Mathematical and numerical study of a Gross-Clark-Schrödinger system

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Abstract

This thesis deals with the study of a Gross-Clark-Schrödinger system which models the motion of an impurity in a Bose condensate.

We have first shown that the Cauchy problem for this system is globally well posed in the associated energy space. The approach used is quite classical and is based on Strichartz type estimates and on a fixed point theorem.

In a second step, we are interested in the travelling waves of this system. These special solutions have been studied since 1974 by physicists using formal asymptotic developments and some numerical simulations. In one dimension space the existence of these solutions and some properties have been rigorously established in 2006. Despite several attempts, there is no rigorous proof in the literature of the existence of travelling waves in dimension greater than or equal to two. We have used several approaches to show the existence, based on ideas and tools recently developed in Calculus of Variations. One of them consists in minimizing the energy associated to the system under two constraints, at constant mass and constant momentum. We have shown that minimizing travelling waves exist for any pair (moment, mass) that verifies a strict subadditivity condition of the minimal energy as a function of two variables.

In parallel, we have performed numerical simulations that have well highlighted the travelling waves in the cases that correspond to the physical applications, we have obtained their profiles and we have calculated their energy levels.

We have also studied other types of special solutions, including zero-momentum ground states and bubble-vortex solutions.

Résumé

Cette thèse porte sur l'étude d'un système de Gross-Clark-Schrödinger qui modélise le mouvement d'une impureté dans un condensat de Bose.

Nous avons d'abord montré que le problème de Cauchy pour ce système est globalement bien posé dans l'espace d'énergie associé. L'approche utilisée est assez classique et est basée sur des estimations de type Strichartz ainsi que sur l'utilisation d'un théorème de point fixe.

Dans un second temps nous nous sommes intéressés aux ondes progressives de ce système. Ces solutions spéciales ont été étudiées dès 1974 par des physiciens à l'aide des développements asymptotiques formels et de quelques simulations numériques. En dimension un d'espace l'existence de ces solutions et quelques propriétés ont été établies rigoureusement en 2006. Malgré plusieurs tentatives, il n'existe dans la littérature aucune preuve rigoureuse de l'existence des ondes progressives en dimension supérieure ou égale à deux. Nous avons utilisé plusieurs approches pour montrer l'existence, basées sur des idées et des outils récemment développés en Calcul des Variations. Une d'elles consiste à minimiser l'énergie associée au système sous deux contraintes, à masse constante et à moment constant. Nous avons montré que les ondes progressives minimisantes existent pour tout couple (moment, masse) qui vérifie une condition de stricte sous-additivité de l'énergie minimale comme fonction de deux variables.

En parallèle, nous avons effectué des simulations numériques qui ont bien mis en évidence les ondes progressives dans les cas qui correspondent aux applications physiques, nous avons obtenu leurs profils et nous avons calculé leurs niveaux d'énergie.

Nous avons étudié également d'autres types de solutions spéciales, notamment les états fondamentaux de moment nul et les solutions de type bulle-vortex.

Contents

Chapter 1

Introduction

1.1 English version

1.1.1 Presentation of the problem

We consider the following Gross-Clark system:

$$\text{(GC)} \qquad \begin{cases} i\partial_t \Psi &= -\Delta\Psi + \frac{1}{\varepsilon^2}(\frac{1}{\varepsilon^2}|\Phi|^2 + |\Psi|^2 - 1)\Psi \\ \\ i\delta\partial_t \Phi &= -\Delta\Phi + \frac{1}{\varepsilon^2}(q^2|\Psi|^2 - \varepsilon^2 k^2)\Phi \end{cases} \qquad \text{in } \mathbb{R}\times\mathbb{R}^N,$$

with the "boundary conditions"

$$|\Psi| \to 1, \Phi \to 0 \text{ as } |x| \to \infty.$$

This system, is originally introduced by Clark and Gross, has been studied by Grant and Roberts (see [12]). It models the movement of an impurity in a Bose-Einstein condensate. The functions Ψ and Φ are the wavefunctions for the bosons and for the impurity, respectively. Several physical parameters are relevant for the system (GC): In the second equation in (GC) we have put $\delta = \frac{\mu}{M}$, where μ is the mass of the impurity, and M is the boson mass; since μ is small compared to M, δ is supposed to be small. We have denoted $q^2 = \frac{l}{2d}$, where l is the boson-impurity scattering length, and d the boson diameter, while k is a dimensionless measure for the single-particle impurity energy. The parameter ε is defined by $\varepsilon = \left(\frac{b\mu}{lM}\right)^{\frac{1}{5}}$, where b is the "healing length"; in applications we have $\varepsilon \cong 0.2$. The system (GC) has been studied in [12], where the effective radius and the induced mass of the uncharged impurity were computed using formal asymptotic expansions and some numerical experiments.

When $\Phi = 0$, the system (GC) reduces to the Gross-Pitaevskii equation

$$i\partial_t \Psi = -\Delta\Psi + \frac{1}{\varepsilon^2}(|\Psi|^2 - 1)\Psi \qquad \text{in } \mathbb{R}\times\mathbb{R}^N. \tag{1.1}$$

Both the system (GC) and the Gross-Pitaevskii equation (1.1) are Hamiltonian. The conserved energies are, respectively

$$E(\psi,\varphi) = \int_{\mathbb{R}^N} |\nabla\psi|^2 + \frac{1}{2\varepsilon^2}\big(|\psi|^2 - 1\big)^2 + \frac{1}{\varepsilon^2 q^2}|\nabla\varphi|^2 + \frac{1}{\varepsilon^4}|\psi|^2|\varphi|^2 \mathrm{d}x \tag{1.2}$$

and the *Ginzburg-Landau energy* of Ψ, namely

$$E_1(\psi) = \int_{\mathbb{R}^N} |\nabla\psi|^2 + \frac{1}{2\varepsilon^2}\big(|\psi|^2 - 1\big)^2 \mathrm{d}x. \tag{1.3}$$

Other quantities conserved by the flow of (GC) are the momentum $\mathbf{P}(\Psi) = \big(P_1(\Psi), \cdots, P_N(\Psi)\big)$, and the mass $\mathbf{M}(\Phi) = \int_{\mathbb{R}^N} |\Phi|^2 \mathrm{d}x$. The momentum was rigorously defined in [21] in dimension $N \geq 3$, then in [8] if $N = 2$. We will recall the main ideas of this definition later. If Ψ is a function sufficiently localized in space, the momentum is $P_k(\Psi) = \int_{\mathbb{R}^N} \langle i\Psi, \frac{\partial \Psi}{\partial x_k} \rangle \mathrm{d}x$, for $k \in \{1, \cdots, N\}$, and where $\langle ., . \rangle$ is the usual scalar product in $\mathbb{C} \simeq \mathbb{R}^2$.
We denote by Q the momentum with respect to the x_1 direction, *i.e* $Q(\psi) = P_1(\psi)$. The momentum has a nice behavior with respect to dilations, *i.e* if $\psi \in \mathcal{E}$, $\gamma, \sigma > 0$, $\psi_{\gamma,\sigma} := \psi(\frac{x_1}{\gamma}, \frac{x'}{\sigma})$, then

$$Q(\psi_{\gamma,\sigma}) = \sigma^{N-1} Q(\psi), \tag{1.4}$$

where $\mathcal{E}$ is the function space naturally associated to the Gross-Pitaevskii equation defined by:

$$\mathcal{E} = \big\{\psi : \mathbb{R}^N \longrightarrow \mathbb{C}; \psi \text{ is measurable}, |\psi| - 1 \in L^2(\mathbb{R}^N), \nabla\psi \in L^2(\mathbb{R}^N)\big\}. \tag{1.5}$$

There are several equivalent definitions for $\mathcal{E}$. If $N \in \{2, 3, 4\}$, it can be proved that $\mathcal{E} = \mathbf{E}$, where

$$\mathbf{E} = \big\{\psi \in H^1_{\mathrm{loc}}(\mathbb{R}^N) \mid \nabla\psi \in L^2(\mathbb{R}^N), |\psi|^2 - 1 \in L^2(\mathbb{R}^N)\big\}. \tag{1.6}$$

1.1.2 Existing results

The nonlinear Schrödinger equation

The results corresponding to the nonlinear Schrödinger equation,

$$i\frac{\partial \Psi}{\partial t} + \Delta\psi + F(|\Psi|^2)\Psi = 0 \qquad \text{in } \mathbb{R} \times \mathbb{R}^N, \tag{1.7}$$

with the boundary condition $|\Psi| \to 1$ as $|x| \to \infty$, are several. The most important cases that have been extensively studied by physicists and mathematicians are the Gross-Pitaevskii (GP) equation, where $F(s) = 1 - s$, and the so-called "cubic-quintic" Schrödinger equation, where $F(s) = -\alpha_1 + \alpha_3 s - \alpha_5 s^2$, α_1, α_3 and α_5 being positive constants and F has two positive roots.

Denoting $V(s) = \int_s^1 F(\tau)\mathrm{d}\tau$, we define the conserved energy E_1 with a general nonlinearity by

$$E_1(\Psi) = \int_{\mathbb{R}^N} |\nabla\Psi|^2 \mathrm{d}x + \int_{\mathbb{R}^N} V(|\Psi|^2)\mathrm{d}x. \tag{1.8}$$

For this equation, a particular attention has been paid to the travelling waves solutions in a series of papers (see, e.g., [1], [2], [12], [15], [16], [17]). These are solutions of the form $\Psi(t, x) = \psi(x_1 - ct, x')$ that must satisfy the equation

$$-ic\frac{\partial \psi}{\partial x_1} + \Delta\psi + F(|\psi|^2)\psi = 0, \tag{1.9}$$

where $x' = (x_2, \cdots, x_N)$ and c is the speed of the travelling wave. In [21], it is proved that there exist nontrivial finite energy travelling waves moving with speed c in any space dimension $N \geq 3$, for a large class of nonlinear Schrödinger equations with nonzero conditions at infinity, for any speed c less than the sound velocity, and under general conditions on the nonlinearity F. The sound velocity is defined as follows: if F is C^1 near 1, $F'(1) < 0$, the sound velocity at infinity associated to (1.1) is $v_s = \sqrt{-2F'(1)}$. These solutions are critical points of the energy E_1 when the momentum is fixed.

C. A. Jones, C. J. Putterman and P. H. Roberts computed the energy and the momentum of the travelling waves they have found numercially for the Gross-Pitevskii equation. The rigorous proof of the existence of travelling waves for (GP) in space dimension $N = 2$ was done in [5] for all speed in

some interval $(0, \eta)$ where η is small. Minimizing the energy at fixed momentum has been used first in [4] to construct a sequence of travelling waves with speeds tending to 0 in dimension $N \geq 3$. The same result was established in [7] for small speeds by using a mountain-pass argument. In space dimension $N = 2$ and $N = 3$ the existence of travelling waves for the Gross-Pitaevskii equation was done in [3] by minimizing the energy at fixed momentum. In this case, the Lagrange multiplier associated to the obtained minimizers corresponds to the propagation speed, and if $N = 2$, the result is obtained for any speed in a set $A \subset (0, v_s)$ where v_s is the sound velocity at infinity, and A contains points very close to 0 and v_s. Later on, it was shown in [8] that the minimization of the energy at fixed momentum can be used in any dimension $N \geq 2$ for general nonlinearities under the assumption that V is nonegative. Moreover, the set of solutions is orbitally stable. Unfortunately, in this case it is not clear that the set of speeds of travelling waves constructed forms an interval. Also, it was proved in [3] and in [18] that in space dimension $N \geq 3$ there exists $v_0 \in (0, v_s)$ such that minimizing the energy at fixed momentum can not give travelling waves of speed $c \in (v_0, v_s)$.

In [21], we have an existence result that covers the whole range $(0, v_s)$ of possible speeds if $N \geq 3$, for general nonlinearities. For the (GP) equation the nonexistence of travelling waves for $c > v_s$ was done in [13]. For a large class of nonlinearities, it was done in [20].

The travelling waves have the best regularity allowed by the nonlinearity F, for example in the case of (GP), the solutions are analytic. In [16], Jones, Putterman and Roberts predicted formally the asymptotic behavior of travelling waves as $|x| \to \infty$. In [14], Gravejat considered the (GP) equation to study the asymptotic behavior of the travelling waves. His proof could be adapted for the general nonlinearities.

It was conjectured in [16] and in [17] that there exists a critical speed c_v such that the travelling waves of speed smaller than c_v present vortices, while those of speed greater than c_v do not.

For the (GP) equation, the small velocity solutions solutions of small velocity solutions constructed in [4], [5] and [7] have vortices. For general nonlinearities, it was proved in [9] that the travelling waves do not have vortices if $N = 2$ or 3 and c close to v_s.

Travelling waves solutions for the (GP) equation were obtained in [3] by minimizing the energy at fixed and small momentum and have velocities close to v_s. The energy-momentum diagram of these solutions shows that they are of arbitrary small energy and momentum in dimension two. In higher dimensions, the energy and the momentum of the three-dimensional travelling waves for (GP) are bounded from below. This result was noticed in [16]. If $N = 3$, it was proved in [3] that (GP) does not admit small energy travelling waves solutions. This result was extended to higher dimensions by De Laire, in [18].

For general nonlinearities, finite energy travelling waves of speed close to v_s were provided in [21] if $N \geq 3$ and in [8] if $N = 2$. In the two and three dimensional cases, the travelling waves solutions have modulus close to 1.

It was proved in [10] and [11] that if $N \leq 3$, the Cauchy problem for the (GP) is globally well-posed for all initial data in the energy space. If $N = 4$, the result is true for initial datas with small energy. The method used by Gerard can be adapted for other nonlinearities.

The assumptions used to prove the existence of travelling waves for the nonlinear Schrödinger equation with general nonlinearity F (see (1.7)) are:

1. the function F is continuous on $[0,\infty)$, C^1 in a neighborhood of 1, $F(1)=0$ and $F'(1)<0$;
2. there exist $C>0$ and $p_0<\frac{2}{N-2}$ (with $p_0<\infty$ if $N=2$) such that $|F(s)|\leq C(1+s^{p_0})$ for any $s\geq 0$;
3. there exist $C,\alpha_0>0$ and $r_*>1$ such that $F(s)\leq -Cs^{\alpha_0}$ for any $s\geq r_*$;
4. F is C^2 near 1 and

$$F(s)=-(s-1)+\frac{1}{2}F''(1)(s-1)^2+\mathcal{O}((s-1)^3) \qquad \text{for } s \text{ close to } 1.$$

The Gross-Clark system

The results about the Gross-Clark system are very rare. In [19], it is proven that in space dimension one, there exist travelling-waves moving with velocity c if and only if c is less than the sound velocity at infinity. In this case, the structure of the set of travelling waves is investigated and it is showed that it contains global subcontinua in appropriate Sobolev spaces.

Bouchel [6] showed decay estimates for finite energy travelling waves of (GC) and the nonexistence of supersonic travelling waves in dimension 3.

On the other hand, it was proved in [20] that in space dimension $N\geq 2$, travelling waves do not exist for c greater than the sound velocity at infinity.

If $N\in\{3,4\}$ Nguyen proved in [22] that the travelling waves exist, by minimizing some action under a Pohozaev constraint under general conditions on the nonlinearity F and for any speed $c\in(0,v_s)$ satisfying $\varepsilon^2(c^2\delta^2+k^2)<q^2$. Moreover, minimizing the energy E at fixed momentum Q has given the trivial solution $(\Psi,0)$, where Ψ is a solution of (1.1) obtained by minimizing E_1 at fixed momentum Q as in [8] and [21].

1.1.3 Main results

- The aim of Chapter 2 is to prove that the Cauchy problem for a Gross-Clark-Schrödinger system is globally well-posed in the energy space naturally associated to the system. The proof of this result is standard and based on the method used in [10]: Strichartz estimates and Banach fixed-point theorem.

- In Chapter 3, we prove the existence of travelling waves of (GC). These are solutions of the form

$$\Psi(t,x)=\psi(x-ct\omega),$$

$$\Phi(t,x)=\varphi(x-ct\omega),$$

where $\omega\in S^{N-1}$ is the direction of propagation, and $c\in\mathbb{R}^*$ is the travelling wave speed. Since (GC) is invariant by rotation, we may assume that $\omega=(1,0,\cdots,0)$. Then travelling waves of speed c satisfy the following equations in $\mathbb{R}^N$

$$\begin{cases} -ic\frac{\partial\psi}{\partial x_1} &= -\Delta\psi+\frac{1}{\varepsilon^2}(\frac{1}{\varepsilon^2}|\varphi|^2+|\psi|^2-1)\psi \\ -ic\delta\frac{\partial\varphi}{\partial x_1} &= -\Delta\varphi+\frac{1}{\varepsilon^2}(q^2|\psi|^2-\varepsilon^2k^2)\varphi, \end{cases} \tag{1.10}$$

with a fixed mass m of φ. It is easy to see that (ψ, φ) satisfies (1.10) for some velocity c if and only if $(\psi, \varphi)(-x_1, x')$ satisfies (1.10) with $-c$ instead of c. Hence, we may assume that $c \geq 0$.

It is obvious that $\varphi \in H^1(\mathbb{R}^N)$ if and only if $e^{iax_1}\varphi \in H^1(\mathbb{R}^N)$ for some $a \in \mathbb{R}$. Therefore, we can replace φ by $e^{iax_1}\varphi$ in the second equation of (1.10) to get

$$c\delta a\varphi - ic\delta\frac{\partial\varphi}{\partial x_1} = a^2\varphi - 2ia\frac{\partial\varphi}{\partial x_1} - \Delta\varphi + \frac{1}{\varepsilon^2}(q^2|\psi|^2 - \varepsilon^2k^2)\varphi.$$

Then choosing $a = \frac{c\delta}{2}$, we see that travelling waves for (GC) have to satisfy a system of two equations given by

$$\text{(TW)} \qquad \begin{cases} -ic\frac{\partial\psi}{\partial x_1} & = -\Delta\psi + \frac{1}{\varepsilon^2}(\frac{1}{\varepsilon^2}|\varphi|^2 + |\psi|^2 - 1)\psi \\ \big(-\Delta + \frac{q^2}{\varepsilon^2}|\psi|^2\big)\varphi & = \lambda\varphi, \end{cases}$$

where $\lambda = \frac{c^2\delta^2}{4} + k^2$ is unknown, and k calculated once λ found.

Formally, travelling waves are critical points of the functional $E(\psi, \varphi) - cQ(\psi) - \lambda\mathbf{M}(\varphi)$. Therefore, it is a natural idea to look for solutions as minimizers of the energy at fixed momentum and mass. The speed c and λ being the Lagrange multipliers associated to the minimization problem.

To prove the existence of non trivial travelling waves solutions for (GC), we choose first to prove the existence of the ground states by minimizing the energy at fixed mass or L^2-norm for φ, then we study the minimization at fixed momentum for φ. We have also minimized the energy at fixed mixed momentum $P_\beta(\psi, \varphi)$ where

$$P_\beta(\psi, \varphi) = \beta Q(\psi) + (1 - \beta)Q(\varphi) \qquad \text{for some } \beta \in [0, 1].$$

Finally, we chose to minimize the energy (1.2) under two constraints: fixed momentum p for ψ and fixed mass m for φ. Therefore, we define the quantity:

$$\text{(1.11)} \qquad E_{min}(p, m) = \inf\big\{E(\psi, \varphi) \mid \psi \in \mathcal{E}, \varphi \in H^1(\mathbb{R}^N), Q(\psi) = p, \int_{\mathbb{R}^N} |\varphi|^2 \mathrm{d}x = m\big\}.$$

The first results gave us some concavity properties for E_{min} and the existence of minimizers for some values of p and m. Indeed, we could not prove the strict subadditivity of E_{min}, for all values of p and m. The method used to prove these results is mainly the concentration-compactness principle, by eliminating the "vanishing" and the "dichotomy" case and concluding that the "compactness" case holds.

- Chapter 4 is devoted to the numerical approximation in dimension 2 of the results obtained in Chapter 3. Indeed, we look numerically for travelling waves solutions for (GC) that have small mass. For ψ, the solution looks like the vortex branch of travelling wave of (1.1). For φ, it looks like a sum of two Gaussians functions of small mass concentrated near the vortices of ψ. The energy-momentum diagram of these solutions forms a concave curve (see Figure 1.1). Moreover, the numerical value of λ is also presented in this chapter.

- In Chapter 5, we use some numerical methods to approximate the solutions of mass 4π for the system (GC) in dimension 2. We could find two branches of solutions: the ground state branch,

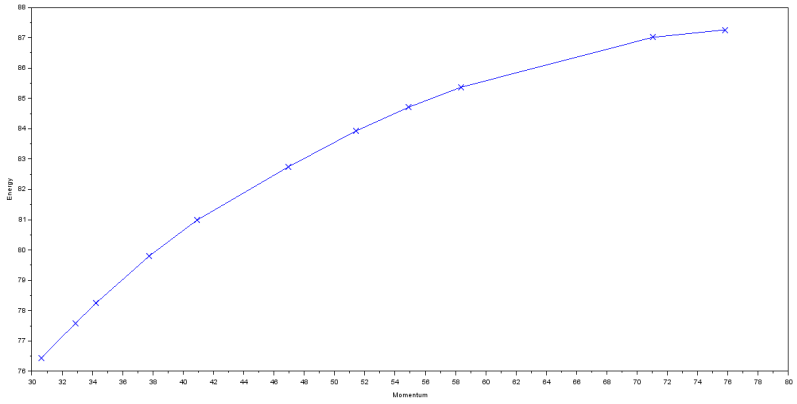

Figure 1.1: The energy-momentum diagram for the small mass solutions with $m = 0.1$

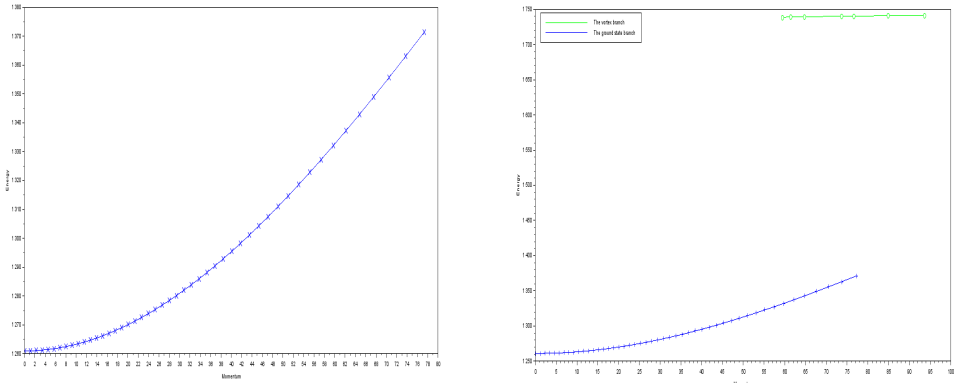

Figure 1.2: The energy-momentum diagram for (a) left: the ground state branch; (b) right: both the vortex and the ground state branches

and the vortex branch. For each of these branches, we start to find the stationary solutions: for the first branch, the stationary solutions were found in dimensions 2 and 3. For the second one, they were obtained in dimension 2. Then we could approximate the solutions with small speeds c for both of the branches. We could plot the energy-momentum diagram corresponding to each type of these solutions. For the ground state branch, the diagram is obtained for speeds $c \in [0, 2.35]$, while it is found for speeds $c \in [0.12, 0.19]$ for the vortex branch (see Figure 1.2). For the vortex branch, if $c \in [0.7, 2.35]$, the algorithm converges, but the used ansatz do not present any vortex, they look like the ansatz of the ground state branch, and the obtained solutions are the same as the ones of the ground state branch. In other words, the ground state branch for $c \in [0.7, 2.35]$ could be obtained in two ways: starting from the ground state solution and starting from the ansatz of two vortices that superpose. The last part of Chapter 5 is devoted to a rigorous proof for the existence of the ground state solution.

1.2 Version française

1.2.1 Présentation du problème

On considère le système de Gross-Clark:

$$\text{(GC)} \qquad \begin{cases} i\partial_t \Psi &= -\Delta\Psi + \frac{1}{\varepsilon^2}(\frac{1}{\varepsilon^2}|\Phi|^2 + |\Psi|^2 - 1)\Psi \\ \\ i\delta\partial_t \Phi &= -\Delta\Phi + \frac{1}{\varepsilon^2}(q^2|\Psi|^2 - \varepsilon^2 k^2)\Phi \end{cases} \qquad \text{dans } \mathbb{R}\times\mathbb{R}^N,$$

avec les "conditions aux bords"

$$|\Psi| \to 1, \Phi \to 0 \text{ quand } |x| \to \infty.$$

Ce système, introduit par Clark et Gross, a été étudié par Grant et Roberts (voir [12]). Ce système modélise le mouvement d'une impureté dans un condensat de Bose-Einstein. Les fonctions Ψ and Φ représentent les fonctions d'ondes des bosons et de l' impureté, respectivement. Plusieurs paramètres physiques sont appropriés au système (GC): dans la seconde équation de (GC) on choisit $\delta = \frac{\mu}{M}$, où μ est la masse de l'impureté, et M celle du boson; comme μ est petite comparée à M, δ est supposé être petit. On note $q^2 = \frac{l}{2d}$, où l est la longueur de diffusion du boson et de l'impureté, et d le diamètre du boson, tandis que k est une mesure sans dimension de l'énergie de l'impureté à une seule particule. Le paramètre ε est défini comme suit $\varepsilon = \left(\frac{b\mu}{lM}\right)^{\frac{1}{5}}$, où b est la "longueur de cohérence"; dans les applications, on a $\varepsilon \cong 0,2$. Le système (GC) a été étudié dans [12], où le rayon effectif et la masse induite de l'impureté non chargée ont été calculés en utilisant des développements asymptotiques formels et quelques expériences numériques.

Lorsque $\Phi = 0$, le système (GC) se réduit à l'équation de Gross-Pitaevskii

$$i\partial_t \Psi = -\Delta\Psi + \frac{1}{\varepsilon^2}(|\Psi|^2 - 1)\Psi \qquad \text{dans } \mathbb{R}\times\mathbb{R}^N. \tag{1.12}$$

Le système (GC) et l'équation de Gross-Pitaevskii (1.12) sont tous les deux hamiltoniens. Les énergies conservées sont, respectivement

$$E(\psi,\varphi) = \int_{\mathbb{R}^N} |\nabla\psi|^2 + \frac{1}{2\varepsilon^2}\big(|\psi|^2 - 1\big)^2 + \frac{1}{\varepsilon^2 q^2}|\nabla\varphi|^2 + \frac{1}{\varepsilon^4}|\psi|^2|\varphi|^2 \mathrm{d}x \tag{1.13}$$

et l'énergie de *Ginzburg-Landau energy* de Ψ, donnée par

$$E_1(\psi) = \int_{\mathbb{R}^N} |\nabla\psi|^2 + \frac{1}{2\varepsilon^2}\big(|\psi|^2 - 1\big)^2 \mathrm{d}x. \tag{1.14}$$

Les autres quantités conservées par le flux de (GC) sont le moment $\mathbf{P}(\Psi) = \big(P_1(\Psi), \cdots, P_N(\Psi)\big)$, et la masse $\mathbf{M}(\Phi) = \int_{\mathbb{R}^N} |\Phi|^2 \mathrm{d}x$. Le moment a été rigoureusement défini dans [21] en dimension $N \geq 3$, puis dans [8] si $N = 2$. On rappellera plus tard les idées principales de cette définition. Si Ψ est une fonction suffisamment localisée dans l'espace, le moment est $P_k(\Psi) = \int_{\mathbb{R}^N} \langle i\Psi, \frac{\partial\Psi}{\partial x_k}\rangle \mathrm{d}x$, pour $k \in \{1, \cdots, N\}$, et où $\langle ., . \rangle$ est le produit scalaire habituel dans $\mathbb{C} \simeq \mathbb{R}^2$.

On désigne par Q le moment par rapport à la direction x_1, *i.e* $Q(\psi) = P_1(\psi)$. Le moment a un bon comportement par rapport aux dilatations, *i.e* si $\psi \in \mathcal{E}$, $\gamma, \sigma > 0$, $\psi_{\gamma,\sigma} := \psi(\frac{x_1}{\gamma}, \frac{x'}{\sigma})$, alors

$$Q(\psi_{\gamma,\sigma}) = \sigma^{N-1} Q(\psi), \tag{1.15}$$

où $\mathcal{E}$ est l'espace d'énergie naturellement associé à l'équation de Gross-Pitaevskii défini par :

$$\mathcal{E} = \big\{\psi : \mathbb{R}^N \longrightarrow \mathbb{C}; \psi \text{ est mesurable}, |\psi| - 1 \in L^2(\mathbb{R}^N), \nabla\psi \in L^2(\mathbb{R}^N)\big\}. \tag{1.16}$$

Il existe plusieurs définitions équivalentes pour $\mathcal{E}$. Si $N \in \{2, 3, 4\}$, on peut prouver que $\mathcal{E} = \mathbf{E}$, où

$$\mathbf{E} = \big\{\psi \in H^1_{\mathrm{loc}}(\mathbb{R}^N) \mid \nabla\psi \in L^2(\mathbb{R}^N), |\psi|^2 - 1 \in L^2(\mathbb{R}^N)\big\}. \tag{1.17}$$

1.2.2 Résultats existant

L'équation de Schrödinger non-linéaire

Les résultats correspondant à l'équation de Schrödinger non-linéaire,

$$i\frac{\partial \Psi}{\partial t} + \Delta\psi + F(|\Psi|^2)\Psi = 0 \qquad \text{dans } \mathbb{R}\times\mathbb{R}^N, \tag{1.18}$$

avec les conditions aux bords $|\Psi| \to 1$ quand $|x| \to \infty$, sont nombreux. Les cas les plus importants qui ont été largement étudiés par les physiciens et les mathématiciens sont l'équation de Gross-Pitaevskii (GP), où $F(s) = 1 - s$, et l'équation de Schrödinger dite "cubique-quintique", où $F(s) = -\alpha_1 + \alpha_3 s - \alpha_5 s^2$, α_1, α_3 et α_5 étant des constantes positives et F a deux racines positives.

On note $V(s) = \int_s^1 F(\tau)\mathrm{d}\tau$, on définit l'énergie conservée E_1 avec une non-linéarité générale par

$$E_1(\Psi) = \int_{\mathbb{R}^N} |\nabla\Psi|^2 \mathrm{d}x + \int_{\mathbb{R}^N} V(|\Psi|^2)\mathrm{d}x. \tag{1.19}$$

Pour cette équation, une attention particulière a été accordée aux solutions d'ondes progressives dans une série de travaux (voir, par exemple, [1], [2], [12], [15], [16], [17]). Ce sont des solutions de la forme $\Psi(t,x) = \psi(x_1 - ct, x')$ qui doivent satisfaire l'équation

$$-ic\frac{\partial \psi}{\partial x_1} + \Delta\psi + F(|\psi|^2)\psi = 0, \tag{1.20}$$

où $x' = (x_2, \cdots, x_N)$ et c est la vitesse de l'onde progressive. Dans [21], il est prouvé qu'il existe des ondes progressives non triviales d'énergie finie se déplaçant à la vitesse c dans tout espace de dimension $N \geq 3$, pour une grande classe d'équations non-linéaires de Schrödinger avec des conditions non nulles à l'infini, pour toute vitesse c inférieure à la vitesse du son, et sous des conditions générales sur la non-linéarité F. La vitesse du son est définie comme suit : si F est C^1 près de 1, $F'(1) < 0$, la vitesse du son à l'infini associée à (1.12) est $v_s = \sqrt{-2F'(1)}$. Ces solutions sont des points critiques de l'énergie E_1 lorsque le moment est fixé.

C. A. Jones, C. J. Putterman et P. H. Roberts ont calculé numériquement l'énergie et le moment des ondes progressives qu'ils ont trouvées pour l'équation de Gross-Pitaevskii. La preuve rigoureuse de l'existence d'ondes progressives pour (GP) en dimension $N = 2$ d'espace a été faite dans [5] pour toute vitesse dans un certain intervalle $(0, \eta)$ où η est petit. La minimisation de l'énergie à moment fixé a été utilisée en premier lieu dans [4] pour construire une séquence d'ondes progressives avec des vitesses tendant vers 0 en dimension $N \geq 3$. Le même résultat a été établi dans [7] pour les petites vitesses en utilisant un argument de type "mountain-pass". En dimension $N = 2$ et $N = 3$, l'existence d'ondes progressives pour l'équation de Gross-Pitaevskii a été établie dans [3] en minimisant l'énergie à moment fixé. Dans ce cas, le multiplicateur de Lagrange associé aux minimiseurs obtenus correspond à la vitesse de propagation, et si $N = 2$, le résultat est obtenu pour toute vitesse dans un ensemble $A \subset (0, v_s)$ où v_s est la vitesse du son à l'infini, et A contient des points très proches de 0 et v_s. Par la suite, il a été montré dans [8] que la minimisation de l'énergie à moment fixé peut être utilisée dans toute dimension $N \geq 2$ pour des non-linéarités générales sous l'hypothèse que V est positif. De plus, l'ensemble des solutions est orbitalement stable. Malheureusement, dans ce cas, il n'est pas clair que l'ensemble des vitesses des ondes progressives construites forme un intervalle. De plus, il a été prouvé dans [3] et dans [18] qu'en dimension $N \geq 3$, il existe $v_0 \in (0, v_s)$ de telle sorte que la minimisation de l'énergie à moment fixé ne peut pas donner des ondes progressives de vitesse $c \in (v_0, v_s)$.

Dans [21], on a un résultat d'existence qui couvre tout l'intervalle $(0, v_s)$ des vitesses possibles si $N \geq 3$, pour des non-linéarités générales. Pour l'équation (GP), la non-existence des ondes progressives pour $c > v_s$ a été faite dans [13]. Pour une grande classe de non-linéarités, elle a été faite dans [20].

Les ondes progressives ont la meilleure régularité permise par la non-linéarité F, par exemple dans le cas de (GP), les solutions sont analytiques. Dans [16], Jones, Putterman et Roberts ont prédit formellement le comportement asymptotique des ondes progressives en tant que $|x| \to \infty$. Dans [14], Gravejat a considéré l'équation (GP) pour étudier le comportement asymptotique des ondes progressives. Sa preuve pourrait être adaptée aux non-linéarités générales.

Dans [16] et [17], la conjecture suivante a été émise : il existe une vitesse critique c_v telle que les ondes progressives de vitesse inférieure à c_v présentent des vortex, tandis que celles de vitesse supérieure à c_v n'en présentent pas.

Pour l'équation (GP), les solutions de petites vitesses construites dans [4], [5] et [7] ont des vortex. Pour des non-linéarités générales, il a été prouvé dans [9] que les ondes progressives n'ont pas de vortex si $N = 2$ ou 3 et c proche de v_s.

Des solutions de type ondes progressives pour l'équation (GP) ont été obtenues dans [3] en minimisant l'énergie à moment fixé et petit et ont des vitesses proches de v_s. Le diagramme énergie-moment de ces solutions montre qu'elles ont une énergie et un moment arbitrairement petits en dimension deux. En dimension supérieure, l'énergie et le moment des ondes progressives tridimensionnelles pour (GP) sont minorés. Ce résultat a été remarqué dans [16]. Si $N = 3$, il a été prouvé dans [3] que (GP) n'admet pas de solutions de type ondes progressives de petite énergie. Ce résultat a été étendu aux dimensions supérieures par De Laire, dans [18].

Pour les non-linéarités générales, des ondes progressives d'énergie finie et de vitesse proche de v_s ont été fournies dans [21] si $N \geq 3$ et dans [8] si $N = 2$. Dans les cas à deux et trois dimensions, les solutions d'ondes progressives ont un module proche de 1.

Il a été prouvé dans [10] et [11] que si $N \leq 3$, le problème de Cauchy pour le (GP) est globalement bien posé pour toutes les données initiales dans l'espace d'énergie. Si $N = 4$, le résultat est vrai pour les données initiales de petite énergie. La méthode utilisée par Gérard peut être adaptée à d'autres non-linéarités.

Les hypothèses utilisées pour prouver l'existence d'ondes progressives pour l'équation non linéaire de Schrödinger avec une non-linéarité générale F (voir (1.18)) sont:

1. la fonction F est continue sur $[0, \infty)$, C^1 dans un voisinage de 1, $F(1) = 0$ et $F'(1) < 0$;
2. il existe $C > 0$ et $p_0 < \frac{2}{N-2}$ (avec $p_0 < \infty$ si $N = 2$) tels que $|F(s)| \leq C(1 + s^{p_0})$ pour tout $s \geq 0$;
3. il existe $C, \alpha_0 > 0$ et $r_* > 1$ tels que $F(s) \leq -Cs^{\alpha_0}$ pour tout $s \geq r_*$;
4. F est C^2 proche de 1 et

$$F(s) = -(s-1) + \frac{1}{2}F''(1)(s-1)^2 + \mathcal{O}((s-1)^3) \qquad \text{pour } s \text{ proche to } 1.$$

Le système de Gross-Clark

Les résultats concernant le système de Gross-Clark sont très rares. Dans [19], il est prouvé qu'en dimension un, il existe des ondes progressives se déplaçant avec une vitesse c si et seulement si c est

inférieur à la vitesse du son à l'infini. Dans ce cas, on étudie la structure de l'ensemble des ondes progressives et on montre qu'il contient des sous-continus globaux dans des espaces de Sobolev appropriés.

Bouchel [6] a montré des estimations de décroissance pour les ondes progressives d'énergie finie de (GC) et la non-existence d'ondes progressives supersoniques en dimension 3.

D'autre part, il a été prouvé dans [20] qu'en dimension $N \geq 2$, les ondes progressives n'existent pas pour c supérieur à la vitesse du son à l'infini.

Si $N \in \{3, 4\}$, Nguyen a prouvé dans [22] que les ondes progressives existent, en minimisant une certaine action sous une contrainte de Pohozaev sous des conditions générales sur la non-linéarité F et pour toute vitesse $c \in (0, v_s)$ satisfaisant $\varepsilon^2(c^2\delta^2 + k^2) < q^2$. De plus, la minimisation de l'énergie E à moment fixé Q a donné la solution triviale $(\Psi, 0)$, où Ψ est une solution de (1.12) obtenue en minimisant E_1 à moment fixé Q comme dans [8] et [21].

1.2.3 Résultats principaux

- Le but du Chapitre 2 est de prouver que le problème de Cauchy pour le système de Gross-Clark-Schrödinger est globalement bien posé dans l'espace d'énergie naturellement associé au système. La preuve de ce résultat est standard et repose sur la méthode utilisée dans [10] : estimations de Strichartz et théorème du point fixe de Banach.

- Dans le Chapitre 3, on prouve l'existence d'ondes progressives de (GC). Ce sont des solutions de la forme

$$\Psi(t, x) = \psi(x - ct\omega),$$
$$\Phi(t, x) = \varphi(x - ct\omega),$$

où $\omega \in S^{N-1}$ est la direction de propagation, et $c \in \mathbb{R}^*$ est la vitesse de l'onde progressive. Puisque (GC) est invariant par rotation, on peut supposer que $\omega = (1, 0, \cdots, 0)$. Les ondes progressives de vitesse c satisfont alors les équations suivantes dans $\mathbb{R}^N$.

$$\begin{cases} -ic\frac{\partial \psi}{\partial x_1} & = -\Delta\psi + \frac{1}{\varepsilon^2}(\frac{1}{\varepsilon^2}|\varphi|^2 + |\psi|^2 - 1)\psi \\ -ic\delta\frac{\partial \varphi}{\partial x_1} & = -\Delta\varphi + \frac{1}{\varepsilon^2}(q^2|\psi|^2 - \varepsilon^2 k^2)\varphi, \end{cases} \tag{1.21}$$

avec une masse fixée m de φ. Il est facile de voir que (ψ, φ) satisfait (1.21) pour une certaine vitesse c si et seulement si $(\psi, \varphi)(-x_1, x')$ satisfait (1.21) avec $-c$ au lieu de c. On peut donc supposer que $c \geq 0$.

Il est clair que $\varphi \in H^1(\mathbb{R}^N)$ si et seulement si $e^{iax_1}\varphi \in H^1(\mathbb{R}^N)$ pour $a \in \mathbb{R}$. On peut alors remplacer φ par $e^{iax_1}\varphi$ dans la seconde équation de (1.21) pour avoir

$$c\delta a\varphi - ic\delta\frac{\partial \varphi}{\partial x_1} = a^2\varphi - 2ia\frac{\partial \varphi}{\partial x_1} - \Delta\varphi + \frac{1}{\varepsilon^2}(q^2|\psi|^2 - \varepsilon^2 k^2)\varphi.$$

On choisit $a = \frac{c\delta}{2}$, on voit que les ondes progressives pour (GC) doivent satisfaire un système de deux équations données par

$$\begin{cases} -ic\frac{\partial \psi}{\partial x_1} & = -\Delta\psi + \frac{1}{\varepsilon^2}(\frac{1}{\varepsilon^2}|\varphi|^2 + |\psi|^2 - 1)\psi \\ \big(-\Delta + \frac{q^2}{\varepsilon^2}|\psi|^2\big)\varphi & = \lambda\varphi, \end{cases} \tag{TW}$$

où $\lambda = \frac{c^2\delta^2}{4} + k^2$ est inconnu, et k calculé une fois λ trouvé.

Formellement, les ondes progressives sont des points critiques de la fonction $E(\psi,\varphi) - cQ(\psi) - \lambda\mathbf{M}(\varphi)$. Par conséquent, il est naturel de chercher des solutions comme minimiseurs de l'énergie à moment et masse fixés. La vitesse c et λ étant les multiplicateurs de Lagrange associés au problème de minimisation.

Pour prouver l'existence de solutions d'ondes progressives non triviales pour (GC), on a choisi en premier lieu de prouver l'existence des états fondamentaux en minimisant l'énergie (1.13) sous masse ou norme L^2 de φ fixée égale à m, ensuite on étudie la minimization sous moment fixé pour φ. On a également minimisé l'énergie à moment mixte fixé $P_\beta(\psi,\varphi)$ où

$$P_\beta(\psi,\varphi) = \beta Q(\psi) + (1-\beta)Q(\varphi) \qquad \text{pour un } \beta \in [0,1].$$

Finalement, on a choisi de minimiser l'énergie (1.14) sous deux contraintes: moment fixé p de ψ et masse fixée m pour φ. Par conséquent, on définit la quantité:

$$E_{min}(p,m) = \inf\big\{E(\psi,\varphi) \mid \psi \in \mathcal{E}, \varphi \in H^1(\mathbb{R}^N), Q(\psi) = p, \int_{\mathbb{R}^N} |\varphi|^2 \mathrm{d}x = m\big\}. \tag{1.22}$$

Les premiers résultats nous ont donné quelques propriétés de concavité pour E_{min} et l'existence de minimiseurs pour quelques valeurs de p et m. En effet, on n'a pas pu prouver la sous-additivité stricte de E_{min}, pour toutes les valeurs de p et de m. La méthode utilisée pour prouver ces résultats est principalement le principe de concentration-compacité, en éliminant le cas évanescence et le cas dichotomie et en concluant que le cas compacité tient.

- Le Chapitre 4 est consacré à l'approximation numérique en dimension 2 des résultats obtenus au Chapitre 3. En effet, on cherche numériquement des solutions d'ondes progressives pour (GC) qui ont une petite masse. Pour ψ, la solution ressemble à la branche vortex de l'onde progressive de (1.12). Pour φ, elle ressemble à une somme de deux fonctions de Gauss de petites masses concentrées près des vortex de ψ. Le diagramme énergie-moment de ces solutions forme une courbe concave (voir Figure 1.3). De plus, la valeur numérique de λ est également présentée dans ce Chapitre.

- Dans le Chapitre 5, on utilise certaines méthodes numériques pour approcher les solutions de masse 4π pour le système (GC) en dimension 2. On a pu trouver deux branches de solutions : la branche de l'état fondamental, et la branche des vortex. Pour chacune de ces branches, on commence par trouver les solutions stationnaires: pour la première branche, les solutions stationnaires ont été trouvées en dimensions 2 et 3. Pour la deuxième branche, elles ont été obtenues en dimension 2. Ensuite, on a pu approximer les solutions avec de petites vitesses c pour les deux branches. On a pu tracer le diagramme énergie-moment correspondant à chaque type de ces solutions. Pour la branche de l'état fondamental, le diagramme est obtenu pour des vitesses $c \in [0, 2.35]$, alors qu'il est trouvé pour des vitesses $c \in [0.12, 0.19]$ pour la branche des vortex (voir Figure 1.4). Pour la branche des vortex, si $c \in [0.7, 2.35]$, l'algorithme converge, mais les ansatz utilisés ne présentent pas de vortex, ils ressemblent aux ansatz de la branche de l'état fondamental, et les solutions obtenues sont les mêmes que celles de la branche de l'état fondamental. En d'autres termes, la branche de l'état fondamental pour $c \in [0.7, 2.35]$ pourrait être obtenue de deux façons : soit à partir de la solution de l'état fondamental, soit à partir de l'ansatz de deux vortex qui se superposent. La dernière partie du Chapitre 5 est consacrée à une preuve rigoureuse de l'existence de la solution de l'état fondamental.

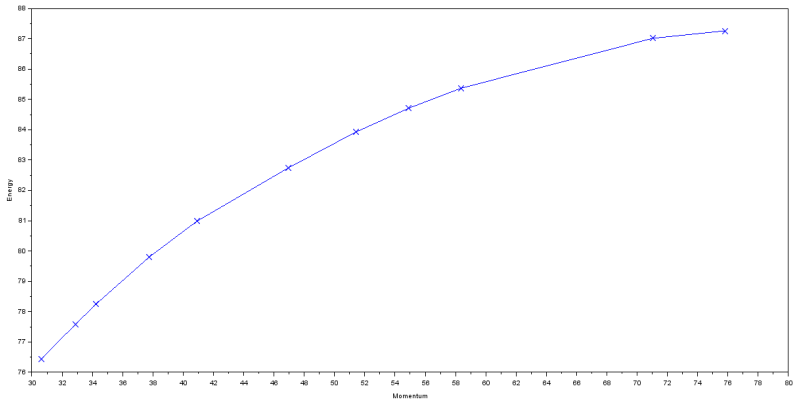

Figure 1.3: Le diagramme énergie-moment pour les solutions à petite masse avec $m = 0.1$

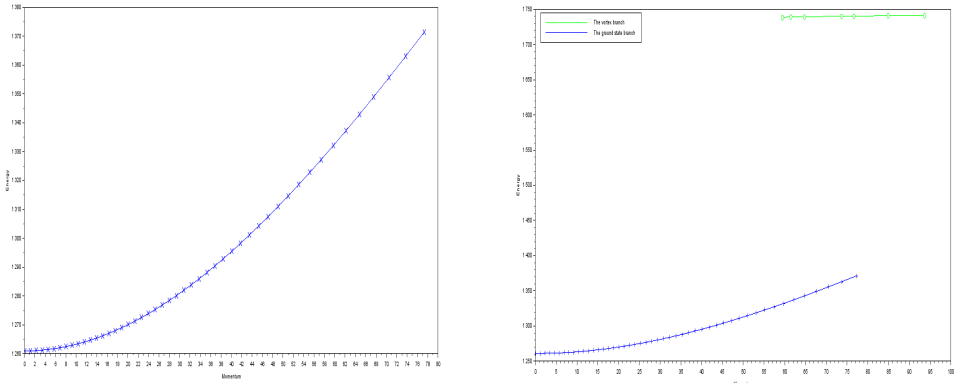

Figure 1.4: Le diagramme énergie-moment pour (a) à gauche : la branche de l'état fondamental ; (b) à droite : les branches des vortex et de l'état fondamental

Bibliography

[1] Barashenkov, I. V. & Gocheva, A.D. & Makhan'kov, V.G. & Puzynin, I.V. (1989). *Stability of the soliton-like "bubbles".* Phys. D 34, 240-254.

[2] Barashenkov, I. V. & Makhan'kov, V.G. & Puzynin, I.V. (1988). *Soliton-like "bubbles" in a system of interacting bosons.* Phys. Lett. A 128, 52-56.

[3] Béthuel, F. & Gravejat, P. & Saut, J.-C. (2009). *Travelling waves for the Gross-Pitaevskii equation.* II, Comm. Math. Phys. 285, 567-651.

[4] Béthuel, F. & Orlandi, G. & Smets, D. (2004). *Vortex rings for the Gross-Pitaevskii equation.* J. Eur. Math. Soc. (JEMS) 6, 17-94.

[5] Béthuel, F. & Saut, J.-C. (1999). *Travelling waves for the Gross-Pitaevskii equation.* I, Ann. Inst. H. Poincaré Phys. Théor. 70, 147-238.

[6] Bouchel, O. (2006) *Quelques équations et systèmes d'équations de Schrödinger non linéaire.* PhD thesis Université Paris-Sud 11.

[7] Chiron, D. (2004) *Travelling waves for the Gross-Pitaevskii equation in dimension larger than two.* Nonlinear Anal. 58, 175-204.

[8] Chiron, D. & Mariş, M. (2017). *Traveling Waves for Nonlinear Schrödinger Equations with Nonzero Conditions at Infinity.* Archive for Rational Mechanics and Analysis. 226 (1). 143-242.

[9] Chiron, D. & Mariş, M. (2014). *Rarefaction pulses for the nonlinear Schrödinger equation in the transonic limit.* Communications in Mathematical Physics. 326 (2). 329-392.

[10] Gerard, P. (2006). *The Cauchy problem for the Gross-Pitaevskii equation.* Annales De L'Institut Henri Poincare-analyse Non Lineaire. 23. 765-779.

[11] Gerard, P. (2008). *The Gross-Pitaevskii equation in the energy space.* in Stationary and Time Dependent Gross-Pitaevskii Equations, Contemp. Math. 473, Amer. Math. Soc., Providence, RI, 2008, pp. 129-148.

[12] Grant, J. & Roberts, P.H. (1974). *Motions in a Bose condensate III. The structure and effective masses of charged and uncharged impurities.* Journal of Physics A: Mathematical, Nuclear and General. 7. 260 - 279.

[13] Gravejat, P. (2003). *A non existence result for supersonic travelling waves in te Gross-Pitaevskii equation.* Comm. Math. Phys. 243, 93-103.

[14] Gravejat, P. (2005). *Asymptotics for the travelling waves in te Gross Pitaevskii equation*, Asymptot. Anal. 45, 227-299.

[15] Iordanskii, S.V. & Smirnov, A.V. (1978). *Three-dimensional solitons in He II.* JETP Lett. 27, 535-538.

[16] Jones, C.A. & Putterman, S.J. & Roberts, P.H. (1986). *Stability of wave solutions of nonlinear Schrödinger equations in two and three dimensions.* J. Phys A: Math. Gen. 19, 2991-3011.

[17] Jones, C.A. & Roberts, P.H. (1982). *Motions in a Bose condensate IV.* Axisymetric solitary waves, J. Phys A: Math. Gen 15, 2599-2619.

[18] De Laire, A. (2009). *Non-existence for travelling waves with small energy for the Gross-Pitaevskii equation in dimension $N \geq 3$.* C. R. Math. Acad. Sci. Paris 347, 375-380.

[19] Mariş, M. (2006). *Global Branches of Travelling-Waves to a Gross–Pitaevskii–Schrödinger System in One Dimension.* SIAM J. Math. Analysis. 37. 1535-1559.

[20] Mariş, M. (2008). *Nonexistence of Supersonic Traveling Waves for Nonlinear Schrödinger Equations with Nonzero Conditions at Infinity.* SIAM J. Math. Analysis. 40. 1076-1103.

[21] Mariş, M. (2009). *Traveling waves for nonlinear Schrödinger equations with nonzero conditions at infinity.* Annals of Mathematics. 178 (1). 107-182 .

[22] Nguyen, T.L. (2014) *Some variational problems arising in the theory of nonlinear waves.* PhD thesis Université Toulouse III-Paul Sabatier.

Chapter 2

On the Cauchy problem for Gross-Clark system

2.1 Introduction

We recall that the Gross-Clark (GC) system is given by:

$$\text{(GC)} \qquad \begin{cases} i\partial_t \Psi &= -\Delta\Psi + \frac{1}{\varepsilon^2}(\frac{1}{\varepsilon^2}|\Phi|^2 + |\Psi|^2 - 1)\Psi \\ \\ i\delta\partial_t \Phi &= -\Delta\Phi + \frac{1}{\varepsilon^2}(q^2|\Psi|^2 - \varepsilon^2 k^2)\Phi \end{cases} \qquad \text{in } \mathbb{R}\times\mathbb{R}^N,$$

with the "boundary conditions"

$$|\Psi| \to 1, \Phi \to 0 \text{ as } |x| \to \infty.$$

The functions Ψ and Φ are the wavefunctions for the bosons and for the impurity, respectively. δ, q, and ε are some physical parameters.

We recall that when $\Phi = 0$, the system (GC) reduces to the Gross-Pitaevskii equation

$$i\partial_t \Psi = -\Delta\Psi + \frac{1}{\varepsilon^2}(|\Psi|^2 - 1)\Psi \qquad \text{in } \mathbb{R}\times\mathbb{R}^N,$$

and that both the system (GC) and the Gross-Pitaevskii equation are Hamiltonian. As we mentioned in the Introduction, the conserved energies are, respectively

$$E(\psi,\varphi) = \int_{\mathbb{R}^N} |\nabla\psi|^2 + \frac{1}{2\varepsilon^2}\big(|\psi|^2 - 1\big)^2 + \frac{1}{\varepsilon^2 q^2}|\nabla\varphi|^2 + \frac{1}{\varepsilon^4}|\psi|^2|\varphi|^2 \mathrm{d}x \tag{2.1}$$

and the *Ginzburg-Landau energy* of ψ, namely

$$E_1(\psi) = \int_{\mathbb{R}^N} |\nabla\psi|^2 + \frac{1}{2\varepsilon^2}\big(|\psi|^2 - 1\big)^2 \mathrm{d}x. \tag{2.2}$$

The function space naturally associated to the Gross-Pitaevskii equation is (see [5, 2, 7])

$$\mathbf{E} = \big\{\psi \in H^1_{\mathrm{loc}}(\mathbb{R}^N) \mid \nabla\psi \in L^2(\mathbb{R}^N), |\psi|^2 - 1 \in L^2(\mathbb{R}^N)\big\}. \tag{2.3}$$

The aim of this chapter is to prove that the Cauchy problem for the system (GC) is globally well posed in $\mathbf{E}\times H^1(\mathbb{R}^N)$ in space dimension $N = 1, 2$ or 3. Our main result is

Theorem 1. *Assume that $N \in \{1,2,3\}$. For any $\Psi_0 \in \mathbf{E}$ and any $\Phi_0 \in H^1(\mathbb{R}^N)$, there exists a unique global solution $(\Psi,\Phi) \in C(\mathbb{R}, \mathbf{E} \times H^1(\mathbb{R}^N))$ of (GC) such that $\Psi(0) = \Psi_0$ and $\Phi(0) = \Phi_0$. Moreover, for any $t \in \mathbb{R}$ we have $E(\Psi(t),\Phi(t)) = E(\Psi_0,\Phi_0)$ and $\|\Phi(t)\|_{L^2} = \|\Phi_0\|_{L^2}$. The solution depends continuously on the initial data uniformly on compact intervals.* [1]

In the next section we give some useful properties of the energy space. Theorem 1 is proven in section 3. In the Appendix we prove an "abstract" result that we needed in the proofs and which could be useful elsewhere.

2.2 The energy space

For any integer $k \geq 1$, we consider the function space (sometimes called *Zhidkov space,* see [3, 4, 8, 9, 10])

$$X^k(\mathbb{R}^N) = \{u \in L^\infty(\mathbb{R}^N) \mid \partial^\alpha u \in L^2(\mathbb{R}^N), 1 \leq |\alpha| \leq k\} \tag{2.4}$$

endowed with the natural norm

$$\|u\|_{X^k} = \|u\|_{L^\infty} + \sum_{1\leq|\alpha|\leq k} \|\partial^\alpha u\|_{L^2}. \tag{2.5}$$

Fix a C^∞, nonincreasing function $\tilde{\chi} : \mathbb{R} \longrightarrow \mathbb{R}$ such that $\tilde{\chi} = 1$ on $(-\infty, 2]$ and $\tilde{\chi} = 0$ on $[3,\infty)$. Define $\chi : \mathbb{C} \longrightarrow \mathbb{R}$ by $\chi(z) = \tilde{\chi}(|z|)$. For any $u \in \mathbf{E}$, define

$$P_1(u) = \chi(u)u \qquad \text{and} \qquad P_2(u) = (1-\chi(u))u. \tag{2.6}$$

Lemma 2 and 4 have been proven in [5]. For the convenience of the reader we recall here the proofs.

Lemma 2. *([5]) Let $N \in \mathbb{N}^*$. For any $u \in \mathbf{E}$ we have*

(i) $P_1(u) \in L^\infty(\mathbb{R}^N)$ and $\|P_1(u)\|_{L^\infty} \leq 3$.

(ii) $P_2(u) \in L^2(\mathbb{R}^N)$ and $\|P_2(u)\|_{L^2} \leq \big\| |u|^2 - 1 \big\|_{L^2} \leq \varepsilon\sqrt{2E_1(u)}$.

(iii) There is $C > 0$, depending only on χ, such that $\|\nabla(P_i(u))\|_{L^2} \leq C\|\nabla u\|_{L^2}$ for $i = 1,2$.

In particular, we have $P_1(u) \in X^1(\mathbb{R}^N)$ with $\|P_1(u)\|_{X^1} \leq 3 + C\sqrt{E_1(u)}$, and $P_2(u) \in H^1(\mathbb{R}^N)$ with $\|P_2(u)\|_{H^1} \leq C\sqrt{E_1(u)}$.

Proof. (i) Since $|z\chi(z)| = 0$ if $|z| \geq 3$, and $|z\chi(z)| = |z||\chi(z)| \leq 3$ if $|z| \leq 3$, we have $\|P_1(u)\|_{L^\infty} \leq 3$.

(ii) It is obvious that $|u(x)| \geq 2$ whenever $P_2(u(x)) \neq 0$. For all $z \in \mathbb{C}$ satisfying $|z| \geq 2$ we have $|(1-\chi(z))z| \leq |z| \leq |z|^2 - 1$. We infer that $|P_2(u)| \leq \big||u|^2 - 1\big|$ and (ii) follows.

(iii) We have

$$\nabla\big(\chi(u)u\big) = \chi(u)\nabla u + u\left[\partial_1\chi(u)\cdot\nabla\left(\mathrm{Re}(u)\right) + \partial_2\chi(u)\cdot\nabla\left(\mathrm{Im}(u)\right)\right]$$

and

$$\begin{aligned}\nabla\Big(\big(1-\chi(u)\big)u\Big) =& \big(1-\chi(u)\big)\nabla u \\ &- u\left[\partial_1\chi(u)\cdot\nabla\left(\mathrm{Re}(u)\right) + \partial_2\chi(u)\cdot\nabla\left(\mathrm{Im}(u)\right)\right].\end{aligned}$$

Then (iii) holds with $C = 1 + 6\|\nabla\chi\|_{L^\infty}$. □

[1] See (2.27) for a more precise statement.

Remark 3. Recall that, given two Banach spaces X, Y of distributions on $\mathbb{R}^N$, the space $X+Y$ is also a Banach space with the norm

$$\|v\|_{X+Y} = \inf\Big\{\|v_1\|_X + \|v_2\|_Y \mid v = v_1+v_2,\ v_1\in X,\ v_2\in Y\Big\}.$$

If X and Y are continuously embedded into $\tilde{X}$ and $\tilde{Y}$, respectively, then $X+Y$ is continuously embedded into $\tilde{X}+\tilde{Y}$.

By Lemma 2 we have $\mathbf{E}\subset X^1(\mathbb{R}^N)+H^1(\mathbb{R}^N)$ and there is $c>0$ such that

$$\|u\|_{X^1+H^1} \le \|P_1(u)\|_{X^1} + \|P_2(u)\|_{H^1} \le c\big(1+\sqrt{E_1(u)}\big) \qquad \text{for any } u\in\mathbf{E}.$$

If $H^1(\mathbb{R}^N)$ is continuously embedded into $L^p(\mathbb{R}^N)$, then $X^1(\mathbb{R}^N)+H^1(\mathbb{R}^N)$ is embedded into $L^\infty + L^p(\mathbb{R}^N)$.

Given $u_0, u_1\in\mathbf{E}$, we define

$$d_{\mathbf{E}}(u_0,u_1) = \|u_0-u_1\|_{X^1+H^1} + \big\|\,|u_0|^2-|u_1|^2\,\big\|_{L^2}. \tag{2.7}$$

It is easy to see that $d_{\mathbf{E}}$ is a distance on $\mathbf{E}$ and $(E,d_{\mathbf{E}})$ is a complete metric space.

Then $\mathbf{E}\times H^1(\mathbb{R}^N)$ can be endowed with a complete metric space structure by defining the distance d as follows: for every $u_0,u_1\in\mathbf{E}$ and for every $v_0,v_1\in H^1(\mathbb{R}^N)$,

$$d\big((u_0,v_0),(u_1,v_1)\big) = d_{\mathbf{E}}(u_0,u_1) + \|v_0-v_1\|_{H^1}. \tag{2.8}$$

Lemma 4. *([5]) Assume that $1\le N\le 4$. Then we have $\boldsymbol{E}+H^1(\mathbb{R}^N)\subset\boldsymbol{E}$, and for any $v\in\boldsymbol{E}$, $w\in H^1(\mathbb{R}^N)$, there holds*

$$\big\|\,|v+w|^2-1\,\big\|_{L^2} \le \big\|\,|v|^2-1\,\big\|_{L^2} + 6\|w\|_{L^2} + C\sqrt{E_1(v)}\|w\|_{L^4} + \|w\|_{L^4}^2. \tag{2.9}$$

Moreover, for all v_1, $v_2\in\boldsymbol{E}$, w_1, $w_2\in H^1(\mathbb{R}^N)$ we have

$$\begin{aligned} d_{\boldsymbol{E}}(v_1+w_1,v_2+w_2) &\le C\big(1+\|w_2\|_{H^1}\big)d_{\boldsymbol{E}}(v_1,v_2)\\ &\quad + C\big(1+\sqrt{E_1(v_1)}+\|w_1\|_{H^1}+\|w_2\|_{H^1}\big)\|w_1-w_2\|_{H^1}. \end{aligned} \tag{2.10}$$

Proof. For $v\in\mathbf{E}$, $w\in H^1(\mathbb{R}^N)$, we have $\nabla(v+w)=\nabla v+\nabla w\in L^2(\mathbb{R}^N)$.

$$|v+w|^2-1 = |v|^2-1+2\mathrm{Re}(\overline{v}w)+|w|^2.$$

Since $v\in\mathbf{E}$, we have $|v|^2-1\in L^2(\mathbb{R}^N)$. By the Sobolev embedding we have $H^1(\mathbb{R}^N)\subset L^4(\mathbb{R}^N)$, hence $w\in L^4(\mathbb{R}^N)$, which implies that $|w|^2\in L^2(\mathbb{R}^N)$. It remains to prove that $\mathrm{Re}(\overline{v}w)\in L^2(\mathbb{R}^N)$. By Lemma 2, v can be written as $v=v_1+v_2$, with $v_1=P_1(v)\in X^1(\mathbb{R}^N)$ and $v_2=P_2(v)\in H^1(\mathbb{R}^N)$. Using the Cauchy-Schwarz inequality, the Sobolev embedding and Lemma 2 (ii) we get

$$\|\overline{v}w\|_{L^2} \le \|\overline{v_1}w\|_{L^2} + \|\overline{v_2}w\|_{L^2} \le 3\|w\|_{L^2} + \|v_2\|_{L^4}\|w\|_{L^4}$$

$$\le 3\|w\|_{L^2} + \|v_2\|_{H^1}\|w\|_{L^4} \le 3\|w\|_{L^2} + C\sqrt{E_1(v)}\|w\|_{L^4}.$$

This gives (2.9). Next we prove (2.10). Obviously,

$$\|v_1+w_1-v_2-w_2\|_{X^1+H^1} \le \|v_1-v_2\|_{X^1+H^1} + \|w_1-w_2\|_{H^1}, \qquad \text{and}$$

$$|v_1+w_1|^2-|v_2+w_2|^2 = |v_1|^2-|v_2|^2+2\mathrm{Re}(\overline{v}_1w_1-\overline{v}_2w_2)+|w_1|^2-|w_2|^2.$$

Using the Cauchy-Schwarz inequality and the Sobolev embedding we get

$$\begin{aligned}\| \, |w_1|^2 - |w_2|^2 \|_{L^2} &\leq \| \, |w_1| + |w_2| \, \|_{L^4} \| \, |w_1| - |w_2| \, \|_{L^4} \\ &\leq C(\|w_1\|_{H^1} + \|w_2\|_{H^1}) \|w_1 - w_2\|_{H^1}\end{aligned}$$

and

$$\begin{aligned}\|\overline{v}_1 w_1 - \overline{v}_2 w_2\|_{L^2} &\leq \|\overline{v}_1 (w_1 - w_2)\|_{L^2} + \|(\overline{v}_1 - \overline{v}_2) w_2\|_{L^2} \\ &\leq \|P_1(v_1)\|_{L^\infty} \|w_1 - w_2\|_{L^2} + \|P_2(v_1)\|_{L^4} \|w_1 - w_2\|_{L^4} \\ &\quad + \|\overline{v}_1 - \overline{v}_2\|_{L^\infty + L^4} \left(\|w_2\|_{L^2} + \|w_2\|_{L^4}\right) \\ &\leq 3 \|w_1 - w_2\|_{L^2} \\ &\quad + C \sqrt{E_1(v_1)} \|w_1 - w_2\|_{H^1} \\ &\quad + C \|v_1 - v_2\|_{X^1 + H^1} \|w_2\|_{H^1}.\end{aligned}$$

From the above estimates we get (2.10). □

Corollary 5. *For every $v_1, v_2 \in \boldsymbol{E}$, $w_1, w_2, u_1, u_2 \in H^1(\mathbb{R}^N)$, we have*

$$\begin{aligned}d\big((v_1 + w_1, u_1), (v_2 + w_2, u_2)\big) &\leq C\big(1 + \|w_2\|_{H^1}\big) d_{\boldsymbol{E}}(v_1, v_2) \\ &+ C\big(1 + \sqrt{E_1(v_1)} + \|w_1\|_{H^1} + \|w_2\|_{H^1}\big) \|w_1 - w_2\|_{H^1} + \|u_1 - u_2\|_{H^1}.\end{aligned} \tag{2.11}$$

It is easily seen that in space dimension $N \geq 4$, the energy E introduced in (2.1) is well-defined and continuous on $\mathbf{E} \times H^1(\mathbb{R}^N)$. It is obvious that $E_1(\Psi) \leq E(\Psi, \Phi)$ and $\|\nabla \Phi\|_{L^2}^2 \leq \varepsilon^2 q^2 E(\Psi, \Phi)$. One can prove that for any $M > 0$, there exists $C(M) > 0$ such that $E(\Psi, \Phi) \leq M$ implies that $\|\Phi\|_{H^1} \leq C(M)$.

