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THÈSE DE DOCTORAT

DE L'UNIVERSITÉ DE NANTES
COMUE UNIVERSITÉ BRETAGNE LOIRE

Ecole Doctorale N° 601
Mathématiques et Sciences et Technologies de l'Information et de la Communication
Spécialité : *Mathématiques et leurs Interactions*
Par

Mohamad RACHID

**Estimations spectrales et limite hydrodynamique pour
l'équation de Landau**

Thèse présentée et soutenue à l'UNIVERSITÉ DE NANTES, le 7 décembre 2021
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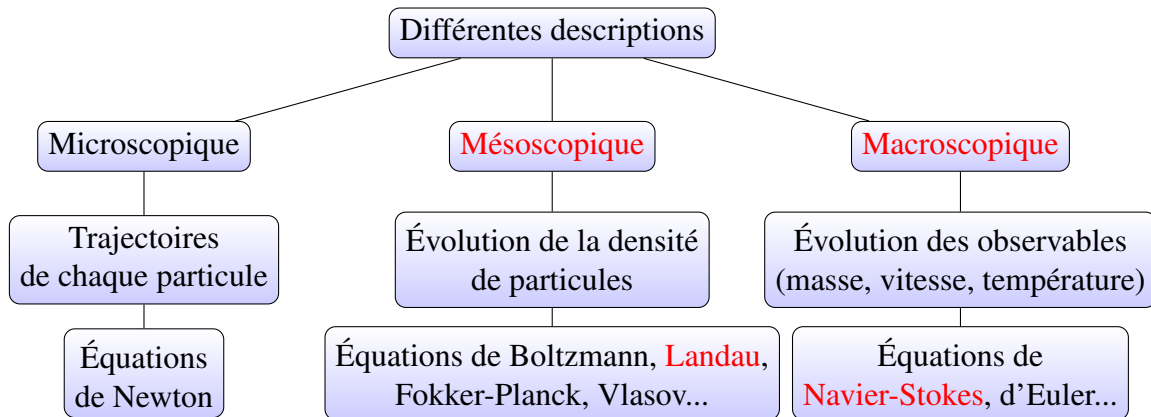
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Introduction

1 Contexte et motivation physique

1.1 Contexte physique

Dans un système de particules comme un gaz, plusieurs niveaux de description existent. La première est la *description microscopique* qui consiste à étudier les trajectoires de chaque particule du système en se basant sur le principe de Newton. Cependant, cette dernière présente des inconvénients, d'une part du fait du grand nombre de particules considérées, d'autre part l'absence d'informations sur les observables macroscopiques du système comme la masse, la vitesse moyenne et la température qui sont des grandeurs physiques intéressantes et mesurables. La deuxième description est la *description macroscopique* qui décrit les grandeurs macroscopiques de ce système (quantités mentionnées précédemment). La troisième description est la *description mésoscopique*, qui se situe entre les descriptions microscopique et macroscopique, c'est à ce niveau que se situent les équations de la théorie cinétique des gaz. Cette théorie a pour objectif de modéliser tout système formé d'un grand nombre de particules. C'est une description statistique, l'inconnue est la densité de probabilité d'une particule, où nous nous intéressons au comportement typique d'une particule plutôt qu'à l'étude détaillée de chacune tout en préservant les informations physiques du système. Le schéma suivant résume les différentes descriptions :



Dans cette thèse, nous nous intéressons dans un premier temps au *niveau mésoscopique*. En particulier, nous nous intéressons à *l'équation de Landau*. Plus précisément, on considère un système formé d'un très grand nombre de particules qui subissent des collisions entre elles. L'état du système est alors décrit par une densité de probabilités $f = f(t, x, v)$ représentant au temps $t \in \mathbb{R}^+$ la densité de particules à la position $x \in \Omega \subset \mathbb{R}^3$ et à une vitesse $v \in \mathbb{R}^3$. Dans un deuxième temps, nous nous intéressons au passage du *niveau mésoscopique au niveau macroscopique*. Ce lien est établi dans un régime où les collisions sont très nombreuses. Plus précisément, nous mettons notre équation cinétique sous une forme adimensionnée en introduisant un paramètre physique très petit, le nombre de Knudsen. C'est l'inverse du nombre moyen de collisions pour chaque particule par unité de temps. Prendre le nombre de Knudsen petit a pour effet de renforcer le rôle des collisions et donc lorsque ce paramètre tend vers zéro, nous obtenons un modèle en mécanique des fluides. Plus précisément, nous justifions le passage de *l'équation de Landau* vers *l'équation de Navier-Stokes incompressible*.

1.2 Quelques modèles cinétiques

Nous présentons ici plusieurs modèles cinétiques qui sont pour des raisons historiques et mathématiques des modèles centraux en théorie cinétique des gaz : les équations de Vlasov, Boltzmann, Landau et Fokker-Planck. La présentation de l'équation de Landau est la plus exhaustive car il s'agit du modèle étudié dans cette thèse.

Équation de Vlasov

C'est l'équation cinétique fondamentale qui décrit un système pour lequel les particules ne subissent ni forces ni chocs. D'après les lois de Newton, les particules se déplacent en ligne droite à vitesse constante :

$$\begin{cases} dx/dt = v \\ dv/dt = 0. \end{cases}$$

La densité de particules f est alors solution de l'équation dite de transport libre :

$$\partial_t f + v \cdot \nabla_x f = 0, \quad (1.1)$$

où $v \cdot \nabla_x$ est l'opérateur de transport. Lorsqu'il y a des forces et des chocs, cette équation doit être modifiée. Lorsque nous considérons une force $F = qE$, où q est la charge (prise égale à 1) et E est un champ auto-consistant de type Poisson associé à la distribution $f(t, x, v)$ donné par :

$$E(t, x) = -\frac{1}{|\mathbb{S}^2|} \frac{x}{|x|^3} *_x \int_{\mathbb{R}^3} f(t, x, v) dv,$$

alors on obtient l'équation de "Vlasov-Poisson", équation non linéaire et donnée (lorsque $q = 1$) par :

$$\partial_t f + v \cdot \nabla_x f - E \cdot \nabla_v f = 0. \quad (1.2)$$

En considérant un système de particules qui subissent des chocs entre elles, modélisés par un noyau de collision Q , le modèle cinétique (sans force auto-consistante) est donné comme suit :

$$\partial_t f + v \cdot \nabla_x f = Q(f), \quad (1.3)$$

où Q agit uniquement sur la variable vitesse. Dans ce cas là, on aboutit alors à des équations cinétiques variées, les plus célèbres étant celles de Boltzmann, Landau (voir [47, 128] par exemple) et Fokker-Planck (voir [114] par exemple). Nous détaillons ci-dessous ces modèles.

Équation de Boltzmann

L'équation de Boltzmann joue un rôle important dans la théorie cinétique, c'est une équation intégro-différentielle qui décrit l'évolution d'un gaz raréfié hors équilibre. C'est la plus ancienne des équations cinétiques dérivée formellement par Boltzmann [14] et Maxwell [100]. L'inconnue est une fonction de distribution positive $f = f(t, x, v) \geq 0$. Elle représente au temps $t \geq 0$ la densité de particules à la position $x \in \Omega \subset \mathbb{R}^3$ avec une vitesse $v \in \mathbb{R}^3$. Cette équation est donnée par

$$\partial_t f + v \cdot \nabla_x f = Q_B(f, f), \quad (1.4)$$

et doit être complétée par une donnée initiale $f|_{t=0} = f_0$. Ici, Q_B est l'opérateur de collision de Boltzmann, c'est un opérateur bilinéaire qui agit uniquement sur la variable de vitesse v et modélise les interactions entre particules. Plus précisément, l'opérateur de Boltzmann est défini par

$$Q_B(g, f) = \int_{\mathbb{R}^3 \times \mathbb{S}^2} B(v - v_*, \sigma) (g'_* f' - g_* f) dv_* d\sigma,$$

où σ est un vecteur unité sur la sphère \mathbb{S}^2 . Nous utilisons les notations standards suivantes : $f = f(t, x, v)$, $f' = f(t, x, v')$, $g_* = g(t, x, v_*)$ et $g'_* = g(t, x, v'_*)$. Ici, v', v'_* sont les vitesses avant collision entre deux particules et v, v_* sont les vitesses après collision de deux particules. Dans le cas où les collisions sont élastiques (il y a conservation de l'énergie cinétique et de la quantité de mouvement) :

$$\begin{cases} v' + v'_* = v + v_* \\ |v'|^2 + |v'_*|^2 = |v|^2 + |v_*|^2, \end{cases}$$

où $|\cdot|$ est la norme canonique euclidienne dans \mathbb{R}^3 . Les vitesses avant collision (v', v'_*) sont alors données par les formules suivantes :

$$\begin{cases} v' = \frac{v + v_*}{2} + \frac{|v - v_*|}{2}\sigma \\ v'_* = \frac{v + v_*}{2} - \frac{|v - v_*|}{2}\sigma. \end{cases}$$

La fonction B est appelée le noyau de collision de Boltzmann. Elle est positive et dépend seulement de la vitesse relative $|v - v_*|$ et du cosinus de l'angle de déviation formé par les deux vecteurs $v - v_*$ et $v' - v'_*$ (voir par exemple [47, 128] pour plus d'informations sur l'équation de Boltzmann).

Équation de Landau

L'équation de Landau est un modèle cinétique qui peut être dérivé de l'équation de Boltzmann. Ce modèle a été introduit par Landau en 1936 et est utilisé pour la modélisation de plasmas. Le terme plasma, appelé aussi "quatrième état de la matière", a été utilisé en physique pour la première fois par le physicien américain Irving Langmuir en 1928 et désigne un gaz ionisé. Ce phénomène n'est visible sur Terre qu'à très haute température, par exemple si nous chauffons assez un gaz et s'il est soumis à un champ électrique assez puissant, les électrons sont arrachés de leur atomes (voir Figure 1.1). On observe alors une sorte de "soupe" dans laquelle baignent des électrons libres et des noyaux d'atomes. Du fait de ces électrons libres, le plasma est un fluide qui conduit bien l'électricité contrairement au gaz. Les plasmas sont extrêmement répandus dans l'univers puisqu'ils représentent plus de 99% de la matière connue.

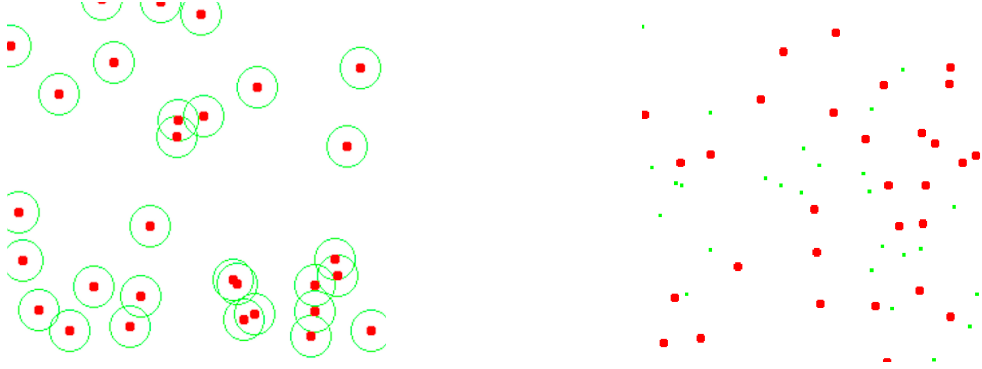


FIGURE 1.1: Transformation d'un gaz chauffé en plasma. Les noyaux des atomes sont représentés en rouge et les électrons en vert.

Source : <https://www.epfl.ch/research/domains/swiss-plasma-center/fr/plasma-et-fusion/#faq-item-057af5a7aed103b311fb9a1b5fd8439a>

Dans [88], Landau a établi par des procédures formelles une asymptotique du noyau de Boltzmann dans le cas de collisions rasantes. On appelle “collision rasante” une collision dans laquelle l'angle de déviation est proche de zéro. La limite de collisions rasantes permet de rendre toutes les collisions rasantes. Plus précisément, nous considérons une suite de noyaux de collision B_ε telle que la partie angulaire se concentre sur les collisions rasantes quand $\varepsilon \rightarrow 0$. Dans ce cas là, l'opérateur de Boltzmann associé à B_ε tend vers l'opérateur de Landau (voir par exemple [33, 31] pour plus de détails sur la limite de collisions rasantes). De plus, l'approximation de Landau est comprise mathématiquement dans le cas d'asymptotique des collisions rasantes (nous renvoyons à [2] pour plus de détails). Passons maintenant à la présentation de l'équation de Landau.

L'équation de Landau est un modèle cinétique en physique des plasmas qui décrit l'évolution de la fonction de densité $f = f(t, x, v)$ à un temps $t \in \mathbb{R}^+$, une position $x \in \Omega \subset \mathbb{R}^3$ et à une vitesse $v \in \mathbb{R}^3$. Cette équation est donnée par

$$\partial_t f + v \cdot \nabla_x f = Q_L(f, f), \quad (1.5)$$

et doit être complétée par une donnée initiale $f|_{t=0} = f_0$. Ici Q_L , noyau de collision de Landau, est un opérateur intégro-différentiel, bilinéaire, non-local qui agit seulement sur la variable v et qui contient de la diffusion en vitesse. Plus précisément, l'opérateur de Landau est défini par

$$Q_L(g, f) = \partial_i \int_{\mathbb{R}^3} a_{ij}(v - v_*) [g_* \partial_j f - f \partial_j g_*] dv_*, \quad (1.6)$$

où nous utilisons la convention de sommation pour les indices répétés et les dérivées sont des dérivées par rapport à la variable de vitesse v i.e. $\partial_i = \partial_{v_i}$. Nous utilisons également les notations $g_* = g(v_*)$, $f = f(v)$, $\partial_j g_* = \partial_{v_{*j}} g(v_*)$, $\partial_j f = \partial_{v_j} f(v)$. La matrice $A(v) =$

$(a_{i,j}(v))_{1 \leq i,j \leq 3}$ est symétrique définie positive et dépend du type d'interaction entre les particules que l'on considère. Elle est donnée par (voir [128] pour plus d'informations) :

$$a_{i,j}(v) = |v|^{\gamma+2} \left(\delta_{ij} - \frac{v_i v_j}{|v|^2} \right), \quad \gamma \in [-3, 1]. \quad (1.7)$$

Par analogie avec l'équation de Boltzmann : on parle de potentiels durs si $\gamma \in (0, 1]$, de molécules Maxwelliennes si $\gamma = 0$, de potentiels faiblement mous si $\gamma \in [-2, 0)$, de potentiels mous si $\gamma \in (-3, -2)$ et de potentiel Coulombien si $\gamma = -3$. Nous notons que le modèle physique correspond au cas $\gamma = -3$ (le cas de potentiel Coulombien), cas le plus difficile compte tenue de la singularité présente dans (1.7). Les autres cas $-3 < \gamma \leq 1$ sont aussi intéressants et importants du point de vue mathématique.

Maintenant, on va discuter des propriétés de l'opérateur de Landau (il est à noter que l'opérateur de Boltzmann vérifie le même type de propriétés excepté la diffusion). Soit $\varphi = \varphi(v)$ une fonction test, en utilisant (1.6) et la symétrie de la matrice $A(v)$ nous obtenons formellement la formulation faible suivante pour l'opérateur de Landau :

$$\int_{\mathbb{R}^3} Q_L(f, f) \varphi(v) dv = -\frac{1}{2} \int_{\mathbb{R}^3 \times \mathbb{R}^3} a_{ij}(v - v_*) \left(\frac{\partial_i f}{f} - \frac{\partial_i f_*}{f_*} \right) (\partial_j \varphi - \partial_j \varphi_*) f f_* dv_* dv. \quad (1.8)$$

Si on prend $\varphi = 1$ ou $\varphi(v) = v_k$ avec $k \in \{1, 2, 3\}$, nous observons facilement que $\partial_j \varphi - \partial_j \varphi_* = 0$ ainsi (1.8) s'annule. De même, si on choisit $\varphi(v) = |v|^2$ on a $\partial_j \varphi - \partial_j \varphi_* = 2(v_j - v_{*j})$ puis en utilisant (1.7), on a $a_{ij}(v - v_*)(v_j - v_{*j}) = 0$. On en déduit ainsi que l'opérateur de collision de Landau Q_L défini dans (1.6) possède la propriété suivante :

$$\int_{\mathbb{R}^3} Q_L(f, f) \varphi(v) dv = 0, \quad \text{pour } \varphi(v) = 1, v, |v|^2. \quad (1.9)$$

Dans le cas où nous travaillons avec $\Omega = \mathbb{T}^3$ (les particules sont confinées dans le tore), en intégrant (1.9) par rapport à $x \in \mathbb{T}^3$, nous obtenons alors que si f solution de (1.5) :

$$\begin{aligned} & \frac{d}{dt} \int_{\mathbb{R}^3 \times \mathbb{T}^3} f(t, x, v) \varphi(v) dx dv \\ &= \int_{\mathbb{R}^3 \times \mathbb{T}^3} \left(Q_L(f(t, x, \cdot), f(t, x, \cdot))(v) - v \cdot \nabla_x f(t, x, v) \right) \varphi(v) dx dv \\ &= 0, \quad \text{pour } \varphi(v) = 1, v, |v|^2. \end{aligned}$$

Donc l'équation de Landau (1.5) conserve respectivement la masse globale, la quantité de mouvement globale et l'énergie globale :

$$\begin{aligned} \int_{\mathbb{T}^3 \times \mathbb{R}^3} f(t, x, v) dx dv &= \int_{\mathbb{T}^3 \times \mathbb{R}^3} f_0(x, v) dx dv, \quad \forall t \geq 0, \\ \int_{\mathbb{T}^3 \times \mathbb{R}^3} v f(t, x, v) dx dv &= \int_{\mathbb{T}^3 \times \mathbb{R}^3} v f_0(x, v) dx dv, \quad \forall t \geq 0, \\ \int_{\mathbb{T}^3 \times \mathbb{R}^3} |v|^2 f(t, x, v) dx dv &= \int_{\mathbb{T}^3 \times \mathbb{R}^3} |v|^2 f_0(x, v) dx dv, \quad \forall t \geq 0. \end{aligned} \quad (1.10)$$

Maintenant, on définit la fonctionnelle d'entropie

$$H(f) := \int_{\mathbb{T}^3 \times \mathbb{R}^3} f \log f \, dx dv \quad (1.11)$$

ainsi que la fonctionnelle de dissipation d'entropie

$$D(f) := - \int_{\mathbb{R}^3} Q_L(f, f) \log f \, dv.$$

Formellement, en prenant $\varphi = \log f$ dans (1.8) et en utilisant le fait que la matrice $A(v)$ est définie positive, on obtient :

$$D(f) = \frac{1}{2} \int_{\mathbb{R}^3 \times \mathbb{R}^3} a_{ij}(v - v_*) \left(\frac{\partial_i f}{f} - \frac{\partial_i f_*}{f_*} \right) \left(\frac{\partial_j f}{f} - \frac{\partial_j f_*}{f_*} \right) f f_* \, dv_* \, dv \geq 0. \quad (1.12)$$

On déduit alors formellement que si f solution de (1.5),

$$\frac{d}{dt} H(t) = - \int_{\mathbb{T}^3} D(f) \, dx \leq 0. \quad (1.13)$$

D'autre part, une distribution f vérifiant $D(f)(t, x) = 0$ est une Maxwellienne locale de la forme (voir par exemple [128]) :

$$f(t, x, v) = \frac{\rho(t, x)}{(2\pi\theta(t, x))^{3/2}} \exp\left(-\frac{|v - u(t, x)|^2}{2\theta(t, x)}\right) := \mathcal{M}_{\rho, u, \theta}(t, x, v), \quad (1.14)$$

où $\rho(t, x)$, $u(t, x)$, $\theta(t, x)$ sont respectivement la densité locale, la vitesse macroscopique locale et la température locale de f au point (t, x) définies comme suit :

$$\begin{aligned} \rho(t, x) &= \int_{\mathbb{R}^3} f(t, x, v) \, dv, & u(t, x) &= \frac{1}{\rho} \int_{\mathbb{R}^3} v f(t, x, v) \, dv, \\ \theta(t, x) &= \frac{1}{3\rho} \int_{\mathbb{R}^3} |v - u(t, x)|^2 f(t, x, v) \, dv. \end{aligned} \quad (1.15)$$

Lorsqu'on cherche des états d'équilibres globaux, il faut étudier les maxwelliennes \mathcal{M} vérifiant

$$\forall x \in \mathbb{T}^3, \quad \forall v \in \mathbb{R}^3, \quad v \cdot \nabla_x \mathcal{M} = 0. \quad (1.16)$$

La résolution de cette équation montre que la solution est donnée par une Maxwellienne $\mathcal{M} = \mathcal{M}(v)$ indépendante des variables de temps t et d'espace x . De plus, \mathcal{M} est définie de manière unique lorsque la masse totale, la quantité de mouvement totale et l'énergie totale sont fixées. Nous notons μ l'unique équilibre global de masse 1, quantité de mouvement nulle et énergie 3 donné par :

$$\mu(v) = (2\pi)^{-3/2} e^{-|v|^2/2}. \quad (1.17)$$

Ces deux propriétés, décroissance de l'entropie et états d'équilibres Maxwelliens sont la version de Landau du célèbre Théorème H de Boltzmann (pour plus de détails voir [36, 127]).

Équation de Fokker-Planck

L'équation de Fokker-Planck est un modèle simplifié de celles de Boltzmann ou de Landau. Ce modèle décrit un système de particules qui subissent un mouvement brownien. Cette équation est donnée par :

$$\partial_t f + v \cdot \nabla_x f = Q_{FP}(f). \quad (1.18)$$

Ici, Q_{FP} est le noyau de collision de Fokker-Planck, c'est un opérateur linéaire qui agit seulement sur la variable v et est défini par :

$$Q_{FP}f = \nabla_v \cdot (\nabla_v + v)f = \Delta_v f + \nabla_v \cdot (vf). \quad (1.19)$$

La masse est la seule quantité conservée par cette équation (voir par exemple [128]). Faisons l'analogie avec l'opérateur de collision de Landau défini au paragraphe précédent. Soit $b(v) := \nabla_v \cdot A(v)$, la divergence de la matrice $A(v)$ (où $A(v)$ est définie dans (1.7)). En utilisant (1.7) on a $b(v) = -2|v|^\gamma v$. Alors nous pouvons réécrire l'opérateur de Landau (1.6) de la manière suivante :

$$Q_L(g, f) = \nabla_v \cdot \{(A(v) *_v g) \nabla_v f - (b(v) *_v g) f\}. \quad (1.20)$$

Si on prend $g = \mu$ (où μ est défini en (1.17)) dans (1.20), on obtient

$$Q_L(f, \mu)(v) = \nabla_v \cdot \{(A(v) *_v \mu) \nabla_v f - (b(v) *_v \mu) f\}. \quad (1.21)$$

Nous remarquons que l'opérateur dans (1.21) ressemble beaucoup à l'opérateur de Fokker-Planck donné par (1.19) mais qu'il est plus compliqué à cause de la matrice de diffusion $A(v) *_v \mu$. Cet opérateur apparaît naturellement lorsqu'on linéarise l'équation de Landau près de l'équilibre μ et sera central dans cette thèse.

Il est important aussi de regarder d'autres versions de cette équation, par exemple une étude en changeant le laplacien par un laplacien fractionnaire est intéressante. Cette dernière équation, peut être vue comme un modèle jouet de l'équation de Boltzmann linéarisée sans troncature angulaire. Une étude de cette équation peut ainsi être considérée comme une première étape pour comprendre un modèle plus compliqué de l'équation de Boltzmann sans cut-off (voir par exemple [3]).

2 Problèmes mathématiques abordés

Nous présentons ici des problèmes historiques importants concernant ces modèles cinétiques ainsi que leur lien avec les équations de la mécanique des fluides.

A) Existence de solutions. Pour faire une étude mathématique de ces modèles, l'étude du problème de Cauchy est la première étape. Nous donnons ici rapidement l'idée de différents cadres possibles pour étudier la théorie de Cauchy.

A-1) Le cadre des *solutions spatialement homogènes* est le premier cadre qui a été étudié dans cette théorie. Il s'agit de considérer une densité de particules $f = f(t, v)$ qui ne dépend pas de la variable position x . Cette hypothèse permet de simplifier le problème étudié, le terme de transport disparaît, ainsi nous évitons les difficultés liées au domaine spatial et aux conditions aux bords associées. Ce problème a été étudié très largement, nous n'entrerons pas ici dans les détails, nous mentionnons par exemple Carleman [21, 20] qui a introduit ce problème pour une équation de Boltzmann avec sphères dures.

A-2) Le deuxième cadre est le cadre des *solutions perturbatives*. Le Théorème H de Boltzmann suggère que les solutions de l'équation convergent lorsque $t \rightarrow \infty$ vers un équilibre global. Il est donc important d'étudier les solutions autour de cet état d'équilibre. Nous citons Hilbert [72, 74], Carleman [21] et Grad [55, 53] pour les études fondamentales de cette théorie. Dans ce cadre, Ukai [124] a construit les premières solutions globales fortes de l'équation de Boltzmann et a obtenu le premier résultat de convergence exponentielle vers l'équilibre global. De plus, des justifications rigoureuses de limites hydrodynamiques ont été aussi obtenues en utilisant ce cadre (voir [40] par exemple).

A-3) Parlons maintenant d'un autre cadre qui est celui des *solutions petites*. Ce cadre a été développé à la suite des travaux de Shinbrot, Kaniel, Illner à partir des années 1980 sur l'équation de Boltzmann. Kaniel et Shinbrot [84] ont construit pour la première fois des solutions régulières en temps petit. Illner et Shinbrot [78] ont travaillé sur les solutions globales proches du vide dans l'espace tout entier ($\Omega = \mathbb{R}^3$).

A-4) Finalement, mentionnons la théorie des *solutions renormalisées* qui a été développée par DiPerna et Lions dans la fin des années 80 (voir l'article fondateur [37]). D'une manière générale, cette méthode fournit une théorie d'existence globale en temps de solutions faibles sans hypothèse de petitesse sur la donnée initiale. Cette méthode s'applique aussi à des variantes de l'équation de Boltzmann, citons par exemple l'article d'Alexandre et Villani [1] où un résultat d'existence de solutions renormalisées à la DiPerna-Lions a été obtenu pour les cas des équations de Boltzmann sans cut-off et de Landau.

B) Comportement en temps long. Une des motivations de Boltzmann quand il a construit son équation était d'étudier le retour vers l'équilibre des solutions, dans le but de donner une base mathématique claire au second principe de la thermodynamique. Le Théorème H de Boltzmann était la solution formelle pour ce problème. La première partie du Théorème H nous informe que la fonctionnelle d'entropie (1.11) est décroissante en temps (1.13). D'autre part, il nous dit également que les distributions Maxwelliennes maximisent cette fonctionnelle. On obtient formellement que la solution converge vers l'équilibre quand il y en a un, lorsque $t \rightarrow \infty$. Le problème du retour à l'équilibre consiste à étudier rigoureusement cette convergence ainsi qu'à quantifier la vitesse de convergence. Il existe plusieurs méthodes pour répondre à ce problème :

B-1) Une des méthodes pour montrer la convergence vers l'équilibre est la *méthode de*

compacité. Cette méthode est basée sur des arguments non constructifs et ces résultats ne donnent pas d'informations sur la vitesse de convergence.

B-2) La deuxième méthode est la *méthode de linéarisation (méthode perturbative)*. L'idée de cette méthode est de linéariser l'équation étudiée autour de l'équilibre (si l'équation n'est pas linéaire) puis d'étudier les propriétés spectrales de l'opérateur de collision linéarisé (plus précisément, une étude sur l'existence du trou spectral). Cette étude se fait dans un espace qui convient avec le problème non linéaire. Ainsi nous pouvons obtenir une vitesse de convergence pour l'équation non linéaire mais seulement dans un certain voisinage de l'équilibre dans lequel les termes non linéaires sont absorbés par les termes linéaires de l'équation.

Soulignons que *l'hypo coercivité* est une sous-méthode de la méthode de linéarisation qui permet d'étudier des équations linéarisées dans un cadre non homogène. Cette méthode a pour but d'étudier le retour à l'équilibre ainsi de déterminer explicitement le taux de convergence. Cette convergence est due aux interactions mathématiques entre l'opérateur de collision (dissipatif en vitesse) et l'opérateur de transport (conservatif). Cette méthode rappelle la problématique de l'hypoellipticité en terme de la régularité des équations aux dérivées partielles (voir par exemple [63, 67, 126]). La notion d'hypo coercivité provient d'un parallèle avec la notion d'hypoellipticité. En effet, l'opérateur de collision fournit de la coercivité dans la variable de vitesse et le but de la théorie de l'hypo coercivité est de comprendre comment l'interaction avec l'opérateur de transport permet de récupérer de la coercivité en variable de position aussi. La théorie hypoelliptique s'intéresse à des opérateurs n'ayant pas d'ellipticité globale c'est à dire la partie elliptique de l'opérateur est dégénérée et a pour but d'obtenir un effet régularisant pour l'équation globale suite à l'interaction entre la partie elliptique et la partie transport. De plus, un opérateur différentiel \mathcal{P} est dit hypoelliptique si pour toute distribution u qui vérifie $\mathcal{P}u = f$ dans le sens des distributions avec f une fonction régulière donnée alors u est régulière également (voir par exemple [75, 86] pour plus de détails sur l'hypoellipticité). Nous donnons plus de détails sur l'hypo coercivité dans la Section 3.

B-3) L'utilisation des *méthodes d'entropie* est une manière pour aborder ce problème. L'idée est d'obtenir une inégalité entre la dissipation d'entropie et l'entropie relative pour obtenir le retour vers l'équilibre. La vitesse de convergence dépend alors de l'inégalité obtenue. Par exemple, la conjecture de Cercignani est l'inégalité suivante :

$$D(f) \geq K_f H(f|\mu)$$

où $D(f)$ la dissipation d'entropie, K_f une constante et l'entropie relative $H(f|\mu) := H(f) - H(\mu)$ (ici μ l'équilibre de l'équation). Cette inégalité permet d'obtenir la convergence exponentielle vers l'équilibre avec un taux déterminé par K_f (voir [35, 128] pour plus de détails sur cette inégalité).

C) Problème de limite hydrodynamique. Une des motivations de l'étude de la limite hydrodynamique est le sixième problème de Hilbert : peut-on obtenir les équations fondamentales de la mécanique des fluides à partir d'un modèle microscopique dirigé par

les lois de Newton ? De nombreux travaux ont été réalisés (voir par exemple [119, 85]) pour étudier ce célèbre problème. Jusqu'à maintenant sa solution n'est connue que pour quelques modèles simplifiés non réalistes. Dans ce contexte, Hilbert [73] suggérait de passer en première étape des équations de Newton à l'équation de Boltzmann puis en deuxième étape de l'équation de Boltzmann aux différentes équations de l'hydrodynamique. Une autre motivation est que l'espace des phases est beaucoup plus grand dans une description cinétique, ce qui rend cette dernière a priori plus complexe qu'une description hydrodynamique. Les équations cinétiques nécessitent des calculs assez coûteux dans les applications pratiques, d'où l'intérêt de remplacer un modèle cinétique par un modèle hydrodynamique lorsque cela est possible.

Pour étudier la limite hydrodynamique, nous devons écrire les équations cinétiques sous une forme adimensionnée en introduisant le nombre de Knudsen (qu'on le notera dans cette thèse par ε). C'est l'inverse du nombre moyen de collisions pour chaque particule par unité de temps (voir [9] par exemple pour plus de détails sur le nombre de Knudsen). Il est important aussi de souligner que le Théorème *H* de Boltzmann suggère que la solution ressemble de plus en plus à un équilibre thermodynamique local (1.14) dans un régime physique où les collisions entre les particules sont très fréquentes, i.e. de faible nombre de Knudsen. L'étude des limites hydrodynamiques pour les équations cinétiques consiste à montrer la validité d'une forme d'équilibre thermodynamique local lorsque le nombre de Knudsen tend vers zéro et obtenir les équations limites (Navier-Stokes, Euler...). Parlons brièvement du contexte des limites hydrodynamiques.

C-1) Dérivation basée sur des développements asymptotiques. Les premières justifications mathématiques des limites hydrodynamiques de l'équation de Boltzmann sont données par Hilbert [73] et Chapman-Enskog [28]. Ces méthodes sont purement formelles, basées sur les développements asymptotiques et permettent d'obtenir formellement les équations d'Euler et Navier-Stokes (compressible ou non). Grad [54, 53] a introduit une autre méthode formelle dite méthode des moments pour dériver les équations hydrodynamiques. Plusieurs résultats de limites hydrodynamiques sont obtenus en utilisant ces méthodes, citons par exemple Caffisch [19] où sont utilisés les développements asymptotiques pour montrer les premières preuves de convergence vers le système d'Euler compressible (voir aussi [87] et [29]). Finalement, mentionnons également qu'il existe un autre type de résultats concernant la connexion aux équations fluides également obtenus dans le cadre de solutions classiques. L'idée générale est d'obtenir d'abord les solutions des équations limites, puis de construire une suite de solutions (autour de la Maxwellienne) pour les équations cinétiques adimensionnées (Boltzmann ou Landau) lorsque le nombre de Knudsen est petit. Citons dans ce contexte le travail de Guo dans [58].

C-2) Dérivation dans un cadre de solutions faibles. Parlons maintenant du programme BGL. Ce programme a été initié par Bardos, Golse et Levermore [9] dans les années 80 et le premier résultat de convergence sans l'hypothèse de compacité a été donné par Golse et Saint-Raymond dans [49], suite à une série de travaux de Bardos, Golse, Levermore, Lions, Masmoudi et Saint-Raymond [10, 96, 50]. La liste donnée ici n'est pas exhaustive. Plus récemment, ce programme a été abordé dans diverses géométries (avec bord

dans [99, 79, 81]). Tous ces résultats sont obtenus dans un cadre de solutions très faibles : les solutions renormalisées pour l'équation de Boltzmann (de DiPerna-Lions [37] ou Mischler [102] pour les domaines bornés) et les solutions de Leray pour les équations de Navier-Stokes. Citons aussi [7] et [115] dans lesquels la théorie de la dérivation des équations macroscopiques des fluides à partir des équations cinétiques a été traitée de manière exhaustive.

C-3) Dérivation dans un cadre de *solutions fortes*. Dans le cadre de solutions fortes, citons par exemple les résultats les plus récents [5, 17, 18] pour l'équation de Boltzmann avec cut-off (avec collisions inélastiques pour [5]) et [83] pour l'équation de Boltzmann sans cut-off dans lesquels un résultat de la convergence faible vers le modèle fluide est obtenu. En général, ces résultats sont obtenus en utilisant des estimations uniformes en nombre de Knudsen sur les équations cinétiques (voir les introductions des Chapitres 3 et 4 pour plus de détails sur les méthodes utilisées). Il existe également des preuves de convergence basées sur l'analyse spectrale dans le cadre de solutions fortes proches de l'équilibre introduites par Grad [52] et Ukai [125] pour l'équation de Boltzmann. Dans ce contexte, nous renvoyons aux travaux dans [108, 11, 44]. Ces résultats utilisent la description du spectre de l'équation de Boltzmann linéarisée dans l'espace de Fourier en variable spatiale x effectuée dans [107, 27, 41].

3 Objectifs de la thèse et outils utilisés

Dans cette thèse, nous nous intéressons au modèle cinétique représenté par l'équation de Landau. En se basant sur les problèmes mathématiques présentés dans la Section 2, nous traitons les deux questions suivantes liées à ces problèmes :

- **Question A** : Localisation du pseudo-spectre et estimations de résolvante [111],
- **Question B** : Étude de la limite hydrodynamique [112, 25].

A) Localisation du pseudo-spectre et estimations de résolvante. La notion de pseudo-spectre a été introduite par des numériciens depuis une dizaine d'années (voir [121] pour plus de détails). Cette notion provient du fait que dans plusieurs problèmes d'ingénierie mathématique où les opérateurs non auto-adjoints apparaissent naturellement, il existe souvent des différences entre les résultats théoriques prédits par l'analyse spectrale et les résultats obtenus par simulation numérique. Ce constat permet de penser que la connaissance du spectre de l'opérateur seule est insuffisante pour comprendre son action. Pour cela, de nouveaux sous-ensembles du plan complexe (pseudo-spectre) ont été introduits pour résoudre ce manque d'information sur le spectre. L'idée principale de cette notion est d'étudier d'une part la région où se trouve le spectre de l'opérateur, d'autre part d'étudier quantitativement l'endroit où la norme de la résolvante est contrôlée.

Dans cette thèse, nous montrons dans [111] une propriété de localisation du pseudo-spectre de l'opérateur de Landau linéarisé associé à l'équation de Landau. Il est de type

cuspidal à l'infini (voir Figure 1.2). De plus, nous donnons une description d'une région très large où la résolvante est contrôlée. Nous répondons à cette question en utilisant une méthode posée par F. Hérau et F. Nier dans l'article [67] puis par B. Helffer et F. Nier dans le livre [62]. Cette méthode est basée sur des outils de calcul pseudo-différentiels (nous renvoyons par exemple à [117] pour une introduction aux opérateurs pseudo-différentiels). Cette méthode leur permet d'obtenir une forme cuspidale du pseudo-spectre pour l'opérateur de Fokker-Planck associé à l'équation de Fokker-Planck dans [62, 67], ainsi pour des opérateurs quadratiques généraux par [109] puis étendue à des opérateurs apparaissant en mécanique statistique dans [39]. Nous soulignons que la motivation principale de cette méthode est de pouvoir comprendre les zones de pseudo-spectre et en déduire le cas échéant le retour à l'équilibre ou la régularisation pour l'équation d'évolution associée grâce aux formules de Cauchy (voir par exemple [67] pour un résultat complet pour le cas Fokker-Planck et [109] pour le cas des opérateurs quadratiques généraux). L'idée générale de cette méthode est de trouver un opérateur pseudo-différentiel auto-adjoint $(\Lambda, D(\Lambda))$ ayant un symbole de Weyl λ , $\Lambda \geq \text{Id}$ et des constantes $C > 0$, $0 \leq m \leq M$ telles que

$$\begin{aligned} \|\Lambda^m f\|^2 &\leq C(\|(\mathcal{P} - i\nu)f\|^2 + \|f\|^2), \quad \forall \nu \in \mathbb{R}, \forall f \in D(\mathcal{P}), \\ \|\mathcal{P}f\|^2 &\leq C\|\Lambda^M f\|^2, \quad \forall f \in D(\Lambda^M), \end{aligned}$$

où $\|\cdot\|$ désigne la norme sur un espace de Hilbert \mathcal{H} et $(\mathcal{P}, D(\mathcal{P}))$ désigne l'opérateur cinétique (sur l'espace de Hilbert \mathcal{H}) linéaire ou linéarisé associé au modèle étudié. Ces estimations hypoelliptiques sont des estimations clés qui nous permettent, en appliquant la Proposition B.1 dans [67], d'obtenir un résultat de localisation du pseudo-spectre de l'opérateur cinétique étudié.

Enfin, notons que dans [111], afin de montrer le résultat de localisation du pseudo-spectre, nous proposons un cadre pseudo-différentiel très robuste et autonome (voir le Théorème 2.40 dans l'Appendix A du Chapitre 2). Notre espoir est qu'il puisse être utilisé pour de nombreux autres modèles cinétiques. Ces outils sont inspirés par de travaux précédents (voir [4], [68], [71]) concernant les équations cinétiques et en suivant les idées fondamentales de [91] sur les calculs pseudo-différentiels de Wick et Weyl. L'idée générale est d'introduire des symboles pseudo-différentiels avec un paramètre $K > 0$ très large qui vérifient certaines hypothèses (voir Théorème 2.40) tels que les opérateurs pseudo-différentiels associés à ces symboles admettent de bonnes propriétés (inversibilité, puissances des opérateurs...). Nous renvoyons le lecteur au Théorème 2.40 dans le Chapitre 2 pour plus d'informations.

B) Étude de la limite hydrodynamique. Nous étudions dans [112, 25] la limite hydrodynamique de l'équation de Landau. Nos études sont basées sur plusieurs méthodes que nous présentons ci-dessous.

(B1) Méthode d'énergie non linéaire. Cette méthode a été développée par Guo (voir [60] par exemple). L'idée de cette méthode est d'utiliser tout d'abord la décomposition microscopique-macroscopique de la solution (voir (1.39)) puis d'établir des estimations

microscopiques et macroscopiques grâce à l'étude des 13 moments classiques de Grad (voir [61] par exemple pour plus de détails) uniformément par rapport au nombre de Knudsen ε dans notre cas. Ensuite, on construit une norme équivalente à la norme usuelle (uniformément par rapport à ε). Les estimations microscopiques et macroscopiques avec l'équivalence des normes nous permettent d'obtenir des estimations d'énergie globales (uniformes en ε) vérifiées par la solution globale et elles sont obtenues directement pour l'équation non linéaire. Dans [112], un processus de convergence faible pour l'équation de Landau dans les cas de potentiels durs, molécules maxwelliennes et potentiels modérément mous vers l'équation de Navier-Stokes-Fourier incompressible avec la relation de Boussinesq a été montré en utilisant cette méthode.

Dans [25], nous améliorons le résultat obtenu dans [112] pour un certain type de données initiales. Plus précisément, nous obtenons un résultat de convergence forte vers l'équation de Navier-Stokes-Fourier incompressible avec la relation de Boussinesq. Pour montrer ce résultat, nous utilisons la méthode d'hypocoercivité- L^2 **(B2)** et des estimations de régularisation en temps court **(B3)** pour l'équation de Landau (adimensionnée) linéaire et non linéaire.

(B2) Parlons maintenant brièvement de la méthode d'hypocoercivité- L^2 . Rappelons que le défi de l'hypocoercivité est de comprendre l'interaction entre l'opérateur de collision qui fournit la dissipativité dans la variable de vitesse et celui de transport qui est conservatif, afin d'obtenir le retour à l'équilibre pour l'ensemble du problème linéaire (voir [126, 64] pour plus de détails sur ce sujet). La méthode d'hypocoercivité- L^2 a été introduite par Hérau [63] (voir aussi [38]) pour l'espace des invariants collisionnels de dimension un et introduite par Guo dans [59] pour un espace d'invariants collisionnels de dimension supérieure à un (y compris les cas de Boltzmann et Landau dans les cas avec symétrie). L'idée de cette méthode est de construire une fonctionnelle de Lyapunov \mathcal{G} sans poids en temps qui est adaptée au modèle étudié, équivalente à la norme L^2 et telle que $\mathcal{G}(f(t))$ décroît en temps si $f = f(t)$ est une solution du problème linéaire. La décroissance de $\mathcal{G}(f(t))$ et l'équivalence avec la norme L^2 nous permettent d'obtenir la convergence de la solution du système étudié dans l'espace L^2 .

Remarque 1.1. Notons qu'il y a un lien fort entre la méthode **(B1)** et la méthode **(B2)**, pourtant on applique la méthode **(B1)** sur les équations non linéaires alors que la méthode **(B2)** s'applique sur les équations linéaires. En effet, dans [112], on a que l'opérateur de Landau linéarisé (voir (1.26)) donne la coercivité sur la partie microscopique (voir (3.32) dans le Chapitre 3). Pour cela nous appliquons la méthode **(B1)** dans le but de récupérer la coercivité sur toute la solution c'est à dire sur la partie microscopique et macroscopique.

(B3) La stratégie que nous utilisons pour prouver des estimations de régularisation est assez classique pour les équations hypoelliptiques linéaires et a été introduite par Hérau [65] pour l'équation cinétique de Fokker-Planck. Cette méthode a été utilisée pour de nombreuses équations, citons par exemple [101] pour l'équation cinétique classique de Fokker-Planck, [71] pour l'équation de Fokker-Planck fractionnaire, [22, 26] pour l'équation de Landau linéarisée, [70] pour l'équation de Boltzmann linéarisée sans cut-off... Dans [25], nous utilisons cette stratégie pour obtenir les estimations de régularisation

pour les équations linéarisée et non linéaire. L'idée de cette méthode est d'introduire une fonctionnelle $\mathcal{E}(f(t), t)$ (pour un temps t petit) avec des poids en temps qui soit une fonctionnelle de Lyapunov adaptée à l'équation étudiée, grâce à laquelle il est possible de récupérer des estimations de régularisation en temps.

Remarque 1.2. Nous notons que les résultats obtenus par la méthode **(B3)** peuvent ressembler à ceux obtenus par la méthode présentée dans **(A)** lorsque ces derniers existent. Plus précisément, en utilisant la méthode **(B3)** nous avons obtenu dans [25] le gain optimal de régularité qui correspond aux estimations hypoelliptiques (1.31) (obtenus en utilisant la méthode présentée dans **(A)**) sur l'opérateur de collision de Landau linéarisé fournies dans [111].

4 Le plan et les résultats obtenus dans la thèse

Le but de cette thèse est d'étudier l'équation de Landau en utilisant les outils mathématiques présentés dans la section précédente. Avant de présenter plus en détails les modèles étudiés, on va introduire les espaces fonctionnels qui seront utilisés dans les Chapitres 3 et 4 respectivement.

- **Espaces fonctionnels introduits dans les Chapitres 3 et 4.** Nous allons considérer des fonctions à deux variables $f = f(x, v)$ avec $x \in \mathbb{T}^3$ et $v \in \mathbb{R}^3$. Nous écrivons pour $p \in [1, +\infty]$

$$L_x^p = L^p(\mathbb{T}^3), \quad L_v^p = L^p(\mathbb{R}^3), \quad L_x^p L_v^p = L^p(\mathbb{T}^3 \times \mathbb{R}^3).$$

Pour $m = m(v)$ une mesure de Borel réelle positive et $1 \leq p, q \leq \infty$, on définit l'espace $L_x^q L_v^p(m)$ comme l'espace de Lebesgue associé à la norme, pour $f = f(x, v)$

$$\begin{aligned} \|f\|_{L_x^q L_v^p(m)} &:= \| \|f\|_{L_v^p(m)} \|_{L_x^q} \\ &= \| \|mf\|_{L_v^p} \|_{L_x^q} \\ &= \left(\int_{\mathbb{T}^3} \left(\int |f(x, v)|^p m(v)^p dv \right)^{\frac{q}{p}} dx \right)^{\frac{1}{q}}. \end{aligned}$$

On définit également l'espace H_x^s , pour $s \in \mathbb{N}$ comme l'espace de Sobolev usuel sur \mathbb{T}^3 et $H_x^s L_v^2(m)$ par la norme

$$\|f\|_{H_x^s L_v^2(m)} = \left(\sum_{|k| \leq s} \|\partial_x^k f\|_{L_x^2 L_v^2(m)}^2 \right)^{\frac{1}{2}}.$$

Nous en venons maintenant à la présentation détaillée des équations étudiées dans cette thèse ainsi que les résultats principaux obtenus. Ces résultats seront présentés d'une manière plus détaillée dans les Chapitres 2, 3 et 4.

4.1 Estimations hypoelliptiques et spectrales pour l'opérateur de Landau linéarisé (Chapitre 2)

Dans cette partie, nous nous intéressons à l'équation cinétique de Landau inhomogène en espace, qui décrit l'évolution de la fonction de densité de particules $F = F(t, x, v)$ à un temps $t \in \mathbb{R}^+$, une position $x \in \mathbb{R}^3$ et à une vitesse $v \in \mathbb{R}^3$. Cette équation est donnée par

$$\begin{cases} \partial_t F + v \cdot \nabla_x F = Q(F, F) \\ F|_{t=0} = F_0, \end{cases} \quad (1.22)$$

avec Q le noyau de collision de Landau qui agit sur la variable v et qui contient de la diffusion en vitesse (voir Section 1.2). Dans le travail présenté dans cette partie, seuls les potentiels durs, les molécules Maxwelliennes sont traités, c'est-à-dire les cas $\gamma \in [0, 1]$. Nous rappelons que μ est l'unique équilibre global de masse 1, quantité de mouvement nulle et énergie 3 donné par :

$$\mu(v) = (2\pi)^{-3/2} e^{-|v|^2/2}.$$

Nous linéarisons alors l'équation de Landau autour de μ avec la perturbation suivante :

$$F = \mu + \mu^{1/2} f.$$

L'équation de Landau (1.22) pour $f = f(t, x, v)$ prend la forme suivante :

$$\begin{cases} \partial_t f + v \cdot \nabla_x f - \mu^{-1/2} Q(\mu^{1/2} f, \mu) - \mu^{-1/2} Q(\mu, \mu^{1/2} f) = \mu^{-1/2} Q(\mu^{1/2} f, \mu^{1/2} f) \\ f|_{t=0} = f_0 = \mu^{-1/2} (F_0 - \mu), \end{cases}$$

(où nous avons utilisé que $Q(\mu, \mu) = 0$). En utilisant la notation suivante :

$$\Gamma(f_1, f_2) = \mu^{-1/2} Q(\mu^{1/2} f_1, \mu^{1/2} f_2), \quad (1.23)$$

nous pouvons réécrire l'équation ci-dessus comme

$$\begin{cases} \partial_t f + \mathcal{P}f = \Gamma(f, f) \\ f|_{t=0} = f_0, \end{cases} \quad (1.24)$$

où \mathcal{P} est l'opérateur de Landau linéaire :

$$\mathcal{P} = v \cdot \nabla_x - \mathcal{L} \quad (1.25)$$

avec

$$\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2, \quad \mathcal{L}_1 = \Gamma(\sqrt{\mu}, f), \quad \mathcal{L}_2 = \Gamma(f, \sqrt{\mu}). \quad (1.26)$$

L'opérateur \mathcal{P} , qui agit uniquement sur les variables (x, v) , est un opérateur non auto-adjoint, qui consiste en une partie de transport (anti-adjointe), une partie de diffusion qui

agit uniquement sur la variable v et une partie compacte (voir par exemple la proposition 2.1 dans [30]).

Le but de l'article [111] est d'étudier les propriétés spectrales de l'opérateur de Landau linéarisé \mathcal{P} . La difficulté de l'étude de l'opérateur \mathcal{P} vient du fait qu'il n'est ni auto-adjoint, ni elliptique, ni sectoriel. Nous montrons une propriété de localisation du pseudo-spectre de l'opérateur de Landau linéarisé \mathcal{P} , c'est-à-dire de localisation de la région du plan complexe où sa résolvante est a priori grande, plus précisément nous donnons une description d'une grande région où la résolvante est contrôlée, qui est incluse dans le complémentaire du pseudo-spectre.

Ce résultat est donné par le théorème suivant :

Théorème 1.3. Soit \mathcal{P} l'opérateur de Landau sur $L^2(\mathbb{R}_x^3 \times \mathbb{R}_v^3)$ défini dans (1.25) avec $\gamma \in [0, 1]$. Alors il existe deux constantes $C_{\mathcal{P}} > 0$ et $Q_{\mathcal{P}} > 0$ telles que :

a) Le spectre de \mathcal{P} vérifie

$$\sigma(\mathcal{P}) \subset S_{\mathcal{P}} \cap \{\Re z \geq 0\},$$

avec

$$S_{\mathcal{P}} = \left\{ z \in \mathbb{C}, |z + 1|^{1/3} \leq C_{\mathcal{P}}(\Re z + 1), \Re z \geq -\frac{1}{2} \right\}. \quad (1.27)$$

b) Pour $z \notin S_{\mathcal{P}}$ avec $\Re z \geq -\frac{1}{2}$, la résolvante est estimée par

$$\|(z - \mathcal{P})^{-1}\|_{\mathcal{B}(L_{x,v}^2)} \leq Q_{\mathcal{P}}|z + 1|^{-1/3}, \quad (1.28)$$

où $\mathcal{B}(L_{x,v}^2)$ est l'espace vectoriel des opérateurs bornés de $L_{x,v}^2$ dans $L_{x,v}^2$.
Notons que si $\Re z \leq -\frac{1}{2}$ alors

$$\|(z - \mathcal{P})^{-1}\|_{\mathcal{B}(L_{x,v}^2)} \leq |\Re z|^{-1}. \quad (1.29)$$

Les résultats du Théorème 1.3 peuvent être illustrés par la figure suivante :

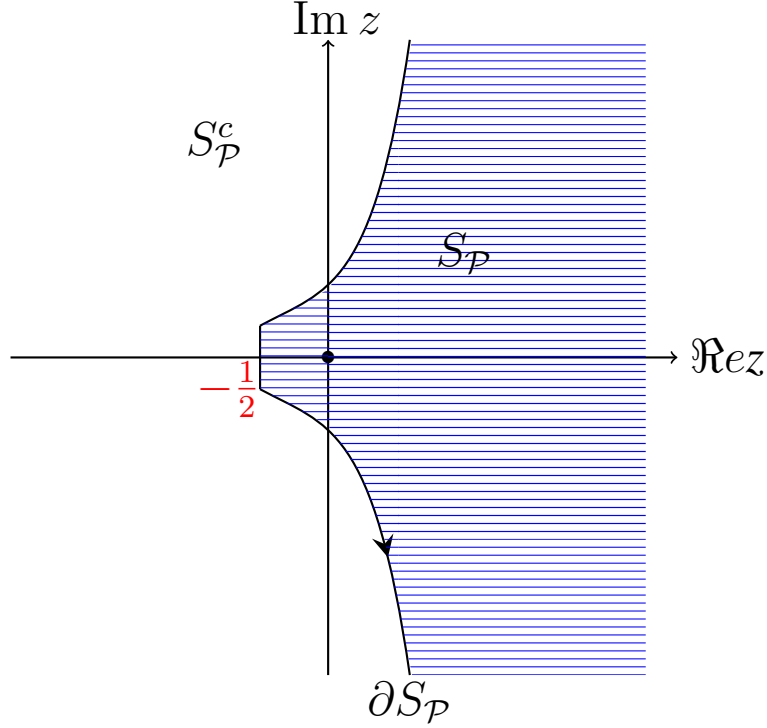


FIGURE 1.2: Localisation du pseudo-spectre de l'opérateur de Landau \mathcal{P} .

Dans cette figure, $\partial S_{\mathcal{P}}$ est orienté de $+i\infty$ à $-i\infty$. La partie hachurée est l'endroit où le spectre est localisé et la partie non hachurée est la zone où l'on a de bonnes estimations de résolvante (voir (1.28)-(1.29)). La stratégie de la preuve pour démontrer le Théorème 1.3 est celle présentée dans (A) dans la Section 3 avec l'opérateur de Landau linéarisé \mathcal{P} défini dans (1.25). Parlons brièvement de la construction de l'opérateur pseudo-différentiel qu'on le notera Λ_K . On définit pour tout $(v, \eta) \in \mathbb{R}^6$, le symbole pseudo-différentiel suivant (voir Appendix A du Chapitre 2 pour plus d'informations sur le calcul pseudo-différentiel)

$$\lambda_K(v, \eta) := \left[\langle v \rangle^\gamma \left(1 + |v|^2 + |\eta|^2 + |\xi|^2 + |v \wedge \eta|^2 + |v \wedge \xi|^2 \right) \right]^{1/2} + K \langle v \rangle^{\frac{\gamma}{2}+1}, \quad (1.30)$$

où le symbole \wedge désigne le produit vectoriel entre deux vecteurs dans \mathbb{R}^3 , $\xi \in \mathbb{R}^3$ joue le rôle d'un paramètre et $K > 0$ une constante très large (uniforme en ξ) construite dans le Théorème 2.40 dans l'Appendix A du Chapitre 2. Ce symbole vérifie les hypothèses du Théorème 2.40, ce qui implique que l'opérateur pseudo-différentiel (voir (2.44)) associé à ce symbole a de bonnes propriétés (voir le résultat du Théorème 2.40). Donc on définit Λ_K comme étant l'opérateur pseudo-différentiel associé au symbole λ_K en considérant la transformée de Fourier inverse par rapport à la variable $x \in \mathbb{R}^3$. Ainsi nous obtenons en utilisant la méthode présentée dans (A) les estimations hypoelliptiques suivantes : il

existe $C > 0$ tel que pour tout $f \in \mathcal{S}(\mathbb{R}_{x,v}^6)$ (espace de Schwartz)

$$\begin{aligned} \|\Lambda_K^{2/3} f\|^2 &\leq C \left(\|(\mathcal{P} - i\nu)f\|^2 + \|f\|^2 \right), \quad \forall \nu \in \mathbb{R}, \\ \|\mathcal{P}f\|^2 &\leq C \|\Lambda_K^2 f\|^2, \end{aligned} \quad (1.31)$$

où $\|\cdot\|$ désigne la norme sur l'espace $L^2(\mathbb{R}_x^3 \times \mathbb{R}_v^3)$. Ces estimations hypoelliptiques sont les estimations clés qui nous permettent d'obtenir l'expression de $S_{\mathcal{P}}$ donné par (1.27) dans le théorème précédent ainsi la localisation du pseudo-spectre de l'opérateur de Landau linéarisé \mathcal{P} comme montre la Figure 1.2. Cette stratégie a été utilisée dans [62], [67] pour l'opérateur de Fokker-Planck (avec un potentiel extérieur), puis étendue aux opérateurs apparaissant en mécanique statistique dans [39]. De plus, des premiers éléments et estimations sous-elliptiques ont été obtenus dans [68] et [66]. Dans notre situation, l'opérateur étudié est plus compliqué à cause de fortes propriétés d'anisotropie.

4.2 La limite faible vers l'équation de Navier-Stokes-Fourier incompressible à partir de l'équation de Landau (Chapitre 3)

Dans cette partie, nous étudions la limite hydrodynamique de l'équation de Landau. Nous mettons cette équation sous une forme adimensionnée en introduisant un paramètre ε . La forme adimensionnée de l'équation de Landau est :

$$\begin{cases} \partial_t f + v \cdot \nabla_x f = \frac{1}{\varepsilon} Q(f, f) \\ f|_{t=0} = f_0, \end{cases} \quad (1.32)$$

où $f = f(t, x, v)$ est la fonction de densité de particules à un temps $t \in \mathbb{R}^+$, une position $x \in \mathbb{T}^3$ le tore de dimension 3 et à une vitesse $v \in \mathbb{R}^3$. Ici $\varepsilon > 0$ est le nombre de Knudsen, quantifiant l'inverse du nombre moyen de collisions pour chaque particule par unité de temps et Q est l'opérateur de collision de Landau (voir Section 1.2). Dans le travail présenté dans cette partie, seuls les potentiels durs, les molécules Maxwelliennes et les potentiels faiblement mous sont traités, c'est-à-dire que nous nous restreignons à $\gamma \in [-2, 1]$. Prendre ε petit a pour effet de renforcer le rôle des collisions : lorsque $\varepsilon \rightarrow 0$, au vu du Théorème H de Boltzmann, la solution ressemble de plus en plus à un équilibre thermodynamique local (1.14).

En utilisant par exemple les travaux dans [9], nous introduisons l'échelle du temps et de l'espace et la densité des particules suivante :

$$f_\varepsilon(t, x, v) = f\left(\frac{t}{\varepsilon^2}, \frac{x}{\varepsilon}, v\right), \quad t \geq 0$$

dans le but de pouvoir exhiber l'équation de Navier-Stokes à la limite. L'équation de Landau adimensionnée prend alors la forme :

$$\begin{cases} \partial_t f_\varepsilon + \frac{1}{\varepsilon} v \cdot \nabla_x f_\varepsilon = \frac{1}{\varepsilon^2} Q(f_\varepsilon, f_\varepsilon) \\ f_\varepsilon|_{t=0} = f_{\varepsilon,0}. \end{cases} \quad (1.33)$$

Comme mentionné précédemment, les équilibres globaux de l'équation de Landau sont des Maxwelliennes locales en vitesse (voir (1.14)-(1.16)-(1.17)). Pour relier l'équation de Landau à l'équation de Navier-Stokes incompressible, nous considérons la linéarisation d'ordre ε suivante autour de la Maxwellienne μ définie en (1.17)

$$f_\varepsilon(t, x, v) = \mu + \varepsilon \mu^{1/2} g_\varepsilon(t, x, v).$$

Si f_ε vérifie (1.33), alors de manière équivalente g_ε vérifie

$$\begin{cases} \partial_t g_\varepsilon + \frac{1}{\varepsilon} v \cdot \nabla_x g_\varepsilon - \frac{1}{\varepsilon^2} \mathcal{L} g_\varepsilon = \frac{1}{\varepsilon} \Gamma(g_\varepsilon, g_\varepsilon) \\ g_\varepsilon|_{t=0} = g_{\varepsilon,0}, \end{cases} \quad (1.34)$$

où $g_{\varepsilon,0}$ est donné par $f_{\varepsilon,0} = \mu + \varepsilon \mu^{1/2} g_{\varepsilon,0}$, l'opérateur de collision non linéaire Γ est défini dans (1.23) et l'opérateur de collision linéarisé \mathcal{L} est défini dans (1.26).

Dans ce travail, nous supposons que la donnée initiale $f_{\varepsilon,0}$ vérifie :

$$\int_{\mathbb{T}^3 \times \mathbb{R}^3} f_{\varepsilon,0}(x, v) \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} dx dv = \int_{\mathbb{T}^3 \times \mathbb{R}^3} \mu(v) \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} dx dv, \quad (1.35)$$

ce qui implique que

$$\int_{\mathbb{T}^3 \times \mathbb{R}^3} g_\varepsilon(t, x, v) \mu^{1/2}(v) \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} dx dv = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \forall t \geq 0,$$

puisque notre équation conserve la masse totale, la quantité de mouvement et l'énergie (voir (1.10)).

Le but de l'article [112] est d'étudier le processus de convergence faible pour l'équation de Landau dans le cas où $\gamma \in [-2, 1]$ et de vérifier sa limite vers l'équation de Navier-Stokes-Fourier incompressible avec la relation de Boussinesq sur le tore. Cette équation a de fortes propriétés de diffusion, ce qui implique des difficultés liées aux espaces fonctionnels. Avant de présenter les résultats obtenus dans ce travail, nous présentons d'abord quelques notations utilisées (voir également le début de la Section 4). Il est bien connu que le noyau de \mathcal{L} est un sous espace vectoriel de L_v^2 de dimension 5 et est engendré par l'ensemble des invariants de collision (voir [60] par exemple) :

$$\text{Ker } \mathcal{L} = \text{Span} \left\{ \sqrt{\mu}, v_1 \sqrt{\mu}, v_2 \sqrt{\mu}, v_3 \sqrt{\mu}, |v|^2 \sqrt{\mu} \right\}. \quad (1.36)$$

On introduit l'espace $H_{v,*}^1$ associé à la norme

$$\|f\|_{H_{v,*}^1}^2 := \|\langle v \rangle^{\frac{\gamma}{2}+1} f\|_{L_v^2}^2 + \|\langle v \rangle^{\frac{\gamma}{2}} P_v \nabla_v f\|_{L_v^2}^2 + \|\langle v \rangle^{\frac{\gamma}{2}+1} (\text{Id} - P_v) \nabla_v f\|_{L_v^2}^2, \quad (1.37)$$

où $\langle v \rangle = (1 + |v|^2)^{1/2}$ et P_v désigne la projection sur la droite engendrée par v , i.e. $P_v \eta = \left(\eta \cdot \frac{v}{|v|} \right) \frac{v}{|v|}$. Cette norme est naturelle dans l'étude de l'équation de Landau. On définit aussi l'espace χ^s pour $s \in \mathbb{N}$ associé à la norme

$$\|f\|_{\chi^s(\mathbb{T}^3 \times \mathbb{R}^3)}^2 = \sum_{|\alpha| \leq s} \int_{\mathbb{T}^3} \|\partial_x^\alpha f\|_{H_{v,*}^1}^2 dx. \quad (1.38)$$

Rappelons la décomposition dite macro-micro des solutions

$$f = \Pi_0 f + (\text{Id} - \Pi_0) f, \quad (1.39)$$

où Π_0 est la projection orthogonale à $\text{Ker } \mathcal{L}$, $\Pi_0 f$ est appelée la projection macroscopique de f et $(\text{Id} - \Pi_0) f$ la partie cinétique ou microscopique de f . On a alors

$$\Pi_0 f(t, x, v) = \{a(t, x) + v \cdot b(t, x) + |v|^2 c(t, x)\} \sqrt{\mu}, \quad (1.40)$$

avec

$$a := \int_{\mathbb{R}^3} f \left(\frac{5}{2} - \frac{|v|^2}{2} \right) \mu^{1/2} dv, \quad b := \int_{\mathbb{R}^3} f v \mu^{1/2} dv, \quad c := \int_{\mathbb{R}^3} f \left(\frac{|v|^2}{6} - \frac{1}{2} \right) \mu^{1/2} dv.$$

Enfin, nous introduisons respectivement la fonctionnelle d'énergie instantanée et la fonctionnelle de dissipation suivantes

$$\begin{aligned} \mathcal{E}^2(f) &= \|f\|_{H_x^3 L_v^2}^2 \\ &= \|\Pi_0 f\|_{H_x^3 L_v^2}^2 + \|(\text{Id} - \Pi_0) f\|_{H_x^3 L_v^2}^2 \\ \mathcal{D}(f) &= \|(\text{Id} - \Pi_0) f\|_{\chi^3(\mathbb{T}^3 \times \mathbb{R}^3)}, \\ \mathcal{C}(f) &= \|\nabla_x \Pi_0 f\|_{H_x^2 L_v^2}. \end{aligned} \quad (1.41)$$

On a alors le résultat d'existence et d'estimation globale d'énergie suivant.

Théorème 1.4. Il existe $M_0 > 0$ tel que pour $\varepsilon \in (0, 1)$ et $\|g_{\varepsilon, 0}\|_{H_x^3 L_v^2} \leq M_0$, le problème de Cauchy (1.34) admet une solution globale unique

$$g_\varepsilon \in L^\infty([0, \infty); H_x^3 L_v^2)$$

avec l'estimation globale d'énergie

$$\sup_{t \geq 0} \mathcal{E}^2(t) + C_0 \int_0^\infty \frac{1}{\varepsilon^2} \mathcal{D}^2(t) dt + C_0 \int_0^\infty \mathcal{C}^2(t) dt \leq C'_0 \mathcal{E}^2(0), \quad (1.42)$$

où $C_0, C'_0 > 0$ sont indépendantes de ε .

La stratégie de la preuve pour démontrer le Théorème 1.4 est d'utiliser la méthode d'énergie non linéaire présentée en (B1) en introduisant une norme (on la notera E dans le Chapitre 3) qui dépend des quantités macroscopiques a, b et c (1.40) et est équivalente à la norme usuelle \mathcal{E} uniformément par rapport au paramètre ε . Le but est d'obtenir l'estimation d'énergie globale (1.42) et ainsi de montrer l'existence de la solution globale (voir (3.57) dans le Chapitre 3 pour plus de détails sur la norme E). Cette stratégie a été utilisée dans [60, 58] pour l'équation de Landau dans le cas $\varepsilon = 1$ où les résultats ont été obtenus dans $H_{x,v}^N$ avec $N \geq 8$ (la régularité sur huit dérivées dans les deux variables x et v y est nécessaire). Dans notre travail nous élargissons l'espace dans lequel un tel résultat existe et nous ne travaillons qu'avec trois dérivées en x et aucune dérivée en v .

Dans la deuxième partie de ce travail, nous étudions la limite vers le système de Navier-Stokes-Fourier incompressible associée à l'équation de Boussinesq qui s'écrit

$$\begin{cases} \partial_t u + u \cdot \nabla_x u + \nabla_x p = \nu \Delta_x u, \\ \partial_t \theta + u \cdot \nabla_x \theta = \kappa \Delta_x \theta, \\ \nabla_x \cdot u = 0, \\ \rho + \theta = 0. \end{cases} \quad (1.43)$$

Dans ce système θ (la température), ρ (la densité) et p (la pression) sont des inconnues scalaires et u est un champ de vecteur vitesse à 3 composantes. Le coefficient ν (resp. κ) est la viscosité (resp. la conductivité thermique). On note que dans le système précédent, le terme "incompressible" correspond à la relation d'incompressibilité $\nabla_x \cdot u = 0$, le terme "Fourier" correspond à la deuxième équation qui est une équation de type chaleur et $\rho + \theta = 0$ est la relation de Boussinesq.

Le deuxième résultat principal de ce travail est le suivant.

Théorème 1.5. Soit M_0 la constante utilisée dans le Théorème 1.4. Pour tout $\varepsilon \in (0, 1)$, supposons que la donnée initiale $g_{\varepsilon,0}$ dans (1.34) vérifie

- 1) $g_{\varepsilon,0} \in H_x^3 L_v^2$,
- 2) $\|g_{\varepsilon,0}\|_{H_x^3 L_v^2} \leq M_0$,
- 3) Il existe des fonctions scalaires $\rho_0, \theta_0 \in H_x^3$ et une fonction vectorielle $u_0 \in H_x^3$ telles que

$$g_{\varepsilon,0} \xrightarrow{\varepsilon \rightarrow 0} g_0, \text{ fortement dans } H_x^3 L_v^2, \quad (1.44)$$

où $g_0(x, v)$ est de la forme

$$g_0(x, v) = \rho_0(x) \sqrt{\mu}(v) + u_0(x) \cdot v \sqrt{\mu}(v) + \theta_0(x) \left(\frac{|v|^2}{2} - \frac{3}{2} \right) \sqrt{\mu}(v). \quad (1.45)$$

Soit maintenant g_ε la famille de solutions de l'équation de Landau (1.34) construite dans le Théorème 1.4. Alors, lorsque $\varepsilon \rightarrow 0$,

$$g_\varepsilon \rightharpoonup \rho \sqrt{\mu} + u \cdot v \sqrt{\mu} + \theta \left(\frac{|v|^2}{2} - \frac{3}{2} \right) \sqrt{\mu} \quad (1.46)$$

faiblement- \star dans $L^\infty([0, \infty); H_x^3 L_v^2)$. Ici

$$(\rho, u, \theta) \in C(\mathbb{R}^+; H_x^2) \cap L^\infty(\mathbb{R}^+; H_x^3) \quad (1.47)$$

est une solution de l'équation de Navier-Stokes-Fourier incompressible (1.43) avec la donnée initiale :

$$u|_{t=0} = \mathcal{P}u_0(x), \quad \theta|_{t=0} = \frac{3}{5}\theta_0(x) - \frac{2}{5}\rho_0(x), \quad (1.48)$$

où \mathcal{P} est la projection de Leray (c'est la projection sur les champs de vecteurs à divergence nulle). De plus, la convergence des moments suivante est vérifiée :

$$\mathcal{P} (g_\varepsilon, v\sqrt{\mu})_{L_v^2} \xrightarrow{\varepsilon \rightarrow 0} u, \quad (1.49)$$

$$\left(g_\varepsilon, \left(\frac{|v|^2}{5} - 1 \right) \sqrt{\mu} \right)_{L_v^2} \xrightarrow{\varepsilon \rightarrow 0} \theta, \quad (1.50)$$

fortement dans $C(\mathbb{R}^+; H_x^2)$, faiblement- \star dans $L^\infty(\mathbb{R}^+; H_x^3)$.

La stratégie de la preuve pour démontrer le Théorème 1.5 est d'utiliser les estimations obtenues dans le Théorème 1.4. Ces estimations sont uniformes en ε et nous permettent également d'obtenir un résultat de la convergence faible vers le système limite (1.43). En termes d'espaces fonctionnels, notre résultat est comparable à celui de [83] dans lequel les auteurs ont utilisé cette méthode pour dériver l'équation de Navier-Stokes-Fourier incompressible mais pour l'équation de Boltzmann sans cut-off dans l'espace tout entier. La différence avec notre cas concerne le choix de l'énergie instantanée et la fonctionnelle de dissipation pour capturer la structure de l'équation adimensionnée (voir (1.37), (1.38) et (1.41)).

4.3 Estimations de régularisation et limite hydrodynamique forte pour l'équation de Landau (Chapitre 4)

Dans cette partie, nous poursuivons l'étude de la limite hydrodynamique de l'équation de Landau en améliorant le résultat présenté dans la partie précédente. Plus précisément, nous donnons un résultat de convergence forte des solutions de l'équation de Landau (présentée dans la partie précédente) vers des solutions fortes et globales du système de Navier-Stokes-Fourier incompressible. Comme dans la partie précédente, seuls les potentiels durs, les molécules Maxwelliennes et les potentiels faiblement mous sont traités, c'est-à-dire les cas $\gamma \in [-2, 1]$. Nous introduisons quelques notations ainsi que des espaces fonctionnels dans lesquels on travaille dans cette partie.

On note par Λ_ε l'opérateur de Landau linéarisé associé à (1.34) et donné par :

$$\Lambda_\varepsilon := \frac{1}{\varepsilon^2} \mathcal{L} - \frac{1}{\varepsilon} v \cdot \nabla_x, \quad (1.51)$$

où \mathcal{L} est défini dans (1.26). En utilisant par exemple la Proposition 3.1 dans [17], le noyau de Λ_ε est donné par :

$$\text{Ker } \Lambda_\varepsilon = \text{Span} \left\{ \sqrt{\mu}, v_1 \sqrt{\mu}, v_2 \sqrt{\mu}, v_3 \sqrt{\mu}, |v|^2 \sqrt{\mu} \right\}.$$

En considérant la base orthogonale dans L_v^2 du noyau $\text{Ker } \mathcal{L}$ (voir (1.36)) suivante :

$$\left\{ \sqrt{\mu}, \sqrt{\mu} v_i, \sqrt{\mu} (|v|^2 - 3)/3 \right\},$$

on peut réécrire Π_0 défini dans (1.40) sous la forme suivante :

$$\begin{aligned} \Pi_0 f(v) = & \left(\int_{\mathbb{R}^3} f(w) \sqrt{\mu}(w) dw + \int_{\mathbb{R}^3} w f(w) \sqrt{\mu}(w) dw \cdot v \right. \\ & \left. + \int_{\mathbb{R}^3} \frac{|w|^2 - 3}{3} f(w) \sqrt{\mu}(w) dw \frac{|v|^2 - 3}{2} \right) \sqrt{\mu}(v). \end{aligned} \quad (1.52)$$

Pour une distribution cinétique donnée $f = f(x, v)$, on note f^\perp sa partie microscopique et donnée par

$$f^\perp := (\text{Id} - \Pi_0) f \quad (1.53)$$

et par (ρ_f, u_f, θ_f) ses quantités macroscopiques définies par

$$\begin{aligned} \rho_f(x) &:= \int_{\mathbb{R}^3} f(x, v) \sqrt{\mu}(v) dv, \\ u_f(x) &:= \int_{\mathbb{R}^3} v f(x, v) \sqrt{\mu}(v) dv, \\ \theta_f(x) &:= \frac{1}{3} \int_{\mathbb{R}^3} (|v|^2 - 3) f(x, v) \sqrt{\mu}(v) dv. \end{aligned} \quad (1.54)$$

Ainsi $\Pi_0 f = \left(\rho_f + u_f \cdot v + \theta_f \frac{|v|^2 - 3}{2} \right) \sqrt{\mu}$. Notons que les notations sont différentes du cas présenté dans la Section 4.2 mais que l'esprit est le même.

On définit les espaces de type Sobolev

$$\mathcal{X} = \mathcal{H}_x^3(L_v^2), \quad \mathcal{Y}_1 = \mathcal{H}_x^3(H_{v,*}^1),$$

comme les espaces associés aux normes suivantes :

$$\begin{aligned} \|f\|_{\mathcal{X}}^2 &:= \|\langle v \rangle^{3(\frac{\gamma}{2}+1)} f\|_{L_{x,v}^2}^2 + \|\langle v \rangle^{2(\frac{\gamma}{2}+1)} \nabla_x f\|_{L_{x,v}^2}^2 \\ &+ \|\langle v \rangle^{\frac{\gamma}{2}+1} \nabla_x^2 f\|_{L_{x,v}^2}^2 + \|\nabla_x^3 f\|_{L_{x,v}^2}^2, \end{aligned} \quad (1.55)$$

et

$$\begin{aligned} \|f\|_{\mathcal{Y}_1}^2 &:= \|\langle v \rangle^{3(\frac{\gamma}{2}+1)} f\|_{L_x^2(H_{v,*}^1)}^2 + \|\langle v \rangle^{2(\frac{\gamma}{2}+1)} \nabla_x f\|_{L_x^2(H_{v,*}^1)}^2 \\ &+ \|\langle v \rangle^{\frac{\gamma}{2}+1} \nabla_x^2 f\|_{L_x^2(H_{v,*}^1)}^2 + \|\nabla_x^3 f\|_{L_x^2(H_{v,*}^1)}^2, \end{aligned} \quad (1.56)$$

où $\langle v \rangle = (1 + |v|^2)^{1/2}$, $|v|$ est la norme euclidienne canonique de v dans \mathbb{R}^3 et l'espace $H_{v,*}^1$ associé à la norme définie dans (1.37). Notons que l'introduction de ces espaces de Sobolev dans lesquels les poids dépendent de l'ordre de la dérivée en x , nous permet d'avoir les estimations non linéaires.

Soient X_1 et X_2 deux espaces de fonctions. On définit l'espace $X_1 + X_2$ comme l'ensemble de fonctions f qui s'écrivent sous la forme $f = f_1 + f_2$ avec $f_1 \in X_1$ et $f_2 \in X_2$ et qui est associé à la norme suivante :

$$\|f\|_{X_1+X_2} := \min_{\substack{f = f_1 + f_2 \\ f_i \in X_i}} \|f_1\|_{X_1} + \|f_2\|_{X_2}.$$

On a alors le résultat d'existence et de régularisation suivant.

Théorème 1.6. Il existe $\eta_0 > 0$ assez petit tel que pour tout $\varepsilon \in (0, 1)$, si $g_{\varepsilon,0} \in \mathcal{X}$ satisfait

$$\int_{\mathbb{T}^3 \times \mathbb{R}^3} g_{\varepsilon,0}(x, v) \varphi(v) dv dx = 0 \quad \text{pour} \quad \varphi(v) = \sqrt{\mu}, v\sqrt{\mu}, |v|^2\sqrt{\mu}, \quad (1.57)$$

et $\|g_{\varepsilon,0}\|_{\mathcal{X}} \leq \eta_0$ alors ce qui suit est vrai :

(i) Il existe une solution globale unique $g_\varepsilon \in L^\infty(\mathbb{R}^+; \mathcal{X}) \cap L^2(\mathbb{R}^+; \mathcal{Y}_1)$ à (1.34) associée à la donnée initiale $g_{\varepsilon,0}$, qui vérifie en plus

$$\sup_{t \geq 0} e^{2\sigma t} \|g_\varepsilon(t)\|_{\mathcal{X}}^2 + \frac{1}{\varepsilon^2} \int_0^\infty e^{2\sigma t} \|(g_\varepsilon(t))^\perp\|_{\mathcal{Y}_1}^2 dt + \int_0^\infty e^{2\sigma t} \|g_\varepsilon(t)\|_{\mathcal{Y}_1}^2 dt \leq C \|g_{\varepsilon,0}\|_{\mathcal{X}}^2, \quad (1.58)$$

où $\sigma, C > 0$ sont indépendantes de ε et $(g_\varepsilon)^\perp$ est défini dans (1.53).

(ii) De plus, la solution satisfait l'estimation de régularisation suivante

$$\|g_\varepsilon(t)\|_{\mathcal{Y}_1} \leq C \frac{e^{-\sigma t}}{\min(1, \sqrt{t})} \|g_{\varepsilon,0}\|_{\mathcal{X}}, \quad \forall t > 0, \quad (1.59)$$

où $C > 0$ est indépendante de ε .

Remarque 1.7. On note que la condition (1.57) est équivalente à $g_{\varepsilon,0} \in (\text{Ker } \Lambda_\varepsilon)^\perp$.

La stratégie de la preuve pour démontrer le Théorème 1.6-(i) est basée sur l'étude de l'équation linéarisée puis du problème non linéaire. Concernant l'étude du problème linéaire, nous utilisons la méthode d'hypo-coercivité- L^2 présentée dans (B2) en introduisant une norme $\|\cdot\|_{L_{x,v}^2}$ (associée au produit scalaire $\langle\langle \cdot, \cdot \rangle\rangle_{L_{x,v}^2}$) qui dépend de ε et est équivalente à la norme usuelle $\|\cdot\|_{L_{x,v}^2}$ uniformément en ε (voir (4.53) dans le Chapitre 4 pour plus de détails sur la construction de la norme $\|\cdot\|_{L_{x,v}^2}$). Le choix de cette norme permet également d'obtenir le gain de $1/\varepsilon$ sur la partie microscopique de la solution (voir (1.58)). Cette méthode nous permet d'obtenir la dissipativité pour l'opérateur linéarisé Λ_ε dans l'espace \mathcal{X} . En combinant la dissipativité avec les estimations du terme non linéaire Γ , nous obtenons (1.58). Concernant l'estimation de régularisation dans le Théorème 1.6-(ii), nous utilisons la méthode présentée dans (B3) pour l'équation non linéaire en introduisant une fonctionnelle (on la notera par $\mathcal{U}_\varepsilon(t)$ dans le Chapitre 4) avec des poids en temps. Nous montrons que $\mathcal{U}_\varepsilon(t)$ est décroissante en temps ce qui nous permet d'obtenir (1.59). À notre connaissance, c'est la première fois que cette stratégie est utilisée pour une équation non linéaire (même dans le cas le plus simple $\varepsilon = 1$) et pour une équation adimensionnée (avec des estimations uniformes en ε). Cette fonctionnelle a été construite particulièrement pour gérer les dépendances de ε . Les différences de comportements entre les parties microscopiques et macroscopiques de la solution ont aussi été prises en compte en introduisant certains termes de la fonctionnelle qui dépendent seulement de la partie

microscopique de la solution (voir (4.114) pour la définition de la fonctionnelle $\mathcal{U}_\varepsilon(t)$). Ce choix convenable de cette fonctionnelle permet également d'obtenir l'estimation de régularisation (1.59) qui est optimal (le gain optimal de régularité obtenu correspond aux estimations hypoelliptiques (1.31) obtenus dans [111] sur l'opérateur de collision de Landau linéarisé).

On va alors étudier la convergence forte de (1.34) lorsque ε tend vers 0 vers le système de Navier-Stokes incompressible (1.43). Pour cela, on introduit g_0 défini de la manière suivante :

$$g_0(x, v) := \sqrt{\mu}(v) \left(\rho_0(x) + u_0(x) \cdot v + \frac{|v|^2 - 3}{2} \theta_0(x) \right) \\ \text{avec } \nabla_x \cdot u_0 = 0 \quad \text{et} \quad \rho_0 + \theta_0 = 0, \quad (1.60)$$

où (ρ_0, u_0, θ_0) sont les quantités macroscopiques associées. Ici g_0 est appelé une donnée bien préparée (voir le Chapitre 4 pour plus d'informations). Notons que dans cette partie, on suppose que la donnée initiale fluide g_0 soit bien préparée (voir (1.60)), cette condition est nécessaire pour obtenir une convergence forte. Tandis que les hypothèses faites sur la donnée initiale fluide dans la Section 4.2 (voir Théorème 1.5) sont un peu moins restrictives puisque cette condition n'est pas supposée et n'est pas nécessaire pour avoir une convergence faible (voir aussi l'introduction du Chapitre 3 pour plus de détails).

Remarque 1.8. Notons qu'il existe $\eta_1 > 0$ tel que si g_0 est de la forme (1.60) et satisfait $\|g_0\|_{\mathcal{X}} \leq \eta_1$, alors il existe $(\rho, u, \theta) \in H_x^3$ une solution du système (1.43) définie sur \mathbb{R}^+ (voir l'introduction du Chapitre 3 pour plus de détails).

On définit aussi la distribution cinétique dans $\text{Ker } \mathcal{L}$ comme suit :

$$g(t, x, v) := \sqrt{\mu}(v) \left(\rho(t, x) + u(t, x) \cdot v + \frac{|v|^2 - 3}{2} \theta(t, x) \right), \quad (1.61)$$

où (ρ, u, θ) sont les quantités macroscopiques associées et une solution du système de Navier-Stokes-Fourier incompressible (1.43). On a alors le résultat de convergence suivant.

Théorème 1.9. Soit $g_{\varepsilon,0} \in \mathcal{X} \cap (\text{Ker } \Lambda_\varepsilon)^\perp$ pour $\varepsilon \in (0, 1)$ tel que $\|g_{\varepsilon,0}\|_{\mathcal{X}} \leq \eta_0$ (où η_0 est définie dans le Théorème 1.6) et $g^\varepsilon \in L_t^\infty(\mathcal{X})$ étant la solution associée à (1.34) avec la donnée initiale $g_{\varepsilon,0}$ construit dans le Théorème 1.6. Considérons aussi $g_0 \in \mathcal{X} \cap (\text{Ker } \Lambda_\varepsilon)^\perp$ tel que $\|g_0\|_{\mathcal{X}} \leq \eta_1$ (où η_1 est définie dans la Remarque 1.8) et g définis respectivement dans (1.60) et (1.61).

Il existe $\eta_2 \in (0, \min(\eta_0, \eta_1))$ tel que si $\max(\|g_{\varepsilon,0}\|_{\mathcal{X}}, \|g_0\|_{\mathcal{X}}) \leq \eta_2$ et

$$\|g_{\varepsilon,0} - g_0\|_{\mathcal{X}} \xrightarrow{\varepsilon \rightarrow 0} 0, \quad (1.62)$$

alors on a

$$\|g_\varepsilon - g\|_{L_t^\infty(\mathcal{X})} \xrightarrow{\varepsilon \rightarrow 0} 0. \quad (1.63)$$

Si

$$\|\Pi_0 g_{\varepsilon,0} - g_0\|_{\mathcal{X}} \xrightarrow{\varepsilon \rightarrow 0} 0, \quad (1.64)$$

alors on a

$$\|g_\varepsilon - g\|_{L_t^1(\mathcal{A}_1) + L_t^\infty(\mathcal{X})} \xrightarrow{\varepsilon \rightarrow 0} 0. \quad (1.65)$$

La stratégie de la preuve pour démontrer le Théorème 1.9 est d'utiliser les estimations de régularisation pour l'équation de Landau linéarisée, les estimations (1.58), (1.59) obtenues dans le Théorème 1.6 sur l'équation de Landau non linéaire (1.34). De plus, nous utilisons l'analyse spectrale réalisée par Yang et Yu dans [130] pour plusieurs équations cinétiques (dont l'équation de Landau), basée sur la description du spectre de l'équation linéarisée (équation de Landau dans notre cas) dans l'espace de Fourier en variable x (voir [107, 27, 41] pour plus de détails sur ce contexte). Cette analyse spectrale nous permet de reformuler l'équation fluide d'une façon cinétique et d'étudier l'équation satisfaite par la différence entre la solution cinétique et la solution fluide. Tous ces outils avec les estimations uniformes en nombre de Knudsen nous permettent d'obtenir une convergence forte vers l'équation de Navier-Stokes incompressible. Cette approche rappelle celle utilisée dans [11, 44] où l'équation de Boltzmann dans le cas des sphères dures a été traité. Notre résultat améliore les résultats obtenus dans [58, 112] en termes de type de convergence ou de cadre fonctionnel. Plus précisément, Guo dans [58] a justifié la limite vers le système de Navier-Stokes à partir de l'équation de Boltzmann et de Landau (également pour le cas de potentiels mous) dans le tore en utilisant la méthode d'énergie non linéaire. Notre résultat sur la limite hydrodynamique rappelle celui obtenu dans [58] dans les cas de potentiels durs, molécules Maxwelliennes et potentiels faiblement mous dans le sens où nous travaillons avec des solutions fortes et nous prouvons un résultat de convergence forte. On note que notre cadre fonctionnel est moins restrictif (nous ne travaillons qu'avec trois dérivées en x et aucune dérivée en v alors que dans [58], la régularité sur huit dérivées dans les deux variables x et v est nécessaire) et il n'y a quasiment pas de perte de régularité dans les estimations quantitatives de convergence (voir Théorème 4.43). Finalement, notre résultat dans [25] améliore le résultat obtenu dans [112] dans lequel la convergence faible- \star vers le système de Navier-Stokes-Fourier incompressible a été obtenue.

5 Liste de travaux rassemblés dans la thèse

Les travaux de cette thèse sont composés de trois chapitres :

- Chapitre 2 : article [111].
- Chapitre 3 : article [112], paru dans *Kinetic and Related Models*.
- Chapitre 4 : article [25], écrit en collaboration avec Kleber Carrapatoso et Isabelle Tristani.

Hypoelliptic and spectral estimates for the linearized Landau operator

Résumé

Dans cette partie, nous nous intéressons à l'équation de Landau inhomogène qui décrit l'évolution d'une densité de particules $f = f(t, x, v)$ représentant au temps $t \geq 0$ la densité de particules à la position $x \in \mathbb{R}^3$ et la vitesse $v \in \mathbb{R}^3$. L'étude est motivée par la linéarisation de l'équation de Landau près de la distribution maxwellienne. Nous établissons des estimations hypoelliptiques, une localisation du spectre et des estimations de la résolvante de l'opérateur de Landau linéarisé dans les cas de potentiels durs et molécules maxwelliennes. La preuve est basée sur une méthode de multiplicateur et nécessite des outils de calcul pseudo-différentiels raffinés.

Abstract

In this part, we are interested in the inhomogeneous Landau equation which describes the evolution of a particle density $f = f(t, x, v)$ representing at time $t \geq 0$, the density of particles at position $x \in \mathbb{R}^3$ and velocity $v \in \mathbb{R}^3$. The study is motivated by the linearization of the Landau equation near Maxwellian distribution. In this article, we establish hypoelliptic estimates, a localization of the spectrum and estimates of the resolvent of the linearized Landau operator with hard potentials and Maxwellian molecules. The proof is based on a multiplier method and requires refined pseudo-differential calculus tools.

1 Introduction

1.1 The model.

In this paper, we study hypoellipticity and spectral properties associated to the spatially inhomogeneous Landau equation. This equation is a kinetic model in plasma physics that describes the evolution of the density function $F = F(t, x, v)$ representing at time $t \in \mathbb{R}^+$, the density of particles at position $x \in \mathbb{R}^3$ and velocity $v \in \mathbb{R}^3$. This equation is given by

$$\begin{cases} \partial_t F + v \cdot \nabla_x F = Q(F, F) \\ F|_{t=0} = F_0, \end{cases} \quad (2.1)$$

where Q is the so-called Landau collision operator which acts on the variable v and which contains diffusion in velocity. More precisely, the Landau operator is defined by

$$Q(G, F) = \partial_i \int_{\mathbb{R}^3} a_{ij}(v - v_*) [G_* \partial_j F - F \partial_j G_*] dv_*, \quad (2.2)$$

where we use the convention of summation of repeated indices, and the derivatives are in the velocity variable v i.e. $\partial_i = \partial_{v_i}$. Hereafter we use the shorthand notations $G_* = G(v_*)$, $F = F(v)$, $\partial_j G_* = \partial_{v_* j} G(v_*)$, $\partial_j F = \partial_{v_j} F(v)$, etc. The matrix $A(v) = (a_{ij}(v))_{1 \leq i, j \leq 3}$ is symmetric, positive, definite, depends on the interaction between particles and is given by

$$a_{ij}(v) = |v|^{\gamma+2} \left(\delta_{ij} - \frac{v_i v_j}{|v|^2} \right), \quad \gamma \in [-3, 1].$$

We recall the standard classification: we call hard potentials if $\gamma \in (0, 1]$, Maxwellian molecules if $\gamma = 0$, moderately soft potentials if $\gamma \in [-2, 0)$, very soft potentials if $\gamma \in (-3, -2)$ and Coulombian potential if $\gamma = -3$. Hereafter we shall consider the cases of hard potentials, Maxwellian molecules, i.e. $\gamma \in [0, 1]$. We denote by

$$\mu(v) = (2\pi)^{-3/2} e^{-|v|^2/2}$$

the normalized Maxwellian which is a global equilibrium. We linearize the Landau equation around μ with the perturbation

$$F = \mu + \mu^{1/2} f.$$

The Landau equation (2.1) for $f = f(t, x, v)$ takes the form

$$\begin{cases} \partial_t f + v \cdot \nabla_x f - \mu^{-1/2} Q(\mu^{1/2} f, \mu) - \mu^{-1/2} Q(\mu, \mu^{1/2} f) = \mu^{-1/2} Q(\mu^{1/2} f, \mu^{1/2} f) \\ f|_{t=0} = f_0 = \mu^{-1/2} (F_0 - \mu), \end{cases} \quad (2.3)$$

since $Q(\mu, \mu) = 0$. Using the notation

$$\Gamma(f, g) = \mu^{-1/2} Q(\mu^{1/2} f, \mu^{1/2} g),$$

we may rewrite the above equation as

$$\begin{cases} \partial_t f + \mathcal{P}f = \Gamma(f, f) \\ f|_{t=0} = f_0, \end{cases} \quad (2.4)$$

where the linearized Landau operator \mathcal{P} takes the form

$$\mathcal{P} = v \cdot \nabla_x - \mathcal{L} \quad (2.5)$$

with

$$\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2, \quad \mathcal{L}_1 = \Gamma(\sqrt{\mu}, f), \quad \mathcal{L}_2 = \Gamma(f, \sqrt{\mu}).$$

Operator \mathcal{P} acts only in variables (x, v) , is non selfadjoint, and consists of a transport part which is skew-adjoint, a diffusion part acting only in the v variable and a compact part (see for example Proposition 2.1 in [30]). Using for example [129], [120], the diffusion part \mathcal{L}_1 is written as follows

$$\mathcal{L}_1 f = \nabla_v \cdot [\mathbf{A}(v) \nabla_v f] - \left(\mathbf{A}(v) \frac{v}{2} \cdot \frac{v}{2} \right) f + \nabla_v \cdot \left[\mathbf{A}(v) \frac{v}{2} \right] f,$$

where $\mathbf{A}(v) = (\bar{a}_{ij}(v))_{1 \leq i, j \leq 3}$ is a symmetric matrix with

$$\bar{a}_{ij} = a_{ij} *_v \mu,$$

and the compact part \mathcal{L}_2 is given by

$$\mathcal{L}_2 f = -\mu^{-1/2} \partial_i \left\{ \mu \left[a_{ij} *_v \left\{ \mu^{1/2} \left[\partial_j f + \frac{v_j}{2} f \right] \right\} \right] \right\}.$$

Remark 2.1. Here we do not follow the same convention as the one in [60] for operators \mathcal{L}_1 and \mathcal{L}_2 .

1.2 Notations.

Throughout the paper we shall adopt the following notations: we work in dimension $n = 3$ and denote by $(x, v) \in \mathbb{R}_x^3 \times \mathbb{R}_v^3$ the space-velocity variables. For $v \in \mathbb{R}^3$ we denote $\langle v \rangle = (1 + |v|^2)^{1/2}$, where we recall that $|v|$ is the canonical Euclidian norm of v in \mathbb{R}^3 . The gradient in velocity (resp. space) will be denoted by ∂_v (resp. ∂_x). We shall also denote $D_v = \frac{1}{i} \partial_v$ (resp. $D_x = \frac{1}{i} \partial_x$), and denote ξ the dual variable of x , η the dual variable of v . For simplicity of notations, $a \sim b$ means that there exist constants $c_1, c_2 > 0$ such that $c_1 b \leq a \leq c_2 b$; we abbreviate “ $\leq C$ ” to “ \lesssim ”, where C is a positive constant depending only on fixed number. Finally, the space of distributions on Ω is denoted by $\mathcal{D}'(\Omega)$ where $\Omega \subseteq \mathbb{R}^n$ is an open set.

1.3 Main results and comments.

In this article, we will show a localization property of the pseudospectrum of the Landau operator \mathcal{P} , that is to say the region of the complex plane where its resolvent is a priori large (in fact we more precisely give a description of a large region where the resolvent is controlled, which is included in the complementary of the pseudospectrum). This result is given by the following theorem.

Theorem 2.2. *Let \mathcal{P} be the Landau operator on $L^2(\mathbb{R}_x^3 \times \mathbb{R}_v^3)$ defined in (2.5) with $\gamma \in [0, 1]$. Then there are two constants $C_{\mathcal{P}} > 0$ and $Q_{\mathcal{P}} > 0$ so that:*

a) *The spectrum of \mathcal{P} verifies*

$$\sigma(\mathcal{P}) \subset S_{\mathcal{P}} \cap \{\Re z \geq 0\},$$

with

$$S_{\mathcal{P}} = \left\{ z \in \mathbb{C}, |z + 1|^{1/3} \leq C_{\mathcal{P}}(\Re z + 1), \Re z \geq -\frac{1}{2} \right\}. \quad (2.6)$$

b) *For any $z \notin S_{\mathcal{P}}$ with $\Re z \geq -\frac{1}{2}$, the resolvent is estimated by*

$$\|(z - \mathcal{P})^{-1}\|_{\mathcal{B}(L_{x,v}^2)} \leq Q_{\mathcal{P}}|z + 1|^{-1/3}. \quad (2.7)$$

Notice that if $\Re z \leq -\frac{1}{2}$ then

$$\|(z - \mathcal{P})^{-1}\|_{\mathcal{B}(L_{x,v}^2)} \leq |\Re z|^{-1}. \quad (2.8)$$

The results of Theorem 2.2 can be illustrated by the following figure:

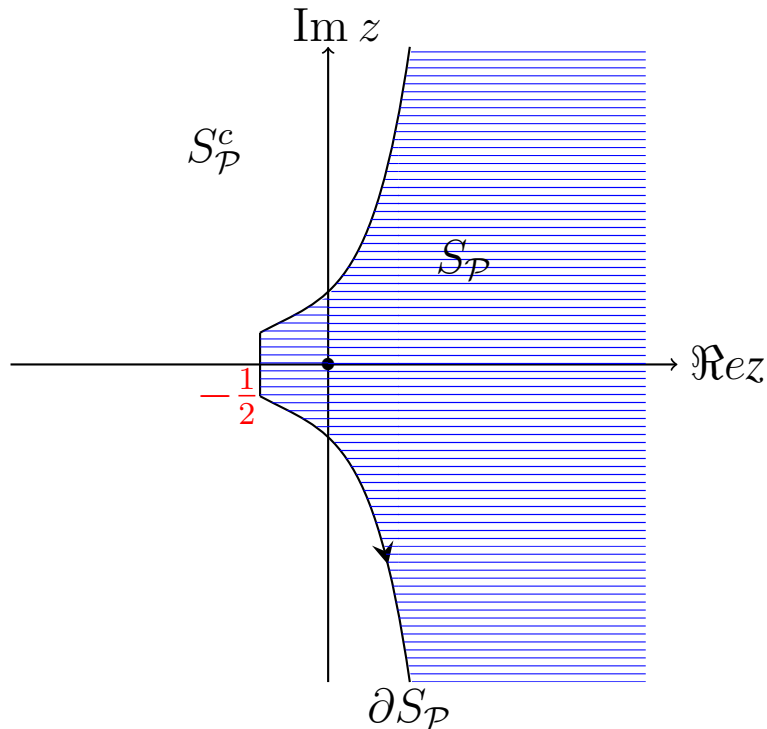


Figure 2.1: Localization of the pseudospectrum of the Landau operator \mathcal{P} .

In this figure $\partial S_{\mathcal{P}}$ oriented from $+i\infty$ to $-i\infty$. The hatched part is where the spectrum is localized and the non hatched part is the zone where we have good resolvent estimates (see (2.7) and (2.8)).

Let us give now some comments and motivations of this result. The cuspidal form of the pseudospectrum of linear or linearized kinetic operators was first shown in [62], [67] for the Fokker-Planck operator, and then extended to operators appearing in statistical mechanics by [39]. The main motivation for this type of study is to be able to understand the so-called pseudospectrum properties and derive possible trend to the equilibrium or regularization properties for the related evolution equation thanks to Cauchy formulae (see e.g. [67] for a complete result in the Fokker-Planck case). Anyway in this article, as a first step of a complete study of properties of this type, we only focus on the pseudospectral localization, but in the case of the much more complicated linearized Landau operator.

The aim of this article is double. First we show that the linearized Landau operator has indeed a pseudospectrum of cuspidal form, which is a good clue for thinking that this is a very general property of general linear or linearized kinetic operators. Second we propose a very robust and self contained pseudo-differential framework (see Section 3.1) in order to show this type of result. Our hope is that it can be used for many other kinetic models. These tools are greatly inspired by previous works (see [4], [68], [71])

concerning kinetic equations, following fundamental ideas in [91] on Wick and Weyl pseudodifferential calculus. Indeed the main remark done in all these works, following preliminary works by Alexandre and Villani [2] is that the linearized Landau operator L is, up to controlled/bounded operators, a pseudodifferential operator, namely that there exists a symbol, in a Hörmander-type class of symbol (see Section 3.1) such that

$$L \equiv a^w + \text{controlled terms},$$

where w stands for the Weyl quantization. The equivalence above can be rigorously stated and we will do it in section 2 below. The understanding of Landau type operators has been greatly improved in the past twenty years, following in particular works by Guo (see [60]).

Using these tools, the proof of the result given in Theorem 2.2 will follow the lines of [68] (see also [4], [67]), where a multiplier method is used for proving regularization (hypoelliptic) properties of the linearized Landau operator. Let us emphasize here that no regularization property is shown in this article, since we only focus on spectral estimates. Mention anyway - and this is a remarkable feature of all these hypoelliptic/hypocoercive techniques - that the same (in spirit) methods give very strong and precise results as the one given in the main theorem.

Let us highlight that a perspective of our work is to use our spectral analysis of the Landau operator to show that the solutions of the Landau equation are asymptotically equivalent in time (with some time decay rate when $t \rightarrow \infty$) to the ones of the compressible Navier–Stokes equations, following ideas used for example by Yang-Yu in [130].

Organization of the article. In Section 2 we give some properties of the Landau operator. In Section 3, we prove hypoelliptic estimates with respect to the velocity variable for a parametric operator. In Section 4 we give hypoelliptic estimates for the linearized Landau operator. Section 5 is devoted to the proof of Theorem 2.2. An appendix is devoted to a short review of some tools used in this work (Weyl–Hörmander quantization, Wick quantization and the proof of the Theorem 2.40 (Basic Theorem)).

2 Properties of the Landau operator

In this section, we first present the decomposition of the linearized Landau operator \mathcal{P} then we exhibit a simpler form of this operator. To end up we show that \mathcal{P} is a generator of a strongly continuous semigroup. Throughout this section, we work with $\gamma \in [-3, 1]$.

2.1 Splitting of the linearized operator.

Consider a smooth positive function $\chi \in C_c^\infty(\mathbb{R}_v^3)$ such that $0 \leq \chi(v) \leq 1$, $\chi(v) = 1$ for $|v| \leq 1$ and $\chi(v) = 0$ for $|v| > 2$. For any $R \geq 1$ we define $\chi_R(v) = \chi(\frac{v}{R})$ and

in the sequel we shall consider the function $M\chi_R$, for some constant $M > 0$. Then, we introduce the decomposition of the operator \mathcal{P} as $\mathcal{P} = \mathcal{A} + \mathcal{K}$ with

$$\mathcal{A} = -\mathcal{L}_1 + v \cdot \nabla_x + M\chi_R, \quad \mathcal{K} = -\mathcal{L}_2 - M\chi_R, \quad (2.9)$$

where $M > 0$ and $R > 0$ will be chosen later.

We define the function $F(v)$ as

$$F(v) = \left(\mathbf{A}(v) \frac{v}{2} \cdot \frac{v}{2} \right) - \nabla_v \cdot \left[\mathbf{A}(v) \frac{v}{2} \right] + M\chi_R. \quad (2.10)$$

Then, we can rewrite \mathcal{A} as follows

$$\mathcal{A}f = v \cdot \nabla_x f - \nabla_v \cdot [\mathbf{A}(v) \nabla_v f] + F(v)f. \quad (2.11)$$

2.2 Preliminaries.

We have the following results concerning the matrix $\mathbf{A}(v)$.

Lemma 2.3. *The following properties hold:*

- a) For $v \in \mathbb{R}^3 \setminus \{0\}$, the matrix $\mathbf{A}(v)$ has a simple eigenvalue $\ell_1(v) > 0$ associated with the eigenvector v and a double eigenvalue $\ell_2(v) > 0$ associated with the eigenspace v^\perp . Moreover, when $|v| \rightarrow +\infty$ we have

$$\ell_1(v) \sim 2\langle v \rangle^\gamma \quad \text{and} \quad \ell_2(v) \sim \langle v \rangle^{\gamma+2}.$$

- b) The function \bar{a}_{ij} is smooth, for any multi-index $\alpha \in \mathbb{N}^3$, there exists $C_\alpha > 0$ such that for all $v \in \mathbb{R}^3$, we have

$$|\partial_v^\alpha \bar{a}_{ij}(v)| + |\partial_v^\alpha (\bar{a}_{ij}(v)v_j)| \leq C_\alpha \langle v \rangle^{\gamma+2-|\alpha|},$$

- c) For $v \in \mathbb{R}^3 \setminus \{0\}$, we have

$$\bar{a}_{ij}(v)v_i v_j = \ell_1(v)|v|^2,$$

$$\bar{a}_{ii}(v) = \text{tr}(\bar{a}(v)) = \ell_1(v) + 2\ell_2(v),$$

$$\bar{a}_{ij}(v)\eta_i \eta_j = \ell_1(v)|P_v \eta|^2 + \ell_2(v)|(I - P_v)\eta|^2,$$

with $\eta \in \mathbb{R}^3$ and P_v is the projection on v , i.e. $P_v \eta = \left(\eta \cdot \frac{v}{|v|} \right) \frac{v}{|v|}$.

- d) For $|v| > 1$, we have

$$|\partial_v^\alpha \ell_1(v)| \leq C_\alpha \langle v \rangle^{\gamma-|\alpha|} \quad \text{and} \quad |\partial_v^\alpha \ell_2(v)| \leq C_\alpha \langle v \rangle^{\gamma+2-|\alpha|}.$$

Proof. See for example [26, Lemma 2.4], [60, Lemma 3] and [129, Proposition 1]. \square

Lemma 2.4. For all $v \in \mathbb{R}^3$ with $|v| > 1$, we have

$$\ell_1(v) \gtrsim \langle v \rangle^\gamma \quad \text{and} \quad \ell_2(v) \gtrsim \langle v \rangle^{\gamma+2}.$$

Proof. Using (a) in Lemma 2.3, when $|v| \rightarrow +\infty$, we have $\ell_1(v) \sim 2\langle v \rangle^\gamma$. In particular, there is a constant $N > 0$ such that for all $|v| > N$, we have

$$\ell_1(v) \geq \langle v \rangle^\gamma.$$

We have that $\ell_1(v)$ is continuous since $\mathbf{A}(v)$ is a positive definite symmetric matrix and continuous (due to convolution with μ), hence the existence of a constant $C > 0$ such that for $1 \leq |v| \leq N$

$$\ell_1(v) \geq C \langle v \rangle^\gamma,$$

and then for all $v \in \mathbb{R}^3$, perhaps with changing C ,

$$\ell_1(v) \geq C \langle v \rangle^\gamma.$$

The proof will be the same for $\ell_2(v)$. □

Lemma 2.5. Let $F(v)$ be defined in (2.10). Then, we can choose M and R big enough such that for all $v \in \mathbb{R}^3$, we have

$$F(v) \gtrsim \langle v \rangle^{\gamma+2}.$$

Proof. Since

$$F(v) \geq \frac{1}{4} \ell_1(v) |v|^2 - \left| \nabla_v \cdot \left[\mathbf{A}(v) \frac{v}{2} \right] \right| + M \chi_R,$$

according to Lemma 2.3, we have

$$\left| \nabla_v \cdot \left[\mathbf{A}(v) \frac{v}{2} \right] \right| \lesssim \langle v \rangle^{\gamma+1}. \quad (2.12)$$

So, using Lemma 2.4 and (2.12), there exist two positive constants C_1, C_2 such that

$$F(v) \geq C_1 \langle v \rangle^{\gamma+2} - C_2 \langle v \rangle^{\gamma+1} + M \chi_R,$$

then there exist M and R such that for all $v \in \mathbb{R}^3$

$$F(v) \gtrsim \langle v \rangle^{\gamma+2}.$$

□

Lemma 2.6. For any multi-index $\alpha \in \mathbb{N}^3$, there exists $C_\alpha > 0$ such that for all $v \in \mathbb{R}^3$, we have

$$|\partial_v^\alpha F(v)| \leq C_\alpha \langle v \rangle^{\gamma+2-|\alpha|}.$$

Proof. For $|v| > 2R$, using Leibniz's formula, we have

$$\partial_v^\alpha F(v) = \frac{1}{4} \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \partial^{\alpha-\beta} \ell_1(v) \partial^\beta \langle v \rangle^2 - \frac{1}{4} \partial^\alpha \ell_1(v) - \partial^\alpha \left(\nabla_v \cdot \left[\mathbf{A}(v) \frac{v}{2} \right] \right),$$

Then, using Lemma 2.3 and Lemma 2.4 we obtain

$$|\partial_v^\alpha F(v)| \leq c_\alpha \langle v \rangle^{\gamma+2-|\alpha|}.$$

The function $F(v)$ being C^∞ on $|v| \leq 2R$, the estimates for $|v| \leq 2R$ are immediate. \square

Lemma 2.7. i) $\mathbf{A}(v)$ is written as follows

$$\mathbf{A}(v) = B^T(v)B(v), \quad (2.13)$$

where $B(v) = (b_{ij}(v))_{1 \leq i, j \leq 3}$ is a matrix with real-valued smooth entries.

ii) For any multi-index $\alpha \in \mathbb{N}^3$, there exists $C_\alpha > 0$ such that for all $v \in \mathbb{R}^3$, we have

$$|\partial_v^\alpha b_{ij}(v)| \leq C_\alpha \langle v \rangle^{\frac{\gamma}{2}+1-|\alpha|}. \quad (2.14)$$

iii) There exists $c, C > 0$ such that for all $v \in \mathbb{R}^3$, for all $\eta \in \mathbb{R}^3$ we have

$$c \langle v \rangle^\gamma (|\eta|^2 + |v \wedge \eta|^2) \leq \mathbf{A}(v) \eta \cdot \eta = |B(v) \eta|^2 \leq C \langle v \rangle^\gamma (|\eta|^2 + |v \wedge \eta|^2) \quad (2.15)$$

Proof. i) As $\mathbf{A}(v)$ is a positive definite symmetric matrix (denoted $\mathbf{S}_3^{++}(\mathbb{R})$) and according to the spectral theorem, there exists \mathbf{Q} an orthogonal matrix such that

$$\mathbf{A}(v) = \mathbf{Q}^T(v) \mathbf{D}(v) \mathbf{Q}(v).$$

On $\{|v| > 1\}$, $v \mapsto \mathbf{Q}(v)$ can be calculated explicitly and can be chosen to be smooth. If we set

$$B(v) = \mathbf{Q}^T(v) \sqrt{\mathbf{D}(v)} \mathbf{Q}(v)$$

with $\sqrt{\mathbf{D}(v)} = \text{diag}(\sqrt{\ell_1(v)}, \sqrt{\ell_2(v)}, \sqrt{\ell_2(v)})$. We have that $v \mapsto \sqrt{\mathbf{D}(v)}$ is of class C^∞ for $|v| > 1$ (because $\ell_1(v), \ell_2(v)$ are of class C^∞ for $|v| > 1$). Then the application $v \mapsto B(v)$ is of class C^∞ on $\{|v| > 1\}$. Regarding the case where $|v| \leq 1$, we consider the following two applications:

$$\begin{aligned} \phi & : \mathbb{R}^3 \rightarrow \mathbf{S}_3^{++}(\mathbb{R}) \\ v & \mapsto \mathbf{A}(v) \end{aligned} ,$$

$$\begin{aligned} \psi & : \mathbf{S}_3^{++}(\mathbb{R}) \rightarrow \mathbf{S}_3^{++}(\mathbb{R}) \\ M & \mapsto \sqrt{M} \end{aligned} ,$$

we note that ϕ, ψ are of class C^∞ , moreover $B(v) = \psi \circ \phi(v)$. Then we have that the application $v \mapsto B(v)$ is of class C^∞ .

ii) For $|v| > 1$, we have $B(v) = \sqrt{\ell_1(v)}P_v + \sqrt{\ell_2(v)}(I - P_v)$, moreover using Lemma 2.3, we have

$$|\partial_v^\alpha \sqrt{\ell_1(v)}| \lesssim \langle v \rangle^{\frac{\gamma}{2}-|\alpha|}, \quad |\partial_v^\alpha \sqrt{\ell_2(v)}| \lesssim \langle v \rangle^{\frac{\gamma+2}{2}-|\alpha|} \quad (2.16)$$

and the fact that $|\partial_v^\alpha P_v| \lesssim 1$ (P_v and all its derivatives are bounded), so we get that for $|v| > 1$,

$$|\partial_v^\alpha b_{ij}(v)| \lesssim \langle v \rangle^{\frac{\gamma}{2}+1-|\alpha|}, \quad (2.17)$$

where the constants in (2.16), (2.17) depend on α .

The function $b_{ij}(v)$ being C^∞ on $|v| \leq 1$, the estimates for $|v| \leq 1$ are immediate. Then, for all $\alpha \in \mathbb{N}^3$, there exists $C_\alpha > 0$ such that for all $v \in \mathbb{R}^3$, we have

$$|\partial_v^\alpha b_{ij}(v)| \leq C_\alpha \langle v \rangle^{\frac{\gamma}{2}+1-|\alpha|}. \quad (2.18)$$

iii) The estimate is immediate on $|v| \leq 1$ because $\mathbf{A}(v)$ is a positive definite symmetric matrix and $\mathbf{A} = B^T B$. For $|v| > 1$, using Lemma 2.3, we have

$$\begin{aligned} \bar{a}_{ij}(v)\eta_i\eta_j &= \ell_1(v)|P_v\eta|^2 + \ell_2(v)|(I - P_v)\eta|^2 \\ &\gtrsim \langle v \rangle^\gamma |\eta|^2 \cos^2(v, \eta) + \langle v \rangle^{\gamma+2} \frac{|v \wedge \eta|^2}{|v|^2} \\ &\gtrsim \langle v \rangle^\gamma |\eta|^2 \cos^2(v, \eta) + \langle v \rangle^\gamma |v \wedge \eta|^2 + \langle v \rangle^\gamma |\eta|^2 \sin^2(v, \eta) \\ &\gtrsim \langle v \rangle^\gamma (|\eta|^2 + |v \wedge \eta|^2), \end{aligned}$$

on the other hand, we have

$$\begin{aligned} \bar{a}_{ij}(v)\eta_i\eta_j &= \ell_1(v)|P_v\eta|^2 + \ell_2(v)|(I - P_v)\eta|^2 \\ &\lesssim \langle v \rangle^\gamma |\eta|^2 + \langle v \rangle^{\gamma+2} \frac{|v \wedge \eta|^2}{|v|^2} \\ &\lesssim \langle v \rangle^\gamma (|\eta|^2 + |v \wedge \eta|^2). \end{aligned}$$

Hence the proof of (iii). □

Using Lemma 2.7, we can rewrite \mathcal{A} in the form

$$\mathcal{A} = v \cdot \nabla_x + (B(v)\nabla_v)^* \cdot B(v)\nabla_v + F(v), \quad (2.19)$$

where $(B(v)D_v)^* = D_v B(v)^T$, is the formal adjoint of $B(v)D_v$.

2.3 Study of the operator \mathcal{P} .

In this part, we will study the following problem:

$$\begin{cases} \partial_t f + \mathcal{P}f = 0 \\ f|_{t=0} = f_0, \end{cases} \quad (2.20)$$

we show that the above problem is well-posed in the space $L^2(\mathbb{R}_x^3 \times \mathbb{R}_v^3)$ in the sense of semi-groups. By Hille-Yosida Theorem, it is sufficient to show that \mathcal{A} is maximal accretive in the space $L^2(\mathbb{R}_x^3 \times \mathbb{R}_v^3)$, then using the Bounded Perturbation Theorem in [42, Theorem 1.3], we get that the operator \mathcal{P} is a generator of a strongly continuous semigroup (for more details on the semi-group theory see also [110]). First, we start by recalling the basic definition of hypoellipticity.

Definition 2.8. Let P be a differential operator with C^∞ coefficients in an open set $\Omega \subset \mathbb{R}^n$. We say that P is a hypoelliptic operator on Ω , if, for any open $\omega \subset \Omega$, any $u \in \mathcal{D}'(\Omega)$, such that $Pu \in C^\infty(\omega)$ belongs to $C^\infty(\omega)$.

Lemma 2.9. Let \mathcal{A} be the operator defined in (2.9). Then, \mathcal{A} is a hypoelliptic operator.

Proof. Using formula (2.19), we can rewrite \mathcal{A} in the following form:

$$\begin{aligned} \mathcal{A} &= \underbrace{v \cdot \partial_x}_{X_0} + \underbrace{\left(\sum_{j=1}^3 b_{1,j}(v) \partial_{v_j} \right)^* \left(\sum_{j=1}^3 b_{1,j}(v) \partial_{v_j} \right)}_{X_1^* X_1} + \underbrace{\left(\sum_{j=1}^3 b_{2,j}(v) \partial_{v_j} \right)^* \left(\sum_{j=1}^3 b_{2,j}(v) \partial_{v_j} \right)}_{X_2^* X_2} \\ &\quad + \underbrace{\left(\sum_{j=1}^3 b_{3,j}(v) \partial_{v_j} \right)^* \left(\sum_{j=1}^3 b_{3,j}(v) \partial_{v_j} \right)}_{X_3^* X_3} + F(v) \\ &= X_0 + \sum_{i=1}^3 X_i^* X_i + F(v). \end{aligned}$$

In addition, the coefficients $b_{ij}(v)$ and $F(v)$ are of class C^∞ , taking the vector field square brackets we get

$$Y_k = [X_k, X_0] = \sum_{j=1}^3 b_{k,j}(v) \partial_{x_j}, \quad \text{for } k \in \{1, 2, 3\}.$$

From the above, \mathcal{A} is a “type II Hörmander’s operators” (see for example [62], [75]). Moreover, the vector fields $\{X_i, Y_i, i = 1, \dots, 3\}$ generate all the space tangent to $\mathbb{R}_{x,v}^6$ so \mathcal{A} is hypoelliptic operator. \square

Theorem 2.1. Let $\gamma \in [-3, 1]$ and \mathcal{A} be the operator defined in (2.9). Then, its closure $\overline{\mathcal{A}}$ on the space $\mathcal{S}(\mathbb{R}_{x,v}^6)$ is maximally accretive.

Proof. We adapt here the proof given in [62, page 44]. We apply the abstract criterion by taking $\mathcal{H} = L^2(\mathbb{R}_{x,v}^6)$ and the domain of \mathcal{A} defined by $D(\mathcal{A}) = \mathcal{S}(\mathbb{R}_{x,v}^6)$. First, we show the accretivity of the operator \mathcal{A} . We want to prove that $\Re(\mathcal{A}u, u)_{\mathcal{H}} \geq 0$ for $u \in D(\mathcal{A})$. Indeed, from (2.19)

$$\begin{aligned} \Re(\mathcal{A}u, u)_{\mathcal{H}} &= \underbrace{\Re(v \cdot \nabla_x u, u)_{\mathcal{H}}}_{=0 \text{ since } v \cdot \nabla_x \text{ is skew-adjoint}} - \Re(\nabla_v \cdot (\mathbf{A}(v)\nabla_v u), u)_{\mathcal{H}} + \Re(F(v)u, u)_{\mathcal{H}} \\ &= \|B(v)\nabla_v u\|_{\mathcal{H}}^2 + \|\sqrt{F(v)}u\|_{\mathcal{H}}^2 \\ &\geq 0. \end{aligned}$$

Since \mathcal{A} is an accretive operator then its closure $\overline{\mathcal{A}}$ exists and it is accretive (see [62, Proposition 5.3]). Let us now show that there exists $\lambda_1 > 0$ such that the operator

$$T = \mathcal{A} + \lambda_1 \text{Id}$$

has dense image in \mathcal{H} . We take $\lambda_1 = 1$. Let $f \in \mathcal{H}$ satisfy

$$(f, Tu)_{\mathcal{H}} = 0, \quad \forall u \in D(\mathcal{A}), \quad (2.21)$$

we want to prove that $f = 0$.

Since T is a differential operator then his formal adjoint T^\sharp exists (in the sense of distributions). According to (2.21) we obtain

$$T^\sharp f = \left(-\nabla_v \cdot \mathbf{A}(v)\nabla_v + F(v) + 1 - X_0 \right) f = 0, \quad \text{in } \mathcal{D}'(\mathbb{R}^6). \quad (2.22)$$

Using Lemma 2.9, we have $-\nabla_v \cdot \mathbf{A}(v)\nabla_v + F(v) + 1 - X_0$ is a hypoelliptic operator, so $f \in C^\infty(\mathbb{R}^6)$ (see [62, Chapter 2]).

