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# Heat content asymptotics in sub-Riemannian geometry

Tommaso Rossi

► **To cite this version:**

Tommaso Rossi. Heat content asymptotics in sub-Riemannian geometry. Analysis of PDEs [math.AP]. Université Grenoble - Alpes; SISSA, Trieste, Italie, 2021. English. NNT: . tel-03372836v1

**HAL Id: tel-03372836**

**<https://hal.science/tel-03372836v1>**

Submitted on 11 Oct 2021 (v1), last revised 11 Jul 2022 (v2)

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## THÈSE

Pour obtenir le grade de

**DOCTEUR DE L'UNIVERSITÉ GRENOBLE ALPES**

**préparée dans le cadre d'une cotutelle entre  
l'Université Grenoble Alpes et SISSA**

Spécialité : **Mathématiques**

Arrêtés ministériels : 6 janvier 2005 - 25 mai 2016

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préparée au sein du **Laboratoire Institut Fourier** et de l'unité de recherche **Mathematical Analysis, Modelling, and Applications**  
dans l'École Doctorale **MSTII** et **SISSA**

# Asymptotiques en temps petit du contenu thermique en géométrie sous-riemannienne

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*A Elena e Luigi*

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## Acknowledgments

*I* would like to express my deepest appreciation to my PhD supervisors Luca Rizzi and Andrei Agrachev, for providing me with unparalleled knowledge and invaluable guidance throughout the duration of this project. It has been a challenging, yet enriching, experience which I will cherish moving on in my professional career. I would like to extend my sincere thanks also to Grégoire Charlot whose help in completing my studies cannot be overestimated.

I'm extremely grateful to all those people, friends and family, who always supported and nurtured me, even when I doubted my own work and abilities. A special mention shall go to my parents, who never wavered in their support and gave helpful insight into the academic and research world, and to my girlfriend Virginia, whose dedication and patience have been of paramount importance in my journey.

Last but not least, everything I have accomplished up to the present, and everything I will accomplish in the future, would not have been possible without the love and encouragement of my grandparents.



# Introduction

In the last decade, sub-Riemannian geometry has undergone a surge of interest, characterized by a great variety of subject matters and techniques. From a theoretical viewpoint, sub-Riemannian manifolds can be found as Gromov-Hausdorff limits of sequences of Riemannian manifolds, as underlying geometries of hypoelliptic diffusion or as principal  $G$ -bundles with a horizontal connection. In mechanics, sub-Riemannian manifolds model non-holonomic constraints providing a good description of various mechanical systems, such as rolling bodies without sliding or systems with conserved angular momentum. In addition, in mathematical modeling, sub-Riemannian geometry proves to be an extremely flexible and effective tool. For example, it has been employed in the problem of image reconstruction to restore partially damaged pictures or to describe a biological model of the vision explaining the pattern recognition in the visual cortex.

Our main interest in sub-Riemannian geometry is in connection with the asymptotic analysis of hypoelliptic diffusion phenomena. Let us firstly illustrate, in the Riemannian setting, how asymptotic analysis of partial differential equations can be a powerful tool to investigate geometrical properties. Let  $(M, g)$  be a complete Riemannian manifold, and consider the classical Cauchy problem for the heat equation on  $M$ :

$$\begin{aligned} (\partial_t - \Delta_g)u(t, x) &= 0, & \forall (t, x) \in (0, \infty) \times M, \\ u(0, \cdot) &= f, & \text{in } L^2(M, \mu_g), \end{aligned} \tag{1}$$

where  $\Delta_g, \mu_g$  denote the Laplace-Beltrami operator and the Riemannian volume respectively. It is well-known that there exists a unique solution  $u(t, x)$  to the problem (1), which can be represented as follows:

$$u(t, x) = \int_M p_t(x, y) f(y) d\mu_g(y), \quad \forall (t, x) \in (0, \infty) \times M,$$

where  $p_t(x, y)$  is a smooth positive function on  $(0, +\infty) \times M \times M$  called heat kernel, see [BGM71, Ros97] for a comprehensive introduction on the topic. In the same spirit of Riemann's definition of curvature of a Riemannian manifold, the small-time asymptotics of the heat kernel encode geometrical information of  $(M, g)$ , unraveling a deep relationship between the heat diffusion and the geometry of the underlying structure. Among numerous interesting results regarding the asymptotic analysis of quantities related to the heat diffusion, see for example [Kac66, Pat71, Cha84], we mention here the seminal result of Minakshisundaram and Pleijel [MP49]: for each  $x \in M$ , there is a complete asymptotic expansion

$$p_t(x, x) = \frac{1}{t^{n/2}} \left( a_0(x) + a_1(x)t + a_2(x)t^2 + \dots \right) \quad \text{as } t \rightarrow 0, \tag{2}$$

where  $a_j(x)$  are smooth functions on  $M$ , and are determined by the Riemannian curvature tensor and its covariant derivatives at  $x$ , therefore they reflect the local geometry of  $M$



close to  $x$ . For example, the zero-order coefficient is just a constant, but already the first-order coefficient  $a_1(x)$  is proportional to the scalar curvature of  $M$  at the point  $x$ .

From a physical viewpoint, the heat equation (1) models an isotropic diffusion on the manifold  $M$ . Hence, a natural generalization of this model can be obtained imposing *non-holonomic constraints* on the manifold, describing in turn an anisotropic diffusion phenomenon. This amounts to replace the Laplace-Beltrami operator  $\Delta_g$  in (1) with a sub-elliptic operator  $\mathcal{L}$  of Hörmander type [Hör67, Str86], generated by a set of non-holonomic constraints  $\{X_1, \dots, X_m\} \subset \Gamma(TM)$ . More precisely,  $\mathcal{L}$  is a second-order differential operator defined as the sum of squares of vector fields  $X_1, \dots, X_m$ , plus possibly a drift term  $X_0 \in \Gamma(TM)$ . Notice that in general,  $X_1, \dots, X_m$  need not be linearly independent at any point, thus the symbol of  $\mathcal{L}$  may be degenerate, making it remarkably different from  $\Delta_g$ , which is instead an elliptic operator. Nevertheless, a fundamental result obtained by Hörmander gives a sufficient condition ensuring hypoellipticity of  $\mathcal{L}$ , i.e. the *bracket-generating condition* (also known as *Hörmander condition*): the Lie algebra generated by the family  $\{X_1, \dots, X_m\}$  at any point  $x \in M$  should coincide with the whole tangent space  $T_x M$ . From a geometrical viewpoint, differently from what happened for the Laplace-Beltrami operator, the underlying geometrical structure of such an operator  $\mathcal{L}$  is no longer Riemannian. Indeed, it is possible to give a notion of length of tangent vectors but only along the admissible directions dictated by the family  $\{X_1, \dots, X_m\}$ . The resulting structure is what is called a sub-Riemannian manifold.

Sub-Riemannian manifolds form a rich class of metric spaces, generalizing the classical concept of Riemannian manifold. A sub-Riemannian structure on  $M$  is defined by a couple  $(\mathcal{D}, g)$ , where  $g$  is a smoothly varying scalar product defined on a subset of admissible directions:

$$\mathcal{D}_x = \text{span}\{X_1(x), \dots, X_m(x)\} \subset T_x M, \quad \forall x \in M,$$

which is called *distribution* (even though its rank can depend on  $x$ ) and must satisfy the bracket-generating condition, see the monographs [Mon02, ABB20]. Since  $g$  is only defined on a subset of preferred directions, not all smooth curves have finite length in a sub-Riemannian manifold. Therefore, we have to restrict ourselves to the set of *horizontal curves*, namely those curves which are tangent to the distribution and have finite length. The Chow-Rashevsky's connectivity theorem ensures that any two points (in the same connected component of  $M$ ) can be connected by a horizontal curve, thereby, the sub-Riemannian distance function  $d_{SR}$  between any two points, obtained as the infimum of the length, among all the horizontal curves joining them, is finite. The resulting metric spaces  $(M, d_{SR})$  exhibits remarkably different phenomena compared to Riemannian case: firstly, there exist points where it is not bi-Lipschitz equivalent to a Euclidean space, and as a consequence its Hausdorff dimension is strictly greater than the topological one in a neighborhood of those points. Secondly,  $(M, d_{SR})$  is a geodesic metric space as in the Riemannian case, however not all geodesics satisfy a Hamilton equation, leading to the existence of *abnormal geodesics*, which are connected to several open problems in the field. Thirdly, there is no canonical way to define an intrinsic volume associated with a sub-Riemannian manifold, in general, since  $g$  is not defined on the whole tangent space, thus the Riemannian procedure fails to produce a well-defined top-dimensional form.

Therefore, for a given hypoelliptic operator  $\mathcal{L}$  on a smooth manifold  $M$ , we consider the underlying sub-Riemannian structure and we study its diffusion: once a smooth outer measure  $\omega$  is fixed, we may consider the heat equation associated with  $\mathcal{L}$ ,

$$\begin{aligned} (\partial_t - \mathcal{L})u(t, x) &= 0, & \forall (t, x) \in (0, \infty) \times M, \\ u(0, \cdot) &= f, & \text{in } L^2(M, \omega). \end{aligned} \tag{3}$$

The hypoellipticity of  $\mathcal{L}$  ensures the existence of a heat kernel, which we still denote by  $p_t(x, y)$ , however, the interplay between the geometry of the sub-Riemannian structure

on  $M$  and the diffusion governed by  $\mathcal{L}$  is more delicate than the Riemannian case. In the first place, the diffusion is affected by the choice of the measure  $\omega$ , as  $\mathcal{L}$  may not be symmetric with respect to the  $L^2(M, \omega)$  scalar product: of particular relevance for us will be the hypoelliptic diffusion governed by a specific hypoelliptic operator, called *sub-Laplacian* and denoted by  $\Delta$ , which is indeed symmetric in  $L^2(M, \omega)$  and directly generalizes the Laplace-Beltrami operator, when  $M$  is Riemannian and  $\omega = \mu_g$ . Secondly, in the general sub-Riemannian setting, an analogue to the Riemannian curvature tensor is not available, making the problem of interest harder to solve. A definition of curvature in the sub-Riemannian setting has been recently proposed in [ABR18]: the approach used by the authors reminds Riemann's one, and so is directly related with the asymptotics of the sub-Riemannian distance. The downside of this approach is that explicit computations are much harder compared to the Riemannian case.

A great effort in the study of the analytical properties of the heat diffusion (3) has been carried out by several authors and a non-exhaustive list is [BA88, Léa92, Nee07, BM09, BBN12, Bar13, BBN19, CdVHT20]. For example, for what concerns the trace heat kernel, the existence of a complete asymptotic expansion is proved in [BA89], however the geometrical interpretation of the coefficients is still lacking. In this regard, a notable result is contained in [Bar13], where the author is able to explicitly compute the first two coefficients of the trace heat kernel asymptotic in 3D-contact sub-Riemannian manifolds:

$$p_t(x, x) = \frac{1}{16t^2} (1 + \kappa(x)t + o(t)), \quad \text{as } t \rightarrow 0, \quad \forall x \in M, \quad (4)$$

where  $\kappa(x)$  is a curvature-type invariant of the sub-Riemannian structure. On the one hand, (4) has a pole in  $t$  of order  $Q/2$ , where  $Q$  is the Hausdorff dimension of the manifold and, in the 3D-contact case, is equal to 4 and greater than the topological dimension. On the other hand, comparing (4) with the corresponding Riemannian result (2), we may expect an invariant which resembles the scalar curvature of a Riemannian manifold, thus we may *define*  $\kappa(x)$  as the scalar sub-Riemannian curvature of  $M$  at the point  $x$ . Moreover, we also mention the recent developments of [CdVHT20], concerning the existence of the small-time asymptotic expansion for hypoelliptic heat kernels, and the description of its coefficients in terms of the nilpotent approximation of the underlying sub-Riemannian structure.

Along these lines of research, our main goal is to further develop the relation between hypoelliptic heat diffusion and geometrical invariants of the sub-Riemannian structure, and we focus in particular on geometrical properties of embedded hypersurfaces. In the Riemannian setting, a relevant quantity to investigate those kind of properties is the so-called *heat content*: let  $\Omega \subset M$  be an open set with smooth boundary and consider  $u_0(t, x)$  the solution to (1), with Dirichlet boundary condition on  $\partial\Omega$ , and initial datum given by the indicator function of the set  $f = \mathbf{1}_\Omega$ . Then, the heat content  $Q_\Omega(t)$  is the total amount of heat contained in the set  $\Omega$  at time  $t$ , assuming that its boundary is perfectly insulated, i.e.

$$Q_\Omega(t) = \int_\Omega u_0(t, x) d\mu_g(x) \quad \forall t > 0. \quad (5)$$

The heat content has been thoroughly studied in the Euclidean and Riemannian setting, see [vdBLG94, vdBD89, vdB91, vdBG94, vdBG99, Sav98, Sav01, Sav04] for the classical heat content (5), see [vdBDG93, DG94] for the case where  $u_0(t, x)$  satisfies Neumann boundary conditions on  $\partial\Omega$  and [vdBG15, Sch21] for the case where boundary conditions are dropped. In the sub-Riemannian setting, Tyson and Wang recently initiated the study of heat content asymptotics in the Heisenberg group, which is a simple, yet non trivial sub-Riemannian manifold, see [TW18]. As highlighted in their work, the presence of a distinguished hypersurface in the sub-Riemannian manifold poses an additional technical

difficulty with respect to the Riemannian case: the presence of *characteristic points*. If  $M$  is sub-Riemannian manifold and  $S \subset M$  is a smooth embedded hypersurface, then a characteristic point of  $S$  is a point where horizontal curves are never transverse to the boundary and can be regarded as an intrinsic singularity of the sub-Riemannian structure. Indeed, on the one hand, the existence of smooth tubular neighborhood is not guaranteed at a characteristic point and, on the other hand, the heat diffusion loses regularity at those points. The latter property heavily affects the behavior of the diffusion, making the asymptotic analysis of quantities such as the heat content (5) difficult to study.

## Structure and contents of the thesis

In Chapter 1, we introduce the basic definitions and we review some essential facts in sub-Riemannian geometry, focusing, in particular, on geodesics in a sub-Riemannian manifold, on the definition and main properties of the sub-Laplacian operator and finally on the nilpotent approximation of a sub-Riemannian manifold. This introductory material is not exhaustive on the topic but adapted to our needs.

In Chapter 2, we study the small-time asymptotics of the heat content of smooth non-characteristic domains of a general rank-varying sub-Riemannian structure, equipped with an arbitrary smooth measure. By adapting to the sub-Riemannian case a technique due to Savo, we establish the existence of the full asymptotic series:

$$Q_{\Omega}(t) = \sum_{k=0}^{\infty} a_k t^{k/2}, \quad \text{as } t \rightarrow 0. \quad (6)$$

As a particular case we recover, using non-probabilistic techniques, the order 2 formula obtained by Tyson and Wang in the Heisenberg group [TW18]. We proceed by analyzing the coefficients appearing in the asymptotics (6). Firstly, presenting the sub-Riemannian manifold  $M$  as a Gromov-Hausdorff limit of a sequence of Riemannian manifolds, we prove that every coefficient can be obtained as the limit of the corresponding Riemannian one. Secondly, we compute explicitly the coefficients up to order  $k = 5$ , in terms of sub-Riemannian invariants of the domain. A consequence of our fifth-order analysis is the evidence of new phenomena in presence of characteristic points. In particular, we prove that the higher-order coefficients in the asymptotics (6) can blow-up in their presence. A key tool for this last result is an exact formula for the distance from a specific surface with an isolated characteristic point in the Heisenberg group, which is of independent interest.

In Chapter 3, we further study characteristic points in the Heisenberg group. In particular, we address the problem of integrability of the sub-Riemannian mean curvature  $\mathcal{H}$  of an embedded hypersurface, around isolated characteristic points, in this setting. The sub-Riemannian mean curvature is a geometrical invariant appearing in the asymptotic expansion of the heat content and its integrability (with respect to the perimeter measure induced by the Euclidean structure) is relevant in many different areas of geometric analysis. The main novelty is the introduction of a concept of a mildly degenerate characteristic point for a smooth surface of the Heisenberg group. This new concept can be regarded as a second-order condition for a characteristic point and it ensures the local integrability of  $\mathcal{H}$ . As a consequence we partially answer to a question posed by Danielli-Garofalo-Nhieu in [DGN12], proving that the mean curvature of a real-analytic surface with discrete characteristic set is locally integrable.

In Chapter 4, we study the small-time asymptotics of the relative heat content associated with smooth non-characteristic domains of a general rank-varying sub-Riemannian

structure, equipped with an arbitrary smooth measure. The relative heat content associated with  $\Omega \subset M$ , denoted by  $H_\Omega(t)$ , is the total amount of heat contained in the set at time  $t$ , assuming that its boundary is no longer insulated, meaning that the heat can flow also outside of  $\Omega$ . In particular, under the assumption of not having characteristic points, we prove the existence of the asymptotic expansion of  $H_\Omega(t)$ , up to order 4 in  $\sqrt{t}$ , as  $t \rightarrow 0$ . The strategy to obtain this result is along the lines of what we did in Chapter 2, however, new technical difficulties arise, the main one being related to the fact that the heat can flow through  $\partial\Omega$  and, as a consequence, we do not know the small-time behavior of the temperature function  $u(t, x)$  at  $\partial\Omega$ . At order zero, using symmetry properties of the diffusion, we are able to describe the behavior of  $u(t, x)$ , which is, in a sense, averaging the heat between  $\Omega$  and its complement. Finally, we remark that the relative heat content asymptotic expansion, although being similar to the one of the classical heat content, is unrelated as showed by comparing the higher-order coefficients.

In Chapter 5, we prove the existence of a tubular neighborhood for submanifolds of any codimension, assuming they do not have characteristic points. The existence of tubular neighborhood for hypersurfaces has been established in [PRS18, FPR20] and has proved to be a crucial tool for developing the results of Chapter 2 and 4. Thus, we generalize this result to any codimension. Next, we propose a definition of relative heat content for submanifolds of codimension  $k \geq 1$  and we build a possible approximation of this quantity, via tubular neighborhoods. For any step of the approximation, the resulting object is the relative heat content associated with an open, smooth and non-characteristic subset of  $M$ , for which we have an asymptotic expansion. However, we show that this approximation fails to recover the asymptotic expansion of the relative heat content of the submanifold, by comparing the two approaches in an example, in which it is possible to carry out explicit computations.

Each chapter starts with a self-contained introduction, summarizing the results contained therein. A summary in French of the thesis can be found in the [résumé](#).

## List of Publications

The research presented in this thesis appears in the following publications and preprints in preparation:

- Rizzi, L., Rossi, T., *Heat content asymptotics for sub-Riemannian manifolds*, Journal de Mathématiques Pures et Appliquées, v. 148, 2021. See [RR21].
- Rossi, T., *Integrability of the sub-Riemannian mean curvature at degenerate characteristic points in the Heisenberg group*, Advances in Calculus of Variations, 2021. See [Ros21].
- Rossi T., *Relative heat content asymptotics for sub-Riemannian manifolds*, in preparation.



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# 1

## Preliminaries of sub-Riemannian geometry

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*In this chapter, we introduce the basic definitions and we review some essential facts in sub-Riemannian geometry. This introductory material is not exhaustive on the topic but tailored to our presentation.*

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### 1.1 Sub-Riemannian geometry

Let  $M$  be a smooth, connected finite-dimensional manifold. A sub-Riemannian structure on  $M$  is defined by a set of  $N$  global smooth vector fields  $X_1, \dots, X_N$ , called a *generating frame*. The generating frame defines a *distribution* of subspaces of the tangent spaces at each point  $x \in M$ , given by

$$\mathcal{D}_x = \text{span}\{X_1(x), \dots, X_N(x)\} \subseteq T_x M. \quad (1.1)$$

We assume that the distribution is *bracket-generating*, namely the Lie algebra of smooth vector fields generated by  $X_1, \dots, X_N$ , evaluated at the point  $x$ , coincides with  $T_x M$ , for all  $x \in M$ . Recall that, by definition, the Lie algebra generated by a family of vector fields  $\{X_1, \dots, X_N\}$  is the smallest sub-algebra of  $\Gamma(TM)$  containing the family itself, i.e.

$$\text{Lie}\{X_1, \dots, X_N\} = \{[X_{j_1}, [\dots [X_{j_{k-1}}, X_{j_k}]]] \mid j_i \in \{1, \dots, N\}, k \in \mathbb{N}\}.$$

Thus, the distribution (1.1) satisfies the bracket-generating condition if

$$\text{Lie}_x(\mathcal{D}) = \{X(x) \mid X \in \text{Lie}(\mathcal{D})\} = T_x M, \quad \forall x \in M. \quad (1.2)$$

We sometimes refer to (1.2) as *Hörmander condition*. The generating frame induces also a norm on the distribution at  $x$ , namely

$$g_x(v, v) = \inf \left\{ \sum_{i=1}^N u_i^2 \mid \sum_{i=1}^N u_i X_i(x) = v \right\}, \quad \forall v \in \mathcal{D}_x, \quad (1.3)$$

which, in turn, defines an inner product on  $\mathcal{D}_x$  by polarization. We use the shorthand  $\|\cdot\|_g$  for the corresponding norm. Notice that the vector fields in the generating frame (1.1) need not be linearly independent at any point, moreover, we can easily observe that if  $\mathcal{D}_x = T_x M$ , for all  $x \in M$ , then  $g$  defined in (1.3) is a Riemannian metric on  $M$ . We say that  $\gamma : [0, T] \rightarrow M$  is a *horizontal curve*, if it is absolutely continuous and

$$\dot{\gamma}(t) \in \mathcal{D}_{\gamma(t)}, \quad \text{for a.e. } t \in [0, T].$$

This implies that there exists a *control function*  $u : [0, T] \rightarrow \mathbb{R}^N$ , such that

$$\dot{\gamma}(t) = \sum_{i=1}^N u_i(t) X_i(\gamma(t)), \quad \text{for a.e. } t \in [0, T].$$

Moreover, we require that  $u \in L^2([0, T], \mathbb{R}^N)$ . We report [ABB20, Lem. 3.12], which ensures the integrability of the map  $t \mapsto \|\dot{\gamma}(t)\|_g$ .

**Lemma 1.1.1.** *Let  $M$  be a sub-Riemannian manifold, and let  $\gamma : [0, T] \rightarrow M$  be a horizontal curve. Define the minimal control of  $\gamma$  as*

$$u^*(t)^2 = \operatorname{argmin} \left\{ \sum_{i=1}^N u_i(t)^2 \mid \dot{\gamma}(t) = \sum_{i=1}^N u_i(t) X_i(\gamma(t)) \right\}, \quad \text{for a.e. } t \in [0, T]. \quad (1.4)$$

Then, the function  $[0, T] \ni t \mapsto u^*(t)$  is measurable and square integrable on  $[0, T]$ .

Notice that, by definition of minimal control,  $\|\dot{\gamma}(t)\|_g = |u^*(t)|$  for a.e.  $t \in [0, T]$ , therefore the map  $t \mapsto \|\dot{\gamma}(t)\|_g$  is integrable on  $[0, T]$  and we can define the *length* of a horizontal curve as follows:

$$\ell(\gamma) = \int_0^T \|\dot{\gamma}(t)\|_g dt.$$

Then, *sub-Riemannian distance* is defined by length-minimization procedure: for any  $x, y \in M$ , we set

$$d_{\text{SR}}(x, y) = \inf \{ \ell(\gamma) \mid \gamma : [0, T] \rightarrow M, \text{ horizontal curve joining } x \text{ and } y \}.$$

By Chow-Rashevsky's Theorem, the bracket-generating assumption ensures that the distance  $d_{\text{SR}} : M \times M \rightarrow \mathbb{R}$  is finite, continuous and it induces the same topology as the manifold one. Furthermore, this implies that the resulting metric space  $(M, d_{\text{SR}})$  is locally compact.

*Remark 1.1.2.* The above definition includes all classical constant-rank sub-Riemannian structures as in [Mon02, Rif14] (where  $\mathcal{D}$  is a vector distribution and  $g$  a symmetric and positive tensor on  $\mathcal{D}$ ), but also general rank-varying sub-Riemannian structures. Moreover, the same sub-Riemannian structure can arise from different generating families.

## 1.2 Geodesics and Hamiltonian flow

We recall some basic facts about length-minimizing curves, and the Hamiltonian formalism.

A *geodesic* is a horizontal curve  $\gamma: [0, T] \rightarrow M$ , parametrized with constant speed, and such that any sufficiently short segment is length-minimizing. We can formulate the problem of finding a length-minimizer joining  $x, y \in M$  as an optimal control problem of the form:

$$\begin{cases} \dot{\gamma}(t) = \sum_{i=1}^N u_i(t) X_i(\gamma(t)), \\ \gamma(0) = x, \quad \gamma(T) = y, \\ \int_0^T \sqrt{\sum u_i^2(t)} dt \rightarrow \min. \end{cases} \quad (1.5)$$

Problem (1.5) involves finding the local minima of a function subject to some constraints, hence we can employ the method of Lagrangian multipliers.

**Example 1.2.1.** Let  $f, f_0: \mathbb{R}^2 \rightarrow \mathbb{R}$  be  $C^1$  functions and consider the constrained optimization problem

$$\begin{cases} f_0(x, y) \rightarrow \min, \\ f(x, y) = 0. \end{cases}$$

Applying the method of Lagrangian multipliers, we introduce the function  $\Lambda = \lambda f + \lambda_0 f_0$ , then the extremal points are obtained solving the equation

$$\nabla_{(x,y),\lambda} \Lambda = 0. \quad (1.6)$$

Therefore, we can distinguish two cases:

- (i)  $\lambda_0 = 0$ , then (1.6) reduces to  $\lambda \nabla f = 0$ , with  $\lambda \neq 0$ , which implies  $\nabla f = 0$ . These are *abnormal* extremals;
- (ii)  $\lambda_0 \neq 0$ , then we can assume  $\lambda_0 = -1$ , hence (1.6) becomes  $\lambda \nabla f - \nabla f_0 = 0$ . These are *normal* extremals.

This kind of difference between the extremals of a constrained optimization problem is typical and appears also when studying an infinite-dimensional problem, such as finding the geodesics joining two fixed points. The characterization of the extremals of Problem (1.5) is given by the so-called *Pontryagin maximum principle*, cf. [ABB20, Sec. 4.3].

Given a sub-Riemannian structure on  $M$  with generating frame  $\{X_1, \dots, X_N\}$ , define the fiber-wise linear functions on  $T^*M$  associated with these vector fields

$$h_i: T^*M \rightarrow \mathbb{R}; \quad h_i(\lambda) = \langle \lambda, X_i \rangle, \quad \forall i = 1, \dots, N,$$

where  $\langle \lambda, \cdot \rangle$  denotes the action of covectors on vectors. For a smooth function  $h \in C^\infty(T^*M)$ , the *Hamiltonian vector field* associated with  $h$  is the vector field  $\vec{h}$  on  $T^*M$ , defined by  $\zeta(\cdot, \vec{h}) = dh$ , where  $\zeta \in \Lambda^2(T^*M)$  is the canonical symplectic 2-form on the cotangent bundle.

**Theorem 1.2.2** (Pontryagin maximum principle). *Let  $M$  be a sub-Riemannian manifold and let  $\gamma: [0, T] \rightarrow M$  be a horizontal curve parametrized by constant speed, which is a length-minimizer and  $u^*(t)$  the corresponding minimal control. Then there exists a lift of  $\gamma$  in  $T^*M$ , i.e. an absolutely continuous curve  $\lambda(t) \in T_{\gamma(t)}^*M$ , such that,*

$$\dot{\lambda}(t) = \sum_{i=1}^N u_i^*(t) \vec{h}_i(\lambda(t)), \quad \text{for a.e. } t \in [0, T], \quad (1.7)$$

and one of the following condition holds:

$$h_i(\lambda(t)) = u_i^*(t), \quad \forall t \in [0, T] \text{ and for any } i = 1, \dots, N, \text{ or} \quad (\mathbf{N})$$

$$h_i(\lambda(t)) = 0, \quad \forall t \in [0, T] \text{ and for any } i = 1, \dots, N. \quad (\mathbf{A})$$

Moreover in case  $(\mathbf{A})$ ,  $\lambda(t) \neq 0$ , for any  $t \in [0, T]$ .

Among the solutions to (1.7), we distinguish between *normal extremals*, satisfying  $(\mathbf{N})$ , and *abnormal extremals*, satisfying  $(\mathbf{A})$ . In particular, normal extremals are solution to a true Hamiltonian system: define the *sub-Riemannian Hamiltonian* as the smooth function  $H : T^*M \rightarrow \mathbb{R}$ , given by

$$H(\lambda) = \frac{1}{2} \sum_{i=1}^N \langle \lambda, X_i \rangle^2, \quad \lambda \in T^*M, \quad (1.8)$$

where  $\{X_1, \dots, X_N\}$  is a generating family for the sub-Riemannian structure. Then, solutions  $\lambda : [0, T] \rightarrow T^*M$  of the *Hamilton equations*

$$\dot{\lambda}(t) = \vec{H}(\lambda(t)), \quad (1.9)$$

are normal extremals in the sense of Theorem 1.2.2. As a consequence, normal extremals are smooth and the Hamiltonian is constant along the motion. Their projections  $\gamma(t) = \pi(\lambda(t))$  on  $M$ , where  $\pi : T^*M \rightarrow M$  is the bundle projection, are locally length-minimizing horizontal curves parametrized with constant speed, and are called *normal geodesics*. If  $\gamma$  is a normal geodesic with normal extremal lift  $\lambda$ , then its speed is given by  $\|\dot{\gamma}\|_g = \sqrt{2H(\lambda)}$ . In particular

$$\ell(\gamma|_{[0,t]}) = t\sqrt{2H(\lambda(0))}, \quad \forall t \in [0, T].$$

The *sub-Riemannian exponential map*  $\exp_x : T_x^*M \rightarrow M$ , with base  $x \in M$  is

$$\exp_x(\lambda) = \pi \circ e^{\vec{H}}(\lambda), \quad \lambda \in T_x^*M, \quad (1.10)$$

where  $e^{\vec{H}}$  denotes the flow of  $\vec{H}$  in the cotangent bundle, and we can assume it is defined up to time 1. This is the case, for example, when  $(M, d_{\text{SR}})$  is a complete metric space. We remark that the sub-Riemannian exponential map (1.10) is a local diffeomorphism at  $\lambda = 0$  if and only if  $\mathcal{D}_x = T_xM$ , thus in a truly sub-Riemannian manifold we will always find points in which the former property fails.

For what concerns abnormal extremals  $\lambda : [0, T] \rightarrow T^*M$ , satisfying (1.7), together with condition  $(\mathbf{A})$ , we define *abnormal geodesics* their projections  $\gamma(t) = \pi(\lambda(t))$  on  $M$ . Differently from normal extremals, the curve  $\lambda(t)$  may not follow the Hamiltonian dynamics (1.9), hence we can not deduce any regularity property. Here we only observe that condition  $(\mathbf{A})$  for an abnormal extremal  $\lambda(t) \in T^*M$  is equivalent to

$$\langle \lambda(t), \mathcal{D}_{\pi(\lambda(t))} \rangle = 0 \quad \text{and} \quad \lambda(t) \neq 0, \quad \forall t \in [0, T],$$

that is  $H(\lambda(t)) \equiv 0$ . Notice that a geodesic may be abnormal and normal at the same time.