2.3 The main result

We consider the Cauchy problem associated to the Gross-Clark system (GC) with initial values $\Psi(0) = \Psi_0 \in \mathbf{E}$ and $\Phi(0) = \Phi_0 \in H^1(\mathbb{R}^N)$. We denote by $S(t) = e^{it\Delta}$ the linear Schrödinger group in $\mathbb{R}^N$. For any given $u_0 \in L^2(\mathbb{R}^N)$, we have $S(t)u_0 = \mathcal{F}^{-1}\left(e^{it|\xi|^2} \mathcal{F}u(\xi)\right)$, where $\mathcal{F}$ is the Fourier transform. Since $e^{it|\xi|^2}$ is of modulus one, $S(t)$ is an isometry from $L^2(\mathbb{R}^N)$ into itself and from $H^s(\mathbb{R}^N)$ into $H^s(\mathbb{R}^N)$ for any $s \in \mathbb{R}$. It is well-known that (Ψ, Φ) is a solution of the Cauchy problem associated to the system (GC) with initial data (Ψ_0, Φ_0) if and only if the functions Ψ and Φ satisfy the Duhamel formula (see e.g. [1], formula (1.6.2) p. 19 or Proposition 3.1.3 p. 57):

$$\Psi(t) = e^{it\Delta}\Psi_0 - i \int_0^t S(t - \tau) \left(\frac{1}{\varepsilon^2}\left(\frac{1}{\varepsilon^2}|\Phi|^2 + |\Psi|^2 - 1\right)\Psi(\tau)\right) \mathrm{d}\tau, \tag{2.12}$$

$$\Phi(t) = e^{i\frac{t}{\delta}\Delta}\Phi_0 - \frac{i}{\delta} \int_0^t S\left(\frac{t - \tau}{\delta}\right) \left(\frac{1}{\varepsilon^2}(q^2|\Psi|^2 - \varepsilon^2 k^2)\Phi(\tau)\right) \mathrm{d}\tau. \tag{2.13}$$

We will study the mappings

$$(\Psi, \Phi) \mapsto F(\Psi, \Phi) = \frac{1}{\varepsilon^2}\left(\frac{1}{\varepsilon^2}|\Phi|^2 + |\Psi|^2 - 1\right)\Psi,$$

and

$$(\Psi, \Phi) \mapsto G(\Psi, \Phi) = \frac{1}{\varepsilon^2}\left(q^2|\Psi|^2 - \varepsilon^2 k^2\right)\Phi.$$

2.3.1 Estimates on the nonlinear drifts

Lemma 6. *Assume that $1 \le N \le 4$. Consider $p \ge 4$ such that $H^1(\mathbb{R}^N)$ is continuously embedded into $L^p(\mathbb{R}^N)$. Then F and G are continuous from $\mathbf{E} \times H^1(\mathbb{R}^N)$ to $L^2 + L^{\frac{2p}{p+2}}(\mathbb{R}^N)$ and the mappings $(\Psi, \Phi) \longmapsto \nabla(F(\Psi, \Phi))$ and $(\Psi, \Phi) \longmapsto \nabla(G(\Psi, \Phi))$ are continuous from $\mathbf{E} \times H^1(\mathbb{R}^N)$ to $L^2 + L^{\frac{2p}{p+4}}(\mathbb{R}^N)$. Moreover, for any $R > 0$ there exists a constant $C(R)$ such that, for every $\Psi_1, \Psi_2 \in \mathbf{E}$, $\Phi_1, \Phi_2 \in H^1(\mathbb{R}^N)$ satisfying $E_1(\Psi_1) \le R$, $E_1(\Psi_2) \le R$ and $\|\Phi_1\|_{H^1} \le R$, $\|\Phi_2\|_{H^1} \le R$, the following inequalities hold:*

$$\begin{aligned} &\|F(\Psi_1, \Phi_1) - F(\Psi_2, \Phi_2)\|_{L^2 + L^{\frac{2p}{p+2}}} + \|\nabla F(\Psi_1, \Phi_1) - \nabla F(\Psi_2, \Phi_2)\|_{L^2 + L^{\frac{2p}{p+4}}} \\ &\qquad \le C(R) d\big((\Psi_1, \Phi_1), (\Psi_2, \Phi_2)\big), \end{aligned} \tag{2.14}$$

$$\begin{aligned} &\|G(\Psi_1, \Phi_1) - G(\Psi_2, \Phi_2)\|_{L^2 + L^{\frac{2p}{p+2}}} + \|\nabla G(\Psi_1, \Phi_1) - \nabla G(\Psi_2, \Phi_2)\|_{L^2 + L^{\frac{2p}{p+4}}} \\ &\qquad \le C(R) d\big((\Psi_1, \Phi_1), (\Psi_2, \Phi_2)\big), \end{aligned} \tag{2.15}$$

Proof. It suffices to prove (2.14) and (2.15). Let $\Psi_1, \Psi_2 \in \mathbf{E}$, and $\Phi_1, \Phi_2 \in H^1(\mathbb{R}^N)$. We have:

$$\begin{aligned} F(\Psi_1, \Phi_1) - F(\Psi_2, \Phi_2) =& \frac{1}{\varepsilon^2}\big(|\Psi_1|^2 - |\Psi_2|^2\big)\Psi_1 + \frac{1}{\varepsilon^2}\big(\Psi_1 - \Psi_2\big)\big(|\Psi_2|^2 - 1\big) \\ &+ \frac{1}{\varepsilon^4}\big(|\Phi_1|^2 - |\Phi_2|^2\big)\Psi_2 + \frac{1}{\varepsilon^4}\big(\Psi_1 - \Psi_2\big)|\Phi_1|^2. \end{aligned}$$

Assume that $N \le 4$ and $H^1(\mathbb{R}^N)$ is continuously embedded into $L^p(\mathbb{R}^N)$, where $p \ge 4$. Then $\mathbf{E} \subset L^\infty + L^p(\mathbb{R}^N)$ and using the above equality, the Sobolev embedding and the fact that $\frac{p}{3} \ge \frac{2p}{p+2}$ we get

$$\begin{aligned} &\|F(\Psi_1, \Phi_1) - F(\Psi_2, \Phi_2)\|_{L^2 + L^{\frac{2p}{p+2}}} \le C\Big(\big\| |\Psi_1|^2 - |\Psi_2|^2 \big\|_{L^2} \big\|\Psi_1\big\|_{L^\infty + L^p} \\ &+ \big\|\Psi_1 - \Psi_2\big\|_{L^\infty + L^p} \big\| |\Psi_2|^2 - 1 \big\|_{L^2} + \big\|\Phi_1 - \Phi_2\big\|_{H^1} \big\|\Psi_2\big\|_{L^\infty + L^p} (\big\|\Phi_1\big\|_{H^1} + \big\|\Phi_2\big\|_{H^1}) \\ &+ \big\|\Psi_1 - \Psi_2\big\|_{L^\infty + L^p} \big\|\Phi_1\big\|_{H^1}^2 \Big) \\ &\le C(R) d\big((\Psi_1, \Phi_1), (\Psi_2, \Phi_2)\big). \end{aligned}$$

It is obvious that

$$\nabla F(\Psi, \Phi) = \frac{1}{\varepsilon^4}\nabla\big(|\Phi|^2\Psi\big) + \frac{1}{\varepsilon^2}\nabla\big((|\Psi|^2)\Psi\big) - \frac{1}{\varepsilon^2}\nabla\Psi.$$

Using the formula

$$\nabla(fgh) = gh\nabla f + fh\nabla g + fg\nabla h$$

with $f = \Phi$, $g = \overline{\Phi}$, $h = \Psi$ and with $f = \Psi$, $g = \overline{\Psi}$, $h = \Psi$ as well as

$$f_1 g_1 h_1 - f_2 g_2 h_2 = (f_1 - f_2) g_1 h_1 + f_2 (g_1 - g_2) h_2 + f_2 g_2 (h_1 - h_2),$$

then using Hölder's inequality for each term we get

$$\|\nabla(F(\Psi_1, \Phi_1)) - \nabla(F(\Psi_2, \Phi_2))\|_{L^2 + L^{\frac{2p}{p+4}}} \le C(R) d\big((\Psi_1, \Phi_1), (\Psi_2, \Phi_2)\big).$$

We have

$$\begin{aligned} &G(\Psi_1, \Phi_1) - G(\Psi_2, \Phi_2) = \\ &\frac{1}{\varepsilon^2}\Big(q^2\Phi_1\big(|\Psi_1|^2 - |\Psi_2|^2\big) + q^2|\Psi_2|^2\big(\Phi_1 - \Phi_2\big) + \varepsilon^2 k^2\big(\Phi_1 - \Phi_2\big)\Big), \end{aligned}$$

and proceeding as above we get (2.15). □

Definition 7. *We say that the pair (p,q) is admissible if $p \geq 2$, $2 \leq q \leq \infty$ if $N = 1$, $2 \leq q < \infty$ if $N = 2$, $2 \leq q \leq \frac{2N}{N-2}$ if $N \geq 3$, and the following equality holds:*

$$\frac{2}{p} + \frac{N}{q} = \frac{N}{2}. \tag{2.16}$$

Given $T > 0$ and $f \in L^1([0,T];\mathscr{X})$, where $\mathscr{X}$ is a Banach space such that $(t,u) \longmapsto S(t)u$ is continuous from $\mathbb{R} \times \mathscr{X}$ to $\mathscr{X}$ (for instance, $\mathscr{X}$ may be any Sobolev space $H^s(\mathbb{R}^N)$ with $s \in \mathbb{R}$), we define $\Lambda_f : [-T,T] \longrightarrow \mathscr{X}$ by

$$\Lambda_f(t) = \int_0^t S(t-\tau) f(\tau)\, d\tau. \tag{2.17}$$

Remark 8. *(Strichartz estimate* - see e.g. [1], Theorem 2.3.3 p. 33). We recall that if (p_1,q_1) and (p_2,q_2) are two admissible pairs, then for every $f \in L^{p_1'}\big([-T,T], L^{q_1'}(\mathbb{R}^N)\big)$, the mapping Λ_f belongs to $L^{p_2}\big([-T,T], L^{q_2}(\mathbb{R}^N)\big)$ and to $C([-T,T], L^2(\mathbb{R}^N))$. Moreover, there exists a constant $C > 0$, independent of T, such that the following inequality holds

$$\|\Lambda_f\|_{L^{p_2}\left([-T,T],L^{q_2}\right)} \leq C \|f\|_{L^{p_1'}\left([-T,T],L^{q_1'}\right)}. \tag{2.18}$$

Let $(\Psi,\Phi) \in C([-T,T], \mathbf{E} \times H^1(\mathbb{R}^N))$. By Lemma 6, $F(\Psi,\Phi)$ and $G(\Psi,\Phi)$ belong to $H^s(\mathbb{R}^N)$ for some $s \in \mathbb{R}$ (for instance, we may take $s = -1$ if $1 \leq N \leq 4$). Hence we may define $\mathcal{A}(\Psi,\Phi)$ and $\mathcal{B}(\Psi,\Phi)$ on $[-T,T]$ by

$$\begin{aligned}
\mathcal{A}(\Psi,\Phi)(t) :&= \int_0^t S(t-\tau) F\big(\Psi(\tau),\Phi(\tau)\big) \mathrm{d}\tau \\
&= \frac{1}{\varepsilon^2} \int_0^t S(t-\tau) \left(\frac{1}{\varepsilon^2} |\Phi(\tau)|^2 + |\Psi(\tau)|^2 - 1 \right) \Psi(\tau) \mathrm{d}\tau,
\end{aligned}$$

$$\begin{aligned}
\mathcal{B}(\Psi,\Phi)(t) :&= \frac{1}{\delta} \int_0^t S\left(\frac{t-\tau}{\delta}\right) G\big(\Psi(\tau),\Phi(\tau)\big) \mathrm{d}\tau \\
&= \frac{1}{\delta\varepsilon^2} \int_0^t S\left(\frac{t-\tau}{\delta}\right) \big(q^2 |\Psi(\tau)|^2 - \varepsilon^2 k^2\big) \Phi(\tau) \mathrm{d}\tau.
\end{aligned}$$

Lemma 9. *Assume that $1 \leq N \leq 3$. For every $\Psi \in C\big([-T,T], \boldsymbol{E}\big)$ and $\Phi \in C\big([-T,T], H^1(\mathbb{R}^N)\big)$, the functions $\mathcal{A}(\Psi,\Phi)$ and $\mathcal{B}(\Psi,\Phi)$ defined above belong to $C\big([-T,T], H^1(\mathbb{R}^N)\big)$.*

Moreover, if $T \leq 1$ then for every $R > 0$, there exists a constant $C(R) > 0$ independent of T such that, if $\Psi_1, \Psi_2, \Phi_1, \Phi_2$ satisfy

$$E_1\big(\Psi_1(t)\big) \leq R,\ \|\Phi_1(t)\|_{H^1} \leq R \quad \text{and} \quad E_1\big(\Psi_2(t)\big) \leq R,\ \|\Phi_2(t)\|_{H^1} \leq R,$$

then

$$\begin{aligned}
&\sup_{|t|\leq T} \big\|\mathcal{A}(\Psi_1,\Phi_1)(t) - \mathcal{A}(\Psi_2,\Phi_2)(t)\big\|_{H^1} \\
&\qquad \leq C(R) T^{\frac{1}{2}} \sup_{|t|\leq T} d\Big(\big(\Psi_1(t),\Phi_1(t)\big), \big(\Psi_2(t),\Phi_2(t)\big)\Big),
\end{aligned} \tag{2.19}$$

and

$$\begin{aligned}
&\sup_{|t|\leq T} \big\|\mathcal{B}(\Psi_1,\Phi_1)(t) - \mathcal{B}(\Psi_2,\Phi_2)(t)\big\|_{H^1} \\
&\qquad \leq C(R) T^{\frac{1}{2}} \sup_{|t|\leq T} d\Big(\big(\Psi_1(t),\Phi_1(t)\big), \big(\Psi_2(t),\Phi_2(t)\big)\Big).
\end{aligned} \tag{2.20}$$

Proof. Let $p \geq 4$ such that $H^1(\mathbb{R}^N)$ is continuously embedded into $L^p(\mathbb{R}^N)$. Obviously, we must have $N \leq 4 \leq p$. Using Lemma 6, then Lemma 12 and Corollary 13 in Appendix, there exist two continuous functions $f \in C([-T,T], L^2(\mathbb{R}^N))$ and $g \in C([-T,T], L^{\frac{2p}{p+2}}(\mathbb{R}^N))$ such that

$$F(\Psi_1, \Phi_1) - F(\Psi_2, \Phi_2) = f + g$$

and

$$\begin{aligned} &\sup_{t\in[-T,T]} \|f(t)\|_{L^2} + \sup_{t\in[-T,T]} \|g(t)\|_{L^{\frac{2p}{p+2}}} \\ &\leq 3 \sup_{t\in[-T,T]} \|F(\Psi_1, \Phi_1) - F(\Psi_2, \Phi_2)\|_{L^2 + L^{\frac{2p}{p+2}}} \\ &\leq C(R) \sup_{t\in[-T,T]} d\big((\Psi_1(t), \Phi_1(t)), (\Psi_2(t), \Phi_2(t)) \big). \end{aligned} \tag{2.21}$$

If (γ, ρ) is an admissible pair such that $\rho' = \frac{2p}{p+2}$, we have necessarily $\rho = \frac{2p}{p-2}$, $\gamma = \frac{2p}{N}$ and $\gamma' = \frac{2p}{2p-N}$. Using Strichartz' estimate with γ and ρ as above, then Hölder's inequality on $[-T,T]$ we see that $\Lambda_f, \Lambda_g \in C([-T,T], L^2(\mathbb{R}^N))$, where Λ_f, Λ_g are as in (2.17), and

$$\sup_{t\in[-T,T]} \|\Lambda_f(t)\|_{L^2} \leq C\|f\|_{L^1([-T,T],L^2)} \leq CT \sup_{t\in[-T,T]} \|f(t)\|_{L^2},$$

$$\sup_{t\in[-T,T]} \|\Lambda_g(t)\|_{L^2} \leq C\|g\|_{L^{\gamma'}([-T,T],L^{\rho'})} \leq CT^{\frac{2p-N}{2p}} \sup_{t\in[-T,T]} \|g(t)\|_{L^{\frac{2p}{p+2}}}.$$

Summing up, we infer that $\mathcal{A}(\Psi_1, \Phi_1) - \mathcal{A}(\Psi_2, \Phi_2) \in C([-T,T], L^2(\mathbb{R}^N))$. From the above estimates and (2.21) we see that there exists $C(R) > 0$ such that if $T \in (0,1]$, we have

$$\begin{aligned} &\sup_{t\in[-T,T]} \|\mathcal{A}(\Psi_1, \Phi_1) - \mathcal{A}(\Psi_2, \Phi_2)\|_{L^2} \\ &\leq C(R) T^{\frac{2p-N}{2p}} \sup_{t\in[-T,T]} d\big((\Psi_1(t), \Phi_1(t)), (\Psi_2(t), \Phi_2(t)) \big). \end{aligned} \tag{2.22}$$

Since $S(t)$ commutes with derivatives with respect to space variables, it is easy to see that

$$\nabla \mathcal{A}(\Psi, \Phi)(t) = \int_0^t S(t-\tau) \Big(\nabla F\big(\Psi(\tau), \Phi(\tau)\big) \Big) \mathrm{d}\tau.$$

From Lemma 6, Lemma 12 and Corollary 13 there exist continuous functions $h \in C\left([-T,T], \left(L^2(\mathbb{R}^N)\right)^N\right)$ and $g \in C\left([-T,T], \left(L^{\frac{2p}{p+4}}(\mathbb{R}^N)\right)^N\right)$ such that

$$\nabla F(\Psi_1, \Phi_1) - \nabla F(\Psi_2, \Phi_2) = h + k$$

and

$$\begin{aligned} &\sup_{t\in[-T,T]} \|h(t)\|_{L^2} + \sup_{t\in[-T,T]} \|k(t)\|_{L^{\frac{2p}{p+4}}} \\ &\leq 3 \sup_{t\in[-T,T]} \|\nabla F(\Psi_1, \Phi_1) - \nabla F(\Psi_2, \Phi_2)\|_{L^2 + L^{\frac{2p}{p+4}}} \\ &\leq C(R) \sup_{t\in[-T,T]} d\big((\Psi_1(t), \Phi_1(t)), (\Psi_2(t), \Phi_2(t)) \big). \end{aligned} \tag{2.23}$$

If (γ,ρ) is an admissible pair such that $\rho'=\frac{2p}{p+4}$, we must have $\rho=\frac{2p}{p-4}$ (respectively $\rho=\infty$ if $p=4$), $\gamma=\frac{p}{N}$ and $\gamma'=\frac{p}{p-N}$. If $N\geq 3$ we need $2\leq\rho\leq\frac{2N}{N-2}$ and this is equivalent to $p\geq 2N$. Hence there is no admissible pair with the desired properties if $N=4$, and in the case $N=3$ the only possible choice is $p=6$, thus $(\gamma,\rho)=(2,6)$ and $(\gamma',\rho')=(2,\frac{6}{5})$. Notice that if $N=3$ we need the *endpoint Strichartz estimate,* that is (2.18) with $(p_1,q_1)=(2,\frac{2N}{N-2})$, see [6]. Using Strichartz' estimate with γ and ρ as above, then Hölder's inequality on $[-T,T]$ we see that $\Lambda_h,\Lambda_k\in C\left([-T,T],\left(L^2(\mathbb{R}^N)\right)^N\right)$ and

$$\sup_{t\in[-T,T]}\|\Lambda_h(t)\|_{L^2}\leq C\|h\|_{L^1([-T,T],L^2)}\leq CT\sup_{t\in[-T,T]}\|h(t)\|_{L^2},$$

$$\sup_{t\in[-T,T]}\|\Lambda_k(t)\|_{L^2}\leq C\|k\|_{L^{\gamma'}([-T,T],L^{\rho'})}\leq CT^{\frac{p-N}{p}}\sup_{t\in[-T,T]}\|k(t)\|_{L^{\frac{2p}{p+4}}}.$$

We infer that $\nabla\left(\mathcal{A}(\Psi_1,\Phi_1)\right)-\nabla\left(\mathcal{A}(\Psi_2,\Phi_2)\right)\in C\left(([-T,T],\left(L^2(\mathbb{R}^N)\right)^N\right)$. From the above estimates and (2.23) we see that there exists $C(R)>0$ such that if $T\in(0,1]$, there holds

$$\begin{aligned}&\sup_{t\in[-T,T]}\|\nabla\left(\mathcal{A}(\Psi_1,\Phi_1)\right)-\nabla\left(\mathcal{A}(\Psi_2,\Phi_2)\right)\|_{L^2}\\&\leq C(R)T^{\frac{p-N}{p}}\sup_{t\in[-T,T]}d\big(\left(\Psi_1(t),\Phi_1(t)\right),\left(\Psi_2(t),\Phi_2(t)\right)\big).\end{aligned}\tag{2.24}$$

We conclude that $\mathcal{A}(\Psi_1,\Phi_1)-\mathcal{A}(\Psi_2,\Phi_2)\in C([-T,T],H^1(\mathbb{R}^N))$. Taking $(\Psi_2,\Phi_2)=(1,0)$ it is obvious that $\mathcal{A}(\Psi_1,\Phi_1)\in C([-T,T],H^1(\mathbb{R}^N))$ whenever $\Psi_1\in C([-T,T],\mathbf{E})$ and $\Phi_1\in C([-T,T],H^1(\mathbb{R}^N))$. Estimate (2.19) follows from (2.22) and (2.24) with $p=6$ if $N=3$, respectively $p=4$ if $N\in\{1,2\}$.

A simple change of variables shows that $\mathcal{B}(\Psi,\Phi)(t)=\Lambda_{G(\Psi(\delta\cdot),\Phi(\delta\cdot))}\left(\frac{t}{\delta}\right)$. Proceeding exactly as above, one proves that $\mathcal{B}(\Psi,\Phi)\in C([-T,T],H^1(\mathbb{R}^N))$ and (2.20) holds. □

2.3.2 Local wellposedness

The Duhamel formula (2.12)-(2.13) can be written as

$$\Psi(t)=S(t)\Psi_0-i\mathcal{A}(\Psi,\Phi)(t),\tag{2.25}$$

$$\Phi(t)=S\left(\frac{t}{\delta}\right)\Phi_0-i\mathcal{B}(\Psi,\Phi)(t).\tag{2.26}$$

Theorem 10. *Assume that $N\in\{1,2,3\}$. For every $R>0$, there exists $T>0$ such that for any $\Psi_0\in\boldsymbol{E}$ and any $\Phi_0\in H^1(\mathbb{R}^N)$ satisfying $E_1(\Psi_0)\leq R$ and $\|\Phi_0\|_{H^1}\leq R$, there exists a unique solution (Ψ,Φ) of the system (GC) with initial values $(\Psi,\Phi)(0)=(\Psi_0,\Phi_0)$, such that*

$$(\Psi,\Phi)\in C\big([-T,T],\boldsymbol{E}\big)\times C\big([-T,T],H^1(\mathbb{R}^N)\big).$$

Moreover, if $(\Psi_{0,1},\Phi_{0,1}),(\Psi_{0,2},\Phi_{0,2})\in\boldsymbol{E}\times H^1(\mathbb{R}^N)$ satisfy $E_1(\Psi_{0,1})\leq R$, $\|\Phi_{0,1}\|_{H^1}\leq R$ and $E_1(\Psi_{0,2})\leq R$, $\|\Phi_{0,2}\|_{H^1}\leq R$ then

$$\sup_{|t|\leq T}d\Big(\big(\Psi_1(t),\Phi_1(t)\big),\big(\Psi_2(t),\Phi_2(t)\big)\Big)\leq C(R)d\big((\Psi_{0,1},\Phi_{0,1}),(\Psi_{0,2},\Phi_{0,2})\big),\tag{2.27}$$

where (Ψ_1,Φ_1) and (Ψ_2,Φ_2) are the solutions of (GC) with initial data $(\Psi_{0,1},\Phi_{0,1})$ and $(\Psi_{0,2},\Phi_{0,2})$, respectively.

Proof. Let $(\Psi_0, \Phi_0) \in \mathbf{E} \times H^1(\mathbb{R}^N)$ be such that $E_1(\Psi_0) \leq R$ and $\|\Phi_0\|_{H^1} \leq R$. We shall prove that for T small enough (depending only on R), the mapping

$$(\Psi, \Phi) \longmapsto \Big(S(\cdot)\Psi_0 - i\mathcal{A}(\Psi, \Phi), S\left(\frac{\cdot}{\delta}\right)\Phi_0 - i\mathcal{B}(\Psi, \Phi)\Big) \tag{2.28}$$

is a contraction on the complete metric space

$$\mathcal{X} = \big\{(\Psi, \Phi) \in C\big([-T,T], \mathbf{E}\big) \times C\big([-T,T], H^1(\mathbb{R}^N)\big) \mid \sup_{|t|\leq T} E_1\big(\Psi(t)\big) \leq 3R, \\ \sup_{|t|\leq T} \|\Phi(t)\|_{H^1} \leq 3R\big\}$$

endowed with the distance

$$\operatorname{dist}\big((\Psi_1, \Phi_1), (\Psi_2, \Phi_2) \big) = \sup_{t\in[-T,T]} d\big((\Psi_1(t), \Phi_1(t)), (\Psi_2(t), \Phi_2(t)) \big).$$

Since $e^{it\Delta}$ is an isometry in $H^1(\mathbb{R}^N)$, we have $\|e^{it\Delta}\Phi_0\|_{H^1} = \|\Phi_0\|_{H^1} \leq R$ for all t. Notice that Proposition 2.3 in [5] is valid for $N = 1, 2, 3$ and it implies that there exists $T_1(R) > 0$ such that for $|t| \leq T_1(R)$ we have $E_1(e^{it\Delta}\Psi_0) \leq 2R$.

Using (2.19) and (2.20) in the particular case when Ψ_2 is constant of modulus 1 and $\Phi_2 = 0$, we get

$$\sup_{|t|\leq T} \|\mathcal{A}(\Psi, \Phi)(t)\|_{H^1} \leq C(R)T^{\frac{1}{2}} \quad \text{and} \quad \sup_{|t|\leq T} \|\mathcal{B}(\Psi, \Phi)(t)\|_{H^1} \leq C(R)T^{\frac{1}{2}}.$$

Then using inequality (2.9) in Lemma 4, we see that for T small enough (depending on R),

$$\sup_{|t|\leq T} E_1\big(e^{it\Delta}\Psi_0 + \mathcal{A}(\Psi, \Phi)(t)\big) \leq 3R.$$

It is obvious that

$$\sup_{|t|\leq T} \|e^{it\Delta}\Phi_0 + \mathcal{B}(\Psi, \Phi)(t)\|_{H^1} \leq R + C(R)T^{\frac{1}{2}} \leq 3R$$

if T is small enough. We conclude that there exists $T_2(R) > 0$ such that for $T \leq T_2(R)$, the mapping (2.28) is from $\mathcal{X}$ to $\mathcal{X}$.

Using estimate (2.10) in Lemma 4, then estimate (2.19) in Lemma 9 we get

$$\begin{aligned} &\sup_{t\in[-T,T]} d_{\mathbf{E}}\left(S(t)\Psi_0 - i\mathcal{A}(\Psi_1(t), \Phi_1(t)), S(t)\Psi_0 - i\mathcal{A}(\Psi_2(t), \Phi_2(t))\right) \\ &\leq C_1(R) \sup_{t\in[-T,T]} \big\|\mathcal{A}(\Psi_1(t), \Phi_1(t)) - \mathcal{A}(\Psi_2(t), \Phi_2(t))\big\|_{H^1} \\ &\leq C_2(R)T^{\frac{1}{2}} \sup_{t\in[-T,T]} d\Big(\big(\Psi_1(t), \Phi_1(t)\big), \big(\Psi_2(t), \Phi_2(t)\big)\Big). \end{aligned} \tag{2.29}$$

Similarly, from estimate (2.20) in Lemma 9 we obtain

$$\begin{aligned} &\sup_{t\in[-T,T]} \left\| \left(S\left(\frac{t}{\delta}\right)\Phi_0 - i\mathcal{B}(\Psi_1(t), \Phi_1(t))\right) - \left(S\left(\frac{t}{\delta}\right)\Phi_0 - i\mathcal{B}(\Psi_2(t), \Phi_2(t))\right) \right\|_{H^1} \\ &= \sup_{t\in[-T,T]} \big\|\mathcal{B}(\Psi_1(t), \Phi_1(t)) - \mathcal{B}(\Psi_2(t), \Phi_2(t))\big\|_{H^1} \\ &\leq C_3(R)T^{\frac{1}{2}} \sup_{t\in[-T,T]} d\Big(\big(\Psi_1(t), \Phi_1(t)\big), \big(\Psi_2(t), \Phi_2(t)\big)\Big). \end{aligned} \tag{2.30}$$

It follows from (2.29) and (2.30) that there is $T(R) > 0$ such that for $T \leq T(R)$, the mapping in (2.28) is a contraction in $\mathcal{X}$; then the Banach-Picard fixed point theorem implies that it has a unique fixed point. Hence (GC) has a unique local solution with initial data (Ψ_0, Φ_0).

Let (Ψ_1, Φ_1) and (Ψ_2, Φ_2) be the solutions of (GC) with initial data $(\Psi_{0,1}, \Phi_{0,1})$ and $(\Psi_{0,2}, \Phi_{0,2})$, respectively, satisfying the assumptions in Theorem 10. Proceeding exactly as in (2.29) and (2.30), we see that there are $C_4(R) > 0$, $C_5(R) > 0$ such that

$$\begin{aligned}
&\sup_{t\in[-T,T]} d\big((\Psi_1(t), \Phi_1(t)), (\Psi_2(t), \Phi_2(t)) \big) \\
&\leq C_4(3R) d\big((\Psi_{0,1}, \Phi_{0,1}), (\Psi_{0,2}, \Phi_{0,2})\big) \\
&\quad + C_5(3R) T^{\frac{1}{2}} \sup_{t\in[-T,T]} d\big((\Psi_1(t), \Phi_1(t)), (\Psi_2(t), \Phi_2(t)) \big).
\end{aligned}$$

Choosing $T_3(R) \leq T(R)$ such that $C_5(3R) T_3(R)^{\frac{1}{2}} \leq \frac{1}{2}$, we see that (2.27) holds on intervals of length $T_3(R)$. Dividing $[-T(R), T(R)]$ into a finite number of intervals of length no greater than $T_3(R)$ and iterating this argument, we see that (2.27) holds on $[-T(R), T(R)]$. □

Since we have a lower bound on the time of existence of the local solution of the Cauchy problem for (GC) only in terms of $E_1(\Psi_0)$ and $\|\Phi_0\|_{H^1}$, a standard argument gives the following:

Corollary 11. *Assume that $1 \leq N \leq 3$. Let $\Psi_0 \in \mathbf{E}$ and $\Phi_0 \in H^1(\mathbb{R}^N)$. Let $(\Psi(t), \Phi(t))$ be the solution of the Cauchy problem with initial data (Ψ_0, Φ_0) and let $(-T_*, T^*)$ be its maximal interval of existence. Then either $T^* = \infty$, or $\lim_{t\uparrow T^*} (E_1(\Psi(t)) + \|\Phi(t)\|_{H^1}) = \infty$. A similar statement holds for T_*.*

2.3.3 Regularity, conservation of energy and global existence

Assume that $N \leq 3$. Then we have $H^2(\mathbb{R}^N) \subset X^2(\mathbb{R}^N)$. Consider $\Psi_0 \in \mathbf{E}$ and $\Phi_0 \in H^1(\mathbb{R}^N)$ such that $\Delta\Psi_0 \in L^2(\mathbb{R}^N)$ and $\Delta\Phi_0 \in L^2(\mathbb{R}^N)$. Let $(\Psi(t), \Phi(t))$ be the solution of (GC) with initial data (Ψ_0, Φ_0) and let $(-T_*, T^*)$ be the maximal interval where this solution exists. Since the mapping $(\Psi, \Phi) \longmapsto \big(F(\Psi, \Phi), G(\Psi, \Phi)\big)$ from $X^2(\mathbb{R}^N) \times H^2(\mathbb{R}^N)$ to itself is Lipschitz continuous on bounded sets of $X^2(\mathbb{R}^N) \times H^2(\mathbb{R}^N)$, proceeding as in [5] p. 772-773 we see that $(\Psi(\cdot), \Phi(\cdot)) \in C((-T_*, T^*), X^2(\mathbb{R}^N) \times H^2(\mathbb{R}^N))$. Then using (GC) we get $\partial_t\Psi(t)$, $\partial_t\Phi(t) \in C((-T_*, T^*), L^2(\mathbb{R}^N))$, and differentiating $\|\Phi(t)\|_{L^2}^2$ and $E((\Psi(t), \Phi(t))$ (where E is given by (2.1)), then integrating by parts and using the system (GC) we find

$$\begin{aligned}
&\frac{d}{dt}\|\Phi(t)\|_{L^2}^2 = 2\mathrm{Re}\langle \Phi(t), \partial_t\Phi(t)\rangle_{L^2} \\
&= 2\mathrm{Re}\int_{\mathbb{R}^N} \overline{\Phi(t)} \left(\frac{-i}{\delta} \left(-\Delta\Phi(t) + \frac{1}{\varepsilon^2}(q^2|\Psi|^2(t) - \varepsilon^2 k^2)\Phi(t) \right) \right) \mathrm{d}x = 0,
\end{aligned}$$

respectively

$$\begin{aligned}
&\frac{d}{dt}E((\Psi(t),\Phi(t))\\
&= 2\mathrm{Re}\int_{\mathbb{R}^N}\overline{\partial_t\Psi(t)}\left(-\Delta\Psi(t)+\frac{1}{\varepsilon^2}(\frac{1}{\varepsilon^2}|\Phi|^2(t)+|\Psi|^2(t)-1)\Psi(t)\right)\,\mathrm{d}x\\
&\quad+\frac{2}{\varepsilon^2q^2}\mathrm{Re}\int_{\mathbb{R}^N}\overline{\partial_t\Phi(t)}\left(-\Delta\Phi(t)+\frac{q^2}{\varepsilon^2}|\Psi|^2(t)\Phi(t)\right)\,\mathrm{d}x\\
&= 2\mathrm{Re}\int_{\mathbb{R}^N}\overline{\partial_t\Psi(t)}\,(i\partial_t\Psi(t))\,\mathrm{d}x\\
&\quad+\frac{2}{\varepsilon^2q^2}\mathrm{Re}\int_{\mathbb{R}^N}\overline{\partial_t\Phi(t)}\left(i\delta\partial_t\Phi(t)+\varepsilon^2k^2\Phi(t)\right)\,\mathrm{d}x\\
&= 0.
\end{aligned}$$

We infer that $\|\Phi(t)\|_{L^2}$ and $E(\Psi(t),\Phi(t))$ do not depend on t.

Let $\Psi_0\in\mathbf{E}$ and $\Phi_0\in H^1(\mathbb{R}^N)$. By Lemma 6 p. 773 in [5], there is a sequence $(\Psi_0^n)_{n\geq1}\subset E\cap X^2(\mathbb{R}^N)$ such that $d_{\mathbf{E}}(\Psi_0^n,\Psi_0)\longrightarrow 0$ as $n\longrightarrow\infty$. Consider a sequence $(\Phi_0^n)_{n\geq1}\subset C_c^\infty(\mathbb{R}^N)$ such that $\Phi_0^n\longrightarrow\Phi_0$ in $H^1(\mathbb{R}^N)$. Denote by $(\Psi(t),\Phi(t))$ and by $(\Psi_n(t),\Phi_n(t))$ the solutions of the Cauchy problem associated to the system (GC) with initial data (Ψ_0,Φ_0) and (Ψ_0^n,Φ_0^n), respectively. Let $(-T_*,T^*)$ be the maximal interval of existence of the solution $(\Psi(t),\Phi(t))$, and let $t_0\in(-T_*,T^*)$. Using (2.27) in Theorem 10, it is standard to see that for all n sufficiently large, t_0 belongs to the existence interval of the solution $(\Psi_n(t),\Phi_n(t))$, and $(\Psi_n(t_0),\Phi_n(t_0))\longrightarrow(\Psi(t_0),\Phi(t_0))$ in $\mathbf{E}\times H^1(\mathbb{R}^N)$ as $n\longrightarrow\infty$. Since $\|\Phi_n(t_0)\|_{L^2}=\|\Phi_0^n\|_{L^2}$ and $E(\Psi_n(t_0),\Phi_n(t_0))=E(\Psi_0^n,\Phi_0^n)$ for all n, passing to the limit we discover that $\|\Phi(t_0)\|_{L^2}=\|\Phi_0\|_{L^2}$ and $E(\Psi(t_0),\Phi(t_0))=E(\Psi_0,\Phi_0)$. We have thus proved that $\|\Phi\|_{L^2}$ and the energy E are conserved quantities for all solutions of (GC) in $\mathbf{E}\times H^1(\mathbb{R}^N)$.

The conservation of the energy and of the L^2-norm of Φ immediately imply that all solutions are global. Indeed, let $(\Psi(t),\Phi(t))$ be a solution with initial data $(\Psi_0,\Phi_0)\in\mathbf{E}\times H^1(\mathbb{R}^N)$, and let $(-T_*,T^*)$ be its maximal interval of existence. We have already seen that if $T^*<\infty$, then $E_1(\Psi(t))+\|\Phi(t)\|_{H^1}\longrightarrow\infty$ as $t\longrightarrow T^*$. Or, we have $E_1(\Psi(t))+\frac{1}{\varepsilon^2q^2}\|\nabla\Phi(t)\|_{L^2}^2\leq E(\Psi(t),\Phi(t))=E(\Psi_0,\Phi_0)$, and $\|\Phi(t)\|_{L^2}=\|\Phi_0\|_{L^2}$, hence $E_1(\Psi(t))+\|\Phi(t)\|_{H^1}$ remains bounded as long as the solution $(\Psi(\cdot),\Phi(\cdot))$ exists, and we conclude that necessarily $T^*=\infty$. Similarly we have $T_*=\infty$.

2.4 Appendix

Let $(X,\|\cdot\|_X)$ and $(Y,\|\cdot\|_Y)$ be two Banach spaces of distributions in $\mathbb{R}^N$. We consider the Banach space $X+Y$ endowed with the norm $\|u\|_{X+Y}=\inf\{\|x\|_X+\|y\|_Y\ \mid\ x\in X,\ y\in Y,\ u=x+y\}$.

Lemma 12. *Let $u:[a,b]\longrightarrow X+Y$ be a continuous function from the compact interval $[a,b]\subset\mathbb{R}$ to $(X+Y,\|\cdot\|_{X+Y})$. Let $\varepsilon>0$. There exist two continuous functions $v:[a,b]\longrightarrow X$ and $w:[a,b]\longrightarrow Y$ such that*

$$u(t)=v(t)+w(t)\qquad \textit{for all } t\in[a,b],$$

$$\begin{aligned}
&\sup_{t\in[a,b]}\|v(t)\|_X\leq\sup_{t\in[a,b]}\|u(t)\|_{X+Y}+\varepsilon\quad \textit{and}\\
&\sup_{t\in[a,b]}\|w(t)\|_Y\leq\sup_{t\in[a,b]}\|u(t)\|_{X+Y}+\varepsilon.
\end{aligned}\tag{2.31}$$

Proof. We divide the proof into two steps.

Step 1. For any $\varepsilon > 0$ there exist two continuous functions $v : [a,b] \longrightarrow X$ and $w : [a,b] \longrightarrow Y$ such that (2.31) holds and

$$\sup_{t\in[a,b]} \|u(t) - v(t) - w(t)\|_{X+Y} \leq \varepsilon. \tag{2.32}$$

Fix $\varepsilon > 0$. Since u is uniformly continuous, there exists $\eta > 0$ such that $\|u(t) - u(s)\|_{X+Y} \leq \frac{\varepsilon}{2}$ for all $s,t \in [a,b]$ such that $|s-t| < \eta$. Consider a finite sequence $a = t_0 < t_1 < t_2 < \cdots < t_{n-1} < t_n = b$ such that $t_k - t_{k-1} < \eta$ for all $k = 1,\ldots,n$. For any $k \in \{0,1,\ldots,n\}$, choose $v_k \in X$ and $w_k \in Y$ such that $u(t_k) = v_k + w_k$ and $\|v_k\|_X + \|w_k\|_Y \leq \|u(t_k)\|_{X+Y} + \frac{\varepsilon}{2}$. Define $v : [a,b] \longrightarrow X$ and $w : [a,b] \longrightarrow Y$ by $v(t_k) = v_k$, $w(t_k) = w_k$ and

$$v(t) = \frac{t_k - t}{t_k - t_{k-1}} v_{k-1} + \frac{t - t_{k-1}}{t_k - t_{k-1}} v_k,$$

$$w(t) = \frac{t_k - t}{t_k - t_{k-1}} w_{k-1} + \frac{t - t_{k-1}}{t_k - t_{k-1}} w_k \qquad \text{for } t \in [t_{k-1}, t_k].$$

In other words, v and w are affine on each of the intervals $[t_{k-1}, t_k]$. It is clear that v and w are continuous on $[a,b]$. For $t \in [t_{k-1}, t_k]$ we have

$$\|v(t)\|_X \leq \frac{t_k - t}{t_k - t_{k-1}} \|v_{k-1}\|_X + \frac{t - t_{k-1}}{t_k - t_{k-1}} \|v_k\|_X \leq \sup_{t\in[a,b]} \|u(t)\|_{X+Y} + \frac{\varepsilon}{2}$$

and a similar estimate holds for w. Hence (2.31) is established.

It is easily seen that

$$\begin{aligned} v(t) - v_{k-1} + w(t) - w_{k-1} &= \frac{t - t_{k-1}}{t_k - t_{k-1}} \left(v_k - v_{k-1} + w_k - w_{k-1}\right) \\ &= \frac{t - t_{k-1}}{t_k - t_{k-1}} \left(u(t_k) - u(t_{k-1})\right) \end{aligned}$$

and consequently

$$\begin{aligned} &\|v(t) + w(t) - u(t_{k-1})\|_{X+Y} \\ &\leq \|u(t_k) - u(t_{k-1})\|_{X+Y} \\ &\leq \frac{\varepsilon}{2} \qquad \text{for all } t \in [t_{k-1}, t_k]. \end{aligned}$$

Therefore for $t \in [t_{k-1}, t_k]$ we have

$$\begin{aligned} &\|u(t) - (v(t) + w(t))\|_{X+Y} \\ &\leq \|u(t) - u(t_{k-1})\|_{X+Y} + \|v(t) + w(t) - u(t_{k-1})\|_{X+Y} \\ &\leq \varepsilon. \end{aligned}$$

Step 2. We iterate the construction in Step 1 to conclude the proof. Given $\varepsilon > 0$, we construct v_1 and w_1 as in Step 1 such that (2.31) and (2.32) hold with $\frac{\varepsilon}{4}$ instead of ε. Let $u_1 = u - v_1 - w_1$, so that $\sup_{t\in[a,b]} \|u_1(t)\|_{X+Y} \leq \frac{\varepsilon}{4}$. Using again Step 1, we construct v_2 and w_2 such that (2.31) and (2.32) hold with u_1 and $\frac{\varepsilon}{8}$ instead of u and ε, respectively. We denote $u_2 = u_1 - v_2 - w_2$, and we

continue the process. If u_k has been constructed such that $\sup_{t\in[a,b]} \|u_k(t)\|_{X+Y} \leq \frac{\varepsilon}{2^{k+1}}$, using Step 1 we find v_{k+1} and w_{k+1} satisfying (2.31) and (2.32) with u_k instead of u and $\frac{\varepsilon}{2^{k+2}}$ instead of ε, and so on. Since $\|v_k\|_{L^\infty([a,b],\,X)} \leq \frac{\varepsilon}{2^k} + \frac{\varepsilon}{2^{k+1}}$ for $k \geq 1$ and $(X, \|\cdot\|_X)$ is complete, the series $\sum_{k\geq 1} v_k$ converges in X. We denote by v its sum. It is clear that v is continuous from $[a,b]$ to X and satisfies (2.31). Similarly $w = \sum_{k\geq 1} w_k$ is continuous from $[a,b]$ to X and satisfies (2.31). We have $\left\|u - \sum_{i=1}^k v_i - \sum_{i=1}^k w_i\right\|_{L^\infty([a,b],X+Y)} \leq \frac{\varepsilon}{2^{k+1}}$ for all $k \in \mathbb{N}^*$ and letting $k \longrightarrow \infty$ we get $u = v + w$. □

Corollary 13. *Let $u : [a,b] \longrightarrow X+Y$ be a continuous function. Let $\varepsilon > 0$. There exist two continuous functions $v : [a,b] \longrightarrow X$ and $w : [a,b] \longrightarrow Y$ such that*

$$u(t) = v(t) + w(t) \qquad \text{for all } t \in [a,b],$$

$$\begin{aligned} &\sup_{t\in[a,b]} \|v(t)\|_X \leq (1+\varepsilon) \sup_{t\in[a,b]} \|u(t)\|_{X+Y} \quad \text{and} \\ &\sup_{t\in[a,b]} \|w(t)\|_Y \leq (1+\varepsilon) \sup_{t\in[a,b]} \|u(t)\|_{X+Y}. \end{aligned} \tag{2.33}$$

Bibliography

[1] Cazenave, T. (2003). *Semilinear Schrödinger Equations*, Courant Lecture Notes in Mathematics vol. 10.

[2] Chiron, D. & Mariş, M. (2017). *Traveling Waves for Nonlinear Schrödinger Equations with Nonzero Conditions at Infinity.* Archive for Rational Mechanics and Analysis. 226 (1). 143-242.

[3] Gallo, C. (2004). *Schrödinger group on Zhidkov spaces.* Adv. Diff. Eq. 9, pp. 509-538.

[4] Gallo, C. (2008) *The Cauchy Problem for Defocusing NonlinearSchrödinger Equations with Non-Vanishing Initial Data at Infinity.* , Communications in PartialDifferential Equations, 33, No. 5, pp. 729-771.

[5] Gerard, P. (2006). *The Cauchy problem for the Gross-Pitaevskii equation.* Annales De L'Institut Henri Poincare-analyse Non Lineaire - ANN INST HENRI POINCARE-ANAL. 23. 765-779.

[6] Keel, M. & Tao, T. (1998). *Endpoint Strichartz estimates.* Amer. J. Math. 120. 955-980.

[7] Mariş, M. (2009). *Traveling waves for nonlinear Schrödinger equations with nonzero conditions at infinity.* Annals of Mathematics. 178 (1). 107-182 .

[8] Zhidkov, P. E. (1987). *The Cauchy Problem for a Nonlinear Schrödinger Equation.* Soobshch. OIYaI, R5-87-373, Dubna.

[9] Zhidkov, P. E. (1989). *Solvability of the Cauchy problems and stability of some solutions ofthe nonlinear Schrödinger equation.* Preprint OIYaI, P5-89-322, Dubna Matem. Model. 1, No. 10, pp. 155-160.

[10] Zhidkov, P. E. (1992). *Existence of solutions to the Cauchy problem and stability of kink-solutions of the nonlinear Schrödinger equation.* Siberian. Math. J. 33, pp. 239-246.

Chapter 3

Travelling waves to the Gross-Clark system

3.1 Introduction

We recall that the Gross-Clark system is given by

$$\text{(GC)} \qquad \begin{cases} i\dfrac{\partial \Psi}{\partial t} &= -\Delta\Psi + \frac{1}{\epsilon^2}(|\Psi|^2 + \frac{1}{\epsilon^2}|\Phi|^2 - 1)\Psi \\ \\ i\delta\dfrac{\partial \Phi}{\partial t} &= -\Delta\Phi + \frac{1}{\epsilon^2}(q^2|\Psi|^2 - \epsilon^2 k^2)\Phi \end{cases}$$

with the "boundary conditions" $|\Psi| \longrightarrow 1$ and $|\Phi| \longrightarrow 0$ as $|x| \longrightarrow \infty$. Here δ, ϵ, q, k are physical parameters, and ϵ is not "small" ($\epsilon \simeq 0.2$ in the literature).

We are interested in traveling waves, that is solutions of the form $\Psi(t,x) = \psi(x_1 - ct, x_2, \ldots, x_N)$, $\Phi(t,x) = \tilde{\varphi}(x_1 - ct, x_2, \ldots, x_N)$. It turns out that it is more interesting to search for $\tilde{\varphi}$ of the form $\tilde{\varphi}(x) = e^{i\delta c x_1}\varphi(x)$; this transform leads finally to $\Phi(t,x) = e^{i\delta c(x_1 - ct)}\varphi(x_1 - ct, x_2, \ldots, x_N)$. Notice that $\tilde{\varphi} \in H^1(\mathbb{R}^N)$ if and only if $\varphi \in H^1(\mathbb{R}^N)$. We find that ψ and φ must satisfy the system

$$\text{(TW)} \qquad \begin{cases} -ic\dfrac{\partial \psi}{\partial x_1} &= -\Delta\psi + \frac{1}{\epsilon^2}(|\psi|^2 + \frac{1}{\epsilon^2}|\varphi|^2 - 1)\psi \\ \\ (\frac{\delta^2 c^2}{4} + k^2)\varphi &= -\Delta\varphi + \frac{q^2}{\epsilon^2}|\psi|^2\varphi. \end{cases}$$

The first equation in (TW) is similar to the equation satisfied by traveling-waves to the Gross-Pitaevskii equation, except that it contains an additional term $\frac{1}{\epsilon^4}|\varphi|^2\psi$ which is linear with respect to ψ. The second equation in (TW) is linear in φ. In fact, φ must be an eigenvalue of the linear operator $L_\psi = -\Delta + \frac{q^2}{\epsilon^2}|\psi|^2$ corresponding to the eigenvalue $\lambda = \frac{\delta^2 c^2}{4} + k^2$. Notice that we have always $\langle L_\psi \phi, \phi\rangle \geq 0$. If $|\psi|$ tends to 1 sufficiently fast as $|x| \longrightarrow \infty$, the essential spectrum of L_ψ is $[1, \infty)$.

We will look only for eigenvectors of L_ψ corresponding to the first eigenvalue. They can be obtained by minimizing $\langle L_\psi\phi, \phi\rangle$ in $H^1(\mathbb{R}^N)$ when $\|\varphi\|_{L^2}$ is kept constant. It is therefore natural to seek for solutions of (TW) by minimizing $E(\psi, \varphi)$ when the momentum of ψ and the L^2-norm of φ are fixed (see below for the definition of the "energy" $E(\psi,\varphi)$ and of the momentum). If (ψ, φ) is a minimizer, the parameters c and $\lambda = \frac{\delta^2 c^2}{4} + k^2$ appearing in (TW) will be the corresponding Lagrange multipliers.

The following "energies" are relevant in the study of (GC) and (TW):

$$E_1(\psi) = \int_{\mathbb{R}^N} |\nabla\psi|^2 + V(|\psi|^2)\,\mathrm{d}x, \qquad \text{and} \tag{3.1}$$

$$
\begin{aligned}
E(\psi,\varphi) &= \int_{\mathbb{R}^N} |\nabla\psi|^2 + V(|\psi|^2) + \tfrac{1}{\epsilon^2 q^2}|\nabla\varphi|^2 + \tfrac{1}{\epsilon^4}|\psi|^2|\varphi|^2 \,\mathrm{d}x \\
&= E_1(\psi) + \int_{\mathbb{R}^N} \tfrac{1}{\epsilon^2 q^2}|\nabla\varphi|^2 + \tfrac{1}{\epsilon^4}|\psi|^2|\varphi|^2 \,\mathrm{d}x = E_1(\psi) + \tfrac{1}{\epsilon^2 q^2}\langle L_\psi\varphi,\varphi\rangle.
\end{aligned} \tag{3.2}
$$

In (GC) - (TW) we have $V(s) = \frac{1}{2\epsilon^2}(s-1)^2$, but we may consider more general potentials V. Throughout this chapter we consider a general potential V, we denote $F(\tau) = -V'(\tau)$, and we work with the same assumptions on F and on V as in [5]:

(A1) The function F is continuous on $[0,\infty)$, C^1 in a neighborhood of 1, $F(1) = 0$ and $F'(1) < 0$.

If **(A1)** is satisfied, the *sound velocity at infinity* associated to (GC) or to (3.3) is $v_s = \sqrt{-2F'(1)}$ (see e.g. [9]). In particular, for $F(s) = \frac{1}{\epsilon^2}(1-s)$ we have $v_s = \frac{\sqrt{2}}{\epsilon}$.

(A2) There exist $C > 0$ and $p_0 < \infty$ if $N = 1$ or $N = 2$, $p_0 < \frac{2}{N-2}$ in $N \geqslant 3$, such that $|F(s)| \leq C(1 + s^{p_0})$ for any $s \geq 0$.

(A3) There exist C, $a_0 > 0$ and $r_* > 1$ such that $F(s) \leq -Cs^{a_0}$ for any $s \geq r_*$.

(A4) F is C^2 near 1 and

$$
F(s) = F'(1)(s-1) + \frac{1}{2}F''(1)(s-1)^2 + \mathcal{O}((s-1)^3) \qquad \text{for } s \text{ close to } 1.
$$

We assume throughout the chapter that **(A1)** and **(A2)** hold. Assumptions **(A3)** and **(A4)** will be needed occasionally. For instance, **(A3)** is useful to obtain the regularity of solutions when a bootstrap cannot be performed, but it is not needed to prove the existence of solutions.

Notice that the energy E is a conserved quantity for the system (GC) (see Section 3.3 in [1]). Similarly, the energy E_1 is a conserved quantity for the Gross-Pitaevskii (GP) equation

$$
i\partial_t\Psi + \Delta\Psi + F(|\Psi|^2)\Psi = 0 \text{ in } \mathbb{R}^N, \qquad |\Psi| \longrightarrow 0 \text{ as } |x| \longrightarrow \infty. \tag{3.3}
$$

The natural *energy space* associated to (GC) is

$$
\mathscr{X} := \{(\psi,\varphi) \in H^1_{loc}(\mathbb{R}^N) \;\big|\; E(\psi,\varphi) < \infty\}.
$$

If $(\psi,\varphi) \in \mathscr{X}$, we have necessarily $E_1(\psi) < \infty$ and we infer that $\psi \in \mathcal{E}$, where $\mathcal{E}$ is the energy space associated to the Gross-Pitaevskii equation. We refer to the introduction of [5] or of [11] for a description of $\mathcal{E}$. Consider a cut-off function $\chi \in C^\infty(\mathbb{R})$ such that χ is nondecreasing, $\chi(t) = t$ if $t \leqslant 2$ and $\chi(t) = 3$ if $t \geqslant 4$. It has been shown in [5] that

$$
\begin{aligned}
\mathcal{E} &= \{\psi \in H^1_{loc}(\mathbb{R}^N) \mid E_1(\psi) < \infty\} \\
&= \{\psi \in H^1_{loc}(\mathbb{R}^N) \mid \nabla\psi \in L^2(\mathbb{R}^N),\ \chi^2(|\psi|) - 1 \in L^2(\mathbb{R}^N)\} \\
&= \{\psi : \mathbb{R}^N \longrightarrow \mathbb{C} \mid \psi \text{ is measurable}, |\psi| - 1 \in L^2(\mathbb{R}^N), \nabla\psi \in L^2(\mathbb{R}^N)\}.
\end{aligned} \tag{3.4}
$$

Lemma 20 below implies that if $(\psi,\varphi) \in \mathscr{X}$, then $\varphi \in H^1(\mathbb{R}^N)$. In the sequel we will work with functions in the space $\mathcal{E} \times H^1(\mathbb{R}^N)$. If $N \geqslant 4$, using the Sobolev embeddings it is easy to see that $E(\psi,\varphi)$ is finite for any $\psi \in \mathcal{E}$ and $\varphi \in H^1(\mathbb{R}^N)$, and $\mathscr{X} = \mathcal{E} \times H^1(\mathbb{R}^N)$. If $N \geqslant 5$ we still have $\mathscr{X} \subset \mathcal{E} \times H^1(\mathbb{R}^N)$, but the inclusion is strict. For the minimization problems considered in this chapter it suffices to consider only functions $(\psi,\varphi) \in \mathcal{E} \times H^1(\mathbb{R}^N)$ such that $E(\psi,\varphi)$ is finite.

Another important quantity for the study of (GC) (as well as for the Gross-Pitaevskii equation) is *the momentum.*

Given any function $\varphi \in H^1(\mathbb{R}^N)$, we define its *momentum* by $Q(\varphi) = \int_{\mathbb{R}^N} \langle i\frac{\partial\varphi}{\partial x_1}, \varphi\rangle \,\mathrm{d}x$. It is obvious that this is well-defined because $\langle i\frac{\partial\varphi}{\partial x_1}, \varphi\rangle \in L^1(\mathbb{R}^N)$ whenever $\varphi \in H^1(\mathbb{R}^N)$ and the Cauchy-Schwarz inequality gives

$$|Q(\varphi)| \leqslant \left\|\frac{\partial\varphi}{\partial x_1}\right\|_{L^2}\|\varphi\|_{L^2} \leqslant \frac{1}{2K}\int_{\mathbb{R}^N}\left|\frac{\partial\varphi}{\partial x_1}\right|^2 + K^2|\varphi|^2\,\mathrm{d}x \qquad \text{for any } K>0. \tag{3.5}$$

If $\psi \in \mathcal{E}$, the function $\langle i\frac{\partial\psi}{\partial x_1}, \psi\rangle$ does not necessarily belong to $L^1(\mathbb{R}^N)$. However, it has been shown in [11], Section 2 and in [5], Section 2 that for any $\psi \in \mathcal{E}$ we may write $\langle i\frac{\partial\psi}{\partial x_1}, \psi\rangle = f + \frac{\partial\phi}{\partial x_1}$, where $f \in L^1(\mathbb{R}^N)$ and $\phi \in \dot{H}^1(\mathbb{R}^N)$, and that it makes sense to define the momentum of ψ by $Q(\psi) = \int_{\mathbb{R}^N} f\,\mathrm{d}x$; see [11, 5] for details.

The momentum $Q(\Psi)$ is a conserved quantity for the Gross-Pitaevskii equation (3.3). The problem of minimizing $E_1(\psi)$ while $Q(\psi)$ is kept fixed has been studied in [5]. This gives traveling waves to (3.3) that are orbitally stable. The following result has been proven in [5]:

Theorem 14. *[5] Assume that $N \geq 2$,* **(A1)** *and* **(A2)** *are satisfied and $V \geq 0$ on $[0,\infty)$. For $p \geq 0$, let*

$$E_{1,min}(p) = \inf\{E_1(\psi) \;\big|\; \psi \in \mathcal{E},\ Q(\psi) = p\}. \tag{3.6}$$

Then:

(i) The function $E_{1,min}$ is concave, increasing on $[0,\infty)$, $E_{1,min}(p) \leqslant v_s p$ for any $p \geq 0$, the right derivative of $E_{1,min}$ at 0 is v_s, $E_{1,min}(p) \longrightarrow \infty$ and $\frac{E_{1,min}(p)}{p} \longrightarrow 0$ as $p \longrightarrow \infty$.

(ii) Let $p_0 = \inf\{p > 0 \mid E_{1,min}(p) < v_s p\}$. For any $p > p_0$, all sequences $(\psi_n)_{n\geq 1} \subset \mathcal{E}$ satisfying $Q(\psi_n) \longrightarrow p$ and $E_1(\psi_n) \longrightarrow E_{1,min}(p)$ are precompact (modulo translations).

The set $\mathcal{S}_p = \{\psi \in \mathcal{E} \mid Q(\psi) = p,\ E(\psi) = E_{1,min}(p)\}$ is not empty and is orbitally stable by the flow associated to (3.3).

(iii) Any $\psi_p \in \mathcal{S}_p$ is a traveling wave for (3.3) of speed $c(\psi_p) \in [d^+E_{1,min}(p), d^-E_{1,min}(p)]$, where we denote by d^- and d^+ the left and right derivatives. We have $c(\psi_p) \longrightarrow 0$ as $p \longrightarrow \infty$.

(iv) If $N \geqslant 3$ we have always $p_0 > 0$. Moreover, if $N = 2$ and assumption **(A4)** *is satisfied, we have $p_0 = 0$ if and only if $F''(1) \neq 3$, in which case $c(\psi_p) \longrightarrow v_s$ as $p \longrightarrow 0$.*

Notice that $Q(\psi)$ and $Q(\varphi)$ are not conserved quantities for (GC). Let

$$P(\psi,\varphi) = Q(\psi) + \frac{\delta}{\epsilon^2 q^2}Q(\varphi). \tag{3.7}$$

It is easily seen that P is (at least formally) a conserved quantity for the system (GC). Therefore it would be natural to seek for traveling waves for (GC) by minimizing E when P is kept fixed. In section 3.3 we consider a more general problem, namely for any $\beta \in (0,1)$ we minimize the energy $E(\psi,\varphi)$ when the mixed momentum $P_\beta(\psi,\varphi) = \beta Q(\psi) + (1-\beta)Q(\varphi)$ is fixed. Our Theorem 26 below is a result analogous to Theorem 14.

Let us give a brief overview of our main results. In Section 3.2 we consider the problem of minimizing $E(\psi,\varphi)$ when the L^2-norm of φ is kept fixed (no constraint is imposed on ψ). We prove the following result:

Theorem 15. *Suppose that* **(A1)** *and* **(A2)** *hold and $V \geq 0$ on $[0,\infty)$. For $m \geq 0$, we define*

$$g_{min}(m) = \inf\left\{E(\psi,\varphi) \;\big|\; \psi \in \mathcal{E}, \varphi \in H^1(\mathbb{R}^N), \int_{\mathbb{R}^N}|\varphi|^2\,\mathrm{d}x = m\right\}. \tag{3.8}$$

Then:

(i) The function g_{min} is concave, increasing on $[0,\infty)$, $g_{min}(m) \leqslant \frac{m}{\epsilon^4}$ for any $m \geqslant 0$, the right derivative of g_{min} at 0 is $\frac{1}{\epsilon^4}$, and $g_{min}(m) \leqslant Cm^{\frac{N}{N+2}}$ for large m.

(ii) Let $m_0 = \inf\left\{m > 0 \mid g_{min}(m) < \frac{m}{\epsilon^4}\right\}$. For any $m > m_0$, all sequences $(\psi_n, \varphi_n)_{n\geq 1} \subset \mathcal{E} \times H^1(\mathbb{R}^N)$ satisfying $\|\varphi_n\|_{L^2}^2 \longrightarrow m$ and $E(\psi_n, \varphi_n) \longrightarrow g_{min}(m)$ are precompact (modulo translations), and there exist minimizers for $g_{min}(m)$.

(iii) Any minimizer (ψ, φ) for $g_{min}(m)$ satisfies the system

$$-\Delta\psi + F(|\psi|^2)\psi + \frac{1}{\epsilon^4}|\varphi|^2\psi = 0, \qquad -\Delta\varphi + \frac{q^2}{\epsilon^2}|\psi|^2\varphi - \epsilon^2 q^2\gamma\varphi = 0 \qquad \text{in } \mathbb{R}^N$$

for some $\gamma \in [d^+g_{min}(m), d^-g_{min}(m)]$. The functions ψ and φ are smooth on $\mathbb{R}^N$ and, after translation, they are radial. After multiplication by complex numbers of modulus one, they are real-valued; moreover, the radial profile of ψ is nondecreasing, and the radial profile of φ is nonincreasing.

(iv) We have $m_0 = 0$ if $N = 1$ and $m_0 > 0$ if $N \geqslant 2$.

In Section 3.3 we consider the problem of minimizing $E(\psi, \varphi)$ when the mixed momentum $P_\beta(\psi, \varphi) = \beta Q(\psi) + (1-\beta)Q(\varphi)$ is fixed. Here is the main result of Section 3.3:

Theorem 16. *Assume that $N \geq 2$, **(A1)** and **(A2)** are satisfied and $V \geq 0$ on $[0,\infty)$. Let*

$$E_{\beta,min}(p) = \inf\left\{E(\psi,\varphi) \;\middle|\; \psi \in \mathcal{E}, \varphi \in H^1(\mathbb{R}^N), P_\beta(\psi,\varphi) = p\right\}.$$

For any $\beta \in (0,1)$, the following holds true.

(i) The function $E_{\beta,min}$ is concave, increasing on $[0,\infty)$, and there exists an explicit constant $S_\beta > 0$ such that $E_{\beta,min}(p) \leqslant S_\beta p$ for any $p \geqslant 0$. The right derivative of $E_{\beta,min}$ at 0 is S_β, and $E_{\beta,min}(p) \longrightarrow \infty$ and $\frac{E_{\beta,min}(p)}{p} \longrightarrow 0$ as $p \longrightarrow \infty$.

(ii) Let $p_\beta = \inf\{p > 0 \mid E_{\beta,min}(p) < S_\beta p\}$. For any $p > p_\beta$, all sequences $(\psi_n, \varphi_n)_{n\geq 1} \subset \mathcal{E} \times H^1(\mathbb{R}^N)$ satisfying $P_\beta(\psi_n, \varphi_n) \longrightarrow p$ and $E(\psi_n, \varphi_n) \longrightarrow E_{\beta,min}(p)$ are precompact (modulo translations), and there exist minimizers for $E_{\beta,min}(p)$.

(iii) Any minimizer (ψ, φ) for $E_{\beta,min}(p)$ satisfies the system

$$\begin{cases} ic\beta\frac{\partial\psi}{\partial x_1} = -\Delta\psi - F(|\psi|^2)\psi + \frac{1}{\epsilon^4}|\varphi|^2\psi \\ \\ ic(1-\beta)\epsilon^2q^2\frac{\partial\varphi}{\partial x_1} = -\Delta\varphi + \frac{q^2}{\epsilon^2}|\psi|^2\varphi \end{cases}$$

for some $c \in [d^+E_{\beta,min}(p), d^-E_{\beta,min}(p)]$. The functions ψ and φ are smooth on $\mathbb{R}^N$ (at least $C^{1,\alpha}$ for any $\alpha \in [0,1)$). After translation, (ψ, φ) is axially symmetric about the axis Ox_1.

In the last section we study a minimization problem with two constraints. More precisely, for $p \in \mathbb{R}$ and $m \geqslant 0$, we define

$$E_{min}(p,m) = \inf\left\{E(\psi,\varphi) \;\middle|\; \psi \in \mathcal{E},\ \varphi \in H^1(\mathbb{R}^N),\ Q(\psi) = p, \int_{\mathbb{R}^N} |\varphi|^2\,\mathrm{d}x = m\right\}. \tag{3.9}$$

The main result of Section 3.4 is as follows.

Theorem 17. *Assume that $N \geqslant 2$, assumptions* **(A1)** *and* **(A2)** *hold and $V \geqslant 0$. Assume that the pair (p, m) satisfies the following strict sub-additivity condition:*

$$E_{1,min}(p') + E_{min}(p - p', m) > E_{min}(p, m) \text{ for any } p' \in \mathbb{R}^*. \tag{3.10}$$

Then there exist minimizers for $E_{min}(p, m)$.

Moreover, any sequence $(\psi_n, \varphi_n) \in \mathcal{E} \times H^1(\mathbb{R}^N)$ satisfying $Q(\psi_n) \longrightarrow p$, $\int_{\mathbb{R}^N} |\varphi_n|^2 \, \mathrm{d}x \longrightarrow m$ and $E(\psi_n, \varphi_n) \longrightarrow E_{min}(p, m)$ has a convergent subsequence (after translations in $\mathbb{R}^N$).

The minimizers solve the system

$$\begin{cases} i\lambda_1 \frac{\partial \psi}{\partial x_1} = -\Delta\psi - F(|\psi|^2)\psi + \frac{1}{\epsilon^4}|\varphi|^2\psi, \\ \\ i\lambda_2 \epsilon^2 q^2 \frac{\partial \varphi}{\partial x_1} = -\Delta\varphi + \frac{q^2}{\epsilon^2}|\psi|^2\varphi \end{cases}$$

for some Lagrange multipliers λ_1, λ_2, and are smooth (at least $C^{1,\alpha}(\mathbb{R}^N)$ for any $\alpha \in [0, 1)$).

3.2 Ground states

For $m > 0$ we consider the minimization problem

$$(GS_m) \qquad \text{minimize } E(\psi, \varphi) \text{ for } \psi \in \mathcal{E},\ \varphi \in H^1(\mathbb{R}^N) \text{ satisfying } \int_{\mathbb{R}^N} |\varphi|^2 \, \mathrm{d}x = m.$$

We define

$$g_{min}(m) = \inf \left\{ E(\psi, \varphi) \;\middle|\; \psi \in \mathcal{E}, \varphi \in H^1(\mathbb{R}^N), \int_{\mathbb{R}^N} |\varphi|^2 \, \mathrm{d}x = m \right\}. \tag{3.11}$$

Proposition 18. *The function g_{min} has the following properties:*

(i) g_{min} is non-decreasing and concave on $(0, \infty)$, and $0 \leqslant g_{min}(m) \leqslant \frac{m}{\epsilon^4}$ for all $m > 0$.

(ii) If $N = 1$ we have $g_{min}(m) < \dfrac{m}{\epsilon^4}$ for any $m > 0$ and $\lim\limits_{m \to 0} \dfrac{g_{min}(m)}{m} = \dfrac{1}{\epsilon^4}$.

(iii) If $N \geqslant 2$, there exists $m_0 > 0$ such that $g_{min}(m) = \dfrac{m}{\epsilon^4}$ for any $m \in (0, m_0]$.

(iv) There exists $C > 0$ such that $g_{min}(m) \leqslant C m^{\frac{N}{N+2}}$.

Proof. (i) Since $E(\psi, \varphi) \geqslant 0$ for any ψ and φ, it is obvious that $g_{min}(m) \geqslant 0$. Consider $\phi \in C_c^\infty(\mathbb{R}^N)$ such that $\int_{\mathbb{R}^N} |\phi|^2 \, \mathrm{d}x = m$ and let $\varphi_\sigma(x) = \frac{1}{\sigma^{N/2}} \phi\left(\frac{x}{\sigma}\right)$. Then $\int_{\mathbb{R}^N} |\varphi_\sigma|^2 \, \mathrm{d}x = m$ and $\int_{\mathbb{R}^N} |\varphi_\sigma|^2 \, \mathrm{d}x = \frac{1}{\sigma^2} \int_{\mathbb{R}^N} |\nabla\phi|^2 \, \mathrm{d}x$. Taking $\psi = 1$ we see that $g_{min}(m) \leqslant E(1, \varphi_\sigma)$ for all $\sigma > 0$, and letting $\sigma \longrightarrow \infty$ we get $g_{min}(m) \leqslant \frac{m}{\epsilon^4}$.

Proceeding as in (3.68) we see that

$$g_{min}(m) = \inf \left\{ \int_{\mathbb{R}^N} |\nabla\psi|^2 + V(|\psi|^2) + \frac{m}{\epsilon^2 q^2}|\nabla\varphi|^2 + \frac{m}{\epsilon^4}|\psi|^2|\varphi|^2 \, \mathrm{d}x \;\middle|\; \right.$$
$$\left. \psi \in \mathcal{E}, \varphi \in H^1(\mathbb{R}^N), \textstyle\int_{\mathbb{R}^N} |\varphi|^2 \, \mathrm{d}x = 1 \right\}. \tag{3.12}$$

For any $(\psi, \varphi) \in \mathcal{E} \times H^1(\mathbb{R}^N)$, the mapping $m \longmapsto \int_{\mathbb{R}^N} |\nabla\psi|^2 + V(|\psi|^2) + \frac{m}{\epsilon^2 q^2}|\nabla\varphi|^2 + \frac{m}{\epsilon^4}|\psi|^2|\varphi|^2 \, \mathrm{d}x$ is concave and non-decreasing, and the infimum of a family of concave and non-decreasing functions is a concave and non-decreasing function, too.