Now we introduce the family of truncation functions ζ_k defined by

$$\zeta_k(x, v) = \zeta\left(\frac{x}{k_1}\right)\zeta\left(\frac{v}{k_2}\right), \quad \forall k = (k_1, k_2) \in (\mathbb{N}^*)^2,$$

where ζ is a C^∞ function satisfying the following conditions:

$$\begin{cases} 0 \leq \zeta \leq 1, \\ \zeta = 1 \text{ on } B(0, 1), \\ \text{supp } \zeta \subset B(0, 2), \\ \zeta \text{ is a radial function.} \end{cases}$$

The expression of $T^\sharp(\zeta_k f)$ is

$$\begin{aligned} T^\sharp(\zeta_k f) &= -\nabla_v \cdot \left(\mathbf{A}(v)\nabla_v(\zeta_k f) \right) + \left(F(v) + 1 \right) \zeta_k f - X_0(\zeta_k f) \\ &= -\nabla_v \cdot \left(\mathbf{A}(v)([\nabla_v \zeta_k]f) \right) - \nabla_v \zeta_k \cdot \mathbf{A}(v)\nabla_v f - X_0(\zeta_k) f - \zeta_k T^\sharp f, \end{aligned}$$

by using (2.22), we obtain

$$T^\sharp(\zeta_k f) = -\nabla_v \cdot (\mathbf{A}(v)([\nabla_v \zeta_k]f)) - \nabla_v \zeta_k \cdot \mathbf{A}(v) \nabla_v f - X_0(\zeta_k) f. \quad (2.23)$$

We note that $T^\sharp(\zeta_k f) \in \mathcal{H}$, taking the scalar product with $\zeta_k f$ we obtain

$$\begin{aligned} (T^\sharp(\zeta_k f), \zeta_k f)_{\mathcal{H}} &= - \iint \nabla_v \cdot (\mathbf{A}(v)([\nabla_v \zeta_k]f)) \zeta_k f \, dx dv - \iint X_0(\zeta_k) \zeta_k |f|^2 \, dx dv \\ &\quad - \iint (\nabla_v \zeta_k \cdot \mathbf{A}(v) \nabla_v f) \zeta_k f \, dx dv, \end{aligned}$$

By doing an integration by parts, we obtain

$$(T^\sharp(\zeta_k f), \zeta_k f)_{\mathcal{H}} = \iint |B \nabla_v(\zeta_k) f|^2 \, dx dv - \iint X_0(\zeta_k) \zeta_k |f|^2 \, dx dv.$$

On the other hand, using (2.22), we obtain

$$\begin{aligned} (T^\sharp(\zeta_k f), \zeta_k f)_{\mathcal{H}} &= \iint |B \nabla_v(\zeta_k f)|^2 \, dx dv + \iint (F(v) + 1) |\zeta_k f|^2 \, dx dv \\ &\quad - \underbrace{\iint X_0(\zeta_k f) \zeta_k f \, dx dv}_{=0 \text{ since } X_0 \text{ is skew-adjoint}} \\ &= \iint |B \nabla_v(\zeta_k f)|^2 \, dx dv + \iint (F(v) + 1) |\zeta_k f|^2 \, dx dv. \end{aligned}$$

Using the fact that

$$\iint |B \nabla_v(\zeta_k f)|^2 \, dx dv \geq 0,$$

we obtain

$$\iint (F(v) + 1) |\zeta_k f|^2 \, dx dv \leq \underbrace{\iint |B \nabla_v(\zeta_k) f|^2 \, dx dv}_{(i)} - \underbrace{\iint X_0(\zeta_k) \zeta_k |f|^2 \, dx dv}_{(ii)}. \quad (2.24)$$

Estimate of (i): Using Lemma 2.3-(b)-(d) and taking into account that the function ζ is a radial function we obtain

$$\begin{aligned} \iint |B \nabla_v(\zeta_k) f|^2 \, dx dv &= \iint \ell_1(v) |P_v \nabla_v \zeta_k|^2 |f|^2 \, dx dv + \underbrace{\iint \ell_2(v) |(I - P_v) \nabla_v \zeta_k|^2 |f|^2 \, dx dv}_{=0 \text{ since } v \text{ is parallel to } \nabla_v \zeta_k} \\ &\leq \frac{C_0}{k_2^2} \iint \langle v \rangle^\gamma \Phi_k |f|^2 \, dx dv, \end{aligned}$$

where $C_0 > 0$ and $\Phi_k = \zeta(\frac{x}{k_1}) \zeta'(\frac{v}{k_2})$. Using the fact that $\gamma \in [-3, 1]$ and the fact that Φ_k is a bounded function we have the existence of a constant $C_1 > 0$ such that,

$$\iint |B \nabla_v(\zeta_k) f|^2 \, dx dv \leq C_1 \left(\frac{1}{k_2^2} + \frac{1}{k_2} \right) \|f\|^2.$$

Estimate of (ii): we have

$$\left| \iint X_0(\zeta_k) \zeta_k |f|^2 dx dv \right| \leq \frac{1}{k_1} \iint |v| \tilde{\Phi}_k \zeta_k |f|^2 dx dv,$$

where $\tilde{\Phi}_k = \zeta'(\frac{x}{k_1}) \zeta(\frac{v}{k_2})$.

Now, taking into account that the functions $\tilde{\Phi}_k$ and ζ_k are bounded, we have the existence of a constant $C_2 > 0$ such that,

$$\left| \iint X_0(\zeta_k) \zeta_k |f|^2 dx dv \right| \leq C_2 \frac{k_2}{k_1} \|f\|^2.$$

Finally, coming back to (2.24) we obtained the existence of a constant $C > 0$ such that,

$$\iint |\zeta_k f|^2 dx dv \leq C \left(\frac{1}{k_2^2} + \frac{1}{k_2} + \frac{k_2}{k_1} \right) \|f\|^2, \quad \forall k. \quad (2.25)$$

Taking $k_1 \rightarrow +\infty$ in (2.25) we obtain

$$\iint \left| \zeta \left(\frac{v}{k_2} \right) f \right|^2 dx dv \leq C \left(\frac{1}{k_2^2} + \frac{1}{k_2} \right) \|f\|^2, \quad (2.26)$$

and taking $k_2 \rightarrow +\infty$ in (2.26) we obtain

$$\iint |f|^2 dx dv = 0,$$

then $f = 0$. □

From now on, we write \mathcal{A} for the closure of the operator \mathcal{A} .

Corollary 2.10. *Let $\gamma \in [-3, 1]$ and \mathcal{P} be the operator defined in (2.5). Then, $-\mathcal{P}$ is a generator of a semi-group $(S(t))_{t \geq 0}$ strongly continuous on $\mathcal{H} = L^2(\mathbb{R}_{x,v}^6)$ verifying*

$$\|S(t)\| \leq e^{\|\mathcal{K}\|t} \text{ for } t \geq 0, \quad (2.27)$$

where \mathcal{K} the operator defined in (2.9).

Proof. Using (2.9), the operator $-\mathcal{P}$ is written as follows

$$-\mathcal{P} = -\mathcal{A} - \mathcal{K}.$$

According to Theorem 2.1, \mathcal{A} is a maximally accretive operator. According to the Hille-Yosida Theorem, $-\mathcal{A}$ is a generator of a strongly continuous semi-group of contraction. On the other hand, $-\mathcal{K}$ is a bounded operator in \mathcal{H} (we have \mathcal{L}_2 a compact operator and $M\chi_R$ is bounded). Using the Bounded Perturbation Theorem in [42, Theorem 1.3], we have that $-\mathcal{P}$ is a generator of a semi-group $(S(t))_{t \geq 0}$ strongly continuous on \mathcal{H} , moreover we have

$$\|S(t)\| \leq e^{\|\mathcal{K}\|t} \text{ for } t \geq 0. \quad (2.28)$$

□

3 Hypocoelliptic estimates for the operator with parameters

In the following discussion, we work with $\gamma \in [0, 1]$. In this section, we will study the operator acting on the velocity variable v :

$$\mathcal{A}_\xi = iv \cdot \xi + (B(v)\nabla_v)^* \cdot B(v)\nabla_v + F(v), \quad (2.29)$$

where ξ is the parameter in \mathbb{R}^3 . The operator \mathcal{A}_ξ is obtained by the partial Fourier transformation in x . The goal of studying the operator \mathcal{A}_ξ and considering ξ as a parameter, is to obtain estimates of the velocity variables v uniformly with respect to ξ . Then by using the inverse Fourier transform with respect to x , we can obtain global estimates in all variables. We note that the operator \mathcal{A}_ξ verifies for all $u \in \mathcal{S}(\mathbb{R}_v^3)$,

$$\|B(v)\nabla_v u\|_{L^2}^2 + \|\sqrt{F(v)}u\|_{L^2}^2 \leq \Re e(\mathcal{A}_\xi u, u)_{L^2}. \quad (2.30)$$

Notations. Throughout this section, we will use $\|\cdot\|_{L^2}$ to denote the norm in the space $L^2(\mathbb{R}_v^3)$ and ξ is a parameter. We use p^{Wick} to denote the Wick quantization of p in the variables (v, η) (for more details on Wick quantization see [92] and Appendix (Section 6.3)).

The main result in this section is Proposition 2.29 and Proposition 2.30.

3.1 Pseudo-differential parts

In this part, we will show several lemmas concerning pseudo-differential symbols. We need to build these symbols, who verify assumptions of Theorem 2.40, such that the pseudo-differential operator associated to these symbols has good properties. These operators play an important role in hypocoelliptic estimates. The standard concepts on pseudo-differential calculus are explained in Appendix (Section 6). We define for $(v, \eta) \in \mathbb{R}^6$ the following symbols, they depend on the parameter ξ but we do not mention it in our notations since ξ is seen as a parameter.

$$\lambda(v, \eta) = \left[\langle v \rangle^\gamma \left(1 + |v|^2 + |\eta|^2 + |\xi|^2 + |v \wedge \eta|^2 + |v \wedge \xi|^2 \right) \right]^{1/2}, \quad (2.31)$$

$$\mathbf{a}(v, \eta) = 1 + |v|^2 + |\eta|^2 + |\xi|^2 + |v \wedge \eta|^2 + |v \wedge \xi|^2, \quad (2.32)$$

$$g_1(v, \eta) = 1 + \langle v \rangle + \langle \eta \rangle, \quad (2.33)$$

$$g_2(v, \eta) = 1 + \langle \xi \rangle^{1/3} + \langle \eta \rangle, \quad (2.34)$$

$$g_3(v, \eta) = 1 + \langle \xi \rangle^{2/3} + \langle \eta \rangle^2. \quad (2.35)$$

Lemma 2.11. *The above symbols are admissible weights in the sense of Definition 2.35 uniformly with respect to the parameter ξ in \mathbb{R}^3 .*

Proof. We have to check λ is an admissible weight. It is sufficient to verify that there exists two constants N and C , both depending only on γ , such that for all $Y = (v, \eta)$, $Y' = (v', \eta')$, we have

$$\lambda(Y) \leq C\lambda(Y') (1 + \Gamma(Y - Y'))^N,$$

where Γ is the metric defined by $\Gamma = dv^2 + d\eta^2$. We have

$$\frac{\lambda^2(v, \eta)}{\lambda^2(v', \eta')} = \frac{\langle v \rangle^\gamma}{\langle v' \rangle^\gamma} \left(\frac{1 + |v|^2 + |\eta|^2 + |\xi|^2 + |v \wedge \eta|^2 + |v \wedge \xi|^2}{1 + |v'|^2 + |\eta'|^2 + |\xi|^2 + |v' \wedge \eta'|^2 + |v' \wedge \xi|^2} \right). \quad (2.36)$$

We now use Peetre's inequality

$$\langle y \rangle^\tau \leq 2^{\frac{|\tau|}{2}} \langle y' \rangle^\tau \langle y - y' \rangle^{|\tau|}, \quad \tau \in \mathbb{R}, \quad (2.37)$$

to get

$$\frac{\langle v \rangle^\gamma}{\langle v' \rangle^\gamma} \leq 2^\gamma \langle v - v' \rangle^\gamma.$$

Using (2.37), we obtain

$$\frac{1 + |v|^2 + |\eta|^2}{1 + |v'|^2 + |\eta'|^2 + |\xi|^2 + |v' \wedge \eta'|^2 + |v' \wedge \xi|^2} \leq \frac{\langle v \rangle^2}{\langle v' \rangle^2} + \frac{\langle \eta \rangle^2}{\langle \eta' \rangle^2} \leq 4(\langle v - v' \rangle + \langle \eta - \eta' \rangle)^2$$

and

$$\frac{|\xi|^2}{1 + |v'|^2 + |\eta'|^2 + |\xi|^2 + |v' \wedge \eta'|^2 + |v' \wedge \xi|^2} \leq 1 \leq (\langle v - v' \rangle + \langle \eta - \eta' \rangle)^2.$$

Using the relation

$$v \wedge \xi = (v - v') \wedge \xi + v' \wedge \xi,$$

we obtain

$$\frac{|v \wedge \xi|^2}{1 + |v'|^2 + |\eta'|^2 + |\xi|^2 + |v' \wedge \eta'|^2 + |v' \wedge \xi|^2} \leq 1 + \frac{\langle v - v' \rangle^2 \langle \xi \rangle^2}{\langle \xi \rangle^2} \leq 2(\langle v - v' \rangle + \langle \eta - \eta' \rangle)^2.$$

Moreover using the relation

$$v \wedge \eta = (v - v') \wedge (\eta - \eta') + (v - v') \wedge \eta' + v' \wedge (\eta - \eta') + v' \wedge \eta',$$

we compute

$$\begin{aligned} & \frac{|v \wedge \eta|^2}{1 + |v'|^2 + |\eta'|^2 + |\xi|^2 + |v' \wedge \eta'|^2 + |v' \wedge \xi|^2} \\ & \leq \frac{4|v - v'|^2 |\eta - \eta'|^2 + 4|v - v'|^2 |\eta'|^2 + 4|v'|^2 |\eta - \eta'|^2 + 4|v' \wedge \eta'|^2}{1 + |v'|^2 + |\eta'|^2 + |\xi|^2 + |v' \wedge \eta'|^2 + |v' \wedge \xi|^2} \\ & \leq 4|v - v'|^2 |\eta - \eta'|^2 + 4|v - v'|^2 + 4|\eta - \eta'|^2 + 4 \\ & \leq 10(\langle v - v' \rangle + \langle \eta - \eta' \rangle)^4. \end{aligned}$$

Combining the above inequalities, we get

$$\frac{\lambda(Y)}{\lambda(Y')} \leq C_\gamma (1 + \Gamma(Y - Y'))^{\frac{4+\gamma}{2}},$$

so λ is an admissible weight. The proof will be the same for a, g_1, g_2 and g_3 . \square

Lemma 2.12. For $m \in \mathbb{R}$,

$$\lambda^m \in S(\lambda^m, \Gamma),$$

uniformly with respect to the parameter ξ in \mathbb{R}^3 , where $S(\lambda^m, \Gamma)$ is defined in Definition 2.36 in Appendix (Section 6.1).

Proof. We can rewrite λ as follows

$$\lambda(v, \eta) = \langle v \rangle^{\frac{\gamma}{2}} \mathbf{a}^{\frac{1}{2}}(v, \eta).$$

To prove the wanted result, we use induction on $|\alpha + \beta|$ to prove that for any $k \in \mathbb{R}$ and any $|\alpha + \beta| \geq 0$,

$$|\partial_v^\alpha \partial_\eta^\beta \mathbf{a}^k(v, \eta)| \lesssim \mathbf{a}^k(v, \eta), \quad (2.38)$$

which obviously holds for $|\alpha + \beta| = 0$. Now suppose $|\alpha + \beta| \geq 1$, then we have either $|\alpha| \geq 1$ or $|\beta| \geq 1$, and suppose $|\beta| \geq 1$ without loss of generality. So we can write $\partial_\eta^\beta = \partial_\eta^{\tilde{\beta}} \partial_{\eta_j}$ with $|\tilde{\beta}| = |\beta| - 1$ and thus

$$\partial_v^\alpha \partial_\eta^\beta \mathbf{a}^k(v, \eta) = \partial_v^\alpha \partial_\eta^{\tilde{\beta}} \left[k \mathbf{a}^{k-1} (2\eta_j + 2(v \wedge \eta) \partial_{\eta_j} (v \wedge \eta)) \right],$$

which along with Leibniz' formula and the induction assumption yields

$$\begin{aligned} |\partial_v^\alpha \partial_\eta^\beta \mathbf{a}^k(v, \eta)| &\lesssim \mathbf{a}^{k-1} (1 + |\eta| + |v||\eta| + |v||v \wedge \eta| + |v|^2) \\ &\lesssim \mathbf{a}^k(v, \eta), \end{aligned}$$

on the other hand, we have for all $k \in \mathbb{R}$,

$$\langle v \rangle^k \in S(\langle v \rangle^k, \Gamma). \quad (2.39)$$

Finally, using (2.38), (2.39) and Leibniz' formula we conclude for all $|\alpha + \beta| \geq 0$,

$$|\partial_v^\alpha \partial_\eta^\beta \lambda^m(v, \eta)| \leq C_{\alpha, \beta} \lambda^m(v, \eta). \quad (2.40)$$

\square

Lemma 2.13. For $m \in \mathbb{R}$,

$$|\xi \cdot \partial_\eta \lambda^m| \lesssim \lambda^m,$$

uniformly with respect to the parameter ξ in \mathbb{R}^3 .

Proof. Lemma 2.13 follows directly from (2.31) and the fact that

$$|\xi \cdot \partial_\eta \lambda^2| = \lambda \langle v \rangle^{\frac{\gamma}{2}} \mathbf{a}^{-1/2} |\xi \cdot \eta + v \wedge \xi \cdot v \wedge \eta| \lesssim \lambda^2,$$

we can conclude that for all $m \in \mathbb{R}$,

$$|\xi \cdot \partial_\eta \lambda^m| \lesssim \lambda^m,$$

uniformly with respect to the parameter ξ in \mathbb{R}^3 . \square

Lemma 2.14. *We have*

$$g_i \in S(g_i, \Gamma), \text{ for } i = 1, \dots, 3,$$

uniformly with respect to the parameter ξ in \mathbb{R}^3 .

Proof. Using the fact that for all $m \in \mathbb{R}$, $\langle v \rangle^m \in S(\langle v \rangle^m, \Gamma)$, $\langle \eta \rangle^m \in S(\langle \eta \rangle^m, \Gamma)$ we obtain

$$\forall \alpha, \beta \in \mathbb{N}^3, \quad |\partial_v^\alpha \partial_\eta^\beta g_i(v, \eta)| \leq C_{\alpha, \beta} g_i,$$

uniformly with respect to the parameter ξ in \mathbb{R}^3 . \square

Lemma 2.15. *For all $\varepsilon > 0$,*

- i) $\partial_\eta \lambda \in S(\varepsilon \lambda + \varepsilon^{-1} \langle v \rangle^{\frac{\gamma}{2}+1}, \Gamma)$,
- ii) $\partial_\eta g_1 \in S(\varepsilon g_1 + \varepsilon^{-1}, \Gamma)$,
- iii) $\partial_\eta g_2 \in S(\varepsilon g_2 + \varepsilon^{-1} \langle v \rangle^{\frac{\gamma}{6}+1}, \Gamma)$,
- iv) $\partial_\eta g_3 \in S(\varepsilon g_3 + \varepsilon^{-1} \langle v \rangle^{\frac{\gamma}{3}+1}, \Gamma)$;

uniformly with respect to the parameter ξ in \mathbb{R}^3 .

Proof. We have

$$\begin{aligned} |\partial_\eta g_1(v, \eta)| &= |\partial_\eta(1 + \langle v \rangle + \langle \eta \rangle)| \lesssim |\eta| \langle \eta \rangle^{-1} \lesssim g_1^{1/2}, \\ |\partial_\eta g_2(v, \eta)| &= |\partial_\eta(1 + \langle \xi \rangle^{1/3} + \langle \eta \rangle)| \lesssim |\eta| \langle \eta \rangle^{-1} \lesssim g_2^{1/2} \langle v \rangle^{\frac{\gamma}{12} + \frac{1}{2}}, \\ |\partial_\eta g_3(v, \eta)| &= |\partial_\eta(1 + \langle \xi \rangle^{2/3} + \langle \eta \rangle^2)| \lesssim \langle \eta \rangle \lesssim g_3^{1/2} \langle v \rangle^{\frac{\gamma}{6} + \frac{1}{2}}. \end{aligned}$$

Using Young's inequality we get for all $\varepsilon > 0$,

$$|\partial_\eta g_1(v, \eta)| \lesssim \varepsilon g_1 + \varepsilon^{-1}, \quad |\partial_\eta g_2(v, \eta)| \lesssim \varepsilon g_2 + \varepsilon^{-1} \langle v \rangle^{\frac{\gamma}{6}+1}, \quad |\partial_\eta g_3(v, \eta)| \lesssim \varepsilon g_3 + \varepsilon^{-1} \langle v \rangle^{\frac{\gamma}{3}+1},$$

then arguing as above we can use induction on $|\alpha| + |\beta|$ to obtain, for $|\alpha| + |\beta| \geq 0$,

$$|\partial_v^\alpha \partial_\eta^\beta \partial_\eta g_1(v, \eta)| \lesssim \varepsilon g_1 + \varepsilon^{-1}, \quad |\partial_v^\alpha \partial_\eta^\beta \partial_\eta g_2(v, \eta)| \lesssim \varepsilon g_2 + \varepsilon^{-1} \langle v \rangle^{\frac{\gamma}{6}+1},$$

$$\text{and } |\partial_v^\alpha \partial_\eta^\beta \partial_\eta g_3(v, \eta)| \lesssim \varepsilon g_3 + \varepsilon^{-1} \langle v \rangle^{\frac{\gamma}{3}+1}.$$

Regarding the symbol λ , we have

$$\begin{aligned} |\partial_\eta \lambda(v, \eta)| &= \langle v \rangle^{\frac{\gamma}{2}} \mathbf{a}^{-1/2} \left| \eta + (v \wedge \eta) \partial_\eta (v \wedge \eta) \right| \\ &\lesssim \langle v \rangle^{\frac{\gamma}{2}} \mathbf{a}^{-1/2} \left(|\eta| + |v| |v \wedge \eta| \right) \\ &\lesssim \langle v \rangle^{\frac{\gamma}{4} + \frac{1}{2}} \lambda^{1/2}. \end{aligned}$$

Using Young's inequality we get for all $\varepsilon > 0$,

$$|\partial_\eta \lambda(v, \eta)| \lesssim \varepsilon \lambda + \varepsilon^{-1} \langle v \rangle^{\frac{\gamma}{2} + 1},$$

then arguing as above we can use induction on $|\alpha| + |\beta|$ to obtain, for $|\alpha| + |\beta| \geq 0$,

$$|\partial_v^\alpha \partial_\eta^\beta \partial_\eta \lambda(v, \eta)| \lesssim \varepsilon \lambda + \varepsilon^{-1} \langle v \rangle^{\frac{\gamma}{2} + 1}.$$

□

From the above, we have shown that the symbols g_1 , g_2 , g_3 and λ verify the hypotheses of Theorem 2.40, so we can apply the results of Theorem 2.40 to the following operators

$$g_{1,K}^w = (g_1 + K)^w, \quad (2.41)$$

$$g_{2,K}^w = (g_2 + K \langle v \rangle^{\frac{\gamma}{6} + 1})^w, \quad (2.42)$$

$$g_{3,K}^w = (g_3 + K \langle v \rangle^{\frac{\gamma}{3} + 1})^w, \quad (2.43)$$

$$\lambda_K^w = (\lambda + K \langle v \rangle^{\frac{\gamma}{2} + 1})^w, \quad (2.44)$$

where K the fixed constant given by Theorem 2.40. In Section 3.2, we will apply the results of Theorem 2.40 on the operators above.

Let ψ be a $C_0^\infty(\mathbb{R}, [0, 1])$ function such that

$$\psi = 1 \text{ on } [-1, 1], \quad \text{supp } \psi \subset [-2, 2]. \quad (2.45)$$

Definition 2.16. Define the real-valued symbol

$$g = -\frac{B(v)\xi \cdot B(v)\eta}{\lambda^{4/3}} \psi \left(\frac{|B(v)\eta|^2 + F(v)}{\lambda^{2/3}} \right), \quad (2.46)$$

where λ is the symbol defined in (2.31).

Lemma 2.17 (Lemma 3.3 in [68]). *We have*

$$\psi \left(\frac{|B(v)\eta|^2 + F(v)}{\lambda^{2/3}} \right) \in S(1, \Gamma),$$

uniformly with respect to the parameter ξ in \mathbb{R}^3 .

Lemma 2.18. *The symbol g belongs to the class $S(1, \Gamma)$ uniformly with respect to the parameter ξ in \mathbb{R}^3 .*

Proof. Notice from (2.45) that

$$|B(v)\eta|^2 + F(v) \leq 2\lambda^{2/3}, \quad (2.47)$$

on the support of the function

$$\psi\left(\frac{|B(v)\eta|^2 + F(v)}{\lambda^{2/3}}\right). \quad (2.48)$$

By recalling (2.31) and using (2.15), we obtain

$$|B(v)\xi| \lesssim \langle v \rangle^{\frac{7}{2}} |\xi| + \langle v \rangle^{\frac{7}{2}} |v \wedge \xi| \lesssim \lambda. \quad (2.49)$$

We deduce from the Cauchy-Schwarz inequality that one can estimate

$$|B(v)\xi \cdot B(v)\eta| \lesssim |B(v)\xi| |B(v)\eta| \lesssim \lambda^{4/3}, \quad (2.50)$$

on the support of ψ . The symbol g is therefore a bounded function uniformly with respect to the parameter ξ in \mathbb{R}^3 . Using (2.14) and (2.15), we can estimate

$$|\partial_v^\alpha B(v)\xi| \lesssim \langle v \rangle^{\frac{7}{2}} |\xi| \lesssim |B(v)\xi| \lesssim \lambda \quad (2.51)$$

where $\alpha \in \mathbb{N}^3$ with $|\alpha| \geq 1$. Using again (2.14) and (2.15), we can estimate

$$|\partial_v^\alpha B(v)\eta| \lesssim \langle v \rangle^{\frac{7}{2}} |\eta| \lesssim |B(v)\eta| \lesssim \lambda^{1/3} \quad (2.52)$$

on the support of ψ . Moreover, one can estimate from above the modulus of all the derivatives of the term $B(v)\eta$ by a constant times $\lambda^{1/3}$ on the support of the function

$$\psi\left(\frac{|B(v)\eta|^2 + F(v)}{\lambda^{2/3}}\right).$$

Using Leibniz's formula, Cauchy-Schwarz inequality (2.51), (2.52) one can estimate from above the modulus of all the derivatives of the term $B(v)\xi \cdot B(v)\eta$ by a constant times $\lambda^{4/3}$ on the support of the function ψ . According to Lemma 2.12 and Lemma 2.17, this proves that the symbol g belongs to the class $S(1, \Gamma)$ uniformly with respect to the parameter ξ in \mathbb{R}^3 . \square

Lemma 2.19. *We have*

$$\left| \xi \cdot \partial_\eta \left[\psi\left(\frac{|B(v)\eta|^2 + F(v)}{\lambda^{2/3}}\right) \right] \right| \lesssim 1 + |B(v)\eta|^2 + F(v),$$

uniformly with respect to the parameter ξ in \mathbb{R}^3 .

Proof. Let $\omega = \frac{|B(v)\eta|^2 + F(v)}{\lambda^{2/3}}$. We may write

$$\xi \cdot \partial_\eta [\psi(\omega)] = \psi'(\omega) \left[\frac{2B(v)\xi \cdot B(v)\eta}{\lambda^{2/3}} + (|B(v)\eta|^2 + F(v))(\xi \cdot \partial_\eta)(\lambda^{-2/3}) \right].$$

Notice from (2.31) and (2.45)

$$\left| \frac{2B(v)\xi \cdot B(v)\eta}{\lambda^{2/3}} \right| \lesssim \frac{|B(v)\xi||B(v)\eta|}{\lambda^{2/3}} \lesssim \frac{\lambda\lambda^{1/3}}{\lambda^{2/3}} \lesssim \lambda^{2/3} \lesssim |B(v)\eta|^2 + F(v),$$

on the support of the function $\psi'(\omega)$. One can then deduce Lemma 2.19 from Lemma 2.13. \square

3.2 Hypoelliptic estimates

We shall consider the multiplier $G = g^{\text{Wick}}$ defined by the Wick quantization of the symbol g . We refer the reader to Appendix (Section 6.3) on Wick calculus. We begin by noticing from (2.133) that there exists a real-valued symbol \tilde{g} belonging to the class $S(1, \Gamma)$ uniformly with respect to the parameter ξ in \mathbb{R}^3 such that

$$G = g^{\text{Wick}} = \tilde{g}^w; \quad (2.53)$$

where \tilde{g}^w denotes the operator obtained by the Weyl quantization of the symbol \tilde{g} given by

$$(\tilde{g}^w u)(v) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^6} e^{i(v-v') \cdot \eta} \tilde{g}\left(\frac{v+v'}{2}, \eta\right) u(v') dv' d\eta. \quad (2.54)$$

We shall sometimes closely follow [68] and refer to appendix (Section 6) for the main features of the Wick and the Weyl quantizations.

We begin by a series of Lemmas whose proof is exactly the same as the one in [68].

Lemma 2.20. *There exists $c_1 > 0$ such that for all $u \in \mathcal{S}(\mathbb{R}_v^3)$,*

$$|(F(v)u, Gu)_{L^2}| + |(\nabla_v \cdot (\mathbf{A}(v)\nabla_v u), Gu)_{L^2}| \leq c_1 \Re(\mathcal{A}_\xi u, u)_{L^2},$$

uniformly with respect to the parameter ξ in \mathbb{R}^3 .

Proof. See Lemma 3.7 in [68]. \square

Lemma 2.21. *There exists $c_2 > 0$ such that for all $u \in \mathcal{S}(\mathbb{R}_v^3)$,*

$$\left| \|B(v)\nabla_v u\|_{L^2}^2 + \|\sqrt{F(v)}u\|_{L^2}^2 - \left([4\pi^2|B(v)\eta|^2 + F(v)]^{\text{Wick}} u, u \right)_{L^2} \right| \leq c_2 \Re(\mathcal{A}_\xi u, u)_{L^2},$$

uniformly with respect to the parameter ξ in \mathbb{R}^3 .

Proof. See Lemma 3.11 in [68]. \square

Let δ be a positive parameter such that $0 < \delta \leq 1$. We use a multiplier method and write that

$$\begin{aligned} \Re(\mathcal{A}_\xi u, (1 - \delta G)u)_{L^2} &= \|B(v)\nabla_v u\|_{L^2}^2 + \|\sqrt{F(v)}u\|_{L^2}^2 - \delta \Re(iv \cdot \xi u, Gu)_{L^2} \\ &\quad + \delta \Re(\nabla_v \cdot (\mathbf{A}(v)\nabla_v u), Gu)_{L^2} - \delta \Re(F(v)u, Gu)_{L^2}. \end{aligned} \quad (2.55)$$

Lemma 2.22. *We have for any $s \in \mathbb{R}$*

$$\|\langle B(v)\xi \rangle^s (1 - \delta G)u\|_{L^2} \lesssim \|\langle B(v)\xi \rangle^s u\|_{L^2},$$

uniformly with respect to the parameter ξ in \mathbb{R}^3 .

Proof. See Lemma 3.8 in [68]. \square

Proposition 2.23. *There exists $C > 0$ such that for all $u \in \mathcal{S}(\mathbb{R}_v^3)$,*

$$((\lambda^{2/3})^{\text{Wick}} u, u)_{L^2} \leq C \left(\Re(\mathcal{A}_\xi u, u)_{L^2} + \Re(\mathcal{A}_\xi u, (1 - \delta G)u)_{L^2} \right),$$

uniformly with respect to the parameter ξ in \mathbb{R}^3 .

Proof. Using (2.55), we have for $0 < \delta \leq 1$ and $u \in \mathcal{S}(\mathbb{R}_v^3)$,

$$\begin{aligned} &\|B(v)\nabla_v u\|_{L^2}^2 + \|\sqrt{F(v)}u\|_{L^2}^2 - \delta \Re(iv \cdot \xi u, Gu)_{L^2} \\ &= \Re(\mathcal{A}_\xi u, (1 - \delta G)u)_{L^2} - \delta \Re(\nabla_v \cdot (\mathbf{A}(v)\nabla_v u), Gu)_{L^2} + \delta \Re(F(v)u, Gu)_{L^2}, \end{aligned} \quad (2.56)$$

uniformly with respect to the parameter ξ in \mathbb{R}^3 . Recalling (2.53) and noticing from (2.133) and (2.134) that $v^{\text{Wick}} = v$, we may rewrite (2.56) as

$$\begin{aligned} &\|B(v)\nabla_v u\|_{L^2}^2 + \|\sqrt{F(v)}u\|_{L^2}^2 - \delta \Re(i\xi \cdot v^{\text{Wick}} u, Gu)_{L^2} \\ &= \Re(\mathcal{A}_\xi u, (1 - \delta G)u)_{L^2} - \delta \Re(\nabla_v \cdot (\mathbf{A}(v)\nabla_v u), Gu)_{L^2} + \delta \Re(F(v)u, Gu)_{L^2}. \end{aligned} \quad (2.57)$$

Using Lemma 2.20, we deduce that there is a constant $c_3 > 0$ such that for all $0 < \delta \leq 1$ and $u \in \mathcal{S}(\mathbb{R}_v^3)$,

$$\begin{aligned} &\|B(v)\nabla_v u\|_{L^2}^2 + \|\sqrt{F(v)}u\|_{L^2}^2 - \delta \Re(i\xi \cdot v^{\text{Wick}} u, Gu)_{L^2} \\ &\leq c_3 \left(\Re(\mathcal{A}_\xi u, u)_{L^2} + \Re(\mathcal{A}_\xi u, (1 - \delta G)u)_{L^2} \right), \end{aligned} \quad (2.58)$$

uniformly with respect to the parameter ξ in \mathbb{R}^3 . We deduce from Lemma 2.18 and (2.135) that

$$\begin{aligned} -\delta \Re e \left(i\xi \cdot v^{\text{Wick}} u, Gu \right)_{L^2} &= -\delta \Re e \left(g^{\text{Wick}} (i\xi \cdot v)^{\text{Wick}} u, u \right)_{L^2} \\ &= \delta \frac{1}{4\pi} \left(\{\xi \cdot v, g\}^{\text{Wick}} u, u \right)_{L^2}. \end{aligned} \quad (2.59)$$

Using (2.46) with a direct computation of the Poisson bracket gives that

$$\begin{aligned} \{\xi \cdot v, g\} &= (B(v)\xi \cdot B(v)\eta)(\xi \cdot \partial_\eta(\lambda^{-4/3}))\psi \left(\frac{|B(v)\eta|^2 + F(v)}{\lambda^{2/3}} \right) \\ &+ \frac{|B(v)\xi|^2}{\lambda^{4/3}} \psi \left(\frac{|B(v)\eta|^2 + F(v)}{\lambda^{2/3}} \right) + \frac{B(v)\xi \cdot B(v)\eta}{\lambda^{4/3}} \xi \cdot \partial_\eta \left[\psi \left(\frac{|B(v)\eta|^2 + F(v)}{\lambda^{2/3}} \right) \right]. \end{aligned} \quad (2.60)$$

We notice from Lemma 2.13, Lemma 2.19, (2.46) and (2.50) that

$$\left| \{\xi \cdot v, g\} - \frac{|B(v)\xi|^2}{\lambda^{4/3}} \psi \left(\frac{|B(v)\eta|^2 + F(v)}{\lambda^{2/3}} \right) \right| \lesssim 1 + |B(v)\eta|^2 + F(v), \quad (2.61)$$

uniformly with respect to the parameter ξ in \mathbb{R}^3 . It follows from (2.59), (2.58) and the fact that the Wick quantization is a positive quantization (2.132) that there exists constants $c_4 > 0$ such that for all $0 < \delta \leq 1$ and $u \in \mathcal{S}(\mathbb{R}_v^3)$,

$$\begin{aligned} &\frac{\delta}{4\pi} \left(\left[\frac{|B(v)\xi|^2}{\lambda^{4/3}} \psi \left(\frac{|B(v)\eta|^2 + F(v)}{\lambda^{2/3}} \right) \right]^{\text{Wick}} u, u \right)_{L^2} + \|B(v)\nabla_v u\|_{L^2}^2 + \|\sqrt{F(v)}u\|_{L^2}^2 \\ &\leq c_3 \left(\Re e (\mathcal{A}_\xi u, u)_{L^2} + \Re e (\mathcal{A}_\xi u, (1 - \delta G)u)_{L^2} \right) \\ &\quad + \delta c_4 \left(\left[1 + |B(v)\eta|^2 + F(v) \right]^{\text{Wick}} u, u \right)_{L^2}, \end{aligned} \quad (2.62)$$

uniformly with respect to the parameter ξ in \mathbb{R}^3 . It follows from Lemma 2.21 that there exists $c_5 > 0$ such that for all $0 < \delta \leq 1$ and $u \in \mathcal{S}(\mathbb{R}_v^3)$

$$\begin{aligned} &\delta \left(\left[\frac{|B(v)\xi|^2}{\lambda^{4/3}} \psi \left(\frac{|B(v)\eta|^2 + F(v)}{\lambda^{2/3}} \right) \right]^{\text{Wick}} u, u \right)_{L^2} + (4\pi^2 \left[|B(v)\eta|^2 + F(v) \right]^{\text{Wick}} u, u)_{L^2} \\ &\quad + \|u\|_{L^2}^2 \leq c_5 \left(\Re e (\mathcal{A}_\xi u, u)_{L^2} + \Re e (\mathcal{A}_\xi u, (1 - \delta G)u)_{L^2} \right), \end{aligned} \quad (2.63)$$

uniformly with respect to the parameter ξ in \mathbb{R}^3 .

Notice from (2.31), (2.45) and (2.15) that

$$\begin{aligned} & \delta \frac{|B(v)\xi|^2}{\lambda^{4/3}} \psi \left(\frac{|B(v)\eta|^2 + F(v)}{\lambda^{2/3}} \right) + 4\pi^2 |B(v)\eta|^2 + F(v) + 1 \geq \\ & \delta \frac{|B(v)\xi|^2 + |B(v)\eta|^2 + F(v) + 1}{\lambda^{4/3}} \psi \left(\frac{|B(v)\eta|^2 + F(v)}{\lambda^{2/3}} \right) \\ & + (|B(v)\eta|^2 + F(v) + 1) \left[1 - \psi \left(\frac{|B(v)\eta|^2 + F(v)}{\lambda^{2/3}} \right) \right] \gtrsim \\ & \delta \frac{\lambda^2}{\lambda^{4/3}} \psi \left(\frac{|B(v)\eta|^2 + F(v)}{\lambda^{2/3}} \right) + \delta \lambda^{2/3} \left[1 - \psi \left(\frac{|B(v)\eta|^2 + F(v)}{\lambda^{2/3}} \right) \right] \gtrsim \delta \lambda^{2/3}, \end{aligned}$$

when $0 < \frac{\delta}{\lambda^{4/3}} \leq 1$; since

$$|B(v)\eta|^2 + F(v) \geq \lambda^{2/3},$$

on the support of the function

$$1 - \psi \left(\frac{|B(v)\eta|^2 + F(v)}{\lambda^{2/3}} \right).$$

By using again that the Wick quantization is a positive quantization (2.132), we deduce that there exists $C > 0$ such that for all $0 < \delta \leq 1$ and $u \in \mathcal{S}(\mathbb{R}_v^3)$,

$$((\lambda^{2/3})^{\text{Wick}} u, u)_{L^2} \leq C \left(\Re e(\mathcal{A}_\xi u, u)_{L^2} + \Re e(\mathcal{A}_\xi u, (1 - \delta G)u)_{L^2} \right),$$

uniformly with respect to the parameter ξ in \mathbb{R}^3 . □

Lemma 2.24. *There exists $C > 0$ such that for all $v \in \mathbb{R}^3$ and $\xi \in \mathbb{R}^3$,*

$$\langle B(v)\xi \rangle^{1/3} \leq C m(v, \xi),$$

with

$$m(v, \xi) = \left(\int_{\mathbb{R}^3} \langle B(v + \tilde{v})\xi \rangle^{2/3} \pi^{-3} e^{-|\tilde{v}|^2} d\tilde{v} \right)^{1/2}.$$

Proof. We have that

$$m(v, \xi)^2 \gtrsim \int_{\mathbb{R}^3} (1 + |B(v + \tilde{v})\xi|^{2/3}) \pi^{-3} e^{-|\tilde{v}|^2} d\tilde{v},$$

therefore by using (2.15), we obtain that

$$m(v, \xi)^2 \gtrsim \int_{\mathbb{R}^3} (1 + \langle v + \tilde{v} \rangle^{\gamma/3} |\xi|^{2/3} + \langle v + \tilde{v} \rangle^{\gamma/3} |(v + \tilde{v}) \wedge \xi|^{2/3}) \pi^{-3} e^{-|\tilde{v}|^2} d\tilde{v},$$

and using Peetre's inequality (2.37), we have

$$\frac{\langle v \rangle^{\gamma/3}}{\langle \tilde{v} \rangle^{\gamma/3}} \lesssim \langle v + \tilde{v} \rangle^{\gamma/3},$$

so we get

$$\begin{aligned} m(v, \xi)^2 &\gtrsim 1 + \langle v \rangle^{\gamma/3} |\xi|^{2/3} + \int_{\mathbb{R}^3} \frac{\langle v \rangle^{\gamma/3}}{\langle \tilde{v} \rangle^{\gamma/3}} |(v + \tilde{v}) \wedge \xi|^{2/3} \pi^{-3} e^{-|\tilde{v}|^2} d\tilde{v} \\ &\gtrsim 1 + \langle v \rangle^{\gamma/3} |\xi|^{2/3} + \int_{B(0,1)} \frac{\langle v \rangle^{\gamma/3}}{\langle \tilde{v} \rangle^{\gamma/3}} |(v + \tilde{v}) \wedge \xi|^{2/3} \pi^{-3} e^{-|\tilde{v}|^2} d\tilde{v} \end{aligned}$$

where $\overline{B(0,1)}$ stands for the closed unit ball in \mathbb{R}^3 . By noticing that we have

$$|(v + \tilde{v}) \wedge \xi| \geq |v \wedge \xi| - |\tilde{v} \wedge \xi| \geq |v \wedge \xi| - |\xi| \geq \frac{1}{2}|v \wedge \xi|,$$

when $|\tilde{v}| \leq 1$ and $2|\xi| \leq |v \wedge \xi|$, it follows that

$$\int_{\mathbb{R}^3} \frac{\langle v \rangle^{\gamma/3}}{\langle \tilde{v} \rangle^{\gamma/3}} |(v + \tilde{v}) \wedge \xi|^{2/3} \pi^{-3} e^{-|\tilde{v}|^2} d\tilde{v} \gtrsim \langle v \rangle^{\gamma/3} |v \wedge \xi|^{2/3},$$

when $2|\xi| \leq |v \wedge \xi|$. Since

$$\langle v \rangle^{\gamma/3} |\xi|^{2/3} \gtrsim \langle v \rangle^{\gamma/3} |v \wedge \xi|^{2/3}$$

when $2|\xi| \geq |v \wedge \xi|$. Based on the above and using (2.15), we obtain

$$m(v, \xi)^2 \gtrsim 1 + \langle v \rangle^{\gamma/3} |\xi|^{2/3} + \langle v \rangle^{\gamma/3} |v \wedge \xi|^{2/3} \gtrsim 1 + |B(v)\xi|^{2/3} \gtrsim \langle B(v)\xi \rangle^{2/3}.$$

□

Lemma 2.25. For all $u \in \mathcal{S}(\mathbb{R}_v^3)$,

$$\|\langle v \rangle^{\frac{5\gamma}{6}+2} u\|_{L^2}^2 + \|\langle v \rangle^{\frac{\gamma}{3}+1} B(v)D_v u\|_{L^2}^2 \lesssim \|\mathcal{A}_\xi u\|_{L^2}^2 + \|u\|_{L^2}^2. \quad (2.64)$$

Proof. See Lemma 3.3 in [66].

□

For the rest, we need to improve this estimate. We have the following Lemma.

Lemma 2.26. For all $u \in \mathcal{S}(\mathbb{R}_v^3)$,

$$\|\langle v \rangle^{\gamma+2} u\|_{L^2}^2 + \|\langle v \rangle^{\frac{\gamma}{2}+1} B(v)D_v u\|_{L^2}^2 \lesssim \|\mathcal{A}_\xi u\|_{L^2}^2 + \|u\|_{L^2}^2. \quad (2.65)$$

Proof. We will start by estimating the term

$$\left([\mathcal{A}_\xi, \langle v \rangle^{\frac{\gamma}{2}+1}] u, \langle v \rangle^{\frac{\gamma}{2}+1} u \right)_{L^2}.$$

Let $u \in \mathcal{S}(\mathbb{R}_v^3)$, we have

$$\begin{aligned} \left| \left(\left[\mathcal{A}_\xi, \langle v \rangle^{\frac{\gamma}{2}+1} \right] u, \langle v \rangle^{\frac{\gamma}{2}+1} u \right)_{L^2} \right| &\leq \left| \left(B(v) \left[D_v, \langle v \rangle^{\frac{\gamma}{2}+1} \right] u, B(v) D_v \langle v \rangle^{\frac{\gamma}{2}+1} u \right)_{L^2} \right| \\ &\quad + \left| \left(B(v) D_v u, B(v) \left[D_v, \langle v \rangle^{\frac{\gamma}{2}+1} \right] \langle v \rangle^{\frac{\gamma}{2}+1} u \right)_{L^2} \right| \\ &\leq \left| \left(B(v) \left[D_v, \langle v \rangle^{\frac{\gamma}{2}+1} \right] u, B(v) \left[D_v, \langle v \rangle^{\frac{\gamma}{2}+1} \right] u \right)_{L^2} \right| \\ &\quad + \left| \left(B(v) \left[D_v, \langle v \rangle^{\frac{\gamma}{2}+1} \right] u, \langle v \rangle^{\frac{\gamma}{2}+1} B(v) D_v u \right)_{L^2} \right| \\ &\quad + \left| \left(B(v) D_v u, B(v) \left[D_v, \langle v \rangle^{\frac{\gamma}{2}+1} \right] \langle v \rangle^{\frac{\gamma}{2}+1} u \right)_{L^2} \right|, \end{aligned}$$

using the fact that the symbol of the operator $\left[D_v, \langle v \rangle^{\frac{\gamma}{2}+1} \right]$ belongs to the class $S(\langle v \rangle^{\frac{\gamma}{2}}, \Gamma)$ uniformly with respect to the parameter ξ and using the following notations

$$\begin{aligned} H_1 &= \underbrace{B(v) \left[D_v, \langle v \rangle^{\frac{\gamma}{2}+1} \right] \langle v \rangle^{-\gamma-1}}_{\in \mathcal{B}(L^2)}, \\ H_2 &= \underbrace{\langle v \rangle^{-\frac{\gamma}{2}-1} B(v) \left[D_v, \langle v \rangle^{\frac{\gamma}{2}+1} \right] \langle v \rangle^{-\frac{\gamma}{2}}}_{\in \mathcal{B}(L^2)}, \end{aligned}$$

we obtain

$$\begin{aligned} \left| \left(\left[\mathcal{A}_\xi, \langle v \rangle^{\frac{\gamma}{2}+1} \right] u, \langle v \rangle^{\frac{\gamma}{2}+1} u \right)_{L^2} \right| &\leq \left| \left(H_1 \langle v \rangle^{\gamma+1} u, H_1 \langle v \rangle^{\gamma+1} u \right)_{L^2} \right| \\ &\quad + \left| \left(H_1 \langle v \rangle^{\gamma+1} u, \langle v \rangle^{\frac{\gamma}{2}+1} B(v) D_v u \right)_{L^2} \right| \\ &\quad + \left| \left(\langle v \rangle^{\frac{\gamma}{2}+1} B(v) D_v u, H_2 \langle v \rangle^{\frac{\gamma}{2}+1} u \right)_{L^2} \right|, \end{aligned}$$

using the fact that $\gamma + 1 \leq 2 + \frac{5\gamma}{6}$ and Lemma 2.25, we obtain for all $\varepsilon > 0$

$$\begin{aligned} \left| \left(\left[\mathcal{A}_\xi, \langle v \rangle^{\frac{\gamma}{2}+1} \right] u, \langle v \rangle^{\frac{\gamma}{2}+1} u \right)_{L^2} \right| &\lesssim \varepsilon \left(\|\langle v \rangle^{\gamma+2} u\|_{L^2}^2 + \|\langle v \rangle^{\frac{\gamma}{2}+1} B(v) D_v u\|_{L^2}^2 \right) \\ &\quad + C_\varepsilon \left(\|\mathcal{A}_\xi u\|_{L^2}^2 + \|u\|_{L^2}^2 \right). \end{aligned}$$

On the other hand, we have

$$\left| \left(\mathcal{A}_\xi \langle v \rangle^{\frac{\gamma}{2}+1} u, \langle v \rangle^{\frac{\gamma}{2}+1} u \right)_{L^2} \right| \leq \left| \left(\mathcal{A}_\xi u, \langle v \rangle^{\gamma+2} u \right)_{L^2} \right| + \left| \left(\left[\mathcal{A}_\xi, \langle v \rangle^{\frac{\gamma}{2}+1} \right] u, \langle v \rangle^{\frac{\gamma}{2}+1} u \right)_{L^2} \right|$$

and

$$\left| \left(\mathcal{A}_\xi u, \langle v \rangle^{\gamma+2} u \right)_{L^2} \right| \leq \varepsilon \|\langle v \rangle^{\gamma+2} u\|_{L^2}^2 + C_\varepsilon \|\mathcal{A}_\xi u\|_{L^2}^2.$$

Now using (2.30), we can write

$$\begin{aligned} &\|\langle v \rangle^{\gamma+2} u\|_{L^2}^2 + \|\langle v \rangle^{\frac{\gamma}{2}+1} B(v) D_v u\|_{L^2}^2 \\ &\leq \|\langle v \rangle^{\frac{\gamma}{2}+1} \langle v \rangle^{\frac{\gamma}{2}+1} u\|_{L^2}^2 + \|B(v) D_v \langle v \rangle^{\frac{\gamma}{2}+1} u\|_{L^2}^2 + \|B(v) \left[D_v, \langle v \rangle^{\frac{\gamma}{2}+1} \right] u\|_{L^2}^2 \\ &\leq \left| \left(\mathcal{A}_\xi \langle v \rangle^{\frac{\gamma}{2}+1} u, \langle v \rangle^{\frac{\gamma}{2}+1} u \right)_{L^2} \right| + \|B(v) \left[D_v, \langle v \rangle^{\frac{\gamma}{2}+1} \right] u\|_{L^2}^2. \end{aligned}$$

For the last term, we have

$$\|B(v) [D_v, \langle v \rangle^{\frac{\gamma}{2}+1}] u\|_{L^2}^2 \leq \|H_1 \langle v \rangle^{\gamma+1} u\|_{L^2}^2 \lesssim \varepsilon \|\langle v \rangle^{\gamma+2} u\|_{L^2}^2 + C_\varepsilon (\|\mathcal{A}_\xi u\|_{L^2}^2 + \|u\|_{L^2}^2).$$

Finally, taking ε small enough, we obtain for all $u \in \mathcal{S}(\mathbb{R}_v^3)$,

$$\|\langle v \rangle^{\gamma+2} u\|_{L^2}^2 + \|\langle v \rangle^{\frac{\gamma}{2}+1} B(v) D_v u\|_{L^2}^2 \lesssim \|\mathcal{A}_\xi u\|_{L^2}^2 + \|u\|_{L^2}^2. \quad (2.66)$$

□

Proposition 2.27. *There exists $C > 0$ such that for all $u \in \mathcal{S}(\mathbb{R}_v^3)$,*

$$\|\langle v \rangle^{\frac{\gamma}{3}} |\xi|^{2/3} u\|_{L^2}^2 \leq C (\|\mathcal{A}_\xi u\|_{L^2}^2 + \|u\|_{L^2}^2),$$

uniformly with respect to the parameter ξ in \mathbb{R}^3 .

Proof. We deduce from Proposition 2.23, Lemma 2.24 and Lemma 2.22 that

$$\|\langle B(v)\xi \rangle^{1/3} u\|_{L^2} \lesssim \|\langle B(v)\xi \rangle^{-1/3} \mathcal{A}_\xi u\|_{L^2} \|\langle B(v)\xi \rangle^{1/3} u\|_{L^2},$$

uniformly with respect to the parameter ξ in \mathbb{R}^3 . By substituting $\langle B(v)\xi \rangle^{1/3} u$ to u in this estimate, we obtain that

$$\|\langle B(v)\xi \rangle^{2/3} u\|_{L^2}^2 \lesssim \|\langle B(v)\xi \rangle^{-1/3} \mathcal{A}_\xi \langle B(v)\xi \rangle^{1/3} u\|_{L^2} \|\langle B(v)\xi \rangle^{2/3} u\|_{L^2}, \quad (2.67)$$

uniformly with respect to the parameter ξ in \mathbb{R}^3 . First, we have

$$\begin{aligned} & \langle B(v)\xi \rangle^{-1/3} [D_v \cdot \mathbf{A}(v) D_v, \langle B(v)\xi \rangle^{1/3}] \\ &= \langle B(v)\xi \rangle^{-1/3} \sum_{j,k=1}^3 D_{v_j} \cdot \bar{a}_{jk}(v) [D_{v_k}, \langle B(v)\xi \rangle^{1/3}] \\ & \quad + \langle B(v)\xi \rangle^{-1/3} \sum_{j,k=1}^3 [D_{v_j}, \langle B(v)\xi \rangle^{1/3}] \cdot \bar{a}_{jk}(v) D_{v_k} \\ &= \langle B(v)\xi \rangle^{-1/3} \sum_{j,k=1}^3 D_{v_j} \bar{a}_{jk}(v) [D_{v_k}, \langle B(v)\xi \rangle^{1/3}] \\ & \quad + \langle B(v)\xi \rangle^{-1/3} \sum_{j,k=1}^3 \bar{a}_{jk}(v) [D_{v_k}, \langle B(v)\xi \rangle^{1/3}] D_{v_j} \\ &+ \langle B(v)\xi \rangle^{-1/3} \sum_{j,k=1}^3 \bar{a}_{jk}(v) D_{v_j} \left([D_{v_k}, \langle B(v)\xi \rangle^{1/3}] \right) \\ & \quad + \langle B(v)\xi \rangle^{-1/3} \sum_{j,k=1}^3 [D_{v_j}, \langle B(v)\xi \rangle^{1/3}] \cdot \bar{a}_{jk}(v) D_{v_k}. \end{aligned}$$

Symbolic calculus shows that

$$\left[D_v, \langle B(v)\xi \rangle^{1/3} \right] = \frac{1}{i} \nabla_v (\langle B(v)\xi \rangle^{1/3}),$$

and using the following notations

$$\begin{aligned} H_3 &= \underbrace{\langle B(v)\xi \rangle^{-1/3} \partial_{v_j} \bar{a}_{jk}(v) \partial_{v_k} \langle B(v)\xi \rangle^{1/3} \langle v \rangle^{-\gamma-1}}_{\in \mathcal{B}(L^2)}, \\ H_4 &= \underbrace{\langle B(v)\xi \rangle^{-1/3} \bar{a}_{jk}(v) \partial_{v_j} \partial_{v_k} \langle B(v)\xi \rangle^{1/3} \langle v \rangle^{-\gamma-2}}_{\in \mathcal{B}(L^2)}, \\ H_5 &= \underbrace{\langle B(v)\xi \rangle^{-1/3} b_{ik}(v) \partial_{v_k} \langle B(v)\xi \rangle^{1/3} \langle v \rangle^{-\frac{\gamma}{2}-1}}_{\in \mathcal{B}(L^2)}, \\ H_6 &= \underbrace{\langle B(v)\xi \rangle^{-1/3} b_{ij}(v) \partial_{v_j} \langle B(v)\xi \rangle^{1/3} \langle v \rangle^{-\frac{\gamma}{2}-1}}_{\in \mathcal{B}(L^2)}, \end{aligned}$$

where we used the fact that $\mathbf{A}(v) = B^T(v)B(v)$. Then, going back to (2.67), we have

$$\begin{aligned} & \| \langle B(v)\xi \rangle^{2/3} u \|_{L^2}^2 \\ & \lesssim \| \langle B(v)\xi \rangle^{-1/3} \mathcal{A}_\xi \langle B(v)\xi \rangle^{1/3} u \|_{L^2} \| \langle B(v)\xi \rangle^{2/3} u \|_{L^2}, \\ & \lesssim \| \langle B(v)\xi \rangle^{-1/3} [\mathcal{A}_\xi, \langle B(v)\xi \rangle^{1/3}] u \|_{L^2}^2 + \| \mathcal{A}_\xi u \|_{L^2}^2, \\ & \lesssim \| \mathcal{A}_\xi u \|_{L^2}^2 + \| H_3 \langle v \rangle^{\gamma+1} u \|_{L^2}^2 + \| H_4 \langle v \rangle^{\gamma+2} u \|_{L^2}^2 \\ & + \sum_{i=1}^3 \| H_5 \left(\sum_{j=1}^3 \langle v \rangle^{\frac{\gamma}{2}+1} b_{ij}(v) D_{v_j} u \right) \|_{L^2}^2 + \sum_{i=1}^3 \| H_6 \left(\sum_{k=1}^3 \langle v \rangle^{\frac{\gamma}{2}+1} b_{ik}(v) D_{v_k} u \right) \|_{L^2}^2, \\ & \lesssim \| \mathcal{A}_\xi u \|_{L^2}^2 + \| \langle v \rangle^{\gamma+1} u \|_{L^2}^2 + \| \langle v \rangle^{\gamma+2} u \|_{L^2}^2 + \| \langle v \rangle^{\frac{\gamma}{2}+1} B(v) D_v u \|_{L^2}^2 \end{aligned}$$

We finally conclude from Lemma 2.26 that for all $u \in \mathcal{S}(\mathbb{R}_v^3)$,

$$\| \langle B(v)\xi \rangle^{2/3} u \|_{L^2}^2 \lesssim \| \mathcal{A}_\xi u \|_{L^2}^2 + \| u \|_{L^2}^2.$$

This ends the proof of Proposition 2.27. \square

Lemma 2.28. *Let λ_K be the symbol defined in (2.44). Then for any $\tilde{\varepsilon} > 0$ there exists a constant $C_{\tilde{\varepsilon}}$, such that for all $u \in \mathcal{S}(\mathbb{R}_v^3)$,*

$$\begin{aligned} & \Re e \left(\mathcal{A}_\xi \left(\lambda_K^{1/3} \right)^w u, \left(\lambda_K^{1/3} \right)^w u \right)_{L^2} + \Re e \left(\mathcal{A}_\xi \left(\lambda_K^{1/3} \right)^w u, (1 - \delta G) \left(\lambda_K^{1/3} \right)^w u \right)_{L^2} \\ & \lesssim \tilde{\varepsilon} \| \left(\lambda_K^{2/3} \right)^w u \|_{L^2}^2 + C_{\tilde{\varepsilon}} \left(\| \mathcal{A}_\xi u \|_{L^2}^2 + \| u \|_{L^2}^2 \right). \end{aligned} \quad (2.68)$$

Proof. As a preliminary step we firstly show that for any $\varepsilon, \tilde{\varepsilon} > 0$ there exists a constant $C_{\varepsilon, \tilde{\varepsilon}}$, such that

$$\begin{aligned} & \Re e \left(\left[\mathcal{A}_\xi, \left(\lambda_K^{1/3} \right)^w \right] u, a^w \left(\lambda_K^{1/3} \right)^w u \right)_{L^2} \\ & \lesssim \tilde{\varepsilon} \left\| \left(\lambda_K^{2/3} \right)^w u \right\|_{L^2}^2 + C_{\varepsilon, \tilde{\varepsilon}} \left(\left\| \mathcal{A}_\xi u \right\|_{L^2}^2 + \|u\|_{L^2}^2 \right) \\ & + \varepsilon \left\{ \Re e \left(\mathcal{A}_\xi \left(\lambda_K^{1/3} \right)^w u, \left(\lambda_K^{1/3} \right)^w u \right)_{L^2} + \Re e \left(\mathcal{A}_\xi \left(\lambda_K^{1/3} \right)^w u, (1 - \delta G) \left(\lambda_K^{1/3} \right)^w u \right)_{L^2} \right\}, \end{aligned} \quad (2.69)$$

where a is an arbitrary symbol belonging to $S(1, \Gamma)$ uniformly with respect to the parameter ξ . Using the notation

$$\mathcal{Z}_1 = \left(\left[D_v \cdot \mathbf{A}(v) D_v, \left(\lambda_K^{1/3} \right)^w \right] u, a^w \left(\lambda_K^{1/3} \right)^w u \right)_{L^2}.$$

We have

$$\begin{aligned} \mathcal{Z}_1 &= \sum_{i,j,k=1}^3 \left(\left[b_{ik}(v) D_{v_k}, \left(\lambda_K^{1/3} \right)^w \right] u, b_{ij}(v) D_{v_j} a^w \left(\lambda_K^{1/3} \right)^w u \right)_{L^2} \\ &+ \sum_{i,j,k=1}^3 \left(b_{ik} D_{v_k} u, \left[\left(\lambda_K^{1/3} \right)^w, b_{ij}(v) D_{v_j} \right] a^w \left(\lambda_K^{1/3} \right)^w u \right)_{L^2}. \end{aligned}$$

Using the fact that

$$\left[B(v) D_v, \left(\lambda_K^{1/3} \right)^w \right] u = B(v) \left[D_v, \left(\lambda_K^{1/3} \right)^w \right] u - \left[\left(\lambda_K^{1/3} \right)^w, B(v) \right] D_v u,$$

we obtain

$$\begin{aligned} |\mathcal{Z}_1| &\leq \left| \sum_{i,j,k=1}^3 \left(b_{ik}(v) \left[D_{v_k}, \left(\lambda_K^{1/3} \right)^w \right] u, b_{ij}(v) D_{v_j} a^w \left(\lambda_K^{1/3} \right)^w u \right)_{L^2} \right| \\ &+ \left| \sum_{i,j,k=1}^3 \left(\left[\left(\lambda_K^{1/3} \right)^w, b_{ik}(v) \right] D_{v_k} u, b_{ij}(v) D_{v_j} a^w \left(\lambda_K^{1/3} \right)^w u \right)_{L^2} \right| \\ &+ \left| \sum_{i,j,k=1}^3 \left(b_{ik}(v) D_{v_k} u, b_{ij}(v) \left[D_{v_j}, \left(\lambda_K^{1/3} \right)^w \right] a^w \left(\lambda_K^{1/3} \right)^w u \right)_{L^2} \right| \\ &+ \left| \sum_{i,j,k=1}^3 \left(b_{ik}(v) D_{v_k} u, \left[\left(\lambda_K^{1/3} \right)^w, b_{ij}(v) \right] D_{v_j} a^w \left(\lambda_K^{1/3} \right)^w u \right)_{L^2} \right| \\ &\leq |\mathcal{Z}_{1,1}| + |\mathcal{Z}_{1,2}| + |\mathcal{Z}_{1,3}| + |\mathcal{Z}_{1,4}|. \end{aligned}$$

Estimate of $\mathcal{Z}_{1,1}$: Observing $a \in S(1, \Gamma)$, $\partial_v \lambda_K^{1/3} \in S(\langle v \rangle^{\gamma/6} g_{2,K}, \Gamma)$ and using (2.14) with symbolic calculus shows that

$$\left[D_{v_j}, a^w \right] \in \Psi(1, \Gamma), \quad [b_{ij}(v), a^w] \in \Psi(\langle v \rangle^{\frac{\gamma}{2}}, \Gamma) \quad \text{and} \quad \left[D_{v_k}, \left(\lambda_K^{1/3} \right)^w \right] \in \Psi(\langle v \rangle^{\frac{\gamma}{6}} g_{2,K}, \Gamma)$$

uniformly with respect to the parameter ξ , where $g_{2,K}^w$ the operator defined in (2.42). Now using the following notation

$$H_7 = \underbrace{b_{ik}(v) \left[D_{v_k}, \left(\lambda_K^{1/3} \right)^w \right] \left(g_{2,K}^w \right)^{-1} \langle v \rangle^{-\gamma/6} \langle v \rangle^{-\frac{\gamma}{2}-1}}_{\in \mathcal{B}(L^2)},$$

we obtain

$$\begin{aligned} |\mathcal{Z}_{1,1}| &\leq \left| \sum_{i,j,k=1}^3 \left(H_7 \langle v \rangle^{\frac{2\gamma}{3}+1} g_{2,K}^w u, b_{ij}(v) \left[D_{v_j}, a^w \right] \langle v \rangle^{-\frac{\gamma}{2}-1} \langle v \rangle^{\frac{\gamma}{2}+1} \left(\lambda_K^{1/3} \right)^w u \right) \right|_{L^2} \\ &+ \left| \sum_{i,j,k=1}^3 \left(H_7 \langle v \rangle^{\frac{2\gamma}{3}+1} g_{2,K}^w u, [b_{ij}(v), a^w] \langle v \rangle^{-\frac{\gamma}{2}} \langle v \rangle^{\frac{\gamma}{2}} D_{v_j} \left(\lambda_K^{1/3} \right)^w u \right) \right|_{L^2} \\ &+ \left| \sum_{i,j,k=1}^3 \left(H_7 \langle v \rangle^{\frac{2\gamma}{3}+1} g_{2,K}^w u, a^w b_{ij}(v) D_{v_j} \left(\lambda_K^{1/3} \right)^w u \right) \right|_{L^2} \\ &\lesssim C_\varepsilon \left(\|\langle v \rangle^{\gamma+2} u\|_{L^2}^2 + \|\langle v \rangle^{\gamma+1} D_v u\|_{L^2}^2 + \|\langle v \rangle^{\frac{\gamma}{3}} |\xi|^{2/3} u\|_{L^2}^2 \right) \\ &+ \varepsilon \Re e \left(\mathcal{A}_\xi \left(\lambda_K^{1/3} \right)^w u, \left(\lambda_K^{1/3} \right)^w u \right)_{L^2}. \end{aligned}$$

Moreover using Lemma 2.26 and Proposition 2.27, we obtain

$$|\mathcal{Z}_{1,1}| \lesssim \varepsilon \Re e \left(\mathcal{A}_\xi \left(\lambda_K^{1/3} \right)^w u, \left(\lambda_K^{1/3} \right)^w u \right)_{L^2} + C_\varepsilon \left(\|\mathcal{A}_\xi u\|_{L^2}^2 + \|u\|_{L^2}^2 \right).$$

Estimate of $\mathcal{Z}_{1,2}$: Observing $\partial_\eta \lambda_K^{1/3} \in S(\langle v \rangle^{\frac{2}{6}+\frac{1}{3}}, \Gamma)$ and using (2.14) with symbolic calculus shows that the symbol of the commutator $\left[\left(\lambda_K^{1/3} \right)^w, b_{ik}(v) \right]$ belongs to $S(\langle v \rangle^{\frac{2\gamma}{3}+\frac{1}{3}}, \Gamma)$ uniformly with respect to the parameter ξ . Now using the following notation

$$H_8 = \underbrace{\left[\left(\lambda_K^{1/3} \right)^w, b_{ik}(v) \right] \langle v \rangle^{-\frac{2\gamma}{3}-\frac{1}{3}}}_{\in \mathcal{B}(L^2)},$$

we obtain

$$\begin{aligned} |\mathcal{Z}_{1,2}| &\leq \left| \sum_{i,j,k=1}^3 \left(H_8 \langle v \rangle^{\frac{2\gamma}{3}+\frac{1}{3}} D_{v_k} u, b_{ij}(v) \left[D_{v_j}, a^w \right] \langle v \rangle^{-\frac{\gamma}{2}-1} \langle v \rangle^{\frac{\gamma}{2}+1} \left(\lambda_K^{1/3} \right)^w u \right) \right|_{L^2} \\ &+ \left| \sum_{i,j,k=1}^3 \left(H_8 \langle v \rangle^{\frac{2\gamma}{3}+\frac{1}{3}} D_{v_k} u, [b_{ij}(v), a^w] \langle v \rangle^{-\frac{\gamma}{2}} \langle v \rangle^{\frac{\gamma}{2}} D_{v_j} \left(\lambda_K^{1/3} \right)^w u \right) \right|_{L^2} \\ &+ \left| \sum_{i,j,k=1}^3 \left(H_8 \langle v \rangle^{\frac{2\gamma}{3}+\frac{1}{3}} D_{v_k} u, a^w b_{ij}(v) D_{v_j} \left(\lambda_K^{1/3} \right)^w u \right) \right|_{L^2} \\ &\lesssim C_\varepsilon \|\langle v \rangle^{\gamma+1} D_v u\|_{L^2}^2 + \varepsilon \Re e \left(\mathcal{A}_\xi \left(\lambda_K^{1/3} \right)^w u, \left(\lambda_K^{1/3} \right)^w u \right)_{L^2}, \end{aligned}$$

moreover using Proposition 2.27, we obtain

$$|\mathcal{Z}_{1,2}| \lesssim \varepsilon \Re e \left(\mathcal{A}_\xi \left(\lambda_K^{1/3} \right)^w u, \left(\lambda_K^{1/3} \right)^w u \right)_{L^2} + C_\varepsilon \left(\|\mathcal{A}_\xi u\|_{L^2}^2 + \|u\|_{L^2}^2 \right).$$

Estimate of $\mathcal{Z}_{1,3}$: Using the following notation

$$H_9 = \underbrace{\langle v \rangle^{-\frac{\gamma}{2}-1} b_{ij}(v) \left[D_{v_j}, \left(\lambda_K^{1/3} \right)^w \right] a^w \left(g_{2,K}^w \right)^{-1} \langle v \rangle^{-\gamma/6}}_{\in \mathcal{B}(L^2)},$$

we obtain

$$\begin{aligned} |\mathcal{Z}_{1,3}| &\leq \left| \sum_{i,j,k=1}^3 \left(\langle v \rangle^{\frac{\gamma}{2}+1} b_{ik} D_{v_k} u, H_9 \langle v \rangle^{\frac{\gamma}{6}} g_{2,K}^w \left(\lambda_K^{1/3} \right)^w u \right)_{L^2} \right| \\ &\lesssim \left| \sum_{i,j,k=1}^3 \left(\langle v \rangle^{\frac{\gamma}{2}+1} b_{ik} D_{v_k} u, \langle v \rangle^{\frac{\gamma}{6}} g_{2,K}^w \left(\lambda_K^{1/3} \right)^w u \right)_{L^2} \right| \\ &\lesssim \varepsilon \left(\|\langle v \rangle^{\frac{\gamma}{6}} \left(\lambda_K^{1/3} \right)^w u\|_{L^2}^2 + \|\langle v \rangle^{\frac{\gamma}{6}} D_v \left(\lambda_K^{1/3} \right)^w u\|_{L^2}^2 + \|\langle v \rangle^{\frac{\gamma}{6}} |\xi|^{1/3} \left(\lambda_K^{1/3} \right)^w u\|_{L^2}^2 \right) \\ &\quad + C_\varepsilon \|\langle v \rangle^{\frac{\gamma}{2}+1} B(v) D_v u\|_{L^2}^2, \end{aligned}$$

moreover using Lemma 2.26 and Proposition 2.23, we obtain

$$\begin{aligned} |\mathcal{Z}_{1,3}| &\lesssim \varepsilon \Re e \left(\mathcal{A}_\xi \left(\lambda_K^{1/3} \right)^w u, \left(\lambda_K^{1/3} \right)^w u \right)_{L^2} + \varepsilon \Re e \left(\mathcal{A}_\xi \left(\lambda_K^{1/3} \right)^w u, (1 - \delta G) \left(\lambda_K^{1/3} \right)^w u \right)_{L^2} \\ &\quad + C_\varepsilon \left(\|\mathcal{A}_\xi u\|_{L^2}^2 + \|u\|_{L^2}^2 \right). \end{aligned}$$

Estimate of $\mathcal{Z}_{1,4}$: Using the following notations

$$H_{10} = \underbrace{\langle v \rangle^{-\frac{\gamma}{3}-1} \left[\left(\lambda_K^{1/3} \right)^w, b_{ij}(v) \right] \left[D_{v_j}, a^w \right] \langle v \rangle^{-\frac{\gamma}{3}}}_{\in \mathcal{B}(L^2)},$$

$$H_{11} = \underbrace{\langle v \rangle^{-\frac{\gamma}{3}-1} \left[\left(\lambda_K^{1/3} \right)^w, b_{ij}(v) \right] a^w \langle v \rangle^{-\frac{\gamma}{3}}}_{\in \mathcal{B}(L^2)},$$

we obtain

$$\begin{aligned} |\mathcal{Z}_{1,4}| &\leq \left| \sum_{i,j,k=1}^3 \left(\langle v \rangle^{\frac{\gamma}{3}+1} b_{ik} D_{v_k} u, H_{10} \langle v \rangle^{\frac{\gamma}{3}} \left(\lambda_K^{1/3} \right)^w u \right)_{L^2} \right| \\ &\quad + \left| \sum_{i,j,k=1}^3 \left(\langle v \rangle^{\frac{\gamma}{3}+1} b_{ik} D_{v_k} u, H_{11} \langle v \rangle^{\frac{\gamma}{3}} D_{v_j} \left(\lambda_K^{1/3} \right)^w u \right)_{L^2} \right| \\ &\lesssim \varepsilon \left(\|\langle v \rangle^{\frac{\gamma}{3}} \left(\lambda_K^{1/3} \right)^w u\|_{L^2}^2 + \|\langle v \rangle^{\frac{\gamma}{3}} D_v \left(\lambda_K^{1/3} \right)^w u\|_{L^2}^2 \right) \\ &\quad + C_\varepsilon \|\langle v \rangle^{\frac{\gamma}{3}+1} B(v) D_v u\|_{L^2}^2, \end{aligned}$$

Moreover using Lemma 2.25, we obtain

$$|\mathcal{Z}_{1,4}| \lesssim \varepsilon \Re e \left(\mathcal{A}_\xi \left(\lambda_K^{1/3} \right)^w u, \left(\lambda_K^{1/3} \right)^w u \right)_{L^2} + C_\varepsilon \left(\|\mathcal{A}_\xi u\|_{L^2}^2 + \|u\|_{L^2}^2 \right),$$

so using the estimates of $\mathcal{Z}_{1,n}$ for $n = 1, \dots, 4$, we obtain

$$|\mathcal{Z}_1| \lesssim \varepsilon \Re e \left(\mathcal{A}_\xi \left(\lambda_K^{1/3} \right)^w u, \left(\lambda_K^{1/3} \right)^w u \right)_{L^2} + C_\varepsilon \left(\|\mathcal{A}_\xi u\|_{L^2}^2 + \|u\|_{L^2}^2 \right).$$

Let's look now

$$\mathcal{Z}_2 = \left(\left[F(v), \left(\lambda_K^{1/3} \right)^w \right] u, a^w \left(\lambda_K^{1/3} \right)^w u \right)_{L^2},$$

observing the symbol $\partial_\eta \lambda_K^{1/3} \in S(\langle v \rangle^{\frac{7}{6} + \frac{1}{3}}, \Gamma)$, and using Lemma 2.6 with symbolic calculus shows that the symbol of the commutator $\left[F(v), \left(\lambda_K^{1/3} \right)^w \right]$ belongs to $S(\langle v \rangle^{\frac{3\gamma}{2} + 2}, \Gamma)$ uniformly with respect to the parameter ξ . Now using the following notation

$$H_{12} = \underbrace{\langle v \rangle^{-\frac{\gamma}{2} - 1} \left[F(v), \left(\lambda_K^{1/3} \right)^w \right] \langle v \rangle^{-\gamma - 1}}_{\in \mathcal{B}(L^2)},$$

we obtain

$$\begin{aligned} |\mathcal{Z}_2| &\leq \left| \left(\langle v \rangle^{\frac{\gamma}{2} + 1} H_{12} \langle v \rangle^{-\gamma - 1} \langle v \rangle^{\gamma + 1} u, a^w \left(\lambda_K^{1/3} \right)^w u \right)_{L^2} \right| \\ &\leq \left| \left(H_{12} \langle v \rangle^{\gamma + 1} u, a^w \langle v \rangle^{\frac{\gamma}{2} + 1} \left(\lambda_K^{1/3} \right)^w u \right)_{L^2} \right| \\ &\quad + \left| \left(H_{12} \langle v \rangle^{\gamma + 1} u, \left[\langle v \rangle^{\frac{\gamma}{2} + 1}, a^w \right] \langle v \rangle^{-\frac{\gamma}{2} - 1} \langle v \rangle^{\frac{\gamma}{2} + 1} \left(\lambda_K^{1/3} \right)^w u \right)_{L^2} \right|, \end{aligned}$$

in addition, we have that the symbol of the commutator $\left[\langle v \rangle^{\frac{\gamma}{2} + 1}, a^w \right]$ belongs to $S(\langle v \rangle^{\frac{\gamma}{2} + 1}, \Gamma)$ and using Lemma 2.26, we obtain

$$|\mathcal{Z}_2| \lesssim \varepsilon \Re e \left(\mathcal{A}_\xi \left(\lambda_K^{1/3} \right)^w u, \left(\lambda_K^{1/3} \right)^w u \right)_{L^2} + C_\varepsilon \left(\|\mathcal{A}_\xi u\|_{L^2}^2 + \|u\|_{L^2}^2 \right). \quad (2.70)$$

Let's look now

$$\mathcal{Z}_3 = \left(\left[iv \cdot \xi, \left(\lambda_K^{1/3} \right)^w \right] u, a^w \left(\lambda_K^{1/3} \right)^w u \right)_{L^2},$$

using Lemma 2.13 with symbolic calculus shows that the symbol of $\left[iv \cdot \xi, \left(\lambda_K^{1/3} \right)^w \right]$ belongs to $S(\lambda_K^{1/3}, \Gamma)$ uniformly with respect to the parameter ξ . Now using the following notation

$$H_{13} = \underbrace{\left(\lambda_K^{1/3} \right)^w (a^w)^* \left[iv \cdot \xi, \left(\lambda_K^{1/3} \right)^w \right] \left(\left(\lambda_K^{2/3} \right)^w \right)^{-1}}_{\in \mathcal{B}(L^2)},$$

we obtain

$$|\mathcal{Z}_3| \leq \left| \left(H_{13} \left(\lambda_K^{2/3} \right)^w u, u \right)_{L^2} \right| \lesssim \tilde{\varepsilon} \left\| \left(\lambda_K^{2/3} \right)^w u \right\|_{L^2}^2 + C_{\tilde{\varepsilon}} \left(\|\mathcal{A}_\xi u\|_{L^2}^2 + \|u\|_{L^2}^2 \right). \quad (2.71)$$

From the above, using the estimates of \mathcal{Z}_l for $l = 1, \dots, 3$, we obtain (2.69). Next we prove (2.68), we have the following relation

$$\begin{aligned}
 & \Re e \left(\mathcal{A}_\xi \left(\lambda_K^{1/3} \right)^w u, \left(\lambda_K^{1/3} \right)^w u \right)_{L^2} + \Re e \left(\mathcal{A}_\xi \left(\lambda_K^{1/3} \right)^w u, (1 - \delta G) \left(\lambda_K^{1/3} \right)^w u \right)_{L^2} \\
 = & \Re e \left(\mathcal{A}_\xi u, \left(\lambda_K^{1/3} \right)^w \left(\lambda_K^{1/3} \right)^w u \right)_{L^2} + \Re e \left(\left[\mathcal{A}_\xi, \left(\lambda_K^{1/3} \right)^w \right] u, \left(\lambda_K^{1/3} \right)^w u \right)_{L^2} \\
 + & \Re e \left(\mathcal{A}_\xi u, \left(\lambda_K^{1/3} \right)^w (1 - \delta G) \left(\lambda_K^{1/3} \right)^w u \right)_{L^2} + \Re e \left(\left[\mathcal{A}_\xi, \left(\lambda_K^{1/3} \right)^w \right] u, (1 - \delta G) \left(\lambda_K^{1/3} \right)^w u \right)_{L^2} \\
 = & \Re e \left(\mathcal{A}_\xi u, \left(\lambda_K^{1/3} \right)^w \left(\lambda_K^{1/3} \right)^w \left(\left(\lambda_K^{2/3} \right)^w \right)^{-1} \left(\lambda_K^{2/3} \right)^w u \right)_{L^2} + \Re e \left(\left[\mathcal{A}_\xi, \left(\lambda_K^{1/3} \right)^w \right] u, \left(\lambda_K^{1/3} \right)^w u \right)_{L^2} \\
 + & \Re e \left(\mathcal{A}_\xi u, \left(\lambda_K^{1/3} \right)^w (1 - \delta G) \left(\lambda_K^{1/3} \right)^w \left(\left(\lambda_K^{2/3} \right)^w \right)^{-1} \left(\lambda_K^{2/3} \right)^w u \right)_{L^2} \\
 + & \Re e \left(\left[\mathcal{A}_\xi, \left(\lambda_K^{1/3} \right)^w \right] u, (1 - \delta G) \left(\lambda_K^{1/3} \right)^w u \right)_{L^2}
 \end{aligned}$$

gives, with $\tilde{\varepsilon} > 0$ arbitrary,

$$\begin{aligned}
 & \Re e \left(\mathcal{A}_\xi \left(\lambda_K^{1/3} \right)^w u, \left(\lambda_K^{1/3} \right)^w u \right)_{L^2} + \Re e \left(\mathcal{A}_\xi \left(\lambda_K^{1/3} \right)^w u, (1 - \delta G) \left(\lambda_K^{1/3} \right)^w u \right)_{L^2} \\
 & \lesssim \tilde{\varepsilon} \left\| \left(\lambda_K^{2/3} \right)^w u \right\|_{L^2}^2 + C_{\tilde{\varepsilon}} \left(\left\| \mathcal{A}_\xi u \right\|_{L^2}^2 + \|u\|_{L^2}^2 \right) + \Re e \left(\left[\mathcal{A}_\xi, \left(\lambda_K^{1/3} \right)^w \right] u, \left(\lambda_K^{1/3} \right)^w u \right)_{L^2} \\
 & + \Re e \left(\left[\mathcal{A}_\xi, \left(\lambda_K^{1/3} \right)^w \right] u, (1 - \delta G) \left(\lambda_K^{1/3} \right)^w u \right)_{L^2}.
 \end{aligned}$$

We could apply (2.69) with $a = 1$ and $a = 1 - \delta \tilde{g}$ to control the last term in the above inequality; this gives, with $\varepsilon, \tilde{\varepsilon} > 0$ arbitrarily small,

$$\begin{aligned}
 & \Re e \left(\mathcal{A}_\xi \left(\lambda_K^{1/3} \right)^w u, \left(\lambda_K^{1/3} \right)^w u \right)_{L^2} + \Re e \left(\mathcal{A}_\xi \left(\lambda_K^{1/3} \right)^w u, (1 - \delta G) \left(\lambda_K^{1/3} \right)^w u \right)_{L^2} \\
 & \lesssim \tilde{\varepsilon} \left\| \left(\lambda_K^{2/3} \right)^w u \right\|_{L^2}^2 + \varepsilon \Re e \left(\mathcal{A}_\xi \left(\lambda_K^{1/3} \right)^w u, \left(\lambda_K^{1/3} \right)^w u \right)_{L^2} + C_{\varepsilon, \tilde{\varepsilon}} \left(\left\| \mathcal{A}_\xi u \right\|_{L^2}^2 + \|u\|_{L^2}^2 \right) \\
 & + \varepsilon \Re e \left(\mathcal{A}_\xi \left(\lambda_K^{1/3} \right)^w u, (1 - \delta G) \left(\lambda_K^{1/3} \right)^w u \right)_{L^2}.
 \end{aligned}$$

Letting ε small enough yields the desired estimate (2.68). \square

Proposition 2.29. *Let λ_K be the symbol defined in (2.44). Then there exists $C_0 > 0$ such that for all $u \in \mathcal{S}(\mathbb{R}_v^3)$,*

$$\left\| \left(\lambda_K^{2/3} \right)^w u \right\|_{L^2}^2 \leq C_0 \left(\left\| \mathcal{A}_\xi u \right\|_{L^2}^2 + \|u\|_{L^2}^2 \right). \quad (2.72)$$

Proof. Using Proposition 2.23, we have for all $u \in \mathcal{S}(\mathbb{R}_v^3)$,

$$\left(\left(\lambda_K^{2/3} \right)^{\text{Wick}} u, u \right)_{L^2} \lesssim \Re e \left(\mathcal{A}_\xi u, u \right)_{L^2} + \Re e \left(\mathcal{A}_\xi u, (1 - \delta G) u \right)_{L^2}, \quad (2.73)$$

uniformly with respect to the parameter ξ in \mathbb{R}^3 . By substituting $\left(\lambda_K^{1/3} \right)^w u$ to u in the above estimate, we obtain that for all $\tilde{\varepsilon} > 0$,

$$\left(\left(\lambda_K^{2/3} \right)^{\text{Wick}} \left(\lambda_K^{1/3} \right)^w u, \left(\lambda_K^{1/3} \right)^w u \right)_{L^2} \lesssim \tilde{\varepsilon} \left\| \left(\lambda_K^{2/3} \right)^w u \right\|_{L^2}^2 + C_{\tilde{\varepsilon}} \left(\left\| \mathcal{A}_\xi u \right\|_{L^2}^2 + \|u\|_{L^2}^2 \right). \quad (2.74)$$

Notice from (2.133) that we may write

$$(\lambda_K^{2/3})^{\text{Wick}} = (\lambda_K^{2/3})^w + r^w, \quad (2.75)$$

with

$$r(Y) = \int_0^1 \int_{\mathbb{R}^6} (1-\theta) (\lambda_K^{2/3})''(Y + \theta Y_1) Y_1 \cdot Y_1 e^{-|Y_1|^2} dY_1 d\theta,$$

where $Y = (v, \eta)$, $Y_1 = (v_1, \eta_1)$ in \mathbb{R}^6 and $(\lambda_K^{2/3})''(Y)$ is the Hessian of $\lambda_K^{2/3}$ at the point Y . Define

$$r_1 = \pi^{-3} \int_0^1 \int_{\mathbb{R}^6} (1-\theta) \nabla_\eta^2 (\lambda_K^{2/3})(Y + \theta Y_1) \eta_1 \cdot \eta_1 e^{-|Y_1|^2} dY_1 d\theta,$$

$$r_2 = \pi^{-3} \sum_{j,k=1}^3 \int_0^1 \int_{\mathbb{R}^6} (1-\theta) \partial_{v_j} \partial_{\eta_k} (\lambda_K^{2/3})(Y + \theta Y_1) (v_{1j} \eta_{1k} + v_{1k} \eta_{1j}) e^{-|Y_1|^2} dY_1 d\theta,$$

and

$$r_3 = \pi^{-3} \int_0^1 \int_{\mathbb{R}^6} (1-\theta) \nabla_v^2 (\lambda_K^{2/3})(Y + \theta Y_1) v_1 \cdot v_1 e^{-|Y_1|^2} dY_1 d\theta.$$

Using Lemma 2.15 with symbolic calculus shows that the symbol

$$\nabla_\eta^2 (\lambda_K^{2/3}), \nabla_v \nabla_\eta (\lambda_K^{2/3}) \in S(\langle v \rangle^{\gamma+2}, \Gamma),$$

uniformly with respect to the parameter ξ , then r_1, r_2 belong to $S(\langle v \rangle^{\gamma+2}, \Gamma)$. Using the following notations

$$H_{14} = \underbrace{\left((\lambda_K^{2/3})^w \right)^{-1} \left(\lambda_K^{1/3} \right)^w r_1^w \left(\lambda_K^{1/3} \right)^w \langle v \rangle^{-\gamma-2}}_{\in \mathcal{B}(L^2)},$$

$$H_{15} = \underbrace{\left((\lambda_K^{2/3})^w \right)^{-1} \left(\lambda_K^{1/3} \right)^w r_2^w \left(\lambda_K^{1/3} \right)^w \langle v \rangle^{-\gamma-2}}_{\in \mathcal{B}(L^2)},$$

we obtain for all $\tilde{\varepsilon} > 0$,

$$\begin{aligned} & \left| \left(r_1^w \left(\lambda_K^{1/3} \right)^w u, \left(\lambda_K^{1/3} \right)^w u \right)_{L^2} \right| + \left| \left(r_2^w \left(\lambda_K^{1/3} \right)^w u, \left(\lambda_K^{1/3} \right)^w u \right)_{L^2} \right| \\ & \leq \left| \left(H_{14} \langle v \rangle^{\gamma+2} u, \left(\lambda_K^{2/3} \right)^w u \right)_{L^2} \right| + \left| \left(H_{15} \langle v \rangle^{\gamma+2} u, \left(\lambda_K^{2/3} \right)^w u \right)_{L^2} \right| \\ & \lesssim \tilde{\varepsilon} \left\| \left(\lambda_K^{2/3} \right)^w u \right\|_{L^2}^2 + C_{\tilde{\varepsilon}} \left(\left\| \mathcal{A}_\xi u \right\|_{L^2}^2 + \|u\|_{L^2}^2 \right). \end{aligned}$$

Taking into account that the symbol $\nabla_v^2 (\lambda_K^{2/3})$, belongs to $S(\langle v \rangle^{\frac{\gamma}{3}} g_{3,K}, \Gamma)$ uniformly with respect to the parameter ξ , where $g_{3,K}$ the operator defined in (2.43), then the symbol r_3 belongs to $S(\langle v \rangle^{\frac{\gamma}{3}} g_{3,K}, \Gamma)$. Using the following notation

$$H_{16} = \underbrace{\left((\lambda_K^{2/3})^w \right)^{-1} \left(\lambda_K^{1/3} \right)^w r_3^w \left(\lambda_K^{1/3} \right)^w \left(g_{3,K}^w \right)^{-1} \langle v \rangle^{-\frac{\gamma}{3}}}_{\in \mathcal{B}(L^2)},$$

we obtain

$$\begin{aligned} \left| \left(r_3^w \left(\lambda_K^{1/3} \right)^w u, \left(\lambda_K^{1/3} \right)^w u \right)_{L^2} \right| &\leq \left| \left(H_{16} \langle v \rangle^{\frac{2}{3}} g_{3,K}^w u, \left(\lambda_K^{2/3} \right)^w u \right)_{L^2} \right| \\ &\lesssim \tilde{\varepsilon} \left\| \left(\lambda_K^{2/3} \right)^w u \right\|_{L^2}^2 + C_{\tilde{\varepsilon}} \left(\left\| \mathcal{A}_\xi u \right\|_{L^2}^2 + \|u\|_{L^2}^2 \right). \end{aligned}$$

From the above, using the estimates of r_l for $l = 1, \dots, 3$, we obtain

$$\left| \left(r^w \left(\lambda_K^{1/3} \right)^w u, \left(\lambda_K^{1/3} \right)^w u \right)_{L^2} \right| \lesssim \tilde{\varepsilon} \left\| \left(\lambda_K^{2/3} \right)^w u \right\|_{L^2}^2 + C_{\tilde{\varepsilon}} \left(\left\| \mathcal{A}_\xi u \right\|_{L^2}^2 + \|u\|_{L^2}^2 \right). \quad (2.76)$$

By applying Theorem 2.40 with $p = \lambda$, we obtain

$$\left(\left(\lambda_K^{2/3} \right)^w \left(\lambda_K^{1/3} \right)^w u, \left(\lambda_K^{1/3} \right)^w u \right)_{L^2} \sim \left\| \left[\left(\lambda_K^{1/3} \right)^w \right]^2 u \right\|_{L^2}^2 \sim \left\| \left(\lambda_K^{2/3} \right)^w u \right\|_{L^2}^2, \quad (2.77)$$

uniformly with respect to the parameter ξ . Using (2.76), then taking $\tilde{\varepsilon}$ small enough, we get for all $\tilde{\varepsilon} > 0$,

$$\left\| \left(\lambda_K^{2/3} \right)^w u \right\|_{L^2}^2 \lesssim \tilde{\varepsilon} \left\| \left(\lambda_K^{2/3} \right)^w u \right\|_{L^2}^2 + \tilde{C}_{\tilde{\varepsilon}} \left(\left\| \mathcal{A}_\xi u \right\|_{L^2}^2 + \|u\|_{L^2}^2 \right),$$

now taking $\tilde{\varepsilon}$ small enough, we obtain that there is a constant $C_0 > 0$ such that for all $u \in \mathcal{S}(\mathbb{R}_v^3)$,

$$\left\| \left(\lambda_K^{2/3} \right)^w u \right\|_{L^2}^2 \leq C_0 \left(\left\| \mathcal{A}_\xi u \right\|_{L^2}^2 + \|u\|_{L^2}^2 \right). \quad (2.78)$$

□

Proposition 2.30. *Let λ_K be the symbol defined in (2.44). Then there exists $C_0 > 0$ such that for all $u \in \mathcal{S}(\mathbb{R}_v^3)$,*

$$\left\| \mathcal{A}_\xi u \right\|_{L^2}^2 \leq C_0 \left\| \left(\lambda_K^2 \right)^w u \right\|_{L^2}^2. \quad (2.79)$$

Proof. We denote by $\tilde{\sigma}$ the symbol of the operator \mathcal{A}_ξ . We will show that $\tilde{\sigma}$ belongs to $S(\lambda^2, \Gamma)$ uniformly with respect to the parameter ξ .

Using Lemma 3.11 in [68], we can write $\tilde{\sigma}$ as follows

$$\tilde{\sigma} = iv \cdot \xi + |B(v)\eta|^2 + F(v) + R_1 + R_2,$$

where R_1 (resp R_2) is a symbol belongs to $S(\langle v \rangle^{\gamma+1} \langle \eta \rangle, \Gamma)$ (resp $S(\langle v \rangle^\gamma, \Gamma)$) uniformly with respect to the parameter ξ .

For $iv \cdot \xi$: Taking into account the fact that $\gamma \geq 0$, we have

$$\begin{aligned} \text{For } |\alpha| = 0, \quad |iv \cdot \xi| &\lesssim |v|^2 + |\xi|^2 \lesssim \langle v \rangle^{\gamma+2} + \langle v \rangle^\gamma |\xi|^2 \lesssim \lambda^2, \\ \text{for } |\alpha| = 1, \quad |\partial_v^\alpha (iv \cdot \xi)| &= |\xi| \lesssim \langle v \rangle^{\gamma/2} |\xi| \lesssim \lambda^2, \\ \text{for } |\alpha| \geq 2, \quad |\partial_v^\alpha (iv \cdot \xi)| &= 0. \end{aligned}$$

For $|B(v)\eta|^2$: Using (2.14), we have

$$\text{for } |\alpha| \geq 1, \quad |\partial_v^\alpha B(v)\eta|^2 \lesssim \langle v \rangle^\gamma |\eta|^2 \lesssim |B(v)\eta|^2, \quad (2.80)$$

then, using Cauchy-Schwarz we get

$$\forall \alpha \in \mathbb{N}^3, \quad |\partial_v^\alpha (|B(v)\eta|^2)| \lesssim |B(v)\eta|^2 \lesssim \lambda^2. \quad (2.81)$$

On the other hand, also using (2.14), we have

$$\begin{aligned} \text{pour } |\alpha| = 0, \quad & |B(v)\eta|^2 \lesssim \lambda^2, \\ \text{pour } |\alpha| = 1, \quad & |\partial_\eta^\alpha (|B(v)\eta|^2)| \lesssim |B(v)||B(v)\eta| \lesssim \lambda^2, \\ \text{pour } |\alpha| = 2, \quad & |\partial_\eta^\alpha (|B(v)\eta|^2)| \lesssim |B^T(v)B(v)| \lesssim \lambda^2, \\ \text{pour } |\alpha| \geq 3, \quad & |\partial_\eta^\alpha (|B(v)\eta|^2)| = 0, \end{aligned}$$

so we get

$$\forall \alpha \in \mathbb{N}^3, \quad |\partial_\eta^\alpha (|B(v)\eta|^2)| \lesssim \lambda^2. \quad (2.82)$$

Moreover, one can estimate from above the modulus of all the derivatives of the term $|B(v)\eta|^2$ by a constant times λ^2 .

For $F(v)$: Using Lemma 2.6, we have

$$\forall \alpha \in \mathbb{N}^3, \quad |\partial_v^\alpha F(v)| \lesssim \langle v \rangle^{\gamma+2-|\alpha|} \lesssim \lambda^2, \quad (2.83)$$

which gives that $F(v) \in S(\lambda^2, \Gamma)$ uniformly with respect to the parameter ξ .

In addition, we have that $R_1, R_2 \in S(\lambda^2, \Gamma)$. From the above, we can deduce that $\tilde{\sigma}$ belongs to $S(\lambda^2, \Gamma)$ uniformly with respect to the parameter ξ . Using the following notation

$$H_{17} = \underbrace{\mathcal{A}_\xi \left((\lambda_K^2)^w \right)^{-1}}_{\in \mathcal{B}(L^2)},$$

Then there exists $\mathbf{C}_0 > 0$ such that

$$\|H_{17}\varphi\|_{L^2}^2 \leq \mathbf{C}_0 \|\varphi\|_{L^2}^2 \quad \forall \varphi \in L^2(\mathbb{R}_v^3), \quad (2.84)$$

uniformly with respect to the parameter ξ , which implies for all $u \in \mathcal{S}(\mathbb{R}_v^3)$,

$$\|\mathcal{A}_\xi u\|_{L^2}^2 \leq \mathbf{C}_0 \|(\lambda_K^2)^w u\|_{L^2}^2. \quad (2.85)$$

□

4 Hypocoelliptic estimates for the whole linearized Landau operator

In this section, we show hypocoelliptic estimates with respect to the velocity and position variables for the Landau operator \mathcal{P} . These estimates allow us to locate the spectrum and estimate the resolvent of the Landau operator. We denote by Λ_K the operator associated to the symbol λ_K by considering the inverse Fourier transform with respect to the variable x .

Theorem 2.31. *There exists $C > 0$ such that for all $u \in \mathcal{S}(\mathbb{R}_{x,v}^6)$,*

$$\|\Lambda_K^{2/3} u\|_{L_{x,v}^2}^2 \leq C \left(\|\mathcal{P}u\|_{L_{x,v}^2}^2 + \|u\|_{L_{x,v}^2}^2 \right). \quad (2.86)$$

Proof. Using Theorem 2.40, we have for all $u \in \mathcal{S}(\mathbb{R}_v^3)$,

$$\|(\lambda_K^{2/3})^w u\|_{L_v^2}^2 \sim \|(\lambda_K^w)^{2/3} u\|_{L_v^2}^2,$$

so using the Proposition 2.29, there exists a constant $C_1 > 0$ such that for all $u \in \mathcal{S}(\mathbb{R}_v^3)$,

$$\|(\lambda_K^w)^{2/3} u\|_{L_v^2}^2 \leq C_1 \left(\|\mathcal{A}_\xi u\|_{L_v^2}^2 + \|u\|_{L_v^2}^2 \right), \quad (2.87)$$

uniformly with respect to the parameter ξ . By integrating the previous inequality with respect to the parameter ξ in \mathbb{R}^3 and considering the inverse Fourier transform with respect to the variable x , we obtain for all $u \in \mathcal{S}(\mathbb{R}_{x,v}^6)$,

$$\|\Lambda_K^{2/3} u\|_{L_{x,v}^2}^2 \leq C_1 \left(\|\mathcal{A}u\|_{L_{x,v}^2}^2 + \|u\|_{L_{x,v}^2}^2 \right). \quad (2.88)$$

Using (2.9), the operator \mathcal{P} is written as follows

$$\mathcal{P} = \mathcal{A} + \mathcal{K}.$$

Consequently, using (2.88), we have

$$\begin{aligned} \|\Lambda_K^{2/3} u\|_{L_{x,v}^2}^2 &\leq C_1 \left(\|(\mathcal{A} + \mathcal{K} - \mathcal{K})u\|_{L_{x,v}^2}^2 + \|u\|_{L_{x,v}^2}^2 \right) \\ &\leq C_1 \left(\|\mathcal{P}u\|_{L_{x,v}^2}^2 + \|\mathcal{K}u\|_{L_{x,v}^2}^2 + \|u\|_{L_{x,v}^2}^2 \right) \end{aligned}$$

and using the fact that \mathcal{K} is a bounded operator, we obtain that there exists a constant $C > 0$ such that for all $u \in \mathcal{S}(\mathbb{R}_{x,v}^6)$,

$$\|\Lambda_K^{2/3} u\|_{L_{x,v}^2}^2 \leq C \left(\|\mathcal{P}u\|_{L_{x,v}^2}^2 + \|u\|_{L_{x,v}^2}^2 \right). \quad (2.89)$$

□

By adding a term $i\kappa$ to \mathcal{P} with $\kappa \in \mathbb{R}$, the proof doesn't change due to the never changing of the real part of \mathcal{P} as mentioned crucially in Remark 2.2 in [67]. So Theorem 2.31 admits the following extension.

Theorem 2.32. *There exists $C > 0$ such that for all $u \in \mathcal{S}(\mathbb{R}_{x,v}^6)$,*

$$\forall \kappa \in \mathbb{R}, \quad \|\Lambda_K^{2/3} u\|_{L_{x,v}^2}^2 \leq C \left(\|(\mathcal{P} - i\kappa)u\|_{L_{x,v}^2}^2 + \|u\|_{L_{x,v}^2}^2 \right). \quad (2.90)$$

Theorem 2.33. *There exists $C > 0$ such that for all $u \in \mathcal{S}(\mathbb{R}_{x,v}^6)$,*

$$\|\mathcal{P}u\|_{L_{x,v}^2}^2 \leq C \|\Lambda_K^2 u\|_{L_{x,v}^2}^2. \quad (2.91)$$

Proof. Using Theorem 2.40, we have for all $u \in \mathcal{S}(\mathbb{R}_v^3)$,

$$\|(\lambda_K^2)^w u\|_{L_v^2}^2 \sim \|(\lambda_K^w)^2 u\|_{L_v^2}^2,$$

so using the Proposition 2.30, there exists a constant $C_1 > 0$ such that for all $u \in \mathcal{S}(\mathbb{R}_v^3)$,

$$\|\mathcal{A}_\xi u\|_{L_v^2}^2 \leq C_1 \|(\lambda_K^w)^2 u\|_{L_v^2}^2, \quad (2.92)$$

uniformly with respect to the parameter ξ . By integrating the previous inequality with respect to the parameter ξ in \mathbb{R}^3 and considering the inverse Fourier transform with respect to the variable x , we obtain for all $u \in \mathcal{S}(\mathbb{R}_{x,v}^6)$,

$$\|\mathcal{A}u\|_{L_{x,v}^2}^2 \leq C_1 \|\Lambda_K^2 u\|_{L_{x,v}^2}^2. \quad (2.93)$$

Using (2.9), the operator \mathcal{P} is written as follows

$$\mathcal{P} = \mathcal{A} + \mathcal{K}.$$

Consequently, using (2.93), we have

$$\begin{aligned} \|\mathcal{P}u\|_{L_{x,v}^2}^2 &\leq \|\mathcal{A}u\|_{L_{x,v}^2}^2 + \|\mathcal{K}u\|_{L_{x,v}^2}^2 \\ &\leq C_1 \|\Lambda_K^2 u\|_{L_{x,v}^2}^2 + C_2 \|u\|_{L_{x,v}^2}^2, \end{aligned}$$

with $C_2 > 0$. Using the fact that the operator $\Lambda_K^2 \geq \text{Id}$ (see for instance Theorem 2.40 in the Appendix (Section 6.2)), we obtain that there exists a constant $C > 0$ such that for all $u \in \mathcal{S}(\mathbb{R}_{x,v}^6)$,

$$\|\mathcal{P}u\|_{L_{x,v}^2}^2 \leq C \|\Lambda_K^2 u\|_{L_{x,v}^2}^2. \quad (2.94)$$

□

We strongly think the estimate in (2.86) is optimal in term of the index $2/3$ appearing in the left hand side, although we don't prove that. This index $2/3$ is classical for kinetic model. We refer for example to Proposition 5.22 in [62], partial results in this direction in the Fokker-Planck case.

5 Localisation of the spectrum for the Landau operator

In all that follows, we denote by $\sigma(\mathcal{P})$ the spectrum of the operator \mathcal{P} , $\rho(\mathcal{P})$ The resolvent set of \mathcal{P} and $\|\cdot\|_{L^2}$ to denote the norm in the space $L^2(\mathbb{R}_{x,v}^6)$. The following lemma holds for any maximally accretive operator.

Lemma 2.34. *Let $(\mathcal{A}, D(\mathcal{A}))$ be a maximally accretive operator in the Hilbert space \mathcal{H} . For any $\eta \in]0, 1[$, the estimate*

$$|z + 1|^{2\eta} \|u\|^2 \leq 4 \left(((\mathcal{A} + 1)^*(\mathcal{A} + 1))^{\eta} u, u \right)_{\mathcal{H}} + 4 \|(\mathcal{A} - z)u\|^2$$

holds for all $u \in D(\mathcal{A})$ and $z \in \mathbb{C}$ with $\Re z \geq -1$.

Proof. See Proposition B.1 in [67]. □

Proof of Theorem 2.2.

Proof. The proof will be divided into two steps.

First step:

Using (2.90) and the triangle inequality we therefore get for all $z = \nu + i\kappa \in \mathbb{C}$, with $\nu = \Re z \geq -1/2$ and $u \in \mathcal{S}(\mathbb{R}_{x,v}^6)$,

$$\begin{aligned} \|\Lambda_K^{2/3} u\|_{L^2}^2 &\leq C(\|(\mathcal{P} - i\kappa)u\|_{L^2}^2 + \|u\|_{L^2}^2) \\ &\leq C(\|(\mathcal{P} - i\kappa - \nu + \nu)u\|_{L^2}^2 + \|u\|_{L^2}^2) \\ &\leq C(2\|(\mathcal{P} - z)u\|_{L^2}^2 + (2\nu^2 + 1)\|u\|_{L^2}^2). \end{aligned}$$

But we have $\nu \geq -1/2$ implies $2\nu + 2 \geq 1$, so we get

$$\begin{aligned} \|\Lambda_K^{2/3} u\|_{L^2}^2 &\leq C(6\|(\mathcal{P} - z)u\|_{L^2}^2 + 6(\nu + 1)^2\|u\|_{L^2}^2) \\ &\leq 6C(\|(\mathcal{P} - z)u\|_{L^2}^2 + (\nu + 1)^2\|u\|_{L^2}^2), \end{aligned}$$

By taking $\tilde{C} = 6C$, we finally obtain the following estimate

$$\|\Lambda_K^{2/3} u\|_{L^2}^2 \leq \tilde{C} \left(\|(\mathcal{P} - z)u\|_{L^2}^2 + (\Re z + 1)^2 \|u\|_{L^2}^2 \right), \quad (2.95)$$

for all $u \in \mathcal{S}(\mathbb{R}_{x,v}^6)$ and $z \in \mathbb{C}$ with $\Re z \geq -\frac{1}{2}$.

Second step:

First, we will show that there exists a constant \tilde{C}_1 such that for all $u \in \mathcal{S}(\mathbb{R}_{x,v}^6)$, we have

$$0 \leq \left((\mathcal{A} + 1)^*(\mathcal{A} + 1)u, u \right)_{L^2} \leq \tilde{C}_1 \left(\Lambda_K^4 u, u \right)_{L^2}, \quad (2.96)$$

where \mathcal{A} the operator defined in (2.9). Indeed, let $u \in \mathcal{S}(\mathbb{R}_{x,v}^6)$, we have

$$\left((1 + \mathcal{A})^*(1 + \mathcal{A})u, u \right)_{L^2} = \left((1 + \mathcal{A})u, (1 + \mathcal{A})u \right)_{L^2} = \|(1 + \mathcal{A})u\|_{L^2}^2 \geq 0.$$

On the other hand, we have

$$\left((1 + \mathcal{A})^*(1 + \mathcal{A})u, u \right)_{L^2} = \|(1 + \mathcal{A})u\|_{L^2}^2 \leq \|\mathcal{A}u\|_{L^2}^2 + \|u\|_{L^2}^2.$$

Using (2.9), the operator \mathcal{P} is written as follows

$$\mathcal{P} = \mathcal{A} + \mathcal{K},$$

we obtain

$$\left((1 + \mathcal{A})^*(1 + \mathcal{A})u, u \right)_{L^2} \leq \|\mathcal{P}u\|_{L^2}^2 + \|\mathcal{K}u\|_{L^2}^2 + \|u\|_{L^2}^2,$$

finally, using the fact that \mathcal{K} is a bounded operator, $\Lambda_K \geq \text{Id}$ and the estimate (2.91), we obtain

$$\begin{aligned} \left((1 + \mathcal{A})^*(1 + \mathcal{A})u, u \right)_{L^2} &\leq (1 + \nu_0) \|\Lambda_K^2 u\|_{L^2}^2 + \mathbf{C} \|\Lambda_K^2 u\|_{L^2}^2 \\ &\leq (1 + \nu_0 + \mathbf{C}) \|\Lambda_K^2 u\|_{L^2}^2, \end{aligned}$$

where $\nu_0 > 0$. By taking $\tilde{C}_1 = (1 + \nu_0 + \mathbf{C})$, we obtain the estimate (2.96).

According to the monotonicity of the operator functional $\mathcal{A} \rightarrow \mathcal{A}^\alpha$ for $\alpha \in [0, 1]$, in particular with $\alpha = \frac{1}{3}$, we obtain

$$0 \leq \left(((1 + \mathcal{A})^*(1 + \mathcal{A}))^{1/3} u, u \right)_{L^2} \leq \tilde{C}_1^{2/3} \left(\Lambda_K^{4/3} u, u \right)_{L^2}.$$

According to Theorem 2.1, \mathcal{A} is maximally accretive, then by applying Lemma 2.34 with $\eta = \frac{1}{3}$, for $\Re z \geq -1/2$ and $u \in \mathcal{S}(\mathbb{R}_{x,v}^6)$, we obtain

$$\begin{aligned} |z + 1|^{2/3} \|u\|_{L^2}^2 &\leq 4 \left(((1 + \mathcal{A})^*(1 + \mathcal{A}))^{1/3} u, u \right)_{L^2} + 4 \|(\mathcal{A} - z)u\|_{L^2}^2 \\ &\leq 4 \tilde{C}_1^{2/3} \left(\Lambda_K^{4/3} u, u \right)_{L^2} + 4 \|(\mathcal{A} - z)u\|_{L^2}^2 \\ &\leq 4 \tilde{C}_1^{2/3} \|\Lambda_K^{2/3} u\|_{L^2}^2 + 4 \|(\mathcal{A} - z)u\|_{L^2}^2 \\ &\leq 4 \tilde{C}_1^{2/3} \|\Lambda_K^{2/3} u\|_{L^2}^2 + 4 \|(\mathcal{P} - z)u\|_{L^2}^2 + 4\nu_1 \|u\|_{L^2}^2. \end{aligned}$$

With the inequality (2.95), we obtain

$$|z + 1|^{2/3} \|u\|_{L^2}^2 \leq (4\tilde{C}_1^{2/3}\tilde{C} + 4) \|(\mathcal{P} - z)u\|_{L^2}^2 + (4\tilde{C}_1^{2/3}\tilde{C} + 16\nu_1) (\Re z + 1)^2 \|u\|_{L^2}^2.$$

By taking $C_{\mathcal{P}} = \sqrt{8(4\nu_1 + \tilde{C}_1^{2/3}\tilde{C})}$, $Q_{\mathcal{P}} = \sqrt{8(1 + \tilde{C}_1^{2/3}\tilde{C})}$ we finally get $\forall z \in \mathbb{C}$, $\Re z \geq -1/2$, $\forall u \in \mathcal{S}(\mathbb{R}_{x,v}^6)$,

$$|z + 1|^{2/3} \|u\|_{L^2}^2 \leq \frac{Q_{\mathcal{P}}^2}{2} \|(\mathcal{P} - z)u\|_{L^2}^2 + \frac{C_{\mathcal{P}}^2}{2} (\Re z + 1)^2 \|u\|_{L^2}^2. \quad (2.97)$$

Now let $z \notin S_{\mathcal{P}}$ such that $\Re z \geq -1/2$, so according to the definition of $S_{\mathcal{P}}$ given in (2.6) we have

$$(\Re z + 1)^2 \leq \frac{1}{C_{\mathcal{P}}^2} |z + 1|^{2/3},$$

then for all $u \in \mathcal{S}(\mathbb{R}_{x,v}^6)$, inequality (2.97) implies

$$|z + 1|^{2/3} \|u\|_{L^2}^2 \leq Q_{\mathcal{P}}^2 \|(\mathcal{P} - z)u\|_{L^2}^2, \quad (2.98)$$

we deduce that $\mathcal{P} - z$ is injective, moreover we can replace $\mathcal{P} - z$ by $(\mathcal{P} - z)^*$ in (2.98), which gives $(\mathcal{P} - z)^*$ is injective and consequently $\mathcal{P} - z$ is bijective with dense image in L^2 , therefore $z \in \rho(\mathcal{P})$. By taking $v = (\mathcal{P} - z)u$ in the estimate (2.98) we get

$$\|(\mathcal{P} - z)^{-1}v\|_{L^2}^2 \leq Q_{\mathcal{P}}^2 |z + 1|^{-2/3} \|v\|_{L^2}^2 \quad \forall v \in L^2,$$

then we obtain the resolvent estimate (2.7).

On the other hand, we have that if $z \in \sigma(\mathcal{P})$ then $z \in S_{\mathcal{P}}$ and taking into account that the numerical range of the operator \mathcal{P} is the half plan $\{\Re z \geq 0\}$, we deduce that the spectrum $\sigma(\mathcal{P})$ satisfies

$$\sigma(\mathcal{P}) \subset S_{\mathcal{P}} \cap \{\Re z \geq 0\}.$$

Concerning estimate (2.8), we have

$$\Re((\mathcal{P} - z)u, u)_{L^2} \geq -\Re z \|u\|_{L^2}^2 \quad \text{with} \quad \Re z \leq -\frac{1}{2} < 0,$$

which implies that

$$\|(z - \mathcal{P})^{-1}\|_{\mathcal{B}(L_{x,v}^2)} \leq |\Re z|^{-1}.$$

The proof is then complete. □

6 Appendix A

6.1 Weyl-Hörmander calculus

We recall here some notations and basic facts of symbolic calculus, and refer to [91] and [76] for detailed discussions on the pseudo-differential calculus.

We introduce on \mathbb{R}^{2n} the following metric

$$\Gamma = dv^2 + d\eta^2.$$

Definition 2.35. Let $m \geq 1$ be a C^∞ function on \mathbb{R}^{2n} . We say that m is an admissible weight for Γ if there exist two constants $C > 0$ and $N > 0$ such that

$$\forall X, Y \in \mathbb{R}^{2n}, \quad m(X) \leq C \langle X - Y \rangle^N m(Y). \quad (2.99)$$

Definition 2.36. Let m be an admissible function. We denote by $S(m, \Gamma)$ the symbol class of all smooth functions $p(v, \eta)$ (possibly depending on parameter ξ) satisfying

$$\forall \alpha, \beta \in \mathbb{N}^n, \exists C_{\alpha, \beta} > 0; \forall (v, \eta) \in \mathbb{R}^{2n}, |\partial_v^\alpha \partial_\eta^\beta p(v, \eta)| \leq C_{\alpha, \beta} m(v, \eta).$$

The space of symbols $S(m, \Gamma)$ endowed with the semi-norms

$$\|p\|_{k; S(m, \Gamma)} = \sup_{|\alpha + \beta| \leq k} \sup_{(v, \eta) \in \mathbb{R}^{2n}} |m(v, \eta)^{-1} \partial_v^\alpha \partial_\eta^\beta p(v, \eta)|; \quad k \in \mathbb{N} \quad (2.100)$$

becomes a Fréchet space.

For such a symbol p in $S(m, \Gamma)$ we may define its Weyl quantization p^w by

$$\forall u \in \mathcal{S}(\mathbb{R}^n), \quad (p^w u)(v) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^{2n}} e^{i(v-v') \cdot \eta} p\left(\frac{v+v'}{2}, \eta\right) u(v') \, dv' d\eta. \quad (2.101)$$

The Weyl quantization of $S(m, \Gamma)$ is denoted by $\Psi(m, \Gamma)$.

Theorem 2.37. (Calderon-Vaillancourt)

Let p^w be an operator in $\Psi(1, \Gamma)$. We have p^w a continuous operator on $L^2(\mathbb{R}^n)$ and

$$\forall u \in L^2(\mathbb{R}^n), \quad \|p^w u\|_{L^2} \leq C \|p\|_{N; S(1, \Gamma)} \|u\|_{L^2}, \quad (2.102)$$

where $C > 0$ and a positive integer N depending only on the dimension.

Proof. See Section 18 in [76]. □

Theorem 2.38. Let p^w be an invertible operator in $\Psi(m, \Gamma)$, then its inverse $[p^w]^{-1}$ belongs to $\Psi(m^{-1}, \Gamma)$.

Proof. See Lemma A.2 in [67]. □

Definition 2.39. Let $p^w \in \Psi(m, \Gamma)$. We say that p^w is an elliptic operator if there exists $C > 0$ such that

$$|p| \geq Cm.$$

Let us also recall here the composition formula of Weyl quantization. Let $a \in S(m_1, \Gamma)$ and $b \in S(m_2, \Gamma)$, the compositions of the pseudo-differential operators a^w and b^w are pseudo-differential operators whose symbol, denoted $a \sharp b$, belongs to $S(m_1 m_2, \Gamma)$ and has the following development:

$$a \sharp b = ab + \int_0^1 \iint e^{-i\sigma(Y-Y_1, Y-Y_2)/(2\theta)} \frac{i}{2} \sigma(\partial_{Y_1}, \partial_{Y_2}) a(Y_1) b(Y_2) dY_1 dY_2 d\theta / (\theta)^{2n}, \quad (2.103)$$

where σ is the symplectic form in $T^*\mathbb{R}^n = \mathbb{R}^{2n}$ given by

$$\sigma(Z, Z') = \sum_{j=0}^n \zeta_j z'_j - z_j \zeta'_j. \quad (2.104)$$

As we work with pseudo-differential operators which belong to classes associated to the metric Γ , all operators will be defined as continuous operators of $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}(\mathbb{R}^n)$ or from $\mathcal{S}'(\mathbb{R}^n)$ to $\mathcal{S}'(\mathbb{R}^n)$.

6.2 Basic theorem

This theorem aims at giving a uniform statement for Weyl Hörmander tools with a large parameter K . This theorem gives very important results which can use these results to establish estimates which requires pseudo-differential operators. Part (I) in the theorem below has been shown in [4] with $\tau = 1$, but here we have improved this result for all $\tau \in \mathbb{R}$. In general, these results give a general and robust framework to techniques already used as well for work that requires these kinds of properties.

Theorem 2.40. *Let $p, q \geq 1$ be two symbols which verify the following hypotheses:*

- i) p is an admissible weight.
- ii) $p \sim q$.
- iii) $p, q \in S(p, \Gamma)$.
- iv) *There exists $M \in \mathbb{R}$ such that for all $\varepsilon > 0$ we have $\partial_\eta p, \partial_\eta q \in S(\varepsilon p + \varepsilon^{-1} \langle v \rangle^M, \Gamma)$.*

We define the symbols p_K, q_K as $p_K = p + K \langle v \rangle^M$ and $q_K = q + K \langle v \rangle^M$. Then there exists K_0 such that for all $K \geq K_0$:

- I) *For all $\tau \in \mathbb{R}$, $(p_K^\tau)^w$ and $(q_K^\tau)^w$ are invertible.*
- II) *For all $\tau \in \mathbb{R}$, $[(p_K^\tau)^w]^{-1}$ and $[(q_K^\tau)^w]^{-1}$ are pseudo-differential operators that belong to $\Psi(p_K^{-\tau}, \Gamma)$ uniformly in K .*
- III) *For all $\tau \in \mathbb{R}$ and for all $\kappa \geq 0$, $[(p_K^\kappa)^w]^\tau$ and $[(q_K^\kappa)^w]^\tau$ are pseudo-differential operators belong to $\Psi(p_K^{\kappa\tau}, \Gamma)$ uniformly in K .*
- IV) *For all $\tau \in \mathbb{R}$ and for all $\kappa \geq 0$, $\forall u \in \mathcal{S}(\mathbb{R}^n)$, we have*

$$\|[(p_K^\kappa)^w]^\tau u\|_{L^2}^2 \sim \|(p_K^{\kappa\tau})^w u\|_{L^2}^2,$$

uniformly in K .

- V) *For all $\tau \in \mathbb{R}$, $\forall u \in \mathcal{S}(\mathbb{R}^n)$, we have*

$$\|(p_K^w)^\tau u\|_{L^2}^2 \sim \|(p_K^\tau)^w u\|_{L^2}^2 \sim \|(q_K^\tau)^w u\|_{L^2}^2 \sim \|(q_K^w)^\tau u\|_{L^2}^2,$$

uniformly in K .

- VI) *$\forall u \in \mathcal{S}(\mathbb{R}^n)$, we have*

$$(p_K^w u, u)_{L^2} \sim (q_K^w u, u)_{L^2},$$

uniformly in K .

- VII) *If $M \geq 0$, we have $p_K^w, q_K^w \geq Id$.*

Proof. Let $K > 0$ and let p and q two symbols and $M \in \mathbb{R}$ such that the hypotheses from (i) to (iv) are verified. For simplification, we will prove (I, II, III) just for the operator $(p_K^\tau)^w$, and we reason in the same way for $(q_K^\tau)^w$.

I. We will show that $(p_K^\tau)^w$ is invertible for all $\tau \in \mathbb{R}$. We note that p_K is an admissible weight (p is an admissible weight). We have $p_K \in S(p_K, \Gamma)$ uniformly in K . Indeed, using (iii) we have for all $\alpha, \beta \in \mathbb{N}^n$,

$$\begin{aligned} |\partial_v^\alpha \partial_\eta^\beta p_K| &\leq |\partial_v^\alpha \partial_\eta^\beta p| + |\partial_v^\alpha \partial_\eta^\beta K \langle v \rangle^M| \\ &\leq C_{\alpha, \beta} p + K C_{\alpha, \beta, M} \langle v \rangle^M \\ &\leq \tilde{C}_{\alpha, \beta, M} p_K. \end{aligned}$$

More generally we can show, by induction on $|\alpha|$ and Leibnitz's formula that for $\tau \in \mathbb{R}$,

$$\forall \alpha \in \mathbb{N}^{2n}, \quad |\partial_{v, \eta}^\alpha p_K^\tau| \leq C_{\alpha, \beta, \tau} p_K^\tau,$$

which gives $p_K^\tau \in S(p_K^\tau, \Gamma)$ uniformly in K .

Using formula (2.103), we may write

$$(p_K^\tau)^w (p_K^{-\tau})^w = \text{Id} - R_K^w, \quad (2.105)$$

where

$$R_K = - \int_0^1 (\partial_\eta p_K^\tau) \sharp_\theta (\partial_v (p_K^{-\tau})) d\theta + \int_0^1 (\partial_v p_K^\tau) \sharp_\theta (\partial_\eta (p_K^{-\tau})) d\theta$$

with $g \sharp_\theta h$ defined by

$$g \sharp_\theta h(Y) = \iint e^{-2i\sigma(Y-Y_1, Y-Y_2)/\theta} \frac{1}{2i} g(Y_1) h(Y_2) dY_1 dY_2 / (\pi\theta)^{2n}, \quad (2.106)$$

with $Y, Y_1, Y_2 \in \mathbb{R}^{2n}$ and σ a symplectic form defined in 2.104.

