Unipotent Representations of Finite Reductive Groups in Transverse Characteristic

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Représentations unipotentes des groupes réductifs finis en caractéristique transverse

Unipotent representations of finite reductive groups in transverse characteristic

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CHAPTER 1

Introduction

This report presents a selection of results on modular representations of finite reductive groups that the author obtained since his Ph.D. thesis. The problems that we shall focus on in this report fall in three categories:

- *Numerical invariants*. Classify irreducible representations, compute their dimension and their character. Determine the decomposition matrices and the Cartan matrices of blocks.
- *Homological invariants*. Compute extensions between irreducible representations. Describe the category of representations (of each block) up to abelian, derived or stable equivalence.
- *Branching rules*. Find the cuspidal representations, the corresponding Harish–Chandra series and determine the branching rules for parabolic induction and restriction.

The main techniques for studying the representations of finite reductive groups are of a geometric nature. Following the work of Deligne and Lusztig [29] for ordinary (characteristic zero) representations, we will explain how these problems can be tackled using the knowledge of mod-ℓ cohomology groups and cohomology complexes of Deligne–Lusztig varieties. Along the way, we will present results on the cohomology of these varieties, such as the explicit determination of individual cohomology groups (over $\mathbb{Q}_\ell$) or the determination of the torsion part of these groups (over $\mathbb{Z}_\ell$). Most of these results are inspired by the geometric version of Broué's abelian defect group conjecture [18].

1.1. Representations of finite groups

Let G be a finite group and $\mathbb{A}$ be a commutative ring with unit. By a representation of G over $\mathbb{A}$ we mean a finitely generated $\mathbb{A}$-module endowed with an action of G by linear automorphisms. This is equivalent to a structure of finitely generated $\mathbb{A}G$-module. The representations of G over $\mathbb{A}$ form an abelian category which we will denote by $\mathbb{A}G$-mod. The set of isomorphism classes of simple (or irreducible) objects in this category will be denoted by $\mathrm{Irr}_{\mathbb{A}} G$.

The *ordinary* representation theory of G is concerned with the case where $\mathbb{A} = \mathbb{K}$ is a field of characteristic zero which contains enough roots of unity (with respect to the order of G). In that situation the algebra $\mathbb{K}G$ is split semisimple and all the information on the abelian category $\mathbb{K}G$-mod is encoded in its Grothendieck group $K_0(\mathbb{K}G\text{-mod})$. In other words, ordinary representations are determined by their character and the only invariants of representations are of a numerical nature. When $\mathbb{A} = \mathbb{k}$ is a field of positive characteristic ℓ – that is for *modular* representations – the situation is much more complicated. When ℓ divides the order of G, the category $\mathbb{k}G$ is not semisimple and understanding irreducible representations is not enough to understand the whole

category. Other classes of indecomposable representations, such as the projective indecomposable modules (which we will abbreviate by PIMs) play a prominant role in studying the category. In addition, short exact sequences do not split in general and the Grothendieck group $K_0(\Bbbk G\text{-mod})$ gives little information on how a representation decomposes and how the composition factors interact with each other. The extra information needed is of a homological nature and is encoded in the extension groups between representations.

In order to study both ordinary and modular representations at the same time, it is convenient to consider representations of G over a discrete valuation ring $\mathbb{O}$ whose fraction field is $\mathbb{K}$ (a field of characteristic zero) and residue field is $\Bbbk$ (a field of characteristic ℓ). Such a triple $(\mathbb{K}, \mathbb{O}, \Bbbk)$ is called an *ℓ-modular system*. We will usually work with those systems which are obtained as finite extensions of $(\mathbb{Q}_\ell, \mathbb{Z}_\ell, \mathbb{F}_\ell)$ such that $\mathbb{K}G$ is split semisimple. Any $\mathbb{K}G$-module admits an integral form over $\mathbb{O}$, which in turn can be reduced modulo ℓ to give a $\Bbbk G$-module. This construction depends on the choice of the integral form, but it is well-defined at the level of Grothendieck groups. The corresponding map

$$\mathrm{dec} : K_0(\mathbb{K}G\text{-mod}) \longrightarrow K_0(\Bbbk G\text{-mod})$$

is called the *decomposition map* and its matrix in the bases $\mathrm{Irr}_{\mathbb{K}} G$ and $\mathrm{Irr}_{\Bbbk} G$ is the *decomposition matrix* of G. On the other hand, every projective $\Bbbk G$-module lifts to a projective $\mathbb{O}G$-module, yielding an embedding $K_0(\Bbbk G\text{-proj}) \hookrightarrow K_0(\mathbb{K}G\text{-mod})$ obtained after extending scalars from $\mathbb{O}$ to $\mathbb{K}$. By a character of a projective $\Bbbk G$-module P we will often mean its image under this embedding, and we will denote it by $[P]$. Brauer reciprocity implies that the decomposition matrix is also the matrix of the characters of the PIMs decomposed in the basis $\mathrm{Irr}_{\mathbb{K}} G$.

The group algebra $\mathbb{K}G$ is split semisimple and therefore it is isomorphic to a product of matrix algebras over $\mathbb{K}$ (one for each element in $\mathrm{Irr}_{\mathbb{K}} G$). This is no longer the case for $\Bbbk G$ or $\mathbb{O}G$ when ℓ divides the order of G. However, one can still decompose the algebra $\mathbb{O}G$ into a direct sum of minimal two-sided ideals called *ℓ-blocks*

$$\mathbb{O}G = B_0 \oplus B_1 \oplus \cdots \oplus B_r.$$

The augmentation map $\mathbb{O}G \twoheadrightarrow \Bbbk$ factors through a unique block called the *principal block*, here denoted by B_0. Each block B_i is of the form $\mathbb{O}Gb_i$ for some central primitive idempotent b_i in $\mathbb{O}G$. Blocks of $\Bbbk G$ are the ℓ-reduction of blocks of $\mathbb{O}G$. To an ℓ-block B one can associate:

- the block idempotent $b \in Z(\mathbb{O}G)$ such that $B = \mathbb{O}Gb$;
- the ordinary irreducible characters $\mathrm{Irr}_{\mathbb{K}} B$ lying in the block, which are those irreducible characters $\chi \in \mathrm{Irr}_{\mathbb{K}} G$ such that $\chi(b) \neq 0$;
- the category $\mathbb{A}B$-mod of representations over the block, where $\mathbb{A}$ is any ring among $(\mathbb{K}, \mathbb{O}, \Bbbk)$. It is a direct summand of the category $\mathbb{A}G$-mod: non-zero indecomposable representations V of $\mathbb{A}B$ correspond to non-zero indecomposable $\mathbb{A}G$-modules such that $bV \neq 0$; we will say that such representations belong to B;
- a conjugacy class of ℓ-subgroups of G called the *defect groups*. They measure how far the block is from being a simple algebra. Defect groups of the principal ℓ-block are the Sylow ℓ-subgroups of G.

The decomposition map preserves the block decomposition, so we can talk about the decomposition matrix of an ℓ-block. This numerical invariant will be studied in §2.2 and §2.3 in the case of unipotent blocks of finite reductive groups.

1.2. Finite reductive groups and Deligne–Lusztig varieties

Let p be a prime number and let q be a power of p. Let $\mathbf{G}$ a be connected reductive group defined over the finite field $\mathbb{F}_q$ via a Frobenius endomorphism $F : \mathbf{G} \longrightarrow \mathbf{G}$. The group of rational points $\mathbf{G}^F = \mathbf{G}(\mathbb{F}_q)$ is called a *finite reductive group*. Given any F-stable closed subgroup $\mathbf{H}$ of $\mathbf{G}$, we will denote by $H := \mathbf{H}^F$ the corresponding group of rational points. It follows from the classification of finite simple groups that most of them come from finite reductive groups (up to considering their derived subgroup and modding out by their center). Therefore understanding their representation theory is a very important problem.

In their landmark paper [29] Deligne and Lusztig introduced a family of algebraic varieties acted on by the finite reductive group G whose ℓ-adic cohomology contains all the irreducible representations of G (over $\overline{\mathbb{Q}}_\ell$). Before recalling their definition we need to fix some data associated to $\mathbf{G}$. We choose an F-stable maximal torus $\mathbf{T}$ of $\mathbf{G}$ contained in an F-stable Borel subgroup $\mathbf{B}$ of $\mathbf{G}$. We denote by $W = N_{\mathbf{G}}(\mathbf{T})/\mathbf{T}$ the Weyl group and by $S \subset W$ the set of simple reflections corresponding to the pair $(\mathbf{T}, \mathbf{B})$. Given $I \subset S$ we denote by W_I the subgroup of W generated by I. The subgroup $\mathbf{P}_I := \mathbf{B}W_I\mathbf{B}$ is called a standard parabolic subgroup of $\mathbf{G}$. It has a Levi decomposition $\mathbf{P}_I = \mathbf{L}_I \ltimes \mathbf{U}_I$ where $\mathbf{U}_I$ is the unipotent radical of $\mathbf{P}_I$ and $\mathbf{L}_I$ is the unique Levi complement of $\mathbf{P}_I$ containing $\mathbf{T}$. Let $w \in W$ be such that the pair (I, w) satisfies the following properties:

- w is I-reduced, that is it has minimal length in the coset $W_I w$;
- $w^{-1}sw \in F(I)$ for any $s \in I$. (1.2.1)

To the pair (I, w) we associate the *parabolic Deligne–Lusztig varieties*

$$\begin{array}{rcl} \mathbf{Y}(I,w) & = & \{g \in \mathbf{G}/\mathbf{U}_I \mid \ g^{-1}F(g) \in \mathbf{U}_I w \mathbf{U}_{F(I)}\} \\ \Big\downarrow{\scriptstyle /\mathbf{L}_I^{wF}} & & \\ \mathbf{X}(I,w) & = & \{g \in \mathbf{G}/\mathbf{P}_I \mid \ g^{-1}F(g) \in \mathbf{P}_I w \mathbf{P}_{F(I)}\}. \end{array}$$

Note that the definition of $\mathbf{Y}(I, w)$ depends on the choice of a representative of w in $N_{\mathbf{G}}(\mathbf{T})$, but we shall not specify it to avoid cumbersome notation. Both varieties are acted on by the finite reductive group G by left multiplication, and the natural projection $\mathbf{G}/\mathbf{U}_I \twoheadrightarrow \mathbf{G}/\mathbf{P}_I$ induces a G-equivariant isomorphism $\mathbf{Y}(I, w)/\mathbf{L}_I^{wF} \xrightarrow{\sim} \mathbf{X}(I, w)$. Their dimension equals $\ell(w)$, the length of w. Originally, only the case $I = \emptyset$ was considered in [29]. In that case no condition on w is required to define the corresponding Deligne–Lusztig varieties, and we will denote them simply by $\mathbf{Y}(w)$ and $\mathbf{X}(w)$.

Recall that $\mathbb{K}$ is a finite extension of $\mathbb{Q}_\ell$ which is sufficiently large for G. Assume now that $\ell \neq p$. Then one can consider the ℓ-adic cohomology groups of the Deligne–Lusztig varieties with coefficients in $\mathbb{K}$. They yield representations of G over $\mathbb{K}$. Given $\theta \in \mathrm{Irr}_{\mathbb{K}}\mathbf{T}^{wF}$, Deligne and Lusztig defined the virtual character

$$\mathrm{R}_w(\theta) := \sum_{i \in \mathbb{Z}} (-1)^i \left[H_c^i(\mathbf{Y}(w), \mathbb{K}) \otimes_{\mathbf{T}^{wF}} \theta\right] \in K_0(\mathbb{K}G\text{-mod}).$$

They showed that every irreducible representation of G occurs in some $\mathrm{R}_w(\theta)$. The ones occurring in $\mathrm{R}_w(1_{\mathbf{T}^{wF}})$ – where $1_{\mathbf{T}^{wF}}$ is the trivial character of $\mathbf{T}^{wF}$ – are called *unipotent*.

When working with modular representations (over $\mathbb{A} = \mathbb{O}$ or $\Bbbk$), one should consider finer linear invariants. Following the work of Broué [18] and Bonnafé–Rouquier [21] we are rather interested in the cohomology complex with compact support $R\Gamma_c(\mathbf{Y}(I,w),\mathbb{A})$ of the Deligne–Lusztig variety $\mathbf{Y}(I,w)$ with coefficients in $\mathbb{A}$. This is well-defined in $\mathrm{D}^b(\mathbb{A}\mathbf{G}\text{-mod-}\mathbb{A}\mathbf{L}_I^{wF})$, the bounded derived category of finitely generated $(\mathbb{A}\mathbf{G},\mathbb{A}\mathbf{L}_I^{wF})$-bimodules. The construction of Deligne–Lusztig lifts to an exact functor

$$\mathcal{R}_{I,w} := R\Gamma_c(\mathbf{Y}(I,w),\mathbb{O}) \otimes^L_{\mathbb{O}\mathbf{L}_I^{wF}} -$$

between the derived categories $\mathrm{D}^b(\mathbb{O}\mathbf{G}\text{-mod})$ and $\mathrm{D}^b(\mathbb{O}\mathbf{L}_I^{wF}\text{-mod})$. Under the identification of the Grothendieck groups of a module category and its bounded derived category, the linear map R_w corresponds to the map induced by the functor $\mathbb{K}\mathcal{R}_{\emptyset,w}$ at the level of Grothendieck groups. The functor $\mathcal{R}_{I,w}$ has a right adjoint, given by

$${}^*\mathcal{R}_{I,w} := R\mathrm{Hom}_{\mathbb{O}G}\big(R\Gamma_c(\mathbf{Y}(I,w),\mathbb{O}),-\big).$$

The functors $\mathcal{R}_{I,w}$ and ${}^*\mathcal{R}_{I,w}$ are called the *Deligne–Lusztig induction and restriction* functors. Even though the virtual characters $\mathrm{R}_w(\theta)$ were explicitly computed by Lusztig in [64], little is known about the the individual cohomology groups of the Deligne–Lusztig varieties, let alone the induction and restriction functors.