(ii) Consider $\chi \in C_c^\infty(\mathbb{R}^N)$ such that $0 \leqslant \chi \leqslant 1$, $\chi = 1$ on $B(0,1)$ and $\chi = 0$ on $\mathbb{R}^N \setminus B(0,2)$. Denote $A = \|\chi\|_{L^2}^2$, $B = \|\nabla\chi\|_{L^2}^2$, and $D = \int_{\mathbb{R}^N} \chi^4 \,\mathrm{d}x$. Let $\kappa \in (0,1)$ (to be chosen later) and let

$$\psi_{a,b}(x) = 1 - \kappa a^2 \chi^2\left(\frac{x}{b}\right), \qquad \varphi_{a,b}(x) = a\chi\left(\frac{x}{b}\right).$$

Fix $m > 0$. If $a \in (0,1)$ and $b > 0$ are chosen so that $a^2 b^N A = m$, we have $\|\varphi_{a,b}\|_{L^2}^2 = m$. It is clear that $\|\nabla\varphi_{a,b}\|_{L^2}^2 = a^2 b^{N-2} B$ and $\|\nabla\psi_{a,b}\|_{L^2}^2 = \kappa^2 a^4 b^{N-2} \|\nabla(\chi^2)\|_{L^2}^2 \leqslant 4\kappa^2 a^4 b^{N-2} B$. There exists $C > 0$ such that $V(\tau) \leqslant C(1-\tau)^2$ for all $\tau \in [0,2]$, hence

$$V(|\psi_{a,b}|^2) \leqslant C(1-\psi_{a,b}^2)^2 \leqslant 4C\kappa^2 a^4 \chi^4\left(\frac{x}{b}\right)$$

and consequently

$$\int_{\mathbb{R}^N} V(|\psi_{a,b}|^2) \,\mathrm{d}x \leqslant 4C\kappa^2 a^4 b^N D.$$

Using the above estimates and the fact that $a^2 b^N A = m$ we get

$$\begin{aligned}
g_{min}(m) - \frac{m}{\epsilon^4} &\leqslant E(\psi_{a,b}, \varphi_{a,b}) - \frac{1}{\epsilon^4}\int_{\mathbb{R}^N} |\varphi_{a,b}|^2 \,\mathrm{d}x \\
&= \int_{\mathbb{R}^N} |\nabla\psi_{a,b}|^2 + V(|\psi_{a,b}|^2) + \frac{1}{\epsilon^2 q^2}|\nabla\varphi_{a,b}|^2 + \frac{1}{\epsilon^4}\left(|\psi_{a,b}|^2 - 1\right)|\varphi_{a,b}|^2 \,\mathrm{d}x \\
&\leqslant 4\kappa^2 a^4 b^{N-2} B + 4C\kappa^2 a^4 b^N D + \frac{a^2 b^{N-2} B}{\epsilon^2 q^2} - \frac{2\kappa a^4 b^N}{\epsilon^4}\int_{\mathbb{R}^N}\left(1 - \frac{1}{2}\kappa a^2 \chi^2(x)\right)\chi^4(x)\,\mathrm{d}x \\
&\leqslant 4\kappa^2 a^4 b^{N-2} B + 4C\kappa^2 a^4 b^N D + \frac{a^2 b^{N-2} B}{\epsilon^2 q^2} - \frac{\kappa a^4 b^N D}{\epsilon^4} \\
&= m\left(4\kappa^2 m^{-\frac{2}{N}} a^{2+\frac{4}{N}} A^{\frac{2}{N}-1} B + 4\kappa^2 a^2 \tfrac{CD}{A} + \tfrac{1}{\epsilon^2 q^2} m^{-\frac{2}{N}} a^{\frac{4}{N}} A^{\frac{2}{N}-1} B - \kappa a^2 \tfrac{D}{A\epsilon^4}\right).
\end{aligned}$$

Choosing κ sufficiently small (for instance, $\kappa \leqslant \frac{C}{8\epsilon^4}$ will do) we see that there exist constants $C_2, C_3 > 0$ such that $g_{min}(m) - \frac{m}{\epsilon^4} \leqslant m\left(C_2 m^{-\frac{2}{N}} a^{\frac{4}{N}} - C_3 a^2\right)$. If $N = 1$ it suffices to take $a = \ell m$, where $\ell > 0$ is sufficiently small, to see that there exists some $C_4 > 0$ such that $g_{min}(m) - \frac{m}{\epsilon^4} \leqslant -C_4 m^3 < 0$ for all $m > 0$.

If $N = 1$, for any $\eta > 0$ there exists $e(\eta) > 0$ such that for any $\psi \in \mathcal{E}$ satisfying $E_1(\psi) \leqslant e(\eta)$ there holds $\||\psi| - 1\|_{L^\infty(\mathbb{R})} \leqslant \eta$. Fix $\eta > 0$. Let $m \in (0, \frac{1}{2}\epsilon^4 e(\eta)]$. If $\psi \in \mathcal{E}$ and $\varphi \in H^1(\mathbb{R})$ are such that $\|\varphi\|_{L^2}^2 = m$ and $E(\psi,\varphi) \leqslant \frac{2m}{\epsilon^4}$, it is obvious that $E_1(\psi) \leqslant \frac{2m}{\epsilon^4} \leqslant e(\eta)$, hence $|\psi(x)| \in [1-\eta, 1+\eta]$ for all $x \in \mathbb{R}$ and consequently $E(\psi,\varphi) \geqslant \frac{1}{\epsilon^4}\int_{\mathbb{R}} |\psi|^2|\varphi|^2 \,\mathrm{d}x \geqslant \frac{(1-\eta)^2 m}{\epsilon^4}$. We conclude that $\frac{(1-\eta)^2 m}{\epsilon^4} \leqslant \frac{g_{min}(m)}{m} \leqslant \frac{1}{\epsilon^4}$ for any $m \in (0, \frac{1}{2}\epsilon^4 e(\eta)]$, hence $\lim_{m\to 0} \frac{g_{min}(m)}{m} = \frac{1}{\epsilon^4}$.

(iii) We already know that $g_{min}(m) \leqslant \frac{m}{\epsilon^4}$ (see part (i)) and it suffices to show that for m sufficiently small and for any $\psi \in \mathcal{E}$ and any $\varphi \in H^1(\mathbb{R}^N)$ satisfying $\|\varphi\|_{L^2}^2 = m$ and $E(\psi,\varphi) \leqslant \frac{2m}{\epsilon^4}$ there holds $E(\psi,\varphi) \geqslant \frac{m}{\epsilon^4}$. It also suffices to consider the case when ψ and φ are real-valued and $0 \leqslant \psi \leqslant 1$ on $\mathbb{R}^N$. Indeed, denoting $\tilde{\varphi} = |\varphi|$ and $\tilde{\psi} = \min(|\psi|, 1)$ we have $\tilde{\psi} \in \mathcal{E}$, $\tilde{\varphi} \in H^1(\mathbb{R}^N)$, $\|\tilde{\varphi}\|_{L^2}^2 = m$ and $E(\tilde{\psi},\tilde{\varphi}) \leqslant E(\psi,\varphi)$.

Suppose that $\psi \in \mathcal{E}$ and $\varphi \in H^1(\mathbb{R}^N)$ are real-valued, $0 \leqslant \psi \leqslant 1$ on $\mathbb{R}^N$, $\|\varphi\|_{L^2}^2 = m$, and $E(\psi,\varphi) \leqslant \frac{2m}{\epsilon^4}$.

Consider first the case $N \geqslant 3$. Using Hölder's inequality, then Sobolev's inequality we get

$$\Big| \int_{\mathbb{R}^N} \left(|\psi|^2 - 1\right) |\varphi|^2 \, \mathrm{d}x \Big| \leqslant \|1 - |\psi|^2\|_{L^{\frac{N}{2}}} \|\varphi\|_{L^{2^*}}^2 \leqslant C_S^2 \|1 - |\psi|^2\|_{L^{\frac{N}{2}}} \|\nabla\varphi\|_{L^2}^2. \tag{3.13}$$

Since $0 \leqslant |\psi| \leqslant 1$, using Lemma 4.1 p. 171 in [5] (see estimate (4.1) there), we infer that there exists $C_1 > 0$ such that

$$\begin{aligned} \int_{\mathbb{R}^N} \left(|\psi|^2 - 1\right)^2 \, \mathrm{d}x &\leqslant 4 \int_{\mathbb{R}^N} V(|\psi|^2) \, \mathrm{d}x + C_1 \|\nabla\psi\|_{L^2}^{2^*} \\ &\leqslant 4E(\psi,\varphi) + C_1 E(\psi,\varphi)^{\frac{2^*}{2}} \leqslant \frac{8m}{\epsilon^4} + C_1 \left(\frac{8m}{\epsilon^4}\right)^{\frac{2^*}{2}}. \end{aligned} \tag{3.14}$$

If $N \geqslant 4$, by Hölder's inequality we get

$$\|1 - |\psi|^2\|_{L^{\frac{N}{2}}} \leqslant \|1 - |\psi|^2\|_{L^2}^{\frac{4}{N}} \cdot \|1 - |\psi|^2\|_{L^\infty}^{1-\frac{2}{N}} \leqslant \left(\frac{8m}{\epsilon^4} + C_1 \left(\frac{8m}{\epsilon^4}\right)^{\frac{2^*}{2}}\right)^{\frac{2}{N}}$$

Then using (3.13) we infer that there is $C_2 > 0$ such that for any $m \in (0,1]$ and for all ψ and φ as above there holds

$$\Big| \int_{\mathbb{R}^N} \left(|\psi|^2 - 1\right) |\varphi|^2 \, \mathrm{d}x \Big| \leqslant C_2 m^{\frac{2}{N}} \|\nabla\varphi\|_{L^2}^2$$

and consequently

$$E(\psi,\varphi) - \frac{m}{\epsilon^4} = E_1(\psi) + \int_{\mathbb{R}^N} \frac{|\nabla\varphi|^2}{\epsilon^2 q^2} + \frac{1}{\epsilon^4}\left(|\psi|^2 - 1\right)|\varphi|^2 \, \mathrm{d}x \geqslant E_1(\psi) + \left(\frac{1}{\epsilon^2 q^2} - \frac{C_2 m^{\frac{2}{N}}}{\epsilon^4}\right) \|\nabla\varphi\|_{L^2}^2.$$

The last quantity is nonnegative if $m \leqslant \min(1, C_2^{-\frac{N}{2}} \epsilon^N q^{-N})$.

If $N = 3$, using Hölder's inequality, estimate (3.14) above and Sobolev's inequality we get

$$\begin{aligned} &\Big| \int_{\mathbb{R}^N} \left(|\psi|^2 - 1\right) |\varphi|^2 \, \mathrm{d}x \Big| \leqslant \|1 - |\psi|^2\|_{L^2} \|\varphi\|_{L^2}^{\frac{1}{2}} \|\varphi\|_{L^6}^{\frac{3}{2}} \leqslant C_S^{\frac{3}{2}} \|1 - |\psi|^2\|_{L^2} \|\varphi\|_{L^2}^{\frac{1}{2}} \|\nabla\varphi\|_{L^2}^{\frac{3}{2}} \\ &\leqslant C_S^{\frac{3}{2}} \left(4E(\psi,\varphi) + C_1 E(\psi,\varphi)^3\right)^{\frac{1}{2}} \|\varphi\|_{L^2}^{\frac{1}{2}} \|\nabla\varphi\|_{L^2}^{\frac{3}{2}}. \end{aligned}$$

If ψ and φ are as above we have $\frac{1}{\epsilon^2 q^2}\|\nabla\varphi\|_{L^2}^2 \leqslant E(\psi,\varphi) \leqslant \frac{2m}{\epsilon^4}$. If $m \leqslant 1$, we infer that there exist $C_3, C_3' > 0$ such that

$$\Big| \int_{\mathbb{R}^3} \left(|\psi|^2 - 1\right) |\varphi|^2 \, \mathrm{d}x \Big| \leqslant C_3 E_1(\psi)^{\frac{1}{2}} \|\nabla\varphi\|_{L^2}^{\frac{3}{2}} \|\varphi\|_{L^2}^{\frac{1}{2}} \leqslant C_3' \left(E_1(\psi) + \frac{1}{\epsilon^2 q^2}\|\nabla\varphi\|_{L^2}^2\right) \|\nabla\varphi\|_{L^2}^{\frac{1}{2}} \|\varphi\|_{L^2}^{\frac{1}{2}}.$$

Then we get

$$E(\psi,\varphi) - \frac{m}{\epsilon^4} \geqslant \left(E_1(\psi) + \frac{1}{\epsilon^2 q^2}\|\nabla\varphi\|_{L^2}^2\right)\left(1 - \frac{C_3'}{\epsilon^4}\|\nabla\varphi\|_{L^2}^{\frac{1}{2}} \|\varphi\|_{L^2}^{\frac{1}{2}}\right) \geqslant \left(E_1(\psi) + \frac{1}{\epsilon^2 q^2}\|\nabla\varphi\|_{L^2}^2\right)(1 - C_5 m)$$

for some $C_5 > 0$ and the last quantity is non-negative for all m sufficiently small.

Consider next the case $N = 2$. Using Plancherel's theorem and Hölder's inequality, we get

$$\|\varphi\|_{\dot{H}^s} \leqslant \|\varphi\|_{L^2}^{1-s} \|\nabla\varphi\|_{L^2}^s \qquad \text{for all } s \in (0,1) \text{ and all } \varphi \in H^1(\mathbb{R}^N).$$

If $p \in (2,\infty)$ and $s \in (0,1)$ satisfy $\frac{1}{p} = \frac{1}{2} - \frac{s}{2}$ (that is, $s = 1 - \frac{2}{p}$), using the Sobolev inequality we see that there exists $C_p > 0$ such that

$$\|\varphi\|_{L^p} \leqslant C_p \|\varphi\|_{\dot{H}^s} \leqslant C_p \|\varphi\|_{L^2}^{1-s} \|\nabla\varphi\|_{L^2}^{s} \qquad \text{for any } \varphi \in H^1(\mathbb{R}^2).$$

In particular, for $p = 4$ we get $\|\varphi\|_{L^4(\mathbb{R}^2)} \leqslant C_4 \|\varphi\|_{L^2}^{\frac{1}{2}} \|\nabla\varphi\|_{L^2}^{\frac{1}{2}}$ for any $\varphi \in H^1(\mathbb{R}^2)$, hence

$$\Big| \int_{\mathbb{R}^2} \left(|\psi|^2 - 1\right) |\varphi|^2 \, \mathrm{d}x \Big| \leqslant \|1 - |\psi|^2\|_{L^2} \|\varphi\|_{L^4}^2 \leqslant C_4^2 \|1 - |\psi|^2\|_{L^2} \|\nabla\varphi\|_{L^2} \|\varphi\|_{L^2}. \tag{3.15}$$

Using Lemma 4.1 p. 171 in [5] (see estimate (4.2) there) we infer that there exists $C > 0$ such that for any $\psi \in \mathcal{E}$ satisfying $|\psi| \leqslant 2$ we have

$$\left(\frac{1}{4} - C \|\nabla\psi\|_{L^2(\mathbb{R}^2)}^{2p_0+2} \right) \int_{\mathbb{R}^2} \left(|\psi|^2 - 1\right)^2 \, \mathrm{d}x \leqslant \int_{\mathbb{R}^2} V(|\psi|^2) \, \mathrm{d}x.$$

In particular, there exists $m_0 > 0$ such that whenever $|\psi| \leqslant 2$ and $E(\psi,\varphi) \leqslant \frac{2m_0}{\epsilon^4}$, there holds

$$\int_{\mathbb{R}^2} \left(|\psi|^2 - 1\right)^2 \, \mathrm{d}x \leqslant 8 \int_{\mathbb{R}^2} V(|\psi|^2) \, \mathrm{d}x.$$

Coming back to (3.15) we see that there exists $C_7 > 0$ such that for all $m \in (0, m_0]$ and for any $\psi \in \mathcal{E}$ and any $\varphi \in H^1(\mathbb{R}^N)$ with $|\phi| \leqslant 2$, $\|\varphi\|_{L^2}^2 = m$ and $E(\psi,\varphi) \leqslant \frac{2m}{\epsilon^4}$ we have

$$\Big| \int_{\mathbb{R}^2} \left(|\psi|^2 - 1\right) |\varphi|^2 \, \mathrm{d}x \Big| \leqslant C_7 \left(\int_{\mathbb{R}^2} V(|\psi|^2) \, \mathrm{d}x + \frac{1}{\epsilon^2 q^2} \|\nabla\varphi\|_{L^2}^2 \right) \|\varphi\|_{L^2}.$$

As previously, we conclude that

$$E(\psi,\varphi) - \frac{m}{\epsilon^4} \geqslant \left(\int_{\mathbb{R}^2} V(|\psi|^2) \, \mathrm{d}x + \frac{1}{\epsilon^2 q^2} \|\nabla\varphi\|_{L^2}^2 \right) \left(1 - \frac{C_7 m^{\frac{1}{2}}}{\epsilon^4} \right) \geqslant 0$$

if ψ, φ are as above and m is sufficiently small.

(iv) Fix a *radial* function $\chi \in C_c^\infty(\mathbb{R}^N)$ such that $\chi = 1$ on $B(0,1)$, $0 \leqslant \chi \leqslant 1$ and $\mathrm{supp}(\chi) \subset B(0,2)$. For $R > 0$ denote $\psi_R(x) = 1 - \chi\left(\frac{x}{R}\right)$. Denote $A = \int_{\mathbb{R}^N} |\nabla\psi_1|^2 \, \mathrm{d}x$ and $B = \int_{\mathbb{R}^N} V(|\psi_1|^2) \, \mathrm{d}x$. We have $\int_{\mathbb{R}^N} |\nabla\psi_R|^2 \, \mathrm{d}x = AR^{N-2}$ and $\int_{\mathbb{R}^N} V(|\psi_R|^2) \, \mathrm{d}x = BR^N$. Since ψ_R is radial we have $Q(\psi_R) = 0$.

Let ϕ_1 be an optimizer for the Poincaré inequality on $B(0,1)$, that is $\phi_1 \in H_0^1(B(0,1))$, $\int_{B(0,1)} |\phi_1|^2 \, \mathrm{d}x = 1$ and $\int_{B(0,1)} |\nabla\phi_1|^2 \, \mathrm{d}x = C_P$. Extend ϕ_1 by zero outside $B(0,1)$. Let $\varphi_R(x) = \frac{m}{R^N} \phi_1\left(\frac{x}{R}\right)$, so that $\int_{\mathbb{R}^N} |\varphi_R|^2 \, \mathrm{d}x = m$, $\int_{\mathbb{R}^N} |\nabla\varphi_R|^2 \, \mathrm{d}x = \frac{C_P}{R^2} m$ and $\mathrm{supp}(\varphi_R) \subset B(0,R)$, hence $\int_{\mathbb{R}^N} |\psi_R|^2 |\varphi_R|^2 \, \mathrm{d}x = 0$. Then we have

$$E_{min}(0,m) \leqslant E(\psi_R,\varphi_R) = AR^{N-2} + BR^N + \frac{C_P}{R^2} m = f_m(R). \tag{3.16}$$

Notice that (3.16) holds for any $R > 0$, hence we may optimize with respect to R. The function f_m has a unique minimum on $(0,\infty)$ at a point R_m satisfying $f_m'(R_m) = 0$, that is $AN R_m^{N+2} + B(N-2) R_m^N = 2C_P m$. It is easily seen that for large m the unique positive root R_m of this equation is of order of magnitude $R_m \sim m^{\frac{1}{N+2}}$ and there is some constant $C > 0$ such that $f_m(R_m) \leqslant C m^{\frac{N}{N+2}}$. Coming back to (3.16) we conclude that

$$E_{min}(0,m) \leqslant C m^{\frac{N}{N+2}}.$$

□

Theorem 19. *Assume that $g_{min}(m) < \frac{m}{\epsilon^4}$. Then there exist minimizers for the problem (GS_m).*

Moreover, if $(\psi_n, \varphi_n)_{n\geqslant 1}$ is any sequence in $\mathcal{E}\times H^1(\mathbb{R}^N)$ such that $\|\varphi_n\|_{L^2}^2 \longrightarrow m$ and $E(\psi_n, \varphi_n) \longrightarrow g_{min}(m)$, then there exists a subsequence, still denoted the same, there is a sequence $(x_n)_{n\geqslant 1} \subset \mathbb{R}^N$ and there are $\psi \in 1 + H^1(\mathbb{R}^N)$ and $\varphi \in H^1(\mathbb{R}^N)$ such that

$$|\psi_n(\cdot - x_n)| - 1 \longrightarrow \psi - 1 \quad \text{and} \quad |\varphi_n(\cdot - x_n)| \longrightarrow \varphi \quad \text{in } H^1(\mathbb{R}^N),$$

$$V(|\psi_n|^2)(\cdot - x_n) \longrightarrow V(|\psi|^2) \text{ and } |\psi_n|^2|\varphi_n|^2(\cdot - x_n) \longrightarrow |\psi|^2|\varphi|^2 \text{ in } L^1(\mathbb{R}^N) \text{ as } n \longrightarrow \infty.$$

Remark. The conclusion of Theorem 19 is only a statement about $|\psi_n|$ and $|\varphi_n|$. More information should be available here. Indeed, if (ψ_n, φ_n) is a minimizing sequence, then $E(\psi_n, \varphi_n) - E(|\psi_n|, |\varphi_n|) \longrightarrow 0$, hence ψ_n should be "close" to $e^{i\theta_n}|\psi_n|$ and φ_n should be "close" to $e^{i\beta_n}|\varphi_n|$ for some $\theta_n, \beta_n \in \mathbb{R}$.

Proof. Let $(\psi_n, \varphi_n)_{n\geqslant 1}$ be a sequence as in Theorem 19. In particular, $E_1(\psi_n)$ and $\|\varphi_n\|_{H^1(\mathbb{R}^N)}$ are bounded. It follows from Lemma 4.8 p. 177 and Corollary 4.3 p. 172 in [5] that $\| \, |\psi_n| - 1\|_{L^2(\mathbb{R}^N)}$ is bounded. Let p_0 be as in assumption **(A2)**. Denote

$$f_n = |\nabla\psi_n|^2 + | \, |\psi_n| - 1|^2 + | \, |\psi_n| - 1|^{2p_0+2} + |\nabla\varphi_n|^2 + |\varphi_n|^2 + |\psi_n|^2|\varphi_n|^2. \tag{3.17}$$

Obviously, $(f_n)_{n\geqslant 1}$ is a bounded sequence in $L^1(\mathbb{R}^N)$ and $\int_{\mathbb{R}^N} f_n(x)\,\mathrm{d}x \geqslant \|\varphi_n\|_{L^2}^2$. Passing to a subsequence we may assume that $\int_{\mathbb{R}^N} f_n(x)\,\mathrm{d}x \longrightarrow \alpha_0$ as $n \longrightarrow \infty$, where $\alpha_0 \geqslant m > 0$. Let $\Lambda_n : [0,\infty) \longrightarrow [0,\infty)$ be the concentration function of f_n, that is

$$\Lambda_n(t) = \sup_{y\in\mathbb{R}^N} \int_{B(y,t)} f_n(x)\,\mathrm{d}x. \tag{3.18}$$

Proceeding as in [8], it is straightforward to show that there exists a subsequence of $((\psi_n, \varphi_n, \Lambda_n))_{n\geq 1}$, still denoted the same, there is a nondecreasing function $\Lambda : [0,\infty) \longrightarrow \mathbb{R}$ and there is $\alpha \in [0, \alpha_0]$ such that

$$\Lambda_n(t) \longrightarrow \Lambda(t) \text{ a.e on } [0,\infty) \text{ as } n \longrightarrow \infty \qquad \text{and} \qquad \Lambda(t) \longrightarrow \alpha \text{ as } t \longrightarrow \infty. \tag{3.19}$$

As in [11] (see the proof of (5.12) p. 156 there) one can prove that there is a nondecreasing sequence $t_n \longrightarrow \infty$ such that

$$\lim_{n\to\infty} \Lambda_n(t_n) = \lim_{n\to\infty} \Lambda_n\left(\frac{t_n}{2}\right) = \alpha. \tag{3.20}$$

Our aim is to show that $\alpha = \alpha_0$. The next lemma, which we will use several times in the sequel to rule out "vanishing," shows that $\alpha > 0$.

Lemma 20. *Suppose that $N \geqslant 2$, the assumptions* **(A1)** *and* **(A2)** *hold, and $V \geqslant 0$. There exists an increasing function $M : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ such that $\lim_{\tau\to 0} M(\tau) = 0$ and for any $\psi \in \mathcal{E}$ and any $\varphi \in H^1(\mathbb{R}^N)$ we have*

$$\|\nabla\psi\|_{L^2} + \| \, |\psi| - 1\|_{L^2} + \|\varphi\|_{H^1} \leqslant M(E(\psi,\varphi)).$$

Proof. It is obvious that $\|\nabla\psi\|_{L^2}^2 \leqslant E_1(\psi) \leqslant E(\psi,\varphi)$ and $\|\nabla\varphi\|_{L^2}^2 \leqslant \epsilon^2 q^2 E(\psi,\varphi)$. It follows from Lemma 4.1 p. 171, Corollary 4.3 p. 172 and Lemma 4.8 p. 177 in [5] that there exists an increasing function $M_1 : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ such that $M_1(\tau) \longrightarrow 0$ as $\tau \longrightarrow 0$ and $\| \, |\psi| - 1\|_{L^2} \leq M_1(E_1(\psi))$.

It remains to estimate $\|\varphi\|_{L^2}$. Let $A = \left\{x \in \mathbb{R}^N \;\middle|\; |\psi(x)| < \frac{1}{2}\right\}$. We have

$$\int_{\mathbb{R}^N\setminus A} |\varphi|^2\,\mathrm{d}x \leqslant 4\int_{\mathbb{R}^N\setminus A} |\psi|^2|\varphi|^2\,\mathrm{d}x \leqslant 4\epsilon^4 E(\psi,\varphi).$$

On A we have $|\,|\psi|-1|>\frac{1}{2}$, hence $\mathcal{L}^N(A)\leqslant 4\|\,|\psi|-1\|^2_{L^2(A)}\leqslant 4\|\,|\psi|-1\|^2_{L^2}$.

If $N\geqslant 3$, by the Sobolev embedding we have $\|\varphi\|_{L^{2^*}}\leqslant C_S\|\nabla\varphi\|_{L^2}$, where $2^*=\frac{2N}{N-2}$, and then using Hölder's inequality we get

$$\int_A|\varphi|^2\,\mathrm{d}x\leqslant\|\varphi\|^2_{L^{2^*}(A)}\mathcal{L}^N(A)^{1-\frac{2}{2^*}}\leqslant C_S^2\|\nabla\varphi\|^2_{L^2}\mathcal{L}^N(A)^{1-\frac{2}{2^*}}.$$

Consider the case $N=2$. By inequality (3.10) p. 107 in [3], for any $r\in[1,\infty)$ there is $C_r>0$ such that for any $\eta\in L^1_{loc}(\mathbb{R}^2)$ satisfying $\nabla\eta\in L^2(\mathbb{R}^2)$ and $\mathcal{L}^2(\{|\eta|>0\})<\infty$ we have

$$\|\eta\|_{L^r(\mathbb{R}^2)}\leqslant C_r\|\nabla\eta\|_{L^2(\mathbb{R}^2)}\left(\mathcal{L}^2(\{|\eta|>0\})\right)^{\frac{1}{r}}. \tag{3.21}$$

Let $B=\{x\in\mathbb{R}^N\;\big|\;|\varphi(x)|>1\}$ and let $\eta=(|\varphi|-1)_+$. We have $|\nabla\eta|\leqslant|\nabla\varphi|$ a.e. on $\mathbb{R}^N$. Since $|\varphi|>1$ and $|\psi|\geqslant\frac{1}{2}$ on $B\setminus A$, we infer that

$$\mathcal{L}^N(B\setminus A)\leqslant 4\int_{B\setminus A}|\psi|^2|\varphi|^2\,\mathrm{d}x\leqslant 4\epsilon^2E(\psi,\varphi)$$

and consequently

$$\mathcal{L}^N(B)\leqslant\mathcal{L}^N(B\setminus A)+\mathcal{L}^N(A)\leqslant 4\epsilon^2E(\psi,\varphi)+4\|\,|\psi|-1\|^2_{L^2}.$$

Using (3.21) with $r=2$ we get $\|\eta\|_{L^2(\mathbb{R}^2)}\leqslant C_2\|\nabla\varphi\|_{L^2(\mathbb{R}^2)}\mathcal{L}^2(B)^{\frac{1}{2}}$. Since $|\varphi|\leqslant 1+\eta$, we have

$$\int_A|\varphi|^2\,\mathrm{d}x\leqslant 2\int_A 1+\eta^2\,\mathrm{d}x\leqslant 2\mathcal{L}^2(A)+\|\eta\|^2_{L^2(\mathbb{R}^2)}.$$

The conclusion of Lemma 20 follows from the above estimates. □

Lemma 21. *Let $N\geqslant 2$. Suppose that $V\geqslant 0$ and the assumptions* **(A1)** *and* **(A2)** *hold.*

(i) Let $(\psi_n)_{N\geqslant 1}\subset\mathcal{E}$ be a sequence such that $E_1(\psi_n)$ is bounded and

$$\sup_{y\in\mathbb{R}^N}\int_{B(y,1)}(|\psi_n|-1)^2\,\mathrm{d}x\longrightarrow 0\qquad as\quad n\longrightarrow\infty. \tag{3.22}$$

Fix any $d\in(0,1)$. Let

$$A_n=\{x\in\mathbb{R}^N\;\big|\;|\psi_n(x)|<1-d\quad or\quad|\psi_n(x)|>1+d\,\}.$$

Then $\mathcal{L}^N(A_n)\longrightarrow 0$ as $n\longrightarrow\infty$, where $\mathcal{L}^N$ is the Lebesgue measure in $\mathbb{R}^N$.

Assume that $(\psi_n)_{n\geqslant1}\subset\mathcal{E}$ and $(\varphi_n)_{n\geqslant1}\subset H^1(\mathbb{R}^N)$ are two sequences such that $E(\psi_n,\varphi_n)$ is bounded and (3.22) holds. Then:

(ii) If $\displaystyle\int_{\mathbb{R}^N}|\varphi_n(x)|^2\,\mathrm{d}x\longrightarrow m$ as $n\longrightarrow\infty$, we have $\displaystyle\liminf_{n\to\infty}\int_{\mathbb{R}^N}|\psi_n|^2|\varphi_n|^2\,\mathrm{d}x\geqslant m$.

(iii) If $Q(\varphi_n)\longrightarrow p$, where $\displaystyle Q(\varphi)=\int_{\mathbb{R}^N}\langle i\frac{\partial\varphi}{\partial x_1},\varphi\rangle\,\mathrm{d}x$, we have

$$\liminf_{n\to\infty}\int_{\mathbb{R}^N}|\nabla\varphi_n|^2+\frac{q^2}{\epsilon^2}|\psi_n|^2|\varphi_n|^2\,\mathrm{d}x\geqslant\frac{2q}{\epsilon}p.$$

Proof. (i) It is well-known that for any $\psi \in H^1_{loc}(\mathbb{R}^N, \mathbb{R}^k)$ we have $|\psi| \in H^1_{loc}(\mathbb{R}^N, \mathbb{R})$ and $\big|\nabla|\psi|\big| \leqslant |\nabla\psi|$ a.e. Denote $u_n = \big||\psi_n| - 1\big|$. Then we have $u_n \in H^1(\mathbb{R}^N)$, $A_n = \{x \in \mathbb{R}^N \mid u_n(x) > d\}$ and $|\nabla u_n| \leqslant |\nabla\psi_n|$ a.e., hence ∇u_n is bounded in $L^2(\mathbb{R}^N)$. Assume that there is a sequence of $(u_n)_{n\geqslant 1}$, still denoted the same, and there is $\eta_0 > 0$ such that $\mathcal{L}^N(A_n) \geqslant \eta_0$ for all n. Using Lieb's Lemma (see Lemma 6 p. 447 in [7]) we infer that there exists $\eta_1 > 0$ and for any $n \in \mathbb{N}^*$ there is $y_n \in \mathbb{R}^N$ such that

$$\mathcal{L}^N\left(B_n \cap B(y_n, 1)\right) \geqslant \eta_1, \qquad \text{where} \qquad B_n = \left\{x \in \mathbb{R}^N \;\middle|\; |u_n(x)| > \frac{d}{2}\right\}.$$

On B_n we have $\big||\psi_n| - 1\big| > \frac{d}{2}$ and consequently

$$\int_{B(y_n,1)} \big||\psi_n| - 1\big|^2 \,\mathrm{d}x \geqslant \int_{B(y_n,1)\cap B_n} \big||\psi_n| - 1\big|^2 \,\mathrm{d}x \geqslant \frac{d^2}{4}\eta_1 \qquad \text{for all } n,$$

contradicting the fact that $\sup\limits_{y\in\mathbb{R}^N} \int_{B(y,1)} \big||\psi_n| - 1\big|^2 \,\mathrm{d}x \longrightarrow 0$ as $n \longrightarrow \infty$.

(ii) Fix $\varepsilon > 0$. Choose $d \in (0,1)$ such that $(1-d)^2\left(m - \frac{\varepsilon}{2}\right) > m - \varepsilon$. For this choice of d, let A_n be as in part (i) Since $E(\psi_n, \varphi_n)$ is bounded and $\|\varphi_n\|^2_{L^2} \longrightarrow m$, the sequence $(\varphi_n)_{n\geqslant 1}$ is bounded in $H^1(\mathbb{R}^N)$.

Fix $p \in (2, \infty)$ if $N \in \{1, 2\}$, respectively $p \in (2, \frac{2N}{N-2}]$ if $N \geqslant 3$. By the Sobolev embedding, $(\varphi_n)_{n\geqslant 1}$ is bounded in $L^p(\mathbb{R}^N)$. Using Hölder's inequality and part (i), we get

$$\int_{A_n} |\varphi_n|^2 \,\mathrm{d}x \leqslant \|\varphi_n\|^2_{L^p(A_n)} \mathcal{L}^N(A_n)^{1-\frac{2}{p}} \longrightarrow 0 \qquad \text{as } n \longrightarrow \infty. \tag{3.23}$$

Then $\int_{\mathbb{R}^N\setminus A_n} |\varphi_n|^2 \,\mathrm{d}x \longrightarrow m$, hence there exists $n_\varepsilon \in \mathbb{N}$ such that $\int_{\mathbb{R}^N\setminus A_n} |\varphi_n|^2 \,\mathrm{d}x > m - \frac{\varepsilon}{2}$ for all $n \geqslant n_\varepsilon$. Since $|\psi_n| \geqslant 1 - d$ on $\mathbb{R}^N \setminus A_n$, we infer that

$$\int_{\mathbb{R}^N} |\psi_n|^2|\varphi_n|^2 \,\mathrm{d}x \geqslant \int_{\mathbb{R}^N\setminus A_n} |\psi_n|^2|\varphi_n|^2 \,\mathrm{d}x \geqslant (1-d)^2\left(m - \frac{\varepsilon}{2}\right) > m - \varepsilon \qquad \text{for all } n \geqslant n_\varepsilon,$$

and part (ii) follows.

Remark. If $N \leqslant 3$, for a given $\varepsilon > 0$ we may choose $d \in (0,1)$ such that $m - \varepsilon < (1-d)^2\left(m - \frac{\varepsilon}{2}\right)$ and $(1+d)^2\left(m + \frac{\varepsilon}{2}\right) < m + \varepsilon$, then consider the set A_n as above. By the Sobolev embedding $(\varphi_n)_{n\geqslant 1}$ is bounded in $L^p(\mathbb{R}^N)$ for some $p > 4$ and using Hölder's inequality and part (i) we get $\int_{A_n} |\varphi_n|^4 \,\mathrm{d}x \longrightarrow 0$ and $\int_{A_n} |\psi_n|^4 \,\mathrm{d}x \longrightarrow 0$, then the Cauchy-Schwarz inequality implies $\int_{A_n} |\psi_n|^2|\varphi_n|^2 \,\mathrm{d}x \longrightarrow 0$ as $n \longrightarrow \infty$. On the other hand, we have $\int_{A_n} |\varphi_n|^2 \,\mathrm{d}x \longrightarrow 0$ and $\int_{\mathbb{R}^N\setminus A_n} |\varphi_n|^2 \,\mathrm{d}x \longrightarrow m$, hence $m - \frac{\varepsilon}{2} < \int_{\mathbb{R}^N\setminus A_n} |\varphi_n|^2 \,\mathrm{d}x < m + \frac{\varepsilon}{2}$ for all sufficiently large n. For such n we get

$$m - \varepsilon < (1-d)^2\left(m - \frac{\varepsilon}{2}\right) \leqslant \int_{\mathbb{R}^N\setminus A_n} |\psi_n|^2|\varphi_n|^2 \,\mathrm{d}x \leqslant (1+d)^2\left(m + \frac{\varepsilon}{2}\right) < m + \varepsilon$$

and we infer that $\int_{\mathbb{R}^N} |\psi_n|^2|\varphi_n|^2 \,\mathrm{d}x \longrightarrow 0$ as $n \longrightarrow \infty$. This is no longer true if $N \geqslant 4$. Under the assumptions of part (ii), $\int_{\mathbb{R}^N} |\psi_n|^2|\varphi_n|^2 \,\mathrm{d}x$ remains bounded but does not necessarily tend to zero if $N = 4$, and this quantity may be arbitrarily large if $N \geqslant 5$.

(iii) By (3.5) and Proposition 24 we already know that

$$\int_{\mathbb{R}^N} |\nabla\varphi_n|^2 + \frac{q^2}{\epsilon^2}|\varphi_n|^2 \,\mathrm{d}x \geqslant \frac{2q}{\epsilon}|Q(\varphi_n)| \longrightarrow \frac{2q}{\epsilon}p,$$

and it suffices to show that

$$\liminf_{n\to\infty} \int_{\mathbb{R}^N} \left(|\psi_n|^2 - 1\right) |\varphi_n|^2 \,\mathrm{d}x \geqslant 0. \tag{3.24}$$

Since $E(\psi_n, \varphi_n)$ is bounded, Lemma 20 implies that $(\varphi_n)_{n\geqslant 1}$ is bounded in $H^1(\mathbb{R}^N)$.

Fix $\varepsilon \in (0,1)$. Let $d = 1 - \sqrt{1-\varepsilon}$. With this choice of d, let A_n be as in part (i). Then we have $|\psi_n|^2 - 1 \geqslant (1-d)^2 - 1 = -\varepsilon$ on $\mathbb{R}^N \setminus A_n$ and we get

$$\int_{\mathbb{R}^N} \left(|\psi_n|^2 - 1\right) |\varphi_n|^2 \,\mathrm{d}x \geqslant -\varepsilon \int_{\mathbb{R}^N \setminus A_n} |\varphi_n|^2 \,\mathrm{d}x - \int_{A_n} |\varphi_n|^2 \,\mathrm{d}x \geqslant -\varepsilon \|\varphi_n\|_{L^2}^2 - \int_{A_n} |\varphi_n|^2 \,\mathrm{d}x.$$

Since $\|\varphi_n\|_{L^2}$ is bounded, $\int_{A_n} |\varphi_n|^2 \,\mathrm{d}x \longrightarrow 0$ by (3.23) and $\varepsilon \in (0,1)$ is arbitrary, (3.24) follows and (iii) is proven. □

We come back to the proof of Theorem 19. If $\Lambda_n(t) \longrightarrow 0$ for some $t \geqslant 1$ it is obvious that (3.22) holds. Then Lemma 21 (ii) implies that $\liminf_{n\to\infty} \int_{\mathbb{R}^N} |\psi_n|^2 |\varphi_n|^2 \,\mathrm{d}x \geqslant m$ and consequently $\liminf_{n\to\infty} E(\psi_n, \varphi_n) \geqslant \dfrac{m}{\epsilon^4}$, contradicting the fact that $E(\psi_n, \varphi_n) \longrightarrow g_{min}(m) < \frac{m}{\epsilon^4}$. Thus we must have $\Lambda(t) > 0$ and this implies that $\alpha > 0$.

To prove that $\alpha \notin (0, \alpha_0)$ we argue by contradiction and we assume that $0 < \alpha < \alpha_0$. Let $h_n = \Lambda_n(t_n) - \Lambda_n\left(\frac{t_n}{2}\right) + \frac{1}{2^n}$. It is obvious that $h_n \longrightarrow 0$ as $n \longrightarrow \infty$. For each $n \in \mathbb{N}^*$ choose x_n such that $\displaystyle\int_{B(x_n, \frac{t_n}{2})} f_n(x) \,\mathrm{d}x > \Lambda_n\left(\frac{t_n}{2}\right) - \frac{1}{2^n}$ and denote $\Omega_n = B(x_n, t_n) \setminus B(x_n, \frac{t_n}{2})$. It is obvious that

$$\int_{\Omega_n} f_n(x) \,\mathrm{d}x = \int_{B(x_n,t_n)} f_n(x) \,\mathrm{d}x - \int_{B(x_n,\frac{t_n}{2})} f_n(x) \,\mathrm{d}x \leqslant \Lambda_n(t_n) - \Lambda_n\left(\frac{t_n}{2}\right) + \frac{1}{2^n} = h_n. \tag{3.25}$$

Take $\chi \in C_c^\infty(\mathbb{R}^N)$ such that $0 \leqslant \chi \leqslant 1$, $\chi = 1$ on $B(0, \frac{1}{2})$ and $\chi = 0$ on $\mathbb{R}^N \setminus B(0,1)$. Denote

$$\psi_{n,1} = (|\psi_n| - 1)\,\chi\Big(\tfrac{x-x_n}{t_n}\Big) + 1, \quad \psi_{n,2} = (|\psi_n| - 1)\left[1 - \chi\Big(\tfrac{x-x_n}{t_n}\Big)\right] + 1$$

$$\varphi_{n,1} = |\varphi_n|\chi\Big(\tfrac{x-x_n}{t_n}\Big), \qquad \varphi_{n,2} = |\varphi_n|\left[1 - \chi\Big(\tfrac{x-x_n}{t_n}\Big)\right].$$

It is clear that

$$\Big|\int_{\mathbb{R}^N} |\varphi_n|^2 - |\varphi_{n,1}|^2 - |\varphi_{n,2}|^2 \,\mathrm{d}x\Big| \leqslant \int_{\Omega_n} \Big|1 - \chi^2\Big(\frac{x-x_n}{t_n}\Big) - (1-\chi)^2\Big(\frac{x-x_n}{t_n}\Big)\Big| |\varphi_n|^2 \,\mathrm{d}x \leqslant h_n. \tag{3.26}$$

Since $|\psi_{n,i}| \leqslant \max(|\psi_n|, 1)$ and $|\varphi_{n,i}| \leqslant |\varphi_n|$ for $i = 1, 2$, we have

$$\Big|\int_{\mathbb{R}^N} |\psi_n|^2|\varphi_n|^2 - |\psi_{n,1}|^2|\varphi_{n,1}|^2 - |\psi_{n,2}|^2|\varphi_{n,2}|^2 \,\mathrm{d}x\Big| \leqslant 3\int_{\Omega_n} |\varphi_n|^2 + |\psi_n|^2|\varphi_n|^2 \,\mathrm{d}x \leqslant 3h_n. \tag{3.27}$$

By assumptions **(A1)** and **(A2)** there exists $C > 0$ such that $V(s^2) \leqslant C\left((s-1)^2 + (s-1)^{2p_0+2}\right)$. We have $\big|\,|\psi_{n,i}| - 1\big| \leqslant \big|\,|\psi_n| - 1\big|$, $i = 1, 2$, and we infer that

$$\begin{aligned}
&\Big|\int_{\mathbb{R}^N} V(|\psi_n|^2) - V|\psi_{n,1}|^2) - V(|\psi_{n,2}|^2) \,\mathrm{d}x\Big| \\
&\leqslant \int_{\Omega_n} |V(|\psi_n|^2)| + |V|\psi_{n,1}|^2)| + |V(|\psi_{n,2}|^2)| \,\mathrm{d}x \qquad (3.28) \\
&\leqslant 3C\int_{\Omega_n} \big|\,|\psi_n| - 1\big|^2 + \big|\,|\psi_n| - 1\big|^{2p_0+2} \leqslant 3Ch_n.
\end{aligned}$$

We have $\frac{\partial\psi_{n,1}}{\partial x_j} = \frac{\partial|\psi_n|}{\partial x_j}\chi\Big(\frac{x-x_n}{t_n}\Big) + \frac{1}{t_n}(|\psi_n|-1)\frac{\partial\chi}{\partial x_j}\Big(\frac{x-x_n}{t_n}\Big)$ and a similar equality holds for $\psi_{n,2}$. If n is sufficiently large, so that $N|\nabla\chi|^2 \leqslant t_n^2$ on $\mathbb{R}^N$, we get

$$\int_{\Omega_n}\Big|\frac{\partial\psi_{n,1}}{\partial x_j}\Big|^2\,\mathrm{d}x \leqslant 2\int_{\Omega_n}\Big|\frac{\partial|\psi_n|}{\partial x_j}\Big|^2 + \frac{1}{N}\big||\psi_n|-1\big|^2\,\mathrm{d}x$$

and summing up we infer that

$$\begin{aligned}&\Big|\int_{\mathbb{R}^N}|\nabla|\psi_n|\,|^2 - |\nabla\psi_{n,1}|^2 - |\nabla\psi_{n,2}|^2)\,\mathrm{d}x\Big| \leqslant \int_{\Omega_n}|\nabla|\psi_n|\,|^2 + |\nabla\psi_{n,1}|^2 + |\nabla\psi_{n,2}|^2)\,\mathrm{d}x\\ &\leqslant 5\int_{\Omega_n}\big|\nabla|\psi_n|\,\big|^2 + \big||\psi_n|-1\big|^2\,\mathrm{d}x \leqslant 5h_n.\end{aligned} \tag{3.29}$$

It is obvious that a similar estimate holds for φ_n. From (3.27)-(3.29) we infer that there exists a constant $C>0$ such that for all n sufficiently large we have

$$\big|E(\psi_n,\varphi_n) - E(\psi_{n,1},\varphi_{n_1}) - E(\psi_{n,2},\varphi_{n_2})\big| \leqslant Ch_n. \tag{3.30}$$

Passing to a subsequence (still denoted the same) we may assume that $\|\varphi_{n,i}\|_{L^2}^2 \longrightarrow m_i$ as $n\longrightarrow\infty$ for $i=1,2$, and (3.26) implies that $m_1+m_2=m$. Let us show that $m_1>0$ and $m_2>0$. We argue again by contradiction and we assume, for instance, that $m_2=0$. Then we have necessarily $m_1=m$. If there is a subsequence $(n_k)_{k\geqslant 1}$ such that $E(\psi_{n_k,2},\varphi_{n_k,2}) \longrightarrow e>0$ as $k\longrightarrow\infty$, by (3.30) we have $E(\psi_{n_k,1},\varphi_{n_k,1}) \longrightarrow g_{min}(m)-e$. On the other hand, $E(\psi_{n_k,1},\varphi_{n_k,1}) \geqslant g_{min}\big(\|\varphi_{n_k,2}\|_{L^2}^2\big)$, and letting $k\longrightarrow\infty$ and using the continuity of g_{min} we find $\liminf_{k\to\infty} E(\psi_{n_k,1},\varphi_{n_k,1}) \geqslant g_{min}(m)$, a contradiction. Therefore a sequence $(n_k)_{k\geqslant 1}$ as above cannot exist, and this implies that $E(\psi_{n,2},\varphi_{n,2}) \longrightarrow 0$ as $n\longrightarrow\infty$. Then we deduce that $\|\varphi_{n,2}\|_{H^1(\mathbb{R}^N)} \longrightarrow 0$ and $E_1(\psi_{n,2}) \longrightarrow 0$ as $n\longrightarrow\infty$, and using Lemma 4.8 p. 177 and Corollary 4.3 p. 172 in [5] we infer that $\|\,|\psi_{n,2}|-1\|_{L^2(\mathbb{R}^N)} \longrightarrow 0$. The Sobolev embedding gives then $\|\,|\psi_{n,2}|-1\|_{L^{2p_0+2}(\mathbb{R}^N)} \longrightarrow 0$. Since $(\psi_n,\varphi_n) = (\psi_{n,2},\varphi_{n,2})$ on $\mathbb{R}^N\setminus B(x_n,t_n)$, we see that $\int_{\mathbb{R}^N\setminus B(x_n,t_n)} f_n(x)\,\mathrm{d}x \longrightarrow 0$, hence $\int_{B(x_n,t_n)} f_n(x)\,\mathrm{d}x \longrightarrow \alpha_0$, and this implies $\Lambda_n(t_n)\longrightarrow\alpha_0$. Recall that the sequence $(t_n)_{n\geqslant 1}$ has been chosen so that $\Lambda_n(t_n)\longrightarrow\alpha$, thus we get $\alpha=\alpha_0$, contradicting the assumption that $\alpha\in(0,\alpha_0)$. So far we have shown that we cannot have $m_2=0$, and similarly we show that $m_1\neq 0$. We conclude that $m_1,m_2\in(0,m)$.

It is clear that $E(\psi_{n,i},\varphi_{n,i}) \geqslant g_{min}(\|\varphi_{n,i}\|_{L^2}^2)$ and letting $n\longrightarrow\infty$ we find

$$\liminf_{n\to\infty} E(\psi_{n,i},\varphi_{n,i}) \geqslant g_{min}(m_i) \qquad \text{for } i=1,2.$$

Then using (3.30) we get $g_{min}(m) \geqslant g_{min}(m_1)+g_{min}(m_2)$. On the other hand, the concavity of g_{min} implies $g_{min}(m_i) \geqslant \frac{m_i}{m}g_{min}(m)$ and equality may occur in this inequality if and only if g_{min} is linear on $[0,m]$. Summing up the last two inequalities and comparing to the previous inequality we see that necessarily $g_{min}(m_i) = \frac{m_i}{m}g_{min}(m)$ for $i=1,2$, and therefore g_{min} must be linear on $[0,m]$. Then Proposition 18 (ii) and (iii) implies that $g_{min}(m') = \frac{m'}{\epsilon^4}$ for all $m'\in[0,m]$, contradicting the fact that $g_{min}(m) < \frac{m}{\epsilon^4}$. We conclude that we cannot have $\alpha\in(0,\alpha_0)$, and consequently we must have $\alpha=\alpha_0$.

Since $\alpha=\alpha_0$, it is standard to prove that there is a sequence $(x_n)_{n\geqslant 1}\subset\mathbb{R}^N$ such that for any $\varepsilon>0$ there are $R_\varepsilon>0$ and $n_\varepsilon\in\mathbb{N}$ such that

$$\int_{\mathbb{R}^N\setminus B(x_n,R_\varepsilon)} f_n(x)\,\mathrm{d}x < \varepsilon \qquad \text{for all } n\geqslant n_\varepsilon. \tag{3.31}$$

Denoting $\tilde{\psi}_n = |\psi_n|(\cdot - x_n)$ and $\tilde{\varphi}_n = |\varphi_n|(\cdot - x_n)$, it is easily seen that $\tilde{\psi}_n - 1$ and $\tilde{\varphi}_n$ are bounded in $H^1(\mathbb{R}^N)$. Passing again to a subsequence (still denoted the same), we infer that there exist $\varphi \in H^1(\mathbb{R}^N)$ and $\psi \in 1 + H^1(\mathbb{R}^N)$ such that

$$\begin{aligned} &\tilde{\psi}_n - \psi \rightharpoonup 0 \quad \text{and } \tilde{\varphi}_n \rightharpoonup \varphi \quad \text{weakly in } H^1(\mathbb{R}^N), \\ &\tilde{\psi}_n \longrightarrow \psi \ \text{ and } \ \tilde{\varphi}_n \longrightarrow \varphi \ \text{ in } L^p_{loc}(\mathbb{R}^N) \text{ for any } 1 \leqslant p < 2^* \text{ and almost everywhere.} \end{aligned} \tag{3.32}$$

The weak convergence implies $\|\varphi\|_{L^2}^2 \leqslant \liminf_{n\to\infty} \|\varphi_n\|_{L^2}^2 = m$. On the other hand, fix $\varepsilon > 0$. Using (3.31), for $n \geqslant n_\varepsilon$ we have $\displaystyle\int_{\mathbb{R}^N \setminus B(0,R_\varepsilon)} |\tilde{\varphi}_n|^2 \,\mathrm{d}x < \varepsilon$, hence $\displaystyle\int_{B(0,R_\varepsilon)} |\tilde{\varphi}_n|^2 \,\mathrm{d}x > \|\tilde{\varphi}_n\|_{L^2}^2 - \varepsilon$ for all $n \geqslant n_\varepsilon$. Since $\tilde{\varphi}_n \longrightarrow \varphi$ in $L^2(B(0,R_\varepsilon))$, we obtain $\displaystyle\int_{B(0,R_\varepsilon)} |\varphi|^2 \,\mathrm{d}x \geqslant m - \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we infer that $\int_{\mathbb{R}^N} |\varphi|^2 \,\mathrm{d}x \geqslant m$. Thus we have shown that $\|\varphi\|_{L^2}^2 = m$.

By weak convergence we have

$$\|\nabla\psi\|_{L^2}^2 \leqslant \liminf_{n\to\infty} \|\nabla\tilde{\psi}_n\|_{L^2}^2 \qquad \text{and} \qquad \|\nabla\varphi\|_{L^2}^2 \leqslant \liminf_{n\to\infty} \|\nabla\tilde{\varphi}_n\|_{L^2}^2. \tag{3.33}$$

The convergence almost everywhere and Fatou's lemma give

$$\int_{\mathbb{R}^N} V(|\psi|^2) \,\mathrm{d}x \leqslant \liminf_{n\to\infty} \int_{\mathbb{R}^N} V(|\tilde{\psi}_n|^2) \,\mathrm{d}x \quad \text{and} \quad \int_{\mathbb{R}^N} |\psi|^2|\varphi|^2 \,\mathrm{d}x \leqslant \liminf_{n\to\infty} \int_{\mathbb{R}^N} |\tilde{\psi}_n|^2|\tilde{\varphi}_n|^2 \,\mathrm{d}x. \tag{3.34}$$

Since $\psi \in \mathcal{E}$, $\varphi \in H^1(\mathbb{R}^N)$ and $\|\varphi\|_{L^2}^2 = m$, we have $E(\psi,\varphi) \geqslant g_{min}(m)$. From (3.33) and (3.34) we get $E(\psi,\varphi) \leqslant \liminf_{n\to\infty} E(\tilde{\psi}_n, \tilde{\varphi}_n)$. On the other hand, since $\psi \in \mathcal{E}$, $\varphi \in H^1(\mathbb{R}^N)$ and $\|\varphi\|_{L^2}^2 = m$, we have $E(\psi,\varphi) \geqslant g_{min}(m)$. We deduce that necessarily

$$\|\nabla\tilde{\psi}_n\|_{L^2}^2 \longrightarrow \|\nabla\psi\|_{L^2}^2, \quad \text{and} \quad \|\nabla\tilde{\varphi}_n\|_{L^2}^2 \longrightarrow \|\nabla\varphi\|_{L^2}^2 \quad \text{as } n \longrightarrow \infty.$$

Moreover, we must have equalities in (3.34) and the lower limits there are in fact limits. We show that $\|\tilde{\psi}_n - 1\|_{L^2} \longrightarrow \|\psi - 1\|_{L^2}$ in the same way as we proved that $\|\varphi\|_{L^2}^2 = m$. Then the weak convergence and the convergence of norms give $\tilde{\psi}_n - 1 \longrightarrow \psi - 1$ and $\tilde{\varphi}_n \longrightarrow \varphi$ in $H^1(\mathbb{R}^N)$.

The last assertion in Theorem 19 is a consequence of the following well-known and elementary result: if ϕ_n and ϕ are nonnegative integrable functions on a measure space $(X, \mathcal{A}, \mu)$, if $\phi_n \longrightarrow \phi$ almost everywhere and if $\displaystyle\int_X \phi_n \,d\mu \longrightarrow \int_X \phi \,d\mu$, then $\displaystyle\int_X |\phi_n - \phi| \,d\mu \longrightarrow 0$. □

Proposition 22. *Assume that $(\psi,\varphi) \in \mathcal{E} \times H^1(\mathbb{R}^N)$ is a solution of the minimisation problem (GS_m). Then:*

(i) There exists $\gamma \in [g'_{min,r}(m), g'_{min,\ell}(m)]$ (where $g'_{min,\ell}$ and $g'_{min,r}$ are the left and right derivatives of g_{min}, respectively) such that

$$-\Delta\psi + F(|\psi|^2)\psi + \frac{1}{\epsilon^4}|\varphi|^2\psi = 0, \qquad -\Delta\varphi + \frac{q^2}{\epsilon^2}|\psi|^2\varphi - \epsilon^2 q^2 \gamma\varphi = 0 \qquad \text{in } \mathbb{R}^N. \tag{3.35}$$

(ii) We have $|\psi| \leqslant 1$ almost everywhere and $\psi \in W^{2,p}_{loc}(\mathbb{R}^N)$, $\varphi \in W^{2,p}(\mathbb{R}^N)$ for any $p \in [2,\infty)$. In particular, ψ and φ are C^1 functions.

(iii) The function (ψ,φ) is radially symmetric (after translation). That is, there exist $x_0 \in \mathbb{R}^N$ and $\tilde{\psi}, \tilde{\varphi} : [0,\infty) \longrightarrow \mathbb{C}$ such that $\psi(x) = \tilde{\psi}(|x - x_0|)$ and $\varphi(x) = \tilde{\varphi}(|x - x_0|)$ for all $x \in \mathbb{R}^N$.

(iv) If (ψ,φ) is a minimizer and $\tilde{\psi}$, $\tilde{\varphi}$ are as in (iii), then the function $|\tilde{\psi}|$ is nondecreasing on $[0,\infty)$, $|\tilde{\varphi}|$ is nonincreasing on $[0,\infty)$ and there exist constants $\theta_0, \beta_0 \in \mathbb{R}$ such that $\tilde{\psi} = e^{i\theta_0}|\tilde{\psi}|$, $\tilde{\varphi} = e^{i\beta_0}|\tilde{\varphi}|$ on $[0,\infty)$.

Proof. (i) is standard.

(ii) If (ψ,φ) is a minimizer, then $(|\psi|,|\varphi|)$ is also a minimizer. It is clear that $E(\min(1,|\psi|),|\varphi|) \leqslant E(|\psi|,|\varphi|)$. Since $(|\psi|,|\varphi|)$ is a minimizer, we must have $E(\min(1,|\psi|),|\varphi|) \leqslant E(|\psi|,|\varphi|)$. This implies $\nabla|\psi| = 0$ a.e. and $V(|\psi|^2) = 0$ a.e. in the set $\{x \in \mathbb{R}^N \mid |\psi(x)| > 1\}$, and we deduce that $(|\psi|-1)_+ = 0$ a.e. on $\mathbb{R}^N$. Thus $|\psi| \leqslant 1$ a.e. on $\mathbb{R}^N$. Then the second equation in (3.35) and a standard boot-strap argument imply that $\varphi \in W^{2,p}(\mathbb{R}^N)$ for any $p \in [2,\infty)$. In particular, $\varphi \in C^{1,\alpha}(\mathbb{R}^N)$ for any $\alpha \in (0,1)$ and φ is bounded on $\mathbb{R}^N$.

Since $|\psi| \leqslant 1$ a.e., we have $F(|\psi|^2) \in L^\infty(\mathbb{R}^N)$. The first equation in (3.35) can be written as $-\Delta\psi + A(x)\psi = 0$, where $A = F(|\psi|^2) + \frac{1}{\epsilon^4}|\varphi|^2 \in L^\infty(\mathbb{R}^N)$. Standard elliptic regularity theory implies that there exists $C > 0$ such that $\|\psi\|_{W^{2,p}(B(y,1))} \leqslant C$ for any $y \in \mathbb{R}^N$. In particular, ψ is C^1 on $\mathbb{R}^N$.

(iii) Since any minimizer (ψ,φ) for the problem (GS_m) is C^1 in $\mathbb{R}^N$, (iii) follows from Theorem 2 p. 314 in [10].

(iv) Given a non-negative, measurable function $w : \mathbb{R}^N \longrightarrow [0,\infty)$ such that $\mathcal{L}^N(\{x \in \mathbb{R}^N \mid w(x) > t\})$ is finite for any $t > 0$, we denote by w_* the symmetric decreasing rearrangement of w. It is well-known that for $p \in (1,\infty)$ we have $\displaystyle\int_{\mathbb{R}^N} |\nabla w_*|^p \,\mathrm{d}x \leqslant \int_{\mathbb{R}^N} |\nabla w|^p \,\mathrm{d}x$, and equality may occur if and only if for any $t \in (0, \operatorname{sup\,ess}(w))$, the level set $\{x \in \mathbb{R}^N \mid w(x) > t\}$ is equivalent to a ball. The last statement is a consequence of Lemma 3.2 p. 163 in [4]. It is also well-known that for w_1, w_2 as above we have $\displaystyle\int_{\mathbb{R}^N} w_1 w_2 \,\mathrm{d}x \leqslant \int_{\mathbb{R}^N} (w_1)_*(w_2)_* \,\mathrm{d}x$. Obviously, we have $(|\varphi|^2)_* = (|\varphi|_*)^2$. Let $u = 1 - |\psi|$. Since $0 \leqslant u \leqslant 1$ and the mapping $s \longmapsto 2s - s^2$ is increasing on $[0,1]$, we have $(2u-u^2)_* = 2u_* - u_*^2$. Therefore

$$\int_{\mathbb{R}^N} (1-u_*)^2 (|\varphi|_*)^2 \,\mathrm{d}x = \int_{\mathbb{R}^N} (|\varphi|_*)^2 \,\mathrm{d}x - \int_{\mathbb{R}^N} (2u_* - u_*^2)(|\varphi|_*)^2 \,\mathrm{d}x$$

$$= \int_{\mathbb{R}^N} |\varphi|^2 \,\mathrm{d}x - \int_{\mathbb{R}^N} (2u-u^2)_*(|\varphi|^2)_* \,\mathrm{d}x \leqslant \int_{\mathbb{R}^N} |\varphi|^2 \,\mathrm{d}x - \int_{\mathbb{R}^N} (2u-u^2)|\varphi|^2 \,\mathrm{d}x = \int_{\mathbb{R}^N} |\psi|^2 |\varphi|^2 \,\mathrm{d}x.$$

We infer that $E(1-u_*,|\varphi|_*) \leqslant E(|\psi|,|\varphi|) \leqslant E(\psi,\varphi)$. Since (ψ,φ) is a minimizer and $\||\varphi|_*\|_{L^2}^2 = \|\,|\varphi|\,\|_{L^2}^2 = \|\varphi\|_{L^2}^2 = m$, we must have $E(1-u_*,|\varphi|_*) = E(|\psi|,|\varphi|) = E(\psi,\varphi)$, and consequently $\displaystyle\int_{\mathbb{R}^N} |\nabla u_*|^2 \,\mathrm{d}x = \int_{\mathbb{R}^N} |\nabla u|^2 \,\mathrm{d}x$ and $\displaystyle\int_{\mathbb{R}^N} |\nabla|\varphi|_*|^2 \,\mathrm{d}x = \int_{\mathbb{R}^N} |\nabla|\varphi|\,|^2 \,\mathrm{d}x$. The result of Brothers and Ziemer implies that for almost all $t > 0$, the sets $\{x \in \mathbb{R}^N \mid u(x) > t\}$ and $\{x \in \mathbb{R}^N \mid |\varphi|(x) > t\}$ are equivalent either to a ball or to $\emptyset$. Since we already know that u and $|\varphi|$ are radially symmetric, we infer that the functions $\tilde{u} = 1 - |\tilde{\psi}|$ and $|\tilde{\varphi}|$ are non-increasing on $[0,\infty)$.

The fact that $|\tilde{\varphi}|$ is nonincreasing implies that the set $D = \{x \in \mathbb{R}^N \mid |\varphi(x)| > 0\}$ is either a ball or $\mathbb{R}^N$. On this set we have a lifting $\varphi = |\varphi| e^{i\beta(x)}$, where $\beta \in H^1_{loc}$ and $|\nabla\varphi|^2 = |\nabla|\varphi|\,|^2 + |\varphi|^2|\nabla\theta|^2$. Since $\displaystyle\int_{\mathbb{R}^N} |\nabla\varphi|^2 \,\mathrm{d}x = \int_{\mathbb{R}^N} |\nabla|\varphi|\,|^2 \,\mathrm{d}x$ we must have $\nabla\beta = 0$ a.e. and we infer that β is constant, $\beta(x) = \beta_0$ for a.e. $x \in D$. Therefore $\varphi(x) = e^{i\beta_0}|\varphi(x)| = e^{i\beta_0}|\tilde{\varphi}|(|x - x_0|)$. A similar argument holds for ψ. □

The following simple facts will be useful in the sequel. Given any $\varphi \in H^1(\mathbb{R}^N)$ and any $a \in \mathbb{R}$, let $\varphi_a(x) = e^{iax_1}\varphi(x)$. Then $\frac{\partial\varphi_a}{\partial x_1} = iae^{iax_1}\varphi + e^{iax_1}\frac{\partial\varphi}{\partial x_1}$ and

$$\int_{\mathbb{R}^N} |\varphi_a|^2 \,\mathrm{d}x = \int_{\mathbb{R}^N} |\varphi|^2 \,\mathrm{d}x, \qquad Q(\varphi_a) = Q(\varphi) - a\int_{\mathbb{R}^N} |\varphi|^2 \,\mathrm{d}x, \qquad \text{and}$$

$$\int_{\mathbb{R}^N} \Big|\frac{\partial\varphi_a}{\partial x_1}\Big|^2 \,\mathrm{d}x = \int_{\mathbb{R}^N} \Big|\frac{\partial\varphi}{\partial x_1}\Big|^2 \,\mathrm{d}x - 2aQ(\varphi) + a^2\int_{\mathbb{R}^N} |\varphi|^2 \,\mathrm{d}x. \tag{3.36}$$

Notice that for any $\varphi \neq 0$, the mapping $a \longmapsto \int_{\mathbb{R}^N} \left|\frac{\partial \varphi_a}{\partial x_1}\right|^2 \mathrm{d}x$ achieves its minimum at $a_{min} = \frac{Q(\varphi)}{\|\varphi\|_{L^2}^2}$ and $Q(\varphi_{a_{min}}) = 0$.

Next we consider a related minimization problem. Let

$$h_{min}(p) = \inf\{E(\psi,\varphi) \mid \psi \in \mathcal{E}, \varphi \in H^1(\mathbb{R}^N),\, Q(\varphi) = p\}.$$

Proposition 23. *The function h_{min} has the following properties:*

(i) h_{min} is positive, (strictly) increasing and concave on $(0,\infty)$, and $h_{min}(p) \longrightarrow \infty$ as $p \longrightarrow \infty$.

(ii) $h_{min}(p) \leqslant \frac{2}{\epsilon^3 q} p$ for any $p > 0$.

(iii) If $N \geqslant 2$, let $m_0 = \sup\left\{m > 0 \;\middle|\; g_{min}(m) = \frac{m}{\epsilon^4}\right\}$ (see Proposition 18 (iii)). We have $h_{min}(p) = \frac{2}{\epsilon^3 q} p$ for any $p \in \left[0, \frac{q}{2\epsilon} m_0\right]$.

(iv) If $N = 1$, we have $h_{min}(p) < \frac{2}{\epsilon^3 q} p$ for any $p > 0$ and $\frac{h_{min}(p)}{p} \longrightarrow \frac{2}{\epsilon^3 q}$ as $p \longrightarrow 0$.

(v) For any $p > 0$ we have

$$h_{min}(p) = \inf_{m>0} \left(g_{min}(m) + \frac{1}{\epsilon^2 q^2} \frac{p^2}{m} \right). \tag{3.37}$$

The infimum in (3.37) is achieved at some $m_p > 0$ (not necessarily unique). The function g_{min} is differentiable at m_p and $g'_{min}(m_p) = \frac{1}{\epsilon^2 q^2} \frac{p^2}{m_p^2}$.

For any $(\psi,\varphi) \in \mathcal{E} \times H^1(\mathbb{R}^N)$ satisfying $\|\varphi\|_{L^2}^2 = m_p$ and $E(\psi,\varphi) = g_{min}(m_p)$, we have $Q(e^{-i(p/m_p)x_1}\varphi) = p$ and $E(\psi, e^{-i(p/m_p)x_1}\varphi) = h_{min}(p)$.

Conversely, if $(\psi_*,\varphi_*) \in \mathcal{E} \times H^1(\mathbb{R}^N)$ satisfies $Q(\varphi_*) = p$ and $E(\psi_*,\varphi_*) = h_{min}(p)$, then the mapping $m \longmapsto g_{min}(m) + \frac{1}{\epsilon^2 q^2} \frac{p^2}{m}$ achieves its minimum on $(0,\infty)$ at $m_* = \|\varphi_*\|_{L^2}^2$ and $(\psi, e^{i(p/m_*)x_1}\varphi_*)$ is a solution of the minimization problem (GS_{m_*}).

(vi) There exists $C > 0$ such that $h_{min}(p) \leqslant C p^{\frac{N}{N+1}}$ for all $p > 0$.