Let now N be the integer which is given in (2.102). By [15, Proposition 1.1], we can find a constant C_N and a positive integer l_N , both depending only on N but independent of K and θ , such that

$$\|(\partial_\eta p_K^\tau) \sharp_\theta (\partial_v (p_K^{-\tau}))\|_{N; S(1, \Gamma)} \leq C_N \|\partial_\eta p_K^\tau\|_{l_N; S(p_K^\tau, \Gamma)} \|\partial_v (p_K^{-\tau})\|_{l_N; S(p_K^{-\tau}, \Gamma)},$$

where the semi-norm $\|\cdot\|_{k; S(M, \Gamma)}$ is defined by (2.100).

Moreover, using (iv), we have $\partial_\eta p \in S(\varepsilon p + \varepsilon^{-1} \langle v \rangle^M, \Gamma)$, by taking $\varepsilon = K^{-1/2}$, we obtain $\partial_\eta p \in S(K^{-1/2} p_K, \Gamma)$. By writing $\partial_\eta p_K^\tau = \tau p_K^{\tau-1} \partial_\eta p$, we obtain

$$|\partial_\eta p_K^\tau| \leq \tilde{C}_N K^{-1/2} p_K^\tau.$$

Then arguing as above we can use induction on $|\alpha| + |\beta|$ to obtain, for $|\alpha| + |\beta| \geq 0$,

$$|\partial_v^\alpha \partial_\eta^\beta \partial_\eta p_K^\tau| \leq \tilde{C}_N K^{-1/2} p_K^\tau,$$

which gives $\partial_\eta p_K^\tau \in S(K^{-1/2} p_K^\tau, \Gamma)$ uniformly in K , moreover we have

$$\|(\partial_\eta p_K^\tau)\|_{l_N; S(p_K^\tau, \Gamma)} \leq \tilde{C}_N K^{-1/2}.$$

On the other hand we have $p_K^{-\tau} \in S(p_K^{-\tau}, \Gamma)$, and thus

$$\|\partial_v(p_K^{-\tau})\|_{L^N; S(p_K^{-\tau}, \Gamma)} \leq \tilde{C}_N,$$

with \tilde{C}_N a constant depending only on N but independent of K . As a result,

$$\|(\partial_\eta p_K^\tau) \#_\theta (\partial_v(p_K^{-\tau}))\|_{N; S(1, \Gamma)} \leq C_N \tilde{C}_N^2 K^{-1/2}. \quad (2.107)$$

Similarly,

$$\|(\partial_v p_K^\tau) \#_\theta (\partial_\eta(p_K^{-\tau}))\|_{N; S(1, \Gamma)} \leq C_N \tilde{C}_N^2 K^{-1/2}. \quad (2.108)$$

Using the estimates (2.107), (2.108) we will estimate the semi-norm $\|R_K\|_{N; S(1, \Gamma)}$. Indeed,

$$\|R_K\|_{N; S(1, \Gamma)} \leq \int_0^1 \|(\partial_v p_K^\tau) \#_\theta (\partial_\eta(p_K^{-\tau}))\|_{N; S(1, \Gamma)} d\theta + \int_0^1 \|(\partial_\eta p_K^\tau) \#_\theta (\partial_v(p_K^{-\tau}))\|_{N; S(1, \Gamma)} d\theta.$$

Then

$$\|R_K\|_{N; S(1, \Gamma)} \leq 2C_N \tilde{C}_N^2 K^{-1/2},$$

and thus by (2.102)

$$\|R_K^w\|_{\mathcal{B}(L^2)} \leq 2CC_N \tilde{C}_N^2 K^{-1/2}$$

with C a constant depending only on the dimension. Taking $K_1 = (4CC_N \tilde{C}_N^2)^2$, so we get for all $K \geq K_1$

$$\|R_K^w\|_{\mathcal{B}(L^2)} \leq \frac{1}{2} < 1,$$

this implies $\text{Id} - R_K^w$ is invertible in the space $\mathcal{B}(L^2)$. In addition, its inverse is given by

$$(\text{Id} - R_K^w)^{-1} = \sum_{i=0}^{\infty} (R_K^w)^i \in \mathcal{B}(L^2). \quad (2.109)$$

Based on (2.105) we obtain

$$(p_K^\tau)^w \left((p_K^{-\tau})^w (\text{Id} - R_K^w)^{-1} \right) = \text{Id}.$$

Similarly we can find a $\hat{R}_K \in S(1, \Gamma)$ such that

$$\left((\text{Id} - \hat{R}_K^w)^{-1} (p_K^{-\tau})^w \right) (p_K^\tau)^w = \text{Id}.$$

Based on the above $(p_K^\tau)^w$ is invertible and its inverse $[(p_K^\tau)^w]^{-1}$ is written in the form

$$[(p_K^\tau)^w]^{-1} = (p_K^{-\tau})^w (\text{Id} - R_K^w)^{-1} = (\text{Id} - \hat{R}_K^w)^{-1} (p_K^{-\tau})^w.$$

We have proved the conclusion in (I).

II. According to (I), we have $[(p_K^\tau)^w]^{-1}$ has the form

$$[(p_K^\tau)^w]^{-1} = (p_K^{-\tau})^w (\text{Id} - R_K^w)^{-1}.$$

By taking $H_K = (\text{Id} - R_K^w)^{-1}$ and using (2.109), we have H_K is a continuous operator in L^2 uniformly in K . So based on Theorem 2.38, we have H_K is a pseudo-differential operator. We note by $\delta(H_K)$ its symbol. We have $\delta(H_K)$ belongs to $S(1, \Gamma)$ uniformly in K . From the above, we have $[(p_K^\tau)^w]^{-1}$ is a pseudo-differential operator (compositions of pseudo-differential operators). We denote h_K its symbol, h_K has the form $h_K = p_K^{-\tau} \# \delta(H_K)$ and belongs to $S(p_K^{-\tau}, \Gamma)$ uniformly in K .

We have proved the conclusion in (II).

III. Using the Theorem 2.38, we have $[(p_K^\kappa)^w]^{-1}$ is a pseudo-differential operator. Taking into account that the composition of the pseudo-differential operators is a pseudo-differential operator, we obtain

$$\forall n \in \mathbb{Z}, [(p_K^\kappa)^w]^n \in \Psi(p_K^{\kappa n}, \Gamma). \quad (2.110)$$

Now consider the case of the exponents $\tau \in \mathbb{R}$. By pseudo-differential calculus the problem will be reduced to

$$\forall \tau \in I, [(p_K^\kappa)^w]^\tau \in \Psi(p_K^{\kappa \tau}, \Gamma),$$

where I an open interval of \mathbb{R} . Note that the operator $(p_K^\kappa)^w$ is self-adjoint because its symbol p_K^κ is real. Now we will show that $(p_K^\kappa)^w$ is a positive operator for K sufficiently large, which is equivalent to show that

$$\left((p_K^\kappa)^w u, u \right)_{L^2} \geq 0, \quad \forall u \in \mathcal{S}(\mathbb{R}^n). \quad (2.111)$$

Using again the formula (2.103), we can write

$$(p_K^{\kappa/2})^w (p_K^{\kappa/2})^w = (p_K^\kappa)^w - R_K^w, \quad (2.112)$$

where

$$R_K = - \int_0^1 (\partial_\eta p_K^{\kappa/2}) \#_\theta (\partial_v (p_K^{\kappa/2})) \, d\theta + \int_0^1 (\partial_v p_K^{\kappa/2}) \#_\theta (\partial_\eta (p_K^{\kappa/2})) \, d\theta$$

with $g \#_\theta h$ defined in (2.106). Based on the proof of (I), we have

$$\partial_\eta p_K^{\kappa/2} \in S(K^{-1/2} p_K^{\kappa/2}, \Gamma)$$

uniformly in K . Moreover we have $\partial_v p_K^{\kappa/2} \in S(p_K^{\kappa/2}, \Gamma)$, so we get

$$(\partial_\eta p_K^{\kappa/2}) \#_\theta (\partial_v (p_K^{\kappa/2})), (\partial_v p_K^{\kappa/2}) \#_\theta (\partial_\eta (p_K^{\kappa/2})) \text{ and } R_K \in S(K^{-1/2} p_K^{\kappa/2}, \Gamma)$$

uniformly in K . Writing R_K^w in the following form

$$R_K^w = K^{-1/2} (p_K^{\kappa/2})^w \underbrace{K^{1/2} [(p_K^{\kappa/2})^w]^{-1} R_K [(p_K^{\kappa/2})^w]^{-1} (p_K^{\kappa/2})^w}_{\in \mathcal{B}(L^2) \text{ uniformly in } K},$$

we obtain

$$|(R_K^w u, u)_{L^2}| \leq C_0 K^{-1/2} \|(p_K^{\kappa/2})^w u\|_{L^2}^2,$$

with C_0 some constant independent of K . Let $K_2 = 16C_0^2$, then using the relation (2.112) we get for all $K \geq K_2$

$$((p_K^\kappa)^w u, u)_{L^2} \geq \frac{3}{4} \|(p_K^{\kappa/2})^w u\|_{L^2}^2 \geq 0.$$

Then, using the following formula (see for example [131]) we can write

$$[(p_K^\kappa)^w]^\tau = -\frac{\sin(\pi\tau)}{\pi} \int_0^\infty s^\tau (s + (p_K^\kappa)^w)^{-1} ds, \quad \tau \in (-1, 0). \quad (2.113)$$

First, $s + (p_K^\kappa)^w$ is a pseudo-differential operator and its symbol $a_{K,s}$ verifies

$$\forall s \in (0, 1), \quad a_{K,s} \in S(p_K^\kappa, \Gamma)$$

and

$$\forall s \geq 1, \quad a_{K,s} \in S(s(p_K^\kappa), \Gamma) \text{ uniformly in } K \text{ and } s.$$

So using the Theorem 2.38, $(s + (p_K^\kappa)^w)^{-1}$ is a pseudo-differential operator and its symbol $b_{K,s}$ verifies

$$\forall s \in (0, 1), \quad b_{K,s} \in S((p_K^\kappa)^{-1}, \Gamma)$$

and

$$\forall s \geq 1, \quad b_{K,s} \in S(s^{-1}(p_K^\kappa)^{-1}, \Gamma) \text{ uniformly in } K \text{ and } s.$$

Then, taking $u \in \mathcal{S}(\mathbb{R}^n)$ we have

$$[(p_K^\kappa)^w]^\tau u = -\frac{\sin(\pi\tau)}{\pi} \int_0^\infty s^\tau (s + (p_K^\kappa)^w)^{-1} u ds,$$

using the formula (2.101), we get

$$\begin{aligned} [(p_K^\kappa)^w]^\tau u &= -\frac{\sin(\pi\tau)}{\pi} \int_0^\infty s^\tau \left(\frac{1}{(2\pi)^n} \int_{\mathbb{R}^{2n}} e^{i(v-v')\cdot\eta} b_{K,s}\left(\frac{v+v'}{2}, \eta\right) u(v') dv' d\eta \right) ds, \\ &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^{2n}} e^{i(v-v')\cdot\eta} \left(-\frac{\sin(\pi\tau)}{\pi} \int_0^\infty s^\tau b_{K,s}\left(\frac{v+v'}{2}, \eta\right) ds \right) u(v') dv' d\eta. \end{aligned}$$

So $[(p_K^\kappa)^w]^\tau$ is a pseudo-differential operator and its symbol d_K given by

$$d_K = -\frac{\sin(\pi\tau)}{\pi} \int_0^\infty s^\tau b_{K,s} ds.$$

Then we will show that $d_K \in S(p_K^{\kappa\tau}, \Gamma)$. Indeed, let $\alpha, \beta \in \mathbb{N}^n, (v, \eta) \in \mathbb{R}^{2n}$,

$$\begin{aligned} |\partial_v^\alpha \partial_\eta^\beta d_K(v, \eta)| &\leq C \left(\int_0^1 s^\tau |\partial_v^\alpha \partial_\eta^\beta b_{K,s}| ds + \int_1^\infty s^\tau |\partial_v^\alpha \partial_\eta^\beta b_{K,s}| ds \right), \\ &\leq C_{\alpha,\beta} \left(\int_0^1 s^\tau (p_K^\kappa)^{-1} ds + \int_1^\infty s^{\tau-1} (p_K^\kappa)^{-1} ds \right), \\ &\leq \tilde{C}_{\alpha,\beta} (p_K^\kappa)^{-1}, \end{aligned}$$

which gives that $d_K \in S(p_K^{\kappa\tau}, \Gamma)$. Take $I = (-1, 0)$, then we get

$$\forall \tau \in I, [(p_K^{\kappa})^w]^\tau \in \Psi(p_K^{\kappa\tau}, \Gamma). \quad (2.114)$$

In addition, $[(p_K^{\kappa})^w]^\tau$ is a bounded operator in L^2 uniformly in K . Indeed, $(s + (p_K^{\kappa})^w)^{-1} \in \Psi(1, \Gamma)$ uniformly in K , then there exists a constant $C_1 > 0$ such that

$$\forall s \in (0, 1), \|(s + (p_K^{\kappa})^w)^{-1}\|_{\mathcal{B}(L^2)} \leq C_1. \quad (2.115)$$

On the other hand we have

$$\forall u \in \mathcal{S}(\mathbb{R}^n), ((s + (p_K^{\kappa})^w)u, u)_{L^2} \geq s\|u\|_{L^2}^2,$$

which gives

$$\forall s \geq 1, \|(s + (p_K^{\kappa})^w)^{-1}\|_{\mathcal{B}(L^2)} \leq \frac{1}{s}. \quad (2.116)$$

Using (2.115), (2.116) and the formula (2.113), we obtain

$$\|[(p_K^{\kappa})^w]^\tau\|_{\mathcal{B}(L^2)} \leq C_2, \quad (2.117)$$

with C_2 some constant independent of K .

Then based on (2.110), (2.114) we obtain for all $\tau \in I$,

$$\forall n \in \mathbb{Z}, [[(p_K^{\kappa})^w]^\tau]^n \in \Psi(p_K^{\kappa n\tau}, \Gamma), \quad (2.118)$$

so we have

$$\forall \tau \in \mathbb{R}, [(p_K^{\kappa})^w]^\tau \in \Psi(p_K^{\kappa\tau}, \Gamma). \quad (2.119)$$

IV. We will show that

$$\forall u \in \mathcal{S}(\mathbb{R}^n), \|[(p_K^{\kappa})^w]^\tau u\|_{L^2}^2 \sim \|(p_K^{\kappa\tau})^w u\|_{L^2}^2, \quad (2.120)$$

otherwise we will show that there are two constants $c, C > 0$ independent of K such that for all $u \in \mathcal{S}(\mathbb{R}^n)$, we have:

$$\|(p_K^{\kappa\tau})^w u\|_{L^2}^2 \leq c \|[(p_K^{\kappa})^w]^\tau u\|_{L^2}^2, \quad (2.121)$$

and

$$\|[(p_K^{\kappa})^w]^\tau u\|_{L^2}^2 \leq C \|(p_K^{\kappa\tau})^w u\|_{L^2}^2. \quad (2.122)$$

We will start with inequality (2.121), using the conclusion (III), we have that $[(p_K^{\kappa})^w]^\tau$ is a pseudo-differential operator belongs to $\Psi(p_K^{\kappa\tau}, \Gamma)$ uniformly in K . Taking $v = [(p_K^{\kappa})^w]^{-\tau} u$, show inequality (2.121) is equivalent to showing that

$$\|(p_K^{\kappa\tau})^w [(p_K^{\kappa})^w]^{-\tau} v\|_{L^2}^2 \leq C_3 \|v\|_{L^2}^2 \quad \forall v \in L^2. \quad (2.123)$$

Using the Theorem 2.37, we have $(p_K^{\kappa\tau})^w [(p_K^{\kappa})^w]^{-\tau} \in \Psi(1, \Gamma) \hookrightarrow \mathcal{B}(L^2)$ uniformly in K , hence the inequality (2.121).

Similarly, using the fact that $[(p_K^{\kappa})^w]^\tau [(p_K^{\kappa\tau})^w]^{-1} \in \Psi(1, \Gamma) \hookrightarrow \mathcal{B}(L^2)$ uniformly in K , then taking $v = [(p_K^{\kappa\tau})^w]^{-1} u$, we get the inequality (2.122), so the estimate (2.120) is true.

V. We will first show that

$$\forall u \in \mathcal{S}(\mathbb{R}^n), \quad \|(p_K^\tau)^w u\|_{L^2}^2 \sim \|(q_K^\tau)^w u\|_{L^2}^2, \quad (2.124)$$

otherwise we will show that there are two constants $C_1, C_2 > 0$ independent of K such that for all $u \in \mathcal{S}(\mathbb{R}^n)$

$$\|(p_K^\tau)^w u\|_{L^2}^2 \leq C_1 \|(q_K^\tau)^w u\|_{L^2}^2, \quad (2.125)$$

and

$$\|(q_K^\tau)^w u\|_{L^2}^2 \leq C_2 \|(p_K^\tau)^w u\|_{L^2}^2. \quad (2.126)$$

We will start with inequality (2.125), using hypothesis (iii) we have that

$(q_K^\tau)^w \in \Psi(p_K^\tau, \Gamma)$ uniformly in K . Using the conclusion (II), we have $(q_K^\tau)^w$ is an invertible operator, moreover its inverse $[(q_K^\tau)^w]^{-1}$ belongs to $\Psi(p_K^{-\tau}, \Gamma)$ uniformly in K . Taking $v = [(q_K^\tau)^w]^{-1}u$, show inequality (2.125) is equivalent to showing that

$$\|(p_K^\tau)^w [(q_K^\tau)^w]^{-1}v\|_{L^2}^2 \leq C_1 \|v\|_{L^2}^2 \quad \forall v \in L^2. \quad (2.127)$$

Using the Theorem 2.37, we have $(p_K^\tau)^w [(q_K^\tau)^w]^{-1} \in \Psi(1, \Gamma) \hookrightarrow \mathcal{B}(L^2)$ uniformly in K , hence the inequality (2.127).

Similarly, using the fact that $(q_K^\tau)^w [(p_K^\tau)^w]^{-1} \in \Psi(1, \Gamma) \hookrightarrow \mathcal{B}(L^2)$ uniformly in K , then taking $v = [(p_K^\tau)^w]^{-1}u$, we get the inequality (2.126), so the estimate (2.124) is true.

In the same way we have

$$\forall u \in \mathcal{S}(\mathbb{R}^n), \quad \|(q_K^\tau)^w u\|_{L^2}^2 \sim \|(q_K^w)^\tau u\|_{L^2}^2. \quad (2.128)$$

Finally, using conclusion (IV) with $\kappa = 1$, we get that

$$\forall u \in \mathcal{S}(\mathbb{R}^n), \quad \|(p_K^w)^\tau u\|_{L^2}^2 \sim \|(p_K^\tau)^w u\|_{L^2}^2. \quad (2.129)$$

VI. We will show that

$$\forall u \in \mathcal{S}(\mathbb{R}^n), \quad (p_K^w u, u)_{L^2} \sim (q_K^w u, u)_{L^2}. \quad (2.130)$$

Indeed, let $u \in \mathcal{S}(\mathbb{R}^n)$ we have

$$(p_K^w u, u)_{L^2} = ((p_K^w)^{1/2} u, (p_K^w)^{1/2} u)_{L^2} = \|(p_K^w)^{1/2} u\|_{L^2}^2,$$

using the result (IV) with $\tau = \frac{1}{2}$, we obtain that

$$\|(p_K^w)^{1/2} u\|_{L^2}^2 \sim \|(p_K^{1/2})^w u\|_{L^2}^2 \sim \|(q_K^{1/2})^w u\|_{L^2}^2 \sim \|(q_K^w)^{1/2} u\|_{L^2}^2 \sim (q_K^w u, u)_{L^2}.$$

So the estimate (2.130) is true.

VII. We will show that

$$\forall u \in \mathcal{S}(\mathbb{R}^n), \quad (p_K^w u, u)_{L^2} \geq \|u\|_{L^2}^2.$$

Indeed, let $u \in \mathcal{S}(\mathbb{R}^n)$

$$(p_K^w u, u)_{L^2} = (p_{K-1}^w u, u)_{L^2} + (\langle v \rangle^M u, u)_{L^2},$$

based on the proof of (2.111), we can show that there exists a positive constant K_3 such that for all $K \geq K_3$ we have

$$(p_{K-1}^w u, u)_{L^2} \geq 0,$$

Which gives

$$(p_K^w u, u)_{L^2} = \underbrace{(p_{K-1}^w u, u)_{L^2}}_{\geq 0} + (\langle v \rangle^M u, u)_{L^2} \geq \|\langle v \rangle^{M/2} u\|_{L^2}^2.$$

With the condition $M \geq 0$ we finally obtain

$$(p_K^w u, u)_{L^2} \geq \|u\|_{L^2}^2.$$

Finally, for a symbol p which verifies the hypotheses of Theorem 2.40, we can fix

$$K \geq \underbrace{\max(K_1, K_2, K_3)}_{K_0} \quad (2.131)$$

and we apply the results of Theorem 2.40 to the operator p_K^w . □

6.3 Wick quantization

Finally let us recall some basic properties of the Wick quantization, which is also called anti-Wick in [118]. The importance in studying the Wick quantization lies in the facts that positive symbols give rise to positive operators.

Let $Y = (v, \eta)$ be a point in \mathbb{R}^6 . The Wick quantization of a symbol q is given by

$$q^{\text{Wick}} = (2\pi)^{-3} \int_{\mathbb{R}^6} q(Y) \Pi_Y \, dY,$$

where Π_Y is the orthogonal projection associated to the Gaussian φ_Y which is defined by

$$\varphi_Y(z) = \pi^{-3/4} e^{-\frac{1}{2}|z-v|^2} e^{iz \cdot \eta/2}, \quad \forall z \in \mathbb{R}^3.$$

More precisely, for any $u \in \mathcal{S}(\mathbb{R}^3)$ we have

$$(\Pi_Y u)(z) = (u, \varphi_Y)_{L^2(\mathbb{R}^3)} \varphi_Y(z), \quad \forall z \in \mathbb{R}^3.$$

The main property of the Wick quantization is its positivity, i.e.,

$$q(v, \eta) \geq 0, \quad \forall (v, \eta) \in \mathbb{R}^6 \text{ implies } q^{\text{Wick}} \geq 0. \quad (2.132)$$

According to Theorem 24.1 in [118], the Wick and Weyl quantizations of a symbol q are linked by the following identities

$$q^{\text{Wick}} = \left(q * \pi^{-3} e^{-|\cdot|^2} \right)^w = q^w + r^w \quad (2.133)$$

with

$$r(Y) = \pi^{-3} \int_0^1 \int_{\mathbb{R}^6} (1 - \theta) q''(Y + \theta Y_1) |Y_1|^2 e^{-|Y_1|^2} dY_1 d\theta, \quad (2.134)$$

where $q''(Y)$ is the Hessian of q at the point Y . Therefore, according to (2.102), if $q \in S(1, \Gamma)$ then q^{Wick} is a bounded operator in L^2 .

We also recall the following composition formula obtained in the proof of Proposition 3.4 in [93]

$$q_1^{\text{Wick}} q_2^{\text{Wick}} = [q_1 q_2 - q'_1 \cdot q'_2 + \frac{1}{i} \{q_1, q_2\}]^{\text{Wick}} + T, \quad (2.135)$$

with T a bounded operator in $L^2(\mathbb{R}^6)$ and q' is the gradient of q with respect to Y , when $q_1 \in L^\infty(\mathbb{R}^6)$ and q_2 is a smooth symbol

whose derivatives of order ≥ 2 are bounded on \mathbb{R}^6 . The notation $\{q_1, q_2\}$ denotes by the Poisson bracket defined by

$$\{q_1, q_2\} = \frac{\partial q_1}{\partial \eta} \cdot \frac{\partial q_2}{\partial v} - \frac{\partial q_1}{\partial v} \cdot \frac{\partial q_2}{\partial \eta}. \quad (2.136)$$

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Incompressible Navier-Stokes-Fourier limit from the Landau equation

Résumé

Dans cette partie, nous fournissons un résultat sur la dérivation du système de Navier-Stokes-Fourier incompressible à partir de l'équation de Landau pour les potentiels durs, maxwelliens et modérément mous. Pour obtenir ce résultat, nous étudions d'abord la théorie de Cauchy associée à l'équation de Landau adimensionnée pour des petites données initiales. Notre approche est basée sur le fait de prouver des estimations pour quelques normes de Sobolev adaptées à la solution qui sont uniformes en nombre de Knudsen. Ces estimations uniformes nous permettent également d'obtenir un résultat de convergence faible vers le système limite fluide.

Abstract

In this part, we provide a result on the derivation of the incompressible Navier-Stokes-Fourier system from the Landau equation for hard, Maxwellian and moderately soft potentials. To this end, we first investigate the Cauchy theory associated to the rescaled Landau equation for small initial data. Our approach is based on proving estimates of some adapted Sobolev norms of the solution that are uniform in the Knudsen number. These uniform estimates also allow us to obtain a result of weak convergence towards the fluid limit system.

1 Introduction

1.1 The model.

We start by introducing the Landau equation. This equation is a kinetic model in plasma physics that describes the evolution of the density function $f_\varepsilon = f_\varepsilon(t, x, v)$ representing at time $t \in \mathbb{R}^+$ the density of particles at position $x \in \mathbb{T}^3$ the 3-dimensional unit periodic box and velocity $v \in \mathbb{R}^3$. This equation is given by

$$\begin{cases} \partial_t f_\varepsilon + v \cdot \nabla_x f_\varepsilon = \frac{1}{\varepsilon} Q(f_\varepsilon, f_\varepsilon) \\ f_\varepsilon|_{t=0} = f_{\varepsilon,0}, \end{cases} \quad (3.1)$$

where $\varepsilon > 0$ is the Knudsen number which is the inverse of the average number of collisions for each particle per unit time and Q is the so-called Landau collision operator which acts on the variable v and which contains diffusion in velocity. More precisely, the Landau operator is defined by

$$Q(G, F) = \partial_i \int_{\mathbb{R}^3} a_{ij}(v - v_*) [G_* \partial_j F - F \partial_j G_*] dv_*, \quad (3.2)$$

and we use the convention of summation of repeated indices, and the usual derivatives are in the velocity variable v i.e. $\partial_i = \partial_{v_i}$. Hereafter we use the shorthand notations $G_* = G(v_*)$, $F = F(v)$, $\partial_j G_* = \partial_{v_{*j}} G(v_*)$, $\partial_j F = \partial_{v_j} F(v)$, etc. The matrix $A(v) = (a_{ij}(v))_{1 \leq i, j \leq 3}$ is symmetric, positive, definite, depends on the interaction between particles and is given by

$$a_{ij}(v) = |v|^{\gamma+2} \left(\delta_{ij} - \frac{v_i v_j}{|v|^2} \right), \quad \gamma \in [-3, 1].$$

We recall the standard classification: we call hard potentials if $\gamma \in (0, 1]$, Maxwellian molecules if $\gamma = 0$, moderately soft potentials if $\gamma \in [-2, 0)$, very soft potentials if $\gamma \in (-3, -2)$ and Coulombian potential if $\gamma = -3$. Hereafter we shall consider the cases of hard potentials, Maxwellian molecules and moderately soft potentials, i.e. $\gamma \in [-2, 1]$. We consider the fluctuation around the centered normalized Maxwellian distribution

$$\mu(v) = (2\pi)^{-3/2} e^{-|v|^2/2}$$

by setting $f_\varepsilon(t, x, v) = \mu + \varepsilon \mu^{1/2} g_\varepsilon(t, x, v)$, and

$$\begin{aligned} \Gamma(f, g) &= \mu^{-1/2} Q(\mu^{1/2} f, \mu^{1/2} g) \\ &= \partial_i \left[(a_{ij} * \mu^{1/2} f) \partial_j g \right] - (a_{ij} * \frac{v_i}{2} \mu^{1/2} f) \partial_j g \\ &\quad - \partial_i \left[(a_{ij} * \mu^{1/2} \partial_j f) g \right] + (a_{ij} * \frac{v_i}{2} \mu^{1/2} \partial_j f) g, \end{aligned}$$

the homogeneous linearized Landau operator \mathcal{L} takes the form

$$\begin{aligned}\mathcal{L} &= -\Gamma(\sqrt{\mu}, f) - \Gamma(f, \sqrt{\mu}) \\ &:= -\mathcal{L}_1 - \mathcal{L}_2.\end{aligned}$$

The operator \mathcal{L} acts only in variable v , is selfadjoint, and consists of a diffusion part and a compact part. Using for example [60], [129], we show that the diffusion part \mathcal{L}_1 writes as follows

$$\mathcal{L}_1 f = \nabla_v \cdot [\mathbf{A}(v) \nabla_v f] - \left(\mathbf{A}(v) \frac{v}{2} \cdot \frac{v}{2} \right) f + \nabla_v \cdot \left[\mathbf{A}(v) \frac{v}{2} \right] f,$$

where $\mathbf{A}(v) = (\bar{a}_{ij}(v))_{1 \leq i, j \leq 3}$ is a symmetric matrix defined through

$$\bar{a}_{ij} = a_{ij} *_v \mu,$$

and the compact part \mathcal{L}_2 is given by

$$\mathcal{L}_2 f = -\mu^{-1/2} \partial_i \left\{ \mu \left[a_{ij} *_v \left\{ \mu^{1/2} \left[\partial_j f + \frac{v_j}{2} f \right] \right\} \right] \right\}.$$

Now the original problem (3.1) is reduced to the Cauchy problem for the fluctuation g_ε

$$\begin{cases} \partial_t g_\varepsilon + \frac{1}{\varepsilon} v \cdot \nabla_x g_\varepsilon + \frac{1}{\varepsilon^2} \mathcal{L} g_\varepsilon = \frac{1}{\varepsilon} \Gamma(g_\varepsilon, g_\varepsilon) \\ g_\varepsilon|_{t=0} = g_{\varepsilon,0}, \end{cases} \quad (3.3)$$

where $g_{\varepsilon,0}$ is given by $f_{\varepsilon,0} = \mu + \varepsilon \mu^{1/2} g_{\varepsilon,0}$. From now on, we will always assume that

$$\int_{\mathbb{T}^3 \times \mathbb{R}^3} f_{\varepsilon,0}(x, v) \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} dx dv = \int_{\mathbb{T}^3 \times \mathbb{R}^3} \mu(v) \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} dx dv, \quad (3.4)$$

which implies that

$$\int_{\mathbb{T}^3 \times \mathbb{R}^3} g_\varepsilon(t, x, v) \mu^{1/2}(v) \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} dx dv = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{for all } t \geq 0,$$

since our equation preserves the total mass, momentum and energy.

1.2 Notations and functional spaces.

Throughout the paper we shall adopt the following notations. For $v \in \mathbb{R}^3$ we denote $\langle v \rangle = (1 + |v|^2)^{1/2}$, where we recall that $|v|$ is the canonical Euclidian norm of v in \mathbb{R}^3 . The gradient in velocity (resp. space) will be denoted by ∂_v (resp. ∂_x). For simplicity of notations, $a \sim b$ means that there exist constants $c_1, c_2 > 0$ depending only on fixed

number such that $c_1 b \leq a \leq c_2 b$; we abbreviate “ $\leq C$ ” to “ \lesssim ”, where C is a positive constant depending only on fixed number that may change from line to line.

In what follows, we shall write for $p \in [1, +\infty]$

$$L_x^p = L^p(\mathbb{T}^3), \quad L_v^p = L^p(\mathbb{R}^3), \quad L_x^p L_v^p = L^p(\mathbb{T}^3 \times \mathbb{R}^3).$$

For $p = 2$, we use the notations $(\cdot, \cdot)_{L_x^2}$, $(\cdot, \cdot)_{L_v^2}$ and $(\cdot, \cdot)_{L_x^2 L_v^2}$ to represent the inner product on the Hilbert spaces L_x^2 , L_v^2 and $L_x^2 L_v^2$ respectively.

For $m = m(v)$ a positive Borel weight function and $1 \leq p, q \leq \infty$, we define the space $L_x^q L_v^p(m)$ as the Lebesgue space associated to the norm, for $f = f(x, v)$

$$\|f\|_{L_x^q L_v^p(m)} = \| \|f\|_{L_v^p(m)} \|_{L_x^q} = \| \|mf\|_{L_v^p} \|_{L_x^q}.$$

We define for $s \in \mathbb{N}$ the spaces H_x^s to be the usual Sobolev space on \mathbb{T}^3 and $H_x^s L_v^2$ by the norm

$$\|f\|_{H_x^s L_v^2} = \left(\sum_{|k| \leq s} \|\partial_x^k f\|_{L_x^2 L_v^2}^2 \right)^{\frac{1}{2}}.$$

It is well known that the null space \mathcal{N} of \mathcal{L} is spanned by the set of collision invariants:

$$\mathcal{N}(\mathcal{L}) = \text{Span} \left\{ \sqrt{\mu}, v_1 \sqrt{\mu}, v_2 \sqrt{\mu}, v_3 \sqrt{\mu}, |v|^2 \sqrt{\mu} \right\}. \quad (3.5)$$

We also let \mathcal{N}^\perp denote the orthogonal space of \mathcal{N} with respect to the standard inner product $(\cdot, \cdot)_{L_v^2}$. Following [60], we introduce the $H_{v,*}^1$ -norm defined by

$$\|f\|_{H_{v,*}^1}^2 := \|\langle v \rangle^{\frac{\gamma}{2}+1} f\|_{L_v^2}^2 + \|\langle v \rangle^{\frac{\gamma}{2}} P_v \nabla_v f\|_{L_v^2}^2 + \|\langle v \rangle^{\frac{\gamma}{2}+1} (I - P_v) \nabla_v f\|_{L_v^2}^2, \quad (3.6)$$

this norm naturally arises in the study of the Landau equation. P_v is the projection on v , i.e. $P_v w = \left(w \cdot \frac{v}{|v|} \right) \frac{v}{|v|}$. We define the space $H_x^s H_{v,*}^1$ for $s \in \mathbb{N}$ associated to the norm

$$\|f\|_{\chi^s(\mathbb{T}_x^3 \times \mathbb{R}_v^3)}^2 = \sum_{|\alpha| \leq s} \int_{\mathbb{T}^3} \|\partial_x^\alpha f\|_{H_{v,*}^1}^2 dx. \quad (3.7)$$

We also define the space $H_x^s (H_{v,*}^1)'$ (is the dual of $H_x^s H_{v,*}^1$ w.r.t. $H_x^s L_v^2$) in the following way

$$\begin{aligned} \|f\|_{H_x^s (H_{v,*}^1)'} &:= \sup_{\|\phi\|_{H_x^s H_{v,*}^1} \leq 1} (f, \phi)_{H_x^s L_v^2} \\ &:= \sup_{\|\phi\|_{H_x^s H_{v,*}^1} \leq 1} \sum_{|\beta| \leq s} \left(\partial_x^\beta f, \partial_x^\beta \phi \right)_{L_x^2 L_v^2}. \end{aligned} \quad (3.8)$$

Let us recall the so called macro-micro decomposition of solutions

$$g = \Pi_0 g + (I - \Pi_0)g := g_1 + g_2, \quad (3.9)$$

où Π_0 est appelée la projection macroscopique de \mathcal{N} , $g_1 = \Pi_0 g$ is called the macroscopic projection of g and $g_2 = (I - \Pi_0)g$ is called the kinetic or microscopic part of g . Furthermore, we often use the following notation:

$$\Pi_0 g(t, x, v) = \{a(t, x) + v \cdot b(t, x) + |v|^2 c(t, x)\} \sqrt{\mu}, \quad \mathcal{A}(g) = (a, b, c), \quad (3.10)$$

where

$$a := \int_{\mathbb{R}^3} g \left(\frac{5}{2} - \frac{|v|^2}{2} \right) \mu^{1/2} dv, \quad b := \int_{\mathbb{R}^3} g v \mu^{1/2} dv, \quad c := \int_{\mathbb{R}^3} g \left(\frac{|v|^2}{6} - \frac{1}{2} \right) \mu^{1/2} dv.$$

We introduce the following instant energy functional and dissipation rate functional respectively

$$\begin{aligned} \mathcal{E}^2(g) &= \|g\|_{H_x^3 L_v^2}^2 = \|g_1\|_{H_x^3 L_v^2}^2 + \|g_2\|_{H_x^3 L_v^2}^2 \\ &\sim \|\mathcal{A}(g)\|_{H_x^3}^2 + \|g_2\|_{H_x^3 L_v^2}^2, \\ \mathcal{D}(g) &= \|g_2\|_{\chi^3(\mathbb{T}_x^3 \times \mathbb{R}_v^3)}, \\ \mathcal{C}(g) &= \|\nabla_x \Pi_0 g\|_{H_x^2 L_v^2} \sim \|\nabla_x \mathcal{A}(g)\|_{H_x^2}. \end{aligned} \quad (3.11)$$

These quantities will be in the heart of the coming study and the main results we present below.

1.3 Main results.

The first result is about the existence and uniqueness of the solution of the Landau equation. Notice that in the subsequent analysis, the Knudsen number ε is always supposed to be less than 1 and in our Cauchy theory, we have a smallness condition on our initial data, which is independent of ε .

Theorem 3.1. *There exists $M_0 > 0$ such that for $\varepsilon \in (0, 1)$ and $\|g_{\varepsilon,0}\|_{H_x^3 L_v^2} \leq M_0$, the Cauchy problem (3.3) admits a unique global solution*

$$g_\varepsilon \in L^\infty([0, \infty); H_x^3 L_v^2)$$

with the global energy estimate

$$\sup_{t \geq 0} \mathcal{E}^2(t) + C_0 \int_0^\infty \frac{1}{\varepsilon^2} \mathcal{D}^2(t) dt + C_0 \int_0^\infty \mathcal{C}^2(t) dt \leq C'_0 \mathcal{E}^2(0), \quad (3.12)$$

where $C_0, C'_0 > 0$ are independent of ε .

In the second part of this work, we study the limit to the incompressible Navier-Stokes-Fourier system associated with the Boussinesq equation which writes

$$\begin{cases} \partial_t u + u \cdot \nabla_x u + \nabla_x p = \nu \Delta_x u, \\ \partial_t \theta + u \cdot \nabla_x \theta = \kappa \Delta_x \theta, \\ \nabla_x \cdot u = 0, \\ \rho + \theta = 0. \end{cases} \quad (3.13)$$

In this system θ (the temperature), ρ (the density) and p (the pressure) are scalar unknowns and u (the velocity) is a 3-component unknown vector field. The pressure can actually be eliminated from the equations by applying to the momentum equation the Leray projector \mathcal{P} onto the space of divergence free vector field (precisely which is defined in (3.96)). This projector is bounded over H_x^N for all N , and in L_x^p for all $1 < p < \infty$. The viscosity coefficients are fully determined by the linearized Landau operator \mathcal{L} (see Section 4). The derivation theorem from (3.3) to (3.13) is the following.

Theorem 3.2. *Let M_0 be as in Theorem 3.1. For any $\varepsilon \in (0, 1)$, assume that the initial data $g_{\varepsilon,0}$ in (3.3) satisfy*

- 1) $g_{\varepsilon,0} \in H_x^3 L_v^2$,
- 2) $\|g_{\varepsilon,0}\|_{H_x^3 L_v^2} \leq M_0$,
- 3) *there exist scalar functions $\rho_0, \theta_0 \in H_x^3$ and vector-valued function $u_0 \in H_x^3$ such that*

$$g_{\varepsilon,0} \longrightarrow g_0, \quad \text{strongly in } H_x^3 L_v^2 \quad (3.14)$$

as $\varepsilon \longrightarrow 0$, where $g_0(x, v)$ is of the form

$$g_0(x, v) = \rho_0(x) \sqrt{\mu}(v) + u_0(x) \cdot v \sqrt{\mu}(v) + \theta_0(x) \left(\frac{|v|^2}{2} - \frac{3}{2} \right) \sqrt{\mu}(v). \quad (3.15)$$

Let now g_ε be the family of solutions to the Landau equation (3.3) constructed in Theorem 3.1. Then, as $\varepsilon \longrightarrow 0$,

$$g_\varepsilon \longrightarrow \rho \sqrt{\mu} + u \cdot v \sqrt{\mu} + \theta \left(\frac{|v|^2}{2} - \frac{3}{2} \right) \sqrt{\mu} \quad (3.16)$$

weakly- \star in $L^\infty([0, \infty); H_x^3 L_v^2)$. Here

$$(\rho, u, \theta) \in C(\mathbb{R}^+; H_x^2) \cap L^\infty(\mathbb{R}^+; H_x^3) \quad (3.17)$$

is a solution of the incompressible Navier-Stokes-Fourier equation (3.13) with initial data:

$$u|_{t=0} = \mathcal{P}u_0(x), \quad \theta|_{t=0} = \frac{3}{5}\theta_0(x) - \frac{2}{5}\rho_0(x), \quad (3.18)$$

where \mathcal{P} is the Leray projection. Moreover, the following convergence of the moments holds:

$$\mathcal{P} (g_\varepsilon, v \sqrt{\mu})_{L_v^2} \longrightarrow u, \quad (3.19)$$

$$\left(g_\varepsilon, \left(\frac{|v|^2}{5} - 1 \right) \sqrt{\mu} \right)_{L_v^2} \longrightarrow \theta, \quad (3.20)$$

strongly in $C(\mathbb{R}^+; H_x^2)$, weakly- \star in $L^\infty(\mathbb{R}^+; H_x^3)$ as $\varepsilon \longrightarrow 0$.

The history of derivation of Navier-Stokes equation from kinetic equations is very rich. This has been an active research field since the 70's with some major breakthrough. In particular Bardos, Golse and Levermore [9] initiated this program in the 80's, and the first convergence result without compactness assumption was given by Golse and Saint-Raymond in [49], following a series of hard works by Bardos, Golse, Levermore, Lions, Masmoudi and Saint-Raymond [10, 96, 50] the list given here is not exhaustive. More recently this program was tackled in various geometries (with boundary in [99, 79, 81]). All these results are obtained in a framework of weak solutions: the renormalized solutions for the Boltzmann equation (from DiPerna-Lions [37] or Mischler [102] for bounded domains) and the Leray solutions for the Navier-Stokes equations. Let us also mention [7] and [115] in which the theory of derivation of macroscopic fluid equations from kinetic equations has been treated exhaustively (although the Landau equation is not studied there). In what follows, we present previous results obtained in a framework of strong solutions and also comment the results obtained in the present paper.

In this framework of strong solutions, we first refer to [5, 17, 18] for Boltzmann equation with cutoff (with inelastic collisions for [5]) and to [83] for Boltzmann equation without cutoff in which a result of weak convergence to the fluid model is obtained. Briant in [17] justified the convergence from the Boltzmann equation with cutoff in the case of hard or Maxwellian potential to the incompressible Navier-Stokes equations on the torus. He used hypocoercivity to obtain a proof of existence and exponential decay (uniformly in the Knudsen number) for solutions around a global equilibrium in Sobolev spaces $H_{x,v}^N$ for $N \geq 3$. These results allow him to obtain a derivation of the incompressible Navier-Stokes equations as the Knudsen number tends to 0. In addition, he obtained strong convergence by adding additional conditions on the initial data and using a decomposition of the semigroup associated to the linearized equation coming from [40]. Roughly speaking, this analysis is based on the study of the spectrum of the Fourier transform in the space variable of the linearized operator. Let us also mention that all the hypocoercivity theory assumptions hold for several different kinetic models which include the Landau equation with hard, Maxwellian and moderately soft potentials. Briant, Merino-Aceituno and Mouhot in [18] obtained similar results in larger Sobolev spaces with polynomial weights using an "enlargement method" coming from [56] that takes into account the dependencies on the Knudsen number (which allowed them to weaken the assumptions on the data down to Sobolev spaces with polynomial weights). Notice that in the present paper, since our linearization is based on the following decomposition: $f_\varepsilon = \mu + \varepsilon\mu^{1/2}g_\varepsilon$, our original data $f_{\varepsilon,0}$ are supposed to enjoy an exponential decay.

We also mention that there is another type of results about the connection to fluid equations that are also obtained in the context of classical solutions. The general idea is to obtain first the solutions for the limiting fluid equations, then constructing a sequence of solutions (around the Maxwellian) of the scaled Kinetic equations (Boltzmann or Landau) for small Knudsen number ε . These solutions are of the form $f_\varepsilon = \mu + \varepsilon\mu^{1/2}(g^1 + \varepsilon g^2 + \dots + \varepsilon^n g_\varepsilon^n)$, where g^1, g^2, \dots can be determined by the Hilbert expansion, and g_ε^n is the error term. There is a large literature in this context, let us point out the work

[58] in which Guo verified the diffusive expansion (around the Maxwellian) for the Boltzmann equation with angular cutoff (hard and soft potential) and for Landau equation in the Coulombian case on the torus using a nonlinear energy method. This expansion is beyond the Navier–Stokes limit in the following sense: the coefficient g^1 is determined by the incompressible Navier–Stokes–Fourier equations and $g^2 \cdots g^{n-1}$ are determined by the linearized incompressible Navier–Stokes–Fourier equations with source term from the known kinetic part. In addition, he studied the decay in time uniformly in ε for the error term by nonlinear energy method, such a result is obtained in $H_{x,v}^N$ for $N \geq 8$.

Our aim in this article is to study the weak convergence process for the Landau equation in the cases of hard potentials, Maxwellian molecules and moderately soft potentials and verify its limit to the incompressible Navier-Stokes-Fourier equation with the Boussinesq relation on the torus. This equation has strong diffusion properties, which implies difficulties linked to the functional spaces. The convergence proof is based on a nonlinear energy method which is similar to [60, 58]. The idea is to establish microscopic and macroscopic estimates (thanks to the study of the classical 13-moments) uniformly in ε for our equation. These estimates allow us to obtain a global energy estimate (uniformly in ε) in $H_x^3 L_v^2$. Notice that in [60, 58], this type of results was obtained in $H_{x,v}^N$ with $N \geq 8$, we have thus enlarge the space in which such a result is available. In terms of functional spaces, our result is comparable to the one in [83] in which the authors used this method to derive the incompressible Navier-Stokes-Fourier equation but from Boltzmann without cutoff on the whole space. Hence the difference with our case concerns the choice of the instant energy and the dissipation rate functional to capture the structure of the rescaled equation, see (3.6), (3.7) and (3.11) for our case. We shall sometimes follow the line of the proof of [83]. But we pay very much attention to the construction of local solutions (uniformly with respect to the Knudsen number), the linear problem being handled thanks to Lion’s theorem (see [95]). A special focus is also given on the weak- \star limit at the very end of this article using in particular the Aubin-Lions-Simon theorem and some compensated compactness argument from [97] (see [5, 49] for similar approaches).

We finally emphasize that a great challenge in this field would be to get strong convergence, that we hope to do in the future using the hypoelliptic properties of the Landau operator.

Organization of the article. In Section 2, we study the construction of the local solution. In Section 3, we show the global existence and we establish the uniform energy estimate. Section 4 is devoted for the incompressible Navier-Stokes limit. In Appendix (Section 5), we give a proof for the existence of a local solution for the linear Landau equation.

2 Construction of Local Solutions

In this section, we show the existence of a local solution to equation (3.3). First, we start with nonlinear estimates where we use the idea of the proof of Lemma 3.5 in [26]. Then,

we are interested in the construction of local solutions for Landau equation where we use the technique presented in section 2 in [83].

2.1 Nonlinear estimates.

We prove in this section some estimates on the nonlinear operator Γ .

Lemma 3.3. *We have:*

$$\begin{aligned} |(a_{ij} * \mu^{1/2} f)(v)| + \left| \left(a_{ij} * \frac{v_i}{2} \mu^{1/2} f \right) (v) \right| + |(a_{ij} * \mu^{1/2} \partial_j f)(v)| + \left| \left(a_{ij} * \frac{v_i}{2} \mu^{1/2} \partial_j f \right) (v) \right| \\ + |(a_{ij} * \mu^{1/2} f)(v) v_i v_j| + |(a_{ij} * \mu^{1/2} f)(v) v_j| \lesssim \langle v \rangle^{\gamma+2} \|f\|_{L_v^2}. \end{aligned}$$

Proof. For the first term, we have

$$|(a_{ij} * \mu^{1/2} f)(v)| = \left| \int_{v_*} a_{ij}(v - v_*) \mu^{1/2}(v_*) f_* \right| \lesssim \int_{v_*} \langle v \rangle^{\gamma+2} \langle v_* \rangle^{\gamma+2} \mu_*^{1/2} |f_*| \lesssim \langle v \rangle^{\gamma+2} \|f\|_{L_v^2}.$$

In a similar way we get

$$\left| \left(a_{ij} * \frac{v_i}{2} \mu^{1/2} f \right) (v) \right| \lesssim \langle v \rangle^{\gamma+2} \|f\|_{L_v^2}.$$

For the third term, we have

$$a_{ij} * \mu^{1/2} \partial_j f = \partial_j a_{ij} * \mu^{1/2} f - a_{ij} * \partial_j \mu^{1/2} f,$$

then

$$\begin{aligned} |(a_{ij} * \mu^{1/2} \partial_j f)(v)| &\lesssim |\partial_j a_{ij} * \mu^{1/2} f(v)| + |a_{ij} * \partial_j \mu^{1/2} f(v)| \\ &\lesssim \int_{v_*} \langle v \rangle^{\gamma+1} \langle v_* \rangle^{\gamma+1} \mu_*^{1/2} |f_*| + \int_{v_*} \langle v \rangle^{\gamma+2} \langle v_* \rangle^{\gamma+3} \mu_*^{1/2} |f_*| \\ &\lesssim \langle v \rangle^{\gamma+2} \|f\|_{L_v^2}. \end{aligned}$$

In a similar way we get

$$\left| \left(a_{ij} * \frac{v_i}{2} \mu^{1/2} \partial_j f \right) (v) \right| \lesssim \langle v \rangle^{\gamma+2} \|f\|_{L_v^2}.$$

Recall that 0 is an eigenvalue of the matrix $A(v) = (a_{ij}(v))_{1 \leq i, j \leq 3}$ with corresponding eigenvector v so that $a_{ij}(v - v_*) v_j = a_{ij}(v - v_*) v_{*j}$ and $a_{ij}(v - v_*) v_i v_j = a_{ij}(v - v_*) v_{*i} v_{*j}$. Using this we can easily obtain,

$$\begin{aligned} |(a_{ij} * \mu^{1/2} f)(v) v_i v_j| &= \left| \int_{v_*} a_{ij}(v - v_*) v_i v_j \mu_*^{1/2} f_* \right| = \left| \int_{v_*} a_{ij}(v - v_*) v_{*i} v_{*j} \mu_*^{1/2} f_* \right| \\ &\lesssim \int_{v_*} \langle v \rangle^{\gamma+2} \langle v_* \rangle^{\gamma+4} \mu_*^{1/2} |f_*| \\ &\lesssim \langle v \rangle^{\gamma+2} \|f\|_{L_v^2}. \end{aligned}$$

In a similar way we get

$$|(a_{ij} * \mu^{1/2} f)(v)v_j| \lesssim \langle v \rangle^{\gamma+2} \|f\|_{L_v^2}.$$

□

Lemma 3.4. *The following estimate holds:*

$$(\Gamma(f, g), h)_{L_v^2} \lesssim \|f\|_{L_v^2} \|g\|_{H_{v,*}^1} \|h\|_{H_{v,*}^1}. \quad (3.21)$$

Proof. We write

$$\begin{aligned} (\Gamma(f, g), h)_{L_v^2} &= - \int (a_{ij} * \mu^{1/2} f) \partial_j g \partial_i h - \int \left(a_{ij} * \frac{v_i}{2} \mu^{1/2} f \right) \partial_j g h \\ &\quad + \int (a_{ij} * \mu^{1/2} \partial_j f) g \partial_i h + \int \left(a_{ij} * \frac{v_i}{2} \mu^{1/2} \partial_j f \right) g h \\ &= - \int (A(v) * \mu^{1/2} f) \nabla_v g \cdot \nabla_v h - \int X(v) \cdot \nabla_v g h \\ &\quad + \int Y(v) \cdot \nabla_v h g + \int \left(a_{ij} * \frac{v_i}{2} \mu^{1/2} \partial_j f \right) g h \\ &=: T_1 + T_2 + T_3 + T_4. \end{aligned}$$

Here $X(v)$ (resp. $Y(v)$) are vectors with coefficients $X_j(v)$ (resp. $Y_i(v)$) which are written in the following form:

$$X_j(v) = \sum_{i=1}^3 a_{ij} * \frac{v_i}{2} \mu^{1/2} f \quad \text{and} \quad Y_i(v) = \sum_{j=1}^3 a_{ij} * \mu^{1/2} \partial_j f.$$

Step 1. For the first term, since the estimate for $|v| \leq 1$ is evident, we only consider the case $|v| \geq 1$. We decompose $\nabla_v g = P_v \nabla_v g + (I - P_v) \nabla_v g$, and similarly for $\nabla_v h$ where we recall that $P_v \nabla_v g = v|v|^{-2}(v \cdot \nabla_v g)$. We hence write

$$\begin{aligned} T_1 &= \int (A(v) * \mu^{1/2} f) P_v \nabla_v g \cdot P_v \nabla_v h + \int (A(v) * \mu^{1/2} f) P_v \nabla_v g \cdot (I - P_v) \nabla_v h \\ &\quad + \int (A(v) * \mu^{1/2} f) (I - P_v) \nabla_v g \cdot P_v \nabla_v h + \int (A(v) * \mu^{1/2} f) (I - P_v) \nabla_v g \cdot (I - P_v) \nabla_v h \\ &:= T_{11} + T_{12} + T_{13} + T_{14}. \end{aligned}$$

Therefore we have

$$T_{11} = \int (A(v) * \mu^{1/2} f) \frac{(v \cdot \nabla_v g)}{|v|^2} v \cdot \frac{(v \cdot \nabla_v h)}{|v|^2} v,$$

thanks to Lemma 3.3, we obtain

$$\begin{aligned} |T_{11}| &\lesssim \|f\|_{L_v^2} \int \langle v \rangle^{\gamma+2} |v|^{-2} |\nabla_v g| |\nabla_v h| \\ &\lesssim \|f\|_{L_v^2} \|\langle v \rangle^{\frac{\gamma}{2}} \nabla_v g\|_{L_v^2} \|\langle v \rangle^{\frac{\gamma}{2}} \nabla_v h\|_{L_v^2}. \end{aligned}$$

Moreover

$$T_{12} = \int (A(v) * \mu^{1/2} f) \frac{(v \cdot \nabla_v g)}{|v|^2} v \cdot (I - P_v) \nabla_v h,$$

then

$$\begin{aligned} |T_{12}| &\lesssim \|f\|_{L_v^2} \int \langle v \rangle^{\gamma+2} |v|^{-1} |\nabla_v g| |(I - P_v) \nabla_v h| \\ &\lesssim \|f\|_{L_v^2} \|\langle v \rangle^{\frac{\gamma}{2}} \nabla_v g\|_{L_v^2} \|\langle v \rangle^{\frac{\gamma}{2}+1} (I - P_v) \nabla_v h\|_{L_v^2}. \end{aligned}$$

Similarly

$$|T_{13}| \lesssim \|f\|_{L_v^2} \|\langle v \rangle^{\frac{\gamma}{2}} \nabla_v g\|_{L_v^2} \|\langle v \rangle^{\frac{\gamma}{2}+1} (I - P_v) \nabla_v h\|_{L_v^2}.$$

For the term T_{14} we obtain

$$T_{14} = \int (A(v) * \mu^{1/2} f) (I - P_v) \nabla_v g \cdot (I - P_v) \nabla_v h,$$

then

$$\begin{aligned} |T_{14}| &\lesssim \|f\|_{L_v^2} \int \langle v \rangle^{\gamma+2} |(I - P_v) \nabla_v g| |(I - P_v) \nabla_v h| \\ &\lesssim \|f\|_{L_v^2} \|\langle v \rangle^{\frac{\gamma}{2}+1} (I - P_v) \nabla_v g\|_{L_v^2} \|\langle v \rangle^{\frac{\gamma}{2}+1} (I - P_v) \nabla_v h\|_{L_v^2}. \end{aligned}$$

Step 2. Let us investigate the second term T_2 , and again we only consider $|v| > 1$. The same argument as for T_1 gives us

$$\begin{aligned} T_2 &= - \int X(v) \cdot \{P_v \nabla_v g + (I - P_v) \nabla_v g\} h \\ &:= T_{21} + T_{22}. \end{aligned}$$

We have

$$T_{21} = - \int X(v) \cdot v \frac{(v \cdot \nabla_v g)}{|v|^2} h,$$

then

$$\begin{aligned} |T_{21}| &\lesssim \|f\|_{L_v^2} \int \langle v \rangle^{\gamma+2} |v|^{-1} |\nabla_v g| |h| \\ &\lesssim \|f\|_{L_v^2} \|\langle v \rangle^{\frac{\gamma}{2}} \nabla_v g\|_{L_v^2} \|\langle v \rangle^{\frac{\gamma}{2}+1} h\|_{L_v^2}. \end{aligned}$$

For the other term we get

$$T_{22} = - \int X(v) \cdot (I - P_v) \nabla_v g h$$

then

$$\begin{aligned} |T_{22}| &\lesssim \|f\|_{L_v^2} \int \langle v \rangle^{\gamma+2} |(I - P_v) \nabla_v g| |h| \\ &\lesssim \|f\|_{L_v^2} \|\langle v \rangle^{\frac{\gamma}{2}+1} (I - P_v) \nabla_v g\|_{L_v^2} \|\langle v \rangle^{\frac{\gamma}{2}+1} h\|_{L_v^2}. \end{aligned}$$

Step 3. For the term T_3 ,

$$\begin{aligned} T_3 &= - \int Y(v) \cdot \{P_v \nabla_v h + (I - P_v) \nabla_v h\} g \\ &:= T_{31} + T_{32}. \end{aligned}$$

Using the same proof in step 2, we have

$$T_{31} \lesssim \|f\|_{L_v^2} \|\langle v \rangle^{\frac{\gamma}{2}+1} g\|_{L_v^2} \|\langle v \rangle^{\frac{\gamma}{2}} \nabla_v h\|_{L_v^2},$$

and

$$|T_{32}| \lesssim \|f\|_{L_v^2} \|\langle v \rangle^{\frac{\gamma}{2}+1} g\|_{L_v^2} \|\langle v \rangle^{\frac{\gamma}{2}+1} (I - P_v) \nabla_v h\|_{L_v^2}.$$

Step 4. We finally investigate the term T_4 and we have

$$T_4 = - \int \left(a_{ij} * \frac{v_i}{2} \mu^{1/2} \partial_j f \right) h g,$$

then

$$|T_4| \lesssim \|f\|_{L_v^2} \|\langle v \rangle^{\frac{\gamma}{2}+1} g\|_{L_v^2} \|\langle v \rangle^{\frac{\gamma}{2}+1} h\|_{L_v^2}.$$

□

The next lemma gives an estimate on the nonlinear collision operator Γ in terms of instant energy functional and dissipation rate.

Lemma 3.5. Consider f such that $\int_{\mathbb{T}^3} \Pi_0 f \, dx = 0$, we have

$$(\Gamma(f, f), h)_{H_x^3 L_v^2} \lesssim \mathcal{E}(f) \{ \mathcal{C}(f) + \mathcal{D}(f) \} \mathcal{D}(h), \quad (3.22)$$

therefore

$$\|\Gamma(f, f)\|_{H_x^3 (H_{v,*}^1)'} \lesssim \|f\|_{H_x^3 L_v^2} \|f\|_{H_x^3 H_{v,*}^1}, \quad (3.23)$$

where we recall that the space $H_x^3 (H_{v,*}^1)'$ is defined in (3.8) and $\mathcal{E}, \mathcal{D}, \mathcal{C}$ are defined in (3.11).

Proof. We have that

$$(\Gamma(f, f), h)_{H_x^3 L_v^2} = (\Gamma(f, f), h_2)_{L_x^2 L_v^2} + \sum_{1 \leq |\beta| \leq 3} (\partial_x^\beta \Gamma(f, g), \partial_x^\beta h_2)_{L_x^2 L_v^2}$$

because

$$(\Gamma(f, f), \phi)_{L_v^2} = 0 \quad \text{for } \phi = \sqrt{\mu}, v_i \sqrt{\mu}, |v|^2 \sqrt{\mu}$$

and

$$\partial_x^\beta \Gamma(f, f) = \sum_{\beta_1 + \beta_2 = \beta} C_{\beta_1, \beta_2} \Gamma(\partial_x^{\beta_1} f, \partial_x^{\beta_2} f).$$

The proof of the lemma is a consequence of Lemma 3.4 together with the following inequalities, that we shall use in the sequel when integrating in $x \in \mathbb{T}^3$

$$\|u\|_{L^\infty(\mathbb{T}^3)} \lesssim \|u\|_{H^2(\mathbb{T}^3)}, \quad \|u\|_{L^6(\mathbb{T}^3)} \lesssim \|u\|_{H^1(\mathbb{T}^3)}, \quad \|u\|_{L^3(\mathbb{T}^3)} \lesssim \|u\|_{H^1(\mathbb{T}^3)}^{1/2} \|u\|_{L^2(\mathbb{T}^3)}^{1/2}.$$

Step 1. Using Lemma 3.4 we easily get,

$$\begin{aligned} (\Gamma(f, f), h_2)_{L_x^2 L_v^2} &\lesssim \int_{\mathbb{T}^3} (\|f\|_{L_v^2} \|f_2\|_{H_{v,*}^1} \|h_2\|_{H_{v,*}^1} + \|f\|_{L_v^2} \|f_1\|_{L_v^2} \|h_2\|_{H_{v,*}^1}) \\ &\lesssim \|f\|_{H_x^2 L_v^2} \|f_2\|_{L_x^2 H_{v,*}^1} \|h_2\|_{L_x^2 H_{v,*}^1} + \|f\|_{H_x^2 L_v^2} \|f_1\|_{L_x^2 L_v^2} \|h_2\|_{L_x^2 H_{v,*}^1}, \end{aligned}$$

furthermore, $\Pi_0 f$ has zero mean on the torus, thus we can apply Poincaré inequality on the torus and we obtain

$$(\Gamma(f, f), h_2)_{L_x^2 L_v^2} \lesssim \mathcal{E}(f) \{ \mathcal{C}(f) + \mathcal{D}(f) \} \mathcal{D}(h).$$

Step 2. Case $|\beta| = 1$. Arguing as in the previous step, from Lemma 3.4, it follows

$$\begin{aligned} &(\Gamma(f, \partial_x^\beta f), \partial_x^\beta h_2)_{L_x^2 L_v^2} \\ &\lesssim \int_{\mathbb{T}^3} (\|f\|_{L_v^2} \|\nabla_x f_2\|_{H_{v,*}^1} \|\nabla_x h_2\|_{H_{v,*}^1} + \|f\|_{L_v^2} \|\nabla_x f_1\|_{L_v^2} \|\nabla_x h_2\|_{H_{v,*}^1}) \\ &\lesssim \|f\|_{H_x^2 L_v^2} \|\nabla_x f_2\|_{L_x^2 H_{v,*}^1} \|\nabla_x h_2\|_{L_x^2 H_{v,*}^1} + \|f\|_{H_x^2 L_v^2} \|\nabla_x f_1\|_{L_x^2 L_v^2} \|\nabla_x h_2\|_{L_x^2 H_{v,*}^1} \\ &\lesssim \mathcal{E}(f) \{ \mathcal{C}(f) + \mathcal{D}(f) \} \mathcal{D}(h). \end{aligned}$$

Moreover, we have

$$\begin{aligned} &(\Gamma(\partial_x^\beta f, f), \partial_x^\beta h_2)_{L_x^2 L_v^2} \\ &\lesssim \int_{\mathbb{T}^3} (\|\nabla_x f\|_{L_v^2} \|f_2\|_{H_{v,*}^1} \|\nabla_x h_2\|_{H_{v,*}^1} + \|\nabla_x f\|_{L_v^2} \|f_1\|_{L_v^2} \|\nabla_x h_2\|_{H_{v,*}^1}) \\ &\lesssim \|\nabla_x f\|_{H_x^2 L_v^2} \|f_2\|_{L_x^2 H_{v,*}^1} \|\nabla_x h_2\|_{L_x^2 H_{v,*}^1} + \|\nabla_x f\|_{H_x^2 L_v^2} \|f_1\|_{L_x^2 L_v^2} \|\nabla_x h_2\|_{L_x^2 H_{v,*}^1}, \end{aligned}$$

using Poincaré inequality on the torus, we get

$$(\Gamma(\partial_x^\beta f, f), \partial_x^\beta h_2)_{L_x^2 L_v^2} \lesssim \mathcal{E}(f) \{ \mathcal{C}(f) + \mathcal{D}(f) \} \mathcal{D}(h).$$

Step 3. Case $|\beta| = 2$. When $\beta_2 = \beta$ we have

$$\begin{aligned}
 & \left(\Gamma(f, \partial_x^\beta f), \partial_x^\beta h_2 \right)_{L_x^2 L_v^2} \\
 & \lesssim \int_{\mathbb{T}^3} \left(\|f\|_{L_v^2} \|\nabla_x^2 f_2\|_{H_{v,*}^1} \|\nabla_x^2 h_2\|_{H_{v,*}^1} + \|f\|_{L_v^2} \|\nabla_x^2 f_1\|_{L_v^2} \|\nabla_x^2 h_2\|_{H_{v,*}^1} \right) \\
 & \lesssim \|f\|_{H_x^2 L_v^2} \|\nabla_x^2 f_2\|_{L_x^2 H_{v,*}^1} \|\nabla_x^2 h_2\|_{L_x^2 H_{v,*}^1} + \|f\|_{H_x^2 L_v^2} \|\nabla_x^2 f_1\|_{L_x^2 L_v^2} \|\nabla_x^2 h_2\|_{L_x^2 H_{v,*}^1} \\
 & \lesssim \mathcal{E}(f) \{ \mathcal{C}(f) + \mathcal{D}(f) \} \mathcal{D}(h).
 \end{aligned}$$

If $|\beta_1| = |\beta_2| = 1$ then we obtain

$$\begin{aligned}
 & \left(\Gamma(\partial_x^{\beta_1} f, \partial_x^{\beta_2} f), \partial_x^\beta h_2 \right)_{L_x^2 L_v^2} \\
 & \lesssim \int_{\mathbb{T}^3} \left(\|\nabla_x f\|_{L_v^2} \|\nabla_x f_2\|_{H_{v,*}^1} \|\nabla_x^2 h_2\|_{H_{v,*}^1} + \|\nabla_x f\|_{L_v^2} \|\nabla_x f_1\|_{L_v^2} \|\nabla_x^2 h_2\|_{H_{v,*}^1} \right) \\
 & \lesssim \|\nabla_x f\|_{H_x^2 L_v^2} \|\nabla_x f_2\|_{L_x^2 H_{v,*}^1} \|\nabla_x^2 h_2\|_{L_x^2 H_{v,*}^1} + \|\nabla_x f\|_{H_x^2 L_v^2} \|\nabla_x f_1\|_{L_x^2 L_v^2} \|\nabla_x^2 h_2\|_{L_x^2 H_{v,*}^1} \\
 & \lesssim \mathcal{E}(f) \{ \mathcal{C}(f) + \mathcal{D}(f) \} \mathcal{D}(h).
 \end{aligned}$$

Finally, when $\beta_1 = \beta$ we get

$$\begin{aligned}
 & \left(\Gamma(\partial_x^\beta f, f), \partial_x^\beta h_2 \right)_{L_x^2 L_v^2} \\
 & \lesssim \int_{\mathbb{T}^3} \left(\|\nabla_x^2 f\|_{L_v^2} \|f_2\|_{H_{v,*}^1} \|\nabla_x^2 h_2\|_{H_{v,*}^1} + \|\nabla_x^2 f\|_{L_v^2} \|f_1\|_{L_v^2} \|\nabla_x^2 h_2\|_{H_{v,*}^1} \right) \\
 & \lesssim \|\nabla_x^2 f\|_{L_x^6 L_v^2} \|f_2\|_{L_x^3 H_{v,*}^1} \|\nabla_x^2 h_2\|_{L_x^2 H_{v,*}^1} + \|\nabla_x^2 f\|_{L_x^6 L_v^2} \|f_1\|_{L_x^3 L_v^2} \|\nabla_x^2 h_2\|_{L_x^2 H_{v,*}^1} \\
 & \lesssim \|\nabla_x^2 f\|_{H_x^1 L_v^2} \|f_2\|_{L_x^2 H_{v,*}^1}^{1/2} \|f_2\|_{H_x^1 H_{v,*}^1}^{1/2} \|\nabla_x^2 h_2\|_{L_x^2 H_{v,*}^1} \\
 & \quad + \|\nabla_x^2 f\|_{H_x^1 L_v^2} \|f_1\|_{L_x^2 L_v^2}^{1/2} \|f_1\|_{H_x^1 H_{v,*}^1}^{1/2} \|\nabla_x^2 h_2\|_{L_x^2 H_{v,*}^1} \\
 & \lesssim \mathcal{E}(f) \{ \mathcal{C}(f) + \mathcal{D}(f) \} \mathcal{D}(h).
 \end{aligned}$$

Step 4. Case $|\beta| = 3$. When $\beta_2 = \beta$ we have obtained

$$\begin{aligned}
 & \left(\Gamma(f, \partial_x^\beta f), \partial_x^\beta h_2 \right)_{L_x^2 L_v^2} \\
 & \lesssim \|f\|_{H_x^2 L_v^2} \|\nabla_x^3 f_2\|_{L_x^2 H_{v,*}^1} \|\nabla_x^3 h_2\|_{L_x^2 H_{v,*}^1} + \|f\|_{H_x^2 L_v^2} \|\nabla_x^3 f_1\|_{L_x^2 L_v^2} \|\nabla_x^3 h_2\|_{L_x^2 H_{v,*}^1} \\
 & \lesssim \mathcal{E}(f) \{ \mathcal{C}(f) + \mathcal{D}(f) \} \mathcal{D}(h).
 \end{aligned}$$

If $|\beta_1| = 1$ and $|\beta_2| = 2$ then we obtain

$$\begin{aligned}
 & \left(\Gamma(\partial_x^{\beta_1} f, \partial_x^{\beta_2} f), \partial_x^\beta h_2 \right)_{L_x^2 L_v^2} \\
 & \lesssim \int_{\mathbb{T}^3} \left(\|\nabla_x f\|_{L_v^2} \|\nabla_x^2 f_2\|_{H_{v,*}^1} \|\nabla_x^3 h_2\|_{H_{v,*}^1} + \|\nabla_x f\|_{L_v^2} \|\nabla_x^2 f_1\|_{L_v^2} \|\nabla_x^3 h_2\|_{H_{v,*}^1} \right) \\
 & \lesssim \|\nabla_x f\|_{H_x^2 L_v^2} \|\nabla_x^2 f_2\|_{L_x^2 H_{v,*}^1} \|\nabla_x^3 h_2\|_{L_x^2 H_{v,*}^1} + \|\nabla_x f\|_{H_x^2 L_v^2} \|\nabla_x^2 f_1\|_{L_x^2 L_v^2} \|\nabla_x^3 h_2\|_{L_x^2 H_{v,*}^1} \\
 & \lesssim \mathcal{E}(f) \{ \mathcal{C}(f) + \mathcal{D}(f) \} \mathcal{D}(h).
 \end{aligned}$$

When $|\beta_1| = 2$ and $|\beta_2| = 1$ then we get

$$\begin{aligned}
 & \left(\Gamma(\partial_x^{\beta_1} f, \partial_x^{\beta_2} f), \partial_x^{\beta} h_2 \right)_{L_x^2 L_v^2} \\
 & \lesssim \int_{\mathbb{T}^3} \left(\|\nabla_x^2 f\|_{L_v^2} \|\nabla_x f_2\|_{H_{v,*}^1} \|\nabla_x^3 h_2\|_{H_{v,*}^1} + \|\nabla_x^2 f\|_{L_v^2} \|\nabla_x f_1\|_{L_v^2} \|\nabla_x^3 h_2\|_{H_{v,*}^1} \right) \\
 & \lesssim \|\nabla_x^2 f\|_{H_x^1 L_v^2} \|\nabla_x f_2\|_{L_x^2 L_v^2}^{1/2} \|\nabla_x f_2\|_{H_x^1 H_{v,*}^1}^{1/2} \|\nabla_x^3 h_2\|_{L_x^2 H_{v,*}^1} \\
 & \quad + \|\nabla_x f\|_{H_x^1 L_v^2} \|\nabla_x f_1\|_{L_x^2 L_v^2}^{1/2} \|\nabla_x f_1\|_{H_x^1 L_v^2}^{1/2} \|\nabla_x^3 h_2\|_{L_x^2 H_{v,*}^1} \\
 & \lesssim \mathcal{E}(f) \{ \mathcal{C}(f) + \mathcal{D}(f) \} \mathcal{D}(h).
 \end{aligned}$$

Finally, when $\beta_1 = \beta$, it follows

$$\begin{aligned}
 & \left(\Gamma(\partial_x^{\beta} f, f), \partial_x^{\beta} h_2 \right)_{L_x^2 L_v^2} \\
 & \lesssim \int_{\mathbb{T}^3} \left(\|\nabla_x^3 f\|_{L_v^2} \|f_2\|_{H_{v,*}^1} \|\nabla_x^3 h_2\|_{H_{v,*}^1} + \|\nabla_x^3 f\|_{L_v^2} \|f_1\|_{L_v^2} \|\nabla_x^3 h_2\|_{H_{v,*}^1} \right) \\
 & \lesssim \|\nabla_x^3 f\|_{L_x^2 L_v^2} \|f_2\|_{H_x^2 H_{v,*}^1} \|\nabla_x^3 h_2\|_{L_x^2 H_{v,*}^1} + \|\nabla_x^3 f\|_{L_x^2 L_v^2} \|f_1\|_{H_x^2 L_v^2} \|\nabla_x^3 h_2\|_{L_x^2 H_{v,*}^1} \\
 & \lesssim \mathcal{E}(f) \{ \mathcal{C}(f) + \mathcal{D}(f) \} \mathcal{D}(h).
 \end{aligned}$$

Regarding estimate (3.23): Using (3.22), we have

$$\left(\Gamma(f, f), h \right)_{H_x^3 L_v^2} \lesssim \|f\|_{H_x^3 L_v^2} \|f\|_{H_x^3 H_{v,*}^1} \|h\|_{H_x^3 H_{v,*}^1},$$

using now (3.8), we obtain for $\|h\|_{H_x^3 H_{v,*}^1} \leq 1$,

$$\|\Gamma(f, f)\|_{H_x^3 (H_{v,*}^1)'} \lesssim \|f\|_{H_x^3 L_v^2} \|f\|_{H_x^3 H_{v,*}^1}.$$

□

2.2 Linear problem

We consider now the linear Cauchy problem

$$\begin{cases} \partial_t g + \frac{1}{\varepsilon} v \cdot \nabla_x g + \frac{1}{\varepsilon^2} \mathcal{L}g = \frac{1}{\varepsilon} \Gamma(f, f) \\ g|_{t=0} = g_0 \in H_x^3 L_v^2 \end{cases} \quad (3.24)$$

where f is a given function such that

$$\sup_{0 \leq t \leq T} \mathcal{E}^2(f(t)) + \int_0^T \mathcal{D}^2(f(t)) dt < \infty \quad (3.25)$$

for some $T > 0$ and verifies

$$\int_{\mathbb{T}^3} \Pi_0 f \, dx = 0. \quad (3.26)$$

We study the existence of a solution to the equation (3.24) in the space $L^\infty([0, T]; H_x^3 L_v^2) \cap L^2([0, T]; H_x^3 H_{v,*}^1)$.

Remark 3.6. We note that by using (3.23), (3.25) and (3.26), we obtain that $\Gamma(f, f) \in L^2([0, T]; H_x^3(H_{v,*}^1)')$.

Definition 3.7. We call weak solution of (3.24) any function $g \in L^2([0, T]; H_x^3 L_v^2)$ satisfying

$$\int_0^T (g, \mathcal{Q}\phi)_{H_x^3 L_v^2} dt - (g_0, \phi(0))_{H_x^3 L_v^2} = \int_0^T \left\langle \frac{1}{\varepsilon} \Gamma(f, f), \phi \right\rangle_{H_x^3(H_{v,*}^1)', H_x^3 H_{v,*}^1} dt \quad (3.27)$$

for all $\phi \in C_c^\infty([0, T] \times \mathbb{T}_x^3 \times \mathbb{R}_v^3)$, where \mathcal{Q} is defined in (3.142) in Appendix (Section 5).

Proposition 3.8. Let $g_0 \in H_x^3 L_v^2$, and f satisfy (3.25) and (3.26). Then the Cauchy problem (3.24) admits a weak solution

$$g \in L^\infty([0, T]; H_x^3 L_v^2) \cap L^2([0, T]; H_x^3 H_{v,*}^1)$$

which satisfies

$$\sup_{0 \leq t \leq T} \mathcal{E}^2(g(t)) + \frac{C_\gamma}{\varepsilon^2} \int_0^T \mathcal{D}^2(g(t)) dt \leq \frac{4C^2}{C_\gamma} \sup_{0 \leq t \leq T} \mathcal{E}^2(f) \left\{ \sup_{0 \leq t \leq T} \mathcal{E}^2(f) + \int_0^T \mathcal{D}^2(f) dt \right\} + 2\mathcal{E}^2(g_0) \quad (3.28)$$

for some $T > 0$, where C, C_γ are independent of $0 < \varepsilon \leq 1$.

Proof. We will show the existence of a solution to the Cauchy problem (3.24) by using Proposition 3.23 in Appendix (Section 5). By applying Proposition 3.23 with $U_\varepsilon = \frac{1}{\varepsilon} \Gamma(f, f)$ (we have from Remark 3.6 $\frac{1}{\varepsilon} \Gamma(f, f) \in L^2([0, T]; H_x^3(H_{v,*}^1)')$ for ε a fixed parameter, then there exists a solution $g \in L^2([0, T]; H_x^3 H_{v,*}^1)$ such that for any $\phi \in C_c^\infty([0, T] \times \mathbb{T}_x^3 \times \mathbb{R}_v^3)$

$$\int_0^T (g, \mathcal{Q}\phi)_{H_x^3 L_v^2} dt = \int_0^T \left\langle \frac{1}{\varepsilon} \Gamma(f, f), \phi \right\rangle_{H_x^3(H_{v,*}^1)', H_x^3 H_{v,*}^1} dt + (g_0, \phi(0))_{H_x^3 L_v^2}, \quad (3.29)$$

where \mathcal{Q} is defined in (3.142). Then, $g \in L^2([0, T]; H_x^3 H_{v,*}^1)$ is a weak solution of the Cauchy problem (3.24). Now, we will show that the weak solution g satisfies (3.28). We only sketch the proof which could be done using mollifiers of g and supposing therefore $\partial_t g, v \cdot \nabla_x g$ and $\mathcal{L}g$ are in $L^2([0, T]; H_x^3 L_v^2)$ and that g and its derivatives in x have traces in times. For elements of proof see [32, Appendix A], [61] and [83]. We define the operator \mathcal{G} by

$$\mathcal{G}g := \partial_t g + \frac{1}{\varepsilon} v \cdot \nabla_x g + \frac{1}{\varepsilon^2} \mathcal{L}g = \frac{1}{\varepsilon} \Gamma(f, f), \quad (3.30)$$

we also suppose $\mathcal{G}g \in L^2([0, T]; H_x^3 L_v^2)$. We have $g \in L^2([0, T]; H_x^3 L_v^2)$, then

$$(\mathcal{G}g, g)_{L^2([0, T]; H_x^3 L_v^2)} = \frac{1}{\varepsilon} (\Gamma(f, f), g)_{L^2([0, T]; H_x^3 L_v^2)}.$$

Using (3.30) and the fact that $v \cdot \nabla_x$ is skew-adjoint, we have

$$\begin{aligned} (\mathcal{G}g, g)_{H_x^3 L_v^2} &= (\partial_t g, g)_{H_x^3 L_v^2} + \frac{1}{\varepsilon} (v \cdot \nabla_x g, g)_{H_x^3 L_v^2} + \frac{1}{\varepsilon^2} (\mathcal{L}g, g)_{H_x^3 L_v^2} \\ &= \frac{1}{2} \frac{d}{dt} \|g(t)\|_{H_x^3 L_v^2}^2 + \frac{1}{\varepsilon^2} (\mathcal{L}g, g)_{H_x^3 L_v^2}. \end{aligned} \quad (3.31)$$

Returning to Theorem 1.2 in [105], we have the following estimate:

$$(\mathcal{L}f, f)_{L_v^2} \geq C_\gamma \|f\|_{H_{v,*}^1}^2, \quad \forall f \in N(\mathcal{L})^\perp, \quad (3.32)$$

where $C_\gamma > 0$. Then,

$$\frac{1}{\varepsilon^2} (\mathcal{L}g, g)_{H_x^3 L_v^2} \geq \frac{C_\gamma}{\varepsilon^2} \mathcal{D}^2(g). \quad (3.33)$$

From the above, we get for $t \in]0, T[$,

$$\int_0^t (\mathcal{G}g, g)_{H_x^3 L_v^2} ds \geq \frac{1}{2} \|g(t)\|_{H_x^3 L_v^2}^2 - \frac{1}{2} \|g(0)\|_{H_x^3 L_v^2}^2 + \frac{C_\gamma}{\varepsilon^2} \int_0^t \mathcal{D}^2(g) ds.$$

Using Lemma 3.5 and the fact that $\mathcal{C}(f) \leq \mathcal{E}(f)$, we have

$$\begin{aligned} &\frac{1}{2} \|g(t)\|_{H_x^3 L_v^2}^2 + \frac{C_\gamma}{\varepsilon^2} \int_0^t \mathcal{D}^2(g) ds \\ &\leq \frac{C}{\varepsilon} \int_0^t \mathcal{E}(f) \{ \mathcal{E}(f) + \mathcal{D}(f) \} \mathcal{D}(g) ds + \frac{1}{2} \mathcal{E}^2(g_0) \\ &\leq \frac{C}{\varepsilon} \int_0^t \{ \mathcal{E}^2(f) + \mathcal{E}(f) \mathcal{D}(f) \} \mathcal{D}(g) ds + \frac{1}{2} \mathcal{E}^2(g_0) \\ &\leq \frac{C^2}{C_\gamma} \int_0^t \mathcal{E}^2(f) \{ \mathcal{E}^2(f) + \mathcal{D}^2(f) \} ds + \frac{C_\gamma}{2\varepsilon^2} \int_0^t \mathcal{D}^2(g) ds + \frac{1}{2} \mathcal{E}^2(g_0). \end{aligned}$$

Thus, we get

$$\mathcal{E}^2(g(t)) + \frac{C_\gamma}{\varepsilon^2} \int_0^t \mathcal{D}^2(g(t)) ds \leq \frac{2C^2}{C_\gamma} \int_0^T \mathcal{E}^2(f) \{ \mathcal{E}^2(f) + \mathcal{D}^2(f) \} ds + \mathcal{E}^2(g_0).$$

This implies that for $T \leq 1$

$$\begin{aligned} \sup_{0 \leq t \leq T} \mathcal{E}^2(g(t)) + \frac{C_\gamma}{\varepsilon^2} \int_0^T \mathcal{D}^2(g(t)) dt &\leq \frac{4C^2}{C_\gamma} \sup_{0 \leq t \leq T} \mathcal{E}^2(f) \left\{ \sup_{0 \leq t \leq T} \mathcal{E}^2(f) + \int_0^T \mathcal{D}^2(f) dt \right\} \\ &\quad + 2\mathcal{E}^2(g_0). \end{aligned}$$

Finally, (3.28) and (3.29) show that $g \in L^\infty([0, T]; H_x^3 L_v^2) \cap L^2([0, T]; H_x^3 H_{v,*}^1)$ is a weak solution of the Cauchy problem (3.24). \square

2.3 Local existence for nonlinear problem.

We consider now the following iteration

$$\begin{cases} \partial_t g^{n+1} + \frac{1}{\varepsilon} v \cdot \nabla_x g^{n+1} + \frac{1}{\varepsilon^2} \mathcal{L} g^{n+1} = \frac{1}{\varepsilon} \Gamma(g^n, g^n) \\ g^{n+1}|_{t=0} = g_0, \end{cases} \quad (3.34)$$

with $g^0 = 0$.

Proposition 3.9. *There exists $0 < \delta_0 \leq 1$, $0 < T \leq 1$, such that for any $0 < \varepsilon \leq 1$, $g_0 \in H_x^3 L_v^2$ with*

$$\|g_0\|_{H_x^3 L_v^2} \leq \delta_0$$

then the iteration problem (3.34) admits a sequence of solutions $\{g^n\}_{n \geq 1}$ satisfying

$$\sup_{0 \leq t \leq T} \mathcal{E}^2(g^n) + \frac{C_\gamma}{\varepsilon^2} \int_0^T \mathcal{D}^2(g^n) dt \leq 4\delta_0^2. \quad (3.35)$$

Proof. For the linear Cauchy problem (3.34), given g^n satisfying (3.35), the existence of g^{n+1} is obtained by the Proposition 3.8. So that it is enough to prove (3.35) by induction. Using estimate (3.28) with $g^n = f$ and $g^{n+1} = g$, we obtain

$$\begin{aligned} \sup_{0 \leq t \leq T} \mathcal{E}^2(g^{n+1}) + \frac{C_\gamma}{\varepsilon^2} \int_0^T \mathcal{D}^2(g^{n+1}) dt &\leq \frac{4C^2}{C_\gamma} \sup_{0 \leq t \leq T} \mathcal{E}^2(g^n) \left\{ \sup_{0 \leq t \leq T} \mathcal{E}^2(g^n) + \int_0^T \mathcal{D}^2(g^n) dt \right\} \\ &\quad + 2\mathcal{E}^2(g_0). \end{aligned}$$

Then, using (3.35), we obtain

$$\sup_{0 \leq t \leq T} \mathcal{E}^2(g^{n+1}) + \frac{C_\gamma}{\varepsilon^2} \int_0^T \mathcal{D}^2(g^{n+1}) dt \leq \delta_0^2 \left(2 + 64 \delta_0^2 \frac{C^2}{C_\gamma} \right).$$

We complete the proof of the Proposition by choosing δ_0 such that

$$2 + 64 \delta_0^2 \frac{C^2}{C_\gamma} \leq 4.$$

□

Theorem 3.10. *For $T > 0$, such that for any $0 < \varepsilon < 1$, $g_{\varepsilon,0} \in H_x^3 L_v^2$ with*

$$\|g_{\varepsilon,0}\|_{H_x^3 L_v^2} \leq \delta_0,$$

then the Cauchy problem (3.3) admits a unique solution $g \in L^\infty([0, T]; H_x^3 L_v^2)$ satisfying

$$\sup_{t \in [0, T]} \mathcal{E}^2(g_\varepsilon) + \frac{C_\gamma}{\varepsilon^2} \int_0^T \mathcal{D}^2(g_\varepsilon) dt \leq 4\delta_0^2. \quad (3.36)$$

Proof. We will show that $\{g^n\}$ defined in Proposition 3.9 is a Cauchy sequence in $L^\infty([0, T]; L^2(\mathbb{T}_x^3 \times \mathbb{R}_v^3))$. We set $w^n = g^{n+1} - g^n$ and deduce from (3.34),

$$\begin{cases} \partial_t w^n + \frac{1}{\varepsilon} v \cdot \nabla_x w^n + \frac{1}{\varepsilon^2} \mathcal{L} w^n = \frac{1}{\varepsilon} [\Gamma(g^n, g^n) - \Gamma(g^{n-1}, g^{n-1})] \\ w^n|_{t=0} = 0. \end{cases} \quad (3.37)$$

Using the fact that, for any $h \in L^2$

$$\begin{aligned} (\Gamma(g^n, g^n) - \Gamma(g^{n-1}, g^{n-1}), \Pi_0 h)_{L_v^2} &= 0, \\ \Gamma(g^n, g^n) - \Gamma(g^{n-1}, g^{n-1}) &= \Gamma(g^n, g^n - g^{n-1}) + \Gamma(g^n, g^{n-1}) - \Gamma(g^{n-1}, g^{n-1}) \\ &= \Gamma(g^n, w^{n-1}) + \Gamma(w^{n-1}, g^{n-1}), \end{aligned}$$

we obtain

$$(\Gamma(g^n, g^n) - \Gamma(g^{n-1}, g^{n-1}), w^n)_{L_{x,v}^2} = (\Gamma(g^n, w^{n-1}) + \Gamma(w^{n-1}, g^{n-1}), w_2^n)_{L_{x,v}^2},$$

where $w_2^n = (I - \Pi_0)w^n$. Using Lemma 3.4, we have

$$\begin{aligned} & \frac{1}{\varepsilon} \left| (\Gamma(g^n, w^{n-1}) + \Gamma(w^{n-1}, g^{n-1}), w_2^n)_{L_{x,v}^2} \right| \\ & \leq \frac{C}{\varepsilon} \mathcal{E}(g^n) (\|w_1^{n-1}\|_{L_{x,v}^2} + \|w_2^{n-1}\|_{\chi^0}) \|w_2^n\|_{\chi^0} \\ & + \frac{C}{\varepsilon} \|w^{n-1}\|_{L_{x,v}^2} (\|g_1^{n-1}\|_{H_x^3 L_v^2} + \mathcal{D}(g^{n-1})) \|w_2^n\|_{\chi^0} \\ & \leq C_\delta \mathcal{E}^2(g^n) (\|w_1^{n-1}\|_{L_{x,v}^2}^2 + \|w_2^{n-1}\|_{\chi^0}^2) \\ & + C_\delta \|w^{n-1}\|_{L_{x,v}^2}^2 (\|g_1^{n-1}\|_{H_x^3 L_v^2}^2 + \mathcal{D}^2(g^{n-1})) \\ & + \frac{\delta}{\varepsilon^2} \|w_2^n\|_{\chi^0}^2. \end{aligned}$$

Thus, fix a small $\delta > 0$, we get

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|w^n\|_{L_{x,v}^2}^2 + \frac{C_\gamma}{2\varepsilon^2} \|w_2^n\|_{\chi^0}^2 & \leq C_\delta \mathcal{E}^2(g^n) (\|w_1^{n-1}\|_{L_{x,v}^2}^2 + \|w_2^{n-1}\|_{\chi^0}^2) \\ & + C_\delta \|w^{n-1}\|_{L_{x,v}^2}^2 (\|g_1^{n-1}\|_{H_x^3 L_v^2}^2 + \mathcal{D}^2(g^{n-1})). \end{aligned}$$

Using the fact that

$$\|w_1^{n-1}\|_{L_{x,v}^2}^2 = \|\Pi_0 w^{n-1}\|_{L_{x,v}^2}^2 \leq C \|w^{n-1}\|_{L_{x,v}^2}^2, \quad \|g_1^{n-1}\|_{H_x^3 L_v^2}^2 \leq C \mathcal{E}^2(g^{n-1}),$$

we obtain $t \in]0, T[$,

$$\begin{aligned} & \|w^n\|_{L_{x,v}^2}^2 + \frac{1}{\varepsilon^2} \int_0^t \|w_2^n\|_{\chi^0}^2 ds \\ & \leq C \sup_{t \in [0, T]} \mathcal{E}^2(g^n) \left(T \|w^{n-1}\|_{L^\infty([0, T]; L_{x,v}^2)}^2 + \int_0^T \|w_2^{n-1}\|_{\chi^0}^2 ds \right) \\ & + C \|w^{n-1}\|_{L^\infty([0, T]; L_{x,v}^2)}^2 \left(T \sup_{t \in [0, T]} \mathcal{E}^2(g^{n-1}) + \int_0^T \mathcal{D}^2(g^{n-1}) ds \right). \end{aligned}$$

Using now (3.35) with $\delta_0 > 0$ small enough, we get that for any $0 < \varepsilon \leq 1$

$$\|w^n\|_{L^\infty([0,T];L^2_{x,v})}^2 + \frac{1}{\varepsilon^2} \int_0^T \|w_2^n\|_{\chi^0}^2 dt \leq \frac{1}{2} \left(\|w^{n-1}\|_{L^\infty([0,T];L^2_{x,v})}^2 + \frac{1}{\varepsilon^2} \int_0^T \|w_2^{n-1}\|_{\chi^0}^2 dt \right). \quad (3.38)$$

Thus we have proved that g^n is a Cauchy sequence in $L^\infty([0, T]; L^2_{x,v})$. Combining with the estimate (3.35), the limit g is in $L^\infty([0, T]; H_x^3 L_v^2)$. If we note

$$\mathcal{F}(g^n) = \sup_{0 \leq t \leq T} \mathcal{E}^2(g^n) + \frac{C_\gamma}{\varepsilon^2} \int_0^T \mathcal{D}^2(g^n) dt,$$

then from weak lower semicontinuity, we have

$$\mathcal{F}(g) \leq \liminf_{n \rightarrow \infty} \mathcal{F}(g^n) \leq 4\delta_0^2.$$

Then we obtain the estimate (3.36). Now, we will show the uniqueness of the local solution. Let $g_\varepsilon^{(1)}$ and $g_\varepsilon^{(2)}$ be two solutions to (3.3) with same initial data $g_{\varepsilon,0}^{(1)} = g_{\varepsilon,0}^{(2)}$ that satisfy (3.36). The difference $\mathbf{h}_\varepsilon = g_\varepsilon^{(1)} - g_\varepsilon^{(2)}$ satisfies

$$\begin{cases} \partial_t \mathbf{h}_\varepsilon + \frac{1}{\varepsilon} v \cdot \nabla_x \mathbf{h}_\varepsilon + \frac{1}{\varepsilon^2} \mathcal{L} \mathbf{h}_\varepsilon = \frac{1}{\varepsilon} \Gamma(g_\varepsilon^{(2)}, \mathbf{h}_\varepsilon) + \frac{1}{\varepsilon} \Gamma(\mathbf{h}_\varepsilon, g_\varepsilon^{(1)}) \\ \mathbf{h}_\varepsilon|_{t=0} = 0, \end{cases} \quad (3.39)$$

using the same argument in Theorem 3.10 in [26], we can show that $\mathbf{h}_\varepsilon = 0$, hence the uniqueness of the weak solution. \square

3 Uniform estimate and global solutions

In this section, we establish microscopic and macroscopic energy estimates for the Landau equation. These estimates with Theorem 3.10 allow to prove the existence of global solutions in the space $L^\infty([0, \infty); H_x^3 L_v^2)$.

3.1 Microscopic energy estimate.

Proposition 3.11. *Let $g \in L^\infty([0, T]; H_x^3 L_v^2)$ be a solution of the equation (3.3) constructed in Theorem 3.10, then there exists a constant C independent of ε such that the following estimate holds:*

$$\frac{d}{dt} \mathcal{E}^2 + \frac{C_\gamma}{\varepsilon^2} \mathcal{D}^2 \leq C \left\{ \frac{1}{\varepsilon} \mathcal{E} \mathcal{D}^2 + (\mathcal{E} \mathcal{C})^2 \right\}, \quad (3.40)$$

where for simplicity, we note $\mathcal{E} = \mathcal{E}(g)$, $\mathcal{D} = \mathcal{D}(g)$ and $\mathcal{C} = \mathcal{C}(g)$.

Proof. We apply ∂_x^α to (3.3) and take the $L^2(\mathbb{T}^3 \times \mathbb{R}^3)$ inner product with $\partial_x^\alpha g$. Since the innerproduct including $v \cdot \nabla_x g$ vanishes by integration by parts, we get

$$\frac{1}{2} \frac{d}{dt} \mathcal{E}^2 + \frac{1}{\varepsilon^2} \sum_{|\alpha| \leq 3} (\mathcal{L} \partial_x^\alpha g, \partial_x^\alpha g)_{L_{x,v}^2} = \frac{1}{\varepsilon} \sum_{|\alpha| \leq 3} (\partial_x^\alpha \Gamma(g, g), \partial_x^\alpha g)_{L_{x,v}^2}.$$

Returning to Theorem 1.2 in [105], we have

$$\sum_{|\alpha| \leq 3} (\mathcal{L} \partial_x^\alpha g, \partial_x^\alpha g)_{L_{x,v}^2} \geq C_\gamma \|g_2\|_{\chi^3(\mathbb{T}_x^3 \times \mathbb{R}_v^3)}^2 = C_\gamma \mathcal{D}^2.$$

Since $\int_{\mathbb{T}^3} \Pi_0 g \, dx = 0$, Lemma 3.5 implies that,

$$\left| \frac{1}{\varepsilon} \sum_{|\alpha| \leq 3} (\partial_x^\alpha \Gamma(g, g), \partial_x^\alpha g)_{L_{x,v}^2} \right| \leq \frac{C_1}{\varepsilon} \mathcal{E}(\mathcal{C}\mathcal{D} + \mathcal{D}^2) \leq \frac{C_1}{\varepsilon} \mathcal{E}\mathcal{D}^2 + \frac{C_1}{2\eta} (\mathcal{E}\mathcal{C})^2 + \frac{\eta}{2} \frac{C_1}{\varepsilon^2} \mathcal{D}^2.$$

Taking $\eta = \frac{C_\gamma}{C_1}$, we have

$$\frac{1}{2} \frac{d}{dt} \mathcal{E}^2 + \frac{C_\gamma}{2} \frac{1}{\varepsilon^2} \mathcal{D}^2 \leq \frac{C_1}{\varepsilon} \mathcal{E}\mathcal{D}^2 + \frac{C_1^2}{2C_\gamma} (\mathcal{E}\mathcal{C})^2,$$

then

$$\frac{d}{dt} \mathcal{E}^2 + \frac{C_\gamma}{\varepsilon^2} \mathcal{D}^2 \leq C \left\{ \frac{1}{\varepsilon} \mathcal{E}\mathcal{D}^2 + (\mathcal{E}\mathcal{C})^2 \right\}.$$

□

3.2 Macroscopic energy estimates

We study now the energy estimate for the macroscopic part $\Pi_0 g$ where g is a solution of the equation (3.3). First we decompose the equation (3.3) into microscopic and macroscopic parts, i.e. rewrite it into the following equation

$$\begin{aligned} \partial_t \{a + bv + c|v|^2\} \mu^{1/2} + \frac{1}{\varepsilon} v \cdot \nabla_x \{a + bv + c|v|^2\} \mu^{1/2} &= -\partial_t g_2 - \frac{1}{\varepsilon} v \cdot \nabla_x g_2 - \frac{1}{\varepsilon^2} \mathcal{L} g_2 \\ &\quad + \frac{1}{\varepsilon} \Gamma(g, g). \end{aligned} \tag{3.41}$$

Lemma 3.12. *Let $\partial^\alpha = \partial_x^\alpha$, $\alpha \in \mathbb{N}^3$, $|\alpha| \leq 2$. If g is a solution of the Landau equation (3.3), and $\mathcal{A} = (a, b, c)$ defined in (3.10), then*

$$\varepsilon \|\partial_t \partial^\alpha \mathcal{A}\|_{L_x^2} \lesssim \mathcal{C} + \mathcal{D}, \tag{3.42}$$

where $\mathcal{E} = \mathcal{E}(g)$ and $\mathcal{D} = \mathcal{D}(g)$.

Proof. Let g be a solution to solution of the scaled Landau equation (3.3). The second set of equations we consider are the local conservation laws satisfied by (a, b, c) . To derive these we multiply (3.41) by the collision invariants that are the elements of $\mathcal{N}(\mathcal{L})$ in (3.5) and integrate only in the velocity variables to obtain

$$\partial_t(a + 3c) + \frac{1}{\varepsilon} \nabla_x \cdot b = 0, \quad (3.43)$$

$$\partial_t b + \frac{1}{\varepsilon} (\nabla_x a + 5 \nabla_x c) = -\frac{1}{\varepsilon} (v \cdot \nabla_x g_2, v \sqrt{\mu})_{L_v^2}, \quad (3.44)$$

$$\partial_t(3a + 15c) + \frac{5}{\varepsilon} \nabla_x \cdot b = -\frac{1}{\varepsilon} (v \cdot \nabla_x g_2, |v|^2 \sqrt{\mu})_{L_v^2}. \quad (3.45)$$

Taking linear combinations of the first and third local conservation laws results in

$$\begin{aligned} \partial_t a &= \frac{1}{2\varepsilon} (v \cdot \nabla_x g_2, |v|^2 \sqrt{\mu})_{L_v^2}, \\ \partial_t b + \frac{1}{\varepsilon} (\nabla_x a + 5 \nabla_x c) &= -\frac{1}{\varepsilon} (v \cdot \nabla_x g_2, v \sqrt{\mu})_{L_v^2}, \\ \partial_t c + \frac{1}{3\varepsilon} \nabla_x \cdot b &= -\frac{1}{6\varepsilon} (v \cdot \nabla_x g_2, |v|^2 \sqrt{\mu})_{L_v^2}. \end{aligned}$$

These are the local conservation laws that we will study below. From the first equation, we have

$$\|\partial_t \partial^\alpha a\|_{L_x^2} \lesssim \frac{1}{\varepsilon} \|\nabla_x \partial^\alpha g_2\|_{L_x^2 L_v^2}, \quad (3.46)$$

the second equation gives that

$$\begin{aligned} \|\partial_t \partial^\alpha b\|_{L_x^2} &\lesssim \frac{1}{\varepsilon} \|\nabla_x \partial^\alpha a\|_{L_x^2} + \frac{1}{\varepsilon} \|\nabla_x \partial^\alpha c\|_{L_x^2} + \frac{1}{\varepsilon} \|\nabla_x \partial^\alpha g_2\|_{L_x^2 L_v^2} \\ &\lesssim \frac{1}{\varepsilon} \|\nabla_x \partial^\alpha \mathcal{A}\|_{L_x^2} + \frac{1}{\varepsilon} \|\nabla_x \partial^\alpha g_2\|_{L_x^2 L_v^2} \end{aligned}$$

and the last equation implies that

$$\begin{aligned} \|\partial_t \partial^\alpha c\|_{L_x^2} &\lesssim \frac{1}{\varepsilon} \|\nabla_x \partial^\alpha b\|_{L_x^2} + \frac{1}{\varepsilon} \|\nabla_x \partial^\alpha g_2\|_{L_x^2 L_v^2} \\ &\lesssim \frac{1}{\varepsilon} \|\nabla_x \partial^\alpha \mathcal{A}\|_{L_x^2} + \frac{1}{\varepsilon} \|\nabla_x \partial^\alpha g_2\|_{L_x^2 L_v^2}. \end{aligned}$$

Then, we obtain

$$\begin{aligned} \varepsilon \|\partial_t \partial^\alpha \mathcal{A}\|_{L_x^2} &\lesssim \|\nabla_x \partial^\alpha \mathcal{A}\|_{L_x^2} + \|\nabla_x \partial^\alpha g_2\|_{L_x^2 L_v^2} \\ &\lesssim \mathcal{C} + \mathcal{D}. \end{aligned}$$

□

We start with the macroscopic energy estimate where we use the so-called 13-moments. The set of 13-moments is 13-dimensional subspace of $L^2(\mathbb{R}_v^3)$ and given by

$$\{e_j\}_{j=1}^{13} = \left\{ \sqrt{\mu}, v_i \sqrt{\mu}, v_i v_j \sqrt{\mu}, v_i |v|^2 \sqrt{\mu} \right\}.$$

It is well-known [61] that the macroscopic component $g_1 = \Pi_0 g \sim \mathcal{A} = (a, b, c)$, satisfies the following set of equations

$$\begin{cases} v_i |v|^2 \mu^{1/2} & : & \frac{1}{\varepsilon} \nabla_x c = -\partial_t r_c + \frac{1}{\varepsilon} m_c + \frac{1}{\varepsilon^2} \ell_c + \frac{1}{\varepsilon} h_c, \\ v_i^2 \mu^{1/2} & : & \partial_t c + \frac{1}{\varepsilon} \partial_i b_i = -\partial_t r_i + \frac{1}{\varepsilon} m_i + \frac{1}{\varepsilon^2} \ell_i + \frac{1}{\varepsilon} h_i, \\ v_i v_j \mu^{1/2} & : & \frac{1}{\varepsilon} \partial_i b_j + \frac{1}{\varepsilon} \partial_j b_i = -\partial_t r_{ij} + \frac{1}{\varepsilon} m_{ij} + \frac{1}{\varepsilon^2} \ell_{ij} + \frac{1}{\varepsilon} h_{ij}, \quad i \neq j, \\ v_i \mu^{1/2} & : & \partial_t b_i + \frac{1}{\varepsilon} \partial_i a = -\partial_t r_{bi} + \frac{1}{\varepsilon} m_{bi} + \frac{1}{\varepsilon^2} \ell_{bi} + \frac{1}{\varepsilon} h_{bi}, \\ \mu^{1/2} & : & \partial_t a = -\partial_t r_a + \frac{1}{\varepsilon} m_a + \frac{1}{\varepsilon^2} \ell_a + \frac{1}{\varepsilon} h_a \end{cases} \quad (3.47)$$

where

$$\begin{aligned} r &= (g_2, e)_{L_v^2}, \quad m = (-v \cdot \nabla_x g_2, e)_{L_v^2}, \quad h = (\Gamma(g, g), e)_{L_v^2}, \\ \ell &= -(\mathcal{L}g_2, e)_{L_v^2} \end{aligned} \quad (3.48)$$

stand for r_c, \dots, h_a , while

$$e \in \text{Span}\{v_i |v|^2 \mu^{1/2}, v_i^2 \mu^{1/2}, v_i v_j \mu^{1/2}, v_i \mu^{1/2}, \mu^{1/2}\}, \quad \text{for } i, j = 1, 2, 3.$$

Lemma 3.13. *Let r, m, ℓ, h be the ones defined by (3.48), $\partial^\alpha = \partial_x^\alpha$, $\partial_i = \partial_{x_i}$ $\alpha \in \mathbb{N}^3$, $|\alpha| \leq 2$. Then, one has*

$$\|\partial_i \partial^\alpha r\|_{L_x^2} \lesssim \min\{\|g_2\|_{H_x^3 L_v^2}, \mathcal{D}\}, \quad (3.49)$$

$$\|\partial^\alpha m\|_{L_x^2} \lesssim \min\{\|g_2\|_{H_x^3 L_v^2}, \mathcal{D}\}, \quad (3.50)$$

$$\|\partial^\alpha \ell\|_{L_x^2} \lesssim \min\{\|g_2\|_{H_x^2 L_v^2}, \|g_2\|_{\chi^2(\mathbb{T}_x^3 \times \mathbb{R}_v^3)}\}, \quad (3.51)$$

$$\|\partial^\alpha h\|_{L_x^2} \lesssim \|g\|_{H_x^2 L_v^2} (\|g_2\|_{\chi^2(\mathbb{T}_x^3 \times \mathbb{R}_v^3)} + \|\nabla_x \Pi_0 g\|_{H_x^1 L_v^2}). \quad (3.52)$$

Proof. We have

$$\|\partial_i \partial^\alpha r\|_{L_x^2} = \|(\partial_i \partial^\alpha g_2, e)_{L_v^2}\|_{L_x^2} \lesssim \|\partial_i \partial^\alpha g_2\|_{L_x^2 L_v^2} \|e\|_{L_v^2} \lesssim \|g_2\|_{H_x^3 L_v^2},$$

$$\|\partial^\alpha \ell\|_{L_x^2} = \|(\partial^\alpha g_2, \mathcal{L}e)_{L_v^2}\|_{L_x^2} \lesssim \|\partial^\alpha g_2\|_{L_x^2 L_v^2} \|\mathcal{L}e\|_{L_v^2} \lesssim \|g_2\|_{H_x^2 L_v^2},$$

$$\|\partial^\alpha m\|_{L_x^2} = \|(\nabla_x \partial^\alpha g_2, ve)_{L_v^2}\|_{L_x^2} \lesssim \|\nabla_x \partial^\alpha g_2\|_{L_x^2 L_v^2} \|ve\|_{L_v^2} \lesssim \|g_2\|_{H_x^3 L_v^2}.$$

The fact that for $f \in H_{v,*}^1$ and $\gamma \in [-2, 1]$,

$$\|f\|_{L_v^2 \langle v \rangle^{1+\gamma/2}} \leq \|f\|_{H_{v,*}^1},$$

implies (3.49), (3.50) and (3.51). For the estimate (3.52), we have

$$\partial^\alpha h = (\partial^\alpha \Gamma(g, g), e)_{L_v^2},$$

then using the same method of proof of Lemma 3.5, we get (3.52). \square

Lemma 3.14. *Let $|\alpha| \leq 2$, and let g be a solution of the scaled Landau equation (3.3). Then there exists a positive constant \tilde{C} independent of ε , such that the following estimate holds:*

$$\varepsilon \frac{d}{dt} \left\{ \sum_{|\alpha| \leq 2} (\partial^\alpha r, \nabla_x \partial^\alpha (a, b, c))_{L_x^2} + (\partial^\alpha b, \nabla_x \partial^\alpha a)_{L_x^2} \right\} + \mathcal{C}^2 \leq \tilde{C} \left\{ \frac{1}{\varepsilon^2} \mathcal{D}^2 + \mathcal{E}(\mathcal{C}^2 + \mathcal{D}^2) \right\}. \quad (3.53)$$

Proof. Recall that

$$\mathcal{C}^2 = \sum_{|\alpha| \leq 2} \|\nabla_x \partial^\alpha \mathcal{A}\|_{L_x^2}^2 = \sum_{|\alpha| \leq 2} \|\nabla_x \partial^\alpha a\|_{L_x^2}^2 + \|\nabla_x \partial^\alpha b\|_{L_x^2}^2 + \|\nabla_x \partial^\alpha c\|_{L_x^2}^2.$$

(a) **Estimate of $\nabla_x \partial^\alpha a$.** From the macroscopic equations (3.47),

$$\begin{aligned} \|\nabla_x \partial^\alpha a\|_{L_x^2}^2 &= (\nabla_x \partial^\alpha a, \nabla_x \partial^\alpha a)_{L_x^2} \\ &= \left(\partial^\alpha \left(-\varepsilon \partial_t b - \varepsilon \partial_t r + m + \frac{1}{\varepsilon} \ell + h \right), \nabla_x \partial^\alpha a \right)_{L_x^2} \\ &\lesssim \varepsilon R_1 + |(\partial^\alpha m, \nabla_x \partial^\alpha a)_{L_x^2}| + \frac{1}{\varepsilon} |(\partial^\alpha \ell, \nabla_x \partial^\alpha a)_{L_x^2}| + |(\partial^\alpha h, \nabla_x \partial^\alpha a)_{L_x^2}|. \end{aligned}$$

Here,

$$\begin{aligned} \varepsilon R_1 &= -\varepsilon (\partial^\alpha \partial_t b + \partial^\alpha \partial_t r, \nabla_x \partial^\alpha a)_{L_x^2} \\ &= -\varepsilon \frac{d}{dt} (\partial^\alpha (b + r), \nabla_x \partial^\alpha a)_{L_x^2} - \varepsilon (\nabla_x \partial^\alpha (b + r), \partial_t \partial^\alpha a)_{L_x^2}. \end{aligned}$$

Note that, from (3.46) and (3.49) $\varepsilon (\nabla_x \partial^\alpha r, \partial_t \partial^\alpha a)_{L_x^2} \lesssim \mathcal{D}^2$, and

$$\varepsilon (\nabla_x \partial^\alpha b, \partial_t \partial^\alpha a)_{L_x^2} \lesssim \eta \|\nabla_x \partial^\alpha b\|_{L_x^2}^2 + \frac{1}{2\eta} \mathcal{D}^2.$$

Furthermore, Lemma 3.13 implies that

$$\begin{aligned} |(\partial^\alpha m, \nabla_x \partial^\alpha a)_{L_x^2}| &\lesssim \mathcal{D} \|\nabla_x \partial^\alpha a\|_{L_x^2} \lesssim \mathcal{D} \|\nabla_x \partial^\alpha \mathcal{A}\|_{H_x^2} \lesssim \mathcal{D} \mathcal{C}, \\ \frac{1}{\varepsilon} |(\partial^\alpha \ell, \nabla_x \partial^\alpha a)_{L_x^2}| &\lesssim \frac{1}{\varepsilon} \|g_2\|_{\chi^2(\mathbb{T}_x^3 \times \mathbb{R}_v^3)} \|\nabla_x \partial^\alpha \mathcal{A}\|_{H_x^2} \lesssim \frac{1}{\varepsilon} \mathcal{D} \mathcal{C}, \\ |(\partial^\alpha h, \nabla_x \partial^\alpha a)_{L_x^2}| &\lesssim \mathcal{E}(\mathcal{C} + \mathcal{D}) \mathcal{C}. \end{aligned}$$

Hence, for some small $0 < \eta < 1$

$$\begin{aligned} \varepsilon \frac{d}{dt} (\partial^\alpha (b + r), \nabla_x \partial^\alpha a)_{L_x^2} + \|\nabla_x \partial^\alpha a\|_{L_x^2}^2 &\lesssim \eta \|\nabla_x \partial^\alpha b\|_{L_x^2}^2 + \frac{1}{2\eta} \mathcal{D}^2 + \mathcal{D} \mathcal{C} \\ &\quad + \frac{1}{\varepsilon} \mathcal{D} \mathcal{C} + \mathcal{E}(\mathcal{C} + \mathcal{D}) \mathcal{C} \\ &\lesssim \eta \mathcal{C}^2 + \frac{1}{\eta} \left\{ \frac{1}{\varepsilon^2} \mathcal{D}^2 + \mathcal{E}(\mathcal{C}^2 + \mathcal{D}^2) \right\}. \end{aligned} \quad (3.54)$$

(b) **Estimate of $\nabla_x \partial^\alpha b$.** Recall $b = (b_1, b_2, b_3)$. From the macroscopic equations (3.47),

$$\begin{aligned} \Delta_x \partial^\alpha b + \partial_i^2 \partial^\alpha b_i &= \partial^\alpha \left[\sum_{j \neq i} \partial_j (\partial_j b_i + \partial_i b_j) + \partial_i (2\partial_i b_i - \sum_{j \neq i} \partial_j b_j) \right] \\ &= \partial^\alpha \left[\sum_{j \neq i} \partial_j \left(-\varepsilon \partial_t r_{ij} + m_{ij} + \frac{1}{\varepsilon} \ell_{ij} + h_{ij} \right) \right. \\ &\quad \left. + \partial_i \left(-2\varepsilon \partial_t r_i + 2m_i + \frac{2}{\varepsilon} \ell_i + 2h_i + \sum_{j \neq i} \varepsilon \partial_t r_j - m_j - \frac{1}{\varepsilon} \ell_j - h_j \right) \right] \\ &= \partial^\alpha \left[\partial_i \left(-\varepsilon \partial_t r + m + \frac{1}{\varepsilon} \ell + h \right) \right] \end{aligned}$$

where r, ℓ, h stand for linear combinations of r_i, ℓ_i, h_i and $r_{ij}, \ell_{ij}, h_{ij}$ for $i, j = 1, 2, 3$ respectively. Then

$$\begin{aligned} \|\nabla_x \partial^\alpha b_i\|_{L_x^2}^2 + \|\partial_i \partial^\alpha b_i\|_{L_x^2}^2 &= - \left(\Delta_x \partial^\alpha b + \partial_i^2 \partial^\alpha b_i, \partial^\alpha b_i \right)_{L_x^2} \\ &= \varepsilon R_2 + R_3 + R_4 + R_5, \end{aligned}$$

where

$$\begin{aligned} \varepsilon R_2 &= \varepsilon (\partial_i \partial^\alpha \partial_t r, \partial^\alpha b_i)_{L_x^2} \\ &= -\varepsilon \frac{d}{dt} (\partial^\alpha r, \partial_i \partial^\alpha b_i)_{L_x^2} + \varepsilon (\partial^\alpha r, \partial_t \partial_i \partial^\alpha b_i)_{L_x^2}, \end{aligned}$$

moreover, we have from (3.42) and (3.49) that

$$\varepsilon |(\partial_i \partial^\alpha r, \partial_t \partial^\alpha b_i)_{L_x^2}| \lesssim \eta \mathcal{C}^2 + \frac{1}{\eta} \mathcal{D}^2.$$

Furthermore, Lemma 3.13 implies that

$$\begin{aligned} |(\partial^\alpha m, \partial_i \partial^\alpha b_i)_{L_x^2}| &\lesssim \frac{1}{\eta} \|\partial^\alpha m\|_{L_x^2}^2 + \eta \|\partial_i \partial^\alpha b_i\|_{L_x^2}^2 \\ &\lesssim \frac{1}{\eta} \|\partial^\alpha m\|_{L_x^2}^2 + \eta \|\nabla_x \partial^\alpha b_i\|_{L_x^2}^2 \\ &\lesssim \frac{1}{\eta} \mathcal{D}^2 + \eta \|\nabla_x \partial^\alpha b_i\|_{L_x^2}^2, \\ \frac{1}{\varepsilon} |(\partial^\alpha \ell, \partial_i \partial^\alpha b_i)_{L_x^2}| &\lesssim \frac{1}{\varepsilon} \|\partial^\alpha \ell\|_{L_x^2} \|\nabla_x \partial^\alpha \mathcal{A}\|_{H_x^2} \lesssim \frac{1}{\varepsilon} \mathcal{D} \mathcal{C}, \\ |(\partial^\alpha h, \partial_i \partial^\alpha b_i)_{L_x^2}| &\lesssim \mathcal{E} (\mathcal{C} + \mathcal{D}) \mathcal{C} \\ &\lesssim \mathcal{E} (\mathcal{C}^2 + \mathcal{D}^2). \end{aligned}$$

Hence, for some small $0 < \eta < 1$

$$\varepsilon \frac{d}{dt} (\partial^\alpha r, \partial_i \partial^\alpha b_i)_{L_x^2} + \|\nabla_x \partial^\alpha b_i\|_{L_x^2}^2 \lesssim \eta \mathcal{C}^2 + \frac{1}{\eta} \left\{ \frac{1}{\varepsilon^2} \mathcal{D}^2 + \mathcal{E} (\mathcal{C}^2 + \mathcal{D}^2) \right\},$$

summing up for $1 \leq i \leq 3$, we obtain that

$$\varepsilon \frac{d}{dt} (\partial^\alpha r, \nabla_x \partial^\alpha b)_{L_x^2} + \|\nabla_x \partial^\alpha b\|_{L_x^2}^2 \lesssim \eta \mathcal{C}^2 + \frac{1}{\eta} \left\{ \frac{1}{\varepsilon^2} \mathcal{D}^2 + \mathcal{E}(\mathcal{C}^2 + \mathcal{D}^2) \right\}. \quad (3.55)$$

(c) **Estimate of $\nabla_x \partial^\alpha c$.** From the macroscopic equations (3.47),

$$\begin{aligned} \|\nabla_x \partial^\alpha c\|_{L_x^2}^2 &= (\nabla_x \partial^\alpha c, \nabla_x \partial^\alpha c)_{L_x^2} \\ &= \left(\partial^\alpha (-\varepsilon \partial_t r + m + \frac{1}{\varepsilon} \ell + h), \nabla_x \partial^\alpha c \right)_{L_x^2} \\ &\lesssim \varepsilon R_6 + \eta \mathcal{C}^2 + \frac{1}{\eta} \mathcal{D}^2 + \frac{1}{\eta \varepsilon^2} \mathcal{D}^2 + \mathcal{E}(\mathcal{C}^2 + \mathcal{D}^2), \end{aligned}$$

where

$$\begin{aligned} \varepsilon R_6 &= -\varepsilon (\partial^\alpha \partial_t r, \nabla_x \partial^\alpha c)_{L_x^2} \\ &= -\varepsilon \frac{d}{dt} (\partial^\alpha r, \nabla_x \partial^\alpha c)_{L_x^2} - \varepsilon (\nabla_x \partial^\alpha r, \partial_t \partial^\alpha c)_{L_x^2} \\ &\lesssim -\varepsilon \frac{d}{dt} (\partial^\alpha r, \nabla_x \partial^\alpha c)_{L_x^2} + \eta \mathcal{C}^2 + \frac{2}{\eta} \frac{1}{\varepsilon^2} \mathcal{D}^2. \end{aligned}$$

Thus

$$\varepsilon \frac{d}{dt} (\partial^\alpha r, \nabla_x \partial^\alpha c)_{L_x^2} + \|\nabla_x \partial^\alpha c\|_{L_x^2}^2 \lesssim \eta \mathcal{C}^2 + \frac{1}{\eta} \left\{ \frac{1}{\varepsilon^2} \mathcal{D}^2 + \mathcal{E}(\mathcal{C}^2 + \mathcal{D}^2) \right\}. \quad (3.56)$$

By combining the above estimates (3.54), (3.55), (3.56) and taking $\eta > 0$ sufficiently small, then we get the estimate (3.53) uniformly for $0 < \varepsilon < 1$, thus complete the proof of Lemma 3.14. \square

Let

$$E := \left[\mathcal{E}^2 + \eta_1 \varepsilon \left\{ \sum_{|\alpha| \leq 2} (\partial^\alpha r, \nabla_x \partial^\alpha (a, b, c))_{L_x^2} + (\partial^\alpha b, \nabla_x \partial^\alpha a)_{L_x^2} \right\} \right]^{1/2}, \quad (3.57)$$

with $\eta_1 > 0$ to be chosen later.

Theorem 3.15 (Global Energy Estimate). *If g is a solution of the scaled Landau equation (3.3), then there exist constants $c_0, c_3 > 0$ independent of ε such that if $E \leq 1$, then*

$$\frac{d}{dt} E^2 + c_3 \left(\frac{1}{\varepsilon^2} \mathcal{D}^2 + \mathcal{C}^2 \right) \leq c_0 E \left\{ \frac{1}{\varepsilon} \mathcal{D}^2 + \mathcal{C}^2 \right\} \quad (3.58)$$

holds as far as g exists.