**Length-minimizers to the boundary.** Consider now a closed embedded submanifold  $S \subset M$  of positive codimension. Let  $\gamma : [0, T] \rightarrow M$  be a horizontal curve, parametrized with constant speed, such that  $\gamma(0) \in S$ ,  $\gamma(T) = x \in M \setminus S$ , and such that it minimizes the distance to  $S$ , that is

$$\ell(\gamma) = \inf\{d_{\text{SR}}(z, x) \mid z \in S\}.$$

In particular,  $\gamma$  is a geodesic. Any corresponding normal or abnormal lift, say  $\lambda : [0, T] \rightarrow T^*M$ , must satisfy the transversality conditions [AS04, Thm. 12.4]

$$\langle \lambda(0), v \rangle = 0, \quad \forall v \in T_{\gamma(0)}S, \quad (1.11)$$

in other words, the initial covector  $\lambda(0)$  must belong to the annihilator bundle  $\mathcal{A}(S) = \{\lambda \in T^*M \mid \langle \lambda, T_{\pi(\lambda)}S \rangle = 0\}$  of  $S$ . An immediate consequence for the case of codimension 1 is the following, cf. [FPR20, Prop. 2.7].

**Proposition 1.2.3.** *Consider a sub-Riemannian structure on a smooth manifold  $M$ . Let  $S \subset M$  be a closed embedded hypersurface. Let  $\gamma : [0, T] \rightarrow M$  be a horizontal curve such that  $\gamma(0) \in S$ ,  $\gamma(T) = p \in M \setminus S$ , and that minimizes the distance to  $S$ . Then  $\gamma$  is abnormal if and only if  $\gamma(0) \in S$  is a characteristic point, i.e.*

$$\mathcal{D}_{\gamma(0)} \subset T_{\gamma(0)}S.$$

In particular, Proposition 1.2.3 implies that as soon as  $S$  has no characteristic points, curves which are length-minimizing to  $S$  are all normal, that is  $\gamma(t) = \exp_x(t\lambda)$ , for some unique (up to reparametrization)  $\lambda$  satisfying (1.11).

### 1.3 The sub-Laplacian

Let  $M$  be a sub-Riemannian manifold and let  $\omega$  be a smooth measure on  $M$ , defined by a positive tensor density. The *divergence* of a smooth vector field is defined by

$$\operatorname{div}_\omega(X)\omega = \mathcal{L}_X\omega, \quad \forall X \in \Gamma(TM),$$

where  $\mathcal{L}_X$  denotes the Lie derivative in the direction of  $X$ . The *horizontal gradient* of a function  $f \in C^\infty(M)$ , denoted by  $\nabla f$ , is defined as the horizontal vector field (i.e. tangent to the distribution at each point), such that

$$g_x(\nabla f(x), v) = v(f)(x), \quad \forall v \in \mathcal{D}_x, \quad (1.12)$$

where  $v$  acts as a derivation on  $f$ . In terms of a generating frame as in (1.1), one has

$$\nabla f = \sum_{i=1}^N X_i(f)X_i, \quad \forall f \in C^\infty(M).$$

We recall the divergence theorem (we stress that  $M$  is not required to be orientable): let  $\Omega \subset M$  be an open subset with smooth boundary, then

$$\int_{\partial\Omega} fg(X, \nu) d\sigma = \int_{\Omega} (f \operatorname{div}_\omega X + g(\nabla f, X)) d\omega, \quad (1.13)$$

for any smooth function  $f$  and smooth vector field  $X$ , such that the vector field  $fX$  is compactly supported. In (1.13),  $\nu$  is the outward-pointing normal vector field to  $\partial\Omega$  and  $\sigma$  is the induced sub-Riemannian measure on  $\partial\Omega$  (i.e. the one whose density is  $\sigma = |i_\nu\omega|_{\partial\Omega}$ ).

The *sub-Laplacian* is the operator  $\Delta = \operatorname{div}_\omega \circ \nabla$ , acting on  $C^\infty(M)$ . Again, we may write its expression with respect to a generating frame (1.1), obtaining

$$\Delta f = \sum_{i=1}^N \left\{ X_i^2(f) + X_i(f) \operatorname{div}_\omega(X_i) \right\}, \quad \forall f \in C^\infty(M). \quad (1.14)$$

If the distribution  $\mathcal{D}$ , defining the sub-Riemannian structure on  $M$ , does not coincide with the whole tangent space, the sub-Laplacian is not an elliptic operator, thus the classical regularity theory is not available. Nevertheless,  $\Delta$  is hypoelliptic, cf. [Hör67, Str86].

**Theorem 1.3.1.** *Let  $M$  be a smooth manifold and let  $\{Y_0, \dots, Y_N\}$  be a set of bracket-generating vector fields on  $M$ . Then the operator*

$$L = Y_0 + \sum_{i=1}^N Y_i^2$$

*is hypoelliptic, which means that if  $\varphi$  is a distribution defined on an open set  $\Omega \subset M$ , such that  $L\varphi \in C^\infty(\Omega)$ , then  $\varphi \in C^\infty(\Omega)$ .*

In a sub-Riemannian manifold  $M$ , with generating frame (1.1) satisfying the bracket-generating assumption, we immediately observe that the sub-Laplacian is hypoelliptic by (1.14). Moreover, Theorem 1.3.1 also ensures that the heat operator is hypoelliptic, indeed

$$L = \partial_t - \Delta = \partial_t - \sum_{i=1}^N \operatorname{div}_\omega(X_i)X_i - \sum_{i=1}^N X_i^2, \quad (1.15)$$

so choosing  $Y_0 = \partial_t - \sum_{i=1}^N \operatorname{div}_\omega(X_i)X_i$  and  $Y_i = X_i$ , for any  $i = 1, \dots, N$ , we conclude. In particular, a solution to heat equation

$$L\varphi = (\partial_t - \Delta)\varphi = 0, \quad \text{in the sense of distributions on } \Omega$$

is a smooth function in  $\Omega$ , for any open subset  $\Omega \subset M$ .

**The Cauchy problem for the heat equation.** Let  $M$  be a sub-Riemannian manifold, equipped with a smooth measure  $\omega$ . We denote by  $L^2(M, \omega)$  the space of real functions on  $M$  which are square-integrable with respect to the measure  $\omega$ . Let  $f \in L^2(M, \omega)$  and consider the *Cauchy problem for the heat equation*:

$$\begin{aligned} (\partial_t - \Delta)u(t, x) &= 0, & \forall (t, x) \in (0, \infty) \times M, \\ u(0, \cdot) &= f, & \text{in } L^2(M, \omega), \end{aligned} \quad (1.16)$$

where  $u(0, \cdot)$  is a shorthand notation for the  $L^2$ -limit of  $u(t, x)$  as  $t \rightarrow 0$  and  $\Delta$  is the sub-Laplacian (1.14). Notice that  $\Delta$  is symmetric with respect to the  $L^2$ -scalar product and negative, moreover, if  $(M, d_{\text{SR}})$  is complete as a metric space, it is essentially self-adjoint, see [Str86]. Thus, there exists a unique solution to (1.16), and it can be represented as

$$u(t, x) = e^{t\Delta}f(x), \quad \forall x \in M, t > 0,$$

where  $e^{t\Delta}: L^2(M, \omega) \rightarrow L^2(M, \omega)$  denotes the heat semi-group, associated with  $\Delta$ . We remark that for all  $f \in L^2(M, \omega)$ , the function  $e^{t\Delta}f$  is smooth for all  $(t, x) \in (0, \infty) \times M$ , by hypoellipticity of the heat operator, see Theorem 1.3.1, and there exists a heat kernel associated with (1.16), i.e. a positive function  $p_t(x, y) \in C^\infty((0, +\infty) \times M \times M)$ , such that:

$$u(t, x) = \int_M p_t(x, y)f(y)d\omega(y), \quad \forall x \in M, t > 0. \quad (1.17)$$

## 1.4 Nilpotent approximation of $M$

We introduce the notion of nilpotent approximation of a sub-Riemannian manifold, see [Jea14, Bel96] for details. This will be used in Chapter 4.

**Sub-Riemannian flag.** Let  $M$  be a  $n$ -dimensional sub-Riemannian manifold with distribution  $\mathcal{D}$ . We define the *flag* of  $\mathcal{D}$  as the sequence of subsheafs  $\mathcal{D}^k \subset TM$  such that

$$\mathcal{D}^1 = \mathcal{D}, \quad \mathcal{D}^{k+1} = \mathcal{D}^k + [\mathcal{D}, \mathcal{D}^k], \quad \forall k \geq 1,$$

with the convention that  $\mathcal{D}^0 = \{0\}$ . Under the Hörmander condition, the flag of the distribution defines an exhaustion of  $T_x M$ , for any point  $x \in M$ , i.e. there exists  $r(x) \in \mathbb{N}$  such that:

$$\{0\} = \mathcal{D}_x^0 \subset \mathcal{D}_x^1 \subset \dots \subset \mathcal{D}_x^{r(x)-1} \subsetneq \mathcal{D}_x^{r(x)} = T_x M. \quad (1.18)$$

The number  $r(x)$  is called *degree of nonholonomy* at  $x$ . We set  $n_k(x) = \dim \mathcal{D}_x^k$ , for any  $k \geq 0$ , then the collection of  $r(x)$  integers

$$(n_1(x), \dots, n_{r(x)}(x))$$

is called *growth vector* at  $x$ , and we have  $n_{r(x)}(x) = n = \dim M$ . Associated with the growth vector, we can define the *sub-Riemannian weights*  $w_i(x)$  at  $x$ , setting for any  $i \in \{1, \dots, n\}$ ,

$$w_i(x) = j, \quad \text{if and only if} \quad n_{j-1}(x) + 1 \leq i \leq n_j(x). \quad (1.19)$$

A point  $x \in M$  is said to be *regular* if the growth vector is constant in a neighborhood of  $x$ , and *singular* otherwise. The sub-Riemannian structure on  $M$  is said to be *equiregular* if all points of  $M$  are regular. In this case, the weights are constant as well on  $M$ . Finally, given any  $x \in M$ , we define the *homogeneous dimension* of  $M$  at  $x$  as

$$\mathcal{Q}(x) = \sum_{i=1}^{r(x)} i(n_i(x) - n_{i-1}(x)) = \sum_{i=1}^n w_i(x). \quad (1.20)$$

We recall that, if  $x$  is regular, then  $\mathcal{Q}(x)$  coincides with the Hausdorff dimension of  $(M, d_{\text{SR}})$  at  $x$ , cf. [Mit85]. Moreover,  $\mathcal{Q}(x) > n$ , for any  $x \in M$  such that  $\mathcal{D}_x \subsetneq T_x M$ .

**Privileged coordinates.** Let  $M$  be a sub-Riemannian manifold with generating frame (1.1) and  $f$  be the germ of a smooth function  $f$  at  $x \in M$ . We call a *nonholonomic derivative* of order  $k \in \mathbb{N}$  of  $f$ , the quantity

$$X_{j_1} \dots X_{j_k} f(x),$$

for any family of indexes  $\{j_1, \dots, j_k\} \subset \{1, \dots, N\}$ . Then, the *nonholonomic order* of  $f$  at the point  $x$  is

$$\text{ord}_x(f) = \min \{k \in \mathbb{N} \mid \exists \{j_1, \dots, j_k\} \subset \{1, \dots, N\} \text{ s.t. } X_{j_1} \dots X_{j_k} f(x) \neq 0\}.$$

**Definition 1.4.1** (Privileged coordinates). Let  $M$  be a  $n$ -dimensional sub-Riemannian manifold and  $x \in M$ . A system of local coordinates  $(z_1, \dots, z_n)$  centered at  $x$  is said to be *privileged* at  $x$  if

$$\text{ord}_x(z_j) = w_j, \quad \forall j = 1, \dots, n.$$

Notice that privileged coordinates  $(z_1, \dots, z_n)$  at  $x$  satisfy the following property

$$\partial_{z_i}|_x \in \mathcal{D}_x^{w_i}, \quad \partial_{z_i}|_x \notin \mathcal{D}_x^{w_i-1}, \quad \forall i = 1, \dots, n. \quad (1.21)$$

A local frame of  $TM$  consisting of  $n$  vector fields  $\{Z_1, \dots, Z_n\}$  and satisfying (1.21) is said to be *adapted* to the flag (1.18) at  $x$ . Thus, privileged coordinates are always adapted to the flag. In addition, given a local frame adapted to the sub-Riemannian flag at  $x$ , say  $\{Z_1, \dots, Z_n\}$ , we can define a set of privileged coordinates at  $x$ , starting from  $\{Z_1, \dots, Z_n\}$ , i.e.

$$\mathbb{R}^n \ni (z_1, \dots, z_n) \mapsto e^{z_1 Z_1} \circ \dots \circ e^{z_n Z_n}(x). \quad (1.22)$$

Moreover, in these coordinates, the vector field  $Z_1$  is exactly  $\partial_{z_1}$ .

**Nilpotent approximation.** Let  $M$  be a sub-Riemannian manifold and let  $x \in M$  with weights as in (1.19). Consider  $\psi = (z_1, \dots, z_n): U \rightarrow V$  a chart of privileged coordinates at  $x$ , where  $U \subset M$  is a relatively compact neighborhood of  $x$  and  $V \subset \mathbb{R}^n$  is a neighborhood of 0. Then, for any  $\varepsilon \in \mathbb{R}$ , we can define the *dilation* at  $x$  as

$$\delta_\varepsilon: \mathbb{R}^n \rightarrow \mathbb{R}^n; \quad \delta_\varepsilon(z) = (\varepsilon^{w_1(x)} z_1, \dots, \varepsilon^{w_n(x)} z_n). \quad (1.23)$$

Using such dilations, we obtain the nilpotent (or first-order) approximation of the generating frame (1.1). Following the construction of [CdVHT20], let  $W_1, W_2 \subset \mathbb{R}^n$  open and relatively compact subsets of  $\mathbb{R}^n$  such that  $\overline{W}_1 \subset W_2 \subset \overline{W}_2 \subset V$  and  $\delta_\varepsilon(W_i) \subset W_i$ , for  $i = 1, 2$  and for every  $|\varepsilon| \leq 1$ . Consider

$$\chi \in C_c^\infty(\mathbb{R}^n), \quad \text{such that } \chi|_{W_1} \equiv 1 \text{ and } \text{supp}(\chi) \subset W_2, \quad (1.24)$$

and set  $Y_i = \chi(\psi_* X_i)$ , for any  $i = 1, \dots, N$ , where  $X_i$ 's are the vector fields of (1.1). Define the first-order approximation of  $X_i$  as

$$\widehat{X}_i^x = \lim_{\varepsilon \rightarrow 0} \varepsilon \delta_{\frac{1}{\varepsilon}*}(Y_i), \quad \forall i = 1, \dots, N, \quad (1.25)$$

where the limit is taken in the  $C^\infty$ -topology of  $\mathbb{R}^n$ .

**Theorem 1.4.2.** *Let  $M$  be a  $n$ -dimensional sub-Riemannian manifold with generating frame  $\{X_1, \dots, X_N\}$  and consider its first-order approximation at  $x$  as in (1.25). Then, the frame  $\{\widehat{X}_1^x, \dots, \widehat{X}_N^x\}$  of vector fields on  $\mathbb{R}^n$  generates a nilpotent Lie algebra of step  $r(x) = w_n(x)$  and satisfies the Hörmander condition.*

The proof of this theorem can be found in [Jea14]. Recall that a Lie algebra is said to be nilpotent of step  $s$  if  $s$  is the smallest integer such that all the brackets of length greater than  $s$  are zero.

**Definition 1.4.3** (Nilpotent approximation). Let  $M$  be a sub-Riemannian manifold and let  $x \in M$ . Then, Theorem 1.4.2 implies that the frame  $\{\widehat{X}_1^x, \dots, \widehat{X}_N^x\}$  is a generating frame for a sub-Riemannian structure on  $\mathbb{R}^n$ : we denote the resulting sub-Riemannian manifold  $\widehat{M}^x$ . This is the so-called *nilpotent approximation* of  $M$  at the point  $x$ .

Notice that the sub-Riemannian distance of  $\widehat{M}^x$ , denoted by  $\widehat{d}^x$ , is 1-homogeneous with respect to the dilations (1.23).

*Remark 1.4.4.* Up to isometries, the nilpotent approximation of  $M$  at  $x$  coincides with the Gromov-Hausdorff metric tangent space of  $(M, d_{\text{SR}})$  at  $x$ . Moreover,  $\widehat{M}^x$  is isometric to a quotient of a Carnot group. See [Gro96, Bel96, Mon02] for further details.

**Nilpotentized sub-Laplacian.** Let  $M$  be a sub-Riemannian manifold, equipped with a smooth measure  $\omega$ , and let  $\psi = (z_1, \dots, z_n)$  be a set of privileged coordinates at  $x \in M$ . Consider a cutoff function  $\chi \in C_c^\infty(\mathbb{R}^n)$  as in (1.24), and define a smooth measure  $\mu$  on  $\mathbb{R}^n$

$$\langle \mu, f \rangle = \langle \omega, \chi f \rangle, \quad \forall f \in C^0(\mathbb{R}^n). \quad (1.26)$$

The *nilpotentization*  $\widehat{\omega}^x$  of  $\omega$  at  $x$  is defined as follows:

$$\langle \widehat{\omega}^x, f \rangle = \lim_{\varepsilon \rightarrow 0} \frac{1}{|\varepsilon|^{\mathcal{Q}(x)}} \langle \delta_\varepsilon^* \mu, f \rangle, \quad \forall f \in C_c^\infty(\mathbb{R}^n). \quad (1.27)$$

Notice that, denoting by  $dz = dz_1 \dots dz_n$  the Lebesgue measure on  $\mathbb{R}^n$ , we have

$$\delta_\varepsilon^*(dz) = |\varepsilon|^{\mathcal{Q}(x)} dz, \quad \forall \varepsilon \neq 0,$$



thus, the limit in (1.27) exists. Finally, we can define the *nilpotentized sub-Laplacian* according to (1.14), acting on  $C^\infty(\mathbb{R}^n)$ , i.e.

$$\widehat{\Delta}^x = \operatorname{div}_{\widehat{\omega}^x}(\widehat{\nabla}^x) = \sum_{i=1}^N (\widehat{X}_i^x)^2. \quad (1.28)$$

We remark that in (1.28) there is no divergence term, since

$$\operatorname{div}_{\widehat{\omega}^x}(\widehat{X}_i^x) = 0 \quad \forall i \in \{1, \dots, N\}.$$

In the nilpotent approximation  $\widehat{M}^x$ , we may consider the Cauchy heat problem (1.16) in  $L^2(\mathbb{R}^n, \widehat{\omega}^x)$ :

$$\begin{aligned} (\partial_t - \widehat{\Delta}^x)u(t, z) &= 0, & \forall (t, z) \in (0, \infty) \times \mathbb{R}^n, \\ u(0, \cdot) &= f, & \text{in } L^2(\mathbb{R}^n, \widehat{\omega}^x). \end{aligned}$$

We will denote the associated heat kernel as

$$\widehat{p}_t^x(z, z') \in C^\infty((0, +\infty) \times \mathbb{R}^n \times \mathbb{R}^n). \quad (1.29)$$

**Heat kernel asymptotics.** Let  $M$  be a sub-Riemannian manifold, equipped with a smooth measure  $\omega$  and denote by  $p_t(x, y)$  the heat kernel (1.17). We have the following result.

**Theorem 1.4.5** ([CdVHT20, Thm. A]). *Let  $M$  be a sub-Riemannian manifold and let  $\psi: U \rightarrow V$  be a chart of privileged coordinates at  $x \in M$ . Then, for any  $m \in \mathbb{N}$ ,*

$$|\varepsilon|^{\mathcal{Q}(x)} p_{\varepsilon^2 \tau}(\delta_\varepsilon(z), \delta_\varepsilon(z')) = \widehat{p}_\tau^x(z, z') + \sum_{i=1}^m \varepsilon^i f_i^x(\tau, z, z') + o(|\varepsilon|^m), \quad \text{as } \varepsilon \rightarrow 0, \quad (1.30)$$

in the  $C^\infty$ -topology of  $(0, \infty) \times V \times V$ , where  $f_i^x$ 's are smooth functions satisfying the following homogeneity property: for  $i = 0, \dots, m$

$$|\varepsilon|^{\mathcal{Q}(x)} \varepsilon^{-i} f_i^x(\varepsilon^2 \tau, \delta_\varepsilon(z), \delta_\varepsilon(z')) = f_i^x(\tau, z, z'), \quad \forall (\tau, z, z') \in (0, \infty) \times \mathbb{R}^n \times \mathbb{R}^n, \quad (1.31)$$

where, for  $i = 0$ , we set  $f_0^x(\tau, z, z') = \widehat{p}_\tau^x(z, z')$ . In (1.30), we are considering the heat kernel  $p_t$  in coordinates, with a little abuse of notation.

*Remark 1.4.6.* We will drop the dependence on the center of the privileged coordinates if there is no confusion.



# 2

## Heat content asymptotics for sub-Riemannian manifolds

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*We study the small-time asymptotics of the heat content general sub-Riemannian manifold. Under the assumption of non-characteristic domain, we establish the existence of the full asymptotic series:*

$$Q_{\Omega}(t) = \sum_{k=0}^{\infty} a_k t^{k/2}, \quad \text{as } t \rightarrow 0.$$

*We compute explicitly the coefficients up to order  $k = 5$ , in terms of sub-Riemannian invariants of the domain. Furthermore, we prove that every coefficient can be obtained as the limit of the corresponding one for a suitable Riemannian extension. As a particular case we recover, using non-probabilistic techniques, the order 2 formula recently obtained by Tyson and Wang in the Heisenberg group [TW18]. A consequence of our fifth-order analysis is the evidence for new phenomena in presence of characteristic points. In particular, we prove that the higher order coefficients in the asymptotics can blow-up in their presence. A key tool for this last result is an exact formula for the distance from a specific surface with an isolated characteristic point in the Heisenberg group, which is of independent interest.*

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## 2.1 Introduction

Let  $(M, g)$  be a complete Riemannian manifold, and  $\Omega \subset M$  be a relatively compact open domain with smooth boundary. Consider the solution  $u(t, x)$  of the heat equation with Dirichlet boundary conditions and homogeneous initial datum:

$$\begin{aligned} (\partial_t - \Delta)u(t, x) &= 0, & \forall (t, x) \in (0, \infty) \times \Omega, \\ u(t, x) &= 0, & \forall (t, x) \in (0, \infty) \times \partial\Omega, \\ u(0, x) &= 1, & \forall x \in \Omega, \end{aligned}$$

where  $\Delta$  is the Laplace-Beltrami operator of  $(M, g)$ . The Riemannian heat content of  $\Omega$  is the function

$$Q_\Omega(t) = \int_\Omega u(t, x) d\mu_g(x), \quad t \in [0, \infty),$$

where  $\mu_g$  is the Riemannian measure. From a physical viewpoint,  $Q_\Omega(t)$  represents the total heat contained in  $\Omega$  at time  $t$ , corresponding to a uniform initial temperature distribution, and where the boundary  $\partial\Omega$  is kept at zero temperature. It turns out that  $Q_\Omega(t)$  admits an asymptotic expansion as a function of  $\sqrt{t}$  whose coefficients encode geometrical information about  $\Omega$  and its boundary.

For Euclidean domains  $\Omega \subset \mathbb{R}^n$ , the asymptotics of  $Q_\Omega(t)$  at order 1 was computed in [vdBD89], and up to order 2 in [vdBLG94], using probabilistic methods<sup>1</sup>. In particular, under the condition that  $\partial\Omega$  is of class  $C^3$ , it holds:

$$Q_\Omega(t) = |\Omega| - \sqrt{\frac{4t}{\pi}} \sigma(\partial\Omega) + \frac{t}{2} \int_{\partial\Omega} \mathcal{H} d\sigma + O(t^{3/2}), \quad (2.1)$$

where  $|\cdot|$  here denotes the Lebesgue measure,  $\sigma$  is the corresponding surface measure on  $\partial\Omega$ , and  $\mathcal{H}$  is the mean curvature of  $\partial\Omega$ . A first non-flat case was studied in [vdB91], where the authors computed the heat content asymptotics to order 2 for the upper hemisphere, exploiting the explicit knowledge of the heat kernel.

For smooth domains in a Riemannian manifold, the existence of an asymptotic expansion in  $\sqrt{t}$  at arbitrary order was established in [vdBG94], where the authors also computed all coefficients up to order 4. In this case, the volume, the perimeter, and the mean curvature appearing in (2.1) are replaced by their corresponding Riemannian counterparts, while the subsequent terms involve the second fundamental form of  $\partial\Omega$  and

<sup>1</sup>Here and throughout the chapter, the order is computed as a power-series in the variable  $\sqrt{t}$ .

the Riemann curvature tensor. We stress that the existence of a full asymptotic series is non-trivial, as the heat content is not a smooth function of  $\sqrt{t}$  around  $t = 0$  (one can easily verify this fact by computing the heat content of a Euclidean segment). Van den Berg and Gilkey's method in [vdBG94], which heavily exploits the functorial properties of the coefficients and invariance theory for the Riemannian curvature, has been extended to compute the heat content asymptotics up to order 5 in [vdBG99], and to the case of Neumann boundary conditions, see [vdBDG93, DG94].

### 2.1.1 Sub-Riemannian heat content asymptotics

In this chapter we study the asymptotics of the heat content in sub-Riemannian geometry. This study is interesting for several reasons. Firstly, there is no analogue of Levi-Civita connection, curvature, or invariance theory for a general sub-Riemannian structure. These were fundamental tools for the study of the Riemannian problem by Van den Berg and Gilkey, and hence new methods must be used in the sub-Riemannian setting. Secondly, in the general sub-Riemannian case, there is no canonical choice of measure. For this reason, we must work with a general smooth measure  $\omega$ , which is necessary for the definition of the sub-Laplacian, see Section (1.3). Notice that, if  $M$  is an equiregular sub-Riemannian manifold, one can define the canonical Popp's measure [Mon02, BR13], extending the Riemannian volume. This construction is not possible for non-equiregular structures, e.g. rank-varying ones. Thirdly, the study of the sub-Riemannian heat content can improve our understanding of the intrinsic geometry of hypersurfaces, which is well-developed only for the case of the Heisenberg group [Pau04, AF07, AF08, CDPT07, BFF<sup>+</sup>15a, BTV17] and Carnot groups [DGN07a] (see also [Vel20] for a concept of Gaussian curvature for surfaces in three-dimensional contact structures, generalizing [BTV17]). Lastly, a genuinely new phenomenon occurs in the sub-Riemannian case: characteristic points, where the distribution is tangent to  $\partial\Omega$ , and whose presence is source of subtle technical problems.

The study of the small-time heat content asymptotics in the sub-Riemannian setting was initiated recently by Tyson and Wang, in [TW18], where they studied the first Heisenberg group  $\mathbb{H}$ . There, they established the existence of a small-time asymptotic series up to order 2 in  $\sqrt{t}$ , for non-characteristic domains. The approach in [TW18] is probabilistic, based on the interpretation of the solution of the Dirichlet problem in terms of the exit time of the corresponding Markov process. This relation holds up to an error of order  $o(t)$ , cf. [TW18, Prop. 3.2], preventing the access to higher order terms in the heat content asymptotics.

We use here a different method with respect to that of Tyson and Wang, by adapting a technique developed in the Riemannian case by Savo [Sav98, Sav01, Sav04]. This method allows us to prove the existence of an asymptotic expansion at arbitrary order, for non-characteristic domains of general rank-varying sub-Riemannian structures. Our first main result is the following. Precise definitions can be found in Chapter 1 and Section 2.2.

**Theorem 2.1.1.** *Let  $M$  be a sub-Riemannian manifold, equipped with a smooth measure  $\omega$ , and let  $\Omega \subset M$  be an open relatively compact subset whose boundary is smooth and has not characteristic points. Then, there exist  $a_k \in \mathbb{R}$  such that for all  $m \geq 4$  it holds*

$$Q_\Omega(t) = \omega(\Omega) - \sqrt{\frac{4t}{\pi}}\sigma(\partial\Omega) + \frac{t}{2} \int_{\partial\Omega} \mathcal{H}d\sigma + \sum_{k=3}^{m-1} a_k t^{k/2} + O(t^{m/2}), \quad \text{as } t \rightarrow 0,$$

where  $\sigma$  is the sub-Riemannian measure induced by  $\omega$  on  $\partial\Omega$ , and  $\mathcal{H}$  is the sub-Riemannian mean curvature<sup>2</sup> of  $\partial\Omega$ .

---

<sup>2</sup>We recall that letting  $\nu$  the outward pointing horizontal normal of  $\partial\Omega$ , the induced sub-Riemannian

In order to report the first few coefficients, we introduce the operator  $N$ , acting on smooth functions in a neighborhood of  $\partial\Omega$ , given by

$$N\phi = 2g(\nabla\phi, \nabla\delta) + \phi\Delta\delta,$$

where  $g$  is the sub-Riemannian scalar product,  $\nabla$  is the sub-Riemannian gradient,  $\Delta = \operatorname{div}_\omega \circ \nabla$  is the sub-Laplacian (symmetric with respect to the smooth measure  $\omega$ ), and  $\delta$  is the sub-Riemannian distance from  $\partial\Omega$  (which, in absence of characteristic points, is smooth on a neighborhood of  $\partial\Omega$ ). We remark that for the sub-Riemannian horizontal mean curvature it holds  $\mathcal{H} = -\Delta\delta|_{\partial\Omega}$ .

**Proposition 2.1.2.** *With the assumptions and notations of Theorem 2.1.1, for  $k \geq 1$ , we have  $a_k = -\int_{\partial\Omega} D_k(1)d\sigma$ , where  $D_k$  is an homogeneous polynomial of degree  $k-1$  in the operators  $\Delta$  and  $N$ . In particular, it holds*

$$\begin{aligned} a_0 &= \omega(\Omega), & a_1 &= -\sqrt{\frac{4}{\pi}}\sigma(\partial\Omega), \\ a_2 &= -\frac{1}{2}\int_{\partial\Omega} \Delta\delta d\sigma, & a_3 &= -\frac{1}{6\sqrt{\pi}}\int_{\partial\Omega} N\Delta\delta d\sigma, \\ a_4 &= -\frac{1}{16}\int_{\partial\Omega} \Delta^2\delta d\sigma, & a_5 &= \frac{1}{240\sqrt{\pi}}\int_{\partial\Omega} (N^3 - 8N\Delta)\Delta\delta d\sigma. \end{aligned}$$

For all  $k \geq 1$ , the operators  $D_k$  are defined recursively in (2.28)–(2.30).

*Remark 2.1.3.* The integrands of  $a_1$  and  $a_2$  have classical interpretation as the perimeter and the mean curvature of  $\partial\Omega$ . We observe that the integrand of  $a_3$  is the so-called *effective potential*, a quantity introduced in [PRS18, FPR20] to describe the essential self-adjointness properties of sub-Laplacians, in presence of singular measures.

*Remark 2.1.4.* The operators  $D_k$  belong to the algebra generated by  $\Delta$  and  $N$ , and are homogeneous of degree  $k-1$  in the generators, where  $N$  has degree 1 and  $\Delta$  has degree 2. Since  $\Delta(1) = 0$  and  $N(1) = \Delta\delta$ , for  $k \geq 2$ , we have that each  $a_k = -\int_{\partial\Omega} \tilde{D}_k(\Delta\delta)d\sigma$ , where  $\tilde{D}_k$  is an homogeneous polynomial of degree  $k-2$  in  $\Delta$  and  $N$ . In particular, each  $a_k$  depends on the horizontal mean curvature of  $\partial\Omega$  and its derivatives.

*Remark 2.1.5.* The iterative construction of operators  $D_k$ , which is quite involved, has been implemented in the software *Mathematica* in [Ros20], thanks to which the coefficients  $a_k$  can be immediately computed. See Appendix C for details.