Proof. Since $E \geqslant 0$ it is obvious that $h_{min} \geqslant 0$. Taking $\psi = 1$ and using Proposition 24 we get

$$h_{min}(p) \leqslant \inf\left\{E(1,\varphi) \;\middle|\; \varphi \in H^1(\mathbb{R}^N),\, Q(\varphi) = p\right\} = \frac{2}{\epsilon^3 q} p.$$

Since $Q(a\varphi) = a^2 Q(\varphi)$, we have

$$h_{min}(p) = \inf\left\{ E_1(\psi) + p \int_{\mathbb{R}^N} \frac{1}{\epsilon^2 q^2} |\nabla\varphi|^2 + \frac{1}{\epsilon^4} |\psi|^2 |\varphi|^2 \,\mathrm{d}x \;\middle|\; \psi \in \mathcal{E}, \varphi \in H^1(\mathbb{R}^N),\, Q(\varphi) = 1 \right\}.$$

The infimum of a family of affine functions is concave, therefore h_{min} is concave on $(0,\infty)$.

For any $\varphi \in H^1(\mathbb{R}^N)$ such that $Q(\varphi) = p$ we have $\int_{\mathbb{R}^N} |\nabla\varphi|^2 + |\varphi|^2 \,\mathrm{d}x \geqslant 2p$ by (3.5), thus necessarily $\int_{\mathbb{R}^N} |\nabla\varphi|^2 \,\mathrm{d}x \geqslant p$ or $\int_{\mathbb{R}^N} |\varphi|^2 \,\mathrm{d}x \geqslant p$. In the former case we have $E(\psi,\varphi) \geqslant \frac{1}{\epsilon^2 q^2} \int_{\mathbb{R}^N} |\nabla\varphi|^2 \,\mathrm{d}x \geqslant \frac{p}{\epsilon^2 q^2}$, and in the latter case we get $E(\psi,\varphi) \geqslant g_{min}(\int_{\mathbb{R}^N} |\varphi|^2 \,\mathrm{d}x) \geqslant g_{min}(p)$ for all $\psi \in \mathcal{E}$. We conclude that

$$h_{min}(p) \geqslant \min\left(\frac{p}{\epsilon^2 q^2}, g_{min}(p) \right) \longrightarrow \infty \qquad \text{as } p \longrightarrow \infty.$$

Any concave mapping from $(0,\infty)$ to itself that tends to 0 at the origin and to infinity at infinity is necessarily increasing and continuous. Assertions (i) and (ii) are thus proven.

(iii) Let $0 < p < \frac{q}{2\epsilon} m_0$. Choose $\eta > 0$ such that $p < \frac{1}{1+\eta}\frac{q}{2\epsilon} m_0$. We already know that $h_{min}(p) \leqslant \frac{2}{\epsilon^3 q} p$. We will show that for any $\psi \in \mathcal{E}$ and $\varphi \in H^1(\mathbb{R}^N)$ such that $Q(\varphi) = p$ and $E(\psi, \varphi) < (1+\eta)\frac{2}{\epsilon^3 q} p$, we have $E(\psi, \varphi) \geqslant \frac{2}{\epsilon^3 q} p$. This implies that $h_{min}(p) = \frac{2}{\epsilon^3 q} p$. Then the continuity of h_{min} implies that the equality $h_{min}(p) = \frac{2}{\epsilon^3 q} p$ holds for $p = \frac{q}{2\epsilon} m_0$, too.

Let p, η, ψ, φ be as above. Denote $m = \int_{\mathbb{R}^N} |\varphi|^2 \, \mathrm{d}x$. We have

$$g_{min}(m) \leqslant E(\psi, \varphi) < (1+\eta)\frac{2}{\epsilon^3 q} p < \frac{m_0}{\epsilon^4} = g_{min}(m_0).$$

Since g_{min} is non-decreasing, we infer that $m < m_0$ and consequently $g_{min}(m) = \frac{m}{\epsilon^4}$. For any $a \in \mathbb{R}$ we have $\int_{\mathbb{R}^N} |e^{iax_1}\varphi(x)|^2 \, \mathrm{d}x = m$, hence $E(\psi, e^{iax_1}\varphi) \geqslant g_{min}(m) = \frac{m}{\epsilon^4}$. Choosing $a_0 = \frac{p}{m}$ (so that $Q(e^{ia_0 x_1}\varphi) = 0$) and using (3.36) we get

$$\begin{aligned} E(\psi, \varphi) &= E(\psi, e^{ia_0 x_1}\varphi) + \frac{1}{\epsilon^2 q^2}\left(2a_0 Q(\varphi) - a_0^2 \int_{\mathbb{R}^N} |\varphi|^2 \, \mathrm{d}x\right) \\ &= E(\psi, e^{ia_0 x_1}\varphi) + \frac{1}{\epsilon^2 q^2}\frac{p^2}{m} \geqslant \frac{m}{\epsilon^4} + \frac{1}{\epsilon^2 q^2}\frac{p^2}{m} \geqslant \frac{2p}{\epsilon^3 q}, \end{aligned} \tag{3.38}$$

as desired.

(v) For any $\varphi \in H^1(\mathbb{R}^N)$ $\psi \in \mathcal{E}$ and any $a \in \mathbb{R}$, we have $\|e^{iax_1}\varphi\|_{L^2(\mathbb{R}^N)} = \|\varphi\|_{L^2(\mathbb{R}^N)}$. The mapping $a \longmapsto E(\psi, e^{iax_1}\varphi)$ achieves its minimum on $\mathbb{R}$ at $a_{min} = \frac{Q(\varphi)}{\|\varphi\|_{L^2}^2}$ and $Q(\varphi_{a_{min}}) = 0$ (see (3.36)). We conclude that

$$g_{min}(m) = \inf\left\{E(\psi, \varphi) \;\middle|\; \psi \in \mathcal{E}, \varphi \in H^1(\mathbb{R}^N), \int_{\mathbb{R}^N} |\varphi|^2 \, \mathrm{d}x = m \text{ and } Q(\varphi) = 0\right\}.$$

Let $p > 0$ and $m > 0$. Consider any $\varphi \in H^1(\mathbb{R}^N)$ such that $\|\varphi\|_{L^2}^2 = m$ and $Q(\varphi) = 0$. Let $a = \frac{p}{m}$. By (3.36) we have $Q(e^{-iax_1}\varphi) = p$, thus for any $\psi \in \mathcal{E}$ we have

$$h_{min}(p) \leqslant E(\psi, e^{-iax_1}\varphi) = E(\psi, \varphi) + \frac{a^2}{\epsilon^2 q^2}\int_{\mathbb{R}^N} |\varphi|^2 \, \mathrm{d}x = E(\psi, \varphi) + \frac{1}{\epsilon^2 q^2}\frac{p^2}{m}.$$

Passing to the infimum we get

$$h_{min}(p) \leqslant g_{min}(m) + \frac{1}{\epsilon^2 q^2}\frac{p^2}{m}.$$

Since the above inequality holds for any $m > 0$, we infer that

$$h_{min}(p) \leqslant \inf_{m>0}\left(g_{min}(m) + \frac{1}{\epsilon^2 q^2}\frac{p^2}{m}\right).$$

Let $p > 0$ and consider $\varphi \in H^1(\mathbb{R}^N)$ such that $Q(\varphi) = p$. Denote $m_\varphi = \|\varphi\|_{L^2}^2$ and $a_\varphi = \frac{p}{m_\varphi}$. Then $\|e^{ia_\varphi x_1}\varphi\|_{L^2}^2 = \|\varphi\|_{L^2}^2 = m$, hence for any $\psi \in \mathcal{E}$ we have

$$g_{min}(m_\varphi) \leqslant E(\psi, e^{ia_\varphi x_1}\varphi) = E(\psi, \varphi) + \frac{1}{\epsilon^2 q^2}\left(-2a_\varphi Q(\varphi) + a_\varphi^2 \|\varphi\|_{L^2}^2\right) = E(\psi, \varphi) - \frac{1}{\epsilon^2 q^2}\frac{p^2}{m_\varphi}.$$

Thus for any $\varphi \in H^1(\mathbb{R}^N)$ with $Q(\varphi) = p$ and for any $\psi \in \mathcal{E}$ we have

$$E(\psi, \varphi) \geqslant g_{min}(m_\varphi) + \frac{1}{\epsilon^2 q^2}\frac{p^2}{m_\varphi} \geqslant \inf_{m>0}\left(g_{min}(m) + \frac{1}{\epsilon^2 q^2}\frac{p^2}{m}\right).$$

The above inequality shows that $h_{min}(p) \geqslant \inf_{m>0}\left(g_{min}(m) + \frac{1}{\epsilon^2 q^2}\frac{p^2}{m}\right)$. Thus (3.37) is proven.

For any fixed $p > 0$, the mapping $m \longmapsto g_{min}(m) + \frac{1}{\epsilon^2 q^2}\frac{p^2}{m}$ is continuous on $(0,\infty)$ and tends to infinity as $m \longrightarrow 0$, respectively as $m \longrightarrow \infty$. We infer that this mapping achieves its minimum at some $m_p \in (0,\infty)$.

The mapping $m \longmapsto g_{min}(m) + \frac{1}{\epsilon^2 q^2}\frac{p^2}{m}$ admits left and right derivatives at any point $m > 0$ because g_{min} is concave. At a minimum point m_p the left derivative must be non-positive and the right derivative must be non-negative, and this gives

$$g'_{min,\ell}(m_p) \leqslant \frac{1}{\epsilon^2 q^2}\frac{p^2}{m_p^2} \leqslant g'_{min,r}(m_p).$$

On the other hand, by concavity we have $g'_{min,\ell}(m_p) \geqslant g'_{min,r}(m_p)$. Thus $g'_{min,\ell}(m_p) = g'_{min,r}(m_p) = \frac{1}{\epsilon^2 q^2}\frac{p^2}{m_p^2}$ and g_{min} is differentiable at m_p.

The other statements in part (v) follow easily using (3.36) and a computation similar to (3.38).

(iv) Assume that $N = 1$. Let $p > 0$. Let $m_0 = \frac{\epsilon}{q}p$. Using (3.37) and the fact that $g_{min}(m_0) < \frac{m_0}{\epsilon^4}$ (see Proposition 18 (ii)), we get

$$h_{min}(p) \leqslant g_{min}(m_0) + \frac{1}{\epsilon^2 q^2}\frac{p^2}{m_0} < \frac{m_0}{\epsilon^4} + \frac{1}{\epsilon^2 q^2}\frac{p^2}{m_0} = \frac{2p}{\epsilon^3 q}.$$

Fix $\varepsilon > 0$. By Proposition 18 (ii) there is $m_\varepsilon > 0$ such that $\frac{g_{min}(m)}{m} \in \left(\frac{1-\varepsilon}{\epsilon^4}, \frac{1}{\epsilon^4}\right)$ for any $m \in (0, m_\varepsilon)$. Let $p_\varepsilon \in \left(0, \frac{\epsilon^3 q}{2} g_{min}(m_\varepsilon)\right)$. Choose $\eta > 0$ such that $(1+\eta)\frac{2p_\varepsilon}{\epsilon^3 q} < g_{min}(m_\varepsilon)$. Let $p \in (0, p_\varepsilon]$. We know that $h_{min}(p) \leqslant \frac{2p}{\epsilon^3 q}$. For any $(\psi,\varphi) \in \mathcal{E} \times H^1(\mathbb{R}^N)$ satisfying $Q(\varphi) = p$ and $E(\psi,\varphi) < (1+\eta)\frac{2p}{\epsilon^3 q}$ we have $E(\psi,\varphi) < g_{min}(m_\varepsilon)$, thus necessarily $\|\varphi\|_{L^2}^2 < m_\varepsilon$. Denoting $m = \|\varphi\|_{L^2}^2$, $a_0 = \frac{p}{m}$ and proceeding exactly as in (3.38) we get

$$E(\psi,\varphi) = E(\psi, e^{ia_0 x_1}\varphi) + \frac{1}{\epsilon^2 q^2}\frac{p^2}{m} \geqslant g_{min}(m) + \frac{1}{\epsilon^2 q^2}\frac{p^2}{m} > \frac{(1-\varepsilon)m}{\epsilon^4} + \frac{1}{\epsilon^2 q^2}\frac{p^2}{m} \geqslant \frac{2\sqrt{1-\varepsilon}}{\epsilon^3 q}p.$$

We conclude that $h_{min}(p) \geqslant \frac{2\sqrt{1-\varepsilon}}{\epsilon^3 q}p$ for all $p \in (0, p_\varepsilon]$.

(vi) By Proposition 18 (iv) there is $C > 0$ such that $g_{min}(m) \leqslant Cm^{\frac{N}{N+2}}$. Taking $m = p^{\frac{N+2}{N+1}}$ in (3.37) we get

$$h_{min}(p) \leqslant g_{min}\left(p^{\frac{N+2}{N+1}}\right) + \frac{1}{\epsilon^2 q^2}\frac{p^2}{p^{\frac{N+2}{N+1}}} \leqslant \left(C + \frac{1}{\epsilon^2 q^2}\right)p^{\frac{N}{N+1}}.$$

□

3.3 Minimizers of the energy when the mixed momentum is fixed

For $\beta \in [0,1]$, $\psi \in \mathcal{E}$ and $\varphi \in H^1(\mathbb{R}^N)$ we define

$$P_\beta(\psi,\varphi) = \beta Q(\psi) + (1-\beta)Q(\varphi).$$

We suppose that the assumptions **(A1) - (A4)** are satisfied. For any $\beta \in [0,1]$ and $p > 0$ we consider the minimization problem

$(\mathcal{P}_{\beta,p})$ minimize $E(\psi,\varphi)$ for $\psi \in \mathcal{E}$, $\varphi \in H^1(\mathbb{R}^N)$ satisfying $P_\beta(\psi,\varphi) = p$.

Obviously, if $\beta = 1$ the solutions of the problem $(\mathcal{P}_{1,p})$ are precisely of the form $(\psi, 0)$, where ψ minimizes $E_1(\psi)$ under the constraint $Q(\psi) = p$. For the existence of such minimizers and some of

their properties, see Theorem 1.1 in [5]. If $\beta = 0$, the solutions of the problem $(\mathcal{P}_{0,p})$ are the minimizers for $h_{min}(p)$ given by Proposition 23. Denoting $\beta_* = \frac{\epsilon^2 q^2}{\epsilon^2 q^2 + \delta}$, where δ, ϵ, q are as in (GC), it is easily seen that P_{β_*} is a conserved quantity for the system (GC).

We begin with the following simple

Proposition 24. *Fix $K > 0$. For any $p > 0$ we have*

$$\inf\left\{\int_{\mathbb{R}^N} |\nabla\varphi|^2 + K^2|\varphi|^2\,\mathrm{d}x \;\Big|\; \varphi \in H^1(\mathbb{R}^N),\ Q(\varphi) = p\right\} = 2Kp$$

and the infimum is never achieved.

Proof. Consider a real-valued function $\chi \in C_c^\infty(\mathbb{R}^N)$ such that $\chi = 1$ in $B(0,1)$ and $\chi = 0$ in $\mathbb{R}^N \setminus B(0,2)$. Let $\tilde{\phi}_R(x) = \chi\left(\frac{x}{R}\right) e^{-iKx_1}$. A simple computation gives

$$\int_{\mathbb{R}^N} |\tilde{\phi}_R(x)|^2\,\mathrm{d}x = R^N \int_{\mathbb{R}^N} |\chi|^2\,\mathrm{d}x, \qquad Q(\tilde{\phi_R}) = KR^N \int_{\mathbb{R}^N} |\chi|^2\,\mathrm{d}x,$$

$$\int_{\mathbb{R}^N} \Big|\frac{\partial\tilde{\phi}_R(x)}{\partial x_1}\Big|^2\,\mathrm{d}x = K^2R^N \int_{\mathbb{R}^N} |\chi|^2\,\mathrm{d}x + R^{N-2}\int_{\mathbb{R}^N} \Big|\frac{\partial\chi}{\partial x_1}\Big|^2\,\mathrm{d}x,$$

$$\int_{\mathbb{R}^N} \Big|\frac{\partial\tilde{\phi}_R(x)}{\partial x_j}\Big|^2\,\mathrm{d}x = R^{N-2}\int_{\mathbb{R}^N} \Big|\frac{\partial\chi}{\partial x_j}\Big|^2\,\mathrm{d}x \qquad \text{for } j = 2,\dots,N.$$

Let $a_R = K^{-\frac{1}{2}} R^{-\frac{N}{2}} p^{\frac{1}{2}} \|\chi\|_{L^2}^{-1}$ and let $\phi_R = a_R\tilde{\phi}_R$. Then we have $Q(\phi_R) = p$ for all $R > 0$ and

$$\int_{\mathbb{R}^N} |\nabla\phi_R|^2 + K^2|\phi_R|^2\,\mathrm{d}x = 2K^2R^N a_R^2 \int_{\mathbb{R}^N} |\chi|^2\,\mathrm{d}x + R^{N-2}a_R^2\int_{\mathbb{R}^N} |\nabla\chi|^2\,\mathrm{d}x = 2Kp + \frac{p}{KR^2}\frac{\|\nabla\chi\|_{L^2}^2}{\|\chi\|_{L^2}^2}.$$

Letting $R \longrightarrow \infty$ we see that the infimum is smaller than or equal to $2Kp$. On the other hand, by (3.5) we see that the infimum is greater than or equal to $2Kp$. If u is a minimizer, by (3.5) we have $\int_{\mathbb{R}^N} \Big|\frac{\partial u}{\partial x_j}\Big|^2\,\mathrm{d}x = 0$ for $j = 2,\dots,N$, hence u depends only on x_1. Since $u \in H^1(\mathbb{R}^N)$ we must have $u = 0$, and this contradicts the fact that $Q(u) = p$. □

We define

$$E_{\beta,min}(p) = \inf\left\{E(\psi,\varphi) \;\big|\; \psi \in \mathcal{E}, \varphi \in H^1(\mathbb{R}^N), P_\beta(\psi,\varphi) = p\right\}. \tag{3.39}$$

If $\beta = 0$, we have $E_{0,min}(p) = h_{min}(p)$ and the properties of the function h_{min} are given by Proposition 23. The properties of the function $E_{1,min}$ as well as the existence of minimizers for the problem $(\mathcal{P}_{1,p})$ have been studied in [5]. See Theorem 14 in the Introduction.

Some important properties of the function $E_{\beta,min}$ that will be useful in the sequel are given in the next Proposition.

Proposition 25. *Assume that $N \geqslant 2$, $V \geqslant 0$ and the assumptions* **(A1)** *and* **(A2)** *hold.*

For any $\beta \in (0,1)$ we denote

$$S_\beta = \min\left(\frac{\sqrt{2}}{\beta\epsilon}, \frac{2}{(1-\beta)\epsilon^3 q}\right) = \min\left(\frac{v_s}{\beta}, \frac{2}{(1-\beta)\epsilon^3 q}\right).$$

Then:

(i) $E_{\beta,min}(p) \leqslant \min\left(E_{1,min}\left(\frac{p}{\beta}\right), h_{min}\left(\frac{p}{1-\beta}\right)\right) \leqslant S_\beta p$ *for any* $\beta \in (0,1)$ *and any* $p > 0$.

(ii) For all $\beta \in (0,1)$ *we have* $\lim_{p\to 0} \dfrac{E_{\beta,min}(p)}{p} = S_\beta$.

(iii) Suppose in addition that $N = 2$, *assumption* **(A4)** *holds,* $F''(1) \neq -3F'(1)$, *and* $\dfrac{v_s}{\beta} < \dfrac{2}{(1-\beta)\epsilon^3 q}$. *Then we have* $E_{\beta,min}(p) < S_\beta p$ *for all* $p > 0$.

(iv) $E_{\beta,min}$ *is concave, positive and increasing on* $(0,\infty)$, *and* $E_{\beta,min}(p) \longrightarrow \infty$ *as* $p \longrightarrow \infty$.

Proof. (i) Taking "test functions" of the form $(\psi, 0)$ with $Q(\psi) = \frac{p}{\beta}$, it is obvious that $E_{\beta,min}(p) \leqslant E_{1,min}\left(\frac{p}{\beta}\right)$.

If $(\psi_n, \varphi_n)_{n\geqslant 1}$ is a minimizing sequence for $h_{min}\left(\frac{p}{1-\beta}\right)$, that is $Q(\varphi_n) \longrightarrow \frac{p}{1-\beta}$ and $E(\psi_n, \varphi_n) \longrightarrow h_{min}\left(\frac{p}{1-\beta}\right)$, then $(|\psi_n|, \varphi_n)$ is also a minimizing sequence. Since $Q(|\psi|) = 0$ for any $\psi \in \mathcal{E}$ we have $P_\beta(|\psi_n|, \varphi_n) = p$, hence $E_{\beta,min}(p) \leqslant \liminf_{n\to\infty} E(|\psi_n|, \varphi_n) = h_{min}\left(\dfrac{p}{1-\beta}\right)$.

The second inequality follows from the fact that $E_{1,min}(p) \leqslant v_s p = \frac{\sqrt{2}}{\epsilon} p$ and $h_{min}(p) \leqslant \frac{2p}{\epsilon^3 q}$ for all $p > 0$ (see Proposition 23 (ii)).

(ii) We already know that $E_{\beta,min}(p) \leqslant S_\beta p$ for any $p > 0$ and it suffices to show that $\liminf_{p\to 0+} \dfrac{E_{\beta,min}(p)}{p} \geqslant S_\beta$. We argue by contradiction and we assume that $\ell := \liminf_{p\to 0+} \dfrac{E_{\beta,min}(p)}{p} < S_\beta$. We choose $\varepsilon > 0$ and ℓ_* such that $\ell < \ell_* < S_* := \min\left(\frac{v_s-\varepsilon}{\beta}, \frac{1}{1-\beta}\left(\frac{2}{\epsilon^3 q} - \varepsilon\right)\right) < S_\beta$. Then there exists a decreasing sequence $p_n \longrightarrow 0$ such that $E_{\beta,min}(p_n) < \ell_* p_n$ for all n. For each n there exist $\psi_n \in \mathcal{E}$ and $\varphi_n \in H^1(\mathbb{R}^N)$ such that $P_\beta(\psi_n, \varphi_n) = p_n$ and $E(\psi_n, \varphi_n) < \ell_* p_n$. From Lemma 4.6 p. 175 and Lemma 4.5 p. 173 in [5] we have $\lim_{p\to 0+} \dfrac{E_{1,min}(p)}{p} = v_s$, and we may assume that p_1 is sufficiently small, so that $E_{1,min}(p) \geqslant (v_s - \varepsilon)p$ for all $p \in [0, 2\frac{p_1}{\beta}]$.

For simplicity denote $p_{n,1} = Q(\psi_n)$ and $p_{n,2} = Q(\varphi_n)$, so that $\beta p_{n,1} + (1-\beta)p_{n,2} = p_n$. If $p_{n,2} < 0$ (respectively if $p_{n,2} > \frac{2}{1-\beta} p_n$) we have $p_{n,1} > \frac{p_n}{\beta}$ (respectively $p_{n,1} < -\frac{p_n}{\beta}$) and we get

$$E(\psi_n, \varphi_n) \geqslant E_1(\psi_n) \geqslant E_{1,min}(|p_{n,1}|) \geqslant E_{1,min}\left(\frac{p_n}{\beta}\right) \geqslant (v_s - \varepsilon)\frac{p_n}{\beta} \geqslant S_* p_n,$$

contradicting the choice of (ψ_n, φ_n). We infer that $0 \leqslant p_{n,2} \leqslant \frac{2}{1-\beta} p_n$ for all n, and this implies that $-\frac{p_n}{\beta} \leqslant p_{n,1} \leqslant \frac{p_n}{\beta}$.

We claim that $\liminf_{n\to\infty} \dfrac{p_{n,2}}{p_n} > 0$. To prove the claim we argue by contradiction and we assume that there is a subsequence $(p_{n_k})_{k\geqslant 1}$ such that $\frac{p_{n_k,2}}{p_{n_k}} \longrightarrow 0$ as $k \longrightarrow \infty$. Then we get $\frac{p_{n_k,1}}{p_{n_k}} \longrightarrow \frac{1}{\beta}$ and proceeding as above we infer that for all sufficiently large k,

$$E(\psi_{n_k}, \varphi_{n_k}) \geqslant E_1(\psi_{n_k}) \geqslant E_{1,min}(p_{n_k,1}) \geqslant (v_s - \varepsilon)p_{n_k,1}$$

therefore $\liminf_{k\to\infty} \dfrac{E(\psi_{n_k}, \varphi_{n_k})}{p_{n_k}} \geqslant \dfrac{v_s - \varepsilon}{\beta} \geqslant S_* > \ell_*$, contradicting the fact that $E(\psi_n, \varphi_n) < \ell_* p_n$ for all n. We conclude that there is $C > 0$ such that $C \leqslant \frac{p_{n,2}}{p_n} \leqslant \frac{2}{1-\beta}$ for all n sufficiently large.

If n is large enough (so that $p_{n,2} \geqslant Cp_n$), let $\tilde{\varphi}_n = \frac{1}{\sqrt{p_{n,2}}}\varphi_n$. It is obvious that $Q(\tilde{\varphi}_n) = 1$ and $p_{n,2} \geqslant Cp_n \geqslant \frac{C}{\ell_*}E(\psi_n, \varphi_n)$ and we have

$$\begin{aligned} E(\psi_n, \tilde{\varphi}_n) &= E_1(\psi_n) + \frac{1}{p_{n,2}}\frac{1}{\epsilon^2 q^2}\int_{\mathbb{R}^N} |\nabla\varphi_n|^2 + \frac{q^2}{\epsilon^2}|\psi_n|^2|\varphi_n|^2 \,\mathrm{d}x \\ &\leqslant E_1(\psi_n) + \frac{\ell_*}{C}\frac{1}{\epsilon^2 q^2}\frac{1}{E(\psi_n, \varphi_n)}\int_{\mathbb{R}^N} |\nabla\varphi_n|^2 + \frac{q^2}{\epsilon^2}|\psi_n|^2|\varphi_n|^2 \,\mathrm{d}x \\ &\leqslant E(\psi_n, \varphi_n) + \tfrac{\ell_*}{C}. \end{aligned}$$

Therefore $E(\psi_n, \tilde{\varphi}_n)$ is bounded. We have $E_1(\psi_n) \leqslant E(\psi_n, \varphi_n) \leqslant \ell_* p_n \longrightarrow 0$ and Lemma 20 implies that $\| \, |\psi_n| - 1\|_{L^2} \longrightarrow 0$ as $n \longrightarrow \infty$. Then Lemma 21 (iii) gives

$$\liminf_{n\to\infty} \int_{\mathbb{R}^N} |\nabla\tilde{\varphi}_n|^2 + \frac{q^2}{\epsilon^2}|\psi_n|^2|\tilde{\varphi}_n|^2 \,\mathrm{d}x \geqslant \frac{2q}{\epsilon}.$$

Hence for all n sufficiently large we have

$$\frac{1}{\epsilon^2 q^2}\int_{\mathbb{R}^N} |\nabla\varphi_n|^2 + \frac{q^2}{\epsilon^2}|\psi_n|^2|\varphi_n|^2 \,\mathrm{d}x \geqslant \left(\frac{2}{\epsilon^3 q} - \varepsilon\right) p_{n,2}. \tag{3.40}$$

We use the following simple observation: given any $A > 0$, $B > 0$ and $\beta \in (0,1)$, we have

$$\min\left\{As_1 + Bs_2 \;\middle|\; s_1 \geqslant 0,\ s_2 \geqslant 0,\ \beta s_1 + (1-\beta)s_2 = s\right\} = \min\left(\frac{As}{\beta}, \frac{Bs}{1-\beta}\right). \tag{3.41}$$

The minimum is reached for $s_1 = 0, s_2 = \frac{s}{1-\beta}$ if $\frac{A}{\beta} > \frac{B}{1-\beta}$, respectively for $s_1 = \frac{s}{\beta}, s_2 = 0$ if $\frac{A}{\beta} < \frac{B}{1-\beta}$.

Using (3.40) and (3.41), for all sufficiently large n we get

$$E(\psi_n, \varphi_n) = E_1(\psi_n) + \frac{1}{\epsilon^2 q^2}\int_{\mathbb{R}^N} |\nabla\varphi_n|^2 + \frac{q^2}{\epsilon^2}|\psi_n|^2|\varphi_n|^2 \,\mathrm{d}x \geqslant (v_s - \varepsilon)|p_{n,1}| + \left(\frac{2}{\epsilon^3 q} - \varepsilon\right) p_{n,2} \geqslant S_* p_n,$$

contradicting the fact that $E(\psi_n, \varphi_n) < \ell_* p_n < S_* p_n$. This proves assertion (ii).

(iii) Under the assumptions of (iii) we have $S_\beta = \frac{v_s}{\beta}$ and Theorem 4.15 p. 190 in [5] implies that $E_{1,min}(p) < v_s p$ for all $p > 0$. Then using part (i) we get

$$E_{min}(p) \leqslant E_{1,min}\left(\frac{p}{\beta}\right) < v_s\frac{p}{\beta} = S_\beta p.$$

(iv) The proof of (iv) is essentially the same as the proof of Lemma 4.7 p. 175 in [5], so we omit it. □

Theorem 26. *Assume that $N \geqslant 2$, assumptions* **(A1)** *and* **(A2)** *hold, $V \geqslant 0$ on $[0,\infty)$ and $p > 0$ is such that $E_{\beta,min}(p) < S_\beta p$.*

Then there exist minimizers for the problem $(\mathcal{P}_{\beta,p})$.

Moreover, any sequence $(\psi_n, \varphi_n)_{n\geqslant 1} \subset \mathcal{E} \times H^1(\mathbb{R}^N)$ satisfying $P_\beta(\psi_n, \varphi_n) \longrightarrow p$ and $E(\psi_n, \varphi_n) \longrightarrow E_{\beta,min}(p)$ contains a subsequence $(\psi_{n_k})_{k\geq 1}$ having the following property: there are a sequence of points $(x_k)_{k\geq 1} \subset \mathbb{R}^N$, $\psi \in \mathcal{E}$ and $\varphi \in H^1(\mathbb{R}^N)$ such that $P_\beta(\psi, \varphi) = p$, $E(\psi, \varphi) = E_{\beta,min}(p)$, and as $k \longrightarrow \infty$ we have

$$\varphi_{n_k} \longrightarrow \varphi \ \text{ in } \ H^1(\mathbb{R}^N), \qquad (\psi_{n_k}, \varphi_{n_k})(\cdot + x_k) \longrightarrow (\psi, \varphi) \ \text{ a.e. in } \mathbb{R}^N,$$

$$\|\nabla\psi_{n_k}(\cdot + x_k) - \nabla\psi\|_{L^2} \longrightarrow 0, \qquad \| \, |\psi_{n_k}|(\cdot + x_k) - |\psi| \, \|_{L^2} \longrightarrow 0,$$

$$\psi_{n_k}\varphi_{n_k}(\cdot + x_k) \longrightarrow \psi\varphi \ \text{ in } L^2(\mathbb{R}^N), \quad V\left(|\psi_{n_k}(\cdot + x_k)|^2\right) \longrightarrow V(|\psi|^2) \text{ in } L^1(\mathbb{R}^N).$$

Proof. Let $(\psi_n, \varphi_n)_{n\geqslant 1}$ be a sequence as in Theorem 26. Then $E(\psi_n, \varphi_n)$ is bounded. Define f_n as in (3.17) and let Λ_n be the concentration function of f_n, as in (3.18). Lemma 20 and the Sobolev embedding imply that $(f_n)_{n\geqslant 1}$ is a bounded sequence in $L^1(\mathbb{R}^N)$.

Since $\|\varphi_n\|_{H^1}$ is bounded, (3.5) implies that $Q(\varphi_n)$ is bounded. From the fact that $P_\beta(\psi_n, \varphi_n) \longrightarrow p$ we infer that $Q(\psi_n)$ is bounded. Passing to a subsequence we may assume that $Q(\psi_n) \longrightarrow p_1$ and $Q(\varphi_n) \longrightarrow p_2$ as $n \longrightarrow \infty$, where $\beta p_1 + (1-\beta)p_2 = p$. Let us show that $0 \leqslant p_1 \leqslant \frac{p}{\beta}$ and $0 \leqslant p_2 \leqslant \frac{p}{1-\beta}$. We have

$$\limsup_{n\to\infty} E_1(\psi_n) \leqslant \lim_{n\to\infty} E(\psi_n, \varphi_n) = E_{\beta,min}(p) \leqslant E_{1,min}\left(\frac{p}{\beta}\right).$$

Since $E_{1,min}$ is increasing on $[0,\infty)$, this implies $\limsup_{n\to\infty} |Q(\psi_n)| \leqslant \frac{p}{\beta}$, thus $|p_1| \leqslant \frac{p}{\beta}$.

Since $|Q(\varphi_n)| \leqslant \|\varphi_n\|_{L^2} \left\|\frac{\partial \varphi_n}{\partial x_1}\right\|_{L^2}$ and $\|\varphi_n\|_{L^2}$ is bounded by Lemma 20, we have $\ell := \liminf_{n\to\infty} \left\|\frac{\partial \varphi_n}{\partial x_1}\right\|_{L^2} > 0$. If $p_1 < 0$, we have $p_2 = \frac{p-\beta p_1}{1-\beta} > \frac{p}{1-\beta}$ and $d_n := \left(\frac{p}{(1-\beta)Q(\varphi_n)}\right)^{\frac{1}{2}} \longrightarrow \left(\frac{p}{(1-\beta)p_2}\right)^{\frac{1}{2}} < 1$. Then $|\psi_n| \in \mathcal{E}$, $Q(|\psi_n|) = 0$, $Q(d_n\varphi_n) = \frac{p}{1-\beta}$, hence $P_\beta(|\psi_n|, d_n\varphi_n) = p$ and taking $(|\psi_n|, d_n\varphi_n)$ instead of (ψ_n, φ_n) we get

$$E(|\psi_n|, d_n\varphi_n) \leqslant E_1(\psi_n) + \frac{d_n^2}{\epsilon^2 q^2}\int_{\mathbb{R}^N} |\nabla\varphi_n|^2 + \frac{q^2}{\epsilon^2}|\psi_n|^2|\varphi_n|^2 \,\mathrm{d}x \leqslant E(\psi_n, \varphi_n) - \frac{1-d_n^2}{\epsilon^2 q^2}\int_{\mathbb{R}^N} \left|\frac{\partial \varphi_n}{\partial x_1}\right|^2 \mathrm{d}x.$$

Passing to the limit we discover

$$E_{\beta,min}(p) \leqslant \limsup_{n\to\infty} E(|\psi_n|, d_n\varphi_n) \leqslant E_{\beta,min}(p) - \frac{(1-d_n^2)\ell^2}{\epsilon^2 q^2},$$

a contradiction. We conclude that $p_1 \geqslant 0$, thus $0 \leqslant p_1 \leqslant \frac{p}{\beta}$ and this implies $0 \leqslant p_2 \leqslant \frac{p}{1-\beta}$.

We claim that there is $C > 0$ such that $\|f_n\|_{L^1} \geqslant C$ for all n sufficiently large. Otherwise, there is a subsequence $(f_{n_k})_{k\geqslant 1}$ such that $\|f_{n_k}\|_{L^1} \longrightarrow 0$. Clearly, this implies that $\|\varphi_{n_k}\|_{H^1} \longrightarrow 0$ and then (3.5) gives $Q(\varphi_{n_k}) \longrightarrow 0$. On the other hand, we have $\|\nabla\psi_{n_k}\|_{L^2} \longrightarrow 0$ and $\|\,|\psi_{n_k}| - 1\|_{L^2} \longrightarrow 0$, and Lemma 4.1 p. 171 in [5] gives $\int_{\mathbb{R}^N} V(|\psi_{n_k}|^2)\,\mathrm{d}x \longrightarrow 0$, hence $E_1(\psi_{n_k}) \longrightarrow 0$. Since $E_1(\psi_{n_k}) \geqslant E_{1,min}(Q(\psi_{n_k}))$ and $E_{1,min}$ is positive on $\mathbb{R}^*$ and increasing on $\mathbb{R}_+$, we get $Q(\psi_{n_k}) \longrightarrow 0$, and therefore $P_\beta(\psi_{n_k}, \varphi_{n_k}) \longrightarrow 0$, a contradiction. This proves the claim.

Passing to a subsequence we may assume that $\int_{\mathbb{R}^N} f_n\,\mathrm{d}x \longrightarrow \alpha_0 > 0$. We apply the concentration-compactness principle to the sequence $(f_n)_{n\geqslant 1}$. Let Λ_n be the concentration function of f_n, as in (3.18). Proceeding as in [8] and using (5.12) p. 156 in [11], we see that there exist a non-decreasing function $\Lambda : [0,\infty) \longrightarrow [0,\infty)$, $\alpha \in [0, \alpha_0]$ and a sequence $t_n \longrightarrow \infty$ such that (3.19) and (3.20) hold. We will show that $\alpha = \alpha_0$. To do this we rule out the possibilities $\alpha = 0$ ("vanishing") and $\alpha \in (0, \alpha_0)$ ("dichotomy").

If $\alpha = 0$ we have $\Lambda(t) = 0$ for all $t \in [0,\infty)$, in particular $\Lambda(1) = 0$ and condition (3.22) is satisfied. Then Lemma 21 (iii) gives

$$\liminf_{n\to\infty} \int_{\mathbb{R}^N} |\nabla\varphi_n|^2 + \frac{q^2}{\epsilon^2}|\psi_n|^2|\varphi_n|^2 \,\mathrm{d}x \geqslant \frac{2q}{\epsilon}p_2. \tag{3.42}$$

We use the next lemma, which is an immediate consequence of Lemma 4.10 p. 179 in [5].

Lemma 27. *([5]) Assume that $N \geq 2$ and assumptions* **(A1)** *and* **(A2)** *hold. Let $(\psi_n)_{n\geq 1} \subset \mathcal{E}$ be a sequence satisfying:*

(a) $E_1(\psi_n) \leq M$ for some positive constant M.

(b) $\liminf_{n\to\infty} Q(\psi_n) \geq q$.

(c) $\sup_{y\in\mathbb{R}^N}\int_{B(y,1)}|\nabla\psi_n|^2+\big||\psi_n|-1\big|^2\,\mathrm{d}x\longrightarrow 0$ *as* $n\longrightarrow\infty$.

Then $\liminf_{n\to\infty}E_1(\psi_n)\geqslant v_s|q|$.

From Lemma 27 we get $\liminf_{n\to\infty}E_1(\psi_n)\geqslant v_sp_1$ and using (3.42) and (3.41) we obtain

$$\liminf_{n\to\infty}E(\psi_n,\varphi_n)\geqslant v_sp_1+\frac{2}{\epsilon^3q}p_2\geqslant S_\beta p,$$

contradicting the assumption that $E_{min}(p)<S_\beta p$.

Assume that $\alpha\in(0,\alpha_0)$. As in the proof of Theorem 19, let $h_n=\Lambda_n(t_n)-\Lambda_n\left(\frac{t_n}{2}\right)+\frac{1}{2^n}$, so that $h_n\longrightarrow 0$ as $n\longrightarrow\infty$, and for each $n\in\mathbb{N}^*$ choose x_n such that $\int_{B(x_n,\frac{t_n}{2})}f_n(x)\,\mathrm{d}x>\Lambda_n\left(\frac{t_n}{2}\right)-\frac{1}{2^n}$ and denote $\Omega_n=B(x_n,t_n)\setminus B(x_n,\frac{t_n}{2})$. Then (3.25) holds, thus $\int_{\Omega_n}f_n\,\mathrm{d}x\longrightarrow 0$ as $n\longrightarrow\infty$.

Take $\chi_1,\chi_2\in C_c^\infty(\mathbb{R}^N)$ such that $0\leqslant\chi_1\leqslant 1$, $0\leqslant\chi_2\leqslant 1$, $\chi_1=1$ on $B(0,\frac{1}{2})$ and $\mathrm{supp}(\chi_1)\subset B(0,\frac{5}{8})$, $\chi_2=0$ on $B(0,\frac{7}{8})$ and $\chi_2=1$ on $\mathbb{R}^N\setminus B(0,1)$. Let $\varphi_{n,i}(x)=\chi_i\left(\frac{x-x_n}{t_n}\right)\varphi_n(x)$ for $i=1,2$. It is easy to see that

$$\int_{\mathbb{R}^N}\big||\nabla\varphi_n|^2-|\nabla\varphi_{n,1}|^2-|\nabla\varphi_{n,2}|^2\big|\,\mathrm{d}x\leqslant C\int_{\Omega_n}|\nabla\varphi_n|^2+|\varphi_n|^2\,\mathrm{d}x\leqslant Ch_n, \tag{3.43}$$

$$\|\varphi_n-\varphi_{n,1}-\varphi_{n,2}\|_{H^1}^2\leqslant C\int_{\Omega_n}|\nabla\varphi_n|^2+|\varphi_n|^2\,\mathrm{d}x\leqslant Ch_n,\qquad\text{and} \tag{3.44}$$

$$\int_{\mathbb{R}^N}|\psi_n|^2\big||\varphi_n|^2-|\varphi_{n,1}|^2-|\varphi_{n,2}|^2\big|\,\mathrm{d}x\leqslant\int_{\Omega_n}|\psi_n|^2|\varphi_n|^2\,\mathrm{d}x\leqslant h_n. \tag{3.45}$$

It is clear that $\varphi_{n,i}$ is bounded in $H^1(\mathbb{R}^N)$, hence $Q(\varphi_{n,i})$ is bounded for $i=1,2$. Since $\varphi_{n,1}$ and $\varphi_{n,2}$ have disjoint supports we have $Q(\varphi_{n,1}+\varphi_{n,2})=Q(\varphi_{n,1})+Q(\varphi_{n,2})$, and then using (3.44) we get $Q(\varphi_n)-Q(\varphi_{n,1})-Q(\varphi_{n,2})\longrightarrow 0$ as $n\longrightarrow\infty$.

To "separate" the behaviour of ψ_n on $B(x_n,\frac{t_n}{2})$ and on $\mathbb{R}^N\setminus B(x_n,t_n)$ we need a more subtle argument than in the proof of Theorem 19. It is based on the next Lemma, which is a particular case of Lemma 3.3 p. 138 in [11] and of Lemma 3.3 p. 167 in [5].

Lemma 28. *([11, 5]) There exist* $\varepsilon_0>0$ *and* $C_i>0$*, depending only on* N *(and on* F *for (v)) such that for any* $R\geq 2$, $\varepsilon\in(0,\varepsilon_0)$ *and* $\psi\in\mathcal{E}$ *verifying* $\int_{B(0,R)\setminus B(0,\frac{R}{2})}|\nabla\psi|^2+\big||\psi|-1\big|^2\,\mathrm{d}x\leq\varepsilon$*, there exist two functions* ψ_1, $\psi_2\in\mathcal{E}$ *and a constant* $\theta_0\in[0,2\pi)$ *satisfying the following properties:*

(i) $\psi_1=\psi$ *on* $B(0,\frac{5}{8}R)$ *and* $\psi_1=e^{i\theta_0}$ *on* $\mathbb{R}^N\setminus B(0,\frac{3}{4}R)$,

(ii) $\psi_2=\psi$ *on* $\mathbb{R}^N\setminus B(0,\frac{7}{8}R)$ *and* $\psi_2=e^{i\theta_0}=$ *constant on* $B(0,\frac{3}{4}R)$,

(iii) $\int_{\mathbb{R}^N}\Big|\Big|\frac{\partial\psi}{\partial x_j}\Big|^2-\Big|\frac{\partial\psi_1}{\partial x_j}\Big|^2-\Big|\frac{\partial\psi_2}{\partial x_j}\Big|^2\Big|\,\mathrm{d}x\leq C_1\varepsilon$ *for* $j=1,\dots,N$,

(iv) $|Q(\psi)-Q(\psi_1)-Q(\psi_2)|\leq C_3\varepsilon$,

(v) If assumptions (A1) and (A2) hold, then

$$\int_{\mathbb{R}^N}\Big|V(|\psi|^2)-V(|\psi_1|^2)-V(|\psi_2|^2)\Big|\,\mathrm{d}x\leq\begin{cases}C_4\varepsilon+C_5\sqrt{\varepsilon}\,(E_1(\psi))^{\frac{2^*-1}{2}} & \text{if } N\geq 3,\\ C_6\varepsilon+C_7\sqrt{\varepsilon}\,(E_1(\psi))^{p_0+1} & \text{if } N=2.\end{cases}$$

Using Lemma 28 with $\psi = \psi_n(\cdot + x_n)$, $R = t_n$, and $\varepsilon = h_n$, for all sufficiently large n we construct two functions $\psi_{n,1}, \psi_{n,2} \in \mathcal{E}$ such that $\psi_{n,1} = \psi_n$ on $B(x_n, \frac{5}{4}t_n)$ and $\psi_{n,1}$ is constant (of modulus 1) on $\mathbb{R}^N \setminus B(x_n, \frac{3}{4}t_n)$, $\psi_{n,2} = \psi_n$ on $\mathbb{R}^N \setminus B(x_n, \frac{7}{4}t_n)$, $\psi_{n,2}$ is constant on $B(x_n, \frac{3}{4}t_n)$, and all conclusions of Lemma 28 hold. Then we have $|\psi_n|^2\left(|\varphi_{n,1}|^2 + |\varphi_{n,2}|^2\right) = |\psi_{n,1}|^2|\varphi_{n,1}|^2 + |\psi_{n,2}|^2|\varphi_{n,2}|^2$ on $\mathbb{R}^N$ and using (3.45) we get

$$\int_{\mathbb{R}^N} \left| \, |\psi_n|^2|\varphi_n|^2 - |\psi_{n,1}|^2|\varphi_{n,1}|^2 - |\psi_{n,2}|^2|\varphi_{n,2}|^2 \right| \mathrm{d}x \leqslant h_n. \tag{3.46}$$

From Lemma 28 (iii)-(v) we infer that $E_1(\psi_{n,i})$ is bounded, and consequently $Q(\psi_{n,i})$ is bounded, and we have

$$E_1(\psi_n) = E_1(\psi_{n,1}) + E_1(\psi_{n,2}) + o(1) \text{ and } Q(\psi_n) = Q(\psi_{n,1}) + Q(\psi_{n,2}) + o(1) \text{ as } n \longrightarrow \infty. \tag{3.47}$$

Passing to a further subsequence (still denoted the same) we may assume that $P_\beta(\psi_{n,1}, \varphi_{n,1}) \longrightarrow p'$ and $P_\beta(\psi_{n,2}, \varphi_{n,2}) \longrightarrow p''$ as $n \longrightarrow \infty$. It is obvious that $p' + p'' = p$. By (3.47), (3.43) and (3.46) we get

$$E(\psi_n, \varphi_n)) = E(\psi_{n,1}, \varphi_{n,1}) + E(\psi_{n,2}, \varphi_{n,2}) + o(1).$$

Since $E(\psi, \varphi) \geqslant E_{\beta,min}(|P_\beta(\psi, \varphi)|)$ for any $\psi \in \mathcal{E}$ and any $\varphi \in H^1(\mathbb{R}^N)$, we obtain

$$\begin{aligned} E_{\beta,min}(p) &= \lim_{n\to\infty} E(\psi_n, \varphi_n)) \geqslant \liminf_{n\to\infty} E(\psi_{n,1}, \varphi_{n,1}) + \liminf_{n\to\infty} E(\psi_{n,2}, \varphi_{n,2}) \\ &\geqslant E_{\beta,min}(|p'|) + E_{\beta,min}(|p''|). \end{aligned} \tag{3.48}$$

Since $E_{\beta,min}$ is nonnegative and increasing on $[0, \infty)$, the above inequality implies that $|p'| \leqslant p$ and $|p''| \leqslant p$, and this implies that $p', p'' \in [0, p]$. If $p'' = 0$ we must have $p' = p$ and consequently $\liminf_{n\to\infty} E(\psi_{n,1}, \varphi_{n,1}) \geqslant E_{\beta,min}(p)$. On the other hand, by (3.47) we get $\limsup_{n\to\infty} E(\psi_{n,1}, \varphi_{n,1}) \leqslant \lim_{n\to\infty} E(\psi_n, \varphi_n) = E_{\beta,min}(p)$. Thus $E(\psi_{n,1}, \varphi_{n,1}) \longrightarrow E_{\beta,min}(p)$ and this implies that $E(\psi_{n,2}, \varphi_{n,2}) \longrightarrow 0$ as $n \to \infty$. Then Lemma 20 gives

$$\|\nabla\psi_{n,2}\|_{L^2} + \| \, |\psi_{n,2}| - 1\|_{L^2} + \|\varphi_{n,2}\|_{H^1(\mathbb{R}^N)} \longrightarrow 0$$

and we infer that $\int_{\mathbb{R}^N \setminus B(x_n,t_n)} f_n \, \mathrm{d}x \longrightarrow 0$ as $n \longrightarrow \infty$. On the other hand we have

$$\int_{\mathbb{R}^N \setminus B(x_n,t_n)} f_n \, \mathrm{d}x = \int_{\mathbb{R}^N} f_n \, \mathrm{d}x - \int_{B(x_n,t_n)} f_n \, \mathrm{d}x \longrightarrow \alpha_0 - \alpha,$$

a contradiction. Thus $p'' > 0$. Similarly we prove that $p' > 0$, thus $p', p'' \in (0, p)$. The concavity of $E_{\beta,min}$ (see Proposition 25 (iv)) implies that $E_{\beta,min}(p') \geqslant \frac{p'}{p} E_{\beta,min}(p)$ and $E_{\beta,min}(p'') \geqslant \frac{p''}{p} E_{\beta,min}(p)$ and equality may occur if and only if $E_{\beta,min}$ is linear on $[0, p]$. Summing the above inequalities and comparing to (3.48) we see that equality must occur, and thus $E_{\beta,min}$ must be linear on $[0, p]$. Since $\frac{E_{\beta,min}(s)}{s} \longrightarrow S_\beta$ as $s \longrightarrow 0+$ (see Proposition 25 (ii)), we infer that $E_{\beta,min}(s) = S_\beta s$ on $[0, p]$, thus $E_{\beta,min}(p) = S_\beta p$, contradicting the assumption $E_{\beta,min}(p) < S_\beta p$ in Theorem 26.

So far we have shown that $\alpha = \alpha_0$. Then it is standard to prove that there is a sequence $(x_n)_{n\geqslant 1} \subset \mathbb{R}^N$ such that for any $\varepsilon > 0$ there exits $R_\varepsilon > 0$ satisfying $\int_{\mathbb{R}^N \setminus B(x_n,\varepsilon)} f_n \, \mathrm{d}x < \varepsilon$ for all n sufficiently large. Let $\tilde{\psi}_n = \psi_n(\cdot + x_n)$, $\tilde{\varphi}_n = \varphi_n(\cdot + x_n)$ and $\tilde{f}_n = f_n(\cdot + x_n)$ (obviously, $\tilde{f}_n$ is the function associated to $(\tilde{\psi}_n, \tilde{\varphi}_n)$ by (3.17)). For all $\varepsilon > 0$ there are $R_\varepsilon > 0$ and $n_\varepsilon \in \mathbb{N}$ such that

$$\int_{\mathbb{R}^N \setminus B(0,R_\varepsilon)} \tilde{f}_n \, \mathrm{d}x < \varepsilon \qquad \text{for all } n \geqslant n_\varepsilon. \tag{3.49}$$

Clearly, $(\tilde{\varphi}_n)_{n\geqslant 1}$ is bounded in $H^1(\mathbb{R}^N)$, $(\nabla\tilde{\psi}_n)_{n\geq 1}$ is bounded in $L^2(\mathbb{R}^N)$ and $(\tilde{\psi}_n)_{n\geq 1}$ is bounded in $L^2(B(0,R))$ for any $R>0$. By classical compact embeddings in Sobolev spaces and a diagonal extraction argument, there exist functions $\psi \in H^1_{loc}(\mathbb{R}^N)$ such that $\nabla\psi \in L^2(\mathbb{R}^N)$, $\varphi \in H^1(\mathbb{R}^N)$, and a subsequence $(\tilde{\psi}_{n_k}, \tilde{\varphi}_{n_k})_{k\geq 1}$ satisfying

$$
\begin{array}{ll}
\tilde{\varphi}_{n_k} \rightharpoonup \varphi & \text{weakly in } H^1(\mathbb{R}^N),\\
\nabla\tilde{\psi}_{n_k} \rightharpoonup \nabla\psi & \text{weakly in } L^2(\mathbb{R}^N),\\
\tilde{\psi}_{n_k} \rightharpoonup \psi & \text{weakly in } H^1(B(0,R)) \text{ for all } R>0,\\
\tilde{\psi}_{n_k} \longrightarrow \psi \quad \text{and} \quad \tilde{\varphi}_{n_k} \longrightarrow \varphi & \text{strongly in } L^p(B(0,R)) \text{ for } R>0 \text{ and } p\in[1,2^*),\\
\tilde{\psi}_{n_k} \longrightarrow \psi \quad \text{and} \quad \tilde{\varphi}_{n_k} \longrightarrow \varphi & \text{almost everywhere on } \mathbb{R}^N.
\end{array} \tag{3.50}
$$

By weak convergence we get

$$
\int_{\mathbb{R}^N} |\nabla\psi|^2\,\mathrm{d}x \leq \liminf_{k\to\infty} \int_{\mathbb{R}^N} |\nabla\tilde{\psi}_{n_k}|^2\,\mathrm{d}x \quad \text{and} \quad \int_{\mathbb{R}^N} |\nabla\varphi|^2\,\mathrm{d}x \leq \liminf_{k\to\infty} \int_{\mathbb{R}^N} |\nabla\tilde{\varphi}_{n_k}|^2\,\mathrm{d}x. \tag{3.51}
$$

Using the a.e. convergence and Fatou's Lemma we infer that

$$
\int_{\mathbb{R}^N} V(|\psi|^2)\,\mathrm{d}x \leqslant \liminf_{k\to\infty} \int_{\mathbb{R}^N} V(|\tilde{\psi}_{n_k}|^2)\,\mathrm{d}x, \tag{3.52}
$$

$$
\int_{\mathbb{R}^N} (|\psi|-1)^2\,\mathrm{d}x \leqslant \liminf_{k\to\infty} \int_{\mathbb{R}^N} (|\tilde{\psi}_{n_k}|-1)^2\,\mathrm{d}x \quad \text{and} \tag{3.53}
$$

$$
\int_{\mathbb{R}^N} |\psi|^2|\varphi|^2\,\mathrm{d}x \leqslant \liminf_{k\to\infty} \int_{\mathbb{R}^N} |\tilde{\psi}_{n_k}|^2|\tilde{\varphi}_{n_k}|^2\,\mathrm{d}x. \tag{3.54}
$$

From (3.51)-(3.54) we get

$$
E(\psi,\varphi) \leqslant \liminf_{k\to\infty} E(\tilde{\psi}_{n_k}, \tilde{\varphi}_{n_k}) = E_{\beta,min}(p). \tag{3.55}
$$

Let us show that $\tilde{\varphi}_{n_k} \longrightarrow \varphi$ strongly in $L^2(\mathbb{R}^N)$. Fix $\varepsilon>0$. Then choose $R_\varepsilon>0$ such that (3.49) holds. Since $\tilde{\varphi}_{n_k} \longrightarrow \varphi$ a.e., by Fatou's Lemma we get

$$
\int_{\mathbb{R}^N\setminus B(0,R_\varepsilon)} |\varphi|^2\,\mathrm{d}x \leqslant \liminf_{k\longrightarrow\infty} \int_{\mathbb{R}^N\setminus B(0,R_\varepsilon)} |\tilde{\varphi}_{n_k}|^2\,\mathrm{d}x \leqslant \varepsilon.
$$

Since $\tilde{\varphi}_{n_k} \longrightarrow \varphi$ in $L^2(B(0,R_\varepsilon))$, there is $k_\varepsilon\in\mathbb{N}$ such that $\int_{B(0,R_\varepsilon)} |\tilde{\varphi}_{n_k}-\varphi|^2\,\mathrm{d}x \leqslant \varepsilon$ for all $k\geqslant k_\varepsilon$. Then

$$
\int_{\mathbb{R}^N} |\tilde{\varphi}_{n_k}-\varphi|^2\,\mathrm{d}x \leqslant \int_{B(0,R_\varepsilon)} |\tilde{\varphi}_{n_k}-\varphi|^2\,\mathrm{d}x + 2\int_{\mathbb{R}^N\setminus B(0,R_\varepsilon)} |\tilde{\varphi}_{n_k}|^2+|\varphi|^2\,\mathrm{d}x \leqslant 5\varepsilon \quad \text{for all } k\geqslant k_\varepsilon.
$$

Since $\varepsilon>0$ is arbitrary, we infer that $\|\tilde{\varphi}_{n_k}-\varphi\|_{L^2(\mathbb{R}^N)} \longrightarrow 0$.

The weak convergence $\frac{\partial\tilde{\varphi}_{n_k}}{\partial x_1} \rightharpoonup \frac{\partial\varphi}{\partial x_1}$ and the strong convergence $\tilde{\varphi}_{n_k} \longrightarrow \varphi$ in $L^2(\mathbb{R}^N)$ give

$$
Q(\tilde{\varphi}_{n_k}) \longrightarrow Q(\varphi). \tag{3.56}
$$

We need the following result, which is an immediate consequence of Lemmas 4.11 p. 182 and 4.12 p. 184 in [5].

Lemma 29. *([5]) Assume that $N \geq 2$ and assumptions* **(A1)** *and* **(A2)** *hold. Let $(\gamma_n)_{n\geq 1} \subset \mathcal{E}$ be a sequence satisfying:*

(a) $(E_1(\gamma_n))_{n\geq 1}$ is bounded and for any $\varepsilon > 0$ there are $R_\varepsilon > 0$ and $n_\varepsilon \in \mathbb{N}$ such that $\int_{\mathbb{R}^N \setminus B(0,R_\varepsilon)} |\nabla \gamma_n|^2 + \big||\gamma_n| - 1\big|^2 \,\mathrm{d}x < \varepsilon$ for any $n \geq n_\varepsilon$.

(b) There exists $\gamma \in \mathcal{E}$ such that $\nabla\gamma_n \rightharpoonup \nabla\gamma$ weakly in $L^2(\mathbb{R}^N)$, $\gamma_n \longrightarrow \gamma$ strongly in $L^2(B(0,R))$ for any $R > 0$, and $\gamma_n \longrightarrow \gamma$ a.e. on $\mathbb{R}^N$ as $n \longrightarrow \infty$.

Then $\| |\gamma_n| - |\gamma| \|_{L^2} \longrightarrow 0$, $\|V(|\gamma_n|^2) - V(|\gamma|^2)\|_{L^1} \longrightarrow 0$, and $Q(\gamma_n) \longrightarrow Q(\gamma)$ as $n \longrightarrow \infty$.

It follows from (3.49), (3.50) and Lemma 29 that $Q(\tilde{\psi}_{n_k}) \longrightarrow Q(\psi)$ as $k \longrightarrow \infty$. Together with (3.56), this gives $P_\beta(\tilde{\psi}_{n_k}, \tilde{\varphi}_{n_k}) \longrightarrow P_\beta(\psi, \varphi)$, hence $P_\beta(\psi, \varphi) = p$. Then we infer that $E(\psi, \varphi) \geqslant E_{\beta,min}(p)$. Comparing this to (3.51)-(3.55) we get $E(\psi, \varphi) = E_{\beta,min}(p)$, hence (ψ, φ) is a minimizer for $(\mathcal{P}_{\beta,p})$. Moreover, we have

$$\|\nabla\tilde{\psi}_{n_k}\|_{L^2}^2 \longrightarrow \|\nabla\psi\|_{L^2}^2, \qquad \|\nabla\tilde{\varphi}_{n_k}\|_{L^2}^2 \longrightarrow \|\nabla\varphi\|_{L^2}^2, \quad \text{and}$$

$$\int_{\mathbb{R}^N} |\tilde{\psi}_{n_k}|^2 |\tilde{\varphi}_{n_k}|^2 \,\mathrm{d}x \longrightarrow \int_{\mathbb{R}^N} |\psi|^2 |\varphi|^2 \,\mathrm{d}x \quad \text{as } k \longrightarrow \infty. \tag{3.57}$$

The weak convergence and the convergence of norms imply that $\nabla\tilde{\psi}_{n_k} \longrightarrow \nabla\psi$ and $\nabla\tilde{\varphi}_{n_k} \longrightarrow \nabla\varphi$ strongly in $L^2(\mathbb{R}^N)$.

Using (3.57), the fact that $\tilde{\psi}_{n_k}\tilde{\varphi}_{n_k} \longrightarrow \psi\varphi$ a.e. in $\mathbb{R}^N$ and Brezis-Lieb Lemma (see, e.g., Exercise 4.17 (3) p. 123 in [2]), we infer that $\tilde{\psi}_{n_k}\tilde{\varphi}_{n_k} \longrightarrow \psi\varphi$ strongly in $L^2(\mathbb{R}^N)$.

From Lemma 29 it follows that $\| |\tilde{\psi}_{n_k}| - |\psi| \|_{L^2} \longrightarrow 0$ and $\|V(|\tilde{\psi}_{n_k}|^2) - V(|\psi|^2)\|_{L^1(\mathbb{R}^N)} \longrightarrow 0$ as $k \longrightarrow \infty$ and the proof of Theorem 26 is complete. □

Proposition 30. *Let $N \geq 2$, $\beta \in (0,1)$ and $p > 0$. Assume that $V \geqslant 0$ on $[0,\infty)$ and $(\psi, \varphi) \in \mathcal{E} \times H^1(\mathbb{R}^N)$ is a solution of the minimization problem $(\mathcal{P}_{\beta,p})$. Then:*

(i) There is $c \in [d^+E_{\beta,min}(p), d^-E_{\beta,min}(p)]$ such that ψ and φ satisfy

$$\begin{cases} ic\beta \frac{\partial\psi}{\partial x_1} = -\Delta\psi - F(|\psi|^2)\psi + \frac{1}{\epsilon^4}|\varphi|^2\psi \\ \\ ic(1-\beta)\epsilon^2 q^2 \frac{\partial\varphi}{\partial x_1} = -\Delta\varphi + \frac{q^2}{\epsilon^2}|\psi|^2\varphi \end{cases} \quad \text{in } \mathcal{D}'(\mathbb{R}^N). \tag{3.58}$$

(ii) For any $p > 0$ such that $E_{\beta,min}(p) < S_\beta p$ there are $(\psi^+, \varphi^+), (\psi^-, \varphi^-) \in \mathcal{E} \times H^1(\mathbb{R}^N)$ solutions of the minimization problem $(\mathcal{P}_{\beta,p})$ that satisfy (3.58) with speeds $c^+ = d^+E_{\beta,min}(p)$ and $c^- = d^-E_{\beta,min}(p)$, respectively.

If $N \geqslant 3$ we assume, in addition, that **(A3)** *holds. Then:*

(iii) Any solution $(\psi, \varphi) \in \mathcal{E} \times H^1(\mathbb{R}^N)$ of (3.58) satisfies $\psi \in W^{2,p}_{loc}(\mathbb{R}^N)$, $\nabla\psi \in W^{1,p}(\mathbb{R}^N)$, and $\varphi \in W^{2,p}(\mathbb{R}^N)$ for any $p \in [2,\infty)$. Moreover, ψ and $\nabla\psi$ are bounded and $\psi, \varphi \in C^{1,\alpha}(\mathbb{R}^N)$ for any $\alpha \in [0,1)$.

If F is C^∞, then φ and $\nabla\psi$ belong to $W^{k,p}(\mathbb{R}^N)$ for any $k \in \mathbb{N}$ and any $p \in [2,\infty)$. In particular, ψ and φ are C^∞ and bounded on $\mathbb{R}^N$.

(iv) After a translation, the pair (ψ, φ) is axially symmetric with respect to the x_1-axis if $N \geq 3$. The same conclusion holds for $N = 2$ if we assume in addition that F is C^1.

Proof. The proof of assertions (i), (ii) and (iv) is very similar to the proof of Proposition 4.14 (i), (iv) and (iii) p.187 in [5], so we omit it. We only sketch the proof of (iii). Denoting $\tilde{\psi} = e^{i\frac{c\beta}{2}x_1}\psi$ and $\tilde{\varphi} = e^{i\frac{c(1-\beta)\epsilon^2q^2}{2}x_1}\varphi$ it is easily seen that $\tilde{\psi},\ \tilde{\varphi}$ satisfy the system

$$\left\{\begin{array}{l} \Delta\tilde{\psi} + \frac{(c\beta)^2}{4}\tilde{\psi} + F(|\tilde{\psi}|^2)\tilde{\psi} - \frac{1}{\epsilon^4}|\tilde{\varphi}|^2\tilde{\psi} = 0, \\ \\ \Delta\tilde{\varphi} + \frac{[c(1-\beta)\epsilon^2q^2]^2}{4}\tilde{\varphi} - \frac{q^2}{\epsilon^2}|\tilde{\psi}|^2\tilde{\varphi} = 0 \end{array}\right. \qquad \text{in } \mathcal{D}'(\mathbb{R}^N). \tag{3.59}$$

It is clear that $\tilde{\varphi} \in H^1(\mathbb{R}^N)$ and $\tilde{\psi} \in H^1_{loc}(\mathbb{R}^N)$. If $N = 2$, a standard bootstrap argument gives the desired regularity result. If $N \geqslant 4$, bootstrap doesn't work anymore because of the terms $|\tilde{\varphi}|^2\tilde{\psi}$ and $|\tilde{\psi}|^2\tilde{\varphi}$ even if the nonlinearity F is subcritical (that is, assumption **(A2)** is satisfied). In the case $N = 3$, the standard nonlinearity $F(s) = \frac{1}{\epsilon^2}(1-s)$ appearing in (GC) becomes critical and prevents bootstrap to work. If $N \geqslant 3$, we use assumption **(A3)** and Proposition 2.2 (i) p. 1078 in [9] and we infer that $\psi \in L^\infty(\mathbb{R}^N)$. (Notice that the proof in [9] is based on an inequality of Kato and on previous work by Farina [6].) Then the second equation in (3.58) and a classical bootstrap argument give $\varphi \in W^{2,p}(\mathbb{R}^N)$ for all $p \in [2,\infty)$. Now the first equation in (3.58) and a bootstrap give the desired result. See Proposition 4.6 (i) p. 1097 in [9] for a complete proof in the case $F(s) = \frac{1}{\epsilon^2}(1-s)$; that proof easily adapts to general nonlinearities. □

Lemma 31. *Assume that $\psi \in \mathcal{E}$ and $\varphi \in H^1(\mathbb{R}^N)$ satisfy the second equation in (3.58). Then we have*

$$Q(\varphi) = \frac{1}{2}c(1-\beta)\epsilon^2q^2\int_{\mathbb{R}^N}|\varphi|^2\,\mathrm{d}x. \tag{3.60}$$

Proof. Formally we obtain (3.60) by multiplying the second equation in (3.58) by $ix_1\varphi$ and integrating by parts. The computation can be made rigorous by taking $\chi \in C^\infty_c(\mathbb{R}^N)$ such that $\chi = 1$ on $B(0,1)$, multiplying the second equation satisfied in (3.58) by $\chi\left(\frac{x}{R}\right)(ix_1\varphi)$, integrating by parts, then letting $R \longrightarrow \infty$. □

3.4 Minimization of the energy at fixed mass and momentum

In this section we will minimize $E(\psi,\varphi)$ when the momentum of ψ and the L^2-norm of φ are fixed. More precisely, we consider the problem

$$(\mathscr{E}_{p,m}) \qquad \text{minimize } E(\psi,\varphi) \text{ for } \psi \in \mathcal{E},\ \varphi \in H^1(\mathbb{R}^N) \text{ satisfying } Q(\psi) = p \text{ and } \int_{\mathbb{R}^N}|\varphi|^2\,\mathrm{d}x = m.$$

For $p \in \mathbb{R}$ and $m \geqslant 0$, let

$$E_{min}(p,m) = \inf\left\{E(\psi,\varphi) \;\middle|\; \psi \in \mathcal{E},\ \varphi \in H^1(\mathbb{R}^N),\ Q(\psi) = p, \int_{\mathbb{R}^N}|\varphi|^2\,\mathrm{d}x = m\right\}. \tag{3.61}$$

Recall that

$$E_{1,min}(q) = \inf\{E_1(\psi) \mid \psi \in \mathcal{E},\ Q(\psi) = q\} \tag{3.62}$$

and the main properties of the function $E_{1,min}$ are given in Theorem 14. It is obvious that

$$E_{min}(q,m) \geqslant E_{1,min}(q) \qquad \text{for any } q, m \geqslant 0.$$

Proposition 32. *The function E_{min} has the following properties:*

(i) $E_{min}(p,m) = E_{min}(-p,m)$ for any $p \in \mathbb{R}$ and any $m \geqslant 0$.

(ii) $E_{min}(p,m)$ is finite and continuous on $\mathbb{R} \times [0,\infty)$, and for all $p \in \mathbb{R}$ and $m \geqslant 0$ we have $E_{min}(p,0) = E_{1,min}(|p|)$, $E_{min}(0,m) = g_{min}(m)$, and

$$\max(E_{1,min}(|p|), g_{min}(m)) \leqslant E_{min}(p,m) \leqslant E_{1,min}(|p|) + g_{min}(m). \tag{3.63}$$

(iii) E_{min} is sub-additive: $E_{min}(p_1+p_2, m_1+m_2) \leqslant E_{min}(p_1,m_1)+E_{min}(p_2,m_2)$ for all p_1, p_2, m_1, m_2.