Proof. Based on the microscopic estimate (3.40) and the macroscopic estimate (3.53), we can derive the uniform energy estimate. Indeed, estimates (3.40) and (3.53), imply that

$$\begin{aligned} & \frac{d}{dt} \left(\mathcal{E}^2 + \eta_1 \varepsilon \left\{ \sum_{|\alpha| \leq 2} (\partial^\alpha r, \nabla_x \partial^\alpha (a, b, c))_{L_x^2} + (\partial^\alpha b, \nabla_x \partial^\alpha a)_{L_x^2} \right\} \right) + \frac{C_\gamma}{\varepsilon^2} \mathcal{D}^2 + \eta_1 \mathcal{C}^2 \\ & \leq C \left(\frac{1}{\varepsilon} \mathcal{E} \mathcal{D}^2 + (\mathcal{E} \mathcal{C})^2 \right) + \frac{\eta_1 \tilde{C}}{\varepsilon^2} \mathcal{D}^2 + \eta_1 \tilde{C} \mathcal{E} (\mathcal{C}^2 + \mathcal{D}^2). \end{aligned}$$

We first choose η_1 small enough so that $C_\gamma - \eta_1 \tilde{C} > 0$, it gives that for $0 < \varepsilon < 1$

$$\begin{aligned} & \frac{d}{dt} \left\{ \mathcal{E}^2 + \eta_1 \varepsilon \sum_{|\alpha| \leq 2} (\partial^\alpha r, \nabla_x \partial^\alpha (a, b, c))_{L_x^2} + (\partial^\alpha b, \nabla_x \partial^\alpha a)_{L_x^2} \right\} + (C_\gamma - \eta_1 \tilde{C}) \frac{1}{\varepsilon^2} \mathcal{D}^2 + \eta_1 \mathcal{C}^2 \\ & \leq (C + \eta_1 \tilde{C}) \frac{1}{\varepsilon} \mathcal{E} \mathcal{D}^2 + (C \mathcal{E} + \eta_1 \tilde{C}) \mathcal{E} \mathcal{C}^2. \end{aligned} \tag{3.59}$$

Using (3.49), we have

$$\begin{aligned} \left| \sum_{|\alpha| \leq 2} (\partial^\alpha r, \nabla_x \partial^\alpha (a, b, c))_{L_x^2} + (\partial^\alpha b, \nabla_x \partial^\alpha a)_{L_x^2} \right| & \lesssim \|\nabla_x \partial^\alpha r\|_{L_x^2}^2 + \|\mathcal{A}\|_{H_x^3}^2 \\ & \lesssim \|g_2\|_{H_x^3 L_v^2}^2 + \|\mathcal{A}\|_{H_x^3}^2 \lesssim \mathcal{E}^2. \end{aligned}$$

Thus, we can choose $\eta_1 > 0$ small such that, for any $0 < \varepsilon < 1$

$$c_1 \mathcal{E} \leq E \leq c_2 \mathcal{E}, \tag{3.60}$$

for some positive constants c_1 and c_2 . Finally, using (3.59) and (3.60) we have

$$\frac{d}{dt} E^2 + (C_\gamma - \eta_1 \tilde{C}) \frac{1}{\varepsilon^2} \mathcal{D}^2 + \eta_1 \mathcal{C}^2 \leq \left(\frac{C + \eta_1 \tilde{C}}{c_1} \right) \frac{1}{\varepsilon} E \mathcal{D}^2 + \left(\frac{C}{c_1^2} + \frac{\eta_1 \tilde{C}}{c_1} \right) E \mathcal{C}^2,$$

then there exist constants $c_0, c_3 > 0$ independent of ε such that

$$\frac{d}{dt} E^2 + c_3 \left(\frac{1}{\varepsilon^2} \mathcal{D}^2 + \mathcal{C}^2 \right) \leq c_0 E \left\{ \frac{1}{\varepsilon} \mathcal{D}^2 + \mathcal{C}^2 \right\}.$$

□

We recall that δ_0, c_2, c_0, c_3 are defined respectively in Proposition 3.9 and Theorem 3.15.

Lemma 3.16. *If we choose the initial data $g_{\varepsilon,0}$ such that*

$$\mathcal{E}(0) = \|g_{\varepsilon,0}\|_{H_x^3 L_v^2} \leq M,$$

where M is defined as

$$M = \min \left\{ \delta_0, \frac{1}{4c_2}, \frac{c_3}{4c_0c_2} \right\}, \quad (3.61)$$

then,

$$E^2(T) + \frac{c_3}{2} \int_0^T \left\{ \frac{1}{\varepsilon^2} \mathcal{D}^2 + \mathcal{C}^2 \right\} dt \leq E^2(0), \quad (3.62)$$

for some $T > 0$.

Proof. Note that $\mathcal{E}(0) \leq M \leq \delta_0$, then from Theorem 3.10 there exists a solution $g_\varepsilon \in L^\infty([0, T]; H_x^3 L_v^2)$ for some $T > 0$, and from the local estimate (3.36), we have $\mathcal{E}(t) \leq 2M$ for $0 < t < T$. Note that on $[0, T]$, $E(t) \leq c_2 \mathcal{E}(t) \leq 2c_2 M < 1$. Then the global energy estimate (3.58) implies that

$$\frac{d}{dt} E^2 + (c_3 - 2c_0c_2M) \left\{ \frac{1}{\varepsilon^2} \mathcal{D}^2 + \mathcal{C}^2 \right\} \leq 0.$$

From the choice of M , $c_3 - 2c_0c_2M \geq \frac{c_3}{2}$. Thus

$$E^2(T) + \frac{c_3}{2} \int_0^T \left\{ \frac{1}{\varepsilon^2} \mathcal{D}^2 + \mathcal{C}^2 \right\} dt \leq E^2(0).$$

□

Proof of Theorem 3.1. Now, we are ready to prove Theorem 3.1 by the usual continuation argument. We set $M_0 := \frac{c_1}{c_2} M$, where M defined in (3.61). Let

$$\mathcal{E}(0) = \|g_{\varepsilon,0}\|_{H_x^3 L_v^2} \leq M_0,$$

which implies that $\mathcal{E}(0) \leq M \leq \delta_0$. Then from Theorem 3.10 there exists a solution g on $[0, T]$ for some $T > 0$. Furthermore, using (3.63), we have $E(T) \leq c_2 M_0$ then $\mathcal{E}(T) \leq M \leq \delta_0$. Using again Theorem 3.10 with initial data $\mathcal{E}(T)$, we obtain the existence of the solution on $[T, 2T]$ and so on. Finally, the local solution constructed in Theorem 3.10 can be extended globally. Now, we will show the estimate (3.12). Using (3.63) and the fact that we can iterate the process in $[T, 2T], [2T, 3T] \dots$, we get for all $t \in \mathbb{R}^+$

$$E^2(t) + \frac{c_3}{2} \int_0^t \left\{ \frac{1}{\varepsilon^2} \mathcal{D}^2 + \mathcal{C}^2 \right\} ds \leq E^2(0). \quad (3.63)$$

We get therefore

$$\sup_{t \geq 0} E^2(t) + \frac{c_3}{2} \int_0^\infty \left\{ \frac{1}{\varepsilon^2} \mathcal{D}^2 + \mathcal{C}^2 \right\} ds \leq 2E^2(0). \quad (3.64)$$

Next, using (3.60) we get

$$\sup_{t \geq 0} \mathcal{E}^2(t) + \frac{c_3}{2c_1^2} \int_0^\infty \left\{ \frac{1}{\varepsilon^2} \mathcal{D}^2 + \mathcal{C}^2 \right\} ds \leq \frac{2c_2^2}{c_1^2} \mathcal{E}^2(0). \quad (3.65)$$

Finally, we can choose $C_0 = \frac{c_3}{2c_1^2}$ and $C'_0 = \frac{2c_2^2}{c_1^2}$, hence the estimate (3.12) is true.

4 Limit to fluid incompressible Navier-Stokes-Fourier

In this section, we study the convergence of the perturbed Landau equation (3.3) to the fluid incompressible Navier-Stokes-Fourier system (3.13) as $\varepsilon \rightarrow 0$. We will present a well explained and detailed proof. The approach is reminiscent of the one in [83] (see also [5] for a related problem) but here, we aim to provide a fully rigorous proof. Note that most of the arguments have already been used in [49] but our framework of strong solution allows us to develop a simpler proof than the one in [49].

In the rest of the paper, our convergences hold up to extracting subsequences. Note that for a vector function $\mathbf{w} = \mathbf{w}(x) \in \mathbb{R}^3$, $\nabla_x \mathbf{w}$ is a matrix defined by $(\partial_{x_j} \mathbf{w}_i)_{1 \leq i, j \leq 3}$. We also write $(\nabla_x \cdot M)^i = \sum_j \partial_{x_j} M_{ij}(x)$, if M is a matrix defined by $M = (M_{ij}(x))_{1 \leq i, j \leq 3}$.

4.1 Local conservation laws

We first introduce the following fluid variables

$$\rho_\varepsilon = (g_\varepsilon, \sqrt{\mu})_{L_v^2}, \quad u_\varepsilon = (g_\varepsilon, v\sqrt{\mu})_{L_v^2}, \quad \theta_\varepsilon = \left(g_\varepsilon, \left(\frac{|v|^2}{3} - 1 \right) \sqrt{\mu} \right)_{L_v^2}. \quad (3.66)$$

Then we can derive the following local conservation laws from the solutions g_ε constructed in Theorem 3.1.

Lemma 3.17. *Assume that g_ε is the solutions to the perturbed Landau equation (3.3) constructed in Theorem 3.1. Then the following local conservation laws hold*

$$\begin{cases} \partial_t \rho_\varepsilon + \frac{1}{\varepsilon} \nabla_x \cdot u_\varepsilon = 0, \\ \partial_t u_\varepsilon + \frac{1}{\varepsilon} \nabla_x (\rho_\varepsilon + \theta_\varepsilon) + \frac{1}{\varepsilon} \nabla_x \cdot (\hat{A}(v) \sqrt{\mu}, \mathcal{L}(g_\varepsilon))_{L_v^2} = 0, \\ \partial_t \theta_\varepsilon + \frac{2}{3} \frac{1}{\varepsilon} \nabla_x \cdot u_\varepsilon + \frac{2}{3} \frac{1}{\varepsilon} \nabla_x \cdot (\hat{B}(v) \sqrt{\mu}, \mathcal{L}(g_\varepsilon))_{L_v^2} = 0. \end{cases} \quad (3.67)$$

Proof. Step 1. Conservation law of ρ_ε . We multiply the first g_ε -equation of (3.3) by $\sqrt{\mu} \in \mathcal{N}(\mathcal{L})$ and integrate over $v \in \mathbb{R}^3$. Then we obtain

$$\partial_t \rho_\varepsilon + \underbrace{\frac{1}{\varepsilon} (v \cdot \nabla_x g_\varepsilon, \sqrt{\mu})_{L_v^2}}_{I_1} + \underbrace{\frac{1}{\varepsilon^2} (\mathcal{L} g_\varepsilon, \sqrt{\mu})_{L_v^2}}_{I_2=0} = \underbrace{\frac{1}{\varepsilon^2} (\Gamma(g_\varepsilon, g_\varepsilon), \sqrt{\mu})_{L_v^2}}_{I_3=0}. \quad (3.68)$$

For the term I_1 , we have

$$I_1 = \frac{1}{\varepsilon} \nabla_x \cdot (g_\varepsilon, v\sqrt{\mu})_{L_v^2} = \frac{1}{\varepsilon} \nabla_x \cdot u_\varepsilon. \quad (3.69)$$

Collecting the above relations, we deduce that

$$\partial_t \rho_\varepsilon + \frac{1}{\varepsilon} \nabla_x \cdot u_\varepsilon = 0, \quad (3.70)$$

hence the first equation of (3.67) holds.

Step 2. Conservation law of u_ε . We multiply the first g_ε -equation of (3.3) by $v\sqrt{\mu} \in \mathcal{N}(\mathcal{L})$ and integrate over $v \in \mathbb{R}^3$, we have

$$\partial_t u_\varepsilon + \underbrace{\frac{1}{\varepsilon} (v \cdot \nabla_x g_\varepsilon, v\sqrt{\mu})_{L_v^2}}_{\mathbb{II}_1} + \underbrace{\frac{1}{\varepsilon^2} (\mathcal{L}g_\varepsilon, v\sqrt{\mu})_{L_v^2}}_{\mathbb{II}_2=0} = \underbrace{\frac{1}{\varepsilon^2} (\Gamma(g_\varepsilon, g_\varepsilon), v\sqrt{\mu})_{L_v^2}}_{\mathbb{II}_3=0}. \quad (3.71)$$

For the term \mathbb{II}_1 , we introduce the matrix $A(v) := v \otimes v - \frac{|v|^2}{3} \mathbb{I}_3$. We have that $A\sqrt{\mu} \in \mathcal{N}^\perp(\mathcal{L})$ and there exists $\hat{A}(v)$ such that $\mathcal{L}(\sqrt{\mu}\hat{A}) = \sqrt{\mu}A$ with $\hat{A}\sqrt{\mu} \in \mathcal{N}^\perp(\mathcal{L})$ and \mathbb{I}_3 is the 3×3 unitary matrix (see for example [49]). Then we can write that

$$\begin{aligned} \mathbb{II}_1 &= \frac{1}{\varepsilon} \nabla_x \cdot (g_\varepsilon, v \otimes v \sqrt{\mu})_{L_v^2} \\ &= \frac{1}{\varepsilon} \nabla_x \cdot \left(g_\varepsilon, \left(v \otimes v - \frac{|v|^2}{3} \mathbb{I}_3 \right) \sqrt{\mu} \right)_{L_v^2} + \frac{1}{\varepsilon} \nabla_x \cdot \left(g_\varepsilon, \frac{|v|^2}{3} \mathbb{I}_3 \sqrt{\mu} \right)_{L_v^2} \\ &= \frac{1}{\varepsilon} \nabla_x \cdot (g_\varepsilon, A(v)\sqrt{\mu})_{L_v^2} + \frac{1}{\varepsilon} \nabla_x \cdot \left((g_\varepsilon, \sqrt{\mu})_{L_v^2} + \left(g_\varepsilon, \left(\frac{|v|^2}{3} - 1 \right) \sqrt{\mu} \right)_{L_v^2} \right) \\ &= \frac{1}{\varepsilon} \nabla_x \cdot (g_\varepsilon, A(v)\sqrt{\mu})_{L_v^2} + \frac{1}{\varepsilon} \nabla_x \theta_\varepsilon + \frac{1}{\varepsilon} \nabla_x \rho_\varepsilon. \end{aligned} \quad (3.72)$$

Finally, using that $\mathcal{L}(\sqrt{\mu}\hat{A}) = \sqrt{\mu}A$ and that \mathcal{L} is self-adjoint in L_v^2 , we have

$$\mathbb{II}_1 = \frac{1}{\varepsilon} \nabla_x \cdot \left(\hat{A}(v)\sqrt{\mu}, \mathcal{L}(g_\varepsilon) \right)_{L_v^2} + \frac{1}{\varepsilon} \nabla_x (\theta_\varepsilon + \rho_\varepsilon).$$

Collecting the previous calculations, we obtain

$$\partial_t u_\varepsilon + \frac{1}{\varepsilon} \nabla_x (\rho_\varepsilon + \theta_\varepsilon) + \frac{1}{\varepsilon} \nabla_x \cdot \left(\hat{A}(v)\sqrt{\mu}, \mathcal{L}(g_\varepsilon) \right)_{L_v^2} = 0, \quad (3.73)$$

then the second equations of (3.67) holds.

Step 3. Conservation law of θ_ε . We multiply the first g_ε -equation of (3.3) by $\left(\frac{|v|^2}{3} - 1 \right) \sqrt{\mu} \in \mathcal{N}(\mathcal{L})$ and integrate over $v \in \mathbb{R}^3$, we have

$$\begin{aligned} \partial_t \theta_\varepsilon + \underbrace{\frac{1}{\varepsilon} \left(v \cdot \nabla_x g_\varepsilon, \left(\frac{|v|^2}{3} - 1 \right) \sqrt{\mu} \right)_{L_v^2}}_{\mathbb{III}_1} + \underbrace{\frac{1}{\varepsilon^2} \left(\mathcal{L}g_\varepsilon, \left(\frac{|v|^2}{3} - 1 \right) \sqrt{\mu} \right)_{L_v^2}}_{\mathbb{III}_2=0} \\ = \underbrace{\frac{1}{\varepsilon^2} \left(\Gamma(g_\varepsilon, g_\varepsilon), \left(\frac{|v|^2}{3} - 1 \right) \sqrt{\mu} \right)_{L_v^2}}_{\mathbb{III}_3=0}. \end{aligned} \quad (3.74)$$

For the term III_2 , we introduce $B(v) := v\left(\frac{|v|^2}{2} - \frac{5}{2}\right)$. We have that $B\sqrt{\mu} \in \mathcal{N}^\perp(\mathcal{L})$ and there exists $\hat{B}(v)$ such that $\mathcal{L}(\sqrt{\mu}\hat{B}) = \sqrt{\mu}B$ with $\hat{B}\sqrt{\mu} \in \mathcal{N}^\perp(\mathcal{L})$. Then, we can write

$$\begin{aligned} \text{III}_2 &= \frac{1}{\varepsilon} \nabla_x \cdot \left(g_\varepsilon, v \left(\frac{|v|^2}{3} - 1 \right) \sqrt{\mu} \right)_{L_v^2} \\ &= \frac{2}{3} \frac{1}{\varepsilon} \nabla_x \cdot \left(g_\varepsilon, v \left(\frac{|v|^2}{2} - \frac{5}{2} \right) \sqrt{\mu} \right)_{L_v^2} + \frac{2}{3} \frac{1}{\varepsilon} \nabla_x \cdot (g_\varepsilon, v \sqrt{\mu})_{L_v^2} \\ &= \frac{2}{3} \frac{1}{\varepsilon} \nabla_x \cdot (g_\varepsilon, B(v) \sqrt{\mu})_{L_v^2} + \frac{2}{3} \frac{1}{\varepsilon} \nabla_x \cdot u_\varepsilon \\ &= \frac{2}{3} \frac{1}{\varepsilon} \nabla_x \cdot \left(\hat{B}(v) \sqrt{\mu}, \mathcal{L}(g_\varepsilon) \right)_{L_v^2} + \frac{2}{3} \frac{1}{\varepsilon} \nabla_x \cdot u_\varepsilon. \end{aligned} \quad (3.75)$$

Collecting the above relations, we deduce that

$$\partial_t \theta_\varepsilon + \frac{2}{3} \frac{1}{\varepsilon} \nabla_x \cdot u_\varepsilon + \frac{2}{3} \frac{1}{\varepsilon} \nabla_x \cdot \left(\hat{B}(v) \sqrt{\mu}, \mathcal{L}(g_\varepsilon) \right)_{L_v^2} = 0, \quad (3.76)$$

hence the third equation of (3.67) holds. \square

4.2 Limits from the global energy estimate

Based on Theorem 3.1, the Cauchy problem (3.3) admits a global solution g_ε belonging to $L^\infty([0, \infty); H_x^3 L_v^2)$ which is subject to the global energy estimate (3.12), namely, there is a positive constant, independent of ε , such that

$$\sup_{t \geq 0} \|g_\varepsilon(t)\|_{H_x^3 L_v^2}^2 \leq C \quad (3.77)$$

and

$$\int_0^\infty \|g_{\varepsilon,2}(t)\|_{\chi^3(\mathbb{T}_x^3 \times \mathbb{R}_v^3)}^2 dt \leq C \varepsilon^2. \quad (3.78)$$

From the energy bound (3.77), there exists $g \in L^\infty([0, \infty); H_x^3 L_v^2)$, such that

$$g_\varepsilon \rightharpoonup g \text{ as } \varepsilon \rightarrow 0 \quad (3.79)$$

where the convergence is weak- \star in $L^\infty([0, \infty); H_x^3 L_v^2)$ (the limits may hold for some subsequence). From the energy dissipation bound (3.78), we have

$$\{I - \Pi_0\}g_\varepsilon \rightarrow 0 \text{ strongly in } L^2([0, \infty); H_x^3 L_v^2) \quad (3.80)$$

as $\varepsilon \rightarrow 0$. We thereby deduce from combining the first convergence in (3.79) and (3.80) that

$$\{I - \Pi_0\}g = 0. \quad (3.81)$$

Indeed, we have $\{I - \Pi_0\}g_\varepsilon$ converges in the sense of distributions to $\{I - \Pi_0\}g$ as ε tends to zero. By uniqueness of the limit, we obtain the proof. Then, it immediately gives that there are $(\rho, u, \theta) \in L^\infty([0, \infty); H^3(\mathbb{T}_x^3))$ such that

$$g = \rho(x)\sqrt{\mu} + u(x) \cdot v\sqrt{\mu} + \theta(x)\left(\frac{|v|^2}{2} - \frac{3}{2}\right)\sqrt{\mu}. \quad (3.82)$$

Via the definitions of $\rho_\varepsilon, u_\varepsilon$ and θ_ε in (3.66) and the uniform energy bound (3.77), we obtain

$$\sup_{t \geq 0} \left(\|\rho_\varepsilon\|_{H^3(\mathbb{T}_x^3)}^2 + \|u_\varepsilon\|_{H^3(\mathbb{T}_x^3)}^2 + \|\theta_\varepsilon\|_{H^3(\mathbb{T}_x^3)}^2 \right) \lesssim \sup_{t \geq 0} \|g_\varepsilon\|_{H_x^3 L_v^2}^2 \leq C. \quad (3.83)$$

We thereby deduce the following convergences from the convergence of (3.79) and the limit function $g(t, x, v)$ given in (3.82) that

$$\begin{aligned} \rho_\varepsilon &= (g_\varepsilon, \sqrt{\mu})_{L_v^2} \longrightarrow (g, \sqrt{\mu})_{L_v^2} = \rho, \\ u_\varepsilon &= (g_\varepsilon, v\sqrt{\mu})_{L_v^2} \longrightarrow (g, v\sqrt{\mu})_{L_v^2} = u, \\ \theta_\varepsilon &= \left(g_\varepsilon, \left(\frac{|v|^2}{3} - 1 \right) \sqrt{\mu} \right)_{L_v^2} \longrightarrow \left(g, \left(\frac{|v|^2}{3} - 1 \right) \sqrt{\mu} \right)_{L_v^2} = \theta, \end{aligned} \quad (3.84)$$

weakly- \star in $L^\infty([0, \infty); H^3(\mathbb{T}_x^3))$.

4.3 Convergences to limiting equations

In this subsection, we will derive the incompressible Navier-Stokes-Fourier system (3.13) from the conservation laws (3.67) in Lemma 3.17 and the convergences obtained in the previous subsection.

Incompressibility and Boussinesq relation: From the first equation of (3.67) in Lemma 3.17 and the energy uniform bound (3.77), it is easy to deduce

$$\nabla_x \cdot u_\varepsilon = -\varepsilon \partial_t \rho_\varepsilon \longrightarrow 0 \quad (3.85)$$

in the sense of distributions as $\varepsilon \longrightarrow 0$. By using the convergence (3.84), we have $\nabla_x \cdot u_\varepsilon$ converges in the sense of distributions to $\nabla_x \cdot u$ as ε tends to zero. By uniqueness of the limit, we obtain

$$\nabla_x \cdot u = 0. \quad (3.86)$$

Via the second equation of (3.67), we have

$$\nabla_x (\rho_\varepsilon + \theta_\varepsilon) = -\varepsilon \partial_t u_\varepsilon - \nabla_x \cdot \left(\hat{A}(v)\sqrt{\mu}, \mathcal{L}(g_\varepsilon) \right)_{L_v^2}. \quad (3.87)$$

Notice that

$$\nabla_x \cdot \left(\hat{A}(v)\sqrt{\mu}, \mathcal{L}(g_\varepsilon) \right)_{L_v^2} = \nabla_x \cdot (A(v)\sqrt{\mu}, \{I - \Pi_0\}g_\varepsilon)_{L_v^2}$$

where the self-adjointness of \mathcal{L} and the fact that $A\sqrt{\mu} \in \mathcal{N}^\perp(\mathcal{L})$ are utilized. Then we derive from the Hölder inequality, and the uniform energy dissipation bound (3.78) that

$$\begin{aligned} \int_0^\infty \|\nabla_x \cdot \left(\hat{A}(v)\sqrt{\mu}, \mathcal{L}(g_\varepsilon) \right)_{L_v^2}\|_{H_x^2}^2 dt &= \int_0^\infty \|(A(v)\sqrt{\mu}, \nabla_x \{I - \Pi_0\}g_\varepsilon)_{L_v^2}\|_{H_x^2}^2 dt \\ &\lesssim \int_0^\infty \|\{I - \Pi_0\}g_\varepsilon\|_{H_x^3 L_v^2}^2 dt \\ &\lesssim \int_0^\infty \mathcal{D}^2(t) dt \\ &\leq C\varepsilon^2, \end{aligned}$$

then, $\nabla_x \cdot \left(\hat{A}(v)\sqrt{\mu}, \mathcal{L}(g_\varepsilon) \right)_{L_v^2} \rightarrow 0$ strongly in $L^2([0, \infty); H^2(\mathbb{T}_x^3))$. Consequently, it is easy to deduce that

$$\nabla_x(\rho_\varepsilon + \theta_\varepsilon) \rightarrow 0 \quad (3.88)$$

in the sense of distributions as $\varepsilon \rightarrow 0$. By using the convergence (3.84), we have $\nabla_x(\rho_\varepsilon + \theta_\varepsilon)$ converges in the sense of distributions to $\nabla_x(\rho + \theta)$ as ε tends to zero. By uniqueness of the limit, we obtain the Boussinesq relation

$$\nabla_x(\rho + \theta) = 0. \quad (3.89)$$

Now, we will show that Boussinesq relation can be strengthened. Using (3.4) and (3.66), we have that

$$\int_{\mathbb{T}^3} (\rho_\varepsilon + \theta_\varepsilon) dx = 0.$$

Furthermore, using also (3.84), we obtain

$$\int_{\mathbb{T}^3} (\rho_\varepsilon + \theta_\varepsilon) dx \rightarrow \int_{\mathbb{T}^3} (\rho + \theta) dx \quad \text{in } \mathcal{D}'_t,$$

from which we deduce that

$$\int_{\mathbb{T}^3} (\rho + \theta) dx = 0, \quad \text{for a.e. } t > 0.$$

Finally (3.89) yields the strengthened form

$$\rho + \theta = 0. \quad (3.90)$$

Convergence of $\frac{3}{5}\theta_\varepsilon - \frac{2}{5}\rho_\varepsilon$: Before doing this, we introduce the following Aubin-Lions-Simon Theorem, a fundamental result of compactness in the study of nonlinear evolution problems, which can be found in Theorem II.5.16 in [16].

Lemma 3.18 (Aubin-Lions-Simon Theorem). *Let $B_0 \subset B_1 \subset B_2$ be three Banach spaces. We assume that the embedding of B_1 in B_2 is continuous and that the embedding of B_0 in B_1 is compact. Let p, r be such that $1 \leq p, r \leq +\infty$. For $T > 0$, we define*

$$E_{p,r} = \{u \in L^p(0, T; B_0), \partial_t u \in L^r(0, T; B_2)\}.$$

- 1) *If $p < +\infty$, the embedding of $E_{p,r}$ in $L^p(0, T; B_1)$ is compact.*
- 2) *If $p = +\infty$ and if $r > 1$, the embedding of $E_{p,r}$ in $C(0, T; B_1)$ is compact.*

On the one hand, the third equation of (3.67) multiplied by $\frac{3}{5}$ minus $\frac{2}{5}$ times of the first equation of (3.67) gives

$$\partial_t \left(\frac{3}{5} \theta_\varepsilon - \frac{2}{5} \rho_\varepsilon \right) + \frac{2}{5} \frac{1}{\varepsilon} \nabla_x \cdot \left(\hat{B}(v) \sqrt{\mu}, \mathcal{L}(g_\varepsilon) \right)_{L_v^2} = 0. \quad (3.91)$$

We thus have that

$$\begin{aligned} \left\| \partial_t \left(\frac{3}{5} \theta_\varepsilon - \frac{2}{5} \rho_\varepsilon \right) \right\|_{H_x^2} &= \left\| \frac{2}{5} \frac{1}{\varepsilon} \nabla_x \cdot (B(v) \sqrt{\mu}, \{I - \Pi_0\} g_\varepsilon)_{L_v^2} \right\|_{H_x^2} \\ &\lesssim \frac{1}{\varepsilon} \|B(v) \sqrt{\mu}\|_{L_v^2} \| \{I - \Pi_0\} g_\varepsilon \|_{H_x^3 L_v^2} \\ &\lesssim \frac{1}{\varepsilon} \| \{I - \Pi_0\} g_\varepsilon \|_{H_x^3 L_v^2} \end{aligned}$$

which immediately implies from the uniform energy dissipation bound (3.78) that

$$\left\| \partial_t \left(\frac{3}{5} \theta_\varepsilon - \frac{2}{5} \rho_\varepsilon \right) \right\|_{L^2(0, T; H_x^2)} \lesssim \left(\int_0^\infty \frac{1}{\varepsilon^2} \mathcal{D}^2(t) dt \right)^{1/2} \leq C \quad (3.92)$$

for any $T > 0$ and $0 < \varepsilon \leq 1$. On the other hand, we easily have from (3.83) that

$$\left\| \left(\frac{3}{5} \theta_\varepsilon - \frac{2}{5} \rho_\varepsilon \right) \right\|_{L^\infty(0, T; H_x^3)} \leq C \quad (3.93)$$

for all $T > 0$ and $0 < \varepsilon \leq 1$. One notices that

$$H_x^3 \hookrightarrow H_x^2 \hookrightarrow H_x^1, \quad (3.94)$$

where the embedding of H_x^3 in H_x^2 is compact and the embedding of H_x^2 in H_x^1 is naturally continuous. Then, from Aubin-Lions-Simon Theorem in Lemma 3.18, the bounds (3.92), (3.93) and the embeddings (3.94), we deduce that there is a $\tilde{\theta} \in C(\mathbb{R}^+; H^2(\mathbb{T}_x^3)) \cap L^\infty(\mathbb{R}^+; H^3(\mathbb{T}_x^3))$ such that

$$\left(\frac{3}{5} \theta_\varepsilon - \frac{2}{5} \rho_\varepsilon \right) \longrightarrow \tilde{\theta}$$

strongly in $C(\mathbb{R}^+; H^2(\mathbb{T}_x^3))$ as $\varepsilon \rightarrow 0$. On the other hand, we have that $(\frac{3}{5}\theta_\varepsilon - \frac{2}{5}\rho_\varepsilon)$ converges in the sense of distributions to $(\frac{3}{5}\theta - \frac{2}{5}\rho)$ as ε tends to zero. By uniqueness of the limit, we obtain that $(\frac{3}{5}\theta - \frac{2}{5}\rho) = \tilde{\theta}$. Now we can write $\theta = (\frac{3}{5}\theta - \frac{2}{5}\rho) + \frac{2}{5}(\rho + \theta)$, which gives us $\tilde{\theta} = \theta$, since $\rho + \theta = 0$ according to the relation (3.90). As a result,

$$\left(\frac{3}{5}\theta_\varepsilon - \frac{2}{5}\rho_\varepsilon\right) \rightarrow \theta \quad (3.95)$$

strongly in $C(\mathbb{R}^+; H^2(\mathbb{T}_x^3))$ as $\varepsilon \rightarrow 0$, where $\theta \in C(\mathbb{R}^+; H^2(\mathbb{T}_x^3)) \cap L^\infty(\mathbb{R}^+; H^3(\mathbb{T}_x^3))$. **Convergence of $\mathcal{P}u_\varepsilon$:** Here \mathcal{P} is the Leray projection operator given by

$$\mathcal{P} = \mathcal{I} - \nabla_x \Delta_x^{-1} \nabla_x. \quad (3.96)$$

where \mathcal{I} is the identical mapping. Taking \mathcal{P} on the second equation of (3.67) gives

$$\partial_t \mathcal{P}u_\varepsilon + \frac{1}{\varepsilon} \mathcal{P} \nabla_x \cdot \left(\hat{A}(v) \sqrt{\mu}, \mathcal{L}(g_\varepsilon) \right)_{L_v^2} = 0, \quad (3.97)$$

where we used $\mathcal{P} \nabla_x (\rho_\varepsilon + \theta_\varepsilon) = 0$. We thus have that

$$\begin{aligned} \|\partial_t \mathcal{P}u_\varepsilon\|_{H_x^2} &= \left\| \frac{1}{\varepsilon} \mathcal{P} \nabla_x \cdot (A(v) \sqrt{\mu}, \{I - \Pi_0\} g_\varepsilon)_{L_v^2} \right\|_{H_x^2} \\ &\lesssim \left\| \frac{1}{\varepsilon} \nabla_x \cdot (A(v) \sqrt{\mu}, \{I - \Pi_0\} g_\varepsilon)_{L_v^2} \right\|_{H_x^2} \\ &\lesssim \frac{1}{\varepsilon} \|A(v) \sqrt{\mu}\|_{L_v^2} \|\{I - \Pi_0\} g_\varepsilon\|_{H_x^3 L_v^2} \\ &\lesssim \frac{1}{\varepsilon} \|\{I - \Pi_0\} g_\varepsilon\|_{H_x^3 L_v^2} \end{aligned}$$

which immediately implies from the uniform energy dissipation bound (3.78) that

$$\|\partial_t \mathcal{P}u_\varepsilon\|_{L^2(0, T; H_x^2)} \lesssim \left(\int_0^\infty \frac{1}{\varepsilon^2} \mathcal{D}^2(t) dt \right)^{1/2} \leq C \quad (3.98)$$

for any $T > 0$ and $0 < \varepsilon \leq 1$. We easily have from (3.83) that

$$\|\mathcal{P}u_\varepsilon\|_{L^\infty(0, T; H_x^3)} \lesssim \|u_\varepsilon\|_{L^\infty(0, T; H_x^3)} \leq C. \quad (3.99)$$

Then, from Aubin-Lions-Simon Theorem in Lemma 3.18, the bounds (3.98), (3.99) and the embeddings (3.94), we deduce that there is a $\tilde{u} \in C(\mathbb{R}^+; H^2(\mathbb{T}_x^3)) \cap L^\infty(\mathbb{R}^+; H^3(\mathbb{T}_x^3))$ such that

$$\mathcal{P}u_\varepsilon \rightarrow \tilde{u}$$

strongly in $C(\mathbb{R}^+; H^2(\mathbb{T}_x^3))$ as $\varepsilon \rightarrow 0$. On the other hand, we have $\mathcal{P}u_\varepsilon$ converges in the sense of distributions to $\mathcal{P}u$ as ε tends to zero. By uniqueness of the limit, we obtain that

$\tilde{u} = \mathcal{P}u$. Using that where we used $\nabla_x \cdot u = 0$ according to the incompressibility relation (3.86), we get

$$\mathcal{P}u_\varepsilon \longrightarrow u \quad (3.100)$$

strongly in $C(\mathbb{R}^+; H^2(\mathbb{T}_x^3))$ as $\varepsilon \longrightarrow 0$, where $u \in C(\mathbb{R}^+; H^2(\mathbb{T}_x^3)) \cap L^\infty(\mathbb{R}^+; H^3(\mathbb{T}_x^3))$. Regarding the convergence of \mathcal{P}^\perp . We have

$$\mathcal{P}^\perp u_\varepsilon \longrightarrow 0 \quad (3.101)$$

weakly- \star in $L^\infty([0, \infty); H^3(\mathbb{T}_x^3))$. Indeed, on the one hand we have

$$\|\mathcal{P}^\perp u_\varepsilon\|_{L^\infty(0, T; H_x^3)} \lesssim \|u_\varepsilon\|_{L^\infty(0, T; H_x^3)} \leq C. \quad (3.102)$$

On the other hand, we have $\mathcal{P}^\perp u_\varepsilon$ converges in the sense of distributions to 0 as ε tends to zero, hence the relation (3.101) is true.

Equation of u and θ : We first calculate the term

$$\left(\hat{E}(v) \sqrt{M}, \frac{1}{\varepsilon} \mathcal{L}g_\varepsilon \right)_{L_v^2}$$

where $\hat{E} = \hat{A}$ or \hat{B} . Following the standard formal derivations of fluid dynamic limits of Boltzmann equation (see [9] for instance), we obtain

$$\left(\hat{A}(v) \sqrt{\mu}, \frac{1}{\varepsilon} \mathcal{L}g_\varepsilon \right)_{L_v^2} = u_\varepsilon \otimes u_\varepsilon - \frac{|u_\varepsilon|^2}{3} \mathbb{I}_3 - \nu \Sigma(u_\varepsilon) - R_{\varepsilon, A}, \quad (3.103)$$

and

$$\left(\hat{B}(v) \sqrt{\mu}, \frac{1}{\varepsilon} \mathcal{L}g_\varepsilon \right)_{L_v^2} = \frac{5}{2} u_\varepsilon \theta_\varepsilon - \frac{5}{2} \kappa \nabla_x \theta_\varepsilon - R_{\varepsilon, B}, \quad (3.104)$$

where

$$\Sigma(u_\varepsilon) := \nabla_x u_\varepsilon + \nabla_x u_\varepsilon^\top - \frac{2}{3} \nabla_x \cdot u_\varepsilon \mathbb{I}_3, \quad (3.105)$$

$$\nu := \frac{1}{10} \left(\sqrt{\mu} A, \sqrt{\mu} \hat{A} \right)_{L_v^2}, \quad (3.106)$$

$$\kappa := \frac{2}{15} \left(\sqrt{\mu} B, \sqrt{\mu} \hat{B} \right)_{L_v^2}. \quad (3.107)$$

For $E = A$ or B , $R_{\varepsilon, E}$ are of the form

$$\begin{aligned} R_{\varepsilon, E} &= \varepsilon \left(\hat{E}(v) \sqrt{\mu}, \partial_t g_\varepsilon \right)_{L_v^2} + \left(\hat{E}(v) \sqrt{\mu}, v \cdot \nabla_x \{I - \Pi_0\} g_\varepsilon \right)_{L_v^2} \\ &+ \left(\hat{E}(v) \sqrt{\mu}, \Gamma(\{I - \Pi_0\} g_\varepsilon, \{I - \Pi_0\} g_\varepsilon) \right)_{L_v^2} \\ &+ \left(\hat{E}(v) \sqrt{\mu}, \Gamma(\Pi_0 g_\varepsilon, \{I - \Pi_0\} g_\varepsilon) \right)_{L_v^2} \\ &+ \left(\hat{E}(v) \sqrt{\mu}, \Gamma(\{I - \Pi_0\} g_\varepsilon, \Pi_0 g_\varepsilon) \right)_{L_v^2}. \end{aligned} \quad (3.108)$$

For the vector field u_ε , we decompose $u_\varepsilon = \mathcal{P}u_\varepsilon + \mathcal{P}^\perp u_\varepsilon$. Then, plugging the relation (3.103) into the equation (3.97), we have

$$\partial_t \mathcal{P}u_\varepsilon + \mathcal{P}\nabla_x \cdot (\mathcal{P}u_\varepsilon \otimes \mathcal{P}u_\varepsilon) - \nu \Delta_x \mathcal{P}u_\varepsilon = R_{\varepsilon,u} \quad (3.109)$$

where,

$$R_{\varepsilon,u} = \mathcal{P}\nabla_x \cdot R_{\varepsilon,A} - \mathcal{P}\nabla_x \cdot (\mathcal{P}u_\varepsilon \otimes \mathcal{P}^\perp u_\varepsilon + \mathcal{P}^\perp u_\varepsilon \otimes \mathcal{P}u_\varepsilon + \mathcal{P}^\perp u_\varepsilon \otimes \mathcal{P}^\perp u_\varepsilon). \quad (3.110)$$

Noticing that $\theta_\varepsilon = \left(\frac{3}{5}\theta_\varepsilon - \frac{2}{5}\rho_\varepsilon\right) + \frac{2}{5}(\rho_\varepsilon + \theta_\varepsilon)$, we substitute the relation (3.104) into the equation (3.91) and then obtain

$$\partial_t \left(\frac{3}{5}\theta_\varepsilon - \frac{2}{5}\rho_\varepsilon\right) + \nabla_x \cdot \left[\mathcal{P}u_\varepsilon \left(\frac{3}{5}\theta_\varepsilon - \frac{2}{5}\rho_\varepsilon\right)\right] - \kappa \Delta_x \left(\frac{3}{5}\theta_\varepsilon - \frac{2}{5}\rho_\varepsilon\right) = R_{\varepsilon,\theta} \quad (3.111)$$

where

$$\begin{aligned} R_{\varepsilon,\theta} = & \frac{2}{5}\nabla_x \cdot R_{\varepsilon,B} - \frac{2}{5}\nabla_x \cdot [\mathcal{P}u_\varepsilon(\rho_\varepsilon + \theta_\varepsilon)] - \nabla_x \cdot \left[\mathcal{P}^\perp u_\varepsilon \left(\frac{3}{5}\theta_\varepsilon - \frac{2}{5}\rho_\varepsilon\right)\right] \\ & - \frac{2}{5}\nabla_x \cdot \left[\mathcal{P}^\perp u_\varepsilon(\rho_\varepsilon + \theta_\varepsilon)\right] + \frac{2}{5}\kappa \Delta_x(\rho_\varepsilon + \theta_\varepsilon). \end{aligned} \quad (3.112)$$

Our final goal is now to perform the limit ε to 0 in (3.109) and (3.111) in order to deduce the u and θ equations in (3.13). We first focus on (3.109).

Now, we take the limit from (3.109) to obtain the u -equation of (3.13). For any $T > 0$, let a vector-valued test function $\psi(t, x) \in C^1(0, T; C_c^\infty(\mathbb{T}^3))$ with $\nabla_x \cdot \psi = 0$, $\psi(0, x) = \psi_0(x) \in C_c^\infty(\mathbb{T}^3)$ and $\psi(t, x) = 0$ for $t \geq T'$ where $T' < T$. We multiply (3.109) by $\psi(t, x)$ and integrate by parts over $(t, x) \in [0, T] \times \mathbb{T}^3$. Then we obtain

$$\int_0^T \int_{\mathbb{T}^3} \partial_t \mathcal{P}u_\varepsilon \cdot \psi(t, x) dx dt = - \underbrace{\int_{\mathbb{T}^3} \mathcal{P}u_\varepsilon(0, x) \cdot \psi(0, x) dx}_{\text{IV}_1} - \underbrace{\int_0^T \int_{\mathbb{T}^3} \mathcal{P}u_\varepsilon \cdot \partial_t \psi(t, x) dx dt}_{\text{IV}_2}. \quad (3.113)$$

From the initial condition (3.14) in Theorem 3.2 and the convergence (3.100), we deduce that

$$\int_{\mathbb{T}^3} \mathcal{P}u_\varepsilon \cdot \partial_t \psi(t, x) dx dt \rightarrow \int_{\mathbb{T}^3} u \cdot \partial_t \psi(t, x) dx dt \quad (3.114)$$

and

$$\int_{\mathbb{T}^3} \mathcal{P}u_\varepsilon(0) \cdot \psi_0(x) dx \rightarrow \int_{\mathbb{T}^3} \mathcal{P}u_0 \cdot \psi_0(x) dx \quad (3.115)$$

as $\varepsilon \rightarrow 0$. Indeed, for the term IV_2 , we have

$$\begin{aligned} \left| \int_0^T \int_{\mathbb{T}^3} (\mathcal{P}u_\varepsilon - u) \cdot \partial_t \psi(t, x) dx dt \right| & \lesssim \max_{t \geq 0} \|\mathcal{P}u_\varepsilon - u\|_{L_x^2} \|\partial_t \psi\|_{L^1(0, T; L_x^2)} \\ & \leq C_{\psi, T} \max_{t \geq 0} \|\mathcal{P}u_\varepsilon - u\|_{L_x^2} \rightarrow 0. \end{aligned}$$

For the term \mathbb{IV}_1 we have,

$$\begin{aligned} \left| \int_{\mathbb{T}^3} (\mathcal{P}u_\varepsilon(0) - \mathcal{P}u_0)\psi_0(x)dx \right| &\lesssim \|\mathcal{P}u_\varepsilon(0) - \mathcal{P}u_0\|_{L_x^2} \|\psi_0(x)\|_{L_x^2} \\ &\lesssim \|\mathcal{P}u_\varepsilon(0) - \mathcal{P}u_0\|_{L_x^2} \longrightarrow 0. \end{aligned}$$

As a consequence, we have

$$\int_0^T \int_{\mathbb{T}^3} \partial_t \mathcal{P}u_\varepsilon \cdot \psi(t, x) dx dt \longrightarrow - \int_{\mathbb{T}^3} \mathcal{P}u_0 \cdot \psi_0(x) dx - \int_{\mathbb{T}^3} u \cdot \partial_t \psi(t, x) dx dt \quad (3.116)$$

as $\varepsilon \rightarrow 0$. Now, we will study the convergence of terms $\mathcal{P}\nabla_x \cdot (\mathcal{P}u_\varepsilon \otimes \mathcal{P}u_\varepsilon)$ and $\nu \Delta_x \mathcal{P}u_\varepsilon$ in the following lemma.

Lemma 3.19. *It holds that*

$$\begin{aligned} \mathcal{P}\nabla_x \cdot (\mathcal{P}u_\varepsilon \otimes \mathcal{P}u_\varepsilon) &\rightarrow \mathcal{P}\nabla_x \cdot (u \otimes u) \quad \text{strongly in } C(\mathbb{R}^+; H_x^1), \\ \nu \Delta_x \mathcal{P}u_\varepsilon &\rightarrow \nu \Delta_x u \quad \text{strongly in } C(\mathbb{R}^+; L_x^2), \end{aligned} \quad (3.117)$$

as $\varepsilon \rightarrow 0$, where ν is defined in (3.106).

Proof. Using that $\nabla_x \cdot \mathcal{P}u_\varepsilon = \nabla_x \cdot u = 0$, we have

$$\begin{aligned} &\|\nabla_x \cdot (\mathcal{P}u_\varepsilon \otimes \mathcal{P}u_\varepsilon - u \otimes u)\|_{H_x^1} \\ &\lesssim \|\nabla_x (\mathcal{P}u_\varepsilon - u) \cdot \mathcal{P}u_\varepsilon\|_{L_x^2} + \|\nabla_x u \cdot (\mathcal{P}u_\varepsilon - u)\|_{L_x^2} \\ &\quad + \|\tilde{\nabla}_x \nabla_x (\mathcal{P}u_\varepsilon - u) \cdot \mathcal{P}u_\varepsilon\|_{L_x^2} + \|\nabla_x (\mathcal{P}u_\varepsilon - u) \cdot \nabla_x \mathcal{P}u_\varepsilon\|_{L_x^2} \\ &\quad + \|\tilde{\nabla}_x \nabla_x u \cdot (\mathcal{P}u_\varepsilon - u)\|_{L_x^2} + \|\nabla_x u \cdot \nabla_x (\mathcal{P}u_\varepsilon - u)\|_{L_x^2} \\ &\lesssim \left(\|u\|_{L_t^\infty H_x^3} + \|\mathcal{P}u_\varepsilon\|_{L_t^\infty H_x^3} \right) \|\mathcal{P}u_\varepsilon - u\|_{C(\mathbb{R}^+; H_x^2)} \\ &\lesssim \|\mathcal{P}u_\varepsilon - u\|_{C(\mathbb{R}^+; H_x^2)} \longrightarrow 0, \end{aligned}$$

where we denote by $\tilde{\nabla}_x \Lambda$ with $\Lambda = (\lambda_{ij})_{1 \leq i, j \leq 3}$ the matrix defined by

$$\tilde{\nabla}_x \Lambda = \begin{pmatrix} \partial_{x_1} \lambda_{11} & \partial_{x_1} \lambda_{12} & \partial_{x_1} \lambda_{13} \\ \partial_{x_2} \lambda_{21} & \partial_{x_2} \lambda_{22} & \partial_{x_2} \lambda_{23} \\ \partial_{x_3} \lambda_{31} & \partial_{x_3} \lambda_{32} & \partial_{x_3} \lambda_{33} \end{pmatrix}.$$

In addition, we have

$$\|\mu \Delta_x (\mathcal{P}u_\varepsilon - u)\|_{C(\mathbb{R}^+; L_x^2)} \lesssim \|\mathcal{P}u_\varepsilon - u\|_{C(\mathbb{R}^+; H_x^2)} \longrightarrow 0.$$

□

The limit of the last term $R_{\varepsilon, u}$ is handled in the following lemma.

Lemma 3.20. *In the distributional sense,*

$$R_{\varepsilon,u} \rightarrow 0 \quad (3.118)$$

as $\varepsilon \rightarrow 0$, where $R_{\varepsilon,u}$ is defined in (3.110).

Proof. First, we have

$$\begin{aligned} & \|\mathcal{P}\nabla_x \cdot (\mathcal{P}u_\varepsilon \otimes \mathcal{P}^\perp u_\varepsilon + \mathcal{P}^\perp u_\varepsilon \otimes \mathcal{P}u_\varepsilon)\|_{L_t^\infty H_x^2} \\ & \lesssim \|\mathcal{P}u_\varepsilon\|_{L_t^\infty H_x^3} \|\mathcal{P}^\perp u_\varepsilon\|_{L_t^\infty H_x^3}, \end{aligned}$$

and by employing the convergences (3.100) and (3.101), one can obtain

$$\mathcal{P}\nabla_x \cdot (\mathcal{P}u_\varepsilon \otimes \mathcal{P}^\perp u_\varepsilon + \mathcal{P}^\perp u_\varepsilon \otimes \mathcal{P}u_\varepsilon) \rightarrow 0 \quad (3.119)$$

in the sense of distributions as $\varepsilon \rightarrow 0$. Let us now show that

$$\mathcal{P}\nabla_x \cdot (\mathcal{P}^\perp u_\varepsilon \otimes \mathcal{P}^\perp u_\varepsilon) \rightarrow 0 \quad (3.120)$$

in the sense of distributions as $\varepsilon \rightarrow 0$. To this end, we set

$$\beta_\varepsilon := \rho_\varepsilon + \theta_\varepsilon.$$

One observes that, Equation (3.73) reads

$$\varepsilon \partial_t u_\varepsilon + \nabla_x \beta_\varepsilon = -\nabla_x \cdot \left(\hat{A}(v) \sqrt{\mu}, \mathcal{L}(g_\varepsilon) \right)_{L_v^2}. \quad (3.121)$$

Now, we multiply the first g_ε -equation of (3.3) by $\frac{|v|^2}{2} \sqrt{\mu} \in \mathcal{N}(\mathcal{L})$ and integrate over $v \in \mathbb{R}^3$, we have

$$\varepsilon \partial_t \beta_\varepsilon + \nabla_x \cdot \left(g_\varepsilon, v \frac{|v|^2}{3} \sqrt{\mu} \right)_{L_v^2} = 0, \quad (3.122)$$

where we check easily that

$$\begin{aligned} \nabla_x \cdot \left(g_\varepsilon, v \frac{|v|^2}{3} \sqrt{\mu} \right)_{L_v^2} &= \frac{2}{3} \nabla_x \cdot \left(g_\varepsilon, v \left(\frac{|v|^2}{2} - \frac{5}{2} \right) \sqrt{\mu} \right)_{L_v^2} + \frac{5}{3} \nabla_x \cdot (g_\varepsilon, v \sqrt{\mu})_{L_v^2} \\ &= \frac{2}{3} \nabla_x \cdot \left(\hat{B}(v) \sqrt{\mu}, \mathcal{L}(g_\varepsilon) \right)_{L_v^2} + \frac{5}{3} \nabla_x \cdot \mathcal{P}^\perp u_\varepsilon. \end{aligned}$$

Using [98, Proposition 1.6], we can write

$$\mathcal{P}^\perp u_\varepsilon = \nabla_x \mathbf{U}_\varepsilon$$

with $\mathbf{U}_\varepsilon \in L^\infty([0, \infty); H^4(\mathbb{T}_x^3))$. After applying \mathcal{P}^\perp to (3.121) and reformulating (3.122), we obtain that \mathbf{U}_ε and β_ε satisfy

$$\begin{cases} \varepsilon \partial_t \nabla_x \mathbf{U}_\varepsilon + \nabla_x \beta_\varepsilon = \mathbf{F}_\varepsilon \\ \varepsilon \partial_t \beta_\varepsilon + \frac{5}{3} \Delta_x \mathbf{U}_\varepsilon = \mathbf{G}_\varepsilon \end{cases} \quad (3.123)$$

with

$$\mathbf{F}_\varepsilon = -\mathcal{P}^\perp \nabla_x \cdot (A(v) \sqrt{\mu}, \{I - \Pi_0\} g_\varepsilon)_{L_v^2} \quad \mathbf{G}_\varepsilon = -\frac{2}{3} \nabla_x \cdot (B(v) \sqrt{\mu}, \{I - \Pi_0\} g_\varepsilon)_{L_v^2}.$$

We get from the uniform energy dissipation bound (3.78) that

$$\|\mathbf{F}_\varepsilon\|_{L^2([0, \infty); L^2(\mathbb{T}_x^3))} \lesssim \varepsilon \quad \text{and} \quad \|\mathbf{G}_\varepsilon\|_{L^2([0, \infty); L^2(\mathbb{T}_x^3))} \lesssim \varepsilon$$

then \mathbf{F}_ε and \mathbf{G}_ε converge strongly to 0 in $L^2([0, \infty); L^2(\mathbb{T}_x^3))$ which implies that \mathbf{F}_ε and \mathbf{G}_ε converge strongly to 0 in $L^1_{loc}(dt; L^2_{loc}(dx))$. Moreover, returning to (3.83), (3.102) we have $\mathbf{U}_\varepsilon \in L^\infty([0, \infty); H^4(\mathbb{T}_x^3))$, $\beta_\varepsilon \in L^\infty([0, \infty); H^3(\mathbb{T}_x^3))$. Then, according to [49, Lemma 13.1] we deduce that (3.120) is true. From the above, we conclude that

$$\mathcal{P} \nabla_x \cdot (\mathcal{P} u_\varepsilon \otimes \mathcal{P}^\perp u_\varepsilon + \mathcal{P}^\perp u_\varepsilon \otimes \mathcal{P} u_\varepsilon + \mathcal{P}^\perp u_\varepsilon \otimes \mathcal{P}^\perp u_\varepsilon) \rightarrow 0 \quad (3.124)$$

in the sense of distributions as $\varepsilon \rightarrow 0$. Next we prove that $E = A$ or B

$$R_{\varepsilon, E} \rightarrow 0 \quad (3.125)$$

in the sense of distributions as $\varepsilon \rightarrow 0$, where $R_{\varepsilon, E}$ are defined in (3.108). Indeed, For any $T > 0$, let a vector-valued test function $\psi(t, x) \in C^1(0, T; C_c^\infty(\mathbb{T}^3))$, $\psi(0, x) = \psi_0(x) \in C_c^\infty(\mathbb{T}^3)$ and $\psi(t, x) = 0$ for $t \geq T'$ where $T' < T$. Then, from the uniform bound (3.77) and the initial energy bounds given in Theorem 3.2 that

$$\begin{aligned} & \left| \int_0^T \int_{\mathbb{T}^3} \varepsilon \left(\hat{E}(v) \sqrt{\mu}, \partial_t g_\varepsilon \right)_{L_v^2} \cdot \psi(t, x) dx dt \right| \\ & \lesssim \left| \int_{\mathbb{T}^3} \varepsilon \left(\hat{E}(v) \sqrt{\mu}, g_{\varepsilon, 0} \right)_{L_v^2} \cdot \psi(0, x) dx \right| + \left| \int_0^T \int_{\mathbb{T}^3} \varepsilon \left(\hat{E}(v) \sqrt{\mu}, g_\varepsilon \right)_{L_v^2} \cdot \partial_t \psi(t, x) dx dt \right| \\ & \lesssim \varepsilon \|\hat{E}(v) \sqrt{\mu}\|_{L_v^2} \left(\|g_{\varepsilon, 0}\|_{L_x^2 L_v^2} \|\psi_0\|_{L_x^2} + \|g_\varepsilon\|_{L^\infty(0, T; L_x^2 L_v^2)} \|\partial_t \psi\|_{L^1(0, T; L_x^2)} \right) \\ & \leq C_{\psi, T} \varepsilon \left(\|g_\varepsilon\|_{L^\infty(0, T; H_x^3 L_v^2)} + \|g_{\varepsilon, 0}\|_{H_x^3 L_v^2} \right) \rightarrow 0, \end{aligned} \quad (3.126)$$

as $\varepsilon \rightarrow 0$, which implies that $\varepsilon \left(\hat{E}(v) \sqrt{\mu}, \partial_t g_\varepsilon \right)_{L_v^2} \rightarrow 0$ in the sense of distributions as $\varepsilon \rightarrow 0$. Hölder inequality yields that

$$\left\| \left(\hat{E}(v) \sqrt{\mu}, v \cdot \nabla_x \{I - \Pi_0\} g_\varepsilon \right)_{L_v^2} \right\|_{H_x^2}^2 \lesssim \|\{I - \Pi_0\} g_\varepsilon\|_{H_x^3 L_v^2}^2$$

which immediately derives from the uniform energy dissipation bound (3.78) that

$$\left\| \left(\hat{E}(v) \sqrt{\mu}, v \cdot \nabla_x \{I - \Pi_0\} g_\varepsilon \right)_{L_v^2} \right\|_{L^2(0, T; H_x^2)} \lesssim \left(\int_0^\infty \mathcal{D}^2(t) dt \right)^{1/2} \leq C \varepsilon \rightarrow 0.$$

Then we have

$$\left(\hat{E}(v) \sqrt{\mu}, v \cdot \nabla_x \{I - \Pi_0\} g_\varepsilon \right)_{L_v^2} \rightarrow 0$$

strongly in $L^2(\mathbb{R}^+; H_x^2)$ as $\varepsilon \rightarrow 0$. For any $T > 0$, we take any vector-valued test function $\phi(t, x) \in C_c^\infty([0, T] \times \mathbb{T}^3)$. Then, by employing the uniform bound (3.78) and Lemma 3.17 we get that

$$\begin{aligned} & \left| \int_0^T \int_{\mathbb{T}^3} \left(\hat{E}(v) \sqrt{\mu}, \Gamma \{I - \Pi_0\} g_\varepsilon, \{I - \Pi_0\} g_\varepsilon \right)_{L_v^2} \cdot \phi(t, x) dx dt \right| \\ & \lesssim \|\hat{E}(v) \sqrt{\mu}\|_{L_v^2} \|\phi\|_{L^\infty([0, T] \times \mathbb{T}^3)} \left(\int_0^\infty \mathcal{D}^2(t) dt \right) \\ & \leq C_{\phi, T} \varepsilon^2 \rightarrow 0. \end{aligned}$$

Thus, we know that

$$\left(\hat{E}(v) \sqrt{\mu}, \Gamma \{I - \Pi_0\} g_\varepsilon, \{I - \Pi_0\} g_\varepsilon \right)_{L_v^2} \rightarrow 0$$

in the sense of distributions as $\varepsilon \rightarrow 0$. Analogously, one easily derives that

$$\begin{aligned} & \left| \int_0^T \int_{\mathbb{T}^3} \left(\hat{E}(v) \sqrt{\mu}, \Gamma(\{I - \Pi_0\} g_\varepsilon, \Pi_0 g_\varepsilon) + \Gamma(\Pi_0 g_\varepsilon, \{I - \Pi_0\} g_\varepsilon) \right)_{L_v^2} \cdot \phi(t, x) dx dt \right| \\ & \lesssim \|\hat{E}(v) \sqrt{\mu}\|_{L_v^2} \|\phi\|_{L^\infty([0, T] \times \mathbb{T}^3)} \|g_\varepsilon\|_{L^\infty(0, T; H_x^3 L_v^2)} \left(\int_0^\infty \mathcal{D}^2(t) dt \right)^{1/2} \\ & \leq C_{\phi, T} \varepsilon \rightarrow 0, \end{aligned}$$

which immediately implies that

$$\left(\hat{E}(v) \sqrt{\mu}, \Gamma(\{I - \Pi_0\} g_\varepsilon, \Pi_0 g_\varepsilon) + \Gamma(\Pi_0 g_\varepsilon, \{I - \Pi_0\} g_\varepsilon) \right)_{L_v^2} \rightarrow 0$$

in the sense of distributions as $\varepsilon \rightarrow 0$. □

Collecting the limits (3.116), (3.117) and (3.118) gives that $u \in L^\infty(\mathbb{R}^+; H_x^3) \cap C(\mathbb{R}^+; H_x^2)$ satisfies

$$\partial_t u + \mathcal{P} \nabla_x \cdot (u \otimes u) - \nu \Delta_x u = 0 \quad (3.127)$$

with the initial data

$$u(0, x) = \mathcal{P} u_0(x). \quad (3.128)$$

Finally, we take the limit from (3.111) to the third θ -equation in (3.13) as $\varepsilon \rightarrow 0$. For any $T > 0$, let $\xi(t, x)$ be a test function satisfying $\xi(t, x) \in C^1(0, T; C_c^\infty(\mathbb{T}^3))$ with $\xi(0, x) = \xi_0(x) \in C_c^\infty(\mathbb{T}^3)$ and $\xi(t, x) = 0$ for $t \geq T'$ where $T' < T$. We have that

$$\begin{aligned}
 & \int_0^T \int_{\mathbb{T}^3} \partial_t \left(\frac{3}{5} \theta_\varepsilon - \frac{2}{5} \rho_\varepsilon \right) (t, x) \xi(t, x) dx dt \\
 &= - \underbrace{\int_0^T \int_{\mathbb{T}^3} \left(\frac{3}{5} \theta_\varepsilon - \frac{2}{5} \rho_\varepsilon \right) (t, x) \partial_t \xi(t, x) dx dt}_{\mathbf{V}_1} \\
 & \quad - \underbrace{\int_{\mathbb{T}^3} \left(g_{\varepsilon,0}, \left(\frac{3}{5} \left(\frac{|v|^2}{3} - 1 \right) - \frac{2}{5} \right) \sqrt{\mu} \right)_{L_v^2} \xi_0(x) dx}_{\mathbf{V}_2}.
 \end{aligned} \tag{3.129}$$

From the initial condition (3.14) in Theorem 3.2 and the convergence (3.95), we deduce that

$$\int_0^T \int_{\mathbb{T}^3} \left(\frac{3}{5} \theta_\varepsilon - \frac{2}{5} \rho_\varepsilon \right) (t, x) \partial_t \xi(t, x) dx dt \longrightarrow \int_0^T \int_{\mathbb{T}^3} \theta(t, x) \partial_t \xi(t, x) dx dt$$

and

$$\int_{\mathbb{T}^3} \left(g_{\varepsilon,0}, \left(\frac{3}{5} \left(\frac{|v|^2}{3} - 1 \right) - \frac{2}{5} \right) \sqrt{\mu} \right)_{L_v^2} \xi_0(x) dx \longrightarrow \int_{\mathbb{T}^3} \left(g_0, \left(\frac{3}{5} \left(\frac{|v|^2}{3} - 1 \right) - \frac{2}{5} \right) \sqrt{\mu} \right)_{L_v^2} \xi_0(x) dx.$$

Indeed, for the term \mathbf{V}_1 , we have

$$\begin{aligned}
 & \left| \int_0^T \int_{\mathbb{T}^3} \left(\left(\frac{3}{5} \theta_\varepsilon - \frac{2}{5} \rho_\varepsilon \right) - \theta \right) \cdot \partial_t \xi(t, x) dx dt \right| \\
 & \lesssim \max_{t \geq 0} \left\| \left(\left(\frac{3}{5} \theta_\varepsilon - \frac{2}{5} \rho_\varepsilon \right) - \theta \right) \right\|_{L_x^2} \|\partial_t \xi\|_{L^1(0, T; L_x^2)} \\
 & \leq C_{\xi, T} \max_{t \geq 0} \left\| \left(\left(\frac{3}{5} \theta_\varepsilon - \frac{2}{5} \rho_\varepsilon \right) - \theta \right) \right\|_{L_x^2} \longrightarrow 0.
 \end{aligned}$$

For the term \mathbf{V}_2 we have,

$$\left| \int_{\mathbb{T}^3} \left(g_{\varepsilon,0} - g_0, \left(\frac{3}{5} \left(\frac{|v|^2}{3} - 1 \right) - \frac{2}{5} \right) \sqrt{\mu} \right)_{L_v^2} \xi_0(x) dx \right| \lesssim \|g_{\varepsilon,0} - g_0\|_{L_{x,v}^2} \|\xi_0(x)\|_{L_x^2} \longrightarrow 0.$$

As a consequence, we have

$$\begin{aligned}
 & \int_0^T \int_{\mathbb{T}^3} \partial_t \left(\frac{3}{5} \theta_\varepsilon - \frac{2}{5} \rho_\varepsilon \right) (t, x) \xi(t, x) dx dt \\
 & \longrightarrow - \int_0^T \int_{\mathbb{T}^3} \theta(t, x) \partial_t \xi(t, x) dx dt - \int_{\mathbb{T}^3} \left(\frac{3}{5} \theta_0 - \frac{2}{5} \rho_0 \right) \xi_0(x) dx
 \end{aligned} \tag{3.130}$$

as $\varepsilon \rightarrow 0$. Now, we will study the convergence of terms $\nabla_x \cdot [\mathcal{P}u_\varepsilon \left(\frac{3}{5} \theta_\varepsilon - \frac{2}{5} \rho_\varepsilon \right)]$ and $\kappa \Delta_x \left(\frac{3}{5} \theta_\varepsilon - \frac{2}{5} \rho_\varepsilon \right)$ in the following lemma.

Lemma 3.21. *It holds that*

$$\begin{aligned} \nabla_x \cdot \left[\mathcal{P}u_\varepsilon \left(\frac{3}{5}\theta_\varepsilon - \frac{2}{5}\rho_\varepsilon \right) \right] &\rightarrow \nabla_x \cdot (u\theta) \quad \text{strongly in } C(\mathbb{R}^+; H_x^1), \\ \kappa \Delta_x \left(\frac{3}{5}\theta_\varepsilon - \frac{2}{5}\rho_\varepsilon \right) &\rightarrow \kappa \Delta_x \theta \quad \text{strongly in } C(\mathbb{R}^+; L_x^2), \end{aligned} \quad (3.131)$$

as $\varepsilon \rightarrow 0$, where κ is defined in (3.107).

Proof. Indeed, we have

$$\left\| \kappa \Delta_x \left(\frac{3}{5}\theta_\varepsilon - \frac{2}{5}\rho_\varepsilon - \theta \right) \right\|_{C(\mathbb{R}^+; L_x^2)} \lesssim \left\| \left(\frac{3}{5}\theta_\varepsilon - \frac{2}{5}\rho_\varepsilon - \theta \right) \right\|_{C(\mathbb{R}^+; H_x^2)} \rightarrow 0.$$

In addition, we have

$$\begin{aligned} &\left\| \nabla_x \cdot \left((\mathcal{P}u_\varepsilon - u) \left(\frac{3}{5}\theta_\varepsilon - \frac{2}{5}\rho_\varepsilon \right) \right) + \nabla_x \cdot \left(u \left(\frac{3}{5}\theta_\varepsilon - \frac{2}{5}\rho_\varepsilon - \theta \right) \right) \right\|_{H_x^1} \\ &\lesssim \left\| (\mathcal{P}u_\varepsilon - u) \cdot \nabla_x \left(\frac{3}{5}\theta_\varepsilon - \frac{2}{5}\rho_\varepsilon \right) \right\|_{L_x^2} + \left\| u \cdot \nabla_x \left(\frac{3}{5}\theta_\varepsilon - \frac{2}{5}\rho_\varepsilon - \theta \right) \right\|_{L_x^2} \\ &+ \left\| \nabla_x (\mathcal{P}u_\varepsilon - u)^\top \cdot \nabla_x \left(\frac{3}{5}\theta_\varepsilon - \frac{2}{5}\rho_\varepsilon \right) \right\|_{L_x^2} + \left\| \nabla_x \left(\nabla_x \left(\frac{3}{5}\theta_\varepsilon - \frac{2}{5}\rho_\varepsilon \right) \right)^\top (\mathcal{P}u_\varepsilon - u) \right\|_{L_x^2} \\ &+ \left\| \nabla_x u^\top \cdot \left(\frac{3}{5}\theta_\varepsilon - \frac{2}{5}\rho_\varepsilon - \theta \right) \right\|_{L_x^2} + \left\| \nabla_x \left(\nabla_x \left(\frac{3}{5}\theta_\varepsilon - \frac{2}{5}\rho_\varepsilon \right) \right)^\top \cdot u \right\|_{L_x^2} \\ &\lesssim \|u\|_{L_t^\infty H_x^3} \left\| \left(\frac{3}{5}\theta_\varepsilon - \frac{2}{5}\rho_\varepsilon - \theta \right) \right\|_{C(\mathbb{R}^+; H_x^2)} + \left\| \left(\frac{3}{5}\theta_\varepsilon - \frac{2}{5}\rho_\varepsilon \right) \right\|_{L_t^\infty H_x^3} \|\mathcal{P}u_\varepsilon - u\|_{C(\mathbb{R}^+; H_x^2)} \\ &\lesssim \|\mathcal{P}u_\varepsilon - u\|_{C(\mathbb{R}^+; H_x^2)} + \left\| \left(\frac{3}{5}\theta_\varepsilon - \frac{2}{5}\rho_\varepsilon - \theta \right) \right\|_{C(\mathbb{R}^+; H_x^2)} \rightarrow 0. \end{aligned}$$

□

The limit of the last term $R_{\varepsilon, \theta}$ is handled in the following lemma.

Lemma 3.22. *In the distributional sense,*

$$R_{\varepsilon, \theta} \rightarrow 0 \quad (3.132)$$

as $\varepsilon \rightarrow 0$, where $R_{\varepsilon, \theta}$ is defined in (3.109).

Proof. First, we have

$$\frac{2}{5} \nabla_x \cdot [\mathcal{P}u_\varepsilon (\rho_\varepsilon + \theta_\varepsilon)] + \frac{2}{5} \nabla_x \cdot [\mathcal{P}^\perp u_\varepsilon (\rho_\varepsilon + \theta_\varepsilon)] + \nabla_x \cdot \left[\mathcal{P}^\perp u_\varepsilon \left(\frac{3}{5}\theta_\varepsilon - \frac{2}{5}\rho_\varepsilon \right) \right] \rightarrow 0 \quad (3.133)$$

weakly- \star in $L^\infty([0, \infty); H^2(\mathbb{T}_x^3))$ as $\varepsilon \rightarrow 0$. Indeed, we have

$$\left\| \frac{2}{5} \nabla_x \cdot [\mathcal{P}u_\varepsilon(\rho_\varepsilon + \theta_\varepsilon)] + \frac{2}{5} \nabla_x \cdot [\mathcal{P}^\perp u_\varepsilon(\rho_\varepsilon + \theta_\varepsilon)] + \nabla_x \cdot \left[\mathcal{P}^\perp u_\varepsilon \left(\frac{3}{5} \theta_\varepsilon - \frac{2}{5} \rho_\varepsilon \right) \right] \right\|_{L_t^\infty H_x^2} \leq C$$

and by employing the convergence (3.100) and (3.101) we derive that

$$\frac{2}{5} \nabla_x \cdot [\mathcal{P}u_\varepsilon(\rho_\varepsilon + \theta_\varepsilon)] + \frac{2}{5} \nabla_x \cdot [\mathcal{P}^\perp u_\varepsilon(\rho_\varepsilon + \theta_\varepsilon)] + \nabla_x \cdot \left[\mathcal{P}^\perp u_\varepsilon \left(\frac{3}{5} \theta_\varepsilon - \frac{2}{5} \rho_\varepsilon \right) \right] \rightarrow 0 \quad (3.134)$$

in the sense of distributions as $\varepsilon \rightarrow 0$. Let us now show that

$$\kappa \Delta_x(\rho_\varepsilon + \theta_\varepsilon) \rightarrow 0 \quad (3.135)$$

weakly- \star in $L^\infty([0, \infty); H^1(\mathbb{T}_x^3))$ as $\varepsilon \rightarrow 0$. We have

$$\|\Delta_x(\rho_\varepsilon + \theta_\varepsilon)\|_{L^\infty(\mathbb{R}^+; H_x^1)} \lesssim \|\rho_\varepsilon + \theta_\varepsilon\|_{L^\infty(\mathbb{R}^+; H_x^3)} \leq C, \quad (3.136)$$

and by employing the convergence (3.88) we derive that

$$\nabla_x \cdot \nabla_x(\rho_\varepsilon + \theta_\varepsilon) \rightarrow 0$$

in the sense of distributions as $\varepsilon \rightarrow 0$. Consequently, the convergences (3.125), (3.133) and (3.135) imply the convergence (3.132). \square

Collecting the limits (3.130), (3.131) and (3.132) gives that $\theta \in L^\infty(\mathbb{R}^+; H_x^3) \cap C(\mathbb{R}^+; H_x^2)$ satisfies

$$\partial_t \theta + \nabla_x \cdot (u\theta) = \kappa \Delta_x \theta \quad (3.137)$$

with the initial data

$$\theta(0, x) = \frac{3}{5} \theta_0(x) - \frac{2}{5} \rho_0(x). \quad (3.138)$$

5 Appendix A

5.1 Study of the linear Landau equation

In this section, we show a local existence of the solution of the linear Landau equation. We use the technique introduced by Degond in [32] for the linear Fokker-Planck equation. We consider now the linear Cauchy problem

$$\begin{cases} \partial_t g + \frac{1}{\varepsilon} v \cdot \nabla_x g + \frac{1}{\varepsilon^2} \mathcal{L}g = U_\varepsilon \\ g(0, x, v) = g_0(x, v) \end{cases} \quad (3.139)$$

where $g_0(x, v)$ and $U_\varepsilon(t, x, v)$ are given functions.

For simplicity denote $X := L^2([0, T]; H_x^3 H_{v,*}^1)$ and $X' := L^2([0, T]; H_x^3 (H_{v,*}^1)')$ (see (3.8)), where X' is the dual of X w.r.t. $H_x^3 L_v^2$. We denote by Y the set of functions defined by

$$Y = \left\{ g \in X, \partial_t g + \frac{1}{\varepsilon} v \cdot \nabla_x g \in X' \right\}.$$

We will construct a local solutions of the problem (3.139) on Y . In what follows, we consider ε as a fixed parameter.

Proposition 3.23. *Let $\varepsilon > 0$, we assume that $g_0 \in H_x^3 L_v^2$ and $U_\varepsilon \in X'$. Then the Cauchy problem (3.139) admits a weak solution $g \in Y$.*

Proof. We take the change of unknown as follows

$$\tilde{g}(t, x, v) = e^{-\lambda t} g(t, x, v).$$

The linear Landau equation (3.139) for $\tilde{g} = \tilde{g}(t, x, v)$ takes the following form

$$\begin{cases} \partial_t \tilde{g} + \frac{1}{\varepsilon} v \cdot \nabla_x \tilde{g} + \frac{1}{\varepsilon^2} \mathcal{L} \tilde{g} + \lambda \tilde{g} = e^{-\lambda t} U_\varepsilon := \tilde{U}_\varepsilon \\ \tilde{g}(0, x, v) = g_0(x, v). \end{cases} \quad (3.140)$$

We introduce the following Lions Theorem, that we will use to prove the existence of solution, which can be found in [95].

Theorem 5.1 (Lions Theorem). Let \mathbf{F} be a Hilbert space, provided with a norm $\|\cdot\|_{\mathbf{F}}$, and an inner product $(\cdot, \cdot)_{\mathbf{F}}$. Let \mathbf{V} be a subspace of \mathbf{F} , provided with a prehilbertian norm $\|\cdot\|_{\mathbf{V}}$, such that the injection $\mathbf{V} \hookrightarrow \mathbf{F}$, is continuous. We consider a bilinear form \mathbf{E}

$$\begin{aligned} \mathbf{E} &: \mathbf{F} \times \mathbf{V} \rightarrow \mathbb{R} \\ (g, \phi) &\mapsto \mathbf{E}(g, \phi) \end{aligned}$$

such that $\mathbf{E}(\cdot, \phi)$ is continuous on \mathbf{F} for any fixed $\phi \in \mathbf{V}$, and such that

$$|\mathbf{E}(\phi, \phi)| \geq \alpha \|\phi\|_{\mathbf{V}}^2, \quad \forall \phi \in \mathbf{V}, \quad \text{with } \alpha > 0.$$

Then, given a linear form \mathbf{L} in \mathbf{V}' , there exists a solution g in \mathbf{F} of the problem

$$\mathbf{E}(g, \phi) = \mathbf{L}(\phi), \quad \forall \phi \in \mathbf{V}. \quad (3.141)$$

In the remaining part of the proof, to lighten the notations, we will drop the tildes. Let $\mathbf{F} = X$ be a Hilbert space with norm

$$\|f\|_{\mathbf{X}}^2 = \int_0^T \|f\|_{\chi^3(\mathbb{T}_x^3 \times \mathbb{R}_v^3)}^2 dt,$$

where $\|f\|_{\chi^3(\mathbb{T}_x^3 \times \mathbb{R}_v^3)}$ is defined in (3.7). Let \mathbf{V} be the space $C_c^\infty([0, T] \times \mathbb{T}_x^3 \times \mathbb{R}_v^3)$ of infinitely differentiable functions, with compact support in $[0, T] \times \mathbb{T}_x^3 \times \mathbb{R}_v^3$. \mathbf{V} is provided with a norm defined by

$$\|\phi\|_{\mathbf{V}}^2 = \|\phi\|_X^2 + \frac{1}{2}\|\phi(0)\|_{H_x^3 L_v^2}^2, \quad \forall \phi \in \mathbf{V}.$$

We define the operator \mathcal{Q} as follows:

$$\mathcal{Q} := -\partial_t - \frac{1}{\varepsilon}v \cdot \nabla_x + \frac{1}{\varepsilon^2}\mathcal{L}, \quad (3.142)$$

with domain $C_c^\infty([0, T] \times \mathbb{T}_x^3 \times \mathbb{R}_v^3)$. We also define the bilinear form \mathbf{E} , and the linear form \mathbf{L} as following

$$\begin{aligned} \mathbf{E}(g, \phi) &:= \int_0^T \left((g, \mathcal{Q}\phi)_{H_x^3 L_v^2} + \lambda (g, \phi)_{H_x^3 L_v^2} \right) dt, \\ \mathbf{L}(\phi) &:= \langle U_\varepsilon, \phi \rangle_{X', X} + (g_0, \phi(0))_{H_x^3 L_v^2}. \end{aligned}$$

The mapping $\mathbf{E}(\cdot, \phi)$ is continuous on X : let $g \in X$, we have

$$|\mathbf{E}(g, \phi)| \leq (\|\mathcal{Q}\phi\|_X + \lambda\|\phi\|_X)\|g\|_X \leq C_\phi\|g\|_X.$$

The mapping \mathbf{E} is a bilinear and coercive form on \mathbf{V} : let $\phi \in \mathbf{V}$, using (3.33), we have

$$|\mathbf{E}(\phi, \phi)| \geq \frac{1}{2}\|\phi(0)\|_{H_x^3 L_v^2}^2 + \frac{C_\gamma}{\varepsilon^2} \int_0^T \mathcal{D}^2(\phi) dt + \lambda \int_0^T \|\phi\|_{H_x^3 L_v^2}^2 dt,$$

by using that

$$\|\phi_1\|_{\chi^3(\mathbb{T}_x^3 \times \mathbb{R}_v^3)}^2 \leq C\|\phi\|_{H_x^3 L_v^2}^2,$$

where $C > 0$ and $\phi_1 = \Pi_0\phi$ (see (3.9)), we obtain

$$\begin{aligned} |\mathbf{E}(\phi, \phi)| &\geq \frac{1}{2}\|\phi(0)\|_{H_x^3 L_v^2}^2 + C_\gamma \int_0^T \mathcal{D}^2(\phi) dt + \frac{\lambda}{C} \int_0^T \|\phi_1\|_{\chi^3(\mathbb{T}_x^3 \times \mathbb{R}_v^3)}^2 dt, \\ &\geq \min\left(1, C_\gamma, \frac{\lambda}{C}\right) \left(\int_0^T \|\phi\|_{\chi^3(\mathbb{T}_x^3 \times \mathbb{R}_v^3)}^2 dt + \frac{1}{2}\|\phi(0)\|_{H_x^3 L_v^2}^2 \right), \end{aligned}$$

then

$$|\mathbf{E}(\phi, \phi)| \geq \alpha\|\phi\|_{\mathbf{V}}^2.$$

The mapping \mathbf{L} is a linear continuous form on \mathbf{V} : let $\phi \in \mathbf{V}$, we have

$$\begin{aligned} |\mathbf{L}(\phi)| &\leq \|U_\varepsilon\|_{X'}\|\phi\|_X + \|g_0\|_{H_x^3 L_v^2}\|\phi(0)\|_{H_x^3 L_v^2} \\ &\leq \|U_\varepsilon\|_{X'}\|\phi\|_{\mathbf{V}} + \sqrt{2}\|g_0\|_{H_x^3 L_v^2}\|\phi\|_{\mathbf{V}} \\ &\leq (\|U_\varepsilon\|_{X'} + \sqrt{2}\|g_0\|_{H_x^3 L_v^2})\|\phi\|_{\mathbf{V}}, \end{aligned}$$

then

$$|\mathbf{L}(\phi)| \leq C' \|\phi\|_{\mathbf{V}}.$$

Finally thanks to Lions Theorem, there exists a solution $g \in X$ for the variational problem (3.141) and in particular, we deduce that

$$\partial_t g + \frac{1}{\varepsilon} v \cdot \nabla_x g = U_\varepsilon - \frac{1}{\varepsilon^2} \mathcal{L}g - \lambda g \in X',$$

then g belongs to Y . Now, to prove that g satisfies the initial condition, we use the following Lemma:

Lemma 3.24. 1) If g belongs to Y , g admits (continuous) trace values $g(0, x, v)$, $g(T, x, v)$ in $H_x^3 L_v^2$.

2) For g and \tilde{g} in Y , we have

$$\begin{aligned} \left\langle \partial_t g + \frac{1}{\varepsilon} v \cdot \nabla_x g, \tilde{g} \right\rangle_{X', X} + \left\langle \partial_t \tilde{g} + \frac{1}{\varepsilon} v \cdot \nabla_x \tilde{g}, g \right\rangle_{X', X} \\ = (g(T), \tilde{g}(T))_{H_x^3 L_v^2} - (g(0), \tilde{g}(0))_{H_x^3 L_v^2}. \end{aligned} \quad (3.143)$$

We note that the proof of Lemma 3.24 is similar to the proof of Lemma A.1 in [32] by considering $H_x^3 L_v^2$ as a pivot space. Now, using (3.140) and (3.143), we obtain that the solution g of the variational problem (3.141) satisfies

$$(g(0) - g_0, \phi(0))_{H_x^3 L_v^2} = 0, \quad \forall \phi \in \mathbf{V}.$$

Then, the initial condition is satisfied in $H_x^3 L_v^2$. □

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Regularization and hydrodynamical limit for the Landau equation

Résumé

Dans cette partie, nous étudions l'équation de Landau sous l'échelle de Navier-Stokes dans le tore dans les cas de potentiels durs et modérément mous. Plus précisément, nous étudions la théorie de Cauchy dans un cadre perturbatif et établissons de nouvelles estimations de régularisation en temps court pour notre équation de Landau non linéaire adimensionnée. Ces estimations sont quantifiées en temps et optimales, en effet, on obtient le gain de régularité anisotrope instantané attendu (voir [111] pour les estimations hypoelliptiques correspondantes à l'opérateur de collision de Landau linéarisé). De plus, les estimations donnant le gain de régularité en variable vitesse sont uniformes en nombre de Knudsen. En combinant ces nouvelles estimations sur l'équation de Landau avec des estimations sur le système Navier-Stokes-Fourier, nous sommes alors en mesure d'obtenir un résultat de convergence forte vers ce système fluide.

Abstract

In this part, we study the Landau equation under the Navier-Stokes scaling in the torus for hard and moderately soft potentials. More precisely, we investigate the Cauchy theory in a perturbative framework and establish some new short time regularization estimates for our rescaled nonlinear Landau equation. These estimates are quantified in time and optimal, indeed, we obtain the instantaneous expected anisotropic gain of regularity (see [111] for the corresponding hypoelliptic estimates on the linearized Landau collision operator). Moreover, the estimates giving the gain of regularity in the velocity variable are uniform in the Knudsen number. Intertwining these new estimates on the Landau equation with estimates on the Navier-Stokes-Fourier system, we are then able to obtain a result of strong convergence towards this fluid system.

1 Introduction

In this paper, we are interested in the link between the Landau and Navier-Stokes equations. It has been a major challenge to establish rigorous links between microscopic and macroscopic equations for many years, this problem goes back to Hilbert [73] and the main goal is to obtain a unified description of gas dynamics. The equations of kinetic theory (including the Boltzmann and the Landau equations) can be seen as an intermediate step between the microscopic and macroscopic scales of description. In order to link the Landau and Navier-Stokes equations, we study a suitable rescaling of the Landau equation, as described in Subsection 1.1.

The first part of our paper is dedicated to the study of this rescaled Landau equation thanks to hypocoercivity methods, (linear and nonlinear) regularization estimates and sharp nonlinear estimates on the Landau collision operator. We study the Cauchy theory in a close-to-equilibrium framework for this equation and establish new and sharp regularization estimates in short time. The second part of our paper focuses on the aforementioned hydrodynamical limit problem. More precisely, we give a result of strong convergence of the solutions to the Landau equation constructed in the first part of the paper towards strong and global solutions to the incompressible Navier-Stokes-Fourier system. Our approach is reminiscent of the one used in [11, 44] for the hard spheres Boltzmann equation and improves the result obtained in [58, 112] in terms of type of convergence or functional framework in the case of not too soft potentials. Our analysis heavily relies on the estimates on the Landau equation established in the first part of the paper as well as on results of spectral analysis for the linearized Landau equation performed in [130] and some refined estimates on the fluid problem (as in [44]).

1.1 The kinetic model

We start by introducing the Landau equation which models the evolution of charged particles in a plasma through the evolution of the density of particles $f = f(t, x, v)$ which depends on time $t \in \mathbb{R}^+$, position $x \in \mathbf{T}^3$ the 3-dimensional unit periodic box and velocity $v \in \mathbb{R}^3$, when only binary collisions are taken into account. The Landau equation reads:

$$\partial_t f + v \cdot \nabla_x f = \frac{1}{\varepsilon} Q(f, f),$$

where $\varepsilon > 0$ is the Knudsen number which is the inverse of the average number of collisions for each particle per unit time and Q is the Landau collision operator. It is defined as

$$Q(g, f)(v) = \partial_{v_i} \int_{\mathbb{R}^3} a_{ij}(v - v_*) \left[g(v_*) \partial_{v_j} f(v) - f(v) \partial_{v_j} g(v_*) \right] dv_*, \quad (4.1)$$

where we use the convention of summation of repeated indices. The matrix a_{ij} is symmetric, semi-positive and is given by

$$a_{ij}(v) = |v|^{\gamma+2} \left(\delta_{ij} - \frac{v_i v_j}{|v|^2} \right), \quad -3 \leq \gamma \leq 1. \quad (4.2)$$

We have the following classification: We call hard potentials if $\gamma \in (0, 1]$, Maxwellian molecules if $\gamma = 0$, moderately soft potentials if $\gamma \in [-2, 0)$, very soft potentials if $\gamma \in (-3, -2)$ and Coulomb potential if $\gamma = -3$. Hereafter, we shall consider the cases of hard potentials, Maxwellian molecules and moderately soft potentials, i.e.

$$-2 \leq \gamma \leq 1.$$

The Landau equation preserves mass, momentum and energy. Indeed, at least formally, for any test function φ , we have

$$\begin{aligned} & \int_{\mathbb{R}^3} Q(f, f)(v) \varphi(v) \, dv \\ &= -\frac{1}{2} \int_{\mathbb{R}^3 \times \mathbb{R}^3} a_{ij}(v-v_*) f(v) f(v_*) \left(\frac{\partial_{v_i} f(v)}{f(v)} - \frac{\partial_{v_i} f(v_*)}{f(v_*)} \right) (\partial_{v_j} \varphi(v) - \partial_{v_j} \varphi(v_*)) \, dv_* \, dv, \end{aligned} \quad (4.3)$$

from which we deduce that

$$\begin{aligned} \frac{d}{dt} \int_{\mathbf{T}^3 \times \mathbb{R}^3} f(x, v) \varphi(v) \, dv \, dx &= \int_{\mathbf{T}^3 \times \mathbb{R}^3} \left[\frac{1}{\varepsilon} Q(f, f)(x, v) - v \cdot \nabla_x f(x, v) \right] \varphi(v) \, dv \, dx \\ &= 0 \quad \text{for } \varphi(v) = 1, v, |v|^2. \end{aligned} \quad (4.4)$$

Moreover, the Landau version of the Boltzmann H -theorem asserts that the entropy

$$H(f) := \int_{\mathbf{T}^3 \times \mathbb{R}^3} f \log f \, dv \, dx$$

is non increasing. Indeed, at least formally, since a_{ij} is nonnegative, we have the following inequality for the entropy dissipation $D(f)$:

$$\begin{aligned} D(f) := -\frac{d}{dt} H(f) &= \frac{1}{2} \int_{\mathbf{T}^3 \times \mathbb{R}^3 \times \mathbb{R}^3} a_{ij}(v-v_*) f(v) f(v_*) \\ & \quad \left(\frac{\partial_{v_i} f(v)}{f(v)} - \frac{\partial_{v_i} f(v_*)}{f(v_*)} \right) \left(\frac{\partial_{v_j} f(v)}{f(v)} - \frac{\partial_{v_j} f(v_*)}{f(v_*)} \right) \, dv_* \, dv \, dx \geq 0. \end{aligned}$$

The second part of the H -theorem asserts that local equilibria of the Landau equation are local Maxwellian distributions in velocity. In what follows, we shall consider the following centered normalized Maxwellian independent of time t and space x which is a global equilibrium of our equation defined by

$$M(v) := \frac{1}{(2\pi)^{\frac{3}{2}}} e^{-\frac{|v|^2}{2}}.$$

Taking ε small has the effect of enhancing the role of collisions and thus when ε goes to 0, in view of the above mentioned Landau version of the Boltzmann H -theorem, the solution looks more and more like a local thermodynamical equilibrium. As suggested in previous works (see for example [9]), we consider the following rescaled Landau equation in which an additional dilatation of the macroscopic time scale has been done in order to be able to reach the Navier-Stokes equation in the limit:

$$\partial_t f^\varepsilon + \frac{1}{\varepsilon} v \cdot \nabla_x f^\varepsilon = \frac{1}{\varepsilon^2} Q(f^\varepsilon, f^\varepsilon) \quad \text{in } \mathbb{R}^+ \times \mathbf{T}^3 \times \mathbb{R}^3. \quad (4.5)$$

To relate the Landau equation to the incompressible Navier-Stokes equation, we look at equation (4.5) under the following linearization of order ε :

$$f^\varepsilon(t, x, v) = M(v) + \varepsilon \sqrt{M}(v) g^\varepsilon(t, x, v). \quad (4.6)$$

Let us recall that taking ε small in this linearization corresponds to taking a small Mach number, which enables one to get in the limit the incompressible Navier-Stokes equation. If f^ε solves (4.5), then equivalently g^ε solves

$$\partial_t g^\varepsilon + \frac{1}{\varepsilon} v \cdot \nabla_x g^\varepsilon = \frac{1}{\varepsilon^2} L g^\varepsilon + \frac{1}{\varepsilon} \Gamma(g^\varepsilon, g^\varepsilon) \quad \text{in } \mathbb{R}^+ \times \mathbf{T}^3 \times \mathbb{R}^3 \quad (4.7)$$

where the nonlinear collision operator Γ is defined by

$$\Gamma(f_1, f_2) := \frac{1}{\sqrt{M}} Q(\sqrt{M} f_1, \sqrt{M} f_2) \quad (4.8)$$

and the linearized collision operator L by

$$L f := \Gamma(\sqrt{M}, f) + \Gamma(f, \sqrt{M}). \quad (4.9)$$

Notice that the property (4.3) implies that for any suitable functions f_1 and f_2 ,

$$\int_{\mathbb{R}^3} \Gamma(f_1, f_2)(v) \varphi(v) dv = 0 \quad \text{for } \varphi(v) = \sqrt{M}, v\sqrt{M}, |v|^2\sqrt{M}. \quad (4.10)$$

We also define the full linearized operator Λ_ε as

$$\Lambda_\varepsilon := \frac{1}{\varepsilon^2} L - \frac{1}{\varepsilon} v \cdot \nabla_x. \quad (4.11)$$

It is well known that the kernel of L is given by

$$\text{Ker } L = \text{Span} \left\{ \sqrt{M}, v_1 \sqrt{M}, v_2 \sqrt{M}, v_3 \sqrt{M}, |v|^2 \sqrt{M} \right\},$$

and we shall denote by π the orthogonal projector onto $\text{Ker } L$ which is defined by:

$$\begin{aligned} \pi f(v) = & \left(\int_{\mathbb{R}^3} f(w) \sqrt{M}(w) dw + \int_{\mathbb{R}^3} w f(w) \sqrt{M}(w) dw \cdot v \right. \\ & \left. + \int_{\mathbb{R}^3} \frac{|w|^2 - 3}{3} f(w) \sqrt{M}(w) dw \frac{|v|^2 - 3}{2} \right) \sqrt{M}(v). \end{aligned} \quad (4.12)$$

Throughout the paper, we shall also use the following notation: For a given kinetic distribution $f = f(x, v)$, we denote by f^\perp its microscopic part, namely

$$f^\perp := (\text{Id} - \pi)f \quad (4.13)$$

and by (ρ_f, u_f, θ_f) its first macroscopic quantities defined through

$$\rho_f(x) := \int_{\mathbb{R}^3} f(x, v) \sqrt{M}(v) dv, \quad (4.14)$$

$$u_f(x) := \int_{\mathbb{R}^3} v f(x, v) \sqrt{M}(v) dv, \quad (4.15)$$

and

$$\theta_f(x) := \frac{1}{3} \int_{\mathbb{R}^3} (|v|^2 - 3) f(x, v) \sqrt{M}(v) dv, \quad (4.16)$$

so that $\pi f = \left(\rho_f + u_f \cdot v + \theta_f \frac{|v|^2 - 3}{2} \right) \sqrt{M}$. Using for example Proposition 3.1 from [17], we also have:

$$\text{Ker } \Lambda_\varepsilon = \text{Span} \left\{ \sqrt{M}, v_1 \sqrt{M}, v_2 \sqrt{M}, v_3 \sqrt{M}, |v|^2 \sqrt{M} \right\}$$

and the projector Π onto $\text{Ker } \Lambda_\varepsilon$ is given by

$$\begin{aligned} \Pi f(v) = & \left(\int_{\mathbf{T}^3 \times \mathbb{R}^3} f(x, w) \sqrt{M}(w) dw dx + \int_{\mathbf{T}^3 \times \mathbb{R}^3} f(x, w) w \sqrt{M}(w) dw dx \cdot v \right. \\ & \left. + \int_{\mathbf{T}^3 \times \mathbb{R}^3} \frac{|w|^2 - 3}{3} f(x, w) \sqrt{M}(w) dw dx \frac{|v|^2 - 3}{2} \right) \sqrt{M}(v). \end{aligned} \quad (4.17)$$

Notice that $\Pi f(v) = \int_{\mathbf{T}^3} \pi f(x, v) dx$.

1.2 Cauchy theory, decay and regularization for the Landau equation

We introduce the following H^1 -norm in velocity which naturally arises in the study of the Landau equation:

$$\|f\|_{H_{v,*}^1}^2 := \|\langle v \rangle^{\frac{\gamma}{2}+1} f\|_{L_v^2}^2 + \|\langle v \rangle^{\frac{\gamma}{2}} P_v \nabla_v f\|_{L_v^2}^2 + \|\langle v \rangle^{\frac{\gamma}{2}+1} (\text{Id} - P_v) \nabla_v f\|_{L_v^2}^2, \quad (4.18)$$

where P_v stands for the projection on v , namely, $P_v w = \left(w \cdot \frac{v}{|v|} \right) \frac{v}{|v|}$. We define the weighted Sobolev-type spaces \mathcal{X} and \mathcal{Y}_1 as the spaces associated to the following norms:

$$\|f\|_{\mathcal{X}}^2 := \|\langle v \rangle^{3(\frac{\gamma}{2}+1)} f\|_{L_{x,v}^2}^2 + \|\langle v \rangle^{2(\frac{\gamma}{2}+1)} \nabla_x f\|_{L_{x,v}^2}^2 + \|\langle v \rangle^{\frac{\gamma}{2}+1} \nabla_x^2 f\|_{L_{x,v}^2}^2 + \|\nabla_x^3 f\|_{L_{x,v}^2}^2, \quad (4.19)$$

and

$$\begin{aligned} \|f\|_{\mathcal{Y}_1}^2 &:= \|\langle v \rangle^{3(\frac{\gamma}{2}+1)} f\|_{L_x^2(H_{v,*}^1)}^2 + \|\langle v \rangle^{2(\frac{\gamma}{2}+1)} \nabla_x f\|_{L_x^2(H_{v,*}^1)}^2 \\ &\quad + \|\langle v \rangle^{\frac{\gamma}{2}+1} \nabla_x^2 f\|_{L_x^2(H_{v,*}^1)}^2 + \|\nabla_x^3 f\|_{L_x^2(H_{v,*}^1)}^2. \end{aligned} \quad (4.20)$$

Remark that as in [60, 26], we work with “twisted” Sobolev spaces in which the weights depend on the order of the derivative in x , it allows us to close our nonlinear estimates.

Let us now state our main result on the well-posedness, decay and regularization of the rescaled Landau equation (4.7).

Theorem 4.1. *There is $\eta_0 > 0$ small enough such that for any $\varepsilon \in (0, 1)$, if $g_{\text{in}}^\varepsilon \in \mathcal{X}$ satisfies*

$$\int_{\mathbf{T}^3 \times \mathbb{R}^3} g_{\text{in}}^\varepsilon(x, v) \varphi(v) \, dv \, dx = 0 \quad \text{for } \varphi(v) = \sqrt{M}, v\sqrt{M}, |v|^2\sqrt{M}, \quad (4.21)$$

and $\|g_{\text{in}}^\varepsilon\|_{\mathcal{X}} \leq \eta_0$, then the following holds:

(i) *There is a unique global solution $g^\varepsilon \in L^\infty(\mathbb{R}_+; \mathcal{X}) \cap L^2(\mathbb{R}_+; \mathcal{Y}_1)$ to (4.7) associated to the initial data $g_{\text{in}}^\varepsilon$, which verifies moreover*

$$\sup_{t \geq 0} e^{2\sigma t} \|g^\varepsilon(t)\|_{\mathcal{X}}^2 + \frac{1}{\varepsilon^2} \int_0^\infty e^{2\sigma t} \|(g^\varepsilon(t))^\perp\|_{\mathcal{Y}_1}^2 \, dt + \int_0^\infty e^{2\sigma t} \|g^\varepsilon(t)\|_{\mathcal{Y}_1}^2 \, dt \lesssim \|g_{\text{in}}^\varepsilon\|_{\mathcal{X}}^2, \quad (4.22)$$

for any $0 < \sigma < \sigma_0$, where σ_0 is the decay rate of linearized operator Λ_ε given in Proposition 4.16, and where we recall that $(g^\varepsilon)^\perp$ is defined in (4.13).

(ii) *In addition, the solution satisfies the following regularization estimates, for all $t > 0$,*

$$\|g^\varepsilon(t)\|_{\mathcal{Y}_1} \lesssim \frac{e^{-\sigma t}}{\min(1, \sqrt{t})} \|g_{\text{in}}^\varepsilon\|_{\mathcal{X}}, \quad \text{and} \quad \varepsilon \|\widetilde{\nabla}_x g^\varepsilon(t)\|_{\mathcal{X}} \lesssim \frac{e^{-\sigma t}}{\min(1, t^{3/2})} \|g_{\text{in}}^\varepsilon\|_{\mathcal{X}}, \quad (4.23)$$

where $\widetilde{\nabla}_x$ is a weighted anisotropic gradient defined in (4.39).

Remark 4.2. It is worth noticing that the condition (4.21) is equivalent to $g_{\text{in}}^\varepsilon \in (\text{Ker } \Lambda_\varepsilon)^\perp$.

Remark 4.3. Let us point out that the results obtained in Theorems 4.1 could be obtained in larger spaces of the type $\mathcal{E} := H_x^3 L_v^2(\langle v \rangle^k \sqrt{M})$ for k large enough. More precisely, due to the linearization (4.6), working in spaces like \mathcal{E} means that the original data f^ε lie in polynomially weighted Sobolev spaces, which is more relevant from a physical point of view. We chose to only present the proof in the functional space \mathcal{X} because this functional framework is compatible with the second part of the paper which is about hydrodynamical limit of the Landau equation (4.7). Let us though explain the strategy to perform such an extension of our results from the functional space \mathcal{X} to \mathcal{E} . The strategy is the same as the one used in [18] by Briant, Merino-Aceituno and Mouhot where uniform in ε estimates on solutions to the hard-spheres Boltzmann equation have been obtained.

The trick is to rewrite the equation (4.7) as an equivalent system of two equations thanks to the splitting of the linearized operator $\Lambda_\varepsilon = \mathcal{A}_\varepsilon + \mathcal{B}_\varepsilon$ introduced in Section 5: We write $g^\varepsilon = g_1^\varepsilon + g_2^\varepsilon$ with

$$\partial_t g_1^\varepsilon = \mathcal{B}_\varepsilon g_1^\varepsilon + \frac{1}{\varepsilon} \Gamma(g_1^\varepsilon, g_1^\varepsilon) + \frac{1}{\varepsilon} \Gamma(g_1^\varepsilon, g_2^\varepsilon) + \frac{1}{\varepsilon} \Gamma(g_2^\varepsilon, g_1^\varepsilon) \quad \text{and} \quad g_1^\varepsilon(t=0) = g_{\text{in}}^\varepsilon \in \mathcal{E}$$

and

$$\partial_t g_2^\varepsilon = \Lambda_\varepsilon g_2^\varepsilon + \frac{1}{\varepsilon} \Gamma(g_2^\varepsilon, g_2^\varepsilon) + \mathcal{A}_\varepsilon g_1^\varepsilon \quad \text{and} \quad g_2^\varepsilon(t=0) = 0 \in \mathcal{X}.$$

The first equation can be studied in the large space \mathcal{E} thanks to the nice properties of \mathcal{B}_ε in all type of spaces and the second one can be studied in the smaller space \mathcal{X} since it starts from 0. Moreover, we have some nice estimates on this equation because the operator \mathcal{A}_ε enjoys some regularizing properties, it is bounded from \mathcal{E} into \mathcal{X} , we can thus use the estimates obtained for the first equation satisfied by g_1^ε . Following those ideas, one can obtain some nice nested a priori estimates on the system, which allow to conclude.

Remark 4.4. Our method should be robust enough to also treat the case of very soft and Coulomb potentials $-3 \leq \gamma < -2$ in which the linearized operator does not have a spectral gap (in this case, the inequality (4.43) does not provide coercivity anymore). More precisely, we should be able to obtain a similar result of Theorem 4.1 with the exponential time-decay in (4.22) being replaced by a sub-exponential one, by combining the arguments developed in this paper together with the study made in the case $\varepsilon = 1$ in [22] by Carrapatoso and Mischler. We do not treat this case in the present paper.

The Cauchy theory and the large-time behavior of the Landau equation for $\varepsilon = 1$ have been extensively studied. We here give a small sample of the existing literature: Let us mention [1] for renormalized solutions with defect measure, [34] for the convergence to equilibrium for a priori smooth solutions with general initial data, [60, 104, 26, 22] for strong solutions in a perturbative framework. Concerning the well-posedness of our rescaled equation, it has already been obtained in [58, 111] respectively by Guo and Rachid in Sobolev spaces (respectively in $H_{x,v}^N$ with $N \geq 8$ and in $H_x^3 L_v^2$) thanks to nonlinear energy methods. In [17], Briant has obtained a similar result in $H_{x,v}^N$ with $N \geq 4$ thanks to the so-called H^1 -hypocoercivity method at the linear level.

Our global strategy to prove Theorem 4.1-(i) is based on the study of the linearized equation. And then, we go back to the fully nonlinear problem. This is a standard strategy to develop a Cauchy theory in a close-to-equilibrium regime. However, we have to emphasize here that our study is quite involved as explained below.

At the linear level, our strategy is based on a L^2 -hypocoercivity method which heavily relies on the micro-macro decomposition and is thus particularly adapted to the study of hydrodynamical problems. Recall that the challenge of hypocoercivity is to understand the interplay between the collision operator that provides dissipativity in the velocity

variable and the transport one which is conservative, in order to obtain global dissipativity for the whole linear problem (see [126, 64] for a presentation of this topic). The L^2 -hypocoercivity method has been introduced by Hérau [63] (see also [38]) for one dimensional space of collisional invariants and introduced by Guo in [59] for a space of collisional invariants of dimension larger than one (including the Boltzmann and Landau cases). Let us explain into more details the strategy, we first define a norm $\|\cdot\|_{L^2_{x,v}}$ (associated to the scalar product $\langle\langle\cdot, \cdot\rangle\rangle_{L^2_{x,v}}$) which is equivalent to the usual one $\|\cdot\|_{L^2_{x,v}}$ uniformly in ε and is such that

$$\langle\langle\Lambda_\varepsilon f, f\rangle\rangle_{L^2_{x,v}} \lesssim -\frac{1}{\varepsilon^2} \|f^\perp\|_{L^2_x(H^1_{v,*})}^2 - \|f\|_{L^2_{x,v}}^2.$$

Such a norm is defined in Subsection 3.1 and is inspired by [59] (see also [13]) in which the more complex case of bounded domains with various boundary conditions is treated. Due to the fact that derivatives in x commute with Λ_ε , it is easy to deduce a similar result on the space $H^3_x L^2_v$. However, it is not an easy matter to recover such an energy estimate in larger or smaller spaces than $H^3_x L^2_v$. Actually, the methods presented in [56, 101] by Gualdani, Mischler and Mouhot to develop shrinkage or enlargement arguments at the level of energy estimates is not easily adaptable to rescaled equations if one wants to get uniform estimates in the parameter of rescaling. To develop an enlargement argument, one can use the trick introduced in [56, 101] of splitting the original equation into several ones. This trick was already used in [18, 5] to obtain uniform in ε estimates on the rescaled Boltzmann equation for respectively elastic and inelastic hard spheres in a large class of Sobolev spaces (see also Remark 4.3). However, we do not have such a method of splitting to perform a “shrinkage” argument. In the present paper, we exhibit a norm equivalent to the usual one that provides dissipativity for Λ_ε in a smaller space than $H^3_x L^2_v$ (namely in the space \mathcal{X} defined in (4.19)) and that also preserves the gain of $1/\varepsilon$ on the microscopic part of the solution. This is done in Subsection 3.1. Notice that it is also possible to obtain decay estimates directly on the semigroup associated to Λ_ε thanks to Duhamel formula once one has exhibited a nice splitting of Λ_ε (see Section 5).

We then prove some new and sharp nonlinear estimates on the Landau collision operator (see Subsection 4.1) to be able to develop our Cauchy theory for the whole nonlinear problem in a close-to-equilibrium framework. It is worth mentioning that to prove good a priori estimates on the nonlinear problem, we use the hypocoercive norm defined in Subsection 3.1 and we only perform energy estimates. It is actually important that our analysis does not rely on the use of Duhamel formula because of the rescaling parameter (see the beginning of Section 4 for more details).

The strategy that we use to prove the regularization estimate in Theorem 4.1-(ii) is quite classical for linear hypoelliptic equations and has been introduced by Hérau and Nier [67] for the kinetic Fokker-Planck equation. Such a method has been used for many hypoelliptic equations: In [71] for the fractional kinetic Fokker-Planck equation, in [26, 22] for the linearized Landau equation, in [70] for the linearized Boltzmann equation without cutoff

etc... To our knowledge, it is the first time that such a strategy is used for a nonlinear equation (even in the simpler case $\varepsilon = 1$) and for a rescaled equation (with uniform estimates in the rescaling parameter). Roughly speaking, the idea is to introduce a functional with weights in time which is a Lyapunov functional for our equation for small times. From this property, we are then able to recover some regularization estimates quantified in time as stated in Theorem 4.1-(ii). Here, the difficulties are threefold:

- First, we study a nonlinear equation, our computations are thus much more intricate, the idea behind our computations being that we work with small data which allows us to absorb the nonlinearity. Our proof also requires some new and sharp nonlinear estimates on the collision operator (see Subsection 4.1).
- Then, the functional has to be suitably defined to handle the dependencies in ε . The differences of behaviors between microscopic and macroscopic parts of the solution have to be taken into account and in the spirit of the definition of the H^1 -hypo coercive norm of Briant [17], some terms of the functional only involve the microscopic part of the solution (see (4.114) for the definition of the functional).
- Finally, since we want to obtain the optimal gain of regularity (the corresponding hypoelliptic estimates are provided in [111] by Rachid), our functional has to be defined accordingly. For example, in [26] in which the authors were not interested in getting the optimal gain of regularity (and in which only the case $\varepsilon = 1$ was treated), only classical derivation operators were involved in the definition of the functional. Here, the definition of the functional is much more intricate: We have to work with the anisotropic operators $\widetilde{\nabla}_v$ and $\widetilde{\nabla}_x$ defined in (4.39) and our functional also involves additional terms which are necessary to close our estimates.

To end this part, we mention that our proof also provides a regularization estimate in the space variable which is not uniform in ε (see (4.23)). The non-uniformity in ε of such a gain is expected since the transport operator and the linearized collision operator (which gives the gain of regularity in velocity) do not act at the same scale.

1.3 The fluid model

In the second part of the paper, we shall prove that the hydrodynamical limit of (4.5) as ε goes to zero is the incompressible Navier-Stokes-Fourier system associated with the Boussinesq equation which writes

$$\begin{cases} \partial_t u + u \cdot \nabla_x u - \nu_1 \Delta_x u = -\nabla_x p \\ \partial_t \theta + u \cdot \nabla_x \theta - \nu_2 \Delta_x \theta = 0 \\ \operatorname{div}_x u = 0 \\ \nabla_x (\rho + \theta) = 0. \end{cases} \quad (4.24)$$

In this system, the temperature θ , the density ρ and the pressure p are scalar unknowns, whereas the velocity u is an unknown vector field. The pressure can actually be eliminated

from the equations by applying to the momentum equation the projector \mathbb{P} onto the space of divergence-free vector fields. This projector is bounded over H_x^ℓ for all ℓ , and in L_x^p for all $1 < p < \infty$. To define the viscosity coefficients ν_i , let us introduce the two unique functions Φ (which is a matrix-valued function) and Ψ (which is a vector-valued function) orthogonal to $\text{Ker } L$ such that

$$\frac{1}{\sqrt{M}}L(\sqrt{M}\Phi) = \frac{|v|^2}{3}\text{Id} - v \otimes v \quad \text{and} \quad \frac{1}{\sqrt{M}}L(\sqrt{M}\Psi) = \frac{5 - |v|^2}{2}v.$$

The viscosity coefficients are then defined by

$$\nu_1 := \frac{1}{10} \int_{\mathbb{R}^3} \Phi : L(\sqrt{M}\Phi) \sqrt{M} \, dv \quad \text{and} \quad \nu_2 := \frac{2}{15} \int_{\mathbb{R}^3} \Psi \cdot L(\sqrt{M}\Psi) \sqrt{M} \, dv.$$

In what follows, we call *well-prepared data* the class of functions $f \in \text{Ker } L$ that write

$$f(x, v) = \sqrt{M}(v) \left(\rho_f(x) + u_f(x) \cdot v + \frac{|v|^2 - 3}{2} \theta_f(x) \right) \\ \text{with} \quad \nabla_x \cdot u_f = 0 \quad \text{and} \quad \rho_f + \theta_f = 0 \quad (4.25)$$

where we recall that ρ_f , u_f and θ_f are defined in (4.14), (4.15), (4.16).

It is known that for mean free $(\rho_0, u_0, \theta_0) \in H_x^3$ small enough and satisfying

$$\nabla_x \cdot u_0 = 0 \quad \text{and} \quad \rho_0 + \theta_0 = 0, \quad (4.26)$$

there exists a unique solution $(\rho, u, \theta) \in H_x^3$ to (4.24) defined on \mathbb{R}^+ with associated initial data (ρ_0, u_0, θ_0) (see [43, 89, 90, 44]). For such an initial data, we also define g_0 a well-prepared data with (ρ_0, u_0, θ_0) as associated first macroscopic quantities, namely

$$g_0(x, v) := \sqrt{M}(v) \left(\rho_0(x) + u_0(x) \cdot v + \frac{|v|^2 - 3}{2} \theta_0(x) \right) \\ \text{with} \quad \nabla_x \cdot u_0 = 0 \quad \text{and} \quad \rho_0 + \theta_0 = 0. \quad (4.27)$$

Notice that from the definition of the space \mathcal{X} , we in particular have that $g_0 \in \mathcal{X}$ and the mean-free assumption made on (ρ_0, u_0, θ_0) implies that $g_0 \in (\text{Ker } \Lambda_\varepsilon)^\perp$. Notice also that due to the definition of \mathcal{X} , the smallness assumption made on (ρ_0, u_0, θ_0) can be translated into a smallness assumption on g_0 . Indeed, given the form of g_0 , by triangular inequality, it is clear that

$$\|g_0\|_{\mathcal{X}} \lesssim \|(\rho_0, u_0, \theta_0)\|_{H_x^3}.$$

Moreover, since $\{\sqrt{M}, \sqrt{M}v_i, \sqrt{M}(|v|^2 - 3)/2\}$ is an orthogonal system in L_v^2 , we also have that

$$\|g_0\|_{\mathcal{X}} \gtrsim \|g_0\|_{H_x^3 L_v^2} \gtrsim \|(\rho_0, u_0, \theta_0)\|_{H_x^3}.$$

As a consequence, there exists $\eta_1 > 0$ such that if g_0 is of the form (4.27) and satisfies $\|g_0\|_{\mathcal{X}} \leq \eta_1$, then there exists $(\rho, u, \theta) \in H_x^3$ defined on \mathbb{R}^+ solution to (4.24). We define the kinetic distribution lying in $\text{Ker } L$ with associated macroscopic quantities (ρ, u, θ)

$$g(t, x, v) := \sqrt{M}(v) \left(\rho(t, x) + u(t, x) \cdot v + \frac{|v|^2 - 3}{2} \theta(t, x) \right). \quad (4.28)$$

We also have the following estimate

$$\|g\|_{L_t^\infty(\mathcal{X})} \lesssim C(\|g_0\|_{\mathcal{X}}) \quad (4.29)$$

where $C(\|g_0\|_{\mathcal{X}})$ is a constant only depending on the \mathcal{X} -norm of the data g_0 . The aforementioned results on the system (4.24) can be found in [44, Appendix B.3] in which more details and references on the subject are given.

1.4 Hydrodynamical limit result

For the statement of the main hydrodynamical result, we first introduce the following notation for functional spaces: If X_1 and X_2 are two function spaces, we say that a function f belongs to $X_1 + X_2$ if there are $f_1 \in X_1$ and $f_2 \in X_2$ such that $f = f_1 + f_2$ and we define

$$\|f\|_{X_1+X_2} := \min_{\substack{f = f_1 + f_2 \\ f_i \in X_i}} (\|f_1\|_{X_1} + \|f_2\|_{X_2}).$$

Theorem 4.5. *Let $g_{\text{in}}^\varepsilon \in \mathcal{X} \cap (\text{Ker } \Lambda_\varepsilon)^\perp$ for $\varepsilon \in (0, 1)$ such that $\|g_{\text{in}}^\varepsilon\|_{\mathcal{X}} \leq \eta_0$ (where η_0 is defined in Theorem 4.1) and $g^\varepsilon \in L_t^\infty(\mathcal{X})$ being the associated solutions of (4.7) with initial data $g_{\text{in}}^\varepsilon$ constructed in Theorem 4.1. Consider also $g_0 \in \mathcal{X} \cap (\text{Ker } \Lambda_\varepsilon)^\perp$ such that $\|g_0\|_{\mathcal{X}} \leq \eta_1$ and g defined respectively as in (4.27) and (4.28).*

There exists $\eta_2 \in (0, \min(\eta_0, \eta_1))$ such that if $\max(\|g_{\text{in}}^\varepsilon\|_{\mathcal{X}}, \|g_0\|_{\mathcal{X}}) \leq \eta_2$ and

$$\|g_{\text{in}}^\varepsilon - g_0\|_{\mathcal{X}} \xrightarrow{\varepsilon \rightarrow 0} 0, \quad (4.30)$$

then we have

$$\|g^\varepsilon - g\|_{L_t^\infty(\mathcal{X})} \xrightarrow{\varepsilon \rightarrow 0} 0. \quad (4.31)$$

If $\max(\|g_{\text{in}}^\varepsilon\|_{\mathcal{X}}, \|g_0\|_{\mathcal{X}}) \leq \eta_2$ and

$$\|\pi g_{\text{in}}^\varepsilon - g_0\|_{\mathcal{X}} \xrightarrow{\varepsilon \rightarrow 0} 0, \quad (4.32)$$

then we have

$$\|g^\varepsilon - g\|_{L_t^1(\mathcal{Q}_1) + L_t^\infty(\mathcal{X})} \xrightarrow{\varepsilon \rightarrow 0} 0. \quad (4.33)$$

Remark 4.6. One can get a rate of convergence in (4.31) and (4.33) if we suppose that g_0 has some additional regularity in x , namely a rate of ε^δ if the regularity is supposed to be $H_x^{3+\delta}$ for $\delta \in (0, 1/2]$. We refer to Theorem 4.43 for a quantitative version of this result.

Remark 4.7. As explained in Remark 4.3, the results of Theorem 4.1 could be obtained in larger spaces $\mathcal{E} = H_x^3 L_v^2(\langle v \rangle^k \sqrt{M})$. A similar approach as the one used by Gervais in [46, 45] in which the hard spheres Boltzmann equation is treated in “large” Sobolev spaces, might yield the associated hydrodynamical result.

Remark 4.8. As explained in Remark 4.4, the strategy of the proof of Theorem 4.1 should also work in order to treat the case of very soft and Coulomb potentials. However, in order to obtain the associated hydrodynamical result, our method employs some fine spectral estimates that are known to hold only for the case $-2 \leq \gamma \leq 1$ by [130]. Therefore, if we are able to extend the results of [130] to the case of very soft and Coulomb potentials, we might then be able to obtain the analogous result of Theorem 4.5.

We first give a short overview of the existing literature on the problem of deriving fluid equations from kinetic ones (we refer to the book by Saint-Raymond [115] for a thorough presentation of the topic). The first justifications of the link between kinetic and fluid equations were formal and based on asymptotic expansions by Hilbert, Chapman, Enskog and Grad (see [73, 28, 53]). The first rigorous convergence proofs based also on asymptotic expansions were given by Caffisch [19] (see also [87] and [29]). In those papers, the limit is justified up to the first singular time for the fluid equation. By using the nonlinear energy method introduced by himself in [60], Guo [58] has justified the limit towards the Navier-Stokes equation and beyond in Hilbert’s expansion from Boltzmann and Landau equations (see below for more details on this result).

There has also been some convergence proofs based on spectral analysis in the framework of strong solutions close to equilibrium introduced by Grad [52] and Ukai [125] for the Boltzmann equation. In this respect, we refer to the works by Nishida, Bardos and Ukai, Gallagher and Tristani [108, 11, 44]. These results use the description of the spectrum of the linearized Boltzmann equation in Fourier space in the space variable performed in [107, 27, 41]. Our approach is reminiscent of these ones and relies on the generalization of the paper [41] to several kinetic equations (including the Landau one) made in [130] by Yang and Yu. Notice also that such a spectral result has recently been obtained in [46] by Gervais for the hard-spheres Boltzmann equation in a larger class of Sobolev spaces.

More recently, some uniform in ε estimates on kinetic equations have allowed to prove (at least) weak convergence towards the Navier-Stokes equation. Let us mention [83, 112] in which the cases of Boltzmann equation without cutoff and the Landau equations are treated. In [17, 18], the authors have obtained convergence to equilibrium results for the rescaled Boltzmann equation (and also the Landau equation in [17]) uniformly in the rescaling parameter using respectively hypocoercivity and enlargement methods. In [18], the authors are able to weaken the assumptions on the data down to Sobolev spaces with

polynomial weights. We also refer to [5] in which a similar approach combined with perturbative arguments has been used to derive a fluid system from the inelastic Boltzmann equation. Notice that Briant [17] has combined this with Ellis and Pinsky result [41] to recover strong convergence in the case of the Boltzmann equation.

Finally, let us mention that this problem has been extensively studied in the framework of weak solutions, the goal being to obtain solutions for the fluid models from renormalized solutions introduced by DiPerna and Lions in [37] for the Boltzmann equation. We shall not make an extensive presentation of this program as it is out of the realm of this study, but let us mention that it was started by Bardos, Golse and Levermore at the beginning of the nineties in [9, 8] and was continued by those authors, Saint-Raymond, Masmoudi, Lions among others. We mention here a (non exhaustive) list of papers which are part of this program (see [49, 48, 94, 96, 115]).

Let us focus on the Landau equation for which the literature is scarcer. As mentioned above, in [58], Guo justifies the limit the Navier-Stokes limit and beyond in the Hilbert's expansion from (the Boltzmann and) the Landau equations (even for the case of very soft potentials) in the torus by using his nonlinear energy method. Our result on the hydrodynamical limit is reminiscent of the one obtained in [58] for the hard and moderately soft potentials in the sense that we work with strong solutions and we prove a strong convergence result. It is however worth noticing that our functional framework is less restrictive (we only work with 3 derivatives in x and no derivative in v whereas in [58], regularity on 8 derivatives in both variables x and v is required). Moreover, there is an important loss of regularity in the estimates of convergence proven in [58] whereas we only lose δ derivatives in x and there is no loss in v to get a rate of convergence of ε^δ from Landau to Navier-Stokes equation (see Theorem 4.43). In the present paper and in [58], the fluid initial data are supposed to be well-prepared, namely the divergence free condition and the Boussinesq relation (4.26) are supposed to hold. We refer to [82] by Jiang and Xiong for an extension to the case where the fluid part is not supposed to be well-prepared and the creation and propagation of initial layers is studied. In [58, 82], the kinetic initial data is supposed to have a specific form so that there is no creation of kinetic initial layers. Our presentation is slightly different since we do not use Hilbert expansion to study the limit towards the Navier-Stokes equation, our assumption being the following: The projection of the kinetic initial data onto the kernel of the linearized operator L is supposed to converge towards the well-prepared fluid initial data as $\varepsilon \rightarrow 0$. Finally, in [112], Rachid obtained a result of weak- \star convergence in $L_t^\infty(H_x^3 L_v^2)$ towards the incompressible Navier-Stokes-Fourier system, we have thus strengthen this result for the type of initial data that we consider here. One can notice that the assumptions made on the fluid initial data in [112] are a bit less restrictive since the divergence free condition for u_0 and the Boussinesq relation for ρ_0 and θ_0 are not supposed to hold. The initial layer that such an initial condition creates is absorbed there in the weak convergence. In our framework, we can not hope to absorb it in a strong convergence framework because the initial layer is propagated over time in the periodic domain (see [82]). Note also that

in [44], the authors were able to treat this type of “completely ill-prepared” data only in the case of the whole space since those terms has some dispersive properties in the whole space.

