|\Omega|-\lambda^2(T_*-\epsilon)}\int_{\Sigma^-(\omega)}|\partial_\nu u|^2dxdt\\ &\le C\int_Q|q(x)(h(t)e^{-ix\cdot\xi}+\Psi_\lambda^+))|^2dxdt+e^{\lambda|\Omega|-\lambda^2(T_*-\epsilon)}\|\partial_\nu u\|^2_{H^{\frac{1}{2},\frac{1}{4}}(\Sigma^-(\omega))}\\ &\le C(\|h\|^2_{L^2(\mathbb{R})}+\|\Psi_\lambda^+\|^2_{L^2(Q)}+e^{4\lambda|\Omega|+4\lambda^2\epsilon)}\|\Lambda_1^\sharp-\Lambda_2^\sharp\|^2)\epsilon^{-1}(1+\lambda^{-2}R^2)\\ &\le C(1+\frac{C}{\lambda^2\epsilon}R^4)+e^{4\lambda|\Omega|+4\lambda^2\epsilon}\|\Lambda_1^\sharp-\Lambda_2^\sharp\|^2\epsilon^{-1}(1+\lambda^{-2}R^2),\end{aligned}$$

where we recall that C is a positive constant depending only on M, Ω and T. Then, we have

$$\|\partial_\nu u e^{-\varphi_\lambda}\|_{L^2(\Sigma_{+,\frac{\varrho}{2}})} \leq C(\lambda^{-\frac{1}{2}} + \lambda^{-\frac{3}{2}}\epsilon^{-1}R^2) + e^{2\lambda|\Omega|+2\lambda^2\epsilon}\|\Lambda_1^\sharp - \Lambda_2^\sharp\|\epsilon^{-1}(\lambda^{-\frac{1}{2}} + \lambda^{-1}R). \tag{6.5.86}$$

We choose $\lambda = \epsilon^{-\frac{1}{2}}$. Then, by (6.5.83), (6.5.84), (6.5.85) and (6.5.86), we get

$$|\widehat{q}(\xi)| \leq C(\epsilon^{-\frac{1}{4}}R^2 + e^{2\epsilon^{-\frac{1}{2}}|\Omega|}\|\Lambda_1^\sharp - \Lambda_2^\sharp\|(\epsilon^{-\frac{1}{2}} + \epsilon^{-\frac{1}{2}}R^2).$$

This completes the proof of the lemma. □

We state now the main result of this subsection.

Theorem 6.5.10. *Let $q_j \in \mathscr{M}$, $j = 1, 2$. Then there exists two constants $C > 0$ and $\mu \in (0, 1)$, which depend only on M, Ω and T, such that*

$$\|q_1 - q_2\|_{H^{-1}(\Omega)} \leq C\left(\|\Lambda_1^\sharp - \Lambda_2^\sharp\| + (\log|\log\|\Lambda_1^\sharp - \Lambda_2^\sharp\||)^{-2/\mu}\right)^{\mu/2}.$$

Here, $\|\Lambda_1^\sharp - \Lambda_2'\|$ stands for the norm of $\Lambda_1^\sharp - \Lambda_2^\sharp$ in $\mathscr{B}\left(H^{-1/2,-1/4}(\Sigma_0); H^{1/2,1/4}(\Sigma_{-,\varrho}(\omega_0'))\right)$.

Proof. By Lemma 6.5.9 we have

$$|\widehat{q}(\xi)| \leq C\epsilon^{-\frac{1}{4}}(R^2 + e^{2\epsilon^{-\frac{1}{2}}|\Omega|}\|\Lambda_1^\sharp - \Lambda_2^\sharp\|(\epsilon^{-\frac{1}{2}} + \epsilon^{-\frac{1}{2}}R^2).$$

In this inequality, we take $R = \epsilon^{-s}$, $s > 0$, and obtain

$$\begin{aligned} |\widehat{q}(\xi)| &\leq C(\epsilon^{-2s-\frac{1}{4}} + e^{2\epsilon^{-\frac{1}{2}}|\Omega|}\|\Lambda_1^\sharp - \Lambda_2^\sharp\|(\epsilon^{-\frac{1}{2}} + \epsilon^{-2s-\frac{1}{2}}) \\ &\leq C(\epsilon^{-2s-\frac{1}{4}} + e^{2\epsilon^{-\frac{1}{2}}|\Omega|}\|\Lambda_1^\sharp - \Lambda_2^\sharp\|). \end{aligned} \tag{6.5.87}$$

