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Nicolas Torres

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# Asymptotic behavior of solutions of the elapsed time model for neural assemblies

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"Ni rire, ni pleurer, ni haïr, mais  
comprendre."

---

Baruch Spinoza

"Quand on pensait avoir toutes les  
réponses, toutes les questions ont  
changé."

---

Mario Benedetti

"Strč prst skrz krk."

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Tchécoslovaquie



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**ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF THE ELAPSED TIME MODEL FOR NEURAL ASSEMBLIES****Abstract**

The elapsed time model has been widely studied in the context of mathematical neuroscience with many open questions left. The model consists of an age-structured equation that describes the dynamics of interacting neurons described by their refractory period, i.e. the elapsed time since their last discharge. Our interest lies in weakly connected networks where mathematical tools such as the Entropy method and Doeblin's theory have proved to be useful methods in proving convergence to steady state, and highly connected networks leading to strong nonlinearities where classical perturbation methods do not apply. Moreover, we are interested in extensions of the elapsed time model by incorporating new variables such as the elapsed time since penultimate discharge and the spatial dependence with learning processes via the change in the interconnections.

To deal with this problem of strong interconnections in the elapsed time model, we choose a particular case which can be reduced to delay differential equations and we prove a general convergence result to a stationary state in the inhibitory and the weakly excitatory cases. In addition, we prove the existence of particular periodic solutions with jump discontinuities in the strongly excitatory case.

In the first extension of the classical elapsed time model we incorporate the elapsed since the penultimate discharge, obtaining a more complex system of integro-differential equations. For this new system we prove convergence to stationary state by means of Doeblin's theory in the case of weak nonlinearities in an appropriate functional setting, inspired by the case of the classical elapsed time equation. We also aim to study how the incorporation of this new elapsed time may change the general dynamics of the system.

In the second extension we focus on incorporating the spatial dependence (e.g. position of neurons across the cortex) and some learning processes to the classical elapsed time model. The learning rules that we consider are essentially inspired from the Hebbian rule, which is a classical principle in modeling the process of learning. We then obtain a new system of integro-differential equations, from which we analyze the convergence to stationary states by the means of entropy method and Doeblin's theory in the case of weak interconnections. We also consider the situation where neural activity is faster than the learning process and give conditions where one can approximate the dynamics by a solution with a similar profile of a steady state.

For each model we present some numerical simulations to observe how the parameters of the system can give different behaviors and pattern formations in order to contrast them with the theoretical results.

**Keywords:** mathematical neuroscience, elapsed time, structured equations, neural networks, periodic solutions, delay differential equations, weak interconnections, learning rule, connectivity kernel, convergence to equilibrium, entropy method, doeblin theory.

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### Résumé

Le modèle du temps écoulé a été largement étudié dans le contexte des neurosciences mathématiques, mais de nombreuses questions restent ouvertes. Le modèle consiste en une équation structurée par âge qui décrit la dynamique des neurones en interaction décrite par leur période réfractaire, c'est-à-dire le temps écoulé depuis leur dernière décharge. Nous nous intéressons aux réseaux faiblement connectés pour lesquels des outils mathématiques tels que la méthode de l'entropie et la théorie de Doeblin se sont avérés utiles pour prouver la convergence vers un état stable, et aux réseaux hautement connectés conduisant à de fortes non-linéarités pour lesquels les méthodes classiques de perturbation ne s'appliquent pas. De plus, nous nous intéressons aux extensions du modèle de temps écoulé en incorporant de nouvelles variables telles que le temps écoulé depuis la pénultième décharge et la dépendance spatiale avec les processus d'apprentissage via le changement des interconnexions.

Pour traiter le problème des interconnexions fortes dans le modèle du temps écoulé, nous choisissons un cas particulier qui peut être réduit à des équations différentielles à retard et nous prouvons un résultat général de convergence vers un état stationnaire dans les cas inhibiteurs et faiblement excitateurs. En outre, nous prouvons l'existence de solutions périodiques particulières avec des discontinuités de saut dans le cas fortement excitateur.

Dans la première extension du modèle classique du temps écoulé, nous incorporons le temps écoulé depuis l'avant-dernière décharge, obtenant ainsi un système plus complexe d'équations intégro-différentielles. Pour ce nouveau système, nous prouvons la convergence vers l'état stationnaire au moyen de la théorie de Doeblin dans le cas de non-linéarités faibles dans un cadre fonctionnel approprié, inspiré du cas de l'équation classique du temps écoulé. Nous cherchons également à étudier comment l'incorporation de ce nouveau temps écoulé peut modifier la dynamique générale du système.

Dans la deuxième extension, nous nous concentrons sur l'incorporation de la dépendance spatiale (par exemple, la position des neurones dans le cortex) et de certains processus d'apprentissage au modèle classique du temps écoulé. Les règles d'apprentissage que nous considérons sont essentiellement inspirées de la règle de Hebb, qui est un principe classique dans la modélisation du processus d'apprentissage. Nous obtenons alors un nouveau système d'équations intégro-différentielles, dont nous analysons la convergence vers des états stationnaires au moyen de la méthode de l'entropie et de la théorie de Doeblin dans le cas d'interconnexions faibles. Nous considérons également la situation où l'activité neuronale est plus rapide que le processus d'apprentissage et nous donnons les conditions dans lesquelles on peut approximer la dynamique par une solution avec un profil similaire à un état stationnaire.

Pour chaque modèle, nous présentons quelques simulations numériques afin d'observer comment les paramètres du système peuvent donner des comportements et des formations de modèles différents afin de les contraster avec les résultats théoriques.

**Mots clés :** neurosciences mathématiques, temps écoulé, équations structurées, réseaux de neurones, solutions périodiques, équations différentielles à retard, interconnexions faibles, règle d'apprentissage, noyau de connectivité, convergence à l'équilibre, méthode d'entropie, théorie de doeblin.

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# Introduction

## General background

Understanding the dynamics of neural processes is an interesting challenge from both mathematical and neuroscience viewpoint. The study and modelling of neural networks have been expanded significantly in the past years and still lead to several stimulating open problems. In the case of homogeneous networks, evolution equations describing neural assemblies derived from stochastic processes and microscopic models have become a very active area.

Among these models we mention for example the well-known integrate-and-fire model which is based on the kinetic Fokker-Planck equation and describes the dynamics of a neural network through the membrane potential and has been studied by several authors like Carrillo et al. [7, 10], Perthame et al. [50, 53, 52] and Zhou et al. [40] in different variants and approaches. Another class of population-based which has attracted the attention of many researchers is the elapsed time equation. In this model we consider a neural network where neurons are described by their refractory period as the key variable, i.e. the elapsed time since the last discharge which is caused by changes in the membrane potential. After receiving some stimulation, neurons spike and interact with other neurons leading them to spike as well and establishing the general dynamics of the system. The relation between integrate-and-fire and the elapsed time model was studied in Dumont et al. [22, 21].

In the general context of modeling neural activities, the elapsed time equation is a model at mesoscopic scale. From microscopic cortical data, parameters per population such as the noise, the membrane potential, connection probabilities and number of neurons are extracted in order to construct a microscopic model of spiking neurons, where dynamics are modeled with single-cell resolution. From the multiple spike trains, we can determine a mesoscopic population density which allows to simulate the corresponding mesoscopic activity of the neural network. In this context, population density models has known a growth interest since they have proven to be a useful tool for the simulation and analysis of the behavior of large populations of neurons and they are versatile in linking mesoscopic scale phenomena with the dynamics of a single neuron and capturing microscopic scale fluctuations, in contrast to classical firing rate models which present several problems in this aspect. Connections between population-based models and spike trains statistics were studied in Ly et al. [42], while the connection between the elapsed time model and Poisson processes was established in the work of Chevalier et al. [11, 12]. For a mathematical formalism on the population density approach see Section 6.2 of [31] and for a further discussion on the refractory density method for the elapsed time model see [56]. Other important works on spiking neurons include Brunel [6], Gerstner et al. [31], and Pham et al. [54].

Furthermore, the elapsed time model has been studied from a mathematical and analytical point of view by several authors such as Cañizo et al. in [9], Kang et al. [38], Mischler et al. in [45] and pioneer works of Pakdaman et al. in [48, 49, 47]. In these works, exponential convergence to the equilibrium under a weak interconnections regime was demonstrated through

different techniques such as the entropy method, Doeblin's theory and spectral methods. In addition some results on strong nonlinearities were established, but they are far to be fully understood. In this thesis we will focus on the mathematical aspects of the elapsed time model and its extensions, which include studying well-posedness, asymptotic behavior of solutions and determining if some phenomena of synchronization such as periodic solution solutions arise.

## The elapsed time equation

Now we continue with mathematical description of the elapsed time model. In this setting, neural dynamics are governed by the following nonlinear age-structured equation

$$\begin{cases} \partial_t n + \partial_s n + p(s, N(t))n = 0 & t > 0, s > 0, \\ N(t) = n(t, s = 0) = \int_0^\infty p(s, N(t))n(t, s) ds & t > 0, \\ n(t = 0, s) = n_0(s) \geq 0 & s \geq 0, \end{cases} \quad (1)$$

where  $n(t, \cdot)$  is the probability density of finding a neuron at time  $t$ , whose elapsed time since last discharge is  $s \geq 0$  and  $N(t)$  represents the flux of discharging neurons at time  $t$ , which in this case corresponds to the activity of the network. The function  $p: [0, \infty) \times \mathbb{R} \rightarrow \mathbb{R}$  represents the firing rate of neurons, which is considered as given. The equation for  $n$  is the renewal equation, where the boundary condition corresponds to integrate with respect to  $s$  the term with the firing rate.

Neurons discharge at the rate given by  $p$  and then the elapsed time is immediately reset to zero, as it is stated by the integral boundary condition of  $n$  at  $s = 0$ . In the terminology of age-structured equations the elapsed time corresponds to "age" of neurons, the term  $p(s, N)n$  corresponds to the "death" term and the boundary condition at  $s = 0$  represents the "birth" term when neurons re-enter the cycle.

The firing rate  $p$  depends on the elapsed time  $s$  and the activity  $N(t)$ . We assume that  $p$  is increasing with respect to the elapsed time  $s$ , i.e. when the elapsed time since last discharge increases, neurons are more susceptible to discharge. Moreover, we say that the network is inhibitory if  $p$  is decreasing with respect to the activity and excitatory if  $p$  is increasing.

We also assume for simplicity that  $p \in W^{1,\infty}([0, \infty) \times \mathbb{R})$ , although most of the theoretical results are also valid for firing rates with simple jump discontinuities and the behaviour of solutions does not depend on this regularity assumption as we show in the numerical simulations. If in addition  $\|\partial_N p\|_\infty$  is small, we say that System (1) is under a weak interconnections regime. In other words,  $p$  depends weakly on the global activity so that the system is a slight perturbation of a linear system.

Furthermore, we assume that there exist  $\sigma_*, p_0, p_\infty > 0$  such that

$$p_0 \mathbb{1}_{\{s \geq \sigma_*\}} \leq p \leq p_\infty. \quad (2)$$

A special example is to consider

$$p(s, N) = p_\infty \mathbb{1}_{\{s > \sigma(N)\}} \quad (3)$$

where  $\sigma: [0, \infty) \rightarrow [0, \infty)$  is a bounded and Lipschitz function, which is called the refractory threshold. In this example, neurons fire if the elapsed time attains the value  $\sigma(N(t))$ . When  $s \leq \sigma(N(t))$  the population density satisfies the classical transport equation and when  $s > \sigma(N(t))$  an exponential decay arises due to the neurons discharge.

Finally we assume the initial data  $n_0 \in L^1(0, \infty)$  is a probability density, so that System (1) formally satisfies the mass conservation law, which reads

$$\int_0^\infty n(t, s) ds = \int_0^\infty n_0(s) ds = 1, \quad n(t, s) \geq 0 \quad \forall t \geq 0, \quad (4)$$

and consequently we have the following bounds on  $N$

$$0 \leq N(t) \leq p_\infty, \quad \forall t \geq 0.$$

Observe that  $N(0)$  formally satisfies the following equation

$$N(0) = \int_0^\infty p(s, N(0))n_0(s) ds, \quad (5)$$

thus, if this equation has multiple solutions for  $N(0)$ , then multiple solutions for Equation (1) might arise. Moreover, we remark that in general  $N(0) \neq n_0(0)$ , which might imply that  $n$  has discontinuities along the line  $\{(t, s) \in \mathbb{R}^2 : t = s\}$ .

For the elapsed time equation and its variants studied throughout this thesis, we consider solutions in the weak sense but for simplicity we simply refer to them as solutions.

In the mathematical analysis of the elapsed time mode, aspects such as well-posedness, asymptotic behavior or the existence of periodic solutions are still open problems for a general firing rate. Indeed, most of the theoretical results have been established for the weak interconnections regime. Regarding well-posedness, we have the following result.

**Theorem 1** (Well-posedness for weak interconnections). *Assume that  $n_0 \in L^1(0, \infty)$  is a probability density and that  $p \in W^{1,\infty}((0, \infty) \times \mathbb{R})$  satisfies (2). Then for*

$$\|\partial_N p\|_\infty < 1,$$

*System (1) has a unique solution with  $n \in \mathcal{C}_b([0, \infty), L^1_s)$  and  $N \in \mathcal{C}_b[0, \infty)$ . Moreover  $n$  verifies Condition (4) for all  $t > 0$ .*

For a proof see Cañizo et al. [9]. Some extensions have been studied by Pakdaman et al. [48] and Mischler et al. [44], assuming a growth condition of the firing rate with respect to the activity. Existence of solutions for the strong interconnections case is still far from being fully understood.

Concerning the asymptotic behavior, we are interested in the stationary solutions of the nonlinear System (1) given by the problem

$$\begin{cases} \partial_s n + p(s, N)n = 0 & s > 0, \\ N = n(s=0) := \int_0^\infty p(s, N)n(s) ds, \\ \int_0^\infty n(s) ds = 1, \quad n(s) \geq 0. \end{cases} \quad (6)$$

If the activity  $N$  is given, we can determine the stationary density through the formula

$$n(s) = N e^{-\int_0^s p(u, N) du}. \quad (7)$$

Thus by integrating with respect to  $s$ , we get that  $(n, N)$  corresponds to a stationary solution of System (1) if the activity satisfies the fixed point equation



$$N = F(N) := \left( \int_0^\infty e^{-\int_0^s p(u, N) du} ds \right)^{-1}. \quad (8)$$

Regarding the existence of a steady state, we have the following simple but important lemma.

**Lemma 1.** *Assume that  $p \in W^{1,\infty}([0, \infty) \times \mathbb{R})$ . Then Equation (8) has at least one solution, i.e. System (1) has at least one steady-state. Moreover in the inhibitory or weak interconnections case the steady-state is unique.*

In the case of weak interconnections, we have the following remarkable convergence theorem.

**Theorem 2** (Convergence to equilibrium for weak interconnections). *Assume that  $n_0 \in L^1(0, \infty)$  satisfies Assumption (4) and that  $p$  Lipschitz satisfies Assumption (2). For  $\|\partial_N p\|_\infty$  small enough, let  $(n^*, N^*)$  be the corresponding stationary state of System (1). Then there exist  $C, \lambda > 0$  such that the solution  $n$  of System (1) satisfies*

$$\|n(t) - n^*\|_{L^1_s} \leq C e^{-\lambda t} \|n_0 - n^*\|_{L^1_s}, \quad \forall t \geq 0.$$

Moreover  $|N(t) - N^*|$  converges exponentially to 0 when  $t \rightarrow \infty$ .

Throughout this thesis we will use the notation  $L^1_x$  to indicate that we integrate with respect to the variable  $x$  in its respective domain.

This result on exponential convergence to equilibrium was proved by different authors through different techniques like the entropy method in [38, 48, 49, 47] using the ideas of [43, 51], Doebelin's theory in [9] and spectral methods in [44, 45]. The main idea of these methods consists in studying small perturbations of the linear case, where exponential convergence to the steady state holds. We will discuss in the next section the entropy method and Doebelin's theory for the linear case.

**Remark 1.** *As it was remarked in Cañizo et al. [9], some minimum conditions on the firing rate are required since otherwise System (1) is not well-posed. If we consider for example  $p(s, N) = N$  then a solution satisfies*

$$N(t) = N(t) \int_0^\infty n(t, s) ds,$$

thus we get infinitely many solutions since we could choose any  $N(t)$ . Similar ill-posed examples can be constructed as well.

The case of strong nonlinearities has been investigated in [49] for the firing rate defined in (3), when the neurons only interact via the variation of the refractory threshold. They constructed explicit periodic solutions with one and two discontinuities along the period. Phenomena such as multiplicity of solutions and jump discontinuities have been observed in the work of Caceres et al. [59], which will be discussed in detail in Chapter 1. Furthermore, for a general firing rate  $p$  in System (1) it is conjectured that solutions converge to the unique steady state in the inhibitory case, whereas periodic solutions may arise in the excitatory case.

In this thesis we aim to extend the study of Equation (1) by considering the case of strong interconnections in order to understand the complexity of solutions and the possible behaviors that may arise. Moreover, we introduce two new models based on the elapsed time equation that incorporate new variables to dynamics. The first extension consists in incorporating the elapsed time since the penultimate discharge, so that we study a multiple time renewal equation. The second extension consists in incorporating the spatial dependence (e.g. position of neurons across the cortex) and the process of learning via a connectivity kernel which evolves in time.

The thesis is organized as follows. In Chapter 1 we prove a convergence result in the inhibitory and the weakly excitatory cases for a specific form of the firing rate, by reducing Equation (1) to

a delay differential equation. Moreover, we prove the existence of particular periodic solutions with jump discontinuities in the strongly excitatory case. In Chapter 2 we study a multiple time renewal equation, where we prove exponential convergence to the steady state in the weak interconnections case by means of Doeblin's theory following the results of the classical elapsed time model and we explain the limitations of the entropy method. Finally in Chapter 3 we study the dynamics of the extended elapsed time equation with spatial structure and we prove exponential convergence via both entropy method and Doeblin's theory and we observe different patterns formations depending on the parameters of the model. In addition, we show in the Appendix the numerical methods that were used for the simulations in each model.

## Preliminary results

In this section we review some results on convergence to the steady state for the linear version of the elapsed time model. These results are useful for determining the asymptotic behavior in the nonlinear problem (1) under a weak interconnections regime.

Assume that  $p$  satisfies the bounds (2) and it does not depend on the activity  $N$ , so that we have the following linear system

$$\begin{cases} \partial_t n + \partial_s n + p(s)n = 0 & t > 0, s > 0, \\ N(t) = n(t, s = 0) = \int_0^\infty p(s)n(t, s) ds & t > 0, \\ n(t = 0, s) = n_0(s) \geq 0 & s \geq 0. \end{cases} \quad (9)$$

For a general  $n_0 \in L^1(0, \infty)$  the steady state is given by the formula

$$\bar{n}(s) = \bar{N} e^{-\int_0^s p(u) du}, \quad \bar{N} = \langle n_0 \rangle \left( \int_0^\infty e^{-\int_0^s p(u) du} ds \right)^{-1}, \quad (10)$$

where  $\langle n_0 \rangle = \int_0^\infty n_0(s) ds$  is the conserved mass.

In the context of age-structured models, the entropy method has been a useful tool for proving convergence to the steady state. The main idea consists in finding a Lyapunov's functional  $\mathcal{H}[n]$  and a dissipation functional  $D_{\mathcal{H}}[n]$  such that the solutions of the system satisfy

$$\frac{d}{dt} \mathcal{H}[n] = -D_{\mathcal{H}}[n] \leq 0.$$

Thus if we can find a Poincaré inequality of the type  $\lambda \mathcal{H}[n] \leq D_{\mathcal{H}}[n]$  for some  $\lambda > 0$ , we can deduce the exponential decay of  $\mathcal{H}[n]$  by using the classical Gronwall's inequality, which eventually allows to deduce convergence to the steady state in some convenient norm. This method was developed in the works of [43, 51] with extensions to measure initial data in [33], and it has been applied to different types of models. In the case of the linear Equation (9) we have the following property.

**Theorem 3** (Generalized relative entropy). *Assume that  $p$  bounded and non-negative. Then for all convex functions  $H: [0, \infty) \rightarrow [0, \infty)$  with  $H(0) = 0$ , the solution  $n$  of the linear Equation (9)*

satisfies

$$\begin{aligned} \frac{d}{dt} \int_0^\infty \bar{n}(s) H\left(\frac{n(t,s)}{\bar{n}(s)}\right) ds &= -D_H[n(t,s)] \leq 0 \quad \forall t \geq 0, \\ D_H[n(t,s)] &= \bar{N} \left[ \int_0^\infty H\left(\frac{n(t,s)}{\bar{n}(s)}\right) d\mu(s) - H\left(\int_0^\infty \frac{n(t,s)}{\bar{n}(s)} d\mu(s)\right) \right] \geq 0. \end{aligned} \quad (11)$$

for the probability measure  $d\mu(s) = \frac{\bar{n}(s)}{\bar{N}} p(s) ds$  and  $D_H[\cdot]$  is non-negative by Jensen's inequality.

When the firing rate is strictly positive we have following Poincaré inequality and the subsequent exponential convergence to the steady state.

**Theorem 4** (Poincaré inequality and exponential convergence). *Assume that there exists a constant  $p_0 > 0$  such that  $p \geq p_0$ . Then for all  $h \in L^1(0, \infty)$  such that  $\int_0^\infty h(s) ds = 0$  the following inequality holds*

$$p_0 \int |h(s)| ds \leq \int p(s) |h(s)| ds - \left| \int p(s) h(s) ds \right|. \quad (12)$$

And subsequently the solution of the linear System (9) satisfies

$$\|n(t) - \langle n_0 \rangle \bar{n}\|_{L^1_s} \leq e^{-p_0 t} \|n_0 - \langle n_0 \rangle \bar{n}\|_{L^1_s}, \quad \forall t \geq 0,$$

with  $\langle n_0 \rangle = \int_0^\infty n_0(s) ds$  the conserved mass. Moreover  $|N(t) - \bar{N}|$  converges exponentially to 0 when  $t \rightarrow \infty$ .

**Remark 2.** *This convergence result can be extended when  $p(s)$  satisfies the bounds in (2), but with  $\sigma_*$  small enough. For some examples, see Pakdaman et al. [48].*

Another important approach to study convergence of the linear System (9) is Doeblin's theory, which was first introduced in the context of Markov chains [19, 18] and later developed in the work of Harris [36]. This theory is an alternative to the classical entropy methods to prove convergence to the steady state for a wider class for firing rates. The main argument consists in proving that after a fixed time the solutions are uniformly bounded from below, implying the exponential convergence to equilibrium.

Consider  $(\mathcal{M}(\mathcal{X}), \|\cdot\|_{M^1})$  the space of finite signed measures on the space  $\mathcal{X}$  with the norm of the total variation

$$\|\mu\|_{M^1} := \mu_+(\mathcal{X}) + \mu_-(\mathcal{X}),$$

where  $\mu = \mu_+ - \mu_-$  is the Hahn–Jordan decomposition of the measure  $\mu$  into its positive and negative parts. For simplicity of the computations throughout the chapters, we will treat measures as if they were  $L^1$  functions and we simply write the  $L^1$ -norm instead of  $M^1$ -norm.

We now remind the definition of a Markov semigroup and Doeblin's condition.

**Definition 1** (Markov semigroup). *Let  $(\mathcal{X}, \mathcal{A})$  be a measure space and  $P_t: \mathcal{M}(\mathcal{X}) \rightarrow \mathcal{M}(\mathcal{X})$  be a linear semigroup. We say that  $P_t$  is a Markov semigroup if  $P_t \mu \geq 0$  for all  $\mu \geq 0$  and  $\int_{\mathcal{X}} dP_t \mu = \int_{\mathcal{X}} d\mu$  for all  $\mu \in \mathcal{M}(\mathcal{X})$ . In other words,  $(P_t)$  preserves the subset of probability measures  $\mathcal{P}(\mathcal{X})$ .*

**Definition 2** (Doeblin's condition). *Let  $P_t: \mathcal{M}(\mathcal{X}) \rightarrow \mathcal{M}(\mathcal{X})$  be a Markov semigroup. We say that  $(P_t)$  satisfies Doeblin's condition if there exists  $t_0 > 0$ ,  $\alpha \in (0, 1)$  and  $\nu \in \mathcal{P}(\mathcal{X})$  such that*

$$P_{t_0} \mu \geq \alpha \nu \quad \forall \mu \in \mathcal{P}(\mathcal{X}).$$

Under this functional setting, we are now ready to state Doeblin's theorem as follows.

**Theorem 5** (Doeblin's Theorem). *Let  $P_t: \mathcal{M}(\mathcal{X}) \rightarrow \mathcal{M}(\mathcal{X})$  be a Markov semigroup that satisfies Doeblin's condition. Then the semigroup has a unique equilibrium  $\mu^* \in \mathcal{P}(\mathcal{X})$ . Moreover, for all  $\mu \in \mathcal{M}(\mathcal{X})$  we have*

$$\|P_t\mu - \langle\mu\rangle\mu^*\|_{M^1} \leq \frac{1}{1-\alpha} e^{-\lambda t} \|\mu - \langle\mu\rangle\mu^*\|_{M^1} \quad \forall t \geq 0,$$

with  $\langle\mu\rangle = \int_{\mathcal{X}} d\mu$  and  $\lambda = -\frac{\ln(1-\alpha)}{t_0} > 0$ .

Concerning the linear System (9), we can extend the notion of solution in the distributional sense when the initial data is a measure. Then the solution of Equation (9) allows to define a semigroup acting on the space  $(\mathcal{M}(\mathcal{X}), \|\cdot\|_{M^1})$ . In the work of Cañizo et al. [9], it was proved that this semigroup satisfies Doeblin's condition when the firing rate satisfies the bounds (2) regardless the size of  $\sigma_*$ . Thus, the following improved convergence theorem holds.

**Theorem 6** (Convergence to equilibrium for linear case). *Assume that  $n_0 \in \mathcal{M}(\mathcal{X})$  and that  $p$  Lipschitz satisfies Assumption (2). Let  $(\bar{n}, \bar{N})$  be the corresponding stationary state of System (9). Then there exist  $C, \lambda > 0$  such that the solution  $n$  of System (1) satisfies*

$$\|n(t) - \langle n_0 \rangle \bar{n}\|_{M_s^1} \leq C e^{-\lambda t} \|n_0 - \langle n_0 \rangle \bar{n}\|_{M_s^1}, \quad \forall t \geq 0,$$

with  $\langle n_0 \rangle = \int_0^\infty n_0 ds$ . Moreover  $|N(t) - \bar{N}|$  converges exponentially to 0 when  $t \rightarrow \infty$ .

For a reference on Doeblin's theory, see for example Gabriel et al. [30]. A well-known extension of this theory is Harris' theorem [36], which has inspired several works such as Bansaye et al. [3], Cañizo et al. [8] and Hairer & Mattingly [35].



# Chapter 1

## The case of strong interconnections

This chapter corresponds to the article [59] titled "An elapsed time model for strongly coupled inhibitory and excitatory neural networks".

### 1.1 Introduction

Understanding dynamics for the elapsed time model in the case of strong interconnections is a challenging problem since most of theoretical results concerned only the case of weak nonlinearities. As we mentioned at the introduction, for a general firing rate  $p$  in System (1) it is conjectured that solutions converge to the unique steady state in the inhibitory case, whereas periodic solutions may arise in the excitatory case. We are concerned to prove these conjectures for a specific form of the firing rate in order to obtain a first approximation to general case.

The first approach to strong nonlinearities has been investigated by Pakdaman et al. [49] for the firing rate defined in (3), which has a variable refractory threshold. They constructed explicit periodic solutions with one and two discontinuities along the period with possible flat states in the case of two discontinuities. Moreover, these solutions were piece-wise exponentials.

Now, we fix the refractory threshold and we assume that, after a fixed refractory state, the discharge rate of neurons follows an exponential law whose parameter depends on the total activity via a smooth function  $\varphi$ . This leads to the following particular form the firing rate  $p$ :

$$p(s, u) = \varphi(u)\mathbb{1}_{\{s > \sigma\}}, \quad (1.1)$$

with  $\sigma > 0$  the constant refractory period. We also assume that there exist two constants  $p_0, p_\infty$  such that

$$0 < p_0 \leq \varphi(\cdot) \leq p_\infty. \quad (1.2)$$

For this particular form of the discharge rate  $p$ , the network is inhibitory when  $\varphi' \leq 0$  and excitatory when  $\varphi' > 0$ , in particular, it is weakly excitatory if  $\varphi'$  is small.

This specific form of  $p$  allows us to investigate regimes with strong interactions as in [49], reducing the problem to a delay equation (see [16, 46, 57] for references), which is beyond the scope of the classical methods applied for weak nonlinearities such as entropy method or Doeblin's theory.

The chapter is organized as follows. In Section 1.2, we show that System (1) can be studied through a delay differential equation. Moreover from a given periodic activity  $N(t)$  solving the delay equation, we can recover a solution of System (1) by using the arguments in [49]. In Section

1.3, we prove convergence to equilibrium in the inhibitory case and the weakly excitatory case, while the activity oscillates around the equilibrium value. In addition we prove a monotone convergence result in the excitatory case under certain conditions. Regarding periodic solutions, we prove in Section 1.4 the existence of piece-wise constant  $\sigma$ -periodic and with more elaborate arguments we also prove the existence of  $2\sigma$ -periodic solutions, which are piece-wise monotone. Finally, in Section 1.5 we show numerical examples with several possible behaviors such as multiplicity of solutions, convergence to equilibrium in different ways and periodic solution with jump discontinuities.

## 1.2 Reduction to a delay differential equation

For analysis purposes, we define the following function  $\psi: [0, \infty) \rightarrow [0, \infty)$  as

$$\psi(u) := \frac{u}{\varphi(u)}, \quad \text{therefore} \quad \psi'(u) = \frac{\varphi(u) - u\varphi'(u)}{\varphi^2(u)}. \quad (1.3)$$

This function plays an important role in the study of System (1), because it can be reduced to a delay differential equation, as one can see in the following lemma:

**Lemma 2.** *We assume (1.1) and (1.2). For  $t > \sigma$  the discharge flux  $N(t)$  satisfies:*

$$\int_{t-\sigma}^t N(s) ds + \psi(N(t)) = 1, \quad (1.4)$$

and if  $N(t)$  is smooth for  $t > \sigma$ , the following formula for  $N'(t)$  holds:

$$N(t) - N(t - \sigma) + \psi'(N(t))N'(t) = 0. \quad (1.5)$$

In the same way, for  $0 < t < \sigma$  the discharge flux satisfies

$$\int_0^t N(s) ds + \int_0^{\sigma-t} n_0(s) ds + \psi(N(t)) = 1, \quad (1.6)$$

and if  $N(t)$  is smooth we have the following formula for  $N'(t)$ :

$$N(t) - n_0(\sigma - t) + \psi'(N(t))N'(t) = 0. \quad (1.7)$$

**Proof.** Using assumption (1.1), the equation for  $N(t)$  is rewritten as

$$N(t) = \varphi(N(t)) \int_{\sigma}^{\infty} n(t, s) ds.$$

Moreover, from the method of characteristics we have following equality

$$\begin{aligned} n(t, s) = & n_0(s - t) \exp\left(-\int_0^t p(s + t' - t, N(t')) dt'\right) \mathbf{1}_{\{s > t\}} \\ & + N(t - s) \exp\left(-\int_0^s p(\tau, N(t - s + s')) ds'\right) \mathbf{1}_{\{0 < s < t\}}, \end{aligned} \quad (1.8)$$

and using the mass-conservation property for  $t > \sigma$ , we deduce:

$$\int_{\sigma}^{\infty} n(t, s) ds = 1 - \int_0^{\sigma} n(t, s) ds = 1 - \int_0^{\sigma} N(t - s) ds.$$

Therefore, we get the first part of Lemma 2. The result for  $0 < t < \sigma$  is proved in the same way.  $\square$

The sign of  $\psi'$  plays a crucial role in the behavior of the system (see Equation (1.5)). We will prove that complex dynamics can only occur when  $\psi'$  changes sign. This does not happen for the inhibitory case, because  $\psi' > 0$  on  $[0, p_{\infty}]$ , and, in the excitatory case if  $\psi' > 0$ , on  $[0, p_{\infty}]$ . In this latter case, we say that the network is *weakly excitatory*. For instance, this holds if  $0 < \varphi' \leq p_0/p_{\infty}$ . Otherwise, we say that the network is *strongly excitatory*, if  $\psi'$  changes sign (and  $\varphi' > 0$ ). We also remark that if  $\psi' > 0$  then  $N \in \mathcal{C}_b[0, \infty)$  thanks to Formulas (1.4) and (1.6).

In the following theorem we show how to recover solutions of the original renewal model from a given activity that is a solution of the integral equation (1.4).

**Theorem 7 (Reconstructing a solution of (1) from a general activity).** *Assume (1.2). Let  $N \in L^{\infty}(0, \infty)$  a non-negative function and satisfying  $\psi(N(t)) \in \mathcal{C}([\sigma, \infty)) \cap \mathcal{C}^1((0, \sigma))$  and the following conditions:*

1.  $\tilde{n}(t) := N(\sigma - t) + \frac{d(\psi(N))}{dt}(\sigma - t) \geq 0$  for  $0 < t < \sigma$ ,
2.  $\int_{t-\sigma}^t N(s) ds + \psi(N(t)) = 1$  for  $t \geq \sigma$ , i.e.,  $\int_0^{\sigma} N(s) ds + \psi(N(\sigma)) = 1$  and  $N$  is solution of the integral Equation (1.4).

Then for any initial probability density  $n_0$  satisfying  $n_0(s) = \tilde{n}(s)$  for  $0 < s < \sigma$ , the solution  $n$  of the linear problem

$$\begin{cases} \partial_t n + \partial_s n + p(s, N(t))n = 0 & t > 0, s > 0, \\ n(t, s = 0) = N(t) & t > 0, \\ n(t = 0, s) = n_0(s) & s > 0, \end{cases} \quad (1.9)$$

determines a solution of Equation (1) with  $N(t)$  as activity.

From this theorem, we deduce that the behavior of  $N(t)$  in Equation (1) is determined just by the initial data  $n_0$  on  $(0, \sigma)$  as long as it is a probability density.

**Proof.** In order to prove that the solution  $n$  of Equation (1.9) is actually a solution of Equation (1), we must verify the following conditions for all  $t > 0$ .

$$\psi(N(t)) = \int_{\sigma}^{\infty} n(t, s) ds \quad \text{and} \quad \int_0^{\infty} n(t, s) ds = 1. \quad (1.10)$$

Consider  $M(t) := \int_0^{\infty} n(t, s) ds$  and observe that

$$\begin{aligned} M'(t) &= - \int_0^{\infty} \partial_s n(t, s) ds - \int_0^{\infty} p(s, N(t))n(t, s) ds \\ &= N(t) - \varphi(N(t)) \int_{\sigma}^{\infty} n(t, s) ds \\ &= N(t) - \varphi(N(t))M(t) + \varphi(N(t)) \int_0^{\sigma} n(t, s) ds. \end{aligned} \quad (1.11)$$



For  $0 < t < \sigma$  we deduce from conditions 1 and 2 that Equation (1.6) holds and by the method of characteristics (1.8) we get that  $M(t)$  satisfies that

$$\begin{aligned} M'(t) &= N(t) - \varphi(N(t))M(t) + \varphi(N(t)) \left( \int_0^t n(t, s) ds + \int_t^\sigma n(t, s) ds \right) \\ &= N(t) - \varphi(N(t))M(t) + \varphi(N(t)) \left( \int_0^t N(s) ds + \int_0^{\sigma-t} n_0(s) ds \right) \\ &= N(t) - \varphi(N(t))M(t) + \varphi(N(t)) (1 - \psi(N(t))) \\ &= \varphi(N(t))(1 - M(t)). \end{aligned} \quad (1.12)$$

Similarly for  $t > \sigma$  we deduce from condition 2 and the method of characteristics (1.8) that for a.e.  $t > \sigma$  the following equality holds

$$\begin{aligned} M'(t) &= N(t) - \varphi(N(t))M(t) + \varphi(N(t)) \int_{t-\sigma}^t N(s) ds \\ &= \varphi(N(t)) (1 - M(t)). \end{aligned} \quad (1.13)$$

Therefore we obtain that  $M(t)$  satisfies the following differential equation for a.e.  $t > 0$ .

$$\begin{cases} M'(t) = \varphi(N(t))(1 - M(t)), \\ M(0) = 1. \end{cases}$$

This implies that  $M(t) \equiv 1$  and we conclude (1.10) for all  $t > 0$ , which implies that  $n$  is a solution of (1) with  $N(t)$  as activity.  $\square$

In particular when the activity is periodic we can construct periodic solutions of (1), using only the Lipschitz continuity of  $\psi(N(t))$  induced by (1.2) and (1.4).