Let us describe into more details our strategy to obtain strong convergence. It is inspired by the ones used in [11, 17, 44]. Indeed, as in [44], using the spectral analysis performed in [130] by Yang and Yu, in order to prove our main convergence result, we reformulate the fluid equation in a kinetic fashion and we then study the equation satisfied by the difference satisfied between the kinetic and the fluid solutions. However, let us point out that we are not able to perform a fixed point argument as in the aforementioned paper. This is due to the fact that the structure of the Landau bilinear operator is more complicated than the hard-spheres Boltzmann one. Indeed, there is an anisotropic loss of derivatives and weights in the nonlinear estimates which prevent us from closing a fixed point estimate. To circumvent this difficulty, we use some new a priori estimates on the solution of the linearized rescaled Landau equation and on the nonlinear rescaled Landau equation (4.7) that are uniform in the Knudsen number and that have been presented in Theorem 4.1. By intertwining these refined and sharp kinetic estimates and fluid mechanics ones, we are able to prove a result of strong convergence from the solutions of the Landau equation to the Navier-Stokes one as stated in Theorem 4.5.

1.5 Outline of the paper

In Section 2, we give some technical results on the Landau collision operator that will be used all along the paper. In Section 3, we develop hypocoercivity and regularization estimates for the linearized problem. In Section 4, we develop our perturbative Cauchy theory for the whole nonlinear problem as well as some regularization estimates on it. In Section 5, we develop some new estimates on the linearized problem that are useful to prove our hydrodynamical result, which is proven in Section 6.

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2 Preliminary results

In this section, we present some technical results and tools that will be useful throughout the paper.

2.1 Collision operator

Recalling the definition of the matrix $a = (a_{ij})_{ij}$ in (4.2) and that we use the convention of summation of repeated indices through the paper, we define the following quantities

$$\begin{aligned} b_i(v) &= \partial_j a_{ij}(v) = -2|v|^\gamma v_i, \\ c(v) &= \partial_{ij} a_{ij}(v) = -2(\gamma + 3)|v|^\gamma, \end{aligned}$$

in such a way that one can rewrite the Landau operator (4.1) as

$$\begin{aligned} Q(g, f) &= (a_{ij} * g) \partial_{v_i v_j} f - (c * g) f \\ &= \partial_{v_i} \left\{ (a_{ij} * g) \partial_{v_j} f - (b_i * g) f \right\}, \end{aligned} \quad (4.34)$$

where $*$ denotes the convolution in the velocity variable v .

We now state a technical lemma in which we provide a rewriting of the nonlinear operator Γ defined in (4.8) and the linearized collision operator L defined in (4.9).

Lemma 4.9. *There holds*

$$\begin{aligned} \Gamma(f_1, f_2) &= \partial_{v_i} \left\{ (a_{ij} * [\sqrt{M} f_1]) \partial_{v_j} f_2 \right\} - \partial_{v_i} \left\{ (b_i * [\sqrt{M} f_1]) f_2 \right\} \\ &\quad - (a_{ij} * [\sqrt{M} f_1]) v_i \partial_{v_j} f_2 + \frac{1}{4} (a_{ij} * [\sqrt{M} f_1]) v_i v_j f_2 \\ &\quad - \frac{1}{2} (a_{ii} * [\sqrt{M} f_1]) f_2, \end{aligned} \quad (4.35)$$

and

$$\begin{aligned} Lf &= \partial_{v_i} \left\{ (a_{ij} * M) \partial_{v_j} f \right\} + \left\{ -\frac{1}{4} (a_{ij} * M) v_i v_j + \frac{1}{2} \partial_{v_i} [(a_{ij} * M) v_j] \right\} f \\ &\quad + (a_{ij} * [\sqrt{M} f]) v_i v_j \sqrt{M} - (a_{ii} * [\sqrt{M} f]) \sqrt{M} - (c * [\sqrt{M} f]) \sqrt{M}. \end{aligned} \quad (4.36)$$

Proof. From the definition of Γ in (4.8) and using the formulation (4.34) for Q , we first obtain

$$\Gamma(g_1, g_2) = \frac{1}{\sqrt{M}} \partial_{v_i} \left\{ (a_{ij} * [\sqrt{M} g_1]) \partial_{v_j} (\sqrt{M} g_2) - (b_i * [\sqrt{M} g_1]) \sqrt{M} g_2 \right\}.$$

By writing $\partial_{v_j} (\sqrt{M} g_2) = \sqrt{M} \partial_{v_j} g_2 - \frac{1}{2} v_j \sqrt{M} g_2$ we thus obtain

$$\begin{aligned} \Gamma(g_1, g_2) &= \frac{1}{\sqrt{M}} \partial_{v_i} \left\{ (a_{ij} * [\sqrt{M} g_1]) \sqrt{M} \partial_{v_j} g_2 \right\} - \frac{1}{2} \frac{1}{\sqrt{M}} \partial_{v_i} \left\{ (a_{ij} * [\sqrt{M} g_1]) v_j \sqrt{M} g_2 \right\} \\ &\quad - \frac{1}{\sqrt{M}} \partial_{v_i} \left\{ (b_i * [\sqrt{M} g_1]) \sqrt{M} g_2 \right\}. \end{aligned}$$

Applying the derivative ∂_{v_i} inside the brackets to the term \sqrt{M} we then get

$$\begin{aligned}\Gamma(g_1, g_2) &= \partial_{v_i} \left\{ (a_{ij} * [\sqrt{M}g_1]) \partial_{v_j} g_2 \right\} - \frac{1}{2} (a_{ij} * [\sqrt{M}g_1]) v_i \partial_{v_j} g_2 \\ &\quad - \frac{1}{2} \partial_{v_i} \left\{ (a_{ij} * [\sqrt{M}g_1]) v_j g_2 \right\} + \frac{1}{4} (a_{ij} * [\sqrt{M}g_1]) v_i v_j g_2 \\ &\quad - \partial_{v_i} \left\{ (b_i * [\sqrt{M}g_1]) g_2 \right\} + \frac{1}{2} (b_i * [\sqrt{M}g_1]) v_i g_2.\end{aligned}$$

Finally, we apply the derivative to the third term in the right-hand side of the above equation and thus we get

$$\begin{aligned}\Gamma(g_1, g_2) &= \partial_{v_i} \left\{ (a_{ij} * [\sqrt{M}g_1]) \partial_{v_j} g_2 \right\} - \frac{1}{2} (a_{ij} * [\sqrt{M}g_1]) v_i \partial_{v_j} g_2 \\ &\quad - \frac{1}{2} (b_j * [\sqrt{M}g_1]) v_j g_2 - \frac{1}{2} (a_{ii} * [\sqrt{M}g_1]) g_2 - \frac{1}{2} (a_{ij} * [\sqrt{M}g_1]) v_j \partial_{v_i} g_2 \\ &\quad + \frac{1}{4} (a_{ij} * [\sqrt{M}g_1]) v_i v_j g_2 - \partial_{v_i} \left\{ (b_i * [\sqrt{M}g_1]) g_2 \right\} + \frac{1}{2} (b_i * [\sqrt{M}g_1]) v_i g_2 \\ &= \partial_{v_i} \left\{ (a_{ij} * [\sqrt{M}g_1]) \partial_{v_j} g_2 \right\} - (a_{ij} * [\sqrt{M}g_1]) v_i \partial_{v_j} g_2 \\ &\quad - \frac{1}{2} (a_{ii} * [\sqrt{M}g_1]) g_2 + \frac{1}{4} (a_{ij} * [\sqrt{M}g_1]) v_i v_j g_2 - \partial_{v_i} \left\{ (b_i * [\sqrt{M}g_1]) g_2 \right\}\end{aligned}$$

which completes the proof of (4.35).

We now prove (4.36). From (4.35) we get

$$\begin{aligned}\Gamma(\sqrt{M}, f) &= \partial_{v_i} \left\{ (a_{ij} * M) \partial_{v_j} f \right\} - \partial_{v_i} \left\{ (b_i * M) f \right\} \\ &\quad - (a_{ij} * M) v_i \partial_{v_j} f + \frac{1}{4} (a_{ij} * M) v_i v_j f - \frac{1}{2} (a_{ii} * M) f.\end{aligned}$$

For the second term in the right-hand side of above equation, we observe that

$$-\partial_{v_i} \left\{ (b_i * M) f \right\} = -(c * M) f - (b_i * M) \partial_{v_i} f,$$

as well as

$$-(b_i * M) \partial_{v_i} f = -(\partial_{v_j} a_{ij} * M) \partial_{v_i} f = (a_{ij} * v_j M) \partial_{v_i} f = (a_{ij} * M) v_j \partial_{v_i} f$$

by using that $a_{ij}(v-w)(v_i-w_i) = 0$. We also remark, using that $a_{ij}(v-w)(v_i-w_i)(v_j-w_j) = 0$,

$$\begin{aligned}-(c * M) &= -(a_{ij} * \partial_{v_i} \partial_{v_j} M) = (a_{ij} * \delta_{ij} M) - (a_{ij} * v_i v_j M) \\ &= (a_{ii} * M) - (a_{ij} * M) v_i v_j,\end{aligned}$$

as well as

$$\partial_{v_i} [(a_{ij} * M) v_j] = -(a_{ij} * M) v_i v_j + (a_{ii} * M).$$

Putting together previous equalities, we finally obtain

$$\Gamma(\sqrt{M}, f) = \partial_{v_i} \left\{ (a_{ij} * M) \partial_{v_j} f \right\} + \left\{ -\frac{1}{4} (a_{ij} * M) v_i v_j + \frac{1}{2} \partial_{v_i} [(a_{ij} * M) v_j] \right\} f.$$

Moreover, from (4.35) we get

$$\begin{aligned} \Gamma(f, \sqrt{M}) &= \partial_{v_i} \left\{ (a_{ij} * [\sqrt{M} f]) \partial_{v_j} \sqrt{M} \right\} - \partial_{v_i} \left\{ (b_i * [\sqrt{M} f]) \sqrt{M} \right\} \\ &\quad - (a_{ij} * [\sqrt{M} f_1]) v_i \partial_{v_j} \sqrt{M} + \frac{1}{4} (a_{ij} * [\sqrt{M} f]) v_i v_j \sqrt{M} \\ &\quad - \frac{1}{2} (a_{ii} * [\sqrt{M} f]) \sqrt{M} \end{aligned}$$

so that, developing the derivatives ∂_{v_i} , we get

$$\begin{aligned} \Gamma(f, \sqrt{M}) &= (b_j * [\sqrt{M} f]) \partial_{v_j} \sqrt{M} + (a_{ij} * [\sqrt{M} f]) \partial_{v_i} \partial_{v_j} \sqrt{M} \\ &\quad - (c * [\sqrt{M} f]) \sqrt{M} - (b_i * [\sqrt{M} f]) \partial_{v_i} \sqrt{M} \\ &\quad - (a_{ij} * [\sqrt{M} f]) v_i \partial_{v_j} \sqrt{M} + \frac{1}{4} (a_{ij} * [\sqrt{M} f]) v_i v_j \sqrt{M} \\ &\quad - \frac{1}{2} (a_{ii} * [\sqrt{M} f]) \sqrt{M}. \end{aligned}$$

Observing that

$$\partial_{v_j} \sqrt{M} = -\frac{1}{2} v_j \sqrt{M} \quad \text{and} \quad \partial_{v_i} \partial_{v_j} \sqrt{M} = -\frac{1}{2} \delta_{ij} \sqrt{M} + \frac{1}{4} v_i v_j \sqrt{M},$$

we then get

$$\Gamma(f, \sqrt{M}) = (a_{ij} * [\sqrt{M} f]) v_i v_j \sqrt{M} - (a_{ii} * [\sqrt{M} f]) \sqrt{M} - (c * [\sqrt{M} f]) \sqrt{M},$$

which concludes the proof. \square

For $v \in \mathbb{R}^3$, we define the symmetric matrix $\mathbf{A}(v) = (A_{ij}(v))_{1 \leq i, j \leq 3}$ whose coefficients are given by

$$A_{ij}(v) = (a_{ij} *_{v} M)(v).$$

We can decompose $\mathbf{A}(v)$ as $\mathbf{A}(v) = \mathbf{B}^\top(v) \mathbf{B}(v)$, where $\mathbf{B}(v) = (B_{ij}(v))_{1 \leq i, j \leq 3}$ is a matrix with real-valued smooth entries (see [111]). Recall also that from [30], for $v \in \mathbb{R}^3 \setminus \{0\}$, the matrix $\mathbf{A}(v)$ has a simple eigenvalue $\ell_1(v) > 0$ associated with the eigenvector v and a double eigenvalue $\ell_2(v) > 0$ associated with the eigenspace v^\perp . Moreover, when $|v| \rightarrow +\infty$, we have

$$\ell_1(v) \sim 2\langle v \rangle^\gamma \quad \text{and} \quad \ell_2(v) \sim \langle v \rangle^{\gamma+2}. \quad (4.37)$$

As such, one can write that for any $v \in \mathbb{R}^3 \setminus \{0\}$,

$$\mathbf{A}(v) = \ell_1(v) \frac{v}{|v|} \otimes \frac{v}{|v|} + \ell_2(v) \left(\text{Id} - \frac{v}{|v|} \otimes \frac{v}{|v|} \right),$$

where Id denotes the identity matrix and

$$\mathbf{B}(v) = \sqrt{\ell_1(v)} \frac{v}{|v|} \otimes \frac{v}{|v|} + \sqrt{\ell_2(v)} \left(\text{Id} - \frac{v}{|v|} \otimes \frac{v}{|v|} \right). \quad (4.38)$$

In what follows, we will use the following differential operators

$$\widetilde{\nabla}_v := \mathbf{B}(v) \nabla_v \quad \text{and} \quad \widetilde{\nabla}_x := \mathbf{B}(v) \nabla_x \quad (4.39)$$

as well as their adjoint operators in L_v^2 given by, for $F : \mathbb{R}^d \rightarrow \mathbb{R}^d$,

$$(\widetilde{\nabla}_v)^* F = -\nabla_v \cdot (\mathbf{B}(v) F) \quad \text{and} \quad (\widetilde{\nabla}_x)^* F = -\widetilde{\nabla}_x \cdot F. \quad (4.40)$$

Using the formulation (4.36) of the linearized collision operator L , we can therefore rewrite it as

$$\begin{aligned} Lf &= -\widetilde{\nabla}_v^* \widetilde{\nabla}_v f - \left\{ \frac{1}{4} |\mathbf{B}(v)v|^2 - \frac{1}{2} \nabla_v \cdot [\mathbf{B}^\top(v) \mathbf{B}(v)v] \right\} f \\ &\quad - \left\{ (a_{ij} * \sqrt{M}f) v_i v_j - (a_{ii} * \sqrt{M}f) + (c * \sqrt{M}f) \right\} \sqrt{M}. \end{aligned} \quad (4.41)$$

The functions A_{ij}, B_{ij} verify the following properties (see for example [58, 111]): For any multi-index $\alpha \in \mathbb{N}^3$ we have, for all $v \in \mathbb{R}^3$,

$$|\partial_v^\alpha A_{ij}(v)| \lesssim \langle v \rangle^{\gamma+2-|\alpha|}, \quad |\partial_v^\alpha B_{ij}(v)| \lesssim \langle v \rangle^{\frac{\gamma}{2}+1-|\alpha|}. \quad (4.42)$$

From [30, 60, 103, 105], we also know that L has a spectral gap, more precisely, there is $\sigma_L > 0$ such that

$$\langle Lf, f \rangle_{L_v^2} \leq -\sigma_L \|f - \pi f\|_{H_{v,*}^1}^2 \quad (4.43)$$

for any $f \in \text{Dom}(L)$, where we recall that π is the projector onto the kernel of L defined in (4.12) and the $H_{v,*}^1$ -norm is defined in (4.18). Notice that in the case $-2 \leq \gamma \leq 1$ which we study in the present paper, the previous inequality is indeed a coercivity estimate because $\|\cdot\|_{H_{v,*}^1} \geq \|\cdot\|_{L_v^2}$.

2.2 Functional spaces

We first notice that the $H_{v,*}^1$ -norm in velocity defined in (4.18) is also equal to

$$\|f\|_{H_{v,*}^1}^2 = \|\langle v \rangle^{\frac{\gamma}{2}+1} f\|_{L_v^2}^2 + \|\widetilde{\nabla}_v f\|_{L_v^2}^2$$

where $\widetilde{\nabla}_v$ is defined in (4.39). We also introduce some H^2 norm in velocity defined through the following norm:

$$\|f\|_{H_{v,*}^2}^2 := \|\langle v \rangle^{\gamma+2} f\|_{L_v^2}^2 + \|\widetilde{\nabla}_v(\langle v \rangle^{\frac{\gamma}{2}+1} f)\|_{L_v^2}^2 + \|\langle v \rangle^{\frac{\gamma}{2}+1} \widetilde{\nabla}_v f\|_{L_v^2}^2 + \|\widetilde{\nabla}_v \widetilde{\nabla}_v f\|_{L_v^2}^2.$$

Similarly to the definitions of \mathcal{X} and \mathcal{B}_1 , we define the weighted Sobolev-type space \mathcal{B}_2 as the space associated to the norm

$$\begin{aligned} \|f\|_{\mathcal{B}_2}^2 &:= \|\langle v \rangle^{3(\frac{\gamma}{2}+1)} f\|_{L_x^2(H_{v,*}^2)}^2 + \|\langle v \rangle^{2(\frac{\gamma}{2}+1)} \nabla_x f\|_{L_x^2(H_{v,*}^2)}^2 \\ &\quad + \|\langle v \rangle^{\frac{\gamma}{2}+1} \nabla_x^2 f\|_{L_x^2(H_{v,*}^2)}^2 + \|\nabla_x^3 f\|_{L_x^2(H_{v,*}^2)}^2. \end{aligned} \quad (4.44)$$

We then introduce the spaces $\mathcal{Z}_i^\varepsilon$ for $i = 1, 2$ that involve derivatives in x : $\mathcal{Z}_1^\varepsilon$ is associated with the norm

$$\|f\|_{\mathcal{Z}_1^\varepsilon}^2 := \|f\|_{\mathcal{X}}^2 + \|f\|_{\mathcal{B}_1}^2 + \varepsilon^2 \|\widetilde{\nabla}_x f\|_{\mathcal{X}}^2, \quad (4.45)$$

and $\mathcal{Z}_2^\varepsilon$ is associated with

$$\|f\|_{\mathcal{Z}_2^\varepsilon}^2 := \|f\|_{\mathcal{Z}_1^\varepsilon}^2 + \|f\|_{\mathcal{B}_2}^2 + \varepsilon^4 \|\widetilde{\nabla}_x^2 f\|_{\mathcal{X}}^2. \quad (4.46)$$

For the sequel, it is worth noticing that if $f \in \mathcal{Z}_2^\varepsilon$, then $\varepsilon \widetilde{\nabla}_v \widetilde{\nabla}_x f \in \mathcal{X}$. Indeed, a simple computation based on integrations by parts shows that

$$\varepsilon \|\widetilde{\nabla}_v \widetilde{\nabla}_x f\|_{\mathcal{X}} \lesssim \varepsilon^2 \|\widetilde{\nabla}_x^2 f\|_{\mathcal{X}} + \|f\|_{\mathcal{B}_2}.$$

For $i = 1, 2$, we also define the associated dual spaces $(\mathcal{B}_i)'$ and $(\mathcal{Z}_i^\varepsilon)'$ with \mathcal{X} as a pivot space, more precisely, they are associated with the following norms:

$$\|f\|_{\mathcal{B}_i'} := \sup_{\|\varphi\|_{\mathcal{B}_i} \leq 1} \langle f, \varphi \rangle_{\mathcal{X}} \quad (4.47)$$

and

$$\|f\|_{(\mathcal{Z}_i^\varepsilon)'} := \sup_{\|\varphi\|_{\mathcal{Z}_i^\varepsilon} \leq 1} \langle f, \varphi \rangle_{\mathcal{X}} \quad (4.48)$$

where $\langle \cdot, \cdot \rangle_{\mathcal{X}}$ is the scalar product associated to $\|\cdot\|_{\mathcal{X}}$ defined in (4.19). Notice that we have the following interpolation result:

$$[\mathcal{X}, (\mathcal{Z}_2^\varepsilon)']_{1/2,2} = (\mathcal{Z}_1^\varepsilon)'. \quad (4.49)$$

The notation used above is the classical one of real interpolation (see [12]). For sake of completeness, we briefly recall the meaning of this notation. For C and D two Banach spaces which are both embedded in the same Hausdorff topological vector space, for any $z \in C + D$, we define the K -function by

$$K(t, z) := \inf_{z=c+d} (\|c\|_C + t\|d\|_D), \quad \forall t > 0.$$

The space $[C, D]_{\theta,p}$ for $\theta \in (0, 1)$ and $p \in [1, +\infty]$ is then defined by:

$$[C, D]_{\theta,p} := \left\{ z \in C + D, t \mapsto K(t, z)/t^\theta \in L^p(dt/t) \right\}.$$

2.3 Basic estimates

We gather in this subsection some basic estimates concerning the collision operator L that will be useful in the sequel.

In order to simplify, we recall the formulation of the operator L in (4.41) and we introduce the function

$$\psi(v) := \frac{1}{4} |\mathbf{B}(v)v|^2 - \frac{1}{2} \nabla_v \cdot [\mathbf{B}^\top(v)\mathbf{B}(v)v] \quad (4.50)$$

as well as the operators

$$L_1 f := -\widetilde{\nabla}_v^* \widetilde{\nabla}_v f - \psi f \quad (4.51)$$

and

$$L_2 f := -\left\{ (a_{ij} * \sqrt{M}f) v_i v_j - (a_{ii} * \sqrt{M}f) + (c * \sqrt{M}f) \right\} \sqrt{M} \quad (4.52)$$

so that we have $L = L_1 + L_2$.

We start with some basic commutator estimates.

Lemma 4.10. *For any suitable function $f = f(x, v)$ and any $(x, v) \in \mathbf{T}^3 \times \mathbb{R}^3$, there holds:*

(i) $[\widetilde{\nabla}_v, v \cdot \nabla_x] f(x, v) = \widetilde{\nabla}_x f(x, v).$

(ii) *For any $\alpha \in \mathbb{R}$ and $1 \leq j \leq 3$, one has*

$$|[\langle v \rangle^\alpha, \widetilde{\nabla}_{v_j}] f|(x, v) = |[\langle v \rangle^\alpha, \widetilde{\nabla}_{v_j}^*] f|(x, v) \lesssim \langle v \rangle^{\frac{\alpha}{2} + \alpha - 1} |f|(x, v).$$

(iii) *For any $1 \leq i, j \leq 3$, one has*

$$|[\widetilde{\nabla}_{v_i}, \widetilde{\nabla}_{v_j}] f|(x, v) \lesssim \langle v \rangle^{\gamma+1} |\nabla_v f|(x, v).$$

(iv) *For any $1 \leq i, j \leq 3$, one has*

$$|[\widetilde{\nabla}_{v_i}, \widetilde{\nabla}_{v_j}^*] f|(x, v) \lesssim \langle v \rangle^{\gamma+1} |\nabla_v f|(x, v) + \langle v \rangle^\gamma |f|(x, v).$$

(v) *For any $1 \leq i, j \leq 3$, one has*

$$|[\widetilde{\nabla}_{v_i}, \widetilde{\nabla}_{x_j}] f|(x, v) = |[\widetilde{\nabla}_{v_i}^*, \widetilde{\nabla}_{x_j}] f|(x, v) \lesssim \langle v \rangle^{\gamma+1} |\nabla_x f|(x, v).$$

(vi) *For any $\alpha \in \mathbb{R}$, one has*

$$|[[\langle v \rangle^\alpha, \widetilde{\nabla}_{v_j}^*], \widetilde{\nabla}_{v_j}] f|(x, v) \lesssim \langle v \rangle^{\gamma + \alpha - 1} |f|(x, v).$$

(vii) For any $1 \leq i, j, k \leq 3$, one has

$$\left| \left[[\widetilde{\nabla}_{v_i}, \widetilde{\nabla}_{v_j}], \widetilde{\nabla}_{v_k} \right] f \right| (x, v) \lesssim \langle v \rangle^{\frac{3\gamma}{2}+1} |\nabla_v f|(x, v).$$

and

$$\left| \left[[\widetilde{\nabla}_{v_i}, \widetilde{\nabla}_{v_j}^*], \widetilde{\nabla}_{v_k} \right] f \right| (x, v) \lesssim \langle v \rangle^{\frac{3\gamma}{2}+1} |\nabla_v f|(x, v) + \langle v \rangle^{\frac{3\gamma}{2}} |f|(x, v).$$

(viii) For any $1 \leq i, j, k \leq 3$, one has

$$\left| \left[[\widetilde{\nabla}_{x_i}, \widetilde{\nabla}_{v_j}], \widetilde{\nabla}_{v_k} \right] f \right| (x, v) = \left| \left[[\widetilde{\nabla}_{x_i}, \widetilde{\nabla}_{v_j}^*], \widetilde{\nabla}_{v_k} \right] f \right| (x, v) \lesssim \langle v \rangle^{\frac{3\gamma}{2}+1} |\nabla_x f|(x, v).$$

Proof. Recall that we denote $\mathbf{B}(v) = (B_{ij})_{1 \leq i, j \leq 3}$ and that we have

$$\begin{aligned} \widetilde{\nabla}_{v_i} f &= B_{im} \partial_{v_m} f, & \widetilde{\nabla}_{x_i} f &= B_{im} \partial_{x_m} f, & \widetilde{\nabla}_{v_i}^* f &= -\partial_{v_m} (B_{im} f), \\ \widetilde{\nabla}_{x_i}^* f &= -B_{im} \partial_{x_m} f = -\widetilde{\nabla}_{x_i} f. \end{aligned}$$

(i) We have

$$\begin{aligned} [\widetilde{\nabla}_{v_i}, v_\ell \partial_{x_\ell}] f &= B_{im} \partial_{v_m} (v_\ell \partial_{x_\ell} f) - v_\ell \partial_{x_\ell} (B_{im} \partial_{v_m} f) \\ &= B_{i\ell} \partial_{x_\ell} f + B_{im} v_\ell \partial_{v_m} \partial_{x_\ell} f - v_\ell B_{im} \partial_{x_\ell} \partial_{v_m} f \\ &= \widetilde{\nabla}_{x_i} f. \end{aligned}$$

(ii) We easily compute

$$[\langle v \rangle^\alpha, \widetilde{\nabla}_{v_j}] f = -(\widetilde{\nabla}_{v_j} \langle v \rangle^\alpha) f$$

as well as

$$[\langle v \rangle^\alpha, \widetilde{\nabla}_{v_j}^*] f = (\widetilde{\nabla}_{v_j} \langle v \rangle^\alpha) f.$$

We conclude the proof by remarking that $\widetilde{\nabla}_{v_j} \langle v \rangle^\alpha = \alpha (\mathbf{B}(v)v)_j \langle v \rangle^{\alpha-2}$ and using that $|\mathbf{B}(v)v| \lesssim \langle v \rangle^{\frac{\gamma}{2}+1}$ thanks to the definition (4.38) of $\mathbf{B}(v)$.

(iii) We easily compute

$$\begin{aligned} [\widetilde{\nabla}_{v_i}, \widetilde{\nabla}_{v_j}] f &= B_{im} \partial_{v_m} (B_{j\ell} \partial_{v_\ell} f) - B_{j\ell} \partial_{v_\ell} (B_{im} \partial_{v_m} f) \\ &= B_{im} B_{j\ell} \partial_{v_m} \partial_{v_\ell} f + B_{im} (\partial_{v_m} B_{j\ell}) \partial_{v_\ell} f - B_{j\ell} B_{im} \partial_{v_\ell} \partial_{v_m} f - B_{j\ell} (\partial_{v_\ell} B_{im}) \partial_{v_m} f \\ &= B_{im} (\partial_{v_m} B_{j\ell}) \partial_{v_\ell} f - B_{j\ell} (\partial_{v_\ell} B_{im}) \partial_{v_m} f, \end{aligned}$$

and we conclude the proof using (4.42).

(iv) We have

$$\begin{aligned}
 [\widetilde{\nabla}_{v_i}, \widetilde{\nabla}_{v_j}^*]f &= -B_{im}\partial_{v_m}\partial_{v_\ell}(B_{j\ell}f) + \partial_{v_\ell}(B_{j\ell}B_{im}\partial_{v_m}f) \\
 &= -B_{im}B_{j\ell}\partial_{v_m}\partial_{v_\ell}f - B_{im}(\partial_{v_m}B_{j\ell})\partial_{v_\ell}f \\
 &\quad - B_{im}(\partial_{v_m}\partial_{v_\ell}B_{j\ell})f - B_{im}(\partial_{v_\ell}B_{j\ell})\partial_{v_m}f \\
 &\quad + B_{im}(\partial_{v_\ell}B_{j\ell})\partial_{v_m}f + B_{j\ell}(\partial_{v_\ell}B_{im})\partial_{v_m}f + B_{j\ell}B_{im}\partial_{v_\ell}\partial_{v_m}f \\
 &= -B_{im}(\partial_{v_m}B_{j\ell})\partial_{v_\ell}f + B_{j\ell}(\partial_{v_\ell}B_{im})\partial_{v_m}f - B_{im}(\partial_{v_m}\partial_{v_\ell}B_{j\ell})f.
 \end{aligned}$$

We can simplify last expression by relabelling the indices m and ℓ of the second term, which gives

$$[\widetilde{\nabla}_{v_i}, \widetilde{\nabla}_{v_j}^*]f = [B_{jm}(\partial_{v_m}B_{i\ell}) - B_{im}(\partial_{v_m}B_{j\ell})]\partial_{v_\ell}f - B_{im}(\partial_{v_m}\partial_{v_\ell}B_{j\ell})f.$$

We then conclude the proof by using using (4.42).

(v) We have

$$\begin{aligned}
 [\widetilde{\nabla}_{v_i}, \widetilde{\nabla}_{x_j}]f &= B_{im}\partial_{v_m}(B_{j\ell}\partial_{x_\ell}f) - B_{j\ell}\partial_{x_\ell}(B_{im}\partial_{v_m}f) \\
 &= B_{im}B_{j\ell}\partial_{v_m}\partial_{x_\ell}f + B_{im}(\partial_{v_m}B_{j\ell})\partial_{x_\ell}f - B_{j\ell}B_{im}\partial_{x_\ell}\partial_{v_m}f \\
 &= B_{im}(\partial_{v_m}B_{j\ell})\partial_{x_\ell}f
 \end{aligned}$$

as well as

$$\begin{aligned}
 -[\widetilde{\nabla}_{v_i}^*, \widetilde{\nabla}_{x_j}]f &= \partial_{v_m}(B_{im}B_{j\ell}\partial_{x_\ell}f) - B_{j\ell}\partial_{x_\ell}\partial_{v_m}(B_{im}f) \\
 &= B_{j\ell}\partial_{x_\ell}\partial_{v_k}(B_{im}f) + B_{im}(\partial_{v_m}B_{j\ell})\partial_{x_\ell}f - B_{j\ell}\partial_{x_\ell}\partial_{v_m}(B_{im}f) \\
 &= B_{im}(\partial_{v_m}B_{j\ell})\partial_{x_\ell}f.
 \end{aligned}$$

We then obtain the estimate by using (4.42).

(vi) Thanks to the proof of item (ii), we write

$$\begin{aligned}
 [\langle v \rangle^\alpha, \widetilde{\nabla}_{v_j}^*]\widetilde{\nabla}_{v_j}f &= (\widetilde{\nabla}_{v_j}\langle v \rangle^\alpha)\widetilde{\nabla}_{v_j}f \\
 &= \widetilde{\nabla}_{v_j}((\widetilde{\nabla}_{v_j}\langle v \rangle^\alpha)f) - (\widetilde{\nabla}_{v_j}(\widetilde{\nabla}_{v_j}\langle v \rangle^\alpha))f \\
 &= \widetilde{\nabla}_{v_j}[\langle v \rangle^\alpha, \widetilde{\nabla}_{v_j}^*]f - (\widetilde{\nabla}_{v_j}(\widetilde{\nabla}_{v_j}\langle v \rangle^\alpha))f.
 \end{aligned}$$

We conclude the proof by using $\widetilde{\nabla}_{v_j}\langle v \rangle^\alpha = \alpha(\mathbf{B}(v)v)_j\langle v \rangle^{\alpha-2}$ and the upper bound (4.42).

(vii) Thanks to the proof of item (iii), we first write

$$\begin{aligned}
 [\widetilde{\nabla}_{v_i}, \widetilde{\nabla}_{v_j}]\widetilde{\nabla}_{v_k}f &= B_{im}(\partial_{v_m}B_{j\ell})\partial_{v_\ell}(B_{kp}\partial_{v_p}f) - B_{j\ell}(\partial_{v_\ell}B_{im})\partial_{v_m}(B_{kp}\partial_{v_p}f) \\
 &= B_{im}(\partial_{v_m}B_{j\ell})\widetilde{\nabla}_{v_k}\partial_{v_\ell}f + B_{im}(\partial_{v_m}B_{j\ell})(\partial_{v_\ell}B_{kp})\partial_{v_p}f \\
 &\quad - B_{j\ell}(\partial_{v_\ell}B_{im})\widetilde{\nabla}_{v_k}\partial_{v_m}f - B_{j\ell}(\partial_{v_\ell}B_{im})(\partial_{v_m}B_{kp})\partial_{v_p}f.
 \end{aligned}$$

We then obtain

$$\begin{aligned}
 [\widetilde{\nabla}_{v_i}, \widetilde{\nabla}_{v_j}] \widetilde{\nabla}_{v_k} f &= \widetilde{\nabla}_{v_k} [B_{im}(\partial_{v_m} B_{j\ell}) \partial_{v_\ell} f] - (\widetilde{\nabla}_{v_k} [B_{im}(\partial_{v_m} B_{j\ell})]) \partial_{v_\ell} f \\
 &\quad - \widetilde{\nabla}_{v_k} [B_{j\ell}(\partial_{v_\ell} B_{im}) \partial_{v_m} f] + (\widetilde{\nabla}_{v_k} [B_{j\ell}(\partial_{v_\ell} B_{im})]) \partial_{v_m} f \\
 &\quad + [B_{im}(\partial_{v_m} B_{j\ell})(\partial_{v_\ell} B_{kp}) - B_{j\ell}(\partial_{v_\ell} B_{im})(\partial_{v_m} B_{kp})] \partial_{v_p} f \\
 &= \widetilde{\nabla}_{v_k} [\widetilde{\nabla}_{v_i}, \widetilde{\nabla}_{v_j}] f - (\widetilde{\nabla}_{v_k} [B_{im}(\partial_{v_m} B_{j\ell})]) \partial_{v_\ell} f + (\widetilde{\nabla}_{v_k} [B_{j\ell}(\partial_{v_\ell} B_{im})]) \partial_{v_m} f \\
 &\quad + [B_{im}(\partial_{v_m} B_{j\ell})(\partial_{v_\ell} B_{kp}) - B_{j\ell}(\partial_{v_\ell} B_{im})(\partial_{v_m} B_{kp})] \partial_{v_p} f
 \end{aligned}$$

and we conclude the proof of the first estimate using (4.42). The second estimate is obtained in a similar way by using the computation of item (iv), thus we omit it.

(viii) Thanks to the proof of item (v), we write

$$\begin{aligned}
 [\widetilde{\nabla}_{x_i}, \widetilde{\nabla}_{v_j}] \widetilde{\nabla}_{v_k} f &= -B_{jm}(\partial_{v_m} B_{i\ell}) \partial_{x_\ell} \widetilde{\nabla}_{v_k} f \\
 &= -\widetilde{\nabla}_{v_k} (B_{jm}(\partial_{v_m} B_{i\ell}) \partial_{x_\ell} f) - (\widetilde{\nabla}_{v_k} [B_{jm}(\partial_{v_m} B_{i\ell})]) \partial_{x_\ell} f \\
 &= \widetilde{\nabla}_{v_k} [\widetilde{\nabla}_{x_i}, \widetilde{\nabla}_{v_j}^*] f - (\widetilde{\nabla}_{v_k} [B_{jm}(\partial_{v_m} B_{i\ell})]) \partial_{x_\ell} f.
 \end{aligned}$$

The estimate then follows from (4.42). □

Using the above result, we shall now compute some commutators related to the L_1 term defined in (4.51) of the collision operator L .

Lemma 4.11. *There holds*

(i) *For any $\alpha \in \mathbb{R}$, one has*

$$[\langle v \rangle^\alpha, L_1] f = -\widetilde{\nabla}_{v_\ell}^* [\langle v \rangle^\alpha, \widetilde{\nabla}_{v_\ell}] f - \widetilde{\nabla}_{v_\ell} [\langle v \rangle^\alpha, \widetilde{\nabla}_{v_\ell}^*] f - [[\langle v \rangle^\alpha, \widetilde{\nabla}_{v_\ell}^*], \widetilde{\nabla}_{v_\ell}] f.$$

(ii) *For any $1 \leq k \leq 3$, one has*

$$[\widetilde{\nabla}_{v_k}, L_1] f = -\widetilde{\nabla}_{v_\ell}^* [\widetilde{\nabla}_{v_k}, \widetilde{\nabla}_{v_\ell}] f - \widetilde{\nabla}_{v_\ell} [\widetilde{\nabla}_{v_k}, \widetilde{\nabla}_{v_\ell}^*] f - [[\widetilde{\nabla}_{v_k}, \widetilde{\nabla}_{v_\ell}^*], \widetilde{\nabla}_{v_\ell}] f - (\widetilde{\nabla}_{v_k} \psi) f.$$

(iii) *For any $1 \leq k \leq 3$, one has*

$$[\widetilde{\nabla}_{x_k}, L_1] f = -\widetilde{\nabla}_{v_\ell}^* [\widetilde{\nabla}_{x_k}, \widetilde{\nabla}_{v_\ell}] f - \widetilde{\nabla}_{v_\ell} [\widetilde{\nabla}_{x_k}, \widetilde{\nabla}_{v_\ell}^*] f - [[\widetilde{\nabla}_{x_k}, \widetilde{\nabla}_{v_\ell}^*], \widetilde{\nabla}_{v_\ell}] f$$

and

$$[\langle v \rangle^{\frac{\gamma}{2}} \partial_{x_k}, L_1] f = -\widetilde{\nabla}_{v_\ell}^* [\langle v \rangle^{\frac{\gamma}{2}} \partial_{x_k}, \widetilde{\nabla}_{v_\ell}] f - \widetilde{\nabla}_{v_\ell} [\langle v \rangle^{\frac{\gamma}{2}} \partial_{x_k}, \widetilde{\nabla}_{v_\ell}^*] f - [[\langle v \rangle^{\frac{\gamma}{2}} \partial_{x_k}, \widetilde{\nabla}_{v_\ell}^*], \widetilde{\nabla}_{v_\ell}] f.$$

Proof. (i) We first write

$$\langle v \rangle^\alpha (L_1 f) = -\langle v \rangle^\alpha \widetilde{\nabla}_{v_\ell}^* \widetilde{\nabla}_{v_\ell} f - \psi \langle v \rangle^\alpha f$$

and

$$L_1(\langle v \rangle^\alpha f) = -\widetilde{\nabla}_{v_\ell}^* \widetilde{\nabla}_{v_\ell}(\langle v \rangle^\alpha f) - \psi \langle v \rangle^\alpha f.$$

We now observe that

$$\begin{aligned} \langle v \rangle^\alpha \widetilde{\nabla}_{v_\ell}^* \widetilde{\nabla}_{v_\ell} f &= \widetilde{\nabla}_{v_\ell}^* \langle v \rangle^\alpha \widetilde{\nabla}_{v_\ell} f + [\langle v \rangle^\alpha, \widetilde{\nabla}_{v_\ell}^*] \widetilde{\nabla}_{v_\ell} f \\ &= \widetilde{\nabla}_{v_\ell}^* \widetilde{\nabla}_{v_\ell}(\langle v \rangle^\alpha f) + \widetilde{\nabla}_{v_\ell}^* [\langle v \rangle^\alpha, \widetilde{\nabla}_{v_\ell}] f + \widetilde{\nabla}_{v_\ell} [\langle v \rangle^\alpha, \widetilde{\nabla}_{v_\ell}^*] f + [[\langle v \rangle^\alpha, \widetilde{\nabla}_{v_\ell}^*], \widetilde{\nabla}_{v_\ell}] f, \end{aligned}$$

which completes the proof.

(ii) We first compute

$$\widetilde{\nabla}_{v_k} (L_1 f) = -\widetilde{\nabla}_{v_k} \widetilde{\nabla}_{v_\ell}^* \widetilde{\nabla}_{v_\ell} f - \widetilde{\nabla}_{v_k} (\psi f)$$

and

$$L_1(\widetilde{\nabla}_{v_k} f) = -\widetilde{\nabla}_{v_\ell}^* \widetilde{\nabla}_{v_\ell}(\widetilde{\nabla}_{v_k} f) - \psi \widetilde{\nabla}_{v_k} f.$$

We conclude the proof by observing that

$$\begin{aligned} \widetilde{\nabla}_{v_k} \widetilde{\nabla}_{v_\ell}^* \widetilde{\nabla}_{v_\ell} f &= \widetilde{\nabla}_{v_\ell}^* \widetilde{\nabla}_{v_k} \widetilde{\nabla}_{v_\ell} f + [\widetilde{\nabla}_{v_k}, \widetilde{\nabla}_{v_\ell}^*] \widetilde{\nabla}_{v_\ell} f \\ &= \widetilde{\nabla}_{v_\ell}^* \widetilde{\nabla}_{v_\ell} \widetilde{\nabla}_{v_k} f + \widetilde{\nabla}_{v_\ell}^* [\widetilde{\nabla}_{v_k}, \widetilde{\nabla}_{v_\ell}] f + \widetilde{\nabla}_{v_\ell} [\widetilde{\nabla}_{v_k}, \widetilde{\nabla}_{v_\ell}^*] f + [[\widetilde{\nabla}_{v_k}, \widetilde{\nabla}_{v_\ell}^*], \widetilde{\nabla}_{v_\ell}] f \end{aligned}$$

and writing $\widetilde{\nabla}_{v_k}(\psi f) = \psi \widetilde{\nabla}_{v_k} f + (\widetilde{\nabla}_{v_k} \psi) f$.

(iii) We compute

$$\widetilde{\nabla}_{x_k} (L_1 f) = -\widetilde{\nabla}_{x_k} \widetilde{\nabla}_{v_\ell}^* \widetilde{\nabla}_{v_\ell} f - \psi \widetilde{\nabla}_{x_k} f$$

and

$$L_1(\widetilde{\nabla}_{x_k} f) = -\widetilde{\nabla}_{v_\ell}^* \widetilde{\nabla}_{v_\ell} \widetilde{\nabla}_{x_k} f - \psi \widetilde{\nabla}_{x_k} f.$$

We now observe that

$$\begin{aligned} \widetilde{\nabla}_{x_k} \widetilde{\nabla}_{v_\ell}^* \widetilde{\nabla}_{v_\ell} f &= \widetilde{\nabla}_{v_\ell}^* \widetilde{\nabla}_{x_k} \widetilde{\nabla}_{v_\ell} f + [\widetilde{\nabla}_{x_k}, \widetilde{\nabla}_{v_\ell}^*] \widetilde{\nabla}_{v_\ell} f \\ &= \widetilde{\nabla}_{v_\ell}^* \widetilde{\nabla}_{v_\ell} \widetilde{\nabla}_{x_k} f + \widetilde{\nabla}_{v_\ell}^* [\widetilde{\nabla}_{x_k}, \widetilde{\nabla}_{v_\ell}] f + \widetilde{\nabla}_{v_\ell} [\widetilde{\nabla}_{x_k}, \widetilde{\nabla}_{v_\ell}^*] f + [[\widetilde{\nabla}_{x_k}, \widetilde{\nabla}_{v_\ell}^*], \widetilde{\nabla}_{v_\ell}] f \end{aligned}$$

which gives the first estimate.

The computation for $[\langle v \rangle^{\frac{\gamma}{2}} \partial_{x_k}, L_1]$ can be obtained in a similar fashion, thus we omit it. \square

Lemma 4.12. For any suitable function $f = f(v)$ and any $v \in \mathbb{R}^3$ there holds:

(i) For any $i, j \in \{1, 2, 3\}$ one has

$$|(a_{ij} * f)(v)| + |(a_{ij} * f)(v)v_i| + |(a_{ij} * f)(v)v_iv_j| \lesssim \langle v \rangle^{\gamma+2} \|\langle v \rangle^7 f\|_{L_v^2}$$

(ii) For any $i, j, \ell \in \{1, 2, 3\}$ one has

$$|(\partial_{v_\ell} a_{ij} * f)(v)| + |(b_i * f)(v)| \lesssim \langle v \rangle^{\gamma+1} \|\langle v \rangle^4 f\|_{L_v^2}$$

and

$$|(\partial_{v_\ell} a_{ij} * f)(v)v_i| + |(\partial_{v_\ell} a_{ij} * f)(v)v_iv_j| \lesssim \langle v \rangle^{\gamma+2} \|\langle v \rangle^8 f\|_{L_v^2}$$

(iii) If $\gamma \geq 0$, for any $\ell \in \{1, 2, 3\}$ one has

$$|(c * f)(v)| + |(\partial_{v_\ell} b_i * f)(v)| \lesssim \langle v \rangle^\gamma \|\langle v \rangle^3 f\|_{L_v^2}.$$

(iv) If $\gamma \in [-2, 0)$, for any $\ell \in \{1, 2, 3\}$ one has

$$|(c * f)(v)| + |(\partial_{v_\ell} b_i * f)(v)| \lesssim \langle v \rangle^\gamma \|\langle v \rangle^3 f\|_{L_v^4}.$$

Proof. All estimates in point (i) come from [26, Lemma 3.4]. The estimates on the term $|(b_i * f)(v)|$ in point (ii) and on the term $|(c * f)(v)|$ in (iii)–(iv) also come from [26, Lemma 3.4], and the proof of the estimates on $|(\partial_{v_\ell} a_{ij} * f)(v)|$ in (ii) and on $|(\partial_{v_\ell} b_i * f)(v)|$ in (iii)–(iv) follow the same lines since $|\partial_{v_\ell} a_{ij}| \lesssim |v|^{\gamma+1}$ and $|\partial_{v_\ell} b_i| \lesssim |v|^\gamma$.

We thus only prove the second estimate in (ii). Using that $a_{ij}(v - v_*)v_iv_j = a_{ij}(v - v_*)(v_*)_i(v_*)_j$, we observe that

$$\begin{aligned} (\partial_{v_\ell} a_{ij} * f)(v)v_iv_j &= (a_{ij} * \partial_{v_\ell} f)(v)v_iv_j \\ &= (a_{ij} * v_iv_j \partial_{v_\ell} f)(v) \\ &= (\partial_{v_\ell} a_{ij} * [v_iv_j f])(v) - (a_{ij} * [\partial_{v_\ell}(v_iv_j) f])(v). \end{aligned}$$

Using the the estimate in (i) and the first estimate in (ii), we thus deduce

$$|(\partial_{v_\ell} a_{ij} * f)(v)v_iv_j| \lesssim \langle v \rangle^{\gamma+1} \|\langle v \rangle^6 f\|_{L_v^2} + \langle v \rangle^{\gamma+2} \|\langle v \rangle^8 f\|_{L_v^2}.$$

□

Lemma 4.13. For any function $f = f(v)$ smooth enough and $\alpha \in \mathbb{R}$, there holds:

$$\|(c * (\sqrt{M}f)) \langle v \rangle^\alpha \sqrt{M}\|_{L_v^2} \lesssim \|M^{\frac{1}{4}} f\|_{L_v^2}.$$

Proof. See [26, Proof of Lemma 2.12].

□

We are now able to obtain some upper bounds on the term L_2 defined in (4.52) of the collision operator L as follows.

Lemma 4.14. *For any function $f = f(x, v)$ smooth enough, there holds:*

(i) *For any $\alpha \in \mathbb{R}$, one has*

$$\|\langle v \rangle^\alpha L_2 f\|_{L^2_{x,v}} \lesssim \|M^{\frac{1}{4}} f\|_{L^2_{x,v}}.$$

(ii) *For any $1 \leq k \leq 3$, one has*

$$\|\widetilde{\nabla}_{v_k}(L_2 f)\|_{L^2_{x,v}} \lesssim \|M^{\frac{1}{4}} f\|_{L^2_{x,v}} + \|M^{\frac{1}{4}} \widetilde{\nabla}_v f\|_{L^2_{x,v}}.$$

(iii) *For any $1 \leq k \leq 3$, one has*

$$\|\widetilde{\nabla}_{x_k}(L_2 f)\|_{L^2_{x,v}} \lesssim \|M^{\frac{1}{4}} \widetilde{\nabla}_x f\|_{L^2_{x,v}}.$$

Proof. (i) We write

$$\langle v \rangle^\alpha L_2 f = - \left\{ (a_{ij} *_v \sqrt{M} f) v_i v_j - (a_{ii} *_v \sqrt{M} f) + (c *_v \sqrt{M} f) \right\} \langle v \rangle^\alpha \sqrt{M}$$

from which, thanks to Lemma 4.12, we deduce

$$|\langle v \rangle^\alpha L_2 f| \lesssim \|M^{\frac{1}{4}} f\|_{L^2_v} \langle v \rangle^{\gamma+\alpha+2} \sqrt{M} + |(c *_v \sqrt{M} f)| \langle v \rangle^\alpha \sqrt{M}.$$

We conclude the proof by taking the L^2 norm of the last estimate and using Lemma 4.13.

(ii) Writing $\widetilde{\nabla}_{v_k} = B_{kl} \partial_{v_\ell}$ we compute

$$\begin{aligned} \widetilde{\nabla}_{v_k}(L_2 f) &= -B_{kl} \left\{ (a_{ij} *_v \partial_{v_\ell}(\sqrt{M} f)) v_i v_j + (a_{ij} *_v \sqrt{M} f) \partial_{v_\ell}(v_i v_j) \right. \\ &\quad \left. - (a_{ii} *_v \partial_{v_\ell}(\sqrt{M} f)) + (c *_v \partial_{v_\ell}(\sqrt{M} f)) \right\} \sqrt{M} \\ &\quad - \left\{ (a_{ij} *_v \sqrt{M} f) v_i v_j - (a_{ii} *_v \sqrt{M} f) + (c *_v \sqrt{M} f) \right\} \widetilde{\nabla}_{v_k} \sqrt{M}. \end{aligned}$$

Thanks to Lemmas 4.12 and 4.13, we obtain that

$$\|\widetilde{\nabla}_{v_k}(L_2 f)(x, \cdot)\|_{L^2_v} \lesssim \|M^{\frac{1}{4}} f(x, \cdot)\|_{L^2_v} + \|M^{\frac{1}{4}} \widetilde{\nabla}_v f(x, \cdot)\|_{L^2_v},$$

and we conclude by integrating in x this last estimate.

(iii) Writing $\widetilde{\nabla}_{x_k} = B_{kl} \partial_{x_\ell}$ we compute

$$\widetilde{\nabla}_{x_k}(L_2 f) = -B_{kl} \left\{ (a_{ij} *_v \sqrt{M} \partial_{x_\ell} f) v_i v_j - (a_{ii} *_v \sqrt{M} \partial_{x_\ell} f) + (c *_v \sqrt{M} \partial_{x_\ell} f) \right\} \sqrt{M},$$

and we obtain the wanted result thanks to Lemmas 4.12 and 4.13. \square

3 Estimates on the linearized problem

We recall that the functional spaces \mathcal{X} , \mathcal{Y}_1 and $\mathcal{Z}_1^\varepsilon$ are respectively defined in (4.19), (4.20) and (4.45), that the operator Λ_ε is given in (4.11) and that Π is the projector onto the kernel of Λ_ε (see (4.17)). We consider $U^\varepsilon(t)$ the semigroup associated with Λ_ε and study its decay and regularization properties.

Theorem 4.15. *For any $\sigma \in (0, \sigma_0)$, we have:*

$$\|U^\varepsilon(t)(\text{Id} - \Pi)\|_{\mathcal{X} \rightarrow \mathcal{X}} \lesssim e^{-\sigma t},$$

$$\|U^\varepsilon(t)(\text{Id} - \Pi)\|_{\mathcal{X} \rightarrow \mathcal{Y}_1} \lesssim \frac{e^{-\sigma t}}{\min(1, \sqrt{t})} \quad \text{and} \quad \|U^\varepsilon(t)(\text{Id} - \Pi)\|_{\mathcal{X} \rightarrow \mathcal{Z}_1^\varepsilon} \lesssim \frac{e^{-\sigma t}}{\min(1, t^{3/2})},$$

where σ_0 is defined in Proposition 4.16.

Our proof is based on hypocoercivity tricks and thus on direct energy estimates on the whole problem $\partial_t f = \Lambda_\varepsilon f$. Our method to prove this theorem is of particular interest when one wants to extend the analysis to the whole nonlinear problem (see the paragraph at the beginning of Section 4). Indeed, it is based on a micro-macro decomposition of the solution and thus allows to identify the different behaviors of the microscopic and macroscopic parts of the solution. We will use this approach in Section 4 in which we develop a Cauchy theory for the nonlinear Landau equation as well as some regularization estimates on it.

Let us finally notice that we are actually able to prove Theorem 4.15 by using a splitting of Λ_ε as presented in Section 5. By establishing nice estimates on each part of the splitting, we are then able to recover the wanted estimates on the whole semigroup $U^\varepsilon(t)$ thanks to Duhamel formula. Such an analysis does not allow us to develop our Cauchy theory and regularization estimates for the nonlinear problem but will be useful to study our problem of hydrodynamical limit, we thus postpone it to Section 5.

3.1 Hypocoercivity estimates

In this part, we state some hypocoercivity results for our linearized operator Λ_ε defined in (4.11). The first one provides a result of hypocoercivity in $L_{x,v}^2$ and the proof is a mere adaptation of the one provided in [13, Theorem 5.1] in the more complicated case of bounded domains with various boundary conditions. For sake of completeness, we give the proof of Proposition 4.16 in Appendix (Section 7).

Proposition 4.16. *There exists a norm $\|\cdot\|_{L_{x,v}^2}$ on $L_{x,v}^2$ (with associated scalar product $\langle\langle \cdot, \cdot \rangle\rangle_{L_{x,v}^2}$) equivalent to the standard norm $\|\cdot\|_{L_{x,v}^2}$ which satisfies the following property: For any $f \in \text{Dom } \Lambda_\varepsilon \cap (\text{Ker } \Lambda_\varepsilon)^\perp$,*

$$\langle\langle \Lambda_\varepsilon f, f \rangle\rangle_{L_{x,v}^2} \leq -\sigma_0 \|f\|_{L_{x,v}^2}^2 - \kappa_0 \|f\|_{L_x^2(H_{v,*}^1)}^2 - \frac{\kappa_0}{\varepsilon^2} \|f^\perp\|_{L_x^2(H_{v,*}^1)}^2,$$

for some constructive constants $0 < \sigma_0 < \sigma_L$ (where σ_L is defined in (4.43)), $\kappa_0 > 0$ and where f^\perp is defined in (4.13).

Roughly speaking, the norm $\| \cdot \|_{L_{x,v}^2}$ is of the following form

$$\| \| f \| \|_{L_{x,v}^2}^2 = \| f \|_{L_{x,v}^2}^2 + \varepsilon \sum_{i=1}^3 \eta_i \left\langle \partial_{x_i} \Delta_x^{-1} \pi f, \tilde{\pi}_i f \right\rangle_{L_x^2} \quad (4.53)$$

where $\tilde{\pi}_i : L_{x,v}^2 \rightarrow L_x^2$ is some suitable moment operator, the inverse laplacian Δ_x^{-1} is suitably defined and the constants η_i are chosen to be small enough (see (4.166) in the proof). The norm $\| \cdot \|_{L_{x,v}^2}$ thus depends on ε but is equivalent to the usual norm $\| \cdot \|_{L_{x,v}^2}$ uniformly in ε , this explains the fact that we do not mention the dependency in ε in our notation.

In the following proposition, we provide a result of hypocoercivity in \mathcal{X} . Notice that obtaining hypocoercivity in $H_x^3 L_v^2$ is a straightforward consequence of the previous proposition since derivatives in x commute with Λ_ε . Due to the presence of additional weights in the definition of the space \mathcal{X} , we have to exhibit a new norm equivalent to the usual one for which we can recover a suitable energy estimate.

Proposition 4.17. *There exists a norm $\| \cdot \|_{\mathcal{X}}$ on \mathcal{X} (with associated scalar product denoted by $\langle \cdot, \cdot \rangle_{\mathcal{X}}$) equivalent to the standard norm $\| \cdot \|_{\mathcal{X}}$ which satisfies the following property: For any $f \in \text{Dom } \Lambda_\varepsilon \cap (\text{Ker } \Lambda_\varepsilon)^\perp$,*

$$\langle \Lambda_\varepsilon f, f \rangle_{\mathcal{X}} \leq -\sigma \| f \|_{\mathcal{X}}^2 - \kappa \| f \|_{\mathcal{X}_1}^2 - \frac{\kappa}{\varepsilon^2} \| f^\perp \|_{\mathcal{X}_1}^2, \quad (4.54)$$

for any $0 < \sigma < \sigma_0$ (where σ_0 is defined in Proposition 4.16) and for some $\kappa > 0$. As a consequence there holds, for all $t \geq 0$,

$$\| U^\varepsilon(t)(\text{Id} - \Pi) \|_{\mathcal{X} \rightarrow \mathcal{X}} \leq C e^{-\sigma t}. \quad (4.55)$$

Proof. We define the inner product $\langle \cdot, \cdot \rangle_{\mathcal{X}}$ on \mathcal{X} by

$$\begin{aligned} \langle \langle f, g \rangle \rangle_{\mathcal{X}} := & \sum_{i=0}^2 \left(\delta \langle \langle v \rangle^{(3-i)(\frac{\gamma}{2}+1)} \nabla_x^i f^\perp, \langle v \rangle^{(3-i)(\frac{\gamma}{2}+1)} \nabla_x^i g^\perp \rangle_{L_{x,v}^2} + \langle \langle \nabla_x^i f, \nabla_x^i g \rangle \rangle_{L_{x,v}^2} \right) \\ & + \langle \langle \nabla_x^3 f, \nabla_x^3 g \rangle \rangle_{L_{x,v}^2} \end{aligned} \quad (4.56)$$

so that its associated norm is given by

$$\| \| f \| \|_{\mathcal{X}}^2 := \sum_{i=0}^2 \left(\delta \| \langle v \rangle^{(3-i)(\frac{\gamma}{2}+1)} \nabla_x^i f^\perp \|_{L_{x,v}^2}^2 + \| \nabla_x^i f \|_{L_{x,v}^2}^2 \right) + \| \nabla_x^3 f \|_{L_{x,v}^2}^2 \quad (4.57)$$

for some constant $\delta \in (0, 1)$ to be chosen small enough, and where $\langle \cdot, \cdot \rangle_{L_{x,v}^2}$ and $\| \cdot \|_{L_{x,v}^2}$ are defined in Proposition 4.16 (see (4.166)). We first observe that this norm is equivalent to the norm $\| \cdot \|_{\mathcal{X}}$.

Let $\sigma' \in (\sigma, \sigma_0)$ be fixed and $f \in \mathcal{X} \cap (\text{Ker } \Lambda_\varepsilon)^\perp$. We shall prove

$$\langle\langle \Lambda_\varepsilon f, f \rangle\rangle_{\mathcal{X}} \leq -\sigma' \|f\|_{\mathcal{X}}^2 - \frac{\kappa'}{\varepsilon^2} \|f^\perp\|_{\mathcal{X}_1}^2 \quad (4.58)$$

for some constant $\kappa' > 0$, which readily implies (4.54) with some constant $\kappa \in (0, \min(\sigma' - \sigma, \kappa'))$ by decomposing the first term with $\sigma' = \sigma + (\sigma' - \sigma)$ and using the fact that $\|\pi f\|_{\mathcal{X}_1} \lesssim \|\pi f\|_{\mathcal{X}}$. Estimate (4.55) is then a direct consequence of (4.54).

Step 1. We first compute

$$\begin{aligned} \langle\langle \Lambda_\varepsilon f, f \rangle\rangle_{\mathcal{X}} &= \sum_{i=0}^2 \delta \left\langle \langle v \rangle^{(3-i)(\frac{\gamma}{2}+1)} \nabla_x^i (\Lambda_\varepsilon f)^\perp, \langle v \rangle^{(3-i)(\frac{\gamma}{2}+1)} \nabla_x^i f^\perp \right\rangle_{L_{x,v}^2} \\ &\quad + \sum_{i=0}^3 \left\langle \langle \nabla_x^i \Lambda_\varepsilon f, \nabla_x^i f \rangle \right\rangle_{L_{x,v}^2}, \end{aligned}$$

and we observe that, thanks to Proposition 4.16 and the fact that ∇_x commutes with Λ_ε , we already have

$$\begin{aligned} &\sum_{i=0}^3 \left\langle \langle \nabla_x^i \Lambda_\varepsilon f, \nabla_x^i f \rangle \right\rangle_{L_{x,v}^2} \\ &\leq - \sum_{i=0}^3 \left\{ \sigma_0 \|\nabla_x^i f\|_{L_{x,v}^2}^2 + \kappa_0 \|\nabla_x^i f\|_{L_x^2(H_{v,*}^1)}^2 + \frac{\kappa_0}{\varepsilon^2} \|\nabla_x^i f^\perp\|_{L_x^2(H_{v,*}^1)}^2 \right\}. \end{aligned}$$

Step 2. We first observe that

$$(\Lambda_\varepsilon f)^\perp = (\text{Id} - \pi) \Lambda_\varepsilon f = \frac{1}{\varepsilon^2} (\text{Id} - \pi) Lf - \frac{1}{\varepsilon} (\text{Id} - \pi) (v \cdot \nabla_x f),$$

from which we obtain, since $\pi L = 0$ and $Lf = Lf^\perp$, that

$$(\Lambda_\varepsilon f)^\perp = \frac{1}{\varepsilon^2} Lf^\perp - \frac{1}{\varepsilon} \left\{ v \cdot \nabla_x f^\perp + v \cdot \nabla_x (\pi f) - \pi (v \cdot \nabla_x f) \right\}. \quad (4.59)$$

We therefore get, for any $i \in \{0, 1, 2\}$ and using that the transport operator $v \cdot \nabla_x$ is skew-adjoint,

$$\left\langle \langle v \rangle^{(3-i)(\frac{\gamma}{2}+1)} \nabla_x^i (\Lambda_\varepsilon f)^\perp, \langle v \rangle^{(3-i)(\frac{\gamma}{2}+1)} \nabla_x^i f^\perp \right\rangle_{L_{x,v}^2} =: \frac{1}{\varepsilon^2} R_1^i - \frac{1}{\varepsilon} R_2^i + \frac{1}{\varepsilon} R_3^i$$

with

$$\begin{aligned} R_1^i &:= \left\langle \langle v \rangle^{(3-i)(\frac{\gamma}{2}+1)} \nabla_x^i Lf^\perp, \langle v \rangle^{(3-i)(\frac{\gamma}{2}+1)} \nabla_x^i f^\perp \right\rangle_{L_{x,v}^2}, \\ R_2^i &:= \left\langle \langle v \rangle^{(3-i)(\frac{\gamma}{2}+1)} \nabla_x^i (v \cdot \nabla_x (\pi f)), \langle v \rangle^{(3-i)(\frac{\gamma}{2}+1)} \nabla_x^i f^\perp \right\rangle_{L_{x,v}^2}, \\ R_3^i &:= \left\langle \langle v \rangle^{(3-i)(\frac{\gamma}{2}+1)} \nabla_x^i (\pi (v \cdot \nabla_x f)), \langle v \rangle^{(3-i)(\frac{\gamma}{2}+1)} \nabla_x^i f^\perp \right\rangle_{L_{x,v}^2}, \end{aligned}$$

and we treat each term separately. For simplicity we denote $\omega_i = \langle v \rangle^{(3-i)(\frac{\gamma}{2}+1)}$ in the sequel.

Step 3. We deal with the term R_1^i . Since ∇_x commutes with L we have

$$\begin{aligned} R_1^i &= \left\langle L(\omega_i \nabla_x^i f^\perp), \omega_i \nabla_x^i f^\perp \right\rangle_{L_{x,v}^2} + \left\langle [\omega_i, L_1] \nabla_x^i f^\perp, \omega_i \nabla_x^i f^\perp \right\rangle_{L_{x,v}^2} \\ &\quad + \left\langle [\omega_i, L_2] \nabla_x^i f^\perp, \omega_i \nabla_x^i f^\perp \right\rangle_{L_{x,v}^2} \\ &=: R_{11}^i + R_{12}^i + R_{13}^i, \end{aligned}$$

where we recall that L_1 and L_2 are defined in (4.51) and (4.52), respectively. Thanks to the spectral gap estimate (4.43), we have

$$\begin{aligned} R_{11}^i &\leq -\sigma_L \|\omega_i \nabla_x^i f^\perp - \pi(\omega_i \nabla_x^i f^\perp)\|_{L_x^2(H_{v,*}^1)}^2 \\ &\leq -\sigma_L \|\omega_i \nabla_x^i f^\perp\|_{L_x^2(H_{v,*}^1)}^2 + C \|\nabla_x^i f^\perp\|_{L_{x,v}^2}^2, \end{aligned}$$

for some constant $C > 0$. From Lemma 4.11 we get

$$\begin{aligned} R_{12}^i &= -\left\langle [\omega_i, \widetilde{\nabla}_{v_\ell}] \nabla_x^i f^\perp, \widetilde{\nabla}_{v_\ell}(\omega_i \nabla_x^i f^\perp) \right\rangle_{L_{x,v}^2} - \left\langle [\omega_i, \widetilde{\nabla}_{v_\ell}^*] \nabla_x^i f^\perp, \widetilde{\nabla}_{v_\ell}^*(\omega_i \nabla_x^i f^\perp) \right\rangle_{L_{x,v}^2} \\ &\quad - \left\langle [\omega_i, \widetilde{\nabla}_{v_\ell}^*], \widetilde{\nabla}_{v_\ell} \right\rangle \nabla_x^i f^\perp, \omega_i \nabla_x^i f^\perp \Big|_{L_{x,v}^2}. \end{aligned}$$

Using Lemma 4.10 and observing that $\|\widetilde{\nabla}_{v_\ell} g\|_{L_{x,v}^2} + \|\widetilde{\nabla}_{v_\ell}^* g\|_{L_{x,v}^2} \lesssim \|g\|_{L_x^2(H_{v,*}^1)}$, we then obtain

$$\begin{aligned} |R_{12}^i| &\leq C \|\omega_i \nabla_x^i f^\perp\|_{L_x^2(H_{v,*}^1)} \|\langle v \rangle^{\frac{\gamma}{2}-1} \omega_i \nabla_x^i f^\perp\|_{L_{x,v}^2} + C \|\langle v \rangle^{\frac{\gamma}{2}-\frac{1}{2}} \omega_i \nabla_x^i f^\perp\|_{L_{x,v}^2}^2 \\ &\leq (\sigma_L - \sigma'') \|\omega_i \nabla_x^i f^\perp\|_{L_x^2(H_{v,*}^1)}^2 + C \|\nabla_x^i f^\perp\|_{L_{x,v}^2}^2 \end{aligned}$$

for any $\sigma'' \in (\sigma', \sigma_0)$, where we have used Young's inequality in last line. For the term R_{13}^i , we use Lemma 4.14 to obtain

$$|R_{13}^i| \leq C \|\nabla_x^i f^\perp\|_{L_{x,v}^2}^2.$$

Gathering previous estimates we finally get

$$R_1^i \leq -\sigma'' \|\omega_i \nabla_x^i f^\perp\|_{L_x^2(H_{v,*}^1)}^2 + C \|\nabla_x^i f^\perp\|_{L_{x,v}^2}^2.$$

Step 4. We deal with the terms R_2^i and R_3^i . Observing that

$$\|\omega_i \nabla_x^i (v \cdot \nabla_x (\pi f))\|_{L_{x,v}^2} + \|\omega_i \nabla_x^i (\pi (v \cdot \nabla_x f))\|_{L_{x,v}^2} \lesssim \|\nabla_x^{i+1} f\|_{L_{x,v}^2},$$

we obtain

$$\begin{aligned} |R_2^i| + |R_3^i| &\leq C \|\nabla_x^{i+1} f\|_{L_{x,v}^2} \|\omega_i \nabla_x^i f^\perp\|_{L_{x,v}^2} \\ &\leq \frac{(\sigma'' - \sigma''')}{\varepsilon} \|\omega_i \nabla_x^i f^\perp\|_{L_x^2(H_{v,*}^1)}^2 + C \varepsilon \|\nabla_x^{i+1} f\|_{L_{x,v}^2}^2, \end{aligned}$$

for any $\sigma''' \in (\sigma', \sigma'')$, where we have used Young's inequality and that $\|\cdot\|_{L_{x,v}^2}$ and $\|\|\cdot\|\|_{L_{x,v}^2}$ are equivalent in last line.

Step 5. Gathering previous estimates we then obtain

$$\begin{aligned} \langle\langle \Lambda_\varepsilon f, f \rangle\rangle_{\mathcal{X}} &\leq - \sum_{i=0}^3 \left\{ \sigma_0 \|\|\nabla_x^i f\|\|_{L_{x,v}^2}^2 + \kappa_0 \|\nabla_x^i f\|_{L_x^2(H_{v,*}^1)}^2 + \frac{\kappa_0}{\varepsilon^2} \|\nabla_x^i f^\perp\|_{L_x^2(H_{v,*}^1)}^2 \right\} \\ &\quad + \sum_{i=0}^2 \left\{ -\frac{\sigma''' \delta}{\varepsilon^2} \|\omega_i \nabla_x^i f^\perp\|_{L_x^2(H_{v,*}^1)}^2 + \frac{C\delta}{\varepsilon^2} \|\nabla_x^i f^\perp\|_{L_{x,v}^2}^2 + C\delta \|\|\nabla_x^{i+1} f\|\|_{L_{x,v}^2}^2 \right\}. \end{aligned}$$

Recalling that $\|\cdot\|_{L_{x,v}^2} \leq \|\|\cdot\|\|_{L_x^2(H_{v,*}^1)}$, it follows

$$\begin{aligned} \langle\langle \Lambda_\varepsilon f, f \rangle\rangle_{\mathcal{X}} &\leq -\sigma_0 \|\|f\|\|_{L_{x,v}^2}^2 - (\sigma_0 - C\delta) \sum_{i=1}^3 \|\|\nabla_x^i f\|\|_{L_{x,v}^2}^2 - \frac{\sigma'}{\varepsilon^2} \sum_{i=0}^2 \delta \|\omega_i \nabla_x^i f^\perp\|_{L_{x,v}^2}^2 \\ &\quad - \frac{(\kappa_0 - C\delta)}{\varepsilon^2} \sum_{i=0}^3 \|\nabla_x^i f^\perp\|_{L_x^2(H_{v,*}^1)}^2 - \frac{(\sigma''' - \sigma')}{\varepsilon^2} \sum_{i=0}^2 \delta \|\omega_i \nabla_x^i f^\perp\|_{L_x^2(H_{v,*}^1)}^2. \end{aligned}$$

We then choose $\delta \in (0, 1)$ small enough such that $\sigma_0 - C\delta \geq \sigma'$ and $\kappa_0 - C\delta > 0$, therefore we obtain (4.58) with $\kappa' = \min((\sigma''' - \sigma')\delta, \kappa_0 - C\delta) > 0$, which completes the proof. \square

3.2 Regularization estimates

Proposition 4.18. *The solution $f(t) = U^\varepsilon(t)f_{\text{in}}$ to the equation*

$$\begin{cases} \partial_t f = \Lambda_\varepsilon f \\ f(0) = f_{\text{in}} \in \mathcal{X} \cap (\text{Ker } \Lambda_\varepsilon)^\perp \end{cases}$$

satisfies, for all $t > 0$,

$$\|f(t)\|_{\mathcal{X}_1} \leq C \frac{e^{-\sigma t}}{\min(1, \sqrt{t})} \|f_{\text{in}}\|_{\mathcal{X}} \quad (4.60)$$

and

$$\|f(t)\|_{\mathcal{X}_1^\varepsilon} \leq C \frac{e^{-\sigma t}}{\min(1, t^{3/2})} \|f_{\text{in}}\|_{\mathcal{X}}, \quad (4.61)$$

for any $0 < \sigma < \sigma_0$ (where σ_0 is defined in Proposition 4.16).

Remark 4.19. Notice that thanks to the second inequality, one can in particular recover a gain of one derivative in the spatial variable (with the associated anisotropic gain of weight in velocity), at the price of loosing a $1/\varepsilon$. As already mentioned, this is explained by the fact that the gain comes from the transport operator which does not act as the same scale as the collision operator in velocity. Notice also that in [18], the authors were

facing a similar singularity in ε when wanting to obtain a gain of regularity in the spatial variable for the hard-spheres Boltzmann equation. The latter equation is not hypoelliptic but thanks to a suitable use of averaging lemmas, the authors were also able to obtain regularization properties in the spatial variable with the same singularity in ε .

Proof of Proposition 4.18. We shall prove that for any $t \in (0, 1]$ there holds

$$\|f(t)\|_{\mathcal{A}_1} \leq \frac{C}{\sqrt{t}} \|f_{\text{in}}\|_{\mathcal{X}} \quad (4.62)$$

and

$$\|f(t)\|_{\mathcal{X}_1^\varepsilon} \leq \frac{C}{t^{3/2}} \|f_{\text{in}}\|_{\mathcal{X}}, \quad (4.63)$$

which readily imply (4.60) and (4.61) thanks to the exponential decay of U^ε in \mathcal{X} from Proposition 4.17.

Step 1. Define the functional

$$\begin{aligned} \mathcal{U}_\varepsilon(t, f) = & \|f\|_{\mathcal{X}}^2 + \alpha_1 t \left(\|\widetilde{\nabla}_v f^\perp\|_{\mathcal{X}}^2 + K \|\langle v \rangle^{\frac{7}{2}+1} f^\perp\|_{\mathcal{X}}^2 \right) \\ & + \varepsilon \alpha_2 t^2 \langle \widetilde{\nabla}_v f, \widetilde{\nabla}_x f \rangle_{\mathcal{X}} + \varepsilon^2 \alpha_3 t^3 \left(\|\widetilde{\nabla}_x f\|_{\mathcal{X}}^2 + K \|\langle v \rangle^{\frac{7}{2}} \nabla_x f\|_{\mathcal{X}}^2 \right), \end{aligned}$$

with constants $0 < \alpha_3 \ll \alpha_2 \ll \alpha_1 \ll 1$ so that $\alpha_2 \leq \sqrt{\alpha_1 \alpha_3}$ and $K > 0$. The constants α_i will be chosen small enough and K large enough during the proof.

We easily observe that

$$\|f\|_{\mathcal{X}}^2 + t \left(\|\widetilde{\nabla}_v f^\perp\|_{\mathcal{X}}^2 + \|\langle v \rangle^{\frac{7}{2}+1} f^\perp\|_{\mathcal{X}}^2 \right) + \varepsilon^2 t^3 \|\widetilde{\nabla}_x f\|_{\mathcal{X}}^2 \lesssim \mathcal{U}_\varepsilon(t, f).$$

Remarking that $\|f\|_{\mathcal{A}_1} \lesssim \|f^\perp\|_{\mathcal{A}_1} + \|\pi f\|_{\mathcal{X}}$, we thus obtain

$$\|f\|_{\mathcal{A}_1} \lesssim \|\widetilde{\nabla}_v f^\perp\|_{\mathcal{X}} + \|\langle v \rangle^{\frac{7}{2}+1} f^\perp\|_{\mathcal{X}} + \|\pi f\|_{\mathcal{X}},$$

from which we deduce the following lower bounds: For all $t \in [0, 1]$ there holds

$$t \|f\|_{\mathcal{A}_1}^2 \lesssim \mathcal{U}_\varepsilon(t, f) \quad \text{and} \quad t^3 \|f\|_{\mathcal{X}_1^\varepsilon}^2 \lesssim \mathcal{U}_\varepsilon(t, f). \quad (4.64)$$

Therefore, in order to prove (4.62) and (4.63), it is sufficient to show that

$$\frac{d}{dt} \mathcal{U}_\varepsilon(t, f) \leq 0, \quad \forall t \in [0, 1],$$

which we shall do next. We thus compute

$$\begin{aligned}
 \frac{d}{dt} \mathcal{W}_\varepsilon(t, f) &= \frac{d}{dt} \|f\|_{\mathcal{X}}^2 + \alpha_1 \left(\|\langle v \rangle^{\frac{\gamma}{2}+1} f^\perp\|_{\mathcal{X}}^2 + \|\widetilde{\nabla}_v f^\perp\|_{\mathcal{X}}^2 \right) \\
 &\quad + \alpha_1 t \frac{d}{dt} \left(\|\langle v \rangle^{\frac{\gamma}{2}+1} f^\perp\|_{\mathcal{X}}^2 + \|\widetilde{\nabla}_v f^\perp\|_{\mathcal{X}}^2 \right) \\
 &\quad + 2\alpha_2 \varepsilon t \langle \widetilde{\nabla}_v f, \widetilde{\nabla}_x f \rangle_{\mathcal{X}} + \alpha_2 \varepsilon t^2 \frac{d}{dt} \langle \widetilde{\nabla}_v f, \widetilde{\nabla}_x f \rangle_{\mathcal{X}} \\
 &\quad + 3\alpha_3 \varepsilon^2 t^2 \left(\|\widetilde{\nabla}_x f\|_{\mathcal{X}}^2 + K \|\langle v \rangle^{\frac{\gamma}{2}} \nabla_x f\|_{\mathcal{X}}^2 \right) \\
 &\quad + \alpha_3 \varepsilon^2 t^3 \frac{d}{dt} \left(\|\widetilde{\nabla}_x f\|_{\mathcal{X}}^2 + K \|\langle v \rangle^{\frac{\gamma}{2}} \nabla_x f\|_{\mathcal{X}}^2 \right)
 \end{aligned} \tag{4.65}$$

and we estimate each term separately in the sequel. In order to simplify, we introduce the notations $g_i = \nabla_x^i f$, $g_i^\perp = (\nabla_x^i f)^\perp = \nabla_x^i f^\perp$ and $\omega_i = \langle v \rangle^{(3-i)(\frac{\gamma}{2}+1)}$ so that

$$\|f\|_{\mathcal{X}}^2 = \sum_{i=0}^3 \|\omega_i g_i\|_{L_{x,v}^2}^2, \quad \|f\|_{\mathcal{X}_1}^2 = \sum_{i=0}^3 \|\omega_i g_i\|_{L_{x,v}^2(H_{v,*}^1)}^2, \quad \|f\|_{\mathcal{X}_2}^2 = \sum_{i=0}^3 \|\omega_i g_i\|_{L_{x,v}^2(H_{v,*}^2)}^2.$$

Step 2. From Proposition 4.17, we already have

$$\frac{d}{dt} \|f\|_{\mathcal{X}}^2 \leq -\kappa_0 \|f\|_{\mathcal{X}_1}^2 - \frac{\kappa_0}{\varepsilon^2} \|f^\perp\|_{\mathcal{X}_1}^2, \tag{4.66}$$

for some constant $\kappa_0 > 0$.

Step 3. We deal in this step with the term $(K \|\langle v \rangle^{\frac{\gamma}{2}+1} f^\perp\|_{\mathcal{X}}^2 + \|\widetilde{\nabla}_v f^\perp\|_{\mathcal{X}}^2)$. We split the computations into two parts.

Step 3.1. We first compute

$$\frac{1}{2} \frac{d}{dt} \|\langle v \rangle^{\frac{\gamma}{2}+1} f^\perp\|_{\mathcal{X}}^2 = \frac{1}{2} \sum_{i=0}^3 \frac{d}{dt} \|\omega_i \langle v \rangle^{\frac{\gamma}{2}+1} g_i^\perp\|_{L_{x,v}^2}^2.$$

Observing that f^\perp satisfies the equation $\partial_t f^\perp = (\Lambda_\varepsilon f)^\perp$, with $(\Lambda_\varepsilon f)^\perp$ given by (4.59), using that the transport operator is skew-adjoint and that derivatives in x commute with π and Λ_ε , for any $i \in \{0, 1, 2, 3\}$ we obtain

$$\frac{1}{2} \frac{d}{dt} \|\omega_i \langle v \rangle^{\frac{\gamma}{2}+1} g_i^\perp\|_{L_{x,v}^2}^2 = \langle \omega_i \langle v \rangle^{\frac{\gamma}{2}+1} \partial_t g_i^\perp, \omega_i \langle v \rangle^{\frac{\gamma}{2}+1} g_i^\perp \rangle_{L_{x,v}^2} =: \frac{1}{\varepsilon^2} J_1^i - \frac{1}{\varepsilon} J_2^i + \frac{1}{\varepsilon} J_3^i$$

with

$$\begin{aligned}
 J_1^i &:= \langle \omega_i \langle v \rangle^{\frac{\gamma}{2}+1} L g_i^\perp, \omega_i \langle v \rangle^{\frac{\gamma}{2}+1} g_i^\perp \rangle_{L_{x,v}^2} \\
 J_2^i &:= \langle \omega_i \langle v \rangle^{\frac{\gamma}{2}+1} (v \cdot \nabla_x (\pi g_i)), \omega_i \langle v \rangle^{\frac{\gamma}{2}+1} g_i^\perp \rangle_{L_{x,v}^2} \\
 J_3^i &:= \langle \omega_i \langle v \rangle^{\frac{\gamma}{2}+1} (\pi (v \cdot \nabla_x g_i)), \omega_i \langle v \rangle^{\frac{\gamma}{2}+1} g_i^\perp \rangle_{L_{x,v}^2}.
 \end{aligned}$$

For the first term, we write

$$\begin{aligned} J_1^i &= \left\langle L(\omega_i \langle v \rangle^{\frac{\gamma}{2}+1} g_i^\perp) + [\omega_i \langle v \rangle^{\frac{\gamma}{2}+1}, L_1] g_i^\perp + [\omega_i \langle v \rangle^{\frac{\gamma}{2}+1}, L_2] g_i^\perp, \omega_i \langle v \rangle^{\frac{\gamma}{2}+1} g_i^\perp \right\rangle_{L_{x,v}^2} \\ &=: J_{11}^i + J_{12}^i + J_{13}^i, \end{aligned}$$

where we recall that L_1 and L_2 are defined in (4.51) and (4.52), respectively. Thanks to the spectral gap estimate (4.43), one has

$$\begin{aligned} J_{11}^i &\leq -2\kappa \|\omega_i \langle v \rangle^{\frac{\gamma}{2}+1} g_i^\perp - \pi(\omega_i \langle v \rangle^{\frac{\gamma}{2}+1} g_i^\perp)\|_{L_x^2(H_{v,*}^1)}^2 \\ &\leq -2\kappa \|\omega_i \langle v \rangle^{\frac{\gamma}{2}+1} g_i^\perp\|_{L_x^2(H_{v,*}^1)}^2 + C \|g_i^\perp\|_{L_{x,v}^2}^2, \end{aligned}$$

for some constants $\kappa, C > 0$. From Lemma 4.11, we get

$$\begin{aligned} J_{12}^i &= - \left\langle [\omega_i \langle v \rangle^{\frac{\gamma}{2}+1}, \widetilde{\nabla}_{v_\ell}] g_i^\perp, \widetilde{\nabla}_{v_\ell} (\omega_i \langle v \rangle^{\frac{\gamma}{2}+1} g_i^\perp) \right\rangle_{L_{x,v}^2} \\ &\quad - \left\langle [\omega_i \langle v \rangle^{\frac{\gamma}{2}+1}, \widetilde{\nabla}_{v_\ell}^*] g_i^\perp, \widetilde{\nabla}_{v_\ell}^* (\omega_i \langle v \rangle^{\frac{\gamma}{2}+1} g_i^\perp) \right\rangle_{L_{x,v}^2} \\ &\quad - \left\langle [[\omega_i \langle v \rangle^{\frac{\gamma}{2}+1}, \widetilde{\nabla}_{v_\ell}^*], \widetilde{\nabla}_{v_\ell}] g_i^\perp, \omega_i \langle v \rangle^{\frac{\gamma}{2}+1} g_i^\perp \right\rangle_{L_{x,v}^2}. \end{aligned}$$

Using Lemma 4.10 and observing that $\|\widetilde{\nabla}_v h\|_{L_{x,v}^2} + \|\widetilde{\nabla}_v^* h\|_{L_{x,v}^2} \lesssim \|h\|_{L_x^2(H_{v,*}^1)}$, we then obtain

$$\begin{aligned} |J_{12}^i| &\leq C \|\omega_i \langle v \rangle^{\frac{\gamma}{2}+1} g_i^\perp\|_{L_x^2(H_{v,*}^1)} \|\langle v \rangle^{\frac{\gamma}{2}-1} \omega_i \langle v \rangle^{\frac{\gamma}{2}+1} g_i^\perp\|_{L_{x,v}^2} + C \|\langle v \rangle^{\frac{\gamma}{2}-\frac{1}{2}} \omega_i \langle v \rangle^{\frac{\gamma}{2}+1} g_i^\perp\|_{L_{x,v}^2}^2 \\ &\leq \kappa \|\omega_i \langle v \rangle^{\frac{\gamma}{2}+1} g_i^\perp\|_{L_x^2(H_{v,*}^1)}^2 + C \|\omega_i \langle v \rangle^{\frac{\gamma}{2}+1} g_i^\perp\|_{L_{x,v}^2}^2 \end{aligned}$$

where we have used that $\gamma \leq 1$ and Young's inequality in last line. For the term J_{13}^i , we use Lemma 4.14 to obtain

$$|J_{13}^i| \leq C \|\omega_i g_i^\perp\|_{L_{x,v}^2}^2.$$

We deal with the terms J_2^i and J_3^i . Observing that

$$\|\omega_i \langle v \rangle^{\frac{\gamma}{2}+1} (v \cdot \nabla_x (\pi g_i))\|_{L_{x,v}^2} + \|\omega_i \langle v \rangle^{\frac{\gamma}{2}+1} (\pi (v \cdot \nabla_x g_i))\|_{L_{x,v}^2} \lesssim \|\omega_i \widetilde{\nabla}_x g_i\|_{L_{x,v}^2},$$

we obtain

$$|J_2^i| + |J_3^i| \leq C \|\omega_i \widetilde{\nabla}_x g_i\|_{L_{x,v}^2} \|\omega_i \langle v \rangle^{\frac{\gamma}{2}+1} g_i^\perp\|_{L_{x,v}^2}.$$

Gathering previous estimates and using that $\|\langle v \rangle^{\gamma+\frac{1}{2}} f^\perp\|_{\mathcal{X}} \lesssim \|f^\perp\|_{\mathcal{Y}_1}$, we obtain

$$\frac{1}{2} \frac{d}{dt} \|\langle v \rangle^{\frac{\gamma}{2}+1} f^\perp\|_{\mathcal{X}}^2 \leq -\frac{\kappa}{\varepsilon^2} \|\langle v \rangle^{\frac{\gamma}{2}+1} f^\perp\|_{\mathcal{Y}_1}^2 + \frac{C}{\varepsilon^2} \|f^\perp\|_{\mathcal{Y}_1}^2 + \frac{C}{\varepsilon} \|f^\perp\|_{\mathcal{Y}_1} \|\widetilde{\nabla}_x f\|_{\mathcal{X}} \quad (4.67)$$

for some constants $\kappa, C > 0$.

Step 3.2. We now compute

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\widetilde{\nabla}_v f^\perp\|_{\mathcal{X}}^2 &= \frac{1}{2} \sum_{i=0}^3 \frac{d}{dt} \|\omega_i \widetilde{\nabla}_v g_i^\perp\|_{L_{x,v}^2}^2 \\ &=: \sum_{i=0}^3 \left(\frac{1}{\varepsilon^2} I_1^i - \frac{1}{\varepsilon} I_2^i + \frac{1}{\varepsilon} I_3^i - \frac{1}{\varepsilon} I_4^i \right) \end{aligned}$$

with

$$\begin{aligned} I_1^i &:= \left\langle \omega_i \widetilde{\nabla}_{v_k} L g_i^\perp, \omega_i \widetilde{\nabla}_{v_k} g_i^\perp \right\rangle_{L_{x,v}^2} \\ I_2^i &:= \left\langle \omega_i \widetilde{\nabla}_{v_k} (v \cdot \nabla_x (\pi g_i)), \omega_i \widetilde{\nabla}_{v_k} g_i^\perp \right\rangle_{L_{x,v}^2} \\ I_3^i &:= \left\langle \omega_i \widetilde{\nabla}_{v_k} (\pi (v \cdot \nabla_x g_i)), \omega_i \widetilde{\nabla}_{v_k} g_i^\perp \right\rangle_{L_{x,v}^2} \\ I_4^i &:= \left\langle \omega_i \widetilde{\nabla}_{v_k} (v \cdot \nabla_x g_i^\perp), \omega_i \widetilde{\nabla}_{v_k} g_i^\perp \right\rangle_{L_{x,v}^2}. \end{aligned}$$

For the first term, we write

$$\begin{aligned} I_1^i &= \left\langle L(\omega_i \widetilde{\nabla}_{v_k} g_i^\perp), \omega_i \widetilde{\nabla}_{v_k} g_i^\perp \right\rangle_{L_{x,v}^2} + \left\langle [\omega_i, L_1] \widetilde{\nabla}_{v_k} g_i^\perp, \omega_i \widetilde{\nabla}_{v_k} g_i^\perp \right\rangle_{L_{x,v}^2} \\ &\quad + \left\langle \omega_i [\widetilde{\nabla}_{v_k}, L_1] g_i^\perp, \omega_i \widetilde{\nabla}_{v_k} g_i^\perp \right\rangle_{L_{x,v}^2} + \left\langle [\omega_i, L_2] \widetilde{\nabla}_{v_k} g_i^\perp, \omega_i \widetilde{\nabla}_{v_k} g_i^\perp \right\rangle_{L_{x,v}^2} \\ &\quad + \left\langle \omega_i [\widetilde{\nabla}_{v_k}, L_2] g_i^\perp, \omega_i \widetilde{\nabla}_{v_k} g_i^\perp \right\rangle_{L_{x,v}^2} \\ &=: I_{11}^i + I_{12}^i + I_{13}^i + I_{14}^i + I_{15}^i. \end{aligned}$$

Thanks to the spectral gap estimate (4.43), one has

$$\begin{aligned} I_{11}^i &\leq -2\kappa \|\omega_i \widetilde{\nabla}_{v_k} g_i^\perp - \pi(\omega_i \widetilde{\nabla}_{v_k} g_i^\perp)\|_{L_x^2(H_{v,*}^1)}^2 \\ &\leq -2\kappa \|\omega_i \widetilde{\nabla}_{v_k} g_i^\perp\|_{L_x^2(H_{v,*}^1)}^2 + C \|g_i^\perp\|_{L_{x,v}^2}^2, \end{aligned}$$

for some constants $\kappa, C > 0$. From Lemma 4.11, we get

$$\begin{aligned} I_{12}^i &= - \left\langle [\omega_i, \widetilde{\nabla}_{v_\ell}] \widetilde{\nabla}_{v_k} g_i^\perp, \widetilde{\nabla}_{v_\ell} (\omega_i \widetilde{\nabla}_{v_k} g_i^\perp) \right\rangle_{L_{x,v}^2} - \left\langle [\omega_i, \widetilde{\nabla}_{v_\ell}^*] \widetilde{\nabla}_{v_k} g_i^\perp, \widetilde{\nabla}_{v_\ell}^* (\omega_i \widetilde{\nabla}_{v_k} g_i^\perp) \right\rangle_{L_{x,v}^2} \\ &\quad - \left\langle [[\omega_i, \widetilde{\nabla}_{v_\ell}^*], \widetilde{\nabla}_{v_\ell}] \widetilde{\nabla}_{v_k} g_i^\perp, \omega_i \widetilde{\nabla}_{v_k} g_i^\perp \right\rangle_{L_{x,v}^2}, \end{aligned}$$

and

$$\begin{aligned} I_{13}^i &= - \left\langle [\widetilde{\nabla}_{v_k}, \widetilde{\nabla}_{v_\ell}] g_i^\perp, \widetilde{\nabla}_{v_\ell} (\omega_i^2 \widetilde{\nabla}_{v_k} g_i^\perp) \right\rangle_{L_{x,v}^2} - \left\langle [\widetilde{\nabla}_{v_k}, \widetilde{\nabla}_{v_\ell}^*] g_i^\perp, \widetilde{\nabla}_{v_\ell}^* (\omega_i^2 \widetilde{\nabla}_{v_k} g_i^\perp) \right\rangle_{L_{x,v}^2} \\ &\quad - \left\langle \omega_i [[\widetilde{\nabla}_{v_k}, \widetilde{\nabla}_{v_\ell}^*], \widetilde{\nabla}_{v_\ell}] g_i^\perp, \omega_i \widetilde{\nabla}_{v_k} g_i^\perp \right\rangle_{L_{x,v}^2} - \left\langle \omega_i (\widetilde{\nabla}_{v_k} \psi) g_i^\perp, \omega_i \widetilde{\nabla}_{v_k} g_i^\perp \right\rangle_{L_{x,v}^2}. \end{aligned}$$

For I_{12}^i , using Lemma 4.10 and observing that $\|\widetilde{\nabla}_v h\|_{L_{x,v}^2} + \|\widetilde{\nabla}_v^* h\|_{L_{x,v}^2} \lesssim \|h\|_{L_x^2(H_{v,*}^1)}$, we obtain

$$|I_{12}^i| \leq C \|\omega_i \widetilde{\nabla}_{v_k} g_i^\perp\|_{L_x^2(H_{v,*}^1)} \|\langle v \rangle^{\frac{\gamma}{2}-1} \omega_i \widetilde{\nabla}_{v_k} g_i^\perp\|_{L_{x,v}^2} + C \|\langle v \rangle^{\frac{\gamma}{2}-\frac{1}{2}} \omega_i \widetilde{\nabla}_{v_k} g_i^\perp\|_{L_{x,v}^2}^2.$$

For I_{13}^i , we first observe that writing

$$\begin{aligned}\widetilde{\nabla}_{v_\ell}(\omega_i^2 h) &= \omega_i \widetilde{\nabla}_{v_\ell}(\omega_i h) + (\widetilde{\nabla}_{v_\ell} \omega_i) \omega_i h \\ \widetilde{\nabla}_{v_\ell}^*(\omega_i^2 h) &= -\omega_i \widetilde{\nabla}_{v_\ell}(\omega_i h) - [(\partial_{v_p} B_{\ell p}) \omega_i + (\widetilde{\nabla}_{v_\ell} \omega_i)] \omega_i h\end{aligned}$$

and using the bounds (4.42), we have

$$\|\omega_i^{-1} \widetilde{\nabla}_{v_\ell}(\omega_i^2 h)\|_{L_{x,v}^2} + \|\omega_i^{-1} \widetilde{\nabla}_{v_\ell}^*(\omega_i^2 h)\|_{L_{x,v}^2} \lesssim \|\omega_i h\|_{L_x^2(H_{v,*}^1)}. \quad (4.68)$$

Therefore, using Lemma 4.10 and noticing that

$$|\widetilde{\nabla}_v \psi| \lesssim \langle v \rangle^{\frac{\gamma}{2}+1} |\nabla_v \psi| \lesssim \langle v \rangle^{\frac{\gamma}{2}+1} \langle v \rangle^{\gamma+1},$$

we obtain

$$\begin{aligned}|I_{13}^i| &\leq C \|\omega_i \widetilde{\nabla}_{v_k} g_i^\perp\|_{L_x^2(H_{v,*}^1)} \|\omega_i \langle v \rangle^{\gamma+1} \nabla_v g_i^\perp\|_{L_{x,v}^2} + C \|\omega_i \widetilde{\nabla}_{v_k} g_i^\perp\|_{L_x^2(H_{v,*}^1)} \|\omega_i \langle v \rangle^\gamma g_i^\perp\|_{L_{x,v}^2} \\ &\quad + C \|\omega_i \langle v \rangle^\gamma \nabla_v g_i^\perp\|_{L_{x,v}^2} \|\langle v \rangle^{\frac{\gamma}{2}+1} \omega_i \widetilde{\nabla}_{v_k} g_i^\perp\|_{L_{x,v}^2} \\ &\quad + C \|\omega_i \langle v \rangle^{\gamma-1} g_i^\perp\|_{L_{x,v}^2} \|\langle v \rangle^{\frac{\gamma}{2}+1} \omega_i \widetilde{\nabla}_{v_k} g_i^\perp\|_{L_{x,v}^2} \\ &\quad + C \|\omega_i \langle v \rangle^{\gamma+1} g_i^\perp\|_{L_{x,v}^2} \|\langle v \rangle^{\frac{\gamma}{2}+1} \omega_i \widetilde{\nabla}_{v_k} g_i^\perp\|_{L_{x,v}^2}.\end{aligned}$$

Putting together the estimates for I_{12}^i and I_{13}^i and using Young's inequality, it follows

$$|I_{12}^i| + |I_{13}^i| \leq \kappa \|\omega_i \widetilde{\nabla}_{v_k} g_i^\perp\|_{L_x^2(H_{v,*}^1)}^2 + C \|\omega_i \langle v \rangle^{\gamma+1} g_i^\perp\|_{L_{x,v}^2}^2 + C \|\omega_i \langle v \rangle^{\gamma+1} \nabla_v g_i^\perp\|_{L_{x,v}^2}^2.$$

For the terms I_{14}^i and I_{15}^i , Lemma 4.14 yields

$$|I_{14}^i| + |I_{15}^i| \leq C \|\omega_i g_i^\perp\|_{L_{x,v}^2}^2 + C \|\omega_i \widetilde{\nabla}_v g_i^\perp\|_{L_{x,v}^2}^2.$$

The terms I_2^i and I_3^i can be dealt as before in Step 3.1, and we obtain

$$|I_2^i| + |I_3^i| \leq C \|\omega_i \widetilde{\nabla}_x g_i\|_{L_{x,v}^2} \|\omega_i \widetilde{\nabla}_{v_k} g_i^\perp\|_{L_{x,v}^2}.$$

For the remainder term I_4^i , we use Lemma 4.10 and the fact that the transport operator is skew-symmetric to obtain

$$I_4^i = \left\langle \omega_i \widetilde{\nabla}_{x_k} g_i^\perp, \omega_i \widetilde{\nabla}_{v_k} g_i^\perp \right\rangle_{L_{x,v}^2} \leq \|\omega_i \widetilde{\nabla}_{x_k} g_i^\perp\|_{L_{x,v}^2} \|\omega_i \widetilde{\nabla}_{v_k} g_i^\perp\|_{L_{x,v}^2}.$$

Gathering previous estimates, we obtain

$$\begin{aligned}\frac{1}{2} \frac{d}{dt} \|\widetilde{\nabla}_v f^\perp\|_{\mathcal{X}}^2 &\leq -\frac{\kappa}{\varepsilon^2} \|\widetilde{\nabla}_v f^\perp\|_{\mathcal{X}_1}^2 + \frac{C}{\varepsilon^2} \|\langle v \rangle^{\gamma+1} f^\perp\|_{\mathcal{X}}^2 + \frac{C}{\varepsilon^2} \|\langle v \rangle^{\gamma+1} \nabla_v f^\perp\|_{\mathcal{X}}^2 \\ &\quad + \frac{C}{\varepsilon} \|\widetilde{\nabla}_v f^\perp\|_{\mathcal{X}} \|\widetilde{\nabla}_x f\|_{\mathcal{X}}.\end{aligned} \quad (4.69)$$

Finally, we observe that $\|\langle v \rangle^{\gamma+1} f^\perp\|_{\mathcal{X}}^2 + \|\langle v \rangle^{\gamma+1} \nabla_v f^\perp\|_{\mathcal{X}}^2 \lesssim \|\langle v \rangle^{\frac{\gamma}{2}+1} f^\perp\|_{\mathcal{Y}_1}$ and thus, gathering (4.67) and (4.69) and taking $K > 0$ large enough, we obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \left(K \|\langle v \rangle^{\frac{\gamma}{2}+1} f^\perp\|_{\mathcal{X}}^2 + \|\widetilde{\nabla}_v f^\perp\|_{\mathcal{X}}^2 \right) \\ & \leq -\frac{\kappa_1}{\varepsilon^2} \|f^\perp\|_{\mathcal{Y}_2}^2 + \frac{C}{\varepsilon^2} \|f^\perp\|_{\mathcal{Y}_1}^2 + \frac{C}{\varepsilon} \|f^\perp\|_{\mathcal{Y}_1} \|\widetilde{\nabla}_x f\|_{\mathcal{X}} \end{aligned} \quad (4.70)$$

for some constants $\kappa_1, C > 0$.

Step 4. We deal in this step with the derivative in time of term $\langle \widetilde{\nabla}_v f, \widetilde{\nabla}_x f \rangle_{\mathcal{X}}$. We compute

$$\begin{aligned} \frac{d}{dt} \langle \widetilde{\nabla}_v f, \widetilde{\nabla}_x f \rangle_{\mathcal{X}} &= \sum_{i=0}^3 \frac{d}{dt} \langle \omega_i \widetilde{\nabla}_v g_i, \omega_i \widetilde{\nabla}_x g_i \rangle_{L_{x,v}^2} \\ &=: \sum_{i=0}^3 \left(\frac{1}{\varepsilon^2} R_1^i - \frac{1}{\varepsilon} R_2^i + \frac{1}{\varepsilon^2} S_1^i - \frac{1}{\varepsilon} S_2^i \right) \end{aligned}$$

with

$$\begin{aligned} R_1^i &:= \langle \omega_i \widetilde{\nabla}_{v_k} (Lg_i^\perp), \omega_i \widetilde{\nabla}_{x_k} g_i \rangle_{L_{x,v}^2} \\ S_1^i &:= \langle \omega_i \widetilde{\nabla}_{v_k} g_i, \omega_i \widetilde{\nabla}_{x_k} (Lg_i^\perp) \rangle_{L_{x,v}^2} \\ R_2^i &:= \langle \omega_i \widetilde{\nabla}_{v_k} (v \cdot \nabla_x g_i), \omega_i \widetilde{\nabla}_{x_k} g_i \rangle_{L_{x,v}^2} \\ S_2^i &:= \langle \omega_i \widetilde{\nabla}_{v_k} g_i, \omega_i \widetilde{\nabla}_{x_k} (v \cdot \nabla_x g_i) \rangle_{L_{x,v}^2}. \end{aligned}$$

Recalling that $\widetilde{\nabla}_{x_k}^* = -\widetilde{\nabla}_{x_k}$, we then write

$$\begin{aligned} R_1^i + S_1^i &= \langle Lg_i^\perp, \widetilde{\nabla}_{v_k}^* (\omega_i^2 \widetilde{\nabla}_{x_k} g_i) - \widetilde{\nabla}_{x_k} (\omega_i^2 \widetilde{\nabla}_{v_k} g_i) \rangle_{L_{x,v}^2} \\ &\leq \|\omega_i Lg_i^\perp\|_{L_{x,v}^2} \left(\|\omega_i^{-1} \widetilde{\nabla}_{v_k}^* (\omega_i^2 \widetilde{\nabla}_{x_k} g_i)\|_{L_{x,v}^2} + \|\omega_i^{-1} \widetilde{\nabla}_{x_k} (\omega_i^2 \widetilde{\nabla}_{v_k} g_i)\|_{L_{x,v}^2} \right). \end{aligned}$$

Observe that

$$\omega_i^{-1} \widetilde{\nabla}_{x_k} (\omega_i^2 \widetilde{\nabla}_{v_k} g_i) = \widetilde{\nabla}_{v_k} (\omega_i \widetilde{\nabla}_{x_k} g_i) + [\omega_i, \widetilde{\nabla}_{v_k}] \widetilde{\nabla}_{x_k} g_i + \omega_i [\widetilde{\nabla}_{x_k}, \widetilde{\nabla}_{v_k}] g_i$$

so that using Lemma 4.10, we get

$$\|\omega_i^{-1} \widetilde{\nabla}_{x_k} (\omega_i^2 \widetilde{\nabla}_{v_k} g_i)\|_{L_{x,v}^2} \lesssim \|\omega_i \widetilde{\nabla}_x g_i\|_{L_x^2(H_{v,*}^1)}. \quad (4.71)$$

Writing $\omega_i Lg_i^\perp = \omega_i L_1 g_i^\perp + \omega_i L_2 g_i^\perp$ and using the explicit formula (4.51) of L_1 together with the bound of Lemma 4.14 for L_2 , we obtain

$$\|\omega_i Lg_i^\perp\|_{L_{x,v}^2} \lesssim \|\omega_i g_i^\perp\|_{L_x^2(H_{v,*}^2)}. \quad (4.72)$$

Together with (4.68), we hence get

$$|R_1^i + S_1^i| \leq C \|\omega_i g_i^\perp\|_{L_x^2(H_{v,*}^2)} \|\omega_i \widetilde{\nabla}_x g_i\|_{L_x^2(H_{v,*}^1)}.$$

For the remainder terms, we observe that using Lemma 4.10 and the fact that the transport operator is skew-adjoint,

$$\begin{aligned} R_2^i + S_2^i &= \|\omega_i \widetilde{\nabla}_{x_k} g_i\|_{L_{x,v}^2}^2 + \langle \omega_i v \cdot \nabla_x (\widetilde{\nabla}_{v_k} g_i), \omega_i \widetilde{\nabla}_{x_k} g_i \rangle_{L_{x,v}^2} \\ &\quad + \langle \omega_i \widetilde{\nabla}_{v_k} g_i, \omega_i v \cdot \nabla_x (\widetilde{\nabla}_{x_k} g_i) \rangle_{L_{x,v}^2} = \|\omega_i \widetilde{\nabla}_{x_k} g_i\|_{L_{x,v}^2}^2. \end{aligned}$$

Gathering previous estimates we obtain

$$\frac{d}{dt} \langle \widetilde{\nabla}_v f, \widetilde{\nabla}_x f \rangle_{\mathcal{X}} \leq \frac{C}{\varepsilon^2} \|f^\perp\|_{\mathcal{Y}_2} \|\widetilde{\nabla}_x f\|_{\mathcal{Y}_1} - \frac{1}{\varepsilon} \|\widetilde{\nabla}_x f\|_{\mathcal{X}}^2 \quad (4.73)$$

for some constant $C > 0$.

Step 5. We deal in this step with the derivative in time of the term $(K \|\langle v \rangle^{\frac{\gamma}{2}} \nabla_x f\|_{\mathcal{X}}^2 + \|\widetilde{\nabla}_x f\|_{\mathcal{X}}^2)$. We first compute, using that the transport operator is skew-adjoint,

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\langle v \rangle^{\frac{\gamma}{2}} \nabla_x f\|_{\mathcal{X}}^2 &= \frac{1}{2} \sum_{i=0}^3 \frac{d}{dt} \|\omega_i \langle v \rangle^{\frac{\gamma}{2}} \nabla_x g_i\|_{L_{x,v}^2}^2 \\ &=: \sum_{i=0}^3 \frac{1}{\varepsilon^2} \langle \omega_i \langle v \rangle^{\frac{\gamma}{2}} \partial_{x_k} (L g_i), \omega_i \langle v \rangle^{\frac{\gamma}{2}} \partial_{x_k} g_i \rangle_{L_{x,v}^2}. \end{aligned}$$

Since ∂_{x_k} commutes with L , we can argue as for the term J_1^i in Step 3.1 above to obtain

$$\langle \omega_i \langle v \rangle^{\frac{\gamma}{2}} \partial_{x_k} (L g_i), \omega_i \langle v \rangle^{\frac{\gamma}{2}} \partial_{x_k} g_i \rangle_{L_{x,v}^2} \leq -\kappa \|\omega_i \langle v \rangle^{\frac{\gamma}{2}} \nabla_x g_i\|_{L_x^2(H_{v,*}^1)}^2 + C \|\omega_i \widetilde{\nabla}_x g_i\|_{L_{x,v}^2}^2,$$

for some constants $\kappa, C > 0$, therefore

$$\frac{1}{2} \frac{d}{dt} \|\langle v \rangle^{\frac{\gamma}{2}} \nabla_x f\|_{\mathcal{X}}^2 \leq -\frac{\kappa}{\varepsilon^2} \|\langle v \rangle^{\frac{\gamma}{2}} \nabla_x f\|_{\mathcal{Y}_1}^2 + \frac{C}{\varepsilon^2} \|\widetilde{\nabla}_x f\|_{\mathcal{X}}^2. \quad (4.74)$$

Using that the transport operator is skew-adjoint and commutes with $\widetilde{\nabla}_{x_k}$, we now compute

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\widetilde{\nabla}_x f\|_{\mathcal{X}}^2 &= \frac{1}{2} \sum_{i=0}^3 \frac{d}{dt} \|\omega_i \widetilde{\nabla}_x g_i\|_{L_{x,v}^2}^2 \\ &= \sum_{i=0}^3 \frac{1}{\varepsilon^2} \langle \omega_i \widetilde{\nabla}_{x_k} (L g_i), \omega_i \widetilde{\nabla}_{x_k} g_i \rangle_{L_{x,v}^2} =: \sum_{i=0}^3 \frac{1}{\varepsilon^2} N^i. \end{aligned}$$

We then write

$$\begin{aligned}
 N^i &= \left\langle L(\omega_i \widetilde{\nabla}_{x_k} g_i), \omega_i \widetilde{\nabla}_{x_k} g_i \right\rangle_{L^2_{x,v}} + \left\langle [\omega_i, L_1] \widetilde{\nabla}_{x_k} g_i, \omega_i \widetilde{\nabla}_{v_k} g_i \right\rangle_{L^2_{x,v}} \\
 &\quad + \left\langle \omega_i [\widetilde{\nabla}_{x_k}, L_1] g_i, \omega_i \widetilde{\nabla}_{x_k} g_i \right\rangle_{L^2_{x,v}} + \left\langle [\omega_i, L_2] \widetilde{\nabla}_{x_k} g_i, \omega_i \widetilde{\nabla}_{x_k} g_i \right\rangle_{L^2_{x,v}} \\
 &\quad + \left\langle \omega_i [\widetilde{\nabla}_{x_k}, L_2] g_i, \omega_i \widetilde{\nabla}_{x_k} g_i \right\rangle_{L^2_{x,v}} \\
 &=: N_1^i + N_2^i + N_3^i + N_4^i + N_5^i.
 \end{aligned}$$

Thanks to the spectral gap estimate (4.43), one has

$$\begin{aligned}
 N_1^i &\leq -2\kappa \|\omega_i \widetilde{\nabla}_{x_k} g_i - \pi(\omega_i \widetilde{\nabla}_{x_k} g_i)\|_{L^2_x(H^1_{v,*})}^2 \\
 &\leq -2\kappa \|\omega_i \widetilde{\nabla}_{x_k} g_i^\perp\|_{L^2_x(H^1_{v,*})}^2 + C \|\omega_i \widetilde{\nabla}_{x_k} g_i\|_{L^2_{x,v}}^2,
 \end{aligned}$$

for some constants $\kappa, C > 0$. From Lemma 4.11, we get

$$\begin{aligned}
 N_2^i &= - \left\langle [\omega_i, \widetilde{\nabla}_{v_\ell}] \widetilde{\nabla}_{x_k} g_i, \widetilde{\nabla}_{v_\ell}(\omega_i \widetilde{\nabla}_{x_k} g_i) \right\rangle_{L^2_{x,v}} - \left\langle [\omega_i, \widetilde{\nabla}_{v_\ell}^*] \widetilde{\nabla}_{x_k} g_i, \widetilde{\nabla}_{v_\ell}^*(\omega_i \widetilde{\nabla}_{x_k} g_i) \right\rangle_{L^2_{x,v}} \\
 &\quad - \left\langle [[\omega_i, \widetilde{\nabla}_{v_\ell}^*], \widetilde{\nabla}_{v_\ell}] \widetilde{\nabla}_{x_k} g_i, \omega_i \widetilde{\nabla}_{x_k} g_i \right\rangle_{L^2_{x,v}},
 \end{aligned}$$

and

$$\begin{aligned}
 N_3^i &= - \left\langle [\widetilde{\nabla}_{x_k}, \widetilde{\nabla}_{v_\ell}] g_i, \widetilde{\nabla}_{v_\ell}(\omega_i^2 \widetilde{\nabla}_{x_k} g_i) \right\rangle_{L^2_{x,v}} - \left\langle [\widetilde{\nabla}_{x_k}, \widetilde{\nabla}_{v_\ell}^*] g_i, \widetilde{\nabla}_{v_\ell}^*(\omega_i^2 \widetilde{\nabla}_{x_k} g_i) \right\rangle_{L^2_{x,v}} \\
 &\quad - \left\langle \omega_i [[\widetilde{\nabla}_{x_k}, \widetilde{\nabla}_{v_\ell}^*], \widetilde{\nabla}_{v_\ell}] g_i, \omega_i \widetilde{\nabla}_{x_k} g_i \right\rangle_{L^2_{x,v}}.
 \end{aligned}$$

Arguing in a similar way as in Step 3.2 above (for the term I_{12}^i and I_{13}^i), it follows

$$|N_2^i| + |N_3^i| \leq \kappa \|\omega_i \widetilde{\nabla}_{x_k} g_i\|_{L^2_x(H^1_{v,*})}^2 + C \|\omega_i \langle v \rangle^{\gamma+1} \nabla_x g_i\|_{L^2_{x,v}}^2.$$

For the terms N_4^i and N_5^i , Lemma 4.14 yields

$$|N_4^i| + |N_5^i| \leq C \|\omega_i \widetilde{\nabla}_x g_i\|_{L^2_{x,v}}^2.$$

Gathering previous estimates, we obtain

$$\frac{1}{2} \frac{d}{dt} \|\widetilde{\nabla}_x f\|_{\mathcal{X}}^2 \leq -\frac{\kappa}{\varepsilon^2} \|\widetilde{\nabla}_x f\|_{\mathcal{Y}_1}^2 + \frac{C}{\varepsilon^2} \|\langle v \rangle^{\gamma+1} \nabla_x f\|_{\mathcal{X}}^2. \quad (4.75)$$

Finally, we observe that $\|\langle v \rangle^{\gamma+1} \nabla_x f\|_{\mathcal{X}}^2 \lesssim \|\langle v \rangle^{\frac{\gamma}{2}} \nabla_x f\|_{\mathcal{Y}_1}^2$ and thus, gathering (4.74) and (4.75) and taking $K > 0$ large enough, we obtain

$$\frac{1}{2} \frac{d}{dt} \left(K \|\langle v \rangle^{\frac{\gamma}{2}} \nabla_x f\|_{\mathcal{X}}^2 + \|\widetilde{\nabla}_x f^\perp\|_{\mathcal{X}}^2 \right) \leq -\frac{\kappa_2}{\varepsilon^2} \|\widetilde{\nabla}_x f\|_{\mathcal{Y}_1}^2 + \frac{C}{\varepsilon^2} \|\widetilde{\nabla}_x f\|_{\mathcal{X}}^2 \quad (4.76)$$

for some constants $\kappa_2, C > 0$.

Step 6. Conclusion. Gathering (4.66)–(4.70)–(4.73)–(4.76), we thus obtain

$$\begin{aligned}
 \frac{d}{dt} \mathcal{U}_\varepsilon(t, f) &\leq -\kappa_0 \|f\|_{\mathcal{H}_1}^2 - \frac{\kappa_0}{\varepsilon^2} \|f^\perp\|_{\mathcal{H}_1}^2 + \alpha_1 \left(K \|\langle v \rangle^{\frac{\gamma}{2}+1} f^\perp\|_{\mathcal{X}}^2 + \|\widetilde{\nabla}_v f^\perp\|_{\mathcal{X}}^2 \right) \\
 &\quad + \alpha_1 t \left(-\frac{\kappa_1}{\varepsilon^2} \|f^\perp\|_{\mathcal{H}_2}^2 + \frac{C}{\varepsilon^2} \|f^\perp\|_{\mathcal{H}_1}^2 + \frac{C}{\varepsilon} \|f^\perp\|_{\mathcal{H}_1} \|\widetilde{\nabla}_x f\|_{\mathcal{X}} \right) \\
 &\quad + 2\alpha_2 \varepsilon t \langle \widetilde{\nabla}_v f, \widetilde{\nabla}_x f \rangle_{\mathcal{X}} + \alpha_2 \varepsilon t^2 \left(\frac{C}{\varepsilon^2} \|f^\perp\|_{\mathcal{H}_2} \|\widetilde{\nabla}_x f\|_{\mathcal{H}_1} - \frac{1}{\varepsilon} \|\widetilde{\nabla}_x f\|_{\mathcal{X}}^2 \right) \\
 &\quad + 3\alpha_3 \varepsilon^2 t^2 \left(\|\widetilde{\nabla}_x f\|_{\mathcal{X}}^2 + K \|\langle v \rangle^{\frac{\gamma}{2}} \nabla_x f\|_{\mathcal{X}}^2 \right) \\
 &\quad + \alpha_3 \varepsilon^2 t^3 \left(-\frac{\kappa_2}{\varepsilon^2} \|\widetilde{\nabla}_x f\|_{\mathcal{H}_1}^2 + \frac{C}{\varepsilon^2} \|\widetilde{\nabla}_x f\|_{\mathcal{X}}^2 \right).
 \end{aligned}$$

Observe that

$$K \|\langle v \rangle^{\frac{\gamma}{2}+1} f^\perp\|_{\mathcal{X}}^2 + \|\widetilde{\nabla}_v f^\perp\|_{\mathcal{X}}^2 \lesssim \|f^\perp\|_{\mathcal{H}_1}^2$$

and

$$\|\widetilde{\nabla}_x f\|_{\mathcal{X}}^2 + K \|\langle v \rangle^{\frac{\gamma}{2}} \nabla_x f\|_{\mathcal{X}}^2 \lesssim \|\widetilde{\nabla}_x f\|_{\mathcal{X}}^2$$

and also that, thanks to Young's inequality, there holds

$$\begin{aligned}
 \frac{\alpha_1 t C}{\varepsilon} \|f^\perp\|_{\mathcal{H}_1} \|\widetilde{\nabla}_x f\|_{\mathcal{X}} &\leq \frac{\alpha_2}{4} t^2 \|\widetilde{\nabla}_x f\|_{\mathcal{X}}^2 + C \frac{\alpha_1^2}{\alpha_2} \frac{1}{\varepsilon^2} \|f^\perp\|_{\mathcal{H}_1}^2 \\
 \frac{\alpha_2 t^2 C}{\varepsilon} \|f^\perp\|_{\mathcal{H}_2} \|\widetilde{\nabla}_x f\|_{\mathcal{H}_1} &\leq \frac{\alpha_3 \kappa_2}{2} t^3 \|\widetilde{\nabla}_x f\|_{\mathcal{H}_1}^2 + C \frac{\alpha_2^2}{\alpha_3} \frac{t}{\varepsilon^2} \|f^\perp\|_{\mathcal{H}_2}^2 \\
 2\alpha_2 \varepsilon t \langle \widetilde{\nabla}_x f, \widetilde{\nabla}_v f \rangle_{\mathcal{X}} &\leq \frac{\alpha_2}{4} t^2 \|\widetilde{\nabla}_x f\|_{\mathcal{X}}^2 + C \alpha_2 \varepsilon^2 \|f\|_{\mathcal{H}_1}^2.
 \end{aligned}$$

We therefore deduce that, for any $t \in [0, 1]$, there holds

$$\begin{aligned}
 \frac{d}{dt} \mathcal{U}_\varepsilon(t, f) &\leq -(\kappa_0 - C\alpha_2) \|f\|_{\mathcal{H}_1}^2 - \frac{1}{\varepsilon^2} \left(\kappa_0 - C\alpha_1 - C \frac{\alpha_1^2}{\alpha_2} \right) \|f^\perp\|_{\mathcal{H}_1}^2 \\
 &\quad - \frac{t}{\varepsilon^2} \left(\alpha_1 \kappa_1 - C \frac{\alpha_2^2}{\alpha_3} \right) \|f^\perp\|_{\mathcal{H}_2}^2 - t^2 \left(\frac{\alpha_2}{2} - C\alpha_3 \right) \|\widetilde{\nabla}_x f\|_{\mathcal{X}}^2 \\
 &\quad - \frac{\alpha_3 \kappa_2 t^3}{2} \|\widetilde{\nabla}_x f\|_{\mathcal{H}_1}^2.
 \end{aligned} \tag{4.77}$$

We now choose $\alpha_1 = \eta$, $\alpha_2 = \eta^{\frac{3}{2}}$, and $\alpha_3 = \eta^{\frac{5}{3}}$, with $\eta \in (0, 1)$ small enough such that each quantity appearing inside the parentheses in above inequality is positive. We hence obtain that $\frac{d}{dt} \mathcal{U}_\varepsilon(t, f) \leq 0$ for any $t \in [0, 1]$, which concludes the proof as explained in Step 1. \square

4 Cauchy theory and regularization estimates for the non-linear problem

In this section, we provide a Cauchy theory for (4.7) for small initial data as well as some new regularization estimates for this equation. Notice that our proofs are based on the results developed in Subsection 3.1. It is actually crucial to be able to avoid the use of Duhamel formula to obtain nice estimates on the nonlinear problem because of the singularity in ε that is in front of the nonlinear term in (4.7). Our strategy is to perform direct energy estimates with the norm $\|\cdot\|_{\mathcal{X}}$ introduced in Subsection 3.1 (see (4.166) and (4.57) for the precise definition) and exploit the facts that $\Gamma(f, g) = (\Gamma(f, g))^\perp$ and $\langle \Gamma(f, g), h \rangle_{L^2_{x,v}} = \langle \Gamma(f, g), h^\perp \rangle_{L^2_{x,v}}$ so that

$$\langle \langle \Gamma(f, g), h \rangle \rangle_{L^2_{x,v}} = \langle \Gamma(f, g), h^\perp \rangle_{L^2_{x,v}}$$

and thus

$$\begin{aligned} \langle \langle \Gamma(f, g), h \rangle \rangle_{\mathcal{X}} &= \sum_{i=0}^2 \delta \langle \langle v \rangle^{(3-i)(\frac{\gamma}{2}+1)} \nabla_x^i \Gamma(f, g), \langle v \rangle^{(3-i)(\frac{\gamma}{2}+1)} \nabla_x^i h^\perp \rangle_{L^2_{x,v}} \\ &\quad + \sum_{i=0}^3 \langle \nabla_x^i \Gamma(f, g), \nabla_x^i h^\perp \rangle_{L^2_{x,v}}, \end{aligned}$$

where we recall that $\delta \in (0, 1)$ is a small enough constant chosen in Proposition 4.17. Notice also that we used the particular form of $\|\cdot\|_{L^2_{x,v}}$ defined in (4.166) and the fact that $\pi\Gamma(f, g) = 0$. Rearranging terms, we then deduce

$$\begin{aligned} \langle \langle \Gamma(f, g), h \rangle \rangle_{\mathcal{X}} &= \delta \langle \Gamma(f, g), h^\perp \rangle_{\mathcal{X}} + \sum_{i=0}^2 \langle \nabla_x^i \Gamma(f, g), \nabla_x^i h^\perp \rangle_{L^2_{x,v}} \\ &\quad + (1 - \delta) \langle \nabla_x^3 \Gamma(f, g), \nabla_x^3 h^\perp \rangle_{L^2_{x,v}}. \end{aligned} \tag{4.78}$$

4.1 Bilinear estimates for the Landau operator

In this part, we start by establishing some new and sharp nonlinear estimates on the Landau collision operator, we recall that the matrix $\mathbf{B}(v)$ is defined in (4.38) and that the spaces \mathcal{X} , \mathcal{Y}_1 , \mathcal{Y}_2 and \mathcal{Y}'_1 are respectively defined in (4.19), (4.20), (4.44) and (4.47).