Before presenting further results, let us comment the proof of Theorem 2.1.1. Savo's method amounts to study the quantity

$$F(t, r) = \int_{\Omega(r)} u(t, x)d\omega(x),$$

where  $\Omega(r) = \{\delta > r\}$ . Upon appropriate localization to deal with the non-smoothness of  $\Omega(r)$  for large  $r$ , it turns out that  $F(t, r)$  satisfies a non-homogeneous one-dimensional heat equation on the half-line  $[0, \infty)$ , with Neumann boundary condition at the origin. Then, the whole asymptotics of Theorem 2.1.1 and the expression of the coefficients are obtained by iterating the corresponding Duhamel's formula. Some non-trivial modifications must be implemented to adapt this technique to the sub-Riemannian setting. For example, the Li-Yau estimate for the heat kernel of Riemannian manifolds with Ricci curvature

---

measure  $\sigma$  on  $\partial\Omega$  is the smooth and positive measure whose density is  $|i_\nu\omega|_{\partial\Omega}$ . Furthermore the sub-Riemannian mean curvature of  $\partial\Omega$  is given by  $\mathcal{H} = \operatorname{div}_\omega(\nu)|_{\partial\Omega}$ , where  $\operatorname{div}_\omega(\cdot)$  is the divergence of a vector field, computed with respect to the measure  $\omega$ .

bounded from below are no longer available (sub-Riemannian manifolds have, in a sense, Ricci curvature unbounded from below). Another important ingredient is the description of tubular neighborhoods of  $\partial\Omega$ . If, for the Heisenberg group, this can be achieved through the explicit formulas for geodesics as done in [AF07, AF08, Rit17, AFM17], we must use a different approach for the general case, based on the study of the Hamiltonian flow on the annihilator bundle of  $\partial\Omega$ .

We also remark that the same method can be used, with no modifications, to study the *weighted* heat content associated with a non-characteristic domain  $\Omega$ , i.e.

$$Q_\Omega^\phi(t) = \int_\Omega u(t, x)\phi(x)d\omega(x), \quad t > 0,$$

with  $\phi \in C^\infty(\Omega)$ . In this case, using the same notation of Proposition 2.1.2, one has  $a_k = -\int_{\partial\Omega} D_k(\phi)d\sigma$ .

### 2.1.2 Riemannian approximations

Any sub-Riemannian structure can be obtained as a monotonic limit of Riemannian ones. This approximation scheme can be easily implemented for constant-rank distributions. In this case, a natural approximating sequence is obtained by taking any Riemannian metric  $g$  extending the sub-Riemannian one, and rescaling it by a factor  $1/\varepsilon$  in the transverse directions. This construction yields a one-parameter family of Riemannian structures  $g_\varepsilon$ . The associated Riemannian distance  $d_\varepsilon$  converges, uniformly on compact sets, to the sub-Riemannian one  $d_{\text{SR}}$ . Outside of the sub-Riemannian cut locus, one can actually prove that  $d_\varepsilon \rightarrow d_{\text{SR}}$  in the  $C^\infty$  topology, see for example [BGM19]. For totally geodesic foliations, this scheme is known under the name of *canonical variation* [Bes08]. We remark though that the Riemannian curvature of the approximating sequence is unbounded below, posing some technical difficulties when taking the limit.

We introduce in this chapter a generalization of the canonical variation scheme which works for general rank-varying sub-Riemannian structures. Our second result relates the coefficients of the small-time asymptotics of the Riemannian heat content  $Q_\Omega^\varepsilon(t)$  of the approximating structure with the sub-Riemannian ones.

**Theorem 2.1.6.** *Let  $M$  be a sub-Riemannian manifold, equipped with a smooth measure  $\omega$ , and let  $\Omega \subset M$  be an open relatively compact subset whose boundary is smooth and has not characteristic points. Then, there exists a family of Riemannian metrics  $g_\varepsilon$  such that  $d_\varepsilon \rightarrow d_{\text{SR}}$  uniformly on compact sets of  $M \times M$ , and such that*

$$\lim_{\varepsilon \rightarrow 0} a_k^\varepsilon = a_k, \quad \forall k \in \mathbb{N}, \quad (2.2)$$

where  $a_k$  and  $a_k^\varepsilon$  denote the coefficients of the sub-Riemannian small-time heat content asymptotics, and the corresponding ones for the Riemannian approximating structure.

Even though  $Q_\Omega^\varepsilon(t) \rightarrow Q_\Omega(t)$  in the  $C^\infty$  uniform topology on compact subsets of  $(0, \infty)$ , this fact alone does not imply (2.2). A direct proof of Theorem 2.1.6 would require (i) an a-priori proof of the existence of the small-time sub-Riemannian asymptotics for  $Q_\Omega(t)$  and (ii) a delicate inversion of the order of the two limits  $\varepsilon \rightarrow 0$  and  $t \rightarrow 0$ . It is also important to stress that Theorem 2.1.1 is not proved using an approximation scheme and thus *it is not* a consequence of Theorem 2.1.6. The latter will be rather proved using the explicit iterative formula for the coefficients of the Riemannian and sub-Riemannian heat content expansions.

### 2.1.3 Characteristic points

One main assumption in all our results is that  $\partial\Omega$  does not contain characteristic points. This is quite restrictive for the case of Heisenberg group, where the only non-characteristic domains are homeomorphic to a torus. More generally, for any contact sub-Riemannian manifold, the non-characteristic assumption and the contact structure imply that  $\partial\Omega$  must have vanishing Euler characteristic. On the other hand, the non-characteristic assumption is less restrictive for general structures: it is not hard to prove that for any smooth manifold  $M$  of dimension  $n \geq 4$ , and any smooth relatively compact domain  $\Omega$  with smooth boundary  $\partial\Omega$ , there exists a possibly rank-varying sub-Riemannian structure on  $M$  such that  $\partial\Omega$  has no characteristic points.

The non-characteristic assumption is crucial for the smoothness of the distance from  $\partial\Omega$  and for the existence of smooth tubular neighborhoods, cf. Theorem 2.4.2. Furthermore, even if the existence of solutions in  $L^2(\Omega, \omega)$  to the heat equation with Dirichlet boundary conditions holds from general spectral theory (and thus  $Q_\Omega(t)$  is well-defined), their smoothness up to the boundary may fail close to characteristic points [Jer81a, Jer81b].

Despite all these difficulties, one might wonder whether the small-time heat content asymptotic formula of Theorem 2.1.1 makes sense, at least formally, for domains with characteristic points. Firstly, we note that if  $\Sigma$  is a smooth embedded hypersurface in a sub-Riemannian manifold  $M$ , then the set of characteristic points have zero measure in  $\Sigma$ , see [Bal03]. Secondly, any choice of smooth measure  $\omega$  on  $M$  induces a smooth surface measure  $\sigma$  on  $\Sigma$ , even in presence of characteristic points. It is sufficient to consider the contraction of  $\omega$  with the horizontal unit normal to  $\Sigma$  (see for example [DGN07a, Sec. 8] for the case of Carnot groups, where this notion is related with the horizontal perimeter measure). Thirdly, the sub-Riemannian mean curvature is locally integrable with respect  $\sigma$ , even in presence of characteristic points, cf. [DGN12, Ros21]. As a consequence of all these facts, all terms appearing in the order 2 formula for  $\mathbb{H}$  in [TW18] are well-defined also for characteristic domains. This seems to suggest that the same small-time asymptotic formula might hold also for characteristic domains. Our analysis shows that this cannot be true at higher order.

**Theorem 2.1.7.** *Let  $\mathbb{H}$  be the first Heisenberg group, and consider the plane  $\Sigma = \{z = 0\}$ . Observe that the origin is an isolated characteristic point. Denote with  $\sigma$  the sub-Riemannian surface measure on  $\Sigma$  induced by the Lebesgue measure on  $\mathbb{H}$ . Then the integrand of the coefficient  $a_5$  of the small-time heat content expansion is not locally integrable with respect to  $\sigma$  around the characteristic point of  $\Sigma$ .*

Theorem 2.1.7 shows that the asymptotic formula of Theorem 2.1.1 is false at order  $k \geq 5$  for domains with characteristic points. In the example of Theorem 2.1.7, it turns out that the integrands of the coefficients  $a_3$  and  $a_4$  are still locally integrable with respect to the sub-Riemannian surface measure. We expect, however, that one can build a less symmetric example where also the integrand of  $a_4$  is not integrable close to a characteristic point, cf. Remark 2.7.7. On the other hand, Theorem 2.1.1 might still be true at lower order (it has already been remarked that the coefficients appearing therein remain well-defined for characteristic domains in  $\mathbb{H}$  up to  $k = 2$ ).

**Open problem.** *Is it true that, for smooth domains in  $\mathbb{H}$  with characteristic points, the asymptotic expansion of Theorem 2.1.1 remains valid up to some order  $0 < k < 5$ ?*

To prove Theorem 2.1.7 we derived an exact formula for the sub-Riemannian distance from the  $xy$ -plane in  $\mathbb{H}$ . To our best knowledge, this is the first time such an explicit global formula appears in the literature, and it has independent interest. For example, it can be used to study the loss of regularity of the distance at characteristic points.



**Theorem 2.1.8.** *The distance from the  $xy$ -plane in the first Heisenberg group  $\mathbb{H}$ , for all  $p \in z$ -axis, is given by*

$$\delta(p) = \sqrt{2\pi|z_p|},$$

while for all  $p \notin z$ -axis, it is given in cylindrical coordinates by

$$\delta(p) = r_p \frac{4\xi_p + y_0(\xi_p)}{\sqrt{1 + y_0(\xi_p)^2}}, \quad \xi_p = \frac{|z_p|}{r_p^2},$$

where  $\xi \mapsto y_0(\xi)$  is the unique smooth function such that

$$4\xi + y_0 + (1 + y_0^2) \arctan(y_0) = 0.$$

*Remark 2.1.9.* The proof of Theorem 2.1.8 consists in a non-trivial characterization of minimal geodesics to the  $xy$ -plane, and the corresponding cut-locus. Local formulas are easier to obtain, cf. [BFF<sup>+</sup>15a, Ex. 5.1], using the well-known minimality property of short segments of normal sub-Riemannian geodesics. However, these local formulas and related estimates do not hold uniformly when approaching characteristic points.

**Structure of the chapter.** In Section 2.2 we recall the basic definition of heat content, and we explain how to approximate any sub-Riemannian structure with a Riemannian one. In Section 2.3 and 2.4, we provide the key ingredients to obtain the heat content asymptotic, i.e., the localization to the boundary of the heat content and the sub-Riemannian version of the mean value lemma, respectively. Then, in Section 2.5, we develop the asymptotic expansion of the heat content, concluding the proof of Theorem 2.1.1 and Proposition 2.1.2. In Section 2.6, we show how to obtain the coefficients through a Riemannian approximation, proving Theorem 2.1.6. Finally, in Section 2.7 we show that, in presence of a characteristic point, the integrand of the coefficient  $a_5$  is not locally integrable, proving Theorem 2.1.7.

## 2.2 Preliminaries

In this section, we give the precise definition of heat content and we explain how to build the Riemannian variation of a sub-Riemannian manifold. We refer to Chapter 1 for general facts regarding sub-Riemannian geometry.

### 2.2.1 Dirichlet problem for the heat equation

Let  $M$  be a sub-Riemannian manifold, equipped with a smooth measure  $\omega$ , and let  $\Omega \subset M$  be an open relatively compact set with smooth boundary. This means that the closure  $\bar{\Omega}$  is a compact manifold with smooth boundary. We consider the *Dirichlet problem for the heat equation* on  $\Omega$ , that is we look for functions  $u$  such that

$$\begin{aligned} (\partial_t - \Delta)u(t, x) &= 0, & \forall (t, x) \in (0, \infty) \times \Omega, \\ u(t, x) &= 0, & \forall (t, x) \in (0, \infty) \times \partial\Omega, \\ u(0, x) &= 1, & \forall x \in \Omega, \end{aligned} \tag{2.3}$$

where  $\Delta = \operatorname{div}_\omega \circ \nabla$  is the sub-Laplacian defined in (1.14). We denote by  $L^2(\Omega, \omega)$  the space of real functions on  $\Omega$  which are square-integrable with respect to the measure  $\omega$ .

Notice that we can represent the unique solution to (2.3), as

$$u(t, \cdot) = e^{t\Delta} (1|_{\Omega}), \quad \forall t \geq 0,$$

where  $e^{t\Delta}: L^2(\Omega, \omega) \rightarrow L^2(\Omega, \omega)$  denotes the semi-group generated by the Dirichlet self-adjoint extension of the sub-Laplacian on  $\Omega$ , see Appendix A for further details. We remark that for all  $\varphi \in L^2(\Omega, \omega)$ , the function  $e^{t\Delta}\varphi$  is smooth for all  $(t, x) \in (0, \infty) \times \Omega$ , by hypoellipticity of the heat operator (1.15).

**Definition 2.2.1** (Heat content). Let  $u(t, x)$  be the solution to (2.3). We define the *heat content*, associated with  $\Omega$ , as

$$Q_{\Omega}(t) = \int_{\Omega} u(t, x) d\omega(x), \quad \forall t > 0.$$

We recall here two properties of the heat semi-group that we will use in the sequel. First of all, the solution to (2.3) satisfies a weak maximum principle, meaning that

$$0 \leq u(t, x) \leq 1, \quad \forall x \in \Omega, \quad \forall t > 0. \quad (2.4)$$

Second of all, the domain monotonicity property holds. Let  $\Omega' \subset M$  be a relatively compact domain, such that  $\Omega \subset \Omega'$ . Denoting with  $\Delta'$  the Dirichlet sub-Laplacian on  $\Omega'$ , for any  $\varphi \in L^2(\Omega, \omega)$ , we have

$$e^{t\Delta}\varphi(x) \leq e^{t\Delta'}\varphi(x), \quad \forall x \in \Omega, \quad \forall t > 0, \quad (2.5)$$

where  $e^{t\Delta'}$  is the semi-group generated by  $\Delta'$  on  $L^2(\Omega', \omega)$ . Properties (2.4) and (2.5) can be proven following the blueprint of the Riemannian proofs (see [Gri09, Thm. 5.11, Thm. 5.23]). An alternative proof can be given using Riemannian approximations (see Section 2.2.2 for details) exploiting the respective properties for the Riemannian heat semi-groups and then passing to the limit.

**Definition 2.2.2.** We say that  $x \in \partial\Omega$  is a *characteristic point*, or tangency point, if the distribution is tangent to  $\partial\Omega$  at  $x$ , that is

$$\mathcal{D}_x \subseteq T_x(\partial\Omega).$$

We will assume that  $\partial\Omega$  has no characteristic points. We say in this case that  $\Omega$  is a *non-characteristic domain*. In this case, the solution  $u(t, x)$  of the heat equation with Dirichlet boundary conditions and initial datum  $\phi \in C^\infty(\Omega)$  exists, is unique, and is smooth on  $(0, \infty) \times \bar{\Omega}$ , see [GM18, Thm. 2.5]. This is a consequence of the analogous result for the stationary problem considered in [KN65]. In the Riemannian case, this is a classical textbook result, see [Eva10, Sec. 7, Thm. 7].

## 2.2.2 Approximation via Riemannian structures

We describe an approximation procedure of a sub-Riemannian metric structure via a family of Riemannian ones, extending the classical canonical variation scheme, which we will use in Section 2.6. See also [CC16] for a different approximation scheme.

Let  $(\mathcal{D}, g)$  be a sub-Riemannian structure on  $M$ . Consider a global generating frame for a Riemannian structure, i.e. a set of  $L$  global vector fields  $\tilde{X}_1, \dots, \tilde{X}_L$  such that

$$T_x M = \text{span}\{\tilde{X}_1(x), \dots, \tilde{X}_L(x)\}, \quad \forall x \in M. \quad (2.6)$$

For any  $\varepsilon \in \mathbb{R}$ , consider the following family of global smooth vector fields:

$$\{X_1, \dots, X_N, \varepsilon\tilde{X}_1, \dots, \varepsilon\tilde{X}_L\}. \quad (2.7)$$

This is a global generating frame for the whole tangent space at each point, therefore, it induces a scalar product on  $T_x M$ , defined, as in the sub-Riemannian case, by formula (1.3), which we denote by  $g_\varepsilon$ . Since (2.6) holds,  $g_\varepsilon$  is a Riemannian metric on  $M$  and the corresponding Riemannian distance is denoted by  $d_\varepsilon$ . Furthermore it holds

$$d_\varepsilon \xrightarrow{\varepsilon \rightarrow 0} d_{\text{SR}}, \quad \text{uniformly on the compact sets of } M \times M.$$

An explicit proof of this fact in the constant-rank case can be found in [BGM19, Lem. A.1]. The same proof holds verbatim in the general rank-varying case, replacing local orthonormal frames with the generating frame (2.7), and replacing the controls of minimizing geodesics with the minimal controls, defined in (1.4).

We may call the 1-parameter family of metric spaces  $\{(M, d_\varepsilon)\}_{\varepsilon \in \mathbb{R}}$  a *Riemannian variation* of  $(M, d_{\text{SR}})$ . For the fixed smooth measure  $\omega$  we define the corresponding gradient and the Laplacian, denoted respectively by  $\nabla_\varepsilon$  and  $\Delta_\varepsilon$ . The expression of  $\nabla_\varepsilon$  acting on  $f \in C^\infty(M)$ , with respect to the generating frame (2.7), is given by

$$\nabla_\varepsilon f = \sum_{i=1}^N X_i(f) X_i + \varepsilon^2 \sum_{\alpha=1}^L \tilde{X}_\alpha(f) \tilde{X}_\alpha = \nabla f + \varepsilon^2 \tilde{\nabla} f, \quad \forall f \in C^\infty(M), \quad (2.8)$$

while the expression of  $\Delta_\varepsilon$  is given by

$$\begin{aligned} \Delta_\varepsilon f &= \sum_{i=1}^N \left\{ X_i^2(f) + X_i(f) \operatorname{div}_\omega(X_i) \right\} + \varepsilon^2 \sum_{\alpha=1}^L \left\{ \tilde{X}_\alpha^2(f) + \tilde{X}_\alpha(f) \operatorname{div}_\omega(\tilde{X}_\alpha) \right\} \\ &= \Delta f + \varepsilon^2 \tilde{\Delta} f. \end{aligned} \quad (2.9)$$

Here,  $\nabla$  and  $\Delta$  represent the sub-Riemannian gradient and sub-Laplacian, respectively, while  $\tilde{\nabla}$  and  $\tilde{\Delta}$  represent the gradient and the Laplacian, computed with respect to the Riemannian structure defined by the frame in (2.6), respectively. Fix now an open relatively compact subset  $\Omega \subset M$  with smooth boundary. The sequence of Riemannian structures yields naturally an approximation of the solution  $u(t, x)$  of the Dirichlet problem (2.3) on  $\Omega$ . Indeed, applying Corollary A.0.3, we have:

$$e^{t\Delta_\varepsilon} \varphi \xrightarrow[\|\cdot\|_{L^2(\Omega, \omega)}]{\varepsilon \rightarrow 0} e^{t\Delta} \varphi, \quad \forall \varphi \in L^2(\Omega, \omega), \quad \text{uniformly in } t \in [0, T], \quad (2.10)$$

where  $e^{t\Delta_\varepsilon}$  and  $e^{t\Delta}$ , for  $t > 0$ , denote the semi-groups associated with the Dirichlet self-adjoint extension of  $\Delta_\varepsilon$  and  $\Delta$ , respectively. Integrating (2.10), with  $\varphi = 1|_\Omega$ , we obtain an analogous result for the heat content:

$$Q_\Omega^\varepsilon(t) \xrightarrow{\varepsilon \rightarrow 0} Q_\Omega(t), \quad \text{uniformly on } [0, T],$$

where  $Q_\Omega^\varepsilon(t)$  denotes the Riemannian heat content of  $\Omega$  for the metric  $g_\varepsilon$ .

## 2.3 Localization to the boundary of the heat content

The next result is an application of a hypoelliptic version of Kac's principle of not feeling the boundary [Hsu95]. It states that solutions of the heat equation, at any interior point, are not influenced by the boundary conditions, up to a uniform  $O(t^\infty)$  term. As we detail in the proof, the uniformity we need can be achieved through off-diagonal estimates for Hörmander-type operators as in [JSC86].

**Theorem 2.3.1.** *Let  $M$  be a sub-Riemannian manifold, equipped with a smooth measure  $\omega$ , and let  $\Omega \subset M$  be an open relatively compact subset with smooth boundary. Then for any compact set  $K \subset \Omega$  it holds*

$$1 - u(t, x) = O(t^\infty) \quad \text{as } t \rightarrow 0, \quad \text{uniformly for } x \in K,$$

where  $u(t, x)$  denotes the solution to (2.3).

*Remark 2.3.2.* Here, the boundary  $\partial\Omega$  can contain characteristic points. An alternative proof of Theorem 2.3.1 can be found in Appendix A.

*Proof of Theorem 2.3.1.* Without loss of generality we can assume that  $M$  is compact, as any modification of the structure outside  $\Omega$  does not change  $u(t, x)$ . Denote by  $p_t^M(x, y)$  and  $p_t^\Omega(x, y)$  the heat kernels on  $M$  and  $\Omega$ , respectively (for the case of  $\Omega$  recall that we impose Dirichlet boundary conditions). We have

$$1 - u(t, x) = \int_{M \setminus \Omega} p_t^M(x, y) d\omega(y) + \int_{\Omega} (p_t^M(x, y) - p_t^\Omega(x, y)) d\omega(y). \quad (2.11)$$

Let  $K$  as in the statement, and recall that in (2.11)  $x \in K$ .

To estimate the first term of (2.11) we use the off-diagonal estimate for Hörmander-type operators in [JSC86, Prop. 3], that is

$$p_t^M(x, y) \leq C_a e^{-c_a/t}, \quad \forall x, y \text{ with } d(x, y) \geq a, \quad t < 1, \quad (2.12)$$

for some positive constants  $C_a, c_a > 0$ . Since in that first integral of (2.11) one has  $d(x, y) \geq d(K, \partial\Omega) = a > 0$ , we conclude that

$$\int_{M \setminus \Omega} p_t^M(x, y) d\omega(y) \leq \text{Vol}(M) C_a e^{-c_a/t} = O(t^\infty), \quad \text{uniformly for } x \in K.$$

To estimate the second term of (2.11), let  $K' \subset \Omega$  be another compact subset such that  $K \subset K' \subset \Omega$ , so that  $d(K, \Omega \setminus K') = b > 0$ . We split the second term of (2.11) as

$$\begin{aligned} \int_{\Omega} (p_t^M(x, y) - p_t^\Omega(x, y)) d\omega(y) &= \int_{K'} (p_t^M(x, y) - p_t^\Omega(x, y)) d\omega(y) \\ &\quad + \int_{\Omega \setminus K'} (p_t^M(x, y) - p_t^\Omega(x, y)) d\omega(y). \end{aligned} \quad (2.13)$$

By domain monotonicity (2.5), we have  $0 \leq p_t^\Omega(x, y) \leq p_t^M(x, y)$ , whence

$$\begin{aligned} \left| \int_{\Omega \setminus K'} (p_t^M(x, y) - p_t^\Omega(x, y)) d\omega(y) \right| &\leq 2 \int_{\Omega \setminus K'} p_t^M(x, y) d\omega(y) \\ &\leq 2 \text{Vol}(\Omega) C_b e^{-c_b/t} = O(t^\infty), \end{aligned}$$

uniformly for  $x \in K$ , using again (2.12). For the first term in (2.13), we use a hypoelliptic version of Kac's principle (cf. [JSC86, p. 836] or [CdVHT20, Thm. 3.1]). This means, in particular, that

$$p_t^M(x, y) - p_t^\Omega(x, y) = O(t^\infty), \quad \forall x, y \in K', \quad t < 1,$$

uniformly (we stress that this estimate cannot hold uniformly on  $\bar{\Omega}$ , whence the reason for the restriction to compact sets). We conclude easily.  $\square$

## 2.4 Sub-Riemannian mean value lemma

Let  $M$  be a sub-Riemannian manifold with smooth measure  $\omega$ , and let  $\Omega$  be a relatively compact subset of  $M$  with smooth boundary, and assume that  $\partial\Omega$  has no characteristic points. Denote with  $\delta(\cdot) = d_{\text{SR}}(\partial\Omega, \cdot)$  and define, for  $r > 0$ , the open set (see figure 2.1)

$$\Omega(r) = \{x \in \Omega \mid \delta(x) > r\}.$$

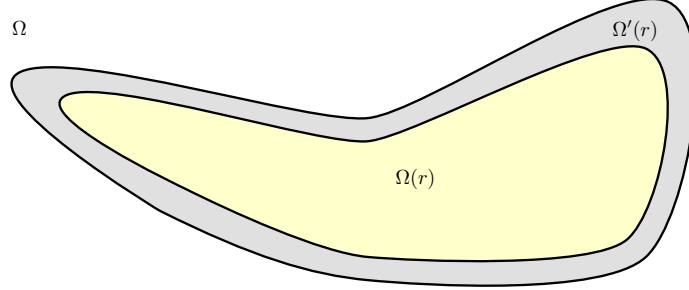


Figure 2.1: The set  $\Omega(r)$  is highlighted in yellow. The gray set is  $\Omega'(r) = \bar{\Omega} \setminus \Omega(r)$ .

Notice that, in general, the distance function is only 1-Lipschitz, therefore the boundary of  $\Omega(r)$ , which we denote by  $\partial\Omega(r)$ , may not be a smooth submanifold. Then, for  $r \in [0, \infty)$  and  $v \in C^\infty(\Omega)$ , consider the function

$$F(r) = \int_{\Omega(r)} v(x) d\omega(x). \quad (2.14)$$

The function  $F$  represents the mean value of the function  $v$  on  $\Omega(r)$ .

**Theorem 2.4.1.** *Let  $M$  be a sub-Riemannian manifold, equipped with a smooth measure  $\omega$ , and let  $\Omega \subset M$  be an open relatively compact subset of  $M$  whose boundary is smooth and has not characteristic points. Let  $\delta: \bar{\Omega} \rightarrow [0, \infty)$  be the distance function from  $\partial\Omega$ . Then there exists  $r_0 > 0$  such that the function  $F$ , defined as in (2.14), is smooth on  $[0, r_0)$  and, for  $r < r_0$*

$$F''(r) = \int_{\Omega(r)} \Delta v(x) d\omega(x) - \int_{\partial\Omega(r)} v(y) \Delta \delta(y) d\sigma(y), \quad (2.15)$$

where  $\Delta$  is the sub-Laplacian associated with  $\omega$  and  $\sigma$  is the induced measure on  $\partial\Omega(r)$ .

Before giving the proof, we establish some basic properties of the distance function  $\delta$  for general sub-Riemannian structures. We recall here the result we need, proved in [FPR20, Prop. 3.1].

**Theorem 2.4.2** (Tubular Neighborhood). *Let  $M$  be a sub-Riemannian manifold, equipped with a smooth measure  $\omega$ , and let  $\Omega \subset M$  be an open relatively compact subset of  $M$  whose boundary is smooth and has not characteristic points. Let  $\delta: \bar{\Omega} \rightarrow [0, \infty)$  be the distance function from  $\partial\Omega$ . Then, we have:*

- i)  $\delta$  is Lipschitz with respect to the sub-Riemannian distance and  $\|\nabla \delta\|_g \leq 1$  a.e.;
- ii) there exists  $r_0 > 0$  such that  $\delta: \Omega'(r_0) \rightarrow [0, \infty)$  is smooth, where  $\Omega'(r) = \bar{\Omega} \setminus \Omega(r)$ ;
- iii) there exists a smooth diffeomorphism  $G: [0, r_0) \times \partial\Omega \rightarrow \Omega'(r_0)$ , such that

$$\delta(G(t, y)) = t \quad \text{and} \quad G_* \partial_t = \nabla \delta, \quad \forall (t, y) \in [0, r_0) \times \partial\Omega.$$

Moreover,  $\|\nabla \delta\|_g \equiv 1$  on  $\Omega'(r_0)$ .

The assumption on the characteristic set of  $\partial\Omega$  is crucial to guarantee the smoothness of the distance around the boundary. On the contrary, if  $\partial\Omega$  has characteristic points,  $\delta$  is not even Lipschitz in coordinates, see [ACS18] for more details.

*Proof of Theorem 2.4.1.* Thanks to Theorem 2.4.2, we get a co-area type formula in the sub-Riemannian case: it is enough to compute the expression of the volume in the coordinates induced by the diffeomorphism  $G$ . Namely, we obtain

$$\int_{\Omega'(r)} v(x) d\omega(x) = \int_0^r \int_{\partial\Omega(s)} v(y) d\sigma(y) ds, \quad (2.16)$$

where  $\sigma$  denotes the measure induced by  $\omega$  on  $\partial\Omega(s)$  via the diffeomorphism  $G$ . Since  $G_*\partial_s = \nabla\delta$ , for the corresponding smooth densities, we have  $\sigma = |i_\nu\omega|$ , where  $\nu = -\nabla\delta$ . Formula (2.16) holds as long as the function  $G$  is a smooth diffeomorphism, i.e. for any  $r \leq r_0$ . Consequently, in the interval  $[0, r_0)$ , we can compute the first derivative of the function  $F$  as follows:

$$\begin{aligned} F(r) &= \int_{\Omega(r)} v(x) d\omega(x) = \int_{\Omega} v(x) d\omega(x) - \int_{\Omega'(r)} v(x) d\omega(x) \\ &= \int_{\Omega} v(x) d\omega(x) - \int_0^r \int_{\partial\Omega(s)} v(y) d\sigma(y) ds. \end{aligned}$$

Then, we have

$$F'(r) = \frac{\partial}{\partial r} \left( \int_{\Omega} v(x) d\omega(x) - \int_0^r \int_{\partial\Omega(s)} v(y) d\sigma(y) ds \right) = - \int_{\partial\Omega(r)} v(y) d\sigma(y).$$

For the second derivative of  $F$ , let us rewrite its first derivative, using the divergence theorem (1.13) on  $\Omega(r)$ , and the fact that  $\|\nabla\delta\|_g = 1$ :

$$F'(r) = - \int_{\partial\Omega(r)} v d\sigma = - \int_{\partial\Omega(r)} v g(\nu, \nu) d\sigma = - \int_{\Omega(r)} (v \operatorname{div}_\omega \nu + g(\nabla v, \nu)) d\omega.$$

Using again the fact that  $\Omega'(r) = \bar{\Omega} \setminus \Omega(r)$ , and applying the co-area formula to the previous expression, we obtain

$$\begin{aligned} F''(r) &= - \frac{\partial}{\partial r} \int_{\Omega(r)} (v \operatorname{div}_\omega \nu + g(\nabla v, \nu)) d\omega \\ &= - \frac{\partial}{\partial r} \left( \int_{\Omega} (v \operatorname{div}_\omega \nu + g(\nabla v, \nu)) d\omega - \int_{\Omega'(r)} (v \operatorname{div}_\omega \nu + g(\nabla v, \nu)) d\omega \right) \\ &= - \frac{\partial}{\partial r} \left( - \int_0^r \int_{\partial\Omega(s)} (v \operatorname{div}_\omega \nu + g(\nabla v, \nu)) d\sigma ds \right) \\ &= \int_{\partial\Omega(r)} (-v \Delta \delta + g(\nabla v, \nu)) d\sigma \\ &= \int_{\Omega(r)} \Delta v(x) d\omega(x) - \int_{\partial\Omega(r)} v(y) \Delta \delta(y) d\sigma(y), \end{aligned}$$

where, in the last passage, we have used again the divergence theorem.  $\square$

**Corollary 2.4.3.** *Under the hypotheses of Theorem 2.4.1, the function  $F$ , defined as*

$$F(t, r) = \int_{\Omega(r)} u(t, x) d\omega(x), \quad (2.17)$$

where  $u(t, x)$  denotes the solution to the heat equation on  $\Omega$  (2.3), satisfies the non-homogeneous one-dimensional heat equation

$$(\partial_t - \partial_r^2) F(t, r) = \int_{\partial\Omega(r)} u(t, y) \Delta \delta(y) d\sigma(y), \quad t > 0, \quad r \in [0, r_0),$$

with Neumann boundary condition  $\partial_r F(t, 0) = 0$ .

