Université Bourgogne Franche-Comté
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HABILITATION À DIRIGER DES RECHERCHES

Spécialité : Mathématiques

présentée par

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Some aspects of group actions on manifolds

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Soutenue publiquement le 17 septembre 2021 devant le jury composé de :

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Abstract

This memoir contains a description of the recent work by the author and collaborators on groups acting on manifolds. We take this as an opportunity to review the state of the art on the study of *locally discrete* groups of real-analytic circle diffeomorphisms.

Remerciements

Tout d'abord, je remercie les trois rapporteurs Danny Calegari, Yves de Cornulier, et Louis Funar pour avoir pris le temps de lire ces pages (et travaux reliés), ainsi que pour leur commentaires. Je remercie aussi les autres membres du jury pour avoir accepté cette tache.

Mes meilleurs remerciements vont aux nombreux collaborateurs. Je garde dans mes meilleurs souvenirs les années passés à Rio de Janeiro, pendant lesquelles j'ai commencé à m'intéresser vraiment aux actions de groupes sur les variétés. Je n'ai pas eu l'opportunité de rendre plus concrets les échanges mathématiques avec d'autres collègues, notamment Bertrand Deroin, Isabelle Liousse, Emmanuel Militon, et Maxime Wolff, mais les discussions avec eux ont été toujours précieuses.

Je remercie aussi les institutions qui m'ont accueilli depuis ma thèse : la PUC de Rio de Janeiro, la UFF, l'Institut de Mathématiques de Bourgogne, l'INRIA Lille, et le Laboratoire Paul Painlevé. J'y ai toujours trouvé des bonnes conditions de travail et de collègues exemplaires. Un remerciement particulier va à Christian Bonatti pour ces dernières années de partage de bureau.

Enfin, un merci tout spécial à Marielle, Lorenzo et Barbara.

List of works by the author

In this memoir we present the main scientific production of the author after his PhD thesis (ENS–Lyon, May 2014), which mainly concerns the study of groups acting on manifolds. Most of these works have been obtained in collaboration with many colleagues, and I heartily thank all of them.

Works on regularity of group actions:

[1] C. Bonatti, Y. Lodha and M. Triestino – Hyperbolicity as an obstruction to smoothability for one-dimensional actions, *Geom. Topol.* **23**, no. **4** (2019), 1841–1876.

[2] Y. Lodha, N. Matte Bon and M. Triestino – Property FW, differentiable structures, and smoothability of singular actions. *J. Topol.* **13**, no. **3** (2020) 1119–1138.

[3] C. Bonatti, S.-h. Kim, T. Koberda and M. Triestino – Small C^1 actions of semidirect products on compact manifolds. *Algebraic & Geometric Topology* **16**, no. **6** (2020) 3183–3203.

Works on orderable groups:

[4] D. Malicet, K. Mann, C. Rivas and M. Triestino – Ping-pong configurations and circular orders on free groups, *Groups Geom. Dyn.* **13**, no. **4** (2019), 1195–1218.

[5] C. Rivas and M. Triestino – One-dimensional actions of Higman's group, *Discrete Analysis* 2019:20, 15pp.

[6] N. Matte Bon and M. Triestino – Groups of piecewise linear homeomorphisms of minimal flows. *Compositio Math.* **156**, no. **8** (2020) 1595–1622.

[7] M. Triestino – On James Hyde's example of non-orderable subgroup of $\mathrm{Homeo}(D, \partial D)$. *L'Enseignement Mathématique (2)* **66**, no. **3/4** (2020), 409–418.

[8] J. Brum, N. Matte Bon, C. Rivas, M. Triestino – Locally moving groups acting on the line and $\mathbb{R}$-focal actions. Preprint `arXiv:2104.14678`

Works on locally discrete groups of circle diffeomorphisms:

[9] S. Alvarez, D. Filimonov, V. Kleptsyn, D. Malicet, C. Meniño Cotón, A. Navas and M. Triestino – Groups with infinitely many ends acting analytically on the circle, *J. Topol.* **12**, no. **4** (2019), 1315–1367.

[10] J. Alonso, S. Alvarez, D. Malicet, C. Meniño Cotón, and M. Triestino – Ping-pong partitions and locally discrete groups of real-analytic circle diffeomorphisms, I: Construction. Preprint `arXiv:1906.03578`

[11] S. Alvarez, P. Barrientos, V. Kleptsyn, D. Malicet, C. Meniño Cotón, and M. Triestino – Ping-pong partitions and locally discrete groups of real-analytic circle diffeomorphisms, II: Applications. Preprint `arXiv:2104.03348`

Other works by the author not discussed in this memoir (these are more related to the author's PhD. Thesis).

[12] A. Navas and M. Triestino – On the invariant distributions of C^2 circle diffeomorphisms of irrational rotation number, *Math. Z.* **274**, no. **1** (2013), 315–321.

[13] M. Triestino – Généricité au sens probabiliste dans les difféomorphismes du cercle, *Ensaios Matemáticos* **27**, Soc. Brasil. Mat. (2014), 1–98.

[14] M. Khristoforov, V. Kleptsyn and M. Triestino – Stationary random metrics on hierarchical graphs *via* $(\min, +)$-type recursive distributional equations, *Commun. Math. Phys.* **345**, no. **1** (2016), 1–76.

[15] V. Kleptsyn and M. Triestino – Cut-off method for endogeny of recursive tree processes. Preprint `arXiv:1610.06946`. *Although the main results are correct, this work contains several erroneous statements, and it needs a general revision.*

(Collective) survey works around Zimmer's program:

[16] *Entropy, Lyapunov exponents, and rigidity of group actions.* Main text by A. W. Brown, with 4 appendices by D. Malicet, D. Obata, B. Santiago and M. Triestino, S. Alvarez and M. Roldán. Edited by M. Triestino. *Ensaios Matemáticos* **33** (by Brazilian Mathematical Society), 1–197.

[17] Arbeitsgemeinschaft: Zimmer's Conjecture, *Oberwolfach Rep.* **16** (2019), 2951–3052. Title of contribution: *Proof of Zimmer's Cocycle Superrigidity: centralizers and finite dimensional invariant subspaces.*

[18] M. Triestino – La conjecture de Zimmer, *La Gazette des Mathématiciens* **169** (Juillet 2021), 8–21.

Part 1. Summary of works

1. Context

Since my post-doc at PUC of Rio de Janeiro (2014) I have been drawing the attention to the study of *groups acting on manifolds*, from the point of view of dynamical systems, which is a subject that has well developed in the last 60 years, with substantial contributions and new methods appearing in these last decades. With this connotation, classical (invertible) dynamical systems fall into this category, as they correspond to actions of $\mathbb{Z}$ or $\mathbb{R}$. The subject has two roots: *foliation theory* and *hyperbolic dynamical systems*, and both have still great influence on the scientific production. Moreover, in the course of its development, the subject has been contaminated by the expansion of *geometric group theory*, and the study of discrete subgroups of Lie groups, and now sits at the interface with all these more classical fields.

There are different approaches to the study of group actions, but the typology of questions mainly are of the following form:

(a) Given a countable group G (typically finitely generated) and a manifold M, *are there faithful actions of G on M?* That is, is there a subgroup of $\mathsf{Homeo}(M)$ which is isomorphic to G? What are the *distinct actions* of G on M, i.e. up to conjugacy in $\mathsf{Homeo}(M)$? What about actions of G that preserve an *extra structure* on M, such as a C^r differentiable structure, a volume form, a foliation, a geometric structure... ?
(b) Suppose that the group G is unknown, but *assumptions on the dynamical behavior of the action* are given, such as equicontinuity properties, or on the entropy, or on the invariant subsets... Is it possible to extrapolate information about the *group structure* of G?

For the particular case of actions on *one-dimensional manifolds* (i.e. circle, closed interval, or real line) there have already been many developments, so that nowadays the theory is well structured and is possible to work on *specific fundamental questions*. This has been possible because of two essential reasons: (1) a totally ordered structure on the manifold and (2) a very satisfactory description of $\mathbb{Z}$-actions. In contrast, the state-of-the-art for actions on higher-dimensional manifolds is still rudimentary.

2. Regularity of group actions

The regularity of the action constitutes a fundamental aspect, which can have a dramatic influence on its qualitative behavior. Although this aspect is somehow well-understood for one-dimensional actions, only sparse results are available for higher-dimensional actions. The general questions are:

(1) Assume a group G admits a faithful C^r action on some manifold M, can we make the action more regular after some change of coordinates? That is, if $G \leq \mathsf{Diff}^r(M)$, can we conjugate G by $h \in \mathsf{Homeo}(M)$ so that $hGh^{-1} \leq \mathsf{Diff}^s(M)$ for some $s > r$?
(2) Similarly, if a group admits a faithful C^r action on some manifold M, can we find an action which is more regular (possibly not conjugate)? That is, if $\rho : G \to \mathsf{Diff}^r(M)$ is a faithful homomorphism, is there a faithful homomorphism $\rho' : G \to \mathsf{Diff}^s(M)$, for some $s > r$?

The classical example confirming that regularity matters is given by Denjoy's theorem, which states that every $\mathbb{Z}$-action on the circle by C^2 diffeomorphisms, without periodic orbits must be minimal (every orbit is dense), whereas counterexamples exist in differentiability class C^r for every $r < 2$ (see [Her79]). Of similar flavor is the classical result by Kopell according to which $\mathbb{Z}$-actions on the closed interval by C^2 diffeomorphisms have small centralizers, which fails in lower regularity (see for instance [BF15] in case of C^1 regularity). This has been exploited to deduce several important results, for instance that the only nilpotent groups which admit C^2 actions on the interval are metabelian (see [Nav11]). Let us also mention the striking recent work by Kim and Koberda [KK20] which provides, for every $r \geq 1$, and example of finitely generated subgroup of $\mathsf{Diff}^r_+(\mathbb{S}^1)$ which does not embed in $\mathsf{Diff}^s_+(\mathbb{S}^1)$ for every $s > r$ (see [MW19] for an alternative construction).

It is already interesting to study the demarcation between C^0 and C^1 regularity. On the one hand C^0 actions on the real line are of particular interest, as they correspond to left-invariant orders on the group (a notion of algebraic nature). These two notions coincide for countable groups (a classical result due to Conrad). Studying such actions provides a dynamical point of view on the theory of *left-orderable groups* (see the monograph [DNR16]). On the other hand, the fact that a group admits a faithful C^1 action on the closed interval entails, by Thurston's Stability Theorem, that the group is *locally indicable*, that is, every nontrivial finitely generated subgroup surjects to $\mathbb{Z}$ (this is again a notion of purely algebraic nature). We will come back later to this algebraic counterpart.

However, Thurston's Stability Theorem does not characterize locally indicable groups, in the sense that there are examples of locally indicable groups which admit no faithful C^1 action on the closed interval. A very concrete and elementary family of examples is given by the solvable Baumslag–Solitar groups $\mathsf{BS}(1,-n)$, with $n \geq 2$, as one can deduce from the work of Cantwell and Conlon [CC02] or Guelman and Liousse [GL11] (although this is not explicitly stated). Interesting examples are described in [Cal08, Nav10]. Motivated by the surprising works by Monod [Mon13] and Lodha–Moore [LM16], pointing out a large class of nonamenable groups without free subgroups, with Bonatti and Lodha, we considered in [1] the family of groups acting by piecewise projective homeomorphisms of the real line (with finitely many breakpoints). These groups are indeed locally indicable, and they include the more classical groups acting by piecewise linear homeomorphisms (the most famous example is represented by Thompson's group F, although this specific group is not covered by the results of [1]). Concretely, we can consider the following groups extensively studied by Bieri–Strebel [BS16].

Definition 2.1. Fix an interval $X \subset \mathbb{R}$, and let $\Lambda \leq \mathbb{R}^*_+$ be a nontrivial multiplicative subgroup and take a nontrivial $\mathbb{Z}[\Lambda]$-submodule $A \subset \mathbb{R}$. Then the *Bieri–Strebel group* $G(X; A, \Lambda)$ is the group of all piecewise linear homeomorphisms of X with finitely many breakpoints, in A, and locally of the form $x \mapsto \lambda x + a$, with $\lambda \in \Lambda$ and $a \in A$.

For instance, the group $G([0,1]; \mathbb{Z}[1/2], \langle 2\rangle)$ is exactly Thompson's group F. When $1 < n_1 < \ldots < n_k$ are natural numbers such that the subgroup $\Lambda = \langle n_i \rangle \leq \mathbb{R}^*_+$ has rank k, and $A = \mathbb{Z}[1/m]$ with $m = \operatorname{lcm}(n_i)$, then we obtain the so-called Brown–Thompson–Stein group $F_{n_1,\ldots,n_k} = G([0,1]; A, \Lambda)$. We are particularly interested in the following particular examples of Bieri–Strebel groups: for $\lambda \in \mathbb{R}^*_+$, let $\Lambda = \langle \lambda \rangle$ and $A = \mathbb{Z}[\Lambda]$, and consider the group $G_\lambda = G(\mathbb{R}; A, \Lambda)$. We can now state some of the main results from our work [1].

Theorem 2.A. *For every $\lambda > 1$, there is no faithful C^1 action of G_λ on the closed interval.*

Remark 2.2. This result was obtained only for a particular class of algebraic $\lambda > 1$ in [1], but it actually holds for every $\lambda > 1$, as we discuss in [8], using different (and more general) methods, on which we will come back later when discussing locally moving groups.

This implies that for every subring $A \subset \mathbb{R}$ containing an invertible $\lambda > 1$, Monod's group $H(A)$ has no C^1 action on the closed interval. Recall that $H(A)$ is defined as the group of piecewise projective homeomorphisms of $\mathbb{R}$ whose breakpoints are fixed points for hyperbolic elements of $\mathsf{PSL}(2, A)$, and are locally given by elements of $\mathsf{PSL}(2, A)$. It also gives that the non-amenable finitely presented subgroup of $H(\mathbb{Z}[\sqrt{2}])$ considered by Lodha–Moore has no faithful C^1 action.

Theorem 2.B. *When $k \geq 2$, the standard action of a Brown–Thompson–Stein group $F_{n_1,\ldots,n_k}$ on the closed interval is not semi-conjugate to any $C^{1+\alpha}$ action.*

Remark 2.3. This result was proved in [1] for C^2 actions, but the same strategy can be improved so that we can simply assume Hölder continuity of derivatives, as pointed out in [8]. Moreover, in [1] we were simply considering actions which are *conjugate* to the standard action.

A technical difference between Theorem 2.B and Theorem 2.A is that Brown–Thompson–Stein groups are defined as groups acting on the closed interval, and thus do not contain affine transformations. It turns out that actions of groups of affine transformations on the line are quite rigid. More precisely, a fundamental work of Bonatti, Monteverde, Navas, and Rivas [BMNR17] describes some form of C^1 rigidity for actions of the solvable Baumslag–Solitar groups $\mathsf{BS}(1, n)$, $|n| \geq 2$ (and more general abelian-by-cyclic groups). This is also a key ingredient for other results in the field (such as the already mentioned [KK20, MW19]). For $\lambda > 1$, consider the affine group A_λ, generated by $a : x \mapsto \lambda x$ and $b : x \mapsto x + 1$. Then in [BMNR17, §4.2] it is proved the following.

Theorem 2.4 (Rigidity of multipliers)**.** *For every $\lambda > 1$, if a C^1 action of A_λ is semi-conjugate to the standard action then it is actually conjugate, and the derivative of the element a at its unique fixed point in $(0, 1)$ is exactly λ.*

A necessary step to obtain Theorem 2.A is to show that the standard action of G_λ on the line cannot be conjugate (and hence semi-conjugate) to any differentiable action on the closed interval. This is a nice consequence of Theorem 2.4: consider the element a_+ in G_λ which coincides with a on $(0, +\infty)$ and is trivial on $(-\infty, 0)$. Then if the standard action of G_λ is conjugate to a C^1 action, then the derivative of a_+ at the point corresponding to 0 should be at the same time λ and 1, which is an absurd.

Let us take the opportunity to present the following related problem raised in [8].