We start by establishing some convolution estimates for the coefficients a_{ij} and b_i :

Lemma 4.20. *For any suitable function $f = f(v)$, vector fields $G = G(v)$, $H = H(v)$ and $\ell \in \{1, 2, 3\}$ there holds, for any $v \in \mathbb{R}^3$:*

$$|(a_{ij} * f)G_i H_j|(v) \lesssim \|\langle v \rangle^\ell f\|_{L^2_v} |\mathbf{B}(v)G(v)| |\mathbf{B}(v)H(v)| \tag{4.79}$$

$$|(\partial_{v_\ell} a_{ij} * f)G_i H_j|(v) \lesssim \|\langle v \rangle^8 f\|_{L_v^2} |\mathbf{B}(v)G(v)| |\mathbf{B}(v)H(v)| \quad (4.80)$$

$$|(\partial_{v_\ell} b_i * f)G_i|(v) \lesssim \|\langle v \rangle^3 f\|_{L_v^2} \langle v \rangle^{\frac{7}{2}} |\mathbf{B}(v)G(v)| \quad \text{if } 0 \leq \gamma \leq 1 \quad (4.81)$$

$$|(\partial_{v_\ell} b_i * f)G_i|(v) \lesssim \left(\|\langle v \rangle^4 f\|_{H_v^1} \langle v \rangle^{-1} + \|\langle v \rangle^4 f\|_{L_v^2} \right) \langle v \rangle^{\frac{7}{2}} |\mathbf{B}(v)G(v)| \quad \text{if } -2 \leq \gamma < 0. \quad (4.82)$$

Proof. We split the proof into three steps.

Step 1. Proof of (4.79). We only prove the estimate for $|v| \geq 1$, the case $|v| < 1$ being trivial. Recalling that P_v denotes the projection onto v , we decompose

$$G_i = (P_v)_i(G) + (\text{Id} - P_v)_i(G) = v_i \left(G \cdot \frac{v}{|v|^2} \right) + (\text{Id} - P_v)_i(G)$$

and

$$H_j = (P_v)_j(H) + (\text{Id} - P_v)_j(H) = v_j \left(H \cdot \frac{v}{|v|^2} \right) + (\text{Id} - P_v)_j(H).$$

We thus obtain

$$\begin{aligned} (a_{ij} * f)(v)G_i(v)H_j(v) &= (a_{ij} * f)(v)v_i v_j \left(G(v) \cdot \frac{v}{|v|^2} \right) \left(H(v) \cdot \frac{v}{|v|^2} \right) \\ &\quad + (a_{ij} * f)(v)v_i \left(G(v) \cdot \frac{v}{|v|^2} \right) (\text{Id} - P_v)_j(H(v)) \\ &\quad + (a_{ij} * f)(v)v_j (\text{Id} - P_v)_i(G(v)) \left(H(v) \cdot \frac{v}{|v|^2} \right) \\ &\quad + (a_{ij} * f)(v)(\text{Id} - P_v)_i(G(v))(\text{Id} - P_v)_j(H(v)). \end{aligned}$$

Using Lemma 4.12, we estimate each term of the previous splitting. First,

$$\begin{aligned} &\left| (a_{ij} * f)(v)v_i v_j \left(G(v) \cdot \frac{v}{|v|^2} \right) \left(H(v) \cdot \frac{v}{|v|^2} \right) \right| \\ &\lesssim \|\langle v \rangle^7 f\|_{L_v^2} \langle v \rangle^{\frac{7}{2}} |G(v)| \langle v \rangle^{\frac{7}{2}} |H(v)|. \end{aligned}$$

Then,

$$\begin{aligned} &\left| (a_{ij} * f)(v)v_i \left(G(v) \cdot \frac{v}{|v|^2} \right) (\text{Id} - P_v)_j(H(v)) \right| \\ &\lesssim \|\langle v \rangle^7 f\|_{L_v^2} \langle v \rangle^{\frac{7}{2}} |G(v)| \langle v \rangle^{\frac{7}{2}+1} |(\text{Id} - P_v)H(v)| \end{aligned}$$

and

$$\begin{aligned} &\left| (a_{ij} * f)(v)v_j (\text{Id} - P_v)_i(G(v)) \left(H(v) \cdot \frac{v}{|v|^2} \right) \right| \\ &\lesssim \|\langle v \rangle^7 f\|_{L_v^2} \langle v \rangle^{\frac{7}{2}+1} |(\text{Id} - P_v)G(v)| \langle v \rangle^{\frac{7}{2}} |H(v)|. \end{aligned}$$

Finally,

$$\begin{aligned} & |(a_{ij} * f)(v)(\text{Id} - P_v)_i(G(v))(\text{Id} - P_v)_j(H(v))| \\ & \lesssim \|\langle v \rangle^\gamma f\|_{L_v^2} \langle v \rangle^{\frac{\gamma}{2}+1} |(\text{Id} - P_v)G(v)| \langle v \rangle^{\frac{\gamma}{2}+1} |(\text{Id} - P_v)H(v)|. \end{aligned}$$

We conclude the proof of (4.79) by gathering previous estimates and recalling that

$$|\mathbf{B}(v)G(v)| \lesssim \langle v \rangle^{\frac{\gamma}{2}} |P_v G(v)| + \langle v \rangle^{\frac{\gamma}{2}+1} |(\text{Id} - P_v)G(v)| \lesssim |\mathbf{B}(v)G(v)|.$$

Step 2. Proof of (4.80). The proof of (4.80) is similar to the one of (4.79) by using the bounds on $(\partial_{v_\ell} a_{ij} * f)$, $(\partial_{v_\ell} a_{ij} * f)v_i$, $(\partial_{v_\ell} a_{ij} * f)v_j$ and $(\partial_{v_\ell} a_{ij} * f)v_i v_j$ given by Lemma 4.12. We thus skip it.

Step 3. Proof of (4.81) and (4.82). Again, we only prove the estimate for $|v| \geq 1$. Recall that $|\partial_{v_\ell} b_i| \lesssim |v|^\gamma$. If $0 \leq \gamma \leq 1$, then using Lemma 4.12, we have

$$|(\partial_{v_\ell} b_i * f)(v)G_i(v)| \lesssim \|\langle v \rangle^3 f\|_{L_v^2} \langle v \rangle^\gamma |G(v)|.$$

If $-2 \leq \gamma < 0$, we use the above decomposition of G to write

$$(\partial_{v_\ell} b_i * f)(v)G_i(v) = (\partial_{v_\ell} b_i * f)(v)v_i \left(G(v) \cdot \frac{v}{|v|^2} \right) + (\partial_{v_\ell} b_i * f)(v)(\text{Id} - P_v)_i(G(v)).$$

Remarking that $(\partial_{v_\ell} b_i * f) = (b_i * \partial_{v_\ell} f)$, we also observe that

$$\begin{aligned} (b_i * \partial_{v_\ell} f)(v)v_i &= (\partial_{v_j} a_{ij} * \partial_{v_\ell} f)(v)v_i \\ &= (a_{ij} * v_i \partial_{v_j} \partial_{v_\ell} f)(v) \\ &= (b_i * (v_i \partial_{v_\ell} f))(v) - (a_{ii} * \partial_{v_\ell} f)(v), \end{aligned}$$

from which we obtain

$$\begin{aligned} (b_i * \partial_{v_\ell} f)(v)v_i &= (b_i * \partial_{v_\ell} [v_i f])(v) - (b_i * (\partial_{v_\ell} v_i) f)(v) - (a_{ii} * \partial_{v_\ell} f)(v) \\ &= (\partial_{v_\ell} b_i * [v_i f])(v) - (b_\ell * f)(v) - (\partial_{v_\ell} a_{ii} * f)(v). \end{aligned}$$

From Lemma 4.12 and using classical Sobolev embeddings, we have

$$|(\partial_{v_\ell} b_i * f)(v)| \lesssim \langle v \rangle^\gamma \|\langle v \rangle^3 f\|_{L_v^4} \lesssim \langle v \rangle^\gamma \|\langle v \rangle^3 f\|_{H_v^1}.$$

Therefore, using once more Lemma 4.12, we obtain

$$|(\partial_{v_\ell} b_i * f)(v)v_i| \lesssim \langle v \rangle^\gamma \|\langle v \rangle^4 f\|_{H_v^1} + \langle v \rangle^{\gamma+1} \|\langle v \rangle^4 f\|_{L_v^2}.$$

Hence,

$$\begin{aligned} & |(\partial_{v_\ell} b_i * f)(v)G_i(v)| \\ & \lesssim \|\langle v \rangle^4 f\|_{H_v^1} \left(\langle v \rangle^\gamma |(\text{Id} - P_v)G(v)| + \langle v \rangle^{\gamma-1} |G(v)| \right) + \|\langle v \rangle^4 f\|_{L_v^2} \langle v \rangle^\gamma |G(v)|, \end{aligned}$$

which concludes the proof. \square

We shall now establish bilinear estimates for the nonlinear operator Γ in Propositions 4.21, 4.22, 4.23 and 4.24 below. Recall from (4.35) that

$$\Gamma(g_1, g_2) = \Gamma_1(g_1, g_2) + \Gamma_2(g_1, g_2) + \Gamma_3(g_1, g_2) + \Gamma_4(g_1, g_2) + \Gamma_5(g_1, g_2)$$

with

$$\Gamma_1(g_1, g_2) = \partial_{v_i} \left\{ \left(a_{ij} * [\sqrt{M}g_1] \right) \partial_{v_j} g_2 \right\}, \quad (4.83)$$

$$\Gamma_2(g_1, g_2) = -\partial_{v_i} \left\{ \left(b_i * [\sqrt{M}g_1] \right) g_2 \right\}, \quad (4.84)$$

$$\Gamma_3(g_1, g_2) = - \left(a_{ij} * [\sqrt{M}g_1] \right) v_i \partial_{v_j} g_2, \quad (4.85)$$

$$\Gamma_4(g_1, g_2) = \frac{1}{4} \left(a_{ij} * [\sqrt{M}g_1] \right) v_i v_j g_2, \quad (4.86)$$

$$\Gamma_5(g_1, g_2) = -\frac{1}{2} \left(a_{ii} * [\sqrt{M}g_1] \right) g_2. \quad (4.87)$$

Proposition 4.21. *Let g_1, g_2 and g_3 be smooth enough functions. For any $\alpha \in \mathbb{R}$, there holds*

$$\langle \langle v \rangle^\alpha \Gamma(g_1, g_2), g_3 \rangle_{\mathcal{X}} \lesssim \|g_1\|_{\mathcal{X}} \|\langle v \rangle^\alpha g_2\|_{\mathcal{X}_1} \|g_3\|_{\mathcal{X}_1}. \quad (4.88)$$

As a consequence, one has by duality

$$\|\Gamma(g_1, g_2)\|_{\mathcal{X}'} \lesssim \|g_1\|_{\mathcal{X}} \|g_2\|_{\mathcal{X}_1}. \quad (4.89)$$

Proof of Proposition 4.21. We shall first prove that for any $\alpha \in \mathbb{R}$, there holds

$$\langle \langle v \rangle^\alpha \Gamma(g_1, g_2), g_3 \rangle_{L_v^2} \lesssim \|M^{\frac{1}{4}} g_1\|_{L_v^2} \|\langle v \rangle^\alpha g_2\|_{H_{v,*}^1} \|g_3\|_{H_{v,*}^1}. \quad (4.90)$$

Once this estimate is established, we shall prove (4.88) in the final step of the proof by integrating it in x and using Sobolev embeddings. Estimate (4.89) is then a direct consequence of (4.88). We thus write $\Gamma(g_1, g_2) = \Gamma_1(g_1, g_2) + \dots + \Gamma_5(g_1, g_2)$ as in (4.83)–(4.87), and we estimate each term separately in the sequel.

Step 1. We write, from (4.83) and making an integration by parts,

$$\begin{aligned} \langle \langle v \rangle^\alpha \Gamma_1(g_1, g_2), g_3 \rangle_{L_v^2} &= - \left\langle \left(a_{ij} * [\sqrt{M}g_1] \right) \partial_{v_j} g_2, \partial_{v_i} (\langle v \rangle^\alpha g_3) \right\rangle_{L_v^2} \\ &= - \left\langle \left(a_{ij} * [\sqrt{M}g_1] \right) \partial_{v_j} g_2, \langle v \rangle^\alpha \partial_{v_i} g_3 \right\rangle_{L_v^2} - \left\langle \left(a_{ij} * [\sqrt{M}g_1] \right) \partial_{v_j} g_2, (\partial_{v_i} \langle v \rangle^\alpha) g_3 \right\rangle_{L_v^2} \\ &=: I_1 + I_2. \end{aligned}$$

For the term I_1 , we use Lemma 4.20, which yields

$$\begin{aligned} I_1 &\lesssim \|M^{\frac{1}{4}} g_1\|_{L_v^2} \left\langle \langle v \rangle^\alpha |\widetilde{\nabla}_v g_2|, |\widetilde{\nabla}_v g_3| \right\rangle_{L_v^2} \\ &\lesssim \|M^{\frac{1}{4}} g_1\|_{L_v^2} \|\langle v \rangle^\alpha \widetilde{\nabla}_v g_2\|_{L_v^2} \|\widetilde{\nabla}_v g_3\|_{L_v^2}. \end{aligned}$$

In a similar way, thanks to Lemma 4.20 and using that $|\widetilde{\nabla}_v \langle v \rangle^\alpha| \lesssim \langle v \rangle^{\frac{\gamma}{2}-1+\alpha}$, we obtain

$$\begin{aligned} I_2 &\lesssim \|M^{\frac{1}{4}}g_1\|_{L_v^2} \left\langle \langle v \rangle^{\frac{\gamma}{2}-1} \langle v \rangle^\alpha |\widetilde{\nabla}_v g_2|, |g_3| \right\rangle_{L_v^2} \\ &\lesssim \|M^{\frac{1}{4}}g_1\|_{L_v^2} \|\langle v \rangle^\alpha \widetilde{\nabla}_v g_2\|_{L_v^2} \|\langle v \rangle^{\frac{\gamma}{2}-1} g_3\|_{L_v^2}. \end{aligned}$$

We therefore obtain

$$\langle \langle v \rangle^\alpha \Gamma_1(g_1, g_2), g_3 \rangle_{L_v^2} \lesssim \|M^{\frac{1}{4}}g_1\|_{L_v^2} \|\langle v \rangle^\alpha \widetilde{\nabla}_v g_2\|_{L_v^2} \|g_3\|_{H_{v,*}^1}. \quad (4.91)$$

Step 2. Starting from (4.84) and making an integration by parts, we get

$$\begin{aligned} \langle \langle v \rangle^\alpha \Gamma_2(g_1, g_2), g_3 \rangle_{L_v^2} &= \left\langle (b_i * [\sqrt{M}g_1]) g_2, \partial_{v_i} (\langle v \rangle^\alpha g_3) \right\rangle_{L_v^2} \\ &= \left\langle (b_i * [\sqrt{M}g_1]) g_2, \langle v \rangle^\alpha \partial_{v_i} g_3 \right\rangle_{L_v^2} + \left\langle (b_i * [\sqrt{M}g_1]) g_2, (\partial_{v_i} \langle v \rangle^\alpha) g_3 \right\rangle_{L_v^2} \\ &=: \Pi_1 + \Pi_2. \end{aligned}$$

For the term Π_1 , we use Lemma 4.12 to obtain

$$\begin{aligned} \Pi_1 &\lesssim \|M^{\frac{1}{4}}g_1\|_{L_v^2} \left\langle \langle v \rangle^{\gamma+1} \langle v \rangle^\alpha |g_2|, |\nabla_v g_3| \right\rangle_{L_v^2} \\ &\lesssim \|M^{\frac{1}{4}}g_1\|_{L_v^2} \|\langle v \rangle^\alpha \langle v \rangle^{\frac{\gamma}{2}+1} g_2\|_{L_v^2} \|\langle v \rangle^{\frac{\gamma}{2}} \nabla_v g_3\|_{L_v^2}. \end{aligned}$$

In a similar fashion, Lemma 4.12 yields

$$\begin{aligned} \Pi_2 &\lesssim \|M^{\frac{1}{4}}g_1\|_{L_v^2} \left\langle \langle v \rangle^{\gamma+1} \langle v \rangle^{\alpha-1} |g_2|, |g_3| \right\rangle_{L_v^2} \\ &\lesssim \|M^{\frac{1}{4}}g_1\|_{L_v^2} \|\langle v \rangle^\alpha \langle v \rangle^{\frac{\gamma}{2}+1} g_2\|_{L_v^2} \|\langle v \rangle^{\frac{\gamma}{2}-1} g_3\|_{L_v^2}. \end{aligned}$$

We thus get

$$\langle \langle v \rangle^\alpha \Gamma_2(g_1, g_2), g_3 \rangle_{L_v^2} \lesssim \|M^{\frac{1}{4}}g_1\|_{L_v^2} \|\langle v \rangle^\alpha \langle v \rangle^{\frac{\gamma}{2}+1} g_2\|_{L_v^2} \|g_3\|_{H_{v,*}^1}. \quad (4.92)$$

Step 3. All the remainder terms associated to Γ_3 , Γ_4 and Γ_5 can be estimated directly thanks to Lemma 4.12 or Lemma 4.20 and Cauchy-Schwarz inequality. Indeed we first have, using Lemma 4.20 and $|\mathbf{B}(v)v| \lesssim \langle v \rangle^{\frac{\gamma}{2}+1}$,

$$\begin{aligned} \langle \langle v \rangle^\alpha \Gamma_3(g_1, g_2), g_3 \rangle_{L_v^2} &= - \left\langle (a_{ij} * [\sqrt{M}g_1]) v_i \partial_{v_j} g_2, \langle v \rangle^\alpha g_3 \right\rangle_{L_v^2} \\ &\lesssim \|M^{\frac{1}{4}}g_1\|_{L_v^2} \left\langle \langle v \rangle^\alpha \langle v \rangle^{\frac{\gamma}{2}+1} |\widetilde{\nabla}_v g_2|, |g_3| \right\rangle_{L_v^2} \\ &\lesssim \|M^{\frac{1}{4}}g_1\|_{L_v^2} \|\langle v \rangle^\alpha \widetilde{\nabla}_v g_2\|_{L_v^2} \|\langle v \rangle^{\frac{\gamma}{2}+1} g_3\|_{L_v^2}. \end{aligned} \quad (4.93)$$

In a similar way, we also get

$$\begin{aligned} \langle \langle v \rangle^\alpha \Gamma_4(g_1, g_2), g_3 \rangle_{L_v^2} &= \frac{1}{4} \left\langle (a_{ij} * [\sqrt{M}g_1]) v_i v_j g_2, \langle v \rangle^\alpha g_3 \right\rangle_{L_v^2} \\ &\lesssim \|M^{\frac{1}{4}}g_1\|_{L_v^2} \left\langle \langle v \rangle^{\gamma+2} \langle v \rangle^\alpha |g_2|, |g_3| \right\rangle_{L_v^2} \\ &\lesssim \|M^{\frac{1}{4}}g_1\|_{L_v^2} \|\langle v \rangle^\alpha \langle v \rangle^{\frac{\gamma}{2}+1} g_2\|_{L_v^2} \|\langle v \rangle^{\frac{\gamma}{2}+1} g_3\|_{L_v^2}. \end{aligned} \quad (4.94)$$

Finally, now using Lemma 4.12, we obtain

$$\begin{aligned}
 \langle \langle v \rangle^\alpha \Gamma_5(g_1, g_2), g_3 \rangle_{L_v^2} &= -\frac{1}{2} \langle \langle v \rangle^\alpha (a_{ii} * [\sqrt{M} g_1]) g_2, g_3 \rangle_{L_v^2} \\
 &\lesssim \|M^{\frac{1}{4}} g_1\|_{L_v^2} \langle \langle v \rangle^\alpha \langle v \rangle^{\gamma+2} |g_2|, |g_3| \rangle_{L_v^2} \\
 &\lesssim \|M^{\frac{1}{4}} g_1\|_{L_v^2} \|\langle v \rangle^\alpha \langle v \rangle^{\frac{\gamma}{2}+1} g_2\|_{L_v^2} \|\langle v \rangle^{\frac{\gamma}{2}+1} g_3\|_{L_v^2}.
 \end{aligned} \tag{4.95}$$

We thus conclude the proof of (4.90) by gathering estimates (4.91)–(4.92)–(4.93)–(4.94)–(4.95) and observing that

$$\|\langle v \rangle^\alpha g_2\|_{H_{v,*}^1}^2 \lesssim \|\langle v \rangle^\alpha \widetilde{\nabla}_v g_2\|_{L_v^2}^2 + \|\langle v \rangle^\alpha \langle v \rangle^{\frac{\gamma}{2}+1} g_2\|_{L_v^2}^2 \lesssim \|\langle v \rangle^\alpha g_2\|_{H_{v,*}^1}^2.$$

Step 4. Proof of (4.88). Recalling the definition of $\langle \cdot, \cdot \rangle_{\mathcal{X}}$ in (4.19), we have

$$\begin{aligned}
 \langle \langle v \rangle^\alpha \Gamma(g_1, g_2), g_3 \rangle_{\mathcal{X}} &= \sum_{i=0}^3 \langle \langle v \rangle^\alpha \langle v \rangle^{(3-i)(\frac{\gamma}{2}+1)} \nabla_x^i \Gamma(g_1, g_2), \langle v \rangle^{(3-i)(\frac{\gamma}{2}+1)} \nabla_x^i g_3 \rangle_{L_{x,v}^2} \\
 &=: T_0 + T_1 + T_2 + T_3.
 \end{aligned}$$

Thanks to (4.90) and the fact that $\|\cdot\|_{L_x^\infty} \lesssim \|\cdot\|_{H_x^2}$, we get

$$\begin{aligned}
 T_0 &\lesssim \int_{\mathbf{T}^3} \|M^{\frac{1}{4}} g_1\|_{L_v^2} \|\langle v \rangle^\alpha \langle v \rangle^{3(\frac{\gamma}{2}+1)} g_2\|_{H_{v,*}^1} \|\langle v \rangle^{3(\frac{\gamma}{2}+1)} g_3\|_{H_{v,*}^1} dx \\
 &\lesssim \|M^{\frac{1}{4}} g_1\|_{H_x^2 L_v^2} \|\langle v \rangle^\alpha \langle v \rangle^{3(\frac{\gamma}{2}+1)} g_2\|_{L_x^2(H_{v,*}^1)} \|\langle v \rangle^{3(\frac{\gamma}{2}+1)} g_3\|_{L_x^2(H_{v,*}^1)} \\
 &\lesssim \|g_1\|_{\mathcal{X}} \|\langle v \rangle^\alpha g_2\|_{\mathcal{X}_1} \|g_3\|_{\mathcal{X}_1}.
 \end{aligned}$$

Using that $\partial_{x_k} \Gamma(g_1, g_2) = \Gamma(\partial_{x_k} g_1, g_2) + \Gamma(g_1, \partial_{x_k} g_2)$, Hölder inequality and the fact that from classical Sobolev embeddings, $\|\cdot\|_{L_x^6} + \|\cdot\|_{L_x^3} \lesssim \|\cdot\|_{H_x^1}$, estimate (4.90) yields

$$\begin{aligned}
 T_1 &\lesssim \int_{\mathbf{T}^3} \|M^{\frac{1}{4}} \nabla_x g_1\|_{L_v^2} \|\langle v \rangle^{2(\frac{\gamma}{2}+1)} \langle v \rangle^\alpha g_2\|_{H_{v,*}^1} \|\langle v \rangle^{2(\frac{\gamma}{2}+1)} \nabla_x g_3\|_{H_{v,*}^1} dx \\
 &\quad + \int_{\mathbf{T}^3} \|M^{\frac{1}{4}} g_1\|_{L_v^2} \|\langle v \rangle^{2(\frac{\gamma}{2}+1)} \langle v \rangle^\alpha \nabla_x g_2\|_{H_{v,*}^1} \|\langle v \rangle^{2(\frac{\gamma}{2}+1)} \nabla_x g_3\|_{H_{v,*}^1} dx \\
 &\lesssim \|M^{\frac{1}{4}} \nabla_x g_1\|_{H_x^1 L_v^2} \|\langle v \rangle^{2(\frac{\gamma}{2}+1)} \langle v \rangle^\alpha g_2\|_{H_x^1(H_{v,*}^1)} \|\langle v \rangle^{2(\frac{\gamma}{2}+1)} \nabla_x g_3\|_{L_x^2(H_{v,*}^1)} \\
 &\quad + \|M^{\frac{1}{4}} g_1\|_{H_x^2 L_v^2} \|\langle v \rangle^{2(\frac{\gamma}{2}+1)} \langle v \rangle^\alpha \nabla_x g_2\|_{L_x^2(H_{v,*}^1)} \|\langle v \rangle^{2(\frac{\gamma}{2}+1)} \nabla_x g_3\|_{L_x^2(H_{v,*}^1)} \\
 &\lesssim \|g_1\|_{\mathcal{X}} \|\langle v \rangle^\alpha g_2\|_{\mathcal{X}_1} \|g_3\|_{\mathcal{X}_1}.
 \end{aligned}$$

Moreover, for the term T_2 , similarly, we have

$$\begin{aligned}
 T_2 &\lesssim \int_{\mathbf{T}^3} \left(\|M^{\frac{1}{4}} \nabla_x^2 g_1\|_{L_v^2} \|\langle v \rangle^{\frac{\gamma}{2}+1} \langle v \rangle^\alpha g_2\|_{H_{v,*}^1} + \|M^{\frac{1}{4}} \nabla_x g_1\|_{L_v^2} \|\langle v \rangle^{\frac{\gamma}{2}+1} \langle v \rangle^\alpha \nabla_x g_2\|_{H_{v,*}^1} \right. \\
 &\quad \left. + \|M^{\frac{1}{4}} g_1\|_{L_v^2} \|\langle v \rangle^{\frac{\gamma}{2}+1} \langle v \rangle^\alpha \nabla_x^2 g_2\|_{H_{v,*}^1} \right) \|\langle v \rangle^{\frac{\gamma}{2}+1} \nabla_x^2 g_3\|_{H_{v,*}^1} dx \\
 &\lesssim \left(\|M^{\frac{1}{4}} \nabla_x^2 g_1\|_{L_x^2 L_v^2} \|\langle v \rangle^{\frac{\gamma}{2}+1} \langle v \rangle^\alpha g_2\|_{H_x^2(H_{v,*}^1)} \right. \\
 &\quad + \|M^{\frac{1}{4}} \nabla_x g_1\|_{H_x^1 L_v^2} \|\langle v \rangle^{\frac{\gamma}{2}+1} \langle v \rangle^\alpha \nabla_x g_2\|_{H_x^1(H_{v,*}^1)} \\
 &\quad \left. + \|M^{\frac{1}{4}} g_1\|_{H_x^2 L_v^2} \|\langle v \rangle^{\frac{\gamma}{2}+1} \langle v \rangle^\alpha \nabla_x^2 g_2\|_{L_x^2(H_{v,*}^1)} \right) \|\langle v \rangle^{\frac{\gamma}{2}+1} \nabla_x^2 g_3\|_{L_x^2(H_{v,*}^1)} \\
 &\lesssim \|g_1\|_{\mathcal{X}} \|\langle v \rangle^\alpha g_2\|_{\mathcal{Y}_1} \|g_3\|_{\mathcal{Y}_1}.
 \end{aligned}$$

Finally, for the term T_3 , we have

$$\begin{aligned}
 T_3 &\lesssim \int_{\mathbf{T}^3} \left(\|M^{\frac{1}{4}} \nabla_x^3 g_1\|_{L_v^2} \|\langle v \rangle^\alpha g_2\|_{H_{v,*}^1} + \|M^{\frac{1}{4}} \nabla_x^2 g_1\|_{L_v^2} \|\langle v \rangle^\alpha \nabla_x g_2\|_{H_{v,*}^1} \right. \\
 &\quad \left. + \|M^{\frac{1}{4}} \nabla_x g_1\|_{L_v^2} \|\langle v \rangle^\alpha \nabla_x^2 g_2\|_{H_{v,*}^1} + \|M^{\frac{1}{4}} g_1\|_{L_v^2} \|\langle v \rangle^\alpha \nabla_x^3 g_2\|_{H_{v,*}^1} \right) \|\nabla_x^3 g_3\|_{H_{v,*}^1} dx \\
 &\lesssim \left(\|M^{\frac{1}{4}} \nabla_x^3 g_1\|_{L_x^2 L_v^2} \|\langle v \rangle^\alpha g_2\|_{H_x^2(H_{v,*}^1)} + \|M^{\frac{1}{4}} \nabla_x^2 g_1\|_{H_x^1 L_v^2} \|\langle v \rangle^\alpha \nabla_x g_2\|_{H_x^1(H_{v,*}^1)} \right. \\
 &\quad \left. + \|M^{\frac{1}{4}} \nabla_x g_1\|_{H_x^2 L_v^2} \|\langle v \rangle^\alpha \nabla_x^2 g_2\|_{L_x^2(H_{v,*}^1)} + \|M^{\frac{1}{4}} g_1\|_{H_x^2 L_v^2} \|\langle v \rangle^\alpha \nabla_x^3 g_2\|_{L_x^2(H_{v,*}^1)} \right) \|\nabla_x^3 g_3\|_{L_x^2(H_{v,*}^1)} \\
 &\lesssim \|g_1\|_{\mathcal{X}} \|\langle v \rangle^\alpha g_2\|_{\mathcal{Y}_1} \|g_3\|_{\mathcal{Y}_1},
 \end{aligned}$$

which concludes the proof of (4.88).

Step 5. Proof of (4.89). The result is immediate using the definition of the norm of \mathcal{Y}'_1 given in (4.47) and (4.88). \square

Proposition 4.22. *Let g_1, g_2 be smooth enough functions and G_3 a smooth enough vector field, then*

$$\left\langle \widetilde{\nabla}_v \Gamma(g_1, g_2), G_3 \right\rangle_{\mathcal{X}} \lesssim \left(\|g_1\|_{\mathcal{X}} \|g_2\|_{\mathcal{Y}_2} + \|g_1\|_{\mathcal{Y}_1} \|g_2\|_{\mathcal{X}} \right) \|G_3\|_{\mathcal{Y}_1}. \quad (4.96)$$

Proof of Proposition 4.22. We shall only prove that for any $\alpha \in \mathbb{R}$, there holds

$$\begin{aligned}
 &\left\langle \langle v \rangle^\alpha \widetilde{\nabla}_v \Gamma(g_1, g_2), G_3 \right\rangle_{L_v^2} \\
 &\lesssim \left(\|M^{\frac{1}{4}} g_1\|_{L_v^2} \|\langle v \rangle^\alpha g_2\|_{H_{v,*}^2} + \|M^{\frac{1}{4}} g_1\|_{H_v^1} \|\langle v \rangle^\alpha g_2\|_{L_v^2} \right) \|G_3\|_{H_{v,*}^1},
 \end{aligned} \quad (4.97)$$

from which we obtain the desired result by integrating in x and arguing as in Step 4 of the proof of Proposition 4.21. We thus write $\Gamma(g_1, g_2) = \Gamma_1(g_1, g_2) + \dots + \Gamma_5(g_1, g_2)$ as in (4.83)–(4.87), and we estimate each term separately in the sequel. We shall use during the proof the following equivalence:

$$\begin{aligned} \|\langle v \rangle^\alpha g_2\|_{H_{v,*}^2} &\lesssim \|\langle v \rangle^\alpha \langle v \rangle^{\gamma+2} g_2\|_{L_v^2} + \|\langle v \rangle^\alpha \langle v \rangle^{\frac{\gamma}{2}+1} \widetilde{\nabla}_v g_2\|_{L_v^2} \\ &\quad + \|\langle v \rangle^\alpha \widetilde{\nabla}_v (\widetilde{\nabla}_v g_2)\|_{L_v^2} \lesssim \|\langle v \rangle^\alpha g_2\|_{H_{v,*}^2}. \end{aligned}$$

Step 1. Term associated to Γ_1 . Writing $\widetilde{\nabla}_{v_k} = B_{k\ell} \partial_{v_\ell}$ and observing that $[\widetilde{\nabla}_{v_k}, \partial_{v_i}] = -(\partial_{v_i} B_{k\ell}) \partial_{v_\ell}$, we first get that for any $k \in \{1, 2, 3\}$,

$$\begin{aligned} \widetilde{\nabla}_{v_k} \Gamma_1(g_1, g_2) &= \partial_{v_i} \left\{ (a_{ij} * [\sqrt{M}g_1]) \partial_{v_j} (\widetilde{\nabla}_{v_k} g_2) \right\} + \partial_{v_i} \left\{ \widetilde{\nabla}_{v_k} (a_{ij} * [\sqrt{M}g_1]) \partial_{v_j} g_2 \right\} \\ &\quad - \partial_{v_i} \left\{ (a_{ij} * [\sqrt{M}g_1]) (\partial_{v_j} B_{k\ell}) \partial_{v_\ell} g_2 \right\} - (\partial_{v_i} B_{k\ell}) \partial_{v_\ell} \left\{ (a_{ij} * [\sqrt{M}g_1]) \partial_{v_j} g_2 \right\} \end{aligned}$$

whence

$$\left\langle \langle v \rangle^\alpha \widetilde{\nabla}_{v_k} \Gamma_1(g_1, g_2), G_{3,k} \right\rangle_{L_v^2} = I_1 + I_2 + I_3 + I_4,$$

where $G_{3,k}$ denotes the k -th component of G_3 .

For the term I_1 , we first make an integration by parts, then we use Lemma 4.20 and the fact that $|\widetilde{\nabla}_v \langle v \rangle^\alpha| \lesssim \langle v \rangle^{\frac{\gamma}{2}-1} \langle v \rangle^\alpha$ together with Cauchy-Schwarz inequality to obtain

$$\begin{aligned} I_1 &= - \left\langle \langle v \rangle^\alpha (a_{ij} * [\sqrt{M}g_1]) \partial_{v_j} (\widetilde{\nabla}_{v_k} g_2), \partial_{v_i} G_{3,k} \right\rangle_{L_v^2} \\ &\quad - \left\langle (a_{ij} * [\sqrt{M}g_1]) \partial_{v_j} (\widetilde{\nabla}_{v_k} g_2) \partial_{v_i} \langle v \rangle^\alpha, G_{3,k} \right\rangle_{L_v^2} \\ &\lesssim \|M^{\frac{1}{4}} g_1\|_{L_v^2} \left(\left\langle \langle v \rangle^\alpha |\widetilde{\nabla}_v (\widetilde{\nabla}_{v_k} g_2)|, |\widetilde{\nabla}_v G_{3,k}| \right\rangle_{L_v^2} + \left\langle \langle v \rangle^{\frac{\gamma}{2}-1} \langle v \rangle^\alpha |\widetilde{\nabla}_v (\widetilde{\nabla}_{v_k} g_2)|, |G_{3,k}| \right\rangle_{L_v^2} \right) \\ &\lesssim \|M^{\frac{1}{4}} g_1\|_{L_v^2} \|\langle v \rangle^\alpha \widetilde{\nabla}_v (\widetilde{\nabla}_{v_k} g_2)\|_{L_v^2} \left(\|\widetilde{\nabla}_v G_3\|_{L_v^2} + \|\langle v \rangle^{\frac{\gamma}{2}-1} G_3\|_{L_v^2} \right). \end{aligned}$$

We argue in a similar fashion for the term I_2 . We first make an integration by parts and write that $\widetilde{\nabla}_{v_k} (a_{ij} * [\sqrt{M}g_1]) = B_{k\ell} (\partial_{v_\ell} a_{ij} * [\sqrt{M}g_1])$, then we use Lemma 4.20 together with $|B_{k\ell}| \lesssim \langle v \rangle^{\frac{\gamma}{2}+1}$, thus we obtain

$$\begin{aligned} I_2 &= - \left\langle \langle v \rangle^\alpha B_{k\ell} (\partial_{v_\ell} a_{ij} * [\sqrt{M}g_1]) \partial_{v_j} g_2, \partial_{v_i} G_{3,k} \right\rangle_{L_v^2} \\ &\quad - \left\langle B_{k\ell} (\partial_{v_\ell} a_{ij} * [\sqrt{M}g_1]) \partial_{v_j} g_2 \partial_{v_i} \langle v \rangle^\alpha, G_{3,k} \right\rangle_{L_v^2} \\ &\lesssim \|M^{\frac{1}{4}} g_1\|_{L_v^2} \left(\left\langle \langle v \rangle^\alpha \langle v \rangle^{\frac{\gamma}{2}+1} |\widetilde{\nabla}_v g_2|, |\widetilde{\nabla}_v G_{3,k}| \right\rangle_{L_v^2} + \left\langle \langle v \rangle^{\frac{\gamma}{2}+1} \langle v \rangle^{\frac{\gamma}{2}-1} \langle v \rangle^\alpha |\widetilde{\nabla}_v g_2|, |G_{3,k}| \right\rangle_{L_v^2} \right) \\ &\lesssim \|M^{\frac{1}{4}} g_1\|_{L_v^2} \|\langle v \rangle^\alpha \langle v \rangle^{\frac{\gamma}{2}+1} \widetilde{\nabla}_v g_2\|_{L_v^2} \left(\|\widetilde{\nabla}_v G_3\|_{L_v^2} + \|\langle v \rangle^{\frac{\gamma}{2}-1} G_3\|_{L_v^2} \right). \end{aligned}$$

For the term I_3 , arguing similarly as above using also that $|\widetilde{\nabla}_v B_{k\ell}| \lesssim \langle v \rangle^{\frac{\gamma}{2}+1} \langle v \rangle^{\frac{\gamma}{2}}$, we get

$$\begin{aligned} I_3 &= \left\langle \langle v \rangle^\alpha \left(a_{ij} * [\sqrt{M} g_1] \right) (\partial_{v_j} B_{k\ell}) \partial_{v_\ell} g_2, \partial_{v_i} G_{3,k} \right\rangle_{L_v^2} \\ &\quad + \left\langle \left(a_{ij} * [\sqrt{M} g_1] \right) (\partial_{v_j} B_{k\ell}) \partial_{v_\ell} g_2 \partial_{v_i} \langle v \rangle^\alpha, G_{3,k} \right\rangle_{L_v^2} \\ &\lesssim \|M^{\frac{1}{4}} g_1\|_{L_v^2} \left(\left\langle \langle v \rangle^\alpha \langle v \rangle^{\frac{\gamma}{2}+1} \langle v \rangle^{\frac{\gamma}{2}} |\nabla_v g_2|, |\widetilde{\nabla}_v G_3| \right\rangle_{L_v^2} \right. \\ &\quad \left. + \left\langle \langle v \rangle^{\frac{\gamma}{2}+1} \langle v \rangle^{\frac{\gamma}{2}} \langle v \rangle^{\frac{\gamma}{2}-1} \langle v \rangle^\alpha |\nabla_v g_2|, |G_3| \right\rangle_{L_v^2} \right) \\ &\lesssim \|M^{\frac{1}{4}} g_1\|_{L_v^2} \|\langle v \rangle^\alpha \langle v \rangle^{\frac{\gamma}{2}+1} \widetilde{\nabla}_v g_2\|_{L_v^2} \left(\|\widetilde{\nabla}_v G_3\|_{L_v^2} + \|\langle v \rangle^{\frac{\gamma}{2}-1} G_3\|_{L_v^2} \right). \end{aligned}$$

We treat the term I_4 in the same way, first performing an integration by parts and using also that $|\partial_{v_i} \partial_{v_\ell} B_{k\ell}| \lesssim \langle v \rangle^{\frac{\gamma}{2}-1}$ and $\langle v \rangle^{\frac{\gamma}{2}} |\nabla_v G_3| \lesssim |\widetilde{\nabla}_v G_3|$, it gives us

$$\begin{aligned} I_4 &= \left\langle \langle v \rangle^\alpha \left(a_{ij} * [\sqrt{M} g_1] \right) \partial_{v_j} g_2 (\partial_{v_i} B_{k\ell}), \partial_{v_\ell} G_{3,k} \right\rangle_{L_v^2} \\ &\quad + \left\langle \left(a_{ij} * [\sqrt{M} g_1] \right) \partial_{v_j} g_2 (\partial_{v_i} B_{k\ell}) \partial_{v_\ell} \langle v \rangle^\alpha, G_{3,k} \right\rangle_{L_v^2} \\ &\quad + \left\langle \langle v \rangle^\alpha \left(a_{ij} * [\sqrt{M} g_1] \right) \partial_{v_j} g_2, (\partial_{v_i} \partial_{v_\ell} B_{k\ell}) G_{3,k} \right\rangle_{L_v^2} \\ &\lesssim \|M^{\frac{1}{4}} g_1\|_{L_v^2} \left(\left\langle \langle v \rangle^\alpha \langle v \rangle^{\frac{\gamma}{2}+1} \langle v \rangle^{\frac{\gamma}{2}} |\widetilde{\nabla}_v g_2|, |\nabla_v G_3| \right\rangle_{L_v^2} + \left\langle \langle v \rangle^\alpha \langle v \rangle^{\frac{\gamma}{2}+1} \langle v \rangle^{\frac{\gamma}{2}-1} |\widetilde{\nabla}_v g_2|, |G_3| \right\rangle_{L_v^2} \right) \\ &\lesssim \|M^{\frac{1}{4}} g_1\|_{L_v^2} \|\langle v \rangle^\alpha \langle v \rangle^{\frac{\gamma}{2}+1} \widetilde{\nabla}_v g_2\|_{L_v^2} \left(\|\widetilde{\nabla}_v G_3\|_{L_v^2} + \|\langle v \rangle^{\frac{\gamma}{2}-1} G_3\|_{L_v^2} \right). \end{aligned}$$

Finally, gathering previous estimates, we get

$$\left\langle \langle v \rangle^\alpha \widetilde{\nabla}_v \Gamma_1(g_1, g_2), G_3 \right\rangle_{L_v^2} \lesssim \|M^{\frac{1}{4}} g_1\|_{L_v^2} \|\langle v \rangle^\alpha g_2\|_{H_{v,*}^2} \|G_3\|_{H_{v,*}^1}. \quad (4.98)$$

Step 2. Term associated to Γ_2 . We write $\widetilde{\nabla}_{v_k} = B_{k\ell} \partial_{v_\ell}$ so that $[\widetilde{\nabla}_{v_k}, \partial_{v_i}] = -(\partial_{v_i} B_{k\ell}) \partial_{v_\ell}$, thus we get, for any $k \in \{1, 2, 3\}$,

$$\begin{aligned} \widetilde{\nabla}_{v_k} \Gamma_2 &= -\widetilde{\nabla}_{v_k} \partial_{v_i} \left\{ \left(b_i * [\sqrt{M} g_1] \right) g_2 \right\} \\ &= -\partial_{v_i} \left\{ \left(b_i * [\sqrt{M} g_1] \right) \widetilde{\nabla}_{v_k} g_2 \right\} - \partial_{v_i} \left\{ \widetilde{\nabla}_{v_k} \left(b_i * [\sqrt{M} g_1] \right) g_2 \right\} \\ &\quad + (\partial_{v_i} B_{k\ell}) \partial_{v_\ell} \left\{ \left(b_i * [\sqrt{M} g_1] \right) g_2 \right\}, \end{aligned}$$

whence

$$\left\langle \langle v \rangle^\alpha \widetilde{\nabla}_{v_k} \Gamma_2(g_1, g_2), G_{3,k} \right\rangle = \Pi_1 + \Pi_2 + \Pi_3.$$

For the term II_1 , we make an integrating by parts and then use Lemma 4.12 together with $\langle v \rangle^{\frac{\gamma}{2}} |\nabla_v G_3| \lesssim |\widetilde{\nabla}_v G_3|$, which yields

$$\begin{aligned} \text{II}_1 &= \left\langle \langle v \rangle^\alpha (b_i * [\sqrt{M}g_1]) \widetilde{\nabla}_{v_k} g_2, \partial_{v_i} G_{3,k} \right\rangle_{L_v^2} + \left\langle (b_i * [\sqrt{M}g_1]) \widetilde{\nabla}_{v_k} g_2 \partial_{v_i} \langle v \rangle^\alpha, G_{3,k} \right\rangle_{L_v^2} \\ &\lesssim \|M^{\frac{1}{4}} g_1\|_{L_v^2} \left(\left\langle \langle v \rangle^\alpha \langle v \rangle^{\gamma+1} |\widetilde{\nabla}_{v_k} g_2|, |\nabla_v G_{3,k}| \right\rangle_{L_v^2} + \left\langle \langle v \rangle^{\gamma+1} |\widetilde{\nabla}_{v_k} g_2| \langle v \rangle^{\alpha-1}, |G_{3,k}| \right\rangle_{L_v^2} \right) \\ &\lesssim \|M^{\frac{1}{4}} g_1\|_{L_v^2} \|\langle v \rangle^\alpha \langle v \rangle^{\frac{\gamma}{2}+1} \widetilde{\nabla}_v g_2\|_{L_v^2} \left(\|\widetilde{\nabla}_v G_3\|_{L_v^2} + \|\langle v \rangle^{\frac{\gamma}{2}-1} G_3\|_{L_v^2} \right). \end{aligned}$$

For the term II_2 , we first make an integration by parts using that $\widetilde{\nabla}_{v_k} (b_i * [\sqrt{M}g_1]) = B_{k\ell} (\partial_{v_\ell} b_i * [\sqrt{M}g_1])$, which yields

$$\begin{aligned} \text{II}_2 &= \left\langle \langle v \rangle^\alpha B_{k\ell} (\partial_{v_\ell} b_i * [\sqrt{M}g_1]) g_2, \partial_{v_i} G_{3,k} \right\rangle_{L_v^2} \\ &\quad + \left\langle B_{k\ell} (\partial_{v_\ell} b_i * [\sqrt{M}g_1]) g_2 \partial_{v_i} \langle v \rangle^\alpha, G_{3,k} \right\rangle_{L_v^2}. \end{aligned}$$

We now split into two cases according to the estimates of Lemma 4.20: If $0 \leq \gamma \leq 1$, using that $|\widetilde{\nabla}_v \langle v \rangle^\alpha| \lesssim \langle v \rangle^{\frac{\gamma}{2}-1+\alpha}$ we get

$$\begin{aligned} \text{II}_2 &\lesssim \|M^{\frac{1}{4}} g_1\|_{L_v^2} \left(\left\langle \langle v \rangle^\alpha \langle v \rangle^{\frac{\gamma}{2}+1} \langle v \rangle^{\frac{\gamma}{2}} |g_2|, |\widetilde{\nabla}_v G_3| \right\rangle_{L_v^2} + \left\langle \langle v \rangle^{\frac{\gamma}{2}+1} \langle v \rangle^{\frac{\gamma}{2}} |g_2| \langle v \rangle^{\frac{\gamma}{2}-1+\alpha}, |G_3| \right\rangle_{L_v^2} \right) \\ &\lesssim \|M^{\frac{1}{4}} g_1\|_{L_v^2} \|\langle v \rangle^\alpha \langle v \rangle^{\gamma+1} g_2\|_{L_v^2} \left(\|\widetilde{\nabla}_v G_3\|_{L_v^2} + \|\langle v \rangle^{\frac{\gamma}{2}-1} G_3\|_{L_v^2} \right), \end{aligned}$$

and if $-2 \leq \gamma < 0$ we get, using also that $\langle v \rangle^\gamma \lesssim 1$,

$$\begin{aligned} \text{II}_2 &\lesssim \|M^{\frac{1}{4}} g_1\|_{L_v^2} \left\langle \langle v \rangle^\alpha \langle v \rangle^{\frac{\gamma}{2}+1} \langle v \rangle^{\frac{\gamma}{2}} |g_2|, |\widetilde{\nabla}_v G_3| \right\rangle_{L_v^2} \\ &\quad + \|M^{\frac{1}{4}} g_1\|_{H_v^1} \left\langle \langle v \rangle^\alpha \langle v \rangle^{\frac{\gamma}{2}+1} \langle v \rangle^{\frac{\gamma}{2}-1} |g_2|, |\widetilde{\nabla}_v G_3| \right\rangle_{L_v^2} \\ &\quad + \|M^{\frac{1}{4}} g_1\|_{L_v^2} \left\langle \langle v \rangle^{\frac{\gamma}{2}+1} \langle v \rangle^{\frac{\gamma}{2}} \langle v \rangle^{\frac{\gamma}{2}-1+\alpha} |g_2|, |G_3| \right\rangle_{L_v^2} \\ &\quad + \|M^{\frac{1}{4}} g_1\|_{H_v^1} \left\langle \langle v \rangle^{\frac{\gamma}{2}+1} \langle v \rangle^{\frac{\gamma}{2}-1} \langle v \rangle^{\frac{\gamma}{2}-1+\alpha} |g_2|, |G_3| \right\rangle_{L_v^2} \\ &\lesssim \|M^{\frac{1}{4}} g_1\|_{L_v^2} \|\langle v \rangle^\alpha \langle v \rangle^{\gamma+1} g_2\|_{L_v^2} \left(\|\widetilde{\nabla}_v G_3\|_{L_v^2} + \|\langle v \rangle^{\frac{\gamma}{2}-1} G_3\|_{L_v^2} \right) \\ &\quad + \|M^{\frac{1}{4}} g_1\|_{H_v^1} \|\langle v \rangle^\alpha g_2\|_{L_v^2} \left(\|\widetilde{\nabla}_v G_3\|_{L_v^2} + \|\langle v \rangle^{\frac{\gamma}{2}-1} G_3\|_{L_v^2} \right). \end{aligned}$$

We deal with the term II_3 by first making an integration by parts and then using Lemma 4.12

together with $|\partial_{v_i} B_{k\ell}| \lesssim \langle v \rangle^{\frac{\gamma}{2}}$ and $|\partial_{v_i} \partial_{v_\ell} B_{k\ell}| \lesssim \langle v \rangle^{\frac{\gamma}{2}-1}$, which yields

$$\begin{aligned} \text{II}_3 &= - \left\langle \langle v \rangle^\alpha \left(b_i * [\sqrt{M} g_1] \right) g_2(\partial_{v_i} B_{k\ell}), \partial_{v_\ell} G_{3,k} \right\rangle_{L_v^2} \\ &\quad - \left\langle \left(b_i * [\sqrt{M} g_1] \right) g_2(\partial_{v_i} B_{k\ell}) \partial_{v_\ell} \langle v \rangle^\alpha, G_{3,k} \right\rangle_{L_v^2} \\ &\quad - \left\langle \langle v \rangle^\alpha \left(b_i * [\sqrt{M} g_1] \right) g_2(\partial_{v_i} \partial_{v_\ell} B_{k\ell}), G_{3,k} \right\rangle_{L_v^2} \\ &\lesssim \|M^{\frac{1}{4}} g_1\|_{L_v^2} \left(\left\langle \langle v \rangle^\alpha \langle v \rangle^{\gamma+1} |g_2| \langle v \rangle^{\frac{\gamma}{2}}, |\nabla_v G_3| \right\rangle_{L_v^2} + \left\langle \langle v \rangle^\alpha \langle v \rangle^{\gamma+1} |g_2| \langle v \rangle^{\frac{\gamma}{2}-1}, |G_3| \right\rangle_{L_v^2} \right) \\ &\lesssim \|M^{\frac{1}{4}} g_1\|_{L_v^2} \|\langle v \rangle^\alpha \langle v \rangle^{\gamma+1} g_2\|_{L_v^2} \left(\|\widetilde{\nabla}_v G_3\|_{L_v^2} + \|\langle v \rangle^{\frac{\gamma}{2}-1} G_3\|_{L_v^2} \right). \end{aligned}$$

Finally, gathering previous estimates, we get

$$\begin{aligned} \left\langle \langle v \rangle^\alpha \widetilde{\nabla}_v \Gamma_2(g_1, g_2), G_3 \right\rangle_{L_v^2} &\lesssim \|M^{\frac{1}{4}} g_1\|_{L_v^2} \|\langle v \rangle^\alpha g_2\|_{H_{v,*}^2} \|G_3\|_{H_{v,*}^1} \\ &\quad + \|M^{\frac{1}{4}} g_1\|_{H_v^1} \|\langle v \rangle^\alpha g_2\|_{L_v^2} \|G_3\|_{H_{v,*}^1}. \end{aligned} \quad (4.99)$$

Step 3. Term associated to Γ_3 . We write $\widetilde{\nabla}_{v_k} = B_{k\ell} \partial_{v_\ell}$ so that $[\widetilde{\nabla}_{v_k}, \partial_{v_i}] = -(\partial_{v_i} B_{k\ell}) \partial_{v_\ell}$, thus we get, for any $k \in \{1, 2, 3\}$,

$$\begin{aligned} \widetilde{\nabla}_{v_k} \Gamma_3(g_1, g_2) &= -\frac{1}{2} \widetilde{\nabla}_{v_k} \left\{ \left(a_{ij} * [\sqrt{M} g_1] \right) v_i \partial_{v_j} g_2 \right\} \\ &= -\frac{1}{2} \left(a_{ij} * [\sqrt{M} g_1] \right) v_i \partial_{v_j} (\widetilde{\nabla}_{v_k} g_2) - \frac{1}{2} \widetilde{\nabla}_{v_k} \left(a_{ij} * [\sqrt{M} g_1] \right) v_i \partial_{v_j} g_2 \\ &\quad - \frac{1}{2} \left(a_{ij} * [\sqrt{M} g_1] \right) (\widetilde{\nabla}_{v_k} v_i) \partial_{v_j} g_2 + \frac{1}{2} \left(a_{ij} * [\sqrt{M} g_1] \right) v_i (\partial_{v_j} B_{k\ell}) \partial_{v_\ell} g_2 \end{aligned}$$

whence

$$\left\langle \langle v \rangle^\alpha \widetilde{\nabla}_{v_k} \Gamma_3(g_1, g_2), G_{3,k} \right\rangle = \text{III}_1 + \text{III}_2 + \text{III}_3 + \text{III}_4.$$

We now bound each term separately using Lemma 4.20. For the term III_1 , using that $|\mathbf{B}(v)v| \lesssim \langle v \rangle^{\frac{\gamma}{2}+1}$, we first obtain,

$$\begin{aligned} \text{III}_1 &= -\frac{1}{2} \left\langle \langle v \rangle^\alpha \left(a_{ij} * [\sqrt{M} g_1] \right) v_i \partial_{v_j} (\widetilde{\nabla}_{v_k} g_2), G_{3,k} \right\rangle_{L_v^2} \\ &\lesssim \|M^{\frac{1}{4}} g_1\|_{L_v^2} \left\langle \langle v \rangle^\alpha \langle v \rangle^{\frac{\gamma}{2}+1} |\widetilde{\nabla}_v (\widetilde{\nabla}_{v_k} g_2)|, |G_{3,k}| \right\rangle_{L_v^2} \\ &\lesssim \|M^{\frac{1}{4}} g_1\|_{L_v^2} \|\langle v \rangle^\alpha \widetilde{\nabla}_v (\widetilde{\nabla}_{v_k} g_2)\|_{L_v^2} \|\langle v \rangle^{\frac{\gamma}{2}+1} G_3\|_{L_v^2}. \end{aligned}$$

Using also that $|B_{k\ell}| \lesssim \langle v \rangle^{\frac{\gamma}{2}+1}$, we then get

$$\begin{aligned} \text{III}_2 &= -\frac{1}{2} \left\langle \langle v \rangle^\alpha B_{k\ell} \left(\partial_{v_\ell} a_{ij} * [\sqrt{M} g_1] \right) v_i \partial_{v_j} g_2, G_{3,k} \right\rangle_{L_v^2} \\ &\lesssim \|M^{\frac{1}{4}} g_1\|_{L_v^2} \left\langle \langle v \rangle^\alpha \langle v \rangle^{\frac{\gamma}{2}+1} \langle v \rangle^{\frac{\gamma}{2}+1} |\widetilde{\nabla}_v g_2|, |G_3| \right\rangle_{L_v^2} \\ &\lesssim \|M^{\frac{1}{4}} g_1\|_{L_v^2} \|\langle v \rangle^\alpha \langle v \rangle^{\frac{\gamma}{2}+1} \widetilde{\nabla}_v g_2\|_{L_v^2} \|\langle v \rangle^{\frac{\gamma}{2}+1} G_3\|_{L_v^2}. \end{aligned}$$

Since $\widetilde{\nabla}_{v_k} v_i = B_{ki}$, we obtain

$$\begin{aligned} \text{III}_3 &= -\frac{1}{2} \left\langle \langle v \rangle^\alpha \left(a_{ij} * [\sqrt{M} g_1] \right) B_{ki} \partial_{v_j} g_2, G_{3,k} \right\rangle_{L_v^2} \\ &\lesssim \|M^{\frac{1}{4}} g_1\|_{L_v^2} \left\langle \langle v \rangle^\alpha \langle v \rangle^{\gamma+2} |\widetilde{\nabla}_{v_j} g_2|, |G_3| \right\rangle_{L_v^2} \\ &\lesssim \|M^{\frac{1}{4}} g_1\|_{L_v^2} \|\langle v \rangle^\alpha \langle v \rangle^{\frac{\gamma}{2}+1} \widetilde{\nabla}_{v_j} g_2\|_{L_v^2} \|\langle v \rangle^{\frac{\gamma}{2}+1} G_3\|_{L_v^2}. \end{aligned}$$

Using now that $|\widetilde{\nabla}_{v_j} B_{k\ell}| \lesssim \langle v \rangle^{\frac{\gamma}{2}+1} \langle v \rangle^{\frac{\gamma}{2}}$ and $\langle v \rangle^{\frac{\gamma}{2}} |\nabla_{v_j} g_2| \lesssim |\widetilde{\nabla}_{v_j} g_2|$, we thus get

$$\begin{aligned} \text{III}_4 &= \frac{1}{2} \left\langle \langle v \rangle^\alpha \left(a_{ij} * [\sqrt{M} g_1] \right) v_i \partial_{v_j} B_{k\ell} \partial_{v_\ell} g_2, G_{3,k} \right\rangle_{L_v^2} \\ &\lesssim \|M^{\frac{1}{4}} g_1\|_{L_v^2} \left\langle \langle v \rangle^\alpha \langle v \rangle^{\frac{\gamma}{2}+1} \langle v \rangle^{\frac{\gamma}{2}+1} \langle v \rangle^{\frac{\gamma}{2}} |\nabla_{v_j} g_2|, |G_3| \right\rangle_{L_v^2} \\ &\lesssim \|M^{\frac{1}{4}} g_1\|_{L_v^2} \|\langle v \rangle^\alpha \langle v \rangle^{\frac{\gamma}{2}+1} \widetilde{\nabla}_{v_j} g_2\|_{L_v^2} \|\langle v \rangle^{\frac{\gamma}{2}+1} G_3\|_{L_v^2}. \end{aligned}$$

Finally, gathering previous estimates, we get

$$\left\langle \langle v \rangle^\alpha \widetilde{\nabla}_{v_j} \Gamma_3(g_1, g_2), G_3 \right\rangle_{L_v^2} \lesssim \|M^{\frac{1}{4}} g_1\|_{L_v^2} \|\langle v \rangle^\alpha g_2\|_{H_{v,*}^2} \|G_3\|_{H_{v,*}^1}. \quad (4.100)$$

Step 4. Term associated to Γ_4 . Writing $\widetilde{\nabla}_{v_k} = B_{k\ell} \partial_{v_\ell}$ and observing that $[\widetilde{\nabla}_{v_k}, \partial_{v_i}] = -(\partial_{v_i} B_{k\ell}) \partial_{v_\ell}$ we first get, for any $k \in \{1, 2, 3\}$,

$$\begin{aligned} \widetilde{\nabla}_{v_k} \Gamma_4 &= \frac{1}{4} \widetilde{\nabla}_{v_k} \left\{ \left(a_{ij} * [\sqrt{M} g_1] \right) v_i v_j g_2 \right\} \\ &= \frac{1}{4} \left(a_{ij} * [\sqrt{M} g_1] \right) v_i v_j \widetilde{\nabla}_{v_k} g_2 + \frac{1}{4} \widetilde{\nabla}_{v_k} \left(a_{ij} * [\sqrt{M} g_1] \right) v_i v_j g_2 \\ &\quad + \frac{1}{4} \left(a_{ij} * [\sqrt{M} g_1] \right) \widetilde{\nabla}_{v_k} (v_i v_j) g_2 \end{aligned}$$

whence

$$\left\langle \langle v \rangle^\alpha \widetilde{\nabla}_{v_k} \Gamma_4(g_1, g_2), G_{3,k} \right\rangle = \text{IV}_1 + \text{IV}_2 + \text{IV}_3.$$

We now bound each term separately using Lemma 4.12. For the term IV_1 , we first obtain

$$\begin{aligned} \text{IV}_1 &= \frac{1}{4} \left\langle \langle v \rangle^\alpha \left(a_{ij} * [\sqrt{M} g_1] \right) v_i v_j \widetilde{\nabla}_{v_k} g_2, G_{3,k} \right\rangle_{L_v^2} \\ &\lesssim \|M^{\frac{1}{4}} g_1\|_{L_v^2} \left\langle \langle v \rangle^\alpha \langle v \rangle^{\gamma+2} |\widetilde{\nabla}_{v_k} g_2|, |G_{3,k}| \right\rangle_{L_v^2} \\ &\lesssim \|M^{\frac{1}{4}} g_1\|_{L_v^2} \|\langle v \rangle^\alpha \langle v \rangle^{\frac{\gamma}{2}+1} \widetilde{\nabla}_{v_k} g_2\|_{L_v^2} \|\langle v \rangle^{\frac{\gamma}{2}+1} G_3\|_{L_v^2}. \end{aligned}$$

For the term IV_2 , we also use that $|B_{k\ell}| \lesssim \langle v \rangle^{\frac{\gamma}{2}+1}$ to obtain

$$\begin{aligned} \text{IV}_2 &= \frac{1}{4} \left\langle \langle v \rangle^\alpha B_{k\ell} \left(\partial_{v_\ell} a_{ij} * [\sqrt{M} g_1] \right) v_i v_j g_2, G_{3,k} \right\rangle_{L_v^2} \\ &\lesssim \|M^{\frac{1}{4}} g_1\|_{L_v^2} \left\langle \langle v \rangle^\alpha \langle v \rangle^{\frac{\gamma}{2}+1} \langle v \rangle^{\gamma+2} |g_2|, |G_3| \right\rangle_{L_v^2} \\ &\lesssim \|M^{\frac{1}{4}} g_1\|_{L_v^2} \|\langle v \rangle^\alpha \langle v \rangle^{\gamma+2} g_2\|_{L_v^2} \|\langle v \rangle^{\frac{\gamma}{2}+1} G_3\|_{L_v^2}. \end{aligned}$$

For the last term IV_3 , we write $\widetilde{\nabla}_{v_k}(v_i v_j) = v_i B_{kj} + v_j B_{ki}$, and thus we get

$$\begin{aligned} IV_3 &= \frac{1}{4} \left\langle \langle v \rangle^\alpha \left(a_{ij} * [\sqrt{M} g_1] \right) [v_i B_{kj} + v_j B_{ki}] g_2, G_{3,k} \right\rangle_{L_v^2} \\ &\lesssim \|M^{\frac{1}{4}} g_1\|_{L_v^2} \left\langle \langle v \rangle^\alpha \langle v \rangle^{\gamma+2} \langle v \rangle^{\frac{\gamma}{2}+1} |g_2|, |G_3| \right\rangle_{L_v^2} \\ &\lesssim \|M^{\frac{1}{4}} g_1\|_{L_v^2} \|\langle v \rangle^\alpha \langle v \rangle^{\gamma+2} g_2\|_{L_v^2} \|\langle v \rangle^{\frac{\gamma}{2}+1} G_3\|_{L_v^2}. \end{aligned}$$

Finally, gathering previous estimates we get

$$\left\langle \langle v \rangle^\alpha \widetilde{\nabla}_v \Gamma_4(g_1, g_2), G_3 \right\rangle_{L_v^2} \lesssim \|M^{\frac{1}{4}} g_1\|_{L_v^2} \|\langle v \rangle^\alpha g_2\|_{H_{v,*}^2} \|G_3\|_{H_{v,*}^1}. \quad (4.101)$$

Step 5. Term associated to Γ_5 . We write $\widetilde{\nabla}_{v_k} = B_{k\ell} \partial_{v_\ell}$ so that $[\widetilde{\nabla}_{v_k}, \partial_{v_i}] = -(\partial_{v_i} B_{k\ell}) \partial_{v_\ell}$, thus we get, for any $k \in \{1, 2, 3\}$,

$$\begin{aligned} \widetilde{\nabla}_{v_k} \Gamma_5(g_1, g_2) &= -\frac{1}{2} \widetilde{\nabla}_{v_k} \left\{ \left(a_{ii} * [\sqrt{M} g_1] \right) g_2 \right\} \\ &= -\frac{1}{2} \left(a_{ii} * [\sqrt{M} g_1] \right) \widetilde{\nabla}_{v_k} g_2 - \frac{1}{2} \widetilde{\nabla}_{v_k} \left(a_{ii} * [\sqrt{M} g_1] \right) g_2 \end{aligned}$$

whence

$$\left\langle \langle v \rangle^\alpha \widetilde{\nabla}_{v_k} \Gamma_5(g_1, g_2), G_{3,k} \right\rangle_{L_v^2} = V_1 + V_2.$$

We now bound each term separately using Lemma 4.12. For the term V_1 , we get

$$\begin{aligned} V_1 &= -\frac{1}{2} \left\langle \langle v \rangle^\alpha \left(a_{ii} * [\sqrt{M} g_1] \right) \widetilde{\nabla}_{v_k} g_2, G_{3,k} \right\rangle_{L_v^2} \\ &\lesssim \|M^{\frac{1}{4}} g_1\|_{L_v^2} \left\langle \langle v \rangle^\alpha \langle v \rangle^{\gamma+2} |\widetilde{\nabla}_{v_k} g_2|, |G_{3,k}| \right\rangle_{L_v^2} \\ &\lesssim \|M^{\frac{1}{4}} g_1\|_{L_v^2} \|\langle v \rangle^\alpha \langle v \rangle^{\frac{\gamma}{2}+1} \widetilde{\nabla}_v g_2\|_{L_v^2} \|\langle v \rangle^{\frac{\gamma}{2}+1} G_3\|_{L_v^2}, \end{aligned}$$

and, for the term V_2 , we use $|B_{k\ell}| \lesssim \langle v \rangle^{\frac{\gamma}{2}+1}$ to deduce

$$\begin{aligned} V_2 &= -\frac{1}{2} \left\langle \langle v \rangle^\alpha B_{k\ell} \left(\partial_{v_\ell} a_{ii} * [\sqrt{M} g_1] \right) g_2, G_{3,k} \right\rangle_{L_v^2} \\ &\lesssim \|M^{\frac{1}{4}} g_1\|_{L_v^2} \left\langle \langle v \rangle^\alpha \langle v \rangle^{\frac{\gamma}{2}+1} \langle v \rangle^{\gamma+1} |g_2|, |G_3| \right\rangle_{L_v^2} \\ &\lesssim \|M^{\frac{1}{4}} g_1\|_{L_v^2} \|\langle v \rangle^\alpha \langle v \rangle^{\gamma+1} g_2\|_{L_v^2} \|\langle v \rangle^{\frac{\gamma}{2}+1} G_3\|_{L_v^2}. \end{aligned}$$

Finally, gathering previous estimates we get

$$\left\langle \langle v \rangle^\alpha \widetilde{\nabla}_v \Gamma_5(g_1, g_2), G_3 \right\rangle_{L_v^2} \lesssim \|M^{\frac{1}{4}} g_1\|_{L_v^2} \|\langle v \rangle^\alpha g_2\|_{H_{v,*}^2} \|G_3\|_{H_{v,*}^1}. \quad (4.102)$$

Step 6. Proof of (4.97). We gather estimates (4.98)–(4.99)–(4.100)–(4.101)–(4.102). \square

Proposition 4.23. *Let g_1, g_2 be smooth enough functions and G_3 a smooth enough vector field, then*

$$\begin{aligned} & \left\langle \widetilde{\nabla}_x \Gamma(g_1, g_2), G_3 \right\rangle_{\mathcal{X}} \\ & \lesssim \left(\|g_1\|_{\mathcal{X}} \|\widetilde{\nabla}_x g_2\|_{\mathcal{Y}_1} + \|\widetilde{\nabla}_x g_1\|_{\mathcal{X}} \|g_2\|_{\mathcal{Y}_1} + \|g_1\|_{\mathcal{X}} \|g_2\|_{\mathcal{Y}_2} \right) \|G_3\|_{\mathcal{Y}_1}. \end{aligned} \quad (4.103)$$

Proof of Proposition 4.23. We shall first prove that for any $\alpha \in \mathbb{R}$, there holds

$$\begin{aligned} & \left\langle \langle v \rangle^\alpha \widetilde{\nabla}_x \Gamma(g_1, g_2), G_3 \right\rangle_{L_v^2} \\ & \lesssim \left(\|M^{\frac{1}{4}} g_1\|_{L_v^2} \|\langle v \rangle^\alpha \widetilde{\nabla}_x g_2\|_{H_{v,*}^1} + \|M^{\frac{1}{4}} \nabla_x g_1\|_{L_v^2} \|\langle v \rangle^\alpha \langle v \rangle^{\frac{7}{2}+1} g_2\|_{H_{v,*}^1} \right) \|G_3\|_{H_{v,*}^1}. \end{aligned} \quad (4.104)$$

We thus write $\Gamma(g_1, g_2) = \Gamma_1(g_1, g_2) + \dots + \Gamma_5(g_1, g_2)$ as in (4.83)–(4.87), and we estimate each term separately in the sequel. The desired result (4.103) is then obtained by integrating in x , the proof is given in Step 7 because it differs from Step 4 of the proof of Proposition 4.21.

Step 1. Term associated to Γ_1 . Writing $\widetilde{\nabla}_{x_k} = B_{k\ell} \partial_{x_\ell}$, observing that $[\widetilde{\nabla}_{x_k}, \partial_{v_i}] = -(\partial_{v_i} B_{k\ell}) \partial_{x_\ell}$ and using that $\widetilde{\nabla}_{x_k} (a_{ij} * [\sqrt{M} g_1]) = B_{k\ell} (a_{ij} * [\sqrt{M} \partial_{x_\ell} g_1])$, we first get that for any $k \in \{1, 2, 3\}$,

$$\begin{aligned} \widetilde{\nabla}_{x_k} \Gamma_1(g_1, g_2) &= \partial_{v_i} \left\{ (a_{ij} * [\sqrt{M} g_1]) \partial_{v_j} (\widetilde{\nabla}_{x_k} g_2) \right\} + \partial_{v_i} \left\{ B_{k\ell} (a_{ij} * [\sqrt{M} \partial_{x_\ell} g_1]) \partial_{v_j} g_2 \right\} \\ &\quad - \partial_{v_i} \left\{ (a_{ij} * [\sqrt{M} g_1]) (\partial_{v_j} B_{k\ell}) \partial_{x_\ell} g_2 \right\} - (\partial_{v_i} B_{k\ell}) (a_{ij} * [\sqrt{M} \partial_{x_\ell} g_1]) \partial_{v_j} g_2 \\ &\quad - (\partial_{v_i} B_{k\ell}) (a_{ij} * [\sqrt{M} g_1]) \partial_{v_j} (\partial_{x_\ell} g_2). \end{aligned}$$

whence

$$\left\langle \langle v \rangle^\alpha \widetilde{\nabla}_{x_k} \Gamma_1(g_1, g_2), G_{3,k} \right\rangle_{L_v^2} = I_1 + I_2 + I_3 + I_4 + I_5.$$

We can then estimate each term separately using Lemma 4.20 and arguing similarly as in Step 1 of the proof of Proposition 4.22, which we brief explain below.

For the term I_1 , we make an integration by parts and use the fact that $|\widetilde{\nabla}_v \langle v \rangle^\alpha| \lesssim \langle v \rangle^{\frac{7}{2}-1} \langle v \rangle^\alpha$ to obtain

$$\begin{aligned} I_1 &= - \left\langle \langle v \rangle^\alpha (a_{ij} * [\sqrt{M} g_1]) \partial_{v_j} (\widetilde{\nabla}_{x_k} g_2), \partial_{v_i} G_{3,k} \right\rangle_{L_v^2} \\ &\quad - \left\langle (a_{ij} * [\sqrt{M} g_1]) \partial_{v_j} (\widetilde{\nabla}_{x_k} g_2) \partial_{v_i} \langle v \rangle^\alpha, G_{3,k} \right\rangle_{L_v^2} \\ &\lesssim \|M^{\frac{1}{4}} g_1\|_{L_v^2} \|\langle v \rangle^\alpha \widetilde{\nabla}_v (\widetilde{\nabla}_x g_2)\|_{L_v^2} \left(\|\widetilde{\nabla}_v G_3\|_{L_v^2} + \|\langle v \rangle^{\frac{7}{2}+1} G_3\|_{L_v^2} \right). \end{aligned}$$

In a similar way, for the term I_2 , we make an integration by parts and use that $|B_{k\ell}| \lesssim \langle v \rangle^{\frac{\gamma}{2}+1}$, which yields

$$\begin{aligned} I_2 &= - \left\langle \langle v \rangle^\alpha B_{k\ell} \left(a_{ij} * [\sqrt{M} \partial_{x_\ell} g_1] \right) \partial_{v_j} g_2, \partial_{v_i} G_{3,k} \right\rangle_{L_v^2} \\ &\quad - \left\langle \langle v \rangle^\alpha B_{k\ell} \left(a_{ij} * [\sqrt{M} \partial_{x_\ell} g_1] \right) \partial_{v_j} g_2 \partial_{v_i} \langle v \rangle^\alpha, G_{3,k} \right\rangle_{L_v^2} \\ &\lesssim \|M^{\frac{1}{4}} \nabla_x g_1\|_{L_v^2} \|\langle v \rangle^\alpha \langle v \rangle^{\frac{\gamma}{2}+1} \widetilde{\nabla}_v g_2\|_{L_v^2} \left(\|\widetilde{\nabla}_v G_3\|_{L_v^2} + \|\langle v \rangle^{\frac{\gamma}{2}+1} G_3\|_{L_v^2} \right). \end{aligned}$$

For the term I_3 , arguing as above and using also that $|\widetilde{\nabla}_v B_{k\ell}| \lesssim \langle v \rangle^{\frac{\gamma}{2}+1} \langle v \rangle^{\frac{\gamma}{2}}$, we get

$$\begin{aligned} I_3 &= \left\langle \langle v \rangle^\alpha \left(a_{ij} * [\sqrt{M} g_1] \right) (\partial_{v_j} B_{k\ell}) \partial_{x_\ell} g_2, \partial_{v_i} G_{3,k} \right\rangle_{L_v^2} \\ &\quad + \left\langle \left(a_{ij} * [\sqrt{M} g_1] \right) (\partial_{v_j} B_{k\ell}) \partial_{x_\ell} g_2 \partial_{v_i} \langle v \rangle^\alpha, G_{3,k} \right\rangle_{L_v^2} \\ &\lesssim \|M^{\frac{1}{4}} g_1\|_{L_v^2} \|\langle v \rangle^\alpha \langle v \rangle^{\frac{\gamma}{2}+1} \widetilde{\nabla}_x g_2\|_{L_v^2} \left(\|\widetilde{\nabla}_v G_3\|_{L_v^2} + \|\langle v \rangle^{\frac{\gamma}{2}+1} G_3\|_{L_v^2} \right). \end{aligned}$$

For the term I_4 , we use that $|\widetilde{\nabla}_v B_{k\ell}| \lesssim \langle v \rangle^{\gamma+1}$ to get

$$\begin{aligned} I_4 &= - \left\langle \langle v \rangle^\alpha \left(a_{ij} * [\sqrt{M} \partial_{x_\ell} g_1] \right) (\partial_{v_i} B_{k\ell}) \partial_{v_j} g_2, G_{3,k} \right\rangle_{L_v^2} \\ &\lesssim \|M^{\frac{1}{4}} \nabla_x g_1\|_{L_v^2} \|\langle v \rangle^\alpha \langle v \rangle^{\frac{\gamma}{2}} \widetilde{\nabla}_v g_2\|_{L_v^2} \|\langle v \rangle^{\frac{\gamma}{2}+1} G_3\|_{L_v^2}. \end{aligned}$$

Performing an integration by parts in the term I_5 , we first obtain

$$\begin{aligned} I_5 &= - \left\langle \langle v \rangle^\alpha \left(a_{ij} * [\sqrt{M} g_1] \right) (\partial_{v_i} B_{k\ell}) \partial_{v_j} (\partial_{x_\ell} g_2), G_{3,k} \right\rangle_{L_v^2} \\ &= \left\langle \langle v \rangle^\alpha \left(a_{ij} * [\sqrt{M} g_1] \right) (\partial_{v_i} B_{k\ell}) \partial_{x_\ell} g_2, \partial_{v_j} G_{3,k} \right\rangle_{L_v^2} \\ &\quad + \left\langle \langle v \rangle^\alpha \left(a_{ij} * [\sqrt{M} g_1] \right) (\partial_{v_j} \partial_{v_i} B_{k\ell}) \partial_{x_\ell} g_2, G_{3,k} \right\rangle_{L_v^2} \\ &\quad + \left\langle \left(a_{ij} * [\sqrt{M} g_1] \right) (\partial_{v_i} B_{k\ell}) (\partial_{x_\ell} g_2) \partial_{v_j} \langle v \rangle^\alpha, G_{3,k} \right\rangle_{L_v^2} \\ &\quad + \left\langle \langle v \rangle^\alpha \left(b_i * [\sqrt{M} g_1] \right) (\partial_{v_i} B_{k\ell}) \partial_{x_\ell} g_2, G_{3,k} \right\rangle_{L_v^2}. \end{aligned}$$

Using Lemmas 4.12 and 4.20 together with $|\partial_{v_j} B_{k\ell}| \lesssim \langle v \rangle^{\frac{\gamma}{2}}$ and $|\partial_{v_\ell} \partial_{v_j} B_{k\ell}| \lesssim \langle v \rangle^{\frac{\gamma}{2}-1}$, we hence obtain

$$\begin{aligned} I_5 &\lesssim \|M^{\frac{1}{4}} g_1\|_{L_v^2} \left\langle \langle v \rangle^\alpha \langle v \rangle^{\gamma+1} |\nabla_x g_2| |\widetilde{\nabla}_v G_3| \right\rangle_{L_v^2} \\ &\quad + \|M^{\frac{1}{4}} g_1\|_{L_v^2} \left\langle \langle v \rangle^\alpha \langle v \rangle^{\gamma+2} \langle v \rangle^{\frac{\gamma}{2}-1} |\nabla_x g_2| |G_3| \right\rangle_{L_v^2} \\ &\quad + \|M^{\frac{1}{4}} g_1\|_{L_v^2} \left\langle \langle v \rangle^{\gamma+1} \langle v \rangle^{\frac{\gamma}{2}-1} \langle v \rangle^\alpha |\nabla_x g_2| |G_3| \right\rangle_{L_v^2} \\ &\quad + \|M^{\frac{1}{4}} g_1\|_{L_v^2} \left\langle \langle v \rangle^\alpha \langle v \rangle^{\gamma+1} \langle v \rangle^{\frac{\gamma}{2}} |\nabla_x g_2| |G_3| \right\rangle_{L_v^2} \\ &\lesssim \|M^{\frac{1}{4}} g_1\|_{L_v^2} \|\langle v \rangle^\alpha \langle v \rangle^{\frac{\gamma}{2}+1} \widetilde{\nabla}_x g_2\|_{L_v^2} \left(\|\widetilde{\nabla}_v G_3\|_{L_v^2} + \|\langle v \rangle^{\frac{\gamma}{2}+1} G_3\|_{L_v^2} \right). \end{aligned}$$

Finally, gathering previous estimates, we get

$$\begin{aligned} & \left\langle \langle v \rangle^\alpha \widetilde{\nabla}_x \Gamma_1(g_1, g_2), G_3 \right\rangle_{L_v^2} \\ & \lesssim \left(\|M^{\frac{1}{4}} g_1\|_{L_v^2} \|\langle v \rangle^\alpha \widetilde{\nabla}_x g_2\|_{H_{v,*}^1} + \|M^{\frac{1}{4}} \nabla_x g_1\|_{L_v^2} \|\langle v \rangle^\alpha \langle v \rangle^{\frac{\gamma}{2}+1} \widetilde{\nabla}_v g_2\|_{L_v^2} \right) \|G_3\|_{H_{v,*}^1}. \end{aligned} \quad (4.105)$$

Step 2. Term associated to Γ_2 . Similarly as for Γ_1 , we have that for any $k \in \{1, 2, 3\}$,

$$\begin{aligned} \widetilde{\nabla}_{x_k} \Gamma_2 &= -\widetilde{\nabla}_{x_k} \partial_{v_i} \left\{ (b_i * [\sqrt{M} g_1]) g_2 \right\} \\ &= -\partial_{v_i} \left\{ (b_i * [\sqrt{M} g_1]) \widetilde{\nabla}_{x_k} g_2 \right\} - \partial_{v_i} \left\{ B_{k\ell} (b_i * [\sqrt{M} \partial_{x_\ell} g_1]) g_2 \right\} \\ &\quad + (\partial_{v_i} B_{k\ell}) (b_i * [\sqrt{M} g_1]) \partial_{x_\ell} g_2 + (\partial_{v_i} B_{k\ell}) (b_i * [\sqrt{M} \partial_{x_\ell} g_1]) g_2. \end{aligned}$$

We can then estimate each term separately using Lemma 4.20 and arguing as in Step 2 of the proof of Proposition 4.22, which yields

$$\begin{aligned} & \left\langle \langle v \rangle^\alpha \widetilde{\nabla}_x \Gamma_2(g_1, g_2), G_3 \right\rangle_{L_v^2} \\ & \lesssim \left(\|M^{\frac{1}{4}} g_1\|_{L_v^2} \|\langle v \rangle^\alpha \widetilde{\nabla}_x g_2\|_{H_{v,*}^1} + \|M^{\frac{1}{4}} \nabla_x g_1\|_{L_v^2} \|\langle v \rangle^\alpha \langle v \rangle^{\gamma+2} g_2\|_{L_v^2} \right) \|G_3\|_{H_{v,*}^1}. \end{aligned} \quad (4.106)$$

Step 3. Term associated to Γ_3 . As previously, we first get that for any $k \in \{1, 2, 3\}$,

$$\begin{aligned} \widetilde{\nabla}_{x_k} \Gamma_3 &= -\widetilde{\nabla}_{x_k} \left\{ (a_{ij} * [\sqrt{M} g_1]) v_i \partial_{v_j} g_2 \right\} \\ &= -\left(a_{ij} * [\sqrt{M} g_1] \right) v_i \partial_{v_j} (\widetilde{\nabla}_{x_k} g_2) - B_{k\ell} \left(a_{ij} * [\sqrt{M} \partial_{x_\ell} g_1] \right) v_i \partial_{v_j} g_2 \\ &\quad + \left(a_{ij} * [\sqrt{M} g_1] \right) v_i (\partial_{v_j} B_{k\ell}) \partial_{x_\ell} g_2. \end{aligned}$$

We can then estimate each term separately using Lemma 4.20 and arguing as in Step 3 of the proof of Proposition 4.22, which yields

$$\begin{aligned} & \left\langle \langle v \rangle^\alpha \widetilde{\nabla}_x \Gamma_3(g_1, g_2), G_3 \right\rangle_{L_v^2} \\ & \lesssim \left(\|M^{\frac{1}{4}} g_1\|_{L_v^2} \|\langle v \rangle^\alpha \widetilde{\nabla}_x g_2\|_{H_{v,*}^1} + \|M^{\frac{1}{4}} \nabla_x g_1\|_{L_v^2} \|\langle v \rangle^\alpha \langle v \rangle^{\frac{\gamma}{2}+1} \widetilde{\nabla}_v g_2\|_{L_v^2} \right) \|G_3\|_{H_{v,*}^1}. \end{aligned} \quad (4.107)$$

Step 4. Term associated to Γ_4 . As previously, we first get that for any $k \in \{1, 2, 3\}$,

$$\begin{aligned} \widetilde{\nabla}_{x_k} \Gamma_4 &= \frac{1}{4} \widetilde{\nabla}_{x_k} \left\{ (a_{ij} * [\sqrt{M} g_1]) v_i v_j g_2 \right\} \\ &= \frac{1}{4} \left(a_{ij} * [\sqrt{M} g_1] \right) v_i v_j \widetilde{\nabla}_{x_k} g_2 + \frac{1}{4} B_{k\ell} \left(a_{ij} * [\sqrt{M} \partial_{x_\ell} g_1] \right) v_i v_j g_2. \end{aligned}$$

We can then estimate each term separately using Lemma 4.12 and arguing as in Step 4 of the proof of Proposition 4.22, which yields

$$\begin{aligned} & \left\langle \langle v \rangle^\alpha \widetilde{\nabla}_x \Gamma_4(g_1, g_2), G_3 \right\rangle_{L_v^2} \\ & \lesssim \left(\|M^{\frac{1}{4}} g_1\|_{L_v^2} \|\langle v \rangle^\alpha \widetilde{\nabla}_x g_2\|_{H_{v,*}^1} + \|M^{\frac{1}{4}} \nabla_x g_1\|_{L_v^2} \|\langle v \rangle^\alpha \langle v \rangle^{\gamma+2} g_2\|_{L_v^2} \right) \|G_3\|_{H_{v,*}^1}. \end{aligned} \quad (4.108)$$

Step 5. Term associated to Γ_5 . As previously, we first get that for any $k \in \{1, 2, 3\}$,

$$\begin{aligned} \widetilde{\nabla}_{x_k} \Gamma_5 &= -\frac{1}{2} \widetilde{\nabla}_{x_k} \left\{ (a_{ii} * [\sqrt{M} g_1]) g_2 \right\} \\ &= -\frac{1}{2} (a_{ii} * [\sqrt{M} g_1]) \widetilde{\nabla}_{x_k} g_2 - \frac{1}{2} B_{k\ell} (a_{ii} * [\sqrt{M} \partial_{x_\ell} g_1]) g_2. \end{aligned}$$

We can then estimate each term separately using Lemma 4.12 and arguing as in Step 5 of the proof of Proposition 4.22, which yields

$$\begin{aligned} & \left\langle \langle v \rangle^\alpha \widetilde{\nabla}_x \Gamma_5(g_1, g_2), G_3 \right\rangle_{L_v^2} \\ & \lesssim \left(\|M^{\frac{1}{4}} g_1\|_{L_v^2} \|\langle v \rangle^\alpha \widetilde{\nabla}_x g_2\|_{H_{v,*}^1} + \|M^{\frac{1}{4}} \nabla_x g_1\|_{L_v^2} \|\langle v \rangle^\alpha \langle v \rangle^{\gamma+2} g_2\|_{L_v^2} \right) \|G_3\|_{H_{v,*}^1}. \end{aligned} \quad (4.109)$$

Step 6. Proof of (4.104). We gather estimates (4.105)–(4.106)–(4.107)–(4.108)–(4.109).

Step 7. Proof of (4.103). We first write

$$\begin{aligned} \left\langle \widetilde{\nabla}_x \Gamma(g_1, g_2), G_3 \right\rangle_{\mathcal{X}} &= \sum_{i=0}^3 \left\langle \langle v \rangle^{(3-i)(\frac{\gamma}{2}+1)} \nabla_x^i \widetilde{\nabla}_x \Gamma(g_1, g_2), \langle v \rangle^{(3-i)(\frac{\gamma}{2}+1)} \nabla_x^i G_3 \right\rangle_{L_{x,v}^2} \\ &=: T_0 + T_1 + T_2 + T_3 \end{aligned}$$

and recall that $\partial_{x_\ell} \Gamma(g_1, g_2) = \Gamma(\partial_{x_\ell} g_1, g_2) + \Gamma(g_1, \partial_{x_\ell} g_2)$ and that ∇_x and $\widetilde{\nabla}_x$ commute. In the remainder of the proof, we shall use (4.104) as well as some classical Sobolev embeddings as in Step 4 of the proof of Proposition 4.21 without no further mention. We here point out that in estimate (4.104), it is important to keep $\|\langle v \rangle^\alpha \langle v \rangle^{\frac{\gamma}{2}+1} g_2\|_{H_{v,*}^1}$ in our estimate instead of bounding this term by $\|\langle v \rangle^\alpha g_2\|_{H_{v,*}^2}$ and also that to close our estimate, we widely use that the weights in our functional spaces depend on the order of the derivatives in x .

For T_0 , we have:

$$\begin{aligned} T_0 &\lesssim \left(\|M^{\frac{1}{4}} g_1\|_{H_x^2 L_v^2} \|\langle v \rangle^{3(\frac{\gamma}{2}+1)} \widetilde{\nabla}_x g_2\|_{L_x^2(H_{v,*}^1)} \right. \\ &\quad \left. + \|M^{\frac{1}{4}} \nabla_x g_1\|_{H_x^2 L_v^2} \|\langle v \rangle^{3(\frac{\gamma}{2}+1)} \langle v \rangle^{\frac{\gamma}{2}+1} g_2\|_{L_x^2(H_{v,*}^1)} \right) \|\langle v \rangle^{3(\frac{\gamma}{2}+1)} G_3\|_{L_x^2(H_{v,*}^1)} \\ &\lesssim \left(\|g_1\|_{\mathcal{X}} \|\widetilde{\nabla}_x g_2\|_{\mathcal{Y}_1} + \|g_1\|_{\mathcal{X}} \|g_2\|_{\mathcal{Y}_2} \right) \|G_3\|_{\mathcal{Y}_1}. \end{aligned}$$

For T_1 , we have:

$$\begin{aligned}
 T_1 &\lesssim \left(\|M^{\frac{1}{4}}g_1\|_{H_x^2L_v^2} \|\langle v \rangle^{2(\frac{\gamma}{2}+1)} \nabla_x(\widetilde{\nabla}_x g_2)\|_{L_x^2(H_{v,*}^1)} \right. \\
 &\quad \left. + \|M^{\frac{1}{4}}\nabla_x g_1\|_{H_x^1L_v^2} \|\langle v \rangle^{2(\frac{\gamma}{2}+1)} \widetilde{\nabla}_x g_2\|_{H_x^1(H_{v,*}^1)} \right) \|\langle v \rangle^{2(\frac{\gamma}{2}+1)} \nabla_x G_3\|_{L_x^2(H_{v,*}^1)} \\
 &\quad + \left(\|M^{\frac{1}{4}}\nabla_x g_1\|_{H_x^2L_v^2} \|\langle v \rangle^{2(\frac{\gamma}{2}+1)} \nabla_x(\langle v \rangle^{\frac{\gamma}{2}+1} g_2)\|_{L_x^2(H_{v,*}^1)} \right. \\
 &\quad \left. + \|M^{\frac{1}{4}}\nabla_x^2 g_1\|_{H_x^1L_v^2} \|\langle v \rangle^{2(\frac{\gamma}{2}+1)} \langle v \rangle^{\frac{\gamma}{2}+1} g_2\|_{H_x^1(H_{v,*}^1)} \right) \|\langle v \rangle^{2(\frac{\gamma}{2}+1)} \nabla_x G_3\|_{L_x^2(H_{v,*}^1)} \\
 &\lesssim \left(\|g_1\|_{\mathcal{X}} \|\widetilde{\nabla}_x g_2\|_{\mathcal{Y}_1} + \|g_1\|_{\mathcal{X}} \|g_2\|_{\mathcal{Y}_2} \right) \|G_3\|_{\mathcal{Y}_1}.
 \end{aligned}$$

For T_2 , we have:

$$\begin{aligned}
 T_2 &\lesssim \left(\|M^{\frac{1}{4}}g_1\|_{H_x^2L_v^2} \|\langle v \rangle^{\frac{\gamma}{2}+1} \nabla_x^2(\widetilde{\nabla}_x g_2)\|_{L_x^2(H_{v,*}^1)} \right. \\
 &\quad + \|M^{\frac{1}{4}}\nabla_x g_1\|_{H_x^1L_v^2} \|\langle v \rangle^{\frac{\gamma}{2}+1} \nabla_x(\widetilde{\nabla}_x g_2)\|_{H_x^1(H_{v,*}^1)} \\
 &\quad \left. + \|M^{\frac{1}{4}}\nabla_x^2 g_1\|_{L_x^2L_v^2} \|\langle v \rangle^{\frac{\gamma}{2}+1} \widetilde{\nabla}_x g_2\|_{H_x^2(H_{v,*}^1)} \right) \|\langle v \rangle^{\frac{\gamma}{2}+1} \nabla_x^2 G_3\|_{L_x^2(H_{v,*}^1)} \\
 &\quad + \left(\|M^{\frac{1}{4}}\nabla_x g_1\|_{H_x^2L_v^2} \|\langle v \rangle^{\frac{\gamma}{2}+1} \nabla_x^2(\langle v \rangle^{\frac{\gamma}{2}+1} g_2)\|_{L_x^2(H_{v,*}^1)} \right. \\
 &\quad + \|M^{\frac{1}{4}}\nabla_x^2 g_1\|_{H_x^1L_v^2} \|\langle v \rangle^{\frac{\gamma}{2}+1} \nabla_x(\langle v \rangle^{\frac{\gamma}{2}+1} g_2)\|_{H_x^1(H_{v,*}^1)} \\
 &\quad \left. + \|M^{\frac{1}{4}}\nabla_x^3 g_1\|_{L_x^2L_v^2} \|\langle v \rangle^{\frac{\gamma}{2}+1} \langle v \rangle^{\frac{\gamma}{2}+1} g_2\|_{H_x^2(H_{v,*}^1)} \right) \|\langle v \rangle^{\frac{\gamma}{2}+1} \nabla_x^2 G_3\|_{L_x^2(H_{v,*}^1)} \\
 &\lesssim \left(\|g_1\|_{\mathcal{X}} \|\widetilde{\nabla}_x g_2\|_{\mathcal{Y}_1} + \|g_1\|_{\mathcal{X}} \|g_2\|_{\mathcal{Y}_2} \right) \|G_3\|_{\mathcal{Y}_1}.
 \end{aligned}$$

For T_3 , we have:

$$\begin{aligned}
 T_3 &\lesssim \left(\|M^{\frac{1}{4}}g_1\|_{H_x^2L_v^2} \|\nabla_x^3(\widetilde{\nabla}_x g_2)\|_{L_x^2(H_{v,*}^1)} + \|M^{\frac{1}{4}}\nabla_x g_1\|_{H_x^1L_v^2} \|\nabla_x^2(\widetilde{\nabla}_x g_2)\|_{H_x^1(H_{v,*}^1)} \right. \\
 &\quad + \|M^{\frac{1}{4}}\nabla_x^2 g_1\|_{L_x^2L_v^2} \|\nabla_x(\widetilde{\nabla}_x g_2)\|_{H_x^2(H_{v,*}^1)} \\
 &\quad \left. + \|M^{\frac{1}{4}}\nabla_x^3 g_1\|_{L_x^2L_v^2} \|\widetilde{\nabla}_x g_2\|_{H_x^2(H_{v,*}^1)} \right) \|\nabla_x^3 G_3\|_{L_x^2(H_{v,*}^1)} \\
 &\quad + \left(\|M^{\frac{1}{4}}\nabla_x g_1\|_{H_x^2L_v^2} \|\nabla_x^3(\langle v \rangle^{\frac{\gamma}{2}+1} g_2)\|_{L_x^2(H_{v,*}^1)} + \|M^{\frac{1}{4}}\nabla_x^2 g_1\|_{H_x^1L_v^2} \|\nabla_x^2(\langle v \rangle^{\frac{\gamma}{2}+1} g_2)\|_{H_x^1(H_{v,*}^1)} \right. \\
 &\quad + \|M^{\frac{1}{4}}\nabla_x^3 g_1\|_{L_x^2L_v^2} \|\nabla_x(\langle v \rangle^{\frac{\gamma}{2}+1} g_2)\|_{H_x^2(H_{v,*}^1)} \\
 &\quad \left. + \|M^{\frac{1}{4}}\nabla_x^4 g_1\|_{L_x^2L_v^2} \|\langle v \rangle^{\frac{\gamma}{2}+1} g_2\|_{H_x^2(H_{v,*}^1)} \right) \|\nabla_x^3 G_3\|_{L_x^2(H_{v,*}^1)} \\
 &\lesssim \left(\|g_1\|_{\mathcal{X}} \|\widetilde{\nabla}_x g_2\|_{\mathcal{Y}_1} + \|g_1\|_{\mathcal{X}} \|g_2\|_{\mathcal{Y}_2} + \|M^{\frac{1}{4}}\nabla_x^4 g_1\|_{L_x^2L_v^2} \|\langle v \rangle^{\frac{\gamma}{2}+1} g_2\|_{H_x^2(H_{v,*}^1)} \right) \|G_3\|_{\mathcal{Y}_1}.
 \end{aligned}$$

We conclude the proof by gathering previous estimates and observing that

$$\|M^{\frac{1}{4}}\nabla_x^4 g_1\|_{L_x^2 L_v^2} \|\langle v \rangle^{\frac{\gamma}{2}+1} g_2\|_{H_x^2(H_{v,*}^1)} \lesssim \|\widetilde{\nabla}_x g_1\|_{\mathcal{X}} \|g_2\|_{\mathcal{Y}_1}.$$

□

Proposition 4.24. *For any smooth enough functions g_1 and g_2 there holds*

$$\|\Gamma(g_1, g_2)\|_{\mathcal{X}} \lesssim \|g_1\|_{\mathcal{X}} \|g_2\|_{\mathcal{Y}_2} + \|g_1\|_{\mathcal{Y}_1} \|g_2\|_{\mathcal{X}}.$$

Proof. We shall only prove that for any $\alpha \in \mathbb{R}$, there holds

$$\|\langle v \rangle^\alpha \Gamma(g_1, g_2)\|_{L_v^2} \lesssim \|M^{\frac{1}{4}} g_1\|_{L_v^2} \|\langle v \rangle^\alpha g_2\|_{H_{v,*}^2} + \|M^{\frac{1}{8}} g_1\|_{H_{v,*}^1} \|\langle v \rangle^\alpha g_2\|_{L_v^2}, \quad (4.110)$$

from which we obtain the desired result by integrating in x and arguing as in Step 4 of the proof of Proposition 4.21.

Starting from the formulation (4.35) of $\Gamma(g_1, g_2)$, we perform the ∂_{v_i} derivative in the first two terms, which gives

$$\begin{aligned} \Gamma(g_1, g_2) &= \left(a_{ij} * [\sqrt{M} g_1] \right) \left\{ \partial_{v_i} \partial_{v_j} g_2 - v_i \partial_{v_j} g_2 + \frac{1}{4} v_i v_j g_2 \right\} - \left(\left[\frac{1}{2} a_{ii} + c \right] * [\sqrt{M} g_1] \right) g_2 \\ &=: \widetilde{\Gamma}_1(g_1, g_2) + \widetilde{\Gamma}_2(g_1, g_2) + \widetilde{\Gamma}_3(g_1, g_2) + \widetilde{\Gamma}_4(g_1, g_2), \end{aligned}$$

and we estimate each term separately in the sequel.

The terms $\widetilde{\Gamma}_2(g_1, g_2)$ and $\widetilde{\Gamma}_3(g_1, g_2)$ can be easily estimated thanks to Lemma 4.20. Indeed, we have

$$\begin{aligned} \|\langle v \rangle^\alpha \widetilde{\Gamma}_2(g_1, g_2)\|_{L_v^2} &= \|\langle v \rangle^\alpha \left(a_{ij} * [\sqrt{M} g_1] \right) v_i \partial_{v_j} g_2\|_{L_v^2} \\ &\lesssim \|M^{\frac{1}{4}} g_1\|_{L_v^2} \|\langle v \rangle^\alpha \langle v \rangle^{\frac{\gamma}{2}+1} \widetilde{\nabla}_v g_2\|_{L_v^2} \lesssim \|M^{\frac{1}{4}} g_1\|_{L_v^2} \|\langle v \rangle^\alpha g_2\|_{H_{v,*}^2}, \end{aligned}$$

and also

$$\begin{aligned} \|\langle v \rangle^\alpha \widetilde{\Gamma}_3(g_1, g_2)\|_{L_v^2} &= \frac{1}{4} \|\langle v \rangle^\alpha \left(a_{ij} * [\sqrt{M} g_1] \right) v_i v_j g_2\|_{L_v^2} \\ &\lesssim \|M^{\frac{1}{4}} g_1\|_{L_v^2} \|\langle v \rangle^\alpha \langle v \rangle^{\gamma+2} g_2\|_{L_v^2} \lesssim \|M^{\frac{1}{4}} g_1\|_{L_v^2} \|\langle v \rangle^\alpha g_2\|_{H_{v,*}^2}. \end{aligned}$$

Moreover, we can easily estimate $\widetilde{\Gamma}_4(g_1, g_2)$ thanks to Lemma 4.12. We have

$$\begin{aligned} \|\langle v \rangle^\alpha \left(a_{ii} * [\sqrt{M} g_1] \right) g_2\|_{L_v^2} &\lesssim \|M^{\frac{1}{4}} g_1\|_{L_v^2} \|\langle v \rangle^\alpha \langle v \rangle^{\gamma+2} g_2\|_{L_v^2} \\ &\lesssim \|M^{\frac{1}{4}} g_1\|_{L_v^2} \|\langle v \rangle^\alpha g_2\|_{H_{v,*}^2}, \end{aligned}$$

as well as, if $\gamma \geq 0$,

$$\begin{aligned} \|\langle v \rangle^\alpha \left(c * [\sqrt{M} g_1] \right) g_2\|_{L_v^2} &\lesssim \|M^{\frac{1}{4}} g_1\|_{L_v^2} \|\langle v \rangle^\alpha \langle v \rangle^\gamma g_2\|_{L_v^2} \\ &\lesssim \|M^{\frac{1}{4}} g_1\|_{L_v^2} \|\langle v \rangle^\alpha g_2\|_{H_{v,*}^2}, \end{aligned}$$

and, if $-2 \leq \gamma < 0$, since $\|\cdot\|_{L_v^4} \lesssim \|\cdot\|_{H_v^1}$, we have

$$\begin{aligned} \|\langle v \rangle^\alpha (c * [\sqrt{M}g_1]) g_2\|_{L_v^2} &\lesssim \|M^{\frac{1}{4}}g_1\|_{H_v^1} \|\langle v \rangle^\alpha \langle v \rangle^\gamma g_2\|_{L_v^2} \\ &\lesssim \|M^{\frac{1}{8}}g_1\|_{H_{v,*}^1} \|\langle v \rangle^\alpha g_2\|_{L_v^2}. \end{aligned}$$

We now prove (4.110) for $\tilde{\Gamma}_1(g_1, g_2)$. When $|v| \leq 1$, the result is straightforward using Lemma 4.12. Consider now $|v| \geq 1$. We first write

$$\partial_{v_j} g_2 = \frac{v_j}{|v|} \frac{v_\ell}{|v|} \partial_{v_\ell} g_2 + (\text{Id} - P_v)_j \nabla_v g_2$$

and

$$\partial_{v_i} \partial_{v_j} g_2 = \frac{v_i}{|v|} \frac{v_k}{|v|} \partial_{v_k} \partial_{v_j} g_2 + (\text{Id} - P_v)_i \nabla_v (\partial_{v_j} g_2),$$

thus we obtain

$$\begin{aligned} \partial_{v_i} \partial_{v_j} g_2 &= \frac{v_i}{|v|} \frac{v_k}{|v|} \partial_{v_k} \left(\frac{v_j}{|v|} \frac{v_\ell}{|v|} \partial_{v_\ell} g_2 \right) + \frac{v_i}{|v|} \frac{v_k}{|v|} \partial_{v_k} ((\text{Id} - P_v)_j \nabla_v g_2) \\ &\quad + (\text{Id} - P_v)_i \nabla_v \left(\frac{v_j}{|v|} \frac{v_\ell}{|v|} \partial_{v_\ell} g_2 \right) + (\text{Id} - P_v)_i \nabla_v ((\text{Id} - P_v)_j \nabla_v g_2). \end{aligned}$$

Thanks to a straightforward computation, we remark that

$$\begin{aligned} &(\text{Id} - P_v)_i \nabla_v \left(\frac{v_j}{|v|} \frac{v_\ell}{|v|} \partial_{v_\ell} g_2 \right) \\ &= \frac{v_j}{|v|} \frac{v_\ell}{|v|} (\text{Id} - P_v)_i \nabla_v (\partial_{v_\ell} g_2) + \partial_{v_k} \left(\frac{v_j}{|v|} \frac{v_\ell}{|v|} \right) \left(1 - \frac{v_i}{|v|} \frac{v_k}{|v|} \right) \partial_{v_\ell} g_2. \end{aligned}$$

and

$$\frac{v_i}{|v|} \frac{v_k}{|v|} \partial_{v_k} \left(\frac{v_j}{|v|} \frac{v_\ell}{|v|} \partial_{v_\ell} g_2 \right) = \frac{v_i}{|v|} \frac{v_j}{|v|} \left(\frac{v_k}{|v|} \frac{v_\ell}{|v|} \partial_{v_k} \partial_{v_\ell} g_2 \right).$$

Therefore, we get

$$\begin{aligned} \langle v \rangle^\alpha \tilde{\Gamma}_1(g_1, g_2)(v) &= \langle v \rangle^\alpha (a_{ij} * [\sqrt{M}g_1]) \frac{v_i}{|v|} \frac{v_j}{|v|} \left(\frac{v_k}{|v|} \frac{v_\ell}{|v|} \partial_{v_k} \partial_{v_\ell} g_2 \right) \\ &\quad + \langle v \rangle^\alpha (a_{ij} * [\sqrt{M}g_1]) \frac{v_i}{|v|} \frac{v_k}{|v|} \partial_{v_k} [(\text{Id} - P_v)_j \nabla_v g_2] \\ &\quad + \langle v \rangle^\alpha (a_{ij} * [\sqrt{M}g_1]) \frac{v_j}{|v|} \frac{v_\ell}{|v|} (\text{Id} - P_v)_i \nabla_v (\partial_{v_\ell} g_2) \\ &\quad + \langle v \rangle^\alpha (a_{ij} * [\sqrt{M}g_1]) \partial_{v_k} \left(\frac{v_j}{|v|} \frac{v_\ell}{|v|} \right) \left(1 - \frac{v_i}{|v|} \frac{v_k}{|v|} \right) \partial_{v_\ell} g_2 \\ &\quad + \langle v \rangle^\alpha (a_{ij} * [\sqrt{M}g_1]) (\text{Id} - P_v)_i \nabla_v ((\text{Id} - P_v)_j \nabla_v g_2) \\ &=: I_1 + I_2 + I_3 + I_4 + I_5. \end{aligned}$$

We can now estimate each of these terms using Lemma 4.12 and Lemma 4.20. We obtain

$$\begin{aligned} \|I_1\|_{L_v^2} &\lesssim \|M^{\frac{1}{4}}g_1\|_{L_v^2} \|\langle v \rangle^\alpha \langle v \rangle^\gamma \nabla_v \nabla_v g_2\|_{L_v^2} \lesssim \|M^{\frac{1}{4}}g_1\|_{L_v^2} \|\langle v \rangle^\alpha g_2\|_{H_{v,*}^2}, \\ \|I_2\|_{L_v^2} &\lesssim \|M^{\frac{1}{4}}g_1\|_{L_v^2} \|\langle v \rangle^\alpha \langle v \rangle^{\gamma+1} \nabla_v [(\text{Id} - P_v) \nabla_v g_2]\|_{L_v^2} \lesssim \|M^{\frac{1}{4}}g_1\|_{L_v^2} \|\langle v \rangle^\alpha g_2\|_{H_{v,*}^2}, \\ \|I_3\|_{L_v^2} &\lesssim \|M^{\frac{1}{4}}g_1\|_{L_v^2} \|\langle v \rangle^\alpha \langle v \rangle^{\gamma+1} (\text{Id} - P_v) \nabla_v (\nabla_v g_2)\|_{L_v^2} \lesssim \|M^{\frac{1}{4}}g_1\|_{L_v^2} \|\langle v \rangle^\alpha g_2\|_{H_{v,*}^2}, \\ \|I_4\|_{L_v^2} &\lesssim \|M^{\frac{1}{4}}g_1\|_{L_v^2} \|\langle v \rangle^\alpha \langle v \rangle^{\gamma+1} \nabla_v g_2\|_{L_v^2} \lesssim \|M^{\frac{1}{4}}g_1\|_{L_v^2} \|\langle v \rangle^\alpha g_2\|_{H_{v,*}^2}, \end{aligned}$$

and finally

$$\|I_5\|_{L_v^2} \lesssim \|M^{\frac{1}{4}}g_1\|_{L_v^2} \|\langle v \rangle^\alpha \langle v \rangle^{\gamma+2} (\text{Id} - P_v) \nabla_v [(\text{Id} - P_v) \nabla_v g_2]\|_{L_v^2} \lesssim \|M^{\frac{1}{4}}g_1\|_{L_v^2} \|\langle v \rangle^\alpha g_2\|_{H_{v,*}^2}.$$

We conclude the proof of (4.110) by gathering previous estimates. \square

4.2 Well-posedness for the Landau equation

In this section, we shall prove the well-posedness part of Theorem 4.1.

Proof of Theorem 4.1-(i). Let g^ε be a solution to (4.7) associated to $g_{\text{in}}^\varepsilon \in \mathcal{X}$ satisfying (4.21). Notice that it implies that $g_{\text{in}}^\varepsilon \in (\text{Ker } \Lambda_\varepsilon)^\perp$ and thus $g^\varepsilon(t) \in (\text{Ker } \Lambda_\varepsilon)^\perp$ for all $t > 0$ from the conservation laws (4.4). We shall use the norm $\|\cdot\|_{\mathcal{X}}$ (and the associated inner product $\langle \cdot, \cdot \rangle_{\mathcal{X}}$) defined in (4.57) during the proof of Proposition 4.17, in order to establish below an a priori estimate for g^ε .

Let $\sigma \in (0, \sigma_0)$ be fixed and compute

$$\frac{1}{2} \frac{d}{dt} \|g^\varepsilon\|_{\mathcal{X}}^2 = \langle \Lambda_\varepsilon g^\varepsilon, g^\varepsilon \rangle_{\mathcal{X}} + \frac{1}{\varepsilon} \langle \Gamma(g^\varepsilon, g^\varepsilon), g^\varepsilon \rangle_{\mathcal{X}}.$$

For the linear part, estimate (4.54) in Proposition 4.17 already gives us

$$\langle \Lambda_\varepsilon g^\varepsilon, g^\varepsilon \rangle_{\mathcal{X}} \leq -\sigma \|g^\varepsilon\|_{\mathcal{X}}^2 - \kappa \|g^\varepsilon\|_{\mathcal{A}_1}^2 - \frac{\kappa}{\varepsilon^2} \|(g^\varepsilon)^\perp\|_{\mathcal{A}_1}^2.$$

For the nonlinear part, from (4.78), we get

$$\begin{aligned} \langle \Gamma(g^\varepsilon, g^\varepsilon), g^\varepsilon \rangle_{\mathcal{X}} &= \delta \langle \Gamma(g^\varepsilon, g^\varepsilon), (g^\varepsilon)^\perp \rangle_{\mathcal{X}} + \sum_{i=0}^2 \langle \nabla_x^i \Gamma(g^\varepsilon, g^\varepsilon), \nabla_x^i (g^\varepsilon)^\perp \rangle_{L_{x,v}^2} \\ &\quad + (1 - \delta) \langle \nabla_x^3 \Gamma(g^\varepsilon, g^\varepsilon), \nabla_x^3 (g^\varepsilon)^\perp \rangle_{L_{x,v}^2}. \end{aligned}$$

Thanks to Proposition 4.21, we have

$$\langle \Gamma(g^\varepsilon, g^\varepsilon), (g^\varepsilon)^\perp \rangle_{\mathcal{X}} \lesssim \|g^\varepsilon\|_{\mathcal{X}} \|g^\varepsilon\|_{\mathcal{A}_1} \|(g^\varepsilon)^\perp\|_{\mathcal{A}_1}$$

and also

$$\left\langle \nabla_x^3 \Gamma(g^\varepsilon, g^\varepsilon), \nabla_x^3 (g^\varepsilon)^\perp \right\rangle_{L_{x,v}^2} \lesssim \|g^\varepsilon\|_{\mathcal{X}} \|g^\varepsilon\|_{\mathcal{Y}_1} \|(g^\varepsilon)^\perp\|_{\mathcal{Y}_1}.$$

Arguing as in Step 4 of the proof of Proposition 4.21, we also obtain

$$\sum_{i=0}^2 \left\langle \nabla_x^i \Gamma(g^\varepsilon, g^\varepsilon), \nabla_x^i (g^\varepsilon)^\perp \right\rangle_{L_{x,v}^2} \lesssim \|g^\varepsilon\|_{\mathcal{X}} \|g^\varepsilon\|_{\mathcal{Y}_1} \|(g^\varepsilon)^\perp\|_{\mathcal{Y}_1}.$$

In summary, and recalling that $\|\cdot\|_{\mathcal{X}}$ and $\|\|\cdot\|\|_{\mathcal{X}}$ are equivalent, we then have

$$\frac{1}{\varepsilon} \langle \Gamma(g^\varepsilon, g^\varepsilon), g^\varepsilon \rangle_{\mathcal{X}} \leq \frac{C}{\varepsilon} \|\|\|g^\varepsilon\|\|\|_{\mathcal{X}} \|g^\varepsilon\|_{\mathcal{Y}_1} \|(g^\varepsilon)^\perp\|_{\mathcal{Y}_1} \quad (4.111)$$

for some constant $C > 0$.

Denoting $e_\sigma : t \mapsto e^{\sigma t}$ and $g_\sigma^\varepsilon = e_\sigma g^\varepsilon$, we therefore obtain

$$\frac{1}{2} \frac{d}{dt} \|\|\|g_\sigma^\varepsilon\|\|\|_{\mathcal{X}}^2 \leq -\kappa \|g_\sigma^\varepsilon\|_{\mathcal{Y}_1}^2 - \frac{\kappa}{\varepsilon^2} \|(g_\sigma^\varepsilon)^\perp\|_{\mathcal{Y}_1}^2 + \frac{C}{\varepsilon} \|g_\sigma^\varepsilon\|_{\mathcal{X}} \|g_\sigma^\varepsilon\|_{\mathcal{Y}_1} \|(g_\sigma^\varepsilon)^\perp\|_{\mathcal{Y}_1}.$$

Thanks to Young's inequality we write

$$\frac{C}{\varepsilon} \|g_\sigma^\varepsilon\|_{\mathcal{X}} \|g_\sigma^\varepsilon\|_{\mathcal{Y}_1} \|(g_\sigma^\varepsilon)^\perp\|_{\mathcal{Y}_1} \leq C \|\|\|g_\sigma^\varepsilon\|\|\|_{\mathcal{X}}^2 \|g_\sigma^\varepsilon\|_{\mathcal{Y}_1}^2 + \frac{\kappa}{2\varepsilon^2} \|(g_\sigma^\varepsilon)^\perp\|_{\mathcal{Y}_1}^2,$$

which then gives the following a priori estimate

$$\frac{1}{2} \frac{d}{dt} \|\|\|g_\sigma^\varepsilon\|\|\|_{\mathcal{X}}^2 \leq -\left(\kappa - C \|\|\|g_\sigma^\varepsilon\|\|\|_{\mathcal{X}}^2\right) \|g_\sigma^\varepsilon\|_{\mathcal{Y}_1}^2 - \frac{\kappa}{2\varepsilon^2} \|(g_\sigma^\varepsilon)^\perp\|_{\mathcal{Y}_1}^2. \quad (4.112)$$

At least formally, from this differential inequality we easily obtain that if $\|\|\|g_{\text{in}}^\varepsilon\|\|\|_{\mathcal{X}}$ is small enough then g^ε satisfies the uniform in time estimate (4.22). The proof of existence and uniqueness of a solution g^ε to (4.7) satisfying (4.22) for small data $\|g_{\text{in}}^\varepsilon\|_{\mathcal{X}} \leq \eta_0$ follows a standard iterative scheme that uses estimate (4.112) (see for example [26]). \square

4.3 Regularity for the Landau equation

In this part, we provide a result of regularization for the solutions to the nonlinear Landau equation which is quantified in time, namely the regularization estimate of Theorem 4.1. Notice here that if we only wanted to handle the case $\varepsilon = 1$, we could have used the triple norm introduced in [56] (see [26] for the Landau equation) which is dissipative for the whole linearized operator and equivalent to the usual one. Here, to handle the ε -dependencies, we have to use our hypocoercive norm defined in Proposition 4.17 and separate carefully the behaviors of microscopic and macroscopic parts of the solution. Some additional remainder terms coming from the fact that the transport operator and the projector π onto the kernel of L (see (4.12)) do not commute have to be treated. The computations are thus much more intricate.

Proof of Theorem 4.1-(ii). Let g^ε be a global solution to (4.7) associated to the initial data $g_{\text{in}}^\varepsilon \in \mathcal{X}$ satisfying (4.21), with $\|g_{\text{in}}^\varepsilon\|_{\mathcal{X}} \leq \eta_0$, provided by Theorem 4.1-(i). As in the proof of Theorem 4.1-(i), we shall only obtain an a priori estimate implying the desired regularization estimates.

We recall that the spaces \mathcal{Y}_1 and $\mathcal{X}_1^\varepsilon$ are defined in (4.20)-(4.45) and we shall prove that for any $t \in (0, 1]$, one has

$$\|g^\varepsilon(t)\|_{\mathcal{Y}_1} \lesssim \frac{1}{\sqrt{t}} \|g_{\text{in}}^\varepsilon\|_{\mathcal{X}} \quad \text{and} \quad \|g^\varepsilon(t)\|_{\mathcal{X}_1^\varepsilon} \lesssim \frac{1}{t^{3/2}} \|g_{\text{in}}^\varepsilon\|_{\mathcal{X}}, \quad (4.113)$$

which readily implies that, for all $t > 0$, there holds

$$\|g^\varepsilon(t)\|_{\mathcal{Y}_1} \lesssim \frac{e^{-\sigma t}}{\min(1, \sqrt{t})} \|g_{\text{in}}^\varepsilon\|_{\mathcal{X}} \quad \text{and} \quad \|g^\varepsilon(t)\|_{\mathcal{X}_1^\varepsilon} \lesssim \frac{e^{-\sigma t}}{\min(1, t^{3/2})} \|g_{\text{in}}^\varepsilon\|_{\mathcal{X}}$$

by using the exponential decay in \mathcal{X} given by Theorem 4.1-(i) and hence concludes the proof of Theorem 4.1-(ii). We split the proof of (4.113) into several steps. We shall use without further mention that since $\pi \in \mathcal{B}(\mathcal{X}, \mathcal{Y}_i)$, we have $\|g^\varepsilon\|_{\mathcal{Y}_i} \lesssim \|g^\varepsilon\|_{\mathcal{X}} + \|(g^\varepsilon)^\perp\|_{\mathcal{Y}_i}$ for $i = 1, 2$.