## 2.5 Heat content asymptotics

In this section, we compute the asymptotic expansion of the sub-Riemannian heat content, associated with the Dirichlet problem on a non-characteristic domain.

Since  $Q_\Omega(0) = \omega(\Omega)$ , we can reduce ourselves to the study of the quantity

$$G(t, r) = \int_{\Omega(r)} (1 - u(t, x)) d\omega(x). \quad (2.18)$$

The small-time asymptotics of the heat content is then recovered by the asymptotics series, as  $t \rightarrow 0$ , of  $G(t, 0)$ , since we have  $G(t, 0) = \omega(\Omega) - Q_\Omega(t)$ . Compare (2.18) with (2.17). While  $F$  satisfies a non-homogeneous heat equation on  $[0, r_0)$ , cf. Corollary 2.4.3, the advantage of  $G$  is that, upon localization, it satisfies a non-homogeneous heat equation on the *whole* half-line. To this purpose, let  $\phi, \eta : \bar{\Omega} \rightarrow \mathbb{R}$  smooth functions with compact support and such that

$$\phi + \eta \equiv 1, \quad \text{supp}(\phi) \subset \Omega'(r_0), \quad \text{supp}(\eta) \subset \Omega(r_0/2),$$

where  $r_0$  is radius of the tubular neighborhood of  $\partial\Omega$  on which the sub-Riemannian distance  $\delta : \bar{\Omega} \rightarrow \mathbb{R}$  is smooth, cf. Theorem 2.4.2. We have then

$$\begin{aligned} G(t, r) &= \int_{\Omega(r)} (1 - u(t, x)) \phi(x) d\omega(x) + \int_{\Omega(r)} (1 - u(t, x)) \eta(x) d\omega(x) \\ &= \int_{\Omega(r)} (1 - u(t, x)) \phi(x) d\omega(x) + \int_{\Omega} (1 - u(t, x)) \eta(x) d\omega(x) \\ &= \int_{\Omega(r)} (1 - u(t, x)) \phi(x) d\omega(x) + O(t^\infty), \end{aligned}$$

where we used Theorem 2.3.1 to deal with the second term, having set  $K = \text{supp}(\eta)$ . For this reason, we focus on the quantity

$$I\phi(t, r) = \int_{\Omega(r)} (1 - u(t, x)) \phi(x) d\omega(x).$$

Since the support of  $\phi$  is contained in  $\Omega'(r_0)$ , it turns out that  $I\phi(t, r)$  is smooth for  $t > 0$  and  $r \geq 0$ , and it is compactly supported in the second variable. This discussion motivates the following definition.

**Definition 2.5.1.** For all  $t \geq 0$ , we define the one-parameter families of operators  $I$  and  $\Lambda$  on the space  $C_c^\infty(\Omega'(r_0))$ , by

$$\begin{aligned} I\phi(t, r) &= \int_{\Omega(r)} (1 - u(t, x)) \phi(x) d\omega(x), \\ \Lambda\phi(t, r) &= -\partial_r I\phi(t, r) = \int_{\partial\Omega(r)} (1 - u(t, y)) \phi(y) d\sigma(y), \end{aligned}$$

for any  $\phi \in C_c^\infty(\Omega'(r_0))$ , and where  $\sigma$  denotes the induced measure on  $\partial\Omega(r)$ .

**Lemma 2.5.2.** *Let  $M$  be a sub-Riemannian manifold, equipped with a smooth measure  $\omega$ , and let  $\Omega \subset M$  be an open relatively compact subset whose boundary is smooth and has not characteristic points. Let  $L = \partial_t - \partial_r^2$  the one-dimensional heat operator. Then, as an operator on  $C_c^\infty(\Omega'(r_0))$*

$$LI = \Lambda N + I\Delta,$$

where  $N$  is the operator defined by

$$N\phi = 2g(\nabla\phi, \nabla\delta) + \phi\Delta\delta, \quad \forall \phi \in C_c^\infty(\Omega'(r_0)), \quad (2.19)$$

and  $\delta : \bar{\Omega} \rightarrow \mathbb{R}$  is the sub-Riemannian distance function from  $\partial\Omega$ .



*Proof.* The computation of  $\partial_t I\phi$  is direct, while for  $\partial_r^2 I\phi$  we apply Theorem 2.4.1 with the function  $v = (1 - u)\phi$ . Then apply the divergence formula, recalling that the sub-Riemannian outward unit normal to  $\partial\Omega(r)$  is  $\nu = -\nabla\delta|_{\partial\Omega(r)}$ . We refer the reader to Lemma B.0.2 for a complete proof.  $\square$

Lemma 2.5.2, in particular, implies that  $I\phi(t, r)$  satisfies a non-homogeneous heat equation on the whole half-line, with Neumann boundary conditions.

**Corollary 2.5.3.** *For any  $\phi \in C_c^\infty(\Omega'(r_0))$ , the function  $I\phi$ , defined as*

$$I\phi(t, r) = \int_{\Omega(r)} (1 - u(t, x))\phi(x)d\omega(x),$$

where  $u(t, x)$  denotes the solution to the heat equation on  $\Omega$  (2.3), satisfies the non-homogeneous one-dimensional heat equation

$$(\partial_t - \partial_r^2)I\phi(t, r) = \Lambda N\phi(t, r) + I\Delta\phi(t, r), \quad t > 0, \quad r \geq 0,$$

with Neumann boundary condition  $\partial_r I\phi(t, 0) = 0$ , for all  $t \geq 0$ .

Thanks to Corollary 2.5.3, and an appropriate Duhamel-type principle, we can obtain an explicit formula for  $I\phi(t, r)$ , which yields its asymptotic series for  $t \rightarrow 0$ . The next lemma contains the general form of the Duhamel's principle that we need.

**Lemma 2.5.4** (Duhamel's principle). *Let  $f \in C((0, \infty) \times [0, \infty))$ ,  $v_0, v_1 \in C([0, \infty))$ , such that  $f(t, \cdot)$  and  $v_0$  are compactly supported and assume that*

$$\exists \lim_{t \rightarrow 0} f(t, r). \tag{2.20}$$

Consider the non-homogeneous heat equation on the half-line:

$$\begin{aligned} Lv(t, r) &= f(t, r), & \text{for } t > 0, \quad r > 0, \\ v(0, r) &= v_0(r), & \text{for } r > 0, \\ \partial_r v(t, 0) &= v_1(t), & \text{for } t > 0, \end{aligned} \tag{2.21}$$

where  $L = \partial_t - \partial_r^2$ . Then, for  $t > 0$ , the solution to (2.21) is given by

$$\begin{aligned} v(t, r) &= \int_0^\infty e(t, r, s)v_0(s)ds + \int_0^t \int_0^\infty e(t - \tau, r, s)f(\tau, s)dsd\tau \\ &\quad - \int_0^t e(t - \tau, r, 0)v_1(\tau)d\tau, \end{aligned} \tag{2.22}$$

where  $e(t, r, s)$  is the Neumann heat kernel on the half-line, that is

$$e(t, r, s) = \frac{1}{\sqrt{4\pi t}} \left( e^{-(r-s)^2/4t} + e^{-(r+s)^2/4t} \right). \tag{2.23}$$

*Remark 2.5.5.* The choice of the Neumann heat kernel in (2.22) is due to the fact that  $I\phi(t, r)$  satisfies Neumann boundary conditions, that is  $v_1 = 0$ , and thus it simplifies the first step of the iteration for the computation of the small-time asymptotics. Subsequent steps will need the general form of (2.22).

*Proof of Lemma 2.5.4.* We derive formula (2.22), assuming that all the functions are sufficiently regular so that all steps are justified. Then, one can check that (2.22) holds true under the regularity assumptions in the statement, concluding the proof by uniqueness



of the solution to (2.21). Let us reduce problem (2.21) to a standard Neumann problem, defining  $\tilde{v}(t, r) = v(t, r) - rv_1(t)$ . Now,  $\tilde{v}$  satisfies:

$$\begin{aligned} (\partial_t - \partial_r^2)\tilde{v}(t, r) &= f(t, r) - rv_1'(t), & \text{for } t > 0, r > 0, \\ \tilde{v}(0, r) &= v_0(r) - rv_1(0), & \text{for } r > 0, \\ \partial_r \tilde{v}(t, 0) &= 0, & \text{for } t > 0. \end{aligned}$$

Therefore, we know that  $\tilde{v}$  is given by

$$\begin{aligned} \tilde{v}(t, r) &= \int_0^\infty e(t, r, s) (v_0(s) - sv_1(0)) ds \\ &\quad + \int_0^t \int_0^\infty e(t - \tau, r, s) (f(\tau, s) - sv_1'(\tau)) ds d\tau. \end{aligned}$$

Recalling that  $v(t, r) = \tilde{v}(t, r) + rv_1(t)$ , integrating twice by parts, and using the properties of the Neumann heat kernel  $e(t, s, r)$ , we obtain (2.22).  $\square$

We can now prove Theorem 2.1.1. We first prove it at first order (Theorem 2.5.6), and then we iterate the construction obtaining the whole asymptotics (Theorem 2.5.8 and Proposition 2.1.2).

### 2.5.1 First order asymptotic expansion

**Theorem 2.5.6.** *Let  $M$  be a sub-Riemannian manifold, equipped with a smooth measure  $\omega$ , and let  $\Omega \subset M$  be an open relatively compact subset whose boundary is smooth and has not characteristic points. The heat content  $Q_\Omega(t)$  satisfies*

$$Q_\Omega(t) = \omega(\Omega) - \sqrt{\frac{4t}{\pi}} \sigma(\partial\Omega) + O(t), \quad \text{as } t \rightarrow 0,$$

where  $\sigma$  is the sub-Riemannian measure induced by  $\omega$  on  $\partial\Omega$ .

*Proof.* Let  $\phi \in C_c^\infty(\Omega'(r_0))$  and apply Lemma 2.5.4 to  $v(t, r) = I\phi(t, r)$ . On one hand, using the definitions of the operators  $I$  and  $\Lambda$ , the initial conditions are:

$$\begin{aligned} v_0(r) &= v(0, r) = I\phi(0, r) = 0 \\ v_1(t) &= \partial_r v(t, 0) = -\Lambda\phi(t, 0) = -\int_{\partial\Omega} \phi(y) d\sigma(y), \end{aligned}$$

where we used the fact that  $u(0, \cdot) = 1$  on  $\Omega$  and  $u(t, \cdot)|_{\partial\Omega} = 0$ . On the other hand, for the non-homogeneous term of the heat equation, we obtain using Lemma 2.5.2

$$f(t, r) = LI\phi(t, r) = \Lambda N\phi(t, r) + I\Delta\phi(t, r).$$

Notice that  $v_0, f(t, \cdot)$  are compactly supported, and  $v_1(t)$  is constant. Furthermore  $\lim_{t \rightarrow 0} f(t, r)$  is well-defined for all  $r \geq 0$  since  $N\phi$  and  $\Delta\phi$  belong to  $C_c^\infty(\Omega'(r_0))$ . Having checked the assumptions, we obtain from (2.22):

$$\begin{aligned} I\phi(t, 0) &= \frac{1}{\sqrt{\pi}} \int_{\partial\Omega} \phi(y) d\sigma(y) \int_0^t \frac{d\tau}{\sqrt{t-\tau}} + \int_0^t \int_0^\infty e(t-\tau, s, 0) LI\phi(\tau, s) ds d\tau \\ &= \sqrt{\frac{4t}{\pi}} \int_{\partial\Omega} \phi(y) d\sigma(y) + \int_0^t \int_0^\infty e(t-\tau, s, 0) LI\phi(\tau, s) ds d\tau. \end{aligned}$$

We now prove that the second term in the right-hand side is a remainder of order  $O(t)$ . Notice that

$$|LI\phi(t, s)| = |\Lambda N\phi(t, s) + I\Delta\phi(t, s)| \leq C_1 \|N\phi\|_{L^\infty(\Omega)} + C_2 \|\Delta\phi\|_{L^\infty(\Omega)} = C,$$

having used the inequality  $0 \leq 1 - u(t, x) \leq 1$  by the weak maximum principle (2.4). Thus, we have

$$\begin{aligned} \left| \int_0^t \int_0^\infty e(t-\tau, s, 0) LI\phi(\tau, s) ds d\tau \right| &\leq C \int_0^t \int_0^\infty e(t-\tau, s, 0) ds d\tau \\ &= C \int_0^t \frac{1}{\sqrt{\pi(t-\tau)}} \int_0^\infty e^{-s^2/4(t-\tau)} ds d\tau \\ &= Ct, \end{aligned}$$

implying that:

$$I\phi(t, 0) = \sqrt{\frac{4t}{\pi}} \int_{\partial\Omega} \phi(y) d\sigma(y) + O(t), \quad \text{as } t \rightarrow 0. \quad (2.24)$$

We now apply the argument at the beginning of Section 2.5, by choosing  $\phi$  to be a function with  $\text{supp}(\phi) \subset \Omega'(r_0)$ , and such that  $\phi \equiv 1$  when restricted to  $\Omega'(r_0/2)$ . In particular  $\phi|_{\partial\Omega} = 1$ . In this case  $\omega(\Omega) - Q_\Omega(t)$  has the same small-time asymptotics of  $I\phi(t, 0)$  up to an  $O(t^\infty)$ , and hence

$$Q_\Omega(t) = \omega(\Omega) - I\phi(t, 0) + O(t^\infty) = \omega(\Omega) - \sqrt{\frac{4t}{\pi}} \sigma(\partial\Omega) + O(t), \quad \text{as } t \rightarrow 0,$$

concluding the proof of the first-order asymptotics.  $\square$

### 2.5.2 Full asymptotic expansion

We would like to iterate the Duhamel's principle (2.22) for the function  $I\phi(t, r)$  to obtain an higher order asymptotic expansion of  $I\phi(t, 0)$ . However, the source term  $f(t, r) = L^k\phi(t, r)$  for  $k \geq 2$  may not satisfy the assumption (2.20), due to the non-differentiability of the solution  $u(t, x)$  of the Dirichlet problem at  $t = 0$ . To avoid this technical issue, one can introduce a suitable mollification of the operators  $I$  and  $\Lambda$ , for which one can iterate Duhamel's principle at arbitrary order. There is no substantial difference with respect to the Riemannian case developed by Savo in [Sav98]. In this section we only outline the main passages leading to the proof of Theorem 2.1.1 at arbitrary order, and we included more details, for self-consistence, in Appendix B.

The iterations of the Duhamel's principle lead to define the following family of operators acting on smooth functions on  $\bar{\Omega}$  with compact support around  $\partial\Omega$ . Set

$$R_{00} = \text{Id}, \quad S_{00} = 0,$$

where  $\text{Id}$  denotes the identity operator. Recall the definition of  $N$  in (2.19) and set, for all  $k \geq 1$ , and  $0 \leq j \leq k$

$$R_{kj} = -(N^2 - \Delta)R_{k-1,j} + NS_{k-1,j}, \quad (2.25)$$

$$S_{kj} = NR_{k-1,j-1} - \Delta NR_{k-1,j} + \Delta S_{k-1,j}, \quad (2.26)$$

and  $R_{kj} = S_{kj} = 0$ , for all other values of the indices, i.e.  $k < 0$ ,  $j < 0$  or  $k < j$ . Moreover, for  $k \geq 0$ , define

$$Z_k = \sum_{j=0}^{k-1} \{k, j-1\} R_{k+j-1,j} \quad \text{and} \quad A_k = \sum_{j=0}^{k+1} \{k, j\} S_{k+j,j}, \quad (2.27)$$

with the convention that  $Z_0 = 0$ , and having set

$$\{k, j\} = \frac{\Gamma(k+j+1/2)}{(k+j)!\Gamma(k+1/2)},$$

and  $\Gamma$  denotes the Euler Gamma function. Finally, let  $D_k$  be the operators acting on smooth functions on  $\bar{\Omega}$  with compact support around  $\partial\Omega$ , defined inductively by the formulas:

$$D_1 = \frac{2}{\sqrt{\pi}} \text{Id}, \quad (2.28)$$

$$D_{2n} = \frac{1}{\sqrt{\pi}} \sum_{i=1}^n \frac{\Gamma(i+1/2)\Gamma(n-i+1/2)}{n!} D_{2i-1} A_{n-i}, \quad (2.29)$$

$$D_{2n+1} = \frac{1}{\sqrt{\pi}} Z_{n+1} + \frac{1}{\sqrt{\pi}} \sum_{i=1}^n \frac{\Gamma(i+1)\Gamma(n-i+1/2)}{\Gamma(n+3/2)} D_{2i} A_{n-i}. \quad (2.30)$$

The coefficients of the small-time asymptotic expansion of the heat content are computed using the recursive formulas (2.28)–(2.30). See Appendix C for a *Mathematica* implementation of this algorithm.

*Remark 2.5.7.* The operators  $D_k$  act on smooth functions on  $\bar{\Omega}$ ,  $\phi$  compactly supported around  $\partial\Omega$ . The restriction  $D_k(\phi)|_{\partial\Omega}$  is a smooth function on  $\partial\Omega$ , which depends only on the germ of  $\phi$  at  $\partial\Omega$ . Thus, the integrand  $D_k(1)$  appearing in the definition of the  $a_k$  in Theorem 2.5.8 is a slight abuse of notation to denote the action of  $D_k$  on a smooth function, with compact support, such that  $\phi \equiv 1$  in a neighborhood of  $\partial\Omega$ .

**Theorem 2.5.8.** *Let  $M$  be a sub-Riemannian manifold, equipped with a smooth measure  $\omega$ , and let  $\Omega \subset M$  be an open relatively compact subset whose boundary is smooth and has not characteristic points. Then for all  $m \geq 1$ , the heat content  $Q_\Omega(t)$  satisfies*

$$Q_\Omega(t) = \sum_{k=0}^{m-1} a_k t^{k/2} + O(t^{m/2}), \quad \text{as } t \rightarrow 0, \quad (2.31)$$

where  $a_0 = \omega(\Omega)$  and  $a_k = -\int_{\partial\Omega} D_k(1) d\sigma$ .

*Proof.* As in the proof of Theorem 2.5.6, we apply the argument at the beginning of Section 2.5, by choosing  $\phi$  to be a function with  $\text{supp}(\phi) \subset \Omega'(r_0)$ , and such that  $\phi \equiv 1$  when restricted to  $\Omega'(r_0/2)$ . Hence

$$Q_\Omega(t) = \omega(\Omega) - I\phi(t, 0) + O(t^\infty), \quad \text{as } t \rightarrow 0. \quad (2.32)$$

It is enough to compute the asymptotic expansion of  $I\phi(t, 0)$ . Since  $\phi \in C_c^\infty(\Omega'(r_0))$ , we can apply the iteration of the Duhamel's formula to  $I\phi(t, 0)$ , cf. Theorem B.0.7, obtaining, for any  $m \in \mathbb{N}$

$$I\phi(t, 0) = \sum_{k=1}^m \left( \int_{\partial\Omega} D_k \phi(y) d\sigma(y) \right) t^{k/2} + O(t^{(m+1)/2}), \quad \text{as } t \rightarrow 0. \quad (2.33)$$

Since, by construction,  $\phi \equiv 1$  close to  $\partial\Omega$ , the coefficients in (2.33) simplify to

$$a_k = -\int_{\partial\Omega} D_k(\phi) d\sigma = -\int_{\partial\Omega} D_k(1) d\sigma, \quad \forall k \in \mathbb{N}.$$

We conclude the proof by replacing (2.33) in (2.32).  $\square$

## 2.6 Riemannian approximations and asymptotic series

In this section, we show that the coefficients of the sub-Riemannian heat content asymptotics can be approximated by their Riemannian counterpart (cf. Theorem 2.1.6).

Let  $(\mathcal{D}, g)$  be a sub-Riemannian structure on  $M$  and fix a Riemannian variation,  $\{(M, g_\varepsilon)\}_{\varepsilon \in \mathbb{R}}$ , of the type illustrated in Section 2.2.2. As we explained there, it holds

$$d_\varepsilon \xrightarrow{\varepsilon \rightarrow 0} d_{\text{SR}}, \quad \text{uniformly on the compact sets of } M \times M.$$

We begin with a result, of independent interest, on the corresponding approximation result for the distance from the boundary of a compact set.

**Lemma 2.6.1.** *Let  $M$  be a sub-Riemannian manifold, and let  $\Omega \subset M$  be an open relatively compact subset whose boundary is smooth and has not characteristic points. Let  $\delta, \delta_\varepsilon : \bar{\Omega} \rightarrow \mathbb{R}$  the sub-Riemannian and  $\varepsilon$ -Riemannian distances from  $\partial\Omega$ , and fix  $\bar{\varepsilon} > 0$ . Then, there exists  $U \subset \bar{\Omega}$ , neighborhood of  $\partial\Omega$ , such that  $\delta, \delta_\varepsilon \in C^\infty(U)$  for any  $|\varepsilon| < \bar{\varepsilon}$  and  $\delta_\varepsilon \rightarrow \delta$ , as  $\varepsilon \rightarrow 0$ , uniformly on the compact sets of  $U$ , with all their derivatives.*

*Proof.* Let us consider the annihilator bundle of the tangent bundle to  $\partial\Omega$ , i.e. the 1-dimensional smooth vector bundle  $\mathcal{A}(\partial\Omega)$  over  $\partial\Omega$  with fibers

$$\mathcal{A}_y(\partial\Omega) = \{\lambda \in T_y^*M \mid \langle \lambda, T_y\partial\Omega \rangle = 0\}, \quad \forall y \in \partial\Omega.$$

For all  $\varepsilon \in \mathbb{R}$ , denote by  $E_\varepsilon$  the restriction to  $\mathcal{A}(\partial\Omega)$  of the cotangent exponential map in the  $\varepsilon$ -Riemannian approximant, namely

$$E_\varepsilon(\lambda) = \exp_{\pi(\lambda)}^\varepsilon(\lambda) = \pi \circ e^{\bar{H}_\varepsilon}(\lambda), \quad \lambda \in \mathcal{A}(\partial\Omega).$$

Here,  $H_\varepsilon : T^*M \rightarrow \mathbb{R}$  is the one-parameter family of Hamiltonians for the  $\varepsilon$ -Riemannian structure,  $\bar{H}_\varepsilon$  the corresponding Hamiltonian vector field, and  $e^{\bar{H}_\varepsilon}$  the corresponding flow (cf. Section 1.2). In terms of the generating frame (2.7), we have

$$H_\varepsilon(\lambda) = \frac{1}{2} \sum_{i=1}^N \langle \lambda, X_i \rangle^2 + \varepsilon^2 \frac{1}{2} \sum_{i=1}^L \langle \lambda, \tilde{X}_i \rangle^2, \quad \forall \lambda \in T^*M. \quad (2.34)$$

Notice that the value  $\varepsilon = 0$  corresponds to the corresponding sub-Riemannian quantities, so that the subscript is omitted when  $\varepsilon = 0$ .

Thanks to the non-characteristic assumption, for any  $\lambda \in \mathcal{A}(\partial\Omega)$ , with  $\lambda \neq 0$ , it holds  $\lambda(\mathcal{D}) \neq 0$ , and hence  $H_\varepsilon(\lambda) > 0$ . It follows that the  $H_\varepsilon$ , for all  $\varepsilon \in \mathbb{R}$ , are well-defined norms on the one-dimensional fibers of  $\mathcal{A}(\partial\Omega)$ .

Define the map<sup>3</sup>

$$F : \mathcal{A}(\partial\Omega) \times \mathbb{R} \rightarrow M \times \mathbb{R}, \quad F(\lambda, \varepsilon) = (E_\varepsilon(\lambda), \varepsilon).$$

Let  $i : \partial\Omega \hookrightarrow \mathcal{A}(\partial\Omega)$  be the embedding as the set of zero covectors. Thanks to the non-characteristic assumption, one can show that  $F$  has full rank on  $\partial\Omega \times \mathbb{R}$ , that is around points  $(i(y), 0)$ . Set  $I = (-\bar{\varepsilon}, \bar{\varepsilon})$ . It follows that there exists  $a > 0$  such that, letting

$$V = \{\lambda \in \mathcal{A}(\partial\Omega) \mid \sqrt{2H(\lambda)} < a\},$$

the map  $F$  restricts to a smooth diffeomorphism from  $V \times I$  onto its image. In particular, denoting by  $p_1 : M \times \mathbb{R} \rightarrow M$  the projection onto the first factor, each map  $E_\varepsilon(\cdot) = p_1(F(\cdot, \varepsilon))$  is a smooth diffeomorphism from  $V$  to its image, for all  $\varepsilon \in I$ . Notice that, by (2.34), it holds  $H \leq H_\varepsilon$  for all  $\varepsilon$ , so that, letting

$$V_\varepsilon = \{\lambda \in \mathcal{A}(\partial\Omega) \mid \sqrt{2H_\varepsilon(\lambda)} < a\}, \quad \forall \varepsilon \in I,$$

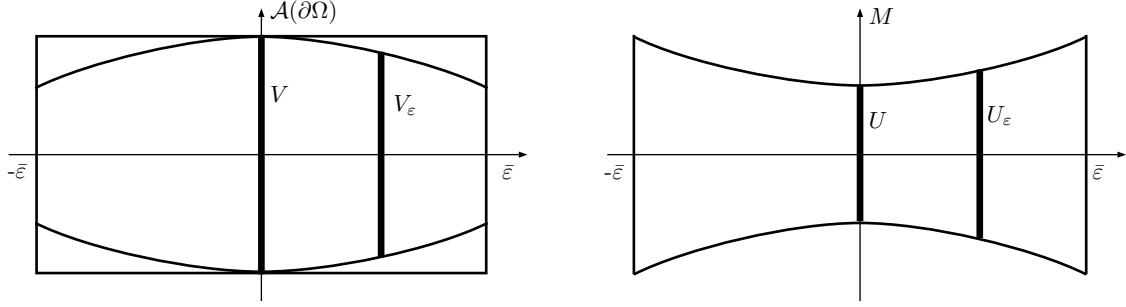


Figure 2.2: Domain and range of the diffeomorphism  $F(\cdot)$ . The sections  $V_\varepsilon$  (resp.  $U_\varepsilon$ ) are monotonically non-increasing (resp. non-decreasing) family of sets as a function of  $|\varepsilon|$ .

it holds  $V_\varepsilon \subset V$  and thus  $E_\varepsilon$  maps diffeomorphically  $V_\varepsilon$  to its image  $U_\varepsilon = E_\varepsilon(V_\varepsilon)$ , see figure 2.2. We claim that  $U_\varepsilon = \{\delta_\varepsilon < a\}$  and  $\delta_\varepsilon$  is smooth on  $\bar{\Omega} \cap U_\varepsilon$ .

In order to prove the claim, fix  $\varepsilon \in I$ . For  $\lambda \in V_\varepsilon$ , let  $\gamma_\lambda : [0, 1] \rightarrow M$  be the  $g_\varepsilon$ -geodesic with initial covector  $\lambda$ , that is  $\gamma_\lambda(t) = \exp_{\pi(\lambda)}^\varepsilon(t\lambda)$ . For the  $g_\varepsilon$ -length it holds  $\ell_\varepsilon(\gamma_\lambda) = \sqrt{2H_\varepsilon(\lambda)}$ . It follows that  $U_\varepsilon \subseteq \{\delta_\varepsilon < a\}$ . Furthermore, for any point  $x \in \{\delta_\varepsilon < a\}$  there exists by compactness at least one geodesic such that its length coincides with the distance  $\delta_\varepsilon(x)$  of  $x$  from  $\partial\Omega$ . Such a geodesic is necessarily normal by Proposition 1.2.3, its initial covector  $\lambda$  must be in  $\mathcal{A}(\partial\Omega)$  by (1.11), and indeed  $\sqrt{2H_\varepsilon(\lambda)} < a$ . There is a unique such a covector in  $V_\varepsilon$ , and any other covector not in  $V_\varepsilon$  yields a longer geodesic. It follows that  $U_\varepsilon = \{\delta_\varepsilon < a\}$ , and furthermore

$$\delta_\varepsilon(E_\varepsilon(\lambda)) = \sqrt{2H_\varepsilon(\lambda)}, \quad \forall \lambda \in V_\varepsilon.$$

In particular  $\delta_\varepsilon : \bar{\Omega} \rightarrow \mathbb{R}$  is smooth on  $U_\varepsilon \cap \bar{\Omega}$ , for all fixed  $\varepsilon \in I$ , proving the claim (notice that  $2H_\varepsilon$  is a homogeneous norm of degree 2, on a one-dimensional space, so that  $\delta_\varepsilon$  is smooth up to  $\partial\Omega$ ).

Let now

$$U := \bigcap_{\varepsilon \in I} U_\varepsilon = \bigcap_{\varepsilon \in I} \{\delta_\varepsilon < a\} = \{\delta < a\},$$

where in the last equality we used the monotonicity of  $\delta_\varepsilon$ , which follows from the analogue property for the approximating distances  $d_\varepsilon$ . Notice also that  $U \times I \subset F(V \times I)$ , in particular  $F^{-1}$  is a well-defined diffeomorphism on  $U \times I$ . Therefore

$$\delta_\varepsilon(q) = \sqrt{2H_\varepsilon \circ F^{-1}(q, \varepsilon)}, \quad \forall q \in U, \varepsilon \in I.$$

The above formula, together with the smoothness of  $H_\varepsilon$  and the fact that it is a well-defined quadratic form on the one-dimensional fibers  $\mathcal{A}_q(\partial\Omega)$ , implies the joint smoothness of  $(q, \varepsilon) \mapsto \delta_\varepsilon(q)$  in both variables and up to the boundary on  $U \cap \bar{\Omega} \times I$ .  $\square$

We equip  $M$  with a smooth measure  $\omega$ , which will be the same for the sub-Riemannian structure  $(\mathcal{D}, g)$ , and for the Riemannian variation  $g_\varepsilon$ . Let  $\Delta_\varepsilon = \operatorname{div}_\omega \circ \nabla_\varepsilon$  be the corresponding weighted Laplace-Beltrami on  $(M, g_\varepsilon)$ , where  $\nabla_\varepsilon$ , for all  $\varepsilon \neq 0$ , is the Riemannian gradient for  $g_\varepsilon$ . Notice that the semi-groups associated with the Dirichlet extensions of  $\Delta$  and  $\Delta_\varepsilon$  are defined on the same  $L^2(\Omega, \omega)$ . Denote by  $Q_\Omega^\varepsilon$  the corresponding  $\varepsilon$ -Riemannian heat content. Recall that, as explained in Section 2.2.2, we have

$$Q_\Omega^\varepsilon(t) \xrightarrow{\varepsilon \rightarrow 0} Q_\Omega(t), \quad \text{uniformly on } [0, T].$$

<sup>3</sup>To avoid complications, we assume here that all the vector fields  $\vec{H}_\varepsilon$  are complete. If this is not the case, the domain of  $F$  should be replaced by a suitable neighborhood of  $i(\partial\Omega) \times \mathbb{R}$ .