When $w = 1$ is the trivial element, the corresponding parabolic Deligne–Lusztig varieties are just the finite sets G/U_I and G/P_I respectively. In that case the cohomology complexes are given by the corresponding permutation modules, placed in degree 0, and the Deligne–Lusztig induction and restriction functors are induced by exact functors between the abelian categories $\mathbb{O}L_I$-mod and $\mathbb{O}G$-mod called the *parabolic* (or *Harish-Chandra*) *induction and restriction* functors. For simplicity we will denote them by $\mathcal{R}_I$ and ${}^*\mathcal{R}_I$ respectively.

When ℓ is not too small and does not divide q, unipotent ℓ-blocks have abelian defect. Their classification, which was determined by Fong–Srinivasan [34, 36] and Broué–Malle–Michel [19], depends only on the order $d_\ell(q)$ of the class of q in $\Bbbk^\times$. The unipotent characters lying in each unipotent block are obtained after a suitable Deligne–Lusztig induction. This provides numerical evidence for the geometric version of Broué's conjecture, see §1.3.

1.3. The abelian defect group conjecture

Let G be any finite group, and let B be an ℓ-block of G with defect group D. To the block B Brauer associates a block B' of $N_G(D)$ with the same defect group, the *Brauer correspondent* of B. There are many results and problems relating the representation theory of the blocks B and B', see for example [67] for some examples of open problems. Most of our work is inspired by one of these conjectures, which was stated by Broué in 1988 [18].

Conjecture 1.3.1 (Broué's abelian defect group conjecture). *Let B be a block of $\mathbb{O}G$ with defect group D and let B' be its Brauer correspondent, a block of $\mathbb{O}N_G(D)$. If D is abelian then*

$$\mathrm{D}^b(B\text{-mod}) \simeq \mathrm{D}^b(B'\text{-mod})$$

as triangulated categories.

Even though it was solved for some classes of groups (e.g. for ℓ-solvable groups [53], symmetric groups or $\mathrm{GL}_n(q)$ [25]) or under some restriction on the defect (e.g.

when D is cyclic [70, 61, 71]), this conjecture remains mostly open in general. However most of its numerical consequences are already known to hold.

When $G = \mathbf{G}(\mathbb{F}_q)$ is a finite reductive group and $\ell \nmid q$, Broué predicted that the derived equivalence between blocks can be induced by the cohomology complex of a suitable Deligne–Lusztig variety. To simplify its statement, we shall consider principal blocks only, and assume that F acts trivially on W. Let D be a Sylow ℓ-subgroup of G. If D is abelian then there exists a pair (I, w) as in (1.2.1) such that $C_G(D) \simeq \mathbf{L}_I^{wF}$. The pair (I, w) is defined only up to conjugation but Broué predicts that there is a choice of such a pair such that the action of $C_G(D)$ on $R\Gamma_c(\mathbf{Y}(I, w), \mathbb{O})$ under the isomorphism $C_G(D) \simeq \mathbf{L}_I^{wF}$ extends to an action of $N_G(D)$ and that the functor

$$R\Gamma_c(\mathbf{Y}(I, w), \mathbb{O}) \otimes^L_{\mathbb{O}N_G(D)} -$$

induces a derived equivalence between the principal blocks of $N_G(D)$ and G, or more generally between $\mathbb{O}N_G(D)$ and the sum of all ℓ-blocks of defect D of G. For this property to hold, the complex $C := R\Gamma_c(\mathbf{Y}(I, w), \mathbb{O})$ should be a tilting complex, therefore satisfying

- $\mathrm{Hom}_{\mathrm{D}^b(\mathbb{O}G\text{-mod})}(C, C[n]) = 0$ for $n \neq 0$;
- $\mathbb{O}N_G(D) \xrightarrow{\sim} \mathrm{End}_{\mathrm{D}^b(\mathbb{O}G\text{-mod})}(C)$ via the natural action map.

For choosing the suitable pairs (I, w) it is convenient to work in the braid group or in the braid monoid attached to W, rather than in the Weyl group. Let $\mathbf{S}$ be a set in bijection with S. The braid monoid B_W^+ is the monoid generated by $\mathbf{S}$ subject to the braid relations $\mathbf{sts}\cdots = \mathbf{tst}\cdots$ coming from the Coxeter presentation of (W, S). It embeds in its fraction group B_W. Both B_W^+ and B_W map naturally onto W. In addition, there is a set-theoretic section $W \hookrightarrow B_W^+$ induced by the bijection $S \xrightarrow{\sim} \mathbf{S}$ on reduced expressions. The lift under this section of an element $w \in W$ will be denoted by $\mathbf{w}$. The definition of Deligne–Lusztig varieties extends to pairs $(I, \mathbf{b})$ where $I \subset S$, and $\mathbf{b} \in B_W^+$ satisfies

- $\mathbf{b}$ is I-reduced, that is $\mathbf{s}^{-1}\mathbf{b} \notin B_W^+$ whenever $s \in I$;
- for all $s \in I$ there exists $t \in I$ such that $\mathbf{b}^{-1}\mathbf{sb} = F(\mathbf{t})$[1]. (1.3.1)

Now, let w_0 be the longest element of W and let $\mathbf{w}_0$ be its lift to B_W^+. The full-twist $\boldsymbol{\pi} := \mathbf{w}_0^2$ is a central element in B_W^+. Then the pairs (I, w) one should consider for Broué's conjecture are those satisfying

$$\mathbf{w}^d = \boldsymbol{\pi}/\boldsymbol{\pi}_I$$

where $d := d_\ell(q)$ is the order of the class of q in $\mathbb{k}^\times$, and $\boldsymbol{\pi}_I$ is the full-twist associated with the parabolic subgroup W_I.

When $I = \emptyset$, that is when the centraliser of a Sylow ℓ-subgroup is a torus, the choice of a d-th root of $\boldsymbol{\pi}$ corresponds to the choice of a specific d-regular element w of W in the sense of Springer [74], and was first considered by Broué–Michel in [20]. In that situation, $C_G(D) \simeq \mathbf{T}^{wF}$ and $N_G(D)/C_G(D) \simeq C_W(w)$ is a complex reflection group whose associated braid monoid is $C_{B_W^+}(\mathbf{w})$. It is expected that the action of $N_G(D)$ on $R\Gamma_c(\mathbf{Y}(w), \mathbb{O})$ comes from a q-deformation of a natural action of $C_W(w) \ltimes \mathbf{T}^{wF}$, which in turn can be defined from a geometric action of $C_{B_W^+}(\mathbf{w})$ on $\mathbf{X}(w)$. The case where I is non-empty was studied by Digne–Michel in [32].

[1] The action of F on B_W is induced by the action of F on S and the bijection $S \xrightarrow{\sim} \mathbf{S}$. However here $F(\mathbf{t}) = \mathbf{t}$ since we assumed for simplicity that F acts trivially on W.

1.4. Outline of the results

The selection of results presented in this report are arranged in four sections, which we detail below.

Cohomology of Deligne–Lusztig varieties. In a joint work with Bonnafé–Rouquier [1] we prove a periodicity conjecture of Digne–Michel–Rouquier relating the ℓ-adic cohomology of the Deligne–Lusztig varieties $\mathbf{X}(\pi\mathbf{w})$ and $\mathbf{X}(\mathbf{w})$. As a byproduct we obtain the explicit ℓ-adic cohomology for the variety $\mathbf{X}(\pi)$ (the case $d = 1$ in Broué's conjecture). Together with the results in [5], this completes the determination of the ℓ-adic cohomology of all the Deligne–Lusztig varieties which are relevant for Broué's conjecture for $\mathrm{GL}_n(q)$.

Brauer trees. With Rouquier and Craven–Rouquier we complete the determination of the planar embedded Brauer trees for the unipotent ℓ-blocks [14, 3]. We first obtain the Brauer trees corresponding to principal ℓ-blocks when $d_\ell(q)$ is the Coxeter number, solving a conjecture of Hiss–Lübeck–Malle. This uses the property that the cohomology groups with coefficients in $\mathbb{Z}_\ell$ of varieties associated to Coxeter elements are torsion-free. We then extend this torsion-freeness result to other characteristics to determine the missing Brauer trees for unipotent ℓ-blocks of groups of type E_7 and E_8.

Decomposition matrices. In a series of papers with Malle [8, 9, 12], we compute explicitly the decomposition matrices of unipotent ℓ-blocks of finite reductive groups of small rank (up to $\mathrm{SU}_{10}(q)$ for finite unitary groups). We also obtain families of decomposition numbers for finite classical groups of any rank, growing with the rank. The new input in our method is the systematic use of the ℓ-adic cohomology of Deligne–Lusztig varieties to obtain small bounds on decomposition numbers. We conjecture that the intersection cohomology of these varieties should even yield "small" non-virtual projective modules [7]. We are also interested in the global shape of the decomposition matrices: using the generalised Gelfand–Graev representations we show with Malle in [10] that unipotent cuspidal characters remain irreducible after ℓ-reduction. A modified version of these representations is used with Brunat–Taylor in [2] to prove that the matrices have uni-triangular shape. Both of these statements were conjectured by Geck in 1991.

Categorical actions. In a joint work with Varagnolo–Vasserot [17, 15] we construct categorical actions of Kac–Moody algebras on the category of unipotent representations of classical groups (with the exception of groups of type D). From our categorification result we deduce the branching rule for parabolic induction and restriction and produce many interesting derived equivalences between blocks. In particular we obtain a proof of Broué's conjecture for unipotent ℓ-blocks at linear primes ℓ. In [16] we show how Deligne–Lusztig induction and restriction functors can also induce interesting categorical actions, and give a conjectural framework for Ennola duality.

CHAPTER 2

Research report

2.1. ℓ-adic cohomology of Deligne–Lusztig varieties

Recall from §1.3 that the parabolic Deligne–Lusztig varieties $\mathbf{Y}(I,\mathbf{w})$ and $\mathbf{X}(I,\mathbf{w})$ which are relevant for the geometric version of Broué's conjecture are associated with roots of π/π_I. The cohomology complexes of these varieties induce induction and restriction functors between the bounded derived categories of representations of the finite reductive groups $\mathbf{L}_I^{\mathbf{w}F}$ and G

$$\mathrm{D}^b(\mathbb{\Lambda}\mathbf{L}_I^{\mathbf{w}F}\text{-mod}) \underset{{}^*\mathcal{R}_{I,\mathbf{w}}}{\overset{\mathcal{R}_{I,\mathbf{w}}}{\rightleftarrows}} \mathrm{D}^b(\mathbb{\Lambda}G\text{-mod}).$$

In this section we present results obtained in [5] and with Bonnafé–Rouquier in [1] under the restriction that we work with representations over $\mathbb{\Lambda}=\mathbb{K}\supset\mathbb{Q}_\ell$ and consider only unipotent representations. In that case the description of the functors $\mathcal{R}_{I,\mathbf{w}}$ and ${}^*\mathcal{R}_{I,\mathbf{w}}$ amounts to the determination of each individual cohomology group of the parabolic Deligne–Lusztig variety $\mathbf{X}(I,\mathbf{w})$, with coefficients in any unipotent $\mathbb{K}$-local system.