For a fixed $\tau > 0$, we set $F_\tau(\xi) = \widehat{q}(\tau\xi))$ for all $\xi \in \mathbb{R}^n$. It is easy to see that F_τ is analytic and that

$$\begin{aligned} |\partial^\gamma F_\tau(\xi)| &= |\partial^\gamma \int_{\mathbb{R}^n} q(x)e^{-ix\tau\cdot\xi}dx| \\ &= \left|\int_{\mathbb{R}^n} q(x)(-i)^{|\gamma|}\tau^{|\gamma|}|x|^\gamma e^{-i\tau x\cdot\xi}dxdt\right|. \end{aligned}$$

This entails that

$$|\partial^\gamma F_\tau(\tau)| \leq \int_{\mathbb{R}^n} |q(x)|\tau^{|\gamma|}(|x|^2)^{\frac{|\gamma|}{2}}dxdt \leq C|\gamma|!(\mathrm{Diam}(\Omega)^{|\gamma|}e^\tau.$$

Then, applying Lemma 4.2.4 in the set $\mathscr{O} = K_{\varrho,R} \cap B(0,1)$ with $M = Ce^{\tau}$ and $\eta = (\mathrm{Diam}(\Omega))^{-1}$, we can find a constant $\mu \in (0,1)$ such that we have for all $\xi \in B(0,1)$,

$$|F_\tau(\xi)| = |\widehat{q}(\tau\xi))| \leq Ce^{\tau(1-\mu)}\|F_\tau\|^{\mu}_{L^\infty(\mathscr{O})}, \quad \xi \in B(0,1).$$

Here, C a positive constant depends on M, Ω and T. Now, we will find an estimate for the Fourier transform of q in a suitable ball. Hence, by the fact that $\tau K_{\varrho,R} = \{\tau\xi,\ \xi \in K_{\varrho,R}\} = K_{\varrho,R}$, we obtain for all $\xi \in B(0,\tau)$

$$\begin{aligned} |\widehat{q}(\xi)| = |F_\tau(\tau^{-1}(\xi))| &\leq Ce^{\tau(1-\mu)}\|F_\tau\|^{\mu}_{L^\infty(\mathscr{O})} \\ &\leq Ce^{\tau(1-\mu)}\|\widehat{q}\|^{\mu}_{L^\infty(B(0,\tau)\cap K_{\varrho,R})} \\ &\leq Ce^{\tau(1-\mu)}\|\widehat{q}\|^{\mu}_{L^\infty(K_{\varrho,R})}. \end{aligned} \tag{6.5.88}$$

Next, we decompose the $H^{-1}(\mathbb{R}^n)$ norm of q into the following sum

$$\begin{aligned} \|q\|^{2/\mu}_{H^{-1}(\mathbb{R}^n)} &= \left(\int_{|\xi|<\tau}(1+|\xi|^2)^{-1}|\widehat{q}(\xi|^2 d\xi + \int_{|\xi|\geq\tau}(1+|\xi|^2)^{-1}|\widehat{q}(\xi)|^2 d\xi\right)^{1/\mu} \\ &\leq C\left(\tau^n\|\widehat{q}\|^2_{L^\infty(B(0,\tau))} + \tau^{-2}\|q\|^2_{L^2(\mathbb{R}^n)}\right)^{1/\mu}. \end{aligned}$$

Therefore, by using (6.5.87) and (6.5.88), we get

$$\begin{aligned} \|q\|^{2/\mu}_{H^{-1}(\mathbb{R}^n)} &\leq C\left(\tau^n e^{2\tau(1-\mu)}\epsilon^{(-2s-\frac{1}{4})\mu} + e^{2\epsilon^{-\frac{1}{2}}|\Omega|\mu}\|\Lambda_1^\sharp - \Lambda_2^\sharp\|^{2\mu})) + \tau^{-2}\right)^{1/\mu} \\ &\leq C\left(\tau^{\frac{n}{\mu}}e^{2\frac{\tau(1-\mu)}{\mu}}\epsilon^{(-2s-\frac{1}{4})} + \tau^{\frac{n}{\mu}}e^{2\frac{\tau(1-\mu)}{\mu}}e^{2\epsilon^{-\frac{1}{2}}|\Omega|}\|\Lambda_1^\sharp - \Lambda_2^\sharp\|^2 + \tau^{-2/\mu}\right), \end{aligned} \tag{6.5.89}$$

where C a positive constant which depends only on M, Ω and T.