We know that, for any  $\varepsilon \neq 0$ , there exists a complete asymptotic series of  $Q_\Omega^\varepsilon(t)$ , namely

$$Q_\Omega^\varepsilon(t) \sim \sum_{k=0}^{\infty} a_k^\varepsilon t^{k/2}, \quad \text{as } t \rightarrow 0.$$

Moreover, a recursive formula for the coefficients is provided in [Sav98] (notice that the results therein hold for the Riemannian measure, but as we have proven in the previous sections one can generalize these formulas for the arbitrary measure  $\omega$ ). Define the operators  $D_k^\varepsilon$  as in (2.28)–(2.30), replacing the sub-Laplacian  $\Delta$  with the  $\varepsilon$ -Riemannian one  $\Delta_\varepsilon$ , and the operator  $N$ , with the corresponding Riemannian one  $N_\varepsilon$ , defined by

$$N_\varepsilon \phi = 2g_\varepsilon(\nabla_\varepsilon \phi, \nabla_\varepsilon \delta_\varepsilon) + \phi \Delta_\varepsilon \delta_\varepsilon,$$

where  $\delta_\varepsilon: \bar{\Omega} \rightarrow [0, \infty)$  is the  $\varepsilon$ -Riemannian distance from  $\partial\Omega$ . In particular,  $D_k^\varepsilon$  belongs to the algebra generated by  $\Delta_\varepsilon$  and  $N_\varepsilon$ . Then, as in the sub-Riemannian case, the  $k$ -th coefficient is given by

$$a_k^\varepsilon = - \int_{\partial\Omega} D_k^\varepsilon(1) d\sigma_\varepsilon, \quad (2.35)$$

where  $\sigma_\varepsilon$  is the induced Riemannian measure on  $\partial\Omega$ .

**Theorem 2.6.2.** *Let  $M$  be a sub-Riemannian manifold, equipped with a smooth measure  $\omega$ , and let  $\Omega \subset M$  be an open relatively compact subset whose boundary is smooth and has not characteristic points. Then, there exists a family of Riemannian metrics  $g_\varepsilon$  such that  $d_\varepsilon \rightarrow d_{\text{SR}}$  uniformly on compact sets of  $M$ , and such that*

$$\lim_{\varepsilon \rightarrow 0} a_k^\varepsilon = a_k, \quad \forall k \in \mathbb{N},$$

where  $a_k$  and  $a_k^\varepsilon$  denote the coefficients of the sub-Riemannian small-time heat content asymptotics, and the corresponding ones for the Riemannian approximating structure.

*Remark 2.6.3.* We believe that Theorem 2.6.2 can be strengthened and that the convergence result actually holds for *any* one-parameter family of Riemannian structures  $\{(M, g_\varepsilon)\}$  such that the corresponding Hamiltonians  $\{H_\varepsilon\}$  converge to the sub-Riemannian one (uniformly on compact sets as  $\varepsilon \rightarrow 0$ ), cf. (2.34). We refrained to prove this more general result as it would not add new ideas. Furthermore, one advantage of the current version of Theorem 2.6.2 is that the construction of the approximating family is explicit.

*Proof.* First of all, we have that  $\Delta_\varepsilon \delta_\varepsilon \rightarrow \Delta \delta$  as  $\varepsilon \rightarrow 0$ , uniformly on the compact sets of  $U$ , with all the derivatives, where  $U$  is the set given by Lemma 2.6.1. Indeed, using the explicit expression of the  $\varepsilon$ -Laplacian provided in (2.9), with respect to a global generating frame for the Riemannian variation, we have

$$\Delta_\varepsilon = \Delta + \varepsilon^2 \tilde{\Delta}, \quad (2.36)$$

where  $\tilde{\Delta}$  is a second order differential operator independent from  $\varepsilon$ . Thanks to Lemma 2.6.1, we know that  $\tilde{\Delta} \delta_\varepsilon \rightarrow \tilde{\Delta} \delta$  as  $\varepsilon \rightarrow 0$ , with all the derivatives, uniformly on the compact subsets of  $U$ . Hence, we get

$$\Delta_\varepsilon \delta_\varepsilon = \Delta \delta_\varepsilon + \varepsilon^2 \tilde{\Delta} \delta_\varepsilon \xrightarrow{\varepsilon \rightarrow 0} \Delta \delta, \quad (2.37)$$

with all the derivatives, uniformly on the compact subsets of  $U$ .

Second of all, we claim that

$$D_k^\varepsilon(\phi) \xrightarrow{\varepsilon \rightarrow 0} D_k(\phi), \quad \forall \phi \in C_c^\infty(U), \quad (2.38)$$

uniformly on the compact subsets of  $U$ . The operators  $D_k^\varepsilon$  and  $D_k$  are generated by finite combinations of elements in  $\{N_\varepsilon, \Delta_\varepsilon\}$  and  $\{N, \Delta\}$ , respectively. Then, it is sufficient to prove that, for any sequence  $\phi_\varepsilon \in C_c^\infty(U)$  such that  $\phi_\varepsilon \rightarrow \phi$  uniformly with all the derivatives on the compact sets of  $U$ , and for any  $s \in \mathbb{N}$ , it holds

$$\Delta_\varepsilon^s \phi_\varepsilon \xrightarrow{\varepsilon \rightarrow 0} \Delta^s \phi \quad \text{and} \quad N_\varepsilon^s \phi_\varepsilon \xrightarrow{\varepsilon \rightarrow 0} N^s \phi, \quad (2.39)$$

uniformly on the compact sets of  $U$ , with all the derivatives. The first statement in (2.39) follows directly from (2.36). To prove the second statement of (2.39), proceed by induction on  $s \in \mathbb{N}$ : for  $s = 1$ , let us write explicitly

$$N_\varepsilon \phi_\varepsilon = 2g_\varepsilon(\nabla_\varepsilon \phi_\varepsilon, \nabla_\varepsilon \delta_\varepsilon) + \phi_\varepsilon \Delta_\varepsilon \delta_\varepsilon. \quad (2.40)$$

From (2.37), the second term in (2.40) converges as required. On the other hand, using the definition of  $\varepsilon$ -gradient, we have

$$g_\varepsilon(\nabla_\varepsilon \phi_\varepsilon, \nabla_\varepsilon \delta_\varepsilon) = d\delta_\varepsilon(\nabla_\varepsilon \phi_\varepsilon). \quad (2.41)$$

Now, thanks to (2.8), we can write the  $\varepsilon$ -Riemannian gradient with respect to the global generating frame (2.7), as

$$\nabla_\varepsilon \phi_\varepsilon = \nabla \phi_\varepsilon + \varepsilon^2 \tilde{\nabla} \phi_\varepsilon, \quad \forall \varepsilon > 0,$$

where the sequence  $\{\tilde{\nabla} \phi_\varepsilon\}$  has coefficients which are uniformly bounded, together with all the derivatives, on the compact subsets of  $U$ . Thus, using (2.41), we obtain

$$g_\varepsilon(\nabla_\varepsilon \phi_\varepsilon, \nabla_\varepsilon \delta_\varepsilon) = d\delta_\varepsilon(\nabla \phi_\varepsilon) + \varepsilon^2 d\delta_\varepsilon(\tilde{\nabla} \phi_\varepsilon) = g(\nabla \delta_\varepsilon, \nabla \phi_\varepsilon) + \varepsilon^2 d\delta_\varepsilon(\tilde{\nabla} \phi_\varepsilon). \quad (2.42)$$

Since the sequence of smooth functions  $\{d\delta_\varepsilon(\tilde{\nabla} \phi_\varepsilon)\}$  is uniformly bounded on the compact subsets of  $U$ , together with all the derivatives, the second term in (2.42) converges to 0. For the first term in (2.42), using Lemma 2.6.1, we have

$$g(\nabla \delta_\varepsilon, \nabla \phi_\varepsilon) \xrightarrow{\varepsilon \rightarrow 0} g(\nabla \phi, \nabla \delta)$$

with all the derivatives, uniformly on the compact sets of  $U$ , implying that (2.40) converges as required. Assume that (2.39) holds for  $s-1$ , for any sequence  $\{\phi_\varepsilon\}$ . Define  $\psi_\varepsilon = N_\varepsilon^{s-1} \phi_\varepsilon$  then, by induction hypothesis  $\psi_\varepsilon \rightarrow N^{s-1} \phi = \psi$ , with all the derivatives, uniformly on the compact sets of  $U$ . Thus, applying the previous step, we have

$$N_\varepsilon \psi_\varepsilon \xrightarrow{\varepsilon \rightarrow 0} N \psi,$$

with all the derivatives, uniformly on the compact sets of  $U$ . This proves claim (2.38).

Finally, Lemma 2.6.1 implies that the sequence of Riemannian measures induced on  $\partial\Omega$ , i.e.  $\{\sigma_\varepsilon\}$  converges weakly to  $\sigma$  and, from the explicit expression of both the Riemannian and sub-Riemannian coefficients, respectively (2.35) and (2.31), we conclude.  $\square$

## 2.7 Blow-up of $a_5$ in a domain with characteristic points

We provide here an explicit example of a domain with an isolated characteristic point, in which the fifth coefficient of the asymptotic expansion (2.31) blows up, proving Theorem

2.1.7 in the Section 2.1. Let us consider the sub-Riemannian structure on  $\mathbb{R}^3$ , defined by the global generating frame

$$X_1 = \partial_x - \frac{y}{2}\partial_z, \quad X_2 = \partial_y + \frac{x}{2}\partial_z,$$

and we set  $\{X_1, X_2\}$  to be an orthonormal frame. The resulting sub-Riemannian manifold is the well-known first Heisenberg group,  $\mathbb{H}$ . We equip it with the standard Lebesgue measure, and the corresponding sub-Laplacian is  $\Delta = X_1^2 + X_2^2$ . Let  $\Sigma$  to be the  $xy$ -plane, i.e. the zero-level set of the function

$$u(x, y, z) = z.$$

The characteristic points are solution to  $X_1(u)(x, y, z) = X_2(u)(x, y, z) = 0$ . Thus,  $\Sigma$  has only one isolated characteristic point at  $(0, 0, 0)$ . The sub-Riemannian distance from  $\Sigma$ , denoted by  $\delta : \mathbb{H} \rightarrow \mathbb{R}$ , remains smooth in a neighborhood of non-characteristic points but, at the origin, it is no longer smooth. Thus, to investigate the behavior of the heat content coefficients, we need an explicit expression for  $\delta$ .

*Remark 2.7.1.* The surface  $\Sigma$  is not compact. Since we are interested in local integrability properties of the coefficient  $a_5$  around a characteristic point, it is not restrictive to work with a non-compact surface.

In order to study the function  $\delta$ , we employ the symmetries of the Heisenberg group, to obtain a substantial dimensional reduction. First of all, the distance function  $d(\cdot, \cdot)$  is 1-homogeneous with respect to the one-parameter family of anisotropic dilations introduced in (1.23). To avoid confusion in the notation, only in this section, we will denote the dilation by  $\chi_t$  and set

$$\chi_t : \mathbb{H} \rightarrow \mathbb{H}; \quad \chi_t(x, y, z) = (tx, ty, t^2z), \quad \forall t > 0. \quad (2.43)$$

This means that

$$d(\chi_t(p), \chi_t(q)) = td(p, q), \quad \forall p, q \in \mathbb{H}.$$

Since the dilations  $\chi_t$  are injective and fix  $\Sigma$ , we obtain the analogous property for  $\delta$ :

$$\delta(\chi_t(p)) = \inf_{q \in \Sigma} d(\chi_t(p), q) = \inf_{\chi_t(q) \in \Sigma} d(\chi_t(p), \chi_t(q)) = \inf_{\chi_t(q) \in \Sigma} td(p, q) = t\delta(p).$$

Second of all, an isometry  $L : \mathbb{H} \rightarrow \mathbb{H}$  preserves the distance from  $\Sigma$  if and only if  $L(\Sigma) = \Sigma$ . Among the isometries of  $\mathbb{H}$ , the rotations around the  $z$ -axis and the reflection

$$L : \mathbb{H} \rightarrow \mathbb{H}, \quad L(x, y, z) = (x, -y, -z)$$

preserve  $\Sigma$ . This means that, when expressed in cylindrical coordinates

$$\begin{cases} x = r \cos \varphi \\ y = r \sin \varphi \\ z = z \end{cases} \quad (2.44)$$

the function  $\delta$  does not depend on the angle  $\varphi$ , nor on the sign of  $z$ .

Thus, letting  $p = (r_p, \varphi_p, z_p)$  and assuming  $r_p \neq 0$ , we have

$$\delta(p) = \delta(r_p, \varphi_p, z_p) = \delta(r_p, 0, |z_p|) = r_p \delta\left(1, 0, \frac{|z_p|}{r_p}\right) = r_p F\left(\frac{|z_p|}{r_p}\right), \quad (2.45)$$

for a suitable function  $F$ .



### 2.7.1 Cut and focal locus from $\Sigma$

We refer to the preliminaries in Section 1.2 for basic facts about geodesics in sub-Riemannian geometry. The singular points of  $\delta$  are related to where the geodesics spreading out from  $\Sigma$  lose optimality.

Recall that  $\mathbb{H}$  has no non-trivial abnormal geodesics. Thus, by Proposition 1.2.3, we may focus on  $\Sigma_0 = \Sigma \setminus \{0\}$ . By the transversality condition (1.11), for any point  $(x_0, y_0, 0) \in \Sigma_0$ , there exists a unique normal geodesic parametrized by arc-length with initial covector

$$\lambda(0) = \frac{2}{\sqrt{x_0^2 + y_0^2}} dz.$$

Writing in coordinates  $\lambda = (\lambda_x, \lambda_y, \lambda_z; x, y, z)$  the Hamiltonian system (1.9), we obtain

$$\begin{aligned} \dot{x} &= \lambda_x - \frac{y}{2} \lambda_z, & \dot{\lambda}_x &= -\frac{1}{2} \lambda_z \left( \lambda_y + \frac{x}{2} \lambda_z \right), \\ \dot{y} &= \lambda_y + \frac{x}{2} \lambda_z, & \dot{\lambda}_y &= \frac{1}{2} \lambda_z \left( \lambda_x - \frac{y}{2} \lambda_z \right), \\ \dot{z} &= \frac{1}{2} (x\dot{y} - \dot{x}y), & \dot{\lambda}_z &= 0. \end{aligned}$$

Imposing the initial condition  $\lambda(0) = (0, 0, \frac{2}{r_0}; x_0, y_0, 0)$ , we can explicitly compute the geodesic starting from the point  $(x_0, y_0, 0) \in \Sigma_0$

$$\gamma_{x_0, y_0}(t) = \frac{1}{2} \begin{pmatrix} x_0 \left( 1 + \cos \left( \frac{2t}{r_0} \right) \right) - y_0 \sin \left( \frac{2t}{r_0} \right) \\ y_0 \left( 1 + \cos \left( \frac{2t}{r_0} \right) \right) + x_0 \sin \left( \frac{2t}{r_0} \right) \\ \frac{r_0^2}{4} \left( \frac{2t}{r_0} + \sin \left( \frac{2t}{r_0} \right) \right) \end{pmatrix}.$$

Define then the map  $E : [0, \infty) \times \Sigma_0 \rightarrow \mathbb{H}$ , by

$$E(t; x_0, y_0) = \gamma_{x_0, y_0}(t). \quad (2.46)$$

We remark that (2.46) corresponds to the restriction of the sub-Riemannian exponential map to the annihilator bundle of  $\Sigma$ , given by (1.11). We define the *focal locus* to be the set of critical values of  $E$ .

**Lemma 2.7.2.** *The focal locus of  $\Sigma$  coincides with the  $z$ -axis, minus the origin.*

*Proof.* Representing the differential  $dE$  in polar coordinates, we get that

$$\det(dE) = \frac{1}{4} \left( r_0 \left( 1 + \cos \left( \frac{2t}{r_0} \right) \right) + t \sin \left( \frac{2t}{r_0} \right) \right),$$

and we look for solutions to the equation  $\det(dE) = 0$ . For any  $(x_0, y_0) \in \Sigma_0$  the critical points are obtained at  $t = t_c = r_0 \left( \frac{\pi}{2} + k\pi \right)$ , for  $k \in \mathbb{Z}$ , with critical values

$$E(t_c; x_0, y_0) = \left( 0, 0, \frac{r_0^2}{8} (\pi + 2k\pi) \right).$$

Thus, critical values are all points  $(0, 0, \xi) \in \mathbb{H}$  with  $\xi \neq 0$ . Notice that for all such points there exists a one-parameter family of geodesics joining  $\Sigma$  to that point.  $\square$

We define the *cut locus* of  $\Sigma$  as the set of all those points in which the map  $E$  fails to be injective, i.e.  $p \in \text{Cut}(\Sigma)$  if and only if  $E(t; x_0, y_0) = E(t; \bar{x}_0, \bar{y}_0) = p$ , for some points  $(x_0, y_0, 0), (\bar{x}_0, \bar{y}_0, 0) \in \Sigma$ .

**Lemma 2.7.3.** *For a point  $p \in \mathbb{H}$ , not lying on the  $z$ -axis, there exists a unique geodesic which realizes the distance from  $\Sigma$ . Thus  $p \notin \text{Cut}(\Sigma)$ .*

*Proof.* First of all, we notice that the cut locus is invariant under isometries which preserve the horizontal plane.

Second of all, proceed by contradiction and assume there exists a point  $p \in \text{Cut}(\Sigma) \setminus \{z = 0\}$ . Then, for any  $t \in \mathbb{R}$ ,  $\chi_t(p) \in \text{Cut}(\Sigma)$ , where  $\chi_t$  is the dilation defined in (2.43). In an analogous way, the rotations around the  $z$ -axis preserve the cut locus. Therefore, we get that if  $p \in \text{Cut}(\Sigma)$ , the cut locus contains a paraboloid with base point the origin, passing through  $p$ , see Figure 2.3. The paraboloid, contained in the half-space containing

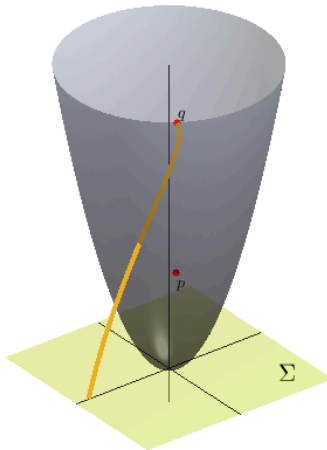


Figure 2.3: The paraboloid passing through  $p = (1, 1, 2)$  is contained in  $\text{Cut}(\Sigma)$ .

$p$ , separates it in two connected components,  $C_1$  to which  $\Sigma$  belongs, and  $C_2$ . Now, pick any point  $q \in C_2 \setminus \text{Cut}(\Sigma)$ : since  $q$  is not in the cut locus, there exists a unique minimizing geodesic joining  $q$  and  $\Sigma$ , however this geodesic must cross the paraboloid, hence the cut locus, losing minimality. This gives a contradiction.

Notice that the existence of a point  $q \in C_2 \setminus \text{Cut}(\Sigma)$  is guaranteed by the fact that the cut locus is a nowhere dense set (see [ACS18] and [Agr09], for further details).  $\square$

**Corollary 2.7.4.** *The cut locus of  $\Sigma$  coincides with  $z$ -axis, minus the origin. Moreover, for all  $p$  on the  $z$ -axis, we have the following formula for the distance function from  $\Sigma$*

$$\delta(p) = \delta(0, 0, z_p) = \sqrt{2\pi|z_p|}, \quad \forall p \in z\text{-axis}.$$

*Proof.* By the last lines in the proof of Lemma 2.7.2, we see that the  $z$ -axis (minus the origin) is contained in the cut locus of  $\Sigma$  and, by Lemma 2.7.3, we conclude that  $\text{Cut}(\Sigma) = \{(0, 0, z) \mid z \neq 0\}$ . To explicitly compute  $\delta(p)$  for a point  $p = (0, 0, z_p)$  in the  $z$ -axis, we look for the smallest positive time  $t = t_{\min}$  for which  $E(t; x_0, y_0) = p$ , where  $E$  is defined in (2.46). We may assume without loss of generality that  $z_p > 0$ . In cylindrical coordinates (2.44),  $E(t_{\min}; x_0, y_0) = p$  is given by

$$\begin{cases} 0 = \left( \cos \varphi_0 + \cos \left( \varphi_0 + \frac{2t_{\min}}{r_0} \right) \right), \\ 0 = \left( \sin \varphi_0 + \sin \left( \varphi_0 + \frac{2t_{\min}}{r_0} \right) \right), \\ z_p = \frac{r_0^2}{8} \left( \frac{2t_{\min}}{r_0} + \sin \left( \frac{2t_{\min}}{r_0} \right) \right). \end{cases} \quad (2.47)$$

Using the first two equations of (2.47), we see that

$$\varphi_0 + \frac{2t_{\min}}{r_0} = \varphi_0 + \pi + 2k\pi \quad \Rightarrow \quad \frac{2t_{\min}}{r_0} = \pi + 2k\pi,$$

with  $k$  a positive integer. In particular, since  $t_{\min}$  has to be minimal,  $k = 0$ , therefore  $t_{\min} = \frac{r_0\pi}{2}$  and  $r_0$  has to satisfy the third equation of (2.47)

$$z_p = \frac{r_0^2\pi}{8}.$$

Notice that the above equation uniquely determines  $r_0$ , while  $\varphi_0$  can vary in the interval  $(0, 2\pi)$ . Thus  $p \in \text{Cut}(\Sigma)$  and

$$\delta(p) = t_{\min} = \frac{r_0\pi}{2} = \sqrt{2z_p\pi},$$

concluding the proof.  $\square$

### 2.7.2 An expression for the distance function from $\Sigma$

We are interested in an explicit expression of the distance function outside the  $z$ -axis, where we expect it to be smooth.

Assume that  $p \in \mathbb{H}$  is not on the  $z$ -axis. This implies that  $r_p \neq 0$  and, exploiting the symmetries of the distance from  $\Sigma$  as in (2.45), we may assume that  $p = (1, 0, \xi)$ , where  $\xi > 0$ . Therefore, we rewrite the system (2.47) for  $p = (1, 0, \xi)$ :

$$\begin{cases} 1 = \frac{r_0}{2} \left( \cos \varphi_0 + \cos \left( \varphi_0 + \frac{2t_{\min}}{r_0} \right) \right), \\ 0 = r_0 \left( \sin \varphi_0 + \sin \left( \varphi_0 + \frac{2t_{\min}}{r_0} \right) \right), \\ \xi = \frac{r_0^2}{8} \left( \frac{2t_{\min}}{r_0} + \sin \left( \frac{2t_{\min}}{r_0} \right) \right), \end{cases} \quad (2.48)$$

using again the cylindrical coordinates (2.44), for the point  $(x_0, y_0)$ . Solving the second equation of the system, we get two possible solutions:

$$\varphi_0 + \frac{2t_{\min}}{r_0} = \varphi_0 + \pi + 2k\pi, \quad \varphi_0 + \frac{2t_{\min}}{r_0} = -\varphi_0 + 2k\pi.$$

In the first case, we don't have any solution of the system (2.48), since its first equation is not verified. In the second case, we obtain

$$\begin{cases} 1 = \frac{r_0}{2} \left( \cos \varphi_0 + \cos \left( \varphi_0 + \frac{2t_{\min}}{r_0} \right) \right) = r_0 \cos \varphi_0, \\ t_{\min} = r_0(k\pi - \varphi_0), \end{cases} \quad (2.49)$$

where  $k$  is an integer parameter to be determined. Replacing the expression for  $t_{\min}$  in the third equation of (2.48), we obtain the following equality for  $k$ :

$$\xi = \frac{r_0^2}{8} (2k\pi - 2\varphi_0 + \sin(-2\varphi_0)) = \frac{r_0}{4} (r_0(k\pi - \varphi_0) - \sin(\varphi_0)).$$

Thus, the integer  $k$ , as a function of the base point  $(1, y_0)$  and the final point  $(1, 0, \xi)$  of the geodesic  $\gamma$ , is given by

$$k(\xi, y_0) = \frac{4\xi + r_0 \sin(\varphi_0) + r_0^2 \varphi_0}{r_0^2 \pi} = \frac{4\xi + y_0 + (1 + y_0^2) \arctan(y_0)}{(1 + y_0^2) \pi}.$$

Here we are using the first equation of (2.48), which tells us that  $r_0 \cos(\varphi_0) = 1$ . Furthermore, using (2.49), we can compute the length of the corresponding geodesic  $\gamma$  as a function of  $\xi$  and  $y_0$ :

$$t(\xi, y_0) = r_0 (k(\xi, y_0)\pi - \varphi_0) = \frac{4\xi + y_0}{\sqrt{1 + y_0^2}}.$$

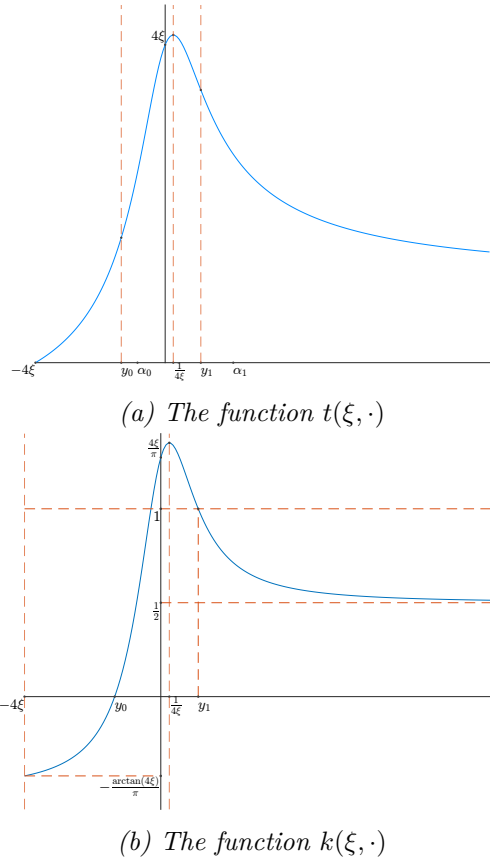


Figure 2.4: Functions occurring in the determination of the distance from  $\Sigma$ .

Finally, to compute  $\delta(p)$ , we have to find  $y_0$ , realizing the minimum

$$t_{\min} = \min \{t(\xi, y) \mid y \in \mathbb{R}, \quad k(\xi, y) \in \mathbb{Z}\}.$$

We will refer to those points for which  $k(\xi, y) \in \mathbb{Z}$ , as admissible starting points.

For fixed  $\xi > 0$ , we plotted in Figures 2.4a and 2.4b the graphs of the functions

$$y \mapsto t(\xi, y), \quad y \mapsto k(\xi, y).$$

First of all, we notice that the admissible starting points must satisfy  $y \geq -4\xi$ . Second of all, we see that the equation  $k(\xi, y) = 0$  has a unique solution for every value of  $\xi$ , and it lies in the interval  $(-4\xi, 0)$ , being  $k(\xi, 0) > 0$  and  $k(\xi, -4\xi) < 0$ . Moreover, since both these functions have a unique maximum at the point  $\frac{1}{4\xi}$ , they are both increasing in the interval  $[-4\xi, \frac{1}{4\xi})$  and decreasing in the interval  $[\frac{1}{4\xi}, +\infty)$ . Thus, among the admissible starting points in  $[-4\xi, \frac{1}{4\xi})$ , the minimum time (for these starting points) is achieved at the point for which  $k(\xi, \cdot)$  is the minimum integer. Therefore, since

$$k(\xi, y) \geq k(\xi, -4\xi) = -\frac{\arctan(4\xi)}{\pi} > -\frac{1}{2}, \quad \forall y \geq -4\xi,$$

we have that in the interval  $[-4\xi, \frac{1}{4\xi})$ , the minimum time is achieved at  $y_0$ , such that  $k(\xi, y_0) = 0$ . On the other hand, in an analogous way, since

$$k(\xi, y) \geq \lim_{y \rightarrow \infty} k(\xi, y) = \frac{1}{2}, \quad \forall y \geq \frac{1}{4\xi},$$

in the interval  $[\frac{1}{4\xi}, +\infty)$ , the minimum time is achieved at  $y_1$ , such that  $k(\xi, y_1) = 1$ , if exists. Hence, we are left to compare, for  $i = 0, 1$

$$t(\xi, y_i) = \frac{4\xi + y_i}{\sqrt{1 + y_i^2}}, \quad \text{with } k(\xi, y_i) = i.$$

The existence of  $y_1$ , such that  $k(\xi, y_1) = 1$ , is guaranteed if the maximum of  $k(\xi, \cdot)$  is  $\geq 1$ . Thus, if this is not the case, i.e. if

$$k\left(\xi, \frac{1}{4\xi}\right) = \frac{4\xi + \arctan(\frac{1}{4\xi})}{\pi} < 1, \quad (2.50)$$

we can immediately conclude that

$$t_{\min} = \frac{4\xi + y_0}{\sqrt{1 + y_0^2}}, \quad \text{with } k(\xi, y_0) = 0.$$

Choosing  $\xi \leq \frac{\pi}{8}$ , the above inequality (2.50) is satisfied and we conclude.

Consider the case  $\xi > \frac{\pi}{8}$ . Let us write explicitly the equation satisfied by  $y_0, y_1$  respectively, defining

$$\begin{aligned} f_0(y) &= 4\xi + y + (1 + y^2) \arctan(y), \\ f_1(y) &= (1 + y^2)\pi - 4\xi - y - (1 + y^2) \arctan(y). \end{aligned}$$

In this way, we will have  $f_i(y_i) = 0$ . Recall that, since we are assuming  $\xi > 0$ , we have  $y_0 < 0$ , and  $y_1 \in [\frac{1}{4\xi}, +\infty)$ , hence, we are looking for a negative zero of  $f_0$  and a positive zero of  $f_1$ . Notice that

$$-\frac{\pi}{2} \leq \arctan(y) \leq 0, \quad \forall y \leq 0,$$

therefore, we have the following estimate for  $f_0$ :

$$-\frac{\pi}{2}y^2 + y + 4\xi - \frac{\pi}{2} \leq f_0(y), \quad \forall y \leq 0. \quad (2.51)$$

Thus, replacing  $y_0$  in (2.51), we are able to compare it with the zero of the parabola

$$y_0 \leq \frac{1 - \sqrt{1 - \pi(\pi - 8\xi)}}{\pi} = \alpha_0 < 0.$$

We remark that we are in the case  $\xi > \frac{\pi}{8}$ , so  $\alpha_0$  is always well-defined, moreover, since  $y_0 \leq \alpha_0 < 0$  and  $t(\xi, \cdot)$  is increasing for negative arguments,  $t(\xi, y_0) \leq t(\xi, \alpha_0)$ , as one can check computing its derivative. See figure 2.4a. Reasoning in an analogous way for  $f_1$ , we obtain

$$\frac{\pi}{2}y^2 - y + \frac{\pi}{2} - 4\xi \leq f_1(y), \quad \forall y \geq 0, \quad (2.52)$$

which implies, for  $y_1$ , the following inequality:

$$\frac{1}{4\xi} < y_1 \leq \frac{1 + \sqrt{1 - \pi(\pi - 8\xi)}}{\pi} = \alpha_1,$$

and, as before  $\alpha_1$  is always well-defined, if  $\xi > \frac{\pi}{8}$ . We remark that  $\alpha_0, \alpha_1$  are the two zeroes of the parabola appearing in the left-hand sides of the estimates (2.51), (2.52). Since  $\frac{1}{4\xi} < y_1 \leq \alpha_1$  and  $t(\xi, \cdot)$  is decreasing for positive arguments in  $[\frac{1}{4\xi}, +\infty)$ , we have  $t(\xi, \alpha_1) \leq t(\xi, y_1)$ , see figure 2.4a.

Finally, to assert which point realizes the minimum time between  $y_0, y_1$ , it's enough to compare  $t(\xi, \alpha_0), t(\xi, \alpha_1)$ . One can check that, for any  $\xi > \frac{\pi}{8}$

$$t(\xi, \alpha_0) < t(\xi, \alpha_1) \quad \Rightarrow \quad t(\xi, y_0) \leq t(\xi, \alpha_0) < t(\xi, \alpha_1) \leq t(\xi, y_1).$$

This proves that, for points of the form  $p = (1, 0, \xi)$ ,

$$\delta(p) = \frac{4\xi + y_0}{\sqrt{1 + y_0^2}} = F(\xi), \quad \text{where } k(\xi, y_0) = 0, \quad (2.53)$$

where  $F$  has been defined in (2.45). Notice that this also provides an analytic proof of the fact that  $p \notin \text{Cut}(\Sigma)$ . Putting together the results of this section and Corollary 2.7.4, we prove Theorem 2.1.8.

