



Espaces des orbites et théorie effective des invariants

Marc Olive

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par

Marc OLIVE

Espaces des orbites et théorie effective des invariants

LAURENT MANIVEL

Rapporteurs :

JOËL MERKER

MAURIZIO VIANELLO

Date de soutenance : 12 Janvier 2021

Composition du jury :

PETER OLVER	(Président du jury)
OLIVIER ALLIX	(Examinateur)
MICHEL BRION	(Examinateur)
IRINA KOGAN	(Examinateuse)
LAURENT MANIVEL	(Rapporteur)
JOËL MERKER	(Rapporteur)
MAURZIO VIANELLO	(Rapporteur)

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- [art1] S. Abramian, B. Desmorat, R. Desmorat, B. Kolev, and M. Olive, *Recovering the normal form and symmetry class of an elasticity tensor* (2020) Journal of Elasticity. [10.1007/s10659-020-09784-7](https://doi.org/10.1007/s10659-020-09784-7) or <https://hal.archives-ouvertes.fr/hal-02410330>
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Première partie

Présentation générale

Introduction

Les travaux présentés ici s'inscrivent dans un cadre qui se situe à l'interface des mathématiques et de la mécanique théorique. Un des objectifs est de s'emparer de problématiques mathématiques fortes qui ont une portée mécanique, à la fois théorique et appliquée. De ce fait, la nature *effective* des résultats reste toujours un élément essentiel de l'approche envisagée.

Une des problématiques centrales de la mécanique, et pour ne pas dire de la physique, est de réussir à décrire des phénomènes indépendamment de l'observateur. Cette indépendance se traduit par le *principe d'indifférence matérielle* [98, 137]. Que ce soit dans les cas linéaires ou non linéaires, on postule l'existence de fonctions tensorielles. Par exemple, on introduit couramment un tenseur qui caractérise la *déformation* et un autre qui caractérise la *contrainte* (tel le tenseur de Cauchy). Le principe d'indifférence matérielle nécessite alors d'introduire des fonctions *équivariantes* de ces tenseurs [129].

C'est à ce stade qu'intervient naturellement la théorie des représentations des groupes et la théorie des invariants, appliquées spécifiquement au cas du groupe $O(3, \mathbb{R})$ des transformations orthogonales de \mathbb{R}^3 , ou bien de l'un de ses sous-groupes (fini ou infini). Dans les années 1950 en mécanique Rivlin [120] initia des questions de *calculs effectifs de bases d'intégrité*, en ne considérant dans un premier temps que des modèles faisant intervenir des tenseurs d'ordre 1 et 2. Pendant une bonne vingtaine d'années, et partant du texte pourtant réputé difficile *The classical groups* de Weyl [148], un grand nombre de résultats effectifs furent obtenus, établissant des listes de bases d'intégrité et de *bases fonctionnelles* (ou bases de séparants) sous l'action du groupe $O(3, \mathbb{R})$ (le cas dit isotrope), du groupe $SO(3, \mathbb{R})$ (le cas dit hémitrope), mais aussi des bases d'intégrité de *covariants*, même si ce n'était pas le vocabulaire employé. Un livre regroupant des travaux de Boehler, Spencer et Betten [15] résume très bien à la fois les problématiques ainsi que l'ensemble des résultats effectifs obtenus. Notons aussi que Smith [127, 128] obtint des bases d'intégrité pour des représentations tensorielles simples des groupes *cristallographiques* [131], sous-groupes finis du groupe $O(3, \mathbb{R})$.

Au-delà des tenseurs d'ordre 2, le cas d'une base d'intégrité pour l'espace des *tenseurs d'élasticité*, d'ordre 4, restait inaccessible, même si quelques tentatives avaient été faites

par Betten [10] ou encore Boehler–Kirilov–Onat [17]. Cela dit, c'est dans ce dernier article [17] que le lien fut établi, via une complexification, entre les représentations tensorielles du groupe $\mathrm{SO}(3, \mathbb{R})$ et les représentations du groupe $\mathrm{SL}(2, \mathbb{C})$ sur les espaces de formes binaires, et donc la *théorie classique des invariants* [108]. Sur cette découverte, il y a d'ailleurs une certaine "ironie" particulière : alors qu'il a fallu une bonne dizaine d'années pour que cette communauté obtienne des résultats effectifs sur des bases d'intégrité d'espaces $n\mathbb{S}^2(\mathbb{R}^3)$ de n tenseurs symétriques d'ordre 2, un même résultat avait été obtenu en 1899 par Young [150] sur des espaces nS_4 de n formes quadriques, traduction en complexe de l'espace $n\mathbb{S}^2(\mathbb{R}^3)$.

C'est donc au cours de ma thèse effectuée entre 2011 et 2014 que j'ai repris ce lien entre les représentations tensorielles du groupe $\mathrm{SO}(3, \mathbb{R})$ et, par complexification, les représentations du groupe $\mathrm{SL}(2, \mathbb{C})$. *Un des résultats majeurs de ce travail a été l'obtention d'une base d'intégrité de l'espace des tenseurs d'élasticité* ([Théorème 2.3.2](#) et [[art8](#), section 7]), de dimension 21 en 3D. Le calcul effectif d'une telle base d'intégrité n'a pu se faire que par une relecture et une réécriture des travaux *des anciens*, les plus actifs ayant été Gordan, Sylvester ou encore Cayley (voir l'article historique [112]). Bien sûr, après les apports d'Hilbert, Noether, Weyl, etc. (pour ne citer que les premiers, tant la liste est longue), la théorie des invariants a acquis une maturité certaine, mais le versant *effectif* a été entièrement mis de côté, alors même que l'effectivité était au cœur des travaux initiaux. En fait, il est possible que la difficulté et la complexité des calculs étaient telles que toute tentative dans ce sens semblait vaine.

Par la suite, il y eut cependant le développement de l'*umbral calculus* par Kung–Rota [84], dont le but était de clarifier la *méthode symbolique* des anciens. Quelques algorithmes issus de l'*umbral calculus* ont été proposés [133], mais leur complexité semblait peu adaptable pour traiter des cas non triviaux. Un passage marquant dans le texte de Kung–Rota précise alors [84, Further works 9] : "*Gordan's method of transvectants for his proof of the finiteness theorem was based on an ingenious method of substitutions of brackets into brackets. After Hilbert's work, Gordan's ideas were abandoned*".

En prenant cette observation au pied de la lettre, j'ai entièrement repris et modernisé ces travaux [[art11](#)], donnant ainsi une nouvelle version des algorithmes de Gordan ([Algorithme 3.2.7](#) et [Théorème 3.1.2](#)). J'ai alors déterminé des bases d'intégrité explicite pour des covariants d'espaces de formes binaires non triviaux : dans [[art11](#)] figurent des bases d'intégrité pour les covariants de $S_6 \oplus S_4$ et $S_6 \oplus S_4 \oplus S_2$, et dans [[art9](#)] figurent notamment des bases d'intégrité de covariants pour S_9 et S_{10} ([Théorème 3.3.5](#)) , obtenues avec Reynald Lercier dans le cadre du groupe de travail TEDI (Théorie Effective des Invariants).

Précisons d'ailleurs que Shioda, dès 1968, put retrouver une base d'intégrité de l'espace S_8 des formes binaires de degré 8, alors qu'une base de covariants avait déjà été obtenue par von Gall [145] en 1880. Par un véritable tour de force, Shioda put aussi préciser la struc-

ture algébrique de l’algèbre des invariants, en exhibant pour la première fois un *système de paramètres* et en effectuant sa résolution libre explicite. Un des points-clés de cette approche est l’utilisation de l’opération de *tranvection*, intimement liée à la *décomposition de Clebsch–Gordan* du produit tensoriel de deux espaces $SL(2, \mathbb{C})$ irréductibles, et par un procédé qui traduit le calcul en un *système linéaire diophantien*, Gordan put obtenir un algorithme de calcul pour les formes binaires, encore efficace aujourd’hui. Il est ainsi possible que cette idée de construire des transvectants en tant que *projecteurs équivariants* d’un produit tensoriel d’irréductibles pour un groupe fini puisse là aussi produire des algorithmes efficaces.

L’obtention de ces bases d’intégrité a alors été le point d’entrée de bien d’autres questions effectives qu’il restait à attaquer autour de cet espace des tenseurs d’élasticité. Et même plus, il s’est avéré que ces questions effectives s’ouvraient naturellement aux représentations linéaires *réelles* (V, G) d’un groupe de Lie compact. Le fait d’être dans le cas réel provient bien sûr des liens qu’il faut conserver avec la mécanique et leurs applications, mais le processus de complexification permet d’étudier des situations classiques en théorie des invariants, à savoir les représentations linéaires complexes de groupes linéairement réductifs. Le point le plus saillant qui a surgi autour de ces études porte sur le lien entre la *stratification isotropiques* de l’espace des orbites [18, 97] et sa *stratification semi-algébrique* [118], tout ceci en restant dans le cas réel.

En fait, l’observation initiale est que l’espace des orbites V/G peut se décrire directement avec l’algèbre des invariants $\mathbb{R}[V]^G$, à la fois pour séparer les orbites [2] mais aussi pour avoir la structure semi-algébrique de V/G par la positivité d’une matrice de Gram [118], du moins théoriquement. Malheureusement, ces idées restent très difficiles à mettre en œuvre : en effet, il faut à la fois se lancer dans le calcul explicite d’une base d’intégrité, et ensuite manipuler une famille d’invariants et une matrice de Gram souvent trop grande (il y a 297 invariants générateurs dans le cas de l’élasticité, et donc une matrice de Gram formelle de taille 297×297).

Une première étude sur ce lien a alors été initiée par Auffray–Kolev–Petitot [5], en se restreignant au cas d’une représentation irréductible (de dimension 9) de $SO(3, \mathbb{R})$, incluse dans l’espace des tenseurs d’élasticité, à savoir l’espace $\mathbb{H}^4(\mathbb{R}^3)$ des *tenseurs harmoniques* d’ordre 4. Pour cet espace, dont le complexifié est équivariant à l’espace $(S_8, SL(2, \mathbb{C}))$, une base d’intégrité de 9 invariants avait été écrite sous forme tensorielle, et la structure isotropique, composée de 8 strates [17], était déjà connue [54]. Ce n’est finalement pas par l’étude d’une matrice de Gram que la stratification isotropique a été étudiée, mais par la détermination d’une *tranche linéaire* associée à chaque strate, et la réduction du problème à la représentation d’un *groupe de monodromie*. Dans les cas d’un groupe de monodromie très simple (allant du groupe cyclique \mathbb{Z}_2 au groupe de permutation S_3), les calculs ont pu aboutir et exhiber, pour les 6 strates isotropes de plus petite dimension, des *équations algébriques explicites*. Malheureusement, même pour une monodromie du type S_3 , les équa-

tions ont été difficiles à obtenir, et toute généralisation aux autres strates isotropiques ainsi qu'à l'espace des tenseurs d'élasticité semblait peu envisageable.

Maintenant, les strates isotropiques d'une représentation $(V, \mathrm{SO}(3, \mathbb{R}))$ sont définies géométriquement et caractérisées par une *classe d'isotropie* explicite, donnée par la classe de conjugaison d'un sous-groupe fermé de $\mathrm{SO}(3, \mathbb{R})$. Un algorithme de calcul effectif de ces classes, développé dans des premiers travaux faits en collaboration avec Auffray [art14, art13] et repris dans [art5], permet à ce jour de connaître exactement la stratification isotropique de tels espaces. L'étape suivante consiste alors à décrire ces strates. Dans le travail commun fait avec Boris Kolev, Rodrigue et Boris Desmorat [105], l'exploitation des covariants, plus riches du point de vue géométrique que les invariants, a permis d'obtenir des équations algébriques explicites des 8 strates de l'espace des tenseurs d'élasticité (Théorème 6.2.1). Cependant, pour pouvoir effectuer des calculs, il a fallu définir une *algèbre de covariants*, en redéfinissant complètement la notion déjà existante de covariant d'une représentation donnée [46, 83]. Dans le même esprit que ce qui existe pour les algèbres de covariants des formes binaires [108], l'algèbre des covariants d'une représentation tensorielle $(V, \mathrm{SO}(3, \mathbb{R}))$ n'est alors rien d'autre que l'algèbre des invariants $\mathbb{R}[V \oplus \mathbb{R}^3]^{\mathrm{SO}(3, \mathbb{R})}$. En allant plus loin, il a été possible d'exhiber un isomorphisme explicite [105] entre les algèbres de covariants d'espaces tenseurs $(V, \mathrm{SO}(3, \mathbb{R}))$ et les algèbres d'invariants d'espace $(V \oplus S_2, \mathrm{SL}(2, \mathbb{C}))$, où V est un espace de formes binaires et S_2 est l'espace des formes quadratiques. En fait, dans des travaux antérieurs réalisés avec Boris Kolev, Boris et Rodrigue Desmorat [art7], le lien entre les représentations de $\mathrm{SO}(3, \mathbb{C})$ et $\mathrm{SL}(2, \mathbb{C})$ avait déjà été rendu explicite (Théorème 5.2.1), et c'est finalement dans [art4] qu'il a été entièrement précisé. Par ces résultats, une base d'intégrité de l'algèbre des covariants de l'espace $(\mathbb{H}^n(\mathbb{R}^3), \mathrm{SO}(3, \mathbb{R}))$ est directement déduite de l'algèbre des invariants de $(S_{2n} \oplus S_2)$, elle-même obtenue à partir d'une base de covariants de S_{2n} (Théorème 5.2.2). Il a ainsi été possible de calculer une base de covariants explicite de l'espace $(\mathbb{H}^4(\mathbb{R}^3), \mathrm{SO}(3, \mathbb{R}))$ (Théorème 5.4.1). Notons d'ailleurs qu'une telle base d'intégrité est écrite sous forme tensorielle [105, Table 2], oubliant ainsi le lien avec les formes binaires (qui n'ont servi que d'intermédiaire théorique). Ce résultat a nécessité de construire une nouvelle opération covariante sur les tenseurs [105] : le produit vectoriel généralisé (Définition 5.3.5). Au final, par cette nouvelle opération tensorielle, les équations algébriques explicites des 8 strates de l'élasticité (Théorème 6.2.1) ont été obtenues à l'aide d'une base d'intégrité des covariants de $(\mathbb{H}^4(\mathbb{R}^3), \mathrm{SO}(3, \mathbb{R}))$. En application de ces résultats, et au cours de l'encadrement d'une étudiante de M2 (Sophie Abramian), Boris Desmorat, Boris Kolev, Rodrigue Desmorat et moi-même [art1] avons établi un algorithme pour déterminer la classe d'isotropie explicite d'un tenseur d'élasticité donné (Figure 6.1). À mes yeux, tous ces résultats sont en fait la première étape d'un projet à plus long terme qui consiste à étudier plus en détail ce lien entre les stratifications isotropiques et semi-algébriques d'une représentation linéaire réelle d'un groupe de Lie compact, en passant d'ailleurs par l'étude de ce lien sur d'autres représentations tensorielles qui interviennent en mécanique des matériaux (en 2D).

ou 3D, en piézoélectricité, piézomagnétisme, etc.).

En parallèle de cette thématique, et pour aborder des questions liées à la détermination de *paramètres physiques*, telle la généralisation du coefficient de raideur d'un matériau élastique en 1D, j'ai aussi travaillé sur la notion de *séparants* d'une représentation (V, G) . Dans le cas complexe, il existe déjà la notion d'*algèbre séparante* [42, 50], et la communauté mécanique des années 1970 avait une notion similaire, appelée *base fonctionnelle* [15]. Encore une fois, dans le cas des espaces de tenseurs d'ordre au plus 2, des tables de familles minimales séparantes pour $\mathrm{SO}(3, \mathbb{R})$ et $\mathrm{O}(3, \mathbb{R})$ ont été obtenus explicitement, à la fois par Smith [126] et Wang [147]. Généraliser ces approches dans le cas de l'élasticité semblait difficile, même si une famille séparante polynomiale était déjà donnée par une base d'intégrité, de cardinal encore trop grand pour être manipulable. Une idée, déjà présente dans l'article de Boehler et al. [17], est alors de chercher des familles d'invariants polynomiaux ou *rationnels* qui ne séparent que des orbites *génériques*. Par cette approche, il est alors possible de connaître une borne minimale pour le cardinal d'une famille de *séparants génériques*. En effet, dans le cas des représentations complexes du groupe $\mathrm{SO}(3, \mathbb{C})$, et donc de son revêtement universel $\mathrm{SL}(2, \mathbb{C})$, on sait que le *corps des invariants* est *rationnel*, c'est-à-dire qu'il s'agit d'une extension transcendante pure de \mathbb{C} , dont le degré de transcendance et donné par la dimension de l'orbite générique (donc 18 dans le cas de l'élasticité). Ce résultat de rationalité a en fait été établi par Katsylo [78] pour toute représentation de dimension finie du groupe $\mathrm{SL}(2, \mathbb{C})$. *Par un travail effectué avec Rodrigue Desmorat, Nicolas Auffray, Boris Desmorat et Boris Kolev [art4], des résultats effectifs de Maeda [93] sur des corps d'invariants de formes binaires ont été traduits sous forme tensorielle. Il a ainsi été possible de construire une famille génératrice du corps des invariants de l'espace complexe $(\mathbb{H}^4(\mathbb{C}^3), \mathrm{SO}(3, \mathbb{C}))$ mais aussi de l'espace réel $(\mathbb{H}^4(\mathbb{R}^3), \mathrm{SO}(3, \mathbb{R}))$* (*Théorème 7.2.1*). En exploitant ce résultat, il est alors possible de construire une famille de 21 séparants génériques polynomiaux des tenseurs d'élasticité (*Théorème 7.3.1*) puis une famille optimale de 18 séparants génériques rationnels (*Corollaire 7.3.2*). Comme précédemment, de telles thématiques peuvent être abordées dans d'autres domaines de la mécanique théorique, et j'envisage aussi de chercher à construire des algorithmes de calcul de familles génératrices d'invariants rationnels, sachant que d'autres pistes ont été envisagées par Hubert et al. [67, 71, 72].

L'organisation de ce mémoire est le suivant. Dans le [Chapitre 1](#), je donne une présentation succincte des espaces d'orbites et de leur lien avec la théorie des invariants, sachant que tous les résultats présentés se trouvent dans la littérature mentionnée. La notion de stratification isotropique est précisée dans la [Section 1.1](#), puis je rappelle sa caractérisation semi-algébrique dans la [Section 1.2](#). Il y a ensuite les notions essentielles issues de la théorie classique des invariants que je résume dans la [Sous-section 1.3.1](#), en définissant notamment l'opération de transvection et son lien avec l'algèbre des covariants de formes binaires. C'est dans la [Sous-section 1.3.2](#) et la [Sous-section 1.3.3](#) que figure une liste de résultats sur les

algèbres et les corps des invariants des représentations linéaires (souvent complexes) de groupes linéairement réductifs, utiles par la suite. Enfin, il y a dans la [Section 1.4](#) des rappels sur les représentations linéaires des groupes $\mathrm{SO}(3, \mathbb{R})$, $\mathrm{SO}(3, \mathbb{C})$ et $\mathrm{SL}(2, \mathbb{C})$, sachant que les représentations de ces deux derniers groupes sont fortement liées. Je conclus alors cette première partie par la [Sous-section 1.4.2](#) qui détaille l'espace des tenseurs d'élasticité, en précisant des formules de décomposition en irréductibles ainsi que sa stratification isotropique.

La [Partie II](#) couvre l'ensemble de mes contributions, en commençant par des résultats de thèse très brièvement présentés dans le [Chapitre 2](#). Il sera ensuite question dans le [Chapitre 3](#) de nouveaux résultats obtenus dans le cadre de la théorie classique des invariants : je donne dans les [Sections 3.1](#) et [3.2](#) ma version des algorithmes de Gordan qui ont été réécrits et réinterprétés, puis une base d'intégrité des covariants de formes binaires de degrés 9 et 10 dans la [Section 3.3](#). Je présente dans le [Chapitre 4](#) ce qui concerne des problèmes de reconstructions équivariantes de tenseurs, l'idée étant de paramétriser des modèles tensoriels avec des tenseurs plus simples à manipuler (tels les tenseurs d'ordre 2). Ce problème est alors abordé dans le cas de l'élasticité, dans la [Section 4.3](#), où des formules explicites de reconstruction sont données pour deux strates parmi les 8. Il y a dans le [Chapitre 5](#) tout ce qui concerne la notion d'algèbre de covariants d'une représentation linéaire, et je précise dans la [Section 5.2](#) le lien explicite entre les algèbres de covariants d'espaces de tenseurs et une algèbre d'invariants de formes binaires. Je donne alors dans la [Section 5.4](#) une base d'intégrité minimale de l'algèbre des covariants de l'espace $(\mathbb{H}^4(\mathbb{R}^3), \mathrm{SO}(3, \mathbb{R}))$, écrite sous forme tensorielle. Dans le [Chapitre 6](#), le lien entre la stratification isotropique d'une représentation réelle (V, G) d'un groupe de Lie compact et sa stratification semi-algébrique est précisé, en se focalisant sur le cas des représentations tensorielles du groupe $\mathrm{SO}(3, \mathbb{R})$. Je donne ainsi dans la [Section 6.2](#) des équations algébriques explicites des 8 classes d'isotropie de l'espace des tenseurs d'élasticité, exploitant pour cela la base d'intégrité de l'algèbre des covariants de l'espace $(\mathbb{H}^4(\mathbb{R}^3), \mathrm{SO}(3, \mathbb{R}))$. Dans le [Chapitre 7](#), la notion de *séparants génériques* d'une représentation linéaire apparaît, et je précise son lien avec une famille génératrice du corps des invariants associé. Une fois traduit sous forme tensorielle des résultats de Maeda sur les formes binaires de degré 8, ce dans la [Section 7.2](#), je donne dans la [Section 7.3](#) deux familles de séparants génériques polynomiaux (l'une de 21 invariants, l'autre de 19), puis une famille de 18 séparants génériques rationnels. Je conclus alors cette partie par le [Chapitre 8](#) qui concerne la détermination des symétries d'ordre 2 de tenseurs totalement symétriques, obtenues à l'aide d'équations algébriques réduites. Ce mémoire se termine finalement par la [Partie III](#) qui précise les différentes questions et perspectives que j'envisage pour la suite, autour de ce qui concerne les espaces d'orbites et la théorie des invariants ([Chapitre 9](#)), puis des applications possibles en mécanique ([Chapitre 10](#)).

Chapitre 1

A survey about orbit space and invariant theory

Here, we recall some important facts about orbit space and invariant algebra associated to a linear representation V of a compact Lie group G , which is the starting point of many questions we develop in [Part II](#). Note that no result here comes from my personal contribution, and I refer in each case to the classical literature on the subject. First of all, the orbit space can be described using its *isotropic stratification*, detailed in [Section 1.1](#). Then, in the real case, Procesi–Schwarz [118], following Abud–Sartori [2], made explicit its *semi-algebraic structure* using a finite integrity basis of the invariant algebra, which is summarized in [Section 1.2](#).

To exploit results from invariant theory, on both theoretical and effective grounds, the complexification of the real representation leads us to consider a complex representation of a linearly reductive group. We thus recall in [Section 1.3](#) some important results about the associated algebra of polynomial invariants and the associated field of rational invariants, with special interest in *classical invariant theory* quickly summarized in [Subsection 1.3.1](#).

Our research work in the field of continuum mechanics makes us consider linear representations of the rotation group $\mathrm{SO}(3, \mathbb{R})$ in \mathbb{R}^3 , related to the special linear group in two dimensions, where some models are presented in [Section 1.4](#). Finally, we mention in [Subsection 1.4.2](#) the main effective results known about the space of elasticity tensors.

1.1 Isotropic stratification of orbit space

Let us first recall some *finiteness* results about isotropy classes of the action of a compact Lie group G on a locally compact manifold X [60, 97]. For each $g \in G$, such an action will

simply be denoted

$$x \in X \mapsto g \star x \in X$$

and for any $x \in X$ we define its isotropy subgroup as

$$G_x := \{g \in G, g \star x = x\}.$$

We say that two points $x, y \in X$ belong to the same *isotropy class* if both G_x and G_y are conjugate. Taking $[H] := \{gHg^{-1}, g \in G\}$ to be the conjugacy class of a subgroup $H \subset G$, this translates into $G_y \in [G_x]$. The question now arises of the finiteness of different isotropy classes, widely studied in the 60's [18, 97, 101]. The main result is :

Theorem 1.1.1 (Mann 1962 [97]). *Let X be an orientable manifold whose homology groups $H_i(M, \mathbb{Z})$ are finitely generated. Then the number of different isotropy classes of any action of a compact Lie group is finite.*

From now on, we only focus on a *linear representation*

$$\rho : V \longrightarrow \mathrm{GL}(V) \tag{1.1}$$

of a compact Lie group G on a finite dimensional real vector space V . In such a specific case, finiteness is directly deduced from Mann's theorem, while weaker results can be used (see for instance [111, Corollary 1.7.26] or [60, Theorem 1.9]).

Let us now define for each isotropy class $[H]$ the associated *stratum*

$$\Sigma_{[H]} := \{\mathbf{v} \in V, G_{\mathbf{v}} \in [H]\}$$

which is a *smooth submanifold* of V [2, 18], also G -stable. Taking $N(H) := \{g \in G, gHg^{-1} = H\}$ to be the normalizer of H and $\mathrm{Fix}(H)$ the *fixed point set* of H defined by

$$\mathrm{Fix}(H) := \{\mathbf{v} \in V, \rho(h)\mathbf{v} = \mathbf{v}, \forall h \in H\},$$

we have [5]

$$\begin{aligned} \dim(\Sigma_{[H]}) &= \dim(\mathrm{Fix}(H)) + \dim(G) - \dim(N(H)), \\ \dim(\Sigma_{[H]}/G) &= \dim(\mathrm{Fix}(H)) - \dim(\Gamma(H)) \end{aligned}$$

with $\Gamma(H) := N(H)/H$ the *monodromy group*. In these formulae, the dimension of the fixed point set can be computed using the character χ of the representation and the trace formula [59]

$$\dim(\mathrm{Fix}(H)) = \frac{1}{|H|} \sum_{h \in H} \chi_{\rho}(h)$$

for a finite group G , with an analogue using Haar integral for an infinite compact group G .