2.1.1. Translation by the full twist. We start with the case where $I=\emptyset$, corresponding to the case of non-parabolic Deligne–Lusztig varieties. We are interested in computing the ℓ-adic cohomology groups of the varieties $\mathbf{X}(\mathbf{w})$ where $\mathbf{w}$ is a root of the full twist π in the braid monoid, or more generally a root of a power of π, therefore satisfying $\mathbf{w}^d=\pi^r$ in B_W^+. In general there are no interesting relations between the varieties $\mathbf{X}(\mathbf{w})$, $\mathbf{X}(\mathbf{w}')$ and $\mathbf{X}(\mathbf{w}\mathbf{w}')$. For that reason one should rather work with sheaves on the double flag variety, for which such relations will exist. More precisely, if $\mathbf{F}\ell$ denotes the flag variety of $\mathbf{G}$ (the variety of all Borel subgroups of $\mathbf{G}$) and $\iota: x\mapsto(x,F(x))$ is the inclusion of the graph of F in the double flag variety $\mathbf{F}\ell\times\mathbf{F}\ell$ then one can consider the functor

$$\begin{array}{rccc}\mathrm{Ind}: & \mathrm{D}^b_{\mathbf{G}}(\mathbf{F}\ell\times\mathbf{F}\ell) & \longrightarrow & \mathrm{D}^b(\mathbb{K}G\text{-mod}).\\ & \mathscr{F} & \longmapsto & R\Gamma_c(\mathbf{F}\ell,\iota^*\mathscr{F})\end{array}$$

Here $\mathrm{D}^b_{\mathbf{G}}(\mathbf{F}\ell\times\mathbf{F}\ell)$ denotes the $\mathbf{G}$-equivariant bounded derived category of constructible $\mathbb{K}$-sheaves on the double flag variety $\mathbf{F}\ell\times\mathbf{F}\ell$. The category $\mathrm{D}^b_{\mathbf{G}}(\mathbf{F}\ell\times\mathbf{F}\ell)$ has a very rich structure which was intensively studied over the past decades. For our purpose, let us just recall that to each element $\mathbf{w}$ of the braid group one can associate an element $\mathscr{F}(\mathbf{w})$ of that category such that

$$\mathrm{Ind}\,\mathscr{F}(\mathbf{w})\simeq R\Gamma_c(\mathbf{X}(\mathbf{w}),\mathbb{K})[\dim\mathbf{F}\ell+\ell(\mathbf{w})]$$

whenever $\mathbf{w}$ lies in the braid monoid B_W^+. In addition, there is a convolution on $\mathrm{D}^b_{\mathbf{G}}(\mathbf{F}\ell\times\mathbf{F}\ell)$ satisfying $\mathscr{F}(\mathbf{w})\odot\mathscr{F}(\mathbf{w}')\simeq\mathscr{F}(\mathbf{w}\mathbf{w}')$. Therefore one can hope to deduce results

on the cohomology of $\mathbf{X}(\mathbf{w})$ using properties of the object $\mathscr{F}(\pi)$ and the relation $\mathscr{F}(\mathbf{w})^{\odot d} \simeq \mathscr{F}(\pi)^{\odot r}$ whenever $\mathbf{w}$ satisfies $\mathbf{w}^d = \pi^r$.

In [1] we focused our attention on $\mathscr{F}(\pi)$. There is a filtration on the category $\mathrm{D}^b_{\mathbf{G}}(\mathbf{F}\ell \times \mathbf{F}\ell)$ coming from Kazhdan–Lusztig theory such that the convolution by $\mathscr{F}(\pi)$ on each subquotient is isomorphic to a shift of the identity functor. Using results of Lusztig [66] on the functor Ind defined above, we were able to transfer that property to the category $\mathrm{D}^b(\mathbb{K}G\text{-mod})$ and to show that the cohomology of the Deligne–Lusztig varieties $\mathbf{X}(\pi\mathbf{w})$ and $\mathbf{X}(\mathbf{w})$ differ only by shifts on each isotypic component.

THEOREM 2.1.1 (Bonnafé–D.–Rouquier [1, Thm. B]). *Let ρ be a unipotent character of G and A_ρ be the degree of the degree polynomial of ρ. Then for every $\mathbf{w} \in B_W^+$ we have*

$$\Big\langle \rho\,;\, H_c^{i+4\dim \mathbf{F}\ell - 2A_\rho}\big(\mathbf{X}(\pi\mathbf{w}), \mathbb{K}\big)\Big\rangle_G = \Big\langle \rho\,;\, H_c^{i}\big(\mathbf{X}(\mathbf{w}), \mathbb{K}\big)\Big\rangle_G.$$

This result was conjectured by Digne–Michel–Rouquier in [33] based on explicit calculations for groups of small rank. Applied to the element $\mathbf{w} = 1$, it gives the individual ℓ-adic cohomology groups of the variety $\mathbf{X}(\pi)$, or more generally of the variety $\mathbf{X}(\pi^r)$, and shows that odd-degree cohomology groups vanish. This was originally conjectured by Broué–Michel in [20].

Another consequence of the properties of the functor Ind is that the cohomology of $\mathbf{X}(\mathbf{w})$ is invariant under conjugation by the braid group. This was also conjectured by Digne–Michel–Rouquier in [33].

THEOREM 2.1.2 (Bonnafé–D.–Rouquier [1, Thm. A]). *Let $\mathbf{w}, \mathbf{w}' \in B_W^+$ and $\mathbf{x} \in B_W$. Assume that $\mathbf{w} = \mathbf{x}^{-1}\mathbf{w}'F(\mathbf{x})$. Then for every $i \in \mathbb{Z}$ there is an isomorphism of $\mathbb{K}G$-modules*

$$H_c^i\big(\mathbf{X}(\mathbf{w}), \mathbb{K}\big) \simeq H_c^i\big(\mathbf{X}(\mathbf{w}'), \mathbb{K}\big).$$

Such a result was already obtained by Deligne–Lusztig [29] when $\mathbf{x}$ divides $\mathbf{w}'$ on the left in the braid monoid. In that case the element $\mathbf{x}$ defines an equivariant homomorphism $\mathbf{X}(\mathbf{w}) \to \mathbf{X}(\mathbf{w}')$ inducing the equivalence at the level of the cohomology. We used properties of the functor Ind to drop the assumption on $\mathbf{x}$. We plan to show in a subsequent work that our construction gives an action of $C_{B_W}(\mathbf{w})$ on $H_c^i\big(\mathbf{X}(\mathbf{w}), \mathbb{K}\big)$. The construction of such actions is part of the strategy towards a proof of the geometric version of Broué's conjecture as we explained in §1.3.

2.1.2. Deligne–Lusztig varieties for $\mathrm{GL}_n(q)$. Let $\mathbf{G} = \mathrm{GL}_n$ be the general linear group over $\overline{\mathbb{F}}_p$, let q be a power of p and let $F : (a_{ij}) \longmapsto (a_{ij}^q)$ be the Frobenius map associated to the natural $\mathbb{F}_q$-structure on $\mathbf{G}$, so that $G = \mathrm{GL}_n(q)$. The unipotent representations of $\mathrm{GL}_n(q)$ all lie in the principal series. They are therefore parametrised by irreducible representations of the symmetric group $\mathfrak{S}_n$, which are in turn labelled by partitions of n. We write

$$\begin{array}{ccc} \{\text{Partitions of } n\} & \overset{\sim}{\longleftrightarrow} & \{\text{Unipotent representations of } \mathrm{GL}_n(q)\} \\ \lambda & \longmapsto & \Delta(\lambda) \end{array}$$

with the convention that $\Delta(n)$ is the trivial representation whereas $\Delta(1^n)$ is the Steinberg representation.

The finite Levi subgroups $\mathbf{L}_I^{\mathbf{w}F}$ which are relevant for Broué's abelian defect group conjecture are the ones which possess a unipotent representation with central ℓ-defect.

These subgroups are all of the form

$$\mathbf{L}_I^{\mathbf{w}F} \simeq \mathrm{GL}_{n-ad}(q) \times \underbrace{\mathrm{GL}_1(q^d) \times \cdots \times \mathrm{GL}_1(q^d)}_{a \text{ factors}}$$

where $d := d_\ell(q)$ is the order of q in $\mathbb{F}_\ell^\times$. In that case the unipotent constituents of $\mathcal{R}_{I,\mathbf{w}}(\Delta(\mu))$ are the unipotent representations $\Delta(\lambda)$ where λ is obtained by adding successively a d-hooks to μ. When μ is a d-core and $\ell > n$, these are exactly the unipotent representations in an ℓ-block. A natural choice of $\mathbf{w}$ is given in [5]: it is a d-th root of π/π_I and behaves well with respect to composition of Lusztig induction and restriction (in other words, with respect to increasing or decreasing the integer a). Note that other roots are conjugate under the braid group, and therefore other choices should not affect the cohomology of $\mathbf{X}(I, \mathbf{w})$ once we have shown a suitable generalisation of Theorem 2.1.2 to the parabolic setting.

THEOREM 2.1.3 (D. [5, Cor. 3.2]). *Let $(I, \mathbf{w})$ be as above, so that in particular $\mathbf{L}_I^{\mathbf{w}F} \simeq \mathrm{GL}_{n-ad}(q) \times \mathrm{GL}_1(q^d)^a$ for some $a, d \geq 1$. Let μ be a partition of $n - ad$. Then each individual ℓ-adic cohomology group with coefficients in the local system associated with $\Delta(\mu)$*

$$H_c^i(\mathbf{X}(I, \mathbf{w}), \Delta(\mu))$$

can be explicitly computed. Moreover, if $i \neq j$ then

$$\left\langle H_c^i(\mathbf{X}(I, \mathbf{w}), \Delta(\mu)) \, ; H_c^j(\mathbf{X}(I, \mathbf{w}), \Delta(\mu)) \right\rangle_{\mathrm{GL}_n(q)} = 0.$$

This theorem was previously known to hold in very specific cases, namely when $\mathbf{L}_I^{\mathbf{w}F}$ is isomorphic to one of the tori $\mathrm{GL}_1(q^d)$ [63] or $\mathrm{GL}_1(q^d) \times \mathrm{GL}_1(q)$ [31]. In that case only non-parabolic varieties and trivial local systems are involved, and the description of $H_c^i(\mathbf{X}(\mathbf{w}), \mathbb{K})$ is easy to state. However, when I is non-empty, the degree of the cohomology groups in which $\Delta(\lambda)$ occurs depends on the various d-hooks that are added to μ. These can be read off from the dimension of $\Delta(\lambda)$ and $\Delta(\mu)$. Conjectural formulae for the cohomology of Deligne–Lusztig varieties were stated by Craven in [27], involving indeed the degrees of the various unipotent characters involved. The stronger form of [5, Cor. 3.2] shows that Craven's conjecture holds for $\mathrm{GL}_n(q)$.

The proof of the theorem relies on the relation between parabolic induction/restriction and Lusztig induction/restriction. It can be seen as a version of the Mackey formula at the level of the functors (not just the virtual characters). Since all unipotent representations of $\mathrm{GL}_n(q)$ lie in the principal series, this is enough to get an inductive strategy to prove Theorem 2.1.3. The limit cases, where $d = n$ and $d = 1$, were settled respectively by Lusztig in [63] and by Bonnafé–Rouquier and the author (see Theorem 2.1.1).

REMARK 2.1.4. A version of Theorem 2.1.3 over $\mathbb{Z}_\ell$ was proved by the author's Ph.D. student Parisa Ghazizadeh, in the case where μ is the trivial partition, and the Sylow ℓ-subgroups of $\mathrm{GL}_n(q)$ are cyclic [38].

2.2. Brauer trees of unipotent blocks

When the defect groups of a block are cyclic, the category of representations of the block is encoded in a combinatorial object, called the *Brauer tree*. The tree together with its planar embedding describes the block up to Morita equivalence, and the structure of any indecomposable module can be read off from the tree.

Let us quickly recall how the planar embedded tree is constructed. Given an ℓ-block B with cyclic defect groups, there is a set $\mathrm{Exc}_{\Bbbk}B \subset \mathrm{Irr}_{\Bbbk}B$ called the set of *exceptional characters* of B such that if we define $\chi_{\mathrm{exc}} := \sum_{\chi\in\mathrm{Exc}_{\Bbbk}B}\chi$ then the character of any projective indecomposable $\Bbbk G$-module P in $\Bbbk B$ is given by

$$[P] = \chi + \chi'$$

with $\chi \neq \chi'$ and $\chi, \chi' \in \{\chi_{\mathrm{exc}}\} \sqcup (\mathrm{Irr}_{\Bbbk}B \smallsetminus \mathrm{Exc}_{\Bbbk}B)$. The Brauer tree Γ_B of B is the graph with vertices labelled by $\{\chi_{\mathrm{exc}}\} \sqcup (\mathrm{Irr}_{\Bbbk}B \smallsetminus \mathrm{Exc}_{\Bbbk}B)$ and edges $\chi \text{ —— } \chi'$ for every PIM P such that $[P] = \chi + \chi'$. The edges of the Brauer tree are therefore labelled by PIMs or equivalently by simple $\Bbbk B$-modules (via their projective cover). The knowledge of the Brauer tree, together with the *multiplicity* $m = \langle \chi_{\mathrm{exc}}; \chi_{\mathrm{exc}} \rangle$ of the exceptional vertex, is equivalent to the knowledge of the decomposition matrix. In addition, there exists a unique ordering around each vertex of Γ_B such that if V and V' are two simple $\Bbbk B$-modules labelling edges incident to a given vertex then V' follows immediately V (in the anti-clockwise order) if and only if $\mathrm{Ext}^1_{\Bbbk G}(V, V') \neq 0$. This defines a planar embedding of the tree.