Conjecture 2.5. *Let $G \leq \mathsf{PL}_+([0, 1])$ be a subgroup. Then the action of G on $[0, 1]$ is C^0 conjugate to a C^1 action if and only if G is isomorphic to a subgroup of Thompson's group F.*

Note that the action of Thompson's F is C^0 conjugate to a C^∞ action on the closed interval [GS87].

Part of the motivation for Theorem 2.B, which leads to the presentation of the content of [2], was the well-known problem of finding a left-orderable group admitting *Kazhdan's property* (T), which can be considered as part of Zimmer's program, which is about actions of "large" groups on low-dimensional manifolds (although Kazhdan groups are not necessarily "large", for instance they can be hyperbolic). Observe that by Thurston's Stability Theorem, such groups do not admit faithful actions on the closed interval by C^1 diffeomorphisms; more generally, locally indicable groups cannot have Kazhdan's property (T). Also, Navas proved [Nav02] that infinite Kazhdan groups admit no faithful actions on the circle by $C^{3/2}$ diffeomorphisms. Navas proposed that good candidates for such examples could be found among groups acting by piecewise linear homeomorphisms of the circle (and their lifts to the real line), such as

Brown–Stein–Thompson's groups. The fact that the actions of these groups do not admit more regular realizations, as proved in [1], was keeping Navas's question open. In collaboration with Lodha and Matte Bon, we found a different approach to the question, which revealed to be of broad generalization. Recall that a countable group G has Kazhdan's property (T) if every isometric action on a Hilbert space has a fixed point. In [2], a very explicit (and very natural) example of isometric action of $\mathsf{PL}_+(\mathbb{S}^1)$ on the Hilbert space $\ell^2(\mathbb{S}^1)$ is given, with the property that every non-abelian subgroup acts without fixed points. This was enough to give a negative answer to Navas' question, but the construction was using heavily the fact that the group is of piecewise linear homeomorphisms. To generalize the result to a broader class of groups, we considered a different fixed-point property, called FW, which is for actions on spaces with walls, or equivalently, for actions on CAT(0) cube complexes. The equivalent notion that we consider is in terms of *commensurating actions*. Given a countable group G acting on a set X, we say that a subset $A \subset X$ is *commensurated* if the symmetric difference $gA \triangle A$ is finite for very $g \in G$. We also say that A is *transfixed* if there exists a G-invariant subset $B \subset X$ such that the symmetric difference $A \triangle B$ is finite. The group G has property FW if for every action on a set X, every commensurated subset is transfixed. Observe that this notion is purely set-theoretical. Countable groups with Kazhdan's property (T) also have property FW, but the converse is not true (for instance the group $\mathsf{SL}(2, \mathbb{Z}[\sqrt{2}])$ has FW).

Property FW turned out to be well-suited to study actions of countable groups on closed manifolds (of any dimension!). For the statement, we say that a homeomorphism $h : M \to M$ is a *countably singular C^r diffeomorphism* if there exists an open subset $U \subset M$ whose complement is countable, such that the restriction $h|_U$ is a C^r diffeomorphism. We denote by $\Omega\mathsf{Diff}^r(M)$ the group of such homeomorphisms. A nontechnical version of the main result of [2] can be stated as follows.

Theorem 2.C. *Let M be a closed manifold and let G be a finitely generated group with property* FW*. For every homomorphism $\rho : G \to \Omega\mathsf{Diff}^r(M)$, one of the following holds:*

(1) the action of $\rho(G)$ on M has a finite orbit;
(2) there exists a closed manifold N and a countably singular C^r diffeomorphism $\varphi : M \to N$ such that $\varphi\rho(G)\varphi^{-1} \subset \mathsf{Diff}^r(N)$.

Roughly speaking, for aperiodic countably singular actions of a group with property FW, upon changing the differentiable structure of the manifold, we can remove all singularities. We have to mention that similar ideas have been developed independently by Cornulier [Cor18], who largely worked on property FW.

We finally mention the work in collaboration with Bonatti, Kim, and Koberda [3]. It deals with actions on compact manifolds M (of any dimension!) of a particular class of groups, those which are semidirect products $G = H \rtimes_\psi \mathbb{Z}$, with H finitely generated, so that the generator of the $\mathbb{Z}$ factor acts on H by some automorphism $\psi \in \mathsf{Aut}(H)$. (As motivating example one can take as H the fundamental group of a compact surface Σ, so that H identifies with the fibered 3-manifold defined by the suspension of a homeomorphism of Σ.) Usually a group G of this form should admit many different actions on compact manifolds M by C^1 diffeomorphisms. Quite surprisingly, we prove that in some circumstances, the trivial action admits very few C^1 deformations.

Theorem 2.D. *Let $G = H \rtimes_\psi \mathbb{Z}$, with H finitely generated, be generated by a finite subset $S \subset G$, and assume that ψ induces a nontrivial hyperbolic automorphism $\psi^* : H^1(H, \mathbb{R}) \to H^1(H, \mathbb{R})$ (that is, as a linear map, ψ^* has no eigenvalue on the unit circle). Then there* *exists a C^1 neighborhood $\mathcal{U}$ of the identity map in $\mathsf{Diff}^1(M)$ such that for any representation $\rho : G \to \mathsf{Diff}^1(M)$ with $\rho(s) \in \mathcal{U}$ for every $s \in S$, one has* $\ker \rho \supset H$.

In other words, if the automorphism ψ^* is hyperbolic, all sufficiently C^1 small deformation of the trivial action, quotient through the action of a cyclic group. This extends a previous result by McCarthy [McC10], who proved the same for H finite rank abelian. A quite striking comment is that the result above fails to be true for $G = H \rtimes_\psi \mathbb{Z}$ with H residually torsion-free nilpotent and $\psi^* : H^1(H, \mathbb{R}) \to H^1(H, \mathbb{R})$ unipotent. In this case the group G itself is residually torsion-free nilpotent and such groups admit faithful actions on the closed interval by C^1 diffeomorphisms, which are arbitrarily closed to the identity [FF03].

The result above relies on an estimate of Bonatti, that we call *approximate linearization*, which we present here as it appears in [BMNR17]:

Lemma 2.6. *Let $M \subset \mathbb{R}^N$ be a compact submanifold. For any $\eta > 0$ and $k \in \mathbb{N}$ there exists a neighborhood $\mathcal{V}$ of the identity in $\mathsf{Diff}^1(M)$ (with respect to the C^1 topology), such that for any $x \in M$, and $f_1, \ldots, f_k \in \mathcal{V}$, and $\epsilon_1, \ldots, \epsilon_k \in \{-1, 1\}$, one has*

$$\left\| f_k^{\epsilon_k} \cdots f_1^{\epsilon_1}(x) - x - \sum_{i=1}^{k} \epsilon_i \left(f_i(x) - x\right) \right\| \le \eta \max_{i=1,\ldots,k} \|f_i(x) - x\| . \tag{2.1}$$

In other words, for diffeomorphisms sufficiently close to the identity, the displacement of a composition is comparable with the sum of the single displacements.

Instead of presenting the proof of Theorem 2.D in full generality, let us explain the main mechanism, and in particular how hyperbolicity of ψ^* is used. We will discuss the slightly different, but instructive case, of endomorphism ψ which induces the doubling map on cohomology. For this, we consider the group $G = \mathsf{BS}(1, 2) = \langle h, t \mid tht^{-1} = h^2 \rangle$. Observe that, for any $n \in \mathbb{N}$, the relation in the group gives

$$t^n h t^{-n} = h^{2^n}.$$

Using this relation, choosing $k = 2^n$ in Lemma 2.6, for any $\eta > 0$ we can find a neighborhood $\mathcal{V}$ of the identity in $\mathsf{Diff}^1(M)$ such that if $t, h \in \mathcal{V}$, then both conditions hold:

$$\left\| (h^{2^n}(x) - x) - 2^n (h(x) - x) \right\| \le \eta \max\{\|h(x) - x\|, \|t(x) - x\|\},$$
$$\left\| (h^{2^n}(x) - x) - (h(x) - x) \right\| \le \eta \max\{\|h(x) - x\|, \|t(x) - x\|\}.$$

This implies that the displacement $\|h(x) - x\|$ must be comparable to $2^n \|h(x) - x\|$, which, for sufficiently large n, forces $\|h(x) - x\| = 0$. In other words, $h \in \mathcal{V}$ must be the identity.

Following McCarthy [McC10], for the general case when the map ψ^* is hyperbolic, with nontrivial stable and unstable subspaces, then one argues essentially in the same way, using projections to these subspaces.

3. Contributions to the theory of left-orderable groups

Recall that a group G is *left-orderable* (LO for short) if it admits a total order $\le$ which is invariant under left-multiplication: $x \le y$ implies $gx \le gy$ for every x, y and $g \in G$. There is an analogous notion of right-order, and total orders which are at the same time left- and right-invariant are called bi-invariant orders. The theory of LO groups is tightly related to the study of group actions on the real line. The whole group $\mathsf{Homeo}_+(\mathbb{R})$ of orientation-preserving homeomorphisms is LO, as well as its subgroups (a subgroup of a LO group is LO). Conversely, every *countable* LO group is isomorphic to a subgroup of $\mathsf{Homeo}_+(\mathbb{R})$. This correspondence is even richer, as the choice of a left-order on a countable group G determines a faithful action of G on the real line (there are many possible choices for such an action, but the semi-conjugacy class of the action is not affected by the choice), and conversely every action of G on the real line determines a left-order on G (typically the same action determines more than one left-order, as this depends on many choices). A clear obstruction to left-orderability is provided by torsion, but there are no other obvious obstructions. Indeed, it is usually a nontrivial problem to determine whether a given group is left-orderable or not. If classical groups like F_n or $\mathbb{Z}^d$ are easily seen to be left-orderable, the task is in general more difficult. In [Hyd19], James Hyde gave a remarkable example of finitely generated, non-left-orderable subgroup of the group of homeomorphisms of the disk fixing the boundary $\mathsf{Homeo}(D,\partial D)$ (this group is torsion-free); see [7] for a personal dynamical interpretation of Hyde's proof.

As we have already mentioned, groups which are locally indicable are left-orderable, but the converse is not true. There are, for instance, examples of left-orderable groups which are *perfect* (ie with trivial abelianization), although only a very restrictive class of such examples is known. In the work [5] in collaboration with Rivas, we considered the classical *Higman's group*

$$H_4=\left\langle a_1,a_2,a_3,a_4 \mid a_ia_{i+1}a_i^{-1}=a_i^2 \quad \text{for } i\in\mathbb{Z}/4\mathbb{Z}\right\rangle.$$

It is evident from the presentation that H_4 is perfect, but it is not directly clear that H_4 is infinite. Recall that Higman introduced the group H_4 to obtain the first example of finitely generated infinite simple group: although H_4 is not itself simple, it admits no nontrivial finite quotient, so the quotient by a maximal normal subgroup gives the desired example.

Theorem 3.A. *Higman's group H_4 is left-orderable.*

See Figure 3.1 for a picture of an action of H_4 (which also gives a "dynamical" proof that H_4 is infinite!). Note that we are not able to prove that such action is faithful, but anyhow we show, using Kurosh subgroup theorem, that the kernel is a free group, so that it is possible to make a blow up of the action and insert an action of the kernel.

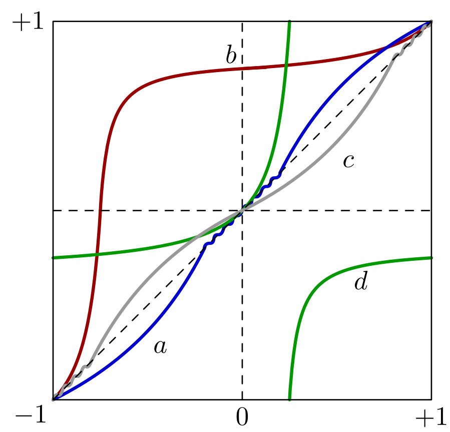

FIGURE 3.1. An action of Higman's group H_4 on the circle $\mathbb{R}/2\mathbb{Z}$ (which lifts to an action on the line).

In fact, finding examples of *finitely generated infinite simple groups* which are left-orderable was an important open problem in the theory of ordered groups, and goes back to Rhemtulla, around 1982. The first examples were manufactured by Hyde–Lodha [HL19]. In the work [6] written in collaboration with Matte Bon, we introduce a systematic construction of finitely generated simple groups of homeomorphisms of the real line that generalizes and clarifies the examples of Hyde–Lodha.

The change of paradigm idea is to consider "quasi-periodic lifts" of Thompson's group T to homeomorphisms of the real line, which is a finitely generated simple group of circle homeomorphisms. For this, let (X,σ) be a minimal subshift and let $Y = X \times \mathbb{R}/_{(\omega,t)\sim(\sigma(\omega),t-1)}$ denote the associated mapping torus, which is a compact connnected topological space, locally homeomorphic to Cantor $\times$ interval. We introduce the group $\mathsf{T}(\sigma)$ as the group of homeomorphisms of Y which are locally (in restriction to charts clopen $\times$ interval) of the form $\mathsf{id}\times$ piecewise linear dyadic homeomorphism. These groups $\mathsf{T}(\sigma)$ are seen to be finitely generated and simple by elementary methods known to experts. Moreover, the group $\mathsf{T}(\sigma)$ preserves every orbit of the vertical flow Φ defined on the mapping torus Y, and the restriction of the action to any Φ-orbit defines a faithful action of the group of the real line.

Theorem 3.B. *For every minimal subshift (X,σ), the group $\mathsf{T}(\sigma)$ is finitely generated, simple, and left-orderable.*

Many further properties for $\mathsf{T}(\sigma)$ are easily obtained.

Theorem 3.C. *Every action of $\mathsf{T}(\sigma)$ on the circle has a fixed point.*

The groups $\mathsf{T}(\sigma)$ constitute the first examples with this property. As a consequence we answer a technical, but fundamental, question of Deroin–Navas–Rivas on the nature of possible orders on/actions on the real line of a left-orderable group (see [DNR16, Question 3.5.11] and [Nav18, Question 6]). This was independently solved by Hyde–Lodha–Navas–Rivas [HLNR21].

Theorem 3.D. *The groups $\mathsf{T}(\sigma)$ are not finitely presented.*

Finding examples of left-orderable finitely presented infinite simple groups is still an open problem. Also, these groups do not help with the problem of finding Kazhdan groups acting on the line (see the discussion in the previous section).

Theorem 3.E. *The groups $\mathsf{T}(\sigma)$ do not have Kazhdan's property (T).*

Let us also point out that these groups definitely constitute a large class of examples.

Theorem 3.F. *Two different groups $\mathsf{T}(\sigma)$ are isomorphic if and only if the corresponding mapping tori are homeomorphic. Therefore the family of groups $\mathsf{T}(\sigma)$ provides uncountably many non-isomorphic examples of finitely generated simple left-orderable groups.*

Beyond these particular results and problems, the work [6] provides a new setting to study actions on the real line: "compactifying" the real line with the mapping torus, allows to use many classical tools which usually require compactness. A similar, but much more abstract, approach was previously developed by Deroin [Der20], about which we will also discuss in the next section. Let us point out the following problem from [6].

Conjecture 3.1. *Every minimal faithful action of $\mathsf{T}(\sigma)$ of the real line is conjugate to the restriction to a Φ-orbit of the standard action on the suspension Y.*

A second natural problem about left-orderable groups is to understand how many different left-orders can be defined on a LO group G. A left-order can be considered as an element of the power set $\{-1,0,1\}^{G\times G}$ (by looking the "sign" of $x^{-1}y$ for $x,y\in G$) and with this identification, the set of left-orders $\mathrm{LO}(G)$ becomes a compact Hausdorff topological space which is moreover totally disconnected. A result by Linnell asserts that the space $\mathrm{LO}(G)$ is always finite or uncountable, and groups admitting finitely many left-orders have been classified by Tararin (they are in particular polycyclic). See the recent monograph on the subject [DNR16]. One of the basic questions is then to understand when the space $\mathrm{LO}(G)$ admits isolated points. The topology of $\mathrm{LO}(G)$ is such that isolated left-orders are exactly those which are *finitely determined*: these are the left-orders which are uniquely determined by looking at the order relations between some finite collection of elements.