The linear representation (V, G) finally induces an *isotropic stratification*

$$V = \Sigma_{[H_1]} \sqcup \dots \sqcup \Sigma_{[H_n]} \quad (1.2)$$

into a finite disjoint union of strata. On the set of conjugacy classes of closed subgroups of G , there is a partial ordering defined as follows (see [18, Proposition 1.9]) :

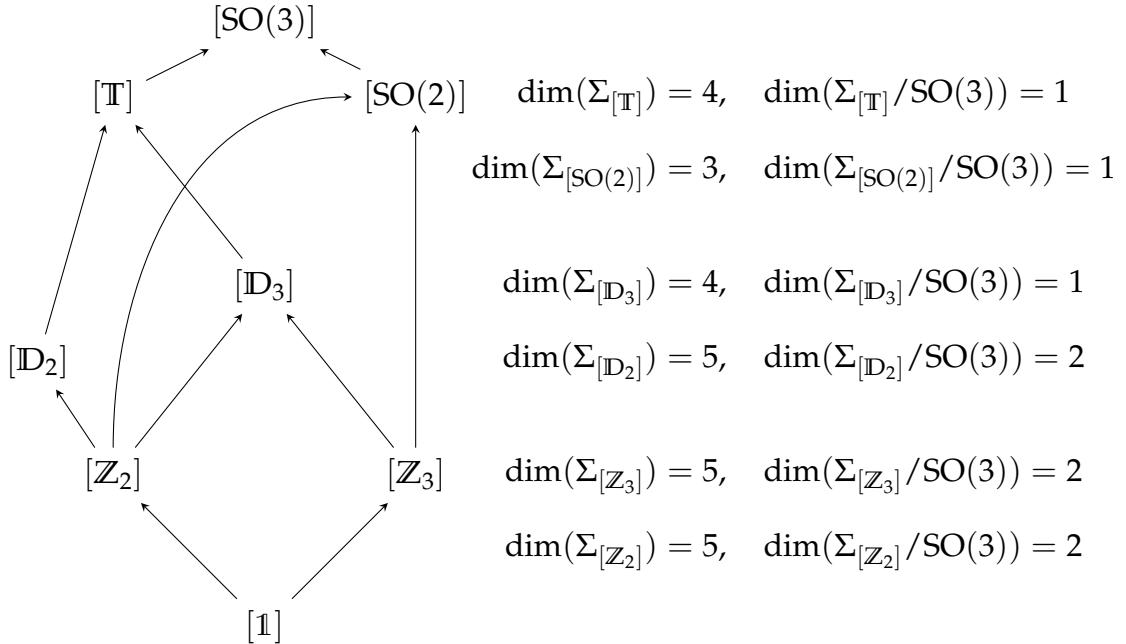
$$[H_1] \preceq [H_2] \iff \exists g \in G, \quad gH_1g^{-1} \subset H_2,$$

and we deduce a (reverse) partial ordering on the strata :

$$\Sigma_{[H_2]} \preceq \Sigma_{[H_1]} \iff [H_1] \preceq [H_2].$$

The maximal stratum Σ_{\max} , corresponding to the minimum isotropy class, is in fact a *dense and open* set in V , and any $\mathbf{v} \in \Sigma_{\max}$ can be said to be *generic*.

Example 1.1.2. Ihrig and Golubitsky obtained the finite set of isotropy classes for all $\mathrm{SO}(3, \mathbb{R})$ and $\mathrm{O}(3, \mathbb{R})$ irreducible representations [73], modelled for instance on the space $\mathcal{H}_3(\mathbb{R}^3)$ of n th degree *harmonic polynomials* [131] (n th degree homogeneous polynomials in \mathbb{R}^3 with null Laplacian). Figure 1.1 (issued from [59]) gives the isotropic stratification of the irreducible representation of $\mathrm{SO}(3, \mathbb{R})$ on the space $\mathcal{H}_3(\mathbb{R}^3)$ of third degree harmonic polynomials, where an arrow $[H_1] \rightarrow [H_2]$ means here that $[H_1] \preceq [H_2]$. The notations are as follow : \mathbb{Z}_n is the cyclic group of order n , D_n is the dihedral group of order $2n$ and \mathbb{T} is the orientation preserving symmetry group of a tetraedron, of order 12. Note also that dimensions of fixed point sets are directly obtained from [73, Theorem 3.2].

FIGURE 1.1 – Isotropic stratification of $(\mathcal{H}_3(\mathbb{R}^3), SO(3))$

1.2 Orbit space and invariant theory

Take some linear representation of a compact Lie group G on a real vector space V . Such a linear representation will simply be denoted $\mathbf{v} \mapsto g\mathbf{v}$ instead of $\rho(g)\mathbf{v}$ and it induces a linear representation on the *coordinate ring* $\mathbb{R}[V]$:

$$(g \star p)(\mathbf{v}) := p(g^{-1}\mathbf{v}),$$

so we can define the *invariant algebra*

$$\mathbb{R}[V]^G := \{ p \in \mathbb{R}[V], \quad g \star p = p \}$$

which is *finitely generated* [69]. A *finite integrity basis* $\{ J_1, \dots, J_N \}$ is defined to be a generating set of such an algebra, so that

$$\mathbb{R}[V]^G = \mathbb{R}[J_1, \dots, J_N].$$

A very important fact which occurs in the real case is that the invariant algebra, and thus a finite integrity basis, always *separates the orbits* [2], meaning that

$$\exists g \in G, \quad \mathbf{v}_1 = g\mathbf{v}_2 \iff \forall k = 1, \dots, N, \quad J_k(\mathbf{v}_1) = J_k(\mathbf{v}_2).$$

Remark 1.2.1. Finiteness in Hilbert's theorem extends to the case of complex representations of *reductive groups*, which can be viewed as complexifications of compact Lie groups [122]. Nevertheless, the separating property is no longer true in this situation [42].

A stronger link has been established between the invariant algebra and the orbit space, initially described by Abud–Sartori [2] and then clearly established by Procesi–Schwarz [118]. More precisely, if we consider the polynomial mapping

$$\mathbf{v} \in V \mapsto J(\mathbf{v}) := (J_1(\mathbf{v}), \dots, J_N(\mathbf{v})) \in \mathbb{R}^N,$$

then the image $X := \text{Im}(J) \subset \mathbb{R}^N$ is a *semi-algebraic set*, and J induces a proper homeomorphism between V/G and X [118].

Take now a G -invariant scalar product $\langle \cdot, \cdot \rangle_G$ on V (which always exists, using for instance Haar integral) and define a *symmetric matrix* $\text{Gram}(\mathbf{v})$ for any $\mathbf{v} \in V$ by :

$$\text{Gram}_{ij}(\mathbf{v}) := \langle dJ_i(\mathbf{v}), dJ_j(\mathbf{v}) \rangle_G \in \mathbb{R}[V]^G, \quad 1 \leq i, j \leq N,$$

where dp is the differential of p . Using the integrity basis, we thus know that $\text{Gram}_{ij}(\mathbf{v})$ can be written as

$$\text{Gram}_{ij}(\mathbf{v}) = p_{ij}(J_1(\mathbf{v}), \dots, J_N(\mathbf{v}))$$

for certain polynomials p_{ij} (not necessary unique), so we obtain a $N \times N$ symmetric matrix $\widetilde{\text{Gram}}(J)$ with coefficient in $\mathbb{R}[J_1, \dots, J_N]$. The main result is finally [118] :

$$X = \{ (J_1, \dots, J_N) \in \mathbb{R}^N, \quad \widetilde{\text{Gram}}(J) \text{ is positive semidefinite} \}.$$

Here, as the symmetric matrix $\widetilde{\text{Gram}}(J)$ is positive semidefinite, it can be expressed using the determinant of its principal minors.

Such a result is, of course, very important, but it relies firstly on the knowledge of an integrity basis and secondly on a *rewriting* procedure in the invariant algebra. The computational aspects for each of these steps can easily become unattainable, as for instance in the case of the 21-dimensional space of elasticity tensors where an integrity basis is given by 297 invariants (see [Subsection 1.4.2](#) and [Section 2.3](#)).

Finally, let us mention here that the semi-algebraic set X has a natural stratification [34], which seems to have strong connection with the isotropic stratification of the orbit space detailed in [section 1.1](#).

Example 1.2.2. Take back here the irreducible representation $(\mathcal{H}_3(\mathbb{R}^3), \text{SO}(3, \mathbb{R}))$ from [Example 1.1.2](#), which is equivalent to the $\text{SO}(3, \mathbb{R})$ representation on the space $\mathbb{H}^3(\mathbb{R}^3)$ of third order *harmonic tensors* (totally symmetric and traceless, see [Subsection 1.4.1](#)). Given

any orthonormal basis $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ of \mathbb{R}^3 , an $\text{SO}(3, \mathbb{R})$ invariant scalar product on $\mathbb{H}^3(\mathbb{R}^3)$ is given by

$$\langle \mathbf{H}^{(1)}, \mathbf{H}^{(2)} \rangle := H_{pqr}^{(1)} H_{pqr}^{(2)}, \quad \mathbf{H}^1, \mathbf{H}^2 \in \mathbb{H}^3(\mathbb{R}^3)$$

where we have used Einstein convention on repeated indices. From Smith–Bao [124], a minimal integrity basis is given by

$$J_2 := H_{ijk} H_{ijk}, \quad J_4 := b_{ij} b_{ij}, \quad J_6 := v_i v_i, \quad J_{10} := H_{ijk} v_i v_j v_k, \quad J_{15} := \varepsilon_{ijk} v_i b_{jp} v_p H_{kqr} v_q v_r$$

where ε_{ijk} is the third order Levi–Civita symbol and $\mathbf{b} = (b_{ij})$ (resp. $\mathbf{v} = (v_i)$) is the second order tensor (resp. first order tensor) defined by $b_{ij} := H_{ipq} H_{jpq}$ (resp. $v_i := H_{ijk} b_{jk}$).

The semi-algebraic set $X \subset \mathbb{R}^5$ is obtained from a 5×5 symmetric matrix defined in $\mathbb{R}[J_2, J_4, J_6, J_{10}, J_{15}]$. Nevertheless, all computations has to be done in the free algebra $\mathbb{R}[h_1, \dots, h_7] = \mathbb{R}[\mathbb{H}^3(\mathbb{R}^3)]$. In this algebra : J_4 is an homogeneous polynomial consisting of 56 monomials, while J_6 , J_{10} and J_{15} respectively consist of 189, 1632 and 10699 monomials.

For the rewriting procedure, we know two strategies (see also [87]) :

1. A first one using an *elimination* procedure from Gröbner bases [35];
2. Another one using linear algebra : each polynomial invariant in the Gram matrix is some k th degree homogeneous invariant in $\mathbb{R}[J_2, J_4, J_6, J_{10}, J_{15}]$, where we can easily obtain a generating set from Diophantine equations. For instance, $\langle dJ_4, dJ_6 \rangle$ is linearly generated by the family $\{ J_2^4, J_2^2 J_4, J_2 J_6, J_4^2 \}$ and we get

$$\langle dJ_4, dJ_6 \rangle = \frac{2}{45} J_2^4 - \frac{2}{9} J_2^2 J_4 + \frac{8}{3} J_2 J_6 + \frac{4}{15} J_4^2.$$

We finally propose in [Table 1.1](#) the computation time, all being done using Macaulay2 software [63].

Invariant	# of monomials	Time, strategy 1	Time, strategy 2
$\langle dJ_2, dJ_4 \rangle = 2J_4^2$	56	2.14289 s	0.00162821 s
$\langle dJ_4, dJ_6 \rangle$	755	N/A	0.067198 s

TABLE 1.1 – Two examples of computation time

1.3 Effective invariant theory

Main questions addressed in *effective invariant theory*, as emphasized by Vinberg [144], are in a way very simple : once we have Hilbert’s finiteness result of a given invariant algebra, how can we obtain an explicit integrity basis, and how to describe this algebra. Such

an effective problematic was particularly present from the beginning of *classical invariant theory*, which is summarized in Subsection 1.3.1. The theoretical framework initiated by Hilbert [69] and enriched by Noether's contribution to algebra [136, 141], then gave rise to important theoretical results that we briefly present in Subsections 1.3.2 and 1.3.3. We also mention here important works on this subject of effective invariant theory from Derksen and al. [39, 41, 42, 79, 81].

1.3.1 Classical invariant theory

Classical invariant theory has been the source of several studies, both from the point of view of the history of science [36, 37, 112] and from the sociological point of view [53]. It carries within it many issues inherent in the more general framework of invariant theory which, among other things, seeks to describe objects independently of their representations [144]. Of course, an abundant literature exists on this subject [46, 52, 58, 62, 108, 133], and we will only give results here that focus on effectiveness.

Let us now consider the standard linear representation of the special linear group

$$\mathrm{SL}(2, \mathbb{C}) := \left\{ \gamma := \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad ad - bc = 1 \right\} \quad (1.3)$$

on a *space of binary forms*. Here, a binary form \mathbf{f} of degree n is a homogeneous complex polynomial in two variables u, v of degree n :

$$\mathbf{f}(\boldsymbol{\xi}) = a_0 u^n + a_1 u^{n-1} v + \cdots + a_{n-1} u v^{n-1} + a_n v^n, \quad \boldsymbol{\xi} := (u, v) \in \mathbb{C}^2, \quad a_k \in \mathbb{C}.$$

The set of all binary forms of degree n is a complex vector space of dimension $n+1$ which will be denoted by S_n . The special linear group $\mathrm{SL}(2, \mathbb{C})$ then acts naturally on \mathbb{C}^2 and induces a left action on S_n , given by

$$(\rho(\gamma)\mathbf{f})(\boldsymbol{\xi}) := \mathbf{f}(\gamma^{-1}\boldsymbol{\xi}),$$

where $\gamma \in \mathrm{SL}(2, \mathbb{C})$. The spaces S_n are all the irreducible holomorphic $\mathrm{SL}(2, \mathbb{C})$ representations (where holomorphic means here that morphism (1.1) is holomorphic, see [82] for instance) and every complex holomorphic linear representation V of $\mathrm{SL}(2, \mathbb{C})$ can be decomposed into a direct sum

$$V \simeq S_{n_1} \oplus \cdots \oplus S_{n_p}.$$

The Clebsch Gordan decomposition in the specific case of $\mathrm{SL}(2, \mathbb{C})$ representations writes

$$S_n \otimes S_p = \bigoplus_{r=0}^{\min(n,p)} S_{n+p-2r} \quad (1.4)$$

and an explicit equivariant projection from $S_n \otimes S_p$ to S_{n+p-2r} is obtained by *transvection* [61] :

Definition 1.3.1. The *transvectant* of index r of two binary forms $\mathbf{f} \in S_n$ and $\mathbf{g} \in S_p$ is defined as

$$(\mathbf{f}, \mathbf{g})_r := \frac{(n-r)!}{n!} \frac{(p-r)!}{p!} \sum_{i=0}^r (-1)^i \binom{r}{i} \frac{\partial^r \mathbf{f}}{\partial^{r-i} u \partial^i v} \frac{\partial^r \mathbf{g}}{\partial^i u \partial^{r-i} v}, \quad (1.5)$$

which is a binary form of degree $n + p - 2r$ (vanishes if $r > \min(n, p)$).

Example 1.3.2. For two binary forms which are n th powers of linear forms

$$(\mathbf{a}\xi)^n := (a_1 u + a_2 v)^n, \quad (\mathbf{b}\xi)^p := (b_1 u + b_2 v)^p, \quad (1.6)$$

we get the particularly simple and classical formula [62]

$$((\mathbf{a}\xi)^n, (\mathbf{b}\xi)^p)_r = [\mathbf{a}\mathbf{b}]^r (\mathbf{a}\xi)^{n-r} (\mathbf{b}\xi)^{p-r},$$

where by definition $[\mathbf{a}\mathbf{b}] := a_1 b_2 - a_2 b_1$. In a way, such an expression is the starting point of the *symbolic method* [65, 133].

This transvection operation makes it possible to obtain equivariant binary forms, using a *polynomial mapping*. More specifically, we consider the *covariant algebra* of a space of binary forms :

Definition 1.3.3. The covariant algebra of V is a bi-graded algebra defined as

$$\mathbf{Cov}(V) := \mathbb{C}[V \oplus \mathbb{C}^2]^{\mathrm{SL}(2, \mathbb{C})} = \bigoplus_{d, m \geq 0} \mathbf{Cov}_{d, m}(V),$$

where $\mathbf{Cov}_{d, m}(V)$ is the homogeneous space of covariants of total degree d in $\mathbf{f} \in V$, and total degree m in $\xi \in \mathbb{C}^2$, called the **order** of a covariant.

Such an algebra is in fact of finite type, the proofs being obtained for the first time and in a constructive way by Gordan [61] (see [art11] for a reformulation of this statement and a modern version of the proof, all summarized in [Chapitre 3](#)). Being of finite type, we can also focus on *minimal bases* of covariant algebras. For a space V of binary forms let us define

$$C_+ := \sum_{d+m>0} \mathbf{Cov}_{d, m}(V)$$

which is an ideal of the graded algebra $\mathbf{Cov}(V)$. Then C_+^2 is itself graded by each (d, m) such that $d + m > 0$. Let now $\delta_{d, m}$ be the codimension of $(C_+^2)_{d, m}$ in $\mathbf{Cov}_{d, m}(V)$. Since the

algebra $\mathbf{Cov}(V)$ is of finite type, there exists an integer p such that $\delta_{d,m} = 0$ for $d + m \geq p$ and we can define the invariant number :

$$n(V) := \sum_{d,m} \delta_{d,m}.$$

Finally we now focus on

Definition 1.3.4. A family $\{\mathbf{h}_1, \dots, \mathbf{h}_s\}$ is a *minimal* integrity basis of $\mathbf{Cov}(V)$ if its image in the vector space C_+/C_+^2 is a basis. In that case, we have $s = n(V)$.

The first challenge of classical invariant theory was in fact to obtain explicit minimal integrity bases of covariant algebras. A website, maintained by Brouwer [22], lists all the last known explicit results of such generating families, and we give only some examples with references in Table 1.2.

Algebra	$n(V)$	Explicit minimal basis
$\mathbf{Cov}(S_5)$	23	Gordan (1868) [61]
$\mathbf{Cov}(S_6)$	26	Gordan (1868) [61]
$\mathbf{Cov}(S_6 \oplus S_2)$	99	von Gall (1888) [146], Brouwer and Popoviciu (2011) [25]
$\mathbf{Cov}(S_7)$	147	Cröni (2002) [38], Bedratyuk (2009) [8]
$\mathbf{Cov}(S_8)$	69	von Gall (non minimal, 1880) [145] Cröni [38], Bedratyuk (2008) [7] Popoviciu (2014) [117]
$\mathbf{Cov}(nS_4)$, $n \geq 1$	in [151]	Young (1899) [151]

TABLE 1.2 – Minimal integrity bases of covariant algebras

Note that the history of classical invariant theory is nowadays very well documented [112]. For the record, it involved an English school [28, 135] as well as a German school [4, 61], and the first issue was the finiteness (or not) of these covariant algebras, from the effective point of view. Gordan's proof, as well as his algorithm, little understood by the English school [112], had led to notable effective results (cf. Table 1.2). All computations used transvection operations and clever *rewriting rules* on iterated transvectants [1, 62, 132]. Moreover, it was in 1968 that Shioda obtained a remarkable result [123], without any computer, on the covariant algebra of a binary form of degree 8, starting with von Gall's results [145].

Many attempts were made to understand and reconsider the symbolic method, starting for instance with Weyl's contribution on invariant theory [148]. It was then followed by

Kung–Rota [84], Dieudonné–Carrell [43], or even more recently by Olver [108]. An algorithm using the symbolic method is mentioned in [84], also reconsidered in [133], but as far as we now, these approaches could not lead to new effective results.

1.3.2 Invariant algebra of linearly reductive groups

Let us consider here some linear group representation (W, Γ) of a linearly reductive group Γ on a finite-dimensional complex vector space W . The invariant algebra $\mathbb{C}[W]^\Gamma$ is a specific case of a graded algebra of finite type. We essentially focus here on some way to obtain an explicit integrity basis.

A first idea is to get a bound on the degrees of the generators, like Noether's bound for finite groups given by its order [133]. For linear algebraic groups, theoretical results exist on this bound [40], but up to our knowledge, associated bounds were not accessible in classical cases : for the space of binary octics [123], this lead to a bound of 262 144. We summarize in [Table 1.3](#) some known results on these bounds, where we have set

$$\beta(W) := \min \{ d, \quad \mathbb{C}[W]^\Gamma \text{ can be generated by polynomials of degree } \leq d \}.$$

Group	Space	Bound	Reference
Γ finite	arbitrary V	$ G $	Noether [102]
$SL(2, \mathbb{C})$	S_n	$\beta(S_n) \leq n^6$	Jordan [75]
$SL(3, \mathbb{C})$	$V_n := S^n(\mathbb{C}^3)$	$\beta(V_n) \leq 27/8(n+2)(n+1)n^{18}$	DerkSEN [40]

TABLE 1.3 – Some degree bounds on generators of $\mathbb{C}[W]^\Gamma$

We now mention some important notions leading to more effective algorithms, all related to graded algebras of finite type, such as the one of *Hilbert series*, *system of parameters* and *Cohen–Macaulay property*.

Let us consider some *graded algebra*

$$\mathcal{A} = \bigoplus_{k \geq 0} \mathcal{A}_k$$

where \mathcal{A}_k is the homogeneous space of k th degree elements of \mathcal{A} . Its *Hilbert series* encode the dimensions of the homogeneous spaces :

$$H_{\mathcal{A}}(z) := \sum_{k \geq 0} \dim(\mathcal{A}_k) z^k.$$

As we are interested in invariant algebra, we mention :

Theorem 1.3.5 (Molien–Weyl formula [88]). *For any finite-dimensional (real or complex) representation (W, G, ρ) of a compact Lie group G , with \mathcal{A} the algebra of G invariant polynomials, one has*

$$H_{\mathcal{A}}(z) = \int_G \frac{1}{\det(\text{Id} - \rho(g)z)} d\mu,$$

where $d\mu$ is the Haar measure on G .

In fact, there is no distinction to make between the invariant algebras of a compact Lie group G and the ones of its complexification $\Gamma := G^{\mathbb{C}}$. We have indeed [art8] :

Lemma 1.3.6. *The complexification of the real algebra $\mathbb{R}[V]^G$ is isomorphic to the invariant algebra $\mathbb{C}[W]^{\Gamma}$, where $W := V^{\mathbb{C}}$ (resp. $\Gamma := G^{\mathbb{C}}$) is the complexification of the real vector space V (resp. of the compact Lie group G).*

Remark 1.3.7. In the case of the special linear group $\text{SL}(2, \mathbb{C})$, this means that we can make use of the special unitary group $\text{SU}(2)$ (this approach is known as the *unitary trick* [46]). From this and the Molien–Weyl formula, Hilbert series of covariant algebras for binary forms are easily obtained, using for instance a Maple package due to Bedratyuk [9].

The Hilbert series of an algebra of finite type is strongly linked to a generating family. Indeed, as a consequence of Hilbert’s syzygy theorem we have [69] :

Lemma 1.3.8. *Let $\{a_1, \dots, a_m\}$ be a generating family of a graded algebra \mathcal{A} , all homogeneous of degrees $\deg(a_i) = d_i$ ($i = 1, \dots, m$). Then the Hilbert series of \mathcal{A} writes*

$$\mathcal{H}_{\mathcal{A}}(z) = \frac{f(z)}{(1 - z^{d_1}) \dots (1 - z^{d_m})},$$

with a certain $f \in \mathbb{Z}[z]$.

In fact, from the use of a *system of parameters* and the Cohen–Macaulay property, it is possible to go one step further. Let first recall [86] :

Lemma 1.3.9. *Let \mathcal{A} be graded algebra over \mathbb{C} of finite type and of transcendence degree s . Then there exists a homogeneous system of parameters $\{\alpha_1, \dots, \alpha_s\}$ of \mathcal{A} , meaning that :*

1. *Each α_i is homogeneous;*
2. *The family $\{\alpha_1, \dots, \alpha_s\}$ is algebraically independent;*
3. *\mathcal{A} is a $\mathbb{C}[\alpha_1, \dots, \alpha_s]$ –module of finite type, so that*

$$\mathcal{A} = p_1 \mathbb{C}[\alpha_1, \dots, \alpha_s] + \dots + p_k \mathbb{C}[\alpha_1, \dots, \alpha_s], \quad p_1, \dots, p_k \in \mathcal{A} \tag{1.7}$$

Remark 1.3.10. For the invariant algebra $\mathbb{R}[V]^G$ of a compact Lie group representation, its transcendence degree is given by the dimension of the *generic stratum*, which is $\dim(V) - \dim(G) + \dim(H_{\min})$, where $[H_{\min}]$ is the minimum isotropy class (see [Section 1.1](#)).

And for invariant algebras $\mathbb{C}[W]^\Gamma$ we have :

Theorem 1.3.11 (Hochster–Roberts [\[70\]](#)). *Suppose (W, Γ) is a linear representation of a linearly reductive group Γ on a finite-dimensional (complex or real) vector space W . Then the invariant algebra $\mathbb{C}[W]^\Gamma$ is Cohen–Macaulay, meaning that there exists a homogeneous system of parameters $\{\theta_1, \dots, \theta_s\}$ such that $\mathbb{C}[W]^\Gamma$ is a free $\mathbb{C}[\theta_1, \dots, \theta_s]$ –module of finite type (i.e. the decomposition [\(1.7\)](#) is a direct sum).*

We thus deduce (see [\[133\]](#) for instance) :

Corollary 1.3.12. *There exists a finite set $\{\eta_1, \dots, \eta_r\}$ of invariants so that $\mathbb{C}[W]^\Gamma$ has the Hironaka decomposition*

$$\mathbb{C}[W]^\Gamma = \eta_1 \mathbb{C}[\theta_1, \dots, \theta_s] \oplus \cdots \oplus \eta_r \mathbb{C}[\theta_1, \dots, \theta_s].$$

Furthermore, writing $d_i := \deg(\theta_i)$ and $e_j := \deg(\eta_j)$, the Hilbert series H of $\mathbb{C}[W]^\Gamma$ is given by

$$H(z) = \frac{z^{e_1} + \cdots + z^{e_r}}{(1 - z^{d_1}) \cdots (1 - z^{d_s})}.$$

Such a corollary is a starting point of an algorithm in effective invariant theory, summarized as follows :

1. Compute the Hilbert series of $\mathbb{C}[W]^\Gamma$ from the Molien–Weyl formula ;
2. Exhibit degrees of a homogeneous system of parameters of the algebra $\mathbb{C}[W]^\Gamma$;
3. Deduce a bound $\beta := \max(d_i, e_j)$ for the generators of $\mathbb{C}[W]^\Gamma$ and compute, for each homogeneous space $(\mathbb{C}[W]^\Gamma)_k$ with $k \leq \beta$, a linear basis.

This was successfully applied in classical invariant theory, where new results were obtained recently for the invariant algebra of binary forms of degrees 9 and 10 [\[21, 24, 117\]](#). However, the ‘trickiest’ point concerns the computation of degrees of a homogeneous system of parameters. There is of course important work by Dixmier in the context of binary forms [\[45\]](#), followed in the same context by some effective results [\[23, 117\]](#), but we do not yet know an effective algorithm, even in the case of binary forms, to obtain an explicit homogeneous system of parameters. Thus, up to our knowledge, even though we know an integrity basis of elasticity tensors (cf. [Section 2.3](#)), a system of parameters has still not been exhibited.