When B is a unipotent block of a finite reductive group G, the non-exceptional characters are exactly the unipotent characters in B. Most of the PIMs in the block are obtained by parabolic induction from Levi subgroups and their character can be computed from the corresponding Hecke algebra. As in §2.3, the difficulty is often to deal with the case of cuspidal modules and their projective covers. In this section we explain how to use the cohomology complexes of Deligne–Lusztig varieties to get information on the character of these PIMs (which gives the tree as a graph) and on extensions between simple modules (which gives the planar embedding). This is based on joint works with Craven and Craven–Rouquier [**14, 3**] which finish the determination of all the planar embedded Brauer trees for unipotent ℓ-blocks.

For simplicity we will assume throughout this section that F defines a split $\mathbb{F}_q$-structure on $\mathbf{G}$, by which we mean that F acts trivially on the Weyl group of $\mathbf{G}$. As in §1.2 we will write $d_\ell(q)$ for the multiplicative order of the image of q in $\Bbbk^\times$. Recall that many properties of unipotent ℓ-blocks depend on $d_\ell(q)$ rather than on the specific prime number ℓ or the prime power q.

2.2.1. Mod-ℓ cohomology of Deligne–Lusztig varieties. Fix a pair (I, w) as in (1.2.1) and denote by $\mathbf{X} := \mathbf{X}(I, w)$ the corresponding parabolic Deligne–Lusztig variety. Since F is assumed to act trivially on W, it fixes the pair (I, w) and therefore induces an endomorphism of $\mathbf{X}$. By considering the cohomology complex $R\Gamma_c(\mathbf{X}, \Bbbk)$ in a suitable category, it decomposes according to the generalised eigenspaces of F

$$R\Gamma_c(\mathbf{X}, \Bbbk) \simeq \bigoplus_{\lambda\in\Bbbk} R\Gamma_c(\mathbf{X}, \Bbbk)_{(\lambda)}.$$

The idea is to choose $\lambda \in \Bbbk$ such that the corresponding eigenspace $C := R\Gamma_c(\mathbf{X}, \Bbbk)_{(\lambda)}$ is "small", that is, has at most two non-zero cohomology groups (see §2.2.2 for the consequences). Checking this property requires to know that

(a) the generalised λ-eigenspaces of F on the ℓ-adic cohomology groups $H^i_c(\mathbf{X}, \mathbb{K})$ vanish outside one or two specific degrees;
(b) the generalised λ-eigenspaces of F on the cohomology groups $H^\bullet_c(\mathbf{X}, \mathbb{O})$ are torsion-free.

It is in general unknown how to determine the invidual ℓ-adic cohomology groups of Deligne–Lusztig varieties outside the case of $G = \mathrm{GL}_n(q)$, unless $I = \emptyset$ and w is a

regular element of small length. In [14, 3] we mostly focused on the case where w is a *Coxeter element* for which $H_c^\bullet(\mathbf{X}, \mathbb{K})$ was computed by Lusztig in [63]. Our contribution to (b) is given by the following theorem. Recall that we assumed that F induces a split $\mathbb{F}_q$-structure on $\mathbf{G}$ and that $d_\ell(q)$ denotes the multiplicative order of q in $\Bbbk^\times$.

THEOREM 2.2.1 (D.–Rouquier [14], Craven–D.–Rouquier [3, §5]). *Let w be a Coxeter element and let $\mathbf{X}(w)$ be the corresponding Deligne–Lusztig variety. The cohomology of $\mathbf{X}(w)$ with coefficients in $\mathbb{Z}_\ell$ is torsion-free in the following cases:*

- *$d_\ell(q) \geq h$ where h is the Coxeter number (the order of w);*
- *$d_\ell(q) \in \{9, 10, 14\}$ and $\mathbf{G}$ is quasi-simple of type E_7;*
- *$d_\ell(q) \in \{9, 12, 15, 18, 20, 24\}$ and $\mathbf{G}$ is quasi-simple of type E_8.*

Under mild assumptions on ℓ, we expect the cohomology of varieties attached to Coxeter elements to always be torsion-free. This is known in the case where $G = \mathrm{GL}_n(q)$ [22]. Here, the restriction on $\mathbf{G}$ and $d_\ell(q)$ in Theorem 2.2.1 comes from the fact that we were specifically interested in the Brauer trees of unipotent ℓ-blocks of exceptional groups of type E_7 and E_8 for these specific values of $d_\ell(q)$, since they were the only ones left undetermined prior to our work. Note that for proving Theorem 2.2.1 one already needs to know the structure of some PIMs, which is why it is easier to work with blocks with cyclic defect groups.

When w is no longer a Coxeter element, one does not know in general the individual ℓ-adic cohomology groups of $\mathbf{X}(w)$. To solve this problem we replaced $\mathbf{X}(w)$ by a smooth compactification $\overline{\mathbf{X}}(w)$ whose ℓ-adic cohomology can be explicitly computed using Lusztig's work [64]. The problem of computing the torsion in the cohomology with coefficients in $\mathbb{Z}_\ell$ can be handled with the following proposition.

PROPOSITION 2.2.2 (Craven–D.–Rouquier [3, Prop. 3.8]). *Let V be a simple $\Bbbk G$-module occurring as a composition factor in the torsion part of $H_c^\bullet(\overline{\mathbf{X}}(w), \mathbb{O})$. Then V lies in a Harish–Chandra series above a Levi subgroup $L_I \subset G$ such that $\ell \mid |L_I|$.*

In particular, if G is an ℓ'-group then the cohomology of $\overline{\mathbf{X}}(w)$ is torsion-free. Under the assumption that $d_\ell(q)$ is large, all the proper standard Levi subgroups $L_I \subset G$ are ℓ'-groups, therefore the torsion part in the cohomology must be cuspidal. Having some control on where the cuspidal representations can occur in the cohomology is then crucial to solve problem (b) for $\overline{\mathbf{X}}(w)$.

When ℓ does not divide the order of $\mathbf{L}_I^{wF}$, the cohomology complex $R\Gamma_c(\mathbf{X}, \Bbbk)$ is perfect, and so is any of the generalised eigenspaces of F. Thus the complex $R\Gamma_c(\mathbf{X}, \Bbbk)_{(\lambda)}$ can be represented by a bounded complex whose terms are finitely generated projective $\Bbbk G$-modules. When ℓ divides $|\mathbf{L}_I^{wF}|$, this is no longer the case in general, but it can be interesting to look at the image of $R\Gamma_c(\mathbf{X}, \Bbbk)$ in the stable category, that is when we mod out by the thick subcategory of perfect complexes. The following result gives an explicit description of this image under some extra assumptions on ℓ and w.

THEOREM 2.2.3 (D.–Rouquier [14, Cor. 2.11]). *Let $w \in W$ and $\mathbf{w} \in B_W^+$ be its lift to the braid group. We assume that*

- *$\mathbf{w}^{d_\ell(q)} = \boldsymbol{\pi}$ where $\boldsymbol{\pi}$ is the full-twist (see §1.3);*
- *$C_W(w) = \langle w \rangle$;*
- *the Sylow ℓ-subgroups of G are cyclic.*

Then for all $m \in \mathbb{Z}$

$$R\Gamma_c(\mathbf{X}(w), \Bbbk)_{(q^m)} \simeq \Bbbk[-2m]$$

in the stable category $\Bbbk G$-stab.

We will explain in the next section how this was used in [14, 3] to determine the missing Brauer trees of unipotent ℓ-blocks.

2.2.2. Application to the determination of Brauer trees. Given a $\Bbbk G$-module M with projective cover P_M we denote by ΩM the module $\mathrm{Ker}(P_M \twoheadrightarrow M)$. The operator Ω is called the *Heller operator*. Even though ΩM is well-defined up to isomorphism, Ω is not functorial in the category of $\Bbbk G$-modules. For $i \geq 1$ we define inductively $\Omega^i M := \Omega(\Omega^{i-1}M)$ with the convention that $\Omega^0 M$ is the minimal submodule of M such that $M/\Omega^0 M$ is projective. For $i < 0$ we set $\Omega^i M := (\Omega^{-i} M^*)^*$.

Now let $B = \mathbb{O}Gb$ be an ℓ-block of G with cyclic defect groups and let Γ_B be its corresponding planar embedded Brauer tree. Leaves of the tree are labelled by those ordinary irreducible characters in the block which remain irreducible after ℓ-reduction. Let χ be such a character and let V be the simple $\Bbbk G$-module corresponding to its ℓ-reduction. Then the $\Bbbk G$-modules $V, \Omega V, \Omega^2 V, \ldots$ lift to characteristic zero, and the sequence of their ordinary characters – the *Green walk* – determines Γ_B [51]. The strategy in [14, 3] consists in computing many terms of this sequence using the cohomology complexes of Deligne–Lusztig varieties studied in §2.2.1. For that purpose, recall that the stable category $\Bbbk G$-stab is obtained from the module category $\Bbbk G$-mod by modding out the morphisms which factor through a projective $\Bbbk G$-module. It has a triangulated structure with suspension given by the inverse of the Heller operator Ω. Equivalently, it is the quotient of $\mathrm{D}^b(\Bbbk G\text{-mod})$ by the thick subcategory of perfect complexes. Let C be a bounded complex of $\mathbb{O}G$-modules coming from the generalised eigenspace of F on $bR\Gamma_c(\mathbf{X}, \mathbb{O})$ for some Deligne–Lusztig variety $\mathbf{X}$. We choose $\mathbf{X}$ such that we are in one of the following three situations:

- C is perfect and $H^\bullet(C)$ vanishes outside the degree r. Then $H^r(C)$ is a projective $\mathbb{O}G$-module with character $H^r(\mathbb{K}C)$. This gives usually one edge in the Brauer tree.
- C is perfect and $H^\bullet(C)$ vanishes outside the degrees r and $s > r$. Then there is a distinguished triangle in $\mathrm{D}^b(\Bbbk G\text{-mod})$

  $$\Bbbk C \longrightarrow H^s(\Bbbk C)[-s] \longrightarrow H^r(\Bbbk C)[-r+1] \rightsquigarrow .$$

  Since $\Bbbk C$ is perfect, the image of this triangle in the stable category yields an isomorphism $\Omega^s H^s(\Bbbk C) \simeq \Omega^{r-1} H^r(\Bbbk C)$ in $\Bbbk G$-stab. If in addition $V = H^s(\Bbbk C)$ is irreducible, then it corresponds to a leaf in the Brauer tree and $H^r(\mathbb{K}C)$ is the character of the lift of $\Omega^{s-r+1}V$. Using the Green walk, this is often enough to locate the vertex labelled by $H^r(\mathbb{K}C)$ in Γ_B.
- C is not perfect but we are in the situation of Theorem 2.2.3, where $\Bbbk C = bR\Gamma_c(\mathbf{X}, \Bbbk)_{(q^m)}$ for some $m \geq \mathbb{Z}$ and $B = \mathbb{O}Gb$ is the principal ℓ-block. If in addition $H^\bullet(C)$ vanishes outside the degree r then we get $H^r(\Bbbk C)[-r] \simeq \Bbbk[-2m]$ in $\Bbbk G$-stab, from which we deduce that the lift of $\Omega^{2m-r}\Bbbk$ has character $H^r(\mathbb{K}C)$. Again, using the Green walk and partial knowledge on the Brauer tree, one can deduce where the vertex labelled by $H^r(\mathbb{K}C)$ is in the tree.

The extra information on the Green walk given by the cohomology complexes of Deligne–Lusztig varieties is enough to determine the planar embedded Brauer trees of all the Brauer trees of unipotent ℓ-blocks of exceptional groups.

THEOREM 2.2.4 (D.–Rouquier [14, Thm. 3.9], Craven–D.–Rouquier [3, Thm. 1.1]). *All planar embedded Brauer trees of unipotent ℓ-blocks of exceptional groups are explicitly known in terms of Lusztig's parametrisation of unipotent characters.*

Prior to our work only algebraic methods had been used to determined the Brauer trees. These were proven enough for classical groups by Fong–Srinivasan [35, 37], and for exceptional groups [23, 73, 41, 54, 75, 55, 56] with the exception of E_7 and E_8, and without the planar embedding in some cases. Most of our work was to handle the case of large trees in E_7 and E_8 using the methods sketched above.

2.3. Decomposition matrices of unipotent blocks

In this section we present a selection of results obtained on ℓ-decomposition numbers of finite reductive groups in transverse characteristic. Recall from §1.1 that these numbers encode how ordinary (*i.e.* characteristic zero) irreducible representations decompose after ℓ-reduction. Using Brauer reciprocity, computing ℓ-decomposition numbers amounts to computing the characters of projective indecomposable modules (PIMs). A standard method for constructing projective modules is to induce projective modules from Levi subgroups using the parabolic induction functor. The decomposition of the induced module into indecomposable summands is governed by the representation theory of its endomorphism algebra, which is well-understood in general. However, this does not account for the cuspidal representations, which cannot be obtained from a proper parabolic induction. To solve this problem we shall consider here three different constructions of projective modules, two of which rely on Deligne–Lusztig theory:

- By considering the Deligne–Lusztig induction of a projective module. This gives a complex of projective modules, whose character is only a virtual projective module in general. For this construction to be helpful one needs to have some control on the multiplicities of the PIMs appearing in this virtual character. This is explained in §2.3.1.
- By inducing "small" representations from ℓ'-groups (such as unipotent groups). We obtain this way the so-called *generalised Gelfand-Graev representations* and their modified version by Kawanaka (see §2.3.2 and §2.3.3). These representations are projective but their irreducible constituents can have very large multiplicities.
- By considering the Alvis–Curtis dual of the intersection cohomology of Deligne–Lusztig varieties. Unlike the usual cohomology we conjecture that the corresponding character is the character of an actual projective module. It has few indecomposable summands, and therefore it gives a powerful (yet conjectural) tool to compute the decomposition numbers, see §2.3.4.