To give just a couple of examples, it is not difficult to see that $\mathrm{LO}(\mathbb{Z}^d)$ is a Cantor set (for $d \geq 2$) and one can also prove the same for the space of left-orders of a nonabelian free group F_n. Finding examples of groups G admitting isolated orders and for which $\mathrm{LO}(G)$ is uncountable, is usually much harder. In collaboration with Malicet, Mann and Rivas, we proved in [4] the quite surprising fact that the direct product $F_n \times \mathbb{Z}$ admits isolated left-orders if and only if n is even. Note that $F_n \times \mathbb{Z} \subset F_m \times \mathbb{Z}$ for n odd and m even, so that a consequence of our result is that the property of admitting isolated left-orders does not behave well when passing to finite-index subgroups. The proof of the main result of [4] passes through the study of the analogous space of *circular orders* $\mathrm{CO}(F_n)$ (which corresponds to actions on the circle). Here the isolated circular orders must be realized by actions on the circle of ping-pong type, which are related to the works [10, 11].

4. Locally moving groups and $\mathbb{R}$-focal actions

4.1. **First results for locally moving groups.** The most recent project [8], in collaboration with Brum, Matte Bon, and Rivas, was vaguely motivated by Conjecture 3.1 (for which now we have a partial solution). The general problem is to study a large class of groups for which it is possible to describe all actions on the real line up to semi-conjugacy. For instance, before this project, very few actions of Thompson's group F were known, and we will see that now we are able to give a rather precise picture. Again, part of the difficulty of the problem is that the real line is non-compact. Indeed, when the group has nontrivial center, it is often possible to reduce the problem to the circle, where the situation is simpler (for instance, the bounded Euler class is a complete invariant). By such methods, we could remark in [6] that the lift $\widetilde{T}$ of Thompson's group T admits only one action up to semi-conjugacy (two, if one takes orientation into account); similarly Militon proved [Mil16] that the group of homeomorphisms of the real line commuting with integer translations (that is, the central lift of $\mathsf{Homeo}_+(\mathbb{S}^1)$) has only one action on the real line.

The class of groups we will consider is the following.

Definition 4.1. A subgroup $G \subset \mathsf{Homeo}_+(\mathbb{R})$ is *locally moving* if for every open interval $I \subset \mathbb{R}$, the subgroup of elements of G supported on I acts on it without fixed points.

As a consequence of deep results by Rubin [Rub89], every group admits at most one locally moving action on the line, up to conjugacy. In fact, studying the structure of actions of locally moving groups, and using the approximate linearization (Lemma 2.6), we were able to prove the following.

Theorem 4.A. *Let $G \subset \mathsf{Homeo}_+(\mathbb{R})$ be a locally moving group, and let $\varphi : G \to \mathsf{Diff}^1_+(\mathbb{R})$ be a minimal faithful C^1 action on the real line. Then φ is conjugate to the standard action of G.*

A more precise version of the statement above also describes C^1 actions which are not minimal. Note that the standard action of G may fail to be conjugate to a C^1 action, as we discussed in Section 2, in which case we obtain as a consequence that the group has no minimal faithful C^1 actions at all.

However, the rigidity displayed in Theorem 4.A fails in the C^0 setting. Perhaps the simplest way to give counterexamples is to consider countable groups of compactly supported homeomorphisms. For such groups, one can always obtain *exotic actions*, in the sense that they are not induced from quotients, nor semi-conjugate to the standard action. For instance, one can consider a left-invariant order on the group of germs of homeomorphisms fixing a point, and declare that an element is positive if the germ at its rightmost point of the support is positive. This defines a left-invariant order on the group and thus an action on the real line.

Proposition 4.2. *Let $G \subset \mathsf{Homeo}_+(\mathbb{R})$ be a countable group of compactly supported homeomorphisms acting minimally. Then G admits actions without fixed points on $\mathbb{R}$ which are not semi-conjugate to its standard action, nor to any non-faithful action of G.*

While this observation is formally sufficient to rule out the C^0 version of Theorem 4.A, it is not fully satisfactory, for instance because a group G as in Proposition 4.2 cannot be finitely generated. In fact, we obtain actions which admit no non-empty closed invariant set on which the group acts minimally (in particular, the action is *not* semi-conjugate to a minimal action nor to a cyclic action); this phenomenon is somewhat degenerate, and cannot arise for a finitely generated group. Much more interesting is the fact that many (finitely generated) locally moving groups admit exotic actions which are *minimal and faithful*. Here we only mention the following existence criteria, which are satisfied by many well-studied groups.

Proposition 4.3 (Criteria for existence of minimal exotic actions)**.** *For $X = (a, b)$, let $G \subset \mathsf{Homeo}_+(X)$ be a finitely generated subgroup. Assume that G acts minimally on X and contains nontrivial elements of relatively compact support in X, and that at least one of the following holds.*

1. *The group G is a subgroup of the group of piecewise projective homeomorphisms of X.*
2. *The groups of germs $\mathsf{Germ}(G, b)$ is abelian and its nontrivial elements have no fixed points in a neighborhood of b.*

Then there exists a faithful minimal action $\varphi\colon G \to \mathsf{Homeo}_+(\mathbb{R})$ which is not topologically conjugate to the action of G on X (nor to any non-faithful action of G).

4.2. Actions on planar real trees and $\mathbb{R}$-focal actions.

A key concept introduced in [8] is the notion of $\mathbb{R}$-*focal action*. This will be the main tool to understand exotic actions on the line of a vast class of locally moving groups (see Theorem 4.C below).

In order to define this notion, we say that a collection $\mathcal{S}$ of open bounded real intervals is a *cross-free cover* if it covers $\mathbb{R}$ and every two intervals in $\mathcal{S}$ with nontrivial intersection are one contained into the other.

Definition 4.4. Let G be a group. An action $\varphi\colon G \to \mathsf{Homeo}_+(\mathbb{R})$ is $\mathbb{R}$-focal if there exists a bounded open interval $I \subset \mathbb{R}$ whose G-orbit is a cross-free cover.

Although the notion of $\mathbb{R}$-focal action is purely given in terms of the action on the real line, it provides an incredibly rich combinatorial structure, which allows to understand the dynamical behavior of the single elements.

Indeed, the terminology comes from group actions on trees (and Gromov hyperbolic spaces). In this classical setting, an isometric group action on a tree $\mathbb{T}$ is called *focal* if it fixes a unique end $\omega \in \partial\mathbb{T}$ and contains hyperbolic elements (which necessarily admit ω as an attracting or repelling fixed point). The key fact is that every $\mathbb{R}$-focal action on the line can be encoded by a focal action on a tree, except that we need to consider group actions on *real trees* (or $\mathbb{R}$-trees) by homeomorphisms (not necessarily isometric). Let us give an overview of this connection.

Recall that a real tree is a metrizable space $\mathbb{T}$ where any two points can be joined by a unique path, and which admits a compatible metric which makes it geodesic. By a *directed tree* we mean a (separable) real tree $\mathbb{T}$ together with a preferred end $\omega \in \partial\mathbb{T}$, called the *focus*. If $\mathbb{T}$ is a directed tree with focus ω, we write $\partial^*\mathbb{T} := \partial\mathbb{T} \setminus \{\omega\}$. An action of a group G on a directed tree $\mathbb{T}$ (by homeomorphisms) is always required to fix the focus. In this topological setting we will say that such an action is *focal* if for every $v \in \mathbb{T}$ there exists a sequence $(g_n) \subset G$ such that $(g_n.v)$ approaches ω along the ray $[v, \omega[$.

By a *planar directed tree* we mean a directed tree $\mathbb{T}$ endowed with a *planar order*, which is the choice of a linear order on the set of directions below every branching point of $\mathbb{T}$ (one can think of $\mathbb{T}$ as embedded in the plane). Note that in this case the set $\partial^*\mathbb{T}$ inherits a linear order $\prec$ in a natural way. Assume that $\mathbb{T}$ is a planar directed tree, and that $\Phi\colon G \to \mathsf{Homeo}(\mathbb{T})$ is a focal action of a countable group which preserves the planar order. Then Φ induces an order-preserving action of G on the ordered space $(\partial^*\mathbb{T}, \prec)$. From this action one can obtain an action $\varphi : G \to \mathsf{Homeo}_+(\mathbb{R})$ on the real line, which we call the *dynamical realization* of the action of Φ. It turns out that such an action is always minimal and $\mathbb{R}$-focal. In fact, we have the following equivalence, which can be taken as an alternative definition of $\mathbb{R}$-focal actions.

Proposition 4.5. *Let G be a countable group. An action $\varphi\colon G \to \mathsf{Homeo}_+(\mathbb{R})$ is minimal and $\mathbb{R}$-focal if and only if it is conjugate to the dynamical realization of a focal action by homeomorphisms of G on some planar directed tree.*

We will say that an $\mathbb{R}$-focal action φ can be represented by an action $\Phi\colon G \to \mathsf{Homeo}(\mathbb{T})$ on a planar directed tree if it is conjugate to the dynamical realization of Φ. Note that in general such an action Φ representing φ is not unique.

Examples of $\mathbb{R}$-focal actions appear naturally in the context of solvable groups. In fact, the notion of $\mathbb{R}$-focal action was largely inspired by an action on the line of the group $\mathbb{Z} \wr \mathbb{Z}$ constructed by Plante [Pla83] to give an example of action of a solvable group on the line which is not semi-conjugate to any action by affine transformations, see Figure 4.1. In fact, for finitely generated solvable groups we obtain the following dichotomy.

Theorem 4.B. *Let G be a finitely generated solvable group. Then every action $\varphi\colon G \to \mathsf{Homeo}_+(\mathbb{R})$ without fixed points is either semi-conjugate to an action by affine transformations, or to a minimal $\mathbb{R}$-focal action.*

A distinctive feature of $\mathbb{R}$-focal actions is that the action of individual elements of the group satisfy a dynamical classification which resembles the classification of isometries of trees into elliptic and hyperbolic elements. Namely if $G \subset \mathsf{Homeo}_+(\mathbb{R})$ is a subgroup whose action is $\mathbb{R}$-focal, then every element $g \in G$ satisfies one of the following:

- Either g is *totally bounded*: its set of fixed points accumulates on both $\pm\infty$.
- Or g is a *pseudohomothety*: it has a non-empty compact set of fixed points $K \subset \mathbb{R}$ and either every $x \notin [\min K, \max K]$ satisfies $|g^n(x)| \to \infty$ as $n \to +\infty$ (in which case we say that g is an *expanding* pseudohomothety), or the same holds as $n \to -\infty$ (in

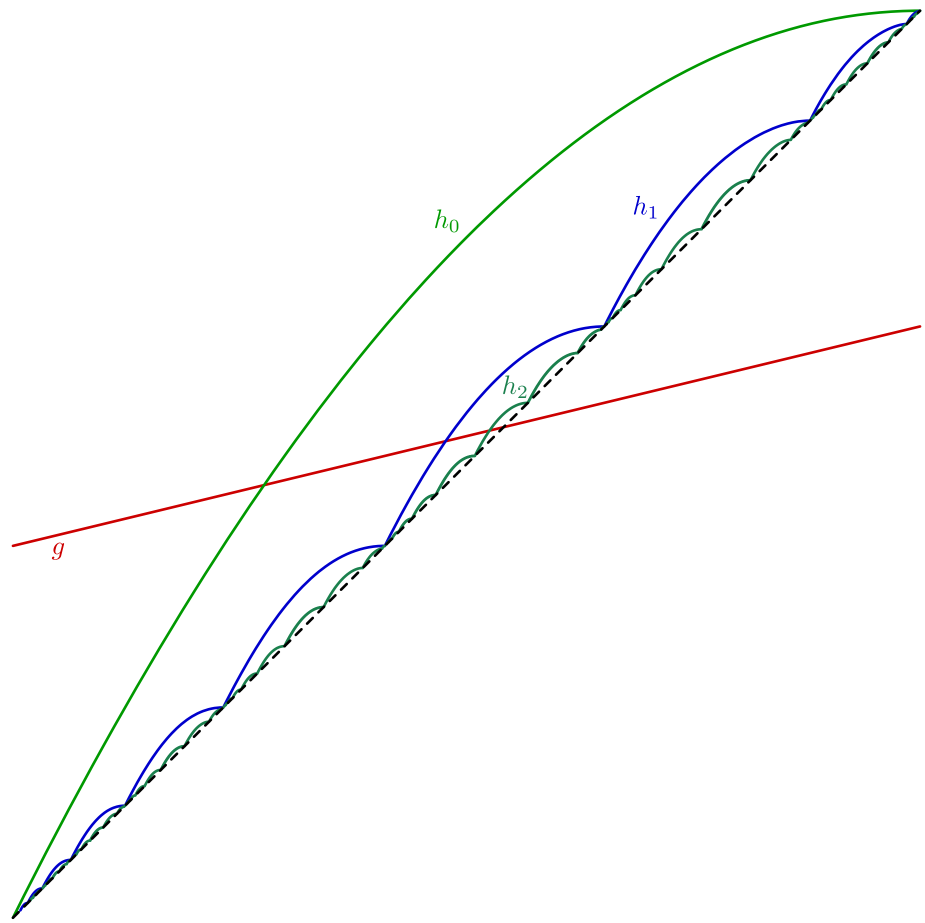

FIGURE 4.1. Plante action of the lamplighter group.

which case we say that g is *contracting*). We simply say that g is a *homothety* when K is reduced to a single point.

Moreover the dynamical type of each element can be explicitly determined from the G-action on a planar directed tree $\mathbb{T}$ representing the $\mathbb{R}$-focal action by looking at the local dynamics near the focus $\omega \in \partial\mathbb{T}$.

We finally discuss another crucial concept: the notion of *horograding* of $\mathbb{R}$-focal action of a group G by another action of G. This will allow us to establish a relation between exotic actions of various locally moving groups and their standard actions. Assume that $\mathbb{T}$ is a directed tree with focus ω. An *increasing horograding* of $\mathbb{T}$ by a real interval $X = (a, b)$ is a map $\pi\colon \mathbb{T} \to X$ such that for every $v \in \mathbb{T}$ the ray $[v, \omega[$ is mapped homeomorphically onto the interval $[\pi(v), b)$. This is a non-metric analogue of the classical metric notion of horofunction associated with ω. A *decreasing horograding* is defined analogously but maps $[v, \omega[$ to $(a, \pi(v)]$, and a *horograding* is an increasing or decreasing horograding. If G is a group acting both on $\mathbb{T}$ and X we say that π is a G-*horograding* if it is G-equivariant, and that its action on $\mathbb{T}$ can be horograded by the action of G on X. This leads to the following definition.

Definition 4.6. Assume $\varphi\colon G \to \mathsf{Homeo}_+(\mathbb{R})$ is a minimal $\mathbb{R}$-focal action, and $j\colon G \to \mathsf{Homeo}_+(X)$ is another action of G on some open interval X. We say that φ can be (increasingly or decreasingly) horograded by j if φ can be represented by an action on a planar directed tree $\Phi\colon G \to \mathsf{Homeo}(\mathbb{T})$ which admits an (increasing or decreasing) G-horograding $\pi\colon \mathbb{T} \to X$.

The existence of such a horograding is a tight relation between φ and j, which is nevertheless quite different from the notion of semi-conjugacy: here the action of G on X plays the role of a hidden "extra-dimensional direction" with respect to the real line on which φ is defined. For instance, in the presence of an increasing G-horograding, the type of each element in φ can be determined from its germ on the rightmost point of X as follows.

Proposition 4.7. *Let $\varphi\colon G \to \mathsf{Homeo}_+(\mathbb{R})$ be a minimal $\mathbb{R}$-focal action, and assume that φ can be increasingly horograded by an action $j\colon G \to \mathsf{Homeo}_+(X)$ on an interval $X = (a, b)$. Then we have the following alternative.*

1. *If the fixed points of $j(g)$ accumulate on b then $\varphi(g)$ is totally bounded.*
2. *Else $\varphi(g)$ is a pseudohomothety, which is expanding if $j(g)(x) > x$ for x in a neighborhood of b, and contracting otherwise. Moreover if $j(g)$ has no fixed points in X, then $\varphi(g)$ is a homothety.*

4.3. **Structure theorems for actions by homeomorphisms.** The notion of $\mathbb{R}$-focal actions can be used to understand exotic actions on the line of a vast class of locally moving groups.