1.3.3 Rational invariants and orbit space

When addressing the question of *separating* the orbits for reductive groups Γ , the complex invariant algebra fails to reflect all features of the orbit space, while for compact groups G Section 1.2 stated it succeeds. To deal with the separating property in the complex and linearly reductive case, one way is to consider the field of *rational invariants* :

$$\mathbb{C}(W)^\Gamma := \{ r \in \mathbb{C}(W), \quad r(\gamma \mathbf{w}) = r(\mathbf{w}), \quad \forall \gamma, \mathbf{w} \in \Gamma \times W \}.$$

An interesting point about the field of rational invariants is that it is always finitely generated (even for non reductive groups), as it is a subfield of the finitely generated field $\mathbb{C}(W)$ [140]. Some recent works [71, 72] addressed the question of an effective computation of a generating set for $\mathbb{C}(W)^\Gamma$, but this effective approach on rational invariants remains to be deepened.

The first interesting theoretical result is that the field of rational invariants is able to separate *orbits in general position*, where we have the following theorem, initially stated in the scope of any *group action* (not necessarily linear) :

Theorem 1.3.13 (Rosenlicht [121]). *Let (W, Γ) be a linear representation of a linearly reductive group on a finite dimensional complex vector space W . Then there exists an invariant dense open subset $X_0 \subset W$ and a finite set $\{r_1, \dots, r_s\} \subset \mathbb{C}(W)^\Gamma$ separating orbits in X_0 :*

$$\forall \mathbf{w}_1, \mathbf{w}_2 \in X_0, \quad \exists \gamma \in \Gamma, \quad \mathbf{w}_1 = \gamma \mathbf{w}_2 \iff r_k(\mathbf{w}_1) = r_k(\mathbf{w}_2), \quad k = 1, \dots, s$$

All vectors in X_0 are said to be in general position (or generic).

Such a theorem is strongly connected to the useful Lemma :

Lemma 1.3.14 (Lemma 2.1 in [116]). *If a finite set $M \subset \mathbb{C}(W)^\Gamma$ separates orbits in general position, then it generates the field $\mathbb{C}(W)^\Gamma$.*

Now, the field of rational invariants contains the *field of quotients* $\mathbb{QC}[W]^\Gamma$ of the integral domain $\mathbb{C}[W]^\Gamma$. More specifically, the following result, which applies to all the groups that concern us here, like $\mathrm{SO}(3, \mathbb{R})$, $\mathrm{SO}(3, \mathbb{C})$, $\mathrm{SL}(2, \mathbb{C})$ and $\mathrm{SU}(2)$:

Theorem 1.3.15 (Theorem 3.3 in [116]). *Let (W, Γ) be a complex linear representation of a linearly reductive group Γ such that the neutral component Γ^0 has no non-trivial characters, then we have $\mathbb{C}(W)^\Gamma = \mathbb{QC}[W]^\Gamma$.*

From all this we also have [116, Proposition 3.4] :

Proposition 1.3.16. *For any complex linear representation (W, Γ) of a linearly reductive group, the algebra of invariant $\mathbb{C}[W]^\Gamma$ separates orbit in general position if and only if $\mathbb{C}(W)^\Gamma = \mathbb{QC}[W]^\Gamma$. In this case there exists a finite set of polynomial invariants that separate orbits in general position, and the transcendence degree of $\mathbb{C}[W]^\Gamma$ is equal to the codimension of an orbit in general position.*

1.4 Linear representations of $\mathrm{SO}(3, \mathbb{k})$ and $\mathrm{SL}(2, \mathbb{C})$

Let us consider the canonical quadratic form $q(\mathbf{x}) := x^2 + y^2 + z^2$ on \mathbb{k}^3 , with $\mathbb{k} = \mathbb{R}$ or $\mathbb{k} = \mathbb{C}$. The orthogonal group $O(3, \mathbb{k})$ is the group of linear transformation preserving this metric, and $\mathrm{SO}(3, \mathbb{k})$ is the subgroup of all elements $g \in O(3, \mathbb{k})$ with $\det(g) = 1$. Note here that the complexification of $\mathrm{SO}(3, \mathbb{R})$ is exactly $\mathrm{SO}(3, \mathbb{R})^\mathbb{C} = \mathrm{SO}(3, \mathbb{C})$.

Taking back the special linear group $\mathrm{SL}(2, \mathbb{C})$ given by (1.3), we can define an explicit two-fold universal covering (see also [131])

$$\pi : \mathrm{SL}(2, \mathbb{C}) \rightarrow \mathrm{SO}(3, \mathbb{C}), \quad \gamma \mapsto \mathrm{Ad}_\gamma, \quad (1.8)$$

where

$$\mathrm{Ad}_\gamma : M \mapsto \gamma M \gamma^{-1}, \quad \gamma \in \mathrm{SL}(2, \mathbb{C}), \quad M \in \mathfrak{sl}(2, \mathbb{C})$$

is the adjoint action of $\mathrm{SL}(2, \mathbb{C})$ on its Lie algebra $\mathfrak{sl}(2, \mathbb{C})$. Note that when restricted to the real Lie group

$$\mathrm{SU}(2) := \{ \gamma \in \mathrm{SL}(2, \mathbb{C}); \bar{\gamma}^t \gamma = I \},$$

it induces the well-known two-fold covering

$$\tilde{\pi} : \mathrm{SU}(2) \rightarrow \mathrm{SO}(3, \mathbb{R}), \quad \gamma \mapsto \mathrm{Ad}_\gamma. \quad (1.9)$$

1.4.1 Irreducible representations of the 3D rotation group

Explicit models for irreducible $\mathrm{SO}(3, \mathbb{k})$ representations are given by the so-called *harmonic tensor spaces* which we now describe.

First of all, n th order tensor may always be considered as a multilinear mapping

$$\mathbf{T} : \mathbb{k}^3 \times \cdots \times \mathbb{k}^3 \rightarrow \mathbb{k}, \quad (\mathbf{x}_1, \dots, \mathbf{x}_n) \mapsto \mathbf{T}(\mathbf{x}_1, \dots, \mathbf{x}_n).$$

Let $\mathbb{T}^n(\mathbb{k}^3)$ be the space of n th order tensors (thanks to the quadratic form, we do not distinguish here between the covariant and contravariant tensors). The standard $\mathrm{SO}(3, \mathbb{k})$ representation on $\mathbb{T}^n(\mathbb{k}^3)$ is given by

$$(\rho_n(g)\mathbf{T})(\mathbf{x}_1, \dots, \mathbf{x}_n) := \mathbf{T}(g^{-1}\mathbf{x}_1, \dots, g^{-1}\mathbf{x}_n), \quad (1.10)$$

where $\mathbf{T} \in \mathbb{T}^n(\mathbb{k}^3)$ and $g \in \mathrm{SO}(3, \mathbb{k})$. Define $\mathbb{S}^n(\mathbb{k}^3)$ to be the $\mathrm{SO}(3, \mathbb{k})$ stable subspace of *totally symmetric tensors* of order n , meaning that for all $\mathbf{S} \in \mathbb{S}^n(\mathbb{k}^3)$, the map

$$(\mathbf{x}_1, \dots, \mathbf{x}_n) \mapsto \mathbf{S}(\mathbf{x}_1, \dots, \mathbf{x}_n)$$

is invariant under any permutation of $(\mathbf{x}_1, \dots, \mathbf{x}_n) \in (\mathbb{k}^3)^n$.

Contracting two indices i, j on a totally symmetric tensor \mathbf{S} does not depend on the particular choice of the pair i, j . Thus, we can refer to this contraction without any reference to a particular choice of indices. We will denote this contraction as $\text{tr } \mathbf{T}$, which is a totally symmetric tensor of order $n - 2$ and is called the *trace* of \mathbf{T} . Iterating the process leads to

$$\text{tr}^k \mathbf{T} := \text{tr}(\text{tr}(\cdots(\text{tr } \mathbf{T})))$$

which is a totally symmetric tensor of order $n - 2k$.

Definition 1.4.1. A *harmonic tensor* of order n is a totally symmetric tensor $\mathbf{T} \in \mathbb{S}^n(\mathbb{k}^3)$ such that $\text{tr } \mathbf{T} = 0$. The space of harmonic tensors of order n will be denoted by $\mathbb{H}^n(\mathbb{k}^3)$. It is a vector subspace of $\mathbb{S}^n(\mathbb{k}^3)$ of dimension $2n + 1$.

The subspace $\mathbb{H}^n(\mathbb{k}^3)$ of $\mathbb{S}^n(\mathbb{k}^3)$ is invariant under the action of $\text{SO}(3, \mathbb{k})$ and we have [59] :

Theorem 1.4.2. For any non-negative integer n , $(\mathbb{H}^n(\mathbb{k}^3), \rho_n)$ is an irreducible $\text{SO}(3, \mathbb{k})$ representation, with ρ_n given by (1.10). Moreover, every irreducible $\text{SO}(3, \mathbb{k})$ representation is equivariantly isomorphic equivariant to $(\mathbb{H}^n(\mathbb{k}^3), \rho_n)$ for some non-negative integer n .

Remark 1.4.3. The fact that the real and the complex irreducible representations of the three-dimensional rotation group are given by the same model is in fact a particular case of a theorem about real and complex irreducible $\text{SO}(2n + 1, \mathbb{k})$ linear representations, which are obtained by a complexification process [3, 20, 74]. As a classical fact, this is not true for $\text{SO}(2)$ irreducible representations.

From the two-fold coverings (1.8) and (1.9), we obtain :

Lemma 1.4.4. For any non-negative integer n , the irreducible representation $(\mathbb{H}^n(\mathbb{C}^3), \text{SO}(3, \mathbb{C}))$ (resp. $(\mathbb{H}^n(\mathbb{R}^3), \text{SO}(3, \mathbb{R}))$) is equivariantly isomorphic equivariant to the irreducible representation $(S_{2n}, \text{SL}(2, \mathbb{C}))$ (resp. $\text{SU}(2)$), where S_p is the space of p th degree binary forms.

We also deduce from Theorem 1.4.2 that every $\text{SO}(3, \mathbb{k})$ -representation V splits into a direct sum of *harmonic tensor spaces* $\mathbb{H}^n(\mathbb{k}^3)$ (see [20, 131]). Such a direct sum is called an *harmonic decomposition* of V .

We now recall a well-known correspondence between totally symmetric tensors of order n and homogeneous polynomials of degree n on \mathbb{k}^3 , which extends the well-known correspondence between a symmetric bilinear form and a quadratic form via polarization. Indeed, let us define

$$\phi : \mathbf{S} \in \mathbb{S}^n(\mathbb{k}^3) \mapsto p(\mathbf{x}) := \mathbf{S}(\mathbf{x}, \dots, \mathbf{x}), \quad \mathbf{x} := (x, y, z) \in \mathbb{k}^3$$

which is a linear equivariant isomorphism between the tensor space $S^n(\mathbb{k}^3)$ and the polynomial space $\mathcal{P}_n(\mathbb{k}^3)$ of homogeneous polynomials of degree n on \mathbb{k}^3 . This equivariance has to be understood with respect to the $\text{SO}(3, \mathbb{k})$ representation on the polynomial space $\mathcal{P}_n(\mathbb{k}^3)$ given by

$$(\rho(g)p)(\mathbf{x}) := p(g^{-1} \cdot \mathbf{x}), \quad g \in \text{SO}(3, \mathbb{k}).$$

The inverse $\mathbf{S} = \phi^{-1}(p)$ can be recovered by *polarization*. More precisely, the expression $p(t_1\mathbf{x}_1 + \dots + t_n\mathbf{x}_n)$ is a homogeneous polynomial in the variables t_1, \dots, t_n and we get

$$\mathbf{S}(\mathbf{x}_1, \dots, \mathbf{x}_n) = \frac{1}{n!} \left. \frac{\partial^n}{\partial t_1 \cdots \partial t_n} \right|_{t_1 = \dots = t_n = 0} p(t_1\mathbf{x}_1 + \dots + t_n\mathbf{x}_n).$$

Remark 1.4.5. Note that the *Laplacian operator* acts on $p = \phi(\mathbf{T})$ as

$$\Delta \phi(\mathbf{T}) = n(n-1)\phi(\text{tr } \mathbf{T}).$$

Thus, *totally symmetric tensors with vanishing trace* correspond via ϕ to *harmonic polynomials*, i.e polynomials with vanishing Laplacian). This justifies the denomination of *harmonic tensors* for elements of $H^n(\mathbb{k}^3)$. More generally, for any non-negative integer k , we have

$$\Delta^k \phi(\mathbf{T}) = \frac{n!}{(n-2k)!} \phi(\text{tr}^k \mathbf{T}). \quad (1.11)$$

The equivariant isomorphism ϕ sends $H^n(\mathbb{k}^3)$ to $\mathcal{H}_n(\mathbb{k}^3)$, the space of homogeneous harmonic polynomials of degree n . Thus, the spaces $\mathcal{H}_n(\mathbb{k}^3)$ provide alternative models for *irreducible representations* of the rotation group $\text{SO}(3, \mathbb{k})$.

1.4.2 The space of elasticity tensors

In the theory of linear elasticity [66, 85, 91], the Cauchy stress tensor $\boldsymbol{\sigma}$ (a second order symmetric tensor) which models the internal forces of a material is a linear function of the infinitesimal strain–displacement tensor $\boldsymbol{\epsilon}$ (also a second order symmetric tensor). The generalized Hook's law then postulates the linear relation

$$\boldsymbol{\sigma} = \mathbf{E} : \boldsymbol{\epsilon}$$

where notation $'.'$ means a double contraction and \mathbf{E} is the fourth order *elasticity tensor*, with the following symmetries

$$\mathbf{E}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4) = \mathbf{E}(\mathbf{x}_2, \mathbf{x}_1, \mathbf{x}_3, \mathbf{x}_4) = \mathbf{E}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_4, \mathbf{x}_3) = \mathbf{E}(\mathbf{x}_3, \mathbf{x}_4, \mathbf{x}_1, \mathbf{x}_2), \quad \forall \mathbf{x}_i \in \mathbb{R}^3.$$

We thus define the space of 3D elasticity tensors as $\mathbb{E}_{\text{Ela}} := \mathbb{S}^2(\mathbb{S}^2(\mathbb{R}^3))$, which is a 21-dimensional vector space. Any tensor \mathbf{E} has a *Kevin matrix representation* : taking an orthonormal basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ of \mathbb{R}^3 we deduce an orthonormal basis $(\mathbf{e}_{11}, \mathbf{e}_{22}, \mathbf{e}_{33}, \mathbf{e}_{23}, \mathbf{e}_{13}, \mathbf{e}_{12})$ of $\mathbb{S}^2(\mathbb{R}^3)$ given by :

$$\mathbf{e}_{ii} := \mathbf{e}_i \otimes \mathbf{e}_i, \quad \mathbf{e}_{ij} := \frac{\sqrt{2}}{2} (\mathbf{e}_i \otimes \mathbf{e}_j + \mathbf{e}_j \otimes \mathbf{e}_i), \quad i \neq j.$$

From this, any elasticity tensor $\mathbf{E} \in \mathbb{S}^2(\mathbb{S}^2(\mathbb{R}^3))$ can be represented by a 6×6 symmetric matrix, denoted $[\mathbf{E}]$.

Any elasticity tensor allows to characterize the elastic behavior of a homogeneous material, once its orientation is given. However, any new orientation of the same material, induced by some rotation $g \in \text{SO}(3)$, will then define another tensor

$$\bar{\mathbf{E}} = \rho_{\text{Ela}}(g)\mathbf{E}, \quad \rho_{\text{Ela}} := \rho_4|_{\mathbb{E}_{\text{Ela}}}, \quad (1.12)$$

where ρ_4 is the standard $\text{SO}(3, \mathbb{R})$ representation (1.10) on the space of fourth order tensor.

Thus, as observed by Boehler–Kirilov–Onat [17], a homogeneous elastic material will not be characterized by a single tensor \mathbf{E} but rather by an *orbit* in the representation $(\mathbb{E}_{\text{Ela}}, \text{SO}(3, \mathbb{R}), \rho_{\text{Ela}})$. In order to reach an understanding of elastic mechanical materials (in the linear case), it is therefore necessary to exploit the mathematical notions developed in previous sections : establish its harmonic decomposition, its isotropic stratification, etc., and this with an effective approach.

An explicit harmonic decomposition was first obtained by Backus [6] :

Theorem 1.4.6 (Harmonic decomposition of \mathbb{E}_{Ela}). *The irreducible decomposition of the space $(\mathbb{E}_{\text{Ela}}, \text{SO}(3, \mathbb{R}), \rho_{\text{Ela}})$ is given by*

$$\mathbb{E}_{\text{Ela}} = \mathbb{H}^0(\mathbb{R}^3) \oplus \mathbb{H}^0(\mathbb{R}^3) \oplus \mathbb{H}^2(\mathbb{R}^3) \oplus \mathbb{H}^2(\mathbb{R}^3) \oplus \mathbb{H}^4(\mathbb{R}^3)$$

and it remains the same when considering the induced $\text{SO}(3, \mathbb{C})$ representation on $\mathbb{E}_{\text{Ela}}^{\mathbb{C}}$:

$$\mathbb{E}_{\text{Ela}}^{\mathbb{C}} = \mathbb{H}^0(\mathbb{C}^3) \oplus \mathbb{H}^0(\mathbb{C}^3) \oplus \mathbb{H}^2(\mathbb{C}^3) \oplus \mathbb{H}^2(\mathbb{C}^3) \oplus \mathbb{H}^4(\mathbb{C}^3).$$

In fact, such a decomposition was obtained with explicit formulae. More precisely, an elasticity tensor \mathbf{E} admits the following harmonic decomposition [6] :

$$\mathbf{E} = \alpha \mathbf{q} \otimes_{(4)} \mathbf{q} + \beta \mathbf{q} \otimes_{(2,2)} \mathbf{q} + \mathbf{q} \odot \mathbf{a} + \mathbf{q} \otimes_{(2,2)} \mathbf{b} + \mathbf{H} \quad (1.13)$$

with \mathbf{q} being the Euclidean canonical bilinear 2-form represented by the components (δ_{ij}) in any orthonormal basis. This defines a linear equivariant mapping

$$(\alpha, \beta, \mathbf{a}, \mathbf{b}, \mathbf{H}) \in \mathbb{H}^0(\mathbb{k}^3) \oplus \mathbb{H}^0(\mathbb{k}^3) \oplus \mathbb{H}^2(\mathbb{k}^3) \oplus \mathbb{H}^2(\mathbb{k}^3) \oplus \mathbb{H}^4(\mathbb{k}^3) \mapsto \mathbf{E} \in \mathbb{E}_{\text{Ela}}(\mathbb{k}).$$

In formula (1.13), \odot is the totally symmetric tensor product, and the Young-symmetrized tensor products $\otimes_{(2,2)}$, between two symmetric second-order tensors \mathbf{c}, \mathbf{d} , is defined as follows :

$$(\mathbf{c} \otimes_{(2,2)} \mathbf{d})_{ijkl} = \frac{1}{6} (2c_{ij}d_{kl} + 2d_{ij}c_{kl} - c_{ik}d_{jl} - c_{il}d_{jk} - d_{ik}c_{jl} - d_{il}c_{jk}).$$

An inverse equivariant mapping is obtained as follows. Let us first define the *dilatation tensor* $\mathbf{d} := \text{tr}_{12} \mathbf{E}$ and the *Voigt tensor* $\mathbf{v} := \text{tr}_{13} \mathbf{E}$, where tr_{ij} is the trace operation contracting indices i and j . From this we have

$$\alpha = \frac{1}{15} (\text{tr } \mathbf{d} + 2 \text{tr } \mathbf{v}), \quad \beta = \frac{1}{6} (\text{tr } \mathbf{d} - \text{tr } \mathbf{v}), \quad \mathbf{a} = \frac{2}{7} (\mathbf{d}' + 2\mathbf{v}'), \quad \mathbf{b} = 2 (\mathbf{d}' - \mathbf{v}')$$

where $\mathbf{d}' := \mathbf{d} - \frac{1}{3} \text{tr}(\mathbf{d}) \mathbf{q}$ and $\mathbf{v}' := \mathbf{v} - \frac{1}{3} \text{tr}(\mathbf{v}) \mathbf{q}$ are said to be the *deviatoric parts* of \mathbf{d} and \mathbf{v} respectively.

The isotropic stratification was first obtained by Forte–Vianello [54], detailed in Figure 1.2. Such an isotropic stratification encodes the different kinds of *anisotropy* in linear elasticity. The subgroups mentioned in this result are : the cyclic (resp. dihedral) subgroup \mathbb{Z}_n (resp. \mathbb{D}_n), the cubic subgroup \mathcal{O} of order 24 and the orthogonal subgroup $O(2)$. All the names used here are the classical names coming from Crystallography.

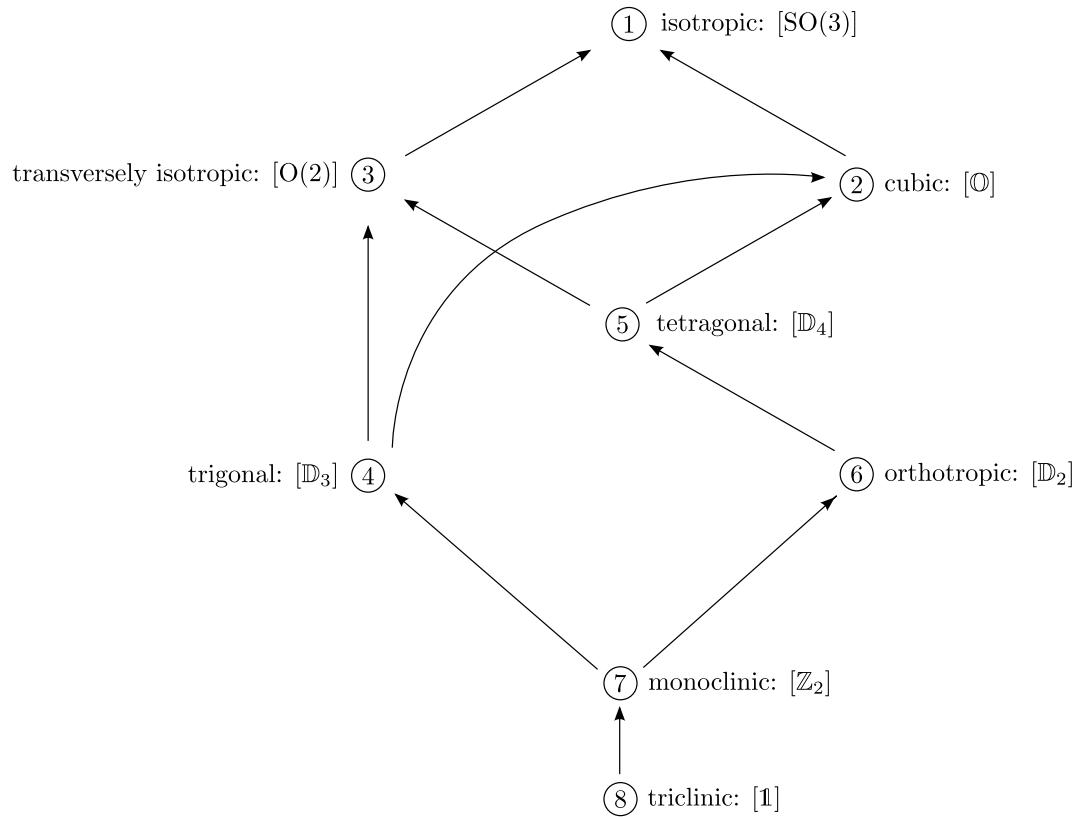


FIGURE 1.2 – Isotropic stratification of $(\text{Ela}, \text{SO}(3, \mathbb{R}))$.

Deuxième partie

Contributions

Chapitre 2

Results issued from my PhD

My PhD thesis work, carried out between 2011 and 2014, gave rise to two essential contributions that I summarize here. *The first one, presented in section 2.1, concerns the implementation of an algorithm for the effective computation of isotropy classes of a linear representation (V, G) of a compact Lie group*, once the isotropy classes of the irreducible representations of the group G are known, as well as the *clips table* (Definition 2.1.1) of the G closed subgroup conjugation classes. This algorithm is made effective in the case of $O(3, \mathbb{R})$ and $SO(3, \mathbb{R})$, for which the results of the isotropy classes of irreducible representations are given in [59], and the clips tables are established in [art5]. *I then present in Section 2.2 results about explicit isotropy classes of standard $O(3, \mathbb{R})$ representations on spaces of totally symmetric tensors of even order (Theorem 2.2.1) and odd order (Theorem 2.2.2).*

I also present in Section 2.3 the main result concerning an integrity basis of the elasticity tensor, directly deduced from an integrity basis of the joint invariants algebra of the space of binary forms (Theorem 2.3.2).

Results presented here come from the publications [art14, art13, art12, art5, art8].

2.1 Clips operation and effective algorithm for isotropy classes

Let first define $\text{Conj}(G)$ to be the set of all conjugacy classes of a given group G :

$$\text{Conj}(G) := \{ [H], \quad H \subset G, \quad H \text{ subgroup} \}.$$

We now define a binary operation, called the *clips operation*, on the set $\text{Conj}(G)$.

Definition 2.1.1. Given two conjugacy classes $[H_1]$ and $[H_2]$ of a group G , we define their *clips* as the following subset of conjugacy classes :

$$[H_1] \odot [H_2] := \{ [H_1 \cap gH_2g^{-1}], \quad g \in G \}.$$

This definition immediately extends to two families (finite or infinite) \mathcal{F}_1 and \mathcal{F}_2 of conjugacy classes :

$$\mathcal{F}_1 \odot \mathcal{F}_2 := \bigcup_{[H_i] \in \mathcal{F}_i} [H_1] \odot [H_2].$$

Remark 2.1.2. The clips operation was already introduced, with no specific name, in [33], the notation being $\mathbf{P}(H_1, H_2)$. In this article, the authors only focus on the $\mathrm{SO}(3, \mathbb{R})$ case, not dealing with a general theory.

This clips operation thus define a binary operation on the set $\mathcal{P}(\mathrm{Conj}(G))$ which is *associative* and *commutative*. We have moreover

$$[\mathbf{1}] \odot [H] = \{ [\mathbf{1}] \} \text{ and } [G] \odot [H] = \{ [H] \},$$

for every conjugacy class $[H]$, where $\mathbf{1} := \{ e \}$ and e is the identity element of G .

Consider now a linear representation (V, G) of a group G and define $\mathfrak{I}(V)$ to be the set of all its *isotropy classes* (or *orbit types*). The central observation is that the isotropy classes of a direct sum of representations is obtained by the clips of their respective isotropy classes [art5, Lemme 2.3] :

Lemma 2.1.3 (Result from PhD). *Let V_1 and V_2 be two linear representations of G . Then*

$$\mathfrak{I}(V_1 \oplus V_2) = \mathfrak{I}(V_1) \odot \mathfrak{I}(V_2).$$

Using this lemma, we deduce a general algorithm to obtain the isotropy classes $\mathfrak{I}(V)$ of a finite dimensional representation of a compact Lie group G , provided we know :

1. a decomposition $V = \bigoplus_i W_i$ into irreducible representations W_i of G ;
2. the isotropy classes $\mathfrak{I}(W_i)$ for the *irreducible representations* W_i ;
3. the tables of *clips operations* $[H_1] \odot [H_2]$ between conjugacy classes of closed subgroups $[H_i]$ of G .