We will focus our attention on unipotent ℓ-blocks. When ℓ is not too small and does not divide q, the unipotent characters form a basic set for these blocks (see [45, 43]) which means that it is enough to study the ℓ-reduction of unipotent characters. Consequently we will restrict our attention to the square matrix D_{uni} of the ℓ-decomposition

matrix whose rows are indexed by the unipotent characters:

$$\left[\begin{array}{ccc} & D_{\mathrm{uni}} & \\ \hline \star & \star & \star \\ \vdots & \vdots & \vdots \\ \star & \star & \star \end{array}\right].$$

More specifically, we will present recent progress made on the following problems:

- Computing the ℓ-decomposition numbers for groups of small rank (including exceptional groups).
- Studying the shape of D_{uni}.
- Bounding decomposition numbers independently from q.

Only limited results on these problems were obtained using purely algebraic methods. As in the case of Brauer trees (see §2.2), our approach relies on a systematic use of the geometric tools, many of them coming from Deligne–Lusztig theory.

Most of our results are valid when ℓ and p satisfy extra assumptions, such as being *good* or *very good* for $\mathbf{G}$. This means one often needs to exclude the small primes 2, 3 and 5, or even the divisors of n when working with groups of type A_{n-1}.

2.3.1. Relations on decomposition numbers with Deligne–Lusztig characters. Let $w \in W$ and let $\mathbf{Y}(w)$ be the corresponding Deligne–Lusztig variety, see §1.2. We work here with representations with coefficients in a finite field $\Bbbk$ of characteristic $\ell \neq p$, in which case the cohomology complex of $\mathbf{Y}(w)$ with coefficients in $\Bbbk$ is perfect. In other words

$$R\Gamma_c(\mathbf{Y}(w), \Bbbk) \simeq C_\bullet = (\cdots 0 \to C_0 \to C_1 \to \cdots \to C_{2\ell(w)} \to 0 \cdots)$$

can be represented by a bounded complex $C_\bullet$ whose terms are finitely generated projective $\Bbbk G$-modules. Bonnafé–Rouquier proved in [21, Prop. 8.10 and 8.12] that if $C_\bullet$ is taken to be minimal up to homotopy equivalence, then "new" projective representations are necessarily located in degree $i = \ell(w)$, the length of w. Here, we are rather interested in the characters of the representations, that is their image in the Grothendieck group $K_0(\Bbbk G\text{-proj})$. From the previous complex one can define

$$P_w := \sum_{i \in \mathbb{Z}} (-1)^{i+\ell(w)} [H_c^i(\mathbf{Y}(w), \Bbbk)] = \sum_{i \in \mathbb{Z}} (-1)^{i+\ell(w)} [C_i]$$

which is only a virtual projective representation in general. As observed in [4], Bonnafé–Rouquier's result gives some control on the multiplicity of "new" PIMs in the virtual representation P_w.

LEMMA 2.3.1. *Let P be a projective indecomposable $\Bbbk G$-module and $[P] \in K_0(\Bbbk G\text{-proj})$ be its character. If $[P]$ occurs in P_w but not in any P_v for $v < w$ then it occurs with positive multiplicity in P_w.*

It turns out that it is often enough to get very small bounds on decomposition numbers, by extracting projective modules from the Deligne–Lusztig characters which are close to being indecomposable. This was first achieved for groups of small rank in [4], such as $G_2(q)$, and shown to be a powerful method by Malle and the author in [8, 9, 12].

THEOREM 2.3.2 (D.–Malle). *Under some mild assumptions on ℓ and up to very few missing entries, the decompositions matrices of the unipotent ℓ-blocks of the following groups are explicitly determined:*

- $G = \mathrm{SU}_n(q)$ *for* $n \leq 10$;
- $G = \mathrm{SO}_9(q)$ *and* $\mathrm{Sp}_8(q)$;
- $G = \mathrm{SO}^{\pm}_{2n}(q)$ *for* $n \leq 6$;
- $G = E_6(q)$ *and* ${}^2E_6(q)$;
- $G = F_4(q)$.

For example, the 30×30 matrices for $\mathrm{SU}_9(q)$ were explictly computed when $\ell > 15$. Note that prior to our work, only the 3×3 matrices for $\mathrm{SU}_3(q)$ were known [69].

In addition to obtaining the full decomposition matrix for groups of small rank, the method in [4] was also used to determine families of decomposition numbers for classical groups. We give here the example of the finite special unitary groups $\mathrm{SU}_n(q)$ to illustrate how fast the numbers can grow with respect to the rank. Their unipotent characters $\Delta(\lambda)$ are labelled by partitions λ of n, where by convention $\Delta(1^n)$ is the Steinberg character. It is usually the unipotent character whose ℓ-reduction is the most difficult to decompose, as it contains many cuspidal simple constituents. There is also a natural parametrisation $\{P(\lambda)\}_{\lambda \vdash n}$ of unipotent PIMs by partitions of n such that the matrix $D_{\mathrm{uni}} = \langle \Delta(\lambda)\,;\, P(\mu)\rangle$ has a lower uni-triangular shape with respect to the dominance order on partitions. Another important result in [8] is the determination of the decomposition numbers for small λ and μ, which corresponds to the bottom-right corner of D_{uni}.

THEOREM 2.3.3 (D.–Malle [8, Thm. B]). *Assume $\ell > n$ and $\ell \mid (q+1)$. Let $b, c \leq \lfloor n/3 \rfloor + 1$. Then*

$$\left\langle \Delta(2^b 1^{n-2b})\,;\, P(2^c 1^{n-2c})\right\rangle_{\mathrm{SU}_n(q)} = \begin{cases} \binom{n-c-b}{c-b} & \text{if } b \leq c, \\ 0 & \text{otherwise.} \end{cases}$$

As a consequence, there are decomposition numbers of $\mathrm{SU}_{3m}(q)$ which equal $\binom{2m}{m}$, and hence cannot be bounded above by any polynomial function in the rank.

This result is compatible with James's row and column removal rule, which is known to hold for $\mathrm{GL}_n(q)$ only [57]. We conjectured in [8, Conj. C] that such a rule also holds for finite unitary groups.

Let us mention another consequence of the work in [9]. Using the explicit knowledge of the ℓ-decomposition matrix for $\mathrm{Sp}_8(q)$, we could exhibit a counter-example to the uniqueness of the supercuspidal support for unipotent representations. This counter-example was lifted to the corresponding p-adic group by Dat in [28], giving a negative answer to a long-standing question in modular representation theory of p-adic groups.

2.3.2. Generalised Gelfand–Graev representations. A second approach for constructing projective representations is to induce representations from ℓ'-groups, which are therefore projective. The generalised Gelfand–Graev representations (GGGRs) are a special kind of such representations which are parametrised by unipotent classes of the finite reductive group considered. Given a unipotent element u of G we denote by Γ_u the corresponding GGGR (see for example [2, §6] for the definition) . Up to isomorphism, it depends only on the G-conjugacy class of u. When u is a regular unipotent element then Γ_u is the usual Gelfand–Graev representation; it is multiplicity-free and

contains only one character in each Lusztig series. The other extreme is the case of the trivial element $u = 1$: the corresponding GGGR is the regular representation $\mathbb{O}G$ of G. In fact, outside of the regular unipotent classes, the GGGRs have indecomposable summands whose multiplicity is a polynomial in q.

There is a notion of cuspidal unipotent elements which parallels that of cuspidal representations, see [49]. With Malle we showed in [10] that a unipotent PIM whose head is cuspidal occurs as a direct summand of some GGGR attached to a cuspidal unipotent element. This proves that the projective module

$$\Gamma := \bigoplus_{\substack{I \subset S \\ u \in (L_I)_{\mathrm{uni}}/\sim \text{ cuspidal}}} \mathcal{R}_I(\Gamma_u)$$

is a progenerator for the category of unipotent representations. Here, I runs over the F-stable subsets of simple reflections, $\mathcal{R}_I$ is the parabolic induction functor (see §1.2) and u runs over the cuspidal unipotent elements of L_I up to conjugation. Even though unipotent characters can have large multiplicity in Γ, this is not the case for cuspidal characters. Indeed, if ρ is a cuspidal unipotent character occurring in $\mathcal{R}_I(\Gamma_u)$ for a cuspidal unipotent element u of L then using a combination of results of Geck–Malle [49], Geck–Hézard [44] and Lusztig [65] we have that

- L must be equal to G since ρ is cuspidal,
- the geometric class $\mathbf{G} \cdot u$ attached to u is the smallest cuspidal unipotent class of $\mathbf{G}$,
- the multiplicity of ρ in Γ_u is 1, and this happens for a unique rational conjugacy class in $(\mathbf{G} \cdot u)^F$.

Consequently, ρ occurs in Γ with multiplicity one. This shows the following result, which was conjectured by Geck in 1991 [40].

THEOREM 2.3.4 (D.–Malle [10, Thm. 3.2]). *Assume that ℓ is a very good prime for $\mathbf{G}$ and that q is a power of a good prime for $\mathbf{G}$. Then every cuspidal unipotent character remains irreducible after ℓ-reduction.*

In other words, all the entries in a row of D_{uni} corresponding to a cuspidal character are zero, except for one entry which equals 1. It is a valuable result for classifying irreducible representations in positive characteristic since it gives a concrete cuspidal representation over $\mathbb{k} \supset \mathbb{F}_\ell$ which is liftable to $\mathbb{K} \supset \mathbb{Q}_\ell$. Note that the result was later generalised in joint work with Malle [11, Thm. 4.3] to unipotent characters which are the "smallest" within their Harish-Chandra series.

2.3.3. Kawanaka characters. When $\mathbf{G} = \mathrm{GL}_n$, both unipotent characters and unipotent classes of G are parametrised by partitions of n. If u_λ is a unipotent element with Jordan blocks of size λ^* (the conjugate partition), then the corresponding GGGR involves only unipotent characters labelled by partitions which are smaller that λ (for the dominance order $\trianglelefteq$). More precisely,

$$[\Gamma_{u_\lambda}] \in \Delta(\lambda) + \sum_{\mu \lhd \lambda} \mathbb{N}\Delta(\mu).$$

This has the following consequence, which was observed by Geck in [42]: one can label the unipotent Brauer characters by partitions in such a way that the ℓ-decomposition matrix of G has a uni-triangular shape with respect to the dominance order on partitions.

With Brunat and Taylor [2], we generalised this result to all finite reductive groups using a modified version of GGGRs due to Kawanaka. To any F-stable *special* unipotent class $\mathcal{O} = \mathbf{G} \cdot u$ of $\mathbf{G}$ Lusztig attached a small finite group $A_{\mathcal{O}}$, called the *canonical quotient* of the component group $C_{\mathbf{G}}(u)/C_{\mathbf{G}}(u)^\circ$. Lusztig's classification of irreducible characters of G gives a labelling of unipotent characters by pairs $(\mathcal{O}, x)$ where $\mathcal{O}$ is an F-stable special unipotent class of $\mathbf{G}$ – the *unipotent support* of the character – and x labels an irreducible representation of the Drinfeld double $\mathrm{Dr}(A_{\mathcal{O}})$ of $A_{\mathcal{O}}$. On the other hand, Kawanaka explained in [58] how one can slightly generalise the definition of GGGRs to form projective modules $\Gamma_{(\mathcal{O},x)}$ which occur as direct summands of a GGGR Γ_v for some unipotent element v lying in the unipotent class dual to $\mathcal{O}$. A similar triangular decomposition holds for the characters of these modules.

THEOREM 2.3.5 (Brunat–D.–Taylor [2, Thm. B]). *Assume $Z(\mathbf{G}) = \{1\}$. If q is a power of a good prime for $\mathbf{G}$ then*

$$[\Gamma_{(\mathcal{O},x)}] \in \Delta(\mathcal{O}, x) + \sum_{\substack{\overline{\mathcal{O}'} \subsetneq \overline{\mathcal{O}} \\ y \in \mathrm{Irr}\,\mathrm{Dr}(A_{\mathcal{O}'})}} \mathbb{N}\Delta(\mathcal{O}', y).$$

Consequently, given any unipotent character Δ with unipotent support $\mathcal{O}$ one can find a PIM whose character involves Δ with multiplicity one and only other unipotent characters with unipotent support $\mathcal{O}'$ such that $\overline{\mathcal{O}'} \subsetneq \overline{\mathcal{O}}$. This shows that the decomposition matrix of unipotent ℓ-blocks has uni-triangular shape, whenever ℓ and p are not too small. This was conjectured by Geck in 1990 [40].