### 2.7.3 Blow-up of $a_5$ and proof of Theorem 2.1.7

Recall that for a domain  $\Omega \subset M$ , whose boundary has no characteristic points, the fifth coefficient in the asymptotic expansion of the heat content is defined by the formula

$$a_5 = \frac{1}{240\sqrt{\pi}} \int_{\partial\Omega} (N^3 - 8N\Delta)\Delta\delta d\sigma, \quad (2.54)$$

where  $d\sigma$  is the sub-Riemannian measure, induced by a fixed volume on  $M$ , and  $N$  is the operator defined in (2.19). We are going to show that, for the horizontal plane in the Heisenberg group, the integrand in (2.54) is not locally integrable around the origin, which proves Theorem 2.1.7.

**Lemma 2.7.5.** *Let  $\Sigma = \{z = 0\} \subset \mathbb{H}$ . Then, the integrand in (2.54) is*

$$\left( \frac{73}{640\sqrt{\pi}} \right) \frac{1}{r^4}, \quad (2.55)$$

whereas the sub-Riemannian measure  $d\sigma = r^2 dr d\varphi$ , in cylindrical coordinates on  $\mathbb{H}$ . In particular the integrand of the coefficient  $a_5$  is not locally integrable with respect to the sub-Riemannian measure on  $\Sigma$ , around the characteristic point.

*Remark 2.7.6.* We explain the heuristic behind the order  $r^{-4}$  in (2.55). In cylindrical coordinates, computing explicitly  $\Delta\delta$ , we have  $\Delta\delta|_{\Sigma_0} = O\left(\frac{1}{r}\right)$ . Moreover, the operator  $(N^3 - 8N\Delta)$  has order 3, and it contains a third-order derivation in the  $r$ -direction.

*Proof.* By the symmetries of  $\mathbb{H}$ , we can write  $\delta$  as in (2.45):

$$\delta(p) = rF\left(\frac{|z|}{r^2}\right),$$

where  $(r, \varphi, z)$  are the cylindrical coordinates for  $p$  and  $F$  has been explicitly computed in Section 2.7.2. In particular, by equation (2.53)

$$F(\xi) = \frac{4\xi + y(\xi)}{\sqrt{1 + y(\xi)^2}}, \quad (2.56)$$

and the function  $y(\xi)$  satisfies

$$4\xi + y(\xi) + (1 + y(\xi)^2) \arctan(y(\xi)) = 0. \quad (2.57)$$

We can now compute explicitly the integrand of  $a_5$ . Rewriting the operators  $N$  and  $\Delta$  in cylindrical coordinates, we obtain

$$(N^3 - 8N\Delta)\Delta\delta|_{\Sigma_0} = \frac{C}{r^4},$$

where  $C$  is a constant depending only on the derivatives of  $F$  computed at the origin, up to order five. Hence, to obtain the integrand of  $a_5$ , it is enough to compute  $F^{(k)}(0)$ , for  $k \leq 5$ , which, by (2.56), amounts to compute  $y^{(k)}(0)$ , for  $k \leq 5$ . But this can be done using equation (2.57) and iterating the formula for the derivative of the implicit function. We omit the long but routine computation.  $\square$

*Remark 2.7.7.* The integrand of  $a_3$ , which is proportional to  $N\mathcal{H}$ ,  $\mathcal{H}$  the sub-Riemannian mean curvature, is locally integrable on  $\Sigma$  with respect to  $d\sigma$ , around the origin. Indeed, it is given by

$$-\left(\frac{3}{8\sqrt{\pi}}\right)\frac{1}{r^2},$$

and  $d\sigma = r^2 dr d\varphi$ . On the other hand, heuristically, each subsequent coefficient  $a_k$  requires an extra derivative of  $1/r$ . We may expect the first non-integrable coefficient to be  $a_4$ , close to a characteristic point. However, for the case of the  $xy$ -plane in  $\mathbb{H}$ , one can check that the integrand of  $a_4$  vanishes. Indeed, in this case, the function  $F$  defining the distance, as in (2.56), is an odd function, therefore the integrands of all coefficients  $a_{2i}$ , which involve only even-order derivatives, vanish.

*Remark 2.7.8.* In general, even in the Heisenberg group, the question of local integrability of the integrand function of  $a_3$ , with respect to sub-Riemannian perimeter measure is more subtle than it appears at first glance. Indeed, as one can check by expressing  $N\mathcal{H}$  using a boundary defining function for  $\partial\Omega$ , the former is almost equivalent to proving that  $\mathcal{H}$  itself is locally integrable with respect to the *Riemannian* perimeter measure. We address this problem in the next chapter.





# 3

## Integrability of the sub-Riemannian mean curvature at degenerate characteristic points in the Heisenberg group

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*We address the problem of integrability of the sub-Riemannian mean curvature  $\mathcal{H}$  of an embedded hypersurface, around isolated characteristic points. The main novelty is the introduction of a concept of a mildly degenerate characteristic point for a smooth surface of the Heisenberg group. This new concept can be regarded as a second-order condition for a characteristic point and it ensures the local integrability of  $\mathcal{H}$ . As a consequence we partially answer to a question posed by Danielli-Garofalo-Nhieu in [DGN12], proving that the mean curvature of a real-analytic surface with discrete characteristic set is locally integrable.*

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### 3.1 Introduction

Let  $M$  be a sub-Riemannian manifold, and let  $\Sigma \subset M$  be an embedded hypersurface. The horizontal mean curvature  $\mathcal{H} : \Sigma \rightarrow \mathbb{R}$  is a geometrical invariant which arises naturally in different areas of geometric analysis. It appears in the theory of minimal surfaces [DGN07b, HP08, Mon15, Pau04, CHMY05], in the study of the heat content asymptotics in sub-Riemannian manifolds, see [TW18] and also Chapter 2, and in Steiner-type formulas for the volume of tubes around hypersurfaces [BFF<sup>+</sup>15b].

Of particular relevance in all aforementioned applications is the (local) integrability of  $\mathcal{H}$ , either with respect to the horizontal perimeter measure or the Riemannian one (cf. Section 3.2 for precise definitions). An important fact is that, even for smooth hypersurfaces  $\Sigma$ , the horizontal mean curvature blows up at characteristic points, where the subspace of horizontal directions is tangent to  $\Sigma$ , making the local integrability problem a non-trivial one.

For what concerns the sub-Riemannian perimeter measure  $\sigma_H$ , as remarked first in [DGN12] for the Heisenberg group, the blow-up of  $\mathcal{H}$  is compensated by the degeneration of  $\sigma_H$ , and thus  $\mathcal{H} \in L^1_{\text{loc}}(\Sigma, \sigma_H)$ .

The aforementioned compensation fails if one replaces the sub-Riemannian perimeter measure  $\sigma_H$  on  $\Sigma$  with the Riemannian one  $\sigma_R$ , and in general  $\mathcal{H} \notin L^1_{\text{loc}}(\Sigma, \sigma_R)$ . In all known examples, however, either  $\Sigma$  is not very smooth, or the set of characteristic points has positive dimension, cf. [DGN12]. On the other hand, if  $\Sigma$  is at least  $C^2$  and the characteristic set is discrete, no counter-examples to local integrability are known. Furthermore, a thorough analysis of many specific cases led the authors in [DGN12] to conjecture that, under these assumptions,  $\mathcal{H} \in L^1_{\text{loc}}(\Sigma, \sigma_R)$ .

In this chapter, we address the question of local integrability with respect to the Riemannian perimeter measure in the Heisenberg group and in the more general context of three-dimensional contact manifolds. The case of non-degenerate and isolated characteristic points is elementary, and we can state the following result (proved in a setting that includes the Heisenberg group).

**Theorem 3.1.1.** *Let  $M$  be a three-dimensional contact sub-Riemannian manifold, equipped with a smooth measure  $\mu$  and let  $\Sigma \subset M$  be a  $C^2$  embedded surface. Assume that all characteristic points of  $\Sigma$  are isolated and non-degenerate. Then,*

$$\mathcal{H} \in L^1_{\text{loc}}(\Sigma, \sigma_R),$$

where  $\sigma_R$  denotes the Riemannian induced measure by  $\mu$  on  $\Sigma$ .

The case of *degenerate* characteristic points is less understood, even for the case of smooth surfaces in the Heisenberg group  $\mathbb{H}$  (to which we restrict to for the rest of this introduction). The main result of this chapter is the definition of a concept of a *mildly degenerate* characteristic point for surfaces in the Heisenberg group, for which we are able to prove local integrability. In the following,  $\mathbb{H}$  is equipped with the Lebesgue measure, which induces the Riemannian measure  $\sigma_R$  on a smooth embedded hypersurface  $\Sigma$ .

**Theorem 3.1.2.** *Let  $\Sigma \subset \mathbb{H}$  be a smooth embedded surface. Assume that all characteristic points of  $\Sigma$  are isolated and mildly degenerate. Then*

$$\mathcal{H} \in L^1_{\text{loc}}(\Sigma, \sigma_R),$$

where  $\sigma_{\mathbb{R}}$  denotes the Riemannian induced measure on  $\Sigma$ .

The concept of *mild degeneration* is based on a finite-order condition along an intrinsic curve  $\mathcal{C} \subset \Sigma$  emanating from degenerate characteristic points, which, to the best of our knowledge, does not appear in previous literature (cf. Definitions 3.5.1 and 3.5.4). In particular, if  $\Sigma$  is *real-analytic*, all degenerate characteristic points are mildly degenerate. As a consequence, we have the following corollary, which answers affirmatively to the conjecture in [DGN12], at least for real-analytic surfaces.

**Theorem 3.1.3.** *Let  $\Sigma \subset \mathbb{H}$  be a real-analytic embedded surface. Assume that all characteristic points of  $\Sigma$  are isolated. Then,*

$$\mathcal{H} \in L^1_{\text{loc}}(\Sigma, \sigma_{\mathbb{R}}),$$

where  $\sigma_{\mathbb{R}}$  denotes the Riemannian induced measure on  $\Sigma$ .

Our results include and unify several previous examples of local integrability of the horizontal mean curvature present in literature. In particular, one can compare our results with Propositions 3.1-3.4 in [DGN12]: Propositions 3.1, 3.2 are included in Theorem 3.1.1, while Proposition 3.3 is covered by Theorem 3.1.3.

*Remark 3.1.4.* Theorems 3.1.1, 3.1.2 and 3.1.3 are proved below in a slightly stronger form, cf. Theorems 3.4.2, 3.5.5 and 3.5.7, respectively. Namely, in each case, we prove the local integrability of  $\frac{1}{W}$ , where  $W$  denotes the norm of horizontal projection of the Riemannian unit normal to  $\Sigma$ . For this stronger result, the mild degeneration assumption is sharp and cannot be improved, cf. Remark 3.5.6. In particular, it implies the local integrability of the horizontal mean curvature, but also the local integrability of the intrinsic horizontal Gaussian curvature as defined in [BTV17], yielding Gauss-Bonnet-type theorems for surfaces with isolated and mildly degenerate (or non-degenerate) characteristic points. We refer to Section 3.5.1 for details.

**Structure of the chapter.** In Section 3.2 we recall the definitions of sub-Riemannian mean curvature of a hypersurface, and of (non-)degenerate characteristic point. In Section 3.3, we provide a general estimate for the sub-Riemannian mean curvature, which can be used to prove its integrability with respect to the sub-Riemannian induced measure. In Section 3.4, we prove Theorem 3.1.1: the key ingredient will be a normal form of a three-dimensional contact manifold. Finally, in Section 3.5, we introduce a notion of a mildly-degenerate characteristic point in the Heisenberg group and we prove Theorem 3.1.2. Then, we show that any characteristic point of an analytical surface in  $\mathbb{H}$  is mildly-degenerate, proving Theorem 3.1.3.

## 3.2 Preliminaries

Let  $M$  be a smooth  $n$ -dimensional manifold and let  $(\mathcal{D}, g)$  be a sub-Riemannian structure on  $M$ . Throughout this chapter, we assume that the distribution of  $M$  is defined by a subbundle of the tangent bundle  $\mathcal{D} \subset TM$  of constant rank  $k < n$ , and  $g$  is a positive, symmetric  $(0, 2)$ -tensor on it. In particular, this means that the generating frame (1.1) can be locally described by a local orthonormal frame, namely in an open set  $U \subset M$ , there exist  $k$  vector fields such that

$$\mathcal{D}_p = \text{span}\{X_1(p), \dots, X_k(p)\} \subset T_p M, \quad \forall p \in U. \quad (3.1)$$

Fix a smooth measure on  $M$ , which we will denote by  $\mu$  throughout the chapter. We can express the horizontal gradient (1.12) and the sub-Laplacian (1.14) of a function  $f \in C^2(M)$  in terms of the local orthonormal frame (3.1), obtaining

$$\nabla f = \sum_{i=1}^k (X_i f) X_i, \quad \Delta f = \sum_{i=1}^k \left( X_i^2 f + X_i f \operatorname{div}_\mu(X_i) \right).$$

Let  $\Sigma \subset M$  be a  $C^1$  embedded hypersurface. We recall that  $p \in \Sigma$  is a *characteristic point* if

$$\mathcal{D}_p \subseteq T_p \Sigma.$$

We denote by  $C(\Sigma)$  the set of characteristic points. Notice that  $C(\Sigma) \subset \Sigma$  is a closed set, and it has zero measure if  $\Sigma$  is at least  $C^2$ . We refer to [Bal03, Der72] for fine results about the size of  $C(\Sigma)$  under suitable assumptions on the regularity of  $\Sigma$ .

The hypersurface  $\Sigma$  can be locally described as follows: at  $p \in M$ , there exists a neighborhood  $U \subset M$  of  $p$  and  $u \in C^1(U)$  such that

$$\Sigma \cap U = \{u = 0\}, \quad du|_{\Sigma \cap U} \neq 0.$$

When  $\Sigma$  is locally given as the zero-locus of  $u$ , then  $p \in C(\Sigma)$  if and only if

$$X_i u(p) = 0, \quad \forall i = 1, \dots, k.$$

### 3.2.1 Horizontal Hessian

Let  $(\mathcal{D}, g)$  be a constant-rank sub-Riemannian structure on  $M$ . We introduce the horizontal Hessian for classifying characteristic points (cf. also [BBCH20]). Fix an affine connection  $\tilde{\nabla}$  on the distribution  $\mathcal{D}$ . Then, the horizontal Hessian of  $u \in C^2(M)$  is the  $(0, 2)$ -tensor on  $\mathcal{D}$ , defined by

$$\operatorname{Hess}_H(u)(V, W) = g(\tilde{\nabla}_V(\nabla u), W), \quad \forall V, W \in \Gamma(\mathcal{D}).$$

While in general the definition of the horizontal Hessian depends on the choice of the connection, it is intrinsic at characteristic points.

**Lemma 3.2.1.** *Let  $(\mathcal{D}, g)$  be a constant-rank sub-Riemannian structure on  $M$  and let  $\Sigma = \{u = 0\} \subset M$ , where  $u: M \rightarrow \mathbb{R}$  is a  $C^2$  submersion on  $\Sigma$ . If  $p \in C(\Sigma)$ , then  $\operatorname{Hess}_H(u)|_p$  does not depend on the choice of the connection and thus is a well-defined bilinear map on  $\mathcal{D}_p$ .*

*Proof.* Let  $\{X_1, \dots, X_k\}$  be a local orthonormal frame around  $p$ . Then, by the definition of the horizontal Hessian, for  $V, W \in \Gamma(\mathcal{D})$ , we have

$$\begin{aligned} \operatorname{Hess}_H(u)(V, W) &= \sum_{i,j=1}^k V^i W^j g\left(\tilde{\nabla}_{X_i} \nabla u, X_j\right) \\ &= \sum_{i,j=1}^k V^i W^j g\left(\tilde{\nabla}_{X_i} \left(\sum_{\ell=1}^k (X_\ell u) X_\ell\right), X_j\right) \\ &= \sum_{i,j,\ell=1}^k V^i W^j g\left((X_i X_\ell u) X_\ell + (X_\ell u) \tilde{\nabla}_{X_i} X_\ell, X_j\right), \end{aligned}$$

using the linearity and the Leibniz formula for the connection. Finally, since  $p \in C(\Sigma)$ , we obtain  $X_\ell u(p) = 0$  for any  $\ell = 1, \dots, k$ . Therefore, we conclude that

$$\operatorname{Hess}_H(u)(V, W)|_p = \sum_{i,j=1}^k V^i W^j X_i X_j u(p),$$

and the right-hand side does not depend on the choice of the connection  $\tilde{\nabla}$ .  $\square$

**Definition 3.2.2.** Let  $(\mathcal{D}, g)$  be a constant-rank sub-Riemannian structure on  $M$  and let  $\Sigma \subset M$  be a  $C^2$  embedded hypersurface in  $M$ . We say that  $p \in C(\Sigma)$  is a *non-degenerate characteristic point* if

$$\det(\text{Hess}_{\mathbb{H}}(u)|_p) \neq 0,$$

where  $u \in C^2$  is a local defining function for  $\Sigma$  in a neighborhood of  $p$ . Notice that this property does not depend on the choice of  $u$ .

*Remark 3.2.3.* This definition appears in literature with different names. In [FMS19, Def. 4.2], the authors introduce a flatness condition on  $u$ , at a characteristic point, while in [BCX09, Def. 1.1], the authors give the notion of *regular* characteristic point of order 2. Both definitions are equivalent to being a non-degenerate characteristic point.

### 3.2.2 Sub-Riemannian mean curvature

In Chapter 2, we encountered the sub-Riemannian mean curvature as the integrand function of the second-order coefficient of the heat content asymptotic expansion, cf. Theorem 2.1.1. In particular, for a smooth and non-characteristic domain  $\Omega \subset M$ , the definition of  $\mathcal{H}$  was given through the distance function from  $\partial\Omega$ , denoted by  $\delta$ . However, when characteristic points are present,  $\delta$  is no longer smooth at the boundary, hence we must adapt accordingly the definition of  $\mathcal{H}$ .

**Definition 3.2.4** (Sub-Riemannian mean curvature). Let  $(\mathcal{D}, g)$  be a constant-rank sub-Riemannian structure on  $M$  and let  $\Sigma \subset M$  be a  $C^2$  embedded hypersurface in  $M$ . The *sub-Riemannian (or horizontal) mean curvature* at  $p \in \Sigma$  is defined by

$$\mathcal{H}(p) = -\text{div}_{\mu} \left( \frac{\nabla u}{\|\nabla u\|} \right) \Big|_p,$$

where  $u \in C^2$  is a local defining function for  $\Sigma$  in a neighborhood of  $p$ . Notice that the value of  $\mathcal{H}$  does not depend on the choice of  $u$ .

Notice that, if  $p \in \Sigma \setminus C(\Sigma)$  the vector field  $\nu = -\nabla u / \|\nabla u\|$  is the (outward-pointing) horizontal unit normal to  $\Sigma$ , thus as one can easily check, we recover the usual notion of horizontal mean curvature as  $-\Delta\delta$ .

### 3.2.3 (Sub-)Riemannian induced measure

Let  $(\mathcal{D}, g)$  be a constant-rank sub-Riemannian structure on  $M$  and let  $\Sigma \subset M$  be a  $C^2$  embedded hypersurface in  $M$ . Let  $\nu$  be the horizontal unit normal to  $\Sigma$ . Then the sub-Riemannian induced measure  $\sigma_{\mathbb{H}}$  on  $\Sigma$  is the positive smooth measure with density  $|i_{\nu}\mu|$ . If  $u$  is a local defining function for  $\Sigma$ , then  $\nu$  is given by

$$\nu = -\frac{\nabla u}{\|\nabla u\|}. \tag{3.2}$$

Analogously, to define the Riemannian induced measure  $\sigma_{\mathbb{R}}$ , consider any Riemannian extension of the sub-Riemannian structure, then replace  $\nu$  with the Riemannian unit normal, which is given by (3.2) with the Riemannian gradient. Notice that  $\sigma_{\mathbb{R}}$  coincides with the  $(n-1)$ -dimensional Hausdorff measure on  $\Sigma$  induced by the Riemannian structure. Moreover, it depends on the choice of a Riemannian extension, but this choice does not play any role concerning the integrability of the horizontal mean curvature.

### 3.3 A general estimate for horizontal mean curvature

We provide here a general estimate for the horizontal mean curvature. Let  $(\mathcal{D}, g)$  be a constant-rank sub-Riemannian structure on  $M$  and let  $\Sigma \subset M$  be a  $C^2$  embedded hypersurface. Without loss of generality, assume that  $u \in C^2(M)$  is a global defining function for  $\Sigma$ , that is  $u: M \rightarrow \mathbb{R}$  is a submersion on  $\Sigma$  and  $\Sigma = \{u = 0\}$ . Having fixed a local orthonormal frame for the sub-Riemannian structure at a point  $p$ , say  $\{X_1, \dots, X_k\}$ , recall that

$$\nabla u = \sum_{i=1}^k (X_i u) X_i, \quad (3.3)$$

and its norm, which we denote by  $W$ , is given by

$$W^2 = \|\nabla u\|^2 = g(\nabla u, \nabla u) = \sum_{i=1}^k (X_i u)^2. \quad (3.4)$$

Therefore, we can write the horizontal mean curvature explicitly in terms of  $\nabla u$  and  $W$

$$\mathcal{H} = -\operatorname{div}_\mu \left( \frac{\nabla u}{W} \right) = -\frac{1}{W} \Delta u + \frac{1}{W^2} g(\nabla u, W).$$

Using formulas (3.3) and (3.4), we obtain

$$\mathcal{H} = -\frac{1}{W} \Delta u + \frac{1}{W^3} \sum_{i,j=1}^k (X_i u)(X_j u)(X_i X_j u),$$

which gives the estimate

$$|\mathcal{H}| \leq \frac{1}{W} \left( \|\Delta u\|_{L^\infty(U)} + \sum_{i,j=1}^k \|X_i X_j u\|_{L^\infty(U)} \right) \leq \frac{C_0}{W}, \quad (3.5)$$

for a suitable constant  $C_0 > 0$ , where we have used the inequality:  $|X_i u| \leq W$ , for any  $i = 1, \dots, k$ . Here  $\|\cdot\|_{L^\infty(U)}$  denotes the supremum norm and  $U$  is a relatively compact neighborhood of  $p$ .

We recover the well-known integrability result for the horizontal mean curvature with respect to the sub-Riemannian perimeter measure (see also [DGN12, Prop. 3.5]).

**Lemma 3.3.1.** *Let  $(\mathcal{D}, g)$  be a constant-rank sub-Riemannian structure on  $M$ , and let  $\mu$  be a smooth measure on  $M$ . Consider a  $C^2$  embedded hypersurface in  $M$ ,  $\Sigma \subset M$ . Then*

$$\mathcal{H} \in L^1_{\text{loc}}(\Sigma, \sigma_{\mathcal{H}}),$$

where  $\sigma_{\mathcal{H}}$  denotes the sub-Riemannian induced measure on  $\Sigma$ .

*Proof.* Let  $U$  be a neighborhood of  $p$  and consider  $u \in C^2$ , a local defining function for  $\Sigma$  on  $U$ . Then, setting  $W = \|\nabla u\|$ , we have  $\sigma_{\mathcal{H}} \propto W \sigma_{\mathcal{R}}$ , up to a smooth never-vanishing function. Therefore, using (3.5), we obtain

$$\left| \int_{\Sigma \cap U} \mathcal{H} d\sigma_{\mathcal{H}} \right| \leq C \int_{\Sigma \cap U} |\mathcal{H}| W d\sigma_{\mathcal{R}} \leq C \int_{\Sigma \cap U} \frac{C_0}{W} W d\sigma_{\mathcal{R}} < +\infty.$$

This concludes the proof. □

### 3.4 Integrability for non-degenerate characteristic points in 3D contact sub-Riemannian manifolds

In this section, we prove the first integrability result on the horizontal mean curvature, Theorem 3.1.1. Let  $M$  be a smooth manifold of dimension 3. Let  $\omega$  be a contact one-form<sup>1</sup>, that is such that  $\omega \wedge d\omega \neq 0$ . Then, the contact distribution is

$$\mathcal{D}_p = \ker(\omega_p) \subset T_p M, \quad \forall p \in M.$$

By the non-degeneracy assumption on  $d\omega$ ,  $\mathcal{D}$  is a subbundle of rank 2 and is bracket-generating. Any metric on  $\mathcal{D}$  defines a sub-Riemannian structure on  $M$ . We will refer to  $M$  as *contact sub-Riemannian manifold*. Recall the following normal form for an orthonormal frame (see [ACEAG98, AG99]).

**Theorem 3.4.1.** *Let  $M$  be a 3D contact sub-Riemannian manifold, with contact 1-form  $\omega$ , and  $\{X_1, X_2\}$  be a local orthonormal frame for  $\mathcal{D} = \ker(\omega)$ . There exists a smooth coordinate system  $(x, y, z)$  such that*

$$\begin{aligned} X_1 &= \partial_x - \frac{y}{2}\partial_z + \beta y(y\partial_x - x\partial_y) + \gamma y\partial_z, \\ X_2 &= \partial_y + \frac{x}{2}\partial_z - \beta x(y\partial_x - x\partial_y) + \gamma x\partial_z, \end{aligned}$$

where  $\beta = \beta(x, y, z)$  and  $\gamma = \gamma(x, y, z)$  are smooth functions satisfying

$$\beta(0, 0, z) = \gamma(0, 0, z) = \partial_x \gamma(0, 0, z) = \partial_y \gamma(0, 0, z) = 0.$$

We now prove the first integrability result for isolated non-degenerate characteristic points on general contact manifolds.

**Theorem 3.4.2.** *Let  $M$  be a 3D contact sub-Riemannian manifold, with contact 1-form  $\omega$ , and let  $\Sigma \subset M$  be a  $C^2$  embedded surface. Let  $p$  be an isolated non-degenerate characteristic point and let  $u \in C^2$  be a local defining function for  $\Sigma$  in a neighborhood of  $p$ . Denote with  $W$  the norm of the horizontal gradient of  $u$ . Then*

$$\frac{1}{W} \in L^1_{\text{loc}}(\Sigma, \sigma_{\mathbb{R}}), \tag{3.6}$$

where  $\sigma_{\mathbb{R}}$  denotes the Riemannian induced measure on  $\Sigma$ . In particular

$$\mathcal{H} \in L^1_{\text{loc}}(\Sigma, \sigma_{\mathbb{R}}).$$

*Proof.* Introducing the normal form given by Theorem 3.4.1, we may assume that the characteristic point is at the origin and that  $\Sigma$  is locally a graph around the origin. Indeed, recall that  $\Sigma \cap U = \{u = 0\}$  and  $du \neq 0$  on  $\Sigma \cap U$ . However, since  $\mathbf{0} \in C(\Sigma)$ , we obtain

$$0 = d_p u(X_1) = \partial_x u(\mathbf{0}) \quad \text{and} \quad 0 = d_p u(X_2) = \partial_y u(\mathbf{0}). \tag{3.7}$$

Therefore,  $\partial_z u(\mathbf{0}) \neq 0$ , which implies that, up to restricting  $U$ ,  $\Sigma \cap U = \{z = g(x, y)\}$ , for some  $C^2$  function  $g: \mathbb{R}^2 \rightarrow \mathbb{R}$ . Moreover, in these coordinates, relations (3.7) give first-order conditions on  $g$

$$g(0, 0) = \partial_x g(0, 0) = \partial_y g(0, 0) = 0.$$

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<sup>1</sup>In literature the contact form is typically denoted with the symbol  $\omega$ . So, only in this chapter, we will adopt the notation  $\omega$  for the contact form, whereas the smooth volume will be denoted by  $\mu$ .

Recall that we want to discuss the finiteness of the following integral

$$\int_{\Sigma \cap U} \frac{1}{W} d\sigma_{\mathbb{R}} = \int_V \frac{1}{W(x, y, g(x, y))} f(x, y) dx dy, \quad (3.8)$$

where  $V$  is a neighborhood of  $(0, 0)$  and  $f$  is the Riemannian density in coordinates. Since the characteristic point is non-degenerate, up to restricting  $V$ , the map

$$\varphi: \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} \quad \text{such that} \quad \begin{cases} \tilde{x} = X_1 u(x, y, g(x, y)) \\ \tilde{y} = X_2 u(x, y, g(x, y)) \end{cases}$$

defines a smooth change coordinate on  $V$ . Indeed, its Jacobian at  $(0, 0)$  equals the determinant of the horizontal Hessian at the origin

$$\det(\mathcal{J}\varphi)|_{(0,0)} = \det \begin{pmatrix} \partial_x X_1 u + g_x \partial_z X_1 u & \partial_y X_1 u + g_y \partial_z X_1 u \\ \partial_x X_2 u + g_x \partial_z X_2 u & \partial_y X_2 u + g_y \partial_z X_2 u \end{pmatrix} \Big|_{(0,0)} = \det(\text{Hess}_{\mathbb{H}}(u)|_{\mathbf{0}}),$$

and by the non-degeneracy assumption is non-zero. Thus, after the change of variables, the integral (3.8) becomes

$$\int_V \frac{1}{W} f(x, y) dx dy = \int_{\varphi(V)} \frac{1}{|\det(\mathcal{J}\varphi)| \sqrt{\tilde{x}^2 + \tilde{y}^2}} \tilde{f}(\tilde{x}, \tilde{y}) d\tilde{x} d\tilde{y} < +\infty.$$

The claim (3.6) follows. Using estimate (3.5), we obtain the local integrability of  $\mathcal{H}$ .  $\square$

*Remark 3.4.3.* A non-degenerate characteristic point need not to be isolated, however, this situation is quite pathological. For example, in the Heisenberg group  $\mathbb{H}$  (cf. Section 3.5), the only situation in which this can occur is when we have a sequence of characteristic points  $(x_n, y_n)$  accumulating at the origin, and not contained in any absolutely continuous curve. Indeed, consider  $\Sigma = \{z = g(x, y)\}$  in  $\mathbb{H}$  and assume we have an absolutely continuous curve  $\gamma: (-\varepsilon, \varepsilon) \rightarrow \Sigma$  of characteristic points with  $\gamma(0) = \mathbf{0}$ , and such that  $\dot{\gamma}(0)$  exists. Then, the origin is a degenerate characteristic point. Indeed for each  $t \in (-\varepsilon, \varepsilon)$ , we have

$$\begin{cases} g_x(\gamma_1(t), \gamma_2(t)) + \frac{\gamma_2(t)}{2} = 0, \\ g_y(\gamma_1(t), \gamma_2(t)) - \frac{\gamma_1(t)}{2} = 0. \end{cases}$$

Differentiating both equations with respect to  $t$ , and evaluating at  $t = 0$ , we have

$$\begin{cases} \dot{\gamma}_1(0)g_{xx}(0, 0) + \dot{\gamma}_2(0)g_{xy}(0, 0) + \frac{\dot{\gamma}_2(0)}{2} = 0, \\ \dot{\gamma}_1(0)g_{xy}(0, 0) + \dot{\gamma}_2(0)g_{yy}(0, 0) - \frac{\dot{\gamma}_1(0)}{2} = 0, \end{cases}$$

thus,  $\begin{pmatrix} \dot{\gamma}_1(0) \\ \dot{\gamma}_2(0) \end{pmatrix} \in \ker(\text{Hess}_{\mathbb{H}}(u)|_{\mathbf{0}})$ , implying that  $\mathbf{0}$  is a degenerate characteristic point.