2.2 Isotropy classes of $\mathrm{O}(3, \mathbb{R})$ linear representations

We can successfully apply the algorithm given above for $\mathrm{O}(3, \mathbb{R})$ linear representations. Indeed, isotropy classes for irreducible $\mathrm{O}(3, \mathbb{R})$ and $\mathrm{SO}(3, \mathbb{R})$ representations were obtained in [32, 73, 100]. Furthermore, clips table for their closed subgroups were established during my PhD thesis [art14, art13, art5, 104], so we obtain an effective algorithm. All subgroups mentioned in the following theorem are defined for instance in [art5, 73].

Recall first that the standard representation ρ_n of $\mathrm{SO}(3, \mathbb{R})$ on the space $\mathbb{T}^n(\mathbb{R}^3)$ of n th order tensor is given by (1.10). Then we have :

Theorem 2.2.1 (PhD result [art14, 104]). Let $(\mathbb{T}^{2n}(\mathbb{R}^3), \text{SO}(3, \mathbb{R}), \rho_{2n})$ be the standard $\text{SO}(3, \mathbb{R})$ representation on $\mathbb{T}^{2n}(\mathbb{R}^3)$ and consider also its induced representation $(\mathbb{S}^{2n}(\mathbb{R}^3), \text{SO}(3, \mathbb{R}), \rho_{2n})$ on the space of totally symmetric tensors. Then we have :

Representation	Isotropy classes
$(\mathbb{T}^2(\mathbb{R}^3), \rho_2)$	$[\mathbf{1}], [\mathbb{Z}_2], [\mathbb{D}_2], [\text{SO}(2)], [\text{O}(2)], [\text{SO}(3)]$
$(\mathbb{T}^4(\mathbb{R}^3), \rho_4)$	$[\mathbf{1}], [\mathbb{Z}_2], \dots, [\mathbb{Z}_4], [\mathbb{D}_2], \dots, [\mathbb{D}_4], [\text{SO}(2)], [\text{O}(2)], [\mathbb{T}], [\mathbb{O}], [\text{SO}(3)]$
$(\mathbb{T}^{2n}(\mathbb{R}^3), \rho_{2n}), n \geq 3$	$[\mathbf{1}], [\mathbb{Z}_2], \dots, [\mathbb{Z}_{2n}], [\mathbb{D}_2], \dots, [\mathbb{D}_{2n}], [\text{SO}(2)], [\text{O}(2)], [\mathbb{T}], [\mathbb{O}], [\mathbb{I}], [\text{SO}(3)]$
$(\mathbb{S}^4(\mathbb{R}^3), \rho_4)$	$[\mathbf{1}], [\mathbb{Z}_2], [\mathbb{D}_2], [\mathbb{D}_3], [\mathbb{D}_4], [\text{O}(2)], [\mathbb{O}], [\text{SO}(3)]$
$(\mathbb{S}^{2n}(\mathbb{R}^3), \rho_{2n}), n \geq 3$	$[\mathbf{1}], [\mathbb{Z}_2], \dots, [\mathbb{Z}_{2(n-1)}], [\mathbb{D}_2], \dots, [\mathbb{D}_{2n}], [\text{O}(2)], [\mathbb{T}], [\mathbb{O}], [\mathbb{I}], [\text{SO}(3)]$

We then propose general results for the group $\text{O}(3)$. For the standard representations of $\text{O}(3, \mathbb{R})$ on tensor spaces, the only relevant case is the one of odd-order tensors, as in the even case, such representations reduce to the $\text{SO}(3, \mathbb{R})$ one.

Theorem 2.2.2 (PhD result [art13, 104]). Let $(\mathbb{T}^{2n+1}(\mathbb{R}^3), \text{O}(3, \mathbb{R}), \rho_{2n+1})$ be the standard $\text{O}(3, \mathbb{R})$ representation on $\mathbb{T}^{2n+1}(\mathbb{R}^3)$. Then we have

Representation	Isotropy classes
$(\mathbb{T}^1(\mathbb{R}^3), \rho_1)$	$[\text{O}(2)^-]$
$(\mathbb{T}^3(\mathbb{R}^3), \rho_3)$	$[\mathbf{1}], [\mathbb{Z}_2], [\mathbb{Z}_3], [\mathbb{D}_2^v], [\mathbb{D}_3^v], [\mathbb{Z}_2^-], [\mathbb{Z}_4^-], [\mathbb{D}_2], [\mathbb{D}_3], [\mathbb{D}_4^h], [\mathbb{D}_6^h], [\mathbb{O}^-], [\text{SO}(2)], [\text{O}(2)], [\text{O}(2)^-], [\text{SO}(3)], [\text{O}(3)]$
$(\mathbb{T}^5(\mathbb{R}^3), \rho_5)$	$[\mathbf{1}], [\mathbb{Z}_2], \dots, [\mathbb{Z}_5], [\mathbb{D}_2^v], \dots, [\mathbb{D}_5^v], [\mathbb{Z}_2^-], [\mathbb{Z}_4^-], \dots, [\mathbb{Z}_8^-], [\mathbb{D}_2], \dots, [\mathbb{D}_5], [\mathbb{D}_4^h], [\mathbb{D}_6^h], \dots, [\mathbb{D}_{10}^h], [\mathbb{T}], [\mathbb{O}^-], [\mathbb{O}], [\text{SO}(2)], [\text{O}(2)], [\text{O}(2)^-], [\text{SO}(3)], [\text{O}(3)]$
$(\mathbb{T}^{2n+1}(\mathbb{R}^3), \rho_{2n+1}), n \geq 3$	$[\mathbf{1}], [\mathbb{Z}_2], \dots, [\mathbb{Z}_{2n+1}], [\mathbb{D}_2^v], \dots, [\mathbb{D}_{2n+1}^v], [\mathbb{Z}_2^-], \dots, [\mathbb{Z}_{4n}^-], [\mathbb{D}_2], \dots, [\mathbb{D}_{2n+1}], [\mathbb{D}_4^h], \dots, [\mathbb{D}_{2(2n+1)}^h], [\mathbb{T}], [\mathbb{O}^-], [\mathbb{O}], [\mathbb{I}], [\text{SO}(2)], [\text{O}(2)], [\text{O}(2)^-], [\text{SO}(3)], [\text{O}(3)]$

It should be noted here that these effective results have important consequences in the field of continuum mechanics. Indeed, since the work of Forte–Vianello [54], anisotropy is directly modelled by the notion of isotropy class of a linear $\text{SO}(3, \mathbb{R})$ or $\text{O}(3, \mathbb{R})$ representation. By our approach, it is thus possible to obtain explicitly the different anisotropies of a given model, this by an algorithm based on clips tables [art14, art13, art5] not shown here.

Remark 2.2.3. The search for isotropy classes, and thus possible physical anisotropies, is equally relevant to the 2D models. The $O(2)$ and $SO(2)$ groups are then involved, and clips operation approaches have been formulated in [art10].

2.3 Integrity basis of elasticity tensors

Let us consider the space of elasticity tensors $\mathbb{E}_{\text{la}} = \mathbb{S}^2(\mathbb{S}^2(\mathbb{R}^3))$ defined in Subsection 1.4.2, endowed with the natural $SO(3, \mathbb{R})$ representation (1.12). As we already mentioned, the invariant algebra $\mathbb{R}[\mathbb{E}_{\text{la}}]^{SO(3, \mathbb{R})}$ can separate the orbits [2] :

$$\exists g \in SO(3, \mathbb{R}), \quad \mathbf{E}_2 = \rho_{\text{Ela}}(g)\mathbf{E}_1 \iff I(\mathbf{E}_1) = I(\mathbf{E}_2), \quad \forall I \in \mathbb{R}[\mathbb{E}_{\text{la}}]^{SO(3, \mathbb{R})}$$

From this, we deduce that any finite integrity basis defines *homogeneous material elastic parameters* (like a 3D generalization of stiffness coefficient in one dimensional elasticity, with classical Hook law). As far as we know, such a finite integrity basis was unknown until now.

Remark 2.3.1. In 2D case, the space of elasticity tensors is given by the 6-dimensional vector space $\mathbb{S}^2(\mathbb{S}^2(\mathbb{R}^2))$. Such a space is endowed with a natural $O(2)$ representation, and a finite integrity basis (with 5 invariants) has already been obtained in many ways [11, 55, 64, 142, 143].

The starting point to obtain such an integrity basis is the harmonic decomposition taken from Lemma 1.4.6. We also know from Lemma 1.3.6 that :

$$(\mathbb{R}[\mathbb{E}_{\text{la}}]^{SO(3, \mathbb{R})})^{\mathbb{C}} \simeq \mathbb{C}[\mathbb{E}_{\text{la}}^{\mathbb{C}}]^{SO(3, \mathbb{C})}, \quad (2.1)$$

and thus we deduce from lemma 1.4.4 that

$$(\mathbb{R}[\mathbb{E}_{\text{la}}]^{SO(3, \mathbb{R})})^{\mathbb{C}} \simeq \mathbb{C}[S_8 \oplus S_4 \oplus S_4 \oplus S_0 \oplus S_0].$$

Finally, as a result of my PhD, I obtained :

Theorem 2.3.2 (PhD result). *An integrity basis of the invariant algebra $\mathbb{R}[\mathbb{E}_{\text{la}}]^{SO(3, \mathbb{R})}$ is given by 297 invariants, deduced from an explicit finite integrity basis of $S_8 \oplus S_4 \oplus S_4 \oplus S_0 \oplus S_0$, obtained by transvection operations.*

Note also that an explicit integrity basis of the real algebra $\mathbb{R}[\mathbb{E}_{\text{la}}]^{SO(3, \mathbb{R})}$ was obtained later in *tensorial form* in [105].

Chapitre 3

Covariant algebra of binary forms

Let us consider a linear representation $(V, \mathrm{SL}(2, \mathbb{C}))$, with $V = S_{n_1} \oplus \cdots \oplus S_{n_p}$. The question addressed here is to obtain an explicit integrity basis for the covariant algebra $\mathbf{Cov}(V)$, defined in [Subsection 1.3.1](#). Up to our knowledge, the last explicit results were established for the invariant algebras $\mathbf{Inv}(S_9)$ and $\mathbf{Inv}(S_{10})$ [[21, 24](#)]. In these articles, computations are done using a degree bound deduced from a homogeneous system of parameters (see [Subsection 1.3.2](#)).

Now, according to us, there is no effective algorithm to obtain a homogeneous system of parameters of a graded algebra of finite type, while some attempts are proposed in [[68, 80, 90](#)]. Furthermore, in the specific cases we had to deal with, as the one of $S_8 \oplus S_4 \oplus S_4$, two difficulties appeared :

1. According to Brion's result from [[19](#)], there is no *multi-homogeneous* system of parameters for this algebra, so the approaches developed by Brouwer–Popoviciu [[21](#)] seem not to be applicable.
2. Even if we know such a homogeneous system of parameters, the degree bound can be very high, which was a serious impediment in our situation of the 19 dimensional space $S_8 \oplus S_4 \oplus S_4$. For instance, in the case of the 11-dimensional space S_{10} , the bound obtained in [[21](#)] is 48.

In fact, taking up some works from the XIXth Century, and especially the memoir of Gordan [[61](#)], it turned out that there were two very efficient algorithms (at least for the computation time). *These two algorithms were rewritten and modernized in [[art11](#)], presented here as [Theorem 3.1.2](#) and [Algorithm 3.2.7](#). Working together with Reynald Lercier [[art9](#)] during the workshop TEDI (Théorie effective des invariants), we obtained new effective results (see [Section 3.3](#)) for integrity basis of the covariant algebras of binary nonic and binary decimic ([Theorem 3.3.5](#)).*

3.1 Gordan's algorithm for joint covariant algebras revisited

I first give Gordan's algorithm to obtain an integrity basis of a joint covariant algebra $\mathbf{Cov}(V_1 \oplus V_2)$, once an integrity basis of each covariant algebra $\mathbf{Cov}(V_1)$ and $\mathbf{Cov}(V_2)$ is known. From now on, let us suppose that

$$A := \{\mathbf{f}_1, \dots, \mathbf{f}_p\} \subset \mathbf{Cov}(V_1), \quad B := \{\mathbf{g}_1, \dots, \mathbf{g}_q\} \subset \mathbf{Cov}(V_2),$$

are finite integrity bases for $\mathbf{Cov}(V_1)$ and $\mathbf{Cov}(V_2)$ respectively. Taking back the transvection operation introduced in [Subsection 1.3.1](#), we first have [[art11](#)] :

Lemma 3.1.1. *The covariant algebra $\mathbf{Cov}(V_1 \oplus V_2)$ is generated by all transvectants*

$$\left(\mathbf{f}_1^{\alpha_1} \cdots \mathbf{f}_p^{\alpha_p}, \mathbf{g}_1^{\beta_1} \cdots \mathbf{g}_q^{\beta_q} \right)_r, \quad \alpha_i, \beta_j \in \mathbb{N},$$

with any non-negative integer r .

Define now a_i (resp. b_j) to be the order of the covariant \mathbf{f}_i (resp. \mathbf{g}_j). To each non-vanishing transvectant

$$\left(\mathbf{f}_1^{\alpha_1} \cdots \mathbf{f}_p^{\alpha_p}, \mathbf{g}_1^{\beta_1} \cdots \mathbf{g}_q^{\beta_q} \right)_r, \tag{3.1}$$

we can associate an integer solution $\kappa := (\alpha, \beta, u, v, r)$ of the linear Diophantine system

$$S(A, B) : \quad \begin{cases} a_1\alpha_1 + \cdots + a_p\alpha_p = u + r, \\ b_1\beta_1 + \cdots + b_q\beta_q = v + r. \end{cases} \tag{3.2}$$

Conversely, to each integer solution κ of $S(A, B)$, we can associate a well defined transvectant (3.1). Recall an integer solution κ of $S(A, B)$ is *reducible* if we can decompose κ as a sum of non-trivial solutions. In fact, we know that there exists a finite family of *irreducible integer solutions* of the system (3.2) (see [[130, 133](#)] for details on linear Diophantine systems), so that we deduce a finite integrity basis of $\mathbf{Cov}(V_1 \oplus V_2)$ [[art11](#)] :

Theorem 3.1.2. *The covariant algebra $\mathbf{Cov}(V_1 \oplus V_2)$ is generated by the (generally non minimal) finite family of transvectants*

$$\left(\mathbf{f}_1^{\alpha_1} \cdots \mathbf{f}_p^{\alpha_p}, \mathbf{g}_1^{\beta_1} \cdots \mathbf{g}_q^{\beta_q} \right)_r$$

corresponding to irreducible solutions of the linear Diophantine system $S(A, B)$ (3.2).

3.2 Gordan's algorithm for simple covariant algebras revisited

There exists a second algorithm, much more subtle, to obtain a finite integrity basis of a simple covariant algebra $\mathbf{Cov}(S_n)$ once we know integrity bases for all $\mathbf{Cov}(S_p)$ with $p < n$. Such an algorithm was successfully applied by Gordan's himself [61] and by von Gall [145, 146] to binary forms of degrees up to 8. It can also be found in Grace–Young [62], while it was modernized and rewritten in [art11]. A key element of this algorithm is to use the *Cayley operator* [108] instead of the transvection operation, so as to obtain another (infinite) generating family issued from the *symbolic method* [108, 148], closely tied with the *bracket algebra* [84, 133]. From a graphical approach initiated by Sylvester [134] and then reconsidered by Olver–Shakiban [110] and myself [art11], I was able to obtain a new formulation and new proof of termination of this algorithm.

Let us now introduce two differential operators, each commuting with the $\mathrm{SL}(2, \mathbb{C})$ action :

- The *Cayley operator* [108], which is a bi-differential operator acting on the tensor product $\mathbf{f}(\mathbf{x}_\alpha)\mathbf{g}(\mathbf{x}_\beta)$ of two complex analytic functions :

$$\Omega_{\alpha\beta}(\mathbf{f}(\mathbf{x}_\alpha)\mathbf{g}(\mathbf{x}_\beta)) := \frac{\partial \mathbf{f}}{\partial x_\alpha} \frac{\partial \mathbf{g}}{\partial y_\beta} - \frac{\partial \mathbf{f}}{\partial y_\alpha} \frac{\partial \mathbf{g}}{\partial x_\beta}, \quad \mathbf{x}_\alpha := (x_\alpha, y_\alpha), \quad \mathbf{x}_\beta := (x_\beta, y_\beta).$$

- The *polarization operator* acting on a complex analytic function $\mathbf{f}(\mathbf{x}_\alpha)$:

$$\sigma_\alpha(\mathbf{f}(\mathbf{x}_\alpha)) = x \frac{\partial \mathbf{f}}{\partial x_\alpha} + y \frac{\partial \mathbf{f}}{\partial y_\alpha}.$$

Transvectant operation can be rewriting using those differential operators :

Lemma 3.2.1. *Given two binary forms $\mathbf{f} \in S_n$ and $\mathbf{g} \in S_p$, their transvectant of index $0 \leq r \leq \min(n, p)$ is given by*

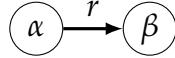
$$(\mathbf{f}, \mathbf{g})_r := \left(\Omega_{\alpha\beta}^r \sigma_\alpha^{n-r} \sigma_\beta^{p-r} \right) [\mathbf{f}(\mathbf{x}_\alpha)\mathbf{g}(\mathbf{x}_\beta)].$$

It is zero if $r > \min(n, p)$.

From this, we deduce a *symbol operator* which defines the Clebsch–Gordan projector π_r associated to an index r transvectant (see Subsection 1.3.1), given by :

$$\Omega_{\alpha\beta}^r \sigma_\alpha^{n-r} \sigma_\beta^{p-r},$$

and such a monomial will be represented by the colored directed graph (colored digraph) :



where the *atom* α (resp. β) is colored by S_n (resp. S_p).

More generally, any monomial

$$\phi_D = \Omega_{\alpha\beta}^{w_1} \cdots \Omega_{\gamma\epsilon}^{w_s} \sigma_\alpha^{n_\alpha - w_1} \sigma_\beta^{n_\beta - w_1} \cdots \sigma_\gamma^{n_\gamma - w_s} \sigma_\epsilon^{n_\epsilon - w_s} \quad (3.3)$$

defines a *molecular* $\mathrm{SL}(2, \mathbb{C})$ equivariant morphism on some tensor product $S_{n_\alpha} \otimes \cdots S_{n_\epsilon}$ of binary forms, so it can be encoded into a colored digraph. We have thus a new kind of covariant :

Definition 3.2.2. A *molecular covariant* \mathbf{M} is defined as a binary form in the image $\mathrm{Im}(\phi_D)$ of a molecular equivariant morphism ϕ_D given by (3.3) :

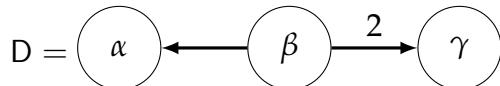
$$\mathbf{M} = \phi_D(\mathbf{f}(x_\alpha)\mathbf{g}(x_\beta) \cdots \mathbf{l}(x_\gamma)\mathbf{h}(x_\epsilon)) \in S_k, \quad k = n_\alpha + n_\beta + \cdots + n_\gamma + n_\epsilon - 2(w_1 + \cdots + w_s)$$

where each $\mathbf{f}(x_\alpha), \dots, \mathbf{h}(x_\epsilon)$ is called an *atom* of \mathbf{M} .

Note that such a molecular covariant is zero if one weight is larger than $\min(n_\alpha, \dots, n_\epsilon)$, and, following Gordan [61], we define :

Definition 3.2.3 (Grade of a molecular covariant). The *grade* of a molecular covariant is the maximum weight of its associated colored digraph.

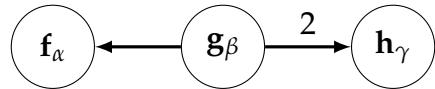
Example 3.2.4. Let us consider the molecule



coloured with $S_{n_\alpha} := S_6$, $S_{n_\beta} := S_5$ and $S_{n_\gamma} := S_7$. This molecule defines the $\mathrm{SL}(2, \mathbb{C})$ -equivariant homomorphism

$$\phi_D := \Omega_{\beta\alpha}\Omega_{\beta\gamma}^2\sigma_\alpha^5\sigma_\beta^2\sigma_\gamma^5.$$

For any product $\mathbf{f}(x_\alpha)\mathbf{g}(x_\beta)\mathbf{h}(x_\gamma) \in S_6 \otimes S_5 \otimes S_7$, we obtain the molecular covariant



which is of grade 2.

The following result is known as the first fundamental theorem for binary forms [84, 108] and define an (infinite) family of generators for $\mathbf{Cov}(V)$.

Theorem 3.2.5. Given a space $V = S_{n_1} \oplus \cdots \oplus S_{n_s}$ of binary forms, the covariant algebra $\mathbf{Cov}(V)$ is generated by the (infinite) family of molecular covariants.

For the specific case of a covariant algebra $\mathbf{Cov}(S_n)$, we thus have to consider molecular covariants where each atom is the same binary form $\mathbf{f} \in S_n$, so we have :

Corollary 3.2.6. The covariant algebra $\mathbf{Cov}(S_n)$ is generated by the (infinite) family of molecular covariants of maximum grade n .

Such a result is the starting point of Gordan's algorithm for the covariant algebra $\mathbf{Cov}(S_n)$, as we only have to construct family of covariants from grade 2 up to grade n , which is the end point of the algorithm. Details are given in [art11], and we only propose here the final algorithm.

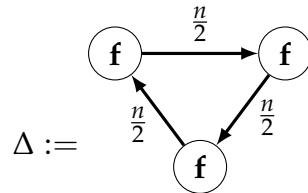
First recall that, for any family of covariants $A = \{\mathbf{f}_1, \dots, \mathbf{f}_p\} \subset \mathbf{Cov}(S_n)$ and $B = \{\mathbf{g}_1, \dots, \mathbf{g}_q\} \subset \mathbf{Cov}(S_n)$, we can obtain a family of covariants $C(A, B)$ deduced from the irreducible solutions of the Diophantine system $S(A, B)$ given by (3.2).

Algorithm 3.2.7 (Gordan's algorithm [art11]). Take S_n to be 'a' space of n th degree binary forms and define

$$\mathbf{H}_{2k} := (\mathbf{f}, \mathbf{f})_{2k} \in S_{2n-4k}, \quad A_0 := \{\mathbf{f}\}, \quad \mathbf{f} \in S_n.$$

From $k = 1$ to $\lfloor n/2 \rfloor - 1$ do one of the following :

1. if $n > 4k$, that is $\mathbf{H}_{2k} \in S_p$ with $p > n$, then define $B_{k-1} := \{\mathbf{H}_{2k}\}$ and $A_k := C(A_{k-1}, B_{k-1})$;
2. if $n = 4k$ then define $B_{k-1} := \{\mathbf{H}_{2k}\}$ and $A_k := C(A_{k-1}, B_{k-1}) \cup \{\Delta\}$, where Δ is the invariant



3. if $n < 4k$, that is $\mathbf{H}_{2k} \in S_p$ with $p < n$, we suppose already known a covariant basis B_{k-1} of S_{2n-4k} . Then we define $A_k := C(A_{k-1}, B_{k-1})$.

Last step Depending on n 's parity :

1. if $n = 2q$ is even then an integrity basis is given by $A_{q-1} \cup \{(\mathbf{f}, \mathbf{f})_n\}$;
2. if $n = 2q + 1$ then define $B_{q-1} := \{\mathbf{H}_{2q}\} \subset S_2$ and an integrity basis is given by $C(A_{q-1}, B_{q-1}) \cup \{(\mathbf{H}_{2q}, \mathbf{H}_{2q})_2\}$.

3.3 Integrity bases for binary nonic and binary decimic covariant algebras

As new results were obtained by Popoviciu–Brouwer [21, 24] about explicit integrity bases for the invariant algebras $\mathbf{Inv}(S_9)$ and $\mathbf{Inv}(S_{10})$, the question about integrity bases of the associated *covariant algebras* was only known as a conjecture. I summarize here how we ‘get rid’ of that conjecture, while all details can be found in [art9]. Note that the simple application of [Algorithm 3.2.7](#) was not enough as the number of solutions of the different diophantine equations was very big. So, we also had to use techniques coming from Cohen–Macaulayness and degree’s bound obtained from a homogeneous system of parameters (see [Subsection 1.3.2](#)). Let us first explain the main ideas, before we give some technical details.

- (1) At each step of [Algorithm 3.2.7](#) for simple covariant algebra, we only focus on the Diophantine equations (and not the computation of the associated binary forms). From this, we obtain all the *degrees* and the *orders* of a finite integrity basis.
- (2) We use a theoretical bound on the *order* of generators, obtained from [Lemma 3.3.1](#), and we can eliminate some solutions from previous Diophantine equations.
- (3) From Cohen–Macaulayness of m th order covariants $\mathbf{Cov}_m(S_n)$ and already known degrees of a homogeneous system of parameters for the invariant algebras $\mathbf{Inv}(S_n)$ ($n = 9, 10$), we deduce a *degree bound* for generators (see [Corollary 1.3.12](#)).
- (4) From points (1) to (3), we deduce all the couples (degrees,orders) of finite integrity bases for $\mathbf{Cov}(S_n)$, but we do not compute the covariants from the associated Diophantine solutions. Instead, we obtain a minimal integrity basis from Olver’s algorithm (see [108] and [87] for an implementation), which produces a minimal generating set by increasing degree, using iterated transvection operations.

Now, let us give the main theoretical results we need to use for this approach. First recall that, for any linear representation $(V, \mathrm{SL}(2, \mathbb{C}))$ of a space V of binary forms, there is a natural bi–graduation of the covariant algebra $\mathbf{Cov}(V)$ using the degree and the order of a covariant (see [Subsection 1.3.1](#)).

There is now an important upper-bound on the *order* of generators. For every integer n , take first λ to be the maximal integer such that $n = 2^\lambda + \nu$ and define

$$\lambda_n := (\lambda - 1)2^\lambda + \nu(\lambda + 1) + 2. \quad (3.4)$$

Then we have [62] :

Lemma 3.3.1. *For every space $V = \bigoplus_{i=0}^s S_{n_i}$ ($n_0 \leq \dots \leq n_s$) of binary forms, the covariant algebra $\mathbf{Cov}(V)$ is generated by covariants of maximum order λ_{n_s} .*

As a corollary, the covariant algebra $\mathbf{Cov}(S_9)$ (resp. $\mathbf{Cov}(S_{10})$) is generated by covariants of maximum order 22 (resp. 26).