**Theorem 8 (Reconstruction of a solution from a periodic activity).** *Assume (1.2). For  $T > 0$ , consider a non-negative  $T$ -periodic solution  $N(t)$  of the integral Equation (1.4) for all  $t \in \mathbb{R}$ . Then, there exists a unique  $T$ -periodic solution  $n$  of the linear Equation (1.9), which corresponds to a solution of the nonlinear System (1).*

**Proof.** By periodicity we can extend the solution of Equation (1.9) for all  $t \in \mathbb{R}$  and from the method of characteristics we find a solution given by

$$n(t, s) = N(t - s) e^{-\int_0^s p(s', N(s'+t-s)) ds'},$$

thus we can consider as initial data

$$n(t = 0, s) = N(-s) e^{-\int_0^s p(s', N(s'-s)) ds'},$$

which determines the unique  $T$ -periodic solution of (1.9). Therefore we can replicate the argument in the proof of Theorem 7 in the case  $t > \sigma$  to conclude the result.  $\square$

### 1.3 Convergence to equilibrium

By using the delay Equation (1.5), we prove that solutions of System (1) converge to a steady state in two cases. In the inhibitory and weakly excitatory case, the steady state is unique and we prove global convergence. In the strongly excitatory case, we prove a local convergence theorem

with specific assumptions on  $\psi$  and for the smooth solution. For the function  $p$  defined in (1.1), these results extend the convergence beyond the standard case of weak interconnections.

We recall here that in both the inhibitory and the weakly excitatory cases  $\psi'$  satisfies:

$$\psi'(u) = \frac{\varphi(u) - u\varphi'(u)}{\varphi^2(u)} > 0, \quad \text{for } u \in [0, p_\infty], \quad (1.14)$$

while in the strongly excitatory case  $\psi'$  changes sign. We know from (1.3) and (1.4) that there exists a unique steady state determined by  $N^*$ , which is the unique solution of the equation

$$\sigma N^* + \psi(N^*) = 1, \quad \text{and therefore, } n^* = N^* e^{-\int_0^s p(s', N^*) ds'}. \quad (1.15)$$

In the following theorem, we prove the convergence to the steady state, in the inhibitory and weakly excitatory cases.

**Theorem 9 (Convergence to equilibrium in the inhibitory or weakly excitatory cases).** *Assume that  $\psi'$  satisfies (1.14). Let  $n$  be the solution of (1). Then  $n(t) \rightarrow n^*$  in  $L^1_s$  and  $N(t) \rightarrow N^*$  when  $t \rightarrow \infty$ , where  $(n^*, N^*)$  corresponds to the unique steady state of Equation (1) given by (1.15).*

In order to achieve the result, we need the following two lemmas. The first one asserts that in the inhibitory and weakly excitatory cases, the activity  $N(t)$  must oscillate infinitely many times.

**Lemma 3.** *Assume that  $\psi$  satisfies (1.14). Then the activity  $N(t)$  can not be strictly monotone over any interval of length larger than  $\sigma$ . In particular there exists a sequence of local maxima  $t_n$  (and resp. minima) such that  $t_n \rightarrow \infty$ .*

**Proof.** Assume the existence an interval  $I \subset (0, \infty)$ , with  $|I| > \sigma$ , such that  $N(t)$  is increasing on  $I$ . Let  $t_0 > \sigma$  such that  $t_0, t_0 - \sigma \in I$ , so we have that  $N(t_0 - \sigma) < N(t_0)$ . However we conclude from (1.5) that  $N'(t_0) < 0$ , which contradicts that  $N(t)$  is increasing. Similarly, if we assume that  $N$  decreases on  $I$  we obtain a contradiction. Therefore, we conclude that  $N(t)$  increases and decreases many infinitely times.  $\square$

**Lemma 4.** *If  $\psi$  satisfies (1.14), then*

$$\max_{t \in I_{k-1}} N(t) \geq \max_{t \in I_k} N(t), \quad \left( \text{resp. } \min_{t \in I_{k-1}} N(t) \leq \min_{t \in I_k} N(t) \right),$$

for  $k \in \mathbb{N}$  and  $I_k := [k\sigma, (k+1)\sigma]$ .

**Proof.** We consider three cases to show that  $\max_{t \in I_{k-1}} N(t) \geq \max_{t \in I_k} N(t)$ :

1. If  $\max_{t \in I_k} N(t) = N(k\sigma)$  then the result is straightforward.
2. If  $\max_{t \in I_k} N(t) = N((k+1)\sigma)$  then  $N'((k+1)\sigma) \geq 0$  and from formula (1.5) we get that  $N(k\sigma) \geq N((k+1)\sigma)$  and the result is proved.
3. If there exists  $t_0 \in (k\sigma, (k+1)\sigma)$  such that  $\max_{I_k} N = N(t_0)$  then  $N'(t_0) = 0$  and by formula (1.5) we have  $N(t_0) = N(t_0 - \sigma)$  and the result is proved.

Analogously it is proved that  $\min_{t \in I_{k-1}} N(t) \leq \min_{t \in I_k} N(t)$ .  $\square$

Now we can prove Theorem 9.

**Proof.** For  $T > 0$  we define the sequence  $N_n(t) = N(t+n)$  for  $t \in [-T, T]$  and  $n \in \mathbb{N}$  such that  $n - T > \sigma$ . From formula (1.5) we get that  $N'$  is uniformly bounded since  $\psi'(u) > 0$ , for  $u \in [0, p_\infty]$ . This allows to conclude that after extraction of a sub-sequence,  $N_n(t)$  converges uniformly in  $[-T, T]$  to some function  $N_\infty(t)$  defined on  $\mathbb{R}$ . Moreover  $N_\infty(t)$  satisfies Equation (1.4) for all  $t \in \mathbb{R}$  so in particular  $N_\infty \in C_b^1(\mathbb{R})$ , and from Lemma 4, we deduce that for all  $k \in \mathbb{Z}$

$$\max_{t \in I_k} N_\infty(t) = \limsup_{t \rightarrow \infty} N(t), \quad \left( \text{resp. } \min_{t \in I_k} N_\infty(t) = \liminf_{t \rightarrow \infty} N(t) \right),$$

and in particular  $N^+ := \max_{t \in \mathbb{R}} N_\infty(t) = \max_{t \in I_k} N_\infty(t)$  (resp.  $N^- := \min_{t \in \mathbb{R}} N_\infty(t) = \min_{t \in I_k} N_\infty(t)$ ).

Next, we prove that  $N_\infty(t)$  is a constant function. Choose  $t^-, t^+ \in \mathbb{R}$ , with  $t^- < t^+$ , such that  $N(t^+) = N^+$ ,  $N(t^-) = N^-$  and  $|t^+ - t^-| \leq \sigma$ . Since  $N'(t^\pm) = 0$ , from Equation (1.5) we get  $N(t^\pm) = N(t^\pm - \sigma)$  and thus by iteration we get

$$N(t^\pm) = N(t^\pm - j\sigma), \quad \forall j \in \mathbb{N}.$$

Consider now  $k \in \mathbb{N}$ . By evaluating formula (1.4) for  $t = t^\pm - j\sigma$  for  $j = 1, \dots, k$  and taking the sum over  $j$ , the following equality holds

$$\int_{t^+ - k\sigma}^{t^+} N_\infty(s) + k\psi(N^+) ds = \int_{t^- - k\sigma}^{t^-} N_\infty(s) ds + k\psi(N^-),$$

and we conclude

$$\psi(N^+) - \psi(N^-) = \frac{1}{k} \left( \int_{t^-}^{t^+} N_\infty(s) ds - \int_{t^- - k\sigma}^{t^+ - k\sigma} N_\infty(s) ds \right).$$

Hence by taking  $k \rightarrow \infty$  we conclude  $\psi(N^+) = \psi(N^-)$  and this implies  $N^+ = N^-$ , since  $\psi$  is strictly increasing. Therefore  $N_\infty$  is a constant function.

Since  $N_\infty$  satisfies formula (1.4), we conclude by uniqueness of the solution of Equation (1.15) that  $N_\infty = N^*$ . In particular we conclude that  $N(t) \rightarrow N^*$  when  $t \rightarrow \infty$ .

Finally we prove the convergence of  $n(t)$ . From the method of characteristics (1.8), we observe that

$$n(t, s) \leq n_0(s-t)e^{-p_0 t} + p_\infty e^{-p_0 s}.$$

Thus  $n(t, \cdot)$  is dominated by the sum of a convergent sequence in  $L^1$  plus a fixed integrable function. By Lebesgue's theorem, we can pass to the limit in Formula (1.8) to get the desired result.  $\square$

In strongly excitatory networks,  $\psi'$  changes sign, therefore the above theorem does not apply. And also there is no steady state uniqueness. In this case, we prove the following local convergence theorem.

**Theorem 10 (Monotone convergence in the strongly excitatory case).** *Let  $N^* > 0$  be a solution of (1.15) (i.e. a steady state activity of (1)), and consider  $N \in \mathcal{C}[0, \sigma]$  satisfying the following conditions:*

1.  $N(t) < N(\sigma) < N^*$  for all  $t \in (0, \sigma)$  (resp.  $N(t) > N(\sigma) > N^*$ ).
2.  $\int_0^\sigma N(s) ds + \psi(N(\sigma)) = 1$ .

3.  $\psi'(u) < 0$  for  $u \in [N_-, N^*]$  with  $N_- = \min_{t \in [0, \sigma]} N(t)$  (resp. for  $u \in [N^*, N_+]$  with  $N_+ = \max_{t \in [0, \sigma]} N(t)$ ).

Then there exists a strictly increasing (resp. decreasing) solution  $N(t)$  of (1.4), which extends the given  $N \in \mathcal{C}[0, \sigma]$ . Moreover if  $N^*$  is the unique steady state activity lying on  $[N(\sigma), N^*]$  (resp. on  $[N^*, N(\sigma)]$ ), then  $N(t) \rightarrow N^*$  and  $n(t) \rightarrow n^*$  in  $L^1$  when  $t \rightarrow \infty$ .

We point out that to be a solution of the time elapsed model (1), we need to find a compatible initial data  $n_0(s)$ . Theorem 7 can help in that direction.

**Proof.** We start with the proof of the case  $N(t)$  increasing. Consider  $\tilde{\psi}: \mathbb{R} \rightarrow \mathbb{R}$  a smooth Lipschitz function such that for some constant  $\alpha > 0$  we have  $\tilde{\psi}'(u) < -\alpha$  and

$$\tilde{\psi} \equiv \psi \quad \text{on} \quad [N_-, N^*].$$

Thus the following delay differential equation

$$\begin{cases} \frac{d}{dt} \tilde{\psi}(u(t)) = u(t - \sigma) - u(t) & t > \sigma, \\ u(t) \equiv N(t) & t \in [0, \sigma], \end{cases} \quad (1.16)$$

has a unique global  $u \in \mathcal{C}^1[\sigma^+, \infty) \cap \mathcal{C}[0, \infty)$  by applying Cauchy-Lipschitz theorem. We call this global solution as  $N(t)$  as well. From Equation (1.16) we observe that

$$\int_{t-\sigma}^t N(s) ds + \tilde{\psi}(N(t))$$

is constant and from condition 2 we have that

$$\int_{t-\sigma}^t N(s) ds + \tilde{\psi}(N(t)) = 1. \quad (1.17)$$

Next we prove that  $N(t) < N^*$  for all  $t > \sigma$ . Suppose that there exists  $t^* > \sigma$  such that  $N(t^*) = N^*$  and  $N(t) < N^*$  for  $t < t^*$ . Thus from Equation (1.17) we observe that

$$1 = \int_{t^*-\sigma}^{t^*} N(s) ds + \psi(N^*) < \sigma N^* + \psi(N^*),$$

which contradicts Equation (1.15) and hence  $N(t) < N^*$  for all  $t > \sigma$ .

Now we prove that  $N'(t) > 0$  for all  $t > \sigma$ . From Equation (1.16) we have that  $N'(\sigma^+) > 0$  and from continuity there exists  $\varepsilon > 0$  such that  $N'(t) > 0$  for  $t \in (\sigma, \sigma + \varepsilon)$ . Suppose that there exists a first local maximum  $t_* > \sigma$ , so that  $N'(t_*) = 0$  and from Equation (1.16) we get  $N(t_* - \sigma) = N(t_*) > N(\sigma)$ . If  $t_* < 2\sigma$  then we contradict condition 1 and if  $t_* \geq 2\sigma$  we contradict the monotonicity of  $N(t)$  on  $[\sigma, t_*]$ . Therefore  $N'(t) > 0$  for all  $t > \sigma$ .

From monotonicity and boundedness of  $N$ , we conclude in particular that  $N$  is a solution of Equation (1.4) for  $t > \sigma$ . Moreover, if  $N^*$  is the unique steady state on the interval  $[N(\sigma), N^*]$ , then it is straightforward that  $N(t) \rightarrow N^*$  when  $t \rightarrow \infty$  and subsequently we get the convergence of  $n(t)$ . The proof of  $N(t)$  decreasing is analogous.  $\square$

Figure 1.1 shows an example of this theorem.

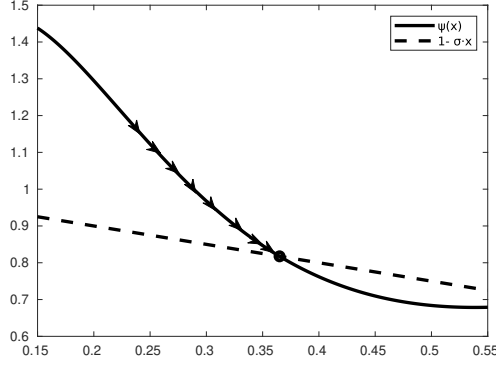


Figure 1.1 – An example of an increasing solution converging to equilibrium. The solution remains in the region where  $\psi$  is strictly decreasing.

## 1.4 Periodic solutions for strongly excitatory networks

From Equation (1.4) we can also construct various periodic solutions for the activity  $N(t)$ , which generate periodic solutions of the System (1) by means of Theorem 8. However, we observe from the integral equation (1.4) that a  $\sigma$ -periodic solution satisfies that  $\int_{t-\sigma}^t N(s)ds$  is constant and thus  $\psi(N(t))$  is also constant. Hence, except when  $\psi$  is locally constant, the only continuous  $\sigma$ -periodic solutions are constant (steady states). When  $\psi'$  changes sign, we build several types of solutions including piece-wise constant discontinuous  $\sigma$ -periodic solutions and piece-wise smooth discontinuous  $2\sigma$ -periodic solutions.

### 1.4.1 Piece-wise constant periodic solutions

Our first goal is to build  $\sigma$  periodic solutions with jump discontinuities which keep the value of  $\psi(N(t))$  constant. As a consequence of Theorem 8, this is possible when  $\psi'$  changes sign and some structure condition is met on  $\psi$ .

**Theorem 11 (Existence of piece-wise constant  $\sigma$ -periodic activities).** *Assume that  $\psi'$  changes sign. Let  $N_1 \neq N_2$  be numbers in  $(0, p_\infty]$  such that  $\psi(N_1) = \psi(N_2)$ . Consider the function defined by*

$$N(t) = \begin{cases} N_1 & \text{when } t \in [0, \alpha), \\ N_2 & \text{when } t \in [\alpha, \sigma), \end{cases} \quad (1.18)$$

and assume there is an  $\alpha \in (0, \sigma)$  such that

$$\alpha N_1 + (\sigma - \alpha)N_2 + \psi(N_1) = 1. \quad (1.19)$$

Then the periodic extension of  $N(t)$  determines a  $\sigma$ -periodic solution of System (1).

**Remark 3.** Notice that, if such an  $\alpha$  exists, there is a steady state between  $N_1$  and  $N_2$ , because if  $N_1 < N_2$  then  $\sigma N_1 + \psi(N_1) < 1 < \sigma N_2 + \psi(N_2)$ . Therefore there exists  $N^*$  such that  $\sigma N^* + \psi(N^*) = 1$ . If  $N_2 < N_1$  we proceed in the same way exchanging  $N_1$  and  $N_2$

**Remark 4.** The same construction can be done for a piece-wise constant  $N(t)$  with more than one jump, as long as it verifies Equation (1.4) and  $\psi$  remains constant for the values taken by

$N(t)$ .

As an example where condition (1.19) is verified, consider  $\sigma = 1$  and  $\varphi(N) = (1 + e^{-9N+3.5})^{-1}$  so that

$$\psi(N) = N(1 + e^{-9N+3.5}).$$

As we see in Figure 1.2,  $\psi'$  changes sign and there are three solutions of the equation  $\psi(u) = 0.9$ .

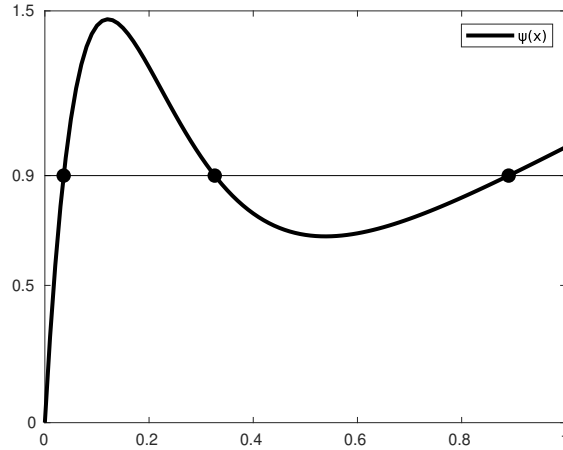


Figure 1.2 – Graphic of  $\psi(u) = u(1 + e^{-9u+3.5})$ .

Take  $N_1$  as the minimal solution and  $N_2$  as the maximal one, so that there exists  $\alpha := \frac{N_2 - 0.1}{N_2 - N_1} \in (0, 1)$  such that

$$\alpha N_1 + (1 - \alpha)N_2 = 0.1,$$

since  $N_1 < 0.1 < N_2$ . Therefore condition (1.19) holds and  $N(t)$  defined in (1.18) determines a periodic solution of (1) by means of Theorem 8.

We can also get continuous periodic solutions for a specific type of firing rates as we state it in the following proposition.

**Proposition 1.** *Let  $\varphi$  be a smooth function such that  $\varphi(u) = Cu$  on some interval  $[a, b] \subseteq [0, p_\infty]$  with  $C > 1$ . Assume moreover that  $\sigma > 0$  satisfies the inequality*

$$a\sigma < 1 - \frac{1}{C} < b\sigma. \quad (1.20)$$

*Let  $u$  be a bounded  $\sigma$  periodic function such that  $\int_0^\sigma u(s) ds = 0$ . Then there exists  $\alpha > 0$  such that*

$$N(t) := \frac{1}{\sigma} \left( 1 - \frac{1}{C} \right) + \alpha u(t)$$

*is a solution of (1.4).*

**Proof.** Let  $u$  be a bounded  $\sigma$  periodic function such that  $\int_0^\sigma u(s) ds = 0$ . For  $\alpha > 0$  small enough,  $N(t)$  defined in Proposition 1 is such that  $N(\mathbb{R}) \subset [a, b]$ , and so,  $N$  satisfies (1.4) because  $\psi(N(\cdot)) = \frac{1}{C}$ .  $\square$

### 1.4.2 Piece-wise monotone $2\sigma$ -periodic solutions

With more elaborate arguments inspired from the works of Haderl et al. in [34] on periodic solutions of differential delay equations, we can also build  $2\sigma$ -periodic solutions of System (1), which are piece-wise monotone. In order to achieve the result, we study the delay differential equation given by

$$\frac{d}{dt}\psi(N(t)) = N(t - \sigma) - N(t). \quad (1.21)$$

**Theorem 12 (Existence of piece-wise monotone  $2\sigma$ -periodic solutions).** *Assume that  $\psi$  is smooth with  $\psi'$  changing sign. Consider  $\underline{N} > 0$  such that it is a local minimum of  $\psi$  and there exists  $\varepsilon > 0$  such that  $\psi$  is strictly convex on  $(\underline{N} - \varepsilon, \underline{N} + \varepsilon)$ . Then for  $\sigma > 0$  small enough there exists a  $2\sigma$  periodic solution  $N(t)$  of (1.21) with  $\psi(N) \in W^{1,\infty}(\mathbb{R})$ , such that  $N(t)$  is strictly decreasing on  $(0, 2\sigma)$  with a discontinuity at  $\sigma$ .*

*If in addition  $\psi(\underline{N}) < 1$  and  $\psi(\underline{N} \pm \varepsilon) > 1$ , then there exists a solution  $n$  of Equation (1) such that the activity  $N(t)$  solves (1.4).*

**Proof.** Since  $\psi$  is strictly convex around the local minimum  $\underline{N}$ , consider  $\bar{N}^-$ ,  $\bar{N}^+$  be two positive constants such that

$$\psi(\bar{N}^-) = \psi(\bar{N}^+), \quad \bar{N}^- < \underline{N} < \bar{N}^+$$

with  $\psi$  strictly decreasing on  $[\bar{N}^-, \underline{N}]$  and strictly increasing on  $[\underline{N}, \bar{N}^+]$ . We construct a periodic solution  $N(t)$  of (1.21) satisfying the following conditions:

1.  $\begin{cases} \underline{N} < N(t) < \bar{N}^+, & N'(t) < 0 & \text{for } t \in (0, \sigma), \\ \bar{N}^- < N(t) < \underline{N}, & N'(t) < 0 & \text{for } t \in (\sigma, 2\sigma). \end{cases}$
2.  $N(0^+) = \bar{N}^+$ ,  $N(2\sigma^-) = \bar{N}^-$ .
3.  $\psi(N(\sigma^-)) = \psi(N(\sigma^+))$ .

The first step is to build a periodic solution solving Equation (1.21) on  $(0, \sigma)$  and  $(\sigma, 2\sigma)$ , which satisfies conditions 1 and 2. Let  $C$  be the closed subset of the Banach space  $\mathcal{C}[0, \sigma]$  defined by

$$C := \{N(t) \in \mathcal{C}[0, \sigma] : N(0) = \bar{N}^+, N(t) \text{ non-increasing, } \psi(N(t)) \geq \psi(\bar{N}^+) - (\bar{N}^+ - \bar{N}^-)\sigma \text{ for } t \in [0, \sigma]\}.$$

Observe that for  $\sigma$  small enough we assure that  $\psi(N(t)) > \psi(\underline{N})$  for all  $t \in [0, \sigma]$ . Our strategy is to build a solution on  $[0, 2\sigma]$  such that its restriction to  $[0, \sigma]$  is a fixed point of an operator in  $C$ .

Now for  $N \in C$  we define  $M(t)$  as the solution of the backward problem

$$\begin{cases} \frac{d}{dt}\psi(M(t)) = N(t - \sigma) - M(t), & t \in (\sigma, 2\sigma). \\ M(2\sigma) = \bar{N}^-. \end{cases} \quad (1.22)$$

For  $\sigma$  small enough, this equation is well-posed in the classical sense on  $[\sigma, 2\sigma)$ . Moreover we have  $\psi'(M(t)) < 0$ ,  $M'(t) < 0$  and  $\frac{d}{dt}\psi(M(t))$  verifies

$$\frac{d}{dt}\psi(M(t)) \leq \bar{N}^+ - \bar{N}^-, \quad \forall t \in (\sigma, 2\sigma).$$

By integrating this inequality, between  $t$  and  $2\sigma$ , we get  $\psi(M(t)) \geq \psi(\bar{N}^+) - (\bar{N}^+ - \bar{N}^-)\sigma > \psi(\underline{N})$ , and since  $M(2\sigma) = \bar{N}^-$  we have  $\bar{N}^- < M(t) < \underline{N}$  for  $t \in (\sigma, 2\sigma)$ . Similarly, we define

$L(t)$  as the solution of problem

$$\begin{cases} \frac{d}{dt}\psi(L(t)) = M(t - \sigma) - L(t), & t \in (2\sigma, 3\sigma). \\ L(2\sigma) = \bar{N}^+, \end{cases} \quad (1.23)$$

so that for  $\sigma$  small enough it is well-posed in the classical sense on  $[2\sigma, 3\sigma]$ . Moreover we have  $\psi'(L(t)) > 0$ ,  $L'(t) < 0$ , and  $\psi(L(t)) \geq \psi(\bar{N}^+) - (\bar{N}^+ - \bar{N}^-)\sigma > \psi(\underline{N})$ , which implies that  $\underline{N} < L(t) < \bar{N}^+$  for  $t \in (2\sigma, 3\sigma)$ , since  $L(2\sigma) = \bar{N}^+$ .

Therefore we define the continuous map  $T: C \rightarrow C$  given by  $T[N](t) = L(t - 2\sigma)$  and we look for a fixed point of the operator  $T$  in order to find a  $2\sigma$ -periodic function  $N(t)$  with  $\psi(N(0^+)) = \psi(N(2\sigma^-))$ , satisfying Equation (1.21) on  $(0, \sigma)$  and  $(\sigma, 2\sigma)$  with jump discontinuities such that  $\psi(N(t))$  is continuous.

Now we proceed to prove that  $T$  is a contraction for  $\sigma$  small enough. Consider  $N_1, N_2 \in C$  with their respective  $M_1, M_2$  and  $L_1, L_2$ . For the difference between  $M_1$  and  $M_2$  we have for  $t \in [\sigma, 2\sigma]$

$$\begin{aligned} |M_2(t) - M_1(t)| &\leq A_\sigma |\psi(M_2(t)) - \psi(M_1(t))| \\ &\leq A_\sigma \left| \int_t^{2\sigma} \frac{d}{dt} (\psi(M_1(t')) - \psi(M_2(t'))) dt' \right| \\ &\leq \sigma A_\sigma \|N_2 - N_1\|_\infty + \sigma A_\sigma \|M_1 - M_2\|_\infty, \end{aligned}$$

with  $A_\sigma := \max \left\{ \left| (\psi^{-1})'(u) \right| : \psi(\bar{N}^+) - (\bar{N}^+ - \bar{N}^-)\sigma \leq u \leq \psi(N^+) \right\}$  and  $\psi^{-1}$  is the local inverse around  $\bar{N}^-$ . Since  $\sigma A_\sigma \rightarrow 0$  when  $\sigma \rightarrow 0$ , we deduce the following estimate for  $\sigma$  small enough

$$\|M_2 - M_1\|_\infty \leq \frac{\sigma A_\sigma}{1 - \sigma A_\sigma} \|N_2 - N_1\|_\infty. \quad (1.24)$$

Analogously for the difference between  $L_1$  and  $L_2$  we get

$$\|L_2 - L_1\|_\infty \leq \frac{\sigma B_\sigma}{1 - \sigma B_\sigma} \|M_2 - M_1\|_\infty, \quad (1.25)$$

with  $B_\sigma := \max \left\{ \left| (\psi^{-1})'(u) \right| : \psi(\bar{N}^+) - (\bar{N}^+ - \bar{N}^-)\sigma \leq u \leq \psi(N^+) \right\}$  and  $\psi^{-1}$  is now considered as the local inverse around  $\bar{N}^+$ . Therefore we conclude from estimates (1.24) and (1.25) that  $T$  is a contraction and we get a unique  $2\sigma$ -periodic function  $N(t)$  satisfying the conditions 1 and 2 and solving (1.21) on  $(0, \sigma)$  and  $(\sigma, 2\sigma)$ .

The next step is to prove that the constructed solution  $N(t)$  verifies the condition 3. From Equation (1.21) we deduce that

$$\mathcal{Q}(t) := \int_{t-\sigma}^t N(s) ds + \psi(N(t))$$

is piece-wise constant and we get the following equalities

$$\begin{aligned} \psi(N(\sigma^-)) - \psi(N(0^+)) &= \int_\sigma^{2\sigma} N(s) ds - \int_0^\sigma N(s) ds, \\ \psi(N(2\sigma^-)) - \psi(N(\sigma^+)) &= \int_0^\sigma N(s) ds - \int_\sigma^{2\sigma} N(s) ds. \end{aligned} \quad (1.26)$$



Since  $\psi(N(0^+)) = \psi(\bar{N}^+) = \psi(\bar{N}^-) = \psi(N(2\sigma^-))$ , we conclude that  $\psi(N(\sigma^-)) = \psi(N(\sigma^+))$ . Moreover, we conclude that  $\psi(N(t))$  is absolutely continuous and thus  $\mathcal{Q}(t)$  is constant and given by

$$\mathcal{Q}(t) \equiv \mathcal{Q}[\bar{N}^+] := \psi(\bar{N}^+) + \int_{\sigma}^{2\sigma} N(s) ds.$$

This proves the first part of theorem.

Assume now the additional hypothesis  $\psi(\underline{N}) < 1$  and  $\psi(\underline{N} \pm \varepsilon) > 1$ , thus there exist two pairs  $(\bar{N}_1^-, \bar{N}_1^+)$ ,  $(\bar{N}_2^-, \bar{N}_2^+)$  of positive numbers such that for  $\sigma$  small enough, the following conditions hold

- $\psi(\bar{N}_1^-) = \psi(\bar{N}_1^+)$  and  $\psi(\bar{N}_2^-) = \psi(\bar{N}_2^+)$ .
- $\bar{N}_1^- < \bar{N}_2^- < \underline{N} < \bar{N}_2^+ < \bar{N}_1^+$ .
- $\psi(\bar{N}_1^+) > 1 - \sigma\bar{N}_1^-$  and  $\psi(\bar{N}_2^-) < 1 - \sigma\bar{N}_2^+$ .

By applying Theorem 8 we can construct two periodic solutions of the System (1) with respective masses  $\mathcal{Q}[\bar{N}_1^+]$  and  $\mathcal{Q}[\bar{N}_2^+]$ , so we need to find a constant  $N_0^+ > 0$  such that  $\mathcal{Q}[N_0^+] = 1$ . Observe we have the following inequalities

$$\begin{aligned} \mathcal{Q}[\bar{N}_1^+] &> \psi(\bar{N}_1^+) + \sigma\bar{N}_1^- > 1 \\ \mathcal{Q}[\bar{N}_2^+] &< \psi(\bar{N}_2^+) + \sigma\bar{N}_2^+ < 1. \end{aligned} \tag{1.27}$$

Since  $\mathcal{Q}[\cdot]$  is continuous with respect to the variable  $\bar{N}^+$ , we conclude by applying the intermediate value theorem the existence of  $N_0^- \in (\bar{N}_1^-, \bar{N}_2^-)$  and  $N_0^+ \in (\bar{N}_2^+, \bar{N}_1^+)$  such that  $\mathcal{Q}[N_0^+] = 1$ . Therefore the corresponding periodic solution  $N(t)$  of the delay Equation (1.21) satisfies Equation (1.4) and the conditions 1, 2 and 3.  $\square$

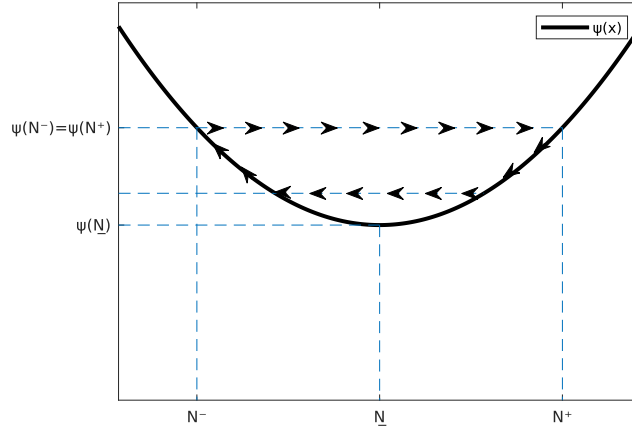


Figure 1.3 – An example of  $\psi$  with a local minimum at  $\underline{N}$ . The trajectory in arrows over the graph of  $\psi$  indicates the variations of periodic solution  $N(t)$  and the horizontal arrows indicate a jump discontinuity, which preserves the value of  $\psi(N)$  at these jumps.

**Remark 5.** We can also construct  $2\sigma$ -periodic solutions  $N(t)$  of (1.21) around a local maximum of  $\psi$ , which are piece-wise strictly increasing and preserve the value of  $\psi$  at jump discontinuities.

## 1.5 Numerical simulations

In order to illustrate the theoretical results of the previous sections, we present numerical results for different networks with multiple steady states, with different types of convergence to equilibrium, and with periodic solutions with jump discontinuities. The numerical illustrations we present below are obtained by solving the equation (1) with a classical first-order upwind scheme (see Scheme 2 in the Appendix of numerical methods).

### 1.5.1 Example 1: Convergence to different steady states

Our first example is a numerical simulation with multiple steady states. For this example we choose

$$\varphi(N) = \frac{1}{1 + e^{-9N+3.5}}, \quad \sigma = \frac{1}{2}. \quad (1.28)$$

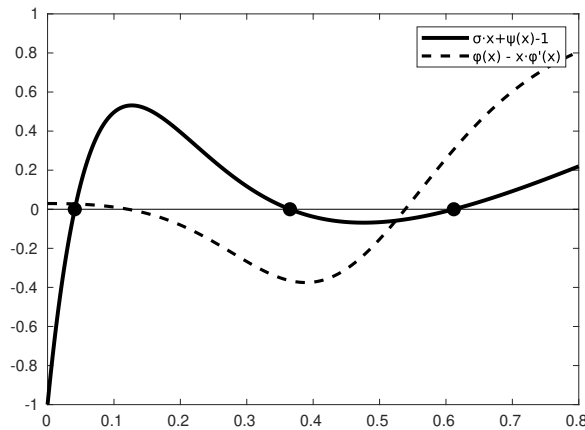


Figure 1.4 – Graphical representation of the three equilibria and the sign of  $\psi'$ , for examples 1 and 2. Two of them are in the region where  $\psi' > 0$  and the other one is the region where  $\psi' < 0$ .

In this case  $\psi'$  changes sign twice since  $\varphi(u) - u\varphi'(u)$  does and from Equation (1.15) we get three steady states given by  $N_*^1 \approx 0.0410$ ,  $N_*^2 \approx 0.3650$  and  $N_*^3 \approx 0.6118$ , as we observe in Figure 1.4.