COROLLARY 2.3.6 (Brunat–D.–Taylor [2, Thm. A]). *Assume that ℓ is a very good prime for $\mathbf{G}$ and q is a power of a good prime for $\mathbf{G}$. Given any total ordering $\mathcal{O}_1 \leqslant \cdots \leqslant \mathcal{O}_r$ of the special unipotent classes of $\mathbf{G}$ refining the closure relations, the irreducible unipotent Brauer characters of G can be ordered such that the decomposition matrix of the unipotent ℓ-blocks of G has the following shape*

$$\begin{pmatrix} D_{\mathcal{O}_r} & 0 & \cdots & 0 \\ \star & D_{\mathcal{O}_{r-1}} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ \star & \cdots & \star & D_{\mathcal{O}_1} \\ \star & \cdots & \cdots & \star \end{pmatrix},$$

where each diagonal block $D_{\mathcal{O}_i}$ is the identity matrix, whose rows are indexed by unipotent characters with unipotent support $\mathcal{O}_i$.

As a byproduct we obtain an $\mathrm{Aut}(G)$-equivariant bijection between unipotent characters and unipotent Brauer characters. This can be helpful in many situations, for example for describing the branching rules of parabolic induction/restriction in terms of the crystal graph of some charged Fock space (see §2.4.1). Another example is the inductive Alperin Weight Conjecture for simple groups (see [67]), where understanding the action of $\mathrm{Aut}(G)$ on Brauer characters is crucial.

2.3.4. Intersection cohomology of Deligne–Lusztig varieties. Finally, in a joint work with Malle [7] we suggested a conjectural construction of projective representations which have very few indecomposable summands, and therefore give a lot of information on D_{uni}. Instead of considering the cohomology of the Deligne–Lusztig

variety $\mathbf{X}(w)$ for some $w \in W$, which gives the usual Deligne–Lusztig character, we consider the intersection cohomology

$$IR_w := \sum_{i\in\mathbb{Z}} (-1)^i IH^i(\overline{\mathbf{X}(w)}, \mathbb{K}).$$

Then IR_w is only a virtual character. Nevertheless, Lusztig showed in [64] that there is a duality D on (signed) characters of finite reductive groups such that $D(IR_w)$ is, up to a sign, the character of an actual representation. Our conjecture predicts that this also holds in positive characteristic.

CONJECTURE 2.3.7 (D.–Malle [7, Conj. 1.2]). *If ℓ is a very good prime for $\mathbf{G}$, the character $D(IR_w)$ is, up to a sign, the unipotent part of the character of a projective $\mathbb{k}G$-module.*

The restriction on ℓ comes from the fact that we want the unipotent characters to form a basic set of characters for the unipotent ℓ-blocks. In that case every integer combination of unipotent characters is the unipotent part of a virtual projective character. Our conjecture predicts that in the case of $D(IR_w)$ one should get an actual projective character.

Many important results can be deduced from the conjecture. For example, it forces the decomposition numbers to be bounded independently from q. Indeed, the multiplicities of unipotent characters in $D(IR_w)$ do not depend on q. They can be computed from the value at 1 of Kazhdan–Lusztig polynomials and the values of irreducible characters of W. It is actually expected that decomposition numbers of unipotent ℓ-blocks do not depend on q when ℓ is not too small. Conjecture 2.3.7 also gives a new proof of the fact that D_{uni} has a uni-triangular shape (see §2.3.3). This comes from the fact that the unipotent characters occurring in $D(IR_w)$ lie in families which are smaller than the 2-sided cell containing w. Note finally that the bounds on decomposition numbers are often sharp in the case where $\ell \mid (q+1)$. This is generally the most difficult case for computing decomposition numbers.

2.4. Categorical actions on unipotent representations of classical groups

In this final section we present results obtained in a series of papers with Varagnolo–Vasserot about the construction of actions of Lie algebras on the category of unipotent representations of finite general unitary groups and finite classical groups of types B and C.

Let $\mathfrak{g}$ be a simply-laced Kac–Moody algebra with Chevalley generators $\{e_i, f_i\}_{i\in I}$. One way to make $\mathfrak{g}$ act on an abelian category $\mathcal{V}$ is to produce biadjoint pairs of exact endofunctors $\{(\mathcal{E}_i, \mathcal{F}_i)\}_{i\in I}$ of $\mathcal{V}$ inducing an integrable action of $\mathfrak{g}$ on the complexified Grothendieck group $\mathbb{C}\otimes_{\mathbb{Z}} K_0(\mathcal{V})$ of $\mathcal{V}$. Chuang–Rouquier [25], Khovanov–Lauda [59, 60], and Rouquier [72] showed that more structure is needed to obtain interesting actions. The extra structure is given by natural transformations of $\mathcal{E}_i$ and $\mathcal{E}_i\mathcal{E}_j$ satisfying the relations of a so-called *quiver-Hecke algebra* of the same type as $\mathfrak{g}$. In our setting $\mathcal{V}$ will be the category of unipotent representations of a series of finite classical groups (with the exception of groups of type D) and the functors $\mathcal{E}_i, \mathcal{F}_i$ will be constructed from the parabolic induction and restriction functors. Our categorification result has two important consequences:

- the branching graph of the parabolic induction and restriction is given by the crystal graph of a well-identified level 2 Fock space;

- two unipotent ℓ-blocks with conjugate weights are derived equivalent.

We will explain how this shows in particular that Broue's abelian defect group conjecture holds for unipotent ℓ-blocks of these classical groups when ℓ is a *linear* prime.

2.4.1. Parabolic induction and restriction. As before we will work with representations over a field $\Bbbk$ of positive characteristic ℓ with $\ell \nmid q$. We will assume in addition that ℓ is odd when working with classical groups of type B and C. We will denote by $\Bbbk G$-umod the category of unipotent representations of the finite reductive group G. Under the assumptions on ℓ and for the groups considered in this section, the unipotent characters form a basic set [45] and therefore their image under the decomposition map yields a $\mathbb{Z}$-basis of $K_0(\Bbbk G\text{-umod})$.

To simplify the exposition of our results, we will work with the finite symplectic groups $G_n = \mathrm{Sp}_{2n}(q)$. Using standard Levi subgroups of G_{n+1} of the form $G_n \times \mathrm{GL}_1(q)$ one obtains a natural chain of subgroups $G_0 \subset G_1 \subset \cdots \subset G_n \subset G_{n+1} \subset \cdots$ together with the parabolic induction and restriction functors

$$\cdots \underset{\mathcal{E}}{\overset{\mathcal{F}}{\rightleftarrows}} \Bbbk G_n\text{-umod} \underset{\mathcal{E}}{\overset{\mathcal{F}}{\rightleftarrows}} \Bbbk G_{n+1}\text{-umod} \underset{\mathcal{E}}{\overset{\mathcal{F}}{\rightleftarrows}} \cdots$$

Here we assume that $\Bbbk = \overline{\mathbb{F}}_\ell$ so that all the group algebras $\Bbbk G_n$ are split. We can form the category

$$\Bbbk G_\bullet\text{-umod} := \bigoplus_{n \in \mathbb{N}} \Bbbk G_n\text{-umod}.$$

When $\ell \nmid (q-1)$ the parabolic induction and restriction functors $(\mathcal{F}, \mathcal{E})$ give a biadjoint pair of exact endofunctors of this abelian category.

From Lusztig's classification of unipotent characters we get an explicit description of the action of $\mathcal{E}$ and $\mathcal{F}$ on the Grothendieck group of these categories. Let us explain how it can be rephrased in terms of level 2 Fock spaces. The group G_n has a cuspidal unipotent character if and only if $n = t^2 + t$ for some $t \in \mathbb{N}$. In that case the characters lying in the Harish-Chandra series above that cuspidal character are parametrised by the irreducible representations of a Hecke algebra of type B, hence by bipartitions. This Hecke algebra has generators $T_0, T_1, \ldots$ satisfying the braid relations of type B and quadratic relations with parameters $Q_t = ((-q)^t, (-q)^{-1-t})$ for T_0 and $(q, -1)$ for the other T_i's. We deduce that each unipotent character of G_n is parametrised by a pair (λ, t) where λ is a bipartition of size $n - t^2 - t$. On the other hand, we can form the level 2 Fock space $\mathrm{F}(Q_t)$ with charge Q_t as the complex vector space equipped with a *standard basis* $|\lambda, Q_t\rangle$ indexed by bipartitions. This yields an isomorphism of vector spaces

$$\bigoplus_{t \in \mathbb{N}} \mathrm{F}(Q_t) \xrightarrow{\sim} \mathbb{C} \otimes_{\mathbb{Z}} K_0(\Bbbk G_\bullet\text{-umod}) \tag{2.4.1}$$

sending $|\lambda, Q_t\rangle$ to the image under the decomposition map of the corresponding unipotent character. Let $\bar{q}$ be the image of q in $\Bbbk^\times$. Each Fock space $\mathrm{F}(Q_t)$ has a natural action of a Kac–Moody algebra $\mathfrak{g} = \langle e_i, f_i \rangle_{i \in I}$ whose quiver has vertices labelled by $I = \bar{q}^{\mathbb{Z}} \cup -\bar{q}^{\mathbb{Z}}$ and arrows given by multiplication by $\bar{q}$. The parabolic induction and restriction maps $[\mathcal{F}]$ and $[\mathcal{E}]$ on the right-hand side of (2.4.1) correspond to the action of $f = \sum_{i \in I} f_i$ and $e = \sum_{i \in I} e_i$ on the Fock spaces.

The main result in [17] and [15] is a categorification of the isomorphism (2.4.1). It holds more generally when $(G_n)_{n\in\mathbb{N}}$ is one of the series $\mathrm{GU}_{2n}(q)$, $\mathrm{GU}_{2n+1}(q)$, $\mathrm{SO}_{2n+1}(q)$ or $\mathrm{Sp}_{2n}(q)$, with a suitable change of the charge Q_t and the Lie algebra $\mathfrak{g}$.

THEOREM 2.4.1 (D.–Varagnolo–Vasserot [17, Thm. 4.21] and [15, Thm. 6.11]). *There is a categorical action of* $\mathfrak{g}$ *on* $\Bbbk G_\bullet$-umod *such that*

$$\bigoplus_{t\in\mathbb{N}} \mathrm{F}(Q_t) \xrightarrow{\sim} \mathbb{C}\otimes_{\mathbb{Z}} K_0(\Bbbk G_\bullet\text{-umod})$$

as $\mathfrak{g}$*-modules.*

A similar result was previously known in the case of finite general linear groups $\mathrm{GL}_n(q)$ by the work of Chuang–Rouquier [25]. In that case, unipotent characters are labelled by partitions and there is only one level 1 Fock space to consider, since all the unipotent characters lie in the principal series.

Let us explain two main consequences of our result. The first one concerns the branching rules for the parabolic induction and restriction $\mathcal{F}$ and $\mathcal{E}$, which was the original motivation for our work. The *branching graph* of $\Bbbk G_\bullet$-umod is the graph whose vertices are labelled by the simple unipotent $\Bbbk G_\bullet$-modules, and whose arrows are given by $T \to T'$ whenever T' appears in the head of the induction $\mathcal{F}(T)$, or equivalently whenever T appears in the socle of the restriction $\mathcal{E}(T')$. By Theorem 2.4.1, the graph can be "colored" by I: we write $T \xrightarrow{i} T'$ whenever T' appears in the head of the induction $\mathcal{F}_i(T)$. On the other hand, one can attach a so-called *crystal graph* to the Fock space $\mathrm{F}(Q_t)$ which depends on the choice of a pair $s_t \in \mathbb{Z}^2$ such that $Q_t = q^{s_t}$. The advantage is that one has an explicit description of this graph in terms of the combinatorics of Young diagrams of charged bipartitions, see for example [48, §6.2]. It was conjectured by Gerber–Hiss–Jacon [50] that s_t can be chosen such that the two graphs match; we showed in [17, 15] how their conjecture can be deduced from our categorification result.

THEOREM 2.4.2 (D.–Varagnolo–Vasserot [17, Thm. 4.37]). *There is a choice of* s_t *for each* $t \in \mathbb{Z}$ *such that the union of the crystal graphs of the Fock spaces* $\mathrm{F}(Q_t)$ *coincides with the colored branching graph of the parabolic induction and restriction.*

Note that by Corollary 2.3.6 the simple unipotent $\Bbbk G_\bullet$-modules are labelled by the unipotent characters, hence by charged bipartitions. The identification of the crystal graph with the branching graph preserves this labelling. This was shown for finite unitary groups in [17], but not for other classical groups since at that time Corollary 2.3.6 was not know to hold in general. This was however later checked in joint work with Norton [13].