Take now \mathcal{M} to be a finitely generated graded \mathcal{R} -module and take $\theta_1, \dots, \theta_s$ to be a homogeneous system of parameters for \mathcal{R} . When the module \mathcal{M} is Cohen–Macaulay (see [Subsection 1.3.2](#)), we know that \mathcal{M} is a free $\mathbb{C}[\theta_1, \dots, \theta_s]$ -module. Thus there exists $\mathfrak{m}_1, \dots, \mathfrak{m}_p \in \mathcal{M}$ such that a Hironaka decomposition of \mathcal{M} is

$$\mathcal{M} = \mathfrak{m}_1 \mathbb{C}[\theta_1, \dots, \theta_s] \oplus \dots \oplus \mathfrak{m}_p \mathbb{C}[\theta_1, \dots, \theta_s]. \quad (3.5)$$

For a covariant algebra $\mathbf{Cov}(V)$, let us observe that for every integer $m > 0$, the space $\mathbf{Cov}_m(V)$ of m -th order covariants is an $\mathbf{Inv}(V)$ -module.

We have an important result due to Van Den Bergh [[138, 139](#)]. For every integer n , let us define

$$\sigma_n := \begin{cases} \frac{(n+1)^2}{4} & \text{if } n \text{ is odd,} \\ \frac{n(n+2)}{4} & \text{otherwise.} \end{cases}$$

Taking now V to be the space of binary forms $\bigoplus_{i=0}^s S_{n_i}$ and letting σ_V be $\sum_{i=0}^s \sigma_{n_i}$, we can state the following.

Theorem 3.3.2. *For every integer $m < \sigma_V - 2$, the $\mathbf{Inv}(V)$ -module $\mathbf{Cov}_m(V)$ of m -th order covariants is Cohen–Macaulay.*

As a corollary, the $\mathbf{Inv}(S_9)$ -module $\mathbf{Cov}_m(S_9)$ is Cohen–Macaulay for every integer $m < 25$ and the $\mathbf{Inv}(S_{10})$ -module $\mathbf{Cov}_m(S_{10})$ is Cohen–Macaulay for every integer $m < 30$. We finally have some known results about homogeneous system of parameters for binary nonic and binary decimic invariant algebras [[21, 24, 44](#)] :

Proposition 3.3.3. (a) *The algebra $\mathbf{Inv}(S_9)$ has a homogeneous system of parameters of degrees 4, 8, 12, 14, 16, 30. It has also homogeneous systems of parameters of degrees 4, 8, 10, 12, 12, 14, 16, degrees 4, 4, 10, 12, 14, 16, 24, degrees 4, 4, 8, 10, 12, 16, 42 and degrees 4, 4, 8, 10, 12, 14, 48.*

(b) *The algebra $\mathbf{Inv}(S_{10})$ has a homogeneous system of parameters of degrees 2, 4, 6, 6, 8, 9, 10, 14.*

As already detailed in [Subsection 1.3.2](#), we thus deduce from Hilbert series and [Lemma 3.3.1](#) a degree bounds for generators of some m th order covariant module :

Lemma 3.3.4. *The $\mathbf{Inv}(S_9)$ -module of m th order covariants $\mathbf{Cov}_m(S_9)$ are generated by covariants of maximum degrees d_m given in the following table, for $m \leq 22$:*

Ord. m	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
Max deg. d_m	66	61	64	63	62	63	64	63	62	65	64	63	62	63	64	63	62	63	64	63	62	63	62

Similarly, the $\mathbf{Inv}(S_{10})$ -module of m th order covariants $\mathbf{Cov}_m(S_{10})$ are generated by covariants of maximum degree d_m given in the following table, for $m \leq 26$:

Ord. m	0	2	4	6	8	10	12	14	16	18	20	22	24	26
Max deg. d_m	59	45	46	45	46	47	46	45	46	45	46	45	45	45

All in all, the calculations that follow, detailed in [art9], enable us to prove :

Theorem 3.3.5. A minimal basis for the covariant algebra $\mathbf{Cov}(S_9)$ (resp. $\mathbf{Cov}(S_{10})$) is given by a family of 476 covariants (resp. 510 covariants) with explicit expression in [art9, Appendix A] (resp. in [art9, Appendix B]).

Chapitre 4

Reconstruction problem for tensorial $\text{SO}(3, \mathbb{R})$ representations

Tensorial $\text{SO}(3)$ representations naturally occurs in continuum mechanics, where we introduce order 3 tensors (in piezoelectricity), order 4 tensors (in linear elasticity), and so on. The idea is to try to use small order tensors (usually up to 2), or tensors with geometrical information (like to encode the anisotropy) so the mechanical model will be simplified. Such tensors are said to be *structural tensors* in the mechanical community [14, 89, 96]. The purpose here is to make clear the mathematical framework of such problems, leading to questions about *decomposition* (not necessary linear) and *equivariant reconstruction* ([Definition 4.0.2](#)). In a work done with Boris Kolev, Boris and Rodrigue Desmorat [[art7](#)], we introduced the harmonic product ([Definition 4.1.2](#)) so to obtain intrinsic equivariant reconstruction formulae for harmonic tensors. When applied to the space of fourth order harmonic tensors, same work [[art7](#)] leads to explicit equivariant reconstruction formulae using second order covariants, in the only possible cases, such as the transversely isotropic case ($[\text{O}(2)]$ isotropy, in [Theorem 4.3.1](#)), and the orthotropic case ($[\mathbb{D}_2]$ isotropy, in [Theorem 4.3.3](#)). All these results could be introduced in the framework of continuum mechanics, as new insight was proposed for damage crack model [[art6](#)].

Let us now provide the following two definitions.

Definition 4.0.1 (Decomposition). Let \mathbf{T} be some n th order tensor. A *decomposition* of \mathbf{T} into other tensors (possibly scalars) $\mathbf{T}_k \in \mathbb{T}^{n_k}(\mathbb{R}^3)$ ($k = 1, \dots, N$) is a mapping

$$\mathcal{F} : (\mathbf{T}_1, \dots, \mathbf{T}_N) \mapsto \mathbf{T} = \mathcal{F}(\mathbf{T}_1, \dots, \mathbf{T}_N).$$

The decomposition is *equivariant* if \mathcal{F} is an equivariant mapping, when considering the standard representations of $\text{SO}(3, \mathbb{R})$.

Definition 4.0.2 (Reconstruction). Given a decomposition

$$\mathbf{T} = \mathcal{F}(\mathbf{T}_1, \dots, \mathbf{T}_N)$$

of a tensor \mathbf{T} into other *tensors* (possibly scalar), a *reconstruction* of \mathbf{T} is a section \mathcal{S} of \mathcal{F} . The reconstruction is *equivariant* if both \mathcal{F} and \mathcal{S} are equivariant mappings.

Note here that for an equivariant reconstruction, all $\mathbf{T}_k = \mathbf{T}_k(\mathbf{T})$ are *covariant tensors* (or invariants if there are scalars). A first equivariant reconstruction can be obtained via the *harmonic components* deduced from the harmonic decomposition (see [Subsection 1.4.1](#)). To go one step further and try non-linear equivariant reconstruction, we need to introduce new operations, such as the *harmonic product* of harmonic tensors (see [Definition 4.1.2](#) below).

When applied to elasticity, we address the question of reconstruction using only covariants up to order 2, all summarized in [Section 4.3](#), after we make clear in [Section 4.2](#) some geometric obstruction about symmetry classes of such tensorial reconstructions.

4.1 Harmonic product

A simple observation here is that, from a theoretical point of view, harmonic tensors and binary forms are isomorphic. Obviously, any product of two binary form is another binary form, but a tensor product of harmonic tensors is not a harmonic tensor. To overcome this problem, we need to build an equivariant projection of tensor products into a space of harmonic tensors. This will be done using total symmetrization of the tensor product, denoted \odot , and an explicit harmonic decomposition of the spaces of totally symmetric tensors. This was in fact initiated in the work of Kanatani [76].

First of all, the harmonic decomposition of the space $\mathbb{S}^n(\mathbb{k}^3)$ is given by [131]

$$\mathbb{S}^n(\mathbb{k}^3) = \mathbb{H}^n(\mathbb{k}^3) \oplus \mathbb{H}^{n-2}(\mathbb{k}^3) \oplus \cdots \oplus \mathbb{H}^{n-2r}(\mathbb{k}^3), \quad r = [n/2].$$

More precisely, we have [[art7](#)] :

Proposition 4.1.1. *Every totally symmetric tensor $\mathbf{S} \in \mathbb{S}^n(\mathbb{k}^3)$ of order n can be decomposed uniquely as*

$$\mathbf{S} = \mathbf{H}_0 + \mathbf{q} \odot \mathbf{H}_1 + \cdots + \mathbf{q}^{\odot r-1} \odot \mathbf{H}_{r-1} + \mathbf{q}^{\odot r} \odot \mathbf{H}_r,$$

where \mathbf{H}_k is a harmonic tensor of order $n - 2k$ and $\mathbf{q}^{\odot k}$ means the symmetrized tensorial product of k copies of the Euclidean quadratic form \mathbf{q} . The harmonic part $(\mathbf{S})_0 := \mathbf{H}_0$ is the orthogonal projection of \mathbf{S} onto $\mathbb{H}^n(\mathbb{k}^3)$.

An explicit expression of such a decomposition can be obtained recursively as follows.
Let first define

$$\mathbf{H}_r := \begin{cases} \frac{1}{2r+1} \operatorname{tr}^r \mathbf{S}, & \text{if } n \text{ is even,} \\ \frac{3}{2r+3} \operatorname{tr}^r \mathbf{S}, & \text{if } n \text{ is odd,} \end{cases}$$

which is a scalar for even n or a vector for odd n . Then we have :

$$\mathbf{H}_k = \mu(k) \frac{n!}{(n-2k)!} \operatorname{tr}^k \left[\mathbf{S} - \sum_{j=k+1}^r \mathbf{q}^{\odot j} \odot \mathbf{H}_j \right],$$

with

$$\mu(k) := \frac{(2n-4k+1)!(n-k)!}{(2n-2k+1)!k!(n-2k)!},$$

which leads to \mathbf{H}_0 after r iterations.

Note that in this equation, $\operatorname{tr}^k \left[\mathbf{S} - \sum_{j=k+1}^r \mathbf{q}^{\odot j} \odot \mathbf{H}_j \right]$ is the orthogonal projection of $\operatorname{tr}^k \mathbf{S}$ on $\mathbb{H}^{n-2k}(\mathbb{k}^3)$. We finally have

Definition 4.1.2 (Harmonic product of harmonic tensors). Let $\mathbf{H}_1 \in \mathbb{H}^{n_1}(\mathbb{k}^3)$ and $\mathbf{H}_2 \in \mathbb{H}^{n_2}(\mathbb{k}^3)$ be two harmonic tensors with n_1, n_2 two non-negative integers. Then the *harmonic product* between \mathbf{H}_1 and \mathbf{H}_2 is defined to be

$$\mathbf{H}_1 * \mathbf{H}_2 := (\mathbf{H}_1 \odot \mathbf{H}_2)_0.$$

Example 4.1.3 (Harmonic product of two vectors). For two vectors $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{H}^1(\mathbb{k}^3)$, we get

$$\mathbf{v}_1 * \mathbf{v}_2 = \frac{1}{2} (\mathbf{v}_1 \otimes \mathbf{v}_2 + \mathbf{v}_2 \otimes \mathbf{v}_1) - \frac{1}{3} (\mathbf{v}_1 \cdot \mathbf{v}_2) \mathbf{q}.$$

Example 4.1.4 (Harmonic product of two second order harmonic tensors). For two second order harmonic tensors $\mathbf{h}_1, \mathbf{h}_2 \in \mathbb{H}^2(\mathbb{k}^3)$, we get

$$\mathbf{h}_1 * \mathbf{h}_2 = \mathbf{h}_1 \odot \mathbf{h}_2 - \frac{2}{7} \mathbf{q} \odot (\mathbf{h}_1 \mathbf{h}_2 + \mathbf{h}_2 \mathbf{h}_1) + \frac{2}{35} \operatorname{tr}(\mathbf{h}_1 \mathbf{h}_2) \mathbf{q} \odot \mathbf{q}.$$

4.2 Obstruction to an equivariant reconstruction

Let us now consider the standard $\operatorname{SO}(3, \mathbb{R})$ representations on tensor spaces $\mathbb{T}^n(\mathbb{R}^3)$, for non-negative integers n . Any equivariant reconstruction of a tensor has consequences on symmetry groups, more precisely [art7] :

Lemma 4.2.1. Consider an equivariant decomposition $\mathbf{T} = \mathcal{F}(\mathbf{T}_1, \dots, \mathbf{T}_N)$ and suppose that there exists an equivariant reconstruction \mathcal{S} , so that \mathbf{T}_k are covariant tensors. Then

$$G_{\mathbf{T}} = \bigcap_k G_{\mathbf{T}_k}, \quad (4.1)$$

where $G_{\mathbf{A}}$ is the symmetry group of the tensor \mathbf{A} .

According to Lemma 4.2.1, the existence of an equivariant reconstruction associated to an equivariant decomposition

$$\mathbf{T} = \mathcal{F}(\kappa_1(\mathbf{T}), \dots, \kappa_N(\mathbf{T}))$$

requires some conditions on the symmetry groups of the involved covariants $\mathbf{T}_k = \kappa_k(\mathbf{T})$. For a decomposition involving symmetric second-order covariant tensors, we have the following result (for details on closed subgroups of $\text{SO}(3, \mathbb{R})$, see [art5, 73] for instance).

Corollary 4.2.2. If $\mathbf{T} = \mathcal{F}(\mathbf{T}_1, \dots, \mathbf{T}_N)$ is an equivariant decomposition of \mathbf{T} into second-order symmetric covariant tensors \mathbf{T}_k , then

$$[G_{\mathbf{T}}] \in \{ [\mathbb{1}], [\mathbb{Z}_2], [\mathbb{D}_2], [\text{O}(2)], [\text{SO}(3, \mathbb{R})] \},$$

where $[H]$ means the conjugacy class of a subgroup $H \subset \text{SO}(3, \mathbb{R})$.

4.3 Reconstructions of elasticity tensors by means of second order covariants

The problem of an equivariant reconstruction of elasticity tensors using second order tensors is important as it can lead to a simpler model, compared to the one using a fourth order tensor. Such a problem is closely related to the so-called *isotropic extension* of anisotropic constitutive functions via structural tensors developed by Boehler [14] and Liu [89] independently (see also [16, 96]). Now, Corollary 4.2.2 implies, in particular, that there is no equivariant reconstruction by means of second order symmetric covariant tensors of a fourth order harmonic tensor \mathbf{H} (and thus of the elasticity tensor \mathbf{E}) belonging to the cubic, the trigonal or the tetragonal symmetry classes (see Subsection 1.4.2 for definition of these classes). Such restriction was already known by the mechanical community [89].

In this section, we provide explicit equivariant reconstructions for transverse-isotropic and orthotropic fourth order harmonic tensors. Note however that to overcome the geometrical constraints in the trigonal and in the tetragonal cases, we establish for these symmetry

classes reconstruction formulas by means of second order covariants up to a single *cubic fourth order covariant remainder* [art7, Section 6].

Let us now define the following second-order (polynomial) covariants introduced in [17] :

$$\begin{aligned} \mathbf{d}_2 &:= \text{tr}_{13}(\mathbf{H}^2), \quad \mathbf{d}_3 := \text{tr}_{13}(\mathbf{H}^3), \quad \mathbf{d}_4 := \mathbf{d}_2^2, \\ \mathbf{d}_5 &:= \mathbf{d}_2(\mathbf{H} : \mathbf{d}_2), \quad \mathbf{d}_6 := \mathbf{d}_2^3, \quad \mathbf{d}_7 := \mathbf{d}_2^2(\mathbf{H} : \mathbf{d}_2), \\ \mathbf{d}_8 &:= \mathbf{d}_2^2(\mathbf{H}^2 \mathbf{d}_2), \quad \mathbf{d}_9 := \mathbf{d}_2^2(\mathbf{H} \mathbf{d}_2^2), \quad \mathbf{d}_{10} := \mathbf{d}_2^2(\mathbf{H}^2 \mathbf{d}_2^2), \end{aligned} \quad (4.2)$$

where $\mathbf{H}^2 := \mathbf{H} : \mathbf{H}$ and $\mathbf{H}^3 := \mathbf{H} : \mathbf{H}^2$. We also make use of the following invariants :

$$J_k := \text{tr } \mathbf{d}_k, \quad k = 2, \dots, 10, \quad (4.3)$$

which constitute an *integrity basis* for $\mathbb{H}^4(\mathbb{R}^3)$ (see [17]).

4.3.1 The transversely isotropic class

A harmonic tensor $\mathbf{H} \in \mathbb{H}^4(\mathbb{R}^3)$ is transversely isotropic ([O(2)] isotropy) if and only if there exists $g \in \text{SO}(3, \mathbb{R})$ such that $\mathbf{H} = g \star \mathbf{H}_0$ where \mathbf{H}_0 has the normal matrix form, in Kelvin representation (see Subsection 1.4.2) :

$$\underline{\mathbf{H}}_0 = \begin{pmatrix} 3\delta & \delta & -4\delta & 0 & 0 & 0 \\ \delta & 3\delta & -4\delta & 0 & 0 & 0 \\ -4\delta & -4\delta & 8\delta & 0 & 0 & 0 \\ 0 & 0 & 0 & -8\delta & 0 & 0 \\ 0 & 0 & 0 & 0 & -8\delta & 0 \\ 0 & 0 & 0 & 0 & 0 & 2\delta \end{pmatrix},$$

where $\delta \neq 0$ and [5, Section 5.2]

$$J_2 \neq 0, \quad \delta = \frac{7J_3}{18J_2}.$$

Thus, by a direct computation of the second order covariant \mathbf{d}_2 (see Eq. (4.2)) and its deviatoric part \mathbf{d}'_2 , we get the following result :

Theorem 4.3.1. *For any transversely isotropic fourth-order harmonic tensor \mathbf{H} , we have*

$$\mathbf{H} = \frac{63}{25} \frac{1}{J_3(\mathbf{H})} \mathbf{d}'_2(\mathbf{H}) * \mathbf{d}'_2(\mathbf{H}).$$

4.3.2 The orthotropic class

A harmonic tensor $\mathbf{H} \in \mathbb{H}^4(\mathbb{R}^3)$ is orthotropic ($[\mathbb{D}_2]$ isotropy) if and only if there exists $g \in \text{SO}(3, \mathbb{R})$ such that $\mathbf{H} = g \star \mathbf{H}_0$ where \mathbf{H}_0 has the following normal matrix form, in Kelvin representation :

$$\underline{\mathbf{H}}_0 = \begin{pmatrix} \lambda_2 + \lambda_3 & -\lambda_3 & -\lambda_2 & 0 & 0 & 0 \\ -\lambda_3 & \lambda_3 + \lambda_1 & -\lambda_1 & 0 & 0 & 0 \\ -\lambda_2 & -\lambda_1 & \lambda_2 + \lambda_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2\lambda_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2\lambda_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2\lambda_3 \end{pmatrix}, \quad (4.4)$$

where $\lambda_1, \lambda_2, \lambda_3$ are three *distinct* real numbers. Note that this normal form is however not unique : any permutation of the λ_k provides an alternative normal form, but this is the only ambiguity. It is therefore useful to introduce the three elementary symmetric functions

$$\begin{aligned} \sigma_1 &:= \lambda_1 + \lambda_2 + \lambda_3, \\ \sigma_2 &:= \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3, \\ \sigma_3 &:= \lambda_1\lambda_2\lambda_3 \end{aligned} \quad (4.5)$$

which are independent of a particular normal form, as being invariant under any permutation of the λ_i . Moreover, it was shown in [5, Section 5.5] that the *discriminant*

$$\Delta_3 := (\lambda_2 - \lambda_1)^2(\lambda_2 - \lambda_3)^2(\lambda_3 - \lambda_1)^2,$$

which is *strictly positive*, belong to the invariant algebra $\mathbb{R}[\mathbb{H}^4]^{\text{SO}(3, \mathbb{R})}$, and that the σ_k are themselves in the field of *rational invariants* $\mathbb{R}(\mathbb{H}^4)^{\text{SO}(3, \mathbb{R})}$, with explicit expression given in [art7].

Consider now the matrix

$$\boldsymbol{\lambda}_0 := \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}.$$

Since both \mathbf{H}_0 and $\boldsymbol{\lambda}_0$ have the same symmetry group, namely \mathbb{D}_2 , the following definition

$$\boldsymbol{\lambda}(\mathbf{H}) := g \star \boldsymbol{\lambda}_0, \quad \text{if } \mathbf{H} = g \star \mathbf{H}_0$$

does not depend on the rotation g and thus defines a covariant mapping from the orthotropic stratum $\Sigma_{[\mathbb{D}_2]}$ in $\mathbb{H}^4(\mathbb{R}^3)$ to the space of symmetric second order tensors. We have then :

Lemma 4.3.2. Let \mathbf{H} be an orthotropic fourth-order harmonic tensor. Then, the deviatoric part $\boldsymbol{\lambda}'(\mathbf{H}) = \boldsymbol{\lambda}(\mathbf{H}) - 1/3 \operatorname{tr}(\boldsymbol{\lambda}(\mathbf{H}))\mathbf{q}$ of the second order covariant $\boldsymbol{\lambda}(\mathbf{H})$ can be written as

$$\boldsymbol{\lambda}'(\mathbf{H}) = \frac{1}{8\Delta_3} (\alpha_2 \mathbf{d}_2'(\mathbf{H}) + \alpha_3 \mathbf{d}_3'(\mathbf{H}) - 54\sigma_3 \mathbf{d}_4'(\mathbf{H}) + 11\sigma_2 \mathbf{d}_5'(\mathbf{H})) \quad (4.6)$$

where

$$\begin{aligned}\alpha_2 &:= 2(112\sigma_1^2\sigma_3 + 21\sigma_1\sigma_2^2 - 270\sigma_2\sigma_3), \\ \alpha_3 &:= 8(14\sigma_1\sigma_3 - 11\sigma_1^2\sigma_2 + 15\sigma_2^2).\end{aligned}$$

Let us now introduce the positive invariant

$$\sigma_{eq} := \sqrt{\sigma_1^2 - 3\sigma_2},$$

and the *Lode invariant* defined by

$$\mathcal{L} := \frac{1}{\sigma_{eq}^3} \left(\sigma_1^3 - \frac{9}{2}\sigma_1\sigma_2 + \frac{27}{2}\sigma_3 \right) \in]-1, 1[.$$

With these notations, we obtain the following result [art7] :

Theorem 4.3.3. For any orthotropic fourth-order harmonic tensor \mathbf{H} , we have

$$\mathbf{H} = h_1 \boldsymbol{\lambda}'(\mathbf{H}) * \boldsymbol{\lambda}'(\mathbf{H}) + 2h_2 \boldsymbol{\lambda}'(\mathbf{H}) * (\boldsymbol{\lambda}'(\mathbf{H})^2)' + h_3 (\boldsymbol{\lambda}'(\mathbf{H})^2)' * (\boldsymbol{\lambda}'(\mathbf{H})^2)', \quad (4.7)$$

where $\boldsymbol{\lambda}'(\mathbf{H})$ is defined by (4.6) and the three invariants h_k are given by

$$h_1 := \frac{5\sigma_1 + 7\mathcal{L}\sigma_{eq}}{2(1 - \mathcal{L}^2)\sigma_{eq}^2}, \quad h_2 := -\frac{3(5\mathcal{L}\sigma_1 + 7\sigma_{eq})}{2(1 - \mathcal{L}^2)\sigma_{eq}^3}, \quad h_3 := \frac{9(5\sigma_1 + 7\mathcal{L}\sigma_{eq})}{2(1 - \mathcal{L}^2)\sigma_{eq}^4}.$$

Both Theorems 4.3.1 and 4.3.3 produce formulae for equivariant reconstruction with second order covariants of any fourth order harmonic tensor belonging to transversely isotropic and orthotropic class. Other classes cannot have similar reconstruction formulae as a consequence of geometric obstruction (Corollary 4.2.2). Nevertheless, it was possible to obtain equivariant reconstruction for the tetragonal ($[\mathbb{D}_4]$ isotropy) and the trigonal class ($[\mathbb{D}_3]$ isotropy), given by [art7, Theorems 6.4, 6.9]. In such cases, there is an indeterminacy that results in the appearance of a *cubic remainder*, coming from the geometry and the link between groups \mathbb{D}_3 , \mathbb{D}_4 and \mathbb{O} .

Chapitre 5

Covariant algebra of a linear representation

When considering classical invariant theory, the initial notion of a polynomial invariant naturally generalizes to the one of *covariant*. According to Parshall [112], Eisenstein was the first to study the geometry of a binary cubic using an associated binary quadratic form [51], which appeared to be its Hessian. The peculiarity here is that it is possible to construct an equivariant object of the same type and, furthermore, it is possible to define an *algebra* of such covariants (see [Subsection 1.3.1](#)). The idea is to generalize this notion to any linear representation (V, G) of a group G , which is proposed for instance by Kraft and Procesi [83], while other one can be found in [46].

In fact, *a definition is proposed in [105]*, with Boris Kolev, Rodrigue and Boris Desmorat as co-authors, of the notion of a covariant algebra of any linear representation ([Definition 5.1.2](#)). The covariant algebras of $\mathrm{SO}(3, \mathbb{k})$ tensorial representations ($\mathbb{k} = \mathbb{R}$ or \mathbb{C}) are of particular interest, and *they can be directly deduced from the standard covariant algebras of binary forms* ([Theorem 5.2.1](#) and [Theorem 5.2.2](#)). Now, to obtain a minimal integrity basis of a covariant algebra for tensorial representations in intrinsic tensorial form, *the same collaboration [105] led to the definition of a new covariant operation, namely the generalized cross product* ([Definition 5.3.5](#)). In [section 5.4](#), all previous results were applied to obtain an explicit minimal integrity basis in tensorial form for the covariant algebra $\mathbf{Cov}(\mathbb{H}^4(\mathbb{R}^3))$ of fourth order harmonic tensors, strongly linked to the space of elasticity tensors ([Theorem 5.4.1](#)).

5.1 Covariant algebra of type W

Let us now recall a definition due to Kraft and Procesi of a type W covariant [83] :

Definition 5.1.1 (Type W covariant). Given two representations (V, ρ) and (W, σ) of a group G , we define $\text{Pol}(V, W)$ to be the space of polynomial mappings p from V to W (*i.e.* each component function is a polynomial expression of the components of $\mathbf{v} \in V$, and such in any basis). A *polynomial covariant of V of type W* is a G -equivariant polynomial mapping $p : V \rightarrow W$, in the sense that

$$p(\rho(g)\mathbf{v}) = \sigma(g)(p(\mathbf{v})), \quad \forall \mathbf{v} \in W, \forall g \in G.$$

The problem with this definition is that the set $\text{Pol}(V, W)^G$ of polynomial covariant of V of type W is only a vector space and not an algebra. Using the natural linear representation of G on $V \oplus W^*$, induced by ρ and σ , we extend this definition as follows, so to obtain an *finitely generated algebra* for compact (or linearly reductive) Lie groups [69].

Definition 5.1.2. Let (V, ρ) and (W, σ) be finite dimensional representations of a group G . The *covariant algebra of V of type W* , noted $\mathbf{Cov}(V, W)$, is defined as the invariant algebra

$$\mathbb{R}[V \oplus W^*]^G,$$

where W^* is the dual vector space of W .