When we take  $n_0(s) = \frac{1}{2}e^{-(s-1)_+}$  as the initial data, we have three different solutions for  $N(0)$  determined by the equation

$$N(0) = \varphi(N(0)) \int_{\sigma}^{\infty} n_0(s) ds, \quad (1.29)$$

that are given by  $N_0^1 \approx 0.0281$ ,  $N_0^2 \approx 0.4089$ ,  $N_0^3 \approx 0.7114$ . These values determine three different branches of solutions, which numerically converge to their respective steady states. In Figure 1.5a we observe that  $N(t)$  is increasing in  $[0, \sigma]$  and then approaches to the value  $N_*^1$ , which corresponds to a convergence to equilibrium according to Theorem 9. Moreover we observe in Figure 1.5b that  $N(t)$  converges monotonically to  $N_*^2$ , which satisfies  $\psi'(N_*^2) < 0$ . This is compatible with Theorem 10 in the case when  $\psi'(N(t))$  remains negative for all  $t \geq 0$ . Finally

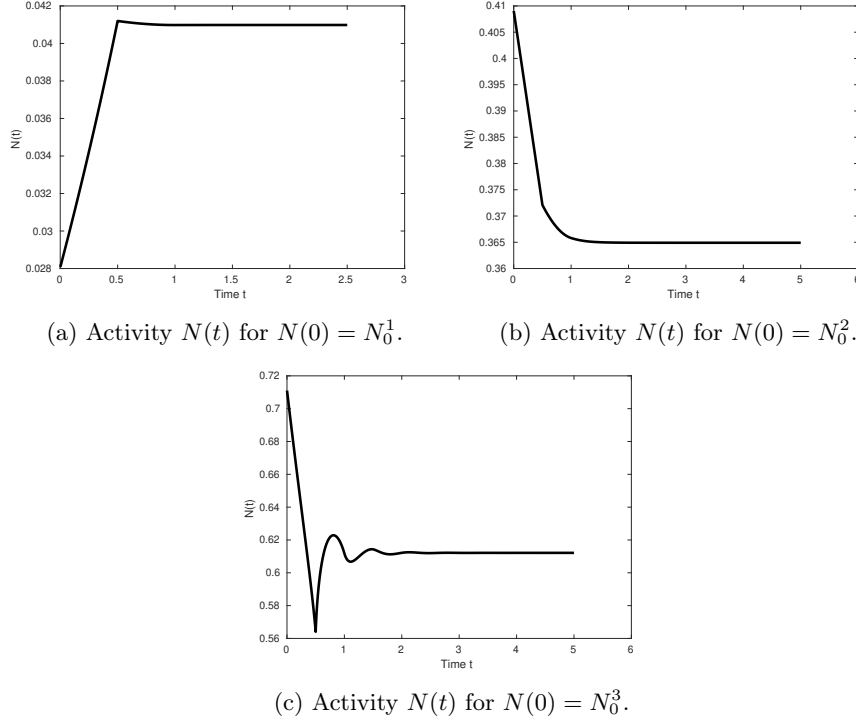


Figure 1.5 – **Example 1.** Activities of different solutions with  $n_0(s) = \frac{1}{2}e^{-(s-1)_+}$  for different values of  $N(0)$ .

in Figure 1.5c we observe that  $N(t)$  converges to  $N_*^3$  in the same way stated in Theorem 9 with  $\psi'(N_*^3) > 0$ .

### 1.5.2 Example 2: Possible jump discontinuities

Under the same  $\varphi$  defined in (1.28), consider now as initial data  $n_0(s) = e^{-(s-0.5)} \mathbf{1}_{\{s>0.5\}}$ . In this case we also get three possible solutions for  $N(0)$  in Equation (1.29), which are given by  $N_0^1 \approx 0.0423$ ,  $N_0^2 \approx 0.2887$ ,  $N_0^3 \approx 0.9958$ .

In Figure 1.6a we observe that  $N(t)$  is decreasing in  $[0, \sigma]$  and then approaches to the value  $N_*^1$ , which corresponds again to a convergence to equilibrium according to Theorem 9. In Figure 1.6b we observe that  $N(t)$  converges monotonically to  $N_*^2$ , which satisfies  $\psi'(N_*^2) < 0$ . In this case the solution is increasing and it corresponds to the behaviour stated in Theorem 10. Moreover in Figure 1.6c we observe that  $N(t)$  has a jump discontinuity at some  $t_0 \in (0, \sigma)$  that causes the solution to change to the branch of  $N_0^1$  and then  $N(t)$  converges to  $N_*^1$  afterwards. At the jump time, the solution preserves the value of  $\psi$  as we show in Figure 1.6d. The horizontal arrows represent the change of  $N(t)$  along the graph of  $\psi$  at this discontinuity.

### 1.5.3 Example 3: Periodic solutions

To describe periodic solutions we simulate two different examples.

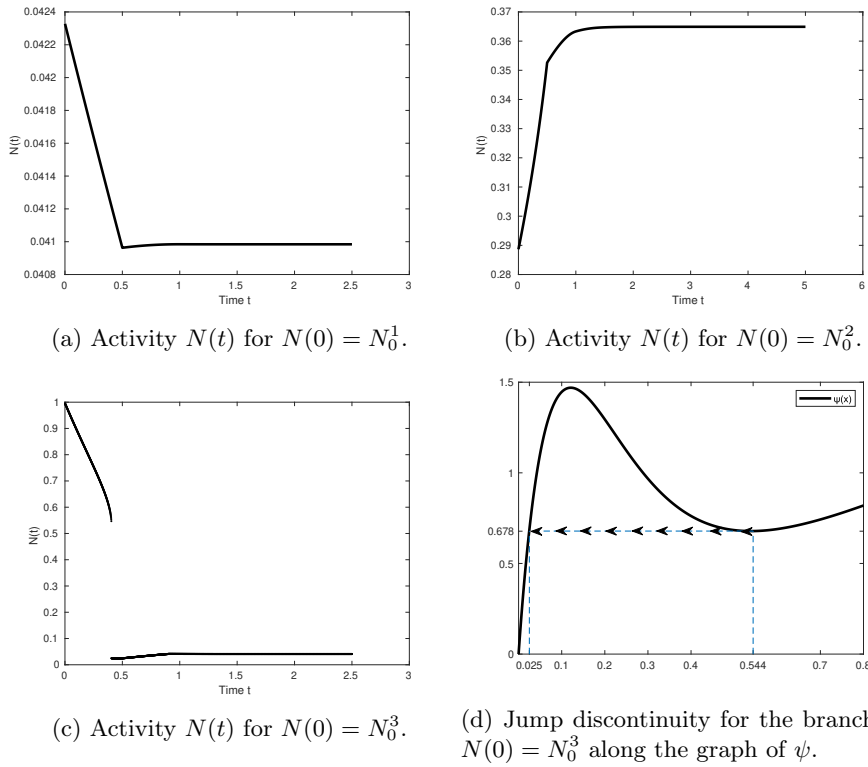


Figure 1.6 – **Example 2.** Activities of different solutions with  $n_0(s) = e^{-(s-0.5)} \mathbf{1}_{\{s>0.5\}}$  for different values of  $N(0)$ .

*Example 3.1.* We consider the firing rate determined by

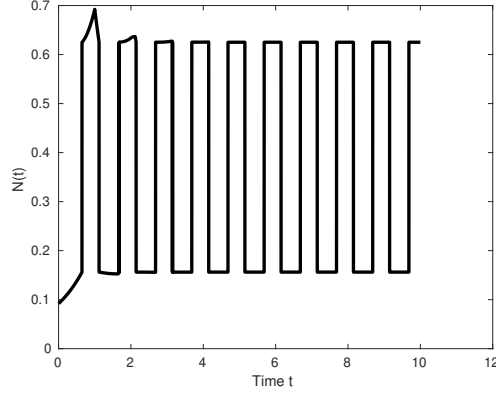
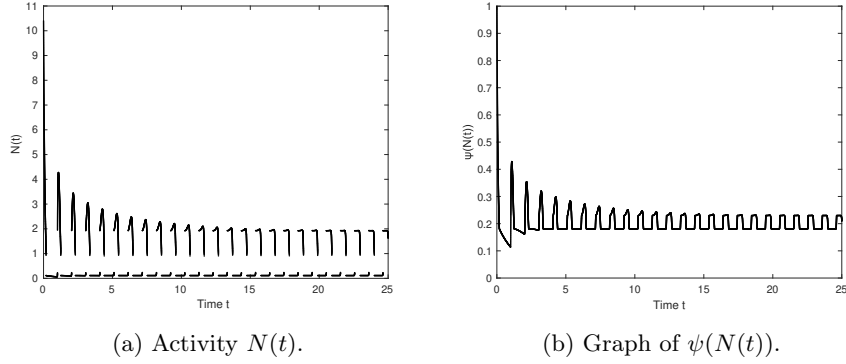
$$\varphi(N) = \max\{\min(1.6N, 1), 0.25\}, \quad \sigma = 1.$$

From Equation (1.15), there exists a unique steady state with  $N_* = 0.375$  and  $\psi'(N_*) = 0$ . Moreover, we observe in Figure 1.7 that the solution with initial data  $n_0(s) = e^{-s}$  converges to a  $\sigma$ -periodic solution of (1.4), which is piece-wise constant whose values oscillate between  $N_1 = \frac{0.25}{1.6}$  and  $N_2 = \frac{1}{1.6}$ . This periodic profile is an example of the type of solutions presented in Theorem 11.

*Example 3.2.* Next, we consider the firing rate determined by

$$\varphi(N) = \frac{10N^2}{N^2 + 1} + 0.5, \quad \sigma = 1,$$

From Equation (1.15) there exists a unique steady state with  $N_* \approx 0.8186$  and  $\psi'(N_*) < 0$ . For the initial data  $n_0(s) = e^{-(s-1)} \mathbf{1}_{\{s>1\}}$ , we observe in Figure 1.8a that the solution is asymptotic to a periodic pattern with jump discontinuities. The period is larger than 1 since  $\psi(N(t))$  is not converging to a constant as we see in Figure 1.8b.

Figure 1.7 – **Example 3.1.** Activity  $N(t)$  for  $n_0(s) = e^{-s}$ .(a) Activity  $N(t)$ .(b) Graph of  $\psi(N(t))$ .Figure 1.8 – **Example 3.2.** Behavior for  $n_0(s) = e^{-(s-1)} \mathbb{1}_{\{s>1\}}$ .

#### 1.5.4 Example 4: A non monotone firing rate

Since dynamics in Equation (1) depend heavily on the function  $\psi$ , theoretical results are valid not only in the strictly excitatory or inhibitory case. This allows to include non monotone examples of functions  $\varphi$ , which represents a more realistic assumption in the model.

For this example we choose the firing rate determined by

$$\varphi(N) = 8e^{-(N-0.1)^2} + 8e^{-(N-3)^2}, \quad \sigma = \frac{1}{5}.$$

Unlike of previous examples, the function  $\varphi$  is non monotone as we see in Figure 1.9.

In this case  $\psi'$  changes sign and there exist three steady states with  $N_*^1 \approx 1.4423$ ,  $N_*^2 \approx 2.0695$ ,  $N_*^3 \approx 3.0711$ , as we note in Figure 1.10.

When we take as initial data  $n_0(s) = \frac{2}{3}(1 + \cos(s))e^{-s}$  we get three possible solutions for  $N(0)$  in Equation (1.29), which are given by  $N_0^1 \approx 1.4976$ ,  $N_0^2 \approx 1.8163$ ,  $N_0^3 \approx 3.7037$ . These values determine three different branches of solutions.

In Figure 1.11a we observe that  $N(t)$  decreasing on  $[0, \sigma]$  and the solution converges to the steady state determined by  $N_*^1$ , which corresponds to the behaviour stated in Theorem 9. In Figure 1.11b we see that  $N(t)$  converges monotonically to  $N_*^2$ , which  $\psi'(N_*^2) < 0$ . In this case the solution is increasing and it corresponds to the behaviour stated in Theorem 10. Furthermore in

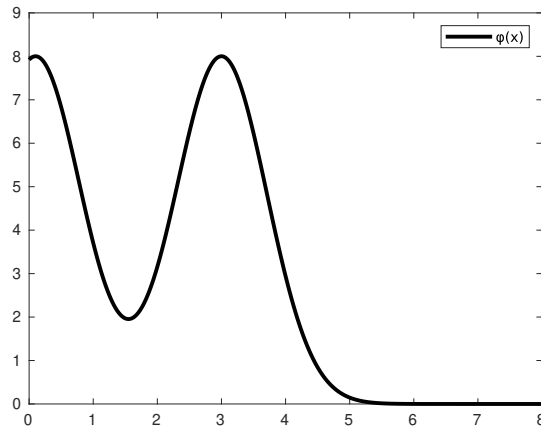


Figure 1.9 – Graphical representation of  $\varphi$ , which changes two times from a excitatory regime to an inhibitory one.

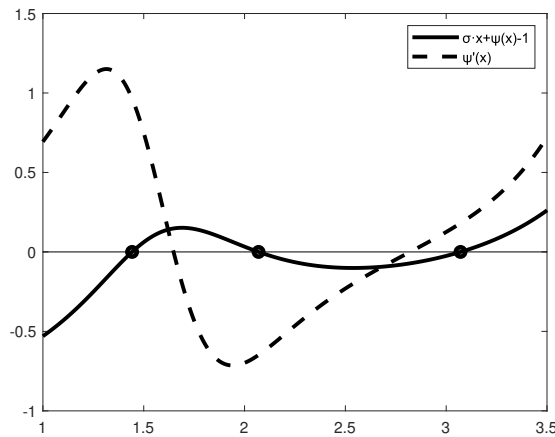


Figure 1.10 – Graphical representation of the three equilibria and the sign of  $\psi'$ , for example 4. Two of them are in the region where  $\psi' > 0$  and the other one is the region where  $\psi' < 0$ .

Figure 1.11c we observe that  $N(t)$  has a jump discontinuity at some  $t_0 \in (0, \sigma)$  that causes the solution to change to the branch of  $N_0^1$  and then  $N(t)$  converges to  $N_*^1$  afterwards. At this jump time, the solution preserves the value of  $\psi$  as we show in Figure 1.11d. The horizontal arrows represent the change of  $N(t)$  along the graph of  $\psi$  at this discontinuity. We observe essentially the same behaviors of Example 2.

## 1.6 Perspectives

In the present analysis of System (1) with the firing rate  $p$  modulated by amplitude as given by (1.1), we have exhibited some possible qualitative behaviours of solutions. Steady state

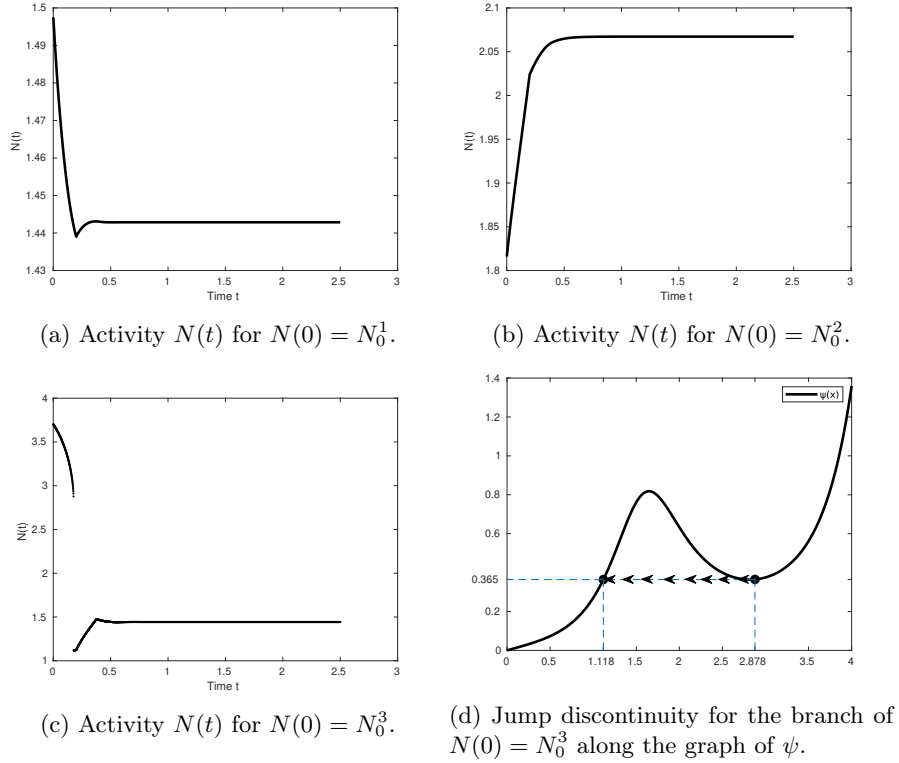


Figure 1.11 – **Example 4.** Activities of solutions with  $n_0(s) = \frac{2}{3}(1 + \cos(s))e^{-s}$  for different values of  $N(0)$ .

convergence always occurs in the inhibitory and the weakly excitatory networks. And this can also occur for strongly excitatory connections, in particular situations. Periodic solutions can be built for strongly excitatory connections. Our method is based on the derivation of a nonlinear delay equation for the network activity. Moreover, numerical simulations are consistent with the theoretical results obtained about the convergence to equilibrium and the existence of jump discontinuities.

This study provides possible behaviors which might arise for a more general firing rate. From this particular example of firing rate, we can think that the model induces an implicit delay, which is consistent with the interpretation of the discharge dynamics in the elapsed time model.

With respect to multiplicity of solutions in the case of a strong nonlinearities, an interesting problem is to determine the continuity of each branch of solutions from the same initial data and to understand how a jump discontinuity arises. Lack of well-posedness due to multiplicities and discontinuities is a hard problem to address.

Regarding convergence to equilibrium, we conjecture that the convergence rate in Theorem 9 for the inhibitory and weakly excitatory case is exponential, as it occurs in [49] for a variant of the firing rate  $p$ , which is also given by an indicator function. Moreover, we expect the convergence in Theorem 10 for monotone solutions in the strongly excitatory case to be exponential as well.

Concerning the existence of periodic solutions, it remains open to prove the existence of periodic continuous solutions when  $p$  is not locally a linear function and also to find periodic solutions with a period other than a multiple of  $\sigma$ . With respect to stability of periodic orbits,

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an interesting question would be to determine what kind of piece-wise monotone solutions for the activity are stable in the excitatory case and if exponential convergence to this type of profile arises. We conjecture that the stable solutions correspond to those with few jump discontinuities in general. Moreover, another interesting problem is to determine if the solutions obtained for the particular firing rate  $p$  in (1.1) are structurally stable with respect to perturbations of  $p$ .





## Chapter 2

# Elapsed time model with two times

This chapter corresponds to the pre-print [60] titled "A multiple time renewal equation for neural assemblies with elapsed time model".

### 2.1 Introduction

We now study a multiple time renewal equation that extends the classical elapsed time model by taking into account the elapsed time since penultimate discharge in addition to the last one, leading to a more complex system of integro-differential equations. Different extensions of the elapsed time model have been considered by incorporating new variables such as spatial dependence and a connectivity kernel in Salort et al. [59] or a leaky memory variable in Fonte et al. [28], with interesting pattern formations. From a biological point of view, the motivation of taking into account the penultimate discharge is to have a deeper understanding of the dependence of system dynamics on the previous history of spike trains. We aim to analyze how the incorporation of the elapsed time since penultimate discharge may change dynamics of the classical elapsed time model.

From a theoretical point of view, multiple time renewal equations have been investigated in Fournier et al. [29] in the linear case, where a non-expanding distance was introduced via a coupling argument. Moreover, these type of equations have multiple potential applications in other domains such as epidemiology, where the evolution of individuals under possible secondary infections has been studied. For an example, see the work of Ferretti et al. [27].

The extended model is described as follows. Let  $n = n(t, s, a)$  the probability density of finding a neuron at time  $t$ , such that the elapsed times since its last and penultimate discharge are respectively  $s$  and  $a$ . For simplicity, we simply call  $s$  as the first elapsed time and  $a$  as the second one. Moreover, we assume that for all  $t \geq 0$  the domain of definition of  $n$  in the elapsed time variables is contained in the domain

$$\mathcal{D} := \{(s, a) \in \mathbb{R}^2 : 0 \leq s \leq a\}.$$

Neural dynamics are modelled through the following non-linear renewal system

$$\begin{cases} \partial_t n + \partial_s n + \partial_a n + p(s, a, X(t))n = 0 & t > 0, a > s > 0, \\ n(t, s = 0, a) = N(t, a) := \int_0^\infty p(a, u, X(t))n(t, a, u) du & t > 0, a > 0, \\ X(t) = \int_0^\infty N(t, a) da & t > 0, \\ n(t = 0, s, a) = n_0(s, a) & a > s > 0. \end{cases} \quad (2.1)$$

As in the classical elapsed time model the function  $p: \mathcal{D} \times \mathbb{R} \rightarrow \mathbb{R}$  is the firing rate of neurons, which depends on the total activity  $X(t)$ . Furthermore, for the firing rate function  $p$ , we assume that there exist  $\sigma, p_0, p_\infty > 0$  such that

$$p_0 \mathbf{1}_{\{a > s \geq \sigma\}} \leq p \leq p_\infty. \quad (2.2)$$

Thus, we get

$$0 \leq X(t) \leq p_\infty, \quad \forall t \geq 0. \quad (2.3)$$

We assume for simplicity that  $p \in W^{1,\infty}(\mathcal{D} \times \mathbb{R})$ , although most of the theoretical results are also valid for firing rates with simple jump discontinuities and the behaviour of solutions does not depend on this regularity assumption as we show in the numerical simulations. As in the classical elapsed time model, we say that the network is inhibitory if  $p$  is decreasing with respect to the total activity  $X$  and excitatory if  $p$  is increasing. If in addition  $\|\partial_X p\|_\infty$  is small, we say that System (2.1) is under a weak interconnection regime.

The function  $N(t, a)$  represents the flux discharging neurons conditioned to elapsed time since penultimate discharge, so that the total activity  $X(t)$  corresponds to integrate with respect to all penultimate times. The boundary condition of  $n$  at  $s = 0$  states that the second elapsed time resets to the first elapsed time. This means when the first elapsed time is zero, the second elapsed time becomes instantaneously the first one, which explains the order change in the evaluation of  $p$  and  $n$  in the term  $\int_0^\infty p(a, u, X(t))n(t, a, u) du$ . The density of neurons with state  $(0, a)$  at time  $t$  corresponds to the marginal density of discharging neurons at time  $t$  that have the value of  $a > 0$  as first elapsed time.

We assume that the initial data  $n_0 \in L^1(\mathcal{D})$  is a probability density so that System (2.1) formally verifies

$$\iint n(t, s, a) da ds = \iint n_0(s, a) da ds = 1, \quad n(t, s, a) \geq 0 \quad \forall t \geq 0. \quad (2.4)$$

Concerning the asymptotic behavior of System (2.1), we follow the same strategy of Cañizo et al. in [9] on the application of Doeblin's theory for the classical elapsed time equation, i.e. finding a uniform lower bound for all the solutions of the linear system to obtain exponential convergence to the steady state. Then by a perturbation argument in the weak interconnections case, we get the exponential convergence for the non-linear case.

Finally, we also remark that when  $p$  does not depend on  $a$ , the probability density  $m(t, s) := \int_s^\infty n(t, s, a) da$  satisfies the equation

$$\begin{cases} \partial_t m + \partial_s m + p(s, X(t))m = 0 & t > 0, s > 0, \\ m(t, s = 0) = X(t) = \int_0^\infty p(u, X(t))m(t, u) du & t > 0, \\ m(t = 0, s) = \int_s^\infty n_0(s, a) da & s > 0. \end{cases} \quad (2.5)$$

In other words, the probability with respect to the last elapsed time is a solution of the classical

elapsed time equation. If in addition we consider a the firing rate of the form

$$p = \varphi(X(t))\mathbb{1}_{\{s>\sigma\}},$$

with  $\varphi \in W^{1,\infty}(\mathbb{R})$  strictly positive and  $\sigma > 0$  a constant, we know from the previous chapter that the total activity  $X(t)$  satisfies the integral equation

$$\int_{t-\sigma}^t X(s) ds + \frac{X(t)}{\varphi(X(t))} = 1, \quad \forall t \geq \sigma. \quad (2.6)$$

Moreover, we know that the solutions of this integral equation may have different behaviors such as periodic solution and jump discontinuities. This gives us an idea of possible asymptotic behaviors that solutions of System (2.1) may exhibit.

The chapter is organized as follows. In Section 2.2 we prove that System (2.1) is well-posed in a suitable space for weak non-linearities. Starting with the asymptotic analysis for the linear case, we prove in Section 2.3 the existence of a stationary state and exponential convergence via Doeblin's theory. For the non-linear problem in the case of weak interconnections, we show in Section 2.4 the uniqueness of the steady state and in Section 2.5 we prove the exponential convergence via a perturbation argument. Finally in Section 2.6 we present some examples of numerical simulations for different initial data and firing rates to contrast them with the classical elapsed time model .

## 2.2 Well-posedness for weak non-linearities

We prove that System (2.1) is well-posed under the weak interconnection regime. In order to do so, we start by studying an auxiliary linear problem where total activity is fixed and then we proceed to prove well-posedness of system (2.1) via a fixed point argument by contraction.

### 2.2.1 The linear problem

Given  $X \in \mathcal{C}_b[0, \infty)$ , we consider the following linear problem

$$\begin{cases} \partial_t n + \partial_s n + \partial_a n + p(s, a, X(t))n = 0 & t > 0, a > s > 0, \\ n(t, s = 0, a) = N(t, a) := \int_0^\infty p(a, u, X(t))n(t, a, u) du & t > 0, a > 0, \\ n(t = 0, s, a) = n_0(s, a) \geq 0 & a > s > 0. \end{cases} \quad (2.7)$$

We look for weak solutions satisfying  $n \in \mathcal{C}_b([0, \infty), L^1(\mathcal{D}))$ , so that  $N \in \mathcal{C}_b([0, \infty), L^1(0, \infty))$  and  $X \in \mathcal{C}_b[0, \infty)$ .

**Lemma 5.** *Assume that  $n_0 \in L^1(\mathcal{D})$  is a probability density and  $p \in W^{1,\infty}((0, \infty) \times \mathbb{R})$  satisfies (2.2). Then for a given  $X \in \mathcal{C}_b[0, \infty)$ , Equation (2.7) has a unique weak solution  $n \in \mathcal{C}_b([0, \infty), L^1(\mathcal{D}))$  with  $N \in \mathcal{C}_b([0, \infty), L^1(0, \infty))$  and  $X \in \mathcal{C}_b[0, \infty)$ . Moreover  $n$  is non-negative and verifies the property (2.4).*

In particular this lemma proves the property (2.4) for the non-linear System (2.1).

**Proof.** From the method of characteristics, we start by noticing that a solution of the linear System (2.7) satisfies the following fixed point equation

$$\begin{aligned} n(t, s, a) = \Psi[n](t, s, a) := & n_0(s - t, a - t) e^{-\int_0^t p(t'+s-t, t'+a-t, X(t')) dt'} \mathbb{1}_{\{a>s>t\}} \\ & + N(t - s, a - s) e^{-\int_s^t p(s'+s'-s, s'+a-s, X(s'+t-s)) ds'} \mathbb{1}_{\{t,a>s\}}, \end{aligned} \quad (2.8)$$

with  $N(t, a) = \int_0^\infty p(a, u, X(t))n(t, a, u) du$  depending on  $n$ .

Let  $T > 0$  and  $\mathcal{X}_T := \{n \in \mathcal{C}_b([0, T], L^1(\mathcal{D})) : n(0) = n_0\}$ , it readily follows that  $\Psi$  maps  $\mathcal{X}_T \rightarrow \mathcal{X}_T$ . We prove by the Picard contraction theorem that  $\Psi$  has a unique fixed point in  $\mathcal{X}_T$  for  $T > 0$  small enough, i.e., there exists a unique weak solution of (2.7) defined on  $[0, T]$ . Consider  $n_1, n_2 \in \mathcal{X}_T$ , we compute

$$\begin{aligned} \iint |\Psi[n_1] - \Psi[n_2]|(t, s, a) ds da &\leq \int_0^t \int_s^\infty |N_1 - N_2|(t-s, a-s) da ds \\ &\leq T \sup_{t \in [0, T]} \int_0^\infty |N_1 - N_2|(t, a) da \\ &\leq T p_\infty \sup_{t \in [0, T]} \|n_1(t, s, a) - n_2(t, s, a)\|_{L^1(\mathcal{D})}, \end{aligned} \quad (2.9)$$

thus for  $T < \frac{1}{p_\infty}$ , we have proved that  $\Psi$  is a contraction and there exists a unique  $n \in \mathcal{X}_T$  such that  $\Psi[n] = n$ . Since the choice of  $T$  is independent of  $n_0$ , we can reiterate this argument to get a unique solution of (2.7), which is defined for all  $t \geq 0$ .

From Formula (2.8) we can extend the notion of a weak solution for Equation (2.7) for an initial data  $n_0 \in (\mathcal{M}(\mathcal{D}), \|\cdot\|_{M^1})$ , the space of finite regular measures on  $\mathcal{D}$  with the norm of the total variation. Therefore we can redo the same argument to prove existence and uniqueness of a weak solution  $n \in \mathcal{C}_b([0, \infty), \mathcal{M}(\mathcal{D}))$  with  $N \in \mathcal{C}_b([0, \infty), \mathcal{M}(0, \infty))$  and  $X \in \mathcal{C}_b[0, \infty)$ .

Next we prove the mass conservation property. For all  $t \geq 0$ , consider  $\mathcal{S}_t : \mathcal{M}(\mathcal{D}) \rightarrow \mathcal{M}(\mathcal{D})$  the semigroup given by

$$\mathcal{S}_t[f](s, a) = f(s-t, a-t) \mathbf{1}_{\{a>s>t\}},$$

whose infinitesimal generator is the operator  $\mathcal{L}f = -\partial_s f - \partial_a f$ . From Duhamel's formula, the solution of the fixed point problem (2.8) also verifies the following equality

$$n(t, s, a) = \mathcal{S}_t[n_0](s, a) + \int_0^t \mathcal{S}_{t-\tau}[\delta_{\{s=0\}}](s, a)N(\tau, a) d\tau - \int_0^t \mathcal{S}_{t-\tau}[p(s, a, X(\tau))n(\tau, s, a)] d\tau, \quad (2.10)$$

where  $\delta_{\{s=0\}}(s, a)$  is the measure along the line  $\{(0, a) : a \geq 0\}$ . This formula is translated as

$$\begin{aligned} n(t, s, a) &= n_0(s-t, a-t) \mathbf{1}_{\{a>s>t\}} + N(t-s, a-s) \mathbf{1}_{\{t, a>s\}} \\ &\quad - \int_0^t p(s-t+\tau, a-t+\tau, X(\tau))n(\tau, s-t+\tau, a-t+\tau) \mathbf{1}_{\{a>s>t-\tau\}} d\tau, \end{aligned} \quad (2.11)$$

and we get the mass conservation property by integrating with respect to  $(s, a)$  on the domain  $\mathcal{D}$ .

Finally, since  $n_0$  is non-negative then  $\Psi$  preserves positivity, so by uniqueness of fixed point the corresponding solution  $n$  must be non-negative.  $\square$

## 2.2.2 The non-linear problem

We are now ready to prove that System (2.1) is well-posed in the case of weak interconnection.

**Theorem 13** (Well-posedness for weak interconnections). *Assume that  $n_0 \in L^1(\mathcal{D})$  is a probability density and that  $p \in W^{1, \infty}(\mathcal{D} \times \mathbb{R})$  satisfies (2.2). Then for*

$$\|\partial_X p\|_\infty < 1,$$

System (2.1) has a unique solution with  $n \in \mathcal{C}_b([0, \infty), L^1(\mathcal{D}))$ ,  $N \in \mathcal{C}_b([0, \infty), L^1(0, \infty))$  and  $X \in \mathcal{C}_b[0, \infty)$ . Moreover  $n$  verifies Condition (2.4) for all  $t > 0$ .

**Proof.** Consider  $T > 0$ . We fix a function  $X \in \mathcal{C}_b[0, \infty)$  and define the functions  $n \in \mathcal{C}_b([0, \infty), L^1(\mathcal{D}))$  and  $N \in \mathcal{C}_b([0, \infty), L^1(0, \infty))$  which are solutions of System (2.7) by Lemma 5. Furthermore, the solution of this linear equation satisfies (2.4).

So we have a solution of System (2.1) defined on  $[0, T]$  if  $X$  satisfies for all  $0 \leq t \leq T$  and  $x \in \Omega$ , the following fixed point condition

$$X(t) = \mathcal{T}[X](t) := \int_0^\infty N[X](t, a) da. \quad (2.12)$$

We prove that  $\mathcal{T}$  defines for all  $T > 0$  an operator that maps  $\mathcal{X}_T \rightarrow \mathcal{X}_T$  with  $\mathcal{X}_T := \mathcal{C}_b([0, T])$ . First, we observe the following estimate

$$\left| \int N(t, a) da \right| \leq p_\infty, \quad \forall t \in [0, T], \quad (2.13)$$

and it is immediate that  $\mathcal{T}[X] \in \mathcal{X}_T$ .

We now prove that for  $T$  small enough,  $\mathcal{T}$  is a contraction. Let  $X_1, X_2 \in \mathcal{X}_T$  with their respective solutions  $(n_1, N_1), (n_2, N_2)$  of System (2.7). For the difference between  $N_1$  and  $N_2$  we have

$$\begin{aligned} \int |N_1 - N_2|(t, a) da &\leq \iint |p(a, u, X_1) n_1(t, a, u) - p(a, u, X_2) n_2(t, a, u)| du da \\ &\leq \iint |p(a, u, X_1) - p(a, u, X_2)| n_1 du da + \iint p(a, u, X_2) |n_1 - n_2|(t, a, u) du da \\ &\leq \|\partial_X p\|_\infty \|X_1 - X_2\|_\infty + p_\infty \|n_1 - n_2\|_{L^1(\mathcal{D})}. \end{aligned} \quad (2.14)$$

Now we have to estimate the difference between  $n_1$  and  $n_2$ . From (2.11) and estimate (2.14), we get

$$\|n_1 - n_2\|_{L^1(\mathcal{D})} \leq 2T \|\partial_X p\|_\infty \|X_1 - X_2\|_\infty + 2Tp_\infty \|n_1 - n_2\|_{L^1(\mathcal{D})}.$$

Then, for  $T < \frac{1}{2p_\infty}$  we obtain

$$\|n_1 - n_2\|_{L^1(\mathcal{D})} \leq \frac{2T \|\partial_X p\|_\infty}{1 - 2Tp_\infty} \|X_1 - X_2\|_\infty. \quad (2.15)$$

Finally by using again estimate (2.14), the operator  $\mathcal{T}$  satisfies

$$\|\mathcal{T}[X_1] - \mathcal{T}[X_2]\|_\infty \leq \|\partial_X p\|_\infty \left( 1 + \frac{2Tp_\infty}{1 - 2Tp_\infty} \right) \|X_1 - X_2\|_\infty \quad (2.16)$$

Hence for  $\|\partial_X p\|_\infty < 1$  and  $T$  small enough,  $\mathcal{T}$  is a contraction.

From Picard's fixed point we get a unique  $X \in \mathcal{X}_T$  such that  $\mathcal{T}[X] = X$ , and this implies the existence of a unique solution of (2.1) defined on  $[0, T]$ . Since estimate (2.13) is uniform in  $T$ , we can iterate this argument to get a unique solution of (2.1) defined for all  $t > 0$ .