Step 1. We consider the same functional \mathcal{U}_ε as defined in Step 1 of Proposition 4.18, which we recall is given by

$$\begin{aligned} \mathcal{U}_\varepsilon(t, g^\varepsilon) &= \|g^\varepsilon\|_{\mathcal{X}}^2 + \alpha_1 t \left(\|\widetilde{\nabla}_v(g^\varepsilon)^\perp\|_{\mathcal{X}}^2 + K \|\langle v \rangle^{\frac{7}{2}+1}(g^\varepsilon)^\perp\|_{\mathcal{X}}^2 \right) \\ &\quad + \varepsilon \alpha_2 t^2 \langle \widetilde{\nabla}_v g^\varepsilon, \widetilde{\nabla}_x g^\varepsilon \rangle_{\mathcal{X}} + \varepsilon^2 \alpha_3 t^3 \left(\|\widetilde{\nabla}_x g^\varepsilon\|_{\mathcal{X}}^2 + K \|\langle v \rangle^{\frac{7}{2}} \nabla_x g^\varepsilon\|_{\mathcal{X}}^2 \right), \end{aligned} \quad (4.114)$$

with constants $K > 0$ and $0 < \alpha_3 \ll \alpha_2 \ll \alpha_1 \ll 1$ so that $\alpha_2 \leq \sqrt{\alpha_1 \alpha_3}$. We recall that the constant $K > 0$ is chosen large enough in the proof of Proposition 4.18. The constants α_i will be chosen small enough during the proof here. We also recall that for any $t \in (0, 1]$, one has the lower bounds

$$t \|g^\varepsilon\|_{\mathcal{Y}_1}^2 \lesssim \mathcal{U}_\varepsilon(t, g^\varepsilon) \quad \text{and} \quad t^3 \|g^\varepsilon\|_{\mathcal{X}_1^\varepsilon}^2 \lesssim \mathcal{U}_\varepsilon(t, g^\varepsilon). \quad (4.115)$$

Step 2. Thanks to the proof of Theorem 4.1-(i) we already have

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|g^\varepsilon\|_{\mathcal{X}}^2 &= \langle \Lambda_\varepsilon g^\varepsilon, g^\varepsilon \rangle_{\mathcal{X}} + \frac{1}{\varepsilon} \langle \Gamma(g^\varepsilon, g^\varepsilon), g^\varepsilon \rangle_{\mathcal{X}} \\ &\leq - \left(\kappa - C \|g^\varepsilon\|_{\mathcal{X}}^2 \right) \|g^\varepsilon\|_{\mathcal{Y}_1}^2 - \frac{\kappa}{2\varepsilon^2} \|(g^\varepsilon)^\perp\|_{\mathcal{Y}_1}^2 \end{aligned} \quad (4.116)$$

for some constants $\kappa, C > 0$.

Step 3. We first observe that since $\pi \Gamma(g^\varepsilon, g^\varepsilon) = 0$, $(g^\varepsilon)^\perp$ satisfies the equation

$$\partial_t (g^\varepsilon)^\perp = (\text{Id} - \pi) \Lambda_\varepsilon g^\varepsilon + \frac{1}{\varepsilon} \Gamma(g^\varepsilon, g^\varepsilon).$$

We then compute

$$\begin{aligned}
 & \frac{1}{2} \frac{d}{dt} \left\{ K \|\langle v \rangle^{\frac{7}{2}+1} (g^\varepsilon)^\perp\|_{\mathcal{X}}^2 + \|\widetilde{\nabla}_v (g^\varepsilon)^\perp\|_{\mathcal{X}}^2 \right\} \\
 &= K \left\langle \langle v \rangle^{\frac{7}{2}+1} (\text{Id} - \pi) \Lambda_\varepsilon g^\varepsilon, \langle v \rangle^{\frac{7}{2}+1} (g^\varepsilon)^\perp \right\rangle_{\mathcal{X}} + \left\langle \widetilde{\nabla}_v (\text{Id} - \pi) \Lambda_\varepsilon g^\varepsilon, \widetilde{\nabla}_v (g^\varepsilon)^\perp \right\rangle_{\mathcal{X}} \\
 & \quad + \frac{K}{\varepsilon} \left\langle \langle v \rangle^{\frac{7}{2}+1} \Gamma(g^\varepsilon, g^\varepsilon), \langle v \rangle^{\frac{7}{2}+1} (g^\varepsilon)^\perp \right\rangle_{\mathcal{X}} + \frac{1}{\varepsilon} \left\langle \widetilde{\nabla}_v \Gamma(g^\varepsilon, g^\varepsilon), \widetilde{\nabla}_v (g^\varepsilon)^\perp \right\rangle_{\mathcal{X}} \\
 &=: I_1 + I_2 + I_3 + I_4.
 \end{aligned}$$

From (4.70) in the proof of Proposition 4.18, we already know that

$$I_1 + I_2 \leq -\frac{\kappa_1}{\varepsilon^2} \|(g^\varepsilon)^\perp\|_{\mathcal{Y}_2}^2 + \frac{C}{\varepsilon^2} \|(g^\varepsilon)^\perp\|_{\mathcal{Y}_1}^2 + \frac{C}{\varepsilon} \|(g^\varepsilon)^\perp\|_{\mathcal{Y}_1} \|\widetilde{\nabla}_x g^\varepsilon\|_{\mathcal{X}}$$

for some constants $\kappa_1, C > 0$. Thanks to (4.88) in Proposition 4.21, we have

$$I_3 \lesssim \frac{1}{\varepsilon} \|g^\varepsilon\|_{\mathcal{X}} \|\langle v \rangle^{\frac{7}{2}+1} g^\varepsilon\|_{\mathcal{Y}_1} \|\langle v \rangle^{\frac{7}{2}+1} (g^\varepsilon)^\perp\|_{\mathcal{Y}_1} \lesssim \frac{1}{\varepsilon} \|g^\varepsilon\|_{\mathcal{X}} \|g^\varepsilon\|_{\mathcal{Y}_2} \|(g^\varepsilon)^\perp\|_{\mathcal{Y}_2}.$$

Moreover, thanks to (4.96) in Proposition 4.22, we have

$$\begin{aligned}
 I_4 &\lesssim \frac{1}{\varepsilon} \|g^\varepsilon\|_{\mathcal{X}} \|g^\varepsilon\|_{\mathcal{Y}_2} \|\widetilde{\nabla}_v (g^\varepsilon)^\perp\|_{\mathcal{Y}_1} \lesssim \frac{1}{\varepsilon} \|g^\varepsilon\|_{\mathcal{X}} \|g^\varepsilon\|_{\mathcal{Y}_2} \|(g^\varepsilon)^\perp\|_{\mathcal{Y}_2} \\
 &\lesssim \frac{1}{\varepsilon} \|g^\varepsilon\|_{\mathcal{X}}^2 \|(g^\varepsilon)^\perp\|_{\mathcal{Y}_2} + \frac{1}{\varepsilon} \|g^\varepsilon\|_{\mathcal{X}} \|(g^\varepsilon)^\perp\|_{\mathcal{Y}_2}^2.
 \end{aligned}$$

We therefore obtain, using that $\|h\|_{\mathcal{X}} \lesssim \|h\|_{\mathcal{Y}_1}$,

$$\begin{aligned}
 & \frac{1}{2} \frac{d}{dt} \left\{ K \|\langle v \rangle^{\frac{7}{2}+1} (g^\varepsilon)^\perp\|_{\mathcal{X}}^2 + \|\widetilde{\nabla}_v (g^\varepsilon)^\perp\|_{\mathcal{X}}^2 \right\} \\
 & \leq -\frac{\kappa_1}{\varepsilon^2} \|(g^\varepsilon)^\perp\|_{\mathcal{Y}_2}^2 + \frac{C}{\varepsilon^2} \|(g^\varepsilon)^\perp\|_{\mathcal{Y}_1}^2 + \frac{C}{\varepsilon} \|(g^\varepsilon)^\perp\|_{\mathcal{Y}_1} \|\widetilde{\nabla}_x g^\varepsilon\|_{\mathcal{X}} \\
 & \quad + \frac{C}{\varepsilon} \|g^\varepsilon\|_{\mathcal{X}} \|g^\varepsilon\|_{\mathcal{Y}_1} \|(g^\varepsilon)^\perp\|_{\mathcal{Y}_2} + \frac{C}{\varepsilon} \|g^\varepsilon\|_{\mathcal{X}} \|(g^\varepsilon)^\perp\|_{\mathcal{Y}_2}^2.
 \end{aligned} \tag{4.117}$$

Step 4. We compute

$$\begin{aligned}
 \frac{d}{dt} \left\langle \widetilde{\nabla}_v g^\varepsilon, \widetilde{\nabla}_x g^\varepsilon \right\rangle_{\mathcal{X}} &= \left\langle \widetilde{\nabla}_v (\Lambda_\varepsilon g^\varepsilon), \widetilde{\nabla}_x g^\varepsilon \right\rangle_{\mathcal{X}} + \left\langle \widetilde{\nabla}_x (\Lambda_\varepsilon g^\varepsilon), \widetilde{\nabla}_v g^\varepsilon \right\rangle_{\mathcal{X}} \\
 & \quad + \frac{1}{\varepsilon} \left\langle \widetilde{\nabla}_v \Gamma(g^\varepsilon, g^\varepsilon), \widetilde{\nabla}_x g^\varepsilon \right\rangle_{\mathcal{X}} + \frac{1}{\varepsilon} \left\langle \widetilde{\nabla}_x \Gamma(g^\varepsilon, g^\varepsilon), \widetilde{\nabla}_v g^\varepsilon \right\rangle_{\mathcal{X}} \\
 &=: J_1 + J_2 + J_3 + J_4.
 \end{aligned}$$

Thanks to (4.73) in Proposition 4.18, we already have

$$J_1 + J_2 \leq \frac{C}{\varepsilon^2} \|(g^\varepsilon)^\perp\|_{\mathcal{Y}_2} \|\widetilde{\nabla}_x g^\varepsilon\|_{\mathcal{Y}_1} - \frac{1}{\varepsilon} \|\widetilde{\nabla}_x g^\varepsilon\|_{\mathcal{X}}^2$$

fo some constant $C > 0$. For the term J_3 , estimate (4.96) in Proposition 4.22 yields

$$\begin{aligned} J_3 &\lesssim \frac{1}{\varepsilon} \|g^\varepsilon\|_{\mathcal{X}} \|g^\varepsilon\|_{\mathcal{Y}_2} \|\widetilde{\nabla}_x g^\varepsilon\|_{\mathcal{Y}_1} \\ &\lesssim \frac{1}{\varepsilon} \|g^\varepsilon\|_{\mathcal{X}}^2 \|\widetilde{\nabla}_x g^\varepsilon\|_{\mathcal{Y}_1} + \frac{1}{\varepsilon} \|g^\varepsilon\|_{\mathcal{X}} \|(g^\varepsilon)^\perp\|_{\mathcal{Y}_2} \|\widetilde{\nabla}_x g^\varepsilon\|_{\mathcal{Y}_1}. \end{aligned}$$

Moreover, for the term J_4 , estimate (4.103) in Proposition 4.23 gives us

$$\begin{aligned} J_4 &\lesssim \frac{1}{\varepsilon} \left(\|g^\varepsilon\|_{\mathcal{X}} \|\widetilde{\nabla}_x g^\varepsilon\|_{\mathcal{Y}_1} + \|\widetilde{\nabla}_x g^\varepsilon\|_{\mathcal{X}} \|g^\varepsilon\|_{\mathcal{Y}_1} + \|g^\varepsilon\|_{\mathcal{X}} \|g^\varepsilon\|_{\mathcal{Y}_2} \right) \|\widetilde{\nabla}_v g^\varepsilon\|_{\mathcal{Y}_1} \\ &\lesssim \frac{1}{\varepsilon} \left(\|g^\varepsilon\|_{\mathcal{X}} \|\widetilde{\nabla}_x g^\varepsilon\|_{\mathcal{Y}_1} + \|\widetilde{\nabla}_x g^\varepsilon\|_{\mathcal{X}} \|g^\varepsilon\|_{\mathcal{Y}_1} + \|g^\varepsilon\|_{\mathcal{X}} \|g^\varepsilon\|_{\mathcal{Y}_2} \right) \|g^\varepsilon\|_{\mathcal{Y}_2} \\ &\lesssim \frac{1}{\varepsilon} \left(\|g^\varepsilon\|_{\mathcal{X}}^2 \|\widetilde{\nabla}_x g^\varepsilon\|_{\mathcal{Y}_1} + \|g^\varepsilon\|_{\mathcal{X}} \|\widetilde{\nabla}_x g^\varepsilon\|_{\mathcal{Y}_1} \|(g^\varepsilon)^\perp\|_{\mathcal{Y}_2} + \|g^\varepsilon\|_{\mathcal{X}} \|\widetilde{\nabla}_x g^\varepsilon\|_{\mathcal{X}} \|g^\varepsilon\|_{\mathcal{Y}_1} \right. \\ &\quad \left. + \|g^\varepsilon\|_{\mathcal{X}} \|\widetilde{\nabla}_x g^\varepsilon\|_{\mathcal{X}} \|(g^\varepsilon)^\perp\|_{\mathcal{Y}_2} + \|(g^\varepsilon)^\perp\|_{\mathcal{Y}_1} \|\widetilde{\nabla}_x g^\varepsilon\|_{\mathcal{X}} \|(g^\varepsilon)^\perp\|_{\mathcal{Y}_2} \right. \\ &\quad \left. + \|g^\varepsilon\|_{\mathcal{X}}^3 + \|g^\varepsilon\|_{\mathcal{X}}^2 \|(g^\varepsilon)^\perp\|_{\mathcal{Y}_2} + \|g^\varepsilon\|_{\mathcal{X}} \|(g^\varepsilon)^\perp\|_{\mathcal{Y}_2}^2 \right). \end{aligned}$$

Putting together previous estimates, we get

$$\begin{aligned} &\frac{d}{dt} \langle \widetilde{\nabla}_v g^\varepsilon, \widetilde{\nabla}_x g^\varepsilon \rangle_{\mathcal{X}} \\ &\leq -\frac{1}{\varepsilon} \|\widetilde{\nabla}_x g^\varepsilon\|_{\mathcal{X}}^2 + \frac{C}{\varepsilon^2} \|\widetilde{\nabla}_x g^\varepsilon\|_{\mathcal{Y}_1} \|(g^\varepsilon)^\perp\|_{\mathcal{Y}_2} + \frac{C}{\varepsilon} \|(g^\varepsilon)^\perp\|_{\mathcal{Y}_1} \|\widetilde{\nabla}_x g^\varepsilon\|_{\mathcal{X}} \|(g^\varepsilon)^\perp\|_{\mathcal{Y}_2} \\ &\quad + \frac{C}{\varepsilon} \|g^\varepsilon\|_{\mathcal{X}} \left(\|\widetilde{\nabla}_x g^\varepsilon\|_{\mathcal{X}} \|(g^\varepsilon)^\perp\|_{\mathcal{Y}_2} + \|g^\varepsilon\|_{\mathcal{Y}_1}^2 + \|g^\varepsilon\|_{\mathcal{Y}_1} \|\widetilde{\nabla}_x g^\varepsilon\|_{\mathcal{Y}_1} \right. \\ &\quad \left. + \|g^\varepsilon\|_{\mathcal{Y}_1} \|(g^\varepsilon)^\perp\|_{\mathcal{Y}_2} + \|\widetilde{\nabla}_x g^\varepsilon\|_{\mathcal{Y}_1} \|(g^\varepsilon)^\perp\|_{\mathcal{Y}_2} + \|(g^\varepsilon)^\perp\|_{\mathcal{Y}_2}^2 \right). \end{aligned} \tag{4.118}$$

Step 5. We compute

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \left\{ \|\widetilde{\nabla}_x g^\varepsilon\|_{\mathcal{X}}^2 + K \|\langle v \rangle^{\frac{\gamma}{2}} \nabla_x g^\varepsilon\|_{\mathcal{X}}^2 \right\} \\ &= \langle \widetilde{\nabla}_x (\Lambda_\varepsilon g^\varepsilon), \widetilde{\nabla}_x g^\varepsilon \rangle_{\mathcal{X}} + K \langle \langle v \rangle^{\frac{\gamma}{2}} \nabla_x (\Lambda_\varepsilon g^\varepsilon), \langle v \rangle^{\frac{\gamma}{2}} \nabla_x g^\varepsilon \rangle_{\mathcal{X}} \\ &\quad + \frac{1}{\varepsilon} \langle \widetilde{\nabla}_x \Gamma(g^\varepsilon, g^\varepsilon), \widetilde{\nabla}_x g^\varepsilon \rangle_{\mathcal{X}} + \frac{K}{\varepsilon} \langle \langle v \rangle^{\frac{\gamma}{2}} \nabla_x \Gamma(g^\varepsilon, g^\varepsilon), \langle v \rangle^{\frac{\gamma}{2}} \nabla_x g^\varepsilon \rangle_{\mathcal{X}} \\ &=: R_1 + R_2 + R_3 + R_4. \end{aligned}$$

From (4.76) in the proof of Proposition 4.18, we already know that

$$R_1 + R_2 \leq -\frac{\kappa_2}{\varepsilon^2} \|\widetilde{\nabla}_x g^\varepsilon\|_{\mathcal{Y}_1}^2 + \frac{C}{\varepsilon^2} \|\widetilde{\nabla}_x g^\varepsilon\|_{\mathcal{X}}^2$$

for some constants $\kappa_2, C > 0$. Thanks to (4.103) in Proposition 4.23 for the term R_3 (and a slight adaptation of it for the term R_4), we obtain

$$\begin{aligned} R_3 + R_4 &\lesssim \frac{1}{\varepsilon} \left(\|g^\varepsilon\|_{\mathcal{X}} \|\widetilde{\nabla}_x g^\varepsilon\|_{\mathcal{Y}_1} + \|\widetilde{\nabla}_x g^\varepsilon\|_{\mathcal{X}} \|g^\varepsilon\|_{\mathcal{Y}_1} + \|g^\varepsilon\|_{\mathcal{X}} \|g^\varepsilon\|_{\mathcal{Y}_2} \right) \|\widetilde{\nabla}_x g^\varepsilon\|_{\mathcal{Y}_1} \\ &\lesssim \frac{1}{\varepsilon} \left(\|g^\varepsilon\|_{\mathcal{X}} \|\widetilde{\nabla}_x g^\varepsilon\|_{\mathcal{Y}_1} + \|\widetilde{\nabla}_x g^\varepsilon\|_{\mathcal{X}} \|g^\varepsilon\|_{\mathcal{X}} + \|\widetilde{\nabla}_x g^\varepsilon\|_{\mathcal{X}} \|(g^\varepsilon)^\perp\|_{\mathcal{Y}_1} \right. \\ &\quad \left. + \|g^\varepsilon\|_{\mathcal{X}}^2 + \|g^\varepsilon\|_{\mathcal{X}} \|(g^\varepsilon)^\perp\|_{\mathcal{Y}_2} \right) \|\widetilde{\nabla}_x g^\varepsilon\|_{\mathcal{Y}_1}. \end{aligned}$$

We therefore obtain

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \left\{ \|\widetilde{\nabla}_x g^\varepsilon\|_{\mathcal{X}}^2 + K \|\langle v \rangle^{\frac{\gamma}{2}} \nabla_x g^\varepsilon\|_{\mathcal{X}}^2 \right\} \\ &\leq -\frac{\kappa_2}{\varepsilon^2} \|\widetilde{\nabla}_x g^\varepsilon\|_{\mathcal{Y}_1}^2 + \frac{C}{\varepsilon^2} \|\widetilde{\nabla}_x g^\varepsilon\|_{\mathcal{X}}^2 + \frac{C}{\varepsilon} \|(g^\varepsilon)^\perp\|_{\mathcal{Y}_1} \|\widetilde{\nabla}_x g^\varepsilon\|_{\mathcal{X}} \|\widetilde{\nabla}_x g^\varepsilon\|_{\mathcal{Y}_1} \\ &\quad + \frac{C}{\varepsilon} \|g^\varepsilon\|_{\mathcal{X}} \left(\|\widetilde{\nabla}_x g^\varepsilon\|_{\mathcal{X}} \|\widetilde{\nabla}_x g^\varepsilon\|_{\mathcal{Y}_1} + \|\widetilde{\nabla}_x g^\varepsilon\|_{\mathcal{Y}_1}^2 + \|\widetilde{\nabla}_x g^\varepsilon\|_{\mathcal{Y}_1} \|(g^\varepsilon)^\perp\|_{\mathcal{Y}_2} \right). \end{aligned} \quad (4.119)$$

Step 6. Conclusion. Gathering estimates (4.116)–(4.117)–(4.118)–(4.119) and using that $\sup_{t \geq 0} \|g^\varepsilon(t)\|_{\mathcal{X}} \lesssim \eta_0$ from Theorem 4.1-(i), we then obtain

$$\begin{aligned} \frac{d}{dt} \mathcal{W}_\varepsilon(t, g^\varepsilon) &\leq -\left(\kappa - C\eta_0^2 - C\alpha_2 t^2 \eta_0 \right) \|g^\varepsilon\|_{\mathcal{Y}_1}^2 - \frac{1}{\varepsilon^2} \left(\kappa - C\alpha_1 t - C\alpha_1 \right) \|(g^\varepsilon)^\perp\|_{\mathcal{Y}_1}^2 \\ &\quad - \frac{t}{\varepsilon^2} \left(\kappa_1 \alpha_1 - C\varepsilon^2 \alpha_2 t \eta_0 - C\varepsilon \alpha_1 \eta_0 \right) \|(g^\varepsilon)^\perp\|_{\mathcal{Y}_2}^2 + \frac{C\alpha_1 t}{\varepsilon} \|(g^\varepsilon)^\perp\|_{\mathcal{Y}_1} \|\widetilde{\nabla}_x g^\varepsilon\|_{\mathcal{X}} \\ &\quad + \frac{C\alpha_1 t}{\varepsilon} \eta_0 \|g^\varepsilon\|_{\mathcal{Y}_1} \|(g^\varepsilon)^\perp\|_{\mathcal{Y}_2} + 2\varepsilon \alpha_2 t \left\langle \widetilde{\nabla}_v g^\varepsilon, \widetilde{\nabla}_x g^\varepsilon \right\rangle_{\mathcal{X}} \\ &\quad - t^2 \left(\alpha_2 - C\varepsilon^2 \alpha_3 - C\alpha_3 t \right) \|\widetilde{\nabla}_x g^\varepsilon\|_{\mathcal{X}}^2 + \frac{C\alpha_2 t^2}{\varepsilon} \|\widetilde{\nabla}_x g^\varepsilon\|_{\mathcal{Y}_1} \|(g^\varepsilon)^\perp\|_{\mathcal{Y}_2} \\ &\quad + C\alpha_2 t^2 \|(g^\varepsilon)^\perp\|_{\mathcal{Y}_1} \|\widetilde{\nabla}_x g^\varepsilon\|_{\mathcal{X}} \|(g^\varepsilon)^\perp\|_{\mathcal{Y}_2} \\ &\quad + C\alpha_2 t^2 \eta_0 \left(\|\widetilde{\nabla}_x g^\varepsilon\|_{\mathcal{X}} \|(g^\varepsilon)^\perp\|_{\mathcal{Y}_2} + \|g^\varepsilon\|_{\mathcal{Y}_1} \|\widetilde{\nabla}_x g^\varepsilon\|_{\mathcal{Y}_1} \right. \\ &\quad \left. + \|g^\varepsilon\|_{\mathcal{Y}_1} \|(g^\varepsilon)^\perp\|_{\mathcal{Y}_2} + \|\widetilde{\nabla}_x g^\varepsilon\|_{\mathcal{Y}_1} \|(g^\varepsilon)^\perp\|_{\mathcal{Y}_2} \right) \\ &\quad - t^3 \left(\kappa_2 \alpha_3 - C\varepsilon \alpha_3 \eta_0 \right) \|\widetilde{\nabla}_x g^\varepsilon\|_{\mathcal{Y}_1}^2 + C\varepsilon \alpha_3 t^3 \|(g^\varepsilon)^\perp\|_{\mathcal{Y}_1} \|\widetilde{\nabla}_x g^\varepsilon\|_{\mathcal{X}} \|\widetilde{\nabla}_x g^\varepsilon\|_{\mathcal{Y}_1} \\ &\quad + C\varepsilon \alpha_3 t^3 \eta_0 \left(\|\widetilde{\nabla}_x g^\varepsilon\|_{\mathcal{X}} \|\widetilde{\nabla}_x g^\varepsilon\|_{\mathcal{Y}_1} + \|g^\varepsilon\|_{\mathcal{Y}_1} \|\widetilde{\nabla}_x g^\varepsilon\|_{\mathcal{Y}_1} + \|\widetilde{\nabla}_x g^\varepsilon\|_{\mathcal{Y}_1} \|(g^\varepsilon)^\perp\|_{\mathcal{Y}_2} \right). \end{aligned}$$

We now use Young's inequality to write

$$\begin{aligned}
 \frac{C\alpha_1 t}{\varepsilon} \|\widetilde{\nabla}_x g^\varepsilon\|_{\mathcal{X}} \|(g^\varepsilon)^\perp\|_{\mathcal{Y}_1} &\leq \frac{\alpha_2 t^2}{4} \|\widetilde{\nabla}_x g^\varepsilon\|_{\mathcal{X}}^2 + C \frac{\alpha_1^2}{\alpha_2 \varepsilon^2} \|(g^\varepsilon)^\perp\|_{\mathcal{Y}_1}^2 \\
 \frac{C\alpha_1 t}{\varepsilon} \eta_0 \|g^\varepsilon\|_{\mathcal{Y}_1} \|(g^\varepsilon)^\perp\|_{\mathcal{Y}_2} &\leq \frac{\kappa_1 \alpha_1 t}{4\varepsilon^2} \|(g^\varepsilon)^\perp\|_{\mathcal{Y}_2}^2 + C\alpha_1 t \eta_0^2 \|g^\varepsilon\|_{\mathcal{Y}_1}^2 \\
 \frac{C\alpha_2 t^2}{\varepsilon} \|\widetilde{\nabla}_x g^\varepsilon\|_{\mathcal{Y}_1} \|(g^\varepsilon)^\perp\|_{\mathcal{Y}_2} &\leq \frac{\kappa_2 \alpha_3}{12} t^3 \|\widetilde{\nabla}_x g^\varepsilon\|_{\mathcal{Y}_1}^2 + C \frac{\alpha_2^2}{\alpha_3 \varepsilon^2} t \|(g^\varepsilon)^\perp\|_{\mathcal{Y}_2}^2 \\
 C\eta_0 \alpha_2 t^2 \|g^\varepsilon\|_{\mathcal{Y}_1} \|\widetilde{\nabla}_x g^\varepsilon\|_{\mathcal{Y}_1} &\leq \frac{\kappa_2 \alpha_3}{12} t^3 \|\widetilde{\nabla}_x g^\varepsilon\|_{\mathcal{Y}_1}^2 + C \frac{\alpha_2^2}{\alpha_3} \eta_0^2 t \|g^\varepsilon\|_{\mathcal{Y}_1}^2 \\
 C\eta_0 \alpha_2 t^2 \|g^\varepsilon\|_{\mathcal{Y}_1} \|(g^\varepsilon)^\perp\|_{\mathcal{Y}_2} &\leq \frac{\kappa_1 \alpha_1 t}{4\varepsilon^2} \|(g^\varepsilon)^\perp\|_{\mathcal{Y}_2}^2 + C\varepsilon \eta_0^2 t^{3/2} \frac{\alpha_2^2}{\alpha_1} \|g^\varepsilon\|_{\mathcal{Y}_1}^2 \\
 C\eta_0 \alpha_2 t^2 \|\widetilde{\nabla}_x g^\varepsilon\|_{\mathcal{Y}_1} \|(g^\varepsilon)^\perp\|_{\mathcal{Y}_2} &\leq \frac{\kappa_2 \alpha_3 t^3}{12} \|\widetilde{\nabla}_x g^\varepsilon\|_{\mathcal{Y}_1}^2 + C\eta_0^2 t \frac{\alpha_2^2}{\alpha_3} \|(g^\varepsilon)^\perp\|_{\mathcal{Y}_2}^2 \\
 C\varepsilon \eta_0 \alpha_3 t^3 \|g^\varepsilon\|_{\mathcal{Y}_1} \|\widetilde{\nabla}_x g^\varepsilon\|_{\mathcal{Y}_1} &\leq \frac{\kappa_2 \alpha_3}{12} t^3 \|\widetilde{\nabla}_x g^\varepsilon\|_{\mathcal{Y}_1}^2 + C\varepsilon^2 \eta_0^2 \alpha_3 t^3 \|g^\varepsilon\|_{\mathcal{Y}_1}^2 \\
 C\varepsilon \eta_0 \alpha_3 t^3 \|\widetilde{\nabla}_x g^\varepsilon\|_{\mathcal{Y}_1} \|(g^\varepsilon)^\perp\|_{\mathcal{Y}_2} &\leq \frac{\kappa_2 \alpha_3 t^3}{12} \|\widetilde{\nabla}_x g^\varepsilon\|_{\mathcal{Y}_1}^2 + C\varepsilon^2 \eta_0^2 \alpha_3 t^3 \|(g^\varepsilon)^\perp\|_{\mathcal{Y}_2}^2 \\
 C\varepsilon \alpha_3 \eta_0 t^3 \|\widetilde{\nabla}_x g^\varepsilon\|_{\mathcal{X}} \|\widetilde{\nabla}_x g^\varepsilon\|_{\mathcal{Y}_1} &\leq C\alpha_3 \eta_0 t^3 \|\widetilde{\nabla}_x g^\varepsilon\|_{\mathcal{X}}^2 + \alpha_3 \eta_0 t^3 \|\widetilde{\nabla}_x g^\varepsilon\|_{\mathcal{Y}_1}^2 \\
 2\varepsilon \alpha_2 t \left| \left\langle \widetilde{\nabla}_x g^\varepsilon, \widetilde{\nabla}_v g^\varepsilon \right\rangle_{\mathcal{X}} \right| &\leq \frac{\alpha_2 t^2}{4} \|\widetilde{\nabla}_x g^\varepsilon\|_{\mathcal{X}}^2 + C\alpha_2 \varepsilon^2 \|g^\varepsilon\|_{\mathcal{Y}_1}^2,
 \end{aligned}$$

as well as

$$\begin{aligned}
 C\alpha_2 t^2 \|\widetilde{\nabla}_x g^\varepsilon\|_{\mathcal{X}} \|(g^\varepsilon)^\perp\|_{\mathcal{Y}_1} \|(g^\varepsilon)^\perp\|_{\mathcal{Y}_2} &\leq C\alpha_2 t^3 \|\widetilde{\nabla}_x g^\varepsilon\|_{\mathcal{X}}^2 \|(g^\varepsilon)^\perp\|_{\mathcal{Y}_1}^2 + C\alpha_2 t \|(g^\varepsilon)^\perp\|_{\mathcal{Y}_2}^2 \\
 C\varepsilon \alpha_3 t^3 \|\widetilde{\nabla}_x g^\varepsilon\|_{\mathcal{X}} \|(g^\varepsilon)^\perp\|_{\mathcal{Y}_1} \|\widetilde{\nabla}_x g^\varepsilon\|_{\mathcal{Y}_1} &\leq C\varepsilon^2 \alpha_3 t^3 \|\widetilde{\nabla}_x g^\varepsilon\|_{\mathcal{X}}^2 \|(g^\varepsilon)^\perp\|_{\mathcal{Y}_1}^2 + \frac{\kappa_2 \alpha_3}{12} t^3 \|\widetilde{\nabla}_x g^\varepsilon\|_{\mathcal{Y}_1}^2.
 \end{aligned}$$

We thus obtain

$$\begin{aligned}
 \frac{d}{dt} \mathcal{U}_\varepsilon(t, g^\varepsilon) \leq & - \left(\kappa - C\eta_0^2 - C\alpha_2 t^2 \eta_0 - C\alpha_2 \varepsilon^2 - C\alpha_1 t \eta_0^2 \right. \\
 & \left. - C \frac{\alpha_2^2}{\alpha_3} \eta_0^2 t - C\varepsilon \eta_0^2 t^{3/2} \frac{\alpha_2^2}{\alpha_1} - C\varepsilon^2 \eta_0^2 \alpha_3 t^3 \right) \|g^\varepsilon\|_{\mathcal{Y}_1}^2 \\
 & - \frac{1}{\varepsilon^2} \left(\kappa - C\alpha_1 t - C \frac{\alpha_1^2}{\alpha_2} \right) \|(g^\varepsilon)^\perp\|_{\mathcal{Y}_1}^2 \\
 & - \frac{t}{\varepsilon^2} \left(\frac{\kappa_1 \alpha_1}{2} - C\varepsilon^2 \alpha_2 t \eta_0 - C\varepsilon \alpha_1 \eta_0 - C \frac{\alpha_2^2}{\alpha_3} \right. \\
 & \left. - C\eta_0^2 \frac{\alpha_2^2}{\alpha_3} - C\varepsilon^4 \eta_0^2 \alpha_3 t^2 - C\varepsilon^2 \alpha_2 \right) \|(g^\varepsilon)^\perp\|_{\mathcal{Y}_2}^2 \\
 & - t^2 \left(\frac{\alpha_2}{2} - C\varepsilon^2 \alpha_3 - C\alpha_3 t - C\alpha_3 \eta_0 t \right) \|\widetilde{\nabla}_x g^\varepsilon\|_{\mathcal{X}}^2 \\
 & - t^3 \left(\frac{\kappa_2 \alpha_3}{2} - C\varepsilon \alpha_3 \eta_0 - C\alpha_3 \eta_0 \right) \|\widetilde{\nabla}_x g^\varepsilon\|_{\mathcal{Y}_1}^2 \\
 & + Ct^3 \left(\alpha_2 + \varepsilon^2 \alpha_3 \right) \|(g^\varepsilon)^\perp\|_{\mathcal{Y}_1}^2 \|\widetilde{\nabla}_x g^\varepsilon\|_{\mathcal{X}}^2.
 \end{aligned}$$

We now choose $\alpha_1 = \eta$, $\alpha_2 = \eta^{\frac{3}{2}}$, and $\alpha_3 = \eta^{\frac{5}{3}}$, with $\eta \in (0, 1)$ small enough as well as η_0 small enough, so that we deduce, for all $t \in [0, 1]$,

$$\begin{aligned}
 \frac{d}{dt} \mathcal{U}_\varepsilon(t, g^\varepsilon) \leq & - \frac{\kappa'}{\varepsilon^2} \|(g^\varepsilon)^\perp\|_{\mathcal{Y}_1}^2 - \kappa' \|g^\varepsilon\|_{\mathcal{Y}_1}^2 - \kappa' \frac{t}{\varepsilon^2} \|(g^\varepsilon)^\perp\|_{\mathcal{Y}_2}^2 - \kappa' t^2 \|\widetilde{\nabla}_x g^\varepsilon\|_{\mathcal{X}}^2 - \kappa' t^3 \|\widetilde{\nabla}_x g^\varepsilon\|_{\mathcal{Y}_1}^2 \\
 & + Ct^3 \|\widetilde{\nabla}_x g^\varepsilon\|_{\mathcal{X}}^2 \|(g^\varepsilon)^\perp\|_{\mathcal{Y}_1}^2,
 \end{aligned}$$

for some constants $\kappa', C > 0$. Integrating in time the last inequality, we hence obtain that for any $t \in [0, 1]$, there holds

$$\begin{aligned}
 \mathcal{U}_\varepsilon(t, g^\varepsilon) & + \frac{\kappa'}{\varepsilon^2} \int_0^t \|(g^\varepsilon)^\perp\|_{\mathcal{Y}_1}^2 ds + \kappa' \int_0^t \|g^\varepsilon\|_{\mathcal{Y}_1}^2 ds \\
 & + \frac{\kappa'}{\varepsilon^2} \int_0^t s \|(g^\varepsilon)^\perp\|_{\mathcal{Y}_2}^2 ds + \kappa' \int_0^t s^2 \|\widetilde{\nabla}_x g^\varepsilon\|_{\mathcal{X}}^2 ds + \kappa' \int_0^t s^3 \|\widetilde{\nabla}_x g^\varepsilon\|_{\mathcal{Y}_1}^2 ds \\
 & \leq \mathcal{U}_\varepsilon(0) + C \int_0^t s^3 \|\widetilde{\nabla}_x g^\varepsilon\|_{\mathcal{X}}^2 \|(g^\varepsilon)^\perp\|_{\mathcal{Y}_1}^2 ds \\
 & \leq \mathcal{U}_\varepsilon(0) + C\varepsilon^2 \left(\sup_{t \in [0, 1]} t^3 \|\widetilde{\nabla}_x g^\varepsilon\|_{\mathcal{X}}^2 \right) \frac{1}{\varepsilon^2} \int_0^t \|(g^\varepsilon)^\perp\|_{\mathcal{Y}_1}^2 ds \\
 & \leq \mathcal{U}_\varepsilon(0) + C\eta_0^2 \varepsilon^2 \left(\sup_{t \in [0, 1]} t^3 \|\widetilde{\nabla}_x g^\varepsilon\|_{\mathcal{X}}^2 \right)
 \end{aligned}$$

where we have used that $\frac{1}{\varepsilon^2} \int_0^t \|(g^\varepsilon)^\perp\|_{\mathcal{Y}_1}^2 ds \lesssim \eta_0^2$ from Theorem 4.1-(i) and we have denoted $\mathcal{U}_\varepsilon(0) = \mathcal{U}_\varepsilon(0, g_{\text{in}}^\varepsilon)$. Since

$$\left(\sup_{t \in [0, 1]} \|g^\varepsilon(t)\|_{\mathcal{X}}^2 \right) + \left(\sup_{t \in [0, 1]} t \|(g^\varepsilon(t))^\perp\|_{\mathcal{Y}_1}^2 \right) + \varepsilon^2 \left(\sup_{t \in [0, 1]} t^3 \|\widetilde{\nabla}_x g^\varepsilon(t)\|_{\mathcal{X}}^2 \right) \lesssim \sup_{t \in [0, 1]} \mathcal{U}_\varepsilon(t, g^\varepsilon)$$

if $\eta_0 > 0$ is small enough (independently of ε), we finally obtain

$$t \|g^\varepsilon(t)\|_{\mathcal{Y}_1}^2 \lesssim \mathcal{U}_\varepsilon(0) = \|g_{\text{in}}^\varepsilon\|_{\mathcal{X}}^2 \lesssim \|g_{\text{in}}^\varepsilon\|_{\mathcal{X}}^2 \quad \forall t \in [0, 1]$$

and

$$t^3 \|g^\varepsilon(t)\|_{\mathcal{Y}_1^\varepsilon}^2 \lesssim \mathcal{U}_\varepsilon(0) = \|g_{\text{in}}^\varepsilon\|_{\mathcal{X}}^2 \lesssim \|g_{\text{in}}^\varepsilon\|_{\mathcal{X}}^2 \quad \forall t \in [0, 1],$$

which gives estimates (4.113) and concludes the proof. \square

5 Refined semigroup estimates on the linearized operator

In this part, we go back to the linearized problem and give some new and refined estimates on it. The first and second subsections are dedicated to the introduction of a nice splitting of the linearized operator Λ_ε coming from [23, 26] and to the proof of dissipativity, regularization and boundedness estimates. Roughly speaking, the linearized operator Λ_ε splits into two parts: $\Lambda_\varepsilon = \mathcal{A}_\varepsilon + \mathcal{B}_\varepsilon$, the first part \mathcal{A}_ε having some good regularizing properties, the second one \mathcal{B}_ε having some nice dissipativity (and also regularizing) properties. It is worth mentioning that the regularization estimates on \mathcal{B}_ε that we develop are sharp (it was not the case in [26] where the authors did not intend to obtain optimal regularization estimates on \mathcal{B}_ε) and only the case $\varepsilon = 1$ was treated in [26]. As already mentioned, from those properties and Duhamel formula, we can give a new proof of Theorem 4.15. More importantly, Duhamel formula applied with this splitting will be used to prove our hydrodynamical limit theorem in Section 6.

In the last three subsections, we study the semigroup $U^\varepsilon(t)$ associated with Λ_ε from another point of view which is based on a careful spectral analysis carried out in Fourier in x of Λ_1 in [130], it in particular allows us to give a decomposition of the semigroup $U^\varepsilon(t)$, study its limit as ε goes to 0 and give another type of estimates on it.

5.1 Splitting of the operator

We now introduce a splitting of the full linearized operator Λ_ε . Let $\chi \in \mathcal{C}_c^\infty(\mathbb{R})$ be a smooth cutoff function such that $0 \leq \chi \leq 1$, $\chi \equiv 1$ on $[-1/2, 1/2]$ and $\chi \equiv 0$ on $\mathbb{R} \setminus [-1, 1]$, consider positive constants $R, \bar{R} > 0$ and define $\chi_{\bar{R}}(v) = \chi(|v|/\bar{R})$ as well as

$$m^2(v) := \frac{1}{4} |\mathbf{B}(v)v|^2 - \frac{1}{2} \nabla_v \cdot [\mathbf{B}^\top(v)\mathbf{B}(v)v] + R\chi_{\bar{R}}(v) \quad (4.120)$$

where $\mathbf{B}(v)$ is defined in (4.38). Recalling the formulation of L in (4.41), we then decompose $\Lambda_\varepsilon = \frac{1}{\varepsilon^2}L - \frac{1}{\varepsilon}v \cdot \nabla_x$ as

$$\Lambda_\varepsilon = \mathcal{A}_\varepsilon + \mathcal{B}_\varepsilon \quad \text{with} \quad \mathcal{A}_\varepsilon := \frac{1}{\varepsilon^2}\mathcal{A} \quad \text{and} \quad \mathcal{B}_\varepsilon := \frac{1}{\varepsilon^2}\mathcal{B} - \frac{1}{\varepsilon}v \cdot \nabla_x \quad (4.121)$$

where

$$\begin{aligned} \mathcal{A}f &:= -\left\{ (a_{ij} * \sqrt{M}f) v_i v_j - (a_{ii} * \sqrt{M}f) + (c * \sqrt{M}f) \right\} \sqrt{M} + R\chi_{\bar{R}}f \\ &= (L_2 + R\chi_{\bar{R}})f \end{aligned} \quad (4.122)$$

and

$$\mathcal{B}f := -\widetilde{\nabla}_v^* \widetilde{\nabla}_v f - m^2(v)f = (L_1 - R\chi_{\bar{R}})f. \quad (4.123)$$

Let us now give a lemma providing estimates on m and its derivatives. We fix

$$\sigma_1 := \begin{cases} +\infty & \text{if } -2 < \gamma \leq 1 \\ \frac{1}{2} & \text{if } \gamma = -2. \end{cases} \quad (4.124)$$

Lemma 4.25. *Let $0 < \sigma < \sigma_1$.*

(i) *There are R_0 and \bar{R}_0 large enough so that for any $R \geq R_0$ and $\bar{R} \geq \bar{R}_0$, one has*

$$m^2(v) \geq \sigma + \kappa \langle v \rangle^{\gamma+2}, \quad \forall v \in \mathbb{R}^3,$$

for some $0 < \kappa < \sigma_1 - \sigma$.

(ii) *For any $\alpha \in \mathbb{R}$, we define*

$$m_\alpha^2(v) := m^2(v) - \langle v \rangle^{-2\alpha} |\widetilde{\nabla}_v \langle v \rangle^\alpha|^2.$$

There are R_0 and \bar{R}_0 large enough so that for any $R \geq R_0$ and $\bar{R} \geq \bar{R}_0$, one has

$$m_\alpha^2(v) \geq \sigma + \kappa \langle v \rangle^{\gamma+2}, \quad \forall v \in \mathbb{R}^3,$$

for some $0 < \kappa < \sigma_1 - \sigma$.

(iii) *For any multi-index $\alpha \in \mathbb{N}^3$ we have, for all $v \in \mathbb{R}^3$,*

$$|\partial_v^\alpha m(v)| \lesssim \langle v \rangle^{\frac{7}{2}+1-|\alpha|}.$$

Hereafter in the paper, for any $0 < \sigma < \sigma_1$ we then fix $R, \bar{R} > 0$ large enough so that the results of Lemma 4.25 are in force.

Proof of Lemma 4.25. (i) See [26, Lemma 2.6].

(ii) We have

$$m_\alpha^2(v) = \frac{1}{4} |\mathbf{B}(v)v|^2 - \frac{1}{2} \nabla_v \cdot [\mathbf{B}^\top(v)\mathbf{B}(v)v] - \alpha^2 |\mathbf{B}(v)v|^2 \langle v \rangle^{-4} + R\chi_{\bar{R}}(v)$$

therefore

$$m_\alpha^2(v) = \frac{1}{4}\ell_1(v)|v|^2 - \frac{1}{2}\nabla_v \cdot [\ell_1(v)v] - \alpha^2\ell_1(v)|v|^2\langle v \rangle^{-4} + R\chi_{\bar{R}}(v).$$

Now observe that

$$\begin{aligned} \frac{1}{4}\ell_1(v)|v|^2 - \frac{1}{2}\nabla_v \cdot [\ell_1(v)v] - \alpha^2\ell_1(v)|v|^2\langle v \rangle^{-4} \\ = \frac{1}{4}\ell_1(v)|v|^2 - \frac{1}{2}\nabla_v \ell_1(v) \cdot v - \frac{3}{2}\ell_1(v) - \alpha^2\ell_1(v)|v|^2\langle v \rangle^{-4}, \end{aligned}$$

and from (4.37), $\ell_1(v)|v|^2\langle v \rangle^{-4} \lesssim \langle v \rangle^{\gamma-2}$, hence if $-2 < \gamma \leq 1$ we have

$$\frac{1}{4}\ell_1(v)|v|^2 - \frac{1}{2}\nabla_v \cdot [\ell_1(v)v] - \alpha^2\ell_1(v)|v|^2\langle v \rangle^{-4} \xrightarrow{|v| \rightarrow \infty} +\infty,$$

and if $\gamma = -2$, one has

$$\frac{1}{4}\ell_1(v)|v|^2 - \frac{1}{2}\nabla_v \cdot [\ell_1(v)v] - \alpha^2\ell_1(v)|v|^2\langle v \rangle^{-4} \xrightarrow{|v| \rightarrow \infty} \frac{1}{2} > 0.$$

We then conclude the proof by taking $R_0, \bar{R}_0 > 0$ large enough.

(iii) Direct consequence of (4.42). □

Arguing as in Lemma 4.11 and recalling that $[\widetilde{\nabla}_{v_i}, v \cdot \nabla_x] = \widetilde{\nabla}_{x_i}$ and $[\widetilde{\nabla}_{x_i}, v \cdot \nabla_x] = 0$, we also obtain

Lemma 4.26. *There holds*

(i) *For any $1 \leq i \leq 3$, one has*

$$\begin{aligned} [\widetilde{\nabla}_{v_i}, \mathcal{B}_\varepsilon]f &= -\frac{1}{\varepsilon^2}\widetilde{\nabla}_{v_j}^*[\widetilde{\nabla}_{v_i}, \widetilde{\nabla}_{v_j}]f - \frac{1}{\varepsilon^2}\widetilde{\nabla}_{v_j}[\widetilde{\nabla}_{v_i}, \widetilde{\nabla}_{v_j}^*]f \\ &\quad - \frac{1}{\varepsilon^2}(\widetilde{\nabla}_{v_i} m^2)f - \frac{1}{\varepsilon}\widetilde{\nabla}_{x_i}f - \frac{1}{\varepsilon^2}[[\widetilde{\nabla}_{v_i}, \widetilde{\nabla}_{v_j}^*], \widetilde{\nabla}_{v_j}]f. \end{aligned}$$

(ii) *For any $1 \leq i \leq 3$, one has*

$$[\widetilde{\nabla}_{x_i}, \mathcal{B}_\varepsilon]f = -\frac{1}{\varepsilon^2}\widetilde{\nabla}_{v_j}^*[\widetilde{\nabla}_{x_i}, \widetilde{\nabla}_{v_j}]f - \frac{1}{\varepsilon^2}\widetilde{\nabla}_{v_j}[\widetilde{\nabla}_{x_i}, \widetilde{\nabla}_{v_j}^*]f - \frac{1}{\varepsilon^2}[[\widetilde{\nabla}_{x_i}, \widetilde{\nabla}_{v_j}^*], \widetilde{\nabla}_{v_j}]f.$$

Let us recall that the spaces \mathcal{X} , \mathcal{Y}_i , $(\mathcal{Y}_i)'$, $\mathcal{X}_i^\varepsilon$ and $(\mathcal{X}_i^\varepsilon)'$ are respectively defined in (4.19), (4.20), (4.44), (4.47), (4.45), (4.46) and (4.48).

We also have the following bounds on the operator \mathcal{A} :

Lemma 4.27. *For any $\alpha \in \mathbb{R}$ one has*

$$\begin{aligned} \|\langle v \rangle^\alpha \mathcal{A}f\|_{L_{x,v}^2} &\lesssim \|M^{\frac{1}{4}}f\|_{L_{x,v}^2} \\ \|\langle v \rangle^\alpha \widetilde{\nabla}_v \mathcal{A}f\|_{L_{x,v}^2} &\lesssim \|M^{\frac{1}{4}}f\|_{L_{x,v}^2} + \|M^{\frac{1}{4}}\widetilde{\nabla}_v f\|_{L_{x,v}^2} \\ \|\langle v \rangle^\alpha \widetilde{\nabla}_v \widetilde{\nabla}_v \mathcal{A}f\|_{L_{x,v}^2} &\lesssim \|M^{\frac{1}{4}}f\|_{L_{x,v}^2} + \|M^{\frac{1}{4}}\widetilde{\nabla}_v f\|_{L_{x,v}^2} + \|M^{\frac{1}{4}}\widetilde{\nabla}_v \widetilde{\nabla}_v f\|_{L_{x,v}^2} \\ \|\langle v \rangle^\alpha \widetilde{\nabla}_x \mathcal{A}f\|_{L_{x,v}^2} &\lesssim \|M^{\frac{1}{4}}\widetilde{\nabla}_x f\|_{L_{x,v}^2} \\ \|\langle v \rangle^\alpha \widetilde{\nabla}_x \widetilde{\nabla}_x \mathcal{A}f\|_{L_{x,v}^2} &\lesssim \|M^{\frac{1}{4}}\widetilde{\nabla}_x \widetilde{\nabla}_x f\|_{L_{x,v}^2}. \end{aligned}$$

In particular, one has $\mathcal{A} \in \mathcal{B}(\mathcal{X})$, $\mathcal{A} \in \mathcal{B}(\mathcal{Y}_i)$ and $\mathcal{A} \in \mathcal{B}(\mathcal{Z}_i^\varepsilon)$ for $i = 1, 2$ and since \mathcal{A} is self-adjoint in $L_{x,v}^2$, we also have $\mathcal{A} \in \mathcal{B}(\mathcal{Y}_i')$ and $\mathcal{A} \in \mathcal{B}((\mathcal{Z}_i^\varepsilon)')$ for $i = 1, 2$.

Proof. The terms coming from $R\chi_{\bar{R}}$ are easily treated because $\chi_{\bar{R}}$ has compact support. The first, second and fourth estimates thus directly come from Lemma 4.14. The proofs of the other estimates are completely similar and rely on Lemmas 4.12 and 4.13. \square

5.2 Decay and regularization estimates for $S_{\mathcal{B}_\varepsilon}$

In this section we provide several results on the dissipation and regularization properties of the operator \mathcal{B}_ε . We start with the dissipative ones.

Lemma 4.28. *Let $\sigma \in (0, \sigma_1)$ (where σ_1 is defined in (4.124)). Then for any $\alpha \in \mathbb{R}$, one has*

$$\langle \langle v \rangle^\alpha f, \langle v \rangle^\alpha \mathcal{B}_\varepsilon f \rangle_{L_{x,v}^2} \leq -\frac{\sigma}{\varepsilon^2} \|\langle v \rangle^\alpha f\|_{L_{x,v}^2}^2 - \frac{\kappa}{\varepsilon^2} \|\langle v \rangle^\alpha f\|_{L_x^2(H_{v,*}^1)}^2, \quad (4.125)$$

for some constant $\kappa > 0$. As a consequence, one has for all $t \geq 0$,

$$\|S_{\mathcal{B}_\varepsilon}(t)\|_{\mathcal{X} \rightarrow \mathcal{X}} \leq e^{-\sigma t/\varepsilon^2}.$$

Proof. We compute:

$$\begin{aligned} \langle \langle v \rangle^{2\alpha} f, \mathcal{B}_\varepsilon f \rangle_{L_{x,v}^2} &= \left\langle \langle v \rangle^{2\alpha} f, \left\{ -\frac{1}{\varepsilon^2} \widetilde{\nabla}_v^* \widetilde{\nabla}_v f - \frac{1}{\varepsilon^2} m^2 f - \frac{1}{\varepsilon} v \cdot \nabla_x f \right\} \right\rangle_{L_{x,v}^2} \\ &= -\frac{1}{\varepsilon^2} \|\langle v \rangle^\alpha m f\|_{L_{x,v}^2}^2 - \frac{1}{\varepsilon^2} \langle \widetilde{\nabla}_v (\langle v \rangle^{2\alpha} f), \widetilde{\nabla}_v f \rangle_{L_{x,v}^2}. \end{aligned}$$

Observing that

$$\begin{aligned} \langle \widetilde{\nabla}_v (\langle v \rangle^{2\alpha} f), \widetilde{\nabla}_v f \rangle_{L_{x,v}^2} &= \langle \widetilde{\nabla}_v (\langle v \rangle^\alpha f), \langle v \rangle^\alpha \widetilde{\nabla}_v f \rangle_{L_{x,v}^2} + \langle (\widetilde{\nabla}_v \langle v \rangle^\alpha) f, \langle v \rangle^\alpha \widetilde{\nabla}_v f \rangle_{L_{x,v}^2} \\ &= \|\widetilde{\nabla}_v (\langle v \rangle^\alpha f)\|_{L_{x,v}^2}^2 - \langle \widetilde{\nabla}_v (\langle v \rangle^\alpha f), (\widetilde{\nabla}_v \langle v \rangle^\alpha) f \rangle_{L_{x,v}^2} + \langle (\widetilde{\nabla}_v \langle v \rangle^\alpha) f, \langle v \rangle^\alpha \widetilde{\nabla}_v f \rangle_{L_{x,v}^2} \\ &= \|\widetilde{\nabla}_v (\langle v \rangle^\alpha f)\|_{L_{x,v}^2}^2 - \|(\widetilde{\nabla}_v \langle v \rangle^\alpha) f\|_{L_{x,v}^2}^2 \end{aligned}$$

and recalling that m_α is defined in Lemma 4.25 so that $\langle v \rangle^{2\alpha} m_\alpha^2 = \langle v \rangle^{2\alpha} m^2 - |\widetilde{\nabla}_v \langle v \rangle^\alpha|^2$, we obtain the estimate

$$\langle \langle v \rangle^{2\alpha} f, \mathcal{B}_\varepsilon f \rangle_{L^2_{x,v}} = -\frac{1}{\varepsilon^2} \|\langle v \rangle^\alpha m_\alpha f\|_{L^2_{x,v}}^2 - \frac{1}{\varepsilon^2} \|\widetilde{\nabla}_v (\langle v \rangle^\alpha f)\|_{L^2_{x,v}}^2.$$

We then conclude the proof of (4.125) by using the bound by below on m_α of Lemma 4.25.

The exponential decay estimate of $S_{\mathcal{B}_\varepsilon}$ on \mathcal{X} is then a consequence of (4.125) together with the fact that ∇_x commutes with \mathcal{B}_ε . \square

In what follows, we prove regularization results for the semigroup $S_{\mathcal{B}_\varepsilon}$.

Lemma 4.29. *Let $\sigma \in (0, \sigma_1)$ (where σ_1 is defined in (4.124)). For any $t > 0$, one has*

$$\|S_{\mathcal{B}_\varepsilon}(t)\|_{\mathcal{X} \rightarrow \mathcal{X}_1} \lesssim \frac{\varepsilon}{\sqrt{t}} e^{-\sigma t/\varepsilon^2} \quad \text{and} \quad \|S_{\mathcal{B}_\varepsilon}(t)\|_{\mathcal{X} \rightarrow \mathcal{X}_1^\varepsilon} \lesssim \frac{\varepsilon^3}{t^{3/2}} e^{-\sigma t/\varepsilon^2}. \quad (4.126)$$

as well as the dual estimates

$$\|S_{\mathcal{B}_\varepsilon}(t)\|_{\mathcal{X}'_1 \rightarrow \mathcal{X}} \lesssim \frac{\varepsilon}{\sqrt{t}} e^{-\sigma t/\varepsilon^2} \quad \text{and} \quad \|S_{\mathcal{B}_\varepsilon}(t)\|_{(\mathcal{X}'_1)^\varepsilon \rightarrow \mathcal{X}} \lesssim \frac{\varepsilon^3}{t^{3/2}} e^{-\sigma t/\varepsilon^2}. \quad (4.127)$$

The proof follows similar ideas as the proof of Proposition 4.18 but is somewhat simpler because, the operator \mathcal{B}_ε provides $1/\varepsilon^2$ -dissipativity and regularity on both macroscopic and microscopic parts of the solution whereas the operator Λ_ε only provided a gain of $1/\varepsilon^2$ on the microscopic part. We thus do not need to separate microscopic and macroscopic parts when defining our functional that will be a Lyapunov functional for our equation $\partial_t f = \mathcal{B}_\varepsilon f$ (see (4.169)). For sake of completeness, the proof is given in Appendix (Section 8).

Using the same method, we can push our previous result up to the next notch of regularity (we only mention the dual results because they are the only ones that will be used in the sequel) and the proof is also postponed to Appendix (Section 8):

Lemma 4.30. *Let $\sigma \in (0, \sigma_1)$ (where σ_1 is defined in (4.124)). For any $t > 0$, one has*

$$\|S_{\mathcal{B}_\varepsilon}(t)\|_{\mathcal{X}'_2 \rightarrow \mathcal{X}} \lesssim \frac{\varepsilon^2}{t} e^{-\sigma t/\varepsilon^2} \quad \text{and} \quad \|S_{\mathcal{B}_\varepsilon}(t)\|_{(\mathcal{X}'_2)^\varepsilon \rightarrow \mathcal{X}} \lesssim \frac{\varepsilon^6}{t^3} e^{-\sigma t/\varepsilon^2}.$$

We also have the following dissipativity properties, the proof of which relies on the same line of proof as the above regularization results. Indeed, the idea is to use the same functionals without the weights in time. We thus skip the proof since the computations are the same. As previously, the result is only given in the dual framework which will be the only one useful in the sequel.

Lemma 4.31. *Let $\sigma \in (0, \sigma_1)$ (where σ_1 is defined in (4.124)). For any $t \geq 0$ and $i = 1, 2$, one has*

$$\|S_{\mathcal{B}_\varepsilon}(t)\|_{(\mathcal{X}_i^\varepsilon)' \rightarrow (\mathcal{X}_i^\varepsilon)'} \lesssim e^{-\sigma t/\varepsilon^2}.$$

In our forthcoming analysis, we will use an iterated Duhamel formula based on the splitting $\Lambda_\varepsilon = \mathcal{A}_\varepsilon + \mathcal{B}_\varepsilon$. We introduce the following definition of convolution of semigroups: If S_1 and S_2 are two semigroups, their convolution product is defined by

$$S_1 * S_2(t) := \int_0^t S_1(s)S_2(t-s) ds.$$

We also introduce the semigroups $V_j^\varepsilon(t)$ defined through:

$$V_0^\varepsilon(t) := S_{\mathcal{B}_\varepsilon}(t), \quad V_{j+1}^\varepsilon(t) := (S_{\mathcal{B}_\varepsilon} * \mathcal{A}_\varepsilon V_j^\varepsilon)(t) = (V_j^\varepsilon * \mathcal{A}_\varepsilon S_{\mathcal{B}_\varepsilon})(t), \quad j \in \mathbb{N}$$

so that for any $n \in \mathbb{N}$ and any $t \geq 0$, we have:

$$U^\varepsilon(t) = \sum_{j=0}^n V_j^\varepsilon(t) + (V_n^\varepsilon * \mathcal{A}_\varepsilon U^\varepsilon)(t). \quad (4.128)$$

As a consequence of the previous results on \mathcal{A}_ε and \mathcal{B}_ε , we obtain:

Corollary 4.32. *Let $\sigma \in (0, \sigma_1)$ (where σ_1 is defined in (4.124)). For any $t > 0$, any $j \in \mathbb{N}$ and $i = 1, 2$, we have:*

$$\|V_j^\varepsilon(t)\|_{\mathcal{X} \rightarrow \mathcal{X}} \lesssim e^{-\sigma t/\varepsilon^2} \quad \text{and} \quad \|V_j^\varepsilon(t)\|_{(\mathcal{X}_i^\varepsilon)' \rightarrow (\mathcal{X}_i^\varepsilon)'} \lesssim e^{-\sigma t/\varepsilon^2}.$$

Proof. Fix $\sigma \in (0, \sigma_1)$. The first estimate for $j = 0$ is given by Lemma 4.28. We then argue by induction and thus suppose that the property is satisfied for some $j \in \mathbb{N}$. Then, we consider $\sigma' \in (\sigma, \sigma_1)$. From Lemmas 4.27 and 4.28 and the induction hypothesis, we have:

$$\begin{aligned} \|V_{j+1}^\varepsilon(t)\|_{\mathcal{X} \rightarrow \mathcal{X}} &\lesssim \frac{1}{\varepsilon^2} \int_0^t \|S_{\mathcal{B}_\varepsilon}(s)\|_{\mathcal{X} \rightarrow \mathcal{X}} \|\mathcal{A}\|_{\mathcal{X} \rightarrow \mathcal{X}} \|V_j^\varepsilon(t-s)\|_{\mathcal{X} \rightarrow \mathcal{X}} ds \\ &\lesssim \frac{1}{\varepsilon^2} \int_0^t e^{-\sigma' s/\varepsilon^2} e^{-\sigma(t-s)/\varepsilon^2} ds \lesssim \frac{1}{\varepsilon^2} e^{-\sigma t/\varepsilon^2} \int_0^t e^{-(\sigma'-\sigma)s/\varepsilon^2} ds \lesssim e^{-\sigma t/\varepsilon^2}. \end{aligned}$$

The second estimate can be proven in a similar way by using Lemmas 4.27 and 4.31. \square

Corollary 4.33. *Let $\sigma \in (0, \sigma_1)$ (where σ_1 is defined in (4.124)). For any $t > 0$ and any $j \in \mathbb{N}$, we have:*

$$\|V_j^\varepsilon(t)\|_{(\mathcal{X}_1^\varepsilon)' \rightarrow \mathcal{X}} \lesssim \frac{\varepsilon^{3-2j}}{t^{\frac{3-2j}{2}}} e^{-\sigma t/\varepsilon^2} \quad \text{and} \quad \|V_j^\varepsilon(t)\|_{(\mathcal{X}_2^\varepsilon)' \rightarrow \mathcal{X}} \lesssim \frac{\varepsilon^{6-2j}}{t^{3-j}} e^{-\sigma t/\varepsilon^2}.$$

Proof. Fix $\sigma \in (0, \sigma_1)$. We focus on the proof of the first estimate, the second one is treated in a similar way. We proceed by induction. The case $j = 0$ is given by Lemma 4.29. Suppose then that the estimate holds for some $j \in \mathbb{N}$ and consider $\sigma' \in (\sigma, \sigma_1)$. From Lemmas 4.27-4.28, Corollary 4.32 and the induction hypothesis, we have:

$$\begin{aligned} \|V_{j+1}^\varepsilon(t)\|_{(\mathcal{X}_1^\varepsilon)' \rightarrow \mathcal{X}} &\lesssim \int_0^{t/2} \|S_{\mathcal{B}_\varepsilon}(s)\|_{\mathcal{X} \rightarrow \mathcal{X}} \|\mathcal{A}_\varepsilon\|_{\mathcal{X} \rightarrow \mathcal{X}} \|V_j^\varepsilon(t-s)\|_{(\mathcal{X}_1^\varepsilon)' \rightarrow \mathcal{X}} ds \\ &\quad + \int_{t/2}^t \|V_j^\varepsilon(s)\|_{(\mathcal{X}_1^\varepsilon)' \rightarrow \mathcal{X}} \|\mathcal{A}_\varepsilon\|_{(\mathcal{X}_1^\varepsilon)' \rightarrow (\mathcal{X}_1^\varepsilon)'} \|S_{\mathcal{B}_\varepsilon}(t-s)\|_{(\mathcal{X}_1^\varepsilon)' \rightarrow (\mathcal{X}_1^\varepsilon)'} ds \\ &\lesssim \frac{1}{\varepsilon^2} \int_0^{t/2} e^{-\sigma' s/\varepsilon^2} \frac{\varepsilon^{3-2j}}{(t-s)^{\frac{3-2j}{2}}} e^{-\sigma(t-s)/\varepsilon^2} ds + \frac{1}{\varepsilon^2} \int_{t/2}^t \frac{\varepsilon^{3-2j}}{s^{\frac{3-2j}{2}}} e^{-\sigma' s/\varepsilon^2} e^{-\sigma(t-s)/\varepsilon^2} ds. \end{aligned}$$

From this, we deduce that

$$\|V_{j+1}^\varepsilon(t)\|_{(\mathcal{X}_1^\varepsilon)' \rightarrow \mathcal{X}} \lesssim e^{-\sigma t/\varepsilon^2} \varepsilon^{1-2j} \left(\int_0^{t/2} \frac{ds}{(t-s)^{\frac{3-2j}{2}}} + \int_{t/2}^t \frac{ds}{s^{\frac{3-2j}{2}}} \right) \lesssim \frac{\varepsilon^{1-2j}}{t^{\frac{1-2j}{2}}} e^{-\sigma t/\varepsilon^2},$$

which yields the wanted result. \square

5.3 Spectral study in Fourier space

We denote by \mathcal{F}_x the Fourier transform in $x \in \mathbb{T}^3$ with $\xi \in \mathbb{Z}^3$ its dual variable. Since we will only be working with Fourier transform in x , we will also interchangeably use the classical “hat” notation. Moreover, to lighten the reading, for any operator that acts only on velocity, with a little abuse of notation, we will omit the “hat” in the notation for its x -wise Fourier transform.

In this part, we are going to look at the Fourier transform in $x \in \mathbb{T}^3$ of the operator Λ_1 :

$$\widehat{\Lambda}_1(\xi) := -i \xi \cdot v + L$$

and study the spectrum of $\widehat{\Lambda}_1(\xi)$ for $\xi \in \mathbb{Z}^3$. This type of analysis was initiated in [107, 27, 41] for the Boltzmann equation for hard spheres and then with hard cutoff potentials (see also [123]). In [130], Yang and Yu were then able to adapt it to more general kinetic equations including the linearized Landau one for hard and moderately soft potentials.

Roughly speaking, for small frequencies, the spectrum of $\widehat{\Lambda}_1(\xi)$ is a perturbation of the one of the homogeneous collision operator L (which acts only on velocity). As already mentioned, in the case of hard and moderately soft potentials ($\gamma \geq -2$), the operator L has a spectral gap and in [130], the authors then prove that for small frequencies ξ , the spectrum of $\widehat{\Lambda}_1(\xi)$ is made of “small” eigenvalues around 0 in the right part of the plan. They also provide Taylor expansions of those eigenvalues as well as for their associated projectors. For large frequencies, they prove that the operator $\widehat{\Lambda}_1(\xi)$ has no spectrum in some suitable right part of the plan. All those spectral results provide a decomposition of the semigroup which is given in Lemma 4.34.

In what follows, we write

$$\widehat{U}^\varepsilon(t) = \mathcal{F}_x U^\varepsilon(t) \mathcal{F}_x^{-1}$$

so that \widehat{U}^ε is the semigroup associated with the operator

$$\widehat{\Lambda}_\varepsilon(\xi) := \frac{1}{\varepsilon^2}(-i\varepsilon \xi \cdot v + L).$$

We also introduce the bilinear operator $\Psi^\varepsilon(t)$ defined by

$$\Psi^\varepsilon(t)(f_1, f_2) := \frac{1}{\varepsilon} \int_0^t U^\varepsilon(t-s) \Gamma(f_1(s), f_2(s)) \, ds. \quad (4.129)$$

We recall that χ is a fixed, compactly supported function of the interval $(-1, 1)$, equal to one on $[-1/2, 1/2]$.

Lemma 4.34. *There exists $\kappa > 0$ such that one can write*

$$U^\varepsilon(t) = \sum_{j=1}^4 U_j^\varepsilon(t) + U^{\varepsilon\#}(t)$$

$$\text{with } \widehat{U}_j^\varepsilon(t, \xi) := \widehat{U}_j\left(\frac{t}{\varepsilon^2}, \varepsilon\xi\right) \quad \text{and} \quad \widehat{U}^{\varepsilon\#}(t, \xi) := \widehat{U}^\# \left(\frac{t}{\varepsilon^2}, \varepsilon\xi\right),$$

where for $1 \leq j \leq 4$,

$$\widehat{U}_j(t, \xi) := \chi\left(\frac{|\xi|}{\kappa}\right) e^{t\lambda_j(\xi)} P_j(\xi)$$

with λ_j satisfying

$$\begin{aligned} \lambda_j(\xi) &= i\alpha_j|\xi| - \beta_j|\xi|^2 + \gamma_j(|\xi|), \\ \alpha_1 > 0, \quad \alpha_2 > 0, \quad \alpha_3 = \alpha_4 = 0, \quad \beta_j > 0, \\ \gamma_j(|\xi|) &=_{|\xi| \rightarrow 0} O(|\xi|^3) \quad \text{and} \quad \gamma_j(|\xi|) \leq \beta_j|\xi|^2/2 \quad \text{for } |\xi| \leq \kappa, \end{aligned} \quad (4.130)$$

and

$$P_j(\xi) = P_j^0\left(\frac{\xi}{|\xi|}\right) + |\xi| P_j^1\left(\frac{\xi}{|\xi|}\right) + |\xi|^2 P_j^2(\xi),$$

with P_j^n bounded linear operators on L_v^2 with operator norms uniform for $|\xi| \leq \kappa$.

We also have that the orthogonal projector π onto $\text{Ker } L$ (see (4.12)) satisfies

$$\pi = \sum_{j=1}^4 P_j^0\left(\frac{\xi}{|\xi|}\right)$$

and is independent of $\xi/|\xi|$.

Moreover, $P_j^0(\xi/|\xi|)$, $P_j^1(\xi/|\xi|)$ and $P_j^2(\xi)$ are bounded from L_v^2 into $L_v^2(\langle v \rangle^\ell)$ uniformly in $|\xi| \leq \kappa$ for any $\ell \geq 0$.

Finally, for any $\ell \geq 0$, $\widehat{U}^\#$ satisfies

$$\|\widehat{U}^\#\|_{L_v^2(\langle v \rangle^\ell) \rightarrow L_v^2(\langle v \rangle^\ell)} \leq C e^{-\alpha t} \quad (4.131)$$

for some positive constants C and α independent of t and ξ .

Proof. The decomposition of $\widehat{U}^\varepsilon(t)$ follows that of $\widehat{U}^1(t)$: We recall that according to [130, Theorem 3.2 and Remark 5.2], one can write

$$\widehat{U}^1(t, \xi) = \sum_{j=1}^4 \widehat{U}_j(t, \xi) + \widehat{U}^\sharp(t, \xi),$$

where for $1 \leq j \leq 4$,

$$\widehat{U}_j(t, \xi) := \chi \left(\frac{|\xi|}{\kappa} \right) e^{t\lambda_j(\xi)} P_j(\xi)$$

and $\lambda_j(\xi) \in \mathbb{C}$ are the eigenvalues of $\widehat{\Lambda}_1(\xi)$ with associated eigenprojections $P_j(\xi)$ on L_v^2 , satisfying the expansions stated in the lemma. The fact that $\pi = \sum_{j=1}^4 P_j^0 \left(\frac{\xi}{|\xi|} \right)$ also comes from [130, Theorem 3.2].

Let us now prove that $P_j^0(\xi/|\xi|)$, $P_j^1(\xi/|\xi|)$ and $P_j^2(\xi)$ are bounded from L_v^2 into $L_v^2(\langle v \rangle^\ell)$ uniformly in $|\xi| \leq \kappa$ for any $\ell \geq 0$. We first prove that this property is satisfied for $P_j(\xi)$. Recall that for $|\xi| \leq \kappa$,

$$\widehat{\Lambda}_1(\xi) P_j(\xi) = \lambda_j(\xi) P_j(\xi).$$

Thanks to the splitting $\Lambda_1 = \mathcal{A}_1 + \mathcal{B}_1$ introduced in Section 5, denoting $\widehat{\mathcal{B}}_1(\xi) := -i\xi \cdot v + \mathcal{B}_1$, we have for $|\xi| \leq \kappa$:

$$P_j(\xi) = (\lambda_j(\xi) - \widehat{\mathcal{B}}_1(\xi))^{-1} \mathcal{A}_1 P_j(\xi).$$

The dissipative properties of \mathcal{B}_1 in $L_v^2(\langle v \rangle^\ell)$ and the regularization properties of \mathcal{A}_1 (from L_v^2 into $L_v^2(\langle v \rangle^\ell)$) established respectively in Lemmas 4.28 and 4.27 and the fact that from [130, Theorem 3.2], we already know that $P_j(\xi)$ is uniformly bounded in $|\xi| \leq \kappa$ from L_v^2 into itself, imply that $P_j(\xi)$ is bounded from L_v^2 into $L_v^2(\langle v \rangle^\ell)$ for any $\ell \geq 0$ and uniformly in $|\xi| \leq \kappa$. To conclude that the same properties hold for $P_j^0(\xi/|\xi|)$, $P_j^1(\xi/|\xi|)$ and $P_j^2(\xi)$, we notice that $P_j^0(\xi/|\xi|)$, $P_j^1(\xi/|\xi|)$ are given by explicit formula (see the proof of Theorem 3.2 in [130]) that clearly define bounded operators from L_v^2 into $L_v^2(\langle v \rangle^\ell)$ for $|\xi| \leq \kappa$.

Finally, the estimate (4.131) comes from [130, Remark 5.2] for $\ell = 0$. We can also prove it for any $\ell \geq 0$ thanks to Duhamel formula applied with the splitting $\Lambda_1 = \mathcal{A}_1 + \mathcal{B}_1$ introduced in Section 5. We write that

$$\begin{aligned} U^\sharp(t) &= U^1(t) \mathcal{F}_x^{-1} \left(\text{Id} - \chi \left(\frac{|\xi|}{\kappa} \right) e^{t\lambda_j(\xi)} P_j(\xi) \right) \mathcal{F}_x \\ &= (S_{\mathcal{B}_1}(t) + (S_{\mathcal{B}_1} * \mathcal{A}_1 U^1)(t)) \mathcal{F}_x^{-1} \left(\text{Id} - \chi \left(\frac{|\xi|}{\kappa} \right) e^{t\lambda_j(\xi)} P_j(\xi) \right) \mathcal{F}_x. \end{aligned}$$

We are able to get the wanted estimate in $L_v^2(\langle v \rangle^\ell)$ thanks to the uniform boundedness in $|\xi| \leq \kappa$ of the projectors $P_j(\xi)$ in L_v^2 , the dissipativity properties of \mathcal{B}_1 in $L_v^2(\langle v \rangle^\ell)$ and the regularization properties of \mathcal{A}_1 (from L_v^2 to $L_v^2(\langle v \rangle^\ell)$) established respectively in Lemmas 4.28 and 4.27. \square

Remark 4.35. Denoting

$$\tilde{P}_j \left(\xi, \frac{\xi}{|\xi|} \right) := P_j^1 \left(\frac{\xi}{|\xi|} \right) + |\xi| P_j^2(\xi) \quad (4.132)$$

for $1 \leq j \leq 4$, we can further split $\widehat{U}_j^\varepsilon(t)$ into four parts (a main part and three remainder terms):

$$U_j^\varepsilon = U_{j0}^\varepsilon + U_{j0}^{\varepsilon\sharp} + U_{j1}^\varepsilon + U_{j2}^\varepsilon$$

where

$$\begin{aligned} \widehat{U}_{j0}^\varepsilon(t, \xi) &:= e^{i\alpha_j |\xi| \frac{t}{\varepsilon} - \beta_j t |\xi|^2} P_j^0 \left(\frac{\xi}{|\xi|} \right), \\ \widehat{U}_{j0}^{\varepsilon\sharp}(t, \xi) &:= \left(\chi \left(\frac{\varepsilon |\xi|}{\kappa} \right) - 1 \right) e^{i\alpha_j |\xi| \frac{t}{\varepsilon} - \beta_j t |\xi|^2} P_j^0 \left(\frac{\xi}{|\xi|} \right), \\ \widehat{U}_{j1}^\varepsilon(t, \xi) &:= \chi \left(\frac{\varepsilon |\xi|}{\kappa} \right) e^{i\alpha_j |\xi| \frac{t}{\varepsilon} - \beta_j t |\xi|^2} \left(e^{t \frac{\gamma_j(\varepsilon |\xi|)}{\varepsilon^2}} - 1 \right) P_j^0 \left(\frac{\xi}{|\xi|} \right), \\ \widehat{U}_{j2}^\varepsilon(t, \xi) &:= \chi \left(\frac{\varepsilon |\xi|}{\kappa} \right) e^{i\alpha_j |\xi| \frac{t}{\varepsilon} - \beta_j t |\xi|^2 + t \frac{\gamma_j(\varepsilon |\xi|)}{\varepsilon^2}} \varepsilon |\xi| \tilde{P}_j \left(\varepsilon \xi, \frac{\xi}{|\xi|} \right). \end{aligned}$$

One can notice that $U_{30} := U_{30}^\varepsilon$ and $U_{40} := U_{40}^\varepsilon$ do not depend on ε since $\alpha_3 = \alpha_4 = 0$. We set

$$U := U_{30} + U_{40}.$$

We shall see that the operator $U(t)$ is in some sense the limit of $U^\varepsilon(t)$ (see Lemma 4.38).

The decomposition of the semigroup $U^\varepsilon(t)$ also gives us a decomposition of the operator $\Psi_\varepsilon(t)$ defined in (4.129) (see [44, Lemma A.4] and its proof).

Lemma 4.36. *The following decomposition holds*

$$\Psi^\varepsilon = \sum_{j=1}^4 \Psi_j^\varepsilon + \Psi^{\varepsilon\sharp}$$

with

$$\widehat{\Psi}^{\varepsilon\sharp}(t)(f_1, f_2) := \frac{1}{\varepsilon} \int_0^t \widehat{U}^{\varepsilon\sharp}(t-s) \widehat{\Gamma}(f_1(s), f_2(s)) \, ds$$

and

$$\Psi_j^\varepsilon = \Psi_{j0}^\varepsilon + \Psi_{j0}^{\varepsilon\sharp} + \Psi_{j1}^\varepsilon + \Psi_{j2}^\varepsilon$$

where

$$\mathcal{F}_x \left(\Psi_{j0}^\varepsilon(t)(f, f) \right) (\xi) := \int_0^t e^{i\alpha_j |\xi| \frac{t-s}{\varepsilon} - \beta_j (t-s) |\xi|^2} |\xi| P_j^1 \left(\frac{\xi}{|\xi|} \right) \widehat{\Gamma}(f_1, f_2)(s) ds,$$

$$\mathcal{F}_x \left(\Psi_{j0}^{\varepsilon\sharp}(t)(f, f) \right) (\xi) := \left(\chi \left(\frac{\varepsilon |\xi|}{\kappa} \right) - 1 \right) \int_0^t e^{i\alpha_j |\xi| \frac{t-s}{\varepsilon} - \beta_j (t-s) |\xi|^2} |\xi| P_j^1 \left(\frac{\xi}{|\xi|} \right) \widehat{\Gamma}(f_1, f_2)(s) ds,$$

$$\begin{aligned} \mathcal{F}_x \left(\Psi_{j1}^\varepsilon(t)(f, f) \right) (\xi) \\ := \chi \left(\frac{\varepsilon |\xi|}{\kappa} \right) \int_0^t e^{i\alpha_j |\xi| \frac{t-s}{\varepsilon} - \beta_j (t-s) |\xi|^2} \left(e^{(t-s) \frac{\gamma_j(\varepsilon |\xi|)}{\varepsilon^2}} - 1 \right) |\xi| P_j^1 \left(\frac{\xi}{|\xi|} \right) \widehat{\Gamma}(f_1, f_2)(s) ds, \end{aligned}$$

$$\begin{aligned} \mathcal{F}_x \left(\Psi_{j2}^\varepsilon(t)(f, f) \right) (\xi) \\ := \chi \left(\frac{\varepsilon |\xi|}{\kappa} \right) \int_0^t e^{i\alpha_j |\xi| \frac{t-s}{\varepsilon} - \beta_j (t-s) |\xi|^2 + (t-s) \frac{\gamma_j(\varepsilon |\xi|)}{\varepsilon^2}} \varepsilon |\xi|^2 P_j^2(\varepsilon \xi) \widehat{\Gamma}(f_1, f_2)(s) ds. \end{aligned}$$

Remark 4.37. Let us notice that as in Remark 4.35, there holds

$$\Psi_{30}^\varepsilon = \Psi_{30} \quad \text{and} \quad \Psi_{40}^\varepsilon = \Psi_{40}$$

and we set

$$\Psi := \Psi_{30} + \Psi_{40}.$$

5.4 Limit operators $U(t)$ and $\Psi(t)$

The following lemma studies the limit of $U^\varepsilon(t)$ as ε goes to 0, its proof is completely similar as the one of [44, Lemma 3.5], the only difference being that we use that the projectors are bounded from L_v^2 into $L_v^2 \left(\langle v \rangle^{3(\frac{\gamma}{2}+1)} \right)$ (see Lemma 4.34), we thus skip the proof.

Lemma 4.38. *Let f be a well-prepared data as defined in (4.25). Then, we have that*

$$\|(U^\varepsilon(t) - U(t))f\|_{L_t^\infty(\mathcal{X})} \lesssim \|f\|_{H_x^3 L_v^2} \quad (4.133)$$

and

$$\|(U^\varepsilon(t) - U(t))f\|_{L_t^\infty(\mathcal{X})} \lesssim \varepsilon \|f\|_{H_x^4 L_v^2}. \quad (4.134)$$

In the following lemma, we study the convergence of $\Psi^\varepsilon(t)(f, f)$ towards $\Psi(t)(f, f)$ when f is a well-prepared data and its associated macroscopic quantities solve the limit system (4.24). The proof is similar to the one of [44, Lemma 4.1], we thus omit the proof (notice that this result relies on refined estimates on quantities related to f that can be found in [44, Lemmas B.6 and B.7]).

Lemma 4.39. Consider f a well-prepared data as defined in (4.25) with associated macroscopic quantities solving the limit system (4.24) on \mathbb{R}^+ and with mean free initial data $(\rho_0, u_0, \theta_0) \in H_x^3$ and associated kinetic distribution $f_0 \in \mathcal{X}$ (as in (4.27)) satisfying $\|f_0\|_{\mathcal{X}} \leq \eta_1$ (so that f is defined globally in time), then

$$\|\Psi^\varepsilon(t)(f, f) - \Psi(t)(f, f)\|_{L_t^\infty(\mathcal{X})} \lesssim \varepsilon C(\|f_0\|_{H_x^3 L_v^2}),$$

where $C(\|f_0\|_{H_x^3 L_v^2})$ is a constant only depending on $\|f_0\|_{H_x^3 L_v^2}$.

5.5 Decay estimates on the linearized Landau semigroup

We recall that π is the projector onto the kernel of L and is given in (4.12) and that the spaces \mathcal{Y}_1 and $(\mathcal{Z}_i^\varepsilon)'$ are respectively defined in (4.20) and (4.48). From Lemma 4.34, as in [44, Lemma 3.2], we can prove some new decay estimates on the linearized Landau semigroup:

Lemma 4.40. Let $\sigma_2 := \min(\alpha, \beta_1, \dots, \beta_4)$ (where α and β_j for $j = 1, \dots, 4$ are defined in Lemma 4.34). Then, for any $\sigma \in (0, \sigma_2)$, we have

$$\|U^\varepsilon(t)(\text{Id} - \pi)\|_{\mathcal{X} \rightarrow \mathcal{X}} \lesssim \varepsilon \frac{e^{-\sigma t}}{\sqrt{t}}, \quad \forall t > 0.$$

Combining this with Corollaries 4.32-4.33, one can deduce the following result which mixes decay and regularization estimates:

Corollary 4.41. For any $\sigma \in (0, \min(\sigma_1, \sigma_2))$, there holds

$$\|U^\varepsilon(t)(\text{Id} - \pi)\|_{\mathcal{X} \rightarrow \mathcal{Y}_1} \lesssim \varepsilon \frac{e^{-\sigma t}}{\sqrt{t}}, \quad \forall t > 0,$$

where σ_1 is defined in (4.124) and σ_2 in Lemma 4.40.

Proof. Let $\sigma \in (0, \min(\sigma_1, \sigma_2))$. From Duhamel formula, we have:

$$U^\varepsilon(t)(\text{Id} - \pi) = S_{\mathcal{B}_\varepsilon}(t)(\text{Id} - \pi) + (S_{\mathcal{B}_\varepsilon} * \mathcal{A}_\varepsilon U^\varepsilon)(t)(\text{Id} - \pi).$$

From Lemma 4.29, since $\sigma < \sigma_1$, we have:

$$\|S_{\mathcal{B}_\varepsilon}(t)(\text{Id} - \pi)\|_{\mathcal{X} \rightarrow \mathcal{Y}_1} \lesssim \varepsilon \frac{e^{-\sigma t}}{\sqrt{t}} \|\text{Id} - \pi\|_{\mathcal{X} \rightarrow \mathcal{X}} \lesssim \varepsilon \frac{e^{-\sigma t}}{\sqrt{t}}.$$

For the second term, we use Lemmas 4.27 and 4.29:

$$\begin{aligned} \|(S_{\mathcal{B}_\varepsilon} * \mathcal{A}_\varepsilon U^\varepsilon)(t)(\text{Id} - \pi)\|_{\mathcal{X} \rightarrow \mathcal{Y}_1} &\lesssim \int_0^t \|S_{\mathcal{B}_\varepsilon}(t-s) \mathcal{A}_\varepsilon U^\varepsilon(s)(\text{Id} - \pi)\|_{\mathcal{X} \rightarrow \mathcal{Y}_1} \, ds \\ &\lesssim \int_0^t \frac{e^{-\sigma(t-s)}}{\sqrt{t-s}} \|U^\varepsilon(s)(\text{Id} - \pi)\|_{\mathcal{X} \rightarrow \mathcal{X}} \, ds. \end{aligned}$$

Finally, from Lemma 4.40,

$$\|(S_{\mathcal{B}_\varepsilon} * \mathcal{A}_\varepsilon U^\varepsilon)(t)(\text{Id} - \pi)\|_{\mathcal{X} \rightarrow \mathcal{Y}_1} \lesssim \varepsilon \int_0^t \frac{e^{-\sigma(t-s)} e^{-\sigma s}}{\sqrt{t-s} \sqrt{s}} ds \lesssim \varepsilon e^{-\sigma t},$$

which yields the conclusion. \square

By using an interpolation argument, one can deduce the following result:

Lemma 4.42. *For any $\sigma \in (0, \min(\sigma_0, \sigma_1, \sigma_2))$, we have:*

$$\|U^\varepsilon(t)(\text{Id} - \pi)\|_{(\mathcal{Z}_1^\varepsilon)' \rightarrow (\mathcal{Z}_1^\varepsilon)'} \lesssim \sqrt{\varepsilon} \frac{e^{-\sigma t}}{t^{1/4}}, \quad \forall t > 0,$$

where σ_0, σ_1 and σ_2 are respectively defined in Proposition 4.16, in (4.124) and Lemma 4.40.

Proof. Step 1. First, by using an enlargement argument (from [56]), we prove that

$$\|U^\varepsilon(t)(\text{Id} - \Pi)\|_{(\mathcal{Z}_2^\varepsilon)' \rightarrow (\mathcal{Z}_2^\varepsilon)'} \lesssim e^{-\sigma t}. \quad (4.135)$$

For sake of completeness and in order to carefully handle the ε -dependencies, we write the proof. From Duhamel formula, we have that

$$U^\varepsilon(t) = \sum_{j=0}^3 V_j^\varepsilon(t) + (U^\varepsilon * V_3^\varepsilon)(t).$$

Moreover, we have $U^\varepsilon(t)(\text{Id} - \Pi) = (\text{Id} - \Pi)U^\varepsilon(t)$. Then, from Corollary 4.32 and the fact that $\Pi \in \mathcal{B}((\mathcal{Z}_2^\varepsilon)'),$ for any $j = 0, \dots, 3$, we have:

$$\|(\text{Id} - \Pi)V_j^\varepsilon(t)\|_{(\mathcal{Z}_2^\varepsilon)' \rightarrow (\mathcal{Z}_2^\varepsilon)'} \lesssim e^{-\sigma t/\varepsilon^2}. \quad (4.136)$$

For the last term, using that $\mathcal{X} \hookrightarrow (\mathcal{Z}_2^\varepsilon)'$ (independently of ε) and Theorem 4.15, we have:

$$\begin{aligned} \|(\text{Id} - \Pi)(U^\varepsilon * V_3^\varepsilon)(t)\|_{(\mathcal{Z}_2^\varepsilon)' \rightarrow (\mathcal{Z}_2^\varepsilon)'} &\lesssim \int_0^t \|(\text{Id} - \Pi)U^\varepsilon(t-s)V_3^\varepsilon(s)\|_{(\mathcal{Z}_2^\varepsilon)' \rightarrow \mathcal{X}} ds \\ &\lesssim \int_0^t e^{-\sigma(t-s)} \|V_3^\varepsilon(s)\|_{(\mathcal{Z}_2^\varepsilon)' \rightarrow \mathcal{X}} ds. \end{aligned}$$

Corollary 4.33 allows us to conclude that

$$\|(\text{Id} - \Pi)(U^\varepsilon * V_3^\varepsilon)(t)\|_{(\mathcal{Z}_2^\varepsilon)' \rightarrow (\mathcal{Z}_2^\varepsilon)'} \lesssim \int_0^t e^{-\sigma(t-s)} e^{-\sigma s/\varepsilon^2} ds \lesssim e^{-\sigma t}. \quad (4.137)$$

From estimates (4.136) and (4.137), we can conclude that (4.135) holds.

Step 2. From (4.135) and the fact that $U^\varepsilon(t)\Pi = \Pi \in \mathcal{B}((\mathcal{Z}_2^\varepsilon)'),$ we deduce

$$\|U^\varepsilon(t)\|_{(\mathcal{Z}_2^\varepsilon)' \rightarrow (\mathcal{Z}_2^\varepsilon)'} \lesssim 1.$$

Since $\pi \in \mathcal{B}((\mathcal{Z}_2^\varepsilon)'),$ it implies that

$$\|U^\varepsilon(t)(\text{Id} - \pi)\|_{(\mathcal{Z}_2^\varepsilon)' \rightarrow (\mathcal{Z}_2^\varepsilon)'} \lesssim 1.$$

Consequently, combining this with Lemma 4.40, we can conclude the proof by interpolation because from (4.49), we have $(\mathcal{Z}_1^\varepsilon)' = [\mathcal{X}, (\mathcal{Z}_2^\varepsilon)']_{1/2,2}.$ \square

6 Hydrodynamical limit

We first state a quantitative result which provides estimates on the difference on the solution g^ε to the Landau equation constructed in Theorem 4.1 and the solution g defined in (4.28) whose first macroscopic quantities are solution to the fluid system (4.24). As explained in Subsection 6.2, this theorem combined with a density argument allows to prove Theorem 4.5. It is important to notice that thanks to the estimates obtained on the kinetic equation in Theorem 4.1, under some suitable smallness assumptions on the initial data of both kinetic and fluid equations, only extra-regularity in x on the initial data of the fluid system is needed to obtain a quantitative rate of convergence in ε , as can be seen in (4.138) and (4.139).

Theorem 4.43. *Let $g_{\text{in}}^\varepsilon \in \mathcal{X} \cap (\text{Ker } \Lambda_\varepsilon)^\perp$ for $\varepsilon \in (0, 1)$ such that $\|g_{\text{in}}^\varepsilon\|_{\mathcal{X}} \leq \eta_0$ (where η_0 is defined in Theorem 4.1) and $g^\varepsilon \in L_t^\infty(\mathcal{X})$ being the associated solutions of (4.7) with initial data $g_{\text{in}}^\varepsilon$ constructed in Theorem 4.1. Consider also $g_0 \in H_x^{3+\delta} L_v^2 \cap (\text{Ker } \Lambda_\varepsilon)^\perp$ for some $\delta \in [0, 1/2]$ such that $\|g_0\|_{\mathcal{X}} \leq \eta_1$ and g defined respectively in (4.27) and (4.28).*

There exists $\eta_2 \in (0, \min(\eta_0, \eta_1))$ such that if $\max(\|g_{\text{in}}^\varepsilon\|_{\mathcal{X}}, \|g_0\|_{\mathcal{X}}) \leq \eta_2$, then we have

$$\|g^\varepsilon - g\|_{L_t^\infty(\mathcal{X})} \lesssim \varepsilon^\delta C \left(\|g_0\|_{H_x^{3+\delta} L_v^2}, \|g_{\text{in}}^\varepsilon\|_{\mathcal{X}} \right) + \|g_{\text{in}}^\varepsilon - g_0\|_{\mathcal{X}} \quad (4.138)$$

and

$$\|g^\varepsilon - g\|_{L_t^1(\mathcal{B}_1) + L_t^\infty(\mathcal{X})} \lesssim \varepsilon^\delta C \left(\|g_0\|_{H_x^{3+\delta} L_v^2}, \|g_{\text{in}}^\varepsilon\|_{\mathcal{X}} \right) + \|\pi g_{\text{in}}^\varepsilon - g_0\|_{\mathcal{X}} \quad (4.139)$$

where $C \left(\|g_0\|_{H_x^{3+\delta} L_v^2}, \|g_{\text{in}}^\varepsilon\|_{\mathcal{X}} \right)$ is a constant only depending on $\|g_0\|_{H_x^{3+\delta} L_v^2}$ and $\|g_{\text{in}}^\varepsilon\|_{\mathcal{X}}$.

Remark 4.44. Since $g_0 \in \text{Ker } L$, it decays better than any polynomial in velocity at infinity, it explains the fact that we only use classical Sobolev spaces for g_0 in the RHS of the above inequalities, as already noticed, we have

$$\|g_0\|_{H_x^3 L_v^2} \lesssim \|g_0\|_{\mathcal{X}} \lesssim \|g_0\|_{H_x^3 L_v^2}$$

and similar inequalities could be obtained for higher order Sobolev spaces.

Remark 4.45. We restrict ourselves to the case $\delta \in [0, 1/2]$ in our estimates but one can of course suppose more regularity on the initial data g_0 , notice however that we will still have a rate of $\sqrt{\varepsilon}$. It should be noted that we did not look for optimality in terms of rate in our estimates.

6.1 Reformulation of the hydrodynamical problem

Before starting the proof of Theorem 4.43, we reformulate the problem. Using the definition of the operator $\Psi^\varepsilon(t)$ in (4.129), we have that the solution g^ε of (4.7) constructed in Theorem 4.1 writes

$$g^\varepsilon(t) = U^\varepsilon(t)g_{\text{in}}^\varepsilon + \Psi^\varepsilon(t)(g^\varepsilon, g^\varepsilon).$$

It also follows from [11] that given a well-prepared data $g_0 \in \mathcal{X}$ of the form (4.27), the function g defined in (4.28) satisfies

$$g(t) = U(t)g_0 + \Psi(t)(g, g), \quad (4.140)$$

where, as explained in Subsection 5.4, the operators $U(t)$ and $\Psi(t)$ (defined respectively in Remarks 4.35 and 4.37) are in some sense the limiting operators of $U^\varepsilon(t)$ and $\Psi^\varepsilon(t)$. Formulation (4.140) is thus a way to reformulate the fluid equation in a kinetic fashion.

L_t^∞ -estimate

We first reformulate the problem in order to prove the estimate (4.138). To this end, we write the relation satisfied by $h^\varepsilon := g^\varepsilon - g$:

$$\begin{aligned} h^\varepsilon &= U^\varepsilon(t)g_{\text{in}}^\varepsilon + \Psi^\varepsilon(t)(g^\varepsilon, g^\varepsilon) - U(t)g_0 - \Psi(t)(g, g) \\ &= (U^\varepsilon(t) - U(t))g_0 + U^\varepsilon(t)(g_{\text{in}}^\varepsilon - g_0) + (\Psi^\varepsilon(t) - \Psi(t))(g, g) \\ &\quad + \Psi^\varepsilon(t)((g^\varepsilon)^\perp, (g^\varepsilon)^\perp) + \Psi^\varepsilon(t)((g^\varepsilon)^\perp, \pi g^\varepsilon) + \Psi^\varepsilon(t)(\pi g^\varepsilon, (g^\varepsilon)^\perp) \\ &\quad + \Psi^\varepsilon(t)(\pi h^\varepsilon, \pi g^\varepsilon) + \Psi^\varepsilon(t)(g, \pi h^\varepsilon). \end{aligned} \quad (4.141)$$

In the next subsection, we are going to study each term in the RHS of the above equality. Some terms are going to vanish in the limit $\varepsilon \rightarrow 0$ and other ones will be absorbed in the LHS under suitable smallness assumptions on the initial data. Let us underline that the singularity in ε in the definition of Ψ^ε is going to be handled thanks to Lemma 4.40 which provides a gain of ε when the semigroup $U^\varepsilon(t)$ acts on microscopic quantities. Using that $\pi\Gamma(f_1, f_2) = 0$ for any suitable functions f_1, f_2 , we are thus going to be able to remove the singularity in ε in the operator Ψ^ε . In what follows, we shall prove that:

- The three first terms tend to 0 as ε go to 0 (see Lemmas 4.46 and 4.47): For this purpose, we use that the limits of $U^\varepsilon(t)$ and $\Psi^\varepsilon(t)$ as $\varepsilon \rightarrow 0$ are $U(t)$ and $\Psi(t)$ (see Lemmas 4.38 and 4.39).
- The third, fourth and fifth terms tend to 0 as $\varepsilon \rightarrow 0$ (see Lemma 4.48) because those three terms involve the microscopic part of the kinetic solution g^ε , which provides us some extra smallness in ε in $L_t^2(\mathcal{D}_1)$ thanks to Theorem 4.1.
- The last two terms are bounded by some quantity that involves the norms of the kinetic and fluid initial data multiplied by the $L_t^\infty(\mathcal{X})$ -norm of h^ε (see Lemma 4.50). It will thus be absorbed in the LHS of the equality if initial data $g_{\text{in}}^\varepsilon$ and g_0 are chosen to be small enough. Notice that one can not hope smallness in ε for those terms because they only involve macroscopic quantities.

$L_t^1 + L_t^\infty$ -estimate

We now reformulate the problem in order to prove (4.139). To this end, we introduce $R^\varepsilon(t) := U^\varepsilon(t)(g_{\text{in}}^\varepsilon)^\perp$ and we write that

$$U^\varepsilon(t)g_{\text{in}}^\varepsilon - U(t)g_0 = (U^\varepsilon(t) - U(t))g_0 + U^\varepsilon(t)(\pi g_{\text{in}}^\varepsilon - g_0) + R^\varepsilon(t).$$

From Corollary 4.41, we have that for $\sigma \in (0, \min(\sigma_1, \sigma_2))$,

$$\|R^\varepsilon(t)\|_{\mathcal{X}_1} \lesssim \frac{\varepsilon}{\sqrt{t}} e^{-\sigma t} \|g_{\text{in}}^\varepsilon\|_{\mathcal{X}} \quad (4.142)$$

and thus

$$\|R^\varepsilon\|_{L_t^1(\mathcal{X}_1)} \lesssim \varepsilon \|g_{\text{in}}^\varepsilon\|_{\mathcal{X}}. \quad (4.143)$$

To conclude, it is thus enough to prove that for $\delta \in [0, 1/2]$,

$$\|g^\varepsilon - g - R^\varepsilon\|_{L_t^\infty(\mathcal{X})} \lesssim \varepsilon^\delta C \left(\|g_0\|_{H_x^{3+\delta} L_v^2}, \|g_{\text{in}}^\varepsilon\|_{\mathcal{X}} \right) + \|\pi g_{\text{in}}^\varepsilon - g_0\|_{\mathcal{X}}.$$

We then write the relation satisfied by $\tilde{h}^\varepsilon := g^\varepsilon - g - R^\varepsilon$:

$$\begin{aligned} \tilde{h}^\varepsilon &= (U^\varepsilon(t) - U(t))g_0 + U^\varepsilon(t)(\pi g_{\text{in}}^\varepsilon - g_0) + (\Psi^\varepsilon(t) - \Psi(t))(g, g) \\ &\quad + \Psi^\varepsilon(t)((g^\varepsilon)^\perp, (g^\varepsilon)^\perp) + \Psi^\varepsilon(t)((g^\varepsilon)^\perp, \pi g^\varepsilon) + \Psi^\varepsilon(t)(\pi g^\varepsilon, (g^\varepsilon)^\perp) \\ &\quad + \Psi^\varepsilon(t)(\pi R^\varepsilon, \pi g^\varepsilon) + \Psi^\varepsilon(t)(g, \pi R^\varepsilon) \\ &\quad + \Psi^\varepsilon(t)(\pi \tilde{h}^\varepsilon, \pi g^\varepsilon) + \Psi^\varepsilon(t)(g, \pi \tilde{h}^\varepsilon). \end{aligned} \quad (4.144)$$

In what follows, we shall study each term in the RHS of this equality, the ideas between this decomposition being the same as the ones explained after (4.141). Notice furthermore that thanks to estimate (4.142), we are also going to be able to prove that the seventh and eighth terms tend to 0 when $\varepsilon \rightarrow 0$ (see Lemma 4.49).

6.2 Proofs of Theorems 4.43 and 4.5

From now on, assumptions of Theorem 4.43 are supposed to hold. As explained above, in order to prove Theorem 4.43, we have to estimate the $L_t^\infty(\mathcal{X})$ -norm of each term of the decompositions (4.141) and (4.144).

In both decompositions, concerning the first and third terms, Lemmas 4.38 and 4.39 immediately give by interpolation the following lemma:

Lemma 4.46. *We have: For any $t \geq 0$ and $\delta \in [0, 1/2]$,*

$$\|(U^\varepsilon(t) - U(t))g_0\|_{\mathcal{X}} + \|(\Psi^\varepsilon(t) - \Psi(t))(g, g)\|_{\mathcal{X}} \lesssim \varepsilon^\delta C \left(\|g_0\|_{H_x^{3+\delta} L_v^2} \right).$$

Concerning the second terms of (4.141) and (4.144), from Theorem 4.15, we have that $U^\varepsilon(t)$ is bounded in \mathcal{X} uniformly in time and ε . As a consequence, we obtain:

Lemma 4.47. *We have:*

$$\|U^\varepsilon(t)(g_{\text{in}}^\varepsilon - g_0)\|_{L_t^\infty(\mathcal{X})} \lesssim \|g_{\text{in}}^\varepsilon - g_0\|_{\mathcal{X}} \quad \text{and} \quad \|U^\varepsilon(t)(\pi g_{\text{in}}^\varepsilon - g_0)\|_{L_t^\infty(\mathcal{X})} \lesssim \|\pi g_{\text{in}}^\varepsilon - g_0\|_{\mathcal{X}}.$$

The third, fourth and fifth terms of decompositions (4.141) and (4.144) are the most difficult ones to estimate. Indeed, they involve microscopic quantities, we thus have to be sharp in terms of regularity in velocity in order to obtain the following lemma:

Lemma 4.48. *We have: For any $t \geq 0$,*

$$\|\Psi^\varepsilon(t)((g^\varepsilon)^\perp, (g^\varepsilon)^\perp)\|_{\mathcal{X}} + \|\Psi^\varepsilon(t)((g^\varepsilon)^\perp, \pi g^\varepsilon)\|_{\mathcal{X}} + \|\Psi^\varepsilon(t)(\pi g^\varepsilon, (g^\varepsilon)^\perp)\|_{\mathcal{X}} \lesssim \sqrt{\varepsilon} \|g_{\text{in}}^\varepsilon\|_{\mathcal{X}}^2.$$

Proof. In the whole proof, we fix $\sigma \in (0, \min(\sigma_0, \sigma_1, \sigma_2))$ where σ_j for $j = 0, 1, 2$ are respectively defined in Proposition 4.16, (4.124) and Lemma 4.40. We focus on the first term which is the most intricate. We are going to use (4.128) with $n = 2$ to decompose $\Psi^\varepsilon(t)$ into several parts, it yields

$$\begin{aligned} \Psi^\varepsilon(t)((g^\varepsilon)^\perp, (g^\varepsilon)^\perp) &= \sum_{j=0}^2 \frac{1}{\varepsilon} \int_0^t V_j^\varepsilon(t-s) \Gamma((g^\varepsilon)^\perp, (g^\varepsilon)^\perp)(s) \, ds \\ &\quad + \frac{1}{\varepsilon} \int_0^t (V_2^\varepsilon * \mathcal{A}_\varepsilon U^\varepsilon)(t-s) \Gamma((g^\varepsilon)^\perp, (g^\varepsilon)^\perp)(s) \, ds \\ &=: \sum_{j=0}^2 \Psi_j^\varepsilon(t)((g^\varepsilon)^\perp, (g^\varepsilon)^\perp) + \Psi_\star^\varepsilon(t)((g^\varepsilon)^\perp, (g^\varepsilon)^\perp). \end{aligned}$$

We first estimate $\Psi_j^\varepsilon(t)((g^\varepsilon)^\perp, (g^\varepsilon)^\perp)$ for $j = 0, 1, 2$. From Propositions 4.21 and Corollary 4.33, we obtain:

$$\|\Psi_j^\varepsilon(t)((g^\varepsilon)^\perp, (g^\varepsilon)^\perp)\|_{\mathcal{X}} \lesssim \frac{1}{\varepsilon} \int_0^t \frac{\varepsilon}{\sqrt{t-s}} e^{-\sigma(t-s)/\varepsilon^2} \|(g^\varepsilon)^\perp(s)\|_{\mathcal{X}} \|(g^\varepsilon)^\perp(s)\|_{\mathcal{X}_1} \, ds.$$

Then, from (4.22)-(4.23) in Theorem 4.1, we have:

$$\|\Psi_j^\varepsilon(t)((g^\varepsilon)^\perp, (g^\varepsilon)^\perp)\|_{\mathcal{X}} \lesssim \int_0^t \frac{e^{-\sigma(t-s)/\varepsilon^2}}{\sqrt{t-s}} \frac{e^{-3\sigma s/2}}{s^{1/4}} \|(g^\varepsilon)^\perp(s)\|_{\mathcal{X}_1}^{1/2} \, ds \|g_{\text{in}}^\varepsilon\|_{\mathcal{X}}^{3/2}.$$

Using Hölder inequality and (4.22), we obtain

$$\begin{aligned} \|\Psi_j^\varepsilon(t)((g^\varepsilon)^\perp, (g^\varepsilon)^\perp)\|_{\mathcal{X}} &\lesssim \left\| \frac{1}{(t-s)^{1/2} s^{1/4}} \right\|_{L_s^{4/3}([0,t])} \left\| \|(g^\varepsilon)^\perp(s)\|_{\mathcal{X}_1}^{1/2} \right\|_{L_s^4} \|g_{\text{in}}^\varepsilon\|_{\mathcal{X}}^{3/2} \\ &\lesssim \left(\int_0^t \frac{1}{(t-s)^{2/3} s^{1/3}} \, ds \right)^{3/4} \|(g^\varepsilon)^\perp\|_{L_t^2(\mathcal{X}_1)}^{1/2} \|g_{\text{in}}^\varepsilon\|_{\mathcal{X}}^{3/2} \\ &\lesssim \sqrt{\varepsilon} \|g_{\text{in}}^\varepsilon\|_{\mathcal{X}}^2. \end{aligned}$$

Let us now deal with $\Psi_\star^\varepsilon(t)((g^\varepsilon)^\perp, (g^\varepsilon)^\perp)$. Performing a change of variable and recalling that $\pi \Gamma((g^\varepsilon)^\perp, (g^\varepsilon)^\perp) = 0$, one can notice that

$$\Psi_\star^\varepsilon(t)((g^\varepsilon)^\perp, (g^\varepsilon)^\perp) = \frac{1}{\varepsilon} \int_0^t \int_0^s V_2^\varepsilon(t-s) \mathcal{A}_\varepsilon U^\varepsilon(s-\tau) (\text{Id} - \pi) \Gamma((g^\varepsilon)^\perp, (g^\varepsilon)^\perp)(\tau) \, d\tau \, ds.$$

Remark then that Corollary 4.33 for $j = 2$ implies that for $\sigma' \in (\sigma, \sigma_1)$, we have

$$\|V_2^\varepsilon(t)\|_{(\mathcal{X}_1^\varepsilon)' \rightarrow \mathcal{X}} \lesssim \frac{\sqrt{t}}{\varepsilon} e^{-\sigma't/\varepsilon^2} \lesssim e^{-\sigma t/\varepsilon^2}.$$

Using now Lemma 4.27, it implies that

$$\begin{aligned} & \|\Psi_\star^\varepsilon(t)((g^\varepsilon)^\perp, (g^\varepsilon)^\perp)\|_{\mathcal{X}} \\ & \lesssim \frac{1}{\varepsilon^3} \int_0^t \int_0^s e^{-\sigma(t-s)/\varepsilon^2} \|U^\varepsilon(s-\tau)(\text{Id} - \pi)\Gamma((g^\varepsilon)^\perp, (g^\varepsilon)^\perp)(\tau)\|_{(\mathcal{X}_1^\varepsilon)'} d\tau ds. \end{aligned}$$

Lemma 4.42 and the fact that $\mathcal{Y}_1' \hookrightarrow (\mathcal{X}_1^\varepsilon)'$ (independently of ε) then imply that

$$\begin{aligned} & \|\Psi_\star^\varepsilon(t)((g^\varepsilon)^\perp, (g^\varepsilon)^\perp)\|_{\mathcal{X}} \\ & \lesssim \frac{1}{\varepsilon^3} \int_0^t \int_0^s e^{-\sigma(t-s)/\varepsilon^2} \frac{\sqrt{\varepsilon}}{(s-\tau)^{1/4}} e^{-\sigma(s-\tau)} \|\Gamma((g^\varepsilon)^\perp, (g^\varepsilon)^\perp)(\tau)\|_{\mathcal{Y}_1'} d\tau ds. \end{aligned}$$

From Proposition 4.21 and the fact that $\|(g^\varepsilon)^\perp\|_{\mathcal{X}} \lesssim \|g^\varepsilon\|_{\mathcal{X}}$, we deduce that

$$\begin{aligned} & \|\Psi_\star^\varepsilon(t)((g^\varepsilon)^\perp, (g^\varepsilon)^\perp)\|_{\mathcal{X}} \\ & \lesssim \frac{1}{\varepsilon^{5/2}} \int_0^t \int_0^s e^{-\sigma(t-s)/\varepsilon^2} \frac{e^{-\sigma(s-\tau)}}{(s-\tau)^{1/4}} \|g^\varepsilon(\tau)\|_{\mathcal{X}} \|(g^\varepsilon)^\perp(\tau)\|_{\mathcal{Y}_1} d\tau ds. \end{aligned}$$

Using Cauchy-Schwarz inequality in the variable τ and (4.22) in Theorem 4.1, we obtain

$$\|\Psi_\star^\varepsilon(t)((g^\varepsilon)^\perp, (g^\varepsilon)^\perp)\|_{\mathcal{X}} \lesssim \frac{1}{\varepsilon^{3/2}} \int_0^t e^{-\sigma(t-s)/\varepsilon^2} ds \|g_{\text{in}}^\varepsilon\|_{\mathcal{X}}^2 \lesssim \sqrt{\varepsilon} \|g_{\text{in}}^\varepsilon\|_{\mathcal{X}}^2,$$

which concludes the proof of the term $\Psi^\varepsilon(t)((g^\varepsilon)^\perp, (g^\varepsilon)^\perp)$.

The proof for the second and third terms $\Psi^\varepsilon(t)((g^\varepsilon)^\perp, \pi g^\varepsilon)$ and $\Psi^\varepsilon(t)(\pi g^\varepsilon, (g^\varepsilon)^\perp)$ is completely similar once one has noticed that from Proposition 4.21,

$$\begin{aligned} & \|\Gamma((g^\varepsilon)^\perp, \pi g^\varepsilon)\|_{\mathcal{Y}_1'} + \|\Gamma(\pi g^\varepsilon, (g^\varepsilon)^\perp)\|_{\mathcal{Y}_1'} \\ & \lesssim \|(g^\varepsilon)^\perp\|_{\mathcal{Y}_1} \|\pi g^\varepsilon\|_{\mathcal{X}} + \|(g^\varepsilon)^\perp\|_{\mathcal{X}} \|\pi g^\varepsilon\|_{\mathcal{Y}_1} \lesssim \|(g^\varepsilon)^\perp\|_{\mathcal{Y}_1} \|g^\varepsilon\|_{\mathcal{X}} \end{aligned}$$

where we used the facts that $\mathcal{Y}_1 \hookrightarrow \mathcal{X}$ and $\pi \in \mathcal{B}(\mathcal{X}, \mathcal{Y}_1)$. □

For the proof of (4.139), we also need the following lemma:

Lemma 4.49. *We have: For any $t \geq 0$,*

$$\|\Psi^\varepsilon(t)(\pi R^\varepsilon, \pi g^\varepsilon)\|_{\mathcal{X}} + \|\Psi^\varepsilon(t)(g, \pi R^\varepsilon)\|_{\mathcal{X}} \lesssim \varepsilon (\|g_{\text{in}}^\varepsilon\|_{\mathcal{X}} + \|g_0\|_{\mathcal{X}}) \|g_{\text{in}}^\varepsilon\|_{\mathcal{X}}.$$

Proof. From Lemma 4.40, Proposition 4.24, the fact that $\pi \in \mathcal{B}(\mathcal{X}, \mathcal{Y}_2)$ and $g = \pi g$, we obtain:

$$\begin{aligned} & \|\Psi^\varepsilon(t)(\pi R^\varepsilon, \pi g^\varepsilon)\|_{\mathcal{X}} + \|\Psi^\varepsilon(t)(g, \pi R^\varepsilon)\|_{\mathcal{X}} \\ & \lesssim \int_0^t \frac{e^{-\sigma(t-s)}}{\sqrt{t-s}} \|R^\varepsilon(s)\|_{\mathcal{X}} (\|g(s)\|_{\mathcal{X}} + \|g^\varepsilon(s)\|_{\mathcal{X}}) \, ds. \end{aligned}$$

Using now (4.142), (4.22) from Theorem 4.1 and (4.29), we obtain:

$$\begin{aligned} & \|\Psi^\varepsilon(t)(\pi R^\varepsilon, \pi g^\varepsilon)\|_{\mathcal{X}} + \|\Psi^\varepsilon(t)(g, \pi R^\varepsilon)\|_{\mathcal{X}} \\ & \lesssim \varepsilon \int_0^t \frac{e^{-\sigma(t-s)}}{\sqrt{t-s}} \frac{e^{-\sigma s}}{\sqrt{s}} \, ds \|g_{\text{in}}^\varepsilon\|_{\mathcal{X}}^2 + \varepsilon \int_0^t \frac{e^{-\sigma(t-s)}}{\sqrt{t-s}} \frac{e^{-\sigma s}}{\sqrt{s}} \, ds \|g_{\text{in}}^\varepsilon\|_{\mathcal{X}} \|g_0\|_{\mathcal{X}}, \end{aligned}$$

which yields the final result. \square

Concerning the last two terms of (4.141) and (4.144), we prove the following lemma:

Lemma 4.50. *For any $f \in L_t^\infty(\mathcal{X})$, we have: For any $t \geq 0$,*

$$\|\Psi^\varepsilon(t)(\pi f, \pi g^\varepsilon)\|_{\mathcal{X}} + \|\Psi^\varepsilon(t)(g, \pi f)\|_{\mathcal{X}} \lesssim (\|g_{\text{in}}^\varepsilon\|_{\mathcal{X}} + \|g_0\|_{\mathcal{X}}) \|f\|_{L_t^\infty(\mathcal{X})}.$$

Proof. The proof is similar to the one of Lemma 4.49. We use Lemma 4.40, Proposition 4.24, the fact that $\pi \in \mathcal{B}(\mathcal{X}, \mathcal{Y}_2)$ and $g = \pi g$, and we conclude thanks to (4.22) and (4.29). \square

End of the proof of Theorem 4.43

Gathering results from Lemmas 4.46, 4.47, 4.48 and 4.50, we can conclude the proof of (4.138) by taking $\|g_{\text{in}}^\varepsilon\|_{\mathcal{X}}$ and $\|g_0\|_{\mathcal{X}}$ small enough. Concerning (4.139), Lemmas 4.46, 4.47, 4.48, 4.49 and 4.50 imply that

$$\|g^\varepsilon - g - R^\varepsilon\|_{L_t^\infty(\mathcal{X})} \lesssim \varepsilon^\delta C \left(\|g_0\|_{H_x^{3+\delta} L_x^2}, \|g_{\text{in}}^\varepsilon\|_{\mathcal{X}} \right) + \|\pi g_{\text{in}}^\varepsilon - g_0\|_{\mathcal{X}}.$$

The estimate (4.143) then allows to conclude the proof of (4.139). \square

Proof of Theorem 4.5

As mentioned above, in order to obtain results of convergence in Theorem 4.5, we are going to use a density argument that is explained in what follows. We consider a smooth family $(g_{0,\eta})_{\eta \in (0,1)}$ such that

$$\|g_0 - g_{0,\eta}\|_{\mathcal{X}} \leq \eta, \quad \forall \eta \in (0,1). \quad (4.145)$$

We also have stability for the Navier-Stokes-Fourier system (see for example [44, Appendix B.3]), we know that

$$g_\eta(t) := U(t)g_{0,\eta} + \Psi(t)(g_\eta, g_\eta)$$

satisfies

$$\lim_{\eta \rightarrow 0} \|g_\eta - g\|_{L_t^\infty(\mathcal{X})} = 0. \quad (4.146)$$

Then, to study the convergence of g^ε towards g , we write

$$g^\varepsilon - g = g^\varepsilon - g_\eta + g_\eta - g. \quad (4.147)$$

We then apply estimate (4.138) from Theorem 4.43 with $g_{0,\eta}$ and g_η instead of g_0 and g . Notice that for g_0 and η small enough, $g_{0,\eta}$ will also satisfy $\|g_{0,\eta}\|_{\mathcal{X}} \leq \eta_2$. Coming back to (4.147), we deduce that

$$\|g^\varepsilon - g\|_{L_t^\infty(\mathcal{X})} \lesssim \|g - g_\eta\|_{L_t^\infty(\mathcal{X})} + \varepsilon^\delta C(\|g_{\text{in}}^\varepsilon\|_{\mathcal{X}}, \|g_{0,\eta}\|_{\mathcal{X}}) + \|g_{\text{in}}^\varepsilon - g_0\|_{\mathcal{X}} + \|g_0 - g_{0,\eta}\|_{\mathcal{X}}.$$

Using (4.146), we can conclude the proof of (4.31). The proof of (4.33) is similar. \square

7 Appendix A. Proof of Proposition 4.16

We start by recalling that for any suitable function g , we write

$$g = g^\perp + \pi g, \quad \pi g(x, v) = \rho_g(x) \sqrt{M}(v) + u_g(x) \cdot v \sqrt{M}(v) + \theta_g(x) \frac{(|v|^2 - 3)}{2} \sqrt{M}(v)$$

with ρ_g , u_g , θ_g defined in (4.14), (4.15) and (4.16). In what follows, we also use the following notations: $\rho[g] = \rho_g$, $u[g] = u_g$, $\theta[g] = \theta_g$.

For $g \in L_x^2(\mathbf{T}^3)$, we introduce the following notation: $\langle g \rangle := \int_{\mathbf{T}^3} g \, dx$. Recall the following classical result. For any $\phi \in L_x^2(\mathbf{T}^3)$ with null mean (i.e. $\langle \phi \rangle = 0$), there is a unique solution $u \in H_x^2(\mathbf{T}^3)$ to the equation

$$-\Delta_x u = \phi \quad \text{in } \mathbf{T}^3 \quad \text{with } \langle u \rangle = 0.$$

Denote by $(-\Delta_x)^{-1}$ the following bounded operator:

$$\begin{aligned} (-\Delta_x)^{-1} : L_x^2(\mathbf{T}^d) \cap \{\langle \cdot \rangle = 0\} &\longrightarrow H_x^2(\mathbf{T}^3) \cap \{\langle \cdot \rangle = 0\} \\ \phi &\longmapsto u. \end{aligned} \quad (4.148)$$

Notice that in particular, we have that the operator $(-\Delta_x)^{-1}$ is also bounded from $L_x^2(\mathbf{T}^d) \cap \{\langle \cdot \rangle = 0\}$ into $H_x^1(\mathbf{T}^3) \cap \{\langle \cdot \rangle = 0\}$.

Let $f \in \text{Dom } \Lambda_\varepsilon \cap (\text{Ker } \Lambda_\varepsilon)^\perp$. Remark that in particular we have for $k = 1, 2, 3$:

$$\langle \rho[f] \rangle = \langle u_k[f] \rangle = \langle \theta[f] \rangle = 0$$

where we have denoted by $u_k[f]$ the k -th coordinate of $u[f]$. We thus deduce that $(-\Delta_x)^{-1}\rho[f]$, $(-\Delta_x)^{-1}u_k[f]$, and $(-\Delta_x)^{-1}\theta[f]$ are well-defined, and more precisely we have

$$\begin{aligned} \|(-\Delta_x)^{-1}\rho[f]\|_{H_x^2} &\lesssim \|\rho[f]\|_{L_x^2} \\ \|(-\Delta_x)^{-1}u_k[f]\|_{H_x^2} &\lesssim \|u_k[f]\|_{L_x^2} \\ \|(-\Delta_x)^{-1}\theta[f]\|_{H_x^2} &\lesssim \|\theta[f]\|_{L_x^2}. \end{aligned}$$

In what follows, we shall use without further mention the following explicit computations: for $k \in \{1, 2, 3\}$,

$$\begin{aligned} \int_{\mathbb{R}^3} v_k^2 M \, dv &= 1, & \int_{\mathbb{R}^3} |v|^2 M \, dv &= 3, \\ \int_{\mathbb{R}^3} v_k^2 |v|^2 M \, dv &= 5, & \int_{\mathbb{R}^3} |v|^4 M \, dv &= 15, \\ \int_{\mathbb{R}^3} v_k^2 |v|^4 M \, dv &= 35, & \int_{\mathbb{R}^3} |v|^6 M \, dv &= 105, \end{aligned}$$

as well as that for any odd polynomial function $p = p(v)$, one has $\int_{\mathbb{R}^3} p(v)M \, dv = 0$.

We split the proof into five steps.