Definition 4.8 (The classes $\mathcal{F}$ and $\mathcal{F}_0$)**.** For $X = (a, b)$, let $G \subset \mathsf{Homeo}_+(X)$ be a subgroup. For an interval $(x, y) \subseteq X$, write $G_{(x,y)} = \{g \in G : \mathsf{supp}(g) \subset (x, y)\}$. Write also $G_+ := \bigcup_{x<b} G_{(a,x)}$ and $G_- := \bigcup_{x>a} G_{(x,b)}$ for the subgroups of elements with trivial germ at a and b respectively. Consider the following conditions.

1. G is locally moving.
2. There exist two finitely generated subgroups $\Gamma_\pm \subset G_\pm$ and $x, y \in X$ such that $G_{(a,x)} \subset \Gamma_+$ and $G_{(y,b)} \subset \Gamma_-$.
3. There exists an element of G without fixed points in X.

We say that G belongs to the class $\mathcal{F}$ if it satisfies (1–2), and that it belongs to the class $\mathcal{F}_0$ if it satisfies (1–3).

Note that condition (2) trivially holds true provided there exist $x, y \in X$ such that $G_{(a,x)}$ and $G_{(y,b)}$ are finitely generated. In practice this weaker condition is satisfied in many examples, but (2) is more flexible and more convenient to handle.

The class $\mathcal{F}_0$ contains many well-studied examples of finitely generated locally moving groups of piecewise linear or projective homeomorphisms, including Thompson's group F and all Thompson–Brown–Stein groups $F_{n_1,\ldots,n_k}$, several other Bieri–Strebel groups, the groups of piecewise projective homeomorphisms of Lodha–Moore. It also contains various groups which are far from the setting of groups of piecewise linear or projective homeomorphisms: for instance every countable subgroup of $\mathsf{Homeo}_+(X)$ is contained in a finitely generated group belonging to $\mathcal{F}_0$.

Probably the main result of [8] is a qualitative description of every exotic action of a group un the class $\mathcal{F}$.

Theorem 4.C (Main structure theorem for actions of groups in $\mathcal{F}$)**.** *Let X be an open interval and $G \subset \mathsf{Homeo}_+(X)$ be a group in the class $\mathcal{F}$. Then every action $\varphi\colon G \to \mathsf{Homeo}_+(\mathbb{R})$ without fixed points is semi-conjugate to an action in one of the following families.*

1. (Non-faithful) *An action which factors a proper quotient.*

(2) (Standard) *An action which is conjugate to the standard action of G on X.*
(3) ($\mathbb{R}$-focal) *A minimal faithful $\mathbb{R}$-focal action which can be G-horograded by the standard action of G on X.*

The main content of Theorem 4.C is that even though a group in $\mathcal{F}$ may admit exotic actions on the line, these are nevertheless tightly related to the standard action of G, which appears at the level of a planar directed tree encoding the $\mathbb{R}$-focal action. This relation can be effectively exploited to study such exotic actions, by means of Proposition 4.7.

4.4. **Description of the space of actions without fixed points.** Although describing all possible $\mathbb{R}$-focal actions of a group in the class $\mathcal{F}$ appears out of the reach in general, we can use Theorem 4.C to get an insight into the space $\mathsf{Hom}_{\mathrm{irr}}(G, \mathsf{Homeo}_+(\mathbb{R}))$ of actions without fixed points. Recall that this space can be endowed with the natural *compact-open topology*, which means that a neighborhood basis of a given action $\varphi \in \mathsf{Hom}_{\mathrm{irr}}(G, \mathsf{Homeo}_+(\mathbb{R}))$ is defined by considering for every $\varepsilon > 0$, finite subset $S \subset G$, and compact subset $K \subset \mathbb{R}$, the subset of actions

$$\left\{\psi \in \mathsf{Hom}_{\mathrm{irr}}(G, \mathsf{Homeo}_+(\mathbb{R}))\colon \max_{g\in S}\max_{x\in K} |\varphi(g)(x) - \psi(g)(x)| < \varepsilon\right\}.$$

We say that an action $\varphi \in \mathsf{Hom}_{\mathrm{irr}}(G, \mathsf{Homeo}_+(\mathbb{R}))$ is *locally rigid* if there exists a neighborhood $\mathcal{U} \subset \mathsf{Hom}_{\mathrm{irr}}(G, \mathsf{Homeo}_+(\mathbb{R}))$ of φ such that every $\psi \in \mathcal{U}$ is semi-conjugate to φ. Otherwise, we say that the action of φ is *flexible*. We show the following result for groups in the class $\mathcal{F}_0$.

Theorem 4.D (Local rigidity of the standard action for groups in $\mathcal{F}_0$)**.** *Let $G \subset \mathsf{Homeo}_+(\mathbb{R})$ be a finitely generated group in the class $\mathcal{F}_0$. Then the standard action of G is locally rigid.*

Possible annoying issues when studying the quotient space of $\mathsf{Hom}_{\mathrm{irr}}(G, \mathsf{Homeo}_+(\mathbb{R}))$ by the semi-conjugacy equivalence relation is that it may fail to be Hausdorff (this is for instance the case for the groups $\mathsf{T}(\sigma)$ from [6]), and that locally rigid actions may correspond to non-isolated semi-conjugacy classes. For finitely generated groups, we consider a space which has been introduced by Deroin [Der20], which contains a representative of every (positive) semi-conjugacy class. One way to construct such space is based on work by Deroin, Kleptsyn, Navas, and Parwani [DKNP13] on symmetric random walks on $\mathsf{Homeo}_+(\mathbb{R})$. Given a probability measure μ on G whose support is finite, symmetric, and generates G, one defines the *Deroin space* $\mathsf{Der}_\mu(G)$ as the subspace of $\mathsf{Hom}_{\mathrm{irr}}(G, \mathsf{Homeo}_+(\mathbb{R}))$ of *harmonic actions*, that is, actions of G for which the Lebesgue measure is μ-stationary. The space $\mathsf{Der}_\mu(G)$ is compact and Hausdorff, with a natural topological flow $\Phi : \mathbb{R} \times \mathsf{Der}_\mu(G) \to \mathsf{Der}_\mu(G)$ defined on it, with the property that two actions in $\mathsf{Der}_\mu(G)$ are (positively semi-)conjugate if and only if the are on the same Φ-orbit. For the proof of Theorem 4.D we use a new criterion which might be of independent interest: in order to check the local rigidity of a harmonic action $\varphi \in \mathsf{Der}_\mu(G)$ among all continuous actions, it is enough to check its local rigidity among actions in the space $\mathsf{Der}_\mu(G)$. This criterion is based on a new description of the space $\mathsf{Der}_\mu(G)$ as a quotient of the space of invariant preorders on G. In particular, this last observation shows that the topology of $\mathsf{Der}_\mu(G)$ does not depend on the choice of the probability measure μ.

4.5. **Some concrete groups.** Many illustrative examples of applications of our results arise as subgroups of $\mathsf{PL}_+(X)$, where $X = (a, b)$ is an open interval. Note that Proposition 4.3 implies that every finitely generated locally moving group $G \subset \mathsf{PL}_+(X)$ admits a minimal faithful exotic action on the real line. It turns out that subgroups of $\mathsf{PL}_+(X)$ exhibit a surprising mixture of rigidity and flexibility properties.

The most famous example of group of PL homeomorphisms is Thompson's group F, which belongs to the class $\mathcal{F}_0$. In particular every faithful action $\varphi\colon F \to \mathsf{Diff}^1_0([0,1])$ without fixed points in $(0,1)$ is semi-conjugate to the standard action (Theorem 4.A), every exotic action $\varphi\colon F \to \mathsf{Homeo}_+(\mathbb{R})$ is $\mathbb{R}$-focal and horograded by the standard action of F on $(0,1)$ (Theorem 4.C), and the standard action of F on $(0,1)$ is locally rigid (Theorem 4.D). From this we can make a picture of the Deroin space of F (Figure 4.2).

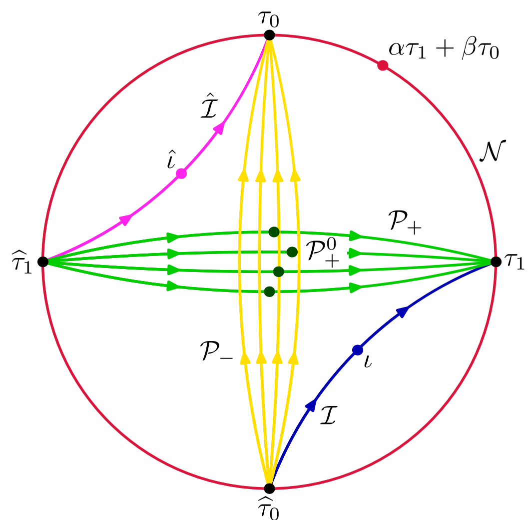

FIGURE 4.2. The circle $\mathcal{N}$ corresponds to the space of actions induced from a proper quotient (which must an action of $\mathbb{Z}^2 \cong F/[F,F]$). The lines $\mathcal{I}$ and $\hat{\mathcal{I}}$ are the orbits of the standard action and the reversed standard action, respectively. The other families correspond to the $\mathbb{R}$-focal actions.

Despite these rigidity results, it turns out that the group F admits a rich and complicated universe of minimal exotic actions (we are able to give explicit examples of $\mathbb{R}$-focal actions with very different behavior).

Theorem 4.E. *Thompson's group F admits uncountably many actions on the line which are faithful, minimal, $\mathbb{R}$-focal and pairwise non-semi-conjugate. Moreover, there are uncountably many such actions whose restrictions to the commutator subgroup $[F,F]$ remain minimal and are pairwise non-semi-conjugate.*

It is interesting to note that this abundance of exotic actions of the group F fails for some tightly related groups of PL homeomorphisms. The Bieri–Strebel group G_λ belongs to $\mathcal{F}_0$ provided λ is algebraic, thus satisfies Theorem 4.C. However, in striking difference with the case of F, we have the following result.

Theorem 4.F (PL groups with finitely many exotic actions)**.** *Let $\lambda > 1$ be an algebraic real number. Then the group $G = G_\lambda$ admits exactly three minimal faithful actions $\varphi\colon G \to$ $\mathsf{Homeo}_+(\mathbb{R})$ on the real line up to conjugacy, namely its standard action and two minimal $\mathbb{R}$-focal actions (which can be horograded by its standard action).*

Building on the proof of Theorem 4.F, we also construct a finitely generated locally moving group G which does not have exotic actions at all.

Theorem 4.G (A finitely generated locally moving group with no exotic actions)**.** *There exists a finitely generated subgroup* $G \subset \mathsf{Homeo}_+(\mathbb{R})$ *in the class* $\mathcal{F}_0$*, such that every faithful minimal action* $\varphi\colon G \to \mathsf{Homeo}_+(\mathbb{R})$ *is conjugate to the standard action.*

5. Classification of locally discrete groups of circle diffeomorphisms

Here we present the content of a series of works, which try to understand a very specific class of groups acting on the circle. More precisely, we will discuss *locally discrete groups of real-analytic circle diffeomorphisms*. This is part of a program that has received contributions by several authors, and which is motivated by old conjectures in foliation theory.

5.1. **State of the art.** Recall that an action of a group G on the circle $\mathbb{S}^1$ gives rise, by suspension of the action, to a codimension-one foliation of a closed manifold. There is a perfect dictionary between the dynamics of the action on $\mathbb{S}^1$ and the dynamics of the leaves of the foliation. Foliations defined by suspensions represent a particular class (e.g. the manifold M must admit a circle bundle structure) however their study is important for developing new techniques, manufacturing examples, and test conjectures. More than 10 years ago Deroin, Kleptsyn and Navas [DKN09] started the study of group actions on the circle with *non-uniformly hyperbolic behavior*. The original motivation for this was the *ergodic theory* of group actions and more precisely to solve long-standing open conjectures by Ghys–Sullivan and Hector, about the relationship between minimality and Lebesgue-ergodicity for C^2 actions on the circle (more generally, for codimension-one foliations). In their original formulations, these conjectures remain unsolved, but many significant cases have been settled, especially in *real-analytic* (C^ω) *regularity* [Reb99, DKN18, FK14, 9].

Conjecture 5.1. *Let* $(M, \mathcal{F})$ *be a closed manifold with a fixed volume form* vol *and a* C^2 *codimension-one foliation* $\mathcal{F}$*, without invariant transverse Borel measure.*

(1) (Ghys–Sullivan) If $\mathcal{F}$ *is minimal (every leaf is dense), then every subset* $E \subset M$ *which is saturated by leaves of* $\mathcal{F}$ *has either full or zero volume.*
(2) (Hector) When $\mathcal{F}$ *is not minimal, every minimal (with respect to inclusion) closed subset saturated by leaves of* $\mathcal{F}$ *has zero volume and its complement in* M *has finitely many connected components.*

The interest goes beyond this precise conjecture: the partial advances are providing very good tools for understanding the dynamics and geometry of finitely generated subgroups of $\mathsf{Diff}^2_+(\mathbb{S}^1)$. Before [DKN09] only the dynamics of *non locally discrete* subgroups of $\mathsf{Diff}^\omega_+(\mathbb{S}^1)$ was well understood [EISV93, Nak94, LR03]. This is a fundamental notion: a subgroup $G \subset \mathsf{Diff}^\omega_+(\mathbb{S}^1)$ is *locally discrete*, if for every interval I, the identity is isolated, in the C^1 topology, among the set of restrictions $\{g|_I\}_{g\in G}$. Let us assume from now on that $G \subset \mathsf{Diff}^\omega_+(\mathbb{S}^1)$ is a finitely generated, locally discrete subgroup. A relatively easy subcase is that of *expanding actions*, which has been treated in [Der13]: a subgroup $G \subset \mathsf{Diff}^\omega_+(\mathbb{S}^1)$ is called *expanding* if for every $x \in \mathbb{S}^1$ there exists an element $g \in G$ such that $|g'(x)| > 1$. On the other hand, understanding non-expanding actions requires heavy work. One wants to prove that a *non-expandable point* (a point $x \in \mathbb{S}^1$ such that $|g'(x)| \le 1$ for every $g \in G$) is fixed by some nontrivial element of the group. After [DKN09], this is called *property* $(\star)$. This is the case for non-expanding discrete subgroups $G \subset \mathsf{PSL}(2,\mathbb{R})$, where a non-expandable point is fixed by a parabolic element, and moreover has a *geometric meaning*: its orbit is identified with a cusp in the quotient $\mathbb{H}^2/G$. If property $(\star)$ holds, then the dynamics has a very nice description, by a non-uniformly hyperbolic Markov partition. The recent collective work [9] goes in the direction of describing the *geometry* of such actions. Let us first state what should be the desired picture.

Conjecture 5.2. *Let $G \subset \mathsf{Diff}^{\omega}_{+}(\mathbb{S}^1)$ be a finitely generated, locally discrete subgroup of orientation-preserving real-analytic circle diffeomorphisms. If the group is expanding, then it is C^{ω} conjugate to a cocompact discrete subgroup of some finite central extension of $\mathsf{PSL}(2,\mathbb{R})$. If the group is non-expanding, then it is virtually free (it contains a free subgroup of finite index) and such actions are classifiable by generalized ping-pong partitions.*

Conjecture 5.2 has been partially validated. As already mentioned, the case of expanding actions has been solved by Deroin [Der13]. The case of non-expanding actions has been addressed in several works:

(1) in [FK14], Filimonov and Kleptsyn proved that if a finitely generated group G has one end and is non-expanding, then either it contains elements of arbitrarily large finite order or it is not finitely presented;
(2) in [DKN18], Deroin, Kleptsyn and Navas were able to obtain a ping-pong partition (see definition below) for every non-expanding free subgroup $G \subset \mathsf{Diff}^{\omega}_{+}(\mathbb{S}^1)$;
(3) in [10], in collaboration with Alonso, Alvarez, Malicet and Meniño, we extended the notion of ping-pong partition to the case of non-expanding *virtually free* subgroups $G \subset \mathsf{Diff}^{\omega}_{+}(\mathbb{S}^1)$, and proved that it is a C^0-conjugacy invariant;
(4) finally, in the collective work [9], we proved that if a locally discrete subgroup $G \subset \mathsf{Diff}^{\omega}_{+}(\mathbb{S}^1)$ has infinitely many ends, then it must be virtually free.