Furthermore, we conclude from this construction that the non-linear System (2.1) satisfies (2.4) like the linear System (2.7).  $\square$

### 2.3 Asymptotic behavior for the linear case

In order to study the behavior of System (2.1), we start by studying the case when  $X \geq 0$  is a fixed constant. Thus we consider the linear problem given by

$$\begin{cases} \partial_t n + \partial_s n + \partial_a n + p(s, a, X)n = 0 & t > 0, a > s > 0, \\ n(t, s = 0, a) = N(t, a) := \int_0^\infty p(a, u, X)n(t, a, u) du & t > 0, a > 0, \\ n(t = 0, s, a) = n_0(s, a) & a > s > 0. \end{cases} \quad (2.17)$$

To determine the behavior of System (2.17), we consider  $(n_X, N_X)$  as the solution of the steady state problem given by

$$\begin{cases} \partial_s n + \partial_a n + p(s, a, X)n = 0 & a > s > 0, \\ n(s = 0, a) = N(a) := \int_0^\infty p(a, u, X)n(a, u) du & t > 0, a > 0. \end{cases} \quad (2.18)$$

In Theorem 3, we observed that classical elapsed time model satisfies the generalized relative entropy inequality for the linear problem. In the same way, we can prove this property for the linear System (2.17).

**Proposition 2** (Generalized relative entropy). *Assume there exists a steady solution of the linear System (2.17) with  $n_X, N_X > 0$ . Then for all convex functions  $H: [0, \infty) \rightarrow [0, \infty)$  with  $H(0) = 0$ , the solution  $n$  of the linear System (2.17) satisfies*

$$\begin{aligned} \frac{d}{dt} \iint n_X(s, a) H\left(\frac{n(t, s, a)}{n_X(s, a)}\right) da ds &= -D_H[n(t, s, a)] \leq 0 \quad \forall t \geq 0, \\ D_H[n(t, s, a)] &= \iint p(s, a, X) n_X(s, a) H\left(\frac{n(t, s, a)}{n_X(s, a)}\right) da ds - \int N_X(a) H\left(\frac{N(t, a)}{N_X(a)}\right) da, \end{aligned} \quad (2.19)$$

and in particular the steady state is unique.

**Proof.** In order to prove the relative entropy property, we follow the arguments in [7]. We start by noticing the following identities

$$\partial_s n = n_X \partial_s \left(\frac{n}{n_X}\right) + \frac{n}{n_X} \partial_s n_X, \quad \partial_a n = n_X \partial_a \left(\frac{n}{n_X}\right) + \frac{n}{n_X} \partial_a n_X, \quad (2.20)$$

and for simplicity we reformulate Equation (2.17) as follows

$$\begin{cases} \partial_t n + \partial_s n + \partial_a n + p(s, a, X)n = \delta_{\{s=0\}}(s, a)N(t, a) & t > 0, a > s > 0, \\ n(t, s = 0, a) = 0 & t > 0, a > 0, \\ n(t = 0, s, a) = n_0(s, a) & a > s > 0, \end{cases} \quad (2.21)$$

where  $\delta_{\{s=0\}}(s, a)$  is the measure along the line  $\{(0, a): a \geq 0\}$ . In the same way, we reformulate the corresponding steady state problem (2.18).

$$\begin{cases} \partial_s n_X + \partial_a n_X + p(s, a, X)n_X = \delta_{\{s=0\}}(s, a)N_X(a) & a > s > 0, \\ n_X(s = 0, a) = 0 & a > 0. \end{cases} \quad (2.22)$$

Hence by using the identities (2.20) along with Equations (2.21) and (2.22), we get the following

equation for  $\frac{n}{n_X}$

$$\partial_t \left( \frac{n}{n_X} \right) + \partial_s \left( \frac{n}{n_X} \right) + \partial_a \left( \frac{n}{n_X} \right) = \delta_{\{s=0\}}(s, a) \frac{N_X}{n_X} \left( \frac{N}{N_X} - \frac{n}{n_X} \right)$$

and if we multiply this equality by  $H' \left( \frac{n}{n_X} \right)$ , we get

$$\partial_t H \left( \frac{n}{n_X} \right) + \partial_s H \left( \frac{n}{n_X} \right) + \partial_a H \left( \frac{n}{n_X} \right) = \delta_{\{s=0\}}(s, a) \frac{N_X}{n_X} \left( \frac{N}{N_X} - \frac{n}{n_X} \right) H' \left( \frac{n}{n_X} \right).$$

Therefore, by multiplying the latter equality by  $n_X$  and using Equation (2.22), we have the corresponding equation for  $u = n_X H \left( \frac{n}{n_X} \right)$

$$\partial_t u + \partial_s u + \partial_a u + p(s, a, X)u = \delta_{\{s=0\}}(s, a) N_X \left[ \left( \frac{N}{N_X} - \frac{n}{n_X} \right) H' \left( \frac{n}{n_X} \right) + H \left( \frac{n}{n_X} \right) \right]. \quad (2.23)$$

Finally, by noticing the following limit

$$\lim_{s \rightarrow 0} \frac{n(t, s, a)}{n_X(s, a)} = \frac{N(t, a)}{N_X(a)}, \quad \text{for a.e. } t, a > 0,$$

we conclude the generalized relative entropy property (2.19) by integrating Equation (2.23) with respect to  $(s, a)$  on the domain  $\mathcal{D}$ . Moreover, we observe that  $D_H[\cdot]$  is non-negative by applying Jensen's inequality with the probability measure  $d\mu = p(a, y) \frac{n_X(a, y)}{N_X(a)} dy$  for each  $a > 0$ . In particular when  $H$  is strictly convex and  $D_H[n] = 0$ , we deduce that  $\frac{n}{n_X}$  is constant and subsequently we get  $n = n_X$ , since both  $n, n_X$  are probability densities. Therefore, the steady state is unique.  $\square$

If we consider the entropy method to prove exponential convergence for the linear Equation (2.17) in  $L^1(\mathcal{D})$ , we have the following equality for  $H(\cdot) = |\cdot|$

$$\frac{d}{dt} \iint |n - n_X| da ds = \int \left| \int p(n - n_X) da \right| ds - \iint p |n - n_X| da ds \leq 0,$$

and the  $L^1$  Poincaré inequality for the right-hand side is not available since the condition  $\int_s^\infty (n - n_X) da = 0$  is not fulfilled.

Furthermore, in Theorem (2) we assumed that  $n_X$  and  $N_X$  are strictly positive, which is not necessarily true. Unlike the classical elapsed time model, there exist solutions where  $n_X$  and  $N_X$  vanish for some values of  $(s, a)$ . Indeed, consider for example  $p(s, a, X) = \mathbb{1}_{\{s > 1\}}$  which satisfies the bounds (2.2) and observe that  $N_X$  satisfies Equation (2.30), implying that  $N_X(a)$  vanishes for  $a < 1$  and subsequently we see from Formula (2.29) that  $n_X$  vanishes when  $a - s < 1$ .

Due to the limitations of the entropy method approach, we will make use of Doeblin's theory. From Lemma 5, the solution of the linear problem (2.17) determines a Markov semigroup acting on  $L^1(\mathcal{D})$ . By means of Doeblin's theorem, the solutions of linear Equation (2.17) converge exponentially to a unique steady state, as we assert in the following theorem.

**Theorem 14.** *Let  $n_0 \in L^1(\mathcal{D})$  be a probability density and assume that  $p$  smooth satisfies Assumption (2.2). Then for a fixed  $X > 0$ , there exists a unique stationary solution  $n_X(s, a) \in L^1(\mathcal{D})$  of the linear Equation (2.17) satisfying  $\iint n_X(s, a) da ds = 1$ . Moreover, the corresponding*



solution of Equation (2.17) satisfies

$$\|n(t) - n_X\|_{L^1_{s,a}} \leq \frac{1}{1-\alpha} e^{-\lambda t} \|n_0 - n_X\|_{L^1_{s,a}} \quad \forall t \geq 0,$$

with  $\alpha = \frac{1}{2} p_0^2 \sigma^2 e^{-3p_\infty \sigma}$  and  $\lambda = -\frac{\log(1-\alpha)}{3\sigma} > 0$ .

In order to obtain the result, we show that after some time the solution of the linear problem is uniformly bounded from below for all probability densities. Thus from Doeblin's theorem we get the exponential convergence to equilibrium.

**Lemma 6.** *Assume (2.4) and (2.2). Let  $n(t, s, a)$  be a solution of (2.17), then there exists  $t_0 > 0$ ,  $\alpha \in (0, 1)$  and a probability density  $\nu \in L^1$  such that*

$$n(t_0, s, a) \geq \alpha \nu(s, a).$$

**Proof.** The main idea of the proof is to control the mass transported along the lines of direction  $(1, 1)$ . Firstly, we observe the transport of the initial data  $n_0$ . From Assumption (2.2) and the characteristics Formula (2.8) the following inequality holds

$$\int_t^\infty \int_s^\infty n(t, s, a) da ds \geq e^{-p_\infty t}, \quad \forall t \geq \sigma. \quad (2.24)$$

Secondly, we see the mass that returns at  $s = 0$ . From (2.24) we get for all  $t \geq \sigma$

$$\int_t^\infty n(t, s = 0, a) da = \int_t^\infty N(t, a) da \geq p_0 \int_t^\infty \int_a^\infty n(t, a, u) du da \geq p_0 e^{-p_\infty t}. \quad (2.25)$$

This means that we reduced by one dimension the problem of finding the uniform lower bound. For  $t \geq \sigma$  the mass of the region  $\{(s, a) : a > s > t\}$  concentrates in the line  $\{(0, a) : a \geq t\}$ , as we see in Figure 2.1.

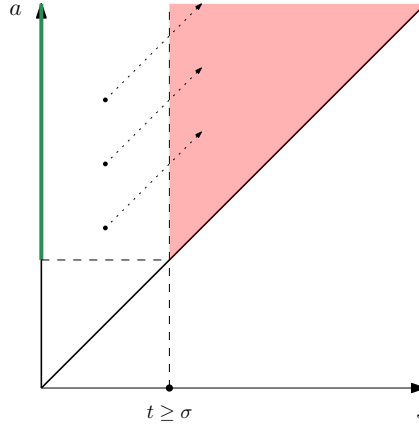


Figure 2.1 – First reduction of dimension. For a  $t \geq \sigma$ , all points in  $\mathcal{D}$  are transported to the red region, which has a total mass of at least  $e^{-p_\infty t}$ . Then a mass of at least  $p_0 e^{-p_\infty t}$  returns to the green line.

Thirdly, in order to control the point values of  $n(t, s, a)$ , we regard the values of  $N(t, a)$ .

Observe that from Formula (2.8) we have

$$\begin{aligned} n(t, s, a) &\geq N(t-s, a-s) e^{-\int_0^s p(s', s'+a-s, X) ds'} \mathbf{1}_{\{t, a > s\}} \\ &\geq N(t-s, a-s) e^{-p_\infty s} \mathbf{1}_{\{t, a > s\}}, \end{aligned} \quad (2.26)$$

thus for  $a \geq \sigma$  and  $t-a > \sigma$ , we obtain by using again Assumption (2.2) that

$$\begin{aligned} N(t, a) &\geq p_0 \int_a^\infty n(t, a, u) du \\ &\geq p_0 e^{-p_\infty a} \int_a^\infty N(t-a, u-a) du = p_0 e^{-p_\infty a} \int_0^\infty N(t-a, u) du \\ &\geq p_0 e^{-p_\infty a} \int_{t-a}^\infty N(t-a, u) du \geq p_0^2 e^{-p_\infty t}. \end{aligned} \quad (2.27)$$

This means we reduced the problem of finding the uniform lower bound by one dimension again, as we see in Figure 2.2.

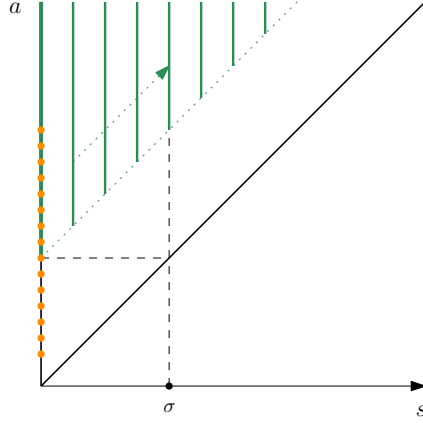


Figure 2.2 – Second reduction of dimension. For  $t \in [\sigma, 2\sigma]$  the green lines are transported to the region where  $s \geq \sigma$  and their mass is of at least  $p_0 e^{-2p_\infty \sigma}$ . Then the mass of each green line is concentrated in the orange points, whose values are at least  $p_0^2 e^{-2p_\infty \sigma}$ .

Finally, once we have estimated  $N(t, a)$  from below, we come back to estimate (2.26) to conclude that for  $a-s \geq \sigma$  and  $t-a > \sigma$  we have

$$\begin{aligned} n(t, a, s) &\geq N(t-s, a-s) e^{-p_\infty s} \mathbf{1}_{\{t, a > s\}} \\ &\geq p_0^2 e^{-p_\infty t} \mathbf{1}_{\{t-a, a-s > \sigma\}}, \end{aligned} \quad (2.28)$$

so that we can choose  $t = 3\sigma$  and conclude that

$$n(3\sigma, a, s) \geq p_0^2 e^{-3p_\infty \sigma} \mathbf{1}_{\{2\sigma > a > s + \sigma\}}.$$

Therefore we get the desired result with  $t_0 = 3\sigma$ ,  $\alpha = \frac{1}{2} \sigma^2 p_0^2 e^{-3p_\infty \sigma} \in (0, 1)$  and  $\nu$  given by

$$\nu(s, a) = \frac{2}{\sigma^2} \mathbf{1}_{\{2\sigma > a > s + \sigma\}},$$

whose support is contained in orange region of Figure 2.3.

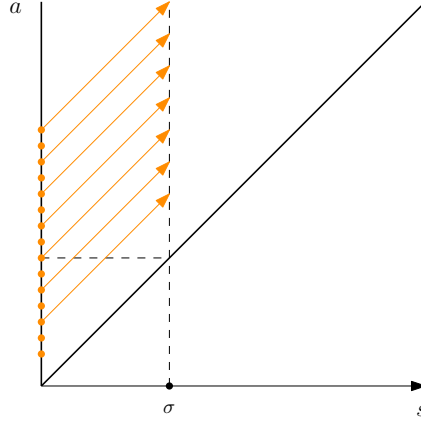


Figure 2.3 – Finally for  $t \in [2\sigma, 3\sigma]$  the orange dots are transported to region where  $s \geq \sigma$ , which allows to construct a minorization function for Doeblin's Theorem.

□

From Lemma 6 the hypothesis of Doeblin's theorem are verified and Theorem 14 readily follows.

Concerning the conditioned activity  $N$  in System (2.18), we conclude from Theorem 14 that for  $X$  fixed, there is a unique stationary  $N_X \in L^1(0, \infty)$  determined by the method of characteristics through the formula

$$n_X(s, a) = N_X(a - s) \exp\left(-\int_0^s p(s', a - s + s', X) ds'\right), \quad a > s. \quad (2.29)$$

Replacing this expression in the boundary condition at  $s = 0$ , we obtain the following integral equation for  $N_X(a)$

$$N_X(a) = \mathcal{T}_X[N_X](a), \quad (2.30)$$

with  $\mathcal{T}_X: L^1(0, \infty) \rightarrow L^1(0, \infty)$  given by

$$\begin{aligned} \mathcal{T}_X[N](a) &:= \int_0^\infty p(a, u + a, X) \exp\left(-\int_0^a p(s', u + s', X) ds'\right) N(u) du \\ &= -\frac{\partial}{\partial a} \int_0^\infty \exp\left(-\int_0^a p(s', u + s', X) ds'\right) N(u) du. \end{aligned}$$

Moreover, by integrating Equation (2.29) we get

$$\int_0^\infty \int_0^\infty N_X(a) \exp\left(-\int_0^s p(s', a + s', X) ds'\right) da ds = 1. \quad (2.31)$$

Therefore we conclude that finding a function  $N \in L^1(0, \infty)$  satisfying Equation (2.30) and Condition (2.31) is equivalent to finding a steady state  $n_X(s, a)$  in Equation (2.18). The integral Equation (2.30) will play an important role in the analysis of the non-linear System (2.1), thus we prove the following two lemmas on the operator  $\mathcal{T}_X$  that will be useful in the sequel.

**Lemma 7.** *Assume that  $p$  Lipschitz satisfies Assumption (2.2). For each  $X > 0$  the operator*

$\mathcal{T}_X$  is compact and it satisfies that  $\dim \ker(I - \mathcal{T}_X) = 1$ , which is generated by a non-negative function, and

$$\text{ran}(I - \mathcal{T}_X) = \left\{ f \in L^1(0, \infty) : \int_0^\infty f(x) dx = 0 \right\}.$$

**Proof.** The first step is to prove that  $\mathcal{T}_X$  is a compact operator. This means we have to prove that the set  $A = \{\mathcal{T}_X[f] : \|f\|_1 \leq 1\}$  is relatively compact in  $L^1$ .

First observe that  $\|\mathcal{T}_X[f]\|_1 \leq p_\infty$  for all  $f$  with  $\|f\|_1 \leq 1$ , so  $A$  is bounded.

Second, we prove that

$$\int_r^\infty |\mathcal{T}_X[f](a)| da \rightarrow 0 \quad \text{uniformly when } r \rightarrow \infty.$$

Indeed for  $r > \sigma$  we have

$$\begin{aligned} \int_r^\infty |\mathcal{T}_X[f](a)| da &\leq p_\infty \int_r^\infty \int_0^\infty |f(u)| e^{-\int_0^a p(s', u+s', X) ds'} du da \\ &\leq p_\infty \int_r^\infty \int_\sigma^\infty |f(u)| e^{-p_0(a-\sigma)} du da \\ &\leq p_\infty e^{p_0\sigma} \int_\sigma^\infty |f(u)| du \int_r^\infty e^{-p_0 a} da \\ &\leq p_\infty e^{p_0\sigma} \frac{e^{-p_0 r}}{p_0} \rightarrow 0. \end{aligned}$$

Now we prove the equicontinuity property. Observe that

$$\begin{aligned} \frac{d}{da} \mathcal{T}_X[f](a) &= \int_0^\infty (\partial_s p + \partial_a p)(a, u+a, X) e^{-\int_0^a p(s', u+s', X) ds'} f(u) du \\ &\quad - \int_0^\infty p(a, u+a, X)^2 e^{-\int_0^a p(s', u+s', X) ds'} f(u) du, \end{aligned}$$

thus for  $f$  with  $\|f\|_1 \leq 1$  we have

$$\int_0^\infty \left| \frac{d}{da} \mathcal{T}_X[f](a) \right| da \leq \|\nabla p\|_\infty + p_\infty^2.$$

Therefore by the Kolmogorov-Frechet theorem we conclude that  $A$  is relatively compact so the operator  $\mathcal{T}_X$  is.

Furthermore, since  $n_X$  is the unique steady state of Equation (2.17) that is a probability density, from the linearity we deduce that any other function in  $\ker(I - \mathcal{T}_X)$  is a multiple of  $n_X$  and thus  $\dim \ker(I - \mathcal{T}_X) = 1$ .

Next, we proceed to determine  $\text{ran}(I - \mathcal{T}_X)$ . Observe that adjoint operator  $\mathcal{T}_X^* : L^\infty \rightarrow L^\infty$  is given by

$$\mathcal{T}_X^*[g](a) = \int_0^\infty p(u, u+a, X) \exp\left(-\int_0^u p(s', a+s', X) ds'\right) g(u) du,$$

and from Fredholm's alternative we get  $\dim \ker(I - \mathcal{T}_X^*) = \dim \ker(I - \mathcal{T}_X) = 1$ . Since  $\mathcal{T}_X^*[g] \equiv 0$  for any constant function, we deduce that  $\ker(I - \mathcal{T}_X^*)$  is the subspace of constant functions.

Finally from orthogonality conditions we conclude that

$$\text{ran}(I - \mathcal{T}_X) = \left\{ f \in L^1(0, \infty) : \int_0^\infty f(x) dx = 0 \right\}.$$

□

A direct consequence of Lemma 7 is the following result

**Lemma 8.** *Assume that  $p$  is smooth respect to variable  $X$ , then  $N_X(a)$  is also smooth with respect to  $X$ .*

**Proof.** Define the  $F: L^1(0, \infty) \times (0, \infty) \rightarrow \text{ran}(I - \mathcal{T}_X) \times \mathbb{R}$  given by

$$F(N, X) = \left( (I - \mathcal{T}_X)[N], \iint N(a) e^{-\int_0^a p(s', u+s', X) ds'} da ds - 1 \right),$$

so that for each  $X$  we have  $F(N_X(a), X) = 0$ . Observe that  $D_N F$  is given by

$$D_N F[h] = \left( (I - \mathcal{T}_X)[h], \iint h(a) e^{-\int_0^a p(s', u+s', X) ds'} da ds \right)$$

Thus by Lemma 7 this operator is an isomorphism and from the implicit function theorem we conclude that  $N_X(a)$  depends smoothly on  $X$ . □

**Remark 6.** *The lower bound condition (2.2) on the firing rate  $p$  is important to verify the existence of a steady state for System (2.1) and Doeblin's condition. For example, when we consider  $X > 0$  and*

$$p(s, a, X) = \mathbb{1}_{\{a-s > X\}},$$

*then there are no steady states of the linear Equation (2.17), besides the zero solution. Indeed, from Equation (2.30) we deduce that the discharging flux  $N$  should satisfy*

$$N(a) = e^{-a} \int_X^\infty N(u) du,$$

*whose unique non-negative solution in  $L^1(0, \infty)$  is  $N \equiv 0$ .*

## 2.4 Steady states

Consider  $n^* = n^*(s, a)$  with support in the set  $\{s \leq a\}$ . We are interested in the stationary solutions of the non-linear System (2.1) given by

$$\begin{cases} \partial_s n + \partial_a n + p(s, a, X)n = 0 & a > s > 0, \\ n(s=0, a) = N(a) := \int_0^\infty p(a, u, X)n(a, u) du & a > 0, \\ X = \int_0^\infty N(a) da, \\ \iint n(s, a) da ds = 1, \quad n(s, a) \geq 0. \end{cases} \quad (2.32)$$

We define  $N_X$  as the respective conditional activity in terms of  $X$ . In order to have a steady state of the non-linear Problem (2.1), we must find  $X > 0$  such that

$$X = \Phi(X) := \int_0^\infty N_X(a) da. \quad (2.33)$$

In the general case this equation has always a solution since the right-hand side is uniformly bounded thanks to estimate (2.3) and  $N_X(a)$  depends continuously on  $X$ . By using the properties of the operator  $\mathcal{T}_X$ , we prove that under the weak interconnections regime the non-linear System (2.1) has a unique steady state.

**Theorem 15.** *Assume (2.4) and that  $p$  smooth satisfies Assumption (2.2). Then for  $\|\partial_X p\|_\infty$  small enough, System (2.1) has a unique steady state  $(n^*, N^*, X^*)$ .*

**Proof.** The goal is to prove that  $\Phi$  is a contraction in order to obtain a unique fixed point. In order to estimate  $\partial_X N$  we make use of the implicit function theorem. By differentiating Equation (2.31) we get

$$\iint \partial_X(N_X) e^{-\int_0^s p(s', a+s', X) ds'} da ds = \iint N_X e^{-\int_0^s p(s', a+s', X) ds'} \left( \int_0^s \partial_X p(s', u' + s', X) ds' \right) da ds. \quad (2.34)$$

Furthermore, if we differentiate with respect to  $X$  the Equation (2.30), we get

$$\begin{aligned} \partial_X N_X(a) &= \int_0^\infty \left( -\partial_a e^{-\int_0^a p(s', u+s', X) ds'} \right) \partial_X N_X(u) du \\ &\quad + \int_0^\infty \left( -\partial_a \partial_X e^{-\int_0^a p(s', u+s', X) ds'} \right) N_X(u) du, \end{aligned}$$

i.e.  $\partial_X N_X(a)$  satisfies the equation

$$\begin{aligned} (I - \mathcal{T}_X)[\partial_X N](a) &= \int_0^\infty p(a, u + a, X) e^{-\int_0^a p(s', u+s', X) ds'} N_X(u) du \\ &\quad - \int_0^\infty p(a, u + a, X) \left( \int_0^a \partial_X p(s', u + s', X) ds' \right) e^{-\int_0^a p(s', u+s', X) ds'} N_X(u) du. \end{aligned} \quad (2.35)$$

By using the implicit function theorem and the Condition (2.34) we can define an inverse of  $I - \mathcal{T}_X$  which depends continuously on  $X$ . Observe that  $\|(I - \mathcal{T}_X)^{-1}\|$  is uniformly bounded on  $X$  in the operator norm, since  $X$  is uniformly bounded. Thus, for the function  $\Phi$  we get

$$\begin{aligned} |\Phi'(X)| &= \left| \int \partial_X N_X(a) da \right| \\ &\leq \|(I - \mathcal{T}_X)^{-1}\| \|\partial_X p\|_\infty \left( \iint (1 + p_\infty a) e^{-\int_0^a p(s', u+s', X) ds} N_X(u) da du \right) \\ &\leq C \|\partial_X p\|_\infty \left( \iint (1 + p_\infty a) e^{-p_0 a} N_X(u) da du \right) \\ &\leq C \|\partial_X p\|_\infty p_\infty \left( \int (1 + p_\infty a) e^{-p_0 a} da \right), \end{aligned}$$

so that for  $\|\partial_X p\|_\infty$  small enough  $\Phi$  is a contraction and we conclude the result.  $\square$

## 2.5 Convergence to equilibrium

After studying the linear case, we are now ready to prove convergence to the steady state under the weak interconnection regime, i.e.  $\|\partial_X p\|_\infty$  small enough, by a perturbation argument.

**Theorem 16** (Convergence to equilibrium). *Assume that  $n_0 \in L^1(\mathcal{D})$  satisfies Assumption (2.4) and that  $p$  Lipschitz satisfies Assumption (2.2). For  $\|\partial_X p\|_\infty$  small enough, let  $(n^*, N^*, X^*)$  be the corresponding stationary state of System (2.1). Then there exist  $C, \lambda > 0$  such that the solution  $n$  of System (2.1) satisfies*

$$\|n(t) - n^*\|_{L^1_{s,a}} \leq C e^{-\lambda t} \|n_0 - n^*\|_{L^1_{s,a}}, \quad \forall t \geq 0.$$

Moreover  $\|N(t) - N^*\|_{L^1}$  and  $|X(t) - X^*|$  converge exponentially to 0 when  $t \rightarrow \infty$ .

**Proof.** Observe that  $n$  satisfies the evolution equation

$$\partial_t n = \mathcal{L}_X[n] := -\partial_s n - \partial_a n - p(s, a, X(t))n + \delta_{\{s=0\}}(s, a) \int_0^\infty p(a, u, X(t))n(t, a, u) du,$$

where  $\delta_{\{s=0\}}(s, a)$  is the measure along the line  $\{(0, a) : a \geq 0\}$ . We can rewrite the evolution equation as

$$\partial_t n = \mathcal{L}_{X^*}[n] + (\mathcal{L}_X[n] - \mathcal{L}_{X^*}[n]) = \mathcal{L}_{X^*}[n] + h. \quad (2.36)$$

with  $h(t, s, a)$  given by

$$h = (p(s, a, X^*) - p(s, a, X(t)))n(t, s, a) + \delta_{\{s=0\}}(s, a) \int_0^\infty (p(a, u, X(t)) - p(a, u, X^*))n(t, a, u) du. \quad (2.37)$$

Let  $P_t: L^1(\mathcal{D}) \rightarrow L^1(\mathcal{D})$  be the linear semigroup associated to operator  $\mathcal{L}_{X^*}$ . As in the proof of Lemma 5,  $P_t$  is extended to the space  $(\mathcal{M}(\mathcal{D}), \|\cdot\|_{M^1})$  in order to be able to evaluate at the measure  $h$ . Since  $P_t n^* = n^*$  for all  $t \geq 0$ , we get that  $n$  satisfies

$$n - n^* = P_t(n_0 - n^*) + \int_0^t P_{t-\tau} h(\tau, s, a) d\tau, \quad (2.38)$$

so we need find an estimate for the measure  $h$ . Observe that we have the following inequalities:

$$\begin{aligned} \|h(t)\|_{L^1_{s,a}} &\leq 2\|\partial_X p\|_\infty |X(t) - X^*|, \\ |X(t) - X^*| &\leq \|N(t) - N^*\|_1, \\ \|N(t) - N^*\|_1 &\leq \frac{\partial p}{\partial X} \|\partial_X p\|_\infty |X(t) - X^*| + p_\infty \|n(t) - n^*\|_{L^1_{s,a}}, \end{aligned}$$

and since  $\|\partial_X p\|_\infty < 1$  we get

$$\begin{aligned} \|h(t)\|_{L^1_{s,a}} &\leq \frac{2p_\infty \|\partial_X p\|_\infty}{1 - \|\partial_X p\|_\infty} \|n(t) - n^*\|_{L^1_{s,a}}, \\ |X(t) - X^*| &\leq \frac{p_\infty}{1 - \|\partial_X p\|_\infty} \|n(t) - n^*\|_{L^1_{s,a}}, \\ \|N(t) - N^*\|_1 &\leq p_\infty \left( \frac{\|\partial_X p\|_\infty}{1 - \|\partial_X p\|_\infty} + 1 \right) \|n(t) - n^*\|_{L^1_{s,a}}, \end{aligned}$$

thus by taking norm in the equality (2.38) and applying Doeblin's Theorem we obtain

$$\begin{aligned} \|n(t) - n_*\|_{L^1_{s,a}} &\leq \|P_t(n_0 - n_*)\|_{L^1_{s,a}} + \int_0^t \|P_{t-\tau}h(\tau)\|_{L^1_{s,a}} d\tau \\ &\leq \frac{e^{-\lambda t}}{1-\alpha} \|n_0 - n_*\|_{L^1_{s,a}} + \frac{1}{1-\alpha} \int_0^t e^{-\lambda(t-\tau)} \|h(\tau)\|_{L^1_{s,a}} d\tau \\ &\leq \frac{e^{-\lambda t}}{1-\alpha} \|n_0 - n_*\|_{L^1_{s,a}} + C \int_0^t e^{-\lambda(t-\tau)} \|n(\tau) - n^*\|_{L^1_{s,a}} d\tau, \end{aligned}$$

with  $C := \frac{1}{1-\alpha} \frac{2p_\infty \|\partial_X p\|_\infty}{1 - \|\partial_X p\|_\infty}$ . By using Gronwall's inequality with respect to the function  $e^{\lambda t} \|n(t) - n^*\|_{L^1_{s,a}}$  we conclude

$$\|n(t) - n^*\|_{L^1_{s,a}} \leq \frac{e^{-(\lambda-C)t}}{1-\alpha} \|n_0 - n^*\|_{L^1_{s,a}},$$

so that for  $\|\partial_X p\|_\infty$  small enough we have  $C < \lambda$  and we deduce the exponential convergence of  $n(t, \cdot, \cdot)$ ,  $N(t, \cdot)$  and  $X(t)$  when  $t \rightarrow \infty$ .  $\square$

## 2.6 Numerical simulations

In order to illustrate the theoretical long time results and other possible behaviors of System (2.1), we present numerical simulations for different firing rates and initial data. The numerical illustrations below are obtained by solving the equation (2.1) with a classical first-order upwind scheme (see Scheme 4 in the Appendix of numerical methods).

We focus in displaying the discharging flux  $N(t, a)$  and the total activity  $X(t)$  since these two elements determine the general behavior of system (2.1).

### 2.6.1 Example 1: Convergence to equilibrium

For our first example, we choose as initial data  $n_0(s, a) = e^{-a}$  and the firing rate is given by

$$p = \mathbf{1}_{\{s > X\}} + \mathbf{1}_{\{s-a > X\}},$$

which corresponds to an inhibitory case since  $p$  is decreasing with respect to  $X$ . Moreover, this particular form of  $p$  is decomposed as the sum of two simple threshold functions with the first one depending only on the first elapsed time and the second one depending on the difference between the last two discharges.

In this case the solution simply converges to a steady state, as we see in Figure 2.4 for the discharging flux  $N$  and the total activity  $X$ . From Equation (2.30) we note that the discharging flux at equilibrium  $N^*$  has a jump discontinuity at  $X^*$ , which is consistent with the numerical solution. This convergence is compatible with Theorem 16.

### 2.6.2 Example 2: Jump discontinuities

We now consider the initial data  $n_0(s, a) = 2 \cdot \mathbf{1}_{\{2 > a > s+1\}}$  and the firing rate is given

$$p = \mathbf{1}_{\{s > e^{-X}\}} + \mathbf{1}_{\{s-a > e^{-X}\}},$$

which corresponds to an excitatory case since  $p$  is increasing with respect to  $X$ .



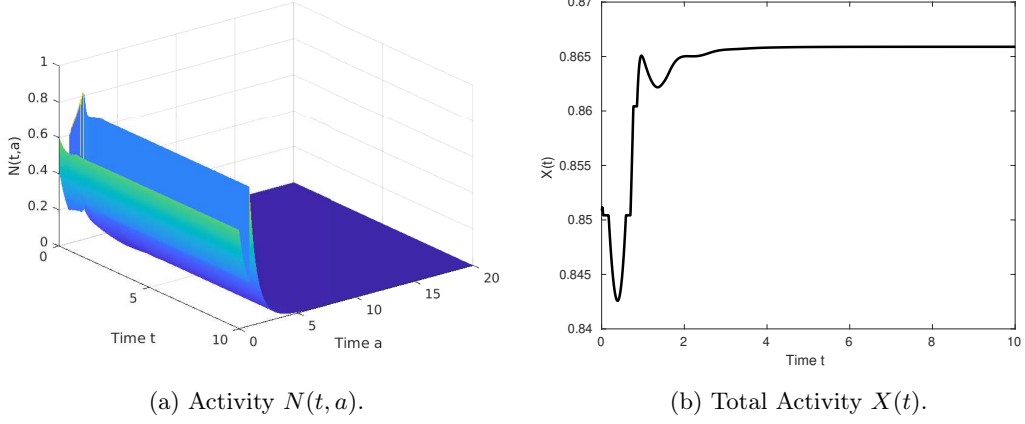


Figure 2.4 – **Example 1.** Case  $n_0(s, a) = e^{-a}$  and  $p = \mathbb{1}_{\{s > X\}} + \mathbb{1}_{\{s-a > X\}}$ .

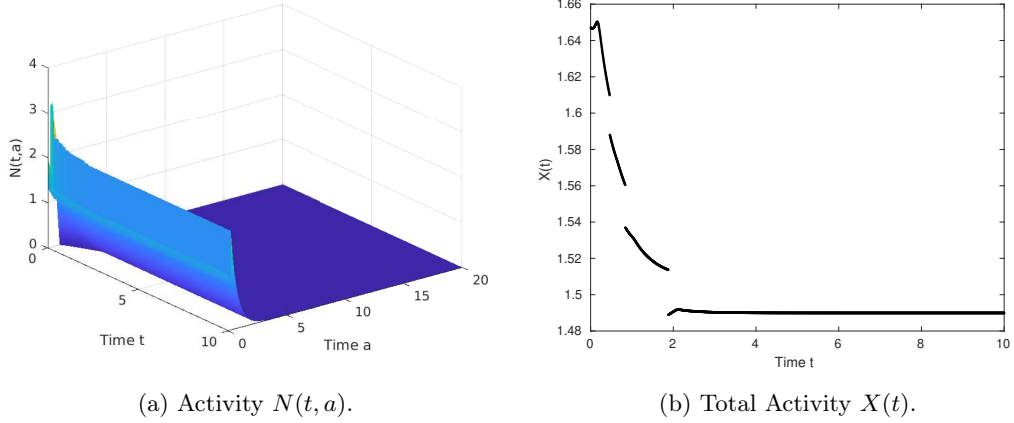


Figure 2.5 – **Example 2.** Case  $n_0(s, a) = 2 \cdot \mathbb{1}_{\{2 > a > s+1\}}$  and  $p = \mathbb{1}_{\{s > e^{-X}\}} + \mathbb{1}_{\{s-a > e^{-X}\}}$ .