Remark 5.1.3. We can define similarly the *contravariant algebra of V of type W* as $\mathbb{R}[V \oplus W]^G$. However, if W and W^* are equivalent representations (for instance if the representation W is unitary), we do not have to distinguish between these two algebras which are canonically isomorphic.

Note that the covariant algebra $\mathbf{Cov}(V, W)$ has a natural bi-graduation. It is the direct sum of the finite dimensional vector spaces $\mathbf{Cov}_{d,k}(V, W)$ of bi-homogeneous polynomial $p(\mathbf{v}, \omega)$:

- of total degree d in $\mathbf{v} \in V$, called the **degree** of the covariant;
- of total degree k in $\omega \in W^*$, called the **order** of the covariant.

Furthermore, the subspace of covariants of order 0 is identical to the invariant algebra of V .

Remark 5.1.4. The vector space of polynomial covariants $\text{Pol}(V, W)^G$ can thus be identified with

$$\mathbf{Cov}_1(V, W) = \bigoplus_{k=0}^{+\infty} \mathbf{Cov}_{k,1}(V, W),$$

the vector space of first-order covariants.

In this memoir, we essentially focus on the case when $G = \mathrm{SO}(3, \mathbb{R})$ and W is the Euclidean space \mathbb{R}^3 (in which case, we do not have to make any difference between the covariant and the contravariant algebras), and we set

$$\mathbf{Cov}(V) := \mathbb{R}[V \oplus \mathbb{R}^3]^{\mathrm{SO}(3, \mathbb{R})}.$$

An element $p \in \mathbf{Cov}(V)$ is thus a polynomial which can be written as

$$p(\mathbf{v}, \mathbf{x}) = \sum_{i,j,k} p_{ijk}(\mathbf{v}) x^i y^j z^k,$$

where each coefficient $p_{ijk}(\mathbf{v})$ is a polynomial function of \mathbf{v} satisfying

$$p(\rho(g)\mathbf{v}, \mathbf{x}) = p(\mathbf{v}, g^{-1}\mathbf{x}),$$

for all $\mathbf{v} \in V$, $\mathbf{x} \in \mathbb{R}^3$ and $g \in \mathrm{SO}(3, \mathbb{R})$. As a consequence, any homogeneous polynomial covariant of $\mathbf{v} \in V$ of degree d and of type $\mathbb{S}^k(\mathbb{R}^3)$ can thus be identified with a polynomial in $\mathbf{Cov}_{d,k}(V)$.

5.2 Covariant algebra of tensors towards binary forms

We now focus on covariant algebras $\mathbb{R}[V \oplus \mathbb{R}^3]^{\mathrm{SO}(3, \mathbb{R})}$ (and its complex counterpart), where we suppose V is given by some $\mathrm{SO}(3, \mathbb{R})$ tensorial representation. The question is to obtain an *explicit integrity basis* of such an algebra. In fact, in the case of irreducible representations (see [section 1.4](#)), it is possible to deduce an integrity basis of $\mathbf{Cov}(\mathbb{H}^n(\mathbb{R}^3))$ from the one of $\mathbf{Cov}(S_{2n})$.

First, let us define an explicit equivariant isomorphism between the spaces $(\mathcal{H}_n(\mathbb{C}^3), \mathrm{SO}(3, \mathbb{C}))$ and $(S_{2n}, \mathrm{SL}(2, \mathbb{C}))$, theoretically given by [Lemma 1.4.4](#). In fact this isomorphism derives from an equivariant mapping introduced first in Cartan's theory of spinors going back to 1913 (see [27, Chapter 3]) and rediscovered later by Backus [6]. More precisely, let us introduce the *Cartan map*

$$\phi : \mathbb{C}^2 \rightarrow \mathfrak{sl}(2, \mathbb{C}), \quad \xi \mapsto \xi \xi^\omega = \begin{pmatrix} -uv & u^2 \\ -v^2 & uv \end{pmatrix}, \quad (5.1)$$

where

$$\xi = \begin{pmatrix} u \\ v \end{pmatrix}, \quad \xi^\omega = (-v \ u),$$

and ξ^ω means the covariant version of the vector ξ , defined using the determinant ω on \mathbb{C}^2 (a nondegenerate bilinear form). The main property of this mapping is that it is $\text{SL}(2, \mathbb{C})$ -equivariant as we have :

$$\phi(\gamma\xi) = \text{Ad}_\gamma \phi(\xi) = \gamma^{-1} \phi(\xi) \gamma, \quad \forall \gamma \in \text{SL}(2, \mathbb{C}).$$

Choosing the following basis

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},$$

of the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ (corresponding to multiplication by i of Pauli matrices), allows us to identify $\mathfrak{sl}(2, \mathbb{C})$ with \mathbb{C}^3 , using the parametrization

$$\begin{pmatrix} iz & x + iy \\ -x + iy & -iz \end{pmatrix}.$$

In this basis, the Cartan map (5.1) writes

$$\phi : \mathbb{C}^2 \rightarrow \mathbb{C}^3, \quad (u, v) \mapsto \left(x = \frac{u^2 + v^2}{2}, y = \frac{u^2 - v^2}{2i}, z = iuv \right). \quad (5.2)$$

Finally we have [105] :

Theorem 5.2.1. *By pull-back, the Cartan map induces a linear isomorphism $\phi^* : \mathcal{H}_n(\mathbb{C}^3) \rightarrow S_{2n}$ given by*

$$(\phi^*(h))(u, v) := h\left(\frac{u^2 + v^2}{2}, \frac{u^2 - v^2}{2i}, iuv\right).$$

It is an $\text{SL}(2, \mathbb{C})$ -equivariant isomorphism.

This equivariant isomorphism produces an explicit isomorphism between the covariant algebras $\mathbb{C}[\mathcal{H}_n(\mathbb{C}^3) \oplus \mathbb{C}^3]^{\text{SO}(3, \mathbb{C})}$ and $\mathbb{C}[S_{2n} \oplus S_2]^{\text{SL}(2, \mathbb{C})}$. It is now possible to directly deduce the algebra $\mathbb{C}[S_{2n} \oplus S_2]^{\text{SL}(2, \mathbb{C})}$ from the one of $\text{Cov}(S_{2n})$, using the covariant linear mapping

$$\varsigma : \text{Cov}(S_{2n}) \rightarrow \mathbb{C}[S_{2n} \oplus S_2]^{\text{SL}(2, \mathbb{C})}, \quad \mathbf{h} \mapsto \varsigma(\mathbf{h}) := \frac{1}{(r!)^2} (\mathbf{h}, \mathbf{w}^r)_{2r}, \quad \mathbf{w} \in S_2 \quad (5.3)$$

so that we have [105] :

Theorem 5.2.2. Let $\{\mathbf{h}_1, \dots, \mathbf{h}_N\}$ be a minimal basis for $\mathbf{Cov}(S_{2n})$. Then a minimal basis for the joint invariant algebra

$$\mathbb{C}[S_{2n} \oplus S_2]^{\mathrm{SL}(2, \mathbb{C})}$$

is given by

$$\{\zeta(\mathbf{h}_1), \dots, \zeta(\mathbf{h}_N), \Delta\}$$

where $\Delta(\mathbf{w}) := b_1^2 - b_0 b_2$, if $\mathbf{w}(\boldsymbol{\zeta}) := b_0 u^2 + b_1 u v + b_2 v^2 \in S_2$.

In fact, a consequence of such a result is that any minimal integrity basis of $\mathbf{Cov}(S_{2n})$ can directly be translated into a minimal integrity basis of $\mathbb{R}[\mathcal{H}_n(\mathbb{R}^3) \oplus \mathbb{R}^3]^{\mathrm{SO}(3, \mathbb{R})}$ (see [105, Appendix B] for more details).

5.3 Covariant operations on tensor spaces

In this section, we introduce three operations on tensors, which *commute with the action of the rotation group $\mathrm{SO}(3, \mathbb{k})$* (and the orthogonal group $\mathrm{O}(3, \mathbb{k})$) and are thus called *covariant operations*. The first one is the classical *symmetric tensor product*.

Definition 5.3.1. The symmetric tensor product between two tensors $\mathbf{T}^1 \in \mathbb{T}^p(\mathbb{k}^3)$ and $\mathbf{T}^2 \in \mathbb{T}^q(\mathbb{k}^3)$ is defined as

$$\mathbf{T}^1 \odot \mathbf{T}^2 := (\mathbf{T}^1 \otimes \mathbf{T}^2)^s \in \mathbb{S}^{p+q}(\mathbb{k}^3),$$

where the total symmetrisation $\mathbf{T}^s \in \mathbb{S}^n(\mathbb{k}^3)$ of a tensor $\mathbf{T} \in \mathbb{T}^n(\mathbb{k}^3)$, is defined as

$$\mathbf{T}^s(\mathbf{x}_1, \dots, \mathbf{x}_n) := \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} \mathbf{T}(\mathbf{x}_{\sigma(1)}, \dots, \mathbf{x}_{\sigma(n)})$$

where \mathfrak{S}_n is the symmetric group on n letters.

Remark 5.3.2. When *restricted to totally symmetric tensors*, the polynomial counterpart of the symmetric tensor product is just the usual product of polynomials. This product is thus associative and commutative. It is equivariant relative to either the rotation group $\mathrm{SO}(3, \mathbb{k})$ and to the full orthogonal group $\mathrm{O}(3, \mathbb{k})$.

The second one is the r -contraction between two tensors $\mathbf{T}^1 \in \mathbb{T}^p(\mathbb{k}^3)$ and $\mathbf{T}^2 \in \mathbb{T}^q(\mathbb{k}^3)$ over one or several subscripts. This operation uses the Euclidean structure represented by the canonical Euclidean metric tensor $\mathbf{q} = (q_{ij})$ and its inverse $\mathbf{q}^{-1} = (q^{ij})$. It is defined as follows :

$$(\mathbf{T}^1 \overset{(r)}{\cdot} \mathbf{T}^2)_{i_1 \dots i_{p-r} j_{r+1} \dots j_q} = q^{i_{p-r+1} j_1} \dots q^{i_p j_r} T^1_{i_1 \dots i_p} T^2_{j_1 \dots j_q}.$$

The r -contraction of two tensors is an $O(3, \mathbb{k})$ -equivariant mapping

$$\mathbb{T}^p(\mathbb{k}^3) \times \mathbb{T}^q(\mathbb{k}^3) \rightarrow \mathbb{T}^{p+q-2r}(\mathbb{k}^3),$$

and for $n = p = q$, the n -contraction corresponds to the canonical scalar product on $\mathbb{T}^n(\mathbb{k}^3)$.

Definition 5.3.3 (Symmetric r -contraction). The *symmetric r -contraction* between two totally symmetric tensors $\mathbf{S}^1 \in \mathbb{S}^p(\mathbb{k}^3)$ and $\mathbf{S}^2 \in \mathbb{S}^q(\mathbb{k}^3)$ is defined as

$$(\mathbf{S}^1 \stackrel{(r)}{\cdot} \mathbf{S}^2)^s.$$

Remark 5.3.4. The polynomial counterpart of the symmetric r -contraction is obtained as follows. If $\mathbf{S}^1, \mathbf{S}^2$ correspond respectively to the polynomials p_1, p_2 , then, $(\mathbf{S}^1 \stackrel{(r)}{\cdot} \mathbf{S}^2)^s$ corresponds to the polynomial

$$p = \frac{(p-r)!}{p!} \frac{(q-r)!}{q!} \sum_{k_1+k_2+k_3=r} \frac{r!}{k_1!k_2!k_3!} \frac{\partial^r p_1}{\partial x^{k_1} \partial y^{k_2} \partial z^{k_3}} \frac{\partial^r p_2}{\partial x^{k_1} \partial y^{k_2} \partial z^{k_3}}.$$

The third covariant operation is the *generalized cross product*, which extends the standard cross product between vectors of \mathbb{k}^3 to symmetric tensors of arbitrary order.

Definition 5.3.5 (Generalized cross product [105]). The generalized cross product (or *Lie–Poisson product*) between two totally symmetric tensors $\mathbf{S}^1 \in \mathbb{S}^p(\mathbb{k}^3)$ and $\mathbf{S}^2 \in \mathbb{S}^q(\mathbb{k}^3)$ is defined as

$$\mathbf{S}^1 \times \mathbf{S}^2 := - (\mathbf{S}^1 \cdot \boldsymbol{\epsilon} \cdot \mathbf{S}^2)^s \in \mathbb{S}^{p+q-1}(\mathbb{k}^3).$$

where $\boldsymbol{\epsilon}$ is the *Levi–Civita tensor*. In any orthonormal basis $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$, we get

$$(\mathbf{S}^1 \times \mathbf{S}^2)_{i_1 \dots i_{p+q-1}} := (\varepsilon_{i_1 j k} S^1_{j i_2 \dots i_p} S^2_{k i_{p+1} \dots i_{p+q-1}})^s, \quad \varepsilon_{ijk} := \det(\mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_k).$$

Remark 5.3.6. The generalized cross product is skew-symmetric :

$$\mathbf{S}^2 \times \mathbf{S}^1 = - \mathbf{S}^1 \times \mathbf{S}^2.$$

Its polynomial counterpart is (up to a scaling factor) the *Lie–Poisson bracket* on $\mathfrak{so}^*(3, \mathbb{k})$, the dual of the Lie algebra of the rotation group (isomorphic to \mathbb{k}^3). More precisely, if p_1, p_2 are the polynomial representatives of $\mathbf{S}^1, \mathbf{S}^2$, then the polynomial representative of $\mathbf{S}^1 \times \mathbf{S}^2$ is

$$\frac{1}{pq} \{p_1, p_2\}_{LP} = \frac{1}{pq} \det(\mathbf{x}, \nabla p_1, \nabla p_2).$$

This product is equivariant relative to the rotation group $SO(3, \mathbb{k})$ but not to the full orthogonal group $O(3, \mathbb{k})$, since in that case :

$$(\rho_p(g)\mathbf{S}^1) \times (\rho_q(g)\mathbf{S}^2) = (\det g) \left(\rho_{p+q-1}(g)(\mathbf{S}^1 \times \mathbf{S}^2) \right).$$

5.4 A minimal integrity basis for the covariant algebra of \mathbb{H}^4

As detailed in [Section 5.2](#), $\mathbf{Cov}(\mathbb{H}^4)$ is connected with the invariant algebra $\mathbf{Inv}(S_8 \oplus S_2)$. This algebra is itself connected to the covariant algebra of binary forms of degree 8, $\mathbf{Cov}(S_8)$ ([Theorem 5.2.2](#)). A minimal covariant basis for $\mathbf{Cov}(S_8)$ is known at least partially since 1880 and was first produced by von Gall [145] (see also [7, 38]). We finally obtained (all details being in [105]) :

Theorem 5.4.1. *The polynomial covariant algebra of \mathbb{H}^4 is generated by a minimal basis of 70 homogeneous covariant polynomials. An explicit basis is given by [105, table 2].*

Chapitre 6

Isotropic stratification of real representations

We know that any real representation (V, G) of a compact Lie group is endowed with an *isotropic stratification* as well as a *semi-algebraic stratification* (see [Section 1.1](#) and [Section 1.2](#)). Here I focus on the specific case of linear $\mathrm{SO}(3, \mathbb{R})$ representations on a tensor space \mathbb{V} , for which I already mentioned an explicit procedure to determine the isotropic stratification (see [section 2.2](#)). Using a common work done with Boris Kolev, Rodrigue and Boris Desmorat [105], I first establish the link between the isotropic classes and tensorial covariant vector spaces ([Propositions 6.1.3 and 6.1.4](#)). The case of elasticity is discussed in [section 6.2](#), where it is possible to establish explicit algebraic equations of its 8 isotropic strata ([Theorem 6.2.1](#)), based on the covariant algebra of $(\mathbb{H}^4(\mathbb{R}^3), \mathrm{SO}(3, \mathbb{R}))$ from [Theorem 5.4.1](#). Finally, I propose an algorithm obtained in [\[art1\]](#), so to get an explicit isotropy class of a given elasticity tensor. Note that explicit algebraic equations of 6 elasticity strata (of smaller dimensions) had been obtained in polynomial form in [\[5\]](#), leading to expressions that are sometimes very difficult to manipulate.

6.1 Tensorial covariant and symmetry classes

Given a linear representation \mathbb{V} of $\mathrm{SO}(3, \mathbb{R})$ and $\mathbf{v} \in \mathbb{V}$, we define $\mathbf{Cov}_k(\mathbf{v})$ as the set of all k -order polynomial covariants of \mathbf{v} (see [chapitre 5](#)). Note that whereas $\mathbf{Cov}(\mathbb{V})$ is a polynomial algebra, and $\mathbf{Cov}_k(\mathbb{V})$ is an infinite dimensional vector space, $\mathbf{Cov}_k(\mathbf{v})$ is the set of all *evaluations* of these covariants on the vector \mathbf{v} . As such, it is a subspace of the finite dimensional real vector space $\mathcal{P}_k(\mathbb{R}^3)$ of homogeneous polynomials of degree k on \mathbb{R}^3 , or equivalently of the space $\mathbb{S}^k(\mathbb{R}^3)$ of totally symmetric tensors of order k . In this section, we

will focus on polynomial covariants of order one and two of a vector $\mathbf{v} \in \mathbb{V}$ and relate the symmetry class of $\mathbf{Cov}_1(\mathbf{v})$ and $\mathbf{Cov}_2(\mathbf{v})$ with their respective dimension.

First, we need to extend the concept of isotropy classes to a finite or infinite family of vectors belonging to different (or same) representations of G .

Definition 6.1.1. Let \mathcal{F} be a finite or infinite family of vectors belonging to different (or same) representations of G . We define the isotropy group of \mathcal{F} as the subgroup

$$G_{\mathcal{F}} := \bigcap_{\mathbf{v} \in \mathcal{F}} G_{\mathbf{v}}.$$

The *symmetry class* of \mathcal{F} is the conjugacy class of $G_{\mathcal{F}}$ in G .

Remark 6.1.2. Note that if \mathcal{F} is a vector space and $(\mathbf{v}_i)_{i \in I}$ is any generating set of \mathcal{F} , then

$$G_{\mathcal{F}} = \bigcap_{i \in I} G_{\mathbf{v}_i}.$$

In particular, if $(\mathbf{v}_1, \dots, \mathbf{v}_p)$ is a basis of \mathcal{F} , then

$$G_{\mathcal{F}} = \bigcap_{j=1}^p G_{\mathbf{v}_j}.$$

Now, from [section 2.2](#) the possible isotropy classes for the space $\mathbf{Cov}_1(\mathbf{v})$ are

$$[\mathbf{1}], \quad [\mathrm{SO}(2)], \quad [\mathrm{SO}(3, \mathbb{R})],$$

whereas, for $\mathbf{Cov}_2(\mathbf{v})$, they are

$$[\mathbf{1}], \quad [\mathbb{Z}_2], \quad [\mathrm{D}_2], \quad [\mathrm{O}(2)], \quad [\mathrm{SO}(3, \mathbb{R})].$$

Proposition 6.1.3. Given $\mathbf{v} \in \mathbb{V}$, $\dim(\mathbf{Cov}_1(\mathbf{v}))$ is either 0, 1 or 3. Moreover, the isotropy class of $\mathbf{Cov}_1(\mathbf{v})$ is :

1. $[\mathrm{SO}(3, \mathbb{R})]$ if and only if $\mathbf{Cov}_1(\mathbf{v}) = \{0\}$;
2. $[\mathrm{SO}(2)]$ if and only if $\dim \mathbf{Cov}_1(\mathbf{v}) = 1$;
3. $[\mathbf{1}]$ if and only if $\dim \mathbf{Cov}_1(\mathbf{v}) = 3$.

The case of $\mathbf{Cov}_2(\mathbf{v})$ is more involved, but we obtain (see details in [\[105, Section 7\]](#)) :

Proposition 6.1.4. Given $\mathbf{v} \in \mathbb{V}$, $\dim \mathbf{Cov}_2(\mathbf{v})$ is either 1, 2, 3, 4 or 6. Moreover, the isotropy class of $\mathbf{Cov}_2(\mathbf{v})$ is :

1. $[\mathrm{SO}(3, \mathbb{R})]$ if and only if $\dim \mathbf{Cov}_2(\mathbf{v}) = 1$;
2. $[\mathrm{O}(2)]$ if and only if $\dim \mathbf{Cov}_2(\mathbf{v}) = 2$;
3. $[\mathrm{D}_2]$ if and only if $\dim \mathbf{Cov}_2(\mathbf{v}) = 3$;
4. $[\mathbb{Z}_2]$ if and only if $\dim \mathbf{Cov}_2(\mathbf{v}) = 4$;
5. $[\mathbf{1}]$ if and only if $\dim \mathbf{Cov}_2(\mathbf{v}) = 6$.

6.2 Characterization of the isotropy class of an elasticity tensor

We now state a theorem which characterizes, using polynomial covariants, the isotropy class of an elasticity tensor, so that it produces explicit algebraic equations of each isotropy stratum. This result comes from, among other things, many lemmas and theorems established for the characterization of a fourth order harmonic tensor using an integrity basis of its covariant algebra (see [Theorem 5.4.1](#)). All technical details are given in [\[105\]](#). Recall finally that the generalized cross product is given by [Definition 5.3.5](#) and that $\text{SO}(3, \mathbb{R})$ subgroups are detailed in [\[art5, 59\]](#).

Theorem 6.2.1. *Let $\mathbf{E} = (\mathbf{H}, \mathbf{a}, \mathbf{b}, \lambda, \mu) \in \mathbb{E}_{\text{Ela}}$ be a harmonic decomposition of an elasticity tensor \mathbf{E} , where $\mathbf{H} \in \mathbb{H}^4$, $\mathbf{a}, \mathbf{b} \in \mathbb{H}^2$ and λ, μ are scalars. Then*

1. \mathbf{E} is isotropic ($[\text{SO}(3, \mathbb{R})]$ isotropy class) if and only if $\mathbf{a} = \mathbf{b} = \mathbf{d}_2 = 0$.
2. \mathbf{E} is cubic ($[\text{O}]$ isotropy class) if and only if $\mathbf{a} = \mathbf{b} = \mathbf{d}'_2 = 0$ and $\mathbf{d}_2 \neq 0$, with $\mathbf{d}'_2 := \mathbf{d}_2 - 1/3 \text{tr}(\mathbf{d}_2)\mathbf{q}$.
3. \mathbf{E} is transversely isotropic ($[\text{SO}(2)]$ isotropy class) if and only if $(\mathbf{d}_2, \mathbf{a}, \mathbf{b})$ is transversely isotropic and

$$\mathbf{H} \times \mathbf{d}_2 = \mathbf{H} \times \mathbf{a} = \mathbf{H} \times \mathbf{b} = 0.$$

4. \mathbf{E} is tetragonal ($[\mathbb{D}_4]$ isotropy class) if and only if $(\mathbf{d}_2, \mathbf{a}, \mathbf{b})$ is transversely isotropic,

$$\text{tr}(\mathbf{H} \times \mathbf{d}_2) = \text{tr}(\mathbf{H} \times \mathbf{a}) = \text{tr}(\mathbf{H} \times \mathbf{b}) = 0,$$

and

$$\mathbf{H} \times \mathbf{d}_2 \neq 0, \quad \text{or} \quad \mathbf{H} \times \mathbf{a} \neq 0, \quad \text{or} \quad \mathbf{H} \times \mathbf{b} \neq 0.$$

5. \mathbf{E} is trigonal ($[\mathbb{D}_3]$ isotropy class) if and only if $(\mathbf{d}_2, \mathbf{a}, \mathbf{b})$ is transversely isotropic,

$$\mathbf{d}_2 \times (\mathbf{H} : \mathbf{d}_2) = \mathbf{a} \times (\mathbf{H} : \mathbf{a}) = \mathbf{b} \times (\mathbf{H} : \mathbf{b}) = 0,$$

and

$$\text{tr}(\mathbf{H} \times \mathbf{d}_2) \neq 0, \quad \text{or} \quad \text{tr}(\mathbf{H} \times \mathbf{a}) \neq 0, \quad \text{or} \quad \text{tr}(\mathbf{H} \times \mathbf{b}) \neq 0.$$

6. \mathbf{E} is orthotropic ($[\mathbb{D}_2]$ isotropy class) if and only if the family of second-order tensors

$$\mathcal{F}_o := \{ \mathbf{d}_2, \mathbf{a}, \mathbf{b}, \mathbf{c}_3, \mathbf{c}_4, \mathbf{H} : \mathbf{a}, \mathbf{H} : \mathbf{b}, \mathbf{H} : \mathbf{a}^2, \mathbf{H} : \mathbf{b}^2 \}$$

is orthotropic.

7. \mathbf{E} is monoclinic ($[\mathbb{Z}_2]$ isotropy class) if and only if the family of second-order tensors

$$\mathcal{F}_m := \left\{ \mathbf{d}_2, \mathbf{a}, \mathbf{b}, \mathbf{c}_3, \mathbf{c}_4, \mathbf{H} : \mathbf{a}, \mathbf{H} : \mathbf{b}, \mathbf{H} : \mathbf{a}^2, \mathbf{H} : \mathbf{b}^2, \mathbf{H} : (\mathbf{ab})^s, \mathbf{H} : (\mathbf{ad}_2)^s, \mathbf{H} : (\mathbf{bd}_2)^s \right\}$$

is monoclinic.

8. \mathbf{E} is triclinic ($[1]$ isotropy class) if and only if none of the preceding conditions holds.

Algorithm for symmetry class detection

Take $\mathbf{E} \in \mathbb{E}_{\text{la}}$ to be any elasticity tensor. We propose here a detailed algorithm to obtain the symmetry class of \mathbf{E} . Note also that there exists an algorithm to obtain its normal form [art1]. First, let us take back the two families \mathcal{F}_o and \mathcal{F}_m defined in [Theorem 6.2.1](#).