Therefore only subgroups $G \subset \mathsf{Diff}^{\omega}_{+}(\mathbb{S}^1)$ with one end, and either unbounded torsion or non finitely presented, are missing in this picture. Let us also mention that we have proved in [10] that virtually free groups acting on the circle are of very special form.

Theorem 5.A. *If $G \subset \mathsf{Homeo}_{+}(\mathbb{S}^1)$ is virtually free, then it contains a normal free subgroup whose quotient is finite cyclic. Conversely, every such abstract virtually free group G can be realized as a locally discrete subgroup of $\mathsf{Diff}^{\omega}_{+}(\mathbb{S}^1)$.*

The two works [9, 10] heavily use the point of view of Bass–Serre theory (the theory of groups acting on trees). By the celebrated Stalling's theorem, every finitely generated group with infinitely many ends admits an action on a tree with finite edge stabilizers. Using an inductive argument on the vertex stabilizers, we prove the following.

Theorem 5.B. *If $G \subset \mathsf{Diff}^{\omega}_{+}(\mathbb{S}^1)$ is locally discrete and with infinitely many ends, then it must have property $(\star)$ and be virtually free.*

This extends (and actually builds on) an old result of Ghys [Ghy87]. Secondly, we prove the following result.

Theorem 5.C. *If a subgroup $G \subset \mathsf{Diff}^{\omega}_{+}(\mathbb{S}^1)$ is locally discrete, non-expanding and has property $(\star)$, then it must have infinitely many ends and therefore be virtually free.*

The proof of this result is inspired by a celebrated result of Duminy (see [Nav11]), on actions on the circle with invariant Cantor set (it actually holds for codimension-one foliations). However, in our case we have to develop more sophisticated analytic tools. The leading idea is that an end of the group $G \subset \mathsf{Diff}^{\omega}_{+}(\mathbb{S}^1)$ determines a local "asymptotically invariant" projective structure, so that if G has only one end, then it would be identified as a subgroup of $\mathsf{PSL}(2,\mathbb{R})$, up to finite error.

On the other hand, the work [10] on generalized ping-pong partitions is more of group-theoretical nature. The hardest part is to find the good notion of *ping-pong partition*. The starting geometric picture to have in mind is the situation of a classical ping-pong for Schottky groups, such as

$$\Gamma = \left\langle \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \right\rangle \subset \mathsf{PSL}(2, \mathbb{R}),$$

whose ping-pong partition is obtained by considering the connected components of the complement of the four vertices $\{-1, 0, 1, \infty\} \in \mathbb{R}\mathsf{P}^1 = \partial\mathbb{H}^2$ of the corresponding ideal square in the hyperbolic plane.

For general virtually free groups, a "generalized" ping-pong partition should correspond to a partition of the circle $\mathbb{S}^1$ into finitely many intervals together with the information of how generators map all of these intervals. We want two properties to be satisfied.

(1) (Ping-pong lemma) If the action of a virtually free group G on $\mathbb{S}^1$ admits a ping-pong partition, then the action is *faithful*.
(2) (Nice conjugacy invariant) The partition is defined by *finitely many data*, and this determines the conjugacy class of the action.

This is motivated by an important application, which is a partial solution to an old conjecture by Dippolito [Dip78]:

Conjecture 5.3 (Dippolito)**.** *Let $(M, \mathcal{F})$ be a codimension-one foliation of a closed manifold which is transversally C^2, with exceptional minimal set Λ. Then there exists a transverse measure supported on Λ for which the Radon-Nikodym derivative of the action of any holonomy pseudogroup is locally constant.*

As Dippolito writes in [Dip78], this conjecture is conditioned on the solution of a major open problem:

Conjecture 5.4 (Dippolito)**.** *Let $(M, \mathcal{F})$ be a codimension-one foliation of a closed manifold which is transversally C^2, with exceptional minimal set Λ. Then there exists a semi-exceptional leaf (that is, in the boundary of $M \setminus \Lambda$), whose germs of holonomy maps form an infinite cyclic subgroup.*

The only result available in this direction goes back to the Ph.D. thesis of Hector (Strasbourg, 1972) (see also [Nav06, Appendix]), which establishes Conjecture 5.4 under the assumption of nontrivial r-jets for elements in the holonomy group, for some $r \geq 1$. This holds for instance in the case of transversally real-analytic foliations.

One of the contributions of [10] is to give strong evidences that indeed Conjecture 5.4 is the unique obstacle towards Conjecture 5.3. Indeed, using ping-pong partitions for virtually free groups of real-analytic diffeomorphisms of the circle we prove in [11]:

Theorem 5.D (Realization)**.** *Let $G \subset \mathsf{Diff}^\omega_+(\mathbb{S}^1)$ be a finitely generated group of real-analytic circle diffeomorphisms acting with an invariant Cantor set. Then the action of G is semi-conjugate to an action by piecewise-linear homeomorphisms. More precisely, every such G is C^0 semi-conjugate to a subgroup of Thompson's group T.*

Let us give a little more details on the notion of ping-pong partitions introduced in [10]. We start with a virtually free group $G \subset \mathsf{Diff}^\omega_+(\mathbb{S}^1)$, and a classical result of Karrass–Pietrowski–Solitar, this is equivalent to consider a group G with a proper isometric action $\alpha : G \to \mathsf{Isom}(X)$ on a tree X (*proper:* finite stabilizers, in this context). The choice of the action $\alpha : G \to \mathsf{Isom}(X)$ and a connected fundamental domain $T \subset X$ plays the role of the choice of a free generating system in the case of free groups. This choice also determines a presentation of G as the *fundamental group of a graph of groups*. For general fundamental groups of graphs of groups, we prove a ping-pong lemma:

Theorem 5.E. *Let $G = \pi_1(\overline{X}; G_v, A_e)$ be the fundamental group of a graph of groups and consider an action on a set $G \curvearrowright \Omega$. If the action satisfies a list of* 10 conditions *(families of containment relations), then the action $G \curvearrowright \Omega$ is faithful.*

This extends the classical Klein's ping-pong lemma for free groups, and the (less) classical Fenchel–Nielsen ping-pong lemma for amalgamated products and HNN extensions, and the list of 10 conditions is a generalization of the conditions required for the classical ping-pong, and they have been deduced by looking at the action of G on the boundary of its Bass–Serre tree. This generalized ping-pong lemma is not a surprising result, but the main, and nontrivial, point is that the list of 10 conditions is *optimal* (a shorter list does not suffice).

Note also that, as the group G is virtually free and acts on the circle, the graph of groups must be a graph of finite cyclic groups, but the result above is valid for general graphs of groups.

When considering a virtually free group $G \subset \mathsf{Homeo}_+(\mathbb{S}^1)$, a *generalized ping-pong partition* Θ is a collection of finitely many disjoint open intervals of the circle, which satisfies the 10 families of requirements for the ping-pong lemma, and a few more (for instance, a generator must map one interval of the partition either inside another one, or to an exact union of intervals of the partition).

We prove in [10] that these finitely many combinatorial data are enough to determine the conjugacy class of the action. Moreover, one of the important properties of ping-pong partitions is that they have a good behavior when passing to finite-index subgroups, so that for many aspects, one can then restrict the attention to ping-pong partitions of free groups. In particular, this allows to use the construction of ping-pong partitions for free groups in [DKN18], which relies on involved analytic estimates, to deduce that any locally discrete virtually free group $G \subset \mathsf{Diff}^\omega_+(\mathbb{S}^1)$ admits a generalized ping-pong partition.

5.2. **Further explorations.**

5.2.1. *A naive conjecture.* The collective works [9, 10] are focused on understanding subgroups G of $\mathsf{Diff}^2_+(\mathbb{S}^1)$ which are locally discrete. My belief (which I state as a "naive conjecture") is that such a subgroup G is either conjugate to a central extension of a cocompact Fuchsian group (cocompact discrete subgroup of $\mathsf{PSL}(2, \mathbb{R})$), or semiconjugate to a subgroup of Thompson's group T.

The results obtained so far require a strong regularity assumption, that is we mainly study locally discrete subgroups of $\mathsf{Diff}^\omega_+(\mathbb{S}^1)$, the group of real-analytic circle diffeomorphisms, but even under this strong assumption the picture is incomplete. As a matter of fact, we have very little knowledge about locally discrete subgroups in less rigid regularity (even in class C^∞). I would not be surprised to see my naive conjecture disproved.

5.2.2. *Expanding subgroups.* Recall that a subgroup $G \subset \mathsf{Diff}^1_+(\mathbb{S}^1)$ is *expanding* if for every point $x \in \mathbb{S}^1$, there exists an element $g \in G$ such that $g'(x) > 1$. Non-locally discrete subgroups $G \subset \mathsf{Diff}^\omega_+(\mathbb{S}^1)$ which preserve no Borel probability measure are always expanding, and locally discrete subgroups $G \subset \mathsf{Diff}^\omega_+(\mathbb{S}^1)$ which are expanding have been classified by Deroin [Der13]:

they are C^ω-conjugate to a cocompact discrete subgroup of some $\mathsf{PSL}^{(k)}(2,\mathbb{R})$ (the k-fold central extension of $\mathsf{PSL}(2,\mathbb{R})$). The result of Deroin is likely to be extended to expanding subgroups in C^2 regularity (but not below C^2), although this requires first to obtain Ghys' differentiable rigidity for Fuchsian groups [Ghy93] in C^2 regularity (after discussions with Deroin and Kleptsyn, we have a strategy for that).

The results of [FK14, 9] suggest that subgroups $G \subset \mathsf{Diff}^\omega_+(\mathbb{S}^1)$ *with one end* should be always expanding. However, proving this remains a challenging problem.

5.2.3. *Virtually-free locally discrete subgroups.* When a locally discrete subgroup $G \subset \mathsf{Diff}^\omega_+(\mathbb{S}^1)$ has infinitely many ends, we proved in [9] that G is actually virtually free. In the work [10], extending [DKN18] to a more involved algebraic setting, we have introduced the notion of ping-pong partition for such subgroups, which is a combinatorial object associated with any such subgroup, which determines the semi-conjugacy class in $\mathsf{Homeo}_+(\mathbb{S}^1)$. In the follow-up paper [11], written in collaboration with S. Alvarez, P. Barrientos, D. Filimonov, V. Kleptsyn, D. Malicet and C. Meniño, we use ping-pong partitions to deduce several properties of the dynamics of such subgroups. I will highlight the most interesting consequences, which do not require too many preliminary notions.

The first one is the realization theorem (Theorem 5.D) mentioned before, which solves the conjecture by Dippolito (Conjecture 5.3), in the restrictive case of real-analytic group actions. For this we prove that given a (virtually-free) group G of circle homeomorphisms acting on $\mathbb{S}^1$ with a ping-pong partition, then it is possible to find another group $\widetilde{G}$ which acts with an equivalent ping-pong partition, but with desired regularity properties: the group $\widetilde{G}$ can be chosen inside Thompson's group T, or even inside $\mathsf{Diff}^\omega_+(\mathbb{S}^1)$. Moreover, this combinatorial framework allowed us to exhibit (the first?) examples of locally discrete subgroups of $\mathsf{Diff}^\omega_+(\mathbb{S}^1)$ acting minimally, but which are not conjugate to a subgroup of some $\mathsf{PSL}^{(k)}(2,\mathbb{R})$.

To state a further consequence of ping-pong partitions, we recall that the C^0 conjugacy class of discrete subgroups of $\mathsf{PSL}(2,\mathbb{R})$ is determined in dynamical terms by the so-called *conevrgence property* (after celebrated works of Tukia [Tuk88], Gabai [Gab92], Casson–Jungreis [CJ94]): a subgroup $G \subset \mathsf{Homeo}_+(\mathbb{S}^1)$ has the convergence property if for every sequence $\{g_n\}_{n\in\mathbb{N}} \subset G$ of distinct elements, there exist $\alpha, \omega \in \mathbb{S}^1$ and a subsequence $\{g_{n_k}\}_{k\in\mathbb{N}}$ such that the sequence of restrictions $\{g_{n_k}|_{\mathbb{S}^1\setminus\{\alpha\}}\}$ pointwise converges, as $k \to \infty$, to the constant map $x \in \mathbb{S}^1 \setminus \{\alpha\} \mapsto \omega$. It turns out that every subgroup $G \subset \mathsf{Homeo}_+(\mathbb{S}^1)$ with a ping-pong partition has a similar behavior:

Theorem 5.F (Multiconvergence property)**.** *If $G \subset \mathsf{Diff}^\omega_+(\mathbb{S}^1)$ is a finitely generated, virtually free, locally discrete subgroup, then it has the multiconvergence property: there exists a uniform $K \in \mathbb{N}$ such that for every infinite sequence $\{g_n\}$ of distinct elements in G, there exist finite subsets A and $R \subset \mathbb{S}^1$, with $\#A, \#R \le K$, such that there exists a subsequence $\{g_{n_k}\}$ such that the sequence of restrictions $\{g_{n_k}|_{\mathbb{S}^1\setminus R}\}$ pointwise converges, as $k \to \infty$, to the locally constant map g_∞, with image $g_\infty(\mathbb{S}^1 \setminus R) = A$, with $\#R$ discontinuity points at R and such that $g_\infty(a) = a$ for every $a \in A \setminus R$.*

From this we deduce directly the following:

Corollary 5.5. *If $G \subset \mathsf{Diff}^\omega_+(\mathbb{S}^1)$ is a finitely generated, virtually free, locally discrete subgroup, then the number of fixed points of a nontrivial element in G is uniformly bounded.*

Using a result by Matsuda [Mat09], the multiconvergence property also gives the following:

Corollary 5.6. *Let G be a finitely generated, virtually free subgroup of $\mathsf{Diff}^{\omega}_{+}(\mathbb{S}^1)$. The following statements are equivalent:*

(1) G is locally discrete;
(2) the rotation spectrum $\mathsf{rot}(G) = \{\mathsf{rot}(g) \colon g \in G\}$ is finite.

Indeed, Matsuda proves that if a subgroup $G \subset \mathsf{Diff}^{\omega}_{+}(\mathbb{S}^1)$ is non-locally discrete, then the rotation spectrum $\mathsf{rot}(G)$ is infinite. It is an interesting problem, which appears in [Mat09], to see if this can be improved to show that a non-locally discrete, finitely generated subgroup $G \subset \mathsf{Diff}^{\omega}_{+}(\mathbb{S}^1)$ contains an element of irrational rotation number (it is well-known that this holds for non-discrete, finitely generated subgroups of $\mathsf{PSL}(2,\mathbb{R})$).

Part 2. Locally discrete groups of real-analytic circle diffeomorphisms

This part is a more detailed historical exposition of what we discussed in the last section. We will not focus too much on personal contribution, but we will try to present the study of groups of real-analytic circle diffeomorphisms and some of the more relevant results in a more pedagogical way. The text is based on notes prepared for two series of lectures given at ENS–Lyon in 2018 and at KIAS in 2019.