Like the previous example the solution converges to the steady state, but the total activity  $X$  shows three jump discontinuities as we see in Figure 2.5. The multiple jump discontinuities are consequence of the contribution of the term depending on the difference between the two elapsed times. Furthermore, solutions convergent to the steady state that present a single jump discontinuity were already observed in Caceres et al. [59] for the classical elapsed time model. The phenomenon of multiple jumps discontinuities in Figure 2.5 is an extension for the case of Equation (2.1).

### 2.6.3 Example 3: Periodic solutions and stabilization

Next, we choose initial data  $n_0(s, a) = \frac{1}{2}e^{-(a-1)}\mathbb{1}_{\{a > \max(s, 1)\}}$  and the firing rate is given by

$$p = \varphi(X)\mathbb{1}_{\{s > 1\}}, \quad \varphi(u) = \frac{10u^2}{u^2 + 1} + 0.5,$$

which corresponds to an excitatory case since  $\varphi'(u) > 0$ . Since  $p$  does not depend on  $a$ , we take advantage by solving the classical elapsed time Equation (2.5) after integrating with respect to  $a$ , as we remarked in the introduction.

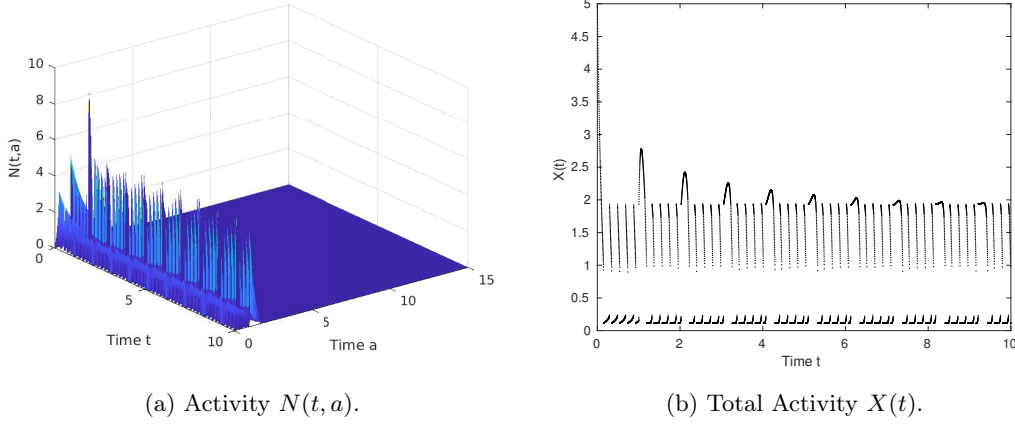


Figure 2.6 – **Example 3.1.** Case  $n_0(s, a) = \frac{1}{2}e^{-(a-1)}\mathbb{1}_{\{a > \max(s, 1)\}}$  and  $p = \varphi(X)\mathbb{1}_{\{s > 1\}}$ .

For these data, both the discharging flux  $N$  and the total activity  $X$  are asymptotic to a periodic pattern as we see in Figure 2.6. Similar examples on periodic solutions were found in Caceres et al. [59] in the classical elapsed time model for the same type of firing rates.

However, when we incorporate the effects of the difference between the elapsed times the periodic regime changes. For the same initial data and

$$p = \varphi(X)\mathbb{1}_{\{s > 1\}} + \mathbb{1}_{\{s - a > X\}},$$

we observe in Figure 2.7 that, with the term depending on the difference between the two elapsed times, the solution of System (2.1) converges to the steady state.

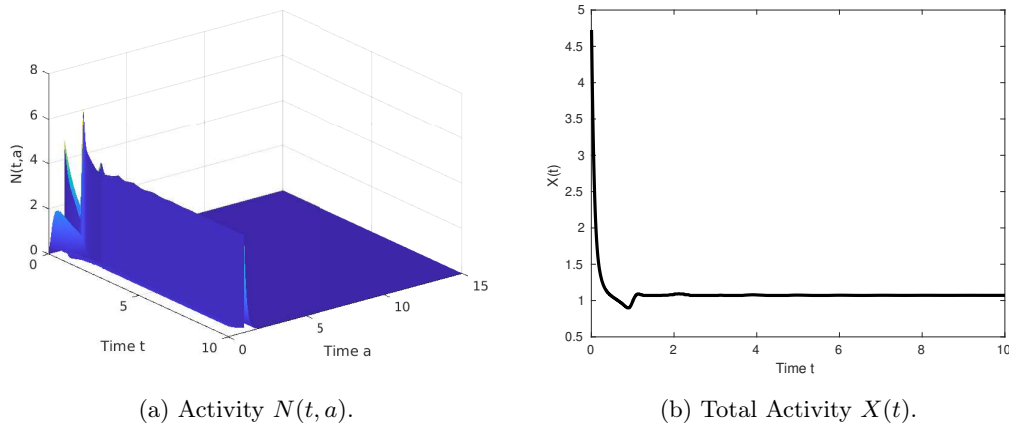


Figure 2.7 – **Example 3.2.** Case  $n_0(s, a) = \frac{1}{2}e^{-(a-1)}\mathbb{1}_{\{a > \max(s, 1)\}}$  and  $p = \varphi(X)\mathbb{1}_{\{s > 1\}} + \mathbb{1}_{\{s - a > X\}}$ .

## 2.7 Perspectives

By means of Doeblin's theory applied to a more complex problem than the classical elapsed time model, we managed to understand the dynamics of System (2.1) for weak non-linearities by adapting the ideas of Cañizo et al. [9]. However, aspects such as well-posedness and the asymptotic behavior for strong interconnections are still an open problem as in the classical elapsed time model.

Concerning the strongly inhibitory case, it remains pending to prove uniqueness of the steady state. Whilst in the classical elapsed time model this problem is reduced to a simple equation, for the model with two elapsed times we have to prove uniqueness for the integral Equation (2.33). Moreover, we conjecture in the general case that the speed of convergence to a steady state must be exponential like it is expected for the classical elapsed equation.

With respect to the existence of periodic solutions, we still have to find or construct a non-trivial example relying on dynamics for two elapsed times. The only examples we have found so far are adaptations of solutions of the classical elapsed time equation that were obtained in Caceres et al. [59] and these types of solutions present jump discontinuities, making them difficult to analyze. Furthermore, it remains as an open problem to find continuous periodic solutions as in the classical elapsed time model.

## Chapter 3

# Elapsed time model with spatial dependence and learning

This chapter corresponds to the article [58] titled "Dynamics of neural networks with elapsed time model and learning processes".

### 3.1 Introduction

Homogeneous network models like the integrate-and-fire or the elapsed time model have proved to be a useful models in connecting stochastic models at microscopic scale with population density models at mesoscopic scale. However, the incorporation of spatial dimension, using these homogeneous models for each unit has not been investigated much yet. Recent works of J. Crevat et al. in [13, 14, 15] consider the case with spatial dimension, where each neuron is described via a kinetic PDE derived from FitzHugh-Nagumo model. Else, the main models used for the incorporation of space variable via integro-differential equations are inspired from the Wilson-Cowan [62] and Amari [2] models, where several theoretical and numerical results has been obtained, see Faye et al. in [24, 25, 26]. The Amari neural field model consists in a scalar equation for the voltage  $u(t, x)$  (or another magnitude like the activity) of the form

$$\partial_t u(t, x) = -u(t, x) + \int k(x, y) f(u(t, y)) dy, \quad (3.1)$$

where  $k(x, y)$  is a connectivity kernel depending on positions across the brain that modulates local activation of neurons represented by the function  $f$ . The sign of the kernel reflects if a inhibition or excitation takes place. The Amari neural field model provides a continuous description of neural networks, which has become widely used in the engineering community. For a review on this model see [55].

In our model we now consider the evolution of interacting neural networks, where each neural network is governed by the time elapsed model and has a position  $x \in \Omega$ , where  $\Omega$  is a bounded domain of  $\mathbb{R}^d$  (with  $d$  the dimension), which models the cortex. Neurons undergo some charging process and then a sudden discharge takes place in response to certain stimulus and this causes other neighboring neurons to discharge, depending on the strength of interconnections in the network represented by a kernel. The time variations of these interconnections determine the learning process of the neural network. For simplicity we assume that for each position  $x$  we

have a homogeneous network that is considered as a single neuron.

We model the neural network through the following nonlinear renewal system

$$\begin{cases} \partial_t n + \partial_s n + p(s, S(t, x))n = 0 & t > 0, s > 0, x \in \Omega, \\ N(t, x) := n(t, s = 0, x) = \int_0^\infty p(s, S(t, x))n ds & t > 0, x \in \Omega, \\ S(t, x) = \int_\Omega w(t, x, y)N(t, y)dy + I(t, x) & t > 0, x \in \Omega, \\ \partial_t w = -w + \gamma G(N(t, x), N(t, y)) & t > 0, x, y \in \Omega, \\ n(t = 0, s, x) = n_0(s, x) \geq 0, w(t = 0, x, y) = w_0(x, y) \geq 0 & s \geq 0, x, y \in \Omega, \end{cases} \quad (3.2)$$

where  $n = n(t, s, x)$  is the probability density of finding a neuron at time  $t$ , such that the elapsed time since its last discharge is  $s \geq 0$  and its position is  $x \in \Omega$ . The equation for  $n$  and the integral boundary condition correspond to the renewal equation, with the function  $p: [0, \infty) \times \mathbb{R} \rightarrow \mathbb{R}$  representing the firing rate of neurons. This function  $p$  depends on the elapsed time  $s$  and  $S(t, x)$ , which is the amplitude of stimulation received by the network at time  $t$  and position  $x$ , and we denote  $I(t, x)$  an external input. The equation satisfied by the amplitude of stimulation is inspired in the neural field Equation (3.1). Like the classical elapsed time model, we assume that  $p$  is increasing with respect to elapsed time  $s$ . Moreover, we say that the system is inhibitory if  $p$  is decreasing with respect to  $S$  and excitatory if  $p$  is increasing.

For the firing rate  $p$ , we deal with the two following cases.

$$p_0 \leq p \leq p_\infty, \text{ for some constants } p_0, p_\infty > 0. \quad (3.3a)$$

$$p_0 \mathbb{1}_{\{s > \sigma_*\}} \leq p \leq p_\infty, \text{ for some constants } p_0, p_\infty, \sigma_* > 0. \quad (3.3b)$$

The hypothesis (3.3b) is an extension of (3.3a), since it allows  $p$  to vanish for values of  $s$  lying on some interval. We mainly deal with the case (3.3b) in Subsection 3.4.2. Like the classical elapsed time model, a special example is to consider

$$p = p_\infty \mathbb{1}_{\{s > \sigma(S)\}} \quad (3.4)$$

where  $p_\infty > 0$  is a constant and  $\sigma: [0, \infty) \rightarrow [0, \infty)$  is a bounded and Lipschitz function. This means that neurons fire if the elapsed time attains the value  $\sigma(S)$ . As in the classical elapsed time model we mostly deal with the case when  $p$  is smooth, but the results are also valid for functions as in example (3.4) as we show in the numerical simulations.

The function  $N(t, x)$  is the activity of a neuron at time  $t$  and position  $x$ . Like the classical elapsed time model, this corresponds to integrate with respect to  $s$  the term with firing rate in the first equation of (3.2) and the integral boundary condition of  $n$  at  $s = 0$  states that the elapsed time is reset to zero after a discharge.

The function  $w \in \mathcal{C}_b([0, \infty) \times \Omega \times \Omega)$  is the connectivity kernel, which depends on the location of neurons. The third equality of (3.2) establishes that the amplitude of stimulation received by the network is the result of connectivity among discharging neurons plus the external input  $I \in \mathcal{C}_b([0, \infty) \times \Omega)$ . For simplicity we assume that connectivity kernel is non-negative, so that an inhibitory or excitatory behavior depends only on the variation of the firing rate.

Furthermore this kernel evolves in time following a learning rule that depends on the smooth function  $G: \mathbb{R}^2 \rightarrow \mathbb{R}$  and the activity  $N$  at locations  $x, y$ . Without loss of generality, we assume for simplicity in computations throughout this chapter that  $G$  that satisfies

$$\|G\|_\infty + \|\nabla G\|_\infty \leq 1. \quad (3.5)$$

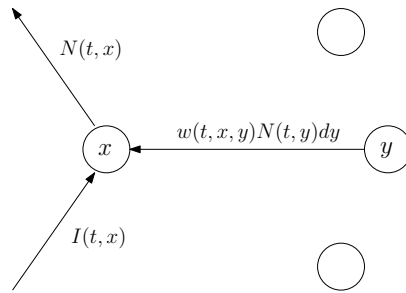


Figure 3.1 – A neuron located at position  $x$  discharges and sends  $N(t, x)$  to rest of the network. At the same time this neuron in  $x$  receives  $I(t, x)$  from the external input and  $w(t, x, y)N(t, y)dy$  from a discharging neuron located at  $y$ .

The impact of the learning is studied in the fourth equation of (3.2), where  $\gamma > 0$  is called the connectivity parameter. If  $\gamma$  and  $\|\partial_{Sp}\|_\infty$  are small, we say that System (3.2) is under a weak interconnection regime.

As an example of a learning rule we have

$$G(N(t, x), N(t, y)) = N(t, x)N(t, y),$$

inspired from the Hebbian learning. In the context of models with a finite population of neurons, the Hebbian rule which essentially consists in assuming that the strength of weights  $w_{i,j}$  between two neurons  $i$  and  $j$  increases when two neurons have high activity simultaneously. For  $M$  interacting neurons, the classical Hebbian rule relates the weights to the activity  $N_i$  of the neuron  $i$

$$\frac{d}{dt}w_{i,j} = -w_{i,j} + k_{i,j}N_iN_j, \quad 1 \leq i, j \leq M.$$

This learning principle has been introduced by Hebb in his seminal work in [37]. Mathematical formulations of this rule have been studied for example by Gerstner and Kistler in [31].

Another learning rule example is to take

$$G(N(t, x), N(t, y)) = \phi(N(t, x)N(t, y)) \exp(-(N(t, x) - N(t, y))^2),$$

with  $\phi$  a sigmoid function. This is inspired from the works of Abbassian et al. in [1] and Amari in [2] on neural fields and membrane potentials. This means that the interconnection of two neurons becomes stronger if their activities are similar and large enough.

Other learning models have been studied in neural networks. In the work of Perthame et al. in [53], they studied the learning process for the leaky integrate-and-fire model (for references about this model, see [7, 10, 50]). They indirectly generalize the Hebbian learning via distributed synaptic weights, which means that there is a total activity distributed throughout the network according to some parameter. In contrast, we present a learning model for the time elapsed dynamics that can generalize directly the Hebbian model via evolution of the connectivity kernel.

Finally,  $(n_0, w_0)$  denotes the initial configuration of the system with

$$n_0 \in \mathcal{C}_b(\Omega, L^1_s), \quad w_0 \in \mathcal{C}_b(\Omega \times \Omega). \quad (3.6)$$

Observe that for each  $x \in \Omega$  the  $L^1$ -norm of  $n(t, \cdot, x)$  is formally preserved, i.e. there exists

$g \in \mathcal{C}_b(\Omega)$  non-negative such that

$$g(x) := \int_0^\infty n_0(s, x) ds = \int_0^\infty n(t, s, x) ds \geq 0 \quad \forall t > 0, x \in \Omega, \quad \int_\Omega g(x) dx = 1. \quad (3.7)$$

The rest of the chapter is organized as follows. In Section 3.2 we prove that System (3.2) is well-posed in a suitable space when the interconnections are weak. Under the same regime of connectivity, we prove in Section 3.3 the existence of stationary states and in Section 3.4 we prove the exponential convergence to equilibrium in two different ways: via the entropy method and via Doeblin's theory. Furthermore in Section 3.5 we study a variant of System (3.2) where the time scale for learning is much slower than that of elapsed time dynamics. Finally in Section 3.6 we present some examples of numerical simulations for different external inputs, connectivity parameters and learning rules.

## 3.2 Well-posedness for the weak interconnection case

We prove that System (3.2) is well-posed under the weak interconnection regime. In order to do so, we start by studying an auxiliary linear problem where the amplitude of stimulation is fixed and then we proceed to prove well-posedness of System (3.2) via a contraction argument.

### 3.2.1 The linear problem

Given  $S \in \mathcal{C}_b([0, \infty) \times \Omega)$ , we consider the following linear problem

$$\begin{cases} \partial_t n + \partial_s n + p(s, S(t, x))n = 0 & t > 0, s > 0, x \in \Omega, \\ N(t, x) := n(t, s = 0, x) = \int_0^\infty p(s, S(t, x))n ds & t > 0, x \in \Omega, \\ n(t = 0, s, x) = n_0(s, x) \geq 0 & s \geq 0, x \in \Omega. \end{cases} \quad (3.8)$$

We look for weak solutions satisfying  $n \in \mathcal{C}_b([0, \infty) \times \Omega, L_s^1)$ , so that  $N \in \mathcal{C}_b([0, \infty) \times \Omega)$ . Furthermore, in this linear system the variable  $x$  is simply a parameter, since there is no derivative or integral term involving the position.

**Lemma 9.** *Assume that  $n_0 \in \mathcal{C}_b(\Omega, L_s^1)$  and  $p \in W^{1, \infty}((0, \infty) \times \mathbb{R})$  satisfies (3.3b). Then for a given  $S \in \mathcal{C}_b([0, \infty) \times \Omega)$ , Equation (3.8) has a unique weak solution  $n \in \mathcal{C}_b([0, \infty) \times \Omega, L_s^1)$  with  $N \in \mathcal{C}_b([0, \infty) \times \Omega)$ . Moreover  $n$  is non-negative and verifies the property (3.7).*

In particular this lemma proves the property (3.7) for the non linear System (3.2). Moreover, this lemma is also valid for  $p$  defined in (3.4) with a similar proof.

**Proof.** From the method of characteristics, we start by noticing that a solution of the linear System (3.8) satisfies the following fixed point equation

$$\begin{aligned} n(t, s, x) = \Psi[n](t, s, x) := & n_0(s - t, x) \exp\left(-\int_0^t p(s + t' - t, S(t', x)) dt'\right) \mathbf{1}_{\{s > t\}} \\ & + N(t - s, x) \exp\left(-\int_0^s p(s', S(t - s + s', x)) ds'\right) \mathbf{1}_{\{0 < s < t\}}, \end{aligned} \quad (3.9)$$

with  $N(t, x) = \int_0^\infty p(u, S(t, x))n(t, u, x) du$ , which depends on  $n$ .

Let  $T > 0$  and  $\mathcal{X}_T := \{n \in \mathcal{C}_b([0, T] \times \Omega, L_s^1) : n(0) = n_0\}$ , it readily follows that  $\Psi$  maps  $\mathcal{X}_T \rightarrow \mathcal{X}_T$ . We prove by the contraction principle that  $\Psi$  has a unique fixed point in  $\mathcal{X}_T$  for

$T > 0$  small enough, i.e. there exists a unique weak solution of (3.8) defined on  $[0, T]$ . Consider  $n_1, n_2 \in \mathcal{X}_T$  so we have

$$\begin{aligned} \int_0^\infty |\Psi[n_1] - \Psi[n_2]|(t, s, x) ds &\leq \int_0^t |N_1 - N_2|(t - s, x) ds \\ &\leq T \sup_{(t, x) \in [0, T] \times \Omega} |N_1 - N_2|(t, x) \\ &\leq T p_\infty \sup_{(t, x) \in [0, T] \times \Omega} \|n_1(t, x) - n_2(t, x)\|_{L_s^1}, \end{aligned} \quad (3.10)$$

thus for  $T < \frac{1}{p_\infty}$ , we have proved that  $\Psi$  is a contraction and there exists a unique  $n \in X_T$  such that  $\Psi[n] = n$ . Since the choice of  $T$  is independent of  $n_0$ , we can reiterate this argument to get a unique solution of (3.8), which is defined for all  $t \geq 0$ .

Next we prove the mass conservation property. Like Equality (2.11) in the elapsed time model with two times, we can redo the argument of Lemma 5 to prove following equality

$$\begin{aligned} n(t, s, x) &= n_0(s - t, x) \mathbf{1}_{\{s > t\}} + N(t - s, x) \mathbf{1}_{\{0 < s < t\}} \\ &\quad - \int_0^t p(s - t + \tau, S(\tau, x)) n(\tau, s - t + \tau, x) \mathbf{1}_{\{s > t - \tau\}} d\tau, \end{aligned} \quad (3.11)$$

and we get the property by integrating with respect to  $s$  on  $(0, \infty)$ .

Finally, since  $n_0$  is non-negative then  $\Psi$  preserves positivity, so by uniqueness of fixed point the corresponding solution  $n$  must be non-negative.  $\square$

### 3.2.2 The nonlinear problem

We are now ready to prove that System (3.2) is well-posed in the case of weak interconnection.

**Theorem 17** (Well-posedness for weak interconnections). *Assume (3.6)-(3.7) and that  $p \in W^{1, \infty}((0, \infty) \times \mathbb{R})$  satisfies (3.3b). Then for*

$$\|g\|_\infty |\Omega| \|\partial_S p\|_\infty \max\{\|w_0\|_\infty, \gamma\} < 1,$$

*System (3.2) has a unique solution with  $n \in \mathcal{C}_b([0, \infty) \times \Omega, L_s^1)$ ,  $N \in \mathcal{C}_b([0, \infty) \times \Omega)$ ,  $S \in \mathcal{C}_b([0, \infty) \times \Omega)$  and  $w \in \mathcal{C}_b([0, \infty) \times \Omega \times \Omega)$ . Moreover,  $n$  is non-negative for all  $t > 0$ .*

**Proof.** Consider  $T > 0$ . We fix a function  $S \in \mathcal{C}_b([0, \infty) \times \Omega)$  and define the functions  $n \in \mathcal{C}_b([0, \infty) \times \Omega, L_s^1)$ ,  $N \in \mathcal{C}_b([0, \infty) \times \Omega)$  which are solutions of (3.8) by Lemma 9. Furthermore, the solution of this linear system preserves positivity and the condition (3.7).

The solution  $w \in \mathcal{C}_b([0, \infty) \times \Omega \times \Omega)$  is obtained through the formula

$$w(t, x, y) = e^{-t} w_0(x, y) + \gamma \int_0^t e^{-(t-\tau)} G(N(\tau, x), N(\tau, y)) d\tau. \quad (3.12)$$

So we have a solution of System (3.2) defined on  $[0, T]$  if  $S$  satisfies for all  $0 \leq t \leq T$  and  $x \in \Omega$ , the following fixed point condition

$$S(t, x) = \mathcal{T}[S](t, x) := \int w(t, x, y) \left( \int_0^\infty p(s, S(t, y)) n(t, s, y) ds \right) dy + I(t, x). \quad (3.13)$$

We prove that  $\mathcal{T}$  defines for all  $T > 0$  an operator that maps  $\mathcal{X}_T \rightarrow \mathcal{X}_T$  with  $\mathcal{X}_T := \mathcal{C}_b([0, T] \times \Omega)$ .



First, we observe the following estimate for the activity

$$|N(t, x)| \leq p_\infty \|g\|_\infty, \quad \forall (t, x) \in [0, T] \times \Omega. \quad (3.14)$$

And from the equation of  $w$ , we get the following uniform estimate

$$|w(t, x, y)| \leq \max\{\|w_0\|_\infty, \gamma\}, \quad \forall (t, x, y) \in [0, T] \times \Omega \times \Omega. \quad (3.15)$$

Let  $A := \max\{\|w_0\|_\infty, \gamma\}$ . This implies that for any  $S \in \mathcal{X}_T$  we have

$$\|\mathcal{T}[S]\|_\infty \leq Ap_\infty + \|I\|_\infty,$$

and it is immediate that  $\mathcal{T}[S]$  is a continuous function, thus  $\mathcal{T}[S] \in \mathcal{X}_T$ .

We now prove that for  $T$  small enough,  $\mathcal{T}$  is a contraction. Consider  $S_1, S_2 \in \mathcal{X}_T$  and observe that the difference between  $w_1$  and  $w_2$  satisfies, by using (3.12),

$$|w_1(t, x, y) - w_2(t, x, y)| \leq 2\gamma T \|N_1 - N_2\|_\infty. \quad (3.16)$$

Next, for the difference between  $N_1$  and  $N_2$  we have

$$\begin{aligned} |N_1 - N_2|(t, x) &\leq \int_0^\infty |p(s, S_1) n_1 - p(s, S_2) n_2| ds \\ &\leq \int_0^\infty |p(s, S_1) - p(s, S_2)| n_1 ds + \int_0^\infty p(s, S_2) |n_1 - n_2| ds \\ &\leq \|g\|_\infty \|\partial_S p\|_\infty \|S_1 - S_2\|_\infty + p_\infty \|n_1 - n_2\|_{L_{t,x}^\infty L_s^1}. \end{aligned} \quad (3.17)$$

Now we have to estimate the difference between  $n_1$  and  $n_2$ . From (3.11) and estimate (3.17), we get

$$\|n_1 - n_2\|_{L_{t,x}^\infty L_s^1} \leq 2T \|g\|_\infty \|\partial_S p\|_\infty \|S_1 - S_2\|_\infty + 2Tp_\infty \|n_1 - n_2\|_{L_{t,x}^\infty L_s^1}.$$

Then, for  $T < \frac{1}{2p_\infty}$  we obtain

$$\|n_1 - n_2\|_{L_{t,x}^\infty L_s^1} \leq \frac{2T \|g\|_\infty \|\partial_S p\|_\infty}{1 - 2Tp_\infty} \|S_1 - S_2\|_\infty. \quad (3.18)$$

Finally by combining the estimates (3.14)-(3.16), the operator  $\mathcal{T}$  satisfies

$$\begin{aligned} |\mathcal{T}[S_1] - \mathcal{T}[S_2]| &\leq \int |w_1 - w_2| N_1 dy + \int |w_2| |N_1 - N_2| dy \\ &\leq (2\gamma Tp_\infty + |\Omega|A) \|N_1 - N_2\|_\infty \\ &\leq C \|S_1 - S_2\|_\infty, \end{aligned} \quad (3.19)$$

with  $C > 0$  given by

$$C := \|g\|_\infty \|\partial_S p\|_\infty (2\gamma Tp_\infty + |\Omega|A) \left(1 + \frac{2Tp_\infty}{1 - 2Tp_\infty}\right).$$

Hence for  $\|g\|_\infty |\Omega| \|\partial_S p\|_\infty A < 1$  and  $T$  small enough we get  $C < 1$ , so  $\mathcal{T}$  is a contraction.

From Picard's fixed point we get a unique  $S \in \mathcal{X}_T$  such that  $\mathcal{T}[S] = S$ , and this implies the existence of a unique solution of (3.2) defined on  $[0, T]$ . Since estimates (3.14) and (3.15) are uniform in  $T$ , we can iterate this argument to get a unique solution of (3.2) defined for all  $t > 0$ .

Furthermore, we conclude from this construction that the nonlinear System (3.2) preserves positivity and satisfies (3.7) like the linear System (3.8).  $\square$

**Remark 7.** From estimate (3.14), we observe that we only need the function  $G$  to be bounded on the set  $[0, p_\infty \|g\|_\infty]^2$ . This justifies that we do not lose generality in assuming  $G$  normalized according to (3.5).

The condition on  $p$  can be relaxed to wider class of functions, as we see in the following theorem.

**Theorem 18.** Consider  $p$  defined in (3.4). Assume in addition that  $n_0 \in L_{s,x}^\infty$  and  $w_0 \in C_b(\Omega \times \Omega)$ , then the same result holds if

$$p_\infty \|\sigma'\|_\infty |\Omega| \max\{\|w_0\|_\infty, \gamma\} (\|n_0\|_\infty + p_\infty \|g\|_\infty) < 1.$$

**Proof.** The proof is the same as for the previous theorem. Let  $\mathcal{T}$  be the operator defined before, we have to verify the contraction principle. The estimates (3.14)-(3.16) for  $N$  and  $w$  remain unchanged. Now, from the solution of linear problem (3.8) we get for  $n$  the uniform estimate

$$|n(t, s, x)| \leq \|n_0\|_\infty + p_\infty \|g\|_\infty, \forall (t, s, x) \in [0, T] \times (0, \infty) \times \Omega.$$

Let  $A := \max\{\|w_0\|_\infty, \gamma\}$  and  $B := \|n_0\|_\infty + p_\infty \|g\|_\infty$ . In this case the difference between  $N_1$  and  $N_2$  in (3.17) is replaced by

$$\begin{aligned} |N_1 - N_2|(t, x) &\leq \int_0^\infty |p(s, S_1) - p(s, S_2)| n_1 ds + \int_0^\infty p(s, S_2) |n_1 - n_2| ds \\ &\leq p_\infty \left| \int_{\sigma(S_1)}^{\sigma(S_2)} n_1 ds \right| + \int_0^\infty p(s, S_2) |n_1 - n_2| ds \\ &\leq p_\infty \|\sigma'\|_\infty B \|S_1 - S_2\|_\infty + p_\infty \|n_1 - n_2\|_{L_{t,x}^\infty L_s^1}. \end{aligned}$$

And from (3.11), the difference between  $n_1$  and  $n_2$  satisfies

$$\|n_1 - n_2\|_{L_{t,x}^\infty L_s^1} \leq 2Tp_\infty \|\sigma'\|_\infty B \|S_1 - S_2\|_\infty + 2Tp_\infty \|n_1 - n_2\|_{L_{t,x}^\infty L_s^1}.$$

Then, for  $T < \frac{1}{2p_\infty}$  we conclude similarly

$$\|n_1 - n_2\|_{L_{t,x}^\infty L_s^1} \leq \frac{2Tp_\infty \|\sigma'\|_\infty B}{1 - 2Tp_\infty} \|S_1 - S_2\|_\infty.$$

Hence, by combining the estimates for  $N_1 - N_2$  and  $n_1 - n_2$ , the operator  $\mathcal{T}$  verifies

$$\begin{aligned} |\mathcal{T}[S_1] - \mathcal{T}[S_2]| &\leq \int |w_1 - w_2| N_1 dy + \int |w_2| |N_1 - N_2| dy \\ &\leq (2\gamma Tp_\infty + |\Omega|A) \|N_1 - N_2\|_\infty \\ &\leq C \|S_1 - S_2\|_\infty, \end{aligned}$$

with  $C > 0$  given by

$$C := p_\infty \|\sigma'\|_\infty B (2\gamma Tp_\infty + |\Omega|A) \left(1 + \frac{2Tp_\infty}{1 - 2Tp_\infty}\right).$$

Thus for  $p_\infty \|\sigma'\|_\infty |\Omega|AB < 1$  and  $T$  small enough we get that  $\mathcal{T}$  is a contraction and this implies

the existence of a unique solution defined on  $[0, T]$ . Finally, we can iterate this argument to get a unique globally defined solution, as we asserted in the previous theorem.  $\square$

### 3.3 Steady states

Assume the input  $I$  depends only on position. We now study the stationary solutions of (3.2), i.e. the system given by

$$\begin{cases} \partial_s n(s, x) + p(s, S(x))n(s, x) = 0 & s > 0, x \in \Omega, \\ N(x) := n(s = 0, x) = \int_0^\infty p(s, S(x))n(s, x) ds & x \in \Omega, \\ S(x) = \int_\Omega w(x, y)N(y)dy + I(x) & x \in \Omega, \\ w(x, y) = \gamma G(N(x), N(y)) & x, y \in \Omega, \end{cases} \quad (3.20)$$

where  $n \in L^1_{s,x}$ ,  $N, S \in C_b(\Omega)$  and  $w \in C_b(\Omega \times \Omega)$ .

If the amplitude  $S$  is given, we can determine  $n, N$  and  $w$  through the formulas

$$\begin{aligned} n(s, x) &= N(x)e^{-\int_0^s p(\tau, S(x)) d\tau}, \\ N(x) &= g(x) \left( \int_0^\infty e^{-\int_0^u p(\tau, S(x)) d\tau} du \right)^{-1}, \\ w(x, y) &= \gamma G(g(x)F(S(x)), g(y)F(S(y))). \end{aligned} \quad (3.21)$$

We define  $F: \mathbb{R} \rightarrow \mathbb{R}_+$  given by

$$F(S) := \left( \int_0^\infty e^{-\int_0^s p(\tau, S) d\tau} ds \right)^{-1}, \quad (3.22)$$

and we get that  $(n, N, S, w)$  in (3.21) corresponds to a stationary solution of (3.2) if  $S$  satisfies the following fixed point condition

$$S(x) = \mathcal{T}[S](x) := \gamma \int G(g(x)F(S(x)), g(y)F(S(y))) g(y)F(S(y)) dy + I(x). \quad (3.23)$$

The following result asserts that there exists a unique steady state for a given  $g \in C_b(\Omega)$ , under weak interconnection regime.

**Theorem 19.** *Assume that  $p \in W^{1,\infty}((0, \infty) \times \mathbb{R})$  satisfies (3.3b) and  $g \in C_b(\Omega)$ . For  $\gamma$  small enough, System (3.2) has a unique stationary state  $(n^*, N^*, S^*, w^*)$ , with  $n^* \in C_b(\Omega, L^1_s)$  satisfying  $\int_0^\infty n^*(s, x) ds = g(x)$  and  $N^* \in C_b(\Omega)$ ,  $w^* \in C_b(\Omega \times \Omega)$ , which are determined by a unique amplitude of stimulation  $S^* \in C_b(\Omega)$  satisfying  $\mathcal{T}[S^*] = S^*$ .*

To prove the result we use the following lemma about the function  $F$ .

**Lemma 10.** *Under the hypothesis of Theorem 19,  $F$  is a bounded and Lipschitz function.*

**Proof.** It readily follows that  $F$  is bounded since it satisfies the following estimate

$$0 < F(S) \leq \left( \int_0^\infty e^{-p_\infty s} ds \right)^{-1} = p_\infty.$$

On the other hand,  $F'$  is given by the formula

$$F'(S) = F(S)^2 \left[ \int_0^\infty e^{-\int_0^s p(\tau, S) d\tau} \left( \int_0^s \partial_S p(\tau, S) d\tau \right) ds \right],$$

so we have the following estimate

$$\begin{aligned} |F'(S)| &\leq p_\infty^2 \|\partial_S p\|_\infty \left[ \int_0^\infty e^{-\int_0^s p(\tau, S) d\tau} s ds \right] \\ &\leq p_\infty^2 \|\partial_S p\|_\infty \left[ \int_0^\infty e^{-p_0(s-\sigma_*)+s} ds \right] \\ &\leq p_\infty^2 \|\partial_S p\|_\infty \left[ \frac{\sigma_*^2}{2} + \frac{\sigma_*}{p_0} + \frac{1}{p_0^2} \right]. \end{aligned}$$

Hence  $F$  is Lipschitz. □

**Remark 8.** *In the case of  $p$  defined in (3.4) we get*

$$F(S) = \frac{1}{p_\infty^{-1} + \sigma(S)},$$

so  $F$  bounded and Lipschitz since  $\sigma$  is. Hence the theorem is also valid for this case.

Next, we conclude the proof of our main theorem.