Another important consequence of Theorem 2.4.1 is the construction of derived equivalences for unipotent blocks. This was one of the original motivations of the work of Chuang–Rouquier, in order to prove Broué's conjecture for finite general linear groups. In [25] it is shown that categorical actions of $\mathfrak{g}$ on $\mathcal{V}$ produce derived equivalences of $\mathcal{V}$ permuting blocks. The permutation of the blocks is given by the action of the affine Weyl group of $\mathfrak{g}$ on the weight spaces of the $\mathfrak{g}$-module $\mathbb{C}\otimes_{\mathbb{Z}} K_0(\mathcal{V})$. This permutation was explicitly computed in [17, 15] for our level 2 Fock spaces. As a byproduct, we obtain a proof of Broué's conjecture in the case of linear primes. Recall that $\bar{q}$ denotes the image of q in $\Bbbk$.

THEOREM 2.4.3 (D.–Varagnolo–Vasserot [17, Thm. 4.34] and [15, Thm. 6.20]). *Assume that we are in one of the following cases:*

- $G_n = \mathrm{GU}_{2n}(q)$ *or* $\mathrm{GU}_{2n+1}(q)$ *and the order of* $-\bar{q}$ *in* $\Bbbk^\times$ *is even;*
- $G_n = \mathrm{SO}_{2n+1}(q)$ *or* $\mathrm{Sp}_{2n}(q)$ *and the order of* $\bar{q}$ *in* $\Bbbk^\times$ *is odd.*

Then any unipotent ℓ-block of G_n with abelian defect groups is derived equivalent to its Brauer correspondent.

The assumption that ℓ is a linear prime (which is given by the assumption on the order of $\bar{q}$ or $-\bar{q}$ in $\Bbbk^\times$) is needed in order to use the fact that "good" blocks (in the sense of [62]) satisfy Broué's conjecture. This is not known to hold yet when ℓ is a unitary prime, but our categorification result shows nevertheless that any unipotent ℓ-block is derived equivalent to a "good" block.

2.4.2. Deligne–Lusztig induction and restriction. In [16] we considered a similar situation to §2.4.1, where parabolic induction and restriction functors $\mathcal{F}$ and $\mathcal{E}$ are replaced by Deligne–Lusztig induction and restriction, which are now exact functors between the bounded derived categories of $\Bbbk G_\bullet$-modules:

$$\cdots \underset{\mathcal{E}}{\overset{\mathcal{F}}{\rightleftarrows}} \mathrm{D}^b(\Bbbk G_n\text{-umod}) \underset{\mathcal{E}}{\overset{\mathcal{F}}{\rightleftarrows}} \mathrm{D}^b(\Bbbk G_{n+1}\text{-umod}) \underset{\mathcal{E}}{\overset{\mathcal{F}}{\rightleftarrows}} \cdots$$

This is particularly suited to the case where $G_n = \mathrm{GU}_n(q)$, since there is no parabolic induction from $\mathrm{GU}_n(q)$ to $\mathrm{GU}_{n+1}(q)$. From the finite reductive groups perspective, this is somehow more natural, as one can expect the natural transformation of $\mathcal{E}$ and $\mathcal{E}^2$ to come from endomorphisms of the corresponding Deligne–Lusztig varieties, such as the Frobenius endormorphisms or more generally from the braid group actions considered in [1].

In [16] we studied the specific example of $G_n = \mathrm{GL}_n(q)$. As mentioned in §2.4.1, the complexified Grothendieck group of the category

$$\Bbbk G_\bullet\text{-umod} := \bigoplus_{n\geq 0} \Bbbk G_n\text{-umod}$$

is naturally isomorphic to a level 1 Fock space. It has a basis labelled by partitions, and it has an action of the Kac–Moody algebra $\mathfrak{g} = \widehat{\mathfrak{sl}_d}$, whenever $d := d_\ell(q) \geq 2$. Chuang–Rouquier showed in [25] that it is categorified by the (truncated) parabolic induction and restriction functors. Here we consider instead Deligne–Lusztig induction and restriction functors given by the cohomology of the Deligne–Lusztig varieties attached to the full-twist. More precisely, for each $n \geq 0$ we set $I = \{s_1, \dots, s_{n-1}\}$ and $\mathbf{w} := \boldsymbol{\pi}/\boldsymbol{\pi}_I = \mathbf{s}_1 \cdots \mathbf{s}_n \mathbf{s}_n \cdots \mathbf{s}_1$. The Levi subgroup $\mathbf{L}_I^{\mathbf{w}F}$ of $\mathrm{GL}_{n+1}(q)$ is isomorphic to $\mathrm{GL}_n(q) \times \mathrm{GL}_1(q)$. When $\ell \nmid |\mathrm{GL}_1(q)|$ (*i.e.* when $d \geq 2$) we can form the exact functors

$$\mathcal{F} := \mathcal{R}_{I,\mathbf{w}} \circ \mathrm{Inf}_{G_n}^{G_n \times \mathrm{GL}_1(q)} = R\Gamma_c\big(\mathbf{Y}(I, \boldsymbol{\pi}/\boldsymbol{\pi}_I)/\mathrm{GL}_1(q), \Bbbk\big) \otimes_{G_n} -$$
$$\mathcal{E} := \mathrm{Inv}_{\mathrm{GL}_1(q)} \circ {}^*\mathcal{R}_{I,\mathbf{w}} = R\mathrm{Hom}_{\Bbbk G_{n+1}}\big(R\Gamma_c(\mathbf{Y}(I, \boldsymbol{\pi}/\boldsymbol{\pi}_I)/\mathrm{GL}_1(q), \Bbbk), -\big)$$

between the derived categories of $\Bbbk G_n$-umod and $\Bbbk G_{n+1}$-umod. Conjecturally, they correspond to a version of parabolic induction and restriction with respect to the heart of a non-standard t-structure of $\mathrm{D}^b(\Bbbk G_\bullet\text{-umod})$. This cannot be seen at the level of the Grothendieck groups, therefore one obtains a different categorification of the same $\mathfrak{g}$-module structure on $K_0(\Bbbk G_\bullet\text{-umod})$.

THEOREM 2.4.4 (D.–Varagnolo–Vasserot [16, Prop. 2.1 and Cor. 2.5]). *Let $d := d_\ell(q)$ be the order of $\bar{q}$ in $\Bbbk^\times$. Assume that $d \geq 2$. Then the functors $\mathcal{F}$ and $\mathcal{E}$ induce a categorical action of $\mathfrak{g} = \widehat{\mathfrak{sl}_d}$ on $\mathrm{D}^b(\Bbbk G_\bullet\text{-umod})$. Under the identification*

$K_0(\mathrm{D}^b(\Bbbk G_\bullet\text{-umod})) \xrightarrow{\sim} K_0(\Bbbk G_\bullet\text{-umod})$, *it categorifies the level 1 Fock space representation of* $\widehat{\mathfrak{sl}_d}$.

The proof of this theorem uses the odd-degree vanishing of the ℓ-adic cohomology of $\mathbf{X}(\pi)$ (see Theorem 2.1.1). A modular analogue of this property would show that there is a perverse equivalence which intertwines the categorification in Theorem 2.4.4 and the original one by Chuang–Rouquier in [25]. Let us mention that Theorem 2.4.4 generalises to the case of Deligne–Lusztig varieties attached to the longest element w_0 of $\mathfrak{S}_n$. In the latter case one obtains functors between derived categories of modules for $\mathrm{GL}_n(q)$ and $\mathrm{GL}_{n+2}(q)$ which categorify a sum of level 2 Fock space representations. This should be obtained from the categorification in §2.4.1 by a change of t-structure and would give a categorical version of Ennola duality between $\mathrm{GL}_n(q)$ and $\mathrm{GU}_n(q)$.

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[75] E. Wings, *Über die unipotenten Charaktere der Gruppen $F_4(q)$*. Dissertation, Lehrstuhl D für Mathematik, RWTH Aachen, 1995.

Curriculum vitae

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Date of birth: March 17, 1983

Nationality: French

Employment

2012– **Chargé de recherche CNRS** (tenured), at the Institut de Mathématiques de Jussieu - Paris Rive Gauche IMJ-PRG.

2010–2012 **EPSRC Research Fellow** at the Oxford Mathematical Institute, UK.

2010–2012 **Junior Research Fellow** at Magdalen College, Oxford, UK.

Education

2006–2010 **Ph.D thesis - Pure Mathematics**, Université de Franche-Comté. Thesis entitled *Geometry of Deligne-Lusztig varieties and modular representation theory of finite reductive groups*. Advisor: Cédric Bonnafé.

2006 **Agrégation (French advanced teaching qualification exam) - Mathematics**, rank 8th (out of about 2000 examinees), July 06.

2005–2006 **DEA (equiv. M.S.) - Pure Mathematics**, Université Paris VII (mention TB), June 06. Dissertation entitled *Hecke algebra associated to a cuspidal representation*. Advisor: Michel Broué.

Magister MMFAI (3rd year), ENS Paris, June 06.

2004–2005 **Magister MMFAI (2nd year)**, ENS Paris, June 05.

2003–2004 **Licence and Maîtrise (equiv B.S.) - Mathematics**, Université Paris XI (mention TB), January 04 and June 04.

Magister MMFAI (1st year), ENS Paris.

2003-2007 **Scholarship at the Ecole Normale Supérieure**, Paris.

Published Papers

[23] (with O. Brunat and J. Taylor), *Unitriangular shape of decomposition matrices of unipotent blocks*, Annals of Mathematics **192** (2020), 583–663.

[22] (with Michela Varagnolo and Éric Vasserot) *Categorical actions on unipotent representations I. Finite unitary groups*, Publications mathmatiques de l'IHÉS **129** (2019), 129–197.

[21] (with C. Bonnafé and R. Rouquier) *Translation by the full twist and Deligne–Lusztig varieties*, J. Algebra **558** (2020), 129–145.

[20] (with David Craven and Raphaël Rouquier) *Brauer trees of unipotent blocks*, JEMS **22** (2020), no. 9, 2821–2877.

[19] (with Gunter Malle) *Bounding Harish-Chandra series*, Transactions of the AMS **371** (2019), 6511–6530.

[18] *Splendeur des variétés de Deligne-Lusztig, d'après Deligne-Lusztig, Broué, Rickard, Bonnafé-Dat-Rouquier*, Séminaire Bourbaki, Exp. 1137, to appear in Astérisque, 2018.

[17] (with Nicolas Jacon) *Alvis-Curtis duality for finite general linear groups and a generalized Mullineux involution*, SIGMA **14**, 2018.

[16] *Non-uniqueness of supercuspidal support for finite reductive groups*, Appendix to the article *Simple subquotients of big parabolically induced representations of p-adic groups* by J.-F. Dat, J. Algebra **510** (2018), 508–512.

[15] *Lectures on modular Deligne-Lusztig theory*, in Local Representation Theory and Simple Groups, 107–177, EMS Series of Lectures in Mathematics, 2018.

[14] (with Gunter Malle) *Modular irreducibility of cuspidal unipotent characters*, Invent. Math. **211** (2018), 579–589.

[13] (with Michela Varagnolo and Éric Vasserot) *Categorical actions from Lusztig induction and restriction on finite general linear groups*, Representation theory - current trends and perspectives, 59–74, EMS Ser. Congr. Rep., Eur. Math. Soc., Zürich, 2017.

[12] (with Michela Varagnolo and Éric Vasserot) *Categorical actions on unipotent representations of finite classical groups*, in Categorification and higher representation theory, 41–104, Contemp. Math. **683**, Amer. Math. Soc., Providence, RI, 2017.

[11] (with Gunter Malle) *Decomposition matrices for exceptional groups at d=4*, J. Pure Appl. Algebra **220** (2016), 1096–1121.

[10] (with Gunter Malle) *Decomposition matrices for low rank unitary groups*, Proc. Lond. Math. Soc. **110** (2015), 1517–1557.

[9] (with Gunter Malle) *Projective modules in the intersection cohomology of Deligne–Lusztig varieties*, C. R. Acad. Sci. Paris Sér. I Math. **352** (2014), 467–471.

[8] (with Raphaël Rouquier) *Coxeter Orbits and Bauer trees III*, J. Amer. Math. Soc. **27** (2014), 1117–1145.

[7] *Cohomology of Deligne-Lusztig varieties for unipotent blocks of* $\mathrm{GL}_n(q)$, Represent. Theory **17** (2014), 647–662.

[6] *Coxeter orbits and Brauer trees II*, Int. Math. Res. Not. IMRN **15** (2014), 4100–4123.

[5] *Note on decomposition numbers of groups of Lie type of small rank*, J. Algebra **388** (2013), 364–373.

[4] *Cohomology of Deligne-Lusztig varieties for short-length regular elements in exceptional groups*, J. Algebra **392** (2013), 276–298.
[3] *Quotient of Deligne-Lusztig varieties*, J. Algebra **381** (2013), 1–20.
[2] *Coxeter orbits and Brauer trees*, Adv. Math. **229** (2012), 3398–3435.
[1] *Deligne-Lusztig restriction of a Gelfand-Graev module*, Ann. Sci. Éc. Norm. Supér. **42** (2009), 653–674

Preprints

[25] (with A. Ivanov) *Orthogonality relations for deep level Deligne–Lusztig schemes of Coxeter type*, `arXiv:2010.15489`, 2020.
[24] (with G. Malle) *Decomposition matrices for groups of Lie type in non-defining characteristic*, `arXiv:2001.06395`, 2020.