6. Preliminaries

6.1. **Topological dynamics.** Let us start recalling fundamental facts about groups acting on the circle (see [Ghy01, Nav11] for a more general introduction). We will assume that all homeomorphisms preserve the orientation. The circle $\mathbb{S}^1$ is identified with the Euclidean torus $\mathbb{R}/\mathbb{Z}$. We denote by $\mathsf{Homeo}_+(\mathbb{S}^1)$ the group of all orientation preserving circle homeomorphisms, and by $\mathsf{Homeo}_\mathbb{Z}(\mathbb{R})$ the group of orientation preserving homeomorphisms of the real line $\mathbb{R}$, which commute with the group of integer translations $\mathbb{Z}$. This identifies with the universal cover of $\mathsf{Homeo}_+(\mathbb{S}^1)$ and is actually a central extension:

$$0 \to \mathbb{Z} \to \mathsf{Homeo}_\mathbb{Z}(\mathbb{R}) \to \mathsf{Homeo}_+(\mathbb{S}^1) \to 1.$$

Given $f \in \mathsf{Homeo}_+(\mathbb{S}^1)$, the *rotation number* $\mathsf{rot}(f) \in \mathbb{S}^1$ is defined as the limit

$$\mathsf{rot}(f) = \lim_{n\to\infty} \frac{F^n(x) - x}{n} \pmod{\mathbb{Z}},$$

where $F \in \mathsf{Homeo}_\mathbb{Z}(\mathbb{R})$ is any lift of f and $x \in \mathbb{R}$ is any point. The rotation number is a semi-conjugacy invariant for f. It is rational of reduced fraction p/q, if and only if f has a periodic orbit of period q (and $F^q(x) = x + p$ for every $x \in \mathbb{R}$). It is irrational $\mathsf{rot}(f) = \alpha \in \mathbb{R}\setminus\mathbb{Q}$ if and only if f is semi-conjugate to the irrational rotation $R_\alpha : x \mapsto x + \alpha$. Here by *semi-conjugacy*, we mean that there exists a continuous, monotone non-decreasing function $h : \mathbb{S}^1 \to \mathbb{S}^1$ of degree 1 such that $hf = R_\alpha h$ (h need not be invertible, that is, we are allowed to collapse f-orbits).

Let $G \subset \mathsf{Homeo}_+(\mathbb{S}^1)$ be a finitely generated subgroup. A subset $\Lambda \subset \mathbb{S}^1$ is a *minimal invariant set* for G if Λ is closed, non-empty and G-invariant, and minimal with respect to inclusion. When Λ is not a finite G-orbit, then Λ is unique, and can only be the whole circle or a Cantor set. In these notes, we say that a subgroup $G \subset \mathsf{Homeo}_+(\mathbb{S}^1)$ is *elementary* if it admits an invariant Borel probability measure μ on $\mathbb{S}^1$. Observe that if G is elementary and with no finite orbits, then G is actually semi-conjugate to a subgroup of the group of rotations (which is actually the image of the function rotation number rot, which defines a homomorphism in the case of an invariant probability measure). When $G \subset \mathsf{Homeo}_+(\mathbb{S}^1)$ is non-elementary then there are *local exponential contractions*: for every $x \in \mathbb{S}^1$ there exists a neighborhood I_x such that a "typical" long composition of generators contracts the size $|I_x|$ at exponential rate (this can be made precise in probabilistic terms [Ant84, Mal17]). One can deduce, as a consequence, that any non-elementary $G \subset \mathsf{Homeo}_+(\mathbb{S}^1)$ contains free subgroups (this is a weak Tits alternative, originally due to Margulis [Ghy01]).

6.2. **Examples from hyperbolic geometry.** We start by describing the canonical examples of groups of circle homeomorphisms. The group $\mathsf{PSL}(2,\mathbb{R})$ is naturally a subgroup of $\mathsf{Homeo}_+(\mathbb{S}^1)$, when considered as the group of (orientation preserving) isometries of the hyperbolic space $\mathbb{H}$ (identify the circle $\mathbb{S}^1$ with the boundary $\partial\mathbb{H}$). Observe that, with this identification, $\mathsf{PSL}(2,\mathbb{R})$ is actually a subgroup of $\mathsf{Diff}^\omega_+(\mathbb{S}^1)$, the group of *real-analytic* circle diffeomorphisms. Let us recall some classical terminology. An element $h \in \mathsf{PSL}(2,\mathbb{R})$ is *elliptic*, *parabolic* or *hyperbolic* if it has respectively 0, 1 or 2 fixed points. In the hyperbolic case, one fixed point is attracting, the other one is repelling (this is usually called a *North-South* dynamics). An elementary subgroup $\Gamma \subset \mathsf{PSL}(2,\mathbb{R})$ is conjugate in $\mathsf{PSL}(2,\mathbb{R})$ either to a subgroup of rotations $\mathsf{SO}(2)$ or to a subgroup of affine transformations $\mathsf{Aff}_+(\mathbb{R})$. A finitely generated subgroup $\Gamma \subset \mathsf{PSL}(2,\mathbb{R})$ is a *Fuchsian group* if it is discrete in the C^0 topology (by the Cauchy inequalities, all C^k topologies on $\mathsf{Diff}^\omega_+(\mathbb{S}^1)$ are equivalent). Elementary Fuchsian groups are virtually cyclic. Fuchsian groups whose minimal invariant set is the whole circle, are called *lattices* (as in this case the quotient $\mathbb{H}/\Gamma$ has finite volume). A lattice is *uniform* (or *cocompact*) if the quotient $\mathbb{H}/\Gamma$ is compact. In general, the quotient $\mathbb{H}/\Gamma$ is a surface of finite type. Dynamically, cocompact lattices are characterized by the property that their actions on $\mathbb{S}^1 \cong \partial\mathbb{H}$ are discrete and *expanding*: for every $x \in \mathbb{S}^1$ there exists an element $h \in \Gamma$ such that $h'(x) > 1$. Moreover, every element $h \in \Gamma$ is either hyperbolic or elliptic (of finite order). When $\Gamma \subset \mathsf{PSL}(2,\mathbb{R})$ is a non-uniform lattice, there are only finitely many orbits of points where the expanding property fails, and every such point has cyclic stabilizer, generated by a parabolic element. Geometrically, these correspond to *cusps* in the quotient $\mathbb{H}/\Gamma$. At the level of group structure, a cocompact lattice is isomorphic to a *closed surface group*, whereas any other Fuchsian group is *virtually free* (ie it contains a free subgroup of finite index). It is a classical fact that every non-elementary, non-discrete, finitely generated subgroup of $\mathsf{PSL}(2,\mathbb{R})$ contains an elliptic element of infinite order, and this element is conjugate in $\mathsf{PSL}(2,\mathbb{R})$ to an irrational rotation $R_\alpha \in \mathsf{SO}(2)$.

The topological conjugacy class of subgroups G of $\mathsf{PSL}(2,\mathbb{R})$ in $\mathsf{Homeo}_+(\mathbb{S}^1)$ is characterized in dynamical terms by important works of Tukia, Casson–Jungreis and Gabai [Tuk88, CJ94, Gab92]. This is the so-called *convergence property*: for every sequence $\{g_n\}$ of distinct elements in G, such that $\{g_{n_k}\}$ is not equicontinuous, there exists a subsequence $\{g_{n_k}\}$ and points $a, b \in \mathbb{S}^1$ such that the sequence of restrictions $g_{n_k}|_{\mathbb{S}^1\setminus\{a\}}$ converges uniformly to the constant map b.

Subgroups of $\mathsf{PSL}(2,\mathbb{R})$ are very classical objects. However, understanding finitely generated, non-discrete subgroups $\Gamma \subset \mathsf{PSL}(2,\mathbb{R})$ is by no means obvious. We mention here the recent monograph by Kim, Koberda, and Mj [KKM19], which highlights how small perturbations of such Γ in $\mathsf{PSL}(2,\mathbb{R})$ can create very different dynamical behavior. Not only the dynamics, but also the group structure of such subgroups can be rather wild: for instance, by arithmetic considerations, one can find subgroups which are isomorphic to closed hyperbolic 3-manifolds groups (see Calegari [Cal06]).

In general, if G is a connected Lie group which is a subgroup of $\mathsf{Homeo}_+(\mathbb{R})$, then G is topologically conjugate to a subgroup of $\mathsf{PSL}^{(k)}(2,\mathbb{R})$, for some $k \geq 1$, the *k-fold central extension* of $\mathsf{PSL}(2,\mathbb{R})$:

$$0 \to \mathbb{Z}_k \to \mathsf{PSL}^{(k)}(2,\mathbb{R}) \to \mathsf{PSL}(2,\mathbb{R}) \to 1.$$

These groups naturally act on the corresponding k-fold cover of $\mathbb{S}^1$, which is still homeomorphic to $\mathbb{S}^1$.

Going beyond these classical examples, a more ambitious program would be to describe finitely generated subgroups of $\mathsf{Homeo}_+(\mathbb{S}^1)$. In this generality, this is a highly difficult task. Assuming some regularity on the action certainly gives more restrictions, but the situation still remains complicated: in the recent beautiful work by Kim and Koberda [KK20], it has been proved that for any $r \geq 1$, there is a finitely generated subgroup $G_r \subset \mathsf{Diff}^r_+(\mathbb{S}^1)$ which is not isomorphic to any subgroup of $\mathsf{Diff}^s_+(\mathbb{S}^1)$, for any $s > r$. Here we will simply focus on subgroups $\mathsf{Diff}^\omega_+(\mathbb{S}^1)$, hoping that the very strong regularity will be enough to grasp a good picture of the situation. More specifically, we will investigate the structure of subgroups that can be seen as the good analogue in $\mathsf{Diff}^\omega_+(\mathbb{S}^1)$ of Fuchsian groups in $\mathsf{PSL}(2,\mathbb{R})$. We will try to convince the reader that these are the *locally discrete* subgroups of $\mathsf{Diff}^\omega_+(\mathbb{S}^1)$.

7. Non-locally discrete groups

7.1. **Elementary subgroups...** Let $G \subset \mathsf{Diff}^\omega_+(\mathbb{S}^1)$ be a finitely generated elementary subgroup. If it has no finite orbit, then G is topologically conjugate to a group of rotations (here we make use of Denjoy theorem), and more precise statements about regularity of the conjugacy are the duty of Herman–Yoccoz theory, that we won't discuss here. If G has finite orbits, then there is a finite index subgroup H with fixed points. Moreover (and this is a first place real-analytic regularity comes strongly into play), every nontrivial element (and hence H) can only have finitely many fixed points. Secondly, because of real-analytic regularity, such a group H is completely determined by its image in the group of real-analytic germs $H_x \subset \mathcal{G}^\omega_+(\mathbb{R},x)$ at a fixed point $x\in\mathbb{S}^1$. As a consequence, we can reduce the problem to the study of the local dynamics defined by H.

7.2. **... and local flows.** After the discussion of the previous paragraph, we now consider subgroups of the group $\mathcal{G}^\omega_+(\mathbb{R},0)$ of real-analytic germs at 0.

Take a nontrivial germ $f\in\mathcal{G}^\omega_+(\mathbb{R},0)$ which is k-flat, for some $k\geq 0$: $f(z)=z+a_{k+1}z^{k+1}+O(z^{k+2})$, with $a_{k+1}\neq 0$. The germ f is the time 1 of the flow of a formal vector field χ at 0, and the time t map f^t $(t\in\mathbb{R})$ is of the form $f^t(z)=z+ta_{k+1}z^{k+1}+O(z^{k+2})$. We claim that every germ g commuting with f belongs to this flow. For this, write the commutation relation $fg=gf$ as a system of equations of the coefficients of f and g, then one sees that g is completely determined by its $(k+1)$-th coefficient, which must equal ta_{k+1} for some t. Proceeding in this way, one succeeds in classifying solvable groups of germs [Ily78, Nak94].

What happens for non-solvable groups of germs? Let us study one particular example.

Example 7.1. Let $f\in\mathcal{G}^\omega_+(\mathbb{R},0)$ be a germ with nontrivial linear part, and choose coordinates (after Poincaré–Kœnigs linearization theorem) so that $f(z)=\lambda z$. Take another germ $h\in\mathcal{G}^\omega_+(\mathbb{R},0)$ which is k-flat, $k\geq 1$, write $h(z)=z+a_{k+1}z^{k+1}+O(z^{k+2})$. For integers $m,n\in\mathbb{Z}$ we have $h^m(z)=z+ma_{k+1}z^{k+1}+O(z^{k+2})$ and

$$f^{-n}h^mf^n(z)=z+\lambda^{nk}ma_{k+1}z^{k+1}+O(\lambda^{n(k+1)}z^{k+2}).$$

Assume without loss of generality $\lambda<1$, then for every fixed $m\in\mathbb{Z}$, the sequence $\{f^{-n}h^mf^n\}$ converges to the identity as $n\to\infty$. Moreover, for fixed t, we can choose a sequence $\{m_n\}$ such that $\lambda^{nk}m_n\to t$ as $n\to\infty$, so that $f^{-n}h^{m_n}f^n\to z+ta_{k+1}z^{k+1}+\dots$ as $n\to\infty$. In other words, the group generated by f and g contains a *local flow* in its closure.

In the general case, the picture is pretty close. Let f be arbitrary; take a germ $g\in\mathcal{G}^\omega_+(\mathbb{R},0)$ which does not commute with f, nor does f commute with the commutator $h=[f,g]$. Then the commutator $h=[f,g]$ is k'-flat for some $k'>k$. Up to taking the inverse of f, we can assume that $a_{k+1}>0$. Then the sequence of conjugates f^nhf^{-n} converges to id as $n\to\infty$. In [Nak94], Nakai proved that there exists an appropriate rescaling $\lambda_n(f^nhf^{-n}-\mathsf{id})$ which converges, as $n\to\infty$, to a vector field $\chi_1=\chi(f,h)$ and the flow of the vector field χ_1 belongs to the closure of the set of conjugates $\{f^nh^mf^{-n}\}_{n,m}$. In the same way, there is a limit vector field $\chi_2=\chi(h,[f,h])$ verifying the analogous property. Moreover, the two vector fields χ_1 and χ_2 are linearly independent, and are preserved under topological conjugacy. In conclusion, the dynamics of the group $\langle f, g\rangle$ on a left neighborhood of 0 is very rich, in the sense that every orbit is dense and it is well described by *local flows* which are a topological invariant of the group.

7.3. **Non-elementary subgroups.** The work of Nakai was later extended by Rebelo [Reb99] to the case of non-elementary subgroups. For this, one has to assume that G contains elements sufficiently close to the identity, more precisely, one wants G to be *non locally discrete* in the following sense:

Definition 7.2. Let $G \subset \mathsf{Diff}^{\omega}_{+}(\mathbb{S}^1)$ be a subgroup and $x \in \mathbb{S}^1$ be a point. One says that G is *non locally discrete at* x (in the C^1 topology) if there exists a neighborhood I of x and a sequence of elements $\{g_n\} \subset G$ such that $g_n|_I \to \mathsf{id}|_I$ in the C^1 topology. If G is non locally discrete at every point of the minimal invariant subset Λ, then one simply says that G is *non locally discrete*.

Observe that by minimality of the action of G on Λ, if G is non locally discrete at a point $x \in \Lambda$, then it is everywhere non locally discrete in Λ.

It is time to recall a fundamental result due to Hector. which unfortunately has no satisfactory analogue in lower regularity (*this issue was pointed out in* [Dip78] *in the 1970s and is fundamental to go beyond real-analytic regularity*).

Theorem 7.3 (Hector's lemma)**.** *Let $G \subset \mathsf{Diff}^{\omega}_{+}(\mathbb{S}^1)$ be a finitely generated subgroup with a minimal invariant Cantor set $\Lambda \subset \mathbb{S}^1$. Let J be a connected component of the complement $\mathbb{S}^1 \setminus \Lambda$. Then the stabilizer $\mathsf{Stab}_G(J)$ is infinite cyclic.*

Remark 7.4. Although not strictly needed in this exposition, let us mention that a weaker version of this result holds in C^2 regularity: $\mathsf{Stab}_G(J)$ is always nontrivial; moreover, writing $J = (a, b)$, there is at most a cyclic group of germs at a (or b) which are nontrivial at order k, for every k. This can be deduced arguing as in Example 7.1. The major problem is to understand whether $\mathsf{Stab}_G(J)$ can contain an element whose germ at a is infinitely flat, but nontrivial on Λ.

From Theorem 7.3, we deduce that when Λ is a Cantor set, then G must be locally discrete at points $x \notin \Lambda$. In fact, the following holds:

Theorem 7.5 (Rebelo)**.** *Let $G \subset \mathsf{Diff}^{\omega}_{+}(\mathbb{S}^1)$ be a finitely generated non-elementary subgroup which is non locally discrete. Then the action of G on $\mathbb{S}^1$ is minimal, that is, $\Lambda = \mathbb{S}^1$.*

The key observation by Rebelo is that such a group contains local flows in the closure, essentially by the same argument as in Example 7.1. It relies on a second fundamental result, which nowadays we can see as a consequence of the local exponential contractions we mentioned before (and which holds in much more general context, see [DKN07]).