**Proof.** It is straightforward that in (3.23)  $\mathcal{T}$  defines an operator that maps  $\mathcal{C}_b(\Omega) \rightarrow \mathcal{C}_b(\Omega)$ . Since  $F$  is bounded and Lipschitz we get for  $S_1, S_2 \in \mathcal{C}_b(\Omega)$

$$\begin{aligned} |\mathcal{T}[S_1] - \mathcal{T}[S_2]|(x) &\leq 2\gamma \|g\|_\infty \|F\|_\infty \|F'\|_\infty \|S_1 - S_2\|_\infty \\ &\quad + \gamma \|F'\|_\infty \|S_1 - S_2\|_\infty, \end{aligned}$$

Thus for  $\gamma$  satisfying

$$\gamma \|F'\|_\infty (2\|g\|_\infty \|F\|_\infty + 1) < 1,$$

the operator  $\mathcal{T}$  is a contraction and there exists a unique  $S^* \in \mathcal{C}_b(\Omega)$  such that  $\mathcal{T}[S^*] = S^*$ . Therefore we get a unique stationary state determined through the formulas in (3.21). □

## 3.4 Convergence to equilibrium

Our next result about System (3.2) is the convergence to equilibrium when  $t \rightarrow \infty$ , under the weak interconnection regime i.e. with  $\gamma$  and  $\|\partial_S p\|_\infty$  small enough. For the proof of this result we present two different approaches: the relative entropy method and the Doeblin theory applied to Markov semigroups.

### 3.4.1 Entropy method approach

Firstly we prove the convergence result when the firing rate  $p$  is strictly positive by means of the relative entropy method studied in [43, 51] and following the ideas in [38].

**Theorem 20** (Long term behavior for the weak interconnection regime). *Assume (3.6)-(3.7) and that  $p \in W^{1,\infty}((0, \infty) \times \mathbb{R})$  satisfies (3.3a). For  $\gamma$  and  $\|\partial_S p\|_\infty$  small enough let  $(n^*, N^*, S^*, w^*)$*

be the corresponding stationary state of (3.2). Then there exist  $C, \lambda > 0$  such that the solution of (3.2) satisfies

$$\|n(t) - n^*\|_{L^1_{s,x}} + \|w(t) - w^*\|_{L^1_{x,y}} \leq C e^{-\lambda t} \left( \|n_0 - n^*\|_{L^1_{s,x}} + \|w_0 - w^*\|_{L^1_{x,y}} \right), \quad \forall t \geq 0. \quad (3.24)$$

Moreover  $\|S(t) - S^*\|_{L^1_x}$  and  $\|N(t) - N^*\|_{L^1_x}$  converge exponentially to 0 when  $t \rightarrow \infty$ .

In other words, if interconnections are weak then solutions converge exponentially to equilibrium.

**Proof.** Observe that  $n - n^*$  and  $w - w^*$  satisfy

$$\begin{aligned} \partial_t(n - n^*) + \partial_s(n - n^*) + p(s, S)(n - n^*) &= -(p(s, S) - p(s, S^*))n^*, \\ \partial_t(w - w^*) &= -(w - w^*) + \gamma G(N(t, x), N(t, y)) - \gamma G(N^*(x), N^*(y)), \end{aligned}$$

so we have the following inequalities

$$\begin{aligned} \partial_t|n - n^*| + \partial_s|n - n^*| + p(s, S)|n - n^*| &\leq \|\partial_S p\|_\infty |S - S^*| n^*, \\ \partial_t|w - w^*| &\leq -|w - w^*| + \gamma(|N(t, x) - N^*(x)| + |N(t, y) - N^*(y)|), \end{aligned}$$

By integrating with respect to the corresponding variables we get

$$\begin{aligned} \frac{\partial}{\partial t} \iint_0^\infty |n - n^*| ds dx + \iint_0^\infty p(s, S)|n - n^*| ds dx &\leq \int |N - N^*| dx + \|g\|_\infty \|\partial_S p\|_\infty \int |S - S^*| dx, \\ \frac{\partial}{\partial t} \iint |w - w^*| dx dy &\leq - \iint |w - w^*| dx dy + 2\gamma|\Omega| \int |N - N^*| dx. \end{aligned} \quad (3.25)$$

Thus we have to estimate the terms in the right-hand side of both inequalities. For the difference between  $N$  and  $N^*$  we get

$$\int |N - N^*| dx \leq \|g\|_\infty \|\partial_S p\|_\infty \int |S - S^*| dx + \int \left| \int_0^\infty p(s, S)(n - n^*) ds \right| dx. \quad (3.26)$$

Next, for the difference between  $S$  and  $S^*$  we obtain

$$\begin{aligned} \int |S - S^*| dx &\leq \iint w^* |N(t, y) - N^*(y)| dx dy + \iint N(t, y) |w - w^*| dx dy \\ &\leq \gamma|\Omega| \int |N - N^*| dx + p_\infty \|g\|_\infty \iint |w - w^*| dx dy, \end{aligned}$$

Hence from (3.26), the following inequality holds

$$\begin{aligned} \int |S - S^*| dx &\leq \gamma|\Omega| \|g\|_\infty \|\partial_S p\|_\infty \int |S - S^*| dx + \gamma|\Omega| p_\infty \iint_0^\infty |n - n^*| ds dx \\ &\quad + p_\infty \|g\|_\infty \iint |w - w^*| dx dy, \end{aligned}$$

and if  $\beta := \gamma|\Omega| \|g\|_\infty \|\partial_S p\|_\infty < 1$ , we deduce the following estimate

$$\int |S - S^*| dx \leq \frac{p_\infty}{1 - \beta} \left( \gamma|\Omega| \iint_0^\infty |n - n^*| ds dx + \|g\|_\infty \iint |w - w^*| dx dy \right). \quad (3.27)$$

Thus from (3.25) we get

$$\begin{aligned} \frac{\partial}{\partial t} \iint_0^\infty |n - n^*| ds dx &\leq - \iint_0^\infty p(s, S) |n - n^*| ds dx + \int \left| \int_0^\infty p(s, S) (n - n^*) ds \right| dx \\ &\quad + \frac{2p_\infty \|g\|_\infty \|\partial_S p\|_\infty}{1 - \beta} \left( \gamma |\Omega| \iint_0^\infty |n - n^*| ds dx + \|g\|_\infty \iint |w - w^*| dx dy \right) \end{aligned} \quad (3.28)$$

Since  $\int_0^\infty (n - n^*) ds = 0$  and  $p \geq p_0$  we may use the argument from [45, 51] to get

$$\begin{aligned} \int \left| \int_0^\infty p(s, S) (n - n^*) ds \right| dx &= \int \left| \int_0^\infty (p(s, S) - p_0) (n - n^*) ds \right| dx \\ &\leq \iint_0^\infty (p(s, S) - p_0) |n - n^*| ds dx. \end{aligned}$$

Therefore we deduce the following inequality for  $n - n^*$

$$\begin{aligned} \frac{\partial}{\partial t} \iint_0^\infty |n - n^*| ds dx &\leq - \left( p_0 - \frac{2\beta p_\infty}{1 - \beta} \right) \iint_0^\infty |n - n^*| ds dx \\ &\quad + \frac{2p_\infty \|g\|_\infty^2 \|\partial_S p\|_\infty}{1 - \beta} \iint |w - w^*| dx dy. \end{aligned} \quad (3.29)$$

On the other hand from the second inequality in (3.25) and estimate (3.26) we get for  $w - w^*$

$$\begin{aligned} \frac{\partial}{\partial t} \iint |w - w^*| dx dy &\leq - \iint |w - w^*| dx dy + \frac{2\beta p_\infty \|g\|_\infty}{1 - \beta} \iint |w - w^*| dx dy \\ &\quad + 2\gamma |\Omega| p_\infty \left( \frac{\beta}{1 - \beta} + 1 \right) \iint_0^\infty |n - n^*| ds dx. \end{aligned} \quad (3.30)$$

If we add these two inequalities we get an expression of the form

$$\begin{aligned} \frac{\partial}{\partial t} \left( \iint_0^\infty |n - n^*| ds dx + \iint |w - w^*| dx dy \right) &\leq -(p_0 - C_1) \iint_0^\infty |n - n^*| ds dx \\ &\quad - (1 - C_2) \iint |w - w^*| dx dy, \end{aligned} \quad (3.31)$$

with  $C_1, C_2 > 0$  given by

$$\begin{aligned} C_1 &= 2\gamma |\Omega| p_\infty \left( \frac{\|g\|_\infty \|\partial_S p\|_\infty}{1 - \beta} + \frac{\beta}{1 - \beta} + 1 \right), \\ C_2 &= \frac{2\beta p_\infty \|g\|_\infty \|\partial_S p\|_\infty}{1 - \beta} (1 + \gamma |\Omega|). \end{aligned}$$

If  $\gamma$  and  $\|\partial_S p\|_\infty$  are such that  $C_1 < p_0$  and  $C_2 < 1$ , we conclude, by solving the corresponding differential inequality, the existence of  $C, \lambda > 0$  satisfying the estimate (3.24). Furthermore the convergence of  $N, S$  and  $w$  readily follows from estimates (3.26) and (3.27).  $\square$

### 3.4.2 Doeblin theory approach

The previous convergence result for System (3.2) can be extended when the firing rate  $p$  satisfies the hypothesis (3.3b) for a  $\sigma_* > 0$  small enough (see [48] for an example). We assert that this result is also valid when  $p$  satisfies the condition (3.3b) with any  $\sigma_* > 0$ . In order to improve the convergence, we follow the ideas of Cañizo et al. in [9] to study the asymptotic behavior of the linear System (3.32) by means of Doeblin's theory.

#### The linear case

Given  $S \in \mathcal{C}_b(\Omega)$ , we consider the linear problem given by

$$\begin{cases} \partial_t n + \partial_s n + p(s, S(x))n = 0 & t > 0, s > 0, x \in \Omega, \\ N(t, x) := n(t, s = 0, x) = \int_0^\infty p(s, S(x))n ds & t > 0, x \in \Omega, \\ n(t = 0, s, x) = n_0(s, x) & s \geq 0, x \in \Omega. \end{cases} \quad (3.32)$$

From Lemma 9 we know that this system has a unique solution  $n \in \mathcal{C}_b([0, \infty) \times \Omega, L_s^1)$ . Since the variable  $x$  is just a parameter, for a fixed  $x \in \Omega$  we define from Equation (3.32) the Markov semigroup  $P_t: L_s^1 \rightarrow L_s^1$  given by

$$P_t n_0(s, x) = n(t, s, x).$$

A key property on the solutions of this system is the exponential convergence to equilibrium as we state in the following theorem:

**Theorem 21.** *Consider  $n_0 \in \mathcal{C}_b(\Omega, L_s^1)$  with its corresponding  $g \in \mathcal{C}_b(\Omega)$  and that  $p$  satisfies (3.3b), then there exists a unique stationary solution  $n^*$  of Equation (3.32) satisfying  $\int_0^\infty n^*(s, x) ds = g(x)$ . Moreover, the corresponding solution of (3.32) satisfies*

$$\|n(t, \cdot, x) - n^*(\cdot, x)\|_{L_s^1} \leq \frac{1}{1 - \alpha} e^{-\lambda t} \|n_0(\cdot, x) - n^*(\cdot, x)\|_{L_s^1} \quad \forall t \geq 0, x \in \Omega.$$

with  $\alpha = p_0 \sigma_* e^{-2p_\infty \sigma_*}$  and  $\lambda = -\frac{\log(1 - \alpha)}{2\sigma_*} > 0$ .

For the sake of completeness, we include the proof of this result done by Cañizo et al. in the theorem 3.12 of [9]. In our case, functions have mass  $g(x)$  instead of having mass 1 with respect to  $L_s^1$ .

**Proof.** Let  $n$  be the solution of (3.32). For fixed  $x \in \Omega$ , we claim  $n$  satisfies the following inequality

$$n(2\sigma_*, s, x) = P_{2\sigma_*} n_0(s, x) \geq p_0 e^{-2p_\infty \sigma_*} \mathbb{1}_{[0, \sigma_*]}(s) g(x) \quad \forall (s, x) \in (0, \infty) \times \Omega. \quad (3.33)$$

This means that the semigroup  $P_t$  associated to Equation (3.32) satisfies Doeblin's condition with  $t_0 = 2\sigma_*$ ,  $\alpha = p_0 \sigma_* e^{-2p_\infty \sigma_*}$  and  $\nu = \frac{1}{\sigma_*} \mathbb{1}_{[0, \sigma_*]}(s)$  for functions  $n_0(\cdot, x) \in L_s^1$  with  $g(x) = 1$ .

Let  $x \in \Omega$  be fixed and consider  $\tilde{P}_t: L_s^1 \rightarrow L_s^1$  the semigroup associated with the problem

$$\begin{cases} \partial_t \tilde{n} + \partial_s \tilde{n} + p(s, S(x))\tilde{n} = 0 & t > 0, s > 0, \\ \tilde{n}(t, s = 0, x) = 0 & t > 0, \\ \tilde{n}(t = 0, s, x) = n_0(s, x) & s \geq 0. \end{cases}$$

In this case the solution is given by

$$\tilde{P}_t n_0(s, x) = n_0(s - t, x) \exp\left(-\int_0^t p(s - t + \tau, S(x)) d\tau\right) \mathbb{1}_{\{s > t\}}. \quad (3.34)$$

Then the solution of (3.32) satisfies

$$n(t, s, x) = \tilde{P}_t n_0(s, x) + \int_0^t \tilde{P}_{t-\tau}(N(\tau, x)\delta_0(s)) d\tau.$$

Moreover we have the following inequalities

$$\begin{aligned} n(t, s, x) &\geq \tilde{P}_t n_0(s, x) = n_0(s - t, x) \exp\left(-\int_0^t p(s - t + \tau, S(x)) d\tau\right) \geq n_0(s - t, x) e^{-p_\infty t} \mathbb{1}_{\{s > t\}}. \\ \tilde{P}_{t-\tau} n_0(s, x) &\geq n_0(s - t + \tau, x) e^{-p_\infty(t-\tau)} \mathbb{1}_{\{s > t-\tau\}}. \end{aligned}$$

Then for  $t > \sigma_*$  we get

$$\begin{aligned} N(t, x) &= \int_0^\infty p(s, S(x)) n(t, s, x) ds \\ &\geq p_0 \int_{\sigma_*}^\infty n(t, s, x) ds \\ &\geq p_0 \int_t^\infty n(t, s, x) ds \\ &\geq p_0 e^{-p_\infty t} \int_t^\infty n_0(s - t, x) ds \\ &\geq p_0 e^{-p_\infty t} g(x). \end{aligned}$$

In that case for any  $s > 0$  and  $t > s + \sigma_*$  we have that

$$\begin{aligned} n(t, s, x) &\geq \int_0^t \tilde{P}_{t-\tau}(N(\tau, x)\delta_0(s)) d\tau \\ &\geq \int_{\sigma_*}^t \tilde{P}_{t-\tau}(p_0 e^{-p_\infty \tau} g(x)\delta_0(s)) d\tau \\ &\geq p_0 \int_{\sigma_*}^t \delta_0(s - t + \tau) e^{-p_\infty \tau} e^{-p_\infty(t-\tau)} g(x) \mathbb{1}_{\{s-t+\tau > 0\}} d\tau \\ &\geq p_0 e^{-p_\infty t} \mathbb{1}_{\{0 < s < t - \sigma_*\}} g(x). \end{aligned}$$

Therefore we get the estimate (3.33) by choosing  $t = 2\sigma_*$ . Finally, the exponential convergence to equilibrium readily follows from Doeblin's theorem with  $\lambda = -\frac{\ln(1-\alpha)}{t_0} > 0$  and from normalizing by  $g(x)$ .  $\square$

**Remark 9.** Doeblin's condition is also verified for the case  $p$  defined in (3.4), even when  $\sigma$  is unbounded. Since the amplitude  $S$  is uniformly bounded in System (3.2), we can relax the condition (3.3b) for  $S$  lying in some bounded interval instead of for all  $S \in \mathbb{R}$ . Therefore the exponential convergence to equilibrium is valid as well.



### The nonlinear case

The linear theory allows to determine the asymptotic behavior of the nonlinear System (3.2) for the weak interconnection regime as well. By using Duhamel's formula, it is possible to conclude the improved version of Theorem 20.

**Theorem 22** (Improved convergence to equilibrium). *Assume (3.6)-(3.7) and that  $p \in W^{1,\infty}((0, \infty) \times \mathbb{R})$  satisfies (3.3b). For  $\gamma$  and  $\|\partial_S p\|_\infty$  small enough let  $(n^*, N^*, S^*, w^*)$  be the corresponding stationary state of (3.2). Then there exist  $C, \lambda > 0$  such that the solution  $n$  of (3.2) satisfies*

$$\|n(t) - n^*\|_{L_x^\infty L_s^1} + \|w(t) - w^*\|_\infty \leq C e^{-\lambda t} (\|n_0 - n^*\|_{L_x^\infty L_s^1} + \|w_0 - w^*\|_\infty), \quad \forall t \geq 0.$$

Moreover  $\|S(t) - S^*\|_\infty$  and  $\|N(t) - N^*\|_\infty$  converge exponentially to 0 when  $t \rightarrow \infty$ .

**Proof.** Observe that  $n$  satisfies the evolution equation

$$\partial_t n = \mathcal{L}_S[n] := -\partial_s n - p(s, S)n + \delta_0(s) \int_0^\infty p(u, S(t, x))n(t, u, x) du.$$

We can rewrite the evolution equation as

$$\partial_t n = \mathcal{L}_{S^*}[n] + (\mathcal{L}_S[n] - \mathcal{L}_{S^*}[n]) = \mathcal{L}_{S^*}[n] + h. \quad (3.35)$$

with  $h(t, s, x)$  given by

$$h = (p(s, S^*(x)) - p(s, S(t, x)))n(t, s, x) + \delta_0(s) \int_0^\infty (p(u, S(t, x)) - p(u, S^*(x)))n(t, u, x) du. \quad (3.36)$$

Let  $P_t$  be the linear semigroup associated to operator  $\mathcal{L}_{S^*}$ . As in the proof of Theorem 16 for the elapsed time model with two times, we can extend  $P_t$  to the space  $(\mathcal{M}(0, \infty), \|\cdot\|_{M^1})$  in order to be able to evaluate at the measure  $h$ . Since  $P_t n^* = n^*$  for all  $t \geq 0$ , we get that  $n$  satisfies

$$n - n^* = P_t(n_0 - n^*) + \int_0^t P_{t-\tau} h(\tau, s, x) d\tau, \quad (3.37)$$

so we need find an estimate for the measure  $h$ . Analogously to the proof of Theorem 20, we have the following inequalities:

$$\begin{aligned} \|S(t) - S^*\|_\infty &\leq p_\infty \|w(t) - w^*\|_\infty + \gamma |\Omega| \|N(t) - N^*\|_\infty, \\ \|N(t) - N^*\|_\infty &\leq \|g\|_\infty \|\partial_S p\|_\infty \|S(t) - S^*\|_\infty + p_\infty \|n(t) - n^*\|_{L_x^\infty L_s^1}, \end{aligned}$$

With  $C_1 := \gamma |\Omega| \|g\|_\infty \|\partial_S p\|_\infty < 1$ , we get from these inequalities

$$\begin{aligned} \|S(t) - S^*\|_\infty &\leq \frac{p_\infty}{1 - C_1} (\|w(t) - w^*\|_\infty + \gamma |\Omega| \|n(t) - n^*\|_{L_x^\infty L_s^1}), \\ \|N(t) - N^*\|_\infty &\leq \frac{p_\infty}{1 - C_1} (\|g\|_\infty \|\partial_S p\|_\infty \|w(t) - w^*\|_\infty + \|n(t) - n^*\|_{L_x^\infty L_s^1}). \end{aligned} \quad (3.38)$$

Thus for  $h$  we get

$$\begin{aligned} \|h(t)\|_{L_x^\infty L_s^1} &\leq 2\|g\|_\infty \|\partial_S p\|_\infty \|S(t) - \sigma_*\|_\infty \\ &\leq \frac{2p_\infty \|g\|_\infty \|\partial_S p\|_\infty}{1 - C_1} (\|w(t) - w^*\|_\infty + \gamma|\Omega| \|n(t) - n^*\|_{L_x^\infty L_s^1}) \\ &\leq C_2 (\|w(t) - w^*\|_\infty + \|n(t) - n^*\|_{L_x^\infty L_s^1}), \end{aligned} \quad (3.39)$$

with  $C_2 := \frac{2p_\infty \|g\|_\infty \|\partial_S p\|_\infty}{1 - C_1} \max\{1, \gamma|\Omega|\}$ . On the one hand, using Theorem 21 and the fact that  $\int_0^\infty h(t, s, x) ds = 0$ , we get from (3.37)

$$\begin{aligned} \|n(t) - n^*\|_{L_x^\infty L_s^1} &\leq \|P_t(n_0 - n^*)\|_{L_x^\infty L_s^1} + \int_0^t \|P_{t-\tau} h(\tau)\|_{L_x^\infty L_s^1} d\tau \\ &\leq \frac{e^{-\lambda t}}{1 - \alpha} \|n_0 - n^*\|_{L_x^\infty L_s^1} + \frac{1}{1 - \alpha} \int_0^t e^{-\lambda(t-\tau)} \|h(\tau)\|_{L_x^\infty L_s^1} d\tau \\ &\leq \frac{e^{-\lambda t}}{1 - \alpha} \|n_0 - n^*\|_{L_x^\infty L_s^1} + \frac{C_2}{1 - \alpha} \int_0^t e^{-\lambda(t-\tau)} (\|w(\tau) - w^*\|_\infty + \|n(\tau) - n^*\|_{L_x^\infty L_s^1}) d\tau, \end{aligned}$$

with  $\alpha = p_0 \sigma_* e^{-2p_\infty \sigma_*}$ ,  $\lambda = -\frac{\ln(1-\alpha)}{2\sigma_*} > 0$ . On the other hand, from the second inequality in (3.38) we deduce

$$\begin{aligned} \|w(t) - w^*\|_\infty &\leq e^{-t} \|w_0 - w^*\|_\infty + 2\gamma \int_0^t e^{-(t-\tau)} \|N(\tau) - N^*\|_\infty d\tau \\ &\leq e^{-t} \|w_0 - w^*\|_\infty + C_3 \int_0^t e^{-(t-\tau)} (\|w(\tau) - w^*\|_\infty + \|n(\tau) - n^*\|_{L_x^\infty L_s^1}) d\tau, \end{aligned}$$

with  $C_3 := \frac{2\gamma p_\infty}{1 - C_1} \max\{\|g\|_\infty \|\partial_S p\|_\infty, 1\}$ . Hence we get

$$\begin{aligned} \|n(t) - n^*\|_{L_x^\infty L_s^1} + \|w(t) - w^*\|_\infty &\leq \frac{e^{-\tilde{\lambda} t}}{1 - \alpha} (\|n_0 - n^*\|_{L_x^\infty L_s^1} + \|w_0 - w^*\|_\infty) \\ &\quad + C_4 e^{-\tilde{\lambda} t} \int_0^t e^{\tilde{\lambda} \tau} (\|w(\tau) - w^*\|_\infty + \|n(\tau) - n^*\|_{L_x^\infty L_s^1}) d\tau, \end{aligned}$$

with  $\tilde{\lambda} := \min\{\lambda, 1\}$ ,  $C_4 := \max\left\{\frac{C_2}{1-\alpha}, C_3\right\}$ . Therefore, by using Gronwall's inequality we have

$$\|n(t) - n^*\|_{L_x^\infty L_s^1} + \|w(t) - w^*\|_\infty \leq \frac{e^{-(\tilde{\lambda} - C_4)t}}{1 - \alpha} (\|n_0 - n^*\|_{L_x^\infty L_s^1} + \|w_0 - w^*\|_\infty).$$

So we get the result if  $\gamma$  and  $\|\partial_S p\|_\infty$  are small enough so that  $C_4 < \tilde{\lambda}$ . The exponential convergence of  $N$  and  $S$  readily follows from the estimates in (3.38).  $\square$

**Remark 10.** *If in addition  $n_0 \in L_{s,x}^\infty$ , the result is also valid for  $p$  defined in (3.4) by replacing the estimates involving  $\|\partial_S p\|_\infty$  by its equivalent with  $\|\sigma'\|_\infty$  small enough.*

### 3.4.3 Effect of large inputs

We now study the asymptotic behavior for a large enough input in System (3.2). For  $k > 0$  consider  $n^k(t, s, x)$  a solution of the system

$$\begin{cases} \partial_t n + \partial_s n + p(s, S(t, x))n = 0 & t > 0, s > 0, x \in \Omega, \\ N(t, x) := n(t, s = 0, x) = \int_0^\infty p(s, S(t, x))n ds & t > 0, x \in \Omega, \\ S(t, x) = \int_\Omega w(t, x, y)N(t, y)dy + kI(x) & t > 0, x \in \Omega, \\ \partial_t w = -w + \gamma G(N(t, x), N(t, y)) & t > 0, x, y \in \Omega, \\ n(t = 0, s, x) = n_0(s, x) \geq 0, w(t = 0, x, y) = w_0(x, y) \geq 0 & s \geq 0, x, y \in \Omega. \end{cases} \quad (3.40)$$

We prove by the means of Deoblin's theory that if  $k$  tends to infinity, then the solutions of (3.40) converge to a solution of linear problem (3.8).

**Theorem 23.** *Assume (3.6)-(3.7) with  $p \in W^{1,\infty}((0, \infty) \times \mathbb{R})$  satisfying (3.3b) and such that  $p(s, \infty) := \lim_{S \rightarrow \infty} p(s, S)$  exists for all  $s \geq 0$ . Moreover suppose that  $I(x) > 0$  almost everywhere in  $\Omega$ . Let  $n^\infty$  be the solution of linear problem*

$$\begin{cases} \partial_t n + \partial_s n + p(s, \infty)n = 0 & t > 0, s > 0, x \in \Omega, \\ N(t, x) := n(t, s = 0, x) = \int_0^\infty p(s, \infty)n ds & t > 0, x \in \Omega, \\ n(t = 0, s, x) = n_0(s, x) & s \geq 0, x \in \Omega. \end{cases} \quad (3.41)$$

Then for all  $t > 0$  we have  $n^k(t) \rightarrow n^\infty(t)$  in  $L^1_{s,x}$  when  $k \rightarrow \infty$ .

**Proof.** Let  $\mathcal{L}_S$  be the operator defined in (3.35). In the same way we define the operator  $\mathcal{L}_\infty$  given by

$$\mathcal{L}_\infty[n] := -\partial_s n - p(s, \infty)n + \delta_0(s) \int_0^\infty p(u, \infty)n(t, u, x) du.$$

Thus we rewrite the evolution equation of  $n^k$  as

$$\partial_t n^k = \mathcal{L}_\infty[n] + (\mathcal{L}_S[n] - \mathcal{L}_\infty[n]) = \mathcal{L}_\infty[n] + h.$$

with  $h(t, s, x)$  given by

$$h = (p(s, \infty) - p(s, S(t, x)))n(t, s, x) + \delta_0(s) \int_0^\infty (p(u, S(t, x)) - p(u, \infty))n(t, u, x) du,$$

so we get

$$\|h\|_{L^1_{s,x}} \leq 2 \iint_0^\infty |p(s, \infty) - p(s, S(t, x))|n(t, s, x) ds dx.$$

Since  $S(t, x) \geq kI(x)$  we get that for all  $t > 0$  and a.e.  $x \in \Omega$  that  $S(t, x) \rightarrow \infty$  when  $k \rightarrow \infty$  and thus for all  $s \geq 0$  we have  $p(s, S(t, x)) \rightarrow p(s, \infty)$ . From the method of characteristics we get that  $n$  satisfies

$$n^k(t, s, x) \leq n_0(t - s, x) + p_\infty g(x) \mathbf{1}_{\{0 < s < t\}},$$

hence by Lebesgue's theorem we conclude for all  $t > 0$  that  $\|h(t)\|_{L^1_{s,x}} \rightarrow 0$  when  $k \rightarrow \infty$ .

Let  $P_t$  be the semigroup associated to  $\mathcal{L}_\infty$ . Since  $P_t[n_0] = n^\infty$  we get that  $n^k$  satisfies

$$n^k - n^\infty = \int_0^t P_{t-\tau} h(\tau, s, x) d\tau.$$

Since  $\int_0^\infty h(t, s, x) ds = 0$  we conclude by Doeblin's theorem that

$$\begin{aligned} \|n^k(t) - n^\infty(t)\| &\leq \int_0^t \|P_{t-\tau}h(\tau)\|_{L^1_{s,x}} d\tau \\ &\leq \int_0^t e^{-(t-\tau)} \|h(\tau)\|_{L^1_{s,x}} d\tau. \end{aligned}$$

And since  $\|h(t)\|_{L^1_{s,x}} \leq 4p_\infty$ , we conclude the result by Lebesgue's theorem.  $\square$

**Remark 11.** In the case of  $p$  defined in (3.4) the same result holds if  $\lim_{S \rightarrow \infty} \sigma(S)$  exists. Moreover for the particular case  $\sigma(S) = S$ , the result is straightforward from the fact that  $p(s, \infty) = 0$  and  $N^k \rightarrow 0$  so  $n^\infty$  is solution of a simple transport equation.

### 3.5 Slow learning dynamics

From a neuroscience viewpoint we can assume that the learning dynamics are much slower than the elapsed time dynamics. This is represented by the rescaled system

$$\begin{cases} \varepsilon \partial_t n + \partial_s n + p(s, S(t, x))n = 0 & t > 0, s > 0, x \in \Omega, \\ N(t, x) := n(t, s = 0, x) = \int_0^\infty p(s, S(t, x))n ds & t > 0, x \in \Omega, \\ S(t, x) = \int w(t, x, y)N(t, y)dy + I(t, x) & t > 0, x \in \Omega, \\ \partial_t w = -w + \gamma G(N(t, x), N(t, y)) & t > 0, x, y \in \Omega, \\ n(t = 0, s, x) = n_0(s, x) \geq 0, w(t = 0, x, y) = w_0(x, y) \geq 0 & s \geq 0, x, y \in \Omega, \end{cases} \quad (3.42)$$

with  $\varepsilon > 0$  small enough. This means that the time scale for  $w$  is of order 1, while  $n$  relaxes very quickly to equilibrium with time scale  $\varepsilon$ . Well-posedness and exponential convergence results are also valid for this system.

Let  $n^\varepsilon(t, s, x)$  be the solution of (3.42), we are interested in the asymptotic behavior of  $n^\varepsilon$  when  $\varepsilon \rightarrow 0$ . In order to do so, consider the formal limit system which corresponds to take  $\varepsilon = 0$  in (3.42)

$$\begin{cases} \partial_s n + p(s, S(t, x))n = 0 & s > 0, x \in \Omega, \\ N(t, x) := n(t, s = 0, x) = \int_0^\infty p(s, S(t, x))n ds & t > 0, x \in \Omega, \\ S(t, x) = \int w(t, x, y)N(t, y)dy + I(t, x) & t > 0, x \in \Omega, \\ \partial_t w = -w + \gamma G(N(t, x), N(t, y)) & t > 0, x, y \in \Omega, \\ w(t = 0, x, y) = w_0(x, y) & x, y \in \Omega. \end{cases} \quad (3.43)$$

The question here is to determine if  $n^\varepsilon$ , the solution of System (3.42), converges to some solution of (3.43) when  $\varepsilon$  vanishes. In order to address this question, we first prove that problem (3.43) is well-posed under the weak interconnection regime.

**Theorem 24** (Existence for System (3.43)). *Consider  $g \in C_b(\Omega)$  and  $F$  be the function defined in (3.22). Then under the condition*

$$\max\{\|w_0\|_\infty, \gamma\} \|F'\|_\infty < 1$$

*System (3.43) has a unique solution satisfying  $\int_0^\infty n(t, s, x) ds = g(x)$  for all  $t \geq 0$ .*

To prove the result we need the following lemma.

**Lemma 11.** Consider  $w \in \mathcal{C}_b([0, \infty) \times \Omega \times \Omega)$  fixed. Then the operator  $\mathcal{T}: \mathcal{C}_b([0, \infty) \times \Omega) \rightarrow \mathcal{C}_b([0, \infty) \times \Omega)$  defined by

$$\mathcal{T}[S](t, x) = \int w(t, x, y)g(y)F(S(t, y)) dy + I(t, x),$$

has a unique fixed point  $\bar{S} \in \mathcal{C}_b([0, \infty))$  if  $\|w\|_\infty \|F'\|_\infty < 1$ . Moreover,  $\bar{S}$  is a locally-Lipschitz function of  $w$ .

**Proof.** We first notice that  $\mathcal{T}$  is a contraction. In fact for  $S_1, S_2 \in \mathcal{C}_b([0, \infty) \times \Omega)$  we have

$$\|\mathcal{T}[S_1] - \mathcal{T}[S_2]\|_\infty \leq \|w\|_\infty \|F'\|_\infty \|S_1 - S_2\|_\infty.$$

Hence by Picard's theorem there is a unique fixed point  $\bar{S}[w] \in \mathcal{C}_b([0, \infty) \times \Omega)$ .

Now consider  $\bar{S}[w_1], \bar{S}[w_2]$  the respective fixed points associated to  $w_1, w_2$ . Then we have the following estimate

$$\|\bar{S}[w_1] - \bar{S}[w_2]\|_\infty \leq \|F\|_\infty \|w_1 - w_2\|_\infty + \|w_2\|_\infty \|F'\|_\infty \|\bar{S}[w_1] - \bar{S}[w_2]\|_\infty$$

and hence

$$\|\bar{S}[w_1] - \bar{S}[w_2]\|_\infty \leq \frac{\|F\|_\infty}{1 - \|w_2\|_\infty \|F'\|_\infty} \|w_1 - w_2\|_\infty$$

so  $\bar{S}$  is a locally Lipschitz function of  $w$ . □

In this setting, we continue with the proof of Theorem 24.

**Proof.** First observe that  $n$  satisfies

$$n(t, s, x) = N(t, x) e^{-\int_0^s p(\tau, S(t, x)) d\tau}.$$

and by integrating with respect to  $s$ , we get the following expression for  $N$

$$N(t, x) = g(x) \left( \int_0^\infty e^{-\int_0^s p(\tau, S(t, x)) d\tau} ds \right)^{-1} = g(x) F(S(t, x)).$$

Hence the problem is reduced to the following system for  $(S, w)$

$$\begin{cases} S(t, x) = \int w(t, x, y)g(y)F(S(t, y))dy + I(t, x) & t > 0, x \in \Omega, \\ \partial_t w = -w + \gamma G(g(x)F(S(t, x)), g(y)F(S(t, y))) & t > 0, x, y \in \Omega, \\ w(t = 0, x, y) = w_0(x, y) & x, y \in \Omega. \end{cases} \quad (3.44)$$

Since we have a uniform estimate for  $w$  in (3.15), we conclude that  $\bar{S}[w]$  is a Lipschitz function restricted to the set

$$U = \{w \in \mathcal{C}_b([0, \infty) \times \Omega \times \Omega) : \|w\|_\infty \leq \max\{\|w_0\|_\infty, \gamma\}\}$$

if  $\max\{\|w_0\|_\infty, \gamma\} \|F'\|_\infty < 1$ . So by applying the Cauchy-Lipschitz-Picard theorem, we conclude that System (3.44) has a unique solution, defined in some time interval  $[0, T]$ . Finally, by noting again that  $w$  is uniformly bounded as in (3.15), we can iterate this argument to get a solution globally defined in time. □

By replicating the proof in Theorem 20, we get for System (3.43) its asymptotic behavior when  $t \rightarrow \infty$ .

**Theorem 25** (Long term behavior for System (3.43)). *Assume (3.6)-(3.7) and that  $p \in W^{1,\infty}((0, \infty) \times \mathbb{R})$  satisfies (3.3a). For  $\gamma$  and  $\|\partial_S p\|_\infty$  small enough, consider  $(n^*, N^*, S^*, w^*)$  the corresponding stationary state of (3.2). Then there exist  $C, \lambda > 0$  such that the solution of (3.43) satisfies*

$$\|n(t) - n^*\|_{L^1_{s,x}} + \|w(t) - w^*\|_{L^1_{x,y}} \leq C e^{-\lambda t} \|w_0 - w^*\|_{L^1_{x,y}}, \quad \forall t \geq 0. \quad (3.45)$$

Moreover  $\|S(t) - S^*\|_{L^1_x}$  and  $\|N(t) - N^*\|_{L^1_x}$  converge exponentially to 0 when  $t \rightarrow \infty$ .