(iv) For any $p > 0$ and $m \geqslant 0$, $E_{min}(p,m)$ is equal to

$$\inf\left\{ p^{\frac{N-2}{N-1}} \int_{\mathbb{R}^N} |\nabla\psi|^2 \,\mathrm{d}x + p^{\frac{N}{N-1}} \int_{\mathbb{R}^N} V(|\psi|^2)\,\mathrm{d}x + \frac{m p^{-\frac{2}{N-1}}}{\epsilon^2 q^2} \int_{\mathbb{R}^N} |\nabla\varphi|^2 \,\mathrm{d}x + \frac{m}{\epsilon^4} \int_{\mathbb{R}^N} |\psi|^2|\varphi|^2 \,\mathrm{d}x \right.$$

$$\left. \Big| \ \psi \in \mathcal{E}, Q(\psi) = 1, \varphi \in H^1(\mathbb{R}^N), \|\varphi\|_{L^2} = 1 \right\}.$$

(v) For any fixed p_0 the mapping $m \longmapsto E_{min}(p_0,m)$ is concave and increasing on $[0,\infty)$.

(vi) If $N \geqslant 3$, for any pair $(p_0,m_0) \neq (0,0)$, $m_0 \geqslant 0$, the mapping $t \longmapsto E_{min}(tp_0, tm_0)$ is concave and increasing on $[0,\infty)$.

(vii) Assume that $p_1, p_2 \in \mathbb{R}$ and $m_1, m_2 \geqslant 0$ are such that

$$E_{min}(p_1,m_1) + E_{min}(p_2,m_2) = E_{min}(p_1+p_2, m_1+m_2). \tag{3.64}$$

Then we have either

$$\begin{aligned} &E_{min}(p_1,0) + E_{min}(p_2, m_1+m_2) = E_{min}(p_1+p_2, m_1+m_2), \qquad \text{or} \\ &E_{min}(p_1, m_1+m_2) + E_{min}(p_2,0) = E_{min}(p_1+p_2, m_1+m_2). \end{aligned} \tag{3.65}$$

Proof. (i) For $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^N$, denote $x' = (x_2, \ldots, x_N) \in \mathbb{R}^{N-1}$. Given any $\psi \in \mathcal{E}$ and $\varphi \in H^1(\mathbb{R}^N)$, let $\tilde\psi(x) = \psi(-x_1, x')$ and $\tilde\varphi(x) = \varphi(-x_1, x')$. It is obvious that $Q(\tilde\psi) = -Q(\psi)$, $\|\tilde\varphi\|_{L^2} = \|\varphi\|_{L^2}$ and $E(\tilde\psi,\tilde\varphi) = E(\psi,\varphi)$, and this implies (i).

From the definition of $E_{1,min}$ and of g_{min} it is clear that

$$E_{min}(p,m) \geqslant E_{1,min}(p), \quad E_{min}(p,m) \geqslant g_{min}(m), \quad \text{and} \quad E_{min}(p,0) = E_{1,min}(p).$$

In particular, we have $E_{min}(0,m) \geqslant g_{min}(m)$ and the first inequality in (3.63) holds. On the other hand, if (ψ_n, φ_n) is a minimizing sequence for $g_{min}(m)$, then $(|\psi_n|, \varphi_n)$ is another minimizing sequence and $Q(|\psi_n|) = 0$. Thus $g_{min}(m) = E_{min}(0,m)$. The second inequality in (3.63) follows from the above and the sub-additivity of E_{min} (see part (iii)).

Fix $q_0 > 0$ and $m_0 > 0$. Let $\psi \in \mathcal{E}$ and $\varphi \in H^1(\mathbb{R}^N)$ be any functions satisfying $Q(\psi) = q_0$ and $\int_{\mathbb{R}^N} |\varphi|^2 \,\mathrm{d}x = m_0$. Let $a,b,c > 0$. Denote $\tilde\psi(x) = \psi(\frac{x_1}{a}, \frac{x'}{b})$ and $\tilde\varphi(x) = c\varphi(\frac{x_1}{a}, \frac{x'}{b})$. An easy computation gives $Q(\tilde\psi) = b^{N-1}Q(\psi)$, $\int_{\mathbb{R}^N} |\tilde\varphi|^2 \,\mathrm{d}x = ab^{N-1}c^2 \int_{\mathbb{R}^N} |\varphi|^2 \,\mathrm{d}x$ and

$$\begin{aligned} E(\tilde\psi,\tilde\varphi) \ &= \ \int_{\mathbb{R}^N} \tfrac{b^{N-1}}{a} \left|\tfrac{\partial\psi}{\partial x_1}\right|^2 + ab^{N-3}|\nabla^\perp\psi|^2 + \tfrac{b^{N-1}c^2}{a\epsilon^2q^2}\left|\tfrac{\partial\varphi}{\partial x_1}\right|^2 + \tfrac{ab^{N-3}c^2}{\epsilon^2q^2}|\nabla^\perp\varphi|^2 \\ &\quad + ab^{N-1}V(|\psi|^2) + \frac{ab^{N-1}c^2}{\epsilon^4}|\psi|^2|\varphi|^2 \,\mathrm{d}x. \end{aligned} \tag{3.66}$$

Let $q > 0$ and $m > 0$. Choose b and c such that $b^{N-1} = \frac{q}{q_0}$ and $ac^2b^{N-1} = \frac{m}{m_0}$. We get $Q(\tilde{\psi}) = q$ and $\int_{\mathbb{R}^N} |\tilde{\varphi}|^2 \, \mathrm{d}x = m$, hence

$$E_{min}(q,m) \leqslant E(\tilde{\psi}, \tilde{\varphi}) \tag{3.67}$$

From (3.66) and (3.67) it is not hard to see that $(q,m) \longmapsto E_{min}(q,m)$ is continuous in the region $(0,\infty)^2$. From (3.63) and the fact that $g_{min}(m) \longrightarrow 0$ as $m \longrightarrow 0$ and $E_{1,min}(p) \longrightarrow 0$ as $p \longrightarrow 0$ we infer that E_{min} is continuous on $\mathbb{R} \times [0,\infty)$.

(iii) Proceeding as in the proof of Theorem 26 and using Lemma 28, it is easy to show that for any given $\psi \in \mathcal{E}$, $\varphi \in H^1(\mathbb{R}^N)$ and $\varepsilon > 0$, there exist $\psi^\sharp \in \mathcal{E}$ and $\varphi^\sharp \in H^1(\mathbb{R}^N)$ such that $\psi^\sharp = 1$ and $\varphi^\sharp = 0$ outside a large ball $B(0,R)$, $|Q(\psi) - Q(\psi^\sharp)| < \varepsilon$, $\|\varphi - \varphi^\sharp\|_{H^1} < \varepsilon$ and $|E(\psi,\varphi) - E(\psi^\sharp,\varphi^\sharp)| < \varepsilon$ (see also Corollary 3.4 p. 169 in [5]).

Then the sub-additivity follows form a classical argument of P.-L. Lions [8]. Given $p_1, p_2 \in \mathbb{R}$, $m_1, m_2 \geqslant 0$ and $\varepsilon > 0$, approximating "almost minimizers" by functions that are constant outside a ball and eventually performing a scaling, we see that there are $(\psi_1,\varphi_1), (\psi_2,\varphi_2) \in \mathcal{E} \times H^1(\mathbb{R}^N)$ and $R > 0$ such that $(\psi_i,\varphi_i) = (1,0)$ on $\mathbb{R}^N \setminus B(0,R)$, $Q(\psi_i) = p_i$, $\|\varphi_i\|_{L^2}^2 = m_i$ and $E(\psi_i,\varphi_i) < E_{min}(p_i,m_i) + \varepsilon$. Take $x_0 \in \mathbb{R}^N$ such that $|x_0| \geqslant 3R$ and define $(\psi,\varphi) = (\psi_1,\varphi_1)$ on $B(0,R)$, $(\psi,\varphi) = (\psi_2,\varphi_2)(\cdot - x_0)$ on $B(x_0,R)$, and $(\psi,\varphi) = (1,0)$ on $\mathbb{R}^N \setminus (B(0,R) \cup B(x_0,R))$. It is then obvious that $Q(\psi) = p_1 + p_2$, $\|\varphi\|_{L^2}^2 = m_1 + m_2$ and

$$E(\psi,\varphi) = E(\psi_1,\varphi_1) + E(\psi_2,\varphi_2) \leqslant E_{min}(p_1,m_1) + E_{min}(p_2,m_2) + 2\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, (iii) follows.

Using (3.66) with $a = b = p^{\frac{1}{N-1}}$ and $c^2 = mp^{-\frac{N}{N-1}}$ we get (iv).

Fix $p_0 > 0$. Let $a = b = 1$ and $c = m^{\frac{1}{2}}$. Using (3.66) and (3.67) we see that

$$\begin{aligned} E_{min}(p_0,m) = \inf \Big\{ \int_{\mathbb{R}^N} |\nabla\psi|^2 + V(|\psi|^2) + \frac{m}{\epsilon^2 q^2}|\nabla\varphi|^2 + \frac{m}{\epsilon^4}|\psi|^2|\varphi|^2 \, \mathrm{d}x \ \Big| \\ \psi \in \mathcal{E}, \varphi \in H^1(\mathbb{R}^N), Q(\psi) = p_0, \textstyle\int_{\mathbb{R}^N} |\varphi|^2 \, \mathrm{d}x = 1 \Big\}. \end{aligned} \tag{3.68}$$

For any $\alpha, \beta \in \mathbb{R}$ the function $m \longmapsto \alpha + \beta m$ is affine, hence concave. The infimum of a family of concave nondecreasing functions is a concave nondecreasing function. We infer that for any fixed $p_0 > 0$ the mapping $m \longmapsto E_{min}(p_0,m)$ is concave and nondecreasing on $(0,\infty)$. By (3.63) it tends to ∞ as $m \longrightarrow \infty$, and since it is concave we infer that it is strictly increasing.

(vi) Assume that $N \geqslant 3$. Fix p_0 and m_0. Let $a = c = 1$ and $b = t^{\frac{1}{N-1}}$, where $t > 0$. From (3.66) we get

$$\begin{aligned} &E_{min}(tp_0, tm_0) \\ &= \inf \Big\{ \int_{\mathbb{R}^N} t\Big|\frac{\partial\psi}{\partial x_1}\Big|^2 + t^{\frac{N-3}{N-1}}|\nabla^\perp\psi|^2 + \frac{t}{\epsilon^2 q^2}\Big|\frac{\partial\varphi}{\partial x_1}\Big|^2 + \frac{t^{\frac{N-3}{N-1}}}{\epsilon^2 q^2}|\nabla^\perp\varphi|^2 + tV(|\psi|^2) + \frac{t}{\epsilon^4}|\psi|^2|\varphi|^2 \, \mathrm{d}x \\ &\qquad \psi \in \mathcal{E}, \varphi \in H^1(\mathbb{R}^N), Q(\psi) = p_0, \textstyle\int_{\mathbb{R}^N} |\varphi|^2 \, \mathrm{d}x = m_0 \Big\}. \end{aligned} \tag{3.69}$$

As previously, $t \longmapsto E_{min}(tp_0,tm_0)$ is the infimum of a family of concave nondecreasing functions, hence it is a concave nondecreasing function. By (3.63) it tends to ∞ as $t \longrightarrow \infty$, thus it must be strictly increasing.

(vii) The mapping $m \longmapsto E_{min}(q_1,m) + E_{min}(q_2, m_1 + m_2 - m)$ is concave on $(0, m_1 + m_2)$. Since E_{min} is sub-additive and (3.64) holds, this mapping reaches its minimum on $[m_1, m_2]$ at $m = m_1$. We

infer that either $m_1 = 0$, or $m_1 = m_1 + m_2$ (which means that $m_2 = 0$), or this mapping is constant on $(0, m_1 + m_2)$, and (vii) follows. □

We are able to prove the following result:

Theorem 33. *Assume that $N \geqslant 2$, assumptions* **(A1)** *and* **(A2)** *hold and $V \geqslant 0$. Assume that the pair (p, m) satisfies the following strict sub-additivity condition:*

$$E_{1,min}(p') + E_{min}(p - p', m) > E_{min}(p, m) \text{ for any } p' \in \mathbb{R}^*. \tag{3.70}$$

Then the minimization problem $(\mathscr{E}_{p,m})$ admits solutions.

Moreover, any sequence $(\psi_n, \varphi_n) \in \mathcal{E} \times H^1(\mathbb{R}^N)$ satisfying $Q(\psi_n) \longrightarrow p$, $\int_{\mathbb{R}^N} |\varphi_n|^2 \,\mathrm{d}x \longrightarrow m$ and $E(\psi_n, \varphi_n) \longrightarrow E_{min}(p, m)$ has a subsequence $(\psi_{n_k}, \varphi_{n_k})_{k\geq 1}$ with the following property: there are a sequence of points $(x_k)_{k\geq 1} \subset \mathbb{R}^N$, $\psi \in \mathcal{E}$ and $\varphi \in H^1(\mathbb{R}^N)$ such that $Q(\psi) = p$, $\|\varphi\|^2_{L^2(\mathbb{R}^N)} = m$, $E(\psi, \varphi) = E_{min}(p, m)$, and

$$\varphi_{n_k} \longrightarrow \varphi \ \text{ in } \ H^1(\mathbb{R}^N), \qquad (\psi_{n_k}, \varphi_{n_k})(\cdot + x_k) \longrightarrow (\psi, \varphi) \ \text{ a.e. in } \mathbb{R}^N,$$

$$\|\nabla\psi_{n_k}(\cdot + x_k) - \nabla\psi\|_{L^2} \longrightarrow 0, \qquad \| \,|\psi_{n_k}|(\cdot + x_k) - |\psi| \,\|_{L^2} \longrightarrow 0,$$

$$\psi_{n_k}\varphi_{n_k}(\cdot + x_k) \longrightarrow \psi\varphi \ \text{ in } L^2(\mathbb{R}^N), \quad V\left(|\psi_{n_k}(\cdot + x_k)|^2\right) \longrightarrow V(|\psi|^2) \text{ in } L^1(\mathbb{R}^N).$$

Proof. Let $(\psi_n, \varphi_n)_{n\geqslant 1}$ be as in Theorem 33. Then $E_1(\psi_n)$ and $\|\varphi_n\|_{H^1}$ are bounded. Lemma 20 and the Sobolev embedding imply that $\| \,|\psi_n| - 1\|_{L^2}$ and $\| \,|\psi_n| - 1\|_{L^{p_0+1}}$ are bounded. Let f_n be as in (3.17). Then the sequence $(f_n)_{n\geqslant 1}$ is bounded in $L^1(\mathbb{R}^N)$ and $\liminf\limits_{n\to\infty} \int_{\mathbb{R}^N} f_n \,\mathrm{d}x \geqslant \lim\limits_{n\to\infty} \int_{\mathbb{R}^N} |\varphi_n|^2 \,\mathrm{d}x = m$. Passing to a subsequence we may assume that $\int_{\mathbb{R}^N} f_n \,\mathrm{d}x \longrightarrow \alpha_0 > 0$ as $n \longrightarrow \infty$.

We proceed as in the proofs of Theorems 41 and 26 and we use the concentration-compactness principle for the sequence $(f_n)_{n\geqslant 1}$. Let Λ_n be the concentration function of f_n, as given by (3.18). Proceeding as in [8] and using (5.12) p. 156 in [11], we infer that there exist a non-decreasing function $\Lambda : [0, \infty) \longrightarrow [0, \infty)$, $\alpha \in [0, \alpha_0]$ and a sequence $t_n \longrightarrow \infty$ such that (3.19) and (3.20) hold. As previously, we rule out the possibilities $\alpha = 0$ ("vanishing") and $\alpha \in (0, \alpha_0)$ ("dichotomy") in order to show that $\alpha = \alpha_0$.

If $\alpha = 0$, by Lemma 21 (ii) we get $\liminf\limits_{n\to\infty} \int_{\mathbb{R}^N} |\psi_n|^2 |\varphi_n|^2 \,\mathrm{d}x \geqslant m$ and Lemma 27 gives $\liminf\limits_{n\to\infty} E_1(\psi_n) \geqslant v_s p$, hence

$$E_{min}(p, m) = \liminf_{n\to\infty} E(\psi_n, \varphi_n) \geqslant v_s p + \frac{m}{\epsilon^4} \geqslant E_{1,min}(p) + g_{min}(m).$$

The above inequality contradicts (3.70). Indeed, for $p' = p$ assumption (3.70) implies that $E_{1,min}(p) + g_{min}(m) > E_{min}(p, m)$ if $p \neq 0$ and $m \neq 0$.

If $\alpha \in (0, \alpha_0)$, proceeding as in the proof of Theorem 26 we construct two sequences $(\psi_{n,1}, \varphi_{n,1})$ and $(\psi_{n,2}, \varphi_{n,2})$ satisfying (3.43) - (3.47) there. We use the same notation as in the proof of Theorem 26. Since $E_1(\psi_{n,i})$ is bounded, we infer that $Q(\psi_{n,i})$ is bounded and passing to a further subsequence we may assume that $Q(\psi_{n,1}) \longrightarrow p_1$ and $Q(\psi_{n,2}) \longrightarrow p_2$ as $n \longrightarrow \infty$. By (3.47) we have $p_1 + p_2 = p$. It is clear that

$$\int_{\mathbb{R}^N} \left| \,|\varphi_n|^2 - |\varphi_{n,1}|^2 - |\varphi_{n,1}|^2\right| \mathrm{d}x \leqslant \int_{\Omega_n} |\varphi_n|^2 \,\mathrm{d}x \leqslant h_n \longrightarrow 0 \quad \text{ as } n \longrightarrow \infty,$$

and passing to a further subsequence we may assume that $\int_{\mathbb{R}^N} |\varphi_{n,i}|^2 \,\mathrm{d}x \longrightarrow m_i$ as $n \longrightarrow \infty$, where $m_1 + m_2 = m$. We have

$$E(\psi_{n,i}, \varphi_{n,i}) \geqslant E_{min}(Q(\psi_{n,i}), \|\varphi_{n,i}\|^2_{L^2})$$

and letting $n \longrightarrow \infty$ we get $\liminf_{n\to\infty} E(\psi_{n,i}, \varphi_{n,i}) \geqslant E_{min}(p_i, m_i)$ for $i = 1, 2$. Using (3.47) we obtain

$$E_{min}(p,m) = \lim_{n\to\infty} E(\psi_n, \varphi_n) \geqslant \sum_{i=1}^{2} \liminf_{n\to\infty} E(\psi_{n,i}, \varphi_{n,i}) \geqslant E_{min}(p_1, m_1) + E_{min}(p_2, m_2).$$

By the sub-additivity of E_{min} (see Proposition 32 (ii)) we must have equality in the above inequality, and then Proposition 32 (vii) implies that (3.65) holds. Assumption (3.70) implies then that either $p_1 = 0$, or $p_2 = 0$.

If $p_1 = 0$, we have $p_2 = p$ and we get

$$E_{min}(0, m_1) + E_{min}(p, m_2) = E_{min}(p, m).$$

By Proposition 32 (iii) and (v), the mapping $\tau \longmapsto E_{min}(0,\tau) + E_{min}(p, m-\tau)$ is concave on $[0, m]$ and is greater than or equal to $E_{min}(p, m)$. If it reaches its minimum at an interior point $m_1 \in (0, m)$, it must be constant on that interval and we infer that

$$E_{min}(0,m) + E_{min}(p,0) = E_{min}(0,m_1) + E_{min}(p, m-m_1) = E_{min}(p,m)$$

contradicting (3.70). The same holds if $m_1 = m$. We infer that necessarily $m_1 = 0$.

Thus if $p_1 = 0$ we must have $m_1 = 0$ and this implies $m_2 = m$, $p_2 = p$. We infer that $\liminf_{n\to\infty} E_(\psi_{n,2}, \varphi_{n,2}) \geqslant E_{min}(p,m)$. Since $E(\psi_n, \varphi_n) = E_(\psi_{n,1}, \varphi_{n,1}) + E_(\psi_{n,2}, \varphi_{n,2})$ and $E(\psi_n, \varphi_n) \longrightarrow E_{min}(p,m)$, we infer that $E_(\psi_{n,1}, \varphi_{n,1}) \longrightarrow 0$ as $n \longrightarrow \infty$. Since $\|\varphi_{n,1}\|_{L^2}^2 \longrightarrow m_1 = 0$, using Lemma 20 we find $\|\varphi_{n,1}\|_{H^1} \longrightarrow 0$, $\|\nabla\psi_{n,1}\|_{L^2} \longrightarrow 0$ and $\| \, |\psi_{n,1}| - 1\|_{L^2} \longrightarrow 0$ as $n \longrightarrow \infty$, and we deduce that $\int_{B(x_n,t_n)} f_n \, dx \longrightarrow 0$, hence $\alpha = 0$, contradicting the fact that $\alpha \in (0, \alpha_0)$. Similarly, if $p_2 = 0$ we get $\alpha_0 - \alpha = 0$, again a contradiction.

We have thus proved that $\alpha = \alpha_0$. By a standard argument we see that there is a sequence $(x_n)_{n\geqslant 1} \subset \mathbb{R}^N$ such that for any $\varepsilon > 0$ there exits $R_\varepsilon > 0$ satisfying $\int_{\mathbb{R}^N \setminus B(x_n,\varepsilon)} f_n \, \mathrm{d}x < \varepsilon$ for all n sufficiently large. Denoting $\tilde{\psi}_n = \psi_n(\cdot + x_n)$, $\tilde{\varphi}_n = \varphi_n(\cdot + x_n)$ and $\tilde{f}_n = f_n(\cdot + x_n)$, we see that (3.49) holds. Then we deduce that there exist a subsequence $(\tilde{\psi}_{n_k}, \tilde{\varphi}_{n_k})_{k\geqslant 1}$, and functions $\psi \in H^1_{loc}(\mathbb{R}^N)$ and $\varphi \in H^1(\mathbb{R}^N)$ such that (3.50) holds. Then (3.51) - (3.54) also hold and we infer that $\psi \in \mathcal{E}$ and

$$E(\psi, \varphi) \leqslant \liminf_{k\to\infty} E(\tilde{\psi}_{n_k}, \tilde{\varphi}_{n_k}) = E_{min}(p,m).$$

On the other hand, as in the proof of Theorem 26 we see that $\|\tilde{\varphi}_{n_k} - \varphi\|_{L^2} \longrightarrow 0$, hence $\|\varphi\|_{L^2}^2 = m$. Lemma 29 implies that $Q(\psi) = \lim_{n\to\infty} Q(\tilde{\psi}_{n_k}) = p$. Thus we have $E(\psi,\varphi) \geqslant E_{min}(p,m)$, and the inequality must be an equality.

For the rest of the proof we proceed exactly as in the proof of Theorem 26. We deduce strong convergence from the weak convergence and the convergence of norms, which follow from (3.51) - (3.54) and the fact that $E(\psi,\varphi) = E_{min}(p,m)$. □

We were not able to rigorously prove (3.70). We suspect that this strict sub-additivity condition is true for all $p > p_0$ and $m > m_0$. We were only able to check (3.70) numerically in some physically relevant situations.

Remark 34. If (ψ,φ) is a minimizer for $E_{min}(p,m)$, there exist $\alpha \in \mathbb{R}$ and a real-valued function φ_0 such that $\varphi = e^{i\alpha}\varphi_0$ (for otherwise, $(\psi, |\varphi|)$ would do better than (ψ,φ)).

Remark 35. If (ψ,φ) solves the minimization problem $(\mathscr{E}_{p,m})$, it is standard to see that there exist Lagrange multipliers $\lambda_1, \lambda_2 \in \mathbb{R}$ such that

$$\left\{ \begin{array}{ll} i\lambda_1 \frac{\partial\psi}{\partial x_1} = -\Delta\psi - F(|\psi|^2)\psi + \frac{1}{\epsilon^4}|\varphi|^2\psi & \\ & \text{in } \mathcal{D}'(\mathbb{R}^N). \\ i\lambda_2\epsilon^2 q^2 \frac{\partial\varphi}{\partial x_1} = -\Delta\varphi + \frac{q^2}{\epsilon^2}|\psi|^2\varphi & \end{array} \right. \tag{3.71}$$

Moreover, λ_2 is between the right and left derivatives at m of the mapping $E_{min}(p,\cdot)$.

If assumption **(A3)** in the Introduction holds, (ψ,φ) satisfies the conclusion of Proposition 30 (iii). In particular, we have $\psi,\varphi\in C^{1,\alpha}(\mathbb{R}^N)$ for any $\alpha\in[0,1)$.

Let $e_1=(1,0,\dots,0)\in\mathbb{R}^N$. If $N\geqslant 4$, Theorem 2' p. 329 in [10] implies that there exists $v\in\mathbb{R}^N$ orthogonal to e_1 such that after translation, (ψ,φ) is cylindrically symmetric around $\mathrm{Span}(e_1,v)$. The general results in [10] do not imply that (ψ,φ) is axially symmetric, although that might be the case (for instance, when $(\psi,e^{-iax_1}\varphi)$ solves a minimization problem $(\mathcal{P}_{\beta,p})$ - see Proposition 36).

Proposition 36. *(i) Assume that (ψ,φ) is a minimizer for the problem $(\mathcal{P}_{\beta,p})$, as given by Theorem 26, and that it solves (3.58) for some $c>0$. Denote $p_1=Q(\psi)$, $m=\|\varphi\|_{L^2}^2$, and $a=\frac{1}{2}c(1-\beta)\epsilon^2q^2$. Then $(\psi,e^{iax_1}\varphi)$ solves the minimization problem $(\mathscr{E}_{p_1,m})$.*

Moreover, if (ψ_1,φ_1) is any solution of the problem $(\mathscr{E}_{p_1,m})$, then $(\psi_1,e^{-iax_1}\varphi_1)$ is a solution of the minimization problem $(\mathcal{P}_{\beta,p})$.

(ii) For any $p\in\mathbb{R}$, $a\in\mathbb{R}$ and $m\geqslant 0$ we have

$$E_{\beta,min}(\beta p+(1-\beta)am)\leqslant E_{min}(p,m)+a^2m. \tag{3.72}$$

Proof. (i) Let $p_1=Q(\psi)$ and $p_2=Q(\varphi)$, so that $\beta p_1+(1-\beta)p_2=p$. By Lemma 31 we have $Q(\varphi)=\frac{1}{2}c(1-\beta)\epsilon^2q^2\int_{\mathbb{R}^N}|\varphi|^2\,\mathrm{d}x$, that is $p_2=am$.

Take any $\tilde\psi\in\mathcal{E}$ such that $Q(\tilde\psi)=p_1$ and any $\tilde\varphi\in H^1(\mathbb{R}^N)$ such that $\|\tilde\varphi\|_{L^2}^2=m$. Let $\varphi^\sharp(x)=e^{-iax_1}|\tilde\varphi(x)|$. We have $Q(|\tilde\varphi|)=0$ because $|\tilde\varphi|$ is real-valued and using (3.36) we get $Q(\varphi^\sharp)=am=p_2$, hence $P_\beta(\tilde\psi,\varphi^\sharp)=\beta p_1+(1-\beta)p_2=p$. We infer that $E(\tilde\psi,\varphi^\sharp)\geqslant E_{\beta,min}(p)=E(\psi,\varphi)$. We have $|\tilde\varphi|=e^{iax_1}\varphi^\sharp$, and using (3.36) again we obtain

$$E(\tilde\psi,\tilde\varphi)\geqslant E(\tilde\psi,|\tilde\varphi|)=E(\tilde\psi,\varphi^\sharp)+\frac{1}{\epsilon^2q^2}\Big(-2aQ(\varphi^\sharp)+a^2m\Big)=E(\tilde\psi,\varphi^\sharp)-\frac{a^2m}{\epsilon^2q^2}$$

and

$$E(\psi,e^{iax_1}\varphi)=E(\psi,\varphi)+\frac{1}{\epsilon^2q^2}\big(-2aQ(\varphi)+a^2m\big)=E(\psi,\varphi)-\frac{a^2m}{\epsilon^2q^2}.$$

We conclude that $E(\psi,e^{iax_1}\varphi)\leqslant E(\tilde\psi,\tilde\varphi)$. Since this is true for any $\tilde\psi$ and any $\tilde\varphi$ as above, we infer that $E(\psi,e^{iax_1}\varphi)=E_{min}(p_1,m)$ and $(\psi,e^{iax_1}\varphi)$ is a minimizer for $E_{min}(p_1,m)$.

The second assertion is obvious because $Q(\psi_1)=p_1=Q(\psi)$, $Q(e^{-iax_1}\varphi_1)=am=p_2=Q(\varphi)$ (here we use the fact that $Q(\varphi_1)=0$ and (3.36)), and

$$E(\psi_1,e^{-iax_1}\varphi_1)=E(\psi_1,\varphi_1)+\frac{a^2m}{\epsilon^2q^2}=E_{min}(p_1,m)+\frac{a^2m}{\epsilon^2q^2}=E(\psi,\varphi).$$

(ii) Consider $\psi\in\mathcal{E}$ such that $Q(\psi)=p$ and $\varphi\in H^1(\mathbb{R}^N)$ such that $\|\varphi\|_{L^2}^2=m$. For $a\in\mathbb{R}$, we have $Q(e^{-iax_1}|\varphi|)=am$ and $P_\beta(\psi,e^{-iax_1}|\varphi|)=\beta p+(1-\beta)am$, hence

$$E_{\beta,min}(\beta p+(1-\beta)am)\leqslant E(\psi,e^{-iax_1}|\varphi|)=E(\psi,|\varphi|)+a^2m.$$

Passing to the infimum we get (3.72). □

Bibliography

[1] Alhelou, J. (2021) *On the Cauchy problem for a Gross-Pitaevskii system,* preprint.

[2] Brezis, H. (2011)*Functional Analysis, Sobolev spaces and partial differential equations,* Springer, Universitext.

[3] Brezis, H. & Lieb, E.H (1984) *Minimum action solutions of some vector field equations,* Commun. Math. Phys. 96, pp. 97-113

[4] Brothers, J.E & Ziemer, W.P. (1988) *Minimal rearrangements of Sobolev functions,* J. reine angew. Math. 384, 153-179.

[5] Chiron, D. & Mariş, M. (2017) *Traveling waves for nonlinear Schrödinger equations with nonzero conditions at infinity,* Arch. Rational Mech. Anal. 226, pp. 143-242.

[6] Farina, A. (2003) *From Ginzburg-Landau to Gross-Pitaevskii,* Monatsh. Math., 139, pp. 265-269.

[7] Lieb, E.H. (1983) *On the lowest eigenvalue of the Laplacian for the intersection of two domains,* Inventiones Mathematicae 74, pp. 441-448.

[8] Lions, P.L. (1984) *The concentration-compactness principle in the calculus of variations. The locally compact case, part I,* Ann. Inst. H. Poincaré, Anal. non linéaire 1, pp. 109-145.

[9] Mariş, M. (2008) *Nonexistence of supersonic traveling waves for nonlinear Schrödinger equations with nonzero conditions at infinity,* SIAM J. Math. Anal. 40 (3), pp. 1076-1103.

[10] Mariş, M. (2009) *On the symmetry of minimizers,* Arch. Rational Mech. Anal. 192 (2), pp. 311-330.

[11] Mariş, M. (2013) *Traveling waves for nonlinear Schrödinger equations with nonzero conditions at infinity,* Ann. of Math. 178 (1), pp. 107-182.

Chapter 4

Travelling waves of small mass to the Gross-Clark system

4.1 Introduction

In this chapter, we will present some numerical results. More precisely, we will approximate the small mass solution of our 2 dimensional system using Newton-Raphson algorithm. We recall that the system is given by

$$\text{(TW)} \qquad \begin{cases} -ic\frac{\partial \psi}{\partial x_1} & = -\Delta\psi + \frac{1}{\varepsilon^2}(\frac{1}{\varepsilon^2}|\varphi|^2 + |\psi|^2 - 1)\psi \\ (-\Delta + \frac{q^2}{\varepsilon^2}|\psi|^2)\varphi & = \lambda\varphi, \end{cases}$$

with the boundary conditions $|\psi| \to 1$ and $\varphi \to 0$ as $|x| \to \infty$. Moreover we fix the L^2-norm of φ: $\int_{\mathbb{R}^2} |\varphi|^2 \mathrm{d}x = \mathfrak{m}$.
Here λ is unknown and will be approximated numerically. These solutions are close to the following approximations: ψ is almost the product of the two Padés approximant for the vortices of the Gross-Pitaevskii equation, and φ is a sum of two functions of small mass concentrated near the vortices of ψ. These results are obtained for several values of the speed c. We are interested also in the energy-momentum diagram for the speeds c for which Newton-Raphson algorithm converges.

4.2 The numerical method

4.2.1 Change of variables

We will look for solutions that respect two symmetries of the problem: ψ is thus assumed to satisfy

$$\psi(x) = \psi(x_1, x_2) = \psi(x_1, -x_2) = \bar{\psi}(-x_1, x_2).$$

Moreover, we can see in [5] that, up to a phase shift, φ is a real function, then it is assumed to satisfy the following symmetries:

$$\varphi(x) = \varphi(x_1, x_2) = \varphi(x_1, -x_2) = \varphi(-x_1, x_2).$$

This allows us to work on the domain $\mathbb{R}_+ \times \mathbb{R}_+$ instead of $\mathbb{R}^2$. Then using the stretched variables used by Jones and Roberts in [4] and by Chiron and Scheid in [2],

$$R_1 x_1 = \tan(\hat{x}_1), \quad R_2 x_2 = \tan(\hat{x}_2),$$

where R_1 and $R_2 > 0$ have to be fixed and are adapted to the lengthscales of the solution, we can choose to work in the bounded domain $[0, \frac{\pi}{2}]^2$. This choice avoids to consider artificial type of boundary

conditions.

Since

$$\frac{\partial h}{\partial x} = R\cos^2(\hat{x})\frac{\partial \hat{h}}{\partial \hat{x}}, \quad \text{and} \quad \frac{\partial^2 h}{\partial x^2} = R^2\big(\cos^4(\hat{x})\frac{\partial^2 \hat{h}}{\partial \hat{x}^2} - 2\sin(\hat{x})\cos^3(\hat{x})\frac{\partial \hat{h}}{\partial \hat{x}}\big),$$

for any function h of the variable x with the change of variables $\hat{x} = \arctan(Rx)$, and setting $\hat{\psi}(\hat{x}) = \psi(x)$, we can rewrite the system (TW) in these variables:

$$(4.1)\qquad \begin{cases} -icR_1\cos^2(\hat{x_1})\frac{\partial \hat{\psi}}{\partial \hat{x}_1} & +R_1^2\big(\cos^4(\hat{x}_1)\frac{\partial^2 \hat{\psi}}{\partial \hat{x}_1^2} - 2\sin(\hat{x}_1)\cos^3(\hat{x}_1)\frac{\partial \hat{\psi}}{\partial \hat{x}_1}\big) \\ & +R_2^2\big(\cos^4(\hat{x}_2)\frac{\partial^2 \hat{\psi}}{\partial \hat{x}_2^2} - 2\sin(\hat{x}_2)\cos^3(\hat{x}_2)\frac{\partial \hat{\psi}}{\partial \hat{x}_2}\big) \\ & -\frac{1}{\varepsilon^2}(\frac{1}{\varepsilon^2}|\hat{\varphi}|^2 + |\hat{\psi}|^2 - 1)\hat{\psi} = 0 \\ -R_1^2\big(\cos^4(\hat{x}_1)\frac{\partial^2 \hat{\varphi}}{\partial \hat{x}_1^2} & -2\sin(\hat{x}_1)\cos^3(\hat{x}_1)\frac{\partial \hat{\varphi}}{\partial \hat{x}_1}\big) \\ & -R_2^2\big(\cos^4(\hat{x}_2)\frac{\partial^2 \hat{\varphi}}{\partial \hat{x}_2^2} - 2\sin(\hat{x}_2)\cos^3(\hat{x}_2)\frac{\partial \hat{\varphi}}{\partial \hat{x}_2}\big) \\ & +\frac{q^2}{\varepsilon^2}|\hat{\psi}|^2\hat{\varphi} - \lambda\hat{\varphi} = 0. \end{cases}$$

4.2.2 Discretization

As in [2] and [4], we discretize the computational domain, the square $[0, \frac{\pi}{2}]^2$, by a cartesian grid, with N_{x_1} points in the direction x_1 and N_{x_2} points in the direction x_2. For simplicity, we will work in a uniform discretization and choose '$\tilde{N} = N_{x_1} = N_{x_2}$. The size of the mesh is denoted by $\mathrm{d}x = \frac{\pi}{2(\tilde{N}-1)}$. We also choose to work in a Finite Difference framework, using central approximations of derivatives that are of order 2.

4.2.3 The equations

We choose to write $\hat{\psi}$ as $\hat{a} + i\hat{b}$, so we can work with real quantities. Dividing the first equation of (4.1) into the real part and the imaginary one, it becomes:

$$(4.2)\qquad \begin{cases} cR_1\cos^2(\hat{x}_1)\frac{\partial \hat{b}}{\partial \hat{x}_1} & +R_1^2\big(\cos^4(\hat{x}_1)\frac{\partial^2 \hat{a}}{\partial \hat{x}_1^2} - 2\sin(\hat{x}_1)\cos^3(\hat{x}_1)\frac{\partial \hat{a}}{\partial \hat{x}_1}\big) \\ & +R_2^2\big(\cos^4(\hat{x}_2)\frac{\partial^2 \hat{a}}{\partial \hat{x}_2^2} - 2\sin(\hat{x}_2)\cos^3(\hat{x}_2)\frac{\partial \hat{a}}{\partial \hat{x}_2}\big) \\ & -\frac{1}{\varepsilon^4}|\hat{\varphi}|^2\hat{a} - \frac{1}{\varepsilon^2}\hat{a}^3 - \frac{1}{\varepsilon^2}\hat{a}\hat{b}^2 + \frac{1}{\varepsilon^2}\hat{a} = 0 \\ -cR_1\cos^2(\hat{x}_1)\frac{\partial \hat{a}}{\partial \hat{x}_1} & +R_1^2\big(\cos^4(\hat{x}_1)\frac{\partial^2 \hat{b}}{\partial \hat{x}_1^2} - 2\sin(\hat{x}_1)\cos^3(\hat{x}_1)\frac{\partial \hat{b}}{\partial \hat{x}_1}\big) \\ & +R_2^2\big(\cos^4(\hat{x}_2)\frac{\partial^2 \hat{b}}{\partial \hat{x}_2^2} - 2\sin(\hat{x}_2)\cos^3(\hat{x}_2)\frac{\partial \hat{b}}{\partial \hat{x}_2}\big) \\ & -\frac{1}{\varepsilon^4}|\hat{\varphi}|^2\hat{b} - \frac{1}{\varepsilon^2}\hat{a}^2\hat{b} - \frac{1}{\varepsilon^2}\hat{b}^3 + \frac{1}{\varepsilon^2}\hat{b} = 0 \\ -R_1^2\big(\cos^4(\hat{x}_1)\frac{\partial^2 \hat{\varphi}}{\partial \hat{x}_1^2} & -2\sin(\hat{x}_1)\cos^3(\hat{x}_1)\frac{\partial \hat{\varphi}}{\partial \hat{x}_1}\big) \\ & -R_2^2\big(\cos^4(\hat{x}_2)\frac{\partial^2 \hat{\varphi}}{\partial \hat{x}_2^2} - 2\sin(\hat{x}_2)\cos^3(\hat{x}_2)\frac{\partial \hat{\varphi}}{\partial \hat{x}_2}\big) \\ & +\frac{q^2}{\varepsilon^2}(\hat{a}^2 + \hat{b}^2)\hat{\varphi} - \lambda\hat{\varphi} = 0. \end{cases}$$

For $i = 0, \ldots, N-1$ and $j = 0, \ldots, N-1$, we denote by $\hat{a}_{i,j}$, $\hat{b}_{i,j}$, and $\hat{\varphi}_{i,j}$ the values of $\hat{a}$, $\hat{b}$, and $\hat{\varphi}$ on the points $(\hat{x}_1, \hat{x}_2) = (i\mathrm{d}x, j\mathrm{d}x)$. We will suppose that φ tends to 0 and ψ tends to a constant of modulus 1 when $|x| \to \infty$, which is not proved for the system, but for the Gross-Pitaevskii equation. Hence on the lines $\hat{x}_1 = \frac{\pi}{2}$ and $\hat{x}_2 = \frac{\pi}{2}$, the values of $\hat{a}$, $\hat{b}$ and $\hat{\varphi}$ are known: $\hat{a} = 1$ and $\hat{b} = 0$, $\hat{\varphi} = 0$. On

the line $\hat{x}_1 = 0$, we have $\hat{b} = 0$, because of the symmetry. In other words, on the line $\hat{x}_1 = 0$, we have $(N-1)$ points: $(0,0), \cdots, (0,(N-2)\mathrm{d}x)$ and in each of them we have two unknowns: $\hat{a}$ and $\hat{\varphi}$. For the other points of the grid, excluding the axis $\hat{x}_1 = \frac{\pi}{2}$, $\hat{x}_2 = \frac{\pi}{2}$, and $\hat{x}_1 = 0$, we have $(N-2)(N-1)$ points containing 3 unknowns for each. Hence, the number of the unknowns in our system is equal to $2(\tilde{N}-1) + 3(\tilde{N}-2)(\tilde{N}-1) = (\tilde{N}-1)(3\tilde{N}-4)$.
Using the central approximations of the derivatives, we can write the equations of (4.2) on every point of the grid, excluding the axis $\hat{x}_1 = \frac{\pi}{2}$, $\hat{x}_2 = \frac{\pi}{2}$ and without the second equation for the points of the line $\hat{x}_1 = 0$. These equations can be written using the symmetries mentioned in the previous section, $\hat{a}(\hat{x}_1, -\hat{x}_2) = \hat{a}(\hat{x}_1, \hat{x}_2)$, $\hat{b}(\hat{x}_1, -\hat{x}_2) = \hat{b}(\hat{x}_1, \hat{x}_2)$, $\hat{a}(-\hat{x}_1, \hat{x}_2) = \hat{a}(\hat{x}_1, \hat{x}_2)$, $\hat{b}(-\hat{x}_1, \hat{x}_2) = -\hat{b}(\hat{x}_1, \hat{x}_2)$, $\hat{\varphi}(\hat{x}_1, -\hat{x}_2) = \hat{\varphi}(\hat{x}_1, \hat{x}_2)$, and $\hat{\varphi}(-\hat{x}_1, \hat{x}_2) = \hat{\varphi}(\hat{x}_1, \hat{x}_2)$, for the critical points on the axis $\hat{x}_1 = 0$, and $\hat{x}_2 = 0$. Hence, the number of the equations is equal to the number of the unknowns.
To control the mass of φ, we will add one more equation, which is the norm L^2 of φ. It is an integral quantity that have to be approximated numerically using a trapezoidal rule. Adding an equation leads us to add an unknown for the system, to have equality between the number of the equations and the one of the unknowns. Since λ is an unknown quantity of the system, we conclude, the number of equations=the number of unknowns=$(\tilde{N}-1)(3\tilde{N}-4)+1$.

4.2.4 Choice of the initializations

For the initialization of ψ, we will choose a function that looks like a product of 2 vortices, one of degree 1 situated at $(0, \frac{1}{c})$ and the other of degree -1 situated at $(0, -\frac{1}{c})$. Hence

$$\psi(x) \approx V_1\Big(\frac{1}{\varepsilon}\big(x - (0, \frac{1}{c})\big)\Big) \times V_{-1}\Big(\frac{1}{\varepsilon}\big(x + (0, \frac{1}{c})\big)\Big).$$

In polar coordinates, we have

$$V_1(x) = \tilde{a}(r)\exp(i\theta) = \bar{V}_{-1}(x)$$

where $\tilde{a} : \mathbb{R}_+ \to \mathbb{R}$ is an increasing function from 0 to 1. Then, we follow the strategy of [2] and choose a Padé approximant. Therefore, we look for an initialization under the form

$$\mathbf{a}_{\text{Padé}}\big(\frac{1}{\varepsilon}|(x_1, x_2 - c^{-1})|\big)\frac{x_1 + i(x_2 - c^{-1})}{|(x_1, x_2 - c^{-1})|} \times \mathbf{a}_{\text{Padé}}\big(\frac{1}{\varepsilon}|(x_1, x_2 + c^{-1})|\big)\frac{x_1 - i(x_2 + c^{-1})}{|(x_1, x_2 + c^{-1})|}, \tag{4.3}$$

where $\mathbf{a}_{\text{Padé}}$ is given by

$$\mathbf{a}_{\text{Padé}}(r) = r\sqrt{\frac{\alpha_1 + \alpha_2 r^2}{1 + \beta_1 r^2 + \beta_2 r^4}}, \tag{4.4}$$

for some coefficients α_1, α_2, β_1, β_2 to be chosen as in [1] or [3], such that $\beta_2 = \alpha_2$ in order to have $\mathbf{a}_{\text{Padé}}(+\infty) = 1$. This Padé is an approximate solution of the ODE

$$a'' + \frac{a'}{r} - \frac{1}{r^2}a + a(1 - a^2) = 0.$$

This initialization was used to approximate the travelling waves of the Gross-Pitaevskii equation satisfying:

$$-ic\frac{\partial \psi}{\partial x_1} = -\Delta\psi + \frac{1}{\varepsilon^2}(|\psi|^2 - 1)\psi. \tag{4.5}$$

To construct the initialization of φ, we consider that the impurity is concentrated near the vortices $(0, \frac{1}{c})$ and $(0, -\frac{1}{c})$. Indeed, in the energy, if ψ is the fixed function of two vortices (4.4), we can see from the kinetic energy for φ and the interaction term of ψ and φ with the mass constraint, that φ

should be put where ψ is small. We look at what happens near one of these vortices. Let us consider V_1. We are brought to the equation

$$-\varepsilon^2\Delta\varphi_0 + q^2|V_1(\frac{x}{\varepsilon})|^2\varphi_0 = \lambda\varepsilon^2\varphi_0, \tag{4.6}$$

setting $\lambda_\varepsilon = \lambda\varepsilon^2$, we see that λ_ε is the smallest eigenvalue of the operator $-\varepsilon^2\Delta + q^2|V_1(\frac{x}{\varepsilon})|^2$. φ_0 is approximated by a Gaussian function of the form

$$\varphi_0(x) = \vartheta \exp\big(\frac{-|x|^2}{2\varepsilon^2\sigma^2}\big),$$

where ϑ is adapted such that the mass of φ_0 is fixed equal to $\frac{\mathfrak{m}}{2}$, in order to have 2 gaussians of mass $\frac{\mathfrak{m}}{2}$ each one. σ is given below.
Our problem is radial, because $(|V_1|^2(x) = \tilde{a}^2(r))$, and we search for a radial φ_0. We choose to approximate φ by the Gaussian function

$$\varphi(x) = \alpha\Big(\exp\big(\frac{-x_1^2 - (x_2 - \frac{1}{c})^2}{2\varepsilon^2\sigma^2}\big) + \exp\big(\frac{-x_1^2 - (x_2 + \frac{1}{c})^2}{2\varepsilon^2\sigma^2}\big)\Big), \tag{4.7}$$

where the coefficient α is calculated using the mass constraint denoted by $\mathfrak{m}$ and given by the formula

$$\mathfrak{m} = \int_{\mathbb{R}^2} \varphi(x)^2 \mathrm{d}x \approx 2\alpha^2 \times 2\pi \times \frac{\sigma^2\varepsilon^2}{2}. \tag{4.8}$$

To introduce these initializations in the code, we are meant to consider the change of variables $u = \frac{1}{R_1\varepsilon}\tan(x_1) = \frac{\hat{x}_1}{\varepsilon}$, $v = \frac{1}{R_2\varepsilon}\tan(x_2) = \frac{\hat{x}_2}{\varepsilon}$, so that the vortices are located in $(0, \frac{1}{c\varepsilon})$ and $(0, -\frac{1}{c\varepsilon})$, in order to work in the stretched variables, and consider the approximation of the solution of the Gross-Pitaevskii equation with the coefficient $\frac{1}{\varepsilon^2}$ in front of the nonlinearity.

We could find some numerical approximations for the eigenvalue λ_ε, and calculate the profile $\tilde{a}$ of the vortex V_1 by a Newton method, starting from the Padé (4.4). After calculating the matrix of the operator $-\varepsilon^2\Delta + q^2\tilde{a}^2(\frac{r}{\varepsilon})$ in radial coordinates, we search for the smallest eigenvalue, using the command "spec" of Scilab and can plot the associated eigenvector φ_0 and fit a Gaussian approximation. The value of σ is $\sqrt{71.219}$, and the graph of φ_0 is given in Figure 4.3 in section 4.3.

4.2.5 Choice of the parameters

We start testing the convergence of our algorithm with $c = 0.2$ for the system (4.2). As in [2], we choose $R_1 = R_2 = 0.2$. This choice can influence the precision of the numerical computations. In fact, the uniform grid in the mapped domain $[0, \frac{\pi}{2}]^2$, is transformed in a non-uniform one in the real domain $(\mathbb{R}^+)^2$. Approaching to infinity, the mesh is dilated and the cells become bigger. In [5], we can see that the physical parameters q and ε can be chosen as

$$q = 0.41, \tag{4.9}$$

$$\varepsilon = 0.187. \tag{4.10}$$

In order to have a better approximation of the vortices, we choose $\tilde{N}$ such that we have enough points in the area where the modulus of the vortices varies. Taking $\tilde{N} \geq 42$, we have almost 5 points over a length of $\varepsilon = 0.187$. Indeed, in this case we have $\frac{5\pi}{2\tilde{N}} \leq \varepsilon$. We fix $\tilde{N} = 60$. For the mass $\mathfrak{m}$ of the function φ, refering to the theoretical results, we have to choose it small enough. We set $\mathfrak{m} = 0.1$. Since we are working on $[0, \frac{\pi}{2}]^2$ that correspond to $(\mathbb{R}^+)^2$, the mass over this domain is in fact $\frac{\mathfrak{m}}{4} = 0.025$.

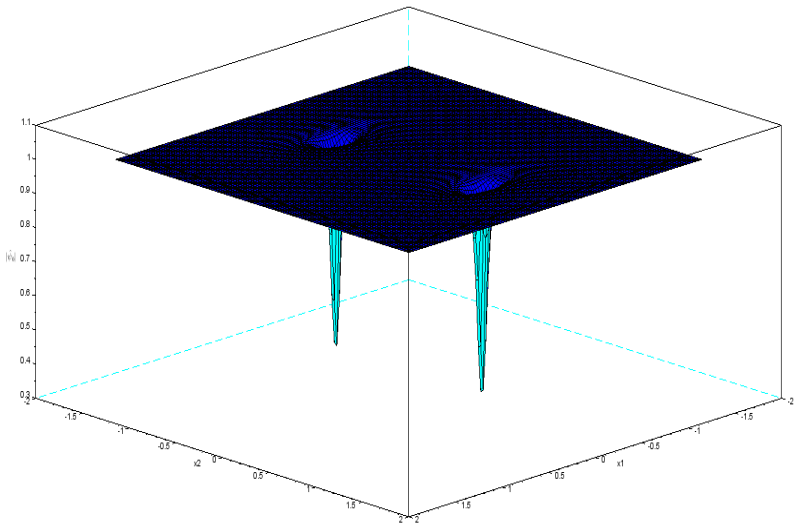

Figure 4.1: The initial data of ψ in the stretched variables for $c = 0.2$

It remains to choose the parameters α_1, α_2, β_1, and β_2 for the initialization of ψ. Two choices are possible: either the Padé found by Berloff in [1] and given by

$$\alpha_1 = 0.3447, \qquad \alpha_2 = 0.0286, \qquad \beta_1 = 0.3333, \qquad \beta_2 = \alpha_2,$$

or, the Padé used in [3], given by

$$\alpha_1 = 0.3350601, \qquad \alpha_2 = 0.0494196, \qquad \beta_1 = 0.3725704, \qquad \beta_2 = \alpha_2.$$

After several tests, it turns out that the choice of the second one leads to the convergence of the algorithm for more values of c. Hence, we choose the second option.

4.3 The results

From numerical approxiations of the eigenvalue, we could find that the initialization of λ_ε must be 0.165 and the variance of the Gauss function is $\sigma^2 = 71.219$. In other words, the code will start by initializing λ with $\frac{0.165399322252}{\varepsilon^2}$ and φ as in (4.7), with $\alpha = \frac{1}{\varepsilon}\sqrt{\frac{\mathrm{m}}{2\sigma^2\pi}}$. Now we can plot these data (see Figure 4.1 and Figure 4.2). For the initialization of λ, we will present the eigenvector φ_0 associated to this eigenvalue in Figure 4.3.

Because of the condition number of the matrix, the Newton-Raphson method diverges on the full coupled system with small values of ε. It converges however very well for $0.8 \leq \varepsilon \leq 1$. Hence, we had to work on separated iterations. In other words, we consider the following algorithm:
Step 1:
Discretize in the stretched variables and divide into real and imaginary part the first equation of the

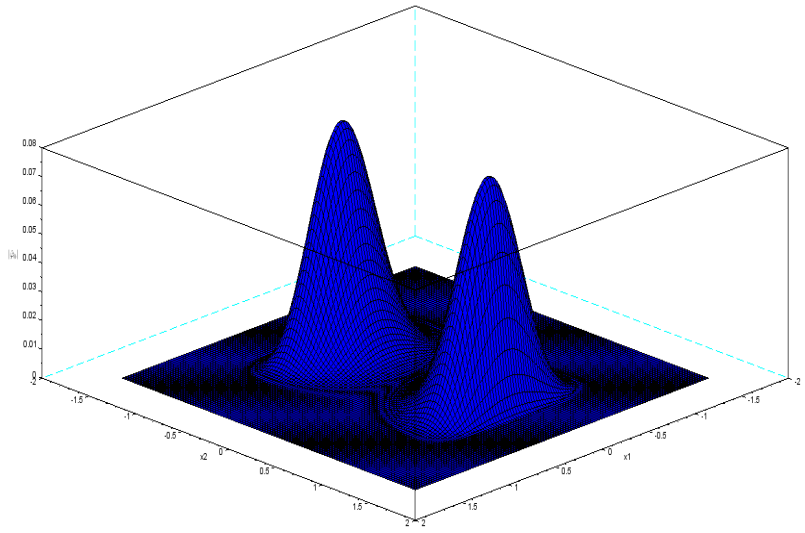

Figure 4.2: The initial data of φ in the stretched variables for $c = 0.2$

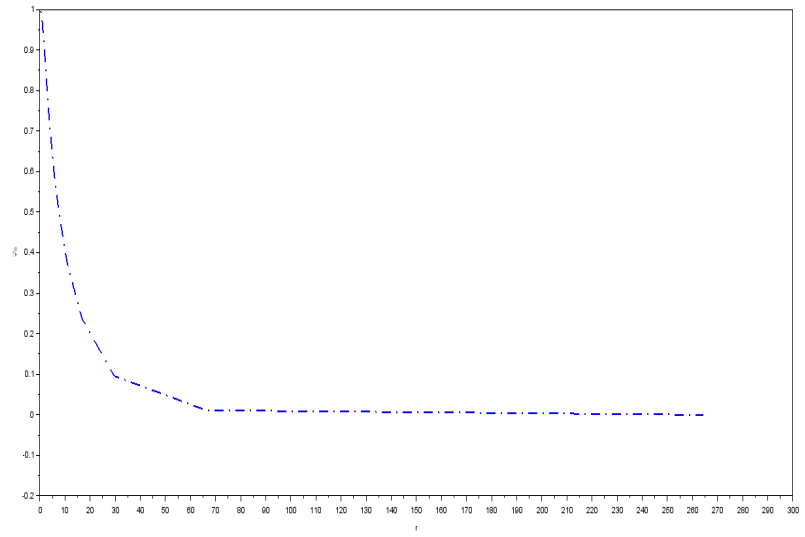

Figure 4.3: The eigenvector φ_0 for the equation (4.6)

system (TW) to get the equations

$$
\begin{cases}
cR_1\cos^2(\hat{x}_1)\frac{\partial\hat{b}}{\partial\hat{x}_1} & +R_1^2\big(\cos^4(\hat{x}_1)\frac{\partial^2\hat{a}}{\partial\hat{x}_1^2}-2\sin(\hat{x}_1)\cos^3(\hat{x}_1)\frac{\partial\hat{a}}{\partial\hat{x}_1}\big)\\
& +R_2^2\big(\cos^4(\hat{x}_2)\frac{\partial^2\hat{a}}{\partial\hat{x}_2^2}-2\sin(\hat{x}_2)\cos^3(\hat{x}_2)\frac{\partial\hat{a}}{\partial\hat{x}_2}\big)\\
& -\frac{1}{\varepsilon^4}|\hat{\varphi}|^2\hat{a}-\frac{1}{\varepsilon^2}\hat{a}^3-\frac{1}{\varepsilon^2}\hat{a}\hat{b}^2+\frac{1}{\varepsilon^2}\hat{a}=0\\
-cR_1\cos^2(\hat{x}_1)\frac{\partial\hat{a}}{\partial\hat{x}_1} & +R_1^2\big(\cos^4(\hat{x}_1)\frac{\partial^2\hat{b}}{\partial\hat{x}_1^2}-2\sin(\hat{x}_1)\cos^3(\hat{x}_1)\frac{\partial\hat{b}}{\partial\hat{x}_1}\big)\\
& +R_2^2\big(\cos^4(\hat{x}_2)\frac{\partial^2\hat{b}}{\partial\hat{x}_2^2}-2\sin(\hat{x}_2)\cos^3(\hat{x}_2)\frac{\partial\hat{b}}{\partial\hat{x}_2}\big)\\
& -\frac{1}{\varepsilon^4}|\hat{\varphi}|^2\hat{b}-\frac{1}{\varepsilon^2}\hat{a}^2\hat{b}-\frac{1}{\varepsilon^2}\hat{b}^3+\frac{1}{\varepsilon^2}\hat{b}=0
\end{cases}
\tag{4.11}
$$

Step 2:
Fix $\hat{\varphi}$ in (4.11) as in (4.7) and solve in $\hat{a}$ and $\hat{b}$ using Newton-Raphson algorithm and starting from the initial data of $\hat{\psi}$ given by (4.3), to get an approximate solution of $\hat{\psi}$ denoted by $\hat{\psi}_1$.
Step 3:
Discretize in the stretched variables the second equation of the system (TW) to get

$$
\begin{aligned}
-R_1^2\big(\cos^4(\hat{x}_1)\frac{\partial^2\hat{\varphi}}{\partial\hat{x}_1^2}-2\sin(\hat{x}_1)\cos^3(\hat{x}_1)\frac{\partial\hat{\varphi}}{\partial\hat{x}_1}\big)-R_2^2\big(\cos^4(\hat{x}_2)\frac{\partial^2\hat{\varphi}}{\partial\hat{x}_2^2}\\
-2\sin(\hat{x}_2)\cos^3(\hat{x}_2)\frac{\partial\hat{\varphi}}{\partial\hat{x}_2}\big)+\frac{q^2}{\varepsilon^2}(\hat{a}^2+\hat{b}^2)\hat{\varphi}-\lambda\hat{\varphi}=0.
\end{aligned}
\tag{4.12}
$$

together with the trapezoidal approximation of the L^2 norm of $\hat{\varphi}$.
Step 4:
Fix $\hat{\psi}=\hat{\psi}_1$ in (4.12), and solve in $\hat{\varphi}$ and λ using Newton-Raphson algorithm and starting from the initial data of $\hat{\varphi}$ and λ given by (4.7) and 4.7298842 respectively. We get an approximate solution denoted by $\hat{\varphi}_1$.
Step 5:
Repeat *Step 2* but with $\hat{\varphi}$ fixed as $\hat{\varphi}_1$, and get an approximate solution of $\hat{\psi}$ denoted by $\hat{\psi}_2$.
Step 6:
Repeat *Step 4* but with $\hat{\psi}=\hat{\psi}_2$, and get a solution $\hat{\varphi}_2$.
We keep repeating *Step 2* and *Step 4* with $\hat{\varphi}_2,\hat{\varphi}_3,\dots$ and $\hat{\psi}_2,\hat{\psi}_3,\dots$ until we reach a relative error of order 10^{-3} between two consecutive solutions $\hat{\psi}_i,\hat{\psi}_{i+1}$ and $\hat{\varphi}_i,\hat{\varphi}_{i+1}$. In this case, the separated iterations method converges and gives the following results: $\lambda\approx 4.1808086$. The modulus of the approximate solutions $\hat{\psi}$ and $\hat{\varphi}$ are represented in Figure 4.4 and Figure 4.5.

In the physical variables, i.e. in $\mathbb{R}^2$, the graphics of $|\psi|$ and $|\varphi|$ are represented in Figure 4.6 and Figure 4.7. We choose also to represent the contour lines of these solutions in a 2D plot, as in Figure 4.8 and Figure 4.9, where we can clearly see that the vortices are radial. To be able to read the values of the countour lines, we changed the scale on the axis x_2 (see Figure 4.10 and Figure 4.11).

In order to have a better look on the vortices and the peak of the Gauss function, Figure 4.12 and Figure 4.13 present the modulus of the obtained solutions in the half-plane $x_1\geq 0$.