Theorem 7.6 (Sacksteder)**.** *Let $G \subset \mathsf{Diff}^{\omega}_{+}(\mathbb{S}^1)$ be a non-elementary, finitely generated subgroup and $\Lambda \subset \mathbb{S}^1$ its minimal invariant subset. Then for every open interval U intersecting Λ, there exists $p \in U \cap \Lambda$ and $g \in G$ such that $g(p) = p$ and $g(p) \neq 1$.*

In the argument revisited by Rebelo, the element f is replaced by a map with an hyperbolic fixed point $p \in I \cap \Lambda$ given by Theorem 7.6, and one plays with a sequence of elements $\{g_n\}$ whose restriction to I converges to the identity in the C^1 topology, as $n \to \infty$. It is at this point that the C^1 topology is needed: the elements g_n do not necessarily fix the point p, and one needs to control their powers. Rebelo proves that for sufficiently small $|t| < \varepsilon$, there exist sequences $\{k_n\}_n$ and $\{\ell_n(t)\}$ such that $f^{-k_n} g_n^{\ell_n(t)} f^{k_n}$ converges, as $n \to \infty$, to the time t of a locally defined flow.

Non locally discrete groups have been extensively studied by Rebelo and collaborators. Let us point out one further consequence of local flows:

Proposition 7.7. *Let $G \subset \mathsf{Diff}^\omega_+(\mathbb{S}^1)$ be a finitely generated non-elementary subgroup which is non locally discrete. Then the action of G on $\mathbb{S}^1$ is expanding: for every $x \in \mathbb{S}^1$ there exists $g \in G$ such that $g'(x) > 1$.*

Proof. Indeed, suppose that for such a group there exists a point x such that $g'(x) \leq 1$ for every $g \in G$. Let I be a neighborhood of x on which local flows are defined, and take a point $p \in I$ and $h \in G$ such that $h(p) = p$, $h'(p) \leq 1$, given by Theorem 7.6. Then using the local flow, we conjugate h to an element h_ε having a hyperbolic fixed point p_ε which is ε-close to x, with derivative $h'_\varepsilon(p_\varepsilon) = h'(p)$ and, which is more important, we keep control on its derivative around p_ε because we conjugate by elements which are close to the identity. By this control on derivative, there exists $\varepsilon > 0$ such that $h'_\varepsilon(p) > 1$, a contradiction. □

8. Introduction to locally discrete groups

The discussion in the previous paragraph indicates that the notion of discreteness is not well-suited for treating groups of real-analytic diffeomorphisms, but rather local discreteness is the appropriate property. Note that from the discussion in Sections 7.1 and 7.2, locally discrete subgroups of $\mathsf{Diff}^\omega_+(\mathbb{S}^1)$ which are elementary are virtually cyclic, and the dynamics is completely determined by periodic orbits. More generally, we believe that even non-elementary locally discrete groups have a simple description.

Conjecture 8.1. *Let $G \subset \mathsf{Diff}^\omega_+(\mathbb{S}^1)$ be a finitely generated, locally discrete subgroup. One of the following holds:*

1. *either G is C^ω conjugate to a cocompact discrete subgroup of $\mathsf{PSL}^{(k)}(2, \mathbb{R})$, for some $k \geq 1$, or*
2. *G is virtually free, and the action is described by a ping-pong partition (the precise definition appears in [10]).*

This conjecture has been validated for groups G which are virtually free [DKN18, 10], with one end, finitely presented and bounded torsion [FK14] and groups with infinitely many ends [9]. The first possibility has been completely described [Der13]:

Theorem 8.2 (Deroin)**.** *Let $G \subset \mathsf{Diff}^\omega_+(\mathbb{S}^1)$ be a finitely generated, locally discrete subgroup. Then G is C^ω conjugate to a cocompact discrete subgroup of $\mathsf{PSL}^{(k)}(2, \mathbb{R})$, for some $k \geq 1$, if and only if the action of G on $\mathbb{S}^1$ is expanding.*

The strategy for working on this conjecture, besides the work of Deroin, is to understand the set of *non-expandable points* $\mathsf{NE}(G) = \{x \in \Lambda : g'(x) \leq 1 \text{ for every } g \in G\}$. In the case of Fuchsian groups, we observed that these points are related to the cusps of the quotient surface. Similarly here one wants to *prove that every $x \in \mathsf{NE}(G)$ is a parabolic fixed point for some element in G*. Control of affine distortion is the key: given an interval J and a C^1 map $f : \mathbb{S}^1 \to \mathbb{S}^1$, one defines the *distortion coefficient*

$$\varkappa(f; J) = \sup_{x,y \in J} \log \frac{f'(x)}{f'(y)}.$$

This is a classical way to measure how f deviates from being an affine map: $\varkappa(f;J) \geq 0$ and $\varkappa(f;J) = 0$ if and only if f is affine on J. Moreover, this coefficient has the nice feature to well-behave under composition:

$$\varkappa(gf;J) \leq \varkappa(g;f(J)) + \varkappa(f;J). \tag{8.1}$$

$$\varkappa(f;I\cup J) \leq \varkappa(f;I) + \varkappa(f;J). \tag{8.2}$$

The second important property is that $\varkappa(f;J)$ is a continuous Lipschitz function with respect to J:

$$\varkappa(f;J) \leq \sup_{\mathbb{S}^1} |(\log f')'| \cdot |J|. \tag{8.3}$$

By the mean value theorem, the distortion coefficient allows to replace the size of an interval with the derivative at one given point, and vice versa: for every $x_0 \in J$ one has

$$e^{-\varkappa(f;J)} f'(x_0) \leq \frac{|f(J)|}{|J|} \leq e^{\varkappa(f;J)} f'(x_0). \tag{8.4}$$

Finally, one important property is that a good control of distortion at some point gives a slightly worse control of distortion on a small neighborhood. This can be made more precise, and it is the so-called Schwartz's lemma [Sch63], emphasized also by Sullivan [Sul83]. Let us give a simple illustration of how this works:

Lemma 8.3 (Schwartz)**.** *Let $\mathcal{G} \subset \mathsf{Diff}^2_+(\mathbb{S}^1)$ be a finite subset and set*

$$C_{\mathcal{G}} = \max_{g\in\mathcal{G}} \sup_{\mathbb{S}^1} |(\log g')'|.$$

Let $\{g_n\}_{n\in\mathbb{N}} \subset \mathsf{Diff}^2_+(\mathbb{S}^1)$ be a sequence of distinct elements with $g_0 = \mathsf{id}$ and $g_n g_{n-1}^{-1} = s_n \in \mathcal{G}$ for every $n \in \mathbb{N}$. Let $J \subset \mathbb{S}^1$ be an interval with

$$S = \sum_{n=0}^{\infty} |g_n(J)| < \infty.$$

Then, for every interval $T \supset J$ such that $|T| \leq (1 + e^{-2C_{\mathcal{G}}S})|J|$, we have $|g_n(T)| \leq 2|g_n(J)|$ for every $n \in \mathbb{N}$.

Proof. Let us prove this by induction on n. When $n = 0$, this holds by assumption. Let us assume $|g_j(T)| \leq 2|g_j(J)|$ for every $j \leq n-1$. Then by sub-addivity (8.1) and Lipschitz control (8.3), we have

$$\varkappa(g_n;T) \leq C_{\mathcal{G}} \sum_{j=0}^{n-1} |g_j(T)| \leq 2C_{\mathcal{G}}S.$$

Take a point $x_n \in J$ such that $g_n'(x_n) = \frac{|g_n(J)|}{|J|}$, then for every $y \in T$ we have

$$g_n'(y) \leq e^{2C_{\mathcal{G}}S} g_n'(x_n) = e^{2C_{\mathcal{G}}S} \frac{|g_n(J)|}{|J|}.$$

We deduce

$$|g_n(T)| = |g_n(J)| + |g_n(T\setminus J)| \leq |g_n(J)| + e^{2C_{\mathcal{G}}S} \frac{|g_n(J)|}{|J|} |T\setminus J| \leq 2,$$

which is the desired inequality. ∎

Let us now explain the approach that has been successful so far [DKN18, FK14, 9]. Take a point $x_0 \in \mathsf{NE}(G)$ and observe that its G-orbit is dense in Λ. This means that there is a sequence of elements $\{g_n\}$ and such that $g_n(x_0) - x_0 = \varepsilon_n \searrow 0$. If one can obtain a good control of distortion for these maps on the intervals $J_n = [x_0, g_n(x_0)]$, then one must have that the derivative of g_n is close to 1 on J_n (the conditions $g_n'(x_0), (g_n^{-1})'(x_0) \leq 1$ imply that there exists a point $y_n \in J_n$ such that $g_n'(y_n) = 1$), and so g_n is close to the identity on J_n (because $g_n(x_0)$ is ε_n-close to x_0). Typically one would like to take as g_n the element in the ball of radius n in G (with respect to some generating system), giving the closest return of x_0, but in practice the choice of g_n has to be adapted to the case under consideration.

9. Actions with minimal invariant Cantor sets

The first structural results for locally discrete groups go back to the end of the 1970s and describe the case of minimal invariant Cantor sets. In what follows, we will consider a finitely generated subgroup $G \subset \mathsf{Diff}_+^\omega(\mathbb{S}^1)$ with a minimal invariant Cantor set $\Lambda \subset \mathbb{S}^1$. By a *gap* of Λ, we mean a connected component of the complement $\mathbb{S}^1 \setminus \Lambda$. Note that by Rebelo's theorem (Theorem 7.5), such a subgroup G is necessarily locally discrete.

9.1. Ends of Schreier graphs.

The first result, although unpublished for a long time, is due to Duminy.

Theorem 9.1 (Duminy)**.** *If a finitely generated subgroup $G \subset \mathsf{Diff}_+^\omega(\mathbb{S}^1)$ has a minimal invariant Cantor set, then G has infinitely many ends.*

Note that such a statement highly requires real-analytic regularity: by a construction of Ghys and Sergiescu [GS87], Thompson's group T, which is an infinite simple group and thus with one end, admits a C^∞ action with minimal invariant Cantor set. But Duminy's original theorem is much more general and holds for transversely C^2, codimension-one foliations with exceptional minimal sets. See [Nav11, §3] for a detailed exposition. In the case of group actions on the circle, it describes the large scale geometry of the orbit of gaps J of Λ. Theorem 9.1 is then obtained as a consequence, using two lemmas of control of affine distortion, one of which is Hector's lemma (the C^2 version of Theorem 7.3). More precisely, given a finite generating system $\mathcal{G} \subset G$, we consider the *Schreier graph* $\mathsf{Sch}(X, \mathcal{G})$ of the orbit $X = G(J_0)$ of a fixed gap J_0: this is the graph whose vertices are gaps in X and whose edges are of the form $(J, s(J))$, where $s \in \mathcal{G}$ is a generator. We consider the Schreier graph as a one-dimensional simplicial complex, and in particular it is a connected topological space. Note that the number of ends of $\mathsf{Sch}(X, \mathcal{G})$ does not depend on the choice of $\mathcal{G}$. The following statement is a more faithful formulation of original Duminy's theorem.

Theorem 9.2 (Duminy)**.** *If a finitely generated subgroup $G \subset \mathsf{Diff}_+^2(\mathbb{S}^1)$ has a minimal invariant Cantor set, then the orbit of every gap has infinitely many ends.*

We give here an outline of the proof of Theorem 9.2, assuming that the action is C^ω. The idea is to show that if $\mathsf{Sch}(X, \mathcal{G})$ has finitely many ends, then G preserves an affine structure an therefore is elementary. One starts with a reference affine structure, by taking an element h with hyperbolic fixed point $p \in \Lambda$, and a linearizing C^ω coordinate in a neighborhood of p (by Poincaré-Kœnigs linearization theorem). That is, we take an interval I containing p, and we assume that h is a linear contraction of I fixing p. We want to show that return maps to I induced by the group action are affine. At infinitesimal level, a map f defined on I is affine if its *nonlinearity* $\mathcal{N}(f) = (\log f')'$ vanishes on I. One key property of the nonlinearity is that it is a cocycle when considered as a differential 1-form:

$$\mathcal{N}(f \circ g) = \mathcal{N}(f) \circ g \cdot g' + \mathcal{N}(g).$$

To get rid of the annoying derivative g' in the cocycle relation, we integrate $\mathcal{N}(f)$ over a gap $J \in X$, and set $N(f) = \int_{J_0} \mathcal{N}(f)$.

If f_0 generates $\mathsf{Stab}_G(J_0)$ (we use Theorem 7.3 here, and hence the C^ω assumption, although not in a crucial way), we set $b = \int_{J_0} \mathcal{N}(f_0)$. Then the map

$$\begin{array}{rl} X \cong G/\langle f_0 \rangle \to & \mathbb{R}/b\mathbb{Z} \\ J = f(J_0) & \mapsto N(J) := N(f) \end{array}$$

is well-defined and satisfies

$$N(f(J)) - N(J) = \int_J \mathcal{N}(f). \tag{9.1}$$

Assume for simplicity that $\mathsf{Sch}(X, \mathcal{G})$ has one end. Take an element $g \in G$, and a gap $J \subset I$ such that $g(J) \subset I$ (this is not restrictive, by minimality). As we are assuming that $\mathsf{Sch}(X, \mathcal{G})$ has one end, for any $\varepsilon > 0$, we can take $n \in \mathbb{N}$ such that $h^n(J)$ and $h^n(g(J))$ are joined in $\mathsf{Sch}(X, \mathcal{G})$ by a path visiting gaps whose length sums up to a quantity $< \varepsilon$. Then, using that $\mathcal{N}(h) = 0$, we deduce from (9.1) that $\int_J \mathcal{N}(g)$ is comparable to the sum of the lengths of these gaps, and therefore arbitrarily close to 0.

In C^ω regularity, we can now take a shortcut to conclude, by considering infinitely many gaps J in I: if $\int_J \mathcal{N}(g) = 0$, then $\mathcal{N}(g)$ admits a zero on J, and thus (by analytic continuation principle), $\mathcal{N}(g)$ is identically zero. In lower regularity, this argument will give that g is locally almost affine, and using Schwartz's lemma on control of distortion, one can put this in good quantitative terms.

9.2. **A first structure theorem.** After the result of Duminy, Ghys obtained in [Ghy87] a very striking improvement.

Theorem 9.3 (Ghys). *If a finitely generated subgroup $G \subset \mathsf{Diff}^\omega_+(\mathbb{S}^1)$ has a minimal invariant Cantor set, then it is virtually free.*

An elementary argument is given by Ghys in the case G is *accessible* in the sense of Dunwoody (note we still don't know whether every locally discrete subgroup $G \subset \mathsf{Diff}^\omega_+(\mathbb{S}^1)$ is accessible). After Theorem 9.1, G has infinitely many ends, and thus, by Stalling's theorem, it splits over a finite subgroup. Let us assume, for simplicity, that $G = H *_A K$ is an amalgamated product over a finite group A (the case of HNN extension is treated similarly). Note that A must be finite cyclic, and that H and K cannot act minimally. If H and K have finite orbits, then they are virtually cyclic, and we conclude. Otherwise we keep applying this argument. Accessibility guarantees that his process ends.

In the general case, the proof of Ghys is of cohomological nature. In our opinion, it would be very enlightening to find a completely dynamical approach to Theorem 9.3.