Step 1. Microscopic part. From (4.43) and the skew-adjointness of the transport operator one has

$$\begin{aligned} \langle \Lambda_\varepsilon f, f \rangle_{L_{x,v}^2} &= \frac{1}{\varepsilon^2} \langle Lf, f \rangle_{L_{x,v}^2} - \frac{1}{\varepsilon} \langle v \cdot \nabla_x f, f \rangle_{L_{x,v}^2} \\ &\leq -\frac{\sigma_L}{\varepsilon^2} \|f^\perp\|_{L_x^2(H_{v,*}^1)}^2. \end{aligned} \quad (4.149)$$

Step 2. Energy estimate. Define for any $k \in \{1, 2, 3\}$ and any suitable function g

$$\psi_k[g] = \int_{\mathbb{R}^3} v_k \frac{(|v|^2 - 5)}{3} g \sqrt{M} \, dv \quad (4.150)$$

and remark that

$$\psi_k[g] = \psi_k[g^\perp]. \quad (4.151)$$

We first compute

$$\begin{aligned} \theta[\Lambda_\varepsilon f] &= \frac{1}{\varepsilon^2} \theta[Lf^\perp] - \frac{1}{\varepsilon} \theta[v \cdot \nabla_x f] \\ &= -\frac{1}{\varepsilon} \nabla_x \cdot \int_{\mathbb{R}^3} \frac{(|v|^2 - 3)}{3} v f \sqrt{M} \, dv \\ &= -\frac{1}{\varepsilon} \frac{2}{3} \nabla_x \cdot \int_{\mathbb{R}^3} v f \sqrt{M} \, dv - \frac{1}{\varepsilon} \nabla_x \cdot \int_{\mathbb{R}^3} \frac{(|v|^2 - 5)}{3} v f \sqrt{M} \, dv \\ &= -\frac{1}{\varepsilon} \frac{2}{3} \nabla_x \cdot u[f] - \frac{1}{\varepsilon} \nabla_x \cdot \psi[f], \end{aligned} \quad (4.152)$$

and using some aforementioned classical results on the moments of M , we also obtain

$$\begin{aligned}
 & \psi_k[\Lambda_\varepsilon f] \\
 &= \frac{1}{\varepsilon^2} \psi_k[Lf^\perp] - \frac{1}{\varepsilon} \psi_k[v \cdot \nabla_x f] \\
 &= \frac{1}{\varepsilon^2} \psi_k[Lf^\perp] - \frac{1}{\varepsilon} \partial_{x_\ell} \int_{\mathbb{R}^d} v_k v_\ell \frac{(|v|^2 - 5)}{3} f \sqrt{M} \, dv \\
 &= \frac{1}{\varepsilon^2} \psi_k[Lf^\perp] - \frac{1}{\varepsilon} \partial_{x_\ell} \int_{\mathbb{R}^d} v_k v_\ell \frac{(|v|^2 - 5)}{3} \left(\rho[f] + u[f] \cdot v + \theta[f] \frac{(|v|^2 - 3)}{2} \right) M \, dv \\
 &\quad - \frac{1}{\varepsilon} \partial_{x_\ell} \int_{\mathbb{R}^d} v_k v_\ell \frac{(|v|^2 - 5)}{3} f^\perp \sqrt{M} \, dv \\
 &= \frac{1}{\varepsilon^2} \psi_k[Lf^\perp] - \frac{1}{\varepsilon} \frac{5}{3} \partial_{x_k} \theta[f] - \frac{1}{\varepsilon} \partial_{x_\ell} \int_{\mathbb{R}^d} v_k v_\ell \frac{(|v|^2 - 5)}{3} f^\perp \sqrt{M} \, dv.
 \end{aligned} \tag{4.153}$$

We now estimate the term

$$\left\langle \nabla_x (-\Delta_x)^{-1} \theta[\Lambda_\varepsilon f], \psi[f] \right\rangle_{L_x^2} + \left\langle \nabla_x (-\Delta_x)^{-1} \theta[f], \psi[\Lambda_\varepsilon f] \right\rangle_{L_x^2} =: I_1 + I_2.$$

We start by estimating I_1 . First, using integrations by parts and (4.152), we have

$$\begin{aligned}
 \|\nabla_x (-\Delta_x)^{-1} \theta[\Lambda_\varepsilon f]\|_{L_x^2}^2 &= \left\langle \theta[\Lambda_\varepsilon f], (-\Delta_x)^{-1} \theta[\Lambda_\varepsilon f] \right\rangle_{L_x^2} \\
 &= -\frac{1}{\varepsilon} \left\langle \frac{2}{3} \nabla_x \cdot u[f] + \nabla_x \cdot \psi[f], (-\Delta_x)^{-1} \theta[\Lambda_\varepsilon f] \right\rangle_{L_x^2} \\
 &= -\frac{1}{\varepsilon} \left\langle \frac{2}{3} u[f] + \psi[f], \nabla_x (-\Delta_x)^{-1} \theta[\Lambda_\varepsilon f] \right\rangle_{L_x^2}.
 \end{aligned}$$

Using now Cauchy-Schwarz inequality in x and (4.151), we obtain:

$$\|\nabla_x (-\Delta_x)^{-1} \theta[\Lambda_\varepsilon f]\|_{L_x^2} \lesssim \frac{1}{\varepsilon} \left(\|u[f]\|_{L_x^2} + \|\psi[f]\|_{L_x^2} \right).$$

Then, from (4.151) and Cauchy-Schwarz inequality in velocity, we have:

$$\|\psi[f]\|_{L_x^2} = \|\psi[f^\perp]\|_{L_x^2} \lesssim \|f^\perp\|_{L_{x,v}^2}.$$

We deduce that

$$|I_1| \lesssim \frac{1}{\varepsilon} \left(\|u[f]\|_{L_x^2} + \|f^\perp\|_{L_{x,v}^2} \right) \|f^\perp\|_{L_{x,v}^2}.$$

Concerning I_2 , from the computation (4.153), we have:

$$\begin{aligned}
 I_2 &= \frac{1}{\varepsilon^2} \left\langle \partial_{x_k} (-\Delta_x)^{-1} \theta[f], \psi_k[Lf^\perp] \right\rangle_{L_x^2} - \frac{1}{\varepsilon} \frac{5}{3} \left\langle \partial_{x_k} (-\Delta_x)^{-1} \theta[f], \partial_{x_k} \theta[f] \right\rangle_{L_x^2} \\
 &\quad - \frac{1}{\varepsilon} \left\langle \partial_{x_k} (-\Delta_x)^{-1} \theta[f], \partial_{x_\ell} \int_{\mathbb{R}^3} v_k v_\ell \frac{(|v|^2 - 5)}{3} f^\perp \sqrt{M} \, dv \right\rangle_{L_x^2} \\
 &=: I_{21} + I_{22} + I_{23}.
 \end{aligned}$$

We treat each term of the above splitting in succession. For I_{21} , we notice first that from the self-adjointness of L in L_v^2 :

$$\|\psi[Lf^\perp]\|_{L_x^2}^2 \lesssim \int_{\mathbf{T}^3} \left(\int_{\mathbb{R}^3} L \left(v_k \frac{(|v|^2 - 5)}{3} \sqrt{M} \right) f^\perp dv \right)^2 dx$$

so that from Cauchy-Schwarz inequality:

$$\|\psi[Lf^\perp]\|_{L_x^2} \lesssim \|f^\perp\|_{L_{x,v}^2}. \quad (4.154)$$

We thus have:

$$I_{21} \lesssim \frac{1}{\varepsilon^2} \|\nabla_x (-\Delta_x)^{-1} \theta[f]\|_{L_x^2} \|f^\perp\|_{L_{x,v}^2} \lesssim \frac{1}{\varepsilon^2} \|\theta[f]\|_{L_x^2} \|f^\perp\|_{L_{x,v}^2}$$

where we used (4.148). The term I_{22} is computed exactly thanks to an integration by parts:

$$I_{22} = \frac{1}{\varepsilon} \frac{5}{3} \left\langle \partial_{x_k} \partial_{x_k} (-\Delta_x)^{-1} (\theta[f]), \theta[f] \right\rangle_{L_x^2} = -\frac{1}{\varepsilon} \frac{5}{3} \|\theta[f]\|_{L_x^2}^2.$$

We bound the last term I_{23} using an integration by parts, (4.148) and Cauchy-Schwarz inequality:

$$\begin{aligned} I_{23} &= \frac{1}{\varepsilon} \left\langle \partial_{x_\ell} \partial_{x_k} (-\Delta_x)^{-1} (\theta[f]), \int_{\mathbb{R}^d} v_k v_\ell \frac{(|v|^2 - 5)}{3} f^\perp \sqrt{M} dv \right\rangle_{L_x^2} \\ &\lesssim \frac{1}{\varepsilon} \|\nabla_x^2 (-\Delta_x)^{-1} \theta[f]\|_{L_x^2} \|f^\perp\|_{L_{x,v}^2} \\ &\lesssim \frac{1}{\varepsilon} \|\theta[f]\|_{L_x^2} \|f^\perp\|_{L_{x,v}^2}. \end{aligned}$$

Finally, we have obtained for some $\kappa, C > 0$:

$$\begin{aligned} &\left\langle \nabla_x (-\Delta_x)^{-1} \theta[\Lambda_\varepsilon f], \psi[f] \right\rangle_{L_x^2} + \left\langle \nabla_x (-\Delta_x)^{-1} \theta[f], \psi[\Lambda_\varepsilon f] \right\rangle_{L_x^2} \\ &\leq -\frac{\kappa}{\varepsilon} \|\theta[f]\|_{L_x^2}^2 + \frac{C}{\varepsilon} \|u[f]\|_{L_x^2} \|f^\perp\|_{L_{x,v}^2} + \frac{C}{\varepsilon^3} \|f^\perp\|_{L_{x,v}^2}^2. \end{aligned} \quad (4.155)$$

Step 3. Momentum estimate. Define, for any $k, \ell \in \{1, 2, 3\}$ and any suitable function g

$$\Theta_{k\ell}[g] = \begin{cases} \frac{1}{7} \int_{\mathbb{R}^3} v_k v_\ell |v|^2 g \sqrt{M} dv & \text{if } k \neq \ell, \\ \int_{\mathbb{R}^3} \left(\frac{1}{2} + v_k^2 - \frac{1}{2} |v|^2 \right) g \sqrt{M} dv & \text{if } k = \ell, \end{cases} \quad (4.156)$$

and remark that, using some classical aforementioned results on the moments of M , we have

$$\Theta_{k\ell}[g] = \Theta_{k\ell}[g^\perp] \quad \text{if } k \neq \ell \quad (4.157)$$

and

$$\begin{aligned}\Theta_{kk}[g] &= \Theta_{kk}[g^\perp] + \int_{\mathbb{R}^3} \left(\frac{1}{2} + v_k^2 - \frac{1}{2}|v|^2 \right) \left(\rho[g] + u[g] \cdot v + \theta[g] \frac{(|v|^2 - 3)}{2} \right) M \, dv \\ &= \Theta_{kk}[g^\perp] - \frac{1}{2}\theta[g].\end{aligned}\tag{4.158}$$

We now compute for any $k \in \{1, 2, 3\}$, writing $f = \pi f + f^\perp$,

$$\begin{aligned}u_k[\Lambda_\varepsilon f] &= \frac{1}{\varepsilon^2} u_k[Lf^\perp] - \frac{1}{\varepsilon} u_k[v \cdot \nabla_x f] \\ &= -\frac{1}{\varepsilon} \partial_{x_\ell} \int_{\mathbb{R}^3} v_k v_\ell \left(\rho[f] + u[f] \cdot v + \theta[f] \frac{(|v|^2 - 3)}{2} \right) M \, dv \\ &\quad - \frac{1}{\varepsilon} \partial_{x_\ell} \int_{\mathbb{R}^3} v_k v_\ell f^\perp \sqrt{M} \, dv \\ &= -\frac{1}{\varepsilon} \partial_{x_k} \rho[f] - \frac{1}{\varepsilon} \partial_{x_k} \theta[f] - \frac{1}{\varepsilon} \partial_{x_\ell} \int_{\mathbb{R}^3} v_k v_\ell f^\perp \sqrt{M} \, dv.\end{aligned}\tag{4.159}$$

We also obtain

$$\begin{aligned}\Theta_{k\ell}[\Lambda_\varepsilon f] &= \frac{1}{\varepsilon^2} \Theta_{k\ell}[Lf^\perp] - \frac{1}{\varepsilon} \Theta_{k\ell}[v \cdot \nabla_x \pi f] - \frac{1}{\varepsilon} \Theta_{k\ell}[v \cdot \nabla_x f^\perp] \\ &= \frac{1}{\varepsilon^2} \Theta_{k\ell}[Lf^\perp] - \frac{1}{\varepsilon} \Theta_{k\ell} \left[v \cdot \nabla_x \left(u[f] \cdot v \sqrt{M} \right) \right] - \frac{1}{\varepsilon} \Theta_{k\ell}[v \cdot \nabla_x f^\perp].\end{aligned}\tag{4.160}$$

If $k \neq \ell$, then

$$\begin{aligned}\Theta_{k\ell} \left[v \cdot \nabla_x \left(u[f] \cdot v \sqrt{M} \right) \right] &= \partial_{x_p} u_q[f] \frac{1}{7} \int_{\mathbb{R}^3} v_k v_\ell v_p v_q |v|^2 M \, dv \\ &= \partial_{x_k} u_\ell[f] \frac{1}{7} \int_{\mathbb{R}^3} v_k^2 v_\ell^2 |v|^2 M \, dv + \partial_{x_\ell} u_k[f] \frac{1}{7} \int_{\mathbb{R}^3} v_k^2 v_\ell^2 |v|^2 M \, dv \\ &= \partial_{x_k} u_\ell[f] + \partial_{x_\ell} u_k[f]\end{aligned}\tag{4.161}$$

because

$$\int_{\mathbb{R}^3} v_k^2 v_\ell^2 |v|^2 M \, dv = \int_{\mathbb{R}^3} v_k^2 v_\ell^2 (v_k^2 + v_\ell^2 + v_m^2) M \, dv = 7.$$

If $k = \ell$, then

$$\begin{aligned}\Theta_{kk} \left[v \cdot \nabla_x \left(u[f] \cdot v \sqrt{M} \right) \right] &= \partial_{x_p} u_q[f] \int_{\mathbb{R}^3} \left(\frac{1}{2} + v_k^2 - \frac{1}{2}|v|^2 \right) v_p v_q M \, dv \\ &= \partial_{x_k} u_k[f] \int_{\mathbb{R}^3} \left(\frac{1}{2} + v_k^2 - \frac{1}{2}|v|^2 \right) v_k^2 M \, dv + \sum_{p \neq k} \partial_{x_p} u_p[f] \int_{\mathbb{R}^3} \left(\frac{1}{2} + v_k^2 - \frac{1}{2}|v|^2 \right) v_p^2 M \, dv \\ &= \partial_{x_k} u_k[f] - \sum_{p \neq k} \partial_{x_p} u_p[f],\end{aligned}\tag{4.162}$$

because

$$\int_{\mathbb{R}^3} \left(\frac{1}{2} + v_k^2 - \frac{1}{2}|v|^2 \right) v_k^2 M \, dv = 1 \quad \text{and} \quad \int_{\mathbb{R}^3} \left(\frac{1}{2} + v_k^2 - \frac{1}{2}|v|^2 \right) v_p^2 M \, dv = -1.$$

We now estimate the term

$$\left\langle \partial_{x_\ell} (-\Delta_x)^{-1} u_k[\Lambda_\varepsilon f], \Theta_{k\ell}[f] \right\rangle_{L_x^2} + \left\langle \partial_{x_\ell} (-\Delta_x)^{-1} u_k[f], \Theta_{k\ell}[\Lambda_\varepsilon f] \right\rangle_{L_x^2} =: J_1 + J_2.$$

For J_1 , we first notice that using (4.157) and (4.158) and Cauchy-Schwarz inequality, one can prove that

$$\|\Theta_{k\ell}[f]\|_{L_x^2} \lesssim \|\theta[f]\|_{L_x^2} + \|f^\perp\|_{L_{x,v}^2}.$$

Performing integrations by parts and using (4.159), we also have that

$$\begin{aligned} \|\nabla_x (-\Delta_x)^{-1} u_k[\Lambda_\varepsilon f]\|_{L_x^2}^2 &= \left\langle u_k[\Lambda_\varepsilon f], (-\Delta_x)^{-1} u_k[\Lambda_\varepsilon f] \right\rangle_{L_x^2} \\ &= -\frac{1}{\varepsilon} \left\langle \partial_{x_k} \rho[f] + \partial_{x_k} \theta[f] + \partial_{x_\ell} \int_{\mathbb{R}^3} v_k v_\ell f^\perp \sqrt{M} \, dv, (-\Delta_x)^{-1} u_k[\Lambda_\varepsilon f] \right\rangle_{L_x^2} \\ &= \frac{1}{\varepsilon} \left\langle \rho[f] + \theta[f], \partial_{x_k} (-\Delta_x)^{-1} u_k[\Lambda_\varepsilon f] \right\rangle_{L_x^2} \\ &\quad + \frac{1}{\varepsilon} \left\langle \int_{\mathbb{R}^3} v_k v_\ell f^\perp \sqrt{M} \, dv, \partial_{x_\ell} (-\Delta_x)^{-1} u_k[\Lambda_\varepsilon f] \right\rangle_{L_x^2}. \end{aligned}$$

From Cauchy-Schwarz inequality (in x and also in v for the third term), we obtain:

$$\|\nabla_x (-\Delta_x)^{-1} u_k[\Lambda_\varepsilon f]\|_{L_x^2} \lesssim \frac{1}{\varepsilon} \left(\|\rho[f]\|_{L_x^2} + \|\theta[f]\|_{L_x^2} + \|f^\perp\|_{L_{x,v}^2} \right).$$

Gathering the two previous estimates, by Cauchy-Schwarz inequality, we deduce

$$\begin{aligned} |J_1| &\lesssim \|(-\Delta_x)^{-1} u_k[\Lambda_\varepsilon f]\|_{H_x^1} \|\Theta_{k\ell}[f]\|_{L_x^2} \\ &\lesssim \frac{1}{\varepsilon} \left(\|\rho[f]\|_{L_x^2} + \|\theta[f]\|_{L_x^2} + \|f^\perp\|_{L_{x,v}^2} \right) \left(\|\theta[f]\|_{L_x^2} + \|f^\perp\|_{L_{x,v}^2} \right). \end{aligned}$$

For the term J_2 , the computation made in (4.160) yields

$$\begin{aligned} J_2 &= \frac{1}{\varepsilon^2} \left\langle \partial_{x_\ell} (-\Delta_x)^{-1} u_k[f], \Theta_{k\ell}[L f^\perp] \right\rangle_{L_x^2} \\ &\quad - \frac{1}{\varepsilon} \left\langle \partial_{x_\ell} (-\Delta_x)^{-1} u_k[f], \Theta_{k\ell} \left[v \cdot \nabla_x (u[f] \cdot v \sqrt{M}) \right] \right\rangle_{L_x^2} \\ &\quad - \frac{1}{\varepsilon} \left\langle \partial_{x_\ell} (-\Delta_x)^{-1} u_k[f], \Theta_{k\ell} [v \cdot \nabla_x f^\perp] \right\rangle_{L_x^2} \\ &=: J_{21} + J_{22} + J_{23}. \end{aligned}$$

To bound J_{21} , we first notice that as in (4.154), one can prove that

$$\|\Theta_{k\ell}[L f^\perp]\|_{L_x^2} \lesssim \|f^\perp\|_{L_{x,v}^2}.$$

Then, (4.148) and Cauchy-Schwarz inequality give

$$J_{21} \lesssim \frac{1}{\varepsilon^2} \|\partial_{x_\ell} (-\Delta_x)^{-1} u_k[f]\|_{L_x^2} \|f^\perp\|_{L_{x,v}^2} \lesssim \frac{1}{\varepsilon^2} \|u[f]\|_{L_x^2} \|f^\perp\|_{L_{x,v}^2}.$$

The term J_{22} is computed explicitly thanks to (4.161) and (4.162) and integrations by parts:

$$\begin{aligned} J_{22} &= -\frac{1}{\varepsilon} \sum_k \sum_{\ell \neq k} \left\langle \partial_{x_\ell} (-\Delta_x)^{-1} u_k[f], \partial_{x_k} u_\ell[f] + \partial_{x_\ell} u_k[f] \right\rangle_{L_x^2} \\ &\quad - \frac{1}{\varepsilon} \sum_k \left\langle \partial_{x_k} (-\Delta_x)^{-1} u_k[f], \partial_{x_k} u_k[f] - \sum_{p \neq k} \partial_{x_p} u_p[f] \right\rangle_{L_x^2} \\ &= -\frac{1}{\varepsilon} \sum_k \sum_{\ell \neq k} \left\langle -\partial_{x_k} \partial_{x_\ell} (-\Delta_x)^{-1} u_k[f], u_\ell[f] \right\rangle_{L_x^2} \\ &\quad - \frac{1}{\varepsilon} \sum_k \sum_{\ell \neq k} \left\langle -\partial_{x_\ell} \partial_{x_\ell} (-\Delta_x)^{-1} u_k[f], u_k[f] \right\rangle_{L_x^2} \\ &\quad - \frac{1}{\varepsilon} \sum_k \left\langle -\partial_{x_k} \partial_{x_k} (-\Delta_x)^{-1} u_k[f], u_k[f] \right\rangle_{L_x^2} \\ &\quad + \frac{1}{\varepsilon} \sum_k \sum_{p \neq k} \left\langle -\partial_{x_p} \partial_{x_k} (-\Delta_x)^{-1} u_k[f], u_p[f] \right\rangle_{L_x^2} \end{aligned}$$

so that

$$J_{22} = -\frac{1}{\varepsilon} \sum_k \left\langle -\Delta_x (-\Delta_x)^{-1} u_k[f], u_k[f] \right\rangle_{L_x^2} = -\frac{1}{\varepsilon} \sum_k \|u_k[f]\|_{L_x^2}^2 = -\frac{1}{\varepsilon} \|u[f]\|_{L_x^2}^2.$$

The term J_{23} is treated thanks to an integration by parts, (4.148) and Cauchy-Schwarz inequality:

$$\begin{aligned} J_{23} &\lesssim \frac{1}{\varepsilon} \|\nabla_x^2 (-\Delta_x)^{-1} u[f]\|_{L_x^2} \|f^\perp\|_{L_{x,v}^2} \\ &\lesssim \frac{1}{\varepsilon} \|u[f]\|_{L_x^2} \|f^\perp\|_{L_{x,v}^2}. \end{aligned}$$

Finally, we have obtained for some constants $\kappa, C > 0$

$$\begin{aligned} &\left\langle \partial_{x_\ell} (-\Delta_x)^{-1} u_k[\Lambda_\varepsilon f], \Theta_{k\ell}[f] \right\rangle_{L_x^2} + \left\langle \partial_{x_\ell} (-\Delta_x)^{-1} u_k[f], \Theta_{k\ell}[\Lambda_\varepsilon f] \right\rangle_{L_x^2} \\ &\leq -\frac{\kappa}{\varepsilon} \|u[f]\|_{L_x^2}^2 + \frac{C}{\varepsilon} \|\rho[f]\|_{L_x^2} \|\theta[f]\|_{L_x^2} + \frac{C}{\varepsilon} \|\rho[f]\|_{L_x^2} \|f^\perp\|_{L_{x,v}^2} \\ &\quad + \frac{C}{\varepsilon} \|\theta[f]\|_{L_x^2}^2 + \frac{C}{\varepsilon^3} \|f^\perp\|_{L_{x,v}^2}^2. \end{aligned} \tag{4.163}$$

Step 4. Mass estimate. We first compute

$$\begin{aligned} \rho[\Lambda_\varepsilon f] &= \frac{1}{\varepsilon^2} \rho[Lf^\perp] - \frac{1}{\varepsilon} \rho[v \cdot \nabla_x f] \\ &= -\frac{1}{\varepsilon} \nabla_x \cdot \int_{\mathbb{R}^d} v f \sqrt{M} dv \\ &= -\frac{1}{\varepsilon} \nabla_x \cdot u[f]. \end{aligned} \tag{4.164}$$

We now estimate the term

$$\left\langle \partial_{x_k}(-\Delta_x)^{-1}\rho[\Lambda_\varepsilon f], u_k[f] \right\rangle_{L_x^2} + \left\langle \partial_{x_k}(-\Delta_x)^{-1}\rho[f], u_k[\Lambda_\varepsilon f] \right\rangle_{L_x^2} =: R_1 + R_2.$$

For R_1 , from the computation (4.164) and (4.148), we have

$$R_1 \lesssim \|(-\Delta_x)^{-1}\rho[\Lambda_\varepsilon f]\|_{H_x^1} \|u[f]\|_{L_x^2} \lesssim \frac{1}{\varepsilon} \|u[f]\|_{L_x^2}^2.$$

From (4.159) we rewrite the term R_2 as

$$\begin{aligned} R_2 &= -\frac{1}{\varepsilon} \left\langle \partial_{x_k}(-\Delta_x)^{-1}\rho[f], \partial_{x_k}\rho[f] \right\rangle_{L_x^2} - \frac{1}{\varepsilon} \frac{2}{3} \left\langle \partial_{x_k}(-\Delta_x)^{-1}\rho[f], \partial_{x_k}\theta[f] \right\rangle_{L_x^2} \\ &\quad - \frac{1}{\varepsilon} \left\langle \partial_{x_k}(-\Delta_x)^{-1}\rho[f], \partial_{x_\ell} \int_{\mathbb{R}^d} v_k v_\ell f^\perp \sqrt{M} dv \right\rangle_{L_x^2} \\ &=: R_{21} + R_{22} + R_{23}. \end{aligned}$$

The first term R_{21} is computed exactly thanks to an integration by parts:

$$R_{21} = -\frac{1}{\varepsilon} \left\langle -\partial_{x_k} \partial_{x_k}(-\Delta_x)^{-1}\rho[f], \rho[f] \right\rangle_{L_x^2} = -\frac{1}{\varepsilon} \|\rho[f]\|_{L_x^2}^2.$$

The second one is estimated thanks to and integration by parts, Cauchy-Schwarz inequality and (4.148):

$$\begin{aligned} R_{22} &\lesssim \frac{1}{\varepsilon} \|\nabla_x^2(-\Delta_x)^{-1}\rho[f]\|_{L_x^2} \|\theta[f]\|_{L_x^2} \\ &\lesssim \frac{1}{\varepsilon} \|\rho[f]\|_{L_x^2} \|\theta[f]\|_{L_x^2}. \end{aligned}$$

Similarly, we obtain

$$\begin{aligned} R_{23} &\lesssim \frac{1}{\varepsilon} \|\nabla_x^2(-\Delta_x)^{-1}\rho[f]\|_{L_x^2} \left\| \int_{\mathbb{R}^d} v_k v_\ell f^\perp \sqrt{M} dv \right\|_{L_x^2} \\ &\lesssim \frac{1}{\varepsilon} \|\rho[f]\|_{L_x^2} \|f^\perp\|_{L_{x,v}^2}. \end{aligned}$$

Gathering the previous estimates, we obtain for some constants $\kappa, C > 0$

$$\begin{aligned} &\left\langle \partial_{x_k}(-\Delta_x)^{-1}\rho[\Lambda_\varepsilon f], u_k[f] \right\rangle_{L_x^2} + \left\langle \partial_{x_k}(-\Delta_x)^{-1}\rho[f], u_k[\Lambda_\varepsilon f] \right\rangle_{L_x^2} \\ &\leq -\frac{\kappa}{\varepsilon} \|\rho[f]\|_{L_x^2}^2 + \frac{C}{\varepsilon} \|u[f]\|_{L_x^2}^2 + \frac{C}{\varepsilon} \|\theta[f]\|_{L_x^2}^2 + \frac{C}{\varepsilon} \|f^\perp\|_{L_{x,v}^2}^2. \end{aligned} \quad (4.165)$$

Step 5. Conclusion. We introduce the following inner product on $L_{x,v}^2$:

$$\begin{aligned} &\langle\langle f, g \rangle\rangle_{L_{x,v}^2} \\ &:= \langle f, g \rangle_{L_{x,v}^2} + \eta_1 \varepsilon \left\langle \partial_{x_k}(-\Delta_x)^{-1}\theta[f], \psi_k[g] \right\rangle_{L_x^2} + \eta_1 \varepsilon \left\langle \partial_{x_k}(-\Delta_x)^{-1}\theta[g], \psi_k[f] \right\rangle_{L_x^2} \\ &\quad + \eta_2 \varepsilon \left\langle \partial_{x_\ell}(-\Delta_x)^{-1}u_k[f], \Theta_{k\ell}[g] \right\rangle_{L_x^2} + \eta_2 \varepsilon \left\langle \partial_{x_\ell}(-\Delta_x)^{-1}u_k[g], \Theta_{k\ell}[f] \right\rangle_{L_x^2} \\ &\quad + \eta_3 \varepsilon \left\langle \partial_{x_k}(-\Delta_x)^{-1}\rho[f], u_k[g] \right\rangle_{L_x^2} + \eta_3 \varepsilon \left\langle \partial_{x_k}(-\Delta_x)^{-1}\rho[g], u_k[f] \right\rangle_{L_x^2} \end{aligned} \quad (4.166)$$

with constants $0 < \eta_3 \ll \eta_2 \ll \eta_1 \ll 1$ to be chosen below, and denote by $\|g\|_{L_{x,v}^2}^2 = \langle\langle g, g \rangle\rangle_{L_{x,v}^2}$ the associated norm. We observe that

$$\|g\|_{L_{x,v}^2} \lesssim \|g\|_{L_{x,v}^2} \lesssim \|g\|_{L_{x,v}^2}$$

where the multiplicative constants are uniform in $\varepsilon \in (0, 1]$. The norms $\|\cdot\|_{L_{x,v}^2}$ and $\|g\|_{L_{x,v}^2}$ are thus equivalent independently of $\varepsilon \in (0, 1]$. Gathering estimates (4.149)–(4.155)–(4.163)–(4.165) and using Young’s inequality, we obtain:

$$\begin{aligned} \langle\langle \Lambda_\varepsilon f, f \rangle\rangle_{L_{x,v}^2} &\leq -\frac{1}{\varepsilon^2} \left(\frac{\sigma_L}{2} - C\eta_1 - C\eta_2 - C\eta_3 \right) \|f^\perp\|_{L_x^2(H_{v,*}^1)}^2 \\ &\quad - \left(\frac{\kappa\eta_1}{2} - C\eta_2 - C\eta_3 \right) \|\theta[f]\|_{L_x^2}^2 \\ &\quad - \left(\kappa\eta_2 - C\eta_1^2 - C\eta_3 \right) \|u[f]\|_{L_x^2}^2 \\ &\quad - \left(\kappa\eta_3 - C\eta_2^2 - C\frac{\eta_2^2}{\eta_1} \right) \|\rho[f]\|_{L_x^2}^2. \end{aligned}$$

By choosing $\eta_1 := \eta$, $\eta_2 := \eta^{\frac{3}{2}}$, $\eta_3 := \eta^{\frac{7}{4}}$ with $\eta > 0$ small enough and recalling that $\|f\|_{L_{x,v}^2}^2 = \|f^\perp\|_{L_{x,v}^2}^2 + \|\rho[f]\|_{L_x^2}^2 + \|u[f]\|_{L_x^2}^2 + \|\theta[f]\|_{L_x^2}^2$, we thus obtain

$$\langle\langle \Lambda_\varepsilon f, f \rangle\rangle_{L_{x,v}^2} \leq -\sigma \|f\|_{L_{x,v}^2}^2 - \frac{\kappa}{\varepsilon^2} \|f^\perp\|_{L_x^2(H_{v,*}^1)}^2$$

for some constants $\sigma, \kappa > 0$, which completes the proof. \square

8 Appendix B. Proofs of Lemmas 4.29 and 4.30

Proof of Lemma 4.29. Let us first recall that $\mathcal{B}_\varepsilon = \varepsilon^{-2}\mathcal{B} - \varepsilon^{-1}v \cdot \nabla_x$ with \mathcal{B} and $v \cdot \nabla_x$ that are respectively self-adjoint and skew-adjoint operators in $L_{x,v}^2$. Proving that $S_{\mathcal{B}_\varepsilon}$ regularizes from \mathcal{Y}'_1 to \mathcal{X} and from $(\mathcal{Z}'_1)^\varepsilon$ to \mathcal{X} is similar to prove that $S_{\mathcal{B}_\varepsilon}$ regularizes from \mathcal{X} to \mathcal{Y}_1 and from \mathcal{X} to $\mathcal{Z}_1^\varepsilon$ with same rates. We will only focus on the proof of (4.126) and explain the adaptation to make to prove (4.127) in the first step of the proof.

Moreover, to prove (4.126), it is sufficient to prove that for any $\alpha \in \mathbb{R}$ and $\langle v \rangle^\alpha f_{\text{in}} \in L_{x,v}^2$, one has: For any $t \in (0, \varepsilon^2]$,

$$\|\langle v \rangle^\alpha S_{\mathcal{B}_\varepsilon}(t) f_{\text{in}}\|_{L_x^2(H_{v,*}^1)} \lesssim \frac{\varepsilon}{\sqrt{t}} \|\langle v \rangle^\alpha f_{\text{in}}\|_{L_{x,v}^2} \quad (4.167)$$

and

$$\begin{aligned} \|\langle v \rangle^\alpha S_{\mathcal{B}_\varepsilon}(t) f_{\text{in}}\|_{L_{x,v}^2} + \|\langle v \rangle^\alpha S_{\mathcal{B}_\varepsilon}(t) f_{\text{in}}\|_{L_x^2(H_{v,*}^1)} + \varepsilon \|\widetilde{\nabla}_x(\langle v \rangle^\alpha S_{\mathcal{B}_\varepsilon}(t) f_{\text{in}})\|_{L_{x,v}^2} \\ \lesssim \frac{\varepsilon^3}{t^{3/2}} \|\langle v \rangle^\alpha f_{\text{in}}\|_{L_{x,v}^2}. \end{aligned} \quad (4.168)$$

Indeed, since ∇_x commutes with \mathcal{B}_ε , from estimate (4.167) (resp. (4.168)), we already obtain the first (resp. second) estimate of (4.126) for $t \in (0, \varepsilon^2]$. Then, fix $\sigma \in (0, \sigma_1)$. We obtain the estimates given in (4.126) for all $t > 0$ by using the exponential decay of $S_{\mathcal{B}_\varepsilon}$ in \mathcal{X} given in Lemma 4.28. More precisely, using that for $t \geq \varepsilon^2$, $S_{\mathcal{B}_\varepsilon}(t) = S_{\mathcal{B}_\varepsilon}(\varepsilon^2)S_{\mathcal{B}_\varepsilon}(t - \varepsilon^2)$, we obtain that for $\sigma' \in (\sigma, \sigma_1)$ and for any $t > 0$,

$$\|S_{\mathcal{B}_\varepsilon}(t)\|_{\mathcal{X} \rightarrow \mathcal{Y}_1} \lesssim \frac{\varepsilon}{\min(\varepsilon, \sqrt{t})} e^{-\sigma' t / \varepsilon^2} \quad \text{and} \quad \|S_{\mathcal{B}_\varepsilon}(t)\|_{\mathcal{X} \rightarrow \mathcal{Y}_1^\varepsilon} \lesssim \frac{\varepsilon^3}{\min(\varepsilon^3, t^{3/2})} e^{-\sigma' t / \varepsilon^2}.$$

It implies the wanted conclusion using that $\sigma' > \sigma$.

For the remainder of the proof, we let $\alpha \in \mathbb{R}$ and f_{in} be such that $\langle v \rangle^\alpha f_{\text{in}} \in L_{x,v}^2$, and consider the solution $f(t) = S_{\mathcal{B}_\varepsilon}(t)f_{\text{in}}$ to the equation $\partial_t f = \mathcal{B}_\varepsilon f$ with initial data f_{in} , and we shall prove (4.167) and (4.168).

Step 1. Define the functional

$$\begin{aligned} \mathcal{E}_\varepsilon(t) := & \|\langle v \rangle^\alpha f\|_{L_{x,v}^2}^2 + \alpha_1 \frac{t}{\varepsilon^2} \left(K \|\langle v \rangle^\alpha \langle v \rangle^{\frac{\gamma}{2}+1} f\|_{L_{x,v}^2}^2 + \|\langle v \rangle^\alpha \widetilde{\nabla}_v f\|_{L_{x,v}^2}^2 \right) \\ & + \varepsilon \alpha_2 \left(\frac{t}{\varepsilon^2} \right)^2 \left\langle \langle v \rangle^\alpha \widetilde{\nabla}_v f, \langle v \rangle^\alpha \widetilde{\nabla}_x f \right\rangle_{L_{x,v}^2} \\ & + \varepsilon^2 \alpha_3 \left(\frac{t}{\varepsilon^2} \right)^3 \left(\|\langle v \rangle^\alpha \widetilde{\nabla}_x f\|_{L_{x,v}^2}^2 + K \|\langle v \rangle^\alpha \langle v \rangle^{\frac{\gamma}{2}} \nabla_x f\|_{L_{x,v}^2}^2 \right), \end{aligned} \quad (4.169)$$

where $\alpha_1, \alpha_2, \alpha_3, K > 0$ are positive constants such that $0 < \alpha_3 \ll \alpha_2 \ll \alpha_1 \ll 1$ and $\alpha_2 \leq \sqrt{\alpha_1 \alpha_3}$. Notice that in order to prove (4.127), one just has to change the sign in front of the term which mixes derivatives in x and v in the definition of the functional \mathcal{E}_ε . It will allow to conclude that (4.126) holds for the adjoint of \mathcal{B}_ε instead of \mathcal{B}_ε and thus imply (4.127). The constants α_i will be chosen small enough in Step 6, and K will be chosen large enough in Step 3 and Step 5. We remark that

$$\|\langle v \rangle^\alpha f\|_{L_x^2(H_{v,*}^1)}^2 \lesssim K \|\langle v \rangle^\alpha \langle v \rangle^{\frac{\gamma}{2}+1} f\|_{L_{x,v}^2}^2 + \|\langle v \rangle^\alpha \widetilde{\nabla}_v f\|_{L_{x,v}^2}^2 \lesssim \|\langle v \rangle^\alpha f\|_{L_x^2(H_{v,*}^1)}^2$$

and

$$\|\langle v \rangle^\alpha \widetilde{\nabla}_x f\|_{L_{x,v}^2}^2 \lesssim \|\langle v \rangle^\alpha \widetilde{\nabla}_x f\|_{L_{x,v}^2}^2 + K \|\langle v \rangle^\alpha \langle v \rangle^{\frac{\gamma}{2}} \nabla_x f\|_{L_{x,v}^2}^2 \lesssim \|\langle v \rangle^\alpha \widetilde{\nabla}_x f\|_{L_{x,v}^2}^2.$$

Therefore, we can already observe that for any $t \in (0, \varepsilon^2]$, one has the following lower bounds

$$\frac{t}{\varepsilon^2} \|\langle v \rangle^\alpha f\|_{L_x^2(H_{v,*}^1)}^2 \lesssim \mathcal{E}_\varepsilon(t)$$

and

$$\left(\frac{t}{\varepsilon^2} \right)^3 \left(\|\langle v \rangle^\alpha f\|_{L_{x,v}^2}^2 + \|\langle v \rangle^\alpha f\|_{L_x^2(H_{v,*}^1)}^2 + \varepsilon^2 \|\langle v \rangle^\alpha \widetilde{\nabla}_x f\|_{L_{x,v}^2}^2 \right) \lesssim \mathcal{E}_\varepsilon(t).$$

Therefore, in order to prove (4.167) and (4.168), it is sufficient to prove that $\frac{d}{dt}\mathcal{E}_\varepsilon(t) \leq 0$ for all $t \in (0, \varepsilon^2]$. We then compute

$$\begin{aligned} \frac{d}{dt}\mathcal{E}_\varepsilon(t) &= \frac{d}{dt}\|\langle v \rangle^\alpha f\|_X^2 + \frac{\alpha_1}{\varepsilon^2} \left(K\|\langle v \rangle^\alpha \langle v \rangle^{\frac{\gamma}{2}+1} f\|_{L_{x,v}^2}^2 + \|\langle v \rangle^\alpha \widetilde{\nabla}_v f\|_{L_{x,v}^2}^2 \right) \\ &\quad + \frac{\alpha_1}{\varepsilon^2} t \frac{d}{dt} \left(K\|\langle v \rangle^\alpha \langle v \rangle^{\frac{\gamma}{2}+1} f\|_{L_{x,v}^2}^2 + \|\langle v \rangle^\alpha \widetilde{\nabla}_v f\|_{L_{x,v}^2}^2 \right) \\ &\quad + 2\frac{\alpha_2}{\varepsilon^3} t \left\langle \langle v \rangle^\alpha \widetilde{\nabla}_v f, \langle v \rangle^\alpha \widetilde{\nabla}_x f \right\rangle_{L_{x,v}^2} + \frac{\alpha_2}{\varepsilon^3} t^2 \frac{d}{dt} \left\langle \langle v \rangle^\alpha \widetilde{\nabla}_v f, \langle v \rangle^\alpha \widetilde{\nabla}_x f \right\rangle_{L_{x,v}^2} \\ &\quad + 3\frac{\alpha_3}{\varepsilon^4} t^2 \left(\|\langle v \rangle^\alpha \widetilde{\nabla}_x f\|_{L_{x,v}^2}^2 + K\|\langle v \rangle^\alpha \langle v \rangle^{\frac{\gamma}{2}} \nabla_x f\|_{L_{x,v}^2}^2 \right) \\ &\quad + \frac{\alpha_3}{\varepsilon^4} t^3 \frac{d}{dt} \left(\|\langle v \rangle^\alpha \widetilde{\nabla}_x f\|_{L_{x,v}^2}^2 + K\|\langle v \rangle^\alpha \langle v \rangle^{\frac{\gamma}{2}} \nabla_x f\|_{L_{x,v}^2}^2 \right) \end{aligned}$$

and we shall estimate each term separately in the sequel.

Step 2. From Lemma 4.28, we already have

$$\frac{1}{2} \frac{d}{dt} \|\langle v \rangle^\alpha f\|_{L_{x,v}^2}^2 \leq -\frac{\kappa}{\varepsilon^2} \|\langle v \rangle^\alpha f\|_{L_x^2(H_{v,*}^1)}^2 \quad (4.170)$$

for some constant $\kappa > 0$.

Step 3. We prove in this step that, choosing $K > 0$ large enough, we have

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \left\{ K\|\langle v \rangle^\alpha \langle v \rangle^{\frac{\gamma}{2}+1} f\|_{L_{x,v}^2}^2 + \|\langle v \rangle^\alpha \widetilde{\nabla}_v f\|_{L_{x,v}^2}^2 \right\} \\ &\leq -\frac{\kappa}{2\varepsilon^2} \|\langle v \rangle^\alpha f\|_{L_x^2(H_{v,*}^2)}^2 + \frac{C}{\varepsilon} \|\langle v \rangle^\alpha f\|_{L_x^2(H_{v,*}^1)} \|\langle v \rangle^\alpha \widetilde{\nabla}_x f\|_X, \end{aligned} \quad (4.171)$$

for some constants $\kappa, C > 0$. From Lemma 4.28, we already have

$$\frac{1}{2} \frac{d}{dt} \|\langle v \rangle^\alpha \langle v \rangle^{\frac{\gamma}{2}+1} f\|_{L_{x,v}^2}^2 \leq -\frac{\kappa}{\varepsilon^2} \|\langle v \rangle^\alpha \langle v \rangle^{\frac{\gamma}{2}+1} f\|_{L_x^2(H_{v,*}^1)}^2$$

for some $\kappa > 0$. Moreover, we have

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\langle v \rangle^\alpha \widetilde{\nabla}_v f\|_{L_{x,v}^2}^2 &= \left\langle \langle v \rangle^{2\alpha} \widetilde{\nabla}_{v_i} f, \widetilde{\nabla}_{v_i} (\mathcal{B}_\varepsilon f) \right\rangle_{L_{x,v}^2} \\ &= \left\langle \langle v \rangle^{2\alpha} \widetilde{\nabla}_{v_i} f, \mathcal{B}_\varepsilon (\widetilde{\nabla}_{v_i} f) \right\rangle_{L_{x,v}^2} + \left\langle \langle v \rangle^{2\alpha} \widetilde{\nabla}_{v_i} f, [\widetilde{\nabla}_{v_i}, \mathcal{B}_\varepsilon] f \right\rangle_{L_{x,v}^2}. \end{aligned}$$

Thanks to Lemma 4.28, there holds

$$\left\langle \langle v \rangle^{2\alpha} \widetilde{\nabla}_{v_i} f, \mathcal{B}_\varepsilon (\widetilde{\nabla}_{v_i} f) \right\rangle_{L_{x,v}^2} \leq -\frac{\kappa}{\varepsilon^2} \|\langle v \rangle^\alpha \widetilde{\nabla}_v f\|_{L_x^2(H_{v,*}^1)}^2,$$

for some constant $\kappa > 0$. From Lemma 4.26-(i), we have

$$\begin{aligned}
 & \left\langle \langle v \rangle^{2\alpha} \widetilde{\nabla}_{v_i} f, [\widetilde{\nabla}_{v_i}, \mathcal{B}_\varepsilon] f \right\rangle_{L_{x,v}^2} \\
 &= -\frac{1}{\varepsilon^2} \left\langle \widetilde{\nabla}_{v_j} (\langle v \rangle^{2\alpha} \widetilde{\nabla}_{v_i} f), [\widetilde{\nabla}_{v_i}, \widetilde{\nabla}_{v_j}] f \right\rangle_{L_{x,v}^2} - \frac{1}{\varepsilon^2} \left\langle \widetilde{\nabla}_{v_j}^* (\langle v \rangle^{2\alpha} \widetilde{\nabla}_{v_i} f), [\widetilde{\nabla}_{v_i}, \widetilde{\nabla}_{v_j}^*] f \right\rangle_{L_{x,v}^2} \\
 &\quad - \frac{1}{\varepsilon^2} \left\langle \langle v \rangle^{2\alpha} \widetilde{\nabla}_{v_i} f, (\widetilde{\nabla}_{v_i} m^2) f \right\rangle_{L_{x,v}^2} - \frac{1}{\varepsilon^2} \left\langle \langle v \rangle^{2\alpha} \widetilde{\nabla}_{v_i} f, [[\widetilde{\nabla}_{v_i}, \widetilde{\nabla}_{v_j}^*], \widetilde{\nabla}_{v_j}] f \right\rangle_{L_{x,v}^2} \\
 &\quad - \frac{1}{\varepsilon} \left\langle \langle v \rangle^{2\alpha} \widetilde{\nabla}_{v_i} f, \widetilde{\nabla}_{x_i} f \right\rangle_{L_{x,v}^2} \\
 &=: -\frac{1}{\varepsilon^2} (T_1 + T_2 + T_3 + T_4) - \frac{1}{\varepsilon} \left\langle \langle v \rangle^{2\alpha} \widetilde{\nabla}_{v_i} f, \widetilde{\nabla}_{x_i} f \right\rangle_{L_{x,v}^2}.
 \end{aligned}$$

Writing $\widetilde{\nabla}_{v_j} (\langle v \rangle^{2\alpha} \widetilde{\nabla}_{v_i} f) = \langle v \rangle^\alpha \widetilde{\nabla}_{v_j} (\langle v \rangle^\alpha \widetilde{\nabla}_{v_i} f) + (\widetilde{\nabla}_{v_j} \langle v \rangle^\alpha) \langle v \rangle^\alpha \widetilde{\nabla}_{v_i} f$, using that $|\widetilde{\nabla}_{v_j} \langle v \rangle^\alpha| \lesssim \langle v \rangle^{\frac{\gamma}{2}-1} \langle v \rangle^\alpha$ and thanks to Lemma 4.10-(iii), we obtain

$$T_1 \leq \frac{\kappa}{8} \|\langle v \rangle^\alpha \widetilde{\nabla}_v f\|_{L_x^2(H_{v,*}^1)}^2 + C \|\langle v \rangle^\alpha \langle v \rangle^{\gamma+1} \nabla_v f\|_{L_{x,v}^2}^2.$$

In a similar way, observing now that

$$\widetilde{\nabla}_{v_j}^* (\langle v \rangle^{2\alpha} \widetilde{\nabla}_{v_i} f) = -\langle v \rangle^\alpha \widetilde{\nabla}_{v_j} (\langle v \rangle^\alpha \widetilde{\nabla}_{v_i} f) - [(\partial_{v_\ell} B_{j\ell}) \langle v \rangle^\alpha + (\widetilde{\nabla}_{v_j} \langle v \rangle^\alpha)] \langle v \rangle^\alpha \widetilde{\nabla}_{v_i} f$$

and using Lemma 4.10-(iv), we get

$$T_2 \leq \frac{\kappa}{8} \|\langle v \rangle^\alpha \widetilde{\nabla}_v f\|_{L_x^2(H_{v,*}^1)}^2 + C \|\langle v \rangle^\alpha \langle v \rangle^\gamma f\|_{L_{x,v}^2}^2 + C \|\langle v \rangle^\alpha \langle v \rangle^{\gamma+1} \nabla_v f\|_{L_{x,v}^2}^2.$$

Using that $|\widetilde{\nabla}_{v_i} m^2| \lesssim \langle v \rangle^{\frac{3\gamma}{2}+2}$ from Lemma 4.25, we also get

$$T_3 \leq \frac{\kappa}{8} \|\langle v \rangle^\alpha \widetilde{\nabla}_v f\|_{L_x^2(H_{v,*}^1)}^2 + C \|\langle v \rangle^\alpha \langle v \rangle^{\gamma+1} f\|_{L_{x,v}^2}^2.$$

For the term T_4 , we use Lemma 4.10-(vii) to obtain

$$T_4 \leq \frac{\kappa}{8} \|\langle v \rangle^\alpha \widetilde{\nabla}_v f\|_{L_x^2(H_{v,*}^1)}^2 + C \|\langle v \rangle^\alpha \langle v \rangle^{\gamma-1} f\|_{L_{x,v}^2}^2 + C \|\langle v \rangle^\alpha \langle v \rangle^\gamma \nabla_v f\|_{L_{x,v}^2}^2.$$

Hence, we obtain

$$\begin{aligned}
 \frac{1}{2} \frac{d}{dt} \|\langle v \rangle^\alpha \widetilde{\nabla}_v f\|_{L_{x,v}^2}^2 &\leq -\frac{\kappa}{2\varepsilon^2} \|\langle v \rangle^\alpha \widetilde{\nabla}_v f\|_{L_x^2(H_{v,*}^1)}^2 + \frac{C}{\varepsilon^2} \|\langle v \rangle^\alpha \langle v \rangle^{\gamma+1} f\|_{L_{x,v}^2}^2 \\
 &\quad + \frac{C}{\varepsilon^2} \|\langle v \rangle^\alpha \langle v \rangle^{\gamma+1} \nabla_v f\|_{L_{x,v}^2}^2 + \frac{C}{\varepsilon} \|\langle v \rangle^\alpha \widetilde{\nabla}_v f\|_{L_{x,v}^2} \|\langle v \rangle^\alpha \widetilde{\nabla}_x f\|_{L_{x,v}^2}.
 \end{aligned}$$

We conclude to (4.171) by gathering previous estimates, observing that $\|\langle v \rangle^\alpha \langle v \rangle^{\gamma+1} f\|_{L_{x,v}^2} + \|\langle v \rangle^\alpha \langle v \rangle^{\gamma+1} \nabla_v f\|_{L_{x,v}^2} \lesssim \|\langle v \rangle^\alpha \langle v \rangle^{\frac{\gamma}{2}+1} f\|_{L_x^2(H_{v,*}^1)}$ and taking $K > 0$ large enough.

Step 4. In this step, we show that

$$\frac{d}{dt} \langle \langle v \rangle^\alpha \widetilde{\nabla}_v f, \langle v \rangle^\alpha \widetilde{\nabla}_x f \rangle_{L_{x,v}^2} \leq -\frac{1}{\varepsilon} \|\langle v \rangle^\alpha \widetilde{\nabla}_x f\|_{L_{x,v}^2}^2 + \frac{C}{\varepsilon^2} \|\langle v \rangle^\alpha f\|_{L_x^2(H_{v,*}^2)} \|\langle v \rangle^\alpha \widetilde{\nabla}_x f\|_{L_x^2(H_{v,*}^1)}, \quad (4.172)$$

for some constant $C > 0$. We compute, using (4.123),

$$\begin{aligned} \frac{d}{dt} \langle \langle v \rangle^\alpha \widetilde{\nabla}_v f, \langle v \rangle^\alpha \widetilde{\nabla}_x f \rangle_{L_{x,v}^2} &= \left\langle \langle v \rangle^{2\alpha} \widetilde{\nabla}_{v_i} f, \widetilde{\nabla}_{x_i} \left\{ \frac{1}{\varepsilon^2} \mathcal{B} f - \frac{1}{\varepsilon} v \cdot \nabla_x f \right\} \right\rangle_{L_{x,v}^2} \\ &\quad + \left\langle \langle v \rangle^{2\alpha} \widetilde{\nabla}_{x_i} f, \widetilde{\nabla}_{v_i} \left\{ \frac{1}{\varepsilon^2} \mathcal{B} f - \frac{1}{\varepsilon} v \cdot \nabla_x f \right\} \right\rangle_{L_{x,v}^2} \\ &= -\frac{1}{\varepsilon^2} \langle \widetilde{\nabla}_{x_i} (\langle v \rangle^{2\alpha} \widetilde{\nabla}_{v_i} f), \mathcal{B} f \rangle_{L_{x,v}^2} + \frac{1}{\varepsilon^2} \langle \widetilde{\nabla}_{v_i}^* (\langle v \rangle^{2\alpha} \widetilde{\nabla}_{x_i} f), \mathcal{B} f \rangle_{L_{x,v}^2} - \frac{1}{\varepsilon} \|\langle v \rangle^\alpha \widetilde{\nabla}_x f\|_{L_{x,v}^2}^2, \end{aligned}$$

where we have used Lemma 4.10-(i). We hence conclude to (4.172) by using the fact that $\|\langle v \rangle^\alpha \mathcal{B} f\|_{L_{x,v}^2} \lesssim \|\langle v \rangle^\alpha f\|_{L_x^2(H_{v,*}^2)}$ (see (4.72) for a similar estimate) together with estimates (4.68) and (4.71).

Step 5. We prove in this step that, choosing $K > 0$ large enough, we have

$$\frac{1}{2} \frac{d}{dt} \left\{ \|\langle v \rangle^\alpha \widetilde{\nabla}_x f\|_{L_{x,v}^2}^2 + K \|\langle v \rangle^\alpha \langle v \rangle^{\frac{\gamma}{2}} \nabla_x f\|_{L_{x,v}^2}^2 \right\} \leq -\frac{\kappa}{2\varepsilon^2} \|\langle v \rangle^\alpha \widetilde{\nabla}_x f\|_{L_x^2(H_{v,*}^1)}^2, \quad (4.173)$$

for some constants $\kappa, C > 0$. We first remark that, since ∇_x commutes with \mathcal{B}_ε , we already have from Lemma 4.28 that

$$\frac{1}{2} \frac{d}{dt} \|\langle v \rangle^\alpha \langle v \rangle^{\frac{\gamma}{2}} \nabla_x f\|_{L_{x,v}^2}^2 \leq -\frac{\kappa}{\varepsilon^2} \|\langle v \rangle^\alpha \langle v \rangle^{\frac{\gamma}{2}} \nabla_x f\|_{L_x^2(H_{v,*}^1)}^2$$

for some constant $\kappa > 0$.

We then write

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\langle v \rangle^\alpha \widetilde{\nabla}_x f\|_{L_{x,v}^2}^2 &= \left\langle \langle v \rangle^{2\alpha} \widetilde{\nabla}_{x_i} f, \widetilde{\nabla}_{x_i} (\mathcal{B}_\varepsilon f) \right\rangle_{L_{x,v}^2} \\ &= \left\langle \langle v \rangle^{2\alpha} \widetilde{\nabla}_{x_i} f, \mathcal{B}_\varepsilon (\widetilde{\nabla}_{x_i} f) \right\rangle_{L_{x,v}^2} + \left\langle \langle v \rangle^{2\alpha} \widetilde{\nabla}_{x_i} f, [\widetilde{\nabla}_{x_i}, \mathcal{B}_\varepsilon] f \right\rangle_{L_{x,v}^2}. \end{aligned}$$

Thanks to Lemma 4.28, there holds

$$\left\langle \langle v \rangle^{2\alpha} \widetilde{\nabla}_{x_i} f, \mathcal{B}_\varepsilon (\widetilde{\nabla}_{x_i} f) \right\rangle_{L_{x,v}^2} \leq -\frac{\kappa}{\varepsilon^2} \|\langle v \rangle^\alpha \widetilde{\nabla}_x f\|_{L_x^2(H_{v,*}^1)}^2,$$

for some constant $\kappa > 0$. From Lemma 4.26-(ii), we have

$$\begin{aligned} &\left\langle \langle v \rangle^{2\alpha} \widetilde{\nabla}_{x_i} f, [\widetilde{\nabla}_{x_i}, \mathcal{B}_\varepsilon] f \right\rangle_{L_{x,v}^2} \\ &= -\frac{1}{\varepsilon^2} \left\langle \widetilde{\nabla}_{v_j} (\langle v \rangle^{2\alpha} \widetilde{\nabla}_{x_i} f), [\widetilde{\nabla}_{x_i}, \widetilde{\nabla}_{v_j}] f \right\rangle_{L_{x,v}^2} - \frac{1}{\varepsilon^2} \left\langle \widetilde{\nabla}_{v_j}^* (\langle v \rangle^{2\alpha} \widetilde{\nabla}_{x_i} f), [\widetilde{\nabla}_{x_i}, \widetilde{\nabla}_{v_j}^*] f \right\rangle_{L_{x,v}^2} \\ &\quad - \frac{1}{\varepsilon^2} \left\langle \langle v \rangle^{2\alpha} \widetilde{\nabla}_{x_i} f, [[\widetilde{\nabla}_{x_i}, \widetilde{\nabla}_{v_j}^*], \widetilde{\nabla}_{v_j}] f \right\rangle_{L_{x,v}^2} \\ &=: -\frac{1}{\varepsilon^2} (R_1 + R_2 + R_3). \end{aligned}$$

Writing $\widetilde{\nabla}_{v_j}(\langle v \rangle^{2\alpha} \widetilde{\nabla}_{x_i} f) = \langle v \rangle^\alpha \widetilde{\nabla}_{v_j}(\langle v \rangle^\alpha \widetilde{\nabla}_{x_i} f) + (\widetilde{\nabla}_{v_j} \langle v \rangle^\alpha) \langle v \rangle^\alpha \widetilde{\nabla}_{x_i} f$, using the fact that $|\widetilde{\nabla}_{v_j} \langle v \rangle^\alpha| \lesssim \langle v \rangle^{\frac{\gamma}{2}-1} \langle v \rangle^\alpha$ and thanks to Lemma 4.10-(v), we obtain

$$R_1 \leq \frac{\kappa}{6} \|\langle v \rangle^\alpha \widetilde{\nabla}_x f\|_{L_x^2(H_{v,*}^1)}^2 + C \|\langle v \rangle^\alpha \langle v \rangle^{\gamma+1} \nabla_x f\|_{L_{x,v}^2}^2.$$

In a similar way, observing now that

$$\widetilde{\nabla}_{v_j}^* (\langle v \rangle^{2\alpha} \widetilde{\nabla}_{x_i} f) = -\langle v \rangle^\alpha \widetilde{\nabla}_{v_j} (\langle v \rangle^\alpha \widetilde{\nabla}_{x_i} f) - [(\partial_{v_\ell} B_{j\ell}) \langle v \rangle^\alpha + (\widetilde{\nabla}_{v_j} \langle v \rangle^\alpha)] \langle v \rangle^\alpha \widetilde{\nabla}_{x_i} f,$$

we get

$$R_2 \leq \frac{\kappa}{6} \|\langle v \rangle^\alpha \widetilde{\nabla}_x f\|_{L_x^2(H_{v,*}^1)}^2 + C \|\langle v \rangle^\alpha \langle v \rangle^{\gamma+1} \nabla_x f\|_{L_{x,v}^2}^2.$$

For the term R_3 , we use Lemma 4.10-(viii) to obtain

$$R_3 \leq \frac{\kappa}{6} \|\langle v \rangle^\alpha \widetilde{\nabla}_x f\|_{L_x^2(H_{v,*}^1)}^2 + C \|\langle v \rangle^\alpha \langle v \rangle^\gamma \nabla_x f\|_{L_{x,v}^2}^2.$$

Hence, we obtain

$$\frac{1}{2} \frac{d}{dt} \|\langle v \rangle^\alpha \widetilde{\nabla}_x f\|_{L_{x,v}^2}^2 \leq -\frac{\kappa}{2\varepsilon^2} \|\langle v \rangle^\alpha \widetilde{\nabla}_x f\|_{L_x^2(H_{v,*}^1)}^2 + \frac{C}{\varepsilon^2} \|\langle v \rangle^\alpha \langle v \rangle^{\gamma+1} \nabla_x f\|_{L_{x,v}^2}^2.$$

We conclude to (4.173) by gathering previous estimates as well as noticing the fact that $\|\langle v \rangle^\alpha \langle v \rangle^{\gamma+1} \nabla_x f\|_{L_{x,v}^2} \lesssim \|\langle v \rangle^\alpha \langle v \rangle^{\frac{\gamma}{2}} \nabla_x f\|_{L_x^2(H_{v,*}^1)}$ and taking $K > 0$ large enough.

Step 6. Proof of (4.167) and (4.168). Gathering (4.170)–(4.171)–(4.172)–(4.173), we obtain

$$\begin{aligned} \frac{d}{dt} \mathcal{E}_\varepsilon(t) &\leq -\frac{2\kappa}{\varepsilon^2} \|\langle v \rangle^\alpha f\|_{L_x^2(H_{v,*}^1)}^2 + \frac{\alpha_1 C}{\varepsilon^2} \|\langle v \rangle^\alpha f\|_{L_x^2(H_{v,*}^1)}^2 \\ &\quad + \frac{\alpha_1}{\varepsilon^2} t \left(-\frac{\kappa}{\varepsilon^2} \|\langle v \rangle^\alpha f\|_{L_x^2(H_{v,*}^2)}^2 + \frac{C}{\varepsilon} \|\langle v \rangle^\alpha f\|_{L_x^2(H_{v,*}^1)} \|\langle v \rangle^\alpha \widetilde{\nabla}_x f\|_X \right) \\ &\quad + 2 \frac{\alpha_2}{\varepsilon^3} t \langle \langle v \rangle^\alpha \widetilde{\nabla}_v f, \langle v \rangle^\alpha \widetilde{\nabla}_x f \rangle_{L_{x,v}^2} \\ &\quad + \frac{\alpha_2}{\varepsilon^3} t^2 \left(-\frac{1}{\varepsilon} \|\langle v \rangle^\alpha \widetilde{\nabla}_x f\|_{L_{x,v}^2}^2 + \frac{C}{\varepsilon^2} \|\langle v \rangle^\alpha f\|_{L_x^2(H_{v,*}^2)} \|\langle v \rangle^\alpha \widetilde{\nabla}_x f\|_{L_x^2(H_{v,*}^1)} \right) \\ &\quad + \frac{\alpha_3 C}{\varepsilon^4} t^2 \|\langle v \rangle^\alpha \widetilde{\nabla}_x f\|_{L_{x,v}^2}^2 - \frac{\alpha_3}{\varepsilon^4} t^3 \frac{\kappa}{\varepsilon^2} \|\langle v \rangle^\alpha \widetilde{\nabla}_x f\|_{L_x^2(H_{v,*}^1)}^2. \end{aligned}$$

Using Young's inequality, we have

$$\begin{aligned} C\alpha_1 \frac{t}{\varepsilon^3} \|\langle v \rangle^\alpha f\|_{L_x^2(H_{v,*}^1)} \|\langle v \rangle^\alpha \widetilde{\nabla}_x f\|_{L_{x,v}^2} &\leq \frac{\alpha_2}{4} \frac{t^2}{\varepsilon^4} \|\langle v \rangle^\alpha \widetilde{\nabla}_x f\|_{L_{x,v}^2}^2 + C \frac{\alpha_1^2}{\alpha_2} \frac{1}{\varepsilon^2} \|\langle v \rangle^\alpha f\|_{L_x^2(H_{v,*}^1)}^2 \\ 2 \frac{\alpha_2}{\varepsilon^3} t \langle \langle v \rangle^\alpha \widetilde{\nabla}_v f, \langle v \rangle^\alpha \widetilde{\nabla}_x f \rangle_{L_{x,v}^2} &\leq \frac{\alpha_2}{4} \frac{t^2}{\varepsilon^4} \|\langle v \rangle^\alpha \widetilde{\nabla}_x f\|_{L_{x,v}^2}^2 + C \frac{\alpha_2}{\varepsilon^2} \|\langle v \rangle^\alpha f\|_{L_x^2(H_{v,*}^1)}^2 \\ C\alpha_2 \frac{t^2}{\varepsilon^5} \|\langle v \rangle^\alpha f\|_{L_x^2(H_{v,*}^2)} \|\langle v \rangle^\alpha \widetilde{\nabla}_x f\|_{L_x^2(H_{v,*}^1)} &\leq \frac{\alpha_3 \kappa}{2} \frac{t^3}{\varepsilon^6} \|\langle v \rangle^\alpha \widetilde{\nabla}_x f\|_{L_x^2(H_{v,*}^1)}^2 \\ &\quad + C \frac{\alpha_2^2}{\alpha_3} \frac{t}{\varepsilon^4} \|\langle v \rangle^\alpha f\|_{L_x^2(H_{v,*}^2)}^2. \end{aligned}$$

We thus deduce, for any $t \in (0, \varepsilon^2]$, that

$$\begin{aligned} \frac{d}{dt} \mathcal{E}_\varepsilon(t) &\leq -\frac{1}{\varepsilon^2} \left(2\kappa - C\alpha_1 - C\alpha_2 - C\frac{\alpha_1^2}{\alpha_2} \right) \|\langle v \rangle^\alpha f\|_{L_x^2(H_{v,*}^1)}^2 \\ &\quad - \frac{t}{\varepsilon^4} \left(\alpha_1\kappa - C\frac{\alpha_2^2}{\alpha_3} \right) \|\langle v \rangle^\alpha f\|_{L_x^2(H_{v,*}^2)}^2 - \frac{t^2}{\varepsilon^4} \left(\frac{\alpha_2}{2} - C\alpha_3 \right) \|\langle v \rangle^\alpha \widetilde{\nabla}_x f\|_{L_{x,v}^2}^2 \\ &\quad - \frac{\alpha_3\kappa t^3}{2\varepsilon^6} \|\langle v \rangle^\alpha \widetilde{\nabla}_x f\|_{L_x^2(H_{v,*}^1)}^2. \end{aligned}$$

We now choose $\alpha_1 = \eta$, $\alpha_2 = \eta^{3/2}$, and $\alpha_3 = \eta^{5/3}$ with $\eta \in (0, 1)$ small enough such that each quantity appearing inside the parentheses in above inequality is positive. Therefore one obtains that $\frac{d}{dt} \mathcal{E}_\varepsilon(t) \leq 0$, for any $t \in (0, \varepsilon^2]$, which concludes the proof as explained in Step 1. \square

Proof of Lemma 4.30. The proof follows the same lines as the proof of Lemma 4.29. More precisely, as explained in the Step 1 of the proof of Lemma 4.29, in order to obtain the desired result it is sufficient to prove that $\frac{d}{dt} \mathcal{E}_\varepsilon^1(t) \leq 0$ for all $t \in (0, \varepsilon^2]$ for some well-chosen functional $\mathcal{E}_\varepsilon^1$.

Let $\alpha \in \mathbb{R}$ and f_{in} be such that $\langle v \rangle^\alpha f_{\text{in}} \in L_{x,v}^2$. We then consider the solution $f(t) = S_{\mathcal{B}_\varepsilon}(t) f_{\text{in}}$ to the equation $\partial_t f = \mathcal{B}_\varepsilon f$ with initial data f_{in} . Recalling that \mathcal{E}_ε is defined in (4.169), we then define the functional

$$\begin{aligned} \mathcal{E}_\varepsilon^1(t) &:= \mathcal{E}_\varepsilon(t) \\ &\quad + \beta_1 \left(\frac{t}{\varepsilon^2} \right)^2 \left(K_1 \|\langle v \rangle^\alpha \langle v \rangle^{\gamma+2} f\|_{L_{x,v}^2}^2 + K_2 \|\langle v \rangle^\alpha \langle v \rangle^{\frac{\gamma}{2}+1} \widetilde{\nabla}_v f\|_{L_{x,v}^2}^2 + \|\langle v \rangle^\alpha \widetilde{\nabla}_v \widetilde{\nabla}_v f\|_{L_{x,v}^2}^2 \right) \\ &\quad + \varepsilon^2 \beta_2 \left(\frac{t}{\varepsilon^2} \right)^4 \left(K_1 \|\langle v \rangle^\alpha \langle v \rangle^{\gamma+1} \nabla_x f\|_{L_{x,v}^2}^2 + K_2 \|\langle v \rangle^\alpha \langle v \rangle^{\frac{\gamma}{2}+1} \widetilde{\nabla}_x f\|_{L_{x,v}^2}^2 \right. \\ &\quad \quad \quad \left. + \|\langle v \rangle^\alpha \widetilde{\nabla}_v \widetilde{\nabla}_x f\|_{L_{x,v}^2}^2 \right) \\ &\quad + \varepsilon^3 \beta_3 \left(\frac{t}{\varepsilon^2} \right)^5 \left\langle \langle v \rangle^\alpha \widetilde{\nabla}_v \widetilde{\nabla}_x f, \langle v \rangle^\alpha \widetilde{\nabla}_x \widetilde{\nabla}_x f \right\rangle_{L_{x,v}^2} \\ &\quad + \varepsilon^4 \beta_4 \left(\frac{t}{\varepsilon^2} \right)^6 \left(K_1 \|\langle v \rangle^\alpha \langle v \rangle^\gamma \nabla_x \nabla_x f\|_{L_{x,v}^2}^2 + K_2 \|\langle v \rangle^\alpha \langle v \rangle^{\frac{\gamma}{2}} \nabla_x \widetilde{\nabla}_x f\|_{L_{x,v}^2}^2 \right. \\ &\quad \quad \quad \left. + \|\langle v \rangle^\alpha \widetilde{\nabla}_x \widetilde{\nabla}_x f\|_{L_{x,v}^2}^2 \right) \end{aligned}$$

where $\beta_1, \beta_2, \beta_3, \beta_4, K_1, K_2 > 0$ are positive constants to be chosen later such that $0 < \beta_4 \ll \beta_3 \ll \beta_2 \ll \beta_1 \ll 1$, $\beta_4 \leq \sqrt{\beta_3 \beta_2}$ and $K_1 \gg K_2 \gg 1$.

Step 1. From the proof of Lemma 4.29 (Step 6), we already have that for any $t \in (0, \varepsilon^2]$,

there holds

$$\begin{aligned} \frac{d}{dt} \mathcal{E}_\varepsilon(t) \leq & -\frac{\kappa}{2\varepsilon^2} \|\langle v \rangle^\alpha f\|_{L_x^2(H_{v,*}^1)}^2 - \frac{\eta\kappa t}{2\varepsilon^4} \|\langle v \rangle^\alpha f\|_{L_x^2(H_{v,*}^2)}^2 \\ & - \frac{\eta^{3/2} t^2}{4\varepsilon^4} \|\langle v \rangle^\alpha \widetilde{\nabla}_x f\|_{L_{x,v}^2}^2 - \frac{\eta^{5/3} \kappa t^3}{2\varepsilon^6} \|\langle v \rangle^\alpha \widetilde{\nabla}_x f\|_{L_x^2(H_{v,*}^1)}^2. \end{aligned} \quad (4.174)$$

for some constant $\kappa > 0$ and where $\eta \in (0, 1)$ is small enough.

Step 2. From Lemma 4.28, we already have

$$\frac{d}{dt} \|\langle v \rangle^\alpha \langle v \rangle^{\gamma+2} f\|_{L_{x,v}^2}^2 \leq -\frac{\kappa}{\varepsilon^2} \|\langle v \rangle^\alpha \langle v \rangle^{\gamma+2} f\|_{L_x^2(H_{v,*}^1)}^2 \quad (4.175)$$

for some $\kappa > 0$. Moreover, from the proof of Lemma 4.29 (Step 3), we already know that

$$\begin{aligned} \frac{d}{dt} \|\langle v \rangle^\alpha \langle v \rangle^{\frac{\gamma}{2}+1} \widetilde{\nabla}_v f\|_{L_{x,v}^2}^2 & \leq -\frac{\kappa}{2\varepsilon^2} \|\langle v \rangle^\alpha \langle v \rangle^{\frac{\gamma}{2}+1} \widetilde{\nabla}_v f\|_{L_x^2(H_{v,*}^1)}^2 + \frac{C}{\varepsilon^2} \|\langle v \rangle^\alpha \langle v \rangle^{\frac{3\gamma}{2}+2} f\|_{L_{x,v}^2}^2 \\ & + \frac{C}{\varepsilon^2} \|\langle v \rangle^\alpha \langle v \rangle^{\frac{3\gamma}{2}+2} \nabla_v f\|_{L_{x,v}^2}^2 + \frac{C}{\varepsilon} \|\langle v \rangle^\alpha \langle v \rangle^{\frac{\gamma}{2}+1} \widetilde{\nabla}_v f\|_{L_{x,v}^2} \|\langle v \rangle^\alpha \langle v \rangle^{\frac{\gamma}{2}+1} \widetilde{\nabla}_x f\|_{L_{x,v}^2} \end{aligned} \quad (4.176)$$

for some constants $\kappa, C > 0$. We now compute

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\langle v \rangle^\alpha \widetilde{\nabla}_v \widetilde{\nabla}_v f\|_{L_{x,v}^2}^2 \\ & = \left\langle \langle v \rangle^\alpha \widetilde{\nabla}_{v_i} \widetilde{\nabla}_{v_j} f, \langle v \rangle^\alpha \widetilde{\nabla}_{v_i} \widetilde{\nabla}_{v_j} \mathcal{B}_\varepsilon f \right\rangle_{L_{x,v}^2} \\ & = \left\langle \langle v \rangle^\alpha \widetilde{\nabla}_{v_i} \widetilde{\nabla}_{v_j} f, \langle v \rangle^\alpha \widetilde{\nabla}_{v_i} \mathcal{B}_\varepsilon(\widetilde{\nabla}_{v_j} f) \right\rangle_{L_{x,v}^2} + \left\langle \langle v \rangle^\alpha \widetilde{\nabla}_{v_i} \widetilde{\nabla}_{v_j} f, \langle v \rangle^\alpha \widetilde{\nabla}_{v_i} [\widetilde{\nabla}_{v_j}, \mathcal{B}_\varepsilon] f \right\rangle_{L_{x,v}^2} \\ & = \left\langle \langle v \rangle^\alpha \widetilde{\nabla}_{v_i} \widetilde{\nabla}_{v_j} f, \langle v \rangle^\alpha \widetilde{\nabla}_{v_i} \mathcal{B}_\varepsilon(\widetilde{\nabla}_{v_j} f) \right\rangle_{L_{x,v}^2} + \left\langle \langle v \rangle^\alpha \widetilde{\nabla}_{v_i} \widetilde{\nabla}_{v_j} f, \langle v \rangle^\alpha [\widetilde{\nabla}_{v_j}, \mathcal{B}_\varepsilon] \widetilde{\nabla}_{v_i} f \right\rangle_{L_{x,v}^2} \\ & \quad + \left\langle \langle v \rangle^\alpha \widetilde{\nabla}_{v_i} \widetilde{\nabla}_{v_j} f, \langle v \rangle^\alpha [\widetilde{\nabla}_{v_i}, [\widetilde{\nabla}_{v_j}, \mathcal{B}_\varepsilon]] f \right\rangle_{L_{x,v}^2} \\ & =: I_1 + I_2 + I_3. \end{aligned}$$

From the proof of Lemma 4.29 (Step 3), we already know that

$$\begin{aligned} I_1 + I_2 \leq & -\frac{\kappa}{2\varepsilon^2} \|\langle v \rangle^\alpha \widetilde{\nabla}_v \widetilde{\nabla}_v f\|_{L_x^2(H_{v,*}^1)}^2 + \frac{C}{\varepsilon^2} \|\langle v \rangle^\alpha \langle v \rangle^{\gamma+1} \widetilde{\nabla}_v f\|_{L_{x,v}^2}^2 \\ & + \frac{C}{\varepsilon^2} \|\langle v \rangle^\alpha \langle v \rangle^{\gamma+1} \nabla_v \widetilde{\nabla}_v f\|_{L_{x,v}^2}^2 + \frac{C}{\varepsilon} \|\langle v \rangle^\alpha \widetilde{\nabla}_v \widetilde{\nabla}_v f\|_{L_{x,v}^2} \|\langle v \rangle^\alpha \widetilde{\nabla}_x \widetilde{\nabla}_v f\|_{L_{x,v}^2}. \end{aligned}$$

For the term I_3 , we use Lemma 4.10 to get

$$\begin{aligned} [\widetilde{\nabla}_{v_i}, [\widetilde{\nabla}_{v_j}, \mathcal{B}_\varepsilon]] f & = -\frac{1}{\varepsilon^2} [\widetilde{\nabla}_{v_i}, \widetilde{\nabla}_{v_k}^* [\widetilde{\nabla}_{v_j}, \widetilde{\nabla}_{v_k}]] f - \frac{1}{\varepsilon^2} [\widetilde{\nabla}_{v_i}, \widetilde{\nabla}_{v_k} [\widetilde{\nabla}_{v_j}, \widetilde{\nabla}_{v_k}^*]] f \\ & \quad - \frac{1}{\varepsilon^2} [\widetilde{\nabla}_{v_i}, [[\widetilde{\nabla}_{v_j}, \widetilde{\nabla}_{v_k}^*], \widetilde{\nabla}_{v_k}]] f - \frac{1}{\varepsilon^2} (\widetilde{\nabla}_{v_i} \widetilde{\nabla}_{v_j} m^2) f \\ & \quad - \frac{1}{\varepsilon} [\widetilde{\nabla}_{v_i}, \widetilde{\nabla}_{x_j}] f. \end{aligned}$$

Expanding the first two terms, we observe that

$$\begin{aligned}
 & [\widetilde{\nabla}_{v_i}, \widetilde{\nabla}_{v_k}^* [\widetilde{\nabla}_{v_j}, \widetilde{\nabla}_{v_k}]] + [\widetilde{\nabla}_{v_i}, \widetilde{\nabla}_{v_k} [\widetilde{\nabla}_{v_j}, \widetilde{\nabla}_{v_k}^*]] \\
 &= \widetilde{\nabla}_{v_i} \widetilde{\nabla}_{v_k}^* [\widetilde{\nabla}_{v_j}, \widetilde{\nabla}_{v_k}] - \widetilde{\nabla}_{v_k}^* [\widetilde{\nabla}_{v_j}, \widetilde{\nabla}_{v_k}] \widetilde{\nabla}_{v_i} + \widetilde{\nabla}_{v_i} \widetilde{\nabla}_{v_k} [\widetilde{\nabla}_{v_j}, \widetilde{\nabla}_{v_k}^*] - \widetilde{\nabla}_{v_k} [\widetilde{\nabla}_{v_j}, \widetilde{\nabla}_{v_k}^*] \widetilde{\nabla}_{v_i} \\
 &= \widetilde{\nabla}_{v_k}^* [\widetilde{\nabla}_{v_i}, [\widetilde{\nabla}_{v_j}, \widetilde{\nabla}_{v_k}]] + \widetilde{\nabla}_{v_k} [\widetilde{\nabla}_{v_i}, [\widetilde{\nabla}_{v_j}, \widetilde{\nabla}_{v_k}^*]] + [\widetilde{\nabla}_{v_i}, \widetilde{\nabla}_{v_k}^*] [\widetilde{\nabla}_{v_j}, \widetilde{\nabla}_{v_k}] \\
 & \quad + [\widetilde{\nabla}_{v_i}, \widetilde{\nabla}_{v_k}] [\widetilde{\nabla}_{v_j}, \widetilde{\nabla}_{v_k}^*],
 \end{aligned}$$

whence

$$\begin{aligned}
 I_3 &= -\frac{1}{\varepsilon^2} \langle \widetilde{\nabla}_{v_k} (\langle v \rangle^{2\alpha} \widetilde{\nabla}_{v_i} \widetilde{\nabla}_{v_j} f), [\widetilde{\nabla}_{v_i}, [\widetilde{\nabla}_{v_j}, \widetilde{\nabla}_{v_k}]] f \rangle_{L_{x,v}^2} \\
 & \quad - \frac{1}{\varepsilon^2} \langle \widetilde{\nabla}_{v_k}^* (\langle v \rangle^{2\alpha} \widetilde{\nabla}_{v_i} \widetilde{\nabla}_{v_j} f), [\widetilde{\nabla}_{v_i}, [\widetilde{\nabla}_{v_j}, \widetilde{\nabla}_{v_k}^*]] f \rangle_{L_{x,v}^2} \\
 & \quad - \frac{1}{\varepsilon^2} \langle \langle v \rangle^\alpha \widetilde{\nabla}_{v_i} \widetilde{\nabla}_{v_j} f, \langle v \rangle^\alpha [\widetilde{\nabla}_{v_i}, \widetilde{\nabla}_{v_k}^*] [\widetilde{\nabla}_{v_j}, \widetilde{\nabla}_{v_k}] f \rangle_{L_{x,v}^2} \\
 & \quad - \frac{1}{\varepsilon^2} \langle \langle v \rangle^\alpha \widetilde{\nabla}_{v_i} \widetilde{\nabla}_{v_j} f, \langle v \rangle^\alpha [\widetilde{\nabla}_{v_i}, \widetilde{\nabla}_{v_k}] [\widetilde{\nabla}_{v_j}, \widetilde{\nabla}_{v_k}^*] f \rangle_{L_{x,v}^2} \\
 & \quad - \frac{1}{\varepsilon^2} \langle \langle v \rangle^\alpha \widetilde{\nabla}_{v_i} \widetilde{\nabla}_{v_j} f, \langle v \rangle^\alpha [\widetilde{\nabla}_{v_i}, [[\widetilde{\nabla}_{v_j}, \widetilde{\nabla}_{v_k}^*], \widetilde{\nabla}_{v_k}]] f \rangle_{L_{x,v}^2} \\
 & \quad - \frac{1}{\varepsilon^2} \langle \langle v \rangle^\alpha \widetilde{\nabla}_{v_i} \widetilde{\nabla}_{v_j} f, \langle v \rangle^\alpha (\widetilde{\nabla}_{v_i} \widetilde{\nabla}_{v_j} m^2) f \rangle_{L_{x,v}^2} \\
 & \quad - \frac{1}{\varepsilon} \langle \langle v \rangle^\alpha \widetilde{\nabla}_{v_i} \widetilde{\nabla}_{v_j} f, \langle v \rangle^\alpha [\widetilde{\nabla}_{v_i}, \widetilde{\nabla}_{x_j}] f \rangle_{L_{x,v}^2} \\
 & =: -\frac{1}{\varepsilon^2} (I_{31} + I_{32} + I_{33} + I_{34} + I_{35} + I_{36}) - \frac{1}{\varepsilon} \langle \langle v \rangle^\alpha \widetilde{\nabla}_{v_i} \widetilde{\nabla}_{v_j} f, \langle v \rangle^\alpha [\widetilde{\nabla}_{v_i}, \widetilde{\nabla}_{x_j}] f \rangle_{L_{x,v}^2}.
 \end{aligned}$$

Now remark that

$$\|\langle v \rangle^{-\alpha} \widetilde{\nabla}_{v_k}^* (\langle v \rangle^{2\alpha} \widetilde{\nabla}_{v_i} \widetilde{\nabla}_{v_j} f)\|_{L_{x,v}^2} + \|\langle v \rangle^{-\alpha} \widetilde{\nabla}_{v_k} (\langle v \rangle^{2\alpha} \widetilde{\nabla}_{v_i} \widetilde{\nabla}_{v_j} f)\|_{L_{x,v}^2} \lesssim \|\langle v \rangle^\alpha \widetilde{\nabla}_v \widetilde{\nabla}_v f\|_{L_x^2(H_{v,*}^1)}$$

and using Lemma 4.10, we obtain

$$I_{31} + I_{32} \lesssim \|\langle v \rangle^\alpha \widetilde{\nabla}_v \widetilde{\nabla}_v f\|_{L_x^2(H_{v,*}^1)} \left(\|\langle v \rangle^\alpha \langle v \rangle^{\frac{3\gamma}{2}+1} \nabla_v f\|_{L_{x,v}^2} + \|\langle v \rangle^\alpha \langle v \rangle^{\frac{3\gamma}{2}} f\|_{L_{x,v}^2} \right).$$

Thanks to Lemma 4.10, we observe that

$$|[\widetilde{\nabla}_{v_i}, \widetilde{\nabla}_{v_k}^*] [\widetilde{\nabla}_{v_j}, \widetilde{\nabla}_{v_k}] f| + |[\widetilde{\nabla}_{v_i}, \widetilde{\nabla}_{v_k}] [\widetilde{\nabla}_{v_j}, \widetilde{\nabla}_{v_k}^*] f| \lesssim \langle v \rangle^{2\gamma+2} |\nabla_v \nabla_v f| + \langle v \rangle^{2\gamma+1} |\nabla_v f|$$

and thus we obtain

$$\begin{aligned}
 I_{33} + I_{34} &\lesssim \|\langle v \rangle^\alpha \langle v \rangle^{\frac{\gamma}{2}+1} \widetilde{\nabla}_v \widetilde{\nabla}_v f\|_{L_{x,v}^2} \left(\|\langle v \rangle^\alpha \langle v \rangle^{\frac{3\gamma}{2}+1} \nabla_v \nabla_v f\|_{L_{x,v}^2} + \|\langle v \rangle^\alpha \langle v \rangle^{\frac{3\gamma}{2}} \nabla_v f\|_{L_{x,v}^2} \right) \\
 &\lesssim \|\langle v \rangle^\alpha \widetilde{\nabla}_v \widetilde{\nabla}_v f\|_{L_x^2(H_{v,*}^1)} \left(\|\langle v \rangle^\alpha \langle v \rangle^{\frac{3\gamma}{2}+1} \nabla_v \nabla_v f\|_{L_{x,v}^2} + \|\langle v \rangle^\alpha \langle v \rangle^{\frac{3\gamma}{2}} \nabla_v f\|_{L_{x,v}^2} \right).
 \end{aligned}$$

For the term I_{35} , we write

$$I_{35} = \left\langle \widetilde{\nabla}_{v_i}^* (\langle v \rangle^{2\alpha} \widetilde{\nabla}_{v_i} \widetilde{\nabla}_{v_j} f), [[\widetilde{\nabla}_{v_j}, \widetilde{\nabla}_{v_k}^*], \widetilde{\nabla}_{v_k}] f \right\rangle_{L_{x,v}^2} \\ - \left\langle \langle v \rangle^\alpha \widetilde{\nabla}_{v_i} \widetilde{\nabla}_{v_j} f, \langle v \rangle^\alpha [[\widetilde{\nabla}_{v_j}, \widetilde{\nabla}_{v_k}^*], \widetilde{\nabla}_{v_k}] \widetilde{\nabla}_{v_i} f \right\rangle_{L_{x,v}^2}$$

and using Lemma 4.10, we thus get

$$I_{35} \lesssim \|\langle v \rangle^\alpha \widetilde{\nabla}_v \widetilde{\nabla}_v f\|_{L_x^2(H_{v,*}^1)} \left(\|\langle v \rangle^\alpha \langle v \rangle^{\frac{3\gamma}{2}+1} \nabla_v f\|_{L_{x,v}^2} + \|\langle v \rangle^\alpha \langle v \rangle^{\frac{3\gamma}{2}} f\|_{L_{x,v}^2} \right. \\ \left. + \|\langle v \rangle^\alpha \langle v \rangle^\gamma \nabla_v \widetilde{\nabla}_v f\|_{L_{x,v}^2} \right).$$

Using that $|\widetilde{\nabla}_{v_i} \widetilde{\nabla}_{v_j} m^2| \lesssim \langle v \rangle^{2\gamma+2}$, we also obtain

$$I_{36} \lesssim \|\langle v \rangle^\alpha \widetilde{\nabla}_v \widetilde{\nabla}_v f\|_{L_x^2(H_{v,*}^1)} \|\langle v \rangle^\alpha \langle v \rangle^{\frac{3\gamma}{2}+1} f\|_{L_{x,v}^2}.$$

Gathering previous estimates and using Young's inequality, we thus get

$$\begin{aligned} \frac{d}{dt} \|\langle v \rangle^\alpha \widetilde{\nabla}_v \widetilde{\nabla}_v f\|_{L_{x,v}^2}^2 &\leq -\frac{\kappa}{4\varepsilon^2} \|\langle v \rangle^\alpha \widetilde{\nabla}_v \widetilde{\nabla}_v f\|_{L_x^2(H_{v,*}^1)}^2 + \frac{C}{\varepsilon^2} \|\langle v \rangle^\alpha \langle v \rangle^{\gamma+1} \widetilde{\nabla}_v f\|_{L_{x,v}^2}^2 \\ &\quad + \frac{C}{\varepsilon^2} \|\langle v \rangle^\alpha \langle v \rangle^{\gamma+1} \nabla_v \widetilde{\nabla}_v f\|_{L_{x,v}^2}^2 + \frac{C}{\varepsilon^2} \|\langle v \rangle^\alpha \langle v \rangle^{\frac{3\gamma}{2}+1} \nabla_v \nabla_v f\|_{L_{x,v}^2}^2 \\ &\quad + \frac{C}{\varepsilon^2} \|\langle v \rangle^\alpha \langle v \rangle^{\frac{3\gamma}{2}+1} \nabla_v f\|_{L_{x,v}^2}^2 + \frac{C}{\varepsilon^2} \|\langle v \rangle^\alpha \langle v \rangle^{\frac{3\gamma}{2}+2} f\|_{L_{x,v}^2}^2 \\ &\quad + \frac{C}{\varepsilon} \|\langle v \rangle^\alpha \widetilde{\nabla}_v \widetilde{\nabla}_v f\|_{L_{x,v}^2} \|\langle v \rangle^\alpha \widetilde{\nabla}_x f\|_{L_x^2(H_{v,*}^1)}. \end{aligned} \tag{4.177}$$

Observing that

$$\|\langle v \rangle^\alpha \langle v \rangle^{\frac{3\gamma}{2}+2} f\|_{L_{x,v}^2} + \|\langle v \rangle^\alpha \langle v \rangle^{\frac{3\gamma}{2}+2} \nabla_v f\|_{L_{x,v}^2} + \|\langle v \rangle^\alpha \langle v \rangle^{\gamma+1} \widetilde{\nabla}_v f\|_{L_{x,v}^2} \\ \lesssim \|\langle v \rangle^\alpha \langle v \rangle^{\gamma+2} f\|_{L_x^2(H_{v,*}^1)}$$

and

$$\|\langle v \rangle^\alpha \langle v \rangle^{\gamma+1} \nabla_v \widetilde{\nabla}_v f\|_{L_{x,v}^2} + \|\langle v \rangle^\alpha \langle v \rangle^{\frac{3\gamma}{2}+1} \nabla_v \nabla_v f\|_{L_{x,v}^2} \lesssim \|\langle v \rangle^\alpha \langle v \rangle^{\frac{\gamma}{2}+1} \widetilde{\nabla}_v f\|_{L_x^2(H_{v,*}^1)},$$

we then gather (4.175)–(4.176)–(4.177) choosing $K_1 \gg K_2 \gg 1$ large enough, which yields

$$\begin{aligned} \frac{d}{dt} &\left(K_1 \|\langle v \rangle^\alpha \langle v \rangle^{\gamma+2} f\|_{L_{x,v}^2}^2 + K_2 \|\langle v \rangle^\alpha \langle v \rangle^{\frac{\gamma}{2}+1} \widetilde{\nabla}_v f\|_{L_{x,v}^2}^2 + \|\langle v \rangle^\alpha \widetilde{\nabla}_v \widetilde{\nabla}_v f\|_{L_{x,v}^2}^2 \right) \\ &\leq -\frac{\kappa}{\varepsilon^2} \left(\|\langle v \rangle^\alpha \langle v \rangle^{\gamma+2} f\|_{L_x^2(H_{v,*}^1)}^2 + \|\langle v \rangle^\alpha \langle v \rangle^{\frac{\gamma}{2}+1} \widetilde{\nabla}_v f\|_{L_x^2(H_{v,*}^1)}^2 + \|\langle v \rangle^\alpha \widetilde{\nabla}_v \widetilde{\nabla}_v f\|_{L_x^2(H_{v,*}^1)}^2 \right) \\ &\quad + \frac{C}{\varepsilon} \|\langle v \rangle^\alpha f\|_{L_x^2(H_{v,*}^2)} \|\langle v \rangle^\alpha \widetilde{\nabla}_x f\|_{L_x^2(H_{v,*}^1)}. \end{aligned} \tag{4.178}$$

Step 3. From the proof of Lemma 4.29 (Step 5) we already have, with K_1 large enough with respect to K_2 ,

$$\begin{aligned} & \frac{d}{dt} \left\{ K_1 \|\langle v \rangle^\alpha \langle v \rangle^{\gamma+1} \nabla_x f\|_{L_{x,v}^2}^2 + K_2 \|\langle v \rangle^\alpha \langle v \rangle^{\frac{\gamma}{2}+1} \widetilde{\nabla}_x f\|_{L_{x,v}^2}^2 \right\} \\ & \leq -\frac{\kappa}{\varepsilon^2} \|\langle v \rangle^\alpha \langle v \rangle^{\frac{\gamma}{2}+1} \widetilde{\nabla}_x f\|_{L_x^2(H_{v,*}^1)}^2. \end{aligned} \quad (4.179)$$

We now compute

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\langle v \rangle^\alpha \widetilde{\nabla}_v \widetilde{\nabla}_x f\|_{L_{x,v}^2}^2 \\ & = \left\langle \langle v \rangle^\alpha \widetilde{\nabla}_{v_i} \widetilde{\nabla}_{x_j} f, \langle v \rangle^\alpha \widetilde{\nabla}_{v_i} \widetilde{\nabla}_{x_j} \mathcal{B}_\varepsilon f \right\rangle_{L_{x,v}^2} \\ & = \left\langle \langle v \rangle^\alpha \widetilde{\nabla}_{v_i} \widetilde{\nabla}_{x_j} f, \langle v \rangle^\alpha \widetilde{\nabla}_{v_i} \mathcal{B}_\varepsilon(\widetilde{\nabla}_{x_j} f) \right\rangle_{L_{x,v}^2} + \left\langle \langle v \rangle^\alpha \widetilde{\nabla}_{v_i} \widetilde{\nabla}_{x_j} f, \langle v \rangle^\alpha \widetilde{\nabla}_{v_i} [\widetilde{\nabla}_{x_j}, \mathcal{B}_\varepsilon] f \right\rangle_{L_{x,v}^2} \\ & = \left\langle \langle v \rangle^\alpha \widetilde{\nabla}_{v_i} \widetilde{\nabla}_{x_j} f, \langle v \rangle^\alpha \widetilde{\nabla}_{v_i} \mathcal{B}_\varepsilon(\widetilde{\nabla}_{x_j} f) \right\rangle_{L_{x,v}^2} + \left\langle \langle v \rangle^\alpha \widetilde{\nabla}_{v_i} \widetilde{\nabla}_{x_j} f, \langle v \rangle^\alpha [\widetilde{\nabla}_{x_j}, \mathcal{B}_\varepsilon] \widetilde{\nabla}_{v_i} f \right\rangle_{L_{x,v}^2} \\ & \quad + \left\langle \langle v \rangle^\alpha \widetilde{\nabla}_{v_i} \widetilde{\nabla}_{x_j} f, \langle v \rangle^\alpha [\widetilde{\nabla}_{v_i}, [\widetilde{\nabla}_{x_j}, \mathcal{B}_\varepsilon]] f \right\rangle_{L_{x,v}^2} \\ & =: J_1 + J_2 + J_3. \end{aligned}$$

From the proof of Lemma 4.29 (Step 3), we already know that

$$\begin{aligned} J_1 + J_2 & \leq -\frac{\kappa}{2\varepsilon^2} \|\langle v \rangle^\alpha \widetilde{\nabla}_v \widetilde{\nabla}_x f\|_{L_x^2(H_{v,*}^1)}^2 + \frac{C}{\varepsilon^2} \|\langle v \rangle^\alpha \langle v \rangle^{\gamma+1} \widetilde{\nabla}_x f\|_{L_{x,v}^2}^2 \\ & \quad + \frac{C}{\varepsilon^2} \|\langle v \rangle^\alpha \langle v \rangle^{\gamma+1} \nabla_v \widetilde{\nabla}_x f\|_{L_{x,v}^2}^2 + \frac{C}{\varepsilon} \|\langle v \rangle^\alpha \widetilde{\nabla}_v \widetilde{\nabla}_x f\|_{L_{x,v}^2} \|\langle v \rangle^\alpha \widetilde{\nabla}_x \widetilde{\nabla}_x f\|_{L_{x,v}^2}. \end{aligned}$$

The term J_3 can be estimated as the term I_3 in Step 2 above. Therefore, we obtain, gathering these estimates and using Young's inequality,

$$\begin{aligned} \frac{d}{dt} \|\langle v \rangle^\alpha \widetilde{\nabla}_v \widetilde{\nabla}_x f\|_{L_{x,v}^2}^2 & \leq -\frac{\kappa}{4\varepsilon^2} \|\langle v \rangle^\alpha \widetilde{\nabla}_v \widetilde{\nabla}_x f\|_{L_x^2(H_{v,*}^1)}^2 + \frac{C}{\varepsilon^2} \|\langle v \rangle^\alpha \langle v \rangle^{\gamma+1} \widetilde{\nabla}_x f\|_{L_{x,v}^2}^2 \\ & \quad + \frac{C}{\varepsilon^2} \|\langle v \rangle^\alpha \langle v \rangle^{\gamma+1} \nabla_v \widetilde{\nabla}_x f\|_{L_{x,v}^2}^2 + \frac{C}{\varepsilon^2} \|\langle v \rangle^\alpha \langle v \rangle^{\frac{3\gamma}{2}+1} \nabla_v \nabla_x f\|_{L_{x,v}^2}^2 \\ & \quad + \frac{C}{\varepsilon^2} \|\langle v \rangle^\alpha \langle v \rangle^{\frac{3\gamma}{2}+1} \nabla_x f\|_{L_{x,v}^2}^2 + \frac{C}{\varepsilon} \|\langle v \rangle^\alpha \widetilde{\nabla}_v \widetilde{\nabla}_x f\|_{L_{x,v}^2} \|\langle v \rangle^\alpha \widetilde{\nabla}_x \widetilde{\nabla}_x f\|_{L_{x,v}^2}. \end{aligned} \quad (4.180)$$

Observing that

$$\begin{aligned} & \|\langle v \rangle^\alpha \langle v \rangle^{\gamma+1} \widetilde{\nabla}_x f\|_{L_{x,v}^2} + \|\langle v \rangle^\alpha \langle v \rangle^{\gamma+1} \nabla_v \widetilde{\nabla}_x f\|_{L_{x,v}^2} \\ & + \|\langle v \rangle^\alpha \langle v \rangle^{\frac{3\gamma}{2}+1} \nabla_v \nabla_x f\|_{L_{x,v}^2} + \|\langle v \rangle^\alpha \langle v \rangle^{\frac{3\gamma}{2}+1} \nabla_x f\|_{L_{x,v}^2} \lesssim \|\langle v \rangle^\alpha \langle v \rangle^{\frac{\gamma}{2}+1} \widetilde{\nabla}_x f\|_{L_x^2(H_{v,*}^1)} \end{aligned}$$

we then gather (4.179)–(4.180) and choose $K_1 \gg K_2 \gg 1$ large enough, which yields

$$\begin{aligned} & \frac{d}{dt} \left(K_1 \|\langle v \rangle^\alpha \langle v \rangle^{\gamma+1} \nabla_x f\|_{L_{x,v}^2}^2 + K_2 \|\langle v \rangle^\alpha \langle v \rangle^{\frac{\gamma}{2}+1} \widetilde{\nabla}_x f\|_{L_{x,v}^2}^2 + \|\langle v \rangle^\alpha \widetilde{\nabla}_v \widetilde{\nabla}_x f\|_{L_{x,v}^2}^2 \right) \\ & \leq -\frac{\kappa}{\varepsilon^2} \left(\|\langle v \rangle^\alpha \langle v \rangle^{\frac{\gamma}{2}+1} \widetilde{\nabla}_x f\|_{L_x^2(H_{v,*}^1)}^2 + \|\langle v \rangle^\alpha \widetilde{\nabla}_v \widetilde{\nabla}_x f\|_{L_x^2(H_{v,*}^1)}^2 \right) \\ & \quad + \frac{C}{\varepsilon} \|\langle v \rangle^\alpha \widetilde{\nabla}_x f\|_{L_x^2(H_{v,*}^1)} \|\langle v \rangle^\alpha \widetilde{\nabla}_x \widetilde{\nabla}_x f\|_{L_{x,v}^2}. \end{aligned} \quad (4.181)$$

Step 4. Arguing as in the proof of Lemma 4.29 (Step 4), we obtain

$$\begin{aligned} & \frac{d}{dt} \left\langle \langle v \rangle^\alpha \widetilde{\nabla}_v \widetilde{\nabla}_x f, \langle v \rangle^\alpha \widetilde{\nabla}_x \widetilde{\nabla}_x f \right\rangle_{L_{x,v}^2} \\ & \leq -\frac{1}{\varepsilon} \|\langle v \rangle^\alpha \widetilde{\nabla}_x \widetilde{\nabla}_x f\|_{L_{x,v}^2}^2 + \frac{C}{\varepsilon^2} \|\langle v \rangle^\alpha \widetilde{\nabla}_x f\|_{L_x^2(H_{v,*}^2)} \|\langle v \rangle^\alpha \widetilde{\nabla}_x \widetilde{\nabla}_x f\|_{L_x^2(H_{v,*}^1)}. \end{aligned} \quad (4.182)$$

Step 5. Since ∇_x commutes with $\widetilde{\nabla}_x$ and \mathcal{B}_ε , we already know from the proof of Lemma 4.29 (Step 5) that, with K_1 large enough with respect to K_2 ,

$$\begin{aligned} & \frac{d}{dt} \left\{ K_1 \|\langle v \rangle^\alpha \langle v \rangle^\gamma \nabla_x \nabla_x f\|_{L_{x,v}^2}^2 + K_2 \|\langle v \rangle^\alpha \langle v \rangle^{\frac{\gamma}{2}} \nabla_x \widetilde{\nabla}_x f\|_{L_{x,v}^2}^2 \right\} \\ & \leq -\frac{\kappa}{\varepsilon^2} \|\langle v \rangle^\alpha \langle v \rangle^{\frac{\gamma}{2}} \nabla_x \widetilde{\nabla}_x f\|_{L_x^2(H_{v,*}^1)}^2 \end{aligned} \quad (4.183)$$

for some constant $\kappa > 0$. We now compute

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\langle v \rangle^\alpha \widetilde{\nabla}_x \widetilde{\nabla}_x f\|_{L_{x,v}^2}^2 \\ & = \left\langle \langle v \rangle^\alpha \widetilde{\nabla}_{x_i} \widetilde{\nabla}_{x_j} f, \langle v \rangle^\alpha \widetilde{\nabla}_{x_i} \widetilde{\nabla}_{x_j} \mathcal{B}_\varepsilon f \right\rangle_{L_{x,v}^2} \\ & = \left\langle \langle v \rangle^\alpha \widetilde{\nabla}_{x_i} \widetilde{\nabla}_{x_j} f, \langle v \rangle^\alpha \widetilde{\nabla}_{x_i} \mathcal{B}_\varepsilon (\widetilde{\nabla}_{x_j} f) \right\rangle_{L_{x,v}^2} + \left\langle \langle v \rangle^\alpha \widetilde{\nabla}_{x_i} \widetilde{\nabla}_{x_j} f, \langle v \rangle^\alpha \widetilde{\nabla}_{x_i} [\widetilde{\nabla}_{x_j}, \mathcal{B}_\varepsilon] f \right\rangle_{L_{x,v}^2} \\ & = \left\langle \langle v \rangle^\alpha \widetilde{\nabla}_{x_i} \widetilde{\nabla}_{x_j} f, \langle v \rangle^\alpha \widetilde{\nabla}_{x_i} \mathcal{B}_\varepsilon (\widetilde{\nabla}_{x_j} f) \right\rangle_{L_{x,v}^2} + \left\langle \langle v \rangle^\alpha \widetilde{\nabla}_{x_i} \widetilde{\nabla}_{x_j} f, \langle v \rangle^\alpha [\widetilde{\nabla}_{x_j}, \mathcal{B}_\varepsilon] \widetilde{\nabla}_{x_i} f \right\rangle_{L_{x,v}^2} \\ & \quad + \left\langle \langle v \rangle^\alpha \widetilde{\nabla}_{x_i} \widetilde{\nabla}_{x_j} f, \langle v \rangle^\alpha [\widetilde{\nabla}_{x_i}, [\widetilde{\nabla}_{x_j}, \mathcal{B}_\varepsilon]] f \right\rangle_{L_{x,v}^2} \\ & =: R_1 + R_2 + R_3. \end{aligned}$$

From the proof of Lemma 4.29 (Step 5), we already know that

$$R_1 + R_2 \leq -\frac{\kappa}{2\varepsilon^2} \|\langle v \rangle^\alpha \widetilde{\nabla}_x \widetilde{\nabla}_x f\|_{L_x^2(H_{v,*}^1)}^2 + \frac{C}{\varepsilon^2} \|\langle v \rangle^\alpha \langle v \rangle^{\gamma+1} \nabla_x \widetilde{\nabla}_x f\|_{L_{x,v}^2}^2.$$

The term R_3 can be estimated in a similar way as the term I_3 in Step 2 above. Therefore we obtain, using Young's inequality,

$$\begin{aligned} \frac{d}{dt} \|\langle v \rangle^\alpha \widetilde{\nabla}_x \widetilde{\nabla}_x f\|_{L_{x,v}^2}^2 & \leq -\frac{\kappa}{2\varepsilon^2} \|\langle v \rangle^\alpha \widetilde{\nabla}_x \widetilde{\nabla}_x f\|_{L_x^2(H_{v,*}^1)}^2 + \frac{C}{\varepsilon^2} \|\langle v \rangle^\alpha \langle v \rangle^{\gamma+1} \nabla_x \widetilde{\nabla}_x f\|_{L_{x,v}^2}^2 \\ & \quad + \frac{C}{\varepsilon^2} \|\langle v \rangle^\alpha \langle v \rangle^{\frac{3\gamma}{2}+1} \nabla_x \nabla_x f\|_{L_{x,v}^2}^2. \end{aligned} \quad (4.184)$$

Observing that

$$\|\langle v \rangle^\alpha \langle v \rangle^{\gamma+1} \nabla_x \widetilde{\nabla}_x f\|_{L_{x,v}^2} + \|\langle v \rangle^\alpha \langle v \rangle^{\frac{3\gamma}{2}+1} \nabla_x \nabla_x f\|_{L_{x,v}^2} \lesssim \|\langle v \rangle^\alpha \langle v \rangle^{\frac{\gamma}{2}} \nabla_x \widetilde{\nabla}_x f\|_{L_x^2(H_{v,*}^1)}$$

we then gather (4.183)–(4.184) and choose $K_1 \gg K_2 \gg 1$ large enough, which yields

$$\begin{aligned} & \frac{d}{dt} \left(K_1 \|\langle v \rangle^\alpha \langle v \rangle^\gamma \nabla_x \nabla_x f\|_{L_{x,v}^2}^2 + K_2 \|\langle v \rangle^\alpha \langle v \rangle^{\frac{\gamma}{2}} \nabla_x \widetilde{\nabla}_x f\|_{L_{x,v}^2}^2 + \|\langle v \rangle^\alpha \widetilde{\nabla}_x \widetilde{\nabla}_x f\|_{L_{x,v}^2}^2 \right) \\ & \leq -\frac{\kappa}{\varepsilon^2} \|\langle v \rangle^\alpha \widetilde{\nabla}_x \widetilde{\nabla}_x f\|_{L_x^2(H_{v,*}^1)}^2. \end{aligned} \tag{4.185}$$

Step 6. We can then conclude the proof as in Step 6 of the proof of Lemma 4.29. \square

Conclusion

Cette thèse est concentrée sur l'étude de l'équation de Landau. Plus précisément, dans le Chapitre 2, nous avons linéarisé cette équation près de la distribution maxwellienne et nous avons établi des estimations hypoelliptiques, une localisation du spectre et des estimations de la résolvante de l'opérateur de Landau linéarisé associé à cette dernière équation dans les cas de potentiels durs et molécules maxwelliennes. La preuve est basée sur une méthode de multiplicateur et nécessite des outils de calcul pseudo-différentiels raffinés. Ensuite, dans le Chapitre 3, nous avons étudié la limite hydrodynamique de l'équation de Landau. Nous avons obtenu un résultat sur la dérivation du système de Navier-Stokes-Fourier incompressible à partir de l'équation de Landau dans les cas de potentiels durs, maxwelliens et modérément mous. Pour obtenir ce résultat, nous avons étudié d'abord la théorie de Cauchy associée à l'équation de Landau adimensionnée pour des petites données initiales, puis nous avons prouvé des estimations pour quelques normes de Sobolev adaptées à la solution qui sont uniformes en nombre de Knudsen en utilisant la méthode d'énergie non linéaire. Ces estimations uniformes nous ont permis d'obtenir un résultat de convergence faible vers le système limite fluide. Finalement, dans le Chapitre 4, nous avons étudié également la limite hydrodynamique pour l'équation de Landau en améliorant le résultat du Chapitre 3. Plus précisément, nous avons montré un résultat de convergence forte des solutions de l'équation de Landau vers des solutions fortes et globales du système de Navier-Stokes-Fourier incompressible. Nous avons étudié d'abord la théorie de Cauchy dans un cadre perturbatif, puis nous avons montré des estimations de régularisation en temps (uniformes en nombre de Knudsen) pour l'équation de Landau (adimensionnée) linéaire et non linéaire où la méthode d'hypo-coercivité- L^2 joue un rôle important. Ces estimations uniformes avec le résultat sur l'analyse spectrale réalisée par Yang et Yu dans [130] pour plusieurs équations cinétiques (dont l'équation de Landau) nous ont permis d'obtenir ce résultat.

Perspectives

I- Estimations hypoelliptiques et spectrales pour l'opérateur de Landau avec un potentiel extérieur

Dans [111], nous avons montré une propriété de localisation du pseudo-spectre de l'opérateur de Landau linéarisé et établi des estimations importantes pour la résolvante. Il est possible également de regarder l'équation de Landau en ajoutant un potentiel extérieur comme suit :

$$\begin{cases} \partial_t f + v \cdot \nabla_x f - \nabla_x V(x) \cdot \nabla_v f = Q(f, f) \\ f|_{t=0} = f_0, \end{cases} \quad (5.1)$$

où $f = f(t, x, v)$ est la fonction de densité de particules à un temps $t \in \mathbb{R}^+$, une position $x \in \mathbb{R}^3$ et à une vitesse $v \in \mathbb{R}^3$. Ici Q est l'opérateur de collision de Landau (voir Section 1.2 sur la présentation complète de l'opérateur Q ainsi que l'équation de Landau) et V désigne le potentiel électrique. Notre objectif est de donner des hypothèses suffisantes sur le potentiel V comme dans les travaux dans [62, 67, 66] et d'utiliser des outils de calculs pseudo-différentiels (comme dans [111]) dans le but de montrer une propriété de localisation du pseudo-spectre pour l'opérateur de Landau linéarisé \mathcal{P} associé à (5.1) (voir la figure ci-dessous) et d'estimer la résolvante.

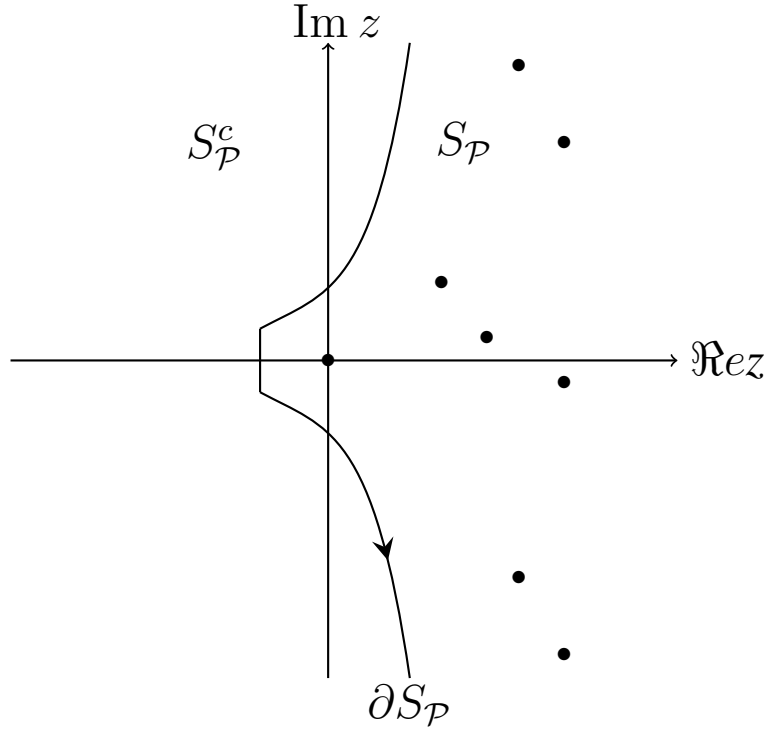


Figure 5.1: Localisation du pseudo-spectre de l'opérateur \mathcal{P} associé à (5.1).

La figure ci-dessus est une conjecture suite au résultat obtenu dans [111] (voir Figure 1.2) dans le cas de l'opérateur de Landau linéarisé sans potentiel extérieur. De plus, grâce aux hypothèses qu'on va supposer sur le potentiel extérieur V comme dans les travaux dans [62, 67], nous envisageons que le spectre de l'opérateur \mathcal{P} associé à (5.1) soit discret comme montre la figure ci-dessus au lieu du spectre continu donné par la Figure 1.2.

II- Étude de la limite hydrodynamique du système de Vlasov-Poisson-Landau

Dans [112, 25], nous avons étudié la limite hydrodynamique de l'équation de Landau et nous avons obtenu les équations limites qui correspondent au système de Navier-Stokes-Fourier incompressible. Une étude intéressante est de coupler l'équation de Landau adimensionnée avec l'équation de Poisson et d'étudier la limite du système obtenu. Plus précisément, nous nous intéressons au système adimensionné suivant :

$$\begin{cases} \varepsilon \partial_t f_\varepsilon + v \cdot \nabla_x f_\varepsilon + \varepsilon \alpha \nabla_x \phi_\varepsilon(x) \cdot \nabla_v f_\varepsilon = \frac{1}{\varepsilon} Q(f_\varepsilon, f_\varepsilon) \\ f_\varepsilon|_{t=0} = f_{\varepsilon,0}, \end{cases} \quad (5.2)$$

où $f_\varepsilon = f_\varepsilon(t, x, v)$ est la fonction de densité de particules à un temps $t \in \mathbb{R}^+$, une position $x \in \mathbb{R}^3$ et à une vitesse $v \in \mathbb{R}^3$. Ici $\varepsilon > 0$ est le nombre de Knudsen qui est l'inverse du nombre moyen de collisions pour chaque particule par unité de temps, $\alpha > 0$ est le rapport de la répulsion électrique suivant la loi de Gauss et le nombre de Knudsen (voir [7] pour plus de détails) et Q est l'opérateur de collision de Landau (ici la présentation complète de l'opérateur Q ainsi que l'équation de Landau sont les mêmes que dans la Section 1.2). On note $\phi_\varepsilon = \phi_\varepsilon(t, x) \in \mathbb{R}$ est le potentiel électrique auto-consistant couplé avec la fonction de densité $f_\varepsilon(t, x, v)$ par l'équation de Poisson

$$\Delta_x \phi_\varepsilon = \frac{\alpha}{\varepsilon} \left(\int_{\mathbb{R}^3} f_\varepsilon \, dv - 1 \right). \quad (5.3)$$

Dans l'article [57], les auteurs ont étudié la limite hydrodynamique du système de Vlasov-Poisson-Boltzmann et ils ont montré que le système limite est le système Navier-Stokes-Fourier-Poisson suivant :

$$\begin{cases} \partial_t u + u \cdot \nabla_x u + \nabla_x p = \nu \Delta_x u + \rho \nabla_x \theta, \\ \partial_t \left(\frac{3}{5} \theta - \frac{2}{5} \rho \right) + u \cdot \nabla_x \left(\frac{3}{5} \theta - \frac{2}{5} \rho \right) = \kappa \Delta_x \theta, \\ \nabla_x \cdot u = 0, \\ \Delta_x (\rho + \theta) = \alpha^2 \rho, \end{cases} \quad (5.4)$$

où dans le système précédent θ (la température), ρ (la densité) et p (la pression) sont des inconnues scalaires et u est un champ de vecteur vitesse à 3 composantes. Le coefficient ν (resp. κ) est la viscosité (resp. la conductivité thermique). Il est envisageable d'obtenir le même système limite (5.4) pour le système de Vlasov-Poisson-Landau (5.2)-(5.3) en combinant les outils utilisés dans [25] ainsi dans [57].

III- Étendre le résultat du Chapitre 4 au cas de potentiels mous et de potentiel Coulombien

La méthode utilisée dans [25] doit être suffisamment robuste pour étendre le résultat au cas de potentiels mous et de potentiel Coulombien $-3 \leq \gamma < -2$. Plus précisément, on doit pouvoir obtenir un résultat similaire au Théorème 1.6 avec une décroissance sous-exponentielle au lieu de la décroissance exponentielle en temps (1.58) en combinant les arguments développés dans [25] avec l'étude faite dans le cas $\varepsilon = 1$ par Carrapatoso et Mischler dans [22]. Pour obtenir le résultat hydrodynamique dans [25] (voir Théorème 1.9), nous avons utilisé des estimations spectrales fines qui ne sont valables que pour le cas $-2 \leq \gamma \leq 1$ par [130]. Si nous pouvons étendre les résultats de [130] au cas $-3 \leq \gamma < -2$, nous pouvons obtenir un résultat analogue au Théorème 1.9 et ainsi étendre le résultat au cas de potentiels mous et de potentiel Coulombien.

IV- Étude du problème de Vlasov-Poisson-Fokker-Planck fractionnaire

Il serait intéressant d'étudier le problème de Cauchy pour le système de Vlasov-Poisson-Fokker-Planck fractionnaire donné sur $\mathbb{R}^+ \times \mathbb{R}_x^d \times \mathbb{R}_v^d$ où $d = 2$ ou $d = 3$ par

$$\begin{cases} \partial_t f + v \cdot \nabla_x f - \varepsilon_0 E \cdot \nabla_v f + (-\Delta_v)^s f - \nabla_v \cdot (vf) = 0 \\ E(t, x) = -\frac{1}{|\mathbb{S}^{d-1}|} \frac{x}{|x|^d} *_x \rho(t, x), \quad \text{où} \quad \rho(t, x) = \int_{\mathbb{R}^d} f(t, x, v) dv, \\ f(0, x, v) = f_0(x, v), \end{cases} \quad (5.5)$$

où $f = f(t, x, v)$ est la distribution de particules, $(-\Delta_v)^s$ est le laplacien fractionnaire en vitesse avec $s \in (0, 1]$ et défini pour $\phi \in \mathcal{S}'(\mathbb{R}_v^d)$ par

$$(-\Delta_v)^s \phi(v) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{iv \cdot \eta} |\eta|^{2s} \hat{\phi}(\eta) d\eta \quad \text{où} \quad \hat{\phi}(\eta) = \int_{\mathbb{R}^d} e^{-iv \cdot \eta} \phi(v) dv. \quad (5.6)$$

La constante $\varepsilon_0 \in \mathbb{R}$ est la charge totale du système.

Dans un travail en cours en collaboration avec Frédéric Hérau et Isabelle Tristani, nous nous intéressons à étudier la régularité nécessaire de la donnée initiale pour pouvoir prouver un résultat d'existence de solutions du problème (5.5). Nous planifions d'utiliser des estimations de régularisation en temps court et le théorème du point fixe en suivant la méthode utilisée par F. Hérau et L.Thomann dans [69] pour étudier le problème de l'existence de solution pour le cas non fractionnaire.

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Titre: Estimations spectrales et limite hydrodynamique pour l'équation de Landau

Mot clés : théorie cinétique; équation de Landau; pseudo-spectre; hypoellipticité, limite hydrodynamique; Navier-Stokes incompressible; nombre de Knudsen.

Resumé : Cette thèse est consacrée à l'étude de l'équation de Landau. Premièrement, nous montrons une propriété de localisation du pseudo-spectre et des estimations de la résolvante de l'opérateur de Landau linéarisé associé à cette dernière équation dans les cas de potentiels durs et molécules maxwelliennes. Deuxièmement, nous étudions la limite hydrodynamique de l'équation de Landau et nous obtenons un résultat de convergence faible vers le système de Navier-Stokes-Fourier incompressible dans les cas de potentiels durs, maxwelliens et modérément mous. Finalement, nous améliorons le résultat précédent sur la limite hydrodynamique et nous montrons un résultat de convergence forte vers le même système limite fluide dans les cas de potentiels durs, maxwelliens et modérément mous.

Title: Spectral estimates and hydrodynamical limit for the Landau equation

Keywords : kinetic theory; Landau equation; pseudo-spectrum; hypoellipticity, hydrodynamical limit; Incompressible Navier-Stokes; Knudsen number.

Abstract : This thesis is devoted to the study of the Landau equation. Firstly, we show a localization property of the pseudospectrum and estimates of the resolvent of the linearized Landau operator associated with the latter equation in the case of hard potentials and Maxwellian molecules. Secondly, we study the hydrodynamical limit of the Landau equation and we obtain a result of weak convergence towards the incompressible Navier-Stokes-Fourier system in the cases of hard potentials, Maxwellian molecules and moderately soft potentials. Finally, we improve the previous result on the hydrodynamical limit and we show a result of strong convergence towards the same fluid limit system in the cases of hard potentials, Maxwellian molecules and moderately soft potentials.