4.4 The energy-momentum diagram

The separated iterations method converges for several values of the speed c. For some c's, this method diverges, i.e. Newton diverges at some point while solving either in ψ or in (φ,λ). When we have convergence, we notice that when c is small, the vortices are well separated from each other, and we calculate the energy $E(\psi,\varphi)$, and the momentum $Q(\psi)$ of the solution, using the trapezoidal rule for the integral. The results are represented in Table 4.1:

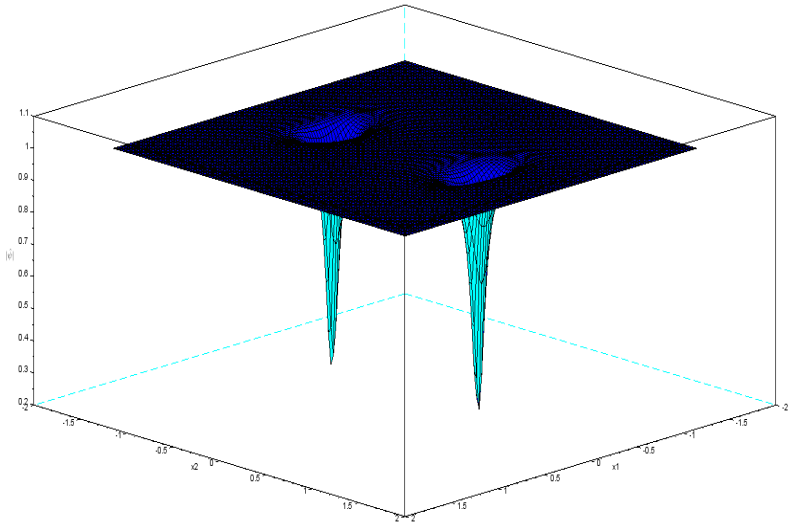

Figure 4.4: The approximate solution of $\hat{\psi}$ in the stretched variables for $c = 0.2$

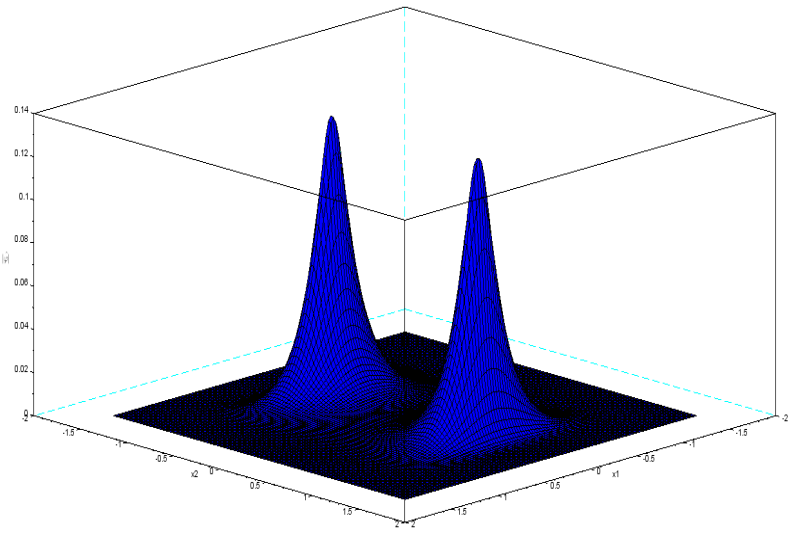

Figure 4.5: The approximate solution of $\hat{\varphi}$ in the stretched variables for $c = 0.2$

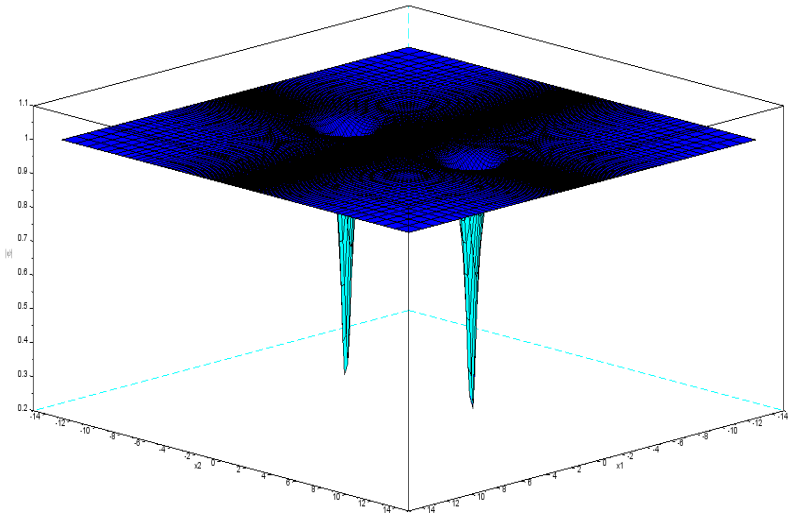

Figure 4.6: The approximate solution of ψ in all $\mathbb{R}^2$ for $c = 0.2$

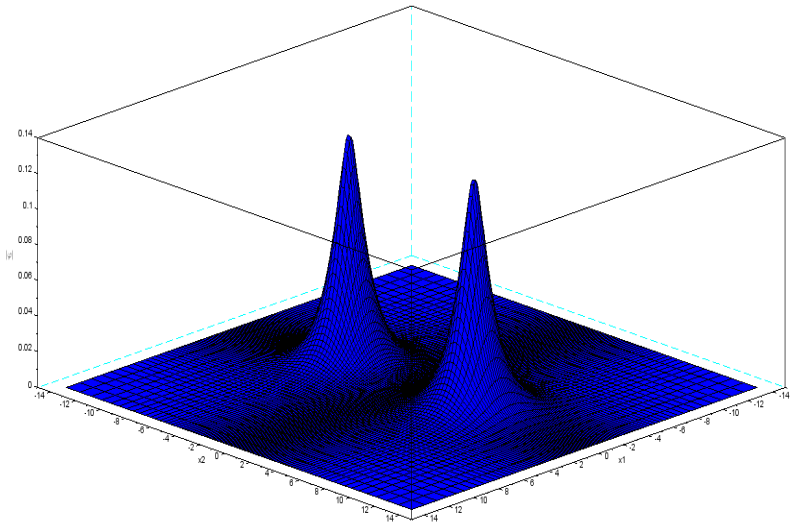

Figure 4.7: The approximate solution of φ in all $\mathbb{R}^2$ for $c = 0.2$

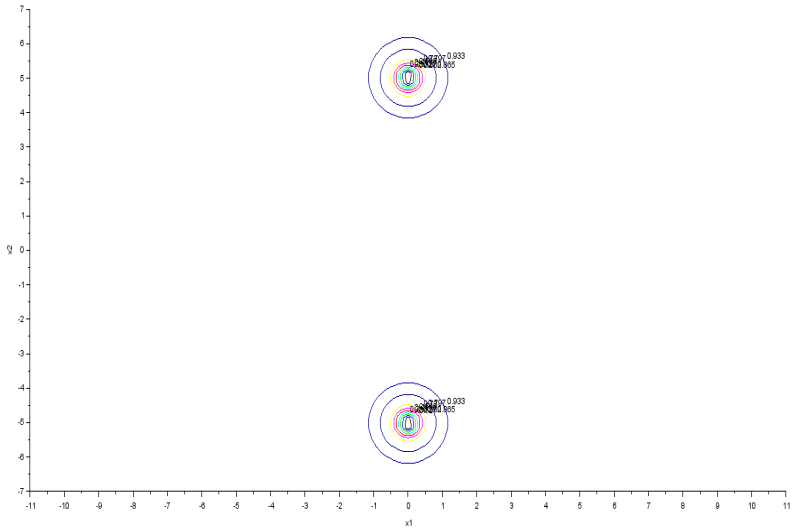

Figure 4.8: The contour line of the approximate solution of ψ for $c = 0.2$

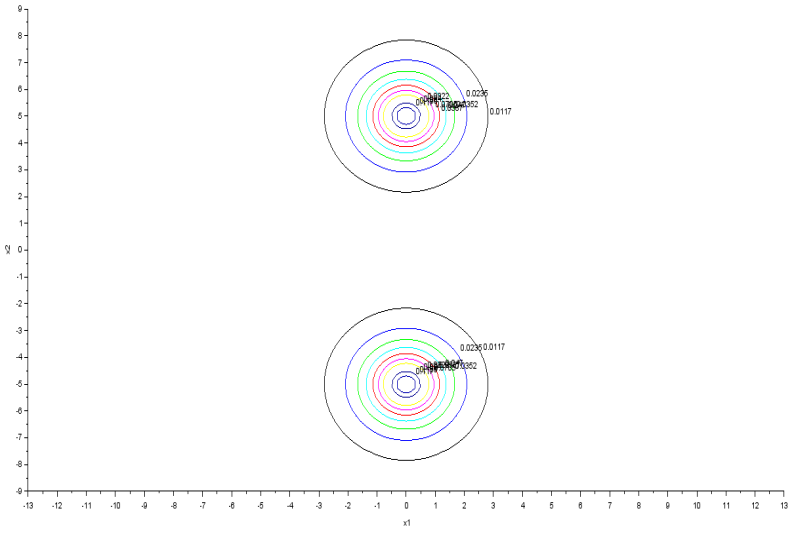

Figure 4.9: The contour line of the approximate solution of φ for $c = 0.2$

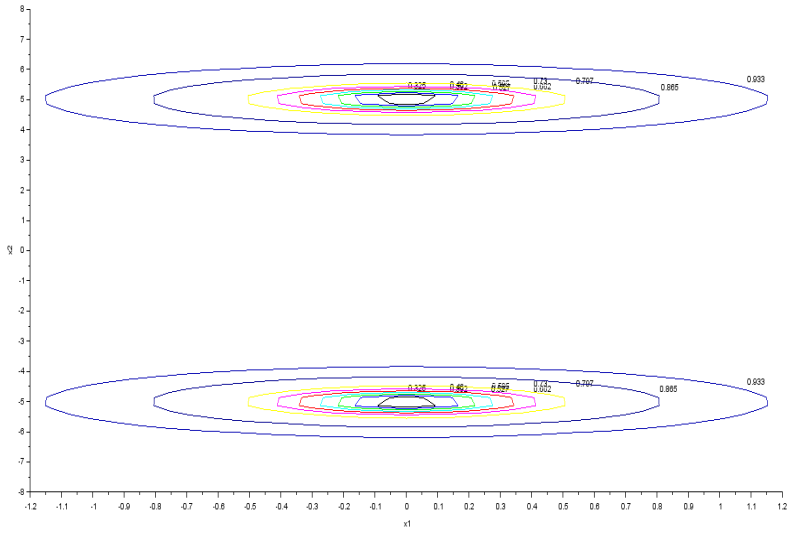

Figure 4.10: The contour line of the approximate solution of ψ for $c = 0.2$

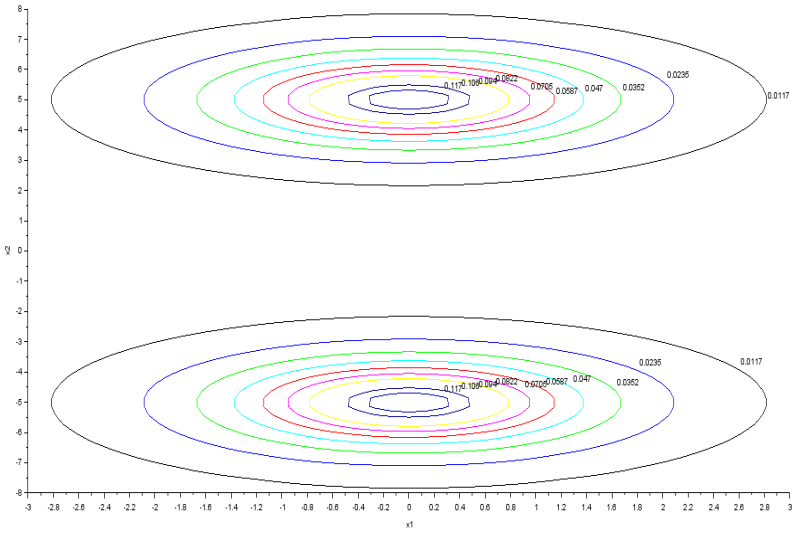

Figure 4.11: The contour line of the approximate solution of φ for $c = 0.2$

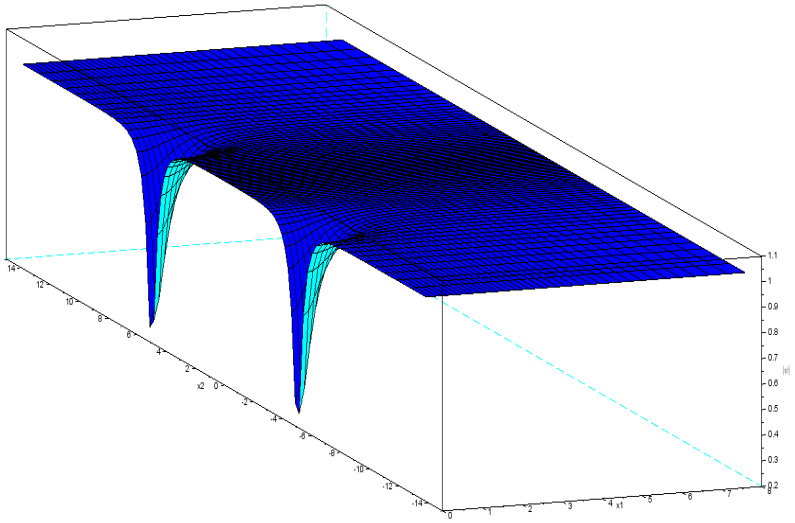

Figure 4.12: The approximate solution of ψ in the half-plane $x_1 \geq 0$ for $c = 0.2$

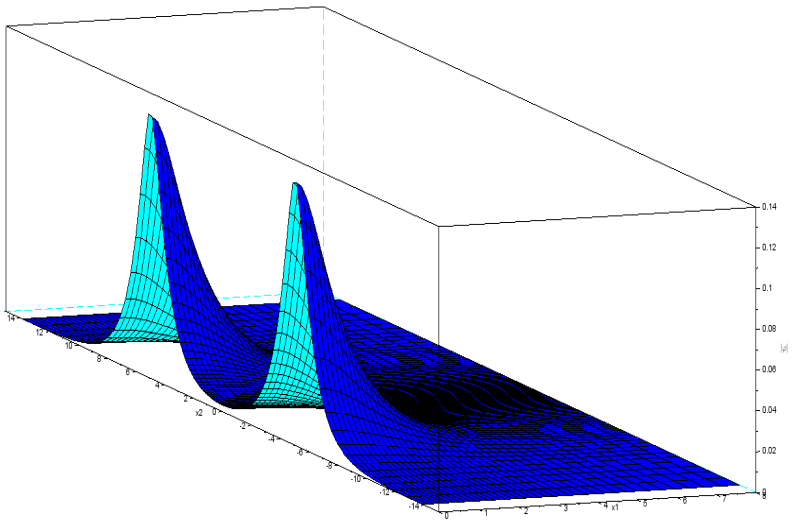

Figure 4.13: The approximate solution of φ in the half-plane $x_1 \geq 0$ for $c = 0.2$

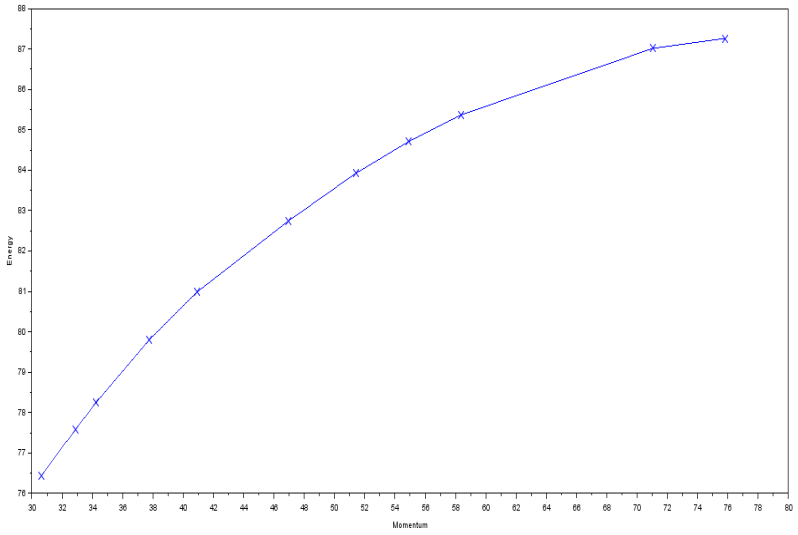

Figure 4.14: The energy-momentum diagram with $\mathfrak{m} = 0.1$

c	Momentum	Energy
0.355	30.622562	76.433487
0.345	32.88967555	77.58745292353
0.335	34.25260209	78.2507466199
0.315	37.73496833	79.79338513551
0.3	40.89830567	80.98027299029
0.27	46.96893883	82.74749393678
0.24	51.39760595	83.92101094292
0.225	54.89109151	84.70760002663
0.215	58.362908	85.370019
0.175	71.032636	87.01579
0.165	75.788422	87.262036

Table 4.1: Numerical values of the momentum and the energy for the small mass solutions of (TW)

Hence the energy-momentum diagram is a concave curve (see Figure 4.14). We could not have any value of the energy or the momentum for $c > 0.355$. The solutions corresponding to ψ and φ for the first and the last points of this curve (i.e. for $c = 0.355$ and $c = 0.165$) are represented in Figure 4.15 - Figure 4.26. For $c = 0.165$, we have $\lambda \approx 4.2082464$. For $c = 0.355$ the algorithm gives $\lambda \approx 4.1518133$.

4.5 Conclusion

The method used numerically is quiet different than the one of the thoeretical proof of Chapter 3 for the existence of solutions for (TW). In Chapter 3, we proved the existence of small mass travelling waves solution for the system by minimizing the energy at fixed momentum and mass. In this chapter, we approximated numerically the solutions of the same system with small mass constraint. We used the Newton-Raphson method, which is of continuation method and not minimization one. We don't have any affirmation that the numerical method used in this chapter captures the minimizers under constraints. They are two different methods. One could minimize the energy numerically under two

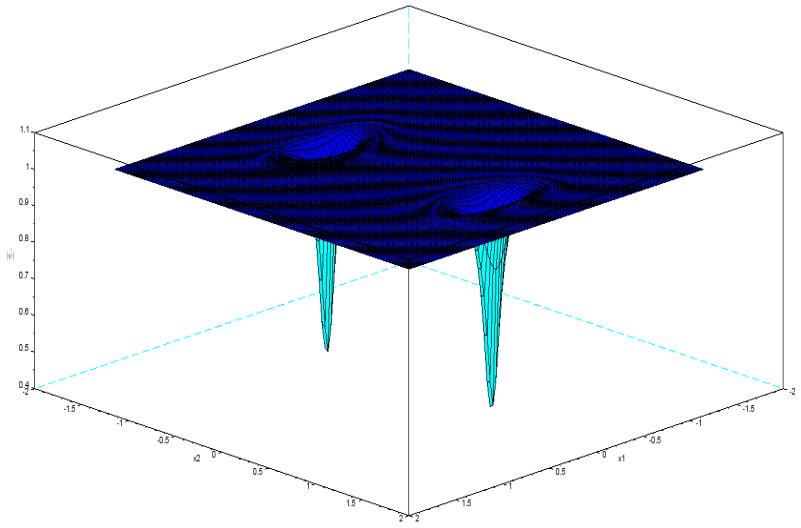

Figure 4.15: The approximate solution of $\hat{\psi}$ in the stretched variables for $c = 0.165$

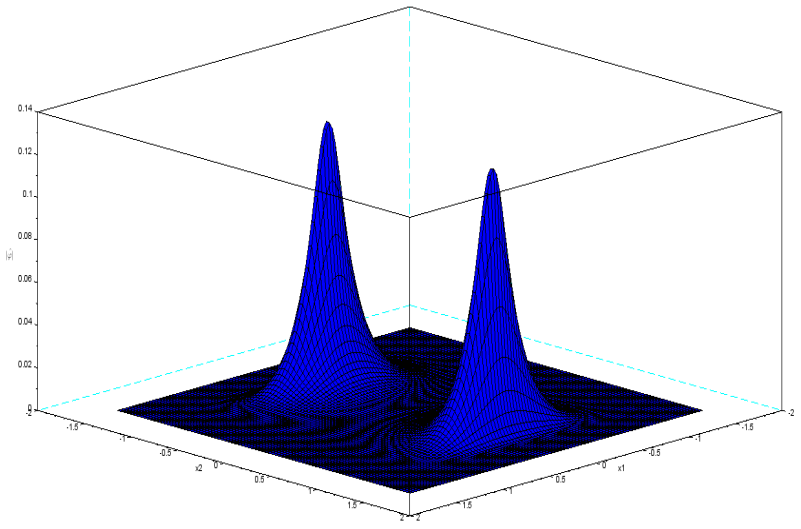

Figure 4.16: The approximate solution of $\hat{\varphi}$ in the stretched variables for $c = 0.165$

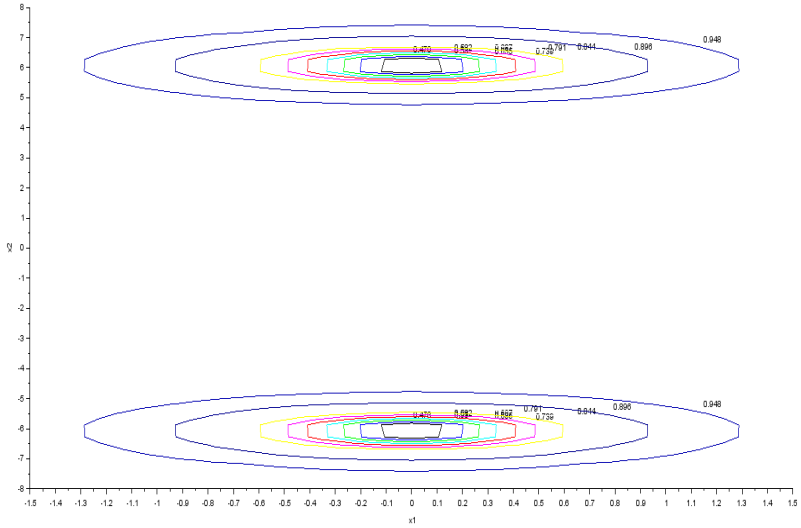

Figure 4.17: The contour line of the approximate solution of ψ for $c = 0.165$

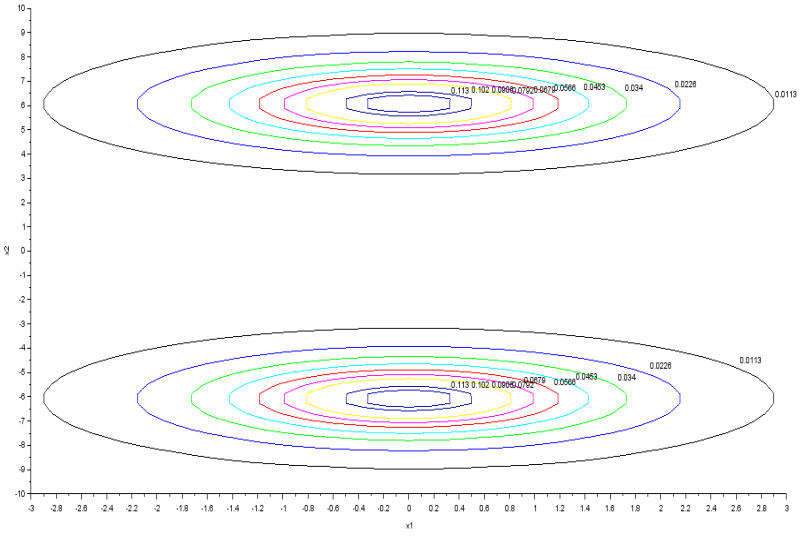

Figure 4.18: The contour line of the approximate solution of φ for $c = 0.165$

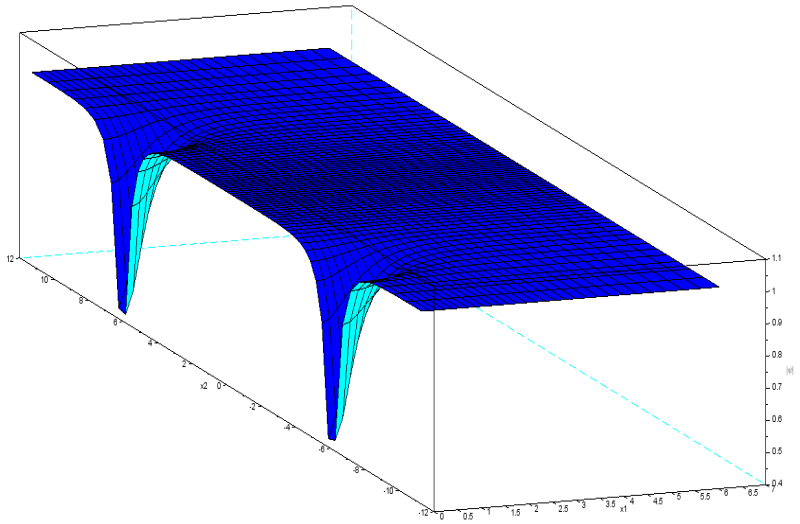

Figure 4.19: The approximate solution of ψ in the half-space $x_1 \geq 0$ for $c = 0.165$

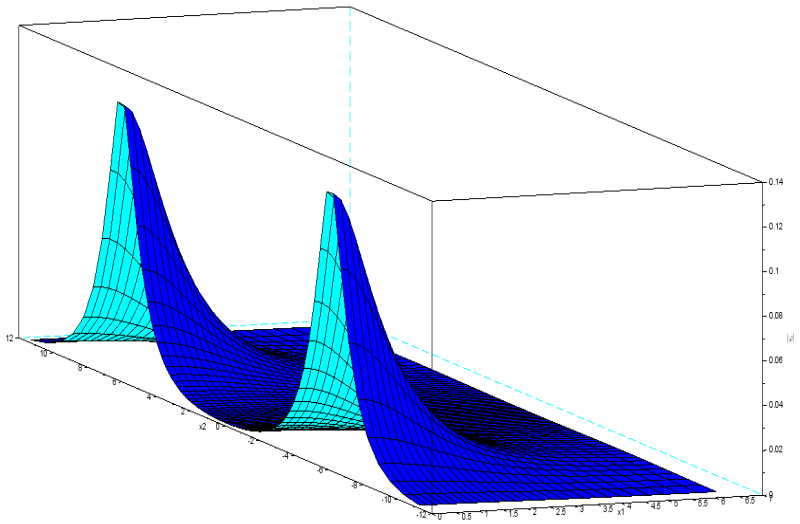

Figure 4.20: The approximate solution of φ in the half-space $x_1 \geq 0$ for $c = 0.165$

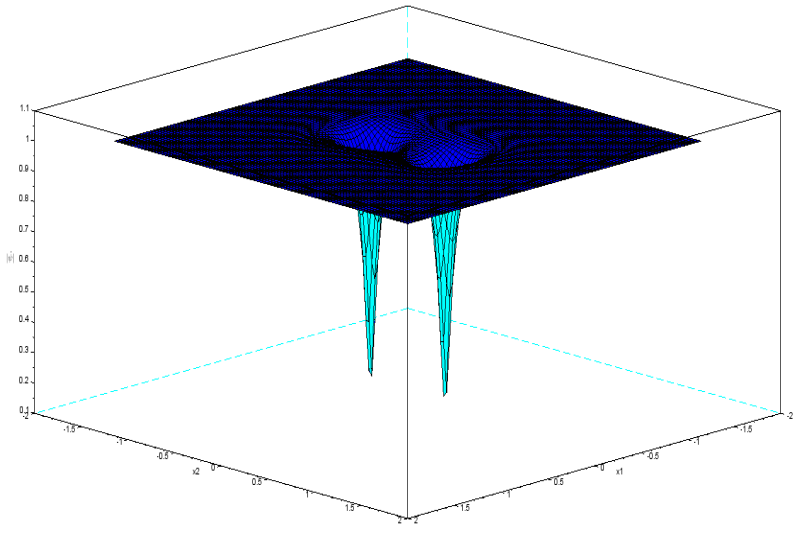

Figure 4.21: The approximate solution of $\hat{\psi}$ in the stretched variables for $c = 0.355$

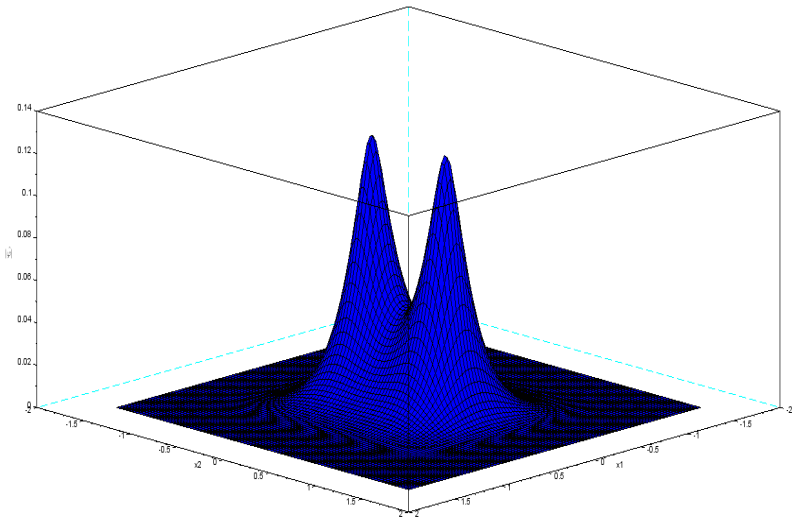

Figure 4.22: The approximate solution of $\hat{\varphi}$ in the stretched variables for $c = 0.355$

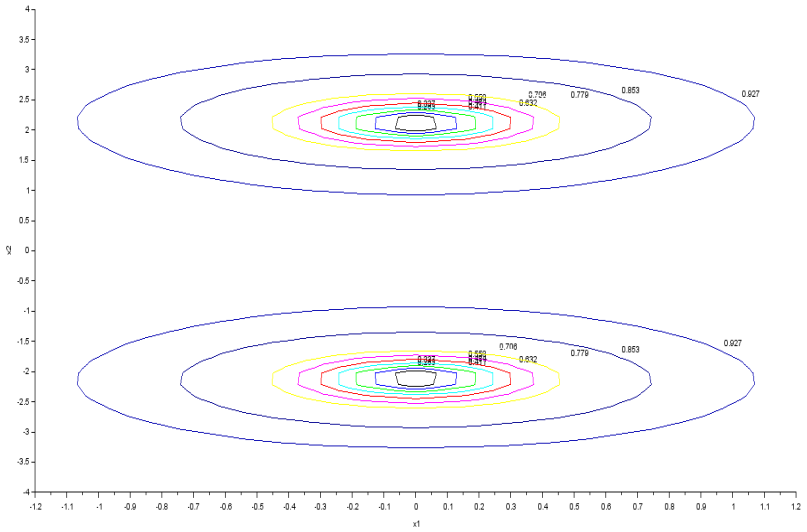

Figure 4.23: The contour line of the approximate solution of ψ for $c = 0.355$

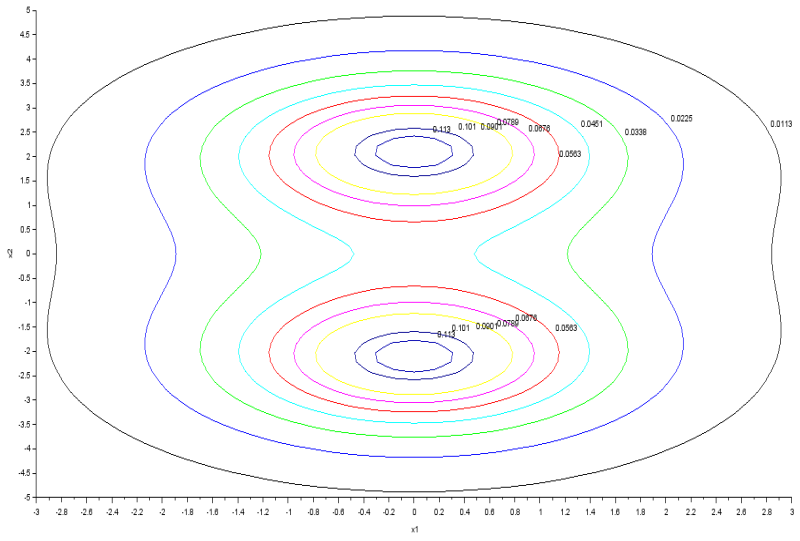

Figure 4.24: The contour line of the approximate solution of φ for $c = 0.355$

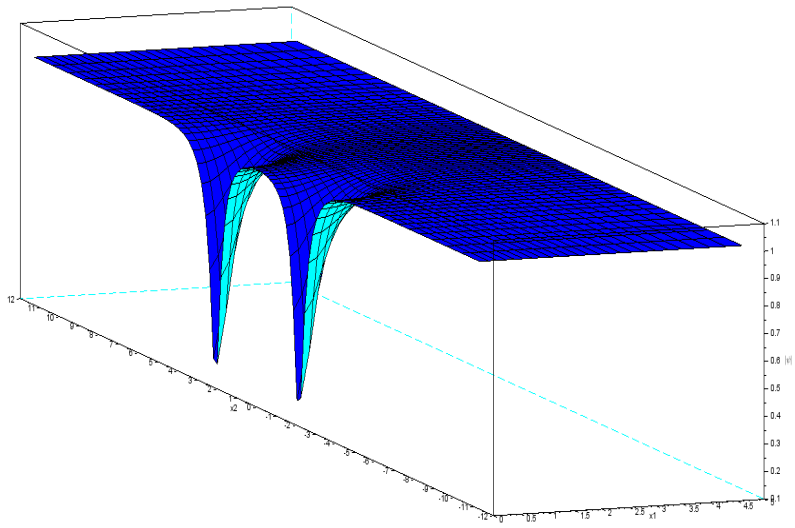

Figure 4.25: The approximate solution of ψ in the half-space $x_1 \geq 0$ for $c = 0.355$

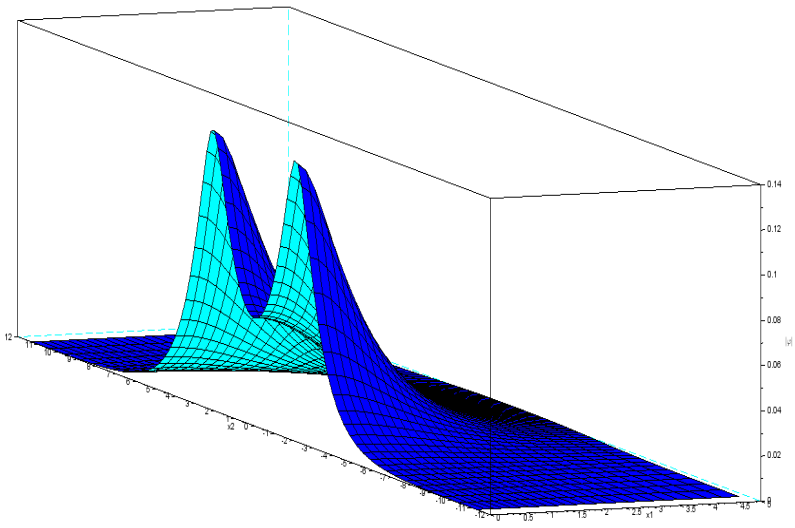

Figure 4.26: The approximate solution of φ in the half-space $x_1 \geq 0$ for $c = 0.355$

constraints, and plot the energy-momentum diagram for the obtained solutions, and conclude from this diagram that these solutions are exactly the minimizers under two constraints. Moreover, the numerical minimization under constraints might give some local minimizers.

Bibliography

[1] Berloff, N. (2004) *Pade approximations of solitary wave solutions of the Gross-Pitaevskii eqation.* J. Phys. A: Math. Gen., 37(5):1617-1632.

[2] Chiron, D. & Scheid, C. (2016). *Travelling Waves for the Nonlinear Schrödinger Equation with General Nonlinearity in Dimension Two.* Journal of Nonlinear Science. 26, no. 1 : 171-231.

[3] Chiron, D. & Scheid, C. (2018). *Multiple branches of travelling waves for the Gross–Pitaevskii equation.* Nonlinearity. 31. 2809-2853.

[4] Jones, C. A. & Roberts, P. H. (1982). *Motions in a Bose condensate: IV. Axisymmetric solitary waves.* J. Phys.A: Math. Gen.15 2599.

[5] Grant, J. & Roberts, P. H. (1974). *Motions in a Bose condensate. III. The structure and effective masses of charged and uncharged impurities.* J. Phys. A: Math. Nucl. Gen. 7 260.

Chapter 5

Stationary and travelling waves to the Gross-Clark system

5.1 Introduction

In this chapter, we will present some numerical and theoretical results concerning the solution of the Gross-Clark system with a large mass of φ equal to 4π. We will present two branches of solutions: the ground state branch and the vortex branch. For each branch, we focus first on some stationary solutions for this system (called ground state solutions and vortex solutions), then we study the travelling waves and construct numerically, through the Newton-Raphson method, the whole branches of solutions for rather small propagation speeds. We recall that this system is given by

$$\text{(TW)} \qquad \begin{cases} -ic\frac{\partial \psi}{\partial x_1} & = -\Delta\psi + \frac{1}{\varepsilon^2}(\frac{1}{\varepsilon^2}|\varphi|^2 + |\psi|^2 - 1)\psi \\ (-\Delta + \frac{q^2}{\varepsilon^2}|\psi|^2)\varphi & = \lambda\varphi, \end{cases}$$

where λ is unknown, and with the boundary conditions $|\psi| \to 1$ and $\varphi \to 0$ as $|x| \to \infty$. We fix the mass of φ by the constraint $\int_{\mathbb{R}^2} |\varphi|^2 \mathrm{d}x = 4\pi$.
For the ground state branch, the solutions of the system with small speeds c are close to the minimizer of the energy corresponding to the stationary solutions. For the vortex branch, each vortex is close to the minimizer of some renormalized energy. These results are obtained for several values of the speed c. We are interested also in the energy-momentum diagram for the speeds c for which Newton-Raphson algorithm converges for the ground state branch and the vortex branch.

5.2 The numerical method

5.2.1 Change of variables

As in the previous chapter, we will look for solutions that respect two symmetries of the problem: ψ is thus assumed to satisfy for all $x \in \mathbb{R}^2$

$$\psi(x) = \psi(x_1, x_2) = \psi(x_1, -x_2) = \bar{\psi}(-x_1, x_2). \tag{5.1}$$

φ is a real function, and is assumed to satisfy the following symmetries:

$$\varphi(x) = \varphi(x_1, x_2) = \varphi(x_1, -x_2) = \varphi(-x_1, x_2). \tag{5.2}$$

Hence we work in $\mathbb{R}_+ \times \mathbb{R}_+$ instead of $\mathbb{R}^2$. Using the change of variables we used in Chapter 4

$$R_1 x_1 = \tan(\hat{x}_1), \quad R_2 x_2 = \tan(\hat{x}_2),$$

we work in the bounded domain $[0, \frac{\pi}{2}]^2$.
We recall that we can rewrite the system (TW) in these variables:

$$\begin{cases} -icR_1\cos^2(\hat{x_1})\frac{\partial\hat{\psi}}{\partial\hat{x}_1} & +R_1^2\big(\cos^4(\hat{x}_1)\frac{\partial^2\hat{\psi}}{\partial\hat{x}_1^2} - 2\sin(\hat{x}_1)\cos^3(\hat{x}_1)\frac{\partial\hat{\psi}}{\partial\hat{x}_1}\big) \\ & +R_2^2\big(\cos^4(\hat{x}_2)\frac{\partial^2\hat{\psi}}{\partial\hat{x}_2^2} - 2\sin(\hat{x}_2)\cos^3(\hat{x}_2)\frac{\partial\hat{\psi}}{\partial\hat{x}_2}\big) \\ & -\frac{1}{\varepsilon^2}(\frac{1}{\varepsilon^2}|\hat{\varphi}|^2 + |\hat{\psi}|^2 - 1)\hat{\psi} = 0 \\ -R_1^2\big(\cos^4(\hat{x}_1)\frac{\partial^2\hat{\varphi}}{\partial\hat{x}_1^2} & -2\sin(\hat{x}_1)\cos^3(\hat{x}_1)\frac{\partial\hat{\varphi}}{\partial\hat{x}_1}\big) \\ & -R_2^2\big(\cos^4(\hat{x}_2)\frac{\partial^2\hat{\varphi}}{\partial\hat{x}_2^2} - 2\sin(\hat{x}_2)\cos^3(\hat{x}_2)\frac{\partial\hat{\varphi}}{\partial\hat{x}_2}\big) \\ & +\frac{q^2}{\varepsilon^2}|\hat{\psi}|^2\hat{\varphi} - \lambda\hat{\varphi} = 0. \end{cases} \tag{5.3}$$

with the Dirichlet conditions

$$\hat{\psi} = 1, \tag{5.4}$$

and

$$\hat{\varphi} = 0 \tag{5.5}$$

on the boundaries $\hat{x}_1 = \frac{\pi}{2}$ and $\hat{x}_2 = \frac{\pi}{2}$.

5.2.2 Discretization

We discretize the computational domain $[0, \frac{\pi}{2}]^2$ as in the previous chapter: N_{x_1} points in the direction x_1 and N_{x_2} points in the direction x_2. and we take $\tilde{N} = N_{x_1} = N_{x_2}$. The size of the mesh is then denoted by $\mathrm{d}x = \frac{\pi}{2(\tilde{N}-1)}$.
We also choose to work in a Finite Difference framework, using central approximations of derivatives that are of order 2.

5.2.3 The equations

We choose to write $\hat{\psi}$ as $\hat{a} + i\hat{b}$, and recall that the system becomes

$$\begin{cases} cR_1\cos^2(\hat{x_1})\frac{\partial\hat{b}}{\partial\hat{x}_1} & +R_1^2\big(\cos^4(\hat{x}_1)\frac{\partial^2\hat{a}}{\partial\hat{x}_1^2} - 2\sin(\hat{x}_1)\cos^3(\hat{x}_1)\frac{\partial\hat{a}}{\partial\hat{x}_1}\big) \\ & +R_2^2\big(\cos^4(\hat{x}_2)\frac{\partial^2\hat{a}}{\partial\hat{x}_2^2} - 2\sin(\hat{x}_2)\cos^3(\hat{x}_2)\frac{\partial\hat{a}}{\partial\hat{x}_2}\big) \\ & -\frac{1}{\varepsilon^4}|\hat{\varphi}|^2\hat{a} - \frac{1}{\varepsilon^2}\hat{a}^3 - \frac{1}{\varepsilon^2}\hat{a}\hat{b}^2 + \frac{1}{\varepsilon^2}\hat{a} = 0 \\ -cR_1\cos^2(\hat{x}_1)\frac{\partial\hat{a}}{\partial\hat{x}_1} & +R_1^2\big(\cos^4(\hat{x}_1)\frac{\partial^2\hat{b}}{\partial\hat{x}_1^2} - 2\sin(\hat{x}_1)\cos^3(\hat{x}_1)\frac{\partial\hat{b}}{\partial\hat{x}_1}\big) \\ & +R_2^2\big(\cos^4(\hat{x}_2)\frac{\partial^2\hat{b}}{\partial\hat{x}_2^2} - 2\sin(\hat{x}_2)\cos^3(\hat{x}_2)\frac{\partial\hat{b}}{\partial\hat{x}_2}\big) \\ & -\frac{1}{\varepsilon^4}|\hat{\varphi}|^2\hat{b} - \frac{1}{\varepsilon^2}\hat{a}^2\hat{b} - \frac{1}{\varepsilon^2}\hat{b}^3 + \frac{1}{\varepsilon^2}\hat{b} = 0 \\ -R_1^2\big(\cos^4(\hat{x}_1)\frac{\partial^2\hat{\varphi}}{\partial\hat{x}_1^2} & -2\sin(\hat{x}_1)\cos^3(\hat{x}_1)\frac{\partial\hat{\varphi}}{\partial\hat{x}_1}\big) \\ & -R_2^2\big(\cos^4(\hat{x}_2)\frac{\partial^2\hat{\varphi}}{\partial\hat{x}_2^2} - 2\sin(\hat{x}_2)\cos^3(\hat{x}_2)\frac{\partial\hat{\varphi}}{\partial\hat{x}_2}\big) \\ & +\frac{q^2}{\varepsilon^2}(\hat{a}^2 + \hat{b}^2)\hat{\varphi} - \lambda\hat{\varphi} = 0. \end{cases} \tag{5.6}$$

For $i = 0, \ldots, \tilde{N} - 1$ and $j = 0, \ldots, \tilde{N} - 1$, we denote by $\hat{a}_{i,j}$, $\hat{b}_{i,j}$, and $\hat{\varphi}_{i,j}$ the values of $\hat{a}$, $\hat{b}$, and $\hat{\varphi}$ on the points $(\hat{x}_1, \hat{x}_2) = (i\mathrm{d}x, j\mathrm{d}x)$. We will suppose that φ tends to 0 and ψ tends to a constant of

modulus 1 when $|x| \to \infty$, which is not proved for (GC), but for the Gross-Pitaevskii equation given by

$$i\frac{\partial \Psi}{\partial t} + \Delta \Psi = \Psi(|\Psi|^2 - 1) \tag{GP}$$

(see [1], [4], and [15]). Hence on the lines $\hat{x}_1 = \frac{\pi}{2}$ and $\hat{x}_2 = \frac{\pi}{2}$, the values of $\hat{a}$, $\hat{b}$ and $\hat{\varphi}$ are known: $\hat{a} = 1$ and $\hat{b} = 0$, $\hat{\varphi} = 0$. On the line $\hat{x}_1 = 0$, we have $\hat{b} = 0$, because of the symmetry (5.1). As in Chapter 4, we have the number of equations=the number of unknowns=$(\tilde{N} - 1)(3\tilde{N} - 4) + 1$.

5.2.4 Choice of the parameters

We choose the parameters used in [6] and in Chapter 4:

$$q = 0.41, \tag{5.7}$$

$$\varepsilon = 0.187, \tag{5.8}$$

5.3 The ground state branch

5.3.1 The ground state solutions. Minimizing the energy

We look for the ground state solutions, these are radial solutions $\psi_{\rm GS}$ and $\varphi_{\rm GS}$ corresponding to (TW) with $c = 0$. In other words, we choose the radial minimizer of the energy E. To minimize this energy in the radial coordinates, we use the projected Gradient method and consider the problem

$$\begin{aligned} &\text{Minimize} \\ &E(\psi_{\rm rad}, \varphi_{\rm rad}) = 2\pi \int_0^\gamma \big(\psi_{\rm rad}'^2 + \frac{1}{2\varepsilon^2}(\psi_{\rm rad}^2 - 1)^2 + \frac{1}{\varepsilon^4}\psi_{\rm rad}^2\varphi_{\rm rad}^2 + \frac{1}{q^2\varepsilon^2}\varphi_{\rm rad}'^2\big) r \mathrm{d}r \\ &\text{under the constraint } 2\pi \int_0^\infty r \mid \varphi_{\rm rad} \mid^2 \mathrm{d}r = 4\pi, \end{aligned} \tag{5.9}$$

and where γ is chosen large enough (we take $\gamma = 30$). Notice that in the energy, $\psi_{\rm rad}$ and $\varphi_{\rm rad}$ are given in the physical variables, and not the stretched ones.

To initialize the constrained minimization problem, we follow the computations of [6] in $2D$ for $\varphi_{\rm GS}$ to get

$$\varphi_{\rm rad,i}(r) = A_0 J_0(kr) \tag{5.10}$$

whenever $kr \leq r_0 \approx 2.4048256$ (the first zero of J_0), and $\varphi_{\rm rad,i}(r) = 0$ elsewhere, and where J_0 is the Bessel function of the first kind, A_0 is a parameter chosen such that $2\pi \int_0^\infty r|\varphi_{\rm rad,i}(r)|^2 \mathrm{d}r = 4\pi$. This last constraint gives us the value of $A_0 = 1.6019872$. For the initialization of $\psi_{\rm GS}$ we choose a continuous function that is equal to zero in $r = 0$ and approaches to 1 when $rk \geq r_0$ and as $r \to \infty$. Hence we initialize $\psi_{\rm GS}$ by the function

$$\psi_{\rm rad,i}(r) = \frac{1 + \tanh(\frac{r - r_0/k}{\varepsilon})}{2}. \tag{5.11}$$

In order to have a good approximation of the minimizers, we put enough points in the interval $[0, \gamma]$, and choose $\tilde{N} = 600$ the number of points so that every subdivision has a size of $h = \gamma/(\tilde{N} - 1) \approx 0.05$.

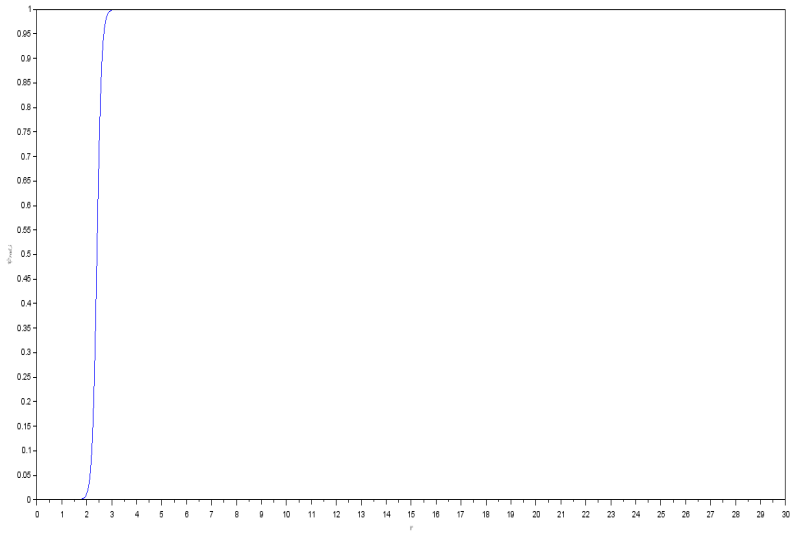

Figure 5.1: The initial data $\psi_{\text{rad,i}}$ for the constrained minimization problem (5.9)

We choose the step size of the projected Gradient method $\rho = h^4$. In the minimization problem, we work with the interval $]0, \gamma[$ so we have $2\tilde{N} - 4$ unknown quantities, and then the values of ψ_{GS} and φ_{GS} on $r = 0$ and $r = \gamma$ are obtained from the Dirichlet conditions (5.4) and (5.5). We minimize this energy by fixing the mass or L^2 norm of φ_{GS} at each iteration.

1. At some iteration n, we know (ψ_n, φ_n);
2. we do the usual steepest descent for the function E that we are minimizing (without considering the constraint)
$$(\psi_{n+1}, \tilde{\varphi}_{n+1}) = (\psi_n, \varphi_n) - \rho E'(\psi_n, \varphi_n);$$
3. we normalize the function $\tilde{\varphi}_n$ to satisfy the constraint : $\varphi_{n+1} = \frac{\tilde{\varphi}_{n+1}}{\alpha_{n+1}}$, where $\alpha_{n+1} > 0$ is such that $\int_{\mathbb{R}^2} \varphi_{n+1}^2 = 4\pi$.

The initial data $\psi_{\text{rad,i}}$ and $\varphi_{\text{rad,i}}$ are given in Figures 5.1 and Figure 5.2 respectively.
After 40 000 iterations, we get an approximation of the minimizers of the energy that are given in Figure 5.3 and a final relative error between two consecutive solutions equal to 1.31×10^{-5}. The value of the energy at each iteration is given in Figure 5.4.
The final value of the coefficient α_{n+1} is equal to 0.9999033. Knowing that the constrained minimizers satisfy $\tilde{\varphi}_{\text{GS}} = \varphi_{\text{GS}} - \frac{2\rho\lambda}{\varepsilon^2}\varphi_{\text{GS}}$, the value of λ is calculated from the minimization algorithm by the formula

$$\lambda_i = \frac{1 - \alpha_{n+1}}{\rho} \times \frac{\varepsilon^2}{2} = 0.268662. \tag{5.12}$$

We could find some approximations of these minimizers, called ground state solutions, that are given by

$$\psi_{\text{GS}}^{\text{app}}(r) = \frac{0.2941 e^{\tilde{r}_\psi} + 65.367 e^{2\tilde{r}_\psi}}{1 + 25.678 e^{\tilde{r}_\psi} + 65.367 e^{2\tilde{r}_\psi}}, \tag{5.13}$$

where $\tilde{r}_\psi = \frac{r - 3.7939}{0.2057}$ and with L^∞-error equal to 0.006, and

$$\varphi_{\text{GS}}^{\text{app}}(r) = \frac{1.8724 e^{-\tilde{r}_\varphi} + 0.19262 e^{-2\tilde{r}_\varphi} - 0.30439 e^{-3\tilde{r_\varphi}} + 2.5802 e^{-4\tilde{r}_\varphi}}{1 + 1.9356 e^{-\tilde{r}_\varphi} + 0.93498 e^{-2\tilde{r}_\varphi} + 1.1382 e^{-3\tilde{r}_\varphi} + 2.9438 e^{-4\tilde{r}_\varphi}}, \tag{5.14}$$

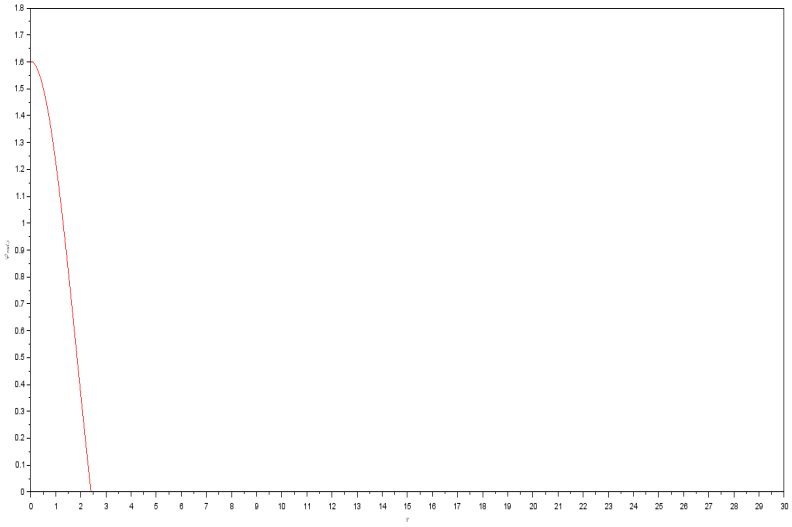

Figure 5.2: The initial data $\varphi_{\mathrm{rad,i}}$ for the constrained minimization problem (5.9)

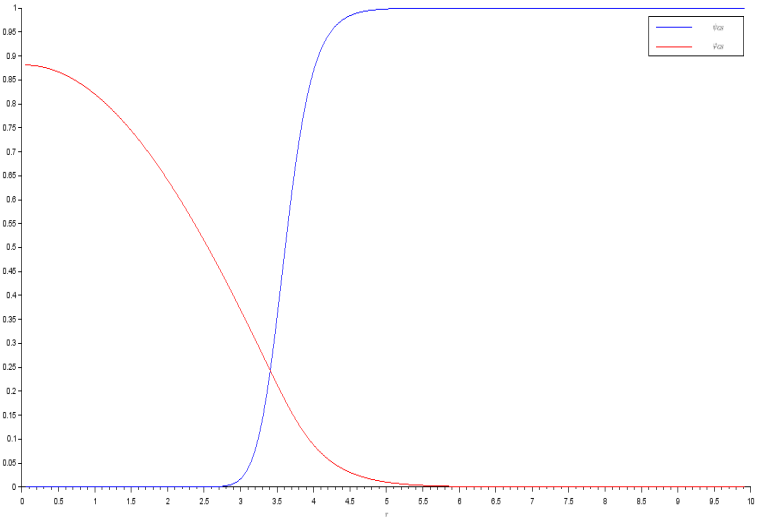

Figure 5.3: The constrained minimizers of the energy E

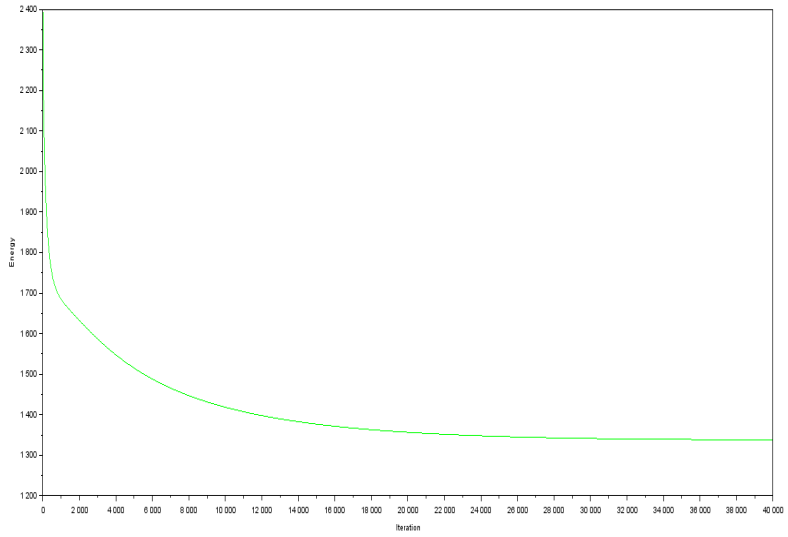

Figure 5.4: The value of the energy at each iteration of the minimization problem (5.9)

where $\tilde{r}_\varphi = \frac{r-2.314}{0.60408}$, with L^∞-error equal to 0.018.
These approximations are obtained by computing a rational function and an exponential that may be called exponential-Padé approximants. To do so, we chose randomly, several times, the coefficients of the exponential Padés. Then, we optimize the coefficients of the Padé, by minimizing the L^∞-norm and starting from the random choice that we did. Once we find the best approximations that conserve the convexity/concavity of the minimizers and the smallest error possible, we make the choice. We plot these approximations with the minimizers in the same graph (see Figure 5.5).

5.3.2 The Newton-Raphson algorithm for the ground state branch

The results of the previous section are some stationary solutions of (TW), given in polar coordinates. These solutions are used as initializations in the Newton-Raphson algorithm to find the whole ground state branch of travelling waves for (GC), at least for small speeds c. The modulus of the stationary ground state solutions in the domain $[0, \frac{\pi}{2}]^2$ are respectively given in Figure 5.6 and Figure 5.7.
First, we choose $c = 0.2$. Newton-Raphson algorithm converges very well for this value. The solutions ψ and φ in the stretched variables are given in Figure 5.8 and 5.9, and in all $\mathbb{R}^2$ in Figure 5.10 and 5.11.
The contour lines for the solutions are presented in Figure 5.12 and 5.13.
To have a better look on the solutions, we choose to present them in the half-space $x_1 \geq 0$ in Figure 5.14 and Figure 5.15.
The Newton-Raphson algorithm gives the approximate value of λ which is 0.3304393, that is close to the value (5.12) obtained in the constrained minimistion problem. These solutions are obtained after 10 iterations with a final relative error equal to 3.8×10^{-8}.

5.3.3 The energy-momentum diagram

The Newton-Raphson Algorithm converged for different values of c. Until $c = 1.5$ we could get the solutions of (TW) by Newton-Raphson starting from the initial conditions that are the ground state of (TW), in other words, the minimizer of the energy, and solution of (TW) with $c = 0$. For the values of c that are greater, we had to start from the solutions of (TW) with c larger than zero and closer to our c. For example, to approximate the solutions of (TW) with $c = 1.55$, we had to initialize our

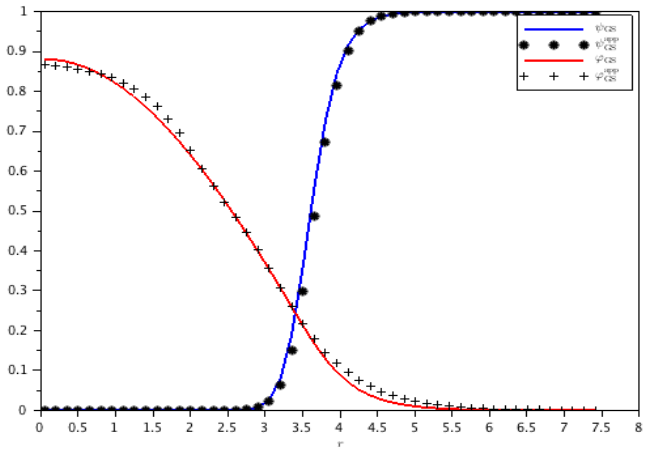

Figure 5.5: The approximation of the constrained minimizers of (5.9)

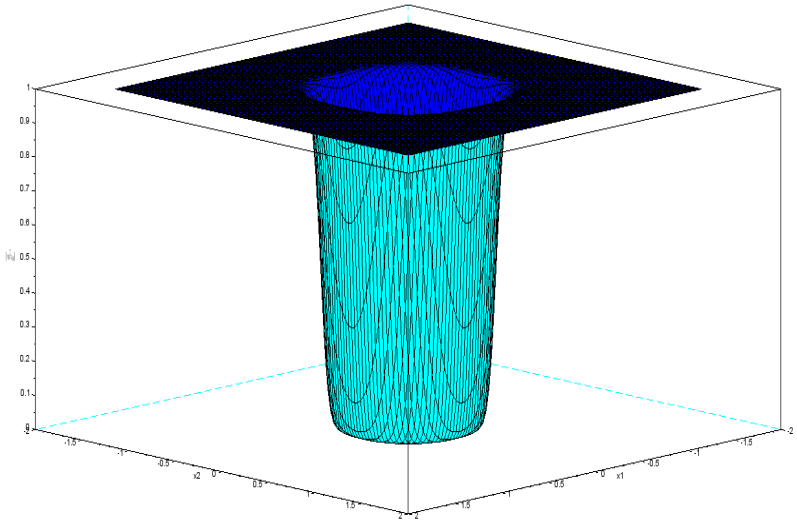

Figure 5.6: The initial data for ψ in the Newton-Raphson algorithm

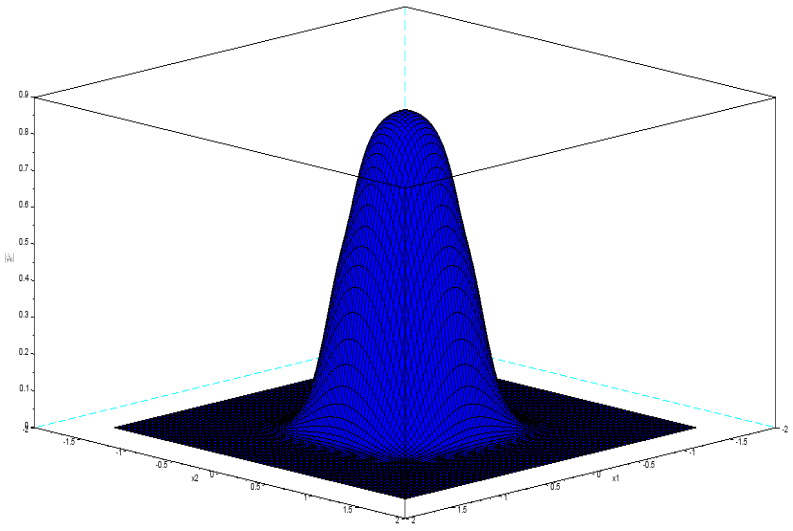

Figure 5.7: The initial data for φ in the Newton-Raphson algorithm

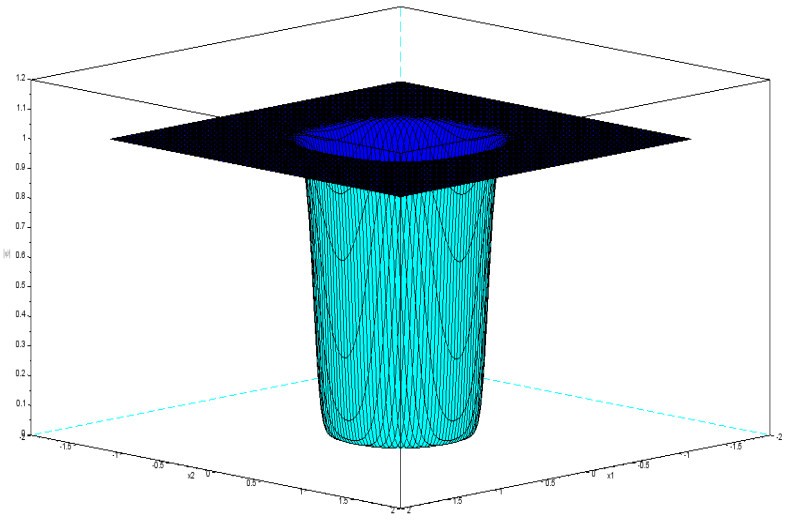

Figure 5.8: The numerical solution for ψ in the stretched variables with $c = 0.2$

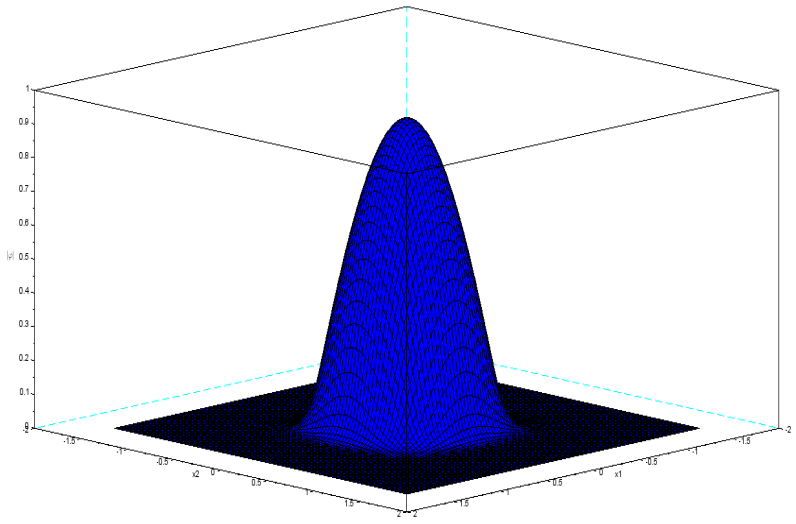

Figure 5.9: The numerical solution for φ in the stretched variables with $c = 0.2$

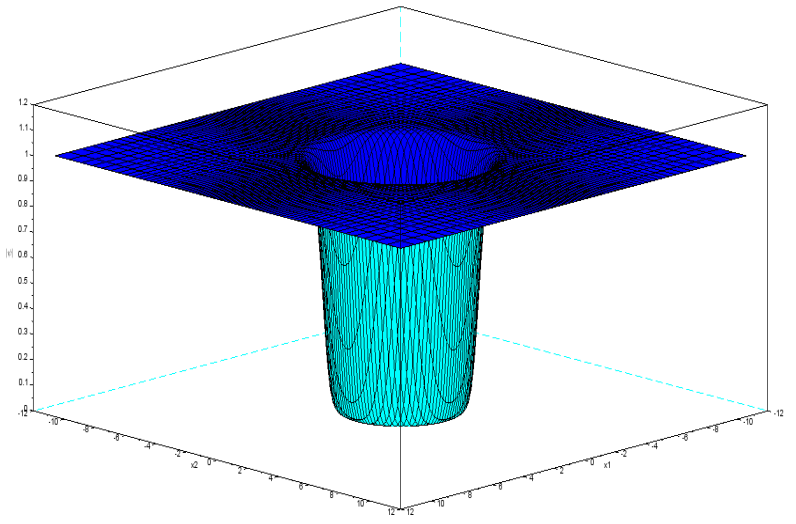

Figure 5.10: The numerical solution for ψ in $\mathbb{R}^2$ with $c = 0.2$

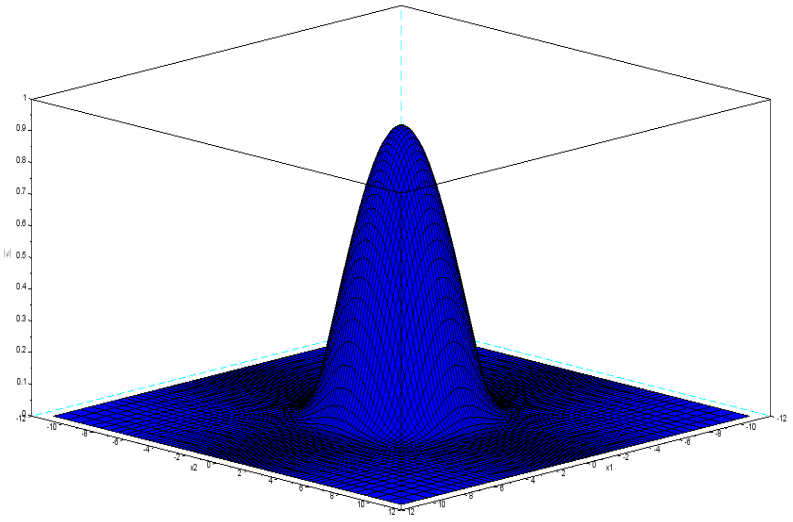

Figure 5.11: The numerical solution for φ in $\mathbb{R}^2$ with $c = 0.2$

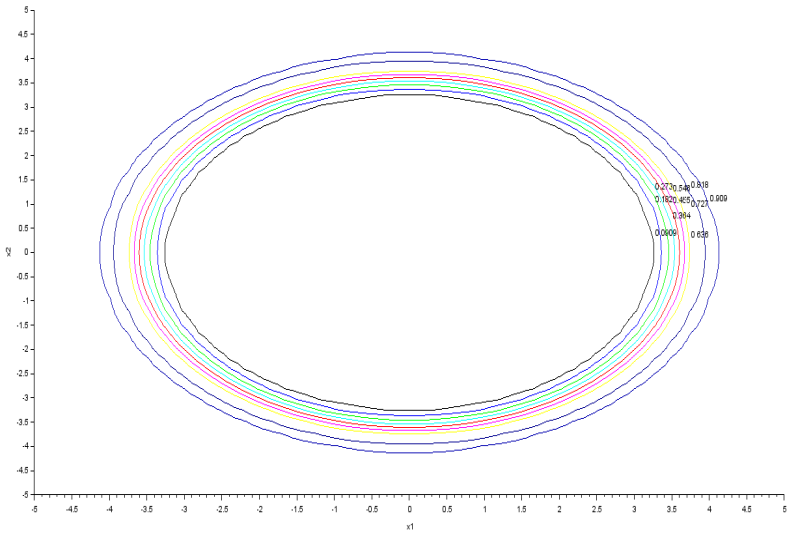

Figure 5.12: The contour lines of the numerical solution of ψ in $\mathbb{R}^2$ with $c = 0.2$

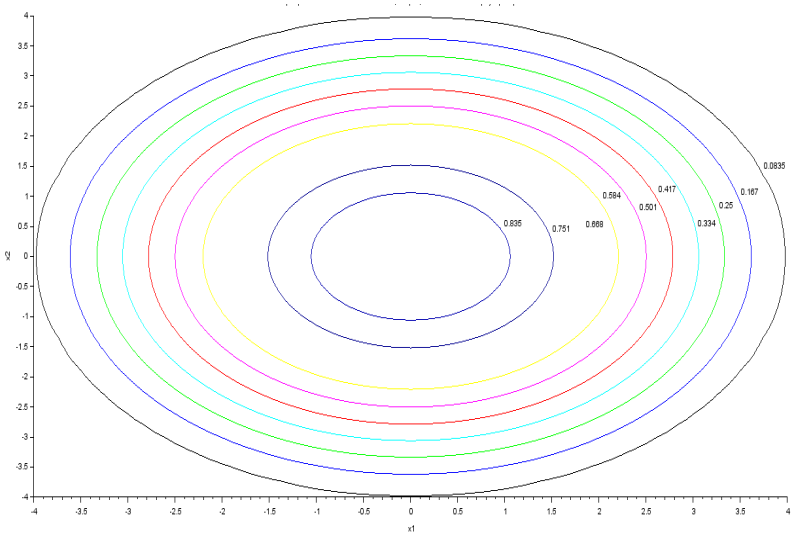

Figure 5.13: The contour lines of the numerical solution of φ in $\mathbb{R}^2$ with $c = 0.2$

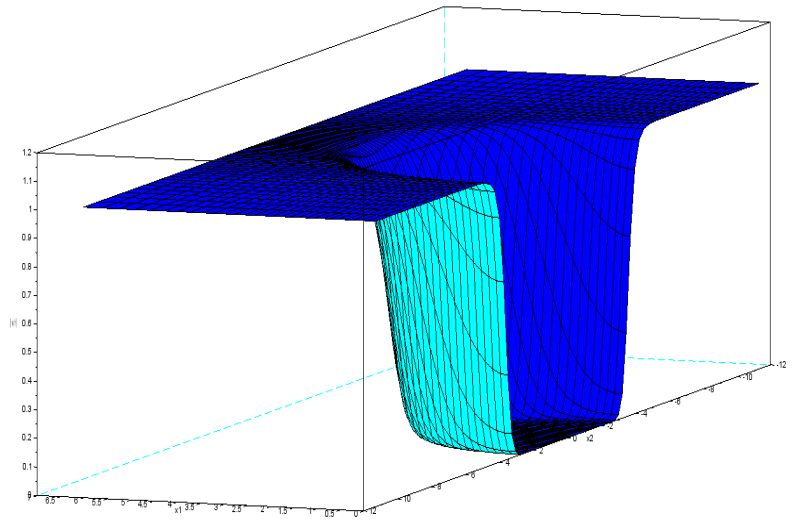

Figure 5.14: The numerical solution of ψ in the half-space $x_1 \geq 0$ with $c = 0.2$

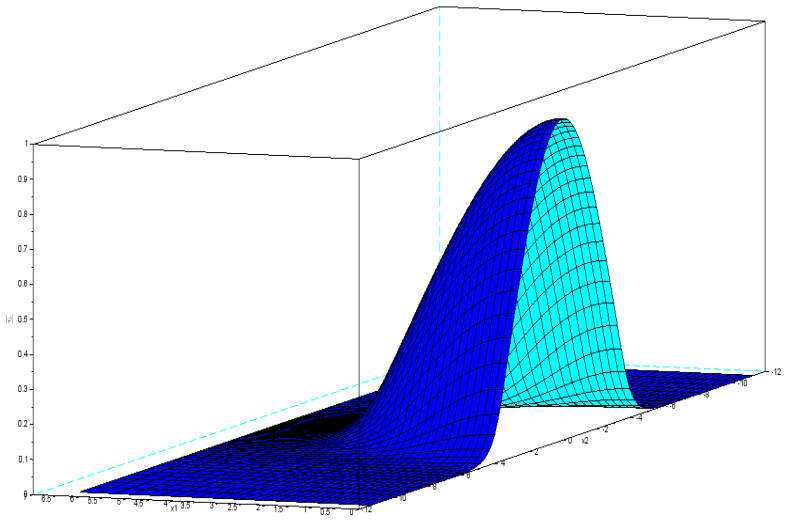

Figure 5.15: The numerical solution of φ in the half-space $x_1 \geq 0$ with $c = 0.2$

Newton-Raphson algorithm with the numerical solutions of (TW) with $c = 1.5$, that are obtained by Newton-Raphson starting from the ground state solutions. Hence, we could reach the value 2.35 for c. For c larger than 2.35, (2.36 for instance), Newton-Raphson algorithm did not converge, even if we initialize our system with the solutions obtained for $c = 2.35$. In this way, we could find the ground state branch. For every value of c for which Newton-Raphson method converges, we calculated the Energy and the Momentum of the numerical solutions. Some of these quantities are represented in Table 5.1 below.

c	Momentum	Energy
0.05	1.1400364	1261.0252
0.35	8.0255292	1262.4817
0.7	16.3374398	1267.0911
0.9	21.341026	1271.2986
1.4	15.288228	1288.1487
1.5	38.441068	1292.8946
1.7	45.280566	1304.221
1.95	55.188544	1322.8241
2.2	67.554076	1349.0038
2.35	77.308418	1371.3961

Table 5.1: Numerical values of the momentum and the energy for the ground state branch of (TW)

The Energy-Momentum diagram is then given in Figure 5.16.
Figure 5.17 and 5.18 represent the solution of (TW) with $c = 2.35$ that is the greatest value of c for which Newton-Raphson algorithm converges.
Comparing Figure 5.10 to Figure 5.17 we see that the modulus of the solution ψ has the same shape and did not vary very much by changing c. We then choose to represent the phase of the obtained solution by Newton-Raphson method for different values of c.
The values of λ for each of these c are given in Table 5.2 below.