Now, the algorithm to obtain symmetry class of a given elasticity tensor $\mathbf{E} \in \mathbb{E}_{\text{la}}$ is summarized in [Figure 6.1](#) and detailed below (all being detailed in [art1]) :

1. If \mathcal{F}_o is triclinic, then \mathcal{F}_m is necessarily triclinic, so that \mathbf{E} is triclinic from [Theorem 6.2.1](#).
2. If \mathcal{F}_o is monoclinic then
 - (a) Either \mathcal{F}_m is monoclinic, and then from [Theorem 6.2.1](#) \mathbf{E} is monoclinic;
 - (b) or \mathcal{F}_m is triclinic, and then \mathbf{E} is triclinic from previous point (1);
3. If \mathcal{F}_o is orthotropic, then \mathbf{E} is orthotropic from [Theorem 6.2.1](#).
4. If \mathcal{F}_o is transversely isotropic, then the triplet $(\mathbf{d}', \mathbf{v}', \mathbf{d}'')$ is transversely isotropic. Compute elasticity tensor $\bar{\mathbf{E}}$ using [Procedure 6.2.2](#).
 - (a) Either the triplet $(\mathbf{d}', \mathbf{v}', \mathbf{d}'')$ is isotropic, but this imply that \mathbf{E} is cubic or isotropic, in which case \mathcal{F}_o is isotropic, which is exclude;
 - (b) Or $(\mathbf{d}', \mathbf{v}', \mathbf{d}'')$ is transverely isotropic, so that \mathbf{E} is either tetragonal, trigonal or transversely isotropic. Using [Procedure 6.2.2](#), we obtain a tensor $\bar{\mathbf{E}}$ so that there exists \mathbf{E}_0 and $\theta \in [0; 2\pi[$ such that $\bar{\mathbf{E}} := \mathbf{r}(\mathbf{e}_3, \theta) * \mathbf{E}$ and \mathbf{E}_0 symmetry group is either $\mathbb{D}_3, \mathbb{D}_4$ or $O(2)$. Now we get three cases :
 - i. If $\bar{E}_{1123}^2 + \bar{E}_{1113}^2 \neq 0$ then \mathbf{E} is trigonal;
 - ii. if $\bar{E}_{1123}^2 + \bar{E}_{1113}^2 = 0$ and $2\bar{E}_{1212} - \bar{E}_{1111} + \bar{E}_{1122} = 0$ then \mathbf{E} is transversely isotropic;
 - iii. if $\bar{E}_{1123}^2 + \bar{E}_{1113}^2 = 0$ and $2\bar{E}_{1212} - \bar{E}_{1111} + \bar{E}_{1122} = 0$ then \mathbf{E} is tetragonal
5. If \mathcal{F}_o is isotropic then $\mathbf{d}'_2 = \mathbf{d}' = \mathbf{v}' = 0$, thus
 - (a) Either $\mathbf{d}_2 = 0$ and $\mathbf{H} = 0$, so that \mathbf{E} is isotropic;

(b) Or $\mathbf{d}_2 \neq 0$, so that \mathbf{H} is cubic.

Procedure 6.2.2 (Transversely isotropic triplet).

Input : $\mathbf{E} \in \mathbb{E}_{\text{la}}$ such that the triplet $(\mathbf{d}', \mathbf{v}', \mathbf{d}'_2)$ is transversely isotropic.

Output : A rotation $g \in \text{SO}(3)$ and an elasticity tensor $\bar{\mathbf{E}} = g \star \mathbf{E}$ such that its symmetry group contains either \mathbb{Z}_3 or \mathbb{Z}_4 .

1. Compute the triplet of covariant deviators $(\mathbf{d}', \mathbf{v}', \mathbf{d}'_2)$ of \mathbf{E} . The triplet $(\mathbf{d}', \mathbf{v}', \mathbf{d}'_2)$ being transversely-isotropic, then one of them is transversely isotropic and call it \mathbf{t} .
2. Let \mathbf{u}_3 be the unit eigenvector corresponding to the single eigenvalue of \mathbf{t} , which is obtained by solving the linear system :

$$\left(\mathbf{t} - 2 \frac{\text{tr}(\mathbf{t}^3)}{\text{tr}(\mathbf{t}^2)} \mathbf{q} \right) \mathbf{u} = 0.$$

(a) If $\mathbf{u}_3 \neq \pm \mathbf{e}_3$, define

$$\mathbf{u}_1 = \frac{\mathbf{e}_3 \times \mathbf{u}_3}{\|\mathbf{e}_3 \times \mathbf{u}_3\|} \quad \text{and} \quad \mathbf{u}_2 = \mathbf{u}_3 \times \mathbf{u}_1.$$

(b) If $\mathbf{u}_3 = \pm \mathbf{e}_3$ then define $\mathbf{u}_1 := \mathbf{e}_1$ and $\mathbf{u}_2 := \mathbf{u}_3 \times \mathbf{u}_1$.

3. Let $\mathcal{B} := (\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$ be the associated orthonormal basis of \mathbb{R}^3 .
4. Define g as the rotation given by

$$g = \begin{pmatrix} \mathbf{u}_1 \cdot \mathbf{e}_1 & \mathbf{u}_1 \cdot \mathbf{e}_2 & \mathbf{u}_1 \cdot \mathbf{e}_3 \\ \mathbf{u}_2 \cdot \mathbf{e}_1 & \mathbf{u}_2 \cdot \mathbf{e}_2 & \mathbf{u}_2 \cdot \mathbf{e}_3 \\ \mathbf{u}_3 \cdot \mathbf{e}_1 & \mathbf{u}_3 \cdot \mathbf{e}_2 & \mathbf{u}_3 \cdot \mathbf{e}_3 \end{pmatrix}$$

and then

$$\bar{\mathbf{E}} := g \star \mathbf{E}$$

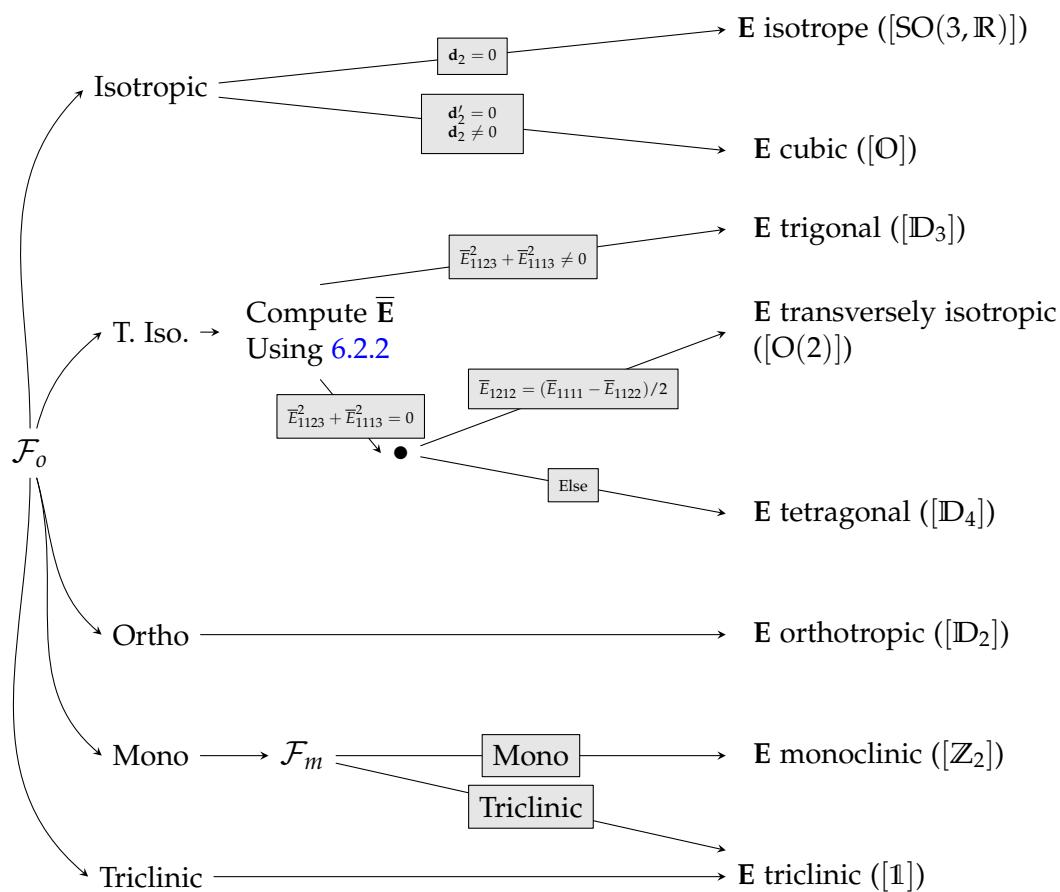


FIGURE 6.1 – Algorithm to obtain the isotropy class of an elasticity tensor

Chapitre 7

Rational invariants and weak separating sets

Even though effective techniques are known to obtain minimal integrity bases of some tensor spaces (or spaces of binary forms), the manipulation of such bases remains difficult in practice. Let us recall for example that the space of elasticity tensors has an integrity basis of 297 invariants, and the isotropic strata equations were obtained, in a very condensed form, using polynomial covariants instead of this integrity basis. Similarly, an integrity basis of the algebra of invariants of binary forms of degree 10 (and thus of $\mathbb{H}^5(\mathbb{C}^3)$) is composed of 106 invariants (with degree up to 21).

Recall now that another question addressed about the orbit space is the one of *separating the orbits*. In the real case, the invariant algebra has such a property, but to reduce the problem, it may be interesting to look for a generating family of a *separating algebra*, introduced by Kemper and Derksen [42, Definition 2.3.8] and further developed by Dufresne [48–50]. Note here that it can also be defined in the complex case. Informally speaking, such an algebra can separate as much as the invariant algebra. Nevertheless, we couldn't get access to an effective algorithm to determine a generating family of such an algebra, even if some polarization techniques exist [47]. It also turns out that this notion of separating algebra has been extensively studied by the mechanical community [15, 126, 147], before the work of Derksen–Kemper and al., but only within the specific framework of tensors of order at most 2. For instance, it has been established that a minimal separating family for the space $(4\mathbb{S}^2(\mathbb{R}^3), \mathrm{SO}(3, \mathbb{R}))$ (four second order symmetric tensors) is composed of 40 polynomial invariants [15, 113], which remains not obvious to manipulate in practice, going up to 98 for 6 second order symmetric tensors.

Passing through *rational invariants* then makes it possible to preserve the property of separation, but by considering only *generic* orbits (see [Subsection 1.3.3](#)). In the case of linear

representations of $\mathrm{SO}(3, \mathbb{C})$, and thus linear representations of $\mathrm{SL}(2, \mathbb{C})$, we know then that the field of invariant is a purely transcendental extension of \mathbb{C} [78]. As a consequence, there exists, at least theoretically a separating family of cardinal the dimension of the generic stratum. This drastically reduces the number of separators, even if this separation applies only to generic vectors. Nevertheless, it remains to explicitly determine such a generating family, which we have been able to obtain for the space of elasticity tensors, while it appears that such a family is also a generating family in the real case.

I will now summarize the essence of what allowed us to achieve this result. In [Section 7.1](#) is defined what is meant by a separating set, different from the one introduced by Kemper–Derksen [42], while more freedom is obtained from a *weak separating set* defined thereafter ([Definition 7.1.3](#)). *In collaboration with Rodrigue Desmorat, Nicolas Auffray, Boris Desmorat and Boris Kolev* [[art4](#)], a generating set of 6 rational invariants of $\mathbb{C}(\mathbb{H}^4(\mathbb{C}^3))^{\mathrm{SO}(3,\mathbb{C})}$ (and its real part) is produced in [Theorem 7.2.1](#). It is the tensorial counterpart of a result by Maeda [93] about a generating family 6 rational invariants of the field of rational invariants $\mathbb{C}(S_8)^{\mathrm{SL}(2,\mathbb{C})}$. Finally in [section 7.3](#), [Theorem 7.3.1](#) gives a weak separating set of 21 polynomial invariants of the space of elasticity tensors, leading to an optimal weak separating set of 18 rational invariants, obtained in [Corollary 7.3.2](#). We also produce another weak separating set of 19 polynomial invariants in [Corollary 7.3.3](#), with higher degrees compare to the ones from [Theorem 7.3.1](#). All such results are coming from [[art4](#)].

7.1 Separating sets and weak separating sets

The weaker concept of *separating set*, often called a *functional basis* in the mechanical community [15, 149], is formulated in invariant theory as follows, where we consider some linear representation (V, G) of a group G .

Definition 7.1.1 (Separating set). A finite set of G -invariant functions $\{s_1, \dots, s_n\}$ over V is a *separating set* if

$$s_i(\mathbf{v}_1) = s_i(\mathbf{v}_2), \quad i = 1, \dots, n \implies \exists g \in G, \mathbf{v}_1 = g\mathbf{v}_2$$

for all $\mathbf{v}_1, \mathbf{v}_2 \in V$. A separating set is minimal if no proper subset of it is a separating set.

Note that this definition is very general and the functions s_1, \dots, s_n are not required to be polynomial. We now recall what is meant by a *generic vector* (also called *vector in general position*) using the Zariski topology on V , where a closed set in the Zariski topology is defined as

$$Z := \{ \mathbf{v} \in V; f(\mathbf{v}) = 0, \forall f \in S \}$$

with S is any set of polynomials in \mathbf{v} . A remarkable fact concerning this topology is that a non-empty closed set is either the whole space or has Lebesgue measure zero [26, 114]. A non-empty Zariski open set is moreover *open and dense* in the usual topology. Now we have :

Definition 7.1.2 (Genericity). A vector \mathbf{v} belonging to some (finite dimensional) vector space V is called *generic* (or as *in general position* by algebraic geometers) if it belongs to a non-empty Zariski open set of V .

Such a definition means informally speaking that, the probability of a randomly chosen vector being generic is 1 and that we omit, in the results, some vectors satisfying certain algebraic relations. Note, however, that this notion of genericity is arbitrary and there is a lot of freedom in the choice of *such a class* of generic vectors. Finally we propose :

Definition 7.1.3 (Weak separating set). Given some non-empty Zariski open set $Z^c \subset V$, a finite set of G -invariant functions $\{s_1, \dots, s_n\}$ over V is called a *weak separating set* (or a *weak functional basis*) if

$$s_i(\mathbf{v}_1) = s_i(\mathbf{v}_2), \quad i = 1, \dots, n \implies \exists g \in G, \mathbf{v}_1 = g\mathbf{v}_2.$$

for all $\mathbf{v}_1, \mathbf{v}_2 \in Z^c$.

Such a notion of a weak separating set is closely related to the one of a generating family of the field of invariant (see [Subsection 1.3.3](#)). We now focus on the cases of fourth order tensors and the one of elasticity tensors.

7.2 Rational invariants of fourth order harmonic tensor

A minimal generating set of 9 generators for the invariant algebra of S_8 is known since at least 1880 (see [123, 145]). In 1990 [99, Theorem B], Maeda produced a system of 6 rational invariants which generate the invariant field $\mathbb{C}(S_8)^{\text{SL}(2, \mathbb{C})}$, so he obtains effective rationality of such a field, following theoretical result from Katsylo about this rationality [77].

Now, using the pull back of the Cartan map introduced in [Section 5.2](#), these rational invariants translate into rational invariants of $\mathbb{C}(\mathbb{H}^4(\mathbb{C}^3))^{\text{SO}(3, \mathbb{C})}$, and they also remain a generating set of the *real field of rational invariants* $\mathbb{R}(\mathbb{H}^4(\mathbb{R}^3))^{\text{SO}(3, \mathbb{R})}$.

In the following result, we have introduced the notation $*^k \mathbf{w}_7$ for the harmonic product of k copies of \mathbf{w}_7 (see [Section 4.1](#) for the harmonic product). All covariant operations used are given in [Section 5.3](#), such as the generalized cross product. We point out, moreover, that the first-order covariant \mathbf{w}_7 , the third-order covariant \mathbf{T}_6 as well as the sixth-order covariant \mathbf{J}_{18} are all harmonic. All indices of polynomial covariants in the list refer to their *degrees* in \mathbf{H} , and details of the proof can be found in [\[art4\]](#).

Theorem 7.2.1. *The field of rational invariants $\mathbb{R}(\mathbb{H}^4(\mathbb{R}^3))^{\text{SO}(3,\mathbb{R})}$ is generated by the following six algebraic independent rational invariants*

$$\begin{aligned} i_2 &= \frac{\|\mathbf{w}_7\|^2}{M_{12}}, \quad i_3 = \frac{\langle *^3 \mathbf{w}_7, \mathbf{T}_6 \rangle}{M_{12}^2}, \quad i_4 = \frac{\langle *^4 \mathbf{w}_7, \mathbf{T}_6 \cdot \mathbf{T}_6 \rangle}{M_{12}^3} \\ k_4 &= \frac{1}{5M_{12}^3} \langle \mathbf{H} * (*^3 \mathbf{w}_7), \mathbf{T}_6 * ((\mathbf{T}_6 \cdot \mathbf{T}_6)^s)_0 \rangle + \frac{1}{7M_{12}^3} \langle \mathbf{H} \times (*^3 \mathbf{w}_7), \mathbf{J}_{18} \rangle \\ &\quad - \frac{7}{99M_{12}^2} \langle \mathbf{H} : (*^3 \mathbf{w}_7), \mathbf{T}_6 \rangle. \\ k_8 &= \frac{\langle \mathbf{w}_7 \times \mathbf{H}, \mathbf{T}_6 \cdot \mathbf{T}_6 \rangle \langle *^6 \mathbf{w}_7, \mathbf{J}_{18} \rangle}{M_{12}^6}, \\ k_9 &= \frac{\langle *^6 \mathbf{w}_7, \mathbf{J}_{18} \rangle}{M_{12}^5} \left(\frac{36}{M_{12}^2} \langle (*^2 \mathbf{w}_7) * \mathbf{H}, \mathbf{J}_{18} \rangle + \frac{28}{5M_{12}} \langle (\mathbf{H} \cdot \mathbf{w}_7) \times \mathbf{w}_7, \mathbf{T}_6 \rangle \right), \end{aligned}$$

where $\mathbf{H} \in \mathbb{H}^4(\mathbb{R}^3)$ is the harmonic tensor, $((\mathbf{T}_6 \cdot \mathbf{T}_6)^s)_0$ is obtained taking first the symmetrization of $\mathbf{T}_6 \cdot \mathbf{T}_6$ (from [Definition 5.3.1](#)) and then its harmonic projection onto $\mathbb{H}^3(\mathbb{R}^3)$ (from [Proposition 4.1.1](#)). Moreover

- $\mathbf{T}_6 := \mathbf{d}_2^2 \times \mathbf{d}_2$ is a third order harmonic covariant of degree 6;
- $\mathbf{w}_7 := \mathbf{H} : \mathbf{T}_6$ is a first order harmonic covariant of degree 7;
- $M_{12} := \|\mathbf{d}_2^2 \times \mathbf{d}_2\|^2$ is an invariant of degree 12;
- $\mathbf{J}_{18} := ((\mathbf{T}_6 \cdot \mathbf{T}_6)^s)_0 \times \mathbf{T}_6$ is a sixth order covariant of degree 18.

7.3 Weak separating sets for elasticity tensors

Take back here the space of elasticity tensors and let us write

$$\mathbf{E} = (\lambda, \mu, \mathbf{d}', \mathbf{v}', \mathbf{H}) \in \mathbb{H}^0(\mathbb{R}^3) \oplus \mathbb{H}^0(\mathbb{R}^3) \oplus \mathbb{H}^2(\mathbb{R}^3) \oplus \mathbb{H}^2(\mathbb{R}^3) \oplus \mathbb{H}^4(\mathbb{R}^3)$$

with $\mathbf{d}' := \mathbf{d} - 1/3 \text{tr}(\mathbf{d})\mathbf{q}$ (the same for \mathbf{v}') and $\lambda := \text{tr } \mathbf{d}, \mu := \text{tr } \mathbf{v}$, where \mathbf{d} and \mathbf{v} are defined in [Subsection 1.4.2](#). We need to recall here that a pair (\mathbf{a}, \mathbf{b}) of second order harmonic tensors is said to be *triclinic* if its symmetry group reduces to identity (see [Definition 6.1.1](#)). The corresponding isotropy stratum in $\mathbb{H}^2(\mathbb{R}^3) \oplus \mathbb{H}^2(\mathbb{R}^3)$ is defined by the Zariski open set (see [105, Theorem 8.6]) :

$$(\mathbf{a}\mathbf{w}) \times \mathbf{w} \neq 0, \quad (\mathbf{b}\mathbf{w}) \times \mathbf{w} \neq 0, \quad \mathbf{w} := \boldsymbol{\epsilon} : [\mathbf{a}, \mathbf{b}]$$

where ϵ is the Levi-Civita symbol in 3 dimensions and $[\mathbf{a}, \mathbf{b}] := \mathbf{ab} - \mathbf{ba}$ the commutator. In the same way, a tensor $\mathbf{a} \in \mathbb{H}^2(\mathbb{R}^3)$ is said to be *orthotropic* if its isotropy class is $[\mathbb{D}_2]$ (with \mathbb{D}_2 the dihedral group of order 4). The corresponding Zariski open set is given by [105, Lemma 8.1] :

$$\mathbf{a}^2 \times \mathbf{a} \neq 0.$$

We will now formulate our main theorem, using the notation $(\mathbf{ab})^s := \mathbf{ab} + \mathbf{ba}$.

Theorem 7.3.1. *Let $\mathbf{E} = (\lambda, \mu, \mathbf{d}', \mathbf{v}', \mathbf{H})$ be an elasticity tensor, where*

$$\mathbf{d} := \text{tr}_{12}(\mathbf{E}), \quad \mathbf{v} := \text{tr}_{13}(\mathbf{E}), \quad \mathbf{d}' := \mathbf{d} - 1/3 \text{tr}(\mathbf{d})\mathbf{q}, \quad \mathbf{v}' := \mathbf{v} - 1/3 \text{tr}(\mathbf{v})\mathbf{q}$$

Let us define the second order symmetric covariants $\mathbf{d}_2 := \text{tr}_{13} \mathbf{H}^2$ and $\mathbf{d}_3 := \text{tr}_{13} \mathbf{H}^3$. Then, the following 21 polynomial invariants, $\lambda = \text{tr } \mathbf{d}$, $\mu = \text{tr } \mathbf{v}$,

$$\begin{aligned} I_2 &:= \text{tr } \mathbf{d}_2, \quad I_3 := \text{tr } \mathbf{d}_3, \quad I_4 := \text{tr } \mathbf{d}_2^2, \quad I_5 := \text{tr}(\mathbf{d}_2 \mathbf{d}_3), \quad I_6 := \text{tr } \mathbf{d}_2^3, \\ I_7 &:= \text{tr}(\mathbf{d}_2^2 \mathbf{d}_3), \quad I_8 := \text{tr}(\mathbf{d}_2 \mathbf{d}_3^2), \quad I_9 := \text{tr } \mathbf{d}_3^3, \quad I_{10} := \text{tr}(\mathbf{d}_2^2 \mathbf{d}_3^2), \\ D_3 &:= \mathbf{d}' : \mathbf{d}_2, \quad D_4 := \mathbf{d}' : \mathbf{d}_3, \quad D_5 := \mathbf{d}' : \mathbf{d}_2^2, \\ D_6 &:= \mathbf{d}' : (\mathbf{d}_2 \mathbf{d}_3)^s, \quad D_{11} := \mathbf{d}' : [\mathbf{d}_2, \mathbf{d}_3]^2, \\ V_3 &:= \mathbf{v}' : \mathbf{d}_2, \quad V_4 := \mathbf{v}' : \mathbf{d}_3, \quad V_5 := \mathbf{v}' : \mathbf{d}_2^2, \\ V_6 &:= \mathbf{v}' : (\mathbf{d}_2 \mathbf{d}_3)^s, \quad V_{11} := \mathbf{v}' : [\mathbf{d}_2, \mathbf{d}_3]^2, \end{aligned}$$

separate generic tensors \mathbf{E} defined by the following conditions : (1) the pair $(\mathbf{d}_2, \mathbf{d}_3)$ is triclinic ; (2) \mathbf{d}_2 is orthotropic.

In fact, we get an optimal result as stated by the corollary below (all details being in [art4]).

Corollary 7.3.2. *The following 18 rational invariants*

$$\begin{gathered} \lambda, \quad \mu, \quad i_2, \quad i_3, \quad i_4, \quad k_4, \quad k_8, \quad k_9, \\ D_3, \quad D_4, \quad D_5, \quad D_6, \quad D_{11}, \quad V_3, \quad V_4, \quad V_5, \quad V_6, \quad V_{11} \end{gathered}$$

separate generic tensors $\mathbf{E} = (\lambda, \mu, \mathbf{d}', \mathbf{v}', \mathbf{H})$ defined by following conditions : (1) the pair $(\mathbf{d}_2, \mathbf{d}_3)$ is triclinic ; (2) \mathbf{d}_2 is orthotropic.

In Theorem 7.2.1, it can be observed that the denominator of each rational invariant

$$i_2, \quad i_3, \quad i_4, \quad k_4, \quad k_8, \quad k_9,$$

is a power of the polynomial invariant of degree 12

$$M_{12} := \|\mathbf{d}_2^2 \times \mathbf{d}_2\|^2.$$

We have thus the following second corollary.

Corollary 7.3.3. *The following 19 polynomial invariants*

$$\begin{aligned} & \lambda, \quad \mu, \quad M_{12} \\ K_{14} &:= M_{12} i_2, \quad K_{27} := M_{12}^2 i_3, \quad K_{40i} := M_{12}^3 i_4, \\ K_{40k} &:= M_{12}^3 k_4, \quad K_{80} := M_{12}^6 k_8, \quad K_{93} := M_{12}^7 k_9, \\ D_3, \quad D_4, \quad D_5, \quad D_6, \quad D_{11}, \quad V_3, \quad V_4, \quad V_5, \quad V_6, \quad V_{11}, \end{aligned}$$

separate generic tensors $\mathbf{E} = (\lambda, \mu, \mathbf{d}', \mathbf{v}', \mathbf{H})$, defined by the following conditions : (1) the pair $(\mathbf{d}_2, \mathbf{d}_3)$ is triclinic; (2) \mathbf{d}_2 is orthotropic.

Chapitre 8

Order two symmetries for tensorial representations

We already know an algorithm to obtain the isotropy stratification of any $\mathrm{SO}(3, \mathbb{R})$ or $\mathrm{O}(3, \mathbb{R})$ linear representation (see [Section 1.1](#)). *From a work done with Boris Desmorat, Boris Kolev and Rodrigue Desmorat* [[art3](#), [art2](#)], optimal algebraic equations were obtained to detect order two symmetries of a given tensor, which can be related directly to its symmetry group. Note for instance that, for the space $(\mathrm{Ela}, \mathrm{SO}(3, \mathbb{R}))$, each isotropy class can be characterized using such order two symmetries [29]. It should also be pointed out that *plane symmetries* of an elasticity tensor have a physical interpretation, as they define propagation directions of longitudinal waves [103].

A general statement is given by [Theorem 8.0.2](#), producing algebraic equations for order two symmetries of a totally symmetric tensor of order n , both for the standard $\mathrm{O}(3, \mathbb{R})$ representation and also the twisted one. This first theorem is established in [[art3](#)] with a very technical proof, which deepens the link between totally symmetric tensors and homogeneous polynomials. The same collaboration leads in [[art2](#)] to application of those results in the case of elasticity and piezoelectricity, related to tensor spaces which are no more spaces of totally symmetric tensors.