Let $\tau_0 \in (0,1)$ and $0 < \tau < \tau_0$. Set

$$\epsilon = \tau^{4\frac{n+2}{(8s+1)\mu}}e^{8\frac{\tau(1-\mu)}{\mu(8s+1)}}.$$

As $\tau < \tau_0$, we can assume that $0 < \epsilon < \epsilon_0$. Then, $\tau^{\frac{n}{\mu}}e^{2\frac{\tau(1-\mu)}{\mu}}\epsilon^{(-2s-\frac{1}{4})} = \tau^{-2/\mu}$ and (6.5.89) yield

$$\|q\|_{H^{-1}(\mathbb{R}^n)} \leq C\left(\tau^{\frac{n}{\mu}}e^{\phi(\tau)}\|\Lambda_1^\sharp - \Lambda_2^\sharp\|^2 + \tau^{-2/\mu}\right)^{\mu/2},$$

where

$$\phi(\tau) = 2\frac{\tau(1-\mu)}{\mu} + 2\tau^{-\frac{n+2}{(4s+3)\mu}}e^{-\frac{\tau(1-\mu)}{\mu(4s+3)}}|\Omega|.$$

Since $\tau^{\frac{n}{\mu}} e^{\phi(\tau)} \leq e^{e^{A\tau}}$, where A depends only on n, μ and Ω_0, we have

$$\|q\|_{H^{-1}(\Omega)} \leq \|q\|_{H^{-1}(\mathbb{R}^n)} \leq C\left(e^{e^{A\tau}}\|\Lambda_1^\sharp - \Lambda_2^\sharp\|^2 + \tau^{-2/\mu}\right)^{\mu/2}.$$

Now, in order to minimize the right-hand side with respect to τ, we set

$$\tau = \frac{1}{A}\log|\log\|\Lambda_1^\sharp - \Lambda_2^\sharp\||.$$

Thus, we have the following estimation

$$\|q\|_{H^{-1}(\Omega)} \leq C\left(\|\Lambda_1^\sharp - \Lambda_2^\sharp\| + (\log|\log\|\Lambda_1^\sharp - \Lambda_2^\sharp\||)^{-2/\mu}\right)^{\mu/2}.$$

□

6.6 Brief comments

Our results of Theorems 6.4.8, 6.4.10, 6.5.8 and 6.5.10 claim that the potential is log-stably determined by either the full or partial parabolic D-to-N map measured on $(T_* - \epsilon, T_* + \epsilon)$, provided ϵ is small enough. In this respect, (Theorems 6.4.8, 6.4.10, resp., 6.5.8 and 6.5.10) are equivalent. The main difference between these results lies in the multiplication constant appearing in the stability estimates. It turns in that this constant blows up as $\epsilon \to 0$ in all cases, making it impossible to claim stable recovery. But, as the prefactor in the stability estimates blows up as ϵ tends to 0, it is not possible to claim stable recovery of the parabolic potential from knowledge of parabolic D-to-N map taken at $T_* \in (0, T)$. We prove this results with two different approaches: the CGO functions are time-shifted in the first method while they are multiplied by a truncate function in the second one. This provides the same type of stability estimate, this was not clear at all at first sight.

Author's publications

The results detailed in the first part of this thesis have given rise to the following publications:

[P1] M .Bellassoued and Y.Boughanja, *An inverse problem for the Linear Boltzmann Equation with time-dependent coefficient,* Inverse Problems, 2019.

[P2] M.Bellassoued and Y.Boughanja, *Recovery of coefficients in the Linear Boltzmann Equation,* Journal of Mathematical Physics, 2019.

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