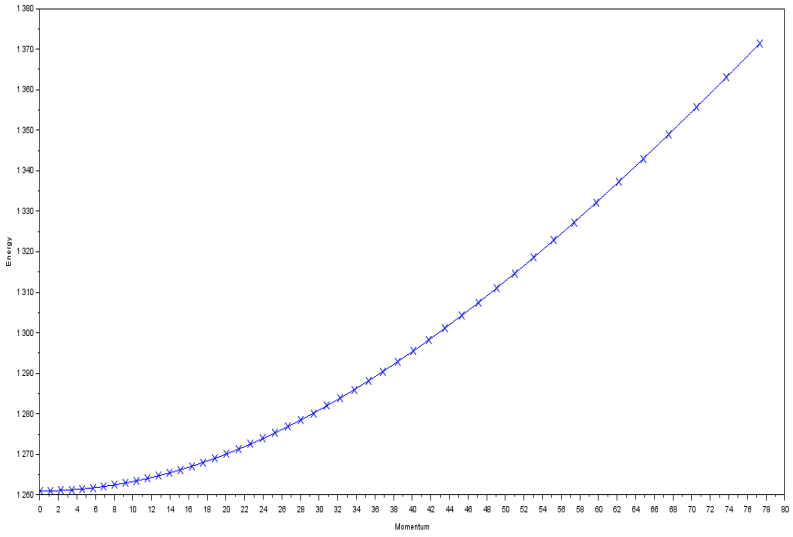

Figure 5.16: The Energy-Momentum diagram for the ground state branch

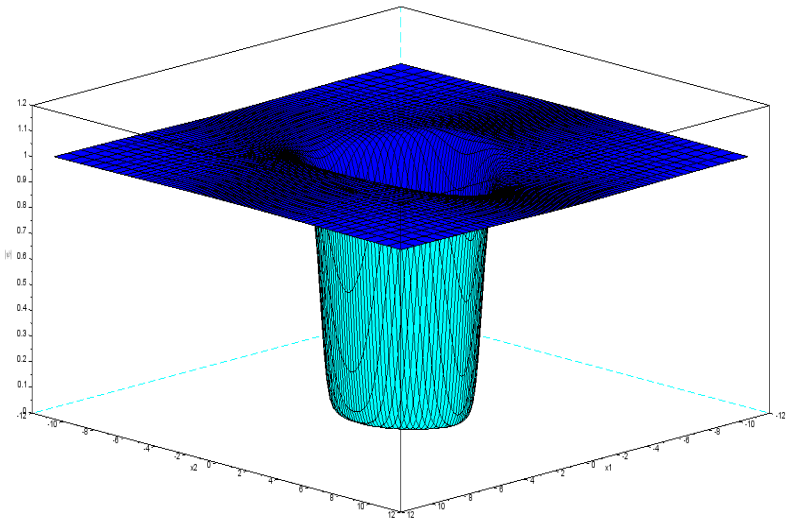

Figure 5.17: The numerical solution for ψ in $\mathbb{R}^2$ with $c = 2.35$

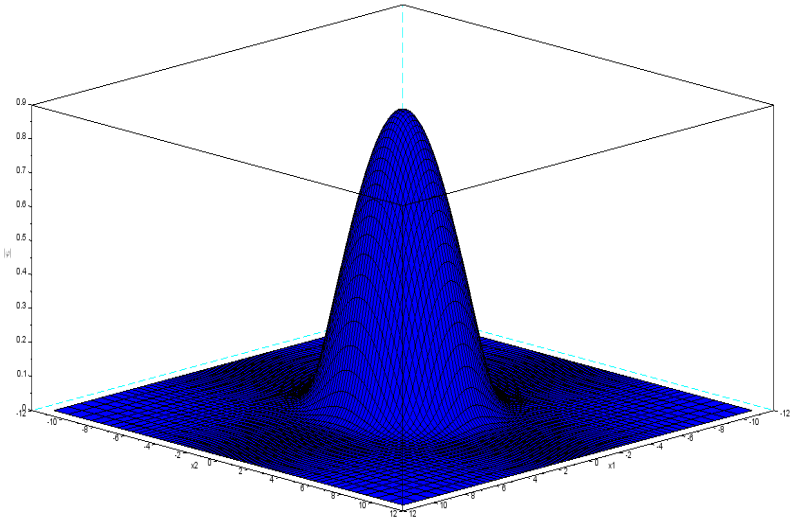

Figure 5.18: The numerical solution for φ in $\mathbb{R}^2$ with $c = 2.35$

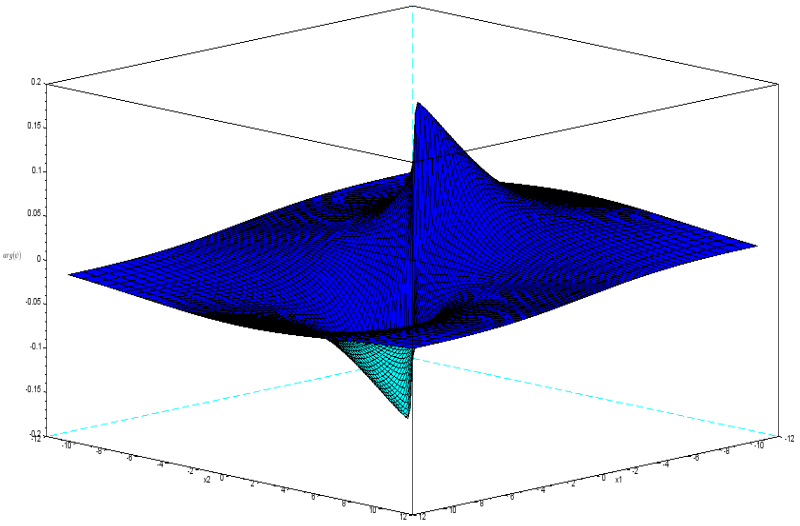

Figure 5.19: The phase of the numerical solution for ψ in $\mathbb{R}^2$ with $c = 0.05$

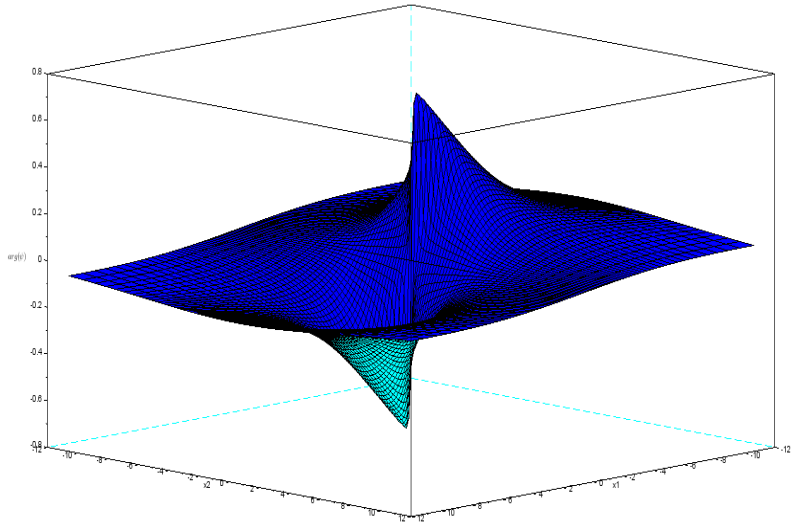

Figure 5.20: The phase of the numerical solution for ψ in $\mathbb{R}^2$ with $c = 0.2$

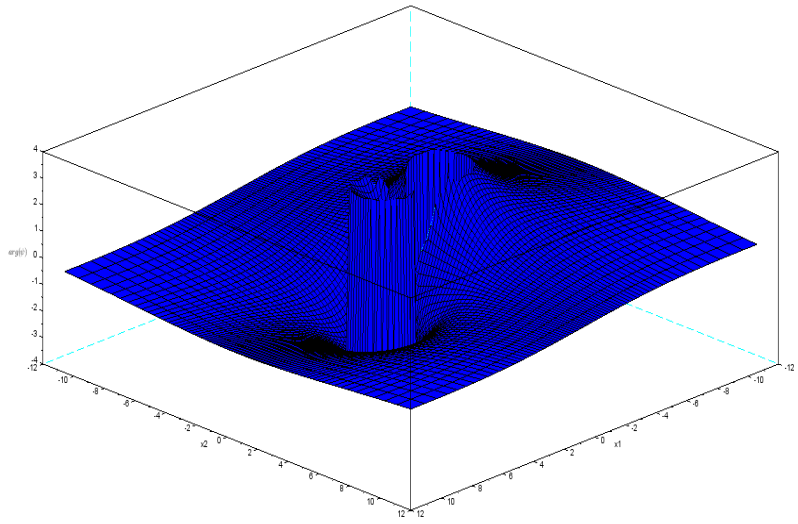

Figure 5.21: The phase of the numerical solution for ψ in $\mathbb{R}^2$ with $c = 1.45$

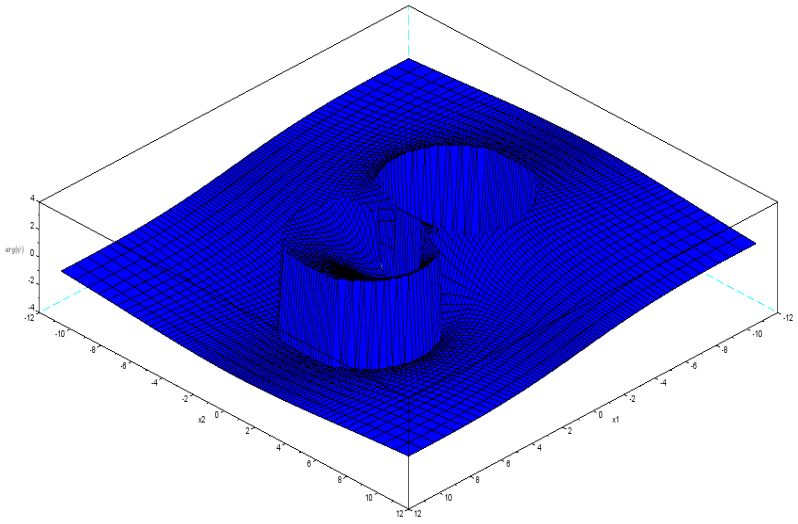

Figure 5.22: The phase of the numerical solution for ψ in $\mathbb{R}^2$ with $c = 2.35$

c	λ obtained by Newton-Raphson
0.05	0.3305474
0.2	0.3304393
1.45	0.3242258
2.35	0.3123219

Table 5.2: Numerical values of λ for the ground state branch

5.4 Ground state in 3D

In dimension 3, the ground state solutions are radial. We consider the minimization problem

Minimize

$$E_{3D}(\psi_{rad,3D}, \varphi_{rad,3D}) = 4\pi \int_0^{\xi} \Big(\psi'^2_{rad,3D} + \frac{1}{2\varepsilon^2}(\psi^2_{rad,3D} - 1)^2 + \frac{1}{\varepsilon^4}\psi^2_{rad,3D}\varphi^2_{rad,3D} + \frac{1}{q^2\varepsilon^2}\varphi'^2_{rad,3D}\Big) r^2 \mathrm{d}r \tag{5.15}$$

under the constraint $4\pi \int_0^{\infty} r^2 |\varphi_{rad,3D}|^2 \mathrm{d}r = 4\pi$.

As we did for the dimension 2, we use the projected Gradient method, choose $\xi = 30$, and initialize the constrained minimization problem, refering to the computations of [6], by

$$\varphi_{rad,3D,i}(r) = A_1 j_0(kr) \tag{5.16}$$

whenever $kr \leq \pi$ (the first zero of j_0), and $\varphi_{rad,3D,i}(r) = 0$ elsewhere, and where $j_0(x) = \frac{\sin(x)}{x}$ is the spherical Bessel function, A_1 is a parameter chosen such that $4\pi \int_0^{\infty} r^2 |\varphi_{rad,3D,i}(r)|^2 \mathrm{d}r = 4\pi$. This last constraint gives us the value of $A_1 = 0.7978846$. For the initialisation of $\psi_{rad,3D}$ we choose a continuous function that is equal to zero in $r = 0$ and approaches to 1 when $rk \geq \pi$ and as $r \to \infty$. Hence we

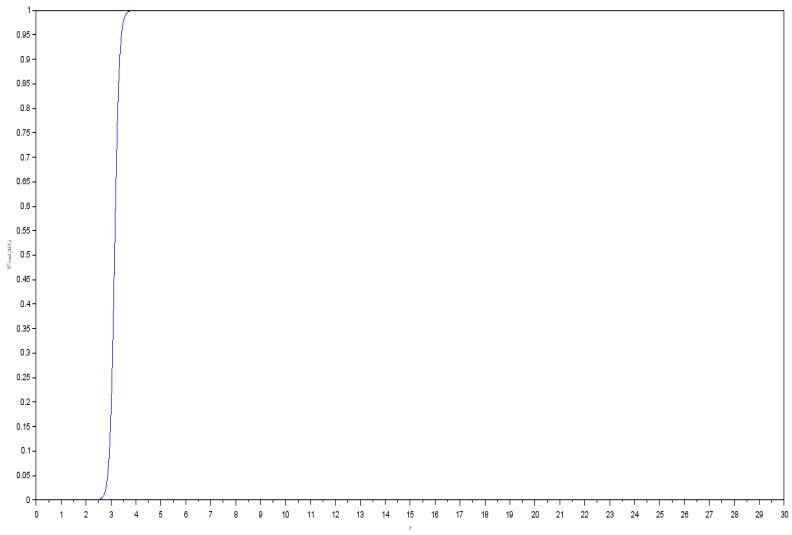

Figure 5.23: The initial data $\psi_{\text{rad,3D,i}}$ for the constrained minimization problem (5.15)

initialize $\psi_{\text{rad,3D}}$ by the function

$$\psi_{\text{rad,3D,i}}(r) = \frac{1+\tanh(\frac{r-\pi/k}{\varepsilon})}{2}. \tag{5.17}$$

We choose $\tilde{N} = 600$ the number of points so that every subdivision has a size of $h = \xi/(\tilde{N}-1) \approx 0.05$, and the step of the projected Gradient method $\rho = h^4$. In the minimization problem, we work with the interval $]0, 30[$ so we have $2\tilde{N}-4$ unknown quantities, and then the values of $\psi_{\text{GS,3D}}$ and $\varphi_{\text{GS,3D}}$ on $r = 0$ and $r = \xi$ are obtained from the Dirichlet conditions. We minimize this energy by fixing the mass or L^2 norm the φ_{GS} at each iteration, as we did in dimension 2. The initial data $\psi_{\text{rad,3D,i}}$ and $\varphi_{\text{rad,3D,i}}$ are given in Figure 5.23 and Figure 5.24 respectively.
After 50 000 iterations, we get an approximation of the minimizers of the energy that are given in Figure 5.25 and a final relative error between two consecutive solutions equal to 2.4×10^{-6}. The value of the energy at each iteration is given in figure 5.26.
The final value of the coefficient α_{n+1} is equal to 0.999855, which gives the numerical value of λ equal to 0.4029429.
We could find some exponential Padé approximations of these minimizers that are given by

$$\psi^{\text{app}}_{\text{GS,3D}}(r) = \frac{-0.0494058e^{\tilde{r}_\psi} + 2.5128964e^{2\tilde{r}_\psi} + 0.3986502e^{3\tilde{r}_\psi} + 1.5034206e^{4\tilde{r}_\psi}}{1 + 26.946355e^{\tilde{r}_\psi} + 4.5772491e^{2\tilde{r}_\psi} + 6.6607466e^{3\tilde{r}_\psi} + 1.5034206e^{4\tilde{r}_\psi}}, \tag{5.18}$$

where $\tilde{r}_\psi = \frac{r-2.5180959}{0.2319942}$ and with L^∞-error equal to 0.007, and

$$\varphi^{\text{app}}_{\text{GS,3D}}(r) = \frac{1.1231657e^{-\tilde{r}_\varphi} + 2.2851536e^{-2\tilde{r}_\varphi} + 0.0008586e^{-3\tilde{r_\varphi}} + 1.8603084e^{-4\tilde{r}_\varphi}}{1 + 4.464733e^{-\tilde{r}_\varphi} + 2.8318444e^{-2\tilde{r}_\varphi} + 3.6186793e^{-3\tilde{r}_\varphi} + 3.0293949e^{-4\tilde{r}_\varphi}}, \tag{5.19}$$

where $\tilde{r}_\varphi = \frac{r-2.0897854}{0.4742224}$, with L^∞-error equal to 0.004. We plot these approximations with the minimizers in the same graph (see Figure 5.27).

5.5 Bubble-vortices solutions

5.5.1 Stationary bubble-vortices

We look now for another branch for the system (TW). We start by the stationary solutions of this branch, called vortex solutions. These are solutions of the system:

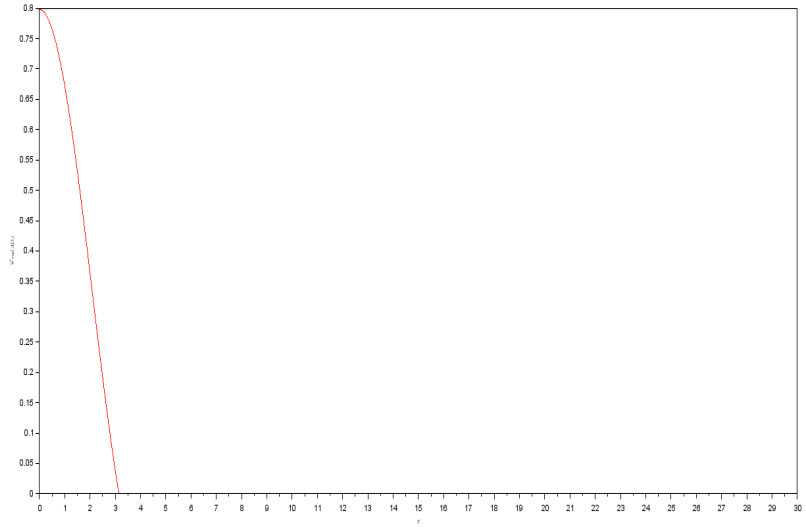

Figure 5.24: The initial data $\varphi_{\mathrm{rad,3D,i}}$ for the constrained minimization problem (5.15)

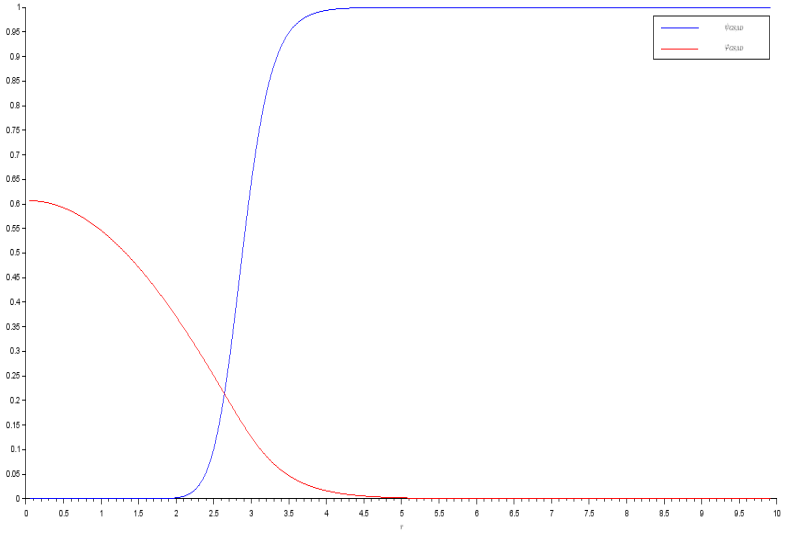

Figure 5.25: The constrained minimizers of the energy E_{3D}

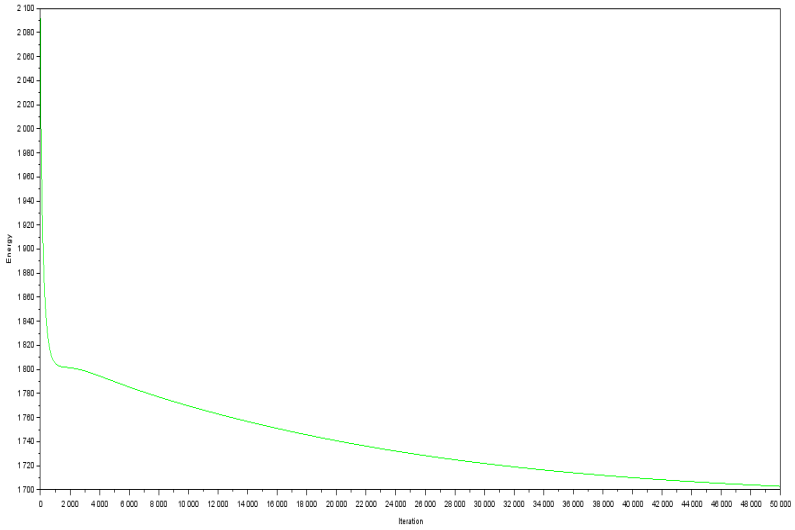

Figure 5.26: The value of the energy at each iteration of the minimization problem (5.15)

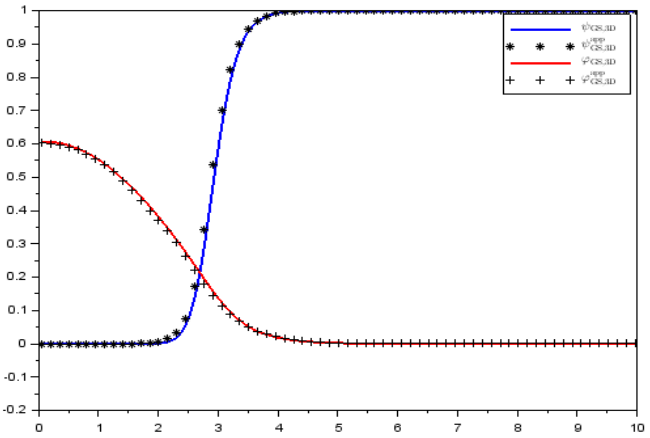

Figure 5.27: The approximation of the constrained minimizers of (5.15)

$$\begin{cases} -\Delta\psi + \frac{1}{\varepsilon^2}(\frac{1}{\varepsilon^2}|\varphi|^2 + |\psi|^2 - 1)\psi & = 0 \\ -\Delta\varphi + \frac{q^2}{\varepsilon^2}|\psi|^2\varphi & = \lambda\varphi \end{cases} \tag{5.20}$$

with

$$\int_{\mathbb{R}^2} \varphi^2 \, \mathrm{d}x = 4\pi.$$

For (GP) (or NLS), there exist some remarkable stationary solutions called vortices. These are stationary solutions to (GP) in 2d of the form

$$\Psi(t,x) = \mathfrak{a}_n(r)\exp(in\theta),$$

in polar coordinates (r,θ), where $n \in \mathbb{Z}$, $n \neq 0$ is the charge, or degree or winding number. Then, Ψ solves the elliptic equation

$$\Delta\Psi = \Psi(|\Psi|^2 - 1), \tag{5.21}$$

which means that the profile $\mathfrak{a}_n : \mathbb{R}_+ \to [0,1]$ is a solution to the ODE

$$\mathfrak{a}_n''(r) + \frac{1}{r}\mathfrak{a}_n'(r) - \frac{n^2}{r^2}\mathfrak{a}_n(r) = \mathfrak{a}_n(r)(\mathfrak{a}_n^2(r) - 1). \tag{5.22}$$

that increases from 0 at $r = 0$ to 1 for $r \gg 1$.
Concerning the (TW) model in 2d, we may look for solutions of (5.20) under the form

$$(\psi,\varphi)(x) = (V_{n,\mathrm{BV}}(x/\varepsilon), \varphi_{n,\mathrm{BV}}(x/\varepsilon)) = (\mathfrak{a}_{n,\mathrm{BV}}(r/\varepsilon)\exp(in\theta), \varphi_{n,\mathrm{BV}}(x/\varepsilon)), \tag{5.23}$$

which yields the system

$$\begin{cases} \Delta V_{n,\mathrm{BV}} = V_{n,\mathrm{BV}}\Big(|V_{n,\mathrm{BV}}|^2 - 1 + \dfrac{|\varphi_{n,\mathrm{BV}}|^2}{\varepsilon^2}\Big) \\ -\Delta\varphi_{n,\mathrm{BV}} + q^2|V_{n,\mathrm{BV}}|^2\varphi_{n,\mathrm{BV}} = \lambda\varphi_{n,\mathrm{BV}}, \end{cases} \tag{5.24}$$

or, in the radial coordinate and with $\varphi_{n,\mathrm{BV}}$ real-valued,

$$\begin{cases} \dfrac{\partial^2\mathfrak{a}_{n,\mathrm{BV}}}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial\mathfrak{a}_{n,\mathrm{BV}}}{\partial r} - \dfrac{n^2}{r^2}\mathfrak{a}_{n,\mathrm{BV}} = \mathfrak{a}_{n,\mathrm{BV}}\Big(\mathfrak{a}_{n,\mathrm{BV}}^2 - 1 + \dfrac{\varphi_{n,\mathrm{BV}}^2}{\varepsilon^2}\Big) \\ -\frac{\partial^2\varphi_{n,\mathrm{BV}}}{\partial r^2} - \frac{1}{r}\frac{\partial\varphi_{n,\mathrm{BV}}}{\partial r} + q^2\mathfrak{a}_{n,\mathrm{BV}}^2\varphi_{n,\mathrm{BV}} = \lambda\varphi_{n,\mathrm{BV}}. \end{cases} \tag{5.25}$$

Here, the mass constraint will be replaced by

$$\int_{\mathbb{R}^2} \varphi_{n,\mathrm{BV}}^2 \, \mathrm{d}x = \mathfrak{m},$$

and λ then depends on $\mathfrak{m}$.

5.5.2 Minimizing the energy

Notice that due to the phase factor, the kinetic energy of $V_{n,\mathrm{BV}}$ is infinite because of the contribution for large r, since we have

$$|\nabla V_{n,\mathrm{BV}}|^2 = |\nabla \mathfrak{a}_{n,\mathrm{BV}}|^2 + \frac{n^2}{r^2}\mathfrak{a}_{n,\mathrm{BV}}^2$$

and the second term will induce in the integral a logarithmic divergence since $\mathfrak{a}_{n,\mathrm{BV}} \to 1$ when r is large. We shall then renormalize the energy by following [16]. Let $\chi = \mathbb{1}_{D(0,1)}$. Then, for $R > 0$, we may write

$$\begin{aligned}
&\int_{D(0,R)} \frac{n^2}{r^2}A^2 + \frac{1}{2\varepsilon^2}(A^2-1)^2\,\mathrm{d}x - 2\pi n^2 \ln R \\
&\quad = \int_{D(0,R)} \chi\Big(\frac{n^2}{r^2}A^2 + \frac{1}{2\varepsilon^2}(A^2-1)^2\Big)\,\mathrm{d}x \\
&\qquad + \int_{D(0,R)} (1-\chi)\Big(\frac{n^2}{r^2}(A^2-1) + \frac{1}{2\varepsilon^2}(A^2-1)^2\Big)\,\mathrm{d}x \\
&\quad = \int_{D(0,R)} \chi\Big(\frac{n^2}{r^2}A^2 + \frac{1}{2\varepsilon^2}(A^2-1)^2\Big)\,\mathrm{d}x \\
&\qquad + \int_{D(0,R)} \frac{1-\chi}{2\varepsilon^2}\Big(A^2 - 1 + \frac{\varepsilon^2 n^2}{r^2}\Big)^2\,\mathrm{d}x - \frac{\varepsilon^2 n^4}{2}\int_{D(0,R)} \frac{1-\chi}{r^4}\,\mathrm{d}x.
\end{aligned}$$

Since $(1-\chi)/r^4 \in L^1(\mathbb{R}^2)$, we therefore define, for $A \in 1 + H^1(\mathbb{R}^2)$ and $\varphi \in H^1(\mathbb{R}^2)$,

$$\begin{aligned}
E_{n,\mathrm{BV}}(A,\varphi) = \int_{\mathbb{R}^2} &|\nabla A|^2 + \frac{1}{\varepsilon^2 q^2}|\nabla\varphi|^2 + \frac{1}{\varepsilon^4}A^2\varphi^2 \\
&+ \chi\Big(\frac{n^2}{r^2}A^2 + \frac{1}{2\varepsilon^2}(A^2-1)^2\Big) + \frac{1-\chi}{2\varepsilon^2}\Big(A^2 - 1 + \frac{\varepsilon^2 n^2}{r^2}\Big)^2 \\
&- \frac{\varepsilon^2 n^4(1-\chi)}{2r^4}\,\mathrm{d}x.
\end{aligned} \tag{5.26}$$

Then, $E_{n,\mathrm{BV}}$ is a renormalization of the energy in the sense that

$$E_{n,\mathrm{BV}}(A,\varphi) = \lim_{R\to+\infty}\Big[E_{D(0,R)}(A\exp(in\theta),\varphi) - 2\pi n^2 \ln R\Big].$$

For (TW), we look for stationary solutions that look like two vortices for ψ and a sum of two Gaussians for φ (as we did for the solutions with small mass). For $\chi = \mathbb{1}_{D(0,1)}$, we consider the constrained minimization problem, for $A \in 1 + H^1(\mathbb{R}^2)$, $\varphi \in H^1(\mathbb{R}^2)$

Minimize

$$\begin{aligned}
E_{n,\mathrm{BV}}(A,\varphi) = \int_{\mathbb{R}^2} &|\nabla A|^2 + \frac{1}{\varepsilon^2 q^2}|\nabla\varphi|^2 + \frac{1}{\varepsilon^4}A^2\varphi^2 \\
&+ \chi\Big(\frac{n^2}{r^2}A^2 + \frac{1}{2\varepsilon^2}(A^2-1)^2\Big) + \frac{1-\chi}{2\varepsilon^2}\Big(A^2 - 1 + \frac{\varepsilon^2 n^2}{r^2}\Big)^2 \\
&- \frac{\varepsilon^2 n^4(1-\chi)}{2r^4}\,\mathrm{d}x
\end{aligned} \tag{5.27}$$

under the constraint $2\pi\displaystyle\int_0^\infty |\,\varphi(x)\,|^2\,\mathrm{d}x = 4\pi$.

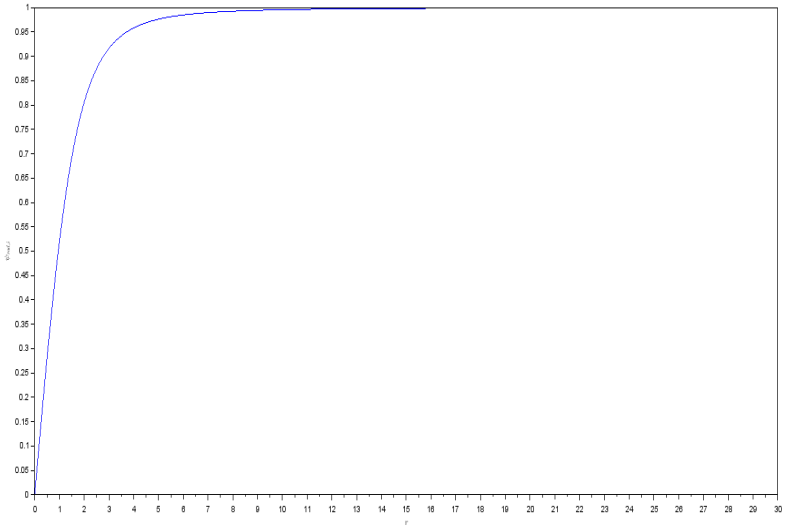

Figure 5.28: The initial data $\psi_{\text{rad,BV,i}}$ for the minimization problem

We will start minimizing this energy to find a numerical stationary solution to (TW). The last term of $E_{n,\text{BV}}$ being constant, we can ignore it in the minimization problem.
To minimize this energy, we start from the same initializations (in radial coordinates) that we used in Newton-Raphson algorithm with separated iterations method with small mass (see Figure 5.28 and 5.29). In other words, we initialize ψ or A by

$$\psi_{\text{rad,BV,i}}(r) = r\sqrt{\frac{\alpha_1 + \alpha_2 r^2}{1 + \beta_1 r^2 + \beta_2 r^4}}$$

where

$$\alpha_1 = 0.3350601; \quad \alpha_2 = 0.0494196; \quad \beta_1 = 0.3725704; \quad \beta_2 = \alpha_2.$$

and φ by

$$\varphi_{\text{rad,BV,i}}(r) = 2\alpha e^{\frac{-r^2}{2\sigma^2}}$$

where

$$\sigma^2 = 71.21960411848; \quad \mathfrak{m} = 4\pi; \quad \alpha = \frac{1}{\varepsilon}\sqrt{\frac{\mathfrak{m}}{2\pi\sigma^2}}.$$

Note that $\mathfrak{m} = 4\pi$, but could be taken 2π or any other value if necessary.
The interval, the parameters and the step for the gradient method are equal to those chosen for the minimization of the energy for the ground state solutions, namely, we work in polar coordinates in the interval $[0, 30]$, with $\varepsilon = 0.187$, $q = 0.41$, and $\rho = 10^{-4}$. We search for vortices of degree 1 so that $n = 1$. We choose $\chi = 1$ to simplify the energy and make some terms vanish, since $0 < r < 30$. The constrained minimizers and the value of the energy at each iteration of the projected Gradient method are given in Figure 5.30 and Figure 5.31.
The final relative error is equal to 0.0000116, and the value of λ is calculated from the constrained minimization problem, as for the ground state solutions, and is equal to

$$\lambda_{1,\text{BV}} = \frac{1 - \alpha_{n+1}}{\rho} \times \frac{\varepsilon^2}{2} = 0.2549954.$$

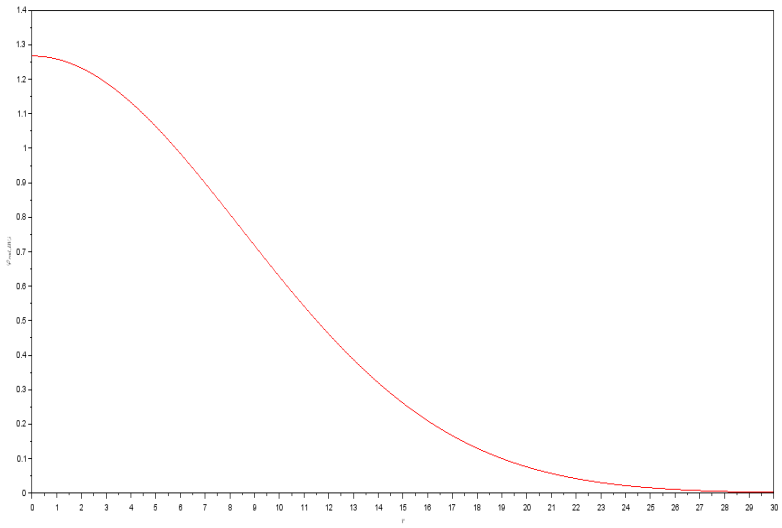

Figure 5.29: The initial data $\varphi_{\mathrm{rad,BV,i}}$ for the minimization problem

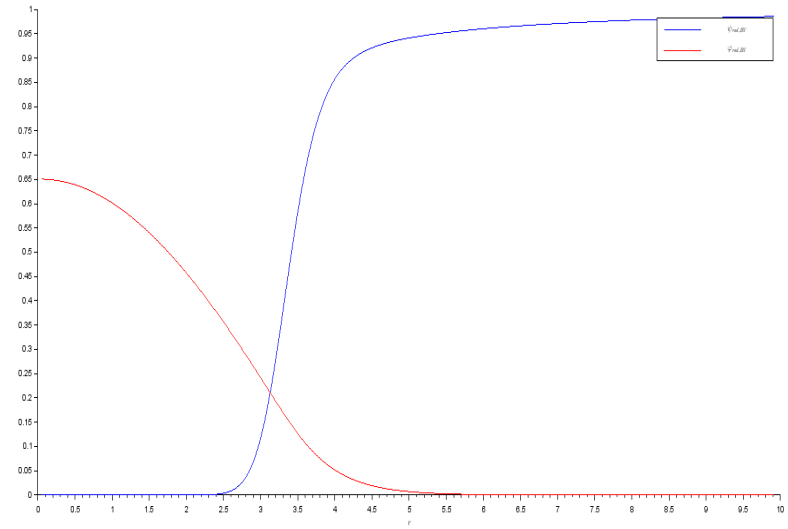

Figure 5.30: The minimizers of the energy $E_{1,\mathrm{BV}}$

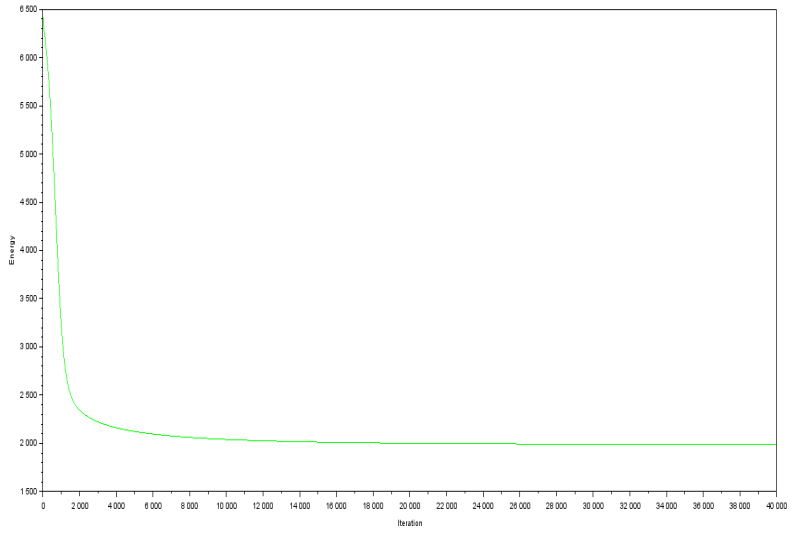

Figure 5.31: The value of the energy $E_{1,\mathrm{BV}}$ at each iteration

As we did for the ground state, we could approximate these minimizers by some exponential-Padés that we used to obtain the two vortices of mass 2π each one (see Figure 5.32). Indeed, we run the code of the minimization of the energy with mass 2π instead of 4π. With the obtained minimizers, we did the same work as for the ground state solutions and get

$$\psi_{1,\mathrm{BV}}^{\mathrm{app}}(r) = \frac{8.7799e^{\tilde{r}_\psi} + 8.6056e^{2\tilde{r}_\psi}}{1 + 10.288e^{\tilde{r}_\psi} + 8.6056e^{2\tilde{r}_\psi}}, \tag{5.28}$$

where $\tilde{r}_\psi = \frac{r-3.5390}{0.19470}$, with L^∞-error equal to 0.012, and

$$\varphi_{1,\mathrm{BV}}^{\mathrm{app}}(r) = \frac{2.2760e^{-\tilde{r}_\varphi} + 4.9087e^{-2\tilde{r}_\varphi} - 0.86256e^{-3\tilde{r_\varphi}} + 1.1311e^{-4\tilde{r}_\varphi}}{1 + 8.9986e^{-\tilde{r}_\varphi} + 6.7775e^{-2\tilde{r}_\varphi} + 0.92214e^{-3\tilde{r}_\varphi} + 1.6446e^{-4\tilde{r}_\varphi}}, \tag{5.29}$$

where $\tilde{r}_\varphi = \frac{r-2.2753439}{0.3978393}$, with L^∞-error equal to 0.017.

5.5.3 The Newton-Raphson algorithm for the vortex branch

We now look for travelling wave solutions to system (GC).
Since we have removed the transport term in the equation for φ, we are allowed to take φ real-valued. To find the vortex branch of (TW) with small $c > 0$, we used the Newton-Raphson Algorithm starting from the stationary solutions obtained in the previous section. In $2D$ the 2 vortices are obtained using the approximate minimizers obtained in the previous section and are presented in Figure 5.33 and Figure 5.34.
For $c = 0.2$, the numerical solutions $\psi_{1,\mathrm{BV}}$ and $\varphi_{1,\mathrm{BV}}$ are obtained by Newton-Raphson algorithm after 28 iterations with a final relative error of order 10^{-5}, and presented in Figure 5.35 and Figure 5.36. The value of λ obtained by Newton-Raphson is equal to 0.4615025.
The contour lines of these numerical solutions are presented in Figure 5.37 and Figure 5.38, and their modulus in the half-space $x_1 \geq 0$ in Figure 5.39 and Figure 5.40.
If we compare the vortices of (GP) obtained in [5] and our vortices for (TW), we see that the ones of the system are much thicker, and the area where ψ is small is much larger.

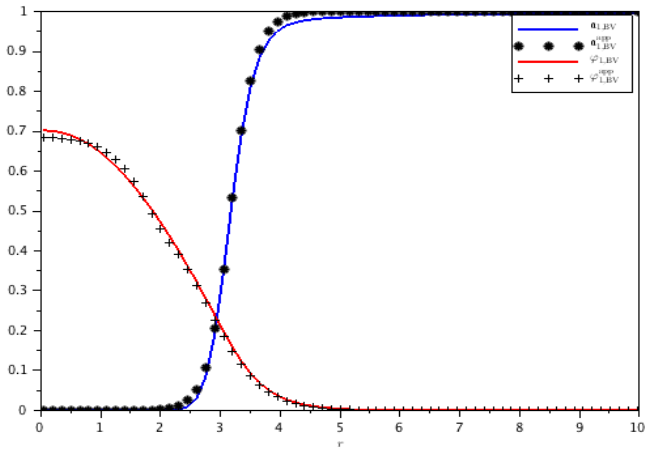

Figure 5.32: The exponential-Padé approximants of the constrained minimizers of the energy $E_{1,BV}$

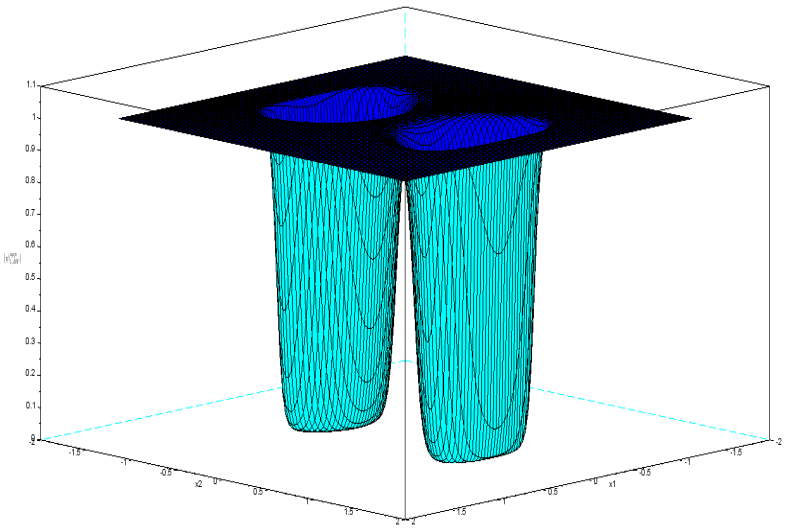

Figure 5.33: The initial data for ψ in the Newton-Raphson algorithm

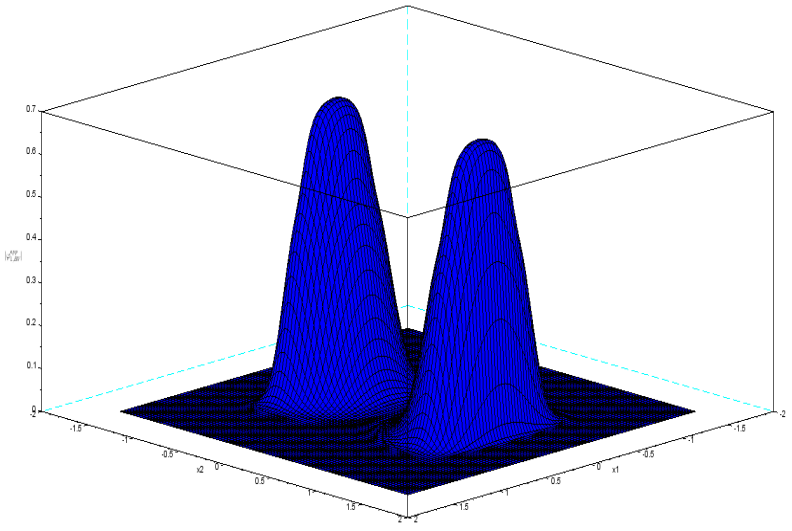

Figure 5.34: The initial data for φ in the Newton-Raphson algorithm

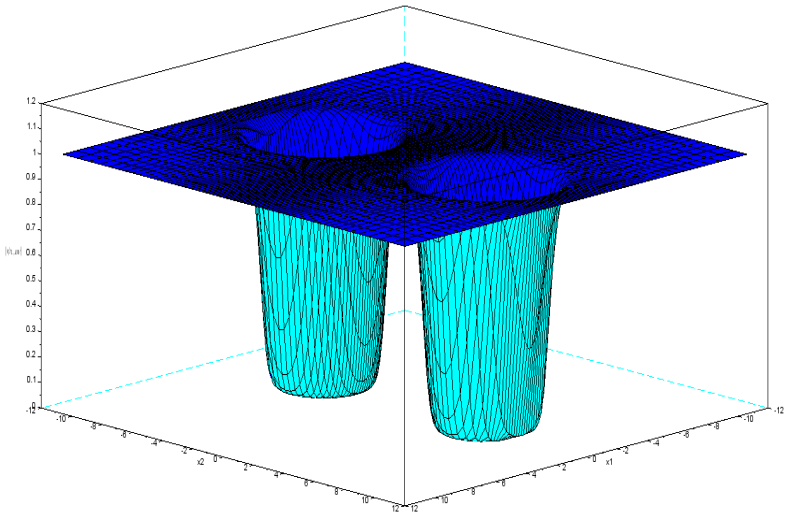

Figure 5.35: The numerical solution for ψ in all $\mathbb{R}^2$ with $c = 0.2$

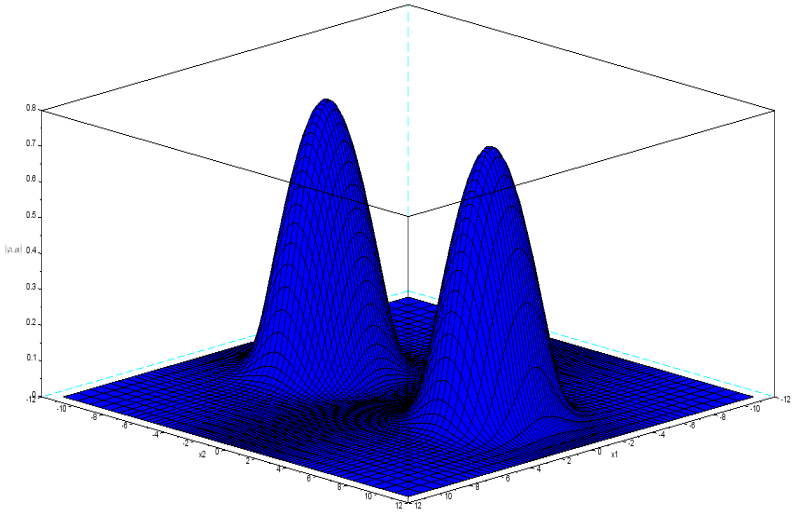

Figure 5.36: The numerical solution for φ in all $\mathbb{R}^2$ with $c = 0.2$

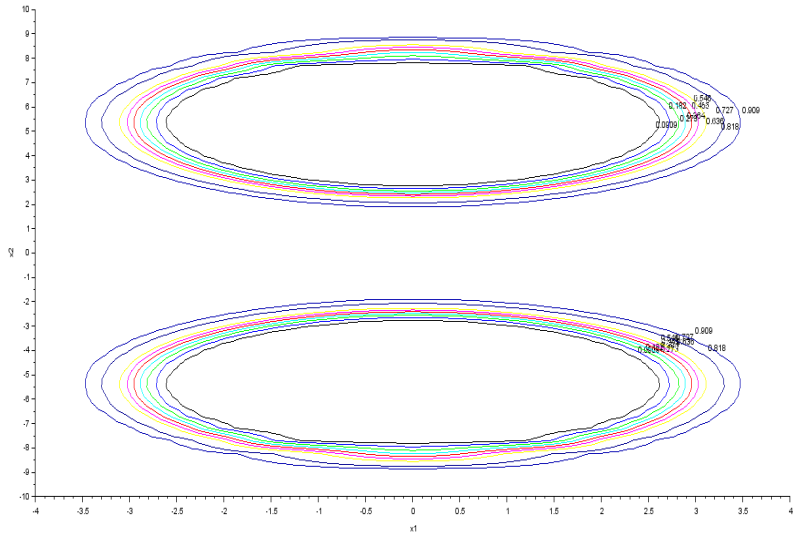

Figure 5.37: The contour lines of the numerical solution of ψ in $\mathbb{R}^2$ with $c = 0.2$

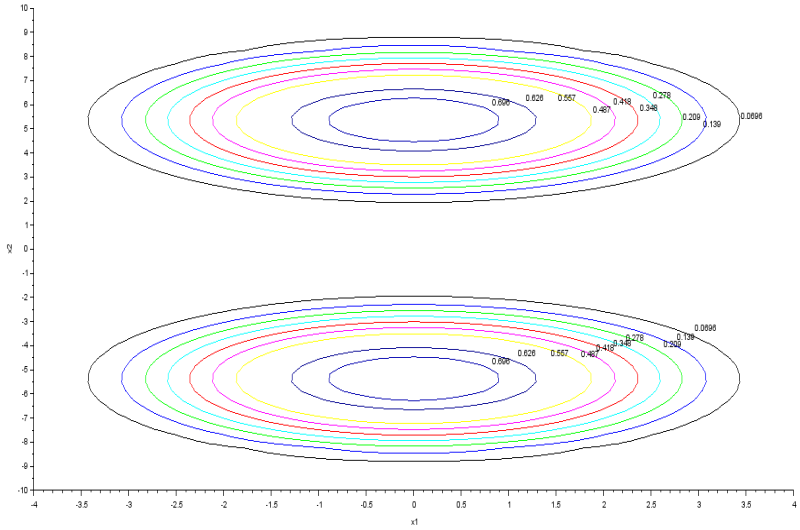

Figure 5.38: The contour lines of the numerical solution of φ in $\mathbb{R}^2$ with $c = 0.2$

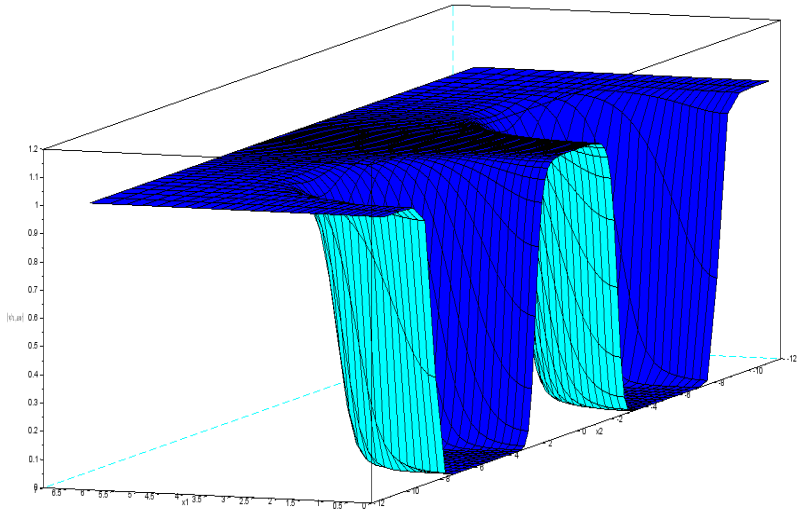

Figure 5.39: The numerical solution of ψ in the half-space $x_1 \geq 0$ with $c = 0.2$

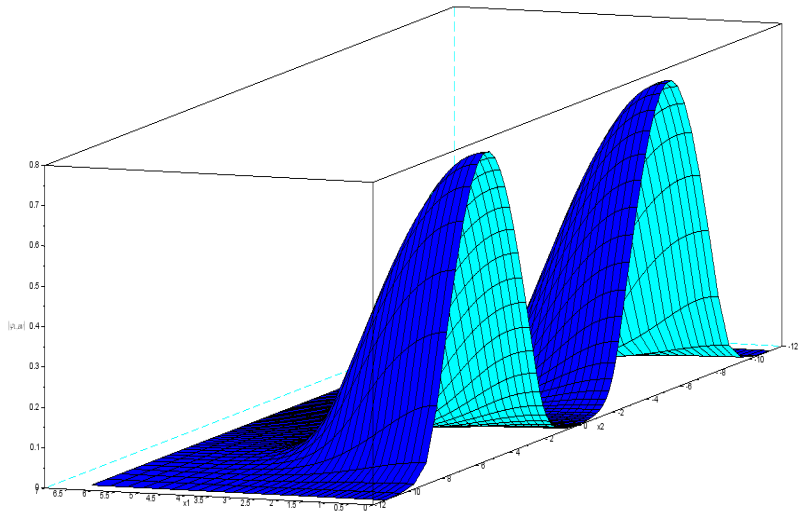

Figure 5.40: The numerical solution of φ in the half-space $x_1 \geq 0$ with $c = 0.2$

5.5.4 The energy-momentum diagram

The Newton-Raphson Algorithm for the vortex branch converges for different values of c. For every value of c for which Newton-Raphson method converges, we calculated the Energy and the Momentum of the numerical solutions. Some of these quantities are represented in Table 5.3:

c	Momentum	Energy
0.12	93.605614	1741.3084
0.137	84.894592	1740.9902
0.15	76.624484	1740.4648
0.157	73.617822	1740.2998
0.168	64.71966	1739.3533
0.18	61.42872	1739.1495
0.19	59.451292	1738.7097
0.7	16.3374398	1267.0911
0.9	21.341026	1271.2986
1.5	38.441068	1292.8946
2.2	67.554076	1349.0038
2.35	77.308418	1371.3961

Table 5.3: Numerical values of the momentum and the energy for the vortex branch

We could not find any value of c between 0.19 and 0.7 for which the algorithm converges. We noticed that when c increases, the two vortices approach, and starting $c = 0.7$, they superpose and the initializations and the obtained solutions for the vortex branch do not present vortices, but are the same as the ground state branch, and have the same energy and momentum. The energy-momentum diagram for the vortex branch is presented in the upper part in Figure 5.41 with the one for the ground state branch. The upper branch correspond to the small values of c $(0.12 - 0.19)$ for the vortex branch, and the lower one corresponds to the great ones $(0.7 - 2.35)$, and similar to the branch obtained in Figure 5.16, and where the solutions are not vortices anymore.

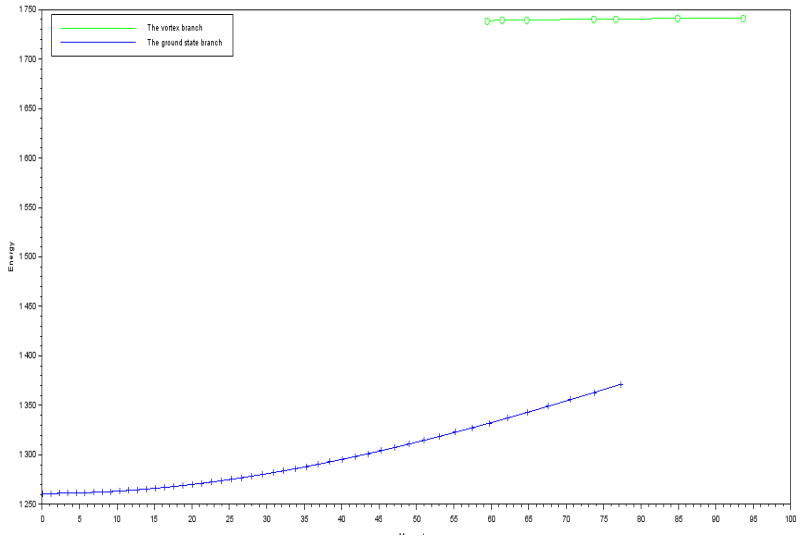

Figure 5.41: The energy-momentum diagram for the vortex and the ground state branches

5.6 Some recalls

We first recall the Pohozaev identities (see [14] Proposition 4.6).

Proposition 37. *Let $c \in \mathbb{R}$ and (ψ, φ) be a solution of (TW) of finite energy. Then the following hold.*

1. *The functions ψ and φ are bounded and C^∞ and $\varphi, \nabla\psi \in W^{k,p}(\mathbb{R}^N)$ for any $k \in \mathbb{N}$ and $p \geqslant 2$.*
2. *There exist $R_* \geqslant 0$ and a real valued function Θ such that $\psi = |\psi| \exp(i\Theta)$ on $\mathbb{R}^N \setminus B(0, R_*)$ and $\nabla\Theta \in W^{k,p}(\mathbb{R}^N)$ for any $k \in \mathbb{N}$ and $p \geq 2$.*
3. *Let $\chi \in C^\infty(\mathbb{R}^N)$ be a cut-off function such that $\chi = 0$ on $B(0, 2R_*)$ and $\chi = 1$ on $\mathbb{R}^N \setminus B(0, 3R_*)$. We have $\psi_1 \frac{\partial \psi_2}{\partial x_1} - \psi_2 \frac{\partial \psi_1}{\partial x_1} - \frac{\partial}{\partial x_1}(\chi\Theta) \in L^1(\mathbb{R}^N)$ and the two following Pohozaev-type identites hold:*

$$\begin{aligned} &\int_{\mathbb{R}^N} -\Big|\frac{\partial \psi}{\partial x_1}\Big|^2 - \frac{1}{\varepsilon^2 q^2}\Big|\frac{\partial \varphi}{\partial x_1}\Big|^2 + \sum_{j=2}^{N}\Big(\Big|\frac{\partial \psi}{\partial x_j}\Big|^2 + \frac{1}{\varepsilon^2 q^2}\Big|\frac{\partial \varphi}{\partial x_j}\Big|^2\Big)\,\mathrm{d}x \\ &+ \int_{\mathbb{R}^N} \frac{1}{2\varepsilon^2}(|\psi|^2 - 1)^2 + \frac{1}{\varepsilon^4}|\psi|^2|\varphi|^2 - \frac{k^2}{\varepsilon^2 q^2}|\varphi|^2 \mathrm{d}x = 0 \end{aligned} \tag{5.30}$$

and

$$\begin{aligned} &(N-2)\int_{\mathbb{R}^N} |\nabla\psi|^2 + \frac{1}{\varepsilon^2 q^2}|\nabla\varphi|^2\,\mathrm{d}x \\ &+ N\int_{\mathbb{R}^N} \frac{1}{2\varepsilon^2}(|\psi|^2 - 1)^2 + \frac{1}{\varepsilon^4}|\psi|^2|\varphi|^2 - \frac{k^2}{\varepsilon^2 q^2}|\varphi|^2\,\mathrm{d}x \\ &= c(N-1)\int_{\mathbb{R}^N} \mathrm{Re}(\psi)\frac{\partial \mathrm{Im}(\psi)}{\partial x_1} - \mathrm{Im}(\psi)\frac{\partial \mathrm{Re}(\psi)}{\partial x_1} - \frac{\partial}{\partial x_1}(\chi\Theta)\,\mathrm{d}x \\ &+ \frac{2c\delta(N-1)}{\varepsilon^2 q^2}\int_{\mathbb{R}^N} \varphi_1 \frac{\partial \varphi_2}{\partial x_1}\,\mathrm{d}x. \end{aligned} \tag{5.31}$$

We have also the following non existence result (see [14], Theorem 4.7).

Theorem 38. *Let $N \geqslant 2$ and let (ψ, φ) be a finite-energy solution of (TW). Assume that either*

1. $c > \frac{2}{\varepsilon^2}$, *or*
2. $N = 2$ *and* $c^2 = \frac{2}{\varepsilon^2}$, *or*
3. $N \geqslant 3$, $c^2 = \frac{2}{\varepsilon^2}$, *and* $\psi_1 \frac{\partial \psi_2}{\partial x_1} - \psi_2 \frac{\partial \psi_1}{\partial x_1} \in L^{\frac{2N-1}{2N-3}}(\mathbb{R}^N)$.

Then $\varphi = 0$ and ψ is constant on $\mathbb{R}^N$.

5.7 Formal aspects

5.7.1 Asymptotic expansion at infinity

In this section, we will focus on the asymptotic expansion at infinity for the travelling waves of the (GP) equation, because φ is considered exponentially small. This study was also done by Bouchel in [2] where he proved that under suitable assumptions on the parameters c, δ, q, and k, the solution (ψ, φ) of the three dimensional system (TW) is in $C^\infty(\mathbb{R}^3) \times C^\infty(\mathbb{R}^3)$, moreover, ψ tends polynomially to 1 and φ tends exponentially to 0. Travelling wave solutions for (GP) have to satisfy the equation

$$ic\frac{\partial \psi}{\partial x} = \Delta\psi + \psi(1 - |\psi|^2) \tag{5.32}$$

In this section we follow the strategy of [4] to study the asymptotic behaviour of the solution to the equation (5.32).

Clearly $\psi = 1$ obeys to (GP) with the boundary condition $\psi \to 1$ as $|(x, y)| \to \infty$. Neighbouring states are obtained by writing

$$\psi = 1 + \psi' = 1 + \psi'_r + i\psi'_i \tag{5.33}$$

substituting into (5.32), linearising these with respect to ψ', and separating their real and imaginary parts, we obtain

$$c\frac{\partial \psi'_i}{\partial x} = -\Delta\psi'_r + 2\psi'_r \tag{5.34}$$

$$c\frac{\partial \psi'_r}{\partial x} = \Delta\psi'_i \tag{5.35}$$

where $(\psi'_r, \psi'_i) \to 0$ for $r = |(x, y)| \to \infty$. Thus ψ'_r and ψ'_i obey

$$(\Delta^2 - 2\Delta + c^2\frac{\partial^2}{\partial x^2})(\psi'_r, \psi'_i) = 0. \tag{5.36}$$

Considering that the Δ^2 term is negligible to leading order, we see that we can discard the term $\Delta\psi'_r$ in (5.34) so that

$$\psi'_r = \frac{c}{2}\frac{\partial \psi'_i}{\partial x} \tag{5.37}$$

$$\Delta\psi'_i = c\frac{\partial \psi'_r}{\partial x} \tag{5.38}$$

and

$$(\Delta - \frac{c^2}{2}\frac{\partial^2}{\partial x^2})(\psi_r', \psi_i') = 0 \tag{5.39}$$

We make the transformation

$$x' = x$$
$$y' = y(1 - \frac{c^2}{2})^{\frac{1}{2}}$$

And this could be real if we consider that $c < \sqrt{2}$. We write ψ_i' as Fourier serie

$$\psi_i' = \frac{a_0}{2} + \Sigma_{n\geq 1} a_n(r')\cos(n\theta') + \Sigma_{n\geq 1} b_n(r')\sin(n\theta'). \tag{5.40}$$

ψ_i' being even in θ' we can suppose that $b_n = 0$ for all $n \geq 1$ and $a_0 = 0$ because $\psi_i' \to 0$ as $r' \to \infty$. Supposing that the terms a_n are of the form $\frac{\alpha_n}{r'^n}$ we can write

$$\psi_i' = \Sigma_{n\geq 1}\frac{\alpha_n}{r'^n}\cos(n\theta').$$

Using this result, we can obtain ψ_r' from (5.37). For sufficiently large r' we may therefore write

$$\psi_r \sim 1 + \frac{c}{2}\frac{\alpha_1(x^2 + y^2(1-\frac{c^2}{2})) - 2\alpha_1 x^2}{(x^2 + y^2(1-\frac{c^2}{2}))^2}$$
$$\psi_i \sim \frac{\alpha_1 x}{x^2 + y^2(1-\frac{c^2}{2})}$$

5.7.2 Hamilton equation

For the travelling waves for the (GP) equation, a branch of solutions has been computed. It may be parametrized by the speed c of propagation, and the following Hamilton equation has been derived (see [8], equation (2.46))

$$c = \frac{\partial E}{\partial Q} \qquad \text{or} \qquad \frac{dE}{dc} = c\frac{dQ}{dc}. \tag{5.41}$$

The computation remains valid in our situation, with Q the momentum of ψ, Let us then assume that we have a smooth curve $c \mapsto (\psi_c, \varphi_c)$ such that, for any c, (ψ_c, φ_c) solves (TW) for some λ depending smoothly on c and with, for some $\mathfrak{m} > 0$ given independent of c,

$$\int_{\mathbb{R}^N} |\varphi_c|^2 \,\mathrm{d}x = \mathfrak{m}.$$

Then, taking the scalar product of the first equation of (TW) by $\partial_c\psi_c$ and taking the scalar product of the second equation of (TW) by $\partial_c\varphi_c$ yields

$$\begin{cases} \dfrac{d}{dc}\displaystyle\int_{\mathbb{R}^N} |\nabla\psi_c|^2 + \frac{1}{2\varepsilon^2}(|\psi_c|^2 - 1)^2 \,\mathrm{d}x + \frac{1}{\varepsilon^4}\int_{\mathbb{R}^N} \partial_c(|\psi_c|^2)|\varphi_c|^2 \,\mathrm{d}x \\ \qquad\qquad = 2c\displaystyle\int_{\mathbb{R}^N} \left\langle \frac{\partial\psi_c}{\partial x_1}, i\partial_c\psi_c \right\rangle \mathrm{d}x \\ \dfrac{d}{dc}\displaystyle\int_{\mathbb{R}^N} |\nabla\varphi_c|^2 \,\mathrm{d}x + \frac{q^2}{\varepsilon^2}\int_{\mathbb{R}^N} |\psi_c|^2\partial_c(|\varphi_c|^2)\,\mathrm{d}x = \lambda_c\frac{d}{dc}\displaystyle\int_{\mathbb{R}^N} |\varphi_c|^2 \,\mathrm{d}x = 0, \end{cases}$$

in view of the normalisation of φ_c. Consequently,

$$\frac{d}{dc}E(\psi_c, \varphi_c) = 2c\int_{\mathbb{R}^N} \left\langle \frac{\partial\psi_c}{\partial x_1}, i\partial_c\psi_c \right\rangle \mathrm{d}x = c\frac{d}{dc}Q(\psi_c),$$

as claimed.

5.7.3 Stability issues

For the (GP) equation, it is conjectured that travelling waves are stable when $\frac{d}{dc}Q(\psi_c) < 0$, that is $Q \mapsto E$ is concave (by the Hamilton equation (5.41)), and unstable when $\frac{d}{dc}Q(\psi_c) > 0$, that is $Q \mapsto E$ is convex. In this direction, we may quote the works [10] (in 1d), [11] (in 3d). This is the usual stability criterion obtained in [17] and [7] for bound states for NLS. In [4], the travelling waves are obtained by minimization of the energy at fixed momentum (when this problem has minimizers), and it follows from this result that the set of minimizers is orbitally stable by the Nonlinear Schrödinger flow. This result assumes in particular that the Hessian of the action at the travelling wave has only one negative eigenvalue.
For our problem, the situation is quite different since on the one hand, the (scalar) momentum $Q(\Psi)$ is not conserved by the (TW) flow, and on the other hand, due to the mass constraint on φ, the Hessian at the travelling wave (ψ_c, φ_c) may have two negative eigenvalues, but it is not very clear to determine what happens on the constraint sphere for φ. Furthermore, the mass constraint $\int_{\mathbb{R}^N} |\varphi|^2 = \mathfrak{m}$ does not enter in the framework of [7]. Therefore, we do not make any claim concerning the stability of the travelling waves we obtain here.

5.8 Rigorous result

The next statement of this chapter is an existence result of a ground state and was done in Chapter 3, but we repost it here for a recall.

Theorem 39. *Assume that $1 \leqslant N \leqslant 3$ and that we work for the physical parameters (5.7) and (5.8). Then, there exists at least one minimizer $(\psi_{\rm GS}, \varphi_{\rm GS})$ for the problem*

$$\inf\{E(\psi,\varphi), \psi \in \mathcal{E}, \varphi \in H^1(\mathbb{R}^N) \ s.t. \ \int_{\mathbb{R}^N} |\varphi|^2 \,\mathrm{d}x = \mathfrak{m}\}.$$

For $\mathfrak{m} > 0$ we consider the minimization problem

$$(GS_{\mathfrak{m}}) \qquad \text{minimize } E(\psi,\varphi) \text{ for } \psi \in \mathcal{E},\ \varphi \in H^1(\mathbb{R}^N) \text{ satisfying } \int_{\mathbb{R}^N} |\varphi|^2 \,\mathrm{d}x = \mathfrak{m}.$$

We define

$$\mathcal{G}_{\min}(\mathfrak{m}) = \inf\left\{ E(\psi,\varphi) \mid \psi \in \mathcal{E}, \varphi \in H^1(\mathbb{R}^N), \int_{\mathbb{R}^N} |\varphi|^2 \,\mathrm{d}x = \mathfrak{m} \right\}. \tag{5.42}$$

Proposition 40. *The function $\mathcal{G}_{\min}$ has the following properties:*

(i) $\mathcal{G}_{\min}$ is non-decreasing and concave on $(0,\infty)$, and $0 \leqslant \mathcal{G}_{\min}(\mathfrak{m}) \leqslant \frac{\mathfrak{m}}{\varepsilon^4}$ for all $\mathfrak{m} > 0$.

(ii) If $N = 1$ we have $\mathcal{G}_{\min}(\mathfrak{m}) < \dfrac{\mathfrak{m}}{\varepsilon^4}$ for any $\mathfrak{m} > 0$ and $\lim\limits_{\mathfrak{m}\to 0} \dfrac{\mathcal{G}_{\min}(\mathfrak{m})}{\mathfrak{m}} = \dfrac{1}{\varepsilon^4}$.

(iii) If $N \geqslant 2$, there exists $\mathfrak{m}_0 > 0$ such that $\mathcal{G}_{\min}(\mathfrak{m}) = \dfrac{\mathfrak{m}}{\varepsilon^4}$ for any $\mathfrak{m} \in (0, \mathfrak{m}_0]$.

(iv) There exists $C > 0$ such that $\mathcal{G}_{\min}(\mathfrak{m}) \leqslant C\mathfrak{m}^{\frac{N}{N+2}}$.