Take now any second order element of $\mathrm{O}(3, \mathbb{R})$, given by some plane symmetry

$$\mathbf{s}(\boldsymbol{\nu}) := \mathbf{q} - \boldsymbol{\nu} \otimes \boldsymbol{\nu},$$

defined by a normal $\boldsymbol{\nu}$ of the plane, or the corresponding symmetry axis $\mathbf{r}(\boldsymbol{\nu}, \pi) := -\mathbf{s}(\boldsymbol{\nu})$. Let $\mathbf{T} \in \mathbb{T}^n(\mathbb{R}^3)$ be some n th order tensor and ρ_n be the standard $\mathrm{O}(3, \mathbb{R})$ representation on $\mathbb{T}^n(\mathbb{R}^3)$ (see [\(1.10\)](#)) or the *twisted* one given by

$$\hat{\rho}_n(g) := \det(g)\rho_n(g), \quad \forall g \in \mathrm{O}(3, \mathbb{R}).$$

Finding a second order symmetry of \mathbf{T} translates into finding a unit vector $\boldsymbol{\nu} \in \mathbb{R}^3$ such that

$$\varrho_n(\mathbf{s}(\boldsymbol{\nu}))\mathbf{T} = \mathbf{T} \text{ or } \varrho_n(\mathbf{r}(\boldsymbol{\nu}, \pi))\mathbf{T} = \mathbf{T}. \quad (8.1)$$

Writing such equations with $\boldsymbol{\nu} = (x, y, z)$ unknown leads to tensorial homogeneous equations of degree $2n$. In fact, such degree can be reduced to degree n equations for totally symmetric tensors, leading to what we call optimal algebraic equations.

In the following, we use the notation

$$\mathbf{w}^k := \underbrace{\mathbf{w} \otimes \mathbf{w} \otimes \cdots \otimes \mathbf{w}}_{k \text{ times}} \in \mathbb{S}^k(\mathbb{R}^3)$$

for any vector $\mathbf{w} \in \mathbb{R}^3$. Furthermore, for any totally symmetric tensor $\mathbf{S} \in \mathbb{S}^n(\mathbb{R}^3)$, r -contraction between \mathbf{S} and \mathbf{w}^k is noted $\mathbf{S}^{(r)} \cdot \mathbf{w}^k$ (see section 5.3). In any orthonormal basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ of \mathbb{R}^3 , for any $r \leq k$, coordinates of $\mathbf{S}^{(r)} \cdot \mathbf{w}^k$ are given by :

$$(\mathbf{S}^{(r)} \cdot \mathbf{w}^k)_{i_1 \dots i_{n-r}} = S_{i_1 \dots i_{n-r} j_1 \dots j_r} w_{j_1} \dots w_{j_r}.$$

A first step to get the reduced algebraic equations is to establish exact algebraic expression (of degree $2n$) of second order symmetries equations (8.1) related to n th order totally symmetric tensors :

Lemma 8.0.1 ([art3]). *Let \mathbf{S} be any totally symmetric n th-order tensor and $\boldsymbol{\nu} \in \mathbb{R}^3$ a unit vector. Then,*

$$\rho_n(\mathbf{s}(\boldsymbol{\nu}))\mathbf{S} = \sum_{k=0}^n \binom{n}{k} (-2)^k \boldsymbol{\nu}^k \odot \left(\mathbf{S}^{(k)} \cdot \boldsymbol{\nu}^k \right), \quad \rho_n(\mathbf{r}(\boldsymbol{\nu}, \pi))\mathbf{S} = (-1)^n \rho_n(\mathbf{s}(\boldsymbol{\nu}))\mathbf{S} \quad (8.2)$$

In order to establish the reduced algebraic equations, we need to introduce the following :

- Given $n \geq 1$, we set $r := \lfloor n/2 \rfloor$, $q := \lfloor (n+1)/2 \rfloor$, and define the $(n+1) \times (n+1)$ matrix B_n by

$$(B_n)_{ij} := \begin{cases} \frac{(i)!}{(i-j)!} & \text{if } i-j \geq 0, \\ 0 & \text{otherwise.} \end{cases} \quad (8.3)$$

where $i, j \in [0, n]$.

- We denote by B_n^1 the $q \times q$ matrix obtained from B_n by deleting the columns

$$0, q+1, q+2, \dots, n,$$

and the rows

$$0, 1, 3, \dots, 2r-1,$$

and by \tilde{B}_n^1 the $(r+1) \times (r+1)$ matrix obtained from B_n by deleting the columns

$$r+1, r+2, \dots, n,$$

and the rows

$$0, 2, 4, \dots, 2q-2.$$

- Finally, we set

$$\lambda_j^{(n)} := (B_n^1)_{jq}^{-1} \quad (j \in [1, q]), \quad \text{and} \quad \mu_j^{(n)} := (\tilde{B}_n^1)_{jr}^{-1} \quad (j \in [0, r]).$$

The subscripts i and j of the matrix $B_n^1 = ((B_n^1)_{ij})$ and of the vector $\lambda^{(n)} = (\lambda_j^{(n)})$ vary from 1 to q . For a matrix $\tilde{B}_n^1 = ((\tilde{B}_n^1)_{ij})$ and vector $\mu^{(n)} = (\mu_j^{(n)})$, i varies from 1 to $r+1$ and j from 0 to r .

Using among other things the generalized cross product and the totally symmetric product (see [section 5.3](#)), we then get (all details and technical proofs can be found in [\[art3\]](#)) :

Theorem 8.0.2. *Let (S^n, ϱ) be a linear representation with $\varrho = \rho_n$ or $\varrho = \hat{\rho}_n$ and*

$$r := \left\lfloor \frac{n}{2} \right\rfloor, \quad q := \left\lfloor \frac{n+1}{2} \right\rfloor,$$

so that $q = r$, if n is even and $q = r+1$, if n is odd.

1. *A unit vector $\boldsymbol{\nu}$ defines a plane/axial symmetry of a tensor $\mathbf{S} \in (S^{2r}, \rho_{2r})$ or an axial symmetry of a tensor $\mathbf{S} \in (S^{2r}, \hat{\rho}_{2r})$ if and only if*

$$\left[\sum_{k=1}^q \frac{n!}{(n-k)!} \lambda_k^{(n)} \boldsymbol{\nu}^{k-1} \odot \left(\mathbf{S} \overset{(k)}{\cdot} \boldsymbol{\nu}^k \right) \right] \times \boldsymbol{\nu} = 0. \quad (8.4)$$

2. *A unit vector $\boldsymbol{\nu}$ defines a plane symmetry of a tensor $\mathbf{S} \in (S^{2r}, \hat{\rho}_{2r})$ if and only if*

$$\sum_{k=0}^r \frac{n!}{(n-k)!} \mu_k^{(n)} \boldsymbol{\nu}^k \odot \left(\mathbf{S} \overset{(k)}{\cdot} \boldsymbol{\nu}^k \right) = 0. \quad (8.5)$$

3. A unit vector $\boldsymbol{\nu}$ define a plane/axial symmetry of a tensor $\mathbf{S} \in (\mathbb{S}^{2r+1}, \hat{\rho}_{2r+1})$ or an axial symmetry of a tensor $\mathbf{S} \in (\mathbb{S}^{2r+1}, \rho_{2r+1})$ if and only if

$$\left[\sum_{k=0}^r \frac{n!}{(n-k)!} \mu_k^{(n)} \boldsymbol{\nu}^k \odot \left(\mathbf{S} \overset{(k)}{\cdot} \boldsymbol{\nu}^k \right) \right] \times \boldsymbol{\nu} = 0. \quad (8.6)$$

4. A unit vector $\boldsymbol{\nu}$ defines a plane symmetry of a tensor $\mathbf{S} \in (\mathbb{S}^{2r+1}, \rho_{2r+1})$ if and only if

$$\sum_{k=1}^q \frac{n!}{(n-k)!} \lambda_k^{(n)} \boldsymbol{\nu}^{k-1} \odot \left(\mathbf{S} \overset{(k)}{\cdot} \boldsymbol{\nu}^k \right) = 0. \quad (8.7)$$

Troisième partie

Questions et perspectives

Chapitre 9

Espace des orbites et théorie effective des invariants

Comme illustré dans la [Partie II](#), une majorité de mes travaux porte sur la *description* de l'espace des orbites associé à une représentation *réelle* (V, G) d'un groupe de Lie compact, ce qui nous amène aussi à considérer, par complexification, une représentation linéaire complexe d'un groupe linéairement réductif. Autour de ces questions, un point de départ peut être ce qui est énoncé par Vinberg lui même dans son article *effective invariant theory* [144], en deux questions :

1. "Classification of the space V up to transformations in the group G ."
2. "Description of the algebra $\mathbb{R}[V]^G$ of invariant polynomials in V ."

Ce type de questions est globalement posé dans le cadre de la géométrie algébrique *complexe*, et nous souhaitons donc creuser le même type de questions dans le cas réel. Il y a donc plusieurs pistes qui restent à explorer et/ou à approfondir, ce que nous détaillons maintenant.

9.1 Stratification semi-algébrique et stratification isotropique

Comme rappelé dans la [Section 1.2](#), les travaux de Abud-Sartori [2] puis de Procesi–Schwarz [118] ont montré qu'il y a un lien très fort entre l'algèbre des invariants $\mathbb{R}[V]^G$ d'une représentation réelle d'un groupe de Lie compact et la structure semi-algébrique de l'espace des orbites V/G , via une *matrice de Gram*. De cette structure semi-algébrique, on sait alors qu'on peut en déduire une stratification associée. Il est énoncé dans [2] un résultat qui nous semble encore conjectural et que nous souhaitons donc creuser :

Conjecture 9.1.1. Etant donnée une représentation linéaire réelle (V, G) d'un groupe de Lie compact et I_1, \dots, I_N une base d'intégrité *minimale* de $\mathbb{R}[V]^G$. En notant

$$\bar{\pi} : V/G \longrightarrow \mathbb{R}^N, \quad \text{Orb}(\mathbf{v}) \mapsto (I_1(\mathbf{v}), \dots, I_N(\mathbf{v}))$$

alors $\bar{\pi}$ met en bijection les composantes connexes de la stratification isotropique de V et celles de la stratification semi-algébrique de $\text{Im}(\bar{\pi})$.

D'autres pistes sont évoquées dans [2], qui mettent en lien le *rang* de la matrice de Gram (voir [Section 1.2](#)) et la *dimension* des strates isotropiques. En fait, au cours de l'étude de l'espace $(\mathbb{E}_{\text{la}}, \text{SO}(3, \mathbb{R}))$ des tenseurs d'élasticité, il s'est avéré que nous avons pu établir explicitement des *équations algébriques* des strates isotropiques, autrement dit, on peut envisager la :

Conjecture 9.1.2. Pour toute représentation linéaire réelle (V, G) d'un groupe de Lie compact, toute strate isotropique est une *variété algébrique réelle*. On conjecture aussi qu'il s'agit d'une variété algébrique affine.

Un exemple, plus simple que celui de l'élasticité, est aussi donné par le cas de l'espace $(\mathbb{H}^2(\mathbb{R}^3), \text{SO}(3, \mathbb{R}))$ des tenseurs harmoniques d'ordre 2, où les équations algébriques des strates sont données dans le tableau

Strate	Équations algébriques
Isotropie $[\mathbb{D}_2]$	$\mathbf{a} \times \mathbf{a}^2 \neq \mathbf{0}$
Isotropie $[\text{O}(2)]$	$\mathbf{a} \times \mathbf{a}^2 = \mathbf{0}$ et $\mathbf{a} \neq \mathbf{0}$
Isotropie $[\text{SO}(3, \mathbb{R})]$	$\mathbf{a} = \mathbf{0}$

TABLE 9.1 – Équations algébriques des strates pour $(\mathbb{H}^2(\mathbb{R}^3), \text{SO}(3, \mathbb{R}))$

Cette conjecture pourrait être vérifiée et rendue explicite dans le cas de l'espace des tenseurs *piézo-électriques*, donné par $\mathbb{P}_{\text{iez}} := \mathbb{S}^2(\mathbb{R}^3) \otimes \mathbb{R}^3$, avec la représentation standard du groupe $(3, \mathbb{R})$:

Conjecture 9.1.3. Chacune des 16 strates isotropes [152] de la stratification isotropique de l'espace $(\mathbb{P}_{\text{iez}}, \text{O}(3, \mathbb{R}))$ est une *variété algébrique affine*, dont on peut établir les équations explicites.

Pour obtenir des équations explicites, une idée consiste à suivre la même piste que celle suivie pour l'élasticité (voir [Section 6.2](#)), à savoir *exploiter une base d'intégrité de l'algèbre des covariants de $\mathbb{H}^3(\mathbb{R}^3)$, laquelle base d'intégrité n'est pas encore connue à ce jour et reste donc à calculer, et cette conjecture peut aussi être envisagée dans le cas du groupe $\text{O}(2)$ (pour les modèles plan en mécanique)*.

9.2 Stratification isotropique et tranche galoisienne

Reprendons ici la stratification isotropique d'une représentation linéaire (V, G) d'un groupe compact G , donnée par (1.2). Pour être en mesure de décrire effectivement chacune de ces strates, on peut exploiter une *tranche linéaire* permettant de donner une *forme normale* pour chaque orbite considérée.

Plus précisément, on montre que chaque orbite ayant l'isotropie $[H]$ rencontre le sous-espace vectoriel

$$\text{Fix}(H) := \{\mathbf{v} \in V, \quad h \cdot \mathbf{v} = \mathbf{v}, \quad \forall h \in H\}$$

et que l'ensemble de ces points forment un ouvert dense dans $\text{Fix}(H)$. Si chaque orbite rencontre $\text{Fix}(H)$ en un point unique, alors $\text{Fix}(H) \cap \Sigma_{[H]}$ est une *section globale* pour l'espace $\Sigma_{[H]}/G$. Dans tous les cas observés, les coordonnées de ce point d'intersection s'exprime rationnellement en fonction des invariants polynomiaux. Dit autrement :

Conjecture 9.2.1. Il existe une correspondance bi-rationnelle entre $\Sigma_{[H]}/G$ et $\text{Fix}(H) \cap \Sigma_{[H]}$.

A notre connaissance, la **justification rigoureuse de cette observation semble une question ouverte**. Si des résultats proches existent dans la littérature [78, 116], c'est toutefois toujours dans le cas complexe mais pas dans le cas réel et le lien entre les deux n'apparaît pas immédiatement.

Dans le cas où l'orbite d'un point de $\Sigma_{[H]}$ rencontre $\text{Fix}(H)$ non plus en un point unique mais un nombre fini de points, cette intersection est décrite par le "groupe de Galois"

$$\Gamma_H := N(H)/H$$

où $N(H)$ est le normalisateur de H dans G . Dans ce cas, on aurait :

Conjecture 9.2.2. Il existe une correspondance bi-rationnelle entre $\Sigma_{[H]}/G$ et $(\text{Fix}(H) \cap \Sigma_{[H]})/\Gamma_H$.

Enfin, il existe toujours des cas où l'orbite d'un point de $\Sigma_{[H]}$ rencontre $\text{Fix}(H)$ en un nombre infini de points. On peut alors réduire la complexité si il existe un sous-espace W de $\text{Fix}(H)$ qui rencontre encore chaque orbite de $\Sigma_{[H]}$ en un nombre fini de points. On est alors amené à considérer les sous-groupes

$$N(W) := \{g \in G, \quad gW = W\} \quad \text{et} \quad Z(W) := \{g \in G, \quad g|_W = \text{id}\}.$$

On pourra noter que $Z(W)$ est distingué dans $N(W)$ et considérer le groupe $\Gamma(W) := N(W)/Z(W)$. Deux questions peuvent alors être abordées :

1. A quelle(s) condition(s) le groupe $\Gamma(W)$ agit-il transitivement sur $W \cap \Sigma_{[H]}$?
2. si c'est le cas, existe-t-il une correspondance bi-rationnelle entre $\Sigma_{[H]}/G$ et $(W \cap \Sigma_{[H]})/\Gamma(W)$?

9.3 Systèmes de paramètres

Une algèbre d'invariants $\mathbb{R}[V]^G$ est un cas particulier d'algèbre graduée Cohen–Macaulay de type fini. Cela se traduit par l'existence d'un famille d'invariants $\{\theta_1, \dots, \theta_s\}$, quelques fois dits primaires [133], ainsi qu'une famille d'invariants secondaires η_1, \dots, η_r dont chacun est *entier* sur l'anneau $\mathbb{R}[V]^G$. Une des conséquences est qu'une fois connues les valeurs des invariants θ_i en un vecteur \mathbf{v} , il n'existe qu'un nombre de valeurs finies possibles pour chaque η_j . Un autre point intéressant est que le cardinal de la famille d'invariants primaires n'est rien d'autre que le degré de transcendance de l'algèbre $\mathbb{R}[V]^G$, et donc la dimension de l'orbite générique, connue *a priori*.

Un problème qui est resté ouvert jusqu'à présent est *la détermination effective d'un système de paramètres pour l'élasticité*. Une telle famille permettrait d'envisager une autre stratégie de séparation que celle passant par les invariants rationnels.

Sur ce thème, il y a aussi certaines conjectures de Dixmier qui pourraient être envisagées du point de vue tensoriel, mais qui s'énoncent dans le cadre des formes binaires. Une première conjecture est la suivante [44, Conjecture 3].

Conjecture 9.3.1. (i) Si d est impair ≤ 15 , alors $\mathbb{C}[S_d]^{\text{SL}(2,\mathbb{C})}$ admet un système de paramètres de degrés $4, 6, 8, \dots, 2d - 2$.

(ii) Si $d \equiv 2 \pmod{4}$ et $d \leq 18$ alors $\mathbb{C}[S_d]^{\text{SL}(2,\mathbb{C})}$ admet un système de paramètres de degrés $2, 3, 5, 6, 6, 7, 8, 9, \dots, d - 1$.

(iii) Si $d \equiv 0 \pmod{4}$, alors $\mathbb{C}[S_d]^{\text{SL}(2,\mathbb{C})}$ admet un système de paramètres de degrés $2, 3, 4, \dots, d - 1$.

Précisons ici que la conjecture 9.3.1 (iii) est immédiate pour $d = 4$, puis obtenue par Shiota [123] pour $d = 8$, et par Dixmier [44] indépendamment. Cette dernière conjecture possède d'ailleurs une autre forme plus précise [44, Conjecture 3'] :

Conjecture 9.3.2. Pour tout $\mathbf{f} \in S_{4n}$, soit $T_{\mathbf{f}}$ l'endomorphisme défini à partir de l'opération de transvection (voir [sous-section 1.3.1](#))

$$\mathbf{g} \in S_{4n-2} \mapsto (\mathbf{f}, \mathbf{g})_{2n} \in S_{4n-2}.$$

En notant $X^{4n-1} + \theta_2(\mathbf{f})X^{4n-3} + \dots + \theta_{4n-1}(\mathbf{f})$ son polynôme caractéristique, la famille $\theta_2, \dots, \theta_{4n-1}$ forme un système de paramètres homogène pour $\mathbb{C}[S_{4n}]^{\text{SL}(2,\mathbb{C})}$.

9.4 Rationalité effective et séparants génériques

Dans le cas complexe, le problème de la rationalité se résume très simplement : pour une représentation linéaire complexe (W, Γ) d'un groupe linéairement réductif, quand est-

ce que le corps des invariants $\mathbb{C}(W)^\Gamma$ est une extension purement transcendante du corps \mathbb{C} ?

Une telle question se pose évidemment pour d'autres types de corps, mais le cadre le plus classique est celui du corps des complexes. Un cas très particulier, qui reste un mystère à ce jour, concerne le corps des invariants $\mathbb{C}(\mathbb{C}^n)^{A_n}$ pour la représentation diagonale du groupe alterné A_n sur l'espace \mathbb{C}^n . L'algèbre des invariants est engendrée par les fonctions symétriques élémentaires $\sigma_1, \dots, \sigma_n$, qui forme un système de paramètres, ainsi que

$$\delta := \prod_{i < j} (x_j - x_i).$$

Le corps des invariants est donné par $\mathbb{C}(\sigma_1, \dots, \sigma_n)[\delta]$, mais on a la conjecture suivante, démontrée jusqu'à $n = 5$ (voir Maeda [92] pour ce dernier cas) :

Conjecture 9.4.1 (Noether). Pour tout entier $n \geq 1$, le corps des invariants $\mathbb{C}(\mathbb{C}^n)^{A_n}$ est une extension purement transcendante de \mathbb{C} : il existe n invariants rationnels r_1, \dots, r_n tels que $\mathbb{C}(\mathbb{C}^n)^{A_n} = \mathbb{C}(r_1, \dots, r_n)$.

Dans les cas des représentations linéaires du groupe $SL(2, \mathbb{C})$, la rationalité a été démontrée par Katsylo : d'abord pour les représentations irréductibles [77], puis pour tout type de représentations [78]. Par la suite, Maeda [93] a précisé ce résultat, le rendant plus constructif (même si Katsylo avait déjà proposé une approche assez constructive), et donnant une famille génératrice minimale de 6 invariants du corps $\mathbb{C}(S_8)^{SL(2, \mathbb{C})}$. Ces résultats ont pu être traduits sous forme tensorielle, et ils sont restés valables dans le cas du corps des réels. Nous souhaiterions généraliser ce résultat et construire des familles génératrices minimales explicites pour le corps des invariants $\mathbb{R}(\mathbb{V})^{SO(3, \mathbb{R})}$, pour tout espace de tenseurs \mathbb{V} . De telles familles donnent alors naturellement des séparants génériques, et le cas de l'espace $\mathbb{P}iez = S^2(\mathbb{R}^3) \otimes \mathbb{R}^3$ reste à traiter en mécanique des milieux continus.

9.5 Projecteurs de Clebsch–Gordan de groupes finis

Le calcul des bases d'intégrités des formes binaires s'appuie sur des opérations de *transvection* (voir [Sous-section 1.3.1](#)) qui s'interprètent en fait comme des *projecteurs de Clebsch–Gordan*. En effet, pour tout entier p, q , le *transvectant* d'indice r (qui se calcule explicitement par la formule (1.5)) est associé à un projecteur équivariant

$$\tau_r : S_p \otimes S_q \longrightarrow S_{p+q-2r}.$$

Une observation faite dès les premiers temps de la théorie classique des invariants est que les covariants de formes binaires s'obtiennent tous par opérations de transvections. C'est

par la suite Gordan [61] qui put obtenir un procédé de réduction de la famille génératrice (infinie) donnée par transvections et finalement donner un algorithme de calcul effectif de bases d'intégrité.

Notre idée est ainsi d'aborder ces questions des **projecteurs de Clebsch–Gordan** dans le cas des groupes finis, à savoir la construction effective de projecteurs équivariants

$$\pi : V_1 \otimes V_2 \longrightarrow V_3$$

pour trois représentations irréductibles données d'un groupe fini. La question serait ensuite de voir dans quelle mesure les approches de Gordan, qui ramène le calcul des invariants à des problèmes d'équations entières, ne peut pas se généraliser dans ces cas. Une première étape consisterait à étudier le cas particulier des groupes de permutations, et notamment le groupe des permutations d'ordre 4, isomorphe au sous-groupe des rotations qui préservent un cube.

Chapitre 10

Applications à la mécanique

Les *lois de comportements* (linéaires ou non linéaires) font naturellement apparaître des espaces de tenseurs, stables par l'action naturelle de $O(3, \mathbb{R})$ ou $SO(3, \mathbb{R})$. Les questions géométriques sont inhérentes à la formulation de ces lois, car le principe d'indifférence matériel impose naturellement de travailler sur des objets équivariants.

Dans le cas des lois linéaires, la notion d'*anisotropie* est traduite par la notion de *classe de symétrie* (ou *classe d'isotropie*). Par les *opérations de clips*, on est maintenant en mesure de connaître simplement les classes d'isotropie d'un espace de tenseurs donné (voir [Section 2.2](#)). Une fois cette classification effectuée, il reste néanmoins un travail à faire pour *déterminer, pour un tenseur donné, quelle est sa classe d'isotropie*, ce qui peut être fait en travaillant sur le lien entre stratification semi-algébrique et stratification isotropique (voir [Section 9.1](#)). Alors que ce travail a été effectué dans le cas de l'élasticité [105], il reste à *appliquer ces méthodes dans le cas de la piézoélectricité notamment*.

Ensuite, pour toute question de *classification des matériaux*, comme nous l'avons déjà vu dans le cas de l'élasticité, il est nécessaire de décrire des espaces d'orbites. Notons que ces questions de classifications restent à aborder dans d'autres cas que celui de l'élasticité, notamment celui de la piézoélectricité (où nous avons aussi déterminé une base d'intégrité finie dans [104]). Cela a aussi été mentionné plus haut : il reste ici à effectuer un *travail de simplification, en essayant par exemple d'exploiter la notion d'invariants rationnels*, pour que nos résultats soient effectivement exploitables.

Les questions de géométrie des espaces de tenseurs appliquées à la mécanique des milieux continus ne se limitent pas au seul cas linéaire. En effet, sous l'impulsion de Rivlin [115, 119, 120, 128] dans les années 1950, il y eut un développement important de théories non-linéaires caractérisées par l'utilisation d'invariants, par exemple dans le cas des matériaux de Mooney–Rivlin [56, 98].

Dans le cas non-linéaire isotrope, l'approche consiste à postuler l'existence d'une éner-

gie de déformation fonction des invariants du tenseur des déformations. On peut tenter de généraliser cette approche : 1) dans le cas où d'autres tenseurs interviennent dans la formulation de l'énergie (pour prendre en compte par exemple la conduction thermique ou des effets électromagnétiques,) ou 2) dans le cas de l'*anisotropie*. Dans toutes ces situations, la théorie dite des *représentations tensorielles* a été développée [12–14, 125, 147], ce qui revient à déterminer des bases d'intégrité finies d'espaces de tenseurs d'ordres inférieurs ou égaux à deux. L'utilisation de *tenseurs de structure* [15], pour modéliser des situations anisotropes, se limite seulement à certaines classes d'isotropie [art7] dans le cas où ces tenseurs sont d'ordre 2 : il serait ainsi intéressant d'*exploiter d'autres approches directement issues de la théorie des invariants*, qui consistent à déterminer une base d'intégrité associée au sous-groupe caractérisant l'*anisotropie étudiée*, puis de comparer les deux méthodes.

Enfin, les approches géométriques interviennent dans le lien existant entre *groupes de symétries et lois de conservation* [106, 107, 109]. Il s'agit d'un champ d'étude très actif, appliqué notamment aux cas non-linéaires [30, 31, 57, 94, 95], champ dans lequel mes compétences et mes connaissances en théorie des invariants, d'un point de vue effectif, pourraient intervenir de façon significative. Le Laboratoire de Mécanique et Technologie (LMT), dont je fais maintenant partie, ouvre aussi plusieurs perspectives de recherches. Pour les questions d'*identification de paramètres*, des collaborations peuvent s'envisager sur des projets dans les bio-matériaux, où des modèles anisotropes linéaires sont mis en jeu, avec des thématiques portées par Christophe Cluzel (LMT). Ensuite, la formulation géométrique de la mécanique (en théorie des coques et en théorie des plaques notamment) nécessite de travailler sur des concepts de géométrie différentielle, le but étant de proposer des lois intrinsèques et de définir proprement les objets mécaniques utilisés dans les modèles. De telles approches, mettant l'accent sur la géométrie différentielle, sont aussi initiées dans des thématiques de *réduction de modèles*, où des problématiques d'*interpolation sur des variétés de Grassmann* sont étudiées en collaboration avec Emmanuel Baranger du LMT.

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