### 3.5 Integrability for mildly degenerate characteristic points in $\mathbb{H}$

In this section, we introduce the definition of a non-degenerate characteristic point in the Heisenberg group  $\mathbb{H}$  and we prove Theorems 3.1.2 and 3.1.3. We restrict ourselves to  $\mathbb{H}$ , since we are going to exploit its group structure.



The *Heisenberg group* is the 3D contact structure on  $\mathbb{R}^3$ , defined by the 1-form

$$\omega = dz - \frac{1}{2}(xdy - ydx).$$

A global frame for the contact distribution is given by  $\{X, Y\}$ , where

$$X = \partial_x - \frac{y}{2}\partial_z, \quad Y = \partial_y + \frac{x}{2}\partial_z. \quad (3.9)$$

Setting  $\{X, Y\}$  to be an orthonormal frame, the resulting sub-Riemannian manifold is the well-known first Heisenberg group,  $\mathbb{H}$ . We equip it with the Lebesgue measure.

Let us consider in  $\mathbb{H}$  a surface  $\Sigma = \{u = 0\}$ , where  $u \in C^\infty(\mathbb{R}^3)$  with  $du \neq 0$  on  $\Sigma$ . Assume that  $p \in C(\Sigma)$  is a degenerate characteristic point, meaning that the horizontal Hessian of  $u$  has zero determinant at  $p$ . Notice that, in  $\mathbb{H}$ , the horizontal Hessian at  $p \in C(\Sigma)$  coincides with the one introduced in [DGN03], and, in terms of the orthonormal basis  $\{X, Y\}$ , takes the form:

$$\text{Hess}_{\mathbb{H}}(u)|_p = \begin{pmatrix} XXu(p) & XYu(p) \\ YXu(p) & YYu(p) \end{pmatrix}.$$

By the bracket-generating assumption, one of the entries of the Hessian must be non-zero at  $p$ , thus, it has a one-dimensional kernel at  $p$ , spanned by some unitary vector, say  $N_p \in \mathcal{D}_p$ , which is unique, up to a sign. We extend  $N_p$  to a left-invariant vector field  $N \in \Gamma(\mathcal{D})$ . Taking an orthogonal vector field to  $N$  in  $\Gamma(\mathcal{D})$ , we obtain an orthonormal frame  $\{N, T\}$  for the distribution, which, up to changing sign, we assume to be co-oriented with the standard one (3.9).

**Definition 3.5.1.** Let  $\Sigma \subset \mathbb{H}$  be a smooth embedded surface and let  $p \in C(\Sigma)$  be degenerate. The *critical curve* of  $p$  is defined as the set of points in  $\Sigma$  where  $N$  is tangent to  $\Sigma$ , i.e.

$$\mathcal{C} = \{q \in \Sigma \mid N(q) \in T_q\Sigma\}. \quad (3.10)$$

We prove now that Definition 3.5.1 is well-posed.

**Lemma 3.5.2.** *Let  $\Sigma \subset \mathbb{H}$  be a smooth embedded surface and let  $p \in C(\Sigma)$  be degenerate. Then, in a neighborhood of  $p$ , the set  $\mathcal{C}$  as in (3.10), is a smooth curve in  $\Sigma$  through  $p$ .*

*Proof.* Consider for  $\Sigma$  a local defining function  $u \in C^\infty$ . Then,  $N(q) \in T_q\Sigma$  if and only if  $d_q u(N) = 0$ , thus

$$\mathcal{C} = \{u = 0\} \cap \{Nu = 0\}.$$

In the orthonormal frame  $\{N, T\}$ ,  $Nu(p) = Tu(p) = 0$  since  $p \in C(\Sigma)$ . However,  $d_p u \neq 0$ , therefore, by the bracket-generating assumption

$$[N, T]u(p) = NTu(p) - TNu(p) \neq 0.$$

We are going to show that  $NTu(p) = 0$ . First of all, since the frame  $\{N, T\}$  is co-oriented with  $\{X, Y\}$ , there exists  $R \in \text{SO}(2)$ , such that

$$\begin{pmatrix} N \\ T \end{pmatrix} = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} \quad (3.11)$$

where we used the shorthands  $a = \cos(\theta)$  and  $b = \sin(\theta)$ , for some  $\theta \in [0, 2\pi)$ . Hence,

$$\begin{aligned} NTu(p) &= (aX + bY)(-bX + aY)u(p) \\ &= -abXXu(p) + a^2XYu(p) - b^2YXu(p) + abYYu(p). \end{aligned}$$

Second of all, by the definition of  $N$ ,  $N_p \in \ker(\text{Hess}_{\mathbb{H}}(u)|_p)$ , thus we obtain

$$0 = \begin{pmatrix} -b & a \end{pmatrix} \begin{pmatrix} XXu(p) & YXu(p) \\ XYu(p) & YYu(p) \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = NTu(p).$$

Finally,  $TNu(p) \neq 0$  so the differential of the map  $\phi = (u, Nu)$  has maximal rank at  $p$ , implying that  $\mathcal{C}$  is a smooth curve in  $\Sigma$ , in a neighborhood of  $p$ .  $\square$

*Remark 3.5.3.* Notice that, in general, the critical curve  $\mathcal{C}$  is not necessarily horizontal, and it is not related to the characteristic foliation induced on  $\Sigma$  by the contact structure.

**Definition 3.5.4.** Let  $\Sigma \subset \mathbb{H}$  be an embedded smooth surface, let  $p \in C(\Sigma)$  be a degenerate characteristic point and let  $u$  be a local defining function for  $\Sigma$  around  $p$ . Let  $\gamma : (-\varepsilon, \varepsilon) \rightarrow \mathcal{C}$  be a regular parametrization of  $\mathcal{C}$ , with  $\gamma(0) = p$ . We say that  $p$  is mildly degenerate if the function

$$s \mapsto Tu(\gamma(s)),$$

has a finite order zero at  $s = 0$ . Notice that this definition does not depend on the choice of  $u$  and the regular parametrization.

**Theorem 3.5.5.** *Let  $\Sigma \subset \mathbb{H}$  be an embedded surface, let  $p$  be an isolated mildly degenerate characteristic point and let  $u \in C^\infty$  be a local defining function for  $\Sigma$  in a neighborhood of  $p$ . Denote with  $W$  the norm of the horizontal gradient of  $u$ . Then*

$$\frac{1}{W} \in L^1_{\text{loc}}(\Sigma, \sigma_{\mathbb{R}}),$$

where  $\sigma_{\mathbb{R}}$  denotes the Riemannian induced measure on  $\Sigma$ . In particular

$$\mathcal{H} \in L^1_{\text{loc}}(\Sigma, \sigma_{\mathbb{R}}).$$

*Proof.* Without loss of generality, we may assume that  $u(x, y, z) = z - g(x, y)$ , i.e.  $\Sigma = \{z = g(x, y)\}$ , where  $g: \mathbb{R}^2 \rightarrow \mathbb{R}$  is smooth, and that  $C(\Sigma) = \{\mathbf{0}\}$ . This implies

$$g(0, 0) = \partial_x g(0, 0) = \partial_y g(0, 0) = 0.$$

Notice that the local integrability of  $W$  is preserved by the action of isometries of  $\mathbb{H}$ . We can exploit this fact to reduce  $g$  to a normal form. Recall first that Heisenberg isometries (preserving the origin and the orientation of the  $z$ -axis) are given by the standard action of  $\text{SO}(2)$  on the  $xy$ -plane. Consider then the isometry

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z} \end{pmatrix} = \begin{pmatrix} R & | & \\ \hline & & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

where  $R \in \text{SO}(2)$  is defined in (3.11). In this way the frame  $\{N, T\}$  is sent to the standard frame  $\{X, Y\}$ . In particular, since  $N_{\mathbf{0}} \in \ker(\text{Hess}_{\mathbb{H}}(u)|_{\mathbf{0}})$ ,  $N$  is given by

$$N = aX + bY = \frac{-YXu(p)X + XXu(p)Y}{\sqrt{YXu(p)^2 + XXu(p)^2}} = \frac{\left(g_{11} + \frac{1}{2}\right)X - g_{20}Y}{\sqrt{\left(g_{11} + \frac{1}{2}\right)^2 + g_{20}^2}},$$

where  $g_{ij} = \partial_x^i \partial_y^j g(0, 0)$ . Hence, the change of coordinates is given by

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{W_0} \begin{pmatrix} g_{11} + \frac{1}{2} & g_{20} \\ -g_{20} & g_{11} + \frac{1}{2} \\ & & 1 \end{pmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z} \end{pmatrix} = \frac{1}{W_0} \begin{pmatrix} (g_{11} + \frac{1}{2}) \tilde{x} + g_{20} \tilde{y} \\ -g_{20} \tilde{x} + (g_{11} + \frac{1}{2}) \tilde{y} \\ \tilde{z} \end{pmatrix},$$

where  $W_0 = \sqrt{(g_{11} + \frac{1}{2})^2 + g_{20}^2}$ . Therefore, expanding  $u$ , we have

$$\begin{aligned} u(x, y, z) &= z - \frac{g_{20}}{2} x^2 - \frac{g_{02}}{2} y^2 - g_{11} xy + O(r^3) \\ &= \tilde{z} - \frac{g_{20}}{2W_0^2} \left( (g_{11} + \frac{1}{2}) \tilde{x} + g_{20} \tilde{y} \right)^2 - \frac{g_{02}}{2W_0^2} \left( -g_{20} \tilde{x} + (g_{11} + \frac{1}{2}) \tilde{y} \right)^2 \\ &\quad - \frac{g_{11}}{W_0^2} \left( (g_{11} + \frac{1}{2}) \tilde{x} + g_{20} \tilde{y} \right) \left( -g_{20} \tilde{x} + (g_{11} + \frac{1}{2}) \tilde{y} \right) + O(r^3), \end{aligned}$$

where  $r = \sqrt{x^2 + y^2}$ , and the coefficients of the second-order terms become

$$\begin{aligned} \tilde{x}^2 &\rightsquigarrow \frac{1}{W_0^2} \left( -\frac{g_{20}}{2} \left( g_{11} + \frac{1}{2} \right)^2 + g_{11} \left( g_{11} + \frac{1}{2} \right) g_{20} - \frac{g_{02}}{2} g_{20}^2 \right) = 0, \\ \tilde{x}\tilde{y} &\rightsquigarrow \frac{1}{W_0^2} \left( -g_{20} \left( g_{11} + \frac{1}{2} \right) g_{20} - g_{11} \left( \left( g_{11} + \frac{1}{2} \right)^2 - g_{20}^2 \right) + g_{02} g_{20} \left( g_{11} + \frac{1}{2} \right) \right) = -\frac{1}{2}, \end{aligned}$$

having used the fact that the characteristic point is degenerate, which gives the condition  $g_{20}g_{02} = g_{11}^2 - \frac{1}{4}$ . Hence, the function  $g$  simplifies to

$$g(\tilde{x}, \tilde{y}) = \frac{1}{2} \tilde{x}\tilde{y} + \frac{\alpha}{2} \tilde{y}^2 + h(\tilde{x}, \tilde{y}), \quad (3.12)$$

where  $\alpha \in \mathbb{R}$  and  $h \in C^\infty(\mathbb{R}^2)$  with order  $\geq 3$ . Notice that the specific value of  $\alpha$  will not play any role in the integrability of  $\mathcal{H}$ .

We can then assume that  $u(x, y, z) = z - g(x, y)$  where  $g$  has the normal form (3.12), and that  $N = X$ ,  $T = Y$ . For such a function  $u$ , the norm of the horizontal gradient is

$$W^2 = (\alpha y + h_y)^2 + (y + h_x)^2,$$

so, since in these coordinates we have  $d\sigma_{\mathbb{R}} = f(x, y) dx dy$ , where  $f$  is a strictly positive and smooth function, we focus on

$$\int_V \frac{1}{W} f(x, y) dx dy, \quad (3.13)$$

where  $V$  is a neighborhood of  $(0, 0)$ . We may set  $f \equiv 1$ , since its explicit expression plays no role in the integrability. From Lemma 3.5.2,  $\mathcal{C} = \{Nu = 0\} \cap \Sigma$  is a smooth curve, whose expression in coordinates is  $\{y + h_x = 0\} \cap \Sigma$ . Thus, we introduce the following smooth change of variables around the origin, rectifying  $\mathcal{C}$

$$\varphi: \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x \\ t \end{pmatrix} \quad \text{such that} \quad \begin{cases} x = x \\ t = y + h_x(x, y) \end{cases} \quad (3.14)$$

and the integral (3.13) becomes

$$\int_{V'} \frac{1}{((\alpha t + (h_y - \alpha h_x))^2 + t^2)^{1/2} |1 + 2h_{xy}|} dt dx, \quad (3.15)$$

where the integrand is evaluated in  $\varphi^{-1}(x, t)$ . Here  $V' = \varphi(V)$ . We expand in Taylor series the function  $h_y - \alpha h_x$ , with respect to the  $t$ -variable at the point  $(x, 0)$ , obtaining

$$h_y(\varphi^{-1}(x, t)) - \alpha h_x(\varphi^{-1}(x, t)) = \xi(x) + tR(x, t),$$

where  $\xi$  and  $R$  are smooth functions of order  $\geq 2$  and  $\geq 1$  respectively, since  $h$  was of order at least 3 and the notion of order in  $t$  is preserved by  $\varphi$ . But now, parametrizing the critical curve by  $x \mapsto \gamma(x) = (x, y(x), g(x, y(x)))$  where  $y(x) + h_x(x, y(x)) = 0$ , we have that  $\varphi^{-1}(x, 0) = (x, y(x))$  and

$$\xi(x) = -\alpha h_x(\varphi^{-1}(x, 0)) + h_y(\varphi^{-1}(x, 0)) = -\alpha h_x(x, y(x)) + h_y(x, y(x)) = Tu(\gamma(x)).$$

Thus, by assumption of mildly degenerate characteristic point,  $\xi$  has a zero of finite order at  $x = 0$ . So, we may write

$$\xi(x) = c_0 x^k (1 + r(x)),$$

where  $k$  is an integer  $\geq 2$ , and  $r$  is a smooth function of order  $\geq 1$ . Thus, we introduce the following weighted polar coordinates in the plane

$$\psi: \begin{pmatrix} x \\ t \end{pmatrix} \mapsto \begin{pmatrix} \rho \\ \theta \end{pmatrix} \quad \text{such that} \quad \begin{cases} c_0 x^k = \rho \cos(\theta) \\ (\alpha^2 + 1)^{1/2} t = \rho \sin(\theta) \end{cases}$$

whose Jacobian is  $\frac{1}{(\alpha^2 + 1)^{1/2} k c_0} \rho^{1/k} |\cos(\theta)|^{1/k-1}$ . In these new coordinates, the function  $W$  becomes

$$W^2 = (\alpha t + \xi(x) + tR(x, t))^2 + t^2 = \rho^2 \left( 1 + \frac{\alpha \sin(2\theta)}{(1 + \alpha^2)^{1/2}} + R_{\text{pol}}(\rho, \theta) \right),$$

where  $R_{\text{pol}}(\rho, \theta)$  is a remainder term vanishing at  $\rho = 0$ . Therefore, the integral (3.15) is controlled by

$$\int_{V''} \frac{|\rho \cos(\theta)|^{1/k-1}}{\sqrt{1 + \frac{\alpha \sin(2\theta)}{(1 + \alpha^2)^{1/2}} + R_{\text{pol}}(\rho, \theta)}} d\rho d\theta,$$

where  $V'' = \psi(\varphi(V))$ . But now this integral is finite, since

$$1 + \frac{\alpha \sin(2\theta)}{(\alpha^2 + 1)^{1/2}} > 1 - \frac{|\alpha|}{(\alpha^2 + 1)^{1/2}} > 0$$

and thus the denominator, up to restricting  $V''$ , is never-vanishing.  $\square$

*Remark 3.5.6.* The mild degeneration assumption is sharp for the local integrability of  $W^{-1}$ . Consider the example taken from [DGN12, Prop. 3.4] where  $\Sigma = \{z = g(x, y)\}$ , with

$$g(x, y) = \frac{1}{2}xy + \frac{1}{2}y^2 + \int_0^x e^{-\tau-2} d\tau.$$

Here  $N = X$  and  $T = Y$ , with  $g$  being in the normal form (3.12). Then, the critical curve of  $\mathbf{0}$  is  $\mathcal{C} = \{y + e^{-x-2} = 0\} \cap \Sigma$ , which can be parametrized by

$$\gamma(x) = \left( x, -e^{-x-2}, g\left(x, e^{-x-2}\right) \right).$$

Thus,  $Tu(\gamma(x)) = -e^{-x-2}$ , which has infinite order at  $x = 0$ . Therefore,  $\mathbf{0}$  is not a mildly degenerate characteristic point and one can check that  $W^{-1}$  is not locally integrable.

Notice, however, that in the previous example,  $\mathcal{H}$  is locally integrable. Thus, in general, to prove the integrability of  $\mathcal{H}$ , one should take into account also its numerator, which vanishes at characteristic points.

**Theorem 3.5.7.** *Let  $\Sigma \subset \mathbb{H}$  be a real-analytic embedded surface, let  $p$  be an isolated characteristic point and let  $u \in C^\omega$  be a local defining function for  $\Sigma$  in a neighborhood of  $p$ . Denote with  $W$  the norm of the horizontal gradient of  $u$ . Then*

$$\frac{1}{W} \in L^1_{\text{loc}}(\Sigma, \sigma_{\mathbb{R}}),$$

where  $\sigma_{\mathbb{R}}$  denotes the Riemannian induced measure on  $\Sigma$ . In particular

$$\mathcal{H} \in L^1_{\text{loc}}(\Sigma, \sigma_{\mathbb{R}}).$$

*Proof.* If  $C(\Sigma)$  consists of non-degenerate characteristic points, the result follows from Theorem 3.4.2. If  $p \in C(\Sigma)$  is degenerate, we show that  $p$  is actually mildly degenerate. We can assume that  $\Sigma = \{z - g(x, y) = 0\}$ , where  $g \in C^\omega(\mathbb{R}^2)$  has the normal form (3.12), and  $\mathbf{0} \in C(\Sigma)$  is degenerate. In this case, in coordinates  $(x, t) = \varphi(x, y)$  defined in (3.14), the critical curve of  $\mathbf{0}$  is  $\mathcal{C} = \{t = 0\}$  and  $p \in C(\Sigma)$  if and only if

$$\begin{cases} \xi(x) = 0, \\ t = 0. \end{cases}$$

Since  $\mathbf{0}$  is an isolated characteristic point,  $\xi$  is not identically zero. Thus, since  $\xi$  is real-analytic it has finite order at  $x = 0$ .  $\square$

### 3.5.1 Integrability of the intrinsic Gaussian curvature

The results that we have proved for the horizontal mean curvature in the 3D contact case and in the Heisenberg group can be extended to another geometrical invariant of surfaces, namely the so-called intrinsic Gaussian curvature. Let  $M$  be a 3D contact manifold and let  $\{X_1, X_2\}$  be a local orthonormal frame for the contact distribution. The intrinsic Gaussian curvature for a  $C^2$  surface  $\Sigma \subset M$  is defined as

$$\mathcal{K}_0 = -\frac{X_0 u}{W^2} - \left(\frac{X_2 u}{W}\right) X_1 \left(\frac{X_0 u}{W}\right) + \left(\frac{X_1 u}{W}\right) X_2 \left(\frac{X_0 u}{W}\right),$$

where  $X_0$  is the Reeb field of the contact structure,  $u$  is a local defining function for  $\Sigma$  at  $p$  and  $W = \|\nabla u\|$ . As was the case for the horizontal mean curvature, the definition of  $\mathcal{K}_0$  restricted at  $\Sigma$  is independent on the choice of the local defining function. The intrinsic Gaussian curvature has been introduced in [BTV17] for the Heisenberg group and in [BBCH20] for the general 3D contact case. We refer to those articles for further details. Reasoning as in the case of the horizontal mean curvature, one can derive the following estimate for  $\mathcal{K}_0$

$$|\mathcal{K}_0| \leq \frac{C_1}{W^2},$$

for a suitable constant  $C_1 > 0$ . Therefore, recalling that the induced sub-Riemannian measure on  $\Sigma$  is proportional to  $W\sigma_{\mathbb{R}}$ , we see that there exists  $C_2 > 0$  such that

$$|\mathcal{K}_0| d\sigma_{\mathbb{H}} \leq \frac{C_2}{W} d\sigma_{\mathbb{R}}. \tag{3.16}$$

Compare (3.16) to the estimate (3.5). By Theorem 3.1.1 and Theorem 3.1.3, we immediately obtain integrability for the intrinsic Gaussian curvature in the 3D contact case for non-degenerate isolated characteristic points and in the Heisenberg group for the analytic case and isolated characteristic points.

This integrability result is more significant for the Heisenberg group, indeed as proved in [BTV17, Thm. 1.1], a Gauss-Bonnet Theorem in  $\mathbb{H}$  for  $C^2$  surfaces holds, provided that the following assumptions are verified:

1.  $\mathcal{H}_E^1(C(\Sigma)) = 0$ ,
2.  $W^{-1}$  is locally summable with respect to the Riemannian induced measure at  $C(\Sigma)$ ,

where  $\mathcal{H}_E^1$  is the 1-dimensional Euclidean Hausdorff measure. Thus, if  $C(\Sigma)$  consists only of isolated points and either  $\Sigma$  is a real-analytic surface, or  $\Sigma$  is  $C^2$  and the characteristic points are also non-degenerate, the assumptions of the Gauss-Bonnet theorem are verified.



# 4

## Relative heat content asymptotics for sub-Riemannian manifolds

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*In this chapter, we study the small-time asymptotics of the relative heat content in sub-Riemannian geometry. This quantity is related, yet different, to the classical heat content. Indeed it is defined as the total amount of heat contained in the domain at time  $t$ , allowing the heat to flow outside the domain. Under the assumption of not having characteristic points, we obtain a fourth-order asymptotic expansion in square root of  $t$ . Significant difficulties emerge, as the boundary behavior of the temperature function is no longer known, as opposed to the case of the classical heat content. We use a symmetry argument of the heat diffusion to obtain information on the small-time behavior of temperature at the boundary of the domain.*

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## 4.1 Introduction

Let  $M$  be a sub-Riemannian manifold, equipped with a smooth measure  $\omega$ , let  $\Omega \subset M$  be an open relatively compact subset of  $M$ , with smooth boundary, and consider the Cauchy problem for the heat equation in this setting:

$$\begin{aligned} (\partial_t - \Delta) u(t, x) &= 0, & \forall (t, x) \in (0, \infty) \times M, \\ u(0, \cdot) &= \mathbb{1}_\Omega, & \text{in } L^2(M, \omega), \end{aligned} \tag{4.1}$$

where  $\mathbb{1}_\Omega$  is the indicator function of the set  $\Omega$ , and  $\Delta$  is the sub-Laplacian, defined with respect to  $\omega$ . By classical spectral theory, there exists a unique solution to (4.1),

$$u(t, x) = e^{t\Delta} \mathbb{1}_\Omega(x), \quad \forall x \in M, t > 0,$$

where  $e^{t\Delta}$  denotes the heat semi-group in  $L^2(M, \omega)$ , associated with  $\Delta$ . The *relative heat content* is the function

$$H_\Omega(t) = \int_\Omega u(t, x) d\omega(x), \quad \forall t > 0.$$

This quantity has been studied in connection with geometric properties of subsets of  $\mathbb{R}^n$ , starting from the seminal work of De Giorgi [DG54], where he introduced the notion of perimeter of a set in  $\mathbb{R}^n$  and proved a characterization of sets of finite perimeter in terms of the heat kernel. His result was subsequently refined, using techniques of functions of bounded variation: it was proven in [Led94] for balls in  $\mathbb{R}^n$ , and in [MPPP07] for general subsets of  $\mathbb{R}^n$ , that a borel set  $\Omega \subset \mathbb{R}^n$  with finite Lebesgue measure has finite perimeter à la De Giorgi if and only if

$$\exists \lim_{t \rightarrow 0} \frac{\sqrt{\pi}}{\sqrt{t}} (|\Omega| - H_\Omega(t)) = P(\Omega), \tag{4.2}$$

where  $|\cdot|$  is the Lebesgue measure and  $P$  is the perimeter measure in  $\mathbb{R}^n$ . Notice that (4.2) is equivalent to a first-order<sup>1</sup> asymptotic expansion of  $H_\Omega(t)$ . A further development in this direction was then obtained in [AMM13], where the authors extended (4.2) to an asymptotic expansion of order 3 in  $\sqrt{t}$ , assuming the boundary of  $\Omega \subset \mathbb{R}^n$  to be a  $C^{1,1}$  set. For simplicity, we state here the result of [AMM13, Thm. 1.1] assuming  $\partial\Omega$  is smooth<sup>2</sup>:

$$\begin{aligned} H_\Omega(t) &= |\Omega| - \frac{1}{\sqrt{\pi}} P(\Omega) t^{1/2} \\ &\quad + \frac{(n-1)^2}{12\sqrt{\pi}} \int_{\partial\Omega} \left( H_{\partial\Omega}^2(x) + \frac{2}{(n-1)^2} c_{\partial\Omega}(x) \right) d\mathcal{H}^{n-1}(x) t^{3/2} + o(t^{3/2}), \end{aligned} \tag{4.3}$$

---

<sup>1</sup>Here and throughout the chapter, the notion of order is computed with respect to  $\sqrt{t}$ .

<sup>2</sup>The statement of Theorem 1.1 in [AMM13] differs from (4.3) by a sign in the third-order coefficient: the correct sign appears a few lines below the statement, in the expansion of the function  $K_t(E, E^c)$ .

as  $t \rightarrow 0$ , where  $\mathcal{H}^{n-1}$  is the Hausdorff measure and, denoting by  $k_i^{\partial\Omega}$  the principal curvatures of  $\partial\Omega$  at the point  $x$ ,

$$H_{\partial\Omega}(x) = \frac{1}{n-1} \sum_{i=1}^{n-1} k_i^{\partial\Omega}(x), \quad c_{\partial\Omega}(x) = \sum_{i=1}^{n-1} k_i^{\partial\Omega}(x)^2,$$

In the Riemannian setting, Van den Berg and Gilkey in [vdBG15] proved the existence of a complete asymptotic expansion for  $H_\Omega(t)$ , generalizing (4.3), when  $\partial\Omega$  is smooth. Moreover, they were able to compute explicitly the coefficients of the expansion up to order 4 in  $\sqrt{t}$ . Their techniques are based on pseudo-differential calculus, and cannot be immediately adapted to the sub-Riemannian setting. In particular, what is missing is a global parametrix estimate for the heat kernel  $p_t(x, y)$ , cf. [vdBG15, Sec. 2.3]: for any  $k \in \mathbb{N}$ , there exist  $J_k, C_k > 0$  such that

$$\left\| p_t(x, y) - \sum_{j=0}^{J_k} p_t^j(x, y) \right\|_{C^k(M \times M)} \leq C_k t^k, \quad \text{as } t \rightarrow 0, \quad (4.4)$$

where  $p_t^j(x, y)$  are suitable smooth functions, given explicitly in terms of the Euclidean heat kernel and iterated convolutions. The closest estimate analogue to (4.4) in the sub-Riemannian setting is the one proved recently in [CdVHT20, Thm. A] (see Theorem 1.4.5 for the precise statement), where the authors show an asymptotic expansion of the heat kernel in an *asymptotic neighborhood* of the diagonal, which is not enough to reproduce (4.4) and thus the argument of Van den Berg and Gilkey. Moreover, in this case,  $p_t^j(x, y)$  is expressed in terms of the heat kernel of the nilpotent approximation and iterated convolutions, thus posing technical difficulties for the explicit computations of the coefficients (which would be no longer “simple” gaussian-type integrals).

In this chapter, under the assumption of not having characteristic points, we prove the existence of the asymptotic expansion of  $H_\Omega(t)$ , up to order 4 in  $\sqrt{t}$ , as  $t \rightarrow 0$ . We remark that we include also the rank-varying case. In order to state our main results, let us introduce the following operator, acting on  $C^\infty(M)$ ,

$$N\phi = 2g(\nabla\phi, \nabla\delta) + \phi\Delta\delta, \quad \forall \phi \in C^\infty(M),$$

where  $\delta: M \rightarrow \mathbb{R}$  denotes the sub-Riemannian signed distance function from  $\partial\Omega$ , see Section 4.4 for precise definitions.

**Theorem 4.1.1.** *Let  $M$  be a compact sub-Riemannian manifold, equipped with a smooth measure  $\omega$ , and let  $\Omega \subset M$  be an open subset whose boundary is smooth and has no characteristic points. Then, as  $t \rightarrow 0$ ,*

$$H_\Omega(t) = \omega(\Omega) - \frac{1}{\sqrt{\pi}} \sigma(\partial\Omega) t^{1/2} - \frac{1}{12\sqrt{\pi}} \int_{\partial\Omega} \left( N(\Delta\delta) - 2(\Delta\delta)^2 \right) d\sigma t^{3/2} + o(t^2), \quad (4.5)$$

where  $\sigma$  denotes the sub-Riemannian perimeter measure.

*Remark 4.1.2.* The compactness assumption in Theorem 4.1.1 is technical and can be relaxed by requiring, instead, global doubling of the measure and a global Poincaré inequality, see section 4.7 and in particular Theorem 4.7.3. Some notable examples satisfying these assumptions are:

- $M$  is a Lie group with polynomial volume growth, the distribution is generated by a family of left-invariant vector fields satisfying the Hörmander condition and  $\omega$  is the Haar measure. This family includes also Carnot groups.

- $M = \mathbb{R}^n$ , equipped with a sub-Riemannian structure induced by a family of vector fields  $\{Y_1, \dots, Y_N\}$  with bounded coefficients together with their derivatives, and satisfying the Hörmander condition.
- $M$  is a complete Riemannian manifold with metric  $g$ , equipped with the Riemannian measure, and with non-negative Ricci curvature.

See Section 4.7.1 for further details. In all these examples, Theorem 4.1.1 holds.

The strategy of the proof of Theorem 4.1.1 follows a similar strategy of what we did in Chapter 2, inspired by the method introduced in [Sav98], used for the classical heat content (4.6). However, as we are going to explain in Section 4.1.1, new technical difficulties arise, the main one being related to the fact that now  $u(t, \cdot)|_{\partial\Omega} \neq 0$ . At order zero, we obtain the following result, see Chapter 1 for precise definitions.

**Theorem 4.1.3.** *Let  $M$  be a sub-Riemannian manifold, equipped with a smooth measure  $\omega$  and let  $\Omega \subset M$  be an open relatively compact subset, whose boundary is smooth and has no characteristic points. Let  $x \in \partial\Omega$  and consider a chart of privileged coordinates  $\psi: U \rightarrow V \subset \mathbb{R}^n$  centered at  $x$ , such that  $\psi(U \cap \Omega) = V \cap \{z_1 > 0\}$ . Then,*

$$\lim_{t \rightarrow 0} u(t, x) = \int_{\{z_1 > 0\}} \hat{p}_1^x(0, z) d\hat{\omega}^x(z) = \frac{1}{2}, \quad \forall x \in \partial\Omega,$$

where  $\hat{\omega}^x$  denotes the nilpotentization of  $\omega$  at  $x$  and  $\hat{p}_t^x$  denotes the heat kernel associated with the nilpotent approximation of  $M$  at  $x$  and measure  $\hat{\omega}^x$ .

This result can be seen as a partial generalization of [CCSGM13, Prop. 3], where the authors proved an asymptotic expansion of  $u(t, x)$  up to order 1 in  $\sqrt{t}$  for  $x \in \partial\Omega$ , for a special class of non-characteristic domains in Carnot groups.