Proof. (i) Since $E(\psi,\varphi) \geqslant 0$ for any ψ and φ, it is obvious that $\mathcal{G}_{\min}(\mathfrak{m}) \geqslant 0$. Consider $\phi \in C_c^\infty(\mathbb{R}^N)$ such that $\int_{\mathbb{R}^N} |\phi|^2 \,\mathrm{d}x = \mathfrak{m}$ and let $\varphi_\sigma(x) = \frac{1}{\sigma^{N/2}}\phi\left(\frac{x}{\sigma}\right)$. Then $\int_{\mathbb{R}^N} |\varphi_\sigma|^2 \,\mathrm{d}x = \mathfrak{m}$ and $\int_{\mathbb{R}^N} |\nabla\varphi_\sigma|^2 \,\mathrm{d}x = \frac{1}{\sigma^2}\int_{\mathbb{R}^N} |\nabla\phi|^2 \,\mathrm{d}x$. Taking $\psi = 1$ we see that $\mathcal{G}_{\min}(\mathfrak{m}) \leqslant E(1,\varphi_\sigma)$ for all $\sigma > 0$, and letting $\sigma \longrightarrow \infty$ we get $\mathcal{G}_{\min}(\mathfrak{m}) \leqslant \frac{\mathfrak{m}}{\varepsilon^4}$.

Writing $\varphi = \sqrt{\mathfrak{m}}\tilde{\varphi}$, we see that

$$\mathcal{G}_{\min}(\mathfrak{m}) = \inf\left\{ \int_{\mathbb{R}^N} |\nabla\psi|^2 + V(|\psi|^2) + \frac{\mathfrak{m}}{\varepsilon^2 q^2}|\nabla\tilde{\varphi}|^2 + \frac{\mathfrak{m}}{\varepsilon^4}|\psi|^2|\tilde{\varphi}|^2 \,\mathrm{d}x \;\Big|\right.$$
$$\left. \psi \in \mathcal{E}, \tilde{\varphi} \in H^1(\mathbb{R}^N), \textstyle\int_{\mathbb{R}^N} |\tilde{\varphi}|^2 \,\mathrm{d}x = 1 \right\}. \tag{5.43}$$

For any $(\psi, \tilde{\varphi}) \in \mathcal{E} \times H^1(\mathbb{R}^N)$, the mapping $\mathfrak{m} \longmapsto \int_{\mathbb{R}^N} |\nabla\psi|^2 + V(|\psi|^2) + \frac{\mathfrak{m}}{\varepsilon^2 q^2}|\nabla\tilde{\varphi}|^2 + \frac{\mathfrak{m}}{\varepsilon^4}|\psi|^2|\tilde{\varphi}|^2 \,\mathrm{d}x$ is concave and non-decreasing, and the infimum of a family of concave and non-decreasing functions is a concave and non-decreasing function, too.

In all what follows, the various constants C_1, C_2, etc. depend on ε.

(ii) Consider $\chi \in C_c^\infty(\mathbb{R}^N)$ such that $0 \leqslant \chi \leqslant 1$, $\chi = 1$ on $B(0,1)$ and $\chi = 0$ on $\mathbb{R}^N \setminus B(0,2)$. Denote $A = \|\chi\|_{L^2}^2$, $B = \|\nabla\chi\|_{L^2}^2$, and $D = \int_{\mathbb{R}^N} \chi^4 \,\mathrm{d}x$. Let $\kappa \in (0,1)$ (to be chosen later) and let

$$\psi_{a,b}(x) = 1 - \kappa a^2 \chi^2\left(\frac{x}{b}\right), \qquad \varphi_{a,b}(x) = a\chi\left(\frac{x}{b}\right).$$

Fix $\mathfrak{m} > 0$. If $a \in (0,1)$ and $b > 0$ are chosen so that $a^2 b^N A = \mathfrak{m}$, we have $\|\varphi_{a,b}\|_{L^2}^2 = \mathfrak{m}$. It is clear that $\|\nabla\varphi_{a,b}\|_{L^2}^2 = a^2 b^{N-2} B$ and $\|\nabla\psi_{a,b}\|_{L^2}^2 = \kappa^2 a^4 b^{N-2}\|\nabla(\chi^2)\|_{L^2}^2 \leqslant 4\kappa^2 a^4 b^{N-2} B$. There exists $C > 0$ such that $V(\tau) \leqslant C(1-\tau)^2$ for all $\tau \in [0,2]$, hence, when $\kappa a^2 \leqslant 1$,

$$V(|\psi_{a,b}|^2) \leqslant C(1-\psi_{a,b}^2)^2 \leqslant 4C\kappa^2 a^4 \chi^4\left(\frac{x}{b}\right)$$

and consequently

$$\int_{\mathbb{R}^N} V(|\psi_{a,b}|^2) \,\mathrm{d}x \leqslant 4C\kappa^2 a^4 b^N D.$$

Using the above estimates and the fact that $a^2 b^N A = \mathfrak{m}$ we get

$$\mathcal{G}_{\min}(\mathfrak{m}) - \frac{\mathfrak{m}}{\varepsilon^4} \leqslant E(\psi_{a,b}, \varphi_{a,b}) - \frac{1}{\varepsilon^4}\int_{\mathbb{R}^N} |\varphi_{a,b}|^2 \,\mathrm{d}x$$

$$= \int_{\mathbb{R}^N} |\nabla\psi_{a,b}|^2 + V(|\psi_{a,b}|^2) + \frac{1}{\varepsilon^2 q^2}|\nabla\varphi_{a,b}|^2 + \frac{1}{\varepsilon^4}\left(|\psi_{a,b}|^2 - 1\right)|\varphi_{a,b}|^2 \,\mathrm{d}x$$

$$\leqslant 4\kappa^2 a^4 b^{N-2} B + 4C\kappa^2 a^4 b^N D + \frac{a^2 b^{N-2} B}{\varepsilon^2 q^2}$$

$$- \tfrac{2\kappa a^4 b^N}{\varepsilon^4}\textstyle\int_{\mathbb{R}^N}\left(1 - \frac{1}{2}\kappa a^2 \chi^2(x)\right)\chi^4(x)\,\mathrm{d}x$$

$$= \mathfrak{m}\left(4\kappa^2 \mathfrak{m}^{-\frac{2}{N}} a^{2+\frac{4}{N}} A^{\frac{2}{N}-1} B + 4\kappa^2 a^2 \tfrac{CD}{A} + \tfrac{1}{\varepsilon^2 q^2}\mathfrak{m}^{-\frac{2}{N}} a^{\frac{4}{N}} A^{\frac{2}{N}-1} B - \kappa a^2 \tfrac{D}{A\varepsilon^4}\right).$$

Choosing κ sufficiently small (for instance, $\kappa \leqslant \min(\frac{C}{8\varepsilon^4}, 1/a^2)$ will do) we see that there exist constants $C_2, C_3 > 0$ such that $\mathcal{G}_{\min}(\mathfrak{m}) - \frac{\mathfrak{m}}{\varepsilon^4} \leqslant \mathfrak{m}\left(C_2 \mathfrak{m}^{-\frac{2}{N}} a^{\frac{4}{N}} - C_3 a^2\right)$. If $N = 1$ it suffices to take $a = \ell\mathfrak{m}$, where $\ell > 0$ is sufficiently small, to see that there exists some $C_4 > 0$ such that $\mathcal{G}_{\min}(\mathfrak{m}) - \frac{\mathfrak{m}}{\varepsilon^4} \leqslant -C_4\mathfrak{m}^3 < 0$ for all $\mathfrak{m} > 0$.

If $N = 1$, for any $\eta > 0$ there exists $e(\eta) > 0$ such that for any $\psi \in \mathcal{E}$ satisfying $E_1(\psi) \leqslant e(\eta)$ there holds $\||\psi| - 1\|_{L^\infty(\mathbb{R})} \leqslant \eta$. Fix $\eta > 0$. Let $\mathfrak{m} \in (0, \frac{1}{2}\varepsilon^4 e(\eta)]$. If $\psi \in \mathcal{E}$ and $\varphi \in H^1(\mathbb{R})$ are such that $\|\varphi\|_{L^2}^2 = \mathfrak{m}$ and $E(\psi, \varphi) \leqslant \frac{2\mathfrak{m}}{\varepsilon^4}$, it is obvious that $E_1(\psi) \leqslant \frac{2\mathfrak{m}}{\varepsilon^4} \leqslant e(\eta)$, hence $|\psi(x)| \in [1-\eta, 1+\eta]$ for all

$x \in \mathbb{R}$ and consequently $E(\psi,\varphi) \geqslant \frac{1}{\varepsilon^4}\int_{\mathbb{R}} |\psi|^2|\varphi|^2\,\mathrm{d}x \geqslant \frac{(1-\eta)^2\mathfrak{m}}{\varepsilon^4}$. We conclude that $\frac{(1-\eta)^2}{\varepsilon^4} \leqslant \frac{\mathcal{G}_{\min}(\mathfrak{m})}{\mathfrak{m}} \leqslant \frac{1}{\varepsilon^4}$ for any $\mathfrak{m} \in (0, \frac{1}{2}\varepsilon^4 e(\eta)]$, hence $\lim_{\mathfrak{m}\to 0}\frac{\mathcal{G}_{\min}(\mathfrak{m})}{\mathfrak{m}} = \frac{1}{\varepsilon^4}$.

(iii) We already know that $\mathcal{G}_{\min}(\mathfrak{m}) \leqslant \frac{\mathfrak{m}}{\varepsilon^4}$ (see part *(i)*) and it suffices to show that for $\mathfrak{m}$ sufficiently small and for any $\psi \in \mathcal{E}$ and any $\varphi \in H^1(\mathbb{R}^N)$ satisfying $\|\varphi\|_{L^2}^2 = \mathfrak{m}$ and $E(\psi,\varphi) \leqslant \frac{2\mathfrak{m}}{\varepsilon^4}$ there holds $E(\psi,\varphi) \geqslant \frac{\mathfrak{m}}{\varepsilon^4}$. It also suffices to consider the case when ψ and φ are real-valued and $0 \leqslant \psi \leqslant 1$ on $\mathbb{R}^N$. Indeed, denoting $\tilde{\varphi} = |\varphi|$ and $\tilde{\psi} = \min(|\psi|, 1)$ we have $\tilde{\psi} \in \mathcal{E}$, $\tilde{\varphi} \in H^1(\mathbb{R}^N)$, $\|\tilde{\varphi}\|_{L^2}^2 = \mathfrak{m}$ and $E(\tilde{\psi},\tilde{\varphi}) \leqslant E(\psi,\varphi)$.

Suppose that $\psi \in \mathcal{E}$ and $\varphi \in H^1(\mathbb{R}^N)$ are real-valued, $0 \leqslant \psi \leqslant 1$ on $\mathbb{R}^N$, $\|\varphi\|_{L^2}^2 = \mathfrak{m}$, and $E(\psi,\varphi) \leqslant \frac{2\mathfrak{m}}{\varepsilon^4}$. Consider first the case $N \geqslant 3$. Using Hölder's inequality, then Sobolev's inequality we get

$$\Big|\int_{\mathbb{R}^N}\left(|\psi|^2-1\right)|\varphi|^2\,\mathrm{d}x\Big| \leqslant \||\psi|^2-1\|_{L^{\frac{N}{2}}}\|\varphi\|_{L^{2^*}}^2 \leqslant C_S^2\||\psi|^2-1\|_{L^{\frac{N}{2}}}\|\nabla\varphi\|_{L^2}^2. \tag{5.44}$$

Since $0 \leqslant |\psi| \leqslant 1$, using Lemma 4.1 p. 171 in [4] (see estimate (4.1) there), we infer that there exists $C_1 > 0$ such that

$$\begin{aligned}\int_{\mathbb{R}^N}\left(|\psi|^2-1\right)^2\,\mathrm{d}x &\leqslant 4\int_{\mathbb{R}^N} V(|\psi|^2)\,\mathrm{d}x + C_1\|\nabla\psi\|_{L^2}^{2^*}\\ &\leqslant 4E(\psi,\varphi) + C_1E(\psi,\varphi)^{\frac{2^*}{2}} \leqslant \frac{8\mathfrak{m}}{\varepsilon^4} + C_1\left(\frac{8\mathfrak{m}}{\varepsilon^4}\right)^{\frac{2^*}{2}}.\end{aligned} \tag{5.45}$$

If $N \geqslant 4$, by Hölder's inequality we get

$$\||\psi|^2-1\|_{L^{\frac{N}{2}}} \leqslant \||\psi|^2-1\|_{L^2}^{\frac{4}{N}}\cdot\||\psi|^2-1\|_{L^\infty}^{1-\frac{4}{N}} \leqslant \left(\frac{8\mathfrak{m}}{\varepsilon^4} + C_1\left(\frac{8\mathfrak{m}}{\varepsilon^4}\right)^{\frac{2^*}{2}}\right)^{\frac{4}{N}}.$$

Then using (3.5) we infer that there is $C_2 > 0$ such that for any $\mathfrak{m} \in (0,1]$ and for all ψ and φ as above there holds

$$\Big|\int_{\mathbb{R}^N}\left(|\psi|^2-1\right)|\varphi|^2\,\mathrm{d}x\Big| \leqslant C_2\mathfrak{m}^{\frac{4}{N}}\|\nabla\varphi\|_{L^2}^2$$

and consequently

$$E(\psi,\varphi) - \frac{\mathfrak{m}}{\varepsilon^4} \geqslant E_1(\psi) + \left(\frac{1}{\varepsilon^2q^2} - \frac{C_2\mathfrak{m}^{\frac{4}{N}}}{\varepsilon^4}\right)\|\nabla\varphi\|_{L^2}^2.$$

The last quantity is nonnegative if $\mathfrak{m} \leqslant \min(1, C_2^{-\frac{N}{4}}\varepsilon^{\frac{N}{2}}q^{-\frac{N}{2}})$.

If $N = 3$, using Hölder's inequality, estimate (5.45) above and Sobolev's inequality we get

$$\begin{aligned}&\Big|\int_{\mathbb{R}^N}\left(|\psi|^2-1\right)|\varphi|^2\,\mathrm{d}x\Big|\\ &\leqslant \||\psi|^2-1\|_{L^2}\|\varphi\|_{L^2}^{\frac{1}{2}}\|\varphi\|_{L^6}^{\frac{3}{2}} \leqslant C_S^{\frac{3}{2}}\||\psi|^2-1\|_{L^2}\|\varphi\|_{L^2}^{\frac{1}{2}}\|\nabla\varphi\|_{L^2}^{\frac{3}{2}}\\ &\leqslant C_S^{\frac{3}{2}}\left(4\int_{\mathbb{R}^N} V(|\psi|^2)\,\mathrm{d}x + C_1\|\nabla\psi\|_{L^2}^6\right)^{\frac{1}{2}}\|\varphi\|_{L^2}^{\frac{1}{2}}\|\nabla\varphi\|_{L^2}^{\frac{3}{2}}.\end{aligned}$$

If ψ and φ are as above we have $\|\nabla\psi\|_{L^2}^2 \leqslant E_1(\psi) \leqslant \frac{2\mathfrak{m}}{\varepsilon^4}$. If $\mathfrak{m} \leqslant 1$, we infer that there exist $C_3, C_3' > 0$ such that

$$\begin{aligned}\Big|\int_{\mathbb{R}^3}\left(|\psi|^2-1\right)|\varphi|^2\,\mathrm{d}x\Big| &\leqslant C_3E_1(\psi)^{\frac{1}{2}}\|\nabla\varphi\|_{L^2}^{\frac{3}{2}}\|\varphi\|_{L^2}^{\frac{1}{2}}\\ &\leqslant C_3'\left(E_1(\psi) + \tfrac{1}{\varepsilon^2q^2}\|\nabla\varphi\|_{L^2}^2\right)\|\nabla\varphi\|_{L^2}^{\frac{1}{2}}\|\varphi\|_{L^2}^{\frac{1}{2}}.\end{aligned}$$

Then we get

$$E(\psi,\varphi)-\tfrac{m}{\varepsilon^4}\geqslant\left(E_1(\psi)+\tfrac{1}{\varepsilon^2q^2}\|\nabla\varphi\|_{L^2}^2\right)\left(1-\tfrac{C_3'}{\varepsilon^4}\|\nabla\varphi\|_{L^2}^{\frac12}\|\varphi\|_{L^2}^{\frac12}\right)$$

$$\geqslant\left(E_1(\psi)+\tfrac{1}{\varepsilon^2q^2}\|\nabla\varphi\|_{L^2}^2\right)(1-C_5\mathfrak{m}^{\frac12})$$

for some $C_5>0$ and the last quantity is non-negative for all $\mathfrak{m}$ sufficiently small.
Consider next the case $N=2$. Using Plancherel's theorem and Hölder's inequality, we get

$$\|\varphi\|_{\dot H^s}\leqslant\|\varphi\|_{L^2}^{1-s}\|\nabla\varphi\|_{L^2}^s\qquad\text{for all } s\in(0,1) \text{ and all } \varphi\in H^1(\mathbb{R}^2).$$

If $p\in(2,\infty)$ and $s\in(0,1)$ satisfy $\frac1p=\frac12-\frac{s}{2}$ (that is, $s=1-\frac2p$), using the Sobolev inequality we see that there exists $C_p>0$ such that

$$\|\varphi\|_{L^p}\leqslant C_p\|\varphi\|_{\dot H^s}\leqslant C_p\|\varphi\|_{L^2}^{1-s}\|\nabla\varphi\|_{L^2}^s\qquad\text{for any } \varphi\in H^1(\mathbb{R}^2).$$

In particular, for $p=4$ we get $\|\varphi\|_{L^4(\mathbb{R}^2)}\leqslant C_4\|\varphi\|_{L^2}^{\frac12}\|\nabla\varphi\|_{L^2}^{\frac12}$ for any $\varphi\in H^1(\mathbb{R}^2)$, hence

$$\left|\int_{\mathbb{R}^2}\left(|\psi|^2-1\right)|\varphi|^2\,\mathrm{d}x\right|\leqslant\|1-|\psi|^2\|_{L^2}\|\varphi\|_{L^4}^2\leqslant C_4^2\|1-|\psi|^2\|_{L^2}\|\nabla\varphi\|_{L^2}\|\varphi\|_{L^2}. \tag{5.46}$$

Using Lemma 4.1 p. 171 in [4] (see estimate (4.2) there) we infer that there exists $C>0$ such that for any $\psi\in\mathcal{E}$ satisfying $|\psi|\leqslant2$ we have

$$\left(\frac{1}{4\varepsilon^2}-C\|\nabla\psi\|_{L^2(\mathbb{R}^2)}^{2p_0+2}\right)\int_{\mathbb{R}^2}\left(|\psi|^2-1\right)^2\,\mathrm{d}x\leqslant\int_{\mathbb{R}^2}V(|\psi|^2)\,\mathrm{d}x.$$

In particular, there exists $\mathfrak{m}_0>0$ such that whenever $|\psi|\leqslant2$ and $E(\psi,\varphi)\leqslant\frac{2\mathfrak{m}_0}{\varepsilon^4}$, there holds

$$\int_{\mathbb{R}^2}\left(|\psi|^2-1\right)^2\,\mathrm{d}x\leqslant8\varepsilon^2\int_{\mathbb{R}^2}V(|\psi|^2)\,\mathrm{d}x.$$

Coming back to (5.46) we see that there exists $C_7>0$ such that for all $\mathfrak{m}\in(0,\mathfrak{m}_0]$ and for any $\psi\in\mathcal{E}$ and any $\varphi\in H^1(\mathbb{R}^N)$ with $|\varphi|\leqslant2$, $\|\varphi\|_{L^2}^2=\mathfrak{m}$ and $E(\psi,\varphi)\leqslant\frac{2\mathfrak{m}}{\varepsilon^4}$ we have

$$\left|\int_{\mathbb{R}^2}\left(|\psi|^2-1\right)|\varphi|^2\,\mathrm{d}x\right|\leqslant C_7\left(\int_{\mathbb{R}^2}V(|\psi|^2)\,\mathrm{d}x+\frac{1}{\varepsilon^2q^2}\|\nabla\varphi\|_{L^2}^2\right)\|\varphi\|_{L^2}.$$

As previously, we conclude that $E(\psi,\varphi)-\frac{\mathfrak{m}}{\varepsilon^4}\geqslant0$ if ψ, φ are as above and $\mathfrak{m}$ is sufficiently small.

(*iv*) Fix a *radial* function $\chi\in C_c^\infty(\mathbb{R}^N)$ such that $\chi=1$ on $B(0,1)$, $0\leqslant\chi\leqslant1$ and $\mathrm{supp}(\chi)\subset B(0,2)$. For $R>0$ denote $\psi_R(x)=1-\chi\left(\frac{x}{R}\right)$. Denote $A=\int_{\mathbb{R}^N}|\nabla\psi_1|^2\,\mathrm{d}x$ and $B=\int_{\mathbb{R}^N}V(|\psi_1|^2)\,\mathrm{d}x$. We have $\int_{\mathbb{R}^N}|\nabla\psi_R|^2\,\mathrm{d}x=AR^{N-2}$ and $\int_{\mathbb{R}^N}V(|\psi_R|^2)\,\mathrm{d}x=BR^N$. Since ψ_R is radial we have $Q(\psi_R)=0$.
Let ϕ_1 be an optimizer for the Poincaré inequality on $B(0,1)$, that is $\phi_1\in H_0^1(B(0,1))$, $\int_{B(0,1)}|\phi_1|^2\,\mathrm{d}x=1$ and $\int_{B(0,1)}|\nabla\phi_1|^2\,\mathrm{d}x=C_P$. Extend ϕ_1 by zero outside $B(0,1)$. Let $\varphi_R(x)=\frac{\sqrt{\mathfrak{m}}}{R^{N/2}}\phi_1\left(\frac{x}{R}\right)$, so that $\int_{\mathbb{R}^N}|\varphi_R|^2\,\mathrm{d}x=\mathfrak{m}$, $\int_{\mathbb{R}^N}|\nabla\varphi_R|^2\,\mathrm{d}x=\frac{C_P}{R^2}\mathfrak{m}$ and $\mathrm{supp}(\varphi_R)\subset B(0,R)$, hence $\int_{\mathbb{R}^N}|\psi_R|^2|\varphi_R|^2\,\mathrm{d}x=0$. Then we have

$$\mathcal{G}_{\min}(\mathfrak{m})\leqslant E(\psi_R,\varphi_R)=AR^{N-2}+BR^N+\frac{C_P}{\varepsilon^2q^2R^2}\mathfrak{m}=f_{\mathfrak{m}}(R). \tag{5.47}$$

Notice that (5.47) holds for any $R>0$, hence we may optimize with respect to R. The function $f_{\mathfrak{m}}$ is convex in $(0,\infty)$ and tends to $+\infty$ when $R\to0^+$ and when $R\to+\infty$; it then has a unique minimum

on $(0,\infty)$ at a point $R_{\mathfrak{m}}$ satisfying $f'_{\mathfrak{m}}(R_{\mathfrak{m}})=0$, that is $A(N-2)R_{\mathfrak{m}}^N + BNR_{\mathfrak{m}}^{N+2} = \frac{2C_P\mathfrak{m}}{\varepsilon^2 q^2}$. It is easily seen that for large $\mathfrak{m}$ the unique positive root $R_{\mathfrak{m}}$ of this equation is of order of magnitude $R_{\mathfrak{m}} \sim \mathfrak{m}^{\frac{1}{N+2}}$ and there is some constant $C>0$ such that $f_{\mathfrak{m}}(R_{\mathfrak{m}}) \leqslant C\mathfrak{m}^{\frac{N}{N+2}}$. Coming back to (5.47) we conclude that

$$\mathcal{G}_{\min}(\mathfrak{m}) \leqslant C\mathfrak{m}^{\frac{N}{N+2}}.$$

Let us now provide some explicit ε-dependent bounds C when $N=1$, 2 or 3 and $F(s)=1-s$, hence $V(s)=(1-s)^2/(2\varepsilon^2)$.
First, we may take, by density, $\chi(x)=1$ if $|x|\leqslant 1$, $\chi(x)=0$ if $|x|\geqslant 2$, $\chi(x)=2-|x|$ if $1\leqslant|x|\leqslant 2$, so that

$$A = \mathcal{H}^{N-1}(\mathbb{S}^{N-1})\int_1^2 r^{N-1}\,dr = \mathcal{L}^N(B(0,1))(2^N-1).$$

Moreover,

$$2\varepsilon^2 B = \mathcal{H}^{N-1}(\mathbb{S}^{N-1})\int_1^2 (r-2)^2 r^{N-1}\,dr.$$

In addition, we know that $C_P=\pi^2/4$ if $N=1$; $C_P = 2.4048^2 < 5.8$ if $N=2$ (≈ 2.4048 is the first zero of the Bessel function J_0) and $C_P=\pi^2$ if $N=3$ (π is the first zero of the modified Bessel function j_0). If $N=2$, then $f_{\mathfrak{m}}(R) = A + BR^2 + \frac{C_P}{\varepsilon^2q^2R^2}\mathfrak{m}$, $R_{\mathfrak{m}} = (C_P\mathfrak{m}/\varepsilon^2q^2B)^{1/2}$ and $f_{\mathfrak{m}}(R_{\mathfrak{m}}) = A + 2(BC_P\mathfrak{m}/\varepsilon^2q^2)^{1/4}$ with $A=3\pi$ and $\varepsilon^2B=5\pi/12$, thus

$$\mathcal{G}_{\min}(\mathfrak{m}) \leqslant f_{\mathfrak{m}}(R_{\mathfrak{m}}) \leqslant 3\pi + \frac{5.502}{\varepsilon^2 q}\sqrt{\mathfrak{m}} \leqslant 9.5 + 383.75\sqrt{\mathfrak{m}}. \tag{5.48}$$

If $N=1$, then $f_{\mathfrak{m}}(R) = A/R + BR + \frac{C_P}{\varepsilon^2q^2R^2}\mathfrak{m}$ with $A=2$, $\varepsilon^2B=1/3$. Choosing $R=(2C_P\mathfrak{m}/\varepsilon^2q^2B)^{1/3}$ as an approximation of $R_{\mathfrak{m}}$, we obtain

$$\mathcal{G}_{\min}(\mathfrak{m}) \leqslant f_{\mathfrak{m}}((2C_P\mathfrak{m}/\varepsilon^2q^2B)^{1/3}) \leqslant \frac{0.45}{\mathfrak{m}^{1/3}} + 63.62\mathfrak{m}^{1/3} \tag{5.49}$$

If $N=3$, then $f_{\mathfrak{m}}(R) = AR + BR^3 + \frac{C_P}{\varepsilon^2q^2R^2}\mathfrak{m}$, with $A=28\pi/3$ and $\varepsilon^2B = 16\pi/15$. Choosing $R=(2C_P\mathfrak{m}/(3\varepsilon^2q^2B))^{1/5} = (\frac{5\pi}{8q^2}\mathfrak{m})^{1/5} = 3.5\mathfrak{m}^{1/5}$ as an approximation of $R_{\mathfrak{m}}$, we obtain

$$\mathcal{G}_{\min}(\mathfrak{m}) \leqslant f_{\mathfrak{m}}((2C_P\mathfrak{m}/(3\varepsilon^2q^2B))^{1/5}) \leqslant 102.63\mathfrak{m}^{1/5} + 4245.71\mathfrak{m}^{3/5}. \tag{5.50}$$

□

Theorem 41. *Assume that $\mathcal{G}_{\min}(\mathfrak{m}) < \frac{\mathfrak{m}}{\varepsilon^4}$. Then there exist minimizers for the problem $(GS_{\mathfrak{m}})$. Moreover, if $(\psi_n,\varphi_n)_{n\geqslant 1}$ is any sequence in $\mathcal{E}\times H^1(\mathbb{R}^N)$ such that*

$$\|\varphi_n\|_{L^2}^2 \longrightarrow \mathfrak{m}$$

and

$$E(\psi_n,\varphi_n) \longrightarrow \mathcal{G}_{\min}(\mathfrak{m}),$$

then there exists a subsequence, still denoted the same, there is a sequence $(x_n)_{n\geqslant 1}\subset\mathbb{R}^N$ and there are $\psi\in 1+H^1(\mathbb{R}^N)$ and $\varphi\in H^1(\mathbb{R}^N)$ such that

$$|\psi_n(\cdot-x_n)|-1 \longrightarrow \psi-1 \quad \text{and} \quad |\varphi_n(\cdot-x_n)| \longrightarrow \varphi \quad \text{in } H^1(\mathbb{R}^N),$$

$$V(|\psi_n|^2)(\cdot-x_n) \longrightarrow V(|\psi|^2) \text{ in } L^1(\mathbb{R}^N) \text{ and}$$

$$|\psi_n|^2|\varphi_n|^2(\cdot-x_n) \longrightarrow |\psi|^2|\varphi|^2 \text{ in } L^1(\mathbb{R}^N) \text{ as } n\longrightarrow\infty.$$

Remark 42. *The conclusion of Theorem 41 is only a statement about $|\psi_n|$ and $|\varphi_n|$. More information should be available here. Indeed, if (ψ_n, φ_n) is a minimizing sequence, then $E(\psi_n, \varphi_n) - E(|\psi_n|, |\varphi_n|) \longrightarrow 0$, hence ψ_n should be "close" to $e^{i\theta_n}|\psi_n|$ and φ_n should be "close" to $e^{i\beta_n}|\varphi_n|$ for some $\theta_n, \beta_n \in \mathbb{R}$.*

Remark 43. *For the physical values (5.7), (5.8) of the parameters, and by using the upper bounds (5.49), (5.48) and (5.50), we may check that the conclusion of Theorem 41 with $F(s) = 1 - s$ holds at least when*
(i) $N = 1$ and $\mathfrak{m} > 0.0245$;
(ii) $N = 2$ and $\mathfrak{m} > 0.25$;
(iii) $N = 3$ and $\mathfrak{m} > 62.14$.

Proof. Let $(\psi_n, \varphi_n)_{n\geqslant 1}$ be a sequence as in Theorem 41. In particular, $E_1(\psi_n)$ and $\|\varphi_n\|_{H^1(\mathbb{R}^N)}$ are bounded. It follows from Lemma 4.8 p. 177 and Corollary 4.3 p. 172 in [4] that $\| |\psi_n| - 1\|_{L^2(\mathbb{R}^N)}$ is bounded. Let p_0 be as in assumption **(A2)**. Denote

$$f_n = |\nabla \psi_n|^2 + | |\psi_n| - 1|^2 + | |\psi_n| - 1|^{2p_0+2} + |\nabla \varphi_n|^2 + |\varphi_n|^2.$$

Obviously, $(f_n)_{n\geqslant 1}$ is a bounded sequence in $L^1(\mathbb{R}^N)$ and $\int_{\mathbb{R}^N} f_n(x)\,\mathrm{d}x \geqslant \|\varphi_n\|_{L^2}^2$. Passing to a subsequence we may assume that $\int_{\mathbb{R}^N} f_n(x)\,\mathrm{d}x \longrightarrow \alpha_0$ as $n \longrightarrow \infty$, where $\alpha_0 \geqslant \mathfrak{m} > 0$. Let $\Lambda_n : [0, \infty) \longrightarrow [0, \infty)$ be the concentration function of f_n, that is

$$\Lambda_n(t) = \sup_{y\in\mathbb{R}^N} \int_{B(y,t)} f_n(x)\,\mathrm{d}x.$$

Proceeding as in [12], it is straightforward to show that there exists a subsequence of $((\psi_n, \varphi_n, \Lambda_n))_{n\geq 1}$, still denoted the same, there is a nondecreasing function $\Lambda : [0, \infty) \longrightarrow \mathbb{R}$ and there is $\alpha \in [0, \alpha_0]$ such that

$$\Lambda_n(t) \longrightarrow \Lambda(t) \text{ a.e on } [0, \infty) \text{ as } n \longrightarrow \infty \qquad \text{and} \qquad \Lambda(t) \longrightarrow \alpha \text{ as } t \longrightarrow \infty. \tag{5.51}$$

As in [15] (see the proof of (5.12) p. 156 there) one can prove that there is a nondecreasing sequence $t_n \longrightarrow \infty$ such that

$$\lim_{n\to\infty} \Lambda_n(t_n) = \lim_{n\to\infty} \Lambda_n\left(\frac{t_n}{2}\right) = \alpha. \tag{5.52}$$

Our aim is to show that $\alpha = \alpha_0$. The next step is to prove that $\alpha > 0$.
Let $\kappa = \mathfrak{m} - \varepsilon^4 \mathcal{G}_{\min}(\mathfrak{m}) > 0$. Choose $d \in (0, 1)$ such that $\mathfrak{m} - \frac{\varepsilon^4}{(1-d)^2}\mathcal{G}_{\min}(\mathfrak{m}) > \frac{\kappa}{2}$. Denote

$$A_n = \{x \in \mathbb{R}^N \mid |\psi_n(x)| < 1 - d\}, \quad m_{1,n} = \textstyle\int_{A_n} |\varphi_n|^2\,\mathrm{d}x \quad \text{and}$$

$$m_{2,n} = \textstyle\int_{\mathbb{R}^N\setminus A_n} |\varphi_n|^2\,\mathrm{d}x.$$

It is obvious that

$$m_{1,n} + m_{2,n} = \|\varphi_n\|_{L^2}^2 \longrightarrow \mathfrak{m}$$

and

$$\begin{aligned} m_{2,n} &\leqslant \frac{1}{(1-d)^2} \int_{\mathbb{R}^N\setminus A_n} |\psi_n|^2 |\varphi_n|^2\,\mathrm{d}x \\ &\leqslant \frac{\varepsilon^4}{(1-d)^2} E(\psi_n, \varphi_n) \longrightarrow \frac{\varepsilon^4}{(1-d)^2}\mathcal{G}_{\min}(\mathfrak{m}) \end{aligned}$$

as $n \longrightarrow \infty$. It follows that

$$\liminf_{n\to\infty} m_{1,n} = \mathfrak{m} - \limsup_{n\to\infty} m_{2,n} \geqslant \mathfrak{m} - \frac{\varepsilon^4}{(1-d)^2}\mathcal{G}_{\min}(\mathfrak{m}) > \frac{\kappa}{2}.$$

Hence there exists $n_0 \in \mathbb{N}$ such that $m_{1,n} > \frac{\kappa}{2}$ for all $n \geqslant n_0$. Fix $p \in (2,\infty)$ if $N \in \{1,2\}$, respectively $p \in (2, \frac{2N}{N-2}]$ if $N \geqslant 3$, so that $(\varphi_n)_{n\geqslant 1}$ is bounded in $L^p(\mathbb{R}^N)$. Denoting by $\mathcal{L}^N$ the Lebesgue measure in $\mathbb{R}^N$ and using Hölder's inequality we get

$$\frac{\kappa}{2} \leqslant \int_{A_n} |\varphi_n|^2 \,\mathrm{d}x \leqslant \|\varphi_n\|^2_{L^p(A_n)} \mathcal{L}^N(A_n)^{1-\frac{2}{p}} \leqslant \mathfrak{m}\mathcal{L}^N(A_n)^{1-\frac{2}{p}} \qquad \text{for all } n \geqslant n_0.$$

The above inequality implies that there exists $\eta_0 > 0$ such that $\mathcal{L}^N(A_n) \geqslant \eta_0$ for all $n \geqslant n_0$. Using Lieb's Lemma for $(1-|\psi_n|)_+$ (see Lemma 6 p. 447 in [9]) we infer that there exists $\eta_1 > 0$ and for any $n \geqslant n_0$ there is $y_n \in \mathbb{R}^N$ such that

$$\mathcal{L}^N\left(B_n \cap B(y_n,1)\right) \geqslant \eta_1, \qquad \text{where} \qquad B_n = \left\{ x \in \mathbb{R}^N \;\middle|\; |\psi_n(x)| < 1 - \frac{d}{2} \right\}.$$

Then we get

$$\Lambda_n(1) \geqslant \int_{B(y_n,1)} f_n(x)\,\mathrm{d}x \geqslant \int_{B(y_n,1)} \left|1-|\psi_n|\right|^2 \mathrm{d}x \geqslant \frac{\eta_1 d^2}{4} \qquad \text{for all } n \geqslant n_0.$$

The above inequality implies that $\Lambda(1) \geqslant \frac{\eta_1 d^2}{4}$, and consequently $\alpha > 0$.
To prove that $\alpha \notin (0,\alpha_0)$ we argue by contradiction and we assume that $0 < \alpha < \alpha_0$. Let $h_n = \Lambda_n(t_n) - \Lambda_n\left(\frac{t_n}{2}\right) + \frac{1}{2^n}$. It is obvious that $h_n \longrightarrow 0$ as $n \longrightarrow \infty$. For each $n \in \mathbb{N}^*$ choose x_n such that $\displaystyle\int_{B(x_n,\frac{t_n}{2})} f_n(x)\,\mathrm{d}x > \Lambda_n\left(\frac{t_n}{2}\right) - \frac{1}{2^n}$ and denote $\Omega_n = B(x_n,t_n) \setminus B(x_n,\frac{t_n}{2})$. It is obvious that

$$\begin{aligned}\int_{\Omega_n} f_n(x)\,\mathrm{d}x &= \int_{B(x_n,t_n)} f_n(x)\,\mathrm{d}x - \int_{B(x_n,\frac{t_n}{2})} f_n(x)\,\mathrm{d}x \\ &\leqslant \Lambda_n(t_n) - \Lambda_n\left(\tfrac{t_n}{2}\right) + \tfrac{1}{2^n} = h_n.\end{aligned}$$

Take $\chi \in C_c^\infty(\mathbb{R}^N)$ such that $0 \leqslant \chi \leqslant 1$, $\chi = 1$ on $B(0,\frac{1}{2})$ and $\chi = 0$ on $\mathbb{R}^N \setminus B(0,1)$. Denote

$$\psi_{n,1} = (|\psi_n|-1)\,\chi\Big(\tfrac{x-x_n}{t_n}\Big) + 1, \quad \psi_{n,2} = (|\psi_n|-1)\left[1-\chi\Big(\tfrac{x-x_n}{t_n}\Big)\right] + 1$$

$$\varphi_{n,1} = |\varphi_n|\chi\Big(\tfrac{x-x_n}{t_n}\Big), \qquad \varphi_{n,2} = |\varphi_n|\left[1-\chi\Big(\tfrac{x-x_n}{t_n}\Big)\right].$$

It is clear that

$$\begin{aligned}&\left|\int_{\mathbb{R}^N} |\varphi_n|^2 - |\varphi_{n,1}|^2 - |\varphi_{n,2}|^2\,\mathrm{d}x\right| \\ &\leqslant \int_{\Omega_n} \left|1-\chi^2\Big(\frac{x-x_n}{t_n}\Big) - (1-\chi)^2\Big(\frac{x-x_n}{t_n}\Big)\right||\varphi_n|^2\,\mathrm{d}x \\ &\leqslant h_n.\end{aligned} \tag{5.53}$$

Since $|\psi_{n,i}| \leqslant \max(|\psi_n|, 1)$ and $|\varphi_{n,i}| \leqslant |\varphi_n|$ for $i = 1, 2$, we have

$$\begin{aligned} &\Big| \int_{\mathbb{R}^N} |\psi_n|^2|\varphi_n|^2 - |\psi_{n,1}|^2|\varphi_{n,1}|^2 - |\psi_{n,2}|^2|\varphi_{n,2}|^2 \,\mathrm{d}x\Big| \\ &\leqslant 3 \int_{\Omega_n} |\varphi_n|^2 + |\psi_n|^2|\varphi_n|^2 \,\mathrm{d}x \\ &\leqslant 3h_n. \end{aligned} \tag{5.54}$$

By assumptions **(A1)** and **(A2)** there exists $C > 0$ such that $V(s) \leqslant C\left((s-1)^2 + (s-1)^{2p_0+2}\right)$. We have $\big|\,|\psi_{n,i}| - 1\big| \leqslant \big|\,|\psi_n| - 1\big|$, $i = 1, 2$, and we infer that

$$\begin{aligned} &\Big| \int_{\mathbb{R}^N} V(|\psi_n|^2) - V|\psi_{n,1}|^2) - V(|\psi_{n,2}|^2) \,\mathrm{d}x\Big| \\ &\leqslant \int_{\Omega_n} |V(|\psi_n|^2)| + |V|\psi_{n,1}|^2)| + |V(|\psi_{n,2}|^2)| \,\mathrm{d}x \\ &\leqslant 3C \int_{\Omega_n} \big|\,|\psi_n| - 1\big|^2 + \big|\,|\psi_n| - 1\big|^{2p_0+2} \leqslant 3Ch_n. \end{aligned} \tag{5.55}$$

We have $\frac{\partial \psi_{n,1}}{\partial x_j} = \frac{\partial |\psi_n|}{\partial x_j}\chi\Big(\frac{x-x_n}{t_n}\Big) + \frac{1}{t_n}(|\psi_n| - 1)\frac{\partial \chi}{\partial x_j}\Big(\frac{x-x_n}{t_n}\Big)$ and a similar equality holds for $\psi_{n,2}$. If n is sufficiently large, so that $N|\nabla\chi|^2 \leqslant t_n^2$ on $\mathbb{R}^N$, we get

$$\int_{\Omega_n} \Big|\frac{\partial \psi_{n,1}}{\partial x_j}\Big|^2 \mathrm{d}x \leqslant 2\int_{\Omega_n} \Big|\frac{\partial |\psi_n|}{\partial x_j}\Big|^2 + \frac{1}{N}\big|\,|\psi_n| - 1\big|^2 \mathrm{d}x$$

and summing up we infer that

$$\begin{aligned} &\Big| \int_{\mathbb{R}^N} |\nabla|\psi_n|\,|^2 - |\nabla\psi_{n,1}|^2 - |\nabla\psi_{n,2}|^2) \,\mathrm{d}x\Big| \\ &\leqslant \int_{\Omega_n} (|\nabla|\psi_n|\,|^2 + |\nabla\psi_{n,1}|^2 + |\nabla\psi_{n,2}|^2) \,\mathrm{d}x \\ &\leqslant 5 \int_{\Omega_n} \big|\nabla|\psi_n|\,\big|^2 + \big|\,|\psi_n| - 1\big|^2 \mathrm{d}x \leqslant 5h_n. \end{aligned} \tag{5.56}$$

It is obvious that a similar estimate holds for φ_n. From (5.54)-(5.56) we infer that there exists a constant $C > 0$ such that for all n sufficiently large we have

$$\big|E(\psi_n, \varphi_n) - E(\psi_{n,1}, \varphi_{n_1}) - E(\psi_{n,2}, \varphi_{n_2})\big| \leqslant Ch_n. \tag{5.57}$$

Passing to a subsequence (still denoted the same) we may assume that $\|\varphi_{n,i}\|_{L^2}^2 \longrightarrow \mathfrak{m}_i$ as $n \longrightarrow \infty$ for $i = 1, 2$, and (5.53) implies that $\mathfrak{m}_1 + \mathfrak{m}_2 = \mathfrak{m}$. Let us show that $\mathfrak{m}_1 > 0$ and $\mathfrak{m}_2 > 0$. We argue again by contradiction and we assume, for instance, that $\mathfrak{m}_2 = 0$. Then we have necessarily $\mathfrak{m}_1 = \mathfrak{m}$. If there is a subsequence $(n_k)_{k\geqslant 1}$ such that $E(\psi_{n_k,2}, \varphi_{n_k,2}) \longrightarrow e > 0$ as $k \longrightarrow \infty$, by (5.57) we have $E(\psi_{n_k,1}, \varphi_{n_k,1}) \longrightarrow \mathcal{G}_{\min}(\mathfrak{m}) - e$. On the other hand, $E(\psi_{n_k,1}, \varphi_{n_k,1}) \geqslant \mathcal{G}_{\min}\big(\|\varphi_{n_k,1}\|_{L^2}^2\big)$, and letting $k \longrightarrow \infty$ and using the continuity of $\mathcal{G}_{\min}$ we find $\liminf\limits_{k\to\infty} E(\psi_{n_k,1}, \varphi_{n_k,1}) \geqslant \mathcal{G}_{\min}(\mathfrak{m})$, a contradiction. Therefore a sequence $(n_k)_{k\geqslant 1}$ as above cannot exist, and this implies that $E(\psi_{n,2}, \varphi_{n,2}) \longrightarrow 0$ as

$n \longrightarrow \infty$. Then we deduce that $\|\varphi_{n,2}\|_{H^1(\mathbb{R}^N)} \longrightarrow 0$ and $E_1(\psi_{n,2}) \longrightarrow 0$ as $n \longrightarrow \infty$, and using Lemma 4.8 p. 177 and Corollary 4.3 p. 172 in [4] we infer that $\| \, |\psi_{n,2}| - 1\|_{L^2(\mathbb{R}^N)} \longrightarrow 0$. The Sobolev embedding gives then $\| \, |\psi_{n,2}| - 1\|_{L^{2p_0+2}(\mathbb{R}^N)} \longrightarrow 0$. Since $(\psi_n, \varphi_n) = (\psi_{n,2}, \varphi_{n,2})$ on $\mathbb{R}^N \setminus B(x_n, t_n)$, we see that $\displaystyle\int_{\mathbb{R}^N \setminus B(x_n,t_n)} f_n(x)\,\mathrm{d}x \longrightarrow 0$, hence $\displaystyle\int_{B(x_n,t_n)} f_n(x)\,\mathrm{d}x \longrightarrow \alpha_0$, and this implies $\Lambda_n(t_n) \longrightarrow \alpha_0$. Recall that the sequence $(t_n)_{n\geqslant 1}$ has been chosen so that $\Lambda_n(t_n) \longrightarrow \alpha$, thus we get $\alpha = \alpha_0$, contradicting the assumption that $\alpha \in (0, \alpha_0)$. So far we have shown that we cannot have $\mathfrak{m}_2 = 0$, and similarly we show that $\mathfrak{m}_1 \neq 0$. We conclude that $\mathfrak{m}_1, \mathfrak{m}_2 \in (0, \mathfrak{m})$.
It is clear that $E(\psi_{n,i}, \varphi_{n,i}) \geqslant \mathcal{G}_{\min}(\|\varphi_{n,i}\|_{L^2}^2)$ and letting $n \longrightarrow \infty$ we find, by continuity of $\mathcal{G}_{\min}$,

$$\liminf_{n\to\infty} E(\psi_{n,i}, \varphi_{n,i}) \geqslant \mathcal{G}_{\min}(\mathfrak{m}_i) \qquad \text{for } i = 1, 2.$$

Then using (5.57) we get $\mathcal{G}_{\min}(\mathfrak{m}) \geqslant \mathcal{G}_{\min}(\mathfrak{m}_1) + \mathcal{G}_{\min}(\mathfrak{m}_2)$. On the other hand, the concavity of $\mathcal{G}_{\min}$ implies $\mathcal{G}_{\min}(\mathfrak{m}_i) \geqslant \frac{\mathfrak{m}_i}{\mathfrak{m}} \mathcal{G}_{\min}(\mathfrak{m})$ and equality may occur in this inequality if and only if $\mathcal{G}_{\min}$ is linear on $[0, \mathfrak{m}]$. Summing up the last two inequalities and comparing to the previous inequality we see that necessarily $\mathcal{G}_{\min}(\mathfrak{m}_i) = \frac{\mathfrak{m}_i}{\mathfrak{m}} \mathcal{G}_{\min}(\mathfrak{m})$ for $i = 1, 2$, and therefore $\mathcal{G}_{\min}$ must be linear on $[0, \mathfrak{m}]$. Then Proposition 40 (*ii*) and (*iii*) implies that $\mathcal{G}_{\min}(\mathfrak{m}') = \frac{\mathfrak{m}'}{\varepsilon^4}$ for all $\mathfrak{m}' \in [0, \mathfrak{m}]$, contradicting the fact that $\mathcal{G}_{\min}(\mathfrak{m}) < \frac{\mathfrak{m}}{\varepsilon^4}$. We conclude that we cannot have $\alpha \in (0, \alpha_0)$, and consequently we must have $\alpha = \alpha_0$. Since $\alpha = \alpha_0$, it is standard to prove that there is a sequence $(x_n)_{n\geqslant 1} \subset \mathbb{R}^N$ such that for any $\varepsilon > 0$ there are $R_\varepsilon > 0$ and $n_\varepsilon \in \mathbb{N}$ such that

$$\int_{\mathbb{R}^N \setminus B(x_n, R_\varepsilon)} f_n(x)\,\mathrm{d}x < \varepsilon \qquad \text{for all } n \geqslant n_\varepsilon. \tag{5.58}$$

Denoting $\tilde{\psi}_n = |\psi_n|(\cdot - x_n)$ and $\tilde{\varphi}_n = |\varphi_n|(\cdot - x_n)$, it is easily seen that $\tilde{\psi}_n - 1$ and $\tilde{\varphi}_n$ are bounded in $H^1(\mathbb{R}^N)$. Passing again to a subsequence (still denoted the same), we infer that there exist $\varphi \in H^1(\mathbb{R}^N)$ and $\psi \in 1 + H^1(\mathbb{R}^N)$ such that

$$\begin{aligned} &\tilde{\psi}_n - \psi \rightharpoonup 0 \quad \text{ and } \tilde{\varphi}_n \rightharpoonup \varphi \quad \text{ weakly in } H^1(\mathbb{R}^N), \\ &\tilde{\psi}_n \longrightarrow \psi \ \text{ and } \ \tilde{\varphi}_n \longrightarrow \varphi \ \text{ in } L^p_{\mathrm{loc}}(\mathbb{R}^N) \text{ for any } 1 \leqslant p < 2^* \\ &\qquad\qquad\qquad\qquad\qquad\qquad \text{and almost everywhere.} \end{aligned} \tag{5.59}$$

The weak convergence implies $\|\varphi\|_{L^2}^2 \leqslant \liminf_{n\to\infty} \|\varphi_n\|_{L^2}^2 = \mathfrak{m}$. On the other hand, fix $\varepsilon > 0$. Using (5.58), for $n \geqslant n_\varepsilon$ we have $\displaystyle\int_{\mathbb{R}^N \setminus B(0,R_\varepsilon)} |\tilde{\varphi}_n|^2\,\mathrm{d}x < \varepsilon$, hence $\displaystyle\int_{B(0,R_\varepsilon)} |\tilde{\varphi}_n|^2\,\mathrm{d}x > \|\tilde{\varphi}_n\|_{L^2}^2 - \varepsilon$ for all $n \geqslant n_\varepsilon$. Since $\tilde{\varphi}_n \longrightarrow \varphi$ in $L^2(B(0, R_\varepsilon))$, we obtain $\displaystyle\int_{B(0,R_\varepsilon)} |\varphi|^2\,\mathrm{d}x \geqslant \mathfrak{m} - \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we infer that $\int_{\mathbb{R}^N} |\varphi|^2\,\mathrm{d}x \geqslant \mathfrak{m}$. Thus we have shown that $\|\varphi\|_{L^2}^2 = \mathfrak{m}$.
By weak convergence we have

$$\|\nabla\psi\|_{L^2}^2 \leqslant \liminf_{n\to\infty} \|\nabla\tilde{\psi}_n\|_{L^2}^2 \qquad \text{and} \qquad \|\nabla\varphi\|_{L^2}^2 \leqslant \liminf_{n\to\infty} \|\nabla\tilde{\varphi}_n\|_{L^2}^2. \tag{5.60}$$

The convergence almost everywhere and Fatou's lemma give

$$\begin{aligned} &\textstyle\int_{\mathbb{R}^N} V(|\psi|^2)\,\mathrm{d}x \leqslant \liminf_{n\to\infty} \int_{\mathbb{R}^N} V(|\tilde{\psi}_n|^2)\,\mathrm{d}x \quad \text{and} \\ &\textstyle\int_{\mathbb{R}^N} |\psi|^2|\varphi|^2\,\mathrm{d}x \leqslant \liminf_{n\to\infty} \int_{\mathbb{R}^N} |\tilde{\psi}_n|^2|\tilde{\varphi}_n|^2\,\mathrm{d}x. \end{aligned} \tag{5.61}$$

Since $\psi \in \mathcal{E}$, $\varphi \in H^1(\mathbb{R}^N)$ and $\|\varphi\|_{L^2}^2 = \mathfrak{m}$, we have $E(\psi,\varphi) \geqslant \mathcal{G}_{\min}(\mathfrak{m})$. From (5.60) and (5.61) we get $E(\psi,\varphi) \leqslant \liminf_{n\to\infty} E(\tilde{\psi}_n, \tilde{\varphi}_n)$. On the other hand, since $\psi \in \mathcal{E}$, $\varphi \in H^1(\mathbb{R}^N)$ and $\|\varphi\|_{L^2}^2 = \mathfrak{m}$, we have $E(\psi,\varphi) \geqslant \mathcal{G}_{\min}(\mathfrak{m})$. We deduce that necessarily

$$\|\nabla\tilde{\psi}_n\|_{L^2}^2 \longrightarrow \|\nabla\psi\|_{L^2}^2, \quad \text{and} \quad \|\nabla\tilde{\varphi}_n\|_{L^2}^2 \longrightarrow \|\nabla\varphi\|_{L^2}^2 \quad \text{as } n \longrightarrow \infty.$$

Moreover, we must have equalities in (5.61) and the lower limits there are in fact limits. We show that $\|\tilde{\psi}_n - 1\|_{L^2} \longrightarrow \|\psi - 1\|_{L^2}$ in the same way as we proved that $\|\varphi\|_{L^2}^2 = m$. Then the weak convergence and the convergence of norms give $\tilde{\psi}_n - 1 \longrightarrow \psi - 1$ and $\tilde{\varphi_n} \longrightarrow \varphi$ in $H^1(\mathbb{R}^N)$.
The last assertion in Theorem 41 is a consequence of the following well-known and elementary result, known as Riesz-Scheffé lemma: if ϕ_n and ϕ are nonnegative integrable functions on a measure space $(X, \mathcal{A}, \mu)$, if $\phi_n \longrightarrow \phi$ almost everywhere and if $\displaystyle\int_X \phi_n \, d\mu \longrightarrow \int_X \phi \, d\mu$, then $\displaystyle\int_X |\phi_n - \phi| \, d\mu \longrightarrow 0$. □

Proposition 44. *Assume that $(\psi,\varphi) \in \mathcal{E} \times H^1(\mathbb{R}^N)$ is a solution of the minimisation problem $(GS_{\mathfrak{m}})$. Then:*
(i) There exists $\gamma \in [\mathcal{G}'_{\min,r}(\mathfrak{m}), \mathcal{G}'_{\min,\ell}(\mathfrak{m})]$ (where $\mathcal{G}'_{\min,\ell}$ and $\mathcal{G}'_{\min,r}$ are the left and right derivatives of the concave function $\mathcal{G}_{\min}$, respectively) such that

$$-\Delta\psi + \frac{1}{\varepsilon^2}F(|\psi|^2)\psi + \frac{1}{\varepsilon^4}|\varphi|^2\psi = 0, \qquad -\Delta\varphi + \frac{q^2}{\varepsilon^2}|\psi|^2\varphi - \varepsilon^2\gamma\varphi = 0 \qquad \text{in } \mathbb{R}^N. \tag{5.62}$$

(ii) We have $|\psi| \leqslant 1$ almost everywhere and $\psi \in W^{2,p}_{\mathrm{loc}}(\mathbb{R}^N)$, $\varphi \in W^{2,p}(\mathbb{R}^N)$ for any $p \in [2,\infty)$. In particular, ψ and φ are C^1 functions.

(iii) The function (ψ,φ) is radially symmetric (after translation). That is, there exist $x_0 \in \mathbb{R}^N$ and $\tilde{\psi}, \tilde{\varphi} : [0,\infty) \longrightarrow \mathbb{C}$ such that $\psi(x) = \tilde{\psi}(|x - x_0|)$ and $\varphi(x) = \tilde{\varphi}(|x - x_0|)$ for all $x \in \mathbb{R}^N$.

(iv) If (ψ,φ) is a minimizer and $\tilde{\psi}$, $\tilde{\varphi}$ are as in (iii), then the function $|\tilde{\psi}|$ is nondecreasing on $[0,\infty)$, $|\tilde{\varphi}|$ is nonincreasing on $[0,\infty)$ and there exist constants $\theta_0, \beta_0 \in \mathbb{R}$ such that $\tilde{\psi} = e^{i\theta_0}|\tilde{\psi}|$, $\tilde{\varphi} = e^{i\beta_0}|\tilde{\varphi}|$ on $[0,\infty)$.

Proof. (i) is standard.
(ii) If (ψ,φ) is a minimizer, then $(|\psi|,|\varphi|)$ is also a minimizer. It is clear that $E(\min(1,|\psi|),|\varphi|) \leqslant E(|\psi|,|\varphi|)$. Since $(|\psi|,|\varphi|)$ is a minimizer, we must have $E(\min(1,|\psi|),|\varphi|) = E(|\psi|,|\varphi|)$. This implies $\nabla|\psi| = 0$ a.e. and $V(|\psi|^2) = 0$ a.e. in the set $\{x \in \mathbb{R}^N \mid |\psi(x)| > 1\}$, and we deduce that $(|\psi|-1)_+ = 0$ a.e. on $\mathbb{R}^N$. Thus $|\psi| \leqslant 1$ a.e. on $\mathbb{R}^N$. Then the second equation in (5.62) and a standard boot-strap argument imply that $\varphi \in W^{2,p}(\mathbb{R}^N)$ for any $p \in [2,\infty)$. In particular, $\varphi \in C^{1,\alpha}(\mathbb{R}^N)$ for any $\alpha \in (0,1)$ and φ is bounded on $\mathbb{R}^N$.
Since $|\psi| \leqslant 1$ a.e., we have $F(|\psi|^2) \in L^\infty(\mathbb{R}^N)$. The first equation in (5.62) can be written as $-\Delta\psi + A(x)\psi = 0$, where $A = F(|\psi|^2) + \frac{1}{\varepsilon^4}|\varphi|^2 \in L^\infty(\mathbb{R}^N)$. Standard elliptic regularity theory implies that there exists $C > 0$ such that $\|\psi\|_{W^{2,p}(B(y,1))} \leqslant C$ for any $y \in \mathbb{R}^N$. In particular, ψ is C^1 on $\mathbb{R}^N$.
(iii) Since any minimizer (ψ,φ) for the problem (GS_m) is C^1 in $\mathbb{R}^N$, (iii) follows from Theorem 2 p. 314 in [13].
(iv) Given a non-negative, measurable function $w : \mathbb{R}^N \longrightarrow [0,\infty)$ such that $\mathcal{L}^N(\{x \in \mathbb{R}^N \mid w(x) > t\})$ is finite for any $t > 0$, we denote by w_* the symmetric decreasing rearrangement of w. It is well-known that for $p \in (1,\infty)$ we have $\displaystyle\int_{\mathbb{R}^N} |\nabla w_*|^p \, \mathrm{d}x \leqslant \int_{\mathbb{R}^N} |\nabla w|^p \, \mathrm{d}x$, and equality may occur if and only if for any $t \in (0, \sup\mathrm{ess}(w))$, the level set $\{x \in \mathbb{R}^N \mid w(x) > t\}$ is equivalent to a ball. The last statement is a consequence of Lemma 3.2 p. 163 in [3]. It is also well-known that for w_1, w_2 as above we have $\displaystyle\int_{\mathbb{R}^N} w_1 w_2 \, \mathrm{d}x \leqslant \int_{\mathbb{R}^N} (w_1)_*(w_2)_* \, \mathrm{d}x$. Obviously, we have $(|\varphi|^2)_* = (|\varphi|_*)^2$. Let $u = 1 - |\psi|$. Since

$0 \leqslant u \leqslant 1$ and the mapping $s \longmapsto 2s - s^2$ is increasing on $[0,1]$, we have $(2u-u^2)_* = 2u_* - u_*^2$. Therefore

$$\begin{aligned}\int_{\mathbb{R}^N} (1-u_*)^2 (|\varphi|_*)^2 \,\mathrm{d}x &= \int_{\mathbb{R}^N} (|\varphi|_*)^2 \,\mathrm{d}x - \int_{\mathbb{R}^N} (2u_* - u_*^2)(|\varphi|_*)^2 \,\mathrm{d}x \\ &= \int_{\mathbb{R}^N} |\varphi|^2 \,\mathrm{d}x - \int_{\mathbb{R}^N} (2u-u^2)_* (|\varphi|^2)_* \,\mathrm{d}x \\ &\leqslant \int_{\mathbb{R}^N} |\varphi|^2 \,\mathrm{d}x - \int_{\mathbb{R}^N} (2u-u^2)|\varphi|^2 \,\mathrm{d}x \\ &= \int_{\mathbb{R}^N} |\psi|^2 |\varphi|^2 \,\mathrm{d}x.\end{aligned}$$

We infer that $E(1-u_*, |\varphi|_*) \leqslant E(|\psi|, |\varphi|) \leqslant E(\psi, \varphi)$. Since (ψ, φ) is a minimizer and $\| |\varphi|_* \|_{L^2}^2 = \| \, |\varphi| \, \|_{L^2}^2 = \|\varphi\|_{L^2}^2 = \mathfrak{m}$, we must have $E(1-u_*, |\varphi|_*) = E(|\psi|, |\varphi|) = E(\psi, \varphi)$, and consequently $\displaystyle\int_{\mathbb{R}^N} |\nabla u_*|^2 \,\mathrm{d}x = \int_{\mathbb{R}^N} |\nabla u|^2 \,\mathrm{d}x$ and $\displaystyle\int_{\mathbb{R}^N} |\nabla |\varphi|_*|^2 \,\mathrm{d}x = \int_{\mathbb{R}^N} |\nabla |\varphi| \, |^2 \,\mathrm{d}x$. The result of Brothers and Ziemer implies that for almost all $t > 0$, the sets $\{x \in \mathbb{R}^N \mid u(x) > t\}$ and $\{x \in \mathbb{R}^N \mid |\varphi|(x) > t\}$ are equivalent either to a ball or to $\emptyset$. Since we already know that u and $|\varphi|$ are radially symmetric, we infer that the functions $\tilde{u} = 1 - |\tilde{\psi}|$ and $|\tilde{\varphi}|$ are non-increasing on $[0, \infty)$.

The fact that $|\tilde{\varphi}|$ is nonincreasing implies that the set $D = \{x \in \mathbb{R}^N \;\big|\; |\varphi(x)| > 0\}$ is either a ball or $\mathbb{R}^N$. On this set we have a lifting $\varphi = |\varphi| e^{i\beta}$, where $\beta \in H^1_{\mathrm{loc}}(\mathbb{R}^N)$ and $|\nabla \varphi|^2 = |\nabla |\varphi| \, |^2 + |\varphi|^2 |\nabla \beta|^2$. Since $\displaystyle\int_{\mathbb{R}^N} |\nabla \varphi|^2 \,\mathrm{d}x = \int_{\mathbb{R}^N} |\nabla |\varphi| \, |^2 \,\mathrm{d}x$ we must have $\nabla \beta = 0$ a.e. and we infer that β is constant, $\beta(x) = \beta_0$ for a.e. $x \in D$. Therefore $\varphi(x) = e^{i\beta_0} |\varphi(x)| = e^{i\beta_0} |\tilde{\varphi}|(|x - x_0|)$. A similar argument holds for ψ. $\square$

Bibliography

[1] Bethuel, F. & Gravejat, J. & Saut, C. (2009). *Travelling waves for the Gross-Pitaevskii equation II.* Comm. Math. Phys. 285, pp.567-651.

[2] Bouchel, O. (2006) *Quelques équations et systèmes d'équations de Schrödinger non linéaire.* PhD thesis Université Paris-Sud 11.

[3] Brothers, J. & Ziemer, W. (1988). *Minimal rearrangements of Sobolev functions.* J. Reine Angew. Math., 384, pp.153–179.

[4] Chiron, D. & Mariş, M. (2017) *Traveling waves for nonlinear Schrödinger equations with nonzero conditions at infinity.* Arch. Rational Mech. Analysis, Vol. 226, No. 1, pp. 143-242.

[5] Chiron, D. & Scheid, C. (2016). *Travelling Waves for the Nonlinear Schrödinger Equation with General Nonlinearity in Dimension Two.* Journal of Nonlinear Science. 26, no. 1 : 171-231.

[6] Grant, J. & Roberts, P. H. (1974). *Motions in a Bose condensate. III. The Structure and Effective Masses of Charged and Uncharged Impurities.* J. Phys. A: Math. Nucl. Gen. 7 260.

[7] Grillakis, M. & Shatah, J. & Strauss, W. (1987) *Stability theory of solitary waves in the presence of symmetry.* I. J. Funct. Anal., 74(1):160-197.

[8] Jones, C.A. & Roberts, P. H. (1982). *Motions in a Bose condensate. VI. Axisymmetric solitary waves.* J. Phys. A: Math. and Gen., 15(8):2599-2619.

[9] Lieb, E.H. (1983) *On the lowest eigenvalue of the Laplacian for the intersection of two domains.* Inventiones Mathematicae 74, pp. 441-448.

[10] Lin, Z. (2002) *Stability and instability of traveling solitonic bubbles.* Adv. Differential Equations, 7(8):897-918.

[11] Lin, Z. & Wang, Z. & Zeng, C. (2016) *Stability of traveling waves of nonlinear Schrödinger equation with nonzero condition at infi*

nity. Arch. Ration. Mech. Anal., 222(1):143-212.

[12] Lions, P.L. (1984) *The concentration-compactness principle in the calculus of variations. The locally compact case. I.* Ann. Inst. H. Poincaré Anal. Non Linéaire, 1(2):109-145.

[13] Mariş, M. (2008) *On the symmetry of minimizers.* Arch. Ration. Mech. Anal., 192(2):311-330.

[14] Mariş, M. (2009) *Nonexistence of supersonic traveling waves for nonlinear Schrödinger equations with nonzero conditions at infinity.* SIAM J. Math. Anal., 40(3):1076-1103.

[15] Mariş, M. (2013) *Traveling waves for nonlinear Schrödinger equations with nonzero conditions at infinity.* Ann. of Math. 178 (1), pp. 107-182.

[16] Ovchinnikov, Y.N. & Sigal, I.M. (1997) *Ginzburg-Landau equation. I. Static vortices.* In Partial differential equations and their applications (Toronto, ON, 1995), volume 12 of CRM Proc. Lecture Notes, pages 199-220. Amer. Math. Soc., Providence, RI.

[17] Weinstein, M.I. (1985) *Modulational stability of ground states of nonlinear Schrödinger equations.* SIAM J. Math. Anal., 16(3):472-491.

Chapter 6

Conclusion and some perspectives

The main goal of this thesis was the analysis of the Gross-Clark-Schrödinger system given by

$$\text{(GC)} \qquad \begin{cases} i\partial_t \Psi &= -\Delta\Psi + \frac{1}{\varepsilon^2}(\frac{1}{\varepsilon^2}|\Phi|^2 + |\Psi|^2 - 1)\Psi \\ \\ i\delta\partial_t \Phi &= -\Delta\Phi + \frac{1}{\varepsilon^2}(q^2|\Psi|^2 - \varepsilon^2 k^2)\Phi \end{cases} \qquad \text{in } \mathbb{R}\times\mathbb{R}^N,$$

with the "boundary conditions"

$$|\Psi| \to 1, \Phi \to 0 \text{ as } |x| \to \infty.$$

First, in Chapter 2, we have shown that the Cauchy problem associated to this system in dimension $N \in \{1,2,3\}$ is locally well posed in $\mathcal{E}\times H^1(\mathbb{R}^N)$ using Strichartz type estimates and a fixed point theorem. The global well posedness was obtained from the conservation of the energy and the L^2 norm of Φ.

Second, the existence of the travelling waves solutions of speed c for the system (GC) was the aim of Chapter 3, where we have studied several minimizing problems to prove the existence of solutions for the system

$$\text{(TW)} \qquad \begin{cases} -ic\frac{\partial\psi}{\partial x_1} &= -\Delta\psi + \frac{1}{\varepsilon^2}(\frac{1}{\varepsilon^2}|\varphi|^2 + |\psi|^2 - 1)\psi \\ (-\Delta + \frac{q^2}{\varepsilon^2}|\psi|^2)\varphi &= \lambda\varphi, \end{cases}$$

The minimizers of the energy at fixed mass for φ has given us the ground states solutions of the system. We have then obtained some minimizers of energy at fixed mixed momentum. The existence of minimizers for the energy at fixed mass for φ and momentum for ψ was obtained under some condition on the strict sub-additivity of E_{min} (see (3.70) in Chapter 3). It would be very fascinating to find the values of p and m for which this condition holds. The existence of these minimizers is restricted to some values of c in terms of the right and the left derivatives of the minimal energy.

Numerically, in Chapter 4 we have found the travelling waves solutions of small mass for (GC) and an approximation for the eigenvalue λ, using a Newton-Raphson algorithm with seperated iterations. These solutions were obtained for speeds $0.165 \leq c \leq 0.355$. One can search for other numerical methods to find an algorithm that converges for a larger speed range.

Finally, in Chapter 5, two branches of solutions for (TW) were studied numerically for small speeds c and with mass equal to 4π: the ground state branch, issued from the stationary solutions of type ground state, and the vortex branch issued from the stationary solutions of type vortex of degree 1. We could plot the energy-momentum diagram corresponding to each type of these solutions. For the ground state branch, the diagram is obtained for speeds $c \in [0, 2.35]$, while it is found for

speeds $c \in [0.12, 0.19]$ for the vortex branch. Moreover, we have noticed that for the vortex branch, if $c \in [0.7, 2.35]$, the algorithm converges, but the obtained solutions are the same as the ones of the ground state branch and do not present vortices anymore. The proof for the existence of the ground states solutions was recalled in this Chapter. A similar one could be adapted to prove rigorously the existence of the stationary solutions of type bubble-vortices. An interesting study is to test the convergence of the algorithm of minimizing the normalized energy for the vortices of degree 2 and 3 and study the issued branch of these minimizers.