10. Ping-pong actions

10.1. **Free groups.** After Theorem 9.3 we see that virtually free groups represent a distinguished class of locally discrete subgroups in $\mathsf{Diff}^\omega_+(\mathbb{S}^1)$, and this motivates a more detailed study of such class of groups. Avoiding technical details (which may require hard algebraic work), we will limit our discussion to *free groups*, following the fundamental work by Deroin, Kleptsyn, and Navas [DKN18]. We recall the main construction in [DKN18] for free groups. We denote by G a rank-n free group. We choose S_0 a system of free generators for G, and write $S = S_0 \cup S_0^{-1}$. We denote by $\| \cdot \|$ the word norm defined by S. Recall that any element g in G may be written in unique way

$$g = g_\ell \cdots g_1, \quad g_i \in S,$$

with the property that if $g_i = s$ then $g_{i+1} \neq s^{-1}$. This is called the *normal form* of g. For any $s \in S$, we define the subset

$$W_s := \{g \in G \mid g = g_\ell \cdots g_1 \text{ in normal form, with } g_1 = s\}\,.$$

If X denotes the Cayley graph of G with respect to the generating set S (which is a $2n$-regular tree), then the subsets W_s are exactly the $2n$ connected components of $X \setminus \{\mathsf{id}\}$, with W_s being the connected component containing s. It is easy to see that the generators S play ping-pong with the subsets W_s: for any $s \in S$, we have $(X \setminus W_{s^{-1}})\, s \subset W_s$ (we consider the right action of G on X). Using the action on the circle, we can push this ping-pong partition of the Cayley graph of G to a partition of the circle into open intervals with very nice dynamical properties. Given $s \in S$, we define

$$U_s = \left\{ x \in \mathbb{S}^1 \,\middle|\, \exists \text{ neighbourhood } I_x \ni x \text{ s.t. } \lim_{n\to\infty} \sup_{g \notin W_s, \|g\| \geq n} |g(I_x)| = 0 \right\}. \tag{10.1}$$

In [DKN18] it is proved the following:

Theorem 10.1 (Deroin, Kleptsyn, and Navas). *Let $G \subset \mathsf{Diff}_+^\omega(\mathbb{S}^1)$ be a finitely generated, locally discrete, free group of real-analytic circle diffeomorphisms. Let S_0 be a system of free generators for G and write $S = S_0 \cup S_0^{-1}$. Consider the collection $\{U_s\}_{s\in S}$ defined in (10.1). We have:*

(1) *every U_s is open;*
(2) *every U_s is the union of finitely many intervals;*
(3) *any two different U_s have empty intersection inside the minimal invariant set Λ_G;*
(4) *the union of the U_s covers all but finitely many points of Λ_G;*
(5) *if $s \in S$, $t \neq s$ then $s(U_t) \subset U_{s^{-1}}$.*

Definition 10.2. Let $G \subset \mathsf{Homeo}_+(\mathbb{S}^1)$ be a finitely generated, free group of circle homeomorphisms and let $S = S_0 \cup S_0^{-1}$ be a symmetric free generating set. A collection $\{U_s\}_{s\in S}$ of subset of $\mathbb{S}^1$ is a *ping-pong partition* for (G, S) if it verifies all conditions (1-5) in Theorem 10.1.

For $s \in S$, denote by U_s^* the subset of U_s which is the union of the connected components of U_s intersecting Λ_G. The *skeleton* of the ping-pong partition is the data consisting of

(1) The cyclic order in $\mathbb{S}^1$ of the intersection of connected components of $\bigcup_{s\in S} U_s$ with Λ_G, and
(2) For each $s \in S$, the assignment of connected components

$$\lambda_s : \pi_0 \left(\bigcup_{t \in S\setminus\{s\}} U_t^* \right) \to \pi_0\left(U_{s^{-1}}\right)$$

induced by the action.

Remark 10.3. In [DKN18] the definition of the sets U_s (there called $\widetilde{\mathcal{M}_\gamma}$) is slightly different based on a control on the sum of derivatives along geodesics in the group. Here the definition that we adopt is simply topological, as we consider how neighbourhoods are contracted along geodesics in the group. This difference in the definition leads to different sets: one can show that U_s contains the corresponding $\widetilde{\mathcal{M}}_s$, and the complement $U_s \setminus \widetilde{\mathcal{M}}_s$ is a finite number of points.

Even with the different definition, the proof of Theorem 10.1 proceeds as in [DKN18]. The hardest part is to prove that property (2), which is Lemma 3.30 in [DKN18]. Property (1) is a direct consequence of the definition, (3) is an easy consequence of Theorem 7.6, (4) can be obtained from minimality (of the pieces of orbits $W_s(x)$, $x \in \mathbb{S}^1$, $s \in S$) and the other properties, as soon as one knows that at least one U_s is *non-empty* (this is not so difficult in the case Λ is a Cantor set, but it is a highly nontrivial statement for minimal actions, which requires to understand the points in $\mathsf{NE}(G)$ as explained before, and here C^ω regularity is used crucially). The ping-pong property (5) comes directly from the definition of U_s and the fact that the generators S play ping-pong with the sets W_s.

Before sketching the proof of property (2), let us state a classical result (see [Mat16, Theorem 4.7]) explaining why ping-pong partitions are important. For this, we first need the following:

Definition 10.4. Let $\rho_\nu : (G,S) \to \mathsf{Homeo}_+(\mathbb{S}^1)$, $\nu \in \{1,2\}$, be two injective representations of a finitely generated, free group with a marked symmetric free generating set $S = S_0 \cup S_0^{-1}$. Let $\{U_s^\nu\}_{s\in S}$, be a ping-pong partition for $\rho_\nu(G,S)$, for $\nu = 1,2$. We say that the two partitions are *equivalent* if they have the same skeleton.

Proposition 10.5. *Let $\rho_\nu : (G,S) \to \mathsf{Homeo}_+(\mathbb{S}^1)$, $\nu \in \{1,2\}$, be two injective representations of a finitely generated, free group with a marked symmetric free generating set $S = S_0 \cup S_0^{-1}$. Suppose that the actions on $\mathbb{S}^1$ have equivalent ping-pong partitions. Then the actions are semi-conjugate.*

In the course of the proof, we will make the simplifying assumption that the action is minimal. Even if we won't not provide full details, we stress that for this part of the proof only C^2 regularity is required. Given an element U_s of the partition, fix one of its connected components $I = (x_-, x_+)$.

Definition 10.6. An element $g \notin W_s \cup \{\mathsf{id}\}$ is *wandering* if, writing $g = g_n \cdots g_1$ in normal form, the intervals $\{g_k \cdots g_1(I)\}_{k=0}^{n-1}$ are all disjoint. We say also that g is a *first return* if $g(I) \cap I \neq \emptyset$.

Remark 10.7. If $g = g_n \cdots g_1$ is wandering, then by Lemma 10.1.5 we have $g_k \cdots g_1(I) \subset U_{g_k^{-1}}$ for every $k \in \{1,\ldots,n\}$. In particular, every first return ends with s^{-1}.

Thus, given a nontrivial element $h \notin W_s$, we can write h in a unique way as the product

$$h = f\pi_\ell \cdots \pi_0,$$

where the π_k are first returns, and f is a wandering element. We call this the *first return decomposition* of h. On the other hand, if g is a first return, then for every $h \in W_s$ there exists a minimal $k \geq 1$ such that $\overline{h} := hg^k \notin W_s$ (see [DKN18, Lemma 3.26]).

Lemma 10.8.

a) *If g and h are two distinct wandering elements, then $g(I) \cap h(I) = \emptyset$.*
b) *If $g = g_n \cdots g_1$ is a first return, then $g_n = s^{-1}$ and $g(I) \subset I$.*
c) *There exists $C > 0$ such that for every wandering element g one has $\varkappa(g; I) \leq C$.*
d) *There exists $\delta > 0$ and $C' > 0$ such that for every admissible element g one has $\varkappa(g; \tilde{I}) \leq C'$, where $\tilde{I}$ denotes the δ-neighborhood of I.*

Proof. Write $g = g_n \cdots g_1$ and $h = h_m \cdots h_1$. By the ping-pong relation (5), we have $g(I) \subset U_{g_n^{-1}}$ and $h(I) \subset U_{h_m^{-1}}$. So the images are disjoint, unless $g_n = h_m$. If this happens, consider $g' = g_{n-1} \cdots g_1$ and $h' = h_{m-1} \cdots h_1$ and repeat the argument until one of the two elements is trivial (as $g \neq h$ the other element is nontrivial). Then statement reduces to the condition of being wandering. This proves a). The second statement b) is a consequence of the ping-pong relation (5) and the fact that I is a connected component. The third statement is a consequence of a) applied to the sequence of wandering elements $\{g_k \cdots g_1\}_{k=1}^n$ and the sub-additivity of the distortion coefficient (8.1). Statement d) is obtained from c) arguing as in Lemma 8.3. □

10.2. **First returns.** We recall the following definition:

Definition 10.9. A map g is a *uniform contraction* on an interval J if there exists $0 < \rho < 1$ such that for any subinterval $E \subset J$ one has $|g(E)| \le \rho |E|$. In this case one says that g is a uniform contraction of ratio $\le \rho$.

Lemma 10.10. *Let $\tilde{I}$ be the δ-neighborhood of I as in Lemma 10.8.d and let R denote the rightmost connected component of $\tilde{I} \setminus I$. If g is a first return which is a uniform contraction on $I \cup R$, then for every $I \subset I' \subset I \cup R$ one has $g(I') \subset I'$.*

Proof. Indeed, by Lemma 10.8, one has $g(I) \subset I$. Let R' be the rightmost connected component of $I' \setminus I$. The image $g(R')$ is adjacent to the right of $g(I)$ and has length $|g(R')| < |R'|$. Therefore, $g(R')$ cannot trespass the rightmost point of I'. □

Lemma 10.11. *For any $0 < \rho < 1$, all but finitely many wandering elements are uniform contractions on $\tilde{I}$, of ratio $\le \rho$.*

Proof. Fix $\varepsilon \in (0,1)$. By Lemma 10.8, there are only finitely many first returns g such that $|g(I)| \ge \varepsilon |I|$. Consider a first return g such that $|g(I)| < \varepsilon |I|$. Then there exists a point $x_0 \in I$ such that $g'(x_0) < \varepsilon$ and hence, by Lemma 10.8.d, we have that for any $x \in \tilde{I}$, $g'(x) \le e^{C'} \varepsilon$. In particular, if $\varepsilon < \rho e^{-C'}$, such a first return is a uniform contraction on $\tilde{I}$, of ratio $\le \rho$. □

Proposition 10.12 ([DKN18], Lemma 3.23)**.** *There exist first returns g_- and g_+ that fix respectively the left and right endpoints x_- and x_+ of I.*

Proof. After Lemma 10.11, there is only a finite collection of first returns $g_1, \dots, g_m$ which are not uniform contractions on $\tilde{I}$ of ratio $\le \frac{1}{2}$. We want to show that one of the g_i fixes x_+ (there is at most one, after Lemma 10.8.a). We shall argue by contradiction. We assume first that none of the g_i fixes x_- neither; supposing that this is not the case, there exists a compact subinterval $J \subset I$, which contains a neighbourhood of all images $g_i(I)$. Let R be a neighbourhood of x_+ which is contained in $\tilde{I}$, and such that

$$g_i(I \cup R) \subset J \text{ for all } i = 1, \dots, m. \tag{10.2}$$

Observe that this condition and Lemma 10.10 give $g(I \cup R) \subset I \cup R$ for every first return g.

We fix $\varepsilon > 0$ and we want to prove that only finitely many elements $h \notin W_s$ satisfy $|h(R)| \ge \varepsilon$: this will imply that $x_+ \in U_s$, which is absurd.

Take $h \notin W_s$ with first return decomposition

$$h = f \pi_\ell \cdots \pi_0.$$

If none of the π_i is one of the fixed first returns $g_1, \dots, g_m$, then

$$|h(R)| \le \frac{|f(R)|}{2^\ell} \le \frac{1}{2^\ell}.$$

In this case ℓ must be bounded. Moreover, after Lemma 10.11, there are only finitely many wandering elements which are not uniform contractions on $\tilde{I}$ of ratio $\rho < \frac{\varepsilon}{|R|}$. We deduce that only finitely many such h can satisfy $|h(R)| \geq \varepsilon$.

Next, suppose that there is some π_i which is the first of the first returns $g_1, \ldots, g_m$ that appears in the first return decomposition of h. Let us write $h = h_1\pi_i h_2$, where no g_i appears in h_2. Note that as we are considering a first return decomposition, we have $h_1 \notin W_s$. From (10.2), we have

$$h(R) = h_1\pi_i h_2(R) \subset h_1(J).$$

Using compactness of $J \subset I$ and the definition (10.1) of U_s, we see that for at most finitely many elements h_1 we have $|h_1(J)| \geq \varepsilon$. As we have finitely many choices for $h_1\pi_i$, fix, by uniform continuity we can fix $\delta > 0$ such that if $E \subset \mathbb{S}^1$ is an interval of length $\leq \delta$, then $|h_1\pi_i(E)| \leq \varepsilon$. For such a $\delta > 0$, by the previous argument, there are only finitely many choices of elements h_2 such that $|h_2(R)| \leq \delta$. This gives the desired contradiction under the additional assumption that no g_i is fixing x_-.

In the case one of the first returns g_i is the first return g_- fixing x_-, we change the choice of the compact subinterval J, requiring that it contains a neighbourhood of all images $g_i(I)$, for $g_i \neq g_-$. Upon taking a larger J, we can take a neighbourhood R such that $g_i(I \cup R) \subset J$ for all $i = 1, \ldots, m$ (including g_-). The proof given for the previous case works *verbatim*, unless in the decomposition $h = h_1\pi_i h_1$ we have $\pi_i = g_-$. For such case, let I_+ be the interval $(I \cup R) \setminus g_-(I)$ (which contains the image of R by every possible h_2). Then $g_-(I_+)$ is compactly contained in I, and we can conclude as in the previous case. □

Note that if g_- and g_+ exist, they must be different elements. Otherwise, there would be a unique first return to I, and it would be possible to find a subinterval $J \subset I$ such that $g(J) \cap J = \emptyset$ for every nontrivial $g \in G$, contradicting minimality of the action (see [DKN18, Lemma 3.24]). More generally, we have the following.

Lemma 10.13 ([DKN18], Lemma 3.29)**.** *With the previous notations, the only fixed point of $g_\pm$ in $\overline{I}$ is $x_\pm$. Moreover, one has $g'_\pm(x_\pm) = 1$ and both $g_\pm$ are topological contractions of I.*

Proof. Assume for contradiction that g_+ fixes a nontrivial subinterval J of I. Note that this implies $g_+(I \setminus J) \subset I \setminus J$. From Lemma 10.8.a, for every wandering element $h \notin W_s$, one has either $h(J) \cap J = \emptyset$ or $h(J) = J$ (in which case $h = g_+$).

Take a nontrivial $h \in G$. Assume first that $h \notin W_s$. If h is a power of g_+, then we have $h(J) = J$. Otherwise, considering the first return decomposition of h, from our preliminary considerations on wandering elements, we get that $h(J) \cap J = \emptyset$. When $h \in W_s$, consider the minimal $k \geq 1$ such that $\overline{h} := hg_+^k \notin W_s$ (Remark 10.7). Then $h(J) = \overline{h}g_+^{-k}(J) = \overline{h}(J)$, and thus, as $\overline{h} \notin W_s$, we see from the previous case that $h(J) \cap J = \emptyset$, unless $\overline{h}$ is a power of g_+ (which implies $h = g_+^{-k}$ by minimality of k).

As a summary, we have proved that for every $g \in G$, one has $g(J) \cap J = \emptyset$, unless g is a power of g_+. As the action of $\langle g_+ \rangle$ on J cannot be minimal, we get a contradiction. Thus x_+ is the unique fixed point of g_+ on $\overline{I}$. The analogue argument works for g_-.

As a consequence, we get that both $g_\pm$ are contractions of I, so that $g'_\pm(x_\pm) \leq 1$. On the other hand, it is not difficult to observe that they are topological expansions on the other side of $x_\pm$ respectively (this is because otherwise we would get control on contractions at points $x_\pm$). This proves $g'_\pm(x_\pm) \geq 1$. □

10.3. **A finite number of connected components.**

Lemma 10.14 ([DKN18], Lemma 3.30)**.** *Each U_s has only finitely many connected components.*

Proof. Fixing one I, we can assume $|g_+(I)| \le \frac{1}{2}|I|$, so that by control of distortion we have $\varkappa(g_+; I) \le \log 2$ and hence $\sum_{i=0}^{k-1} |g_i \cdots g_1(I)| \ge \frac{\log 2}{C}$, where we write $g_+ = g_k \cdots g_1$.

For a given I and the corresponding g_+, write $g_+ = g_k \cdots g_1$, and say that $\Phi_+(I)$ is the connected component of the U_s which contains $g_1(I)$. Similarly one defines Φ_-. This gives a decomposition of the connected components of the U_s into finite disjoint cycles, where the total length along a cycle is bounded from below. Thus there are finitely many cycles. □

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