Next we prove the convergence of  $n^\varepsilon$  for the case of weak interconnection when the firing rate is strictly positive, by means of the entropy method.

**Theorem 26** (Convergence for (3.42) as  $\varepsilon \rightarrow 0$ ). *Assume (3.6)-(3.7) with  $n_0 \in W^{1,1}_{s,x}$  and that  $p \in W^{1,\infty}((0, \infty) \times \mathbb{R})$  satisfies (3.3a). For  $\max\{\|w_0\|_\infty, \gamma\}$  small enough, let  $(n^\varepsilon, N^\varepsilon, S^\varepsilon, w^\varepsilon)$  be the solution of System (3.42) and let  $(\bar{n}, \bar{N}, \bar{S}, \bar{w})$  be the unique solution of System (3.43).*

*Then for all  $T > 0$  we have  $n^\varepsilon \rightarrow \bar{n}$  in  $L^1((0, T) \times (0, \infty) \times \Omega)$  and  $w^\varepsilon \rightarrow \bar{w}$  in  $L^1((0, T) \times \Omega \times \Omega)$ . Moreover  $N^\varepsilon \rightarrow \bar{N}$  and  $S^\varepsilon \rightarrow \bar{S}$  in  $L^1((0, T) \times \Omega)$ .*

**Proof.** Let  $(n^\varepsilon, N^\varepsilon, S^\varepsilon, w^\varepsilon)$  be the solution of System (3.42). We start by reminding the following uniform estimates

$$\|N^\varepsilon(t)\|_\infty \leq p_\infty \|g\|_\infty, \quad \|w^\varepsilon\|_\infty \leq \max\{\|w_0\|_\infty, \gamma\}, \quad \forall t \geq 0, \varepsilon > 0, \quad (3.46)$$

The first step is to estimate  $u = \partial_t n^\varepsilon$ , which satisfies the following equation

$$\varepsilon \partial_t u + \partial_s u + p(s, S^\varepsilon) u + \partial_S p(s, S^\varepsilon) \partial_t S^\varepsilon n^\varepsilon = 0,$$

thus we have the following inequality

$$\varepsilon \partial_t |u| + \partial_s |u| + p(s, S^\varepsilon) |u| \leq \|\partial_S p\|_\infty |\partial_t S^\varepsilon| n^\varepsilon.$$

By integrating with respect to all variables, we get

$$\begin{aligned} \int_0^T \int \int_0^\infty p(s, S^\varepsilon) |u| ds dx dt &\leq \varepsilon \int \int_0^\infty |u|(0, s, x) ds dx + \int_0^T \int |\partial_t N^\varepsilon|(t, x) dx dt \\ &+ \|\partial_S p\|_\infty \int_0^T \|\partial_t S^\varepsilon(t, \cdot)\|_\infty dt. \end{aligned} \quad (3.47)$$

Thus we have to estimate each term in the right-hand side. For the first it readily follows that

$$\varepsilon \int \int_0^\infty |u|(0, s, x) ds dx \leq \int \int_0^\infty |\partial_s n_0| ds dx + \int \int_0^\infty p(s, S^\varepsilon(0, x)) n_0 ds dx \leq \|n_0\|_{W^{1,1}_{s,x}} + p_\infty. \quad (3.48)$$

Next, for  $\partial_t N^\varepsilon$  we have

$$\partial_t N^\varepsilon(t, x) = \partial_t S^\varepsilon(t, x) \int_0^\infty \partial_S p(s, S^\varepsilon(t, x)) n^\varepsilon(t, s, x) ds + \int_0^\infty p(s, S^\varepsilon(t, x)) \partial_t n^\varepsilon(t, s, x) ds.$$

Thus for the second term we get

$$\int_0^T \int |\partial_t N^\varepsilon|(t, x) dx dt \leq \|\partial_S p\|_\infty \int_0^T \|\partial_t S^\varepsilon(t, \cdot)\|_\infty dt + \int_0^T \int \left| \int_0^\infty p(s, S^\varepsilon) u ds \right| dx dt. \quad (3.49)$$

On the other hand, for  $\partial_t S^\varepsilon$  we get

$$\partial_t S^\varepsilon(t, x) = \int \partial_t w^\varepsilon(t, x, y) N^\varepsilon(t, y) dy + \int w^\varepsilon(t, x, y) \partial_t N^\varepsilon(t, y) dy + \partial_t I(t, x).$$

Let  $A := \max\{\|w_0\|_\infty, \gamma\}$ , by using the uniform estimates in (3.46) we obtain

$$\begin{aligned} \int_0^T \|\partial_t S^\varepsilon(t, \cdot)\|_\infty dt &\leq \|\partial_t w^\varepsilon\|_\infty \int_0^T \int N^\varepsilon dy dt + \|w^\varepsilon\|_\infty \int_0^T \int |\partial_t N^\varepsilon| dy dt + \|\partial_t I\|_\infty T \\ &\leq A \left( 2p_\infty T + \int_0^T \int |\partial_t N^\varepsilon| dy dt \right) + \|\partial_t I\|_\infty T. \end{aligned} \quad (3.50)$$

Let  $\beta_1 := \|\partial_S p\|_\infty \max\{\|w_0\|_\infty, \gamma\} < 1$ . Hence from (3.49) we conclude

$$\int_0^T \|\partial_t S^\varepsilon(t, \cdot)\|_\infty dt \leq \frac{1}{1 - \beta_1} \left( 2Ap_\infty T + \|\partial_t I\|_\infty T + Ap_\infty \int_0^T \int \int_0^\infty |u| ds dx dt \right). \quad (3.51)$$

Therefore we can deduce from (3.47) the following estimate

$$\begin{aligned} \int_0^T \int \int_0^\infty p(s, S^\varepsilon) |u| ds dx dt &\leq \|n_0\|_{W_{s,x}^{1,1}} + p_\infty + \frac{2\|\partial_S p\|_\infty T}{1 - \beta_1} (2Ap_\infty + \|\partial_t I\|_\infty) \\ &\quad + \frac{2\beta_1}{1 - \beta_1} p_\infty \int_0^T \int \int_0^\infty |u| ds dx dt + \int_0^T \int \left| \int_0^\infty p(s, S^\varepsilon) u ds \right| dx dt. \end{aligned} \quad (3.52)$$

At this stage we can use again the entropy trick from [43, 51]. Since  $\int_0^\infty u ds = 0$  and  $p \geq p_0$ , we have the following inequality

$$\int_0^T \int \left| \int_0^\infty p(s, S^\varepsilon) u ds \right| dx dt = \int_0^T \int \left| \int_0^\infty (p(s, S^\varepsilon) - p_0) u ds \right| dx dt \leq \int_0^T \int \int_0^\infty (p(s, S^\varepsilon) - p_0) |u| ds dx dt.$$

As  $\beta_1$  is small enough, we conclude the  $L^1$  norm of  $u = \partial_t n^\varepsilon$  is uniformly bounded in  $\varepsilon$ .

$$\begin{aligned} \left( p_0 - \frac{2\beta_1}{1 - \beta_1} p_\infty \right) \int_0^T \int \int_0^\infty |\partial_t n^\varepsilon| ds dx dt &\leq \frac{2\|\partial_S p\|_\infty T}{1 - \beta_1} (2Ap_\infty + \|\partial_t I\|_\infty) \\ &\quad + \|n_0\|_{W_{s,x}^{1,1}} + p_\infty. \end{aligned} \quad (3.53)$$

The next step is to estimate  $n^\varepsilon - \bar{n}$ , by using a similar argument. Let  $\bar{N}, \bar{S}$  and  $\bar{w}$  be the terms associated to  $\bar{n}$  in System (3.43), so that we have

$$\partial_s(n^\varepsilon - \bar{n}) + p(s, S^\varepsilon)(n^\varepsilon - \bar{n}) = -\varepsilon \partial_t n^\varepsilon - (p(s, S^\varepsilon) - p(s, \bar{S}))\bar{n}.$$

Hence we have the following inequality

$$\partial_s |n^\varepsilon - \bar{n}| + p(s, S^\varepsilon) |n^\varepsilon - \bar{n}| \leq \varepsilon |\partial_t n^\varepsilon| + \|\partial_S p\|_\infty |S^\varepsilon - \bar{S}| \bar{n}.$$

By integrating with respect to all variables we get

$$\begin{aligned} \int_0^T \iint \int_0^\infty p(s, S^\varepsilon) |n^\varepsilon - \bar{n}| ds dx dt &\leq \varepsilon \int_0^T \iint \int_0^\infty |\partial_t n^\varepsilon| ds dx dt + \int_0^T \int |N^\varepsilon - \bar{N}| dx dt \\ &+ \|g\|_\infty \|\partial_S p\|_\infty \int_0^T \int |S^\varepsilon - \bar{S}| dx dt. \end{aligned} \quad (3.54)$$

So we have to estimate the respective terms involving  $N$  and  $S$ . For  $N^\varepsilon - \bar{N}$  we have

$$\int_0^T \int |N^\varepsilon - \bar{N}| dx dt \leq \|g\|_\infty \|\partial_S p\|_\infty \int_0^T \int |S^\varepsilon - \bar{S}| dx dt + \int_0^T \int \left| \int_0^\infty p(s, S^\varepsilon) (n^\varepsilon - \bar{n}) ds \right| dx dt. \quad (3.55)$$

In order to estimate  $S^\varepsilon - \bar{S}$ , we need to estimate  $w^\varepsilon - \bar{w}$  first. By using formula (3.12) we obtain

$$\begin{aligned} \int_0^T \iint |w - \bar{w}| dx dy dt &\leq 2\gamma |\Omega| \int_0^T \int_0^t e^{-(t-\tau)} \|N^\varepsilon - \bar{N}\|_{L_x^1}(\tau) d\tau dt \\ &\leq 2\gamma |\Omega| \int_0^T \int_\tau^t e^{-(t-\tau)} \|N^\varepsilon - \bar{N}\|_{L_x^1}(\tau) dt d\tau \\ &\leq 2\gamma |\Omega| \int_0^T \|N^\varepsilon - \bar{N}\|_{L_x^1}(\tau) \left( \int_\tau^T e^{-(t-\tau)} dt \right) d\tau, \end{aligned}$$

so we conclude the following estimate

$$\int_0^T \iint |w - \bar{w}| dx dy dt \leq 2\gamma |\Omega| \int_0^T \int |N^\varepsilon - \bar{N}| dx dt. \quad (3.56)$$

Thus, for  $S^\varepsilon - \bar{S}$  we get

$$\begin{aligned} \int_0^T \int |S^\varepsilon - \bar{S}| dx dt &\leq \int_0^T \iint |w - \bar{w}| N^\varepsilon(t, y) dx dy dt + |\Omega| \|\bar{w}\|_\infty \int_0^T \int |N^\varepsilon - \bar{N}| dy dt \\ &\leq (2\gamma |\Omega| p_\infty \|g\|_\infty + |\Omega| A) \int_0^T \int |N^\varepsilon - \bar{N}| dx dt. \end{aligned}$$

Let  $\beta_2 := 2\gamma |\Omega| p_\infty \|g\|_\infty + |\Omega| A$ . If  $\beta_2 \|g\|_\infty \|\partial_S p\|_\infty < 1$ , from (3.55) we obtain

$$\int_0^T \int |S^\varepsilon - \bar{S}| dx dt \leq \frac{\beta_2 p_\infty}{1 - \beta_2 \|g\|_\infty \|\partial_S p\|_\infty} \int_0^T \iint |n^\varepsilon - \bar{n}| ds dx dt. \quad (3.57)$$

Let  $\beta_3 := \frac{\beta_2 p_\infty}{1 - \beta_2 \|g\|_\infty \|\partial_S p\|_\infty}$ , then from (3.54) we deduce the following inequality

$$\begin{aligned} \int_0^T \iint \int_0^\infty p(s, S^\varepsilon) |n^\varepsilon - \bar{n}| ds dy dt &\leq \varepsilon \int_0^T \iint \int_0^\infty |\partial_t n^\varepsilon| ds dy dt + 2\beta_3 \int_0^T \iint \int_0^\infty |n^\varepsilon - \bar{n}| ds dy dt \\ &+ \int_0^T \int \left| \int_0^\infty p(s, S^\varepsilon) (n^\varepsilon - \bar{n}) ds \right| dy dt. \end{aligned} \quad (3.58)$$



Since  $\int_0^\infty (n^\varepsilon - \bar{n}) ds = 0$  and  $p \geq p_0$ , we have the following inequality

$$\begin{aligned} \int_0^T \int \left| \int_0^\infty p(s, S^\varepsilon)(n^\varepsilon - \bar{n}) ds \right| dy dt &= \int_0^T \int \left| \int_0^\infty (p(s, S^\varepsilon) - p_0)(n^\varepsilon - \bar{n}) ds \right| dy dt \\ &\leq \int_0^T \int \int_0^\infty (p(s, S^\varepsilon) - p_0) |n^\varepsilon - \bar{n}| ds dy dt. \end{aligned}$$

As  $\gamma$  is small enough, we finally conclude the following Poincaré-like estimate for  $n^\varepsilon - \bar{n}$

$$(p_0 - 2\beta_3) \int_0^T \int \int_0^\infty |n^\varepsilon - \bar{n}| ds dy dt \leq \varepsilon \int_0^T \int \int_0^\infty |\partial_t n^\varepsilon| ds dy dt. \quad (3.59)$$

And we obtain the result by taking  $\varepsilon \rightarrow 0$ , since the  $L^1$  norm of  $\partial_t n^\varepsilon$  is uniformly bounded in  $\varepsilon$ . The convergence of  $N, S$  and  $w$  is straightforward from estimates (3.55), (3.56) and (3.57).  $\square$

**Remark 12.** For a firing rate  $p$  satisfying (3.3b) is not evident to apply Doeblin's theory to deduce Theorem 26. Indeed, for a fixed  $S \in \mathcal{C}_b(\Omega)$ , consider  $P_t^\varepsilon$  the semigroup defined by the linear problem

$$\begin{cases} \varepsilon \partial_t n + \partial_s n + p(s, S(x))n = 0 & t > 0, s > 0, x \in \Omega, \\ N(t, x) := n(t, s = 0, x) = \int_0^\infty p(s, S(x))n ds & t > 0, x \in \Omega, \\ n(t = 0, s, x) = n_0(s, x) & s \geq 0, x \in \Omega, \end{cases} \quad (3.60)$$

so that by replicating the proof of Theorem 21 we can prove the following lower bound

$$P_{t_0}^\varepsilon n_0 \geq \varepsilon p_0 e^{-2p_\infty \sigma_*} \mathbb{1}_{[0, \sigma_*]}(s) g(x), \quad t_0 = 2\varepsilon \sigma_*.$$

And we lose Doeblin's condition as  $\varepsilon$  vanishes.

## 3.6 Numerical simulations

### 3.6.1 Elapsed time dynamics

We present numerical simulations of System (3.2) in order to observe the dependence on parameters such as connectivity  $\gamma$  and the input  $I$ . For these simulations the domain for position  $x$  is  $\Omega = (0, 1)$  and the firing rate is given by  $p = \mathbb{1}_{\{s > S\}}$ . We compute numerical solutions with a standard upwind scheme (see Scheme 6 in the Appendix of numerical methods).

We focus in displaying the activity  $N(t, x)$  and the amplitude  $S(t, x)$  since these two elements determine the general behavior of System (3.2). We explore a spatially-homogeneous case and an inhomogeneous one, both with a different learning rule for  $w$ . In every example the initial connectivity kernel is given by

$$w_0(x, y) = 10 \exp(-10(x - y)^2).$$

#### Spatially-homogeneous input

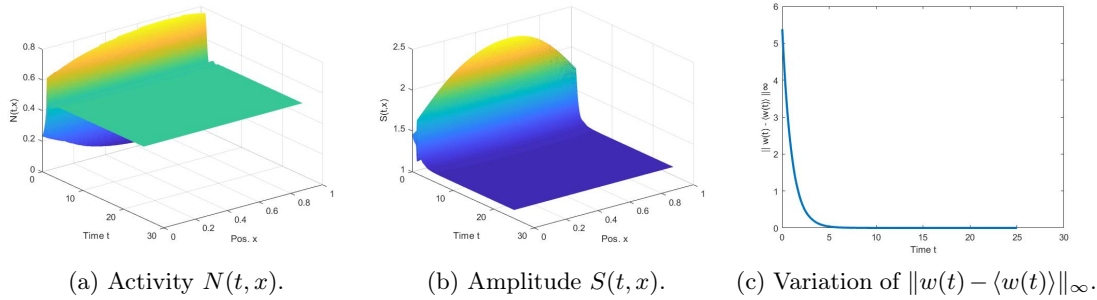
We start with some examples when the external input  $I$  is constant and positive. For this subsection the initial probability density is given by  $n_0(s, x) = (x + 1)e^{-s(x+1)}$ , so that  $g \equiv 1$ . Moreover, we consider a learning rule of Hebbian type with the evolution of the kernel given by

$$\partial_t w = -w + \gamma N(t, x)N(t, y).$$

In this particular example there exists a unique steady state determined, through the formulas in (3.21), by a unique amplitude of stimulation  $S^*$ , which is constant. This is given by the unique positive solution of the equation

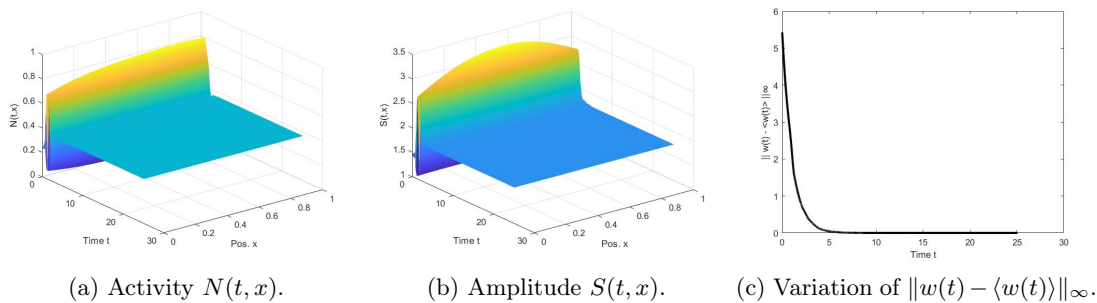
$$S = \frac{\gamma}{(1 + S)^3} + I.$$

Figure 3.2 – Case  $\gamma = 1$  and  $I = 1$ .



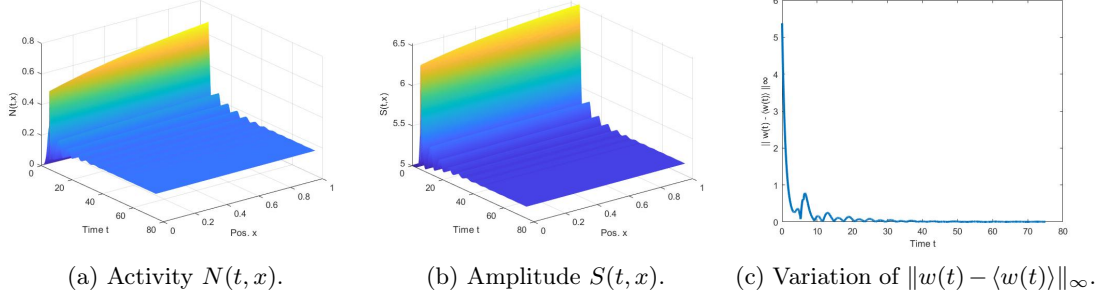
In figure 3.2 we observe that for  $\gamma = 1$  and  $I = 1$  the activity  $N$  and the amplitude  $S$  stabilize very fast in time and become spatially-homogeneous, this means that the numerical solution  $n$  of System (3.2) converges to the equilibrium which is independent of variable  $x$ . Moreover, we observe 3.2c that  $\|w(t) - \langle w(t) \rangle\|_\infty$ , with  $\langle w \rangle := |\Omega|^{-2} \iint w \, dx \, dy$ , decreases to 0 in time so the numerical connectivity kernel  $w$  is converging to a constant. We essentially observe the behavior of Theorem 22.

Figure 3.3 – Case  $\gamma = 15$  and  $I = 1$ .



If we increase the value to  $\gamma = 15$ , we observe in figure 3.3 that  $N$  and  $S$  converge also converge to a steady state and they become spatially-homogeneous. We observe in figure 3.3c that  $\|w(t) - \langle w(t) \rangle\|_\infty$  decreases to 0 with time, so the connectivity kernel  $w$  is converging to a spatially-homogeneous pattern as well.

When we take  $\gamma = 35$  and also increase the value of input  $I$ , the numerical solution exhibits again convergence towards equilibrium when the time is large enough. Like the previous cases, the activity  $N$  and the amplitude  $S$  become spatially-homogeneous in figure 3.4. For the connectivity kernel we have that  $\|w(t) - \langle w(t) \rangle\|_\infty$  decreases to 0 in time as we observe in figure 3.4c, so the numerical connectivity  $w$  is converging to a constant. Moreover, this is also compatible with the large connectivity case studied in the article of Pakdaman et al. [48].

Figure 3.4 – Case  $\gamma = 35$  and  $I = 5$ .

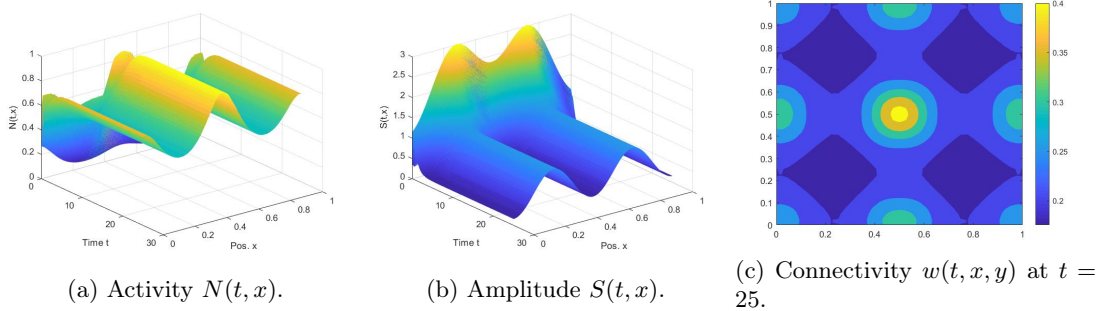
More generally, we can conjecture that when  $g$  and the input  $I$  are constant, then  $N, S$  and  $w$  lose its spatial dependence as time passes.

### Spatially-inhomogeneous input

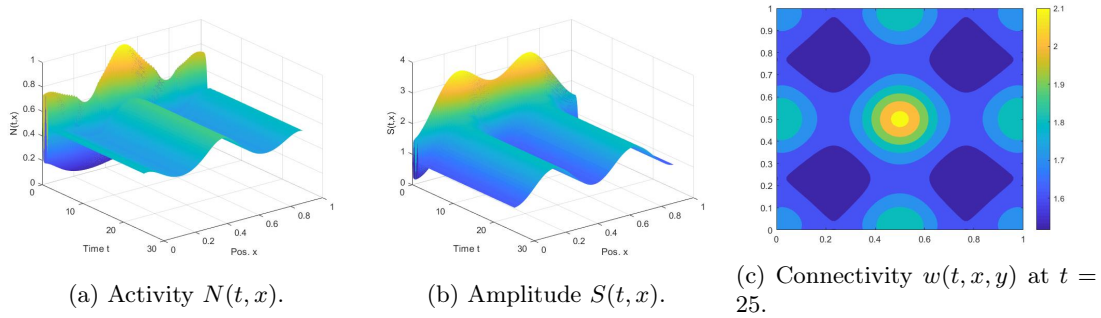
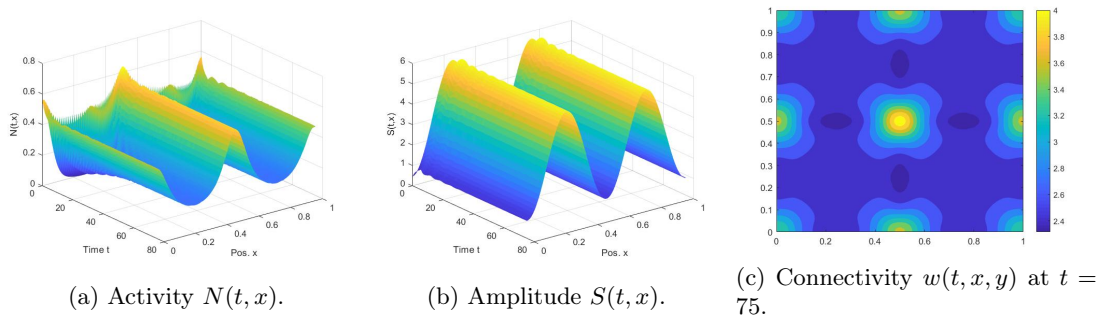
Now we present an example with a non-constant input to see the activity and the connectivity kernel depending strongly on position. For this subsection the initial probability density is given by  $n_0(s, x) = \frac{\exp(-s-(x-0.5)^2)}{\int_0^1 \exp(-(z-0.5)^2) dz}$ . We consider a learning rule with the evolution of the kernel given by

$$\partial_t w = -w + \gamma \frac{\exp(-(N(t, x) - N(t, y))^2)}{1 + \exp(-2N(t, x)N(t, y) + 2)}.$$

Consider first  $I(x) = \sin^2(2\pi x)$ , so for  $\gamma = 1$  we observe in figure 3.5 that both  $N$  and  $S$  converge in time to a stationary state. Moreover in figure 3.5c, we observe that the connectivity kernel converges to a particular pattern that exhibits a symmetric behavior in spatial variable. Like the corresponding spatially-homogeneous example of figure 3.2, we observe again the behavior of Theorem 22.

Figure 3.5 – Case  $\gamma = 1$  and  $I = \sin^2(2\pi x)$ .

As in the previous example, when we increase the connectivity parameter to  $\gamma = 10$ , the behavior of the activity  $N$  and the amplitude  $S$  is essentially the same, as we can see in figure 3.6. The connectivity kernel converge the pattern shown in figure 3.6c and it presents higher values than those in figure 3.5c.

Figure 3.6 – Case  $\gamma = 10$  and  $I = \sin^2(2\pi x)$ .Figure 3.7 – Case  $\gamma = 20$  and  $I = 5 \sin^2(2\pi x)$ .

Finally, in the case of  $\gamma = 20$  and  $I = 5 \sin^2(2\pi x)$ , the numerical solution exhibits convergence towards an equilibrium when the time is large enough as it is presented in figure 3.7. The numerical connectivity kernel  $w$  converge to pattern presented in figure 3.7c.

From these examples, for both spatially-homogeneous and inhomogeneous cases, we conjecture that if the system is inhibitory, then all solutions of System (3.2) converge to a steady state. This result is also conjectured for the classical elapsed time model studied in [48].

Moreover when the input  $I$  is large enough, we expect a similar convergence result. Theorem 23 states that solutions converge pointwise to a solution of a simple linear problem when the external input is large enough in both spatially-homogeneous and inhomogeneous cases. This theorem could be a first approach to verify the general convergence result.

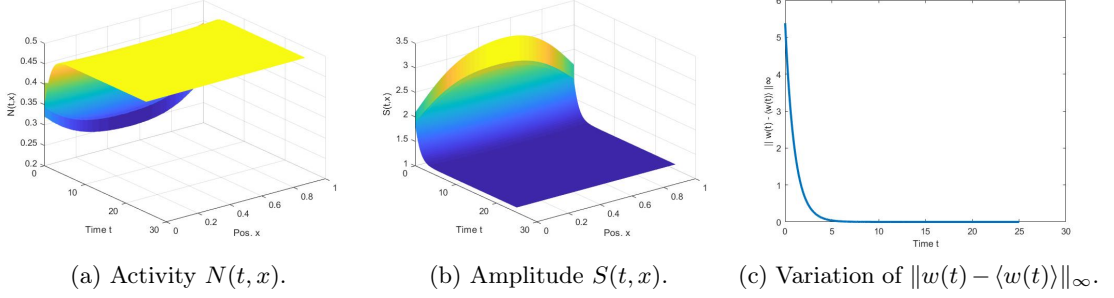
### 3.6.2 Limit system with $\varepsilon = 0$ .

We present some numerical simulations of the limit System (3.43) under the same setting of domain, firing rate and initial kernel. We show the homogeneous and inhomogeneous cases with the same respective initial densities, learning rules and parameter combinations of their counterparts of System (3.2). We compute numerical solutions through Scheme 7 and we contrast the numerical simulations with the convergence Theorem 26 when  $\varepsilon$  vanishes.

### Spatially-homogeneous input

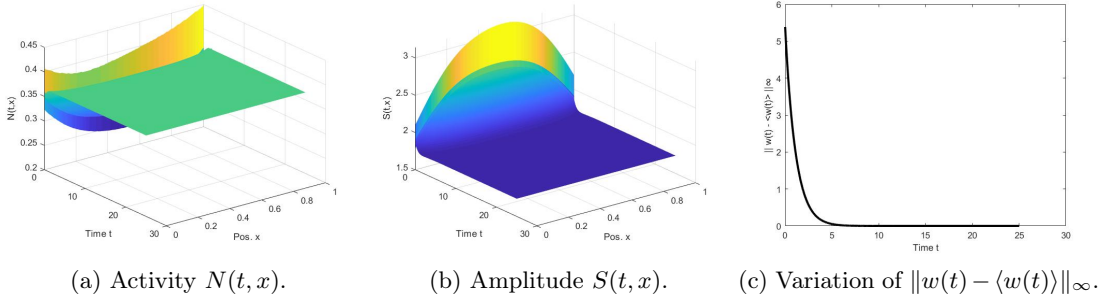
In figure 3.8 we observe that for  $\gamma = 1$  and  $I = 1$  both  $N, S$  converge fast in time to equilibrium and become spatially-homogeneous. Moreover the figure 3.8c shows that  $\|w(t) - \langle w(t) \rangle\|_\infty$  is converging to 0, so  $w$  converges to a constant. This corresponds essentially to the same behavior of the numerical simulations in System (3.2) and it is compatible with the convergence result of Theorem 26.

Figure 3.8 – Case  $\gamma = 1$  and  $I = 1$  for the limit system.



When we increase the value to  $\gamma = 15$  numerical solutions keep the same behavior of convergence to equilibrium and spatial homogeneity as we see in figure 3.9. From figure 3.9c we observe that the numerical connectivity kernel verifies that  $\|w(t) - \langle w(t) \rangle\|_\infty$  is converging to 0 and  $w$  converges to a constant.

Figure 3.9 – Case  $\gamma = 15$  and  $I = 1$  for the limit system.



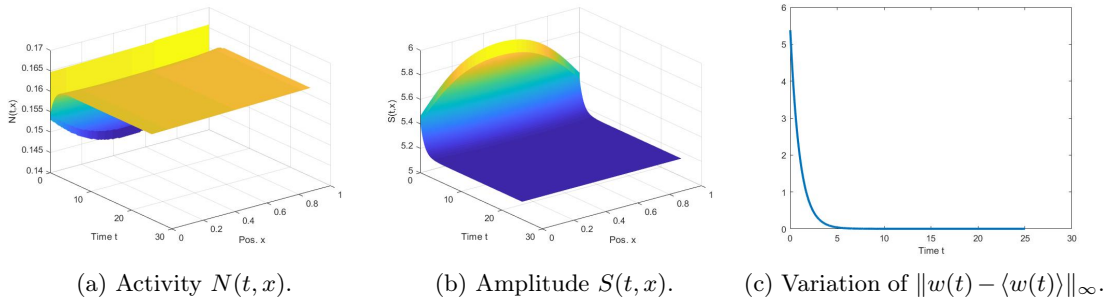
If in addition we take  $\gamma = 35$  and increase the value of input to  $I = 5$ , we observe in figure 3.10 the same behavior for  $N, S$  and  $w$  as in previous cases. Therefore we can conjecture that when  $g$  and the input  $I$  are constant then System (3.43) simply converges to a spatially-homogeneous equilibrium, as we observed in the corresponding numerical simulations of System (3.2).

### Spatially-inhomogeneous input

Now we show some numerical simulations of System (3.43) under the same previously presented non-constant inputs.

If  $I = \sin^2(2\pi x)$  and  $\gamma = 1$  we see in figure 3.11 that both  $N$  and  $S$  converge in time to a stationary state as expected. With respect to the kernel  $w$ , we observe in figure 3.11c a similar

Figure 3.10 – Case  $\gamma = 35$  and  $I = 5$  for the limit system.



pattern formation as in the corresponding simulation for System (3.2) in figure 3.5c. Furthermore, this example is compatible with the result of Theorem 26.

Figure 3.11 – Case  $\gamma = 1$  and  $I = \sin^2(2\pi x)$  for the limit system.

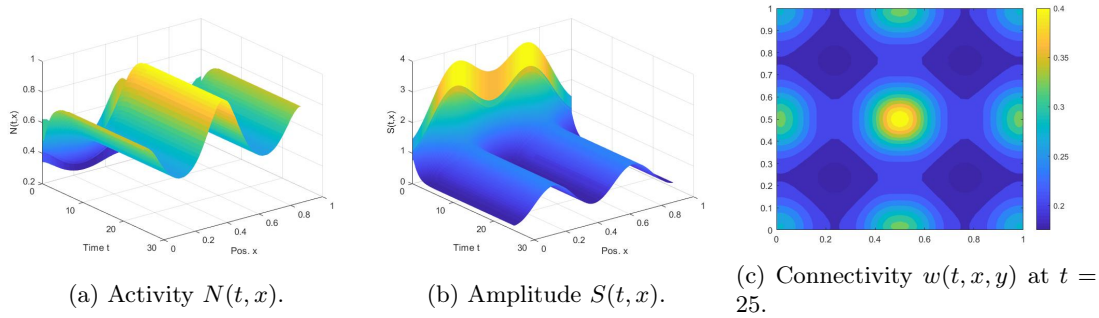
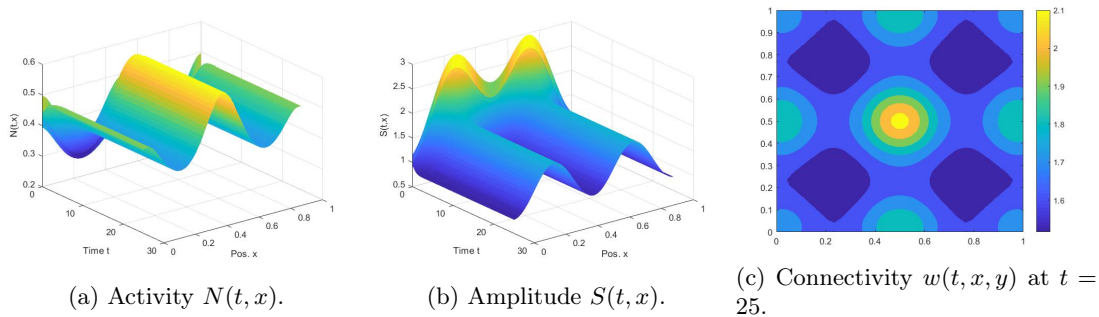
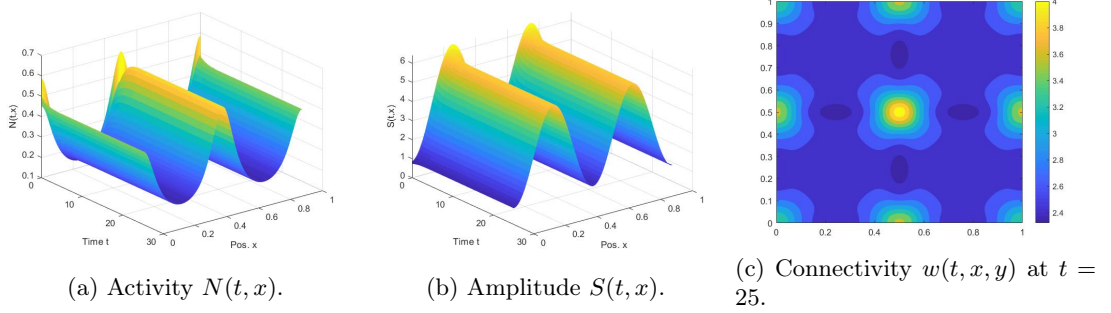


Figure 3.12 – Case  $\gamma = 10$  and  $I = \sin^2(2\pi x)$  for the limit system.