*Remark 4.1.4.* Our proof of Theorem 4.1.3 does not yield an asymptotic series for  $u(t, \cdot)|_{\partial\Omega}$  at order higher than 0. Indeed a complete asymptotic series of this quantity seems difficult to achieve, cf. Section 4.6.

**Structure of the chapter.** In Section 4.2 we recall the basic definition of relative heat content. In Section 4.3, we prove Theorem 4.1.3, using the nilpotent approximation of a sub-Riemannian manifold. In Section 4.4, we use the main tools to obtain the relative heat content asymptotic and we use Theorem 4.1.3 to compute its first-order expansion in a compact sub-Riemannian manifold. Then, in Section 4.5, we introduce the outside contribution to develop the higher-order asymptotic expansion of the relative heat content for a compact sub-Riemannian manifold, proving Theorem 4.1.1. In Section 4.6, we explain why we are not able to adapt [CdVHT20, Thm. A] to obtain a proof of Theorem 4.1.1. Finally, in Section 4.7, we show how to extend the above results when  $M$  is non-compact.

#### 4.1.1 Strategy of the proof of Theorem 4.1.1

To better understand the new technical difficulties in the study of the relative heat content  $H_\Omega(t)$ , let us compare it with the classical heat content  $Q_\Omega(t)$ , cf. Chapter 2, and illustrate the strategy of the proof of Theorem 4.1.1.

**The classical heat content.** We highlight the differences between the relative heat content  $H_\Omega(t)$  and the classical one  $Q_\Omega(t)$ : let  $\Omega \subset M$  an open set in  $M$ , then for all  $t > 0$ , we have

$$H_\Omega(t) = \int_\Omega u(t, x) d\omega(x), \quad Q_\Omega(t) = \int_\Omega u_0(t, x) d\omega(x), \quad (4.6)$$

where  $u(t, x)$  is the solution to (4.1) and  $u_0(t, x)$  is the solution to the Dirichlet problem for the heat equation, associated with  $\Omega$ , i.e.

$$\begin{aligned} (\partial_t - \Delta)u_0(t, x) &= 0, & \forall (t, x) \in (0, \infty) \times \Omega, \\ u_0(t, x) &= 0, & \forall (t, x) \in (0, \infty) \times \partial\Omega, \\ u_0(0, x) &= 1, & \forall x \in \Omega, \end{aligned} \quad (4.7)$$

The crucial difference is that  $u_0(t, \cdot)|_{\partial\Omega} = 0$ , for any  $t > 0$ , whereas  $u(t, \cdot)|_{\partial\Omega} \neq 0$  in general. Thus, there is no a priori relation between  $H_\Omega(t)$  and  $Q_\Omega(t)$ : the only relevant information is given by domain monotonicity, which implies that:

$$Q_\Omega(t) \leq H_\Omega(t), \quad \forall t > 0$$

and clearly this does not give the asymptotics of the latter. See also [vdB13] for other comparison results in the Euclidean setting.

**Failure of Duhamel's principle.** In Chapter 2, we established a complete asymptotic expansion of  $Q_\Omega(t)$ , as  $t \rightarrow 0$ , provided that  $\partial\Omega$  has no characteristic points. The proof of this result relied on an iterated application of the Duhamel's principle and the fact that  $u_0(t, x)|_{\partial\Omega} = 0$ . Following the same strategy, we apply Duhamel's principle to a localized version of  $H_\Omega(t)$ : fix a function  $\phi \in C_c^\infty(M)$ , compactly supported in a tubular neighborhood around  $\partial\Omega$  and such that  $0 \leq \phi \leq 1$  and  $\phi$  is identically 1, close to  $\partial\Omega$ . Then, using off-diagonal estimates for the heat kernel, one can prove that:

$$\omega(\Omega) - H_\Omega(t) = I\phi(t, 0) + O(t^\infty), \quad \text{as } t \rightarrow 0, \quad (4.8)$$

where  $I\phi(t, r)$  is defined for  $t > 0$  and  $r \geq 0$  as

$$I\phi(t, r) = \int_{\Omega_r} (1 - u(t, x))\phi(x)d\omega(x), \quad (4.9)$$

here  $\Omega_r = \{x \in \Omega \mid \delta(x) > r\}$ , with  $\delta: \Omega \rightarrow \mathbb{R}$  denoting the distance function from the boundary. Hence, the small-time behavior of  $H_\Omega(t)$  is captured by  $I\phi(t, 0)$ . By Duhamel's principle and the sub-Riemannian mean value lemma, Theorem 2.4.1, we obtain the following:

$$I\phi(t, 0) = \frac{1}{\sqrt{\pi}} \int_0^t \int_{\partial\Omega} (1 - u(\tau, y))\phi(y)d\sigma(y)(t - \tau)^{-1/2}d\tau + O(t), \quad \text{as } t \rightarrow 0. \quad (4.10)$$

For the classical heat content,  $u_0$  satisfies Dirichlet boundary condition, cf. (4.7), hence (4.10) would give the first-order asymptotics (and then one could iterate). On the contrary, in this case, we do not have prior knowledge of  $u(t, y)$  as  $y \in \partial\Omega$  and  $t \rightarrow 0$ . Thus, already for the first-order asymptotics, Duhamel's principle alone is not enough, and we need some information on the asymptotic behavior of  $u(t, \cdot)|_{\partial\Omega}$ .

**First-order asymptotics.** We study the asymptotics of  $u(t, \cdot)|_{\partial\Omega}$ . Using the notion of nilpotent approximation of a sub-Riemannian manifold, cf. Section 1.4, we deduce the zero-order asymptotic expansion of  $u(t, \cdot)|_{\partial\Omega}$  as  $t \rightarrow 0$ , proving Theorem 4.1.3. This is enough to infer the first-order expansion of  $H_\Omega(t)$ , by means of (4.10). At this point, we iterate the Duhamel's principle to obtain the higher-order terms of the expansion of  $H_\Omega(t)$ . However, already at the first iteration, we obtain the following formula for  $I\phi$ :

$$\begin{aligned} I\phi(t, 0) &= \frac{1}{\sqrt{\pi}} \int_0^t \int_{\partial\Omega} (1 - u(\tau, \cdot))\phi d\sigma (t - \tau)^{-1/2} d\tau \\ &\quad + \frac{1}{2\pi} \int_0^t \int_0^\tau \int_{\partial\Omega} (1 - u(\hat{\tau}, \cdot))N\phi d\sigma ((\tau - \hat{\tau})(t - \tau))^{-1/2} d\hat{\tau} d\tau + O(t^{3/2}), \end{aligned} \quad (4.11)$$

as  $t \rightarrow 0$ . Therefore, the zero-order asymptotic expansion of  $u(t, \cdot)|_{\partial\Omega}$  no longer suffices for obtaining the second-order asymptotics of  $H_\Omega(t)$ .

**The outside contribution  $I^c\phi$ .** We mentioned that the crucial difference between  $H_\Omega(t)$  and  $Q_\Omega(t)$ , defined in (4.6), is related to the fact that  $u(t, \cdot)|_{\partial\Omega} \neq 0$ , whereas  $u_0(t, \cdot)|_{\partial\Omega} = 0$ , for any  $t > 0$ . From a physical viewpoint, this distinction comes from the fact that, since the boundary  $\partial\Omega$  is no longer insulated, the heat governed by the Cauchy problem  $u(t, x)$ , solution to (4.1), can flow also outside of  $\Omega$ , whereas  $u_0(t, x)$ , solution to the Dirichlet problem (4.7), is confined in  $\Omega$ , and the external temperature is 0. Hence, we can imagine that the asymptotic expansion of  $H_\Omega(t)$  is affected by the boundary, both from the inside and from the outside of  $\Omega$ .

Interpreting  $I\phi$  as the *inside contribution* to the asymptotics of  $H_\Omega$ , we are going to formalize the physical intuition of having heat flowing outside of  $\Omega$ , defining an *outside contribution*,  $I^c\phi$  to the asymptotics<sup>3</sup>. The starting observation is the following simple relation: setting

$$K_\Omega(t) = \int_{M \setminus \Omega} u(t, x) d\omega(x), \quad \forall t > 0,$$

we have, by divergence theorem,

$$H_\Omega(t) + K_\Omega(t) = \omega(\Omega), \quad \forall t > 0. \quad (4.12)$$

Similarly to (4.9), for a suitable smooth function  $\phi$ , one may define a localized version of  $K_\Omega(t)$ , which we call  $I^c\phi(t, r)$ , so that

$$K_\Omega(t) = I^c\phi(t, 0) + O(t^\infty), \quad \text{as } t \rightarrow 0, \quad (4.13)$$

see Section 4.5.1 for precise definitions. Using (4.8), (4.12) and (4.13), we show the following relation:

$$I\phi(t, 0) - I^c\phi(t, 0) = O(t^\infty), \quad \text{as } t \rightarrow 0,$$

for a suitable smooth function  $\phi$ . On the other hand, for the localized quantity  $I\phi(t, 0) - I^c\phi(t, 0)$  we have a Duhamel's principle, thanks to which we are able to study the asymptotic expansion, up to order 3, of the *integral* of  $u(t, x)$  over  $\partial\Omega$ , cf. Theorem 4.5.4. The limitation to the order 3 of the asymptotics is technical and seems difficult to overcome, cf. Remark 4.5.5. Inserting this asymptotics in (4.11), we obtain the asymptotics *up to order 3* of the expansion of  $H_\Omega(t)$ , as  $t \rightarrow 0$ .

**Fourth-order asymptotics.** Since we have at disposal only the asymptotics of the integral of  $u(t, x)$  over  $\partial\Omega$ , up to order 3, we need a finer argument to obtain the fourth-order asymptotics of  $H_\Omega(t)$ . The simple but compelling relation is based once again on (4.8), (4.12) and (4.13), thanks to which we can write:

$$\omega(\Omega) - H_\Omega(t) = \frac{1}{2} (I\phi(t, 0) + I^c\phi(t, 0)) + O(t^\infty), \quad \text{as } t \rightarrow 0.$$

Now for the sum of the contributions  $I\phi(t, 0) + I^c\phi(t, 0)$ , the Duhamel's principle implies the following:

$$\begin{aligned} I\phi(t, 0) + I^c\phi(t, 0) &= \frac{2}{\sqrt{\pi}} \sigma(\partial\Omega) t^{1/2} \\ &+ \frac{1}{2\pi} \int_0^t \int_0^\tau \int_{\partial\Omega} (1 - 2u(\hat{\tau}, x) N\phi(y)) d\sigma(y) ((\tau - \hat{\tau})(t - \tau))^{-1/2} d\hat{\tau} d\tau + o(t). \end{aligned}$$

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<sup>3</sup>The notation “superscript  $c$ ” stands for complement. Indeed the outside contribution is the inside contribution of the complement of  $\Omega$ , see Section 4.5.1.

This time notice how the integral of  $u(t, x)$  over  $\partial\Omega$  appears in a first-order term (as opposed to what happened in (4.10) or (4.11)), thus its asymptotic expansion up to order 3 implies a fourth-order expansion for  $H_\Omega(t)$ .

### 4.1.2 From the heat kernel asymptotics to the relative heat content asymptotics

In [CdVHT20, Thm. A], the authors proved the existence of small-time asymptotics of the hypoelliptic heat kernel,  $p_t(x, y)$ , see Theorem 1.4.5 below for the precise statement. In Theorem 4.1.3 we are able to exploit this result to obtain the zero-order asymptotics of the function

$$u(t, x) = e^{t\Delta}\mathbf{1}_\Omega(x) = \int_\Omega p_t(x, y)d\omega(y), \quad \forall t > 0, \quad x \in \partial\Omega.$$

However, we are not able to extend Theorem 4.1.3 to higher-order asymptotics since, roughly speaking, the remainder terms in Theorem 1.4.5 are not uniform as  $t \rightarrow 0$ . If we had a better control on the remainders, we could indeed integrate (in a suitable way) the small-time heat kernel asymptotics to obtain the corresponding expansion for  $u(t, x)$ . Finally, from such an expansion, the relative heat content asymptotics would follow from the localization principle (4.8) and the (iterated) Duhamel's principle (4.10). This is done in Section 4.6.

### 4.1.3 Characteristic points

In order to prove our main results, we need the non-characteristic assumption on the domain  $\Omega$ . We recall that for a subset  $\Omega \subset M$  with smooth boundary,  $x \in \partial\Omega$  is a characteristic point if  $\mathcal{D}_x \subset T_x(\partial\Omega)$ . As was the case for the classical heat content, cf. Chapter 2, the non-characteristic assumption is crucial to follow our strategy, since it guarantees the smoothness of the signed distance function close to  $\partial\Omega$ , cf. Theorem 4.4.1. Nevertheless, one might ask whether Theorem 4.1.1 holds for domains with characteristic points, at least formally.

On the one hand, the coefficients, up to order 2, are well-defined even in presence of characteristic points, cf. [Bal03]. While, for what concerns the integrand of the third-order coefficient, its integrability, with respect to the sub-Riemannian induced measure  $\sigma$ , is related to integrability of the sub-Riemannian mean curvature  $\mathcal{H}$ , with respect to the Riemannian induced measure, cf. Chapter 3 for further details.

On the other hand, differently from what happens in the case of the Dirichlet problem, the heat kernel  $p_t(x, y)$  associated with (4.1) is smooth at the boundary of  $\Omega$ , for positive times, even in presence of characteristic points. Thus, in principle, there is no obstacle in obtaining an asymptotic expansion of  $H_\Omega(t)$  also in that case. Moreover, in Carnot groups of step 2, a similar result to (4.2) holds, cf. [BMP12, GT20]. In particular, the characterization of sets of finite horizontal perimeter in Carnot groups of step 2 is independent of the presence of characteristic points, indicating that an asymptotic expansion such as (4.5) may still hold, dropping the non-characteristic assumption.

### 4.1.3 Notation

Throughout the chapter, for a set  $U \subset M$ , we will use the notation  $C_c^\infty(U)$ , even in the compact case, so that all the statements need not be modified in the non-compact case, when the generalization is possible, cf. Theorem 4.7.3. Moreover, in the non-compact and complete case, the set  $\Omega \subset M$  is assumed to be open and bounded.

## 4.2 The relative heat content

We refer to Chapter 1 for general facts in sub-Riemannian geometry. Here, we recall the precise definition of relative heat content.

Let  $M$  be a sub-Riemannian manifold. Let  $\omega$  be a smooth measure on  $M$ , i.e. by a positive tensor density. Recall that the sub-Laplacian is the operator

$$\Delta f = \operatorname{div}_\omega \circ \nabla f, \quad \forall f \in C^\infty(M).$$

We denote by  $L^2(M, \omega)$  the space of real functions on  $M$  which are square-integrable with respect to the measure  $\omega$ . Let  $\Omega \subset M$  be an open relatively compact set with smooth boundary and  $u(t, x)$  the solution to the Cauchy problem for the heat equation on  $\Omega$ , namely:

$$\begin{aligned} (\partial_t - \Delta) u(t, x) &= 0, & \forall (t, x) \in (0, \infty) \times M, \\ u(0, \cdot) &= \mathbf{1}_\Omega, & \text{in } L^2(M, \omega), \end{aligned} \tag{4.14}$$

where  $u(0, \cdot)$  is a shorthand notation for the  $L^2$ -limit of  $u(t, x)$  as  $t \rightarrow 0$ . We recall that, by standard spectral theory, cf. Section 1.3, there exists a unique solution to (4.14), and it can be represented as

$$u(t, x) = e^{t\Delta} \mathbf{1}_\Omega(x), \quad \forall x \in M, t > 0,$$

where  $e^{t\Delta}: L^2(M, \omega) \rightarrow L^2(M, \omega)$  denotes the heat semi-group, associated with  $\Delta$ . Moreover, there exists a heat kernel associated with (4.14), i.e. a positive function  $p_t(x, y) \in C^\infty((0, +\infty) \times M \times M)$ , such that:

$$u(t, x) = \int_M p_t(x, y) \mathbf{1}_\Omega(y) d\omega(y) = \int_\Omega p_t(x, y) d\omega(y).$$

**Definition 4.2.1** (Relative heat content). Let  $u(t, x)$  be the solution to (4.14). We define the *relative heat content*, associated with  $\Omega$ , as

$$H_\Omega(t) = \int_\Omega u(t, x) d\omega(x), \quad \forall t > 0.$$

*Remark 4.2.2.* If we consider, instead of  $\Omega$ , a set which is the closure of an open set, then the Cauchy problem (4.14) has a unique solution and relative heat content is still well-defined.

We recall here the following property of the solution to (4.14): it satisfies a weak maximum principle, meaning that

$$0 \leq u(t, x) \leq 1, \quad \forall x \in \Omega, \forall t > 0. \tag{4.15}$$

This can be proven following the blueprint of the Riemannian proof (see [Gri09, Thm. 5.11]).

As in Chapter 2, we will assume that  $\partial\Omega$  has no characteristic points. We recall here the definition.

**Definition 4.2.3** (Characteristic point). We say that  $x \in \partial\Omega$  is a *characteristic point*, or tangency point, if the distribution is tangent to  $\partial\Omega$  at  $x$ , that is

$$\mathcal{D}_x \subseteq T_x(\partial\Omega).$$

We will assume that  $\partial\Omega$  has no characteristic points. We say in this case that  $\Omega$  is a *non-characteristic domain*.



### 4.3 Small-time asymptotics of $u(t, x)$ at the boundary

We prove here Theorem 4.1.3, regarding the zero-order asymptotics of  $u(t, \cdot)|_{\partial\Omega}$  as  $t \rightarrow 0$ .

**Theorem 4.3.1.** *Let  $M$  be a compact sub-Riemannian manifold, equipped with a smooth measure  $\omega$  and let  $\Omega \subset M$  be an open subset, whose boundary is smooth and has no characteristic points. Let  $x \in \partial\Omega$  and consider a chart of privileged coordinates  $\psi: U \rightarrow V \subset \mathbb{R}^n$  centered at  $x$ , such that  $\psi(U \cap \Omega) = V \cap \{z_1 > 0\}$ . Then,*

$$\lim_{t \rightarrow 0} u(t, x) = \int_{\{z_1 > 0\}} \hat{p}_1^x(0, z) d\hat{\omega}^x(z) = \frac{1}{2}, \quad \forall x \in \partial\Omega,$$

where  $\hat{\omega}^x$  denotes the nilpotentization of  $\omega$  at  $x$  and  $\hat{p}_t^x$  denotes the heat kernel associated with the nilpotent approximation of  $M$  at  $x$  and measure  $\hat{\omega}^x$ .

*Remark 4.3.2.* A chart of privileged coordinates, such that  $\psi(U \cap \Omega) = V \cap \{z_1 > 0\}$  always exists, provided that  $\partial\Omega$  has no characteristic points. Indeed, in this case, there exists a tubular neighborhood of the boundary, cf. Theorem 4.4.1, which is built through the flow of  $\nabla\delta$ , namely

$$G: (-r_0, r_0) \times \partial\Omega \rightarrow \Omega_{-r_0}^{r_0}; \quad G(t, q) = e^{t\nabla\delta}(q),$$

is a diffeomorphism such that  $G_*\partial_t = \nabla\delta$  and  $\delta(G(t, q)) = t$ . Here  $\delta: M \rightarrow \mathbb{R}$  is the signed distance function and  $\Omega_{-r_0}^{r_0} = \{-r_0 < \delta < r_0\}$ , see Section 4.4.1 for precise definitions. Therefore, choosing an adapted frame for the distribution at  $x$ , say  $\{Z_1, \dots, Z_n\}$  where  $Z_1 = \nabla\delta$ , we can define a set of privileged coordinates as in (1.22):

$$\mathbb{R}^n \ni (z_1, \dots, z_n) \mapsto e^{z_1 Z_1} \circ \underbrace{e^{z_2 Z_2} \circ \dots \circ e^{z_n Z_n}}_{\varphi(z_2, \dots, z_n)}(x) = G(z_1, \varphi(z_2, \dots, z_n)). \quad (4.16)$$

The resulting set of coordinates  $\psi$  satisfies  $\psi_*(\nabla\delta) = \partial_{z_1}$  and, denoting by  $V$  the neighborhood of 0 in  $\mathbb{R}^n$  where  $\psi$  is invertible,  $\psi(U \cap \Omega) = \{z_1 > 0\} \cap V$ . Here,  $e^{sX}(q)$  denotes the flow of the vector field  $X$ , starting at  $q$ , evaluated at time  $s$ .

*Proof of Theorem 4.3.1.* Let  $p_t(x, y)$  be the heat kernel of  $M$ , then we may write

$$u(t, x) = \int_{\Omega} p_t(x, y) d\omega(y), \quad \forall x \in M.$$

For a fixed  $x \in M$ , denoting by  $U$  any relatively compact neighborhood of  $x$ , we have

$$\begin{aligned} u(t, x) &= \int_{U \cap \Omega} p_t(x, y) d\omega(y) + \int_{\Omega \setminus U} p_t(x, y) d\omega(y) \\ &= \int_{U \cap \Omega} p_t(x, y) d\omega(y) + O(t^\infty), \end{aligned}$$

as  $t \rightarrow 0$ . Indeed, since the heat kernel is exponentially decaying outside the diagonal, cf. [JSC86, Prop. 3],

$$\int_{\Omega \setminus U} p_t(x, y) d\omega(y) \leq \omega(\Omega \setminus U) C_U e^{-\frac{cU}{t}} = O(t^\infty), \quad \text{as } t \rightarrow 0. \quad (4.17)$$

as  $t \rightarrow 0$ . Now, for  $x \in \partial\Omega$ , fix the set of privileged coordinates  $\psi: U \rightarrow V \subset \mathbb{R}^n$ , defined as in the statement and assume without loss of generality that  $\delta_\varepsilon(V) \subset V$  for any  $|\varepsilon| \leq 1$ , where  $\delta_\varepsilon$  is the dilation (1.23) of the nilpotent approximation of  $M$ . Also set

$$V_\varepsilon = \delta_\varepsilon(V \cap \{z_1 > 0\}), \quad \forall |\varepsilon| \leq 1.$$



when the limits exist, we have:

$$\lim_{t \rightarrow 0} u(t, x) = \lim_{t \rightarrow 0} \int_{U \cap \Omega} p_t(x, y) d\omega(y) = \lim_{t \rightarrow 0} \int_{V_1} p_t(0, z) d\omega(z), \quad (4.18)$$

where, in the last equation, we are considering the expression of the heat kernel and the measure in coordinates. We want to apply (1.30) at order 1 in  $\varepsilon$ , so let us rephrase the statement as follows: for any compact set  $K \subset V$ ,

$$|\varepsilon|^{\mathcal{Q}} p_{\varepsilon^2 \tau}(0, \delta_\varepsilon(z)) = \hat{p}_\tau(0, z) + \varepsilon R(\varepsilon, \tau, z), \quad \text{as } \varepsilon \rightarrow 0, \quad (4.19)$$

where  $R$  is a smooth function such that

$$\sup_{\substack{\varepsilon \in [-1, 1] \\ z \in K}} |R(\varepsilon, \tau, z)| \leq C(\tau, K), \quad (4.20)$$

with  $C(\tau, K) > 0$ . Notice that (4.20) is not uniform in  $\tau$ , in the sense that  $\tau \mapsto C(\tau, K)$  can explode as  $\tau \rightarrow 0$ , in general. Moreover, without loss of generality and, up to restrictions of  $U$ , we can assume that (4.20) holds globally on  $\bar{V}_1$ . For a fixed parameter  $L > 1$ , we set  $\tau = 1/L$  and  $\varepsilon^2 = tL$  in (4.19), obtaining:

$$|tL|^{\mathcal{Q}/2} p_t(0, \delta_{\sqrt{tL}}(z)) = \hat{p}_{1/L}(0, z) + \sqrt{tL} R(\sqrt{tL}, 1/L, z), \quad \text{as } t \rightarrow 0,$$

where the remainder  $R$  is bounded as  $t \rightarrow 0$  on the compact sets of  $V$ , but with a constant depending on  $L$ . Inserting the above expansion in (4.18), and writing the measure in coordinates  $d\omega(z) = \omega(z) dz$  with  $\omega(\cdot) \in C^\infty(V_1)$ , we have:

$$\begin{aligned} u(t, x) &= \int_{V_1} p_t(0, z) \omega(z) dz + O(t^\infty) \\ &= \int_{V_{\sqrt{tL}}} p_t(0, z) \omega(z) dz + \int_{V_1 \setminus V_{\sqrt{tL}}} p_t(0, z) d\omega(z) + O(t^\infty) \\ &= \int_{V_1} |tL|^{\mathcal{Q}/2} p_t(0, \delta_{\sqrt{tL}}(z)) \omega(\delta_{\sqrt{tL}}(z)) dz + \int_{V_1 \setminus V_{\sqrt{tL}}} p_t(0, z) d\omega(z) + O(t^\infty) \\ &= \int_{V_1} \left( \hat{p}_{1/L}(0, z) + \sqrt{tL} R(\sqrt{tL}, 1/L, z) \right) \omega(\delta_{\sqrt{tL}}(z)) dz \\ &\quad + \int_{V_1 \setminus V_{\sqrt{tL}}} p_t(0, z) d\omega(z) + O(t^\infty) \end{aligned} \quad (4.21)$$

where in the third equality we performed the change of variable  $z \mapsto \delta_{1/\sqrt{tL}}(z)$ . Let us discuss the terms appearing in (4.21) and (4.22). First of all, for any  $L > 1$ , by definition of the nilpotentization of  $\omega$  given in (1.27), we get

$$\lim_{t \rightarrow 0} \int_{V_1} \hat{p}_{1/L}(0, z) \omega(\delta_{\sqrt{tL}}(z)) dz = \int_{V_1} \hat{p}_{1/L}(0, z) d\hat{\omega}(z).$$

Moreover, for a fixed  $L > 1$ , the integral of  $R$  is bounded as  $t \rightarrow 0$ , therefore, using (4.20), we have:

$$\left| \sqrt{tL} \int_{V_1} R(\sqrt{tL}, 1/L, z) d\omega(z) \right| \leq C_L \sqrt{t}, \quad \forall t \leq 1,$$

where  $C_L > 0$  is a constant depending on the fixed  $L$ . Secondly, by an upper Gaussian bound for the heat kernel in compact sub-Riemannian manifold, [JSC86, Thm. 2], we obtain the following estimate for (4.22):

$$\int_{V_1 \setminus V_{\sqrt{tL}}} p_t(0, z) d\omega(z) \leq \int_{V_1 \setminus V_{\sqrt{tL}}} \frac{C_1 e^{-\frac{\beta d_{\text{SR}}^2(0, z)}{t}}}{t^{\mathcal{Q}/2}} d\omega(z), \quad (4.23)$$

where  $C_1, \beta > 0$  are positive constants. Now, by the Ball-Box Theorem [Jea14, Thm. 2.1], the sub-Riemannian distance function at the origin is comparable with the sub-Riemannian distance of  $\widehat{M}^x$ , denoted by  $\hat{d}$ . In particular, there exists a constant  $c > 0$  such that

$$d_{\text{SR}}^2(0, z) \geq c \hat{d}^2(0, z), \quad \forall z \in V.$$

Since in (4.23) we are integrating over the set  $V_1 \setminus V_{\sqrt{tL}}$  and  $\hat{d}$  is 1-homogeneous with respect to  $\delta_\varepsilon$ , we conclude that

$$d_{\text{SR}}^2(0, z) \geq ctL, \quad \forall z \in V_1 \setminus V_{\sqrt{tL}}.$$

Therefore the term (4.23) can be estimated as follows:

$$\int_{V_1 \setminus V_{\sqrt{tL}}} p_t(0, z) d\omega(z) \leq C_1 e^{-\frac{c\beta L}{2}} \int_{V_1} \frac{e^{-\frac{\beta d_{\text{SR}}^2(0, z)}{2t}}}{t^{\mathcal{Q}/2}} d\omega(z) \leq \tilde{C} e^{-\frac{c\beta L}{2}}, \quad (4.24)$$

where  $\tilde{C} > 0$  is independent of  $t$  and  $L$ . The last inequality in (4.24) follows from the fact that, by a lower Gaussian bound for the heat kernel [JSC86, Thm. 4], there exists  $C_2 > 0$  such that

$$\int_{V_1} \frac{e^{-\frac{\beta d_{\text{SR}}^2(0, z)}{2t}}}{t^{\mathcal{Q}/2}} d\omega(z) \leq C_2 \int_{V_1} p_{\tilde{t}}(0, z), \quad (4.25)$$

where  $\tilde{t} > 0$  depends on  $t$ . Then, thanks to the weak maximum principle, we conclude that the integral (4.25) is bounded uniformly with respect to  $t \in [0, \infty)$ .

Therefore, for any  $L > 1$ , we obtain the following estimates for the limit of  $u$ :

$$\begin{aligned} \limsup_{t \rightarrow 0} u(t, x) &\leq \int_{V_1} \hat{p}_{1/L}(0, z) d\hat{\omega}(z) + \tilde{C} e^{-\frac{c\beta L}{2}}, \\ \liminf_{t \rightarrow 0} u(t, x) &\geq \int_{V_1} \hat{p}_{1/L}(0, z) d\hat{\omega}(z) - \tilde{C} e^{-\frac{c\beta L}{2}}. \end{aligned} \quad (4.26)$$

In order to evaluate the limits in (4.26), let us firstly notice that, since  $\hat{p}$  enjoys upper and lower Gaussian bounds (see for example [CdVHT20, App. C]), reasoning as we did for (4.24), we can prove the following:

$$\int_{V_1} \hat{p}_{1/L}(0, z) d\hat{\omega}(z) = \int_{\{z_1 > 0\}} \hat{p}_{1/L}(0, z) d\hat{\omega}(z) + O(e^{-\beta' L}). \quad (4.27)$$

Secondly, thanks to (1.31) for  $\hat{p}$ , we have the following parity property

$$\hat{p}_t(0, z) = \hat{p}_t(0, \delta_{-1}(z)), \quad \forall t > 0, z \in \mathbb{R}^n,$$

and, by the choice of privileged coordinates,  $\delta_{-1}(\{z_1 > 0\}) = \{z_1 < 0\}$ . Thus, using also the stochastic completeness of the nilpotent approximation, we obtain for any  $t \geq 0$ ,

$$\begin{aligned} 1 &= \int_{\mathbb{R}^n} \hat{p}_t(0, z) d\hat{\omega}(z) = \int_{\{z_1 > 0\}} \hat{p}_t(0, z) d\hat{\omega}(z) + \int_{\{z_1 < 0\}} \hat{p}_t(0, z) d\hat{\omega}(z) \\ &= 2 \int_{\{z_1 > 0\}} \hat{p}_t(0, z) d\hat{\omega}(z), \end{aligned}$$

having performed the change of variables  $z \mapsto \delta_{-1}(z)$  in the last equality. Hence, the integral in (4.27) is

$$\int_{V_1} \hat{p}_{1/L}(0, z) d\hat{\omega}(z) = \frac{1}{2} + O(e^{-\beta' L}).$$

Finally, we optimize the inequalities (4.26) with respect to  $L$ , taking  $L \rightarrow \infty$  and concluding the proof.  $\square$

*Remark 4.3.3.* In the non-compact case, if  $M$  is globally doubling and supports a global Poincaré inequality, the proof above is still valid, cf. Theorem 4.7.3. Otherwise, a different proof is needed, see Appendix D for details.

## 4.4 First-order asymptotic expansion of $H_\Omega(t)$

In this section, we introduce the technical tools that allow us to prove the first-order asymptotic expansion of the relative heat content starting from Theorem 4.3.1. The new ingredient is a definition of an operator  $I_\Omega$ , which depends on the base set  $\Omega$ .

### 4.4.1 A mean value lemma

Define  $\delta: M \rightarrow \mathbb{R}$  to be the signed distance function from  $\partial\Omega$ , i.e.

$$\delta(x) = \begin{cases} \delta_{\partial