Awards and Grants

2017–2020 **Mercator Fellowship** in the SFB-TRR Grant Number 195 *Symbolic Tools in Mathematics and their Application*.
2016–2020 **ANR Research Grant (Personal Investigator)** on *Geometric methods in modular representation theory of finite groups*.

Selection of talks (since 2010)

2019-2020 *Modular reduction of unipotent characters*, ARTS, Rome (Nov. 19).
Drinfeld double of braid groups and unipotent characters, Oxford (Sept. 2019) and Paris (Nov. 219).

2018-2019 *Macdonald polynomials and decomposition numbers for finite unitary groups*, Oberwolfach (Nov. 2018).
The abelian defect group conjecture and Deligne–Lusztig varieties, Madrid (Nov. 2018).

2017–2018 *Cohomologie d'intersection des variétés de Deligne-Lusztig*, Montpellier (Oct. 2017).
Splendeurs des variété de Deligne-Lusztig, Séminaire Bourbaki, Paris (Oct. 2017).
Cohomology of Deligne–Lusztig varieties. A change of heart, Sydney, Australia (Dec. 2017).
Shadows of finite groups, Colloquium talk, UCLA (Mar. 2018) and U of Arizona (Fev. 2018).
Fourier matrices for unipotent characters, MSRI, Berkeley, USA (Apr. 2018).

2016–2017 *Morita equivalences for groups of type B and C*, CIB, Lausanne, Switzerland (Sep. 2016).
Alvis-Curtis duality and the Mullineux involution, Amiens, France (Oct. 2016) and Lausanne, Switzerland (Dec. 2016)
Irreducibility of cuspidal characters (for finite groups of Lie type), Aberdeen, Scotland (Feb. 2017), UCLA, USA (Apr. 2017), Wuppertal, Germany (Jun. 2017), Cambridge, UK (Nov. 2017).

2015–2016 *Categorical actions and finite classical groups*, talks in Reims, France (Nov. 2016), Caen, France (Mar. 2016) Oberwolfach (Apr. 2015), Osaka, Japan (Apr. 2015), Aachen, Germany (Jul. 2015).
Around the abelian defect group conjecture, Tours (Jun. 2015).
On the unitriangularity of decomposition matrices, Kaiserslautern, Germany (Dec. 2015).

2014–2015 *Categorical actions and finite unitary groups*, talks in Bad Honnef, Germany (Mar. 2015), Jena, Germany (Jun. 2016).

2013–2014 *Decomposition numbers via Deligne-Lusztig theory*, talks in Beijing, China (Jun. 13), Amiens (Nov. 2013), Caen (Nov. 2013), Aachen (Dec. 2013), Nancy (Jan. 2014), Wuppertal, Germany (Jan. 2014), Banff, Canada (Mar. 2014).

2012–2013 *Modular Deligne-Lusztig Theory*, series of talks in Aachen (Nov. 12).
Soergel bimodules, series of talks in Kaiserslautern (May 2013).

2011–2012 *Quotient of Deligne-Lusztig varieties*, Séminaire Chevalley, Paris (Nov. 2011), Manchester, GB (Dec. 2011) and Oberwolfach, Germany (Mar. 2012).
Brauer trees of finite reductive groups, Colloque tournant, Caen (Jan. 2012), Colloque Solstice, Paris (Jun. 2012) and Conference DMV, Saarbrücken, Germany (Sep. 2012).
Deligne-Lusztig varieties and cyclotomic Hecke algebras, Conference Lie theory and quantum analogues, Luminy (Apr. 2012).

2010–2011 *Modular representation theory of finite reductive groups*, series of talks in Hong Kong UST (Apr. 2011).
Brauer trees of finite reductive groups, Algebra seminar, Oxford and City University, Londres (Nov. 2010), Edinburgh (Dec. 2010), Leeds, UK (Mar. 2011) and Pohang, South Korea (Mar. 2011).

Selection of scientific visits (since 2010)

2019–2020 University of Bonn, Germany, collaboration with Alexander Ivanov (Feb. 20).
University of Kaiserslautern, Germany, collaboration with Gunter Malle (as part of the Mercator Fellowship) (Oct.-Dec. 19).

2018–2019 University of Kaiserslautern, Germany, collaboration with Gunter Malle (as part of the Mercator Fellowship) (Oct.-Dec. 18).

2017–2018 University of Kaiserslautern, collaboration with Gunter Malle (as part of the Mercator Fellowship) (Nov. 17).
Research Membership at the MSRI, Berkeley, USA (Feb.-Apr. 18).
UCLA, USA, and collaboration with Raphaël Rouquier (Mar. 18).
U. of Arizona, USA, and collaboration with J. Taylor (Feb. 18).

2016–2017 UCLA and collaboration with Raphaël Rouquier (Apr. 17).
RWTH Aachen, Germany, collaboration with G. Hiss and T. Gerber (as part of the Mercator Fellowship) (May-Jul. 17).

2015–2016 University of Padova, Italy, collaboration with Jay Taylor (Oct. 2015).

2014–2015 Osaka, Japan, collaboration with Hyohe Miyachi (Apr. 2015).
BICMR, China. Lecture series on representation of finite reductive groups (Jun. 2015)

2013–2014 Frequent invitations to Kaiserslautern, Germany. Collaboration with G. Malle.

2012–2013 RTWH, Aachen, Germany. Lecture series on modular representation theory and collaboration with Gerhard Hiss (Nov. 12).
Kaiserslautern, Germany (April and May 13) and collaboration with Gunter Malle, Britta Spaëth and Jay Taylor.

2011–2012 UCLA, USA, and collaboration with Raphaël Rouquier (Apr.12).

2010–2011 Hong Kong University of Science and Technology (HKUST). Lecture series on modular representation theory and collaboration with Xuhua He (Apr. 11).

Lecture series and mini-courses (by invitation)

2019–2020 *Finite reductive groups and their irreducible characters* (two 1h talks) at the *introductory workshop* of the special semester *Groups, representations and applications: new perspectives*, Newton Institute, Cambridge, UK (Jan. 20).

2018–2019 *Deligne–Lusztig theory*, WARTHOG lectures in the University of Oregon, Eugene, USA (Aug. 18).
Representations of finite reductive groups and character sheaves, mini-course (3x90min) in the conference *Geometric Representation Theory* in Besse-en-Chandesse, France (Sep. 18).

2017–2018 *Representations of finite reductive groups: from characteristic zero to transverse characteristic*, mini-course (3x90min) in the Introductory Workshop at the MSRI, Berkeley, USA (Feb. 2018).

2016–2017 *Deligne-Lusztig theory and modular representations*, lecture series (9x90min) in the summer school *Advanced lectures on local representation theory* at the CIB, Lausanne, Switzerland (Jul. 2016).
Harish-Chandra theory revisited, mini-course (3x90min) in the conference *Representations of finite and p-adic reductive groups* in Luminy, France (Nov. 2016).
Lectures on categorification, mini-course (3x90min) at NYU Abu-Dhabi, UAE (Mar. 2017)
Perverse equivalences, series of 5 lectures at the RWTH Aachen, Germany (May-Jun. 2017).

2014–2015 *Representation of finite reductive groups*, graduate course (18 hours) at the BICMR, Beijing, China (Jun. 2015)

2013–2014 *Representation theory of* $\mathrm{SL}_2(q)$, mini-course (5x60min) at the EPFL for the Young Algebraist Conference (YAC), Lausanne, Switzerland (Jun. 2014)

Teaching (since 2010)

2020–2021 **Graduate course** *Higher representation theory* (48 hours), at University of Paris.
Tutorials for the undergraduate course *Basic algebra and calculus for Chemists - MP1 and IMP1* (40 hours), at University of Paris.

2018–2019 **Graduate course** *Higher representation theory* (48 hours), at Paris 7.
Tutorials for the undergraduate course *Calculus I for Chemists - MP2* (32 hours), at Paris 7.

2016–2017 **Tutorials** for the graduate course *Perverse sheaves* (12 hours), at Paris 7.
Tutorials for the undergraduate course *Calculus I* (54 hours), at Paris 7.

2013–2014 **Graduate course** *Introduction to Deligne-Lusztig theory* (24 hours), at Paris 7.

2011–2012 **Tutorials** in Algebra, Fields, Group Theory et Number Theory, Magdalen College, Oxford.

2010-2011 **Tutorials** in Representation Theory for the exchange program Oxford-Stanford, Oxford.

Ph.D. Students

2016-2020 **Parisa Ghazizadeh**, *Torsion in the cohomology of Deligne-Lusztig varieties* (defended in Dec. 2019).

2015-2020 **Reda Chaneb** (with O. Brunat), *Decomposition matrices of classical groups in bad characteristic* (defended in Oct. 2019).

Graduate Students

2015-2016 **Sébastien Son**, *Unipotent classes and unipotent pieces in positive characteristic.*

2013-2014 **Abel Lacabanne**, *Cherednik algebras and their representation theory.*

Organisation of workshops and conferences

2019–2020 **GRAP** (groupes et représentations entre Amiens et Paris), in Amiens, France (Sept. 2019).

2018–2019 **Journée d'accueil en Mathématiques**, at the Institut Henri Poincaré (IHP) (Apr. 2019).

2017–2018 **Representations in Lie Theory and Interactions** (with Cédric Lecouvey) at the CIRM, Luminy (Nov. 2018).

2016–2017 **Journée d'accueil en Mathématiques**, at the Institut Henri Poincaré (IHP) (Jan. 2017).
Representation theory in Kaiserslautern (with Caroline Lassueur and Alessandro Paolini), at the university of Kaiserslautern, Germany (Jul. 2017).

2014–2015 **Colloque soltice** (with Christian Blanchet), at the institut de mathématiques de Jussieu (Jun. 2015).
Journée d'accueil en Mathématiques, at the Institut Henri Poincaré (IHP) (Jan. 2015).

2013–2014 **Summer school** on quiver Hecke algebras, organized with Gwyn Bellamy, Cargèse, France (Jun. 2014)
Journées du GDR Théorie de Lie, in Amiens, France (Dec. 14).

Academic responsibilities

Since 2020 Member of the *Commission Postes* of the College of Sciences of the Université de Paris.

Since 2019 Member of the *Commission Recherche* of the College of Sciences of the Université de Paris.

Since 2018 Member of the *Conseil du Laboratoire* of the Mathematical Institute IMJ-PRG.

Since 2016 Member of the *Conseil Scientifique de l'UFR*, scientific committee of the Mathematic Department of the University Paris 7.

2016-2019 Member of the *Conseil National des Universités* (CNU), section 25 (Pure Mathematics).

Since 2015 Member of the *Comité Prospective et Recrutement* of the Mathematical Institute IMJ-PRG.

Editorial responsibilities

2021- Member of the editorial board for the **Journal of Group Theory**.

Phd Students

Reda Chaneb. From September 2015 to October 2019, Olivier Brunat and I supervised Reda Chaneb at the Université Paris 7. Reda's project was centred on the study of decomposition matrices in bad characteristic. Until his work, little was known on these matrices when the characteristic of the coefficient field is "small". Only some small rank groups had been studied using algebraic methods, and Reda's work consisted in adapting and applying the geometric methods in [4]. The purpose was twofold: first, to complete partial results on explicit decomposition matrices for small groups (such as $\mathrm{Sp}_4(q)$ or some exceptional groups); second, to obtain general results for classical groups (basic sets, shape of the decomposition matrix...).

While trying to adapt Geck and Hiss's methods to exhibit basic sets, Reda discovered a general formula for the number of irreducible unipotent Brauer characters. His formula generalises the one obtained by Lusztig for unipotent characters in terms of some special unipotent classes and non-abelian Fourier transforms. It gives a strong evidence for a nice parametrisation of the set of unipotent Brauer characters and modular unipotent character sheaves. His results are published in IMRN [24].

Reda defended his thesis in october 2019 and now works as a high school teacher.

Parisa Ghazizadeh. From October 2016 to December 2019 I supervised Parisa Ghazizadeh for a Phd thesis at the Université Paris 7. Parisa's project was originally to study the torsion part of the (intersection) cohomology of Deligne–Lusztig varieties. The methods used are inspired from algebraic geometry and homological algebra but also from modular representation theory of finite groups.

Parisa did several computations on certain families of varieties for the general linear group, and showed that the cohomology groups were all torsion-free in that case. The aim of these computations was to prove a general result for $\mathrm{GL}_n(q)$ which would yield the geometric version of Broué's conjecture, as well as a criterion for the cohomology to be torsion-free, generalising [5] to the modular case. Parisa obtained only a partial result in that direction, which is available as a preprint [39].

She defended her thesis in December 2019 and is now looking for a job as a data scientist.