Next, when we increase the value to  $\gamma = 10$ , we still observe in figure 3.12 the convergence in time for  $N$  and  $S$ . Furthermore, the numerical kernel  $w$  exhibits in figure 3.12c a similar pattern to that observed in figure 3.6c, the corresponding simulation of System (3.2).

Finally in the case of  $\gamma = 20$  and  $I = 5 \sin^2(2\pi x)$ , we observe in figure 3.13 that the numerical

Figure 3.13 – Case  $\gamma = 20$  and  $I = 5 \sin^2(2\pi x)$  for the limit system.

solutions exhibits again a convergent behavior in the variables  $N$  and  $S$ , while the kernel shows essentially in figure 3.13c the same pattern as the corresponding simulation of System (3.2). We conjecture that the general dynamic of the limit System (3.43) is simply a convergence to stationary state. Furthermore, we conjecture that Theorem 26 is also true for a strong interconnection in the inhibitory case or for a large external input.

### 3.7 Perspectives

From the previous theoretical results and numerical simulations we observe that only the case with very weak interconnection begins to be well understood for the Cauchy problem and the asymptotic behavior. More complex dynamics, such as oscillations, that could emerge with stronger interconnections or even convergence to a stationary state for a general case are far from being fully understood.

Concerning well-posedness in System (3.2), it remains unsolved studying the case of a strong connectivity and determine if multiple solutions arise. This means studying the number of solutions for  $S(t, x)$  in the fixed point equation in (3.13). If a multiplicity of solution arises, this might be interpreted as perceiving different reactions from the same input. In the same way, it also remains open the well-posedness for limit System (3.43) with its corresponding fixed point problem.

Regarding steady states, it remains as an open problem to determine the existence beyond the weak interconnections regime. We must remark that even in inhibitory case, the existence or uniqueness of solutions in Equation (3.23) is not evident. With respect to the asymptotic behavior, it is necessary to give a more detailed description of how the size of the kernel  $w$  affects the general behavior of System (3.2) in order to have a clearer idea of the bifurcation diagram in the connectivity parameter  $\gamma$ .

Furthermore, it is pending to study the convergence to equilibrium of System (3.2) for a general large input in order to improve Theorem 23. This include to consider the case when the external input  $I$  goes to infinity in localized regions of  $\Omega$ . Moreover, it remains open to prove when the function  $g$  and the external input are constant then the system approaches to spatially-homogeneous profile as it was observed in the numerical simulations.

Finally for the system with slow learning (3.42), we expect the convergence Theorem 25 for weak interconnection is also true when  $p$  satisfies the lower bound (3.3b). Furthermore, for the limit System (3.43) we expect a simple convergence to equilibrium regardless the value of  $\gamma$ .

# Further extensions

From the present work, we observe that the case of weak nonlinearities for the three studied models is the most accessible case through the techniques of the linear problem such as the entropy method and Doeblin's theory. Regarding strong nonlinearities, we notice that studying pattern formations is still an open problem with some partial results and conjectures, which are inspired in the numerical simulations. In the study of the elapsed time model, other extensions may be considered and we mention some possible research directions.

## Two-population renewal equation

A possible extension of the classical elapsed time model is to consider a system of two sub-population of neurons where one of them is inhibitory and the other one is excitatory. For  $i = 1, 2$ , this extension is represented by the following system

$$\begin{cases} \partial_t n_i + \partial_s n_i + p_i(s, X(t))n_i = 0 & t > 0, s > 0, \\ N_i(t) = n_i(t, s = 0) = \int_0^\infty p_i(s, X(t))n_i(t, s) ds & t > 0, \\ X(t) = \alpha_1 N_1(t) + \alpha_2 N_2(t) & t > 0, \\ n_i(t = 0, s) = n_{0,i}(s) \geq 0 & s \geq 0, \end{cases} \quad (3.61)$$

with  $\partial_N p_1 < 0, \partial_N p_2 > 0$  and the total activity  $X(t)$  is a combination of the discharging flux of neurons of each sub-population. An interesting problem on this model is to study aspects like well-posedness, asymptotic behavior, possible pattern formations and the dependence on the parameters  $\alpha_1, \alpha_2$  in order to contrast the results with the classical elapsed time model.

## Incorporation of a continuous delay

Regarding more realistic assumptions to be considered, we can modify Equation (1) to incorporate of a continuous delay, obtaining the following system

$$\begin{cases} \partial_t n + \partial_s n + p(s, X(t))n = 0 & t > 0, s > 0, \\ N(t) = n(t, s = 0) = \int_0^\infty p(s, X(t))n(t, s) ds & t > 0, \\ X(t) = \int_0^t \alpha(u)N(t-u) du, & t > 0, \\ n(t = 0, s) = n_0(s) \geq 0 & s \geq 0, \end{cases} \quad (3.62)$$

where  $X(t)$  is global activity and  $\alpha(u) \geq 0$  is the distributed delay function. The classical elapsed time model corresponds to the particular case  $\alpha = \delta_0$ , which means that the transmission of a discharge is instantaneous and  $X(t) = N(t)$ , i.e. the global activity coincides with the discharging flux of neurons.



For System (3.62) under the weak interconnections case, well-posedness and convergence to the steady state have been studied by Pakdaman et al. [48] with some numerical simulations. However, it remains open to prove if the exponential convergence to equilibrium holds as in the classical elapsed time equation. An interesting problem is to determine if the results of Chapter 1 may be extended to model with delay. We can also consider the incorporation of the delay to the multiple time renewal equation, the two-population system and the model with spatial dependence.

## Linear stability

Finally, an important issue of the classical elapsed time model is to determine if the principle of linearized stability holds, i.e. the linear system around a steady state determines its stability in the nonlinear problem. Moreover, determining the sign of the corresponding eigenvalues is a difficult problem as we will see below.

Let  $(n^*, N^*)$  be an equilibrium of Equation (1), from Formula (7) we have

$$n^*(s) = N^* e^{-Q(s)}, \quad Q(s) := \int_0^s p(s', N^*) ds',$$

and  $N^*$  is a solution of the fixed point Equation (8). In addition, consider  $\mathcal{F}: L^1(0, \infty) \rightarrow \mathbb{R}$  the map such that for  $g \in L^1$ ,  $\mathcal{F}[g]$  is a solution of the fixed point equation

$$x = \int_0^\infty p(s, x)g(s) ds. \quad (3.63)$$

For simplicity we assume that solutions of this equation are finite and isolated, so that we can choose a branch of solutions depending continuously on  $g \in L^1$  without ambiguity.

In this setting, the linearization of Equation (1) around the equilibrium  $n^*$  is given by the following linear system for  $h = n - n^*$

$$\begin{cases} \partial_t h + \partial_s h + p(s, N^*)h + n^* \partial_N p(s, N^*) D_{n^*} \mathcal{F}[h] = 0 & t > 0, s > 0, \\ h(t, s = 0) = D_{n^*} \mathcal{F}[h] = \int_0^\infty (p(s, N^*)h + n^* \partial_N p(s, N^*) D_{n^*} \mathcal{F}[h]) ds & t > 0, \\ \int_0^\infty h(t, s) ds = 0 & t \geq 0, \end{cases} \quad (3.64)$$

where  $D_{n^*} \mathcal{F}: L^1(0, \infty) \rightarrow \mathbb{R}$  is the differential of the map  $\mathcal{F}$  at the equilibrium  $n^*$ .

Thus, if we look for solution of the form  $h(t, s) = e^{\lambda t} \phi(s)$  and consider the normalization  $D_{n^*} \mathcal{F}[\phi(s)] = 1$ , we obtain the following eigenvalue problem

$$\begin{cases} \partial_s \phi + (p(s, N^*) + \lambda) \phi = -n^* \partial_N p(s, N^*) & s > 0, \\ 1 = \varphi(s = 0) = \int_0^\infty (p(s, N^*) \phi + n^* \partial_N p(s, N^*)) ds, \\ \int_0^\infty \phi(s) ds = 0. \end{cases} \quad (3.65)$$

By solving the differential equation for  $\phi$  and using the formula for  $n^*$ , we get the following equality

$$\varphi(s) = e^{-\lambda s} \frac{n^*(s)}{N^*} \left( 1 - N^* \int_0^s \partial_N p(s', N^*) e^{\lambda s'} ds' \right)$$

and by replacing this formula into the boundary condition in Equation (3.65), we get the nonlinear

equation for  $\lambda$  given by

$$\lambda \int_0^\infty n^*(s) e^{-\lambda s} \left( \frac{1}{N^*} - \int_0^s \partial_{NP}(s', N^*) e^{\lambda s'} ds' \right) ds = 0. \quad (3.66)$$

We observe that the trivial solution  $\lambda = 0$  corresponds to the mass conservation property. Determining the remaining solutions is a hard problem since complex solutions may arise and even the case of pure imaginary eigenvalues is not obvious to deal and the same difficult holds for multiple time renewal equations.

Furthermore, we conjecture that all the steady states solutions in the inhibitory case are linearly stable and the oscillating convergent solutions observed in Chapter 2 are due to complex eigenvalues with negative real part, while the monotone convergent solutions are due to real negative eigenvalues.

### Non-expansion principles

In the recent work of Fournier et al. [29] a non-expanding distance was constructed for several linear age-structured models for both classical and multiple time renewal equation, following the ideas of the coupling method for measures. An interesting question is to determine the existence of a non-expansion principle for the corresponding nonlinear models in order to better understand the dependence on the initial data.

### Connection with stochastic methods

Similar ideas on the coupling method have been developed in the work of Löcherbach et al. [41], which relates finite-size population stochastic models with population density equations that resemble the elapsed time model. Moreover, other techniques on stochastic processes include the work of Faugeras et al. [23] that connects dynamics of spikes trains with the stochastic version of the integrate-and-fire model, including numerical simulations via Monte Carlo method. In this context, an interesting problem is a further study between finite-size populations models and macroscopic continuous models.



# Appendix: Numerical methods

We now present the numerical schemes used in the simulations of previous chapters. We focus on adaptations of the well-known upwind scheme, which is widely applied in advection problems. For further reference on numerical methods for hyperbolic equations, see Bouchut [4], Godlewski et al. [32] and LeVeque [39].

## Classical elapsed time model

We start with the numerical scheme corresponding to the classical elapsed time Equation (1). We truncate the domain for the elapsed time  $s$  to the interval  $[0, L]$  with  $L > 0$  and we consider a uniformly spaced grid

$$s_i = i\Delta s, \quad \text{for } i = 1, \dots, M.$$

The initial data is given by the array  $(n^0)_{i=0}^M$ . The iterations in time are given by

$$t_k = k\Delta t, \quad k > 0$$

and we define  $n_i^k \approx n(t_k, s_i)$  as the numerical solution, with  $n_0^k = N^k \approx N(t_k)$  the approximate activity. From the explicit upwind scheme we get the discretization

$$\frac{n_i^{k+1} - n_i^k}{\Delta t} + \frac{n_i^k - n_{i-1}^k}{\Delta s} + p(s_i, N^{k-1})n_i^k = 0. \quad (67)$$

so that the numerical scheme is as follows.

---

**Algorithm 1** Explicit upwind scheme for classical elapsed time equation

---

```

 $N^{-1} \leftarrow \text{solve} \left( y = \Delta s \sum_{i=1}^M p(s_i, y) n_i^0 \right) \quad \triangleright \text{Initial guess for } N(0)$ 
for  $k = 0 : k_{\max}$  do
   $n_i^{k+1} \leftarrow n_i^k - \frac{\Delta t}{\Delta s} (n_i^k - n_{i-1}^k) - \Delta t p(s_i, N^{k-1}) n_i^k \quad \triangleright \text{Update } n$ 
   $N^k \leftarrow \Delta s \sum_{i=1}^M p(s_i, N^{k-1}) n_i^k + n_M^k \quad \triangleright \text{Update } N$ 
end for

```

---

An important property of this numerical scheme, which is easily verified, is the following.

**Lemma 12.** *The numerical Scheme 1 is mass-conservative i.e.*

$$\Delta s \sum_{i=1}^M (n_i^{k+1} - n_i^k) = 0.$$

In order to have this property, we added the term  $n_M^k$  in the formula for computing  $N^k$ . Thus we need to assure that  $n_M^k$  is small enough, which means that we need  $L$  large enough and assume that solution  $n$  vanishes at infinity.

Moreover, note that for all  $t \geq 0$  the activity  $N(t)$  satisfies the fixed point equation

$$y = \int_0^\infty p(s, y)n(t, s) ds. \quad (68)$$

The fixed point Equation (68) can be solved with Newton's method or gradient descend and this allows to compute in particular the initial guess for  $N(0)$ . In particular, the non linear equation for  $N(0)$  may have multiple solutions that determine different branches of solutions for the elapsed time Equation (1). The argument of fixed point may be extended for each iteration in time, which inspires the following variant of the numerical scheme

---

**Algorithm 2** Variant upwind scheme for classical elapsed time equation

---

```

 $N^{-1} \leftarrow \text{solve} \left( y = \Delta s \sum_{i=1}^M p(s_i, y)n_i^0 \right) \quad \triangleright \text{Calculate } N(0)$ 
for  $k = 0 : k_{\max}$  do
   $n_i^{k+1} \leftarrow n_i^k - \frac{\Delta t}{\Delta s} (n_i^k - n_{i-1}^k) - \Delta t p(s_i, N^k)n_i^k \quad \triangleright \text{Update } n$ 
   $n_i^{k+1} \leftarrow \frac{n_i^{k+1}}{\sum_{j=1}^M n_j^{k+1}} \quad \triangleright \text{Normalization}$ 
   $N^{k+1} \leftarrow \text{solve} \left( y = \Delta s \sum_{i=1}^M p(s_i, y)n_i^{k+1} \right) \quad \triangleright \text{Update } N$ 
end for

```

---

This scheme is more accurate in computing  $N(t)$  and no condition on  $n_M^k$  is required, but we have to normalize the solution for  $n$  at each iteration, so that we still need  $L$  large enough. Moreover, we may consider Scheme 2 with different types of discretizations for the corresponding transport equation, but we are still forced to normalize at each iteration.

## Elapsed time model with two times

We now present the numerical methods for the multiple time renewal Equation (2.1) by extending the ideas of the classical elapsed time model. We truncate the values of  $(s, a)$  to the set  $[0, L] \times [0, L]$  with  $L > 0$  and we consider a uniformly spaced grid

$$(s_i, a_j) = (i\Delta s, j\Delta a) \quad \text{for } 1 \leq i \leq j \leq M, \quad \Delta s = \Delta a.$$

The initial data is given by the matrix  $(n^0)_{i,j}$  of size  $M \times M$ . From the domain  $\mathcal{D}$  we impose the following support conditions

$$\begin{cases} n_{i,j}^0 = 0 & \text{if } j \leq i - 1, \\ n_{i,i-1}^0 = n_{i,M}^0 & \text{for } i = 1, \dots, M. \end{cases} \quad (69)$$

The iterations in time are given by

$$t_k = k\Delta t, \quad k > 0,$$

and we define  $n_{i,j}^k \approx n(t_k, s_i, a_j)$  as the numerical solution, with  $n_{0,j}^k = N_j^k \approx N(t_k, a_j)$  the approximate discharging flux and  $X^k \approx X(t_k)$  the approximate activity.

From the explicit upwind scheme we get the discretization

$$\frac{n_{i,j}^{k+1} - n_{i,j}^k}{\Delta t} + \frac{n_{i,j}^k - n_{i-1,j}^k}{\Delta s} + \frac{n_{i,j}^k - n_{i,j-1}^k}{\Delta a} + p(s_i, N^{k-1})n_i^k = 0, \quad (70)$$

so that the numerical scheme is as follows.

---

**Algorithm 3** Explicit upwind scheme for elapsed time equation with two times

---

```

 $X^{-1} \leftarrow \text{solve} \left( y = \Delta s \Delta a \sum_{i=1}^M \sum_{j=i}^M p(s_i, a_j, y) n_{i,j}^0 \right) \quad \triangleright \text{Initial guess for } X(0)$ 
 $N^{-1} \leftarrow \Delta s \sum_{l=j}^M p(a_j, s_l, X^{-1}) n_{jl}^0 \quad \triangleright \text{Initial guess for } N(0, a)$ 
for  $k = 0 : k_{\max}$  do
  if  $i \leq j$  then
     $n_{i,j}^{k+1} \leftarrow n_{i,j}^k - \frac{\Delta t}{\Delta s} (n_{i,j}^k - n_{i-1,j}^k) - \frac{\Delta t}{\Delta a} (n_{i,j}^k - n_{i,j-1}^k) - \Delta t p(s_i, X^{k-1}) n_{i,j}^k \quad \triangleright \text{Update } n$ 
  else
     $n_{i,j}^{k+1} \leftarrow 0 \quad \triangleright \text{Support condition}$ 
  end if
   $n_{i,i-1}^{k+1} \leftarrow n_{i,M}^{k+1}$ 
   $N_j^k \leftarrow \Delta s \sum_{l=j}^M p(a_j, s_l, X^{k-1}) n_{jl}^k + n_{j,j}^k \quad \triangleright \text{Update } N$ 
   $X^k \leftarrow \Delta a \sum_{j=1}^M N_j^k \quad \triangleright \text{Update } X$ 
end for

```

---

Analogously to Scheme 1, after a direct calculation we have following remarkable property.

**Lemma 13.** *The numerical Scheme 3 is mass-conservative i.e.*

$$\Delta s \Delta a \sum_{i=1}^M \sum_{j=i}^M (n_{i,j}^{k+1} - n_{i,j}^k) = 0.$$

Like Scheme 1, we need to assure that  $n_{j,j}^k$  is small enough i.e. the solution  $n$  vanishes at the diagonal  $\{(s, a) \in \mathcal{D} : s = a\}$ .

In the particular case when the firing rate  $p$  does not depend on the second elapsed time  $a$ , we can establish a relation between Schemes 1 and 3.

**Proposition 3.** *Assume that  $p = p(s, X)$  and let  $n_{i,j}^k$  be a numerical solution for Scheme 3, then  $\bar{n}_i^k := \Delta a \sum_{j=i}^M n_{i,j}^k$  is a numerical solution for Scheme 1.*

As in the classical elapsed time equation, we note that for all  $t \geq 0$  the total activity  $X(t)$  satisfies the fixed point equation

$$y = \int_0^\infty \int_s^\infty p(s, a, y) n(t, s, a) da ds. \quad (71)$$

Like Equation (68), this equation may have different solutions for  $X(0)$ , which determines different branches of solution for the extended system. Similarly to Scheme 2, we may adapt the fixed point argument to the extended model to get the numerical Scheme 4.

Like the variant Scheme 2, this scheme is more accurate in computing  $X(t)$  and no condition on the diagonal and sub-diagonal are required. However, this is at the cost of solving the fixed point Equation 71, which is more expensive computationally, and normalizing the solution for  $n$  at each iteration.

---

**Algorithm 4** Variant upwind scheme for elapsed time equation with two times
 

---

```

 $X^0 \leftarrow \text{solve} \left( y = \Delta s \Delta a \sum_{i=1}^M \sum_{j=i}^M p(s_i, a_j, y) n_{i,j}^0 \right) \quad \triangleright \text{Calculate } X(0)$ 
 $N_j^0 \leftarrow \Delta s \sum_{l=j}^M p(a_j, s_l, X^0) n_{jl}^0 \quad \triangleright \text{Calculate } N(0, a)$ 
for  $k = 0 : k_{\max}$  do
  if  $i \leq j$  then
     $n_{i,j}^{k+1} \leftarrow n_{i,j}^k - \frac{\Delta t}{\Delta s} (n_{i,j}^k - n_{i-1,j}^k) - \frac{\Delta t}{\Delta a} (n_{i,j}^k - n_{i,j-1}^k) - \Delta t p(s_i, X^k) n_{i,j}^k \quad \triangleright \text{Update } n$ 
  else
     $n_{i,j}^{k+1} \leftarrow 0 \quad \triangleright \text{Support condition}$ 
  end if
   $n_{i,j}^{k+1} \leftarrow \frac{n_{i,j}^{k+1}}{\Delta s \Delta a \sum_{i=1}^M \sum_{j=i}^M n_{i,j}^{k+1}} \quad \triangleright \text{Normalization}$ 
   $X^{k+1} \leftarrow \text{solve} \left( y = \Delta s \Delta a \sum_{i=1}^M \sum_{j=i}^M p(s_i, a_j, y) n_{i,j}^{k+1} \right) \quad \triangleright \text{Update } X$ 
   $N_j^{k+1} \leftarrow \Delta s \sum_{l=j}^M p(a_j, s_l, X^{k+1}) n_{jl}^{k+1} \quad \triangleright \text{Update } N$ 
end for

```

---

## Elapsed time model with learning

We now present the numerical methods for the elapsed time system with spatial dependence and learning (3.2). We truncate the domain for the elapsed time  $s$  to the interval  $[0, L]$  with  $L > 0$  and the domain for the position  $x$  is  $\Omega = [0, a]$ . We consider the respective uniformly spaced grids for  $[0, L] \times \Omega$  and  $\Omega \times \Omega$

$$\begin{aligned} (s_i, x_j) &= (i\Delta s, j\Delta x) & \text{for } i = 1, \dots, M, j = 1, \dots, P. \\ (x_j, y_l) &= (i\Delta x, l\Delta x) & \text{for } i, j = 1, \dots, P. \end{aligned}$$

The initial probability density is  $(n_0)_{i,j}$ , which a matrix of size  $M \times P$ , the initial connectivity kernel is  $(w_0)_{j,l}$ , which is a matrix of size  $M \times M$ , and  $I_j := I(x_j)$  is the given external input. The iterations in time are given by

$$t_k = k\Delta t, \quad k > 0.$$

and we define  $n_{i,j}^k \approx n(t_k, s_i, x_j)$  as the numerical solution,  $n_{0,j}^k = N_j^k \approx N(t_k, x_j)$  the approximate activity,  $S_j^k \approx S(t_k, x_j)$  the amplitude of stimulation and  $w_{j,l}^k \approx w(t_k, x_j, y_l)$  the connectivity kernel.

The idea is to solve the transport equation for each position  $x_j$ , adapting Scheme 2. From the explicit upwind scheme we get the following discretization for  $n$

$$\frac{n_{i,j}^{k+1} - n_{i,j}^k}{\Delta t} + \frac{n_{i,j}^k - n_{i-1,j}^k}{\Delta s} + p(s_i, S_j^k) n_{i,j}^k = 0 \quad (72)$$

and we consider the following discretization for  $w$

$$\frac{w_{j,l}^{k+1} - w_{j,l}^k}{\Delta t} = -w_{j,l}^{k+1} + \gamma G(N_j^k, N_l^k), \quad (73)$$

so that we get the numerical Scheme 5.

In order to computer the initial guess for  $S(0, x)$ , we use the fact that for all  $t \geq 0$  the amplitude of stimulation satisfies the following integral equation

**Algorithm 5** Upwind scheme for elapsed time equation with learning

---

```

 $(S^{-1})_{j=1}^P \leftarrow \text{solve} \left( Y_j = \Delta s \Delta x \sum_{i=1}^M \sum_{l=1}^P p(s_i, Y_l) w_{j,l}^0 n_{i,l}^0 + I_j \right) \triangleright \text{Initial guess for } S(0, x)$ 
 $N_j^{-1} \leftarrow \Delta s \sum_{i=1}^M p(s_i, S_j^0) n_{i,j}^0 \triangleright \text{Initial guess for } N(0, x)$ 
for  $k = 0: k_{\max}$  do
   $n_{i,j}^{k+1} \leftarrow n_{i,j}^k - \frac{\Delta t}{\Delta s} (n_{i,j}^k - n_{i-1,j}^k) - \Delta t p(s_i, S_j^{k-1}) n_{i,j}^k \triangleright \text{Update } n$ 
   $N_j^k \leftarrow \Delta s \sum_{i=1}^M p(s_i, S_j^{k-1}) n_{i,j}^k + n_{M,j}^k \triangleright \text{Update } N$ 
   $S_j^k \leftarrow \Delta x \sum_{l=1}^M w_{j,l}^k N_l^k + I_j \triangleright \text{Update } S$ 
   $w_{j,l}^{k+1} \leftarrow \frac{1}{1+\Delta t} \left( w_{j,l}^k + \Delta t \gamma G(N_j^k, N_l^k) \right) \triangleright \text{Update } w$ 
end for

```

---

$$S(t, x) = \int \int_0^\infty p(s, S(t, y)) w(t, x, y) n(t, s, y) ds dy + I(x), \quad (74)$$

so that we have to solve the corresponding system of equations to compute the vector  $S^{-1}$  in Scheme 5. Furthermore, Equation (74) may have multiple solutions, which determine different branches of solutions. We may modify Scheme 5 by solving Equation (74) at each time iteration for computing  $S(t, x)$  and normalizing the solution  $n$ . However, solving a large nonlinear system of equations at each iteration is computationally expensive.

Another property of Scheme 5 is the conservation of mass like Scheme 1.

**Lemma 14.** *The numerical scheme 5 is mass-conservative i.e.*

$$\Delta s \sum_{i=1}^M (n_{i,j}^{k+1} - n_{i,j}^k) = 0, \quad \forall j = 1, \dots, P.$$

Like Scheme 1, we need to assure that  $n_{M,j}^k$  is small enough for all  $j = 1, \dots, P$ . We may avoid this condition in the following numerical scheme at the cost normalizing the solution  $n$  with respect to elapsed time variable  $s$  at each time iteration, as we see in Scheme 6.

**Algorithm 6** Variant upwind scheme for elapsed time equation with learning

---

```

 $g_j \leftarrow \Delta s \sum_{i=1}^M n_{i,j}^0 \triangleright \text{Calculate } g(x) = \int n^0(s, x) ds$ 
 $(S^0)_{j=1}^P \leftarrow \text{solve} \left( Y_j = \Delta s \Delta x \sum_{i=1}^M \sum_{l=1}^P p(s_i, Y_l) w_{j,l}^0 n_{i,l}^0 + I_j \right) \triangleright \text{Calculate } S(0, x)$ 
 $N_j^0 \leftarrow \Delta s \sum_{i=1}^M p(s_i, S_j^0) n_{i,j}^0 \triangleright \text{Calculate } N(0, x)$ 
for  $k = 0: k_{\max}$  do
   $n_{i,j}^{k+1} \leftarrow n_{i,j}^k - \frac{\Delta t}{\Delta s} (n_{i,j}^k - n_{i-1,j}^k) - \Delta t p(s_i, S_j^k) n_{i,j}^k \triangleright \text{Update } n$ 
   $n_{i,j}^{k+1} \leftarrow \frac{g_j}{\Delta s \sum_{i=1}^M n_{i,j}^{k+1}} n_{i,j}^{k+1} \triangleright \text{Normalization}$ 
   $w_{j,l}^{k+1} \leftarrow \frac{1}{1+\Delta t} \left( w_{j,l}^k + \Delta t \gamma G(N_j^k, N_l^k) \right) \triangleright \text{Update } w$ 
   $N_j^{k+1} \leftarrow \Delta s \sum_{i=1}^M p(s_i, S_j^k) n_{i,j}^{k+1} \triangleright \text{Update } N$ 
   $S_j^{k+1} \leftarrow \Delta x \sum_{l=1}^M w_{j,l}^{k+1} N_l^{k+1} + I_j \triangleright \text{Update } S$ 
end for

```

---

Alternatively, we may also consider Scheme 6 with different explicit and implicit numerical method to solve the transport equation for  $n$ .



Concerning the limit System (3.43) of slow learning dynamics with  $\varepsilon = 0$ , we can get explicitly the solution for probability density  $n$  in terms of the activity and the amplitude of stimulation since there is no transport equation to solve. Hence, we may adapt the numerical Method 6 to get the following scheme for the limit System (3.43).

---

**Algorithm 7** Variant upwind scheme for the limit system with  $\varepsilon = 0$

---


$$g_j \leftarrow \Delta s \sum_{i=1}^M n_{i,j}^0 \quad \triangleright \text{Calculate } g(x) = \int n^0(s, x) ds$$

$$(S^0)_{j=1}^P \leftarrow \text{solve} \left( Y_j = \Delta s \Delta x \sum_{i=1}^M \sum_{l=1}^P p(s_i, Y_l) w_{j,l}^0 n_{i,l}^0 + I_j \right) \quad \triangleright \text{Calculate } S(0, x)$$

$$N_j^0 \leftarrow \Delta s \sum_{i=1}^M p(s_i, S_j^0) n_{i,j}^0 \quad \triangleright \text{Calculate } N(0, x)$$

**for**  $k = 1 : k_{\max}$  **do**

$$n_{i,j}^{k+1} \leftarrow N_j^k \exp \left( -\Delta s \sum_{l=1}^i p(s_l, S_j^k) \right) \quad \triangleright \text{Update } n$$

$$n_{i,j}^{k+1} \leftarrow \frac{g_j}{\Delta s \sum_{i=1}^M n_{i,j}^{k+1}} n_{i,j}^{k+1} \quad \triangleright \text{Normalization}$$

$$w_{j,l}^{k+1} \leftarrow \frac{1}{1+\Delta t} \left( w_{j,l}^k + \Delta t \gamma G(N_j^k, N_l^k) \right) \quad \triangleright \text{Update } w$$

$$N_j^{k+1} \leftarrow \Delta s \sum_{i=1}^M p(s_i, S_j^k) n_{i,j}^{k+1} \quad \triangleright \text{Update } N$$

$$S_j^{k+1} \leftarrow \Delta x \sum_{l=1}^M w_{j,l}^{k+1} N_l^{k+1} + I_j \quad \triangleright \text{Update } S$$

**end for**

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## ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF THE ELAPSED TIME MODEL FOR NEURAL ASSEMBLIES

### Abstract

The elapsed time model has been widely studied in the context of mathematical neuroscience with many open questions left. The model consists of an age-structured equation that describes the dynamics of interacting neurons described by their refractory period, i.e. the elapsed time since their last discharge. Our interest lies in weakly connected networks where mathematical tools such as the Entropy method and Doeblin's theory have proved to be useful methods in proving convergence to steady state, and highly connected networks leading to strong nonlinearities where classical perturbation methods do not apply. Moreover, we are interested in extensions of the elapsed time model by incorporating new variables such as the elapsed time since penultimate discharge and the spatial dependence with learning processes via the change in the interconnections.

To deal with this problem of strong interconnections in the elapsed time model, we choose a particular case which can be reduced to delay differential equations and we prove a general convergence result to a stationary state in the inhibitory and the weakly excitatory cases. In addition, we prove the existence of particular periodic solutions with jump discontinuities in the strongly excitatory case.

In the first extension of the classical elapsed time model we incorporate the elapsed time since the penultimate discharge, obtaining a more complex system of integro-differential equations. For this new system we prove convergence to stationary state by means of Doeblin's theory in the case of weak nonlinearities in an appropriate functional setting, inspired by the case of the classical elapsed time equation. We also aim to study how the incorporation of this new elapsed time may change the general dynamics of the system.

In the second extension we focus on incorporating the spatial dependence (e.g. position of neurons across the cortex) and some learning processes to the classical elapsed time model. The learning rules that we consider are essentially inspired from the Hebbian rule, which is a classical principle in modeling the process of learning. We then obtain a new system of integro-differential equations, from which we analyze the convergence to stationary states by the means of entropy method and Doeblin's theory in the case of weak interconnections. We also consider the situation where neural activity is faster than the learning process and give conditions where one can approximate the dynamics by a solution with a similar profile of a steady state.

For each model we present some numerical simulations to observe how the parameters of the system can give different behaviors and pattern formations in order to contrast them with the theoretical results.

**Keywords:** mathematical neuroscience, elapsed time, structured equations, neural networks, periodic solutions, delay differential equations, weak interconnections, learning rule, connectivity kernel, convergence to equilibrium, entropy method, doeblin theory.

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## Résumé

Le modèle du temps écoulé a été largement étudié dans le contexte des neurosciences mathématiques, mais de nombreuses questions restent ouvertes. Le modèle consiste en une équation structurée par âge qui décrit la dynamique des neurones en interaction décrite par leur période réfractaire, c'est-à-dire le temps écoulé depuis leur dernière décharge. Nous nous intéressons aux réseaux faiblement connectés pour lesquels des outils mathématiques tels que la méthode de l'entropie et la théorie de Doeblin se sont avérés utiles pour prouver la convergence vers un état stable, et aux réseaux hautement connectés conduisant à de fortes non-linéarités pour lesquels les méthodes classiques de perturbation ne s'appliquent pas. De plus, nous nous intéressons aux extensions du modèle de temps écoulé en incorporant de nouvelles variables telles que le temps écoulé depuis la pénultième décharge et la dépendance spatiale avec les processus d'apprentissage via le changement des interconnexions.

Pour traiter le problème des interconnexions fortes dans le modèle du temps écoulé, nous choisissons un cas particulier qui peut être réduit à des équations différentielles à retard et nous prouvons un résultat général de convergence vers un état stationnaire dans les cas inhibiteurs et faiblement excitateurs. En outre, nous prouvons l'existence de solutions périodiques particulières avec des discontinuités de saut dans le cas fortement excitateur.

Dans la première extension du modèle classique du temps écoulé, nous incorporons le temps écoulé depuis l'avant-dernière décharge, obtenant ainsi un système plus complexe d'équations intégro-différentielles. Pour ce nouveau système, nous prouvons la convergence vers l'état stationnaire au moyen de la théorie de Doeblin dans le cas de non-linéarités faibles dans un cadre fonctionnel approprié, inspiré du cas de l'équation classique du temps écoulé. Nous cherchons également à étudier comment l'incorporation de ce nouveau temps écoulé peut modifier la dynamique générale du système.

Dans la deuxième extension, nous nous concentrons sur l'incorporation de la dépendance spatiale (par exemple, la position des neurones dans le cortex) et de certains processus d'apprentissage au modèle classique du temps écoulé. Les règles d'apprentissage que nous considérons sont essentiellement inspirées de la règle de Hebb, qui est un principe classique dans la modélisation du processus d'apprentissage. Nous obtenons alors un nouveau système d'équations intégro-différentielles, dont nous analysons la convergence vers des états stationnaires au moyen de la méthode de l'entropie et de la théorie de Doeblin dans le cas d'interconnexions faibles. Nous considérons également la situation où l'activité neuronale est plus rapide que le processus d'apprentissage et nous donnons les conditions dans lesquelles on peut approximer la dynamique par une solution avec un profil similaire à un état stationnaire.

Pour chaque modèle, nous présentons quelques simulations numériques afin d'observer comment les paramètres du système peuvent donner des comportements et des formations de modèles différents afin de les contraster avec les résultats théoriques.

**Mots clés :** neurosciences mathématiques, temps écoulé, équations structurées, réseaux de neurones, solutions périodiques, équations différentielles à retard, interconnexions faibles, règle d'apprentissage, noyau de connectivité, convergence à l'équilibre, méthode d'entropie, théorie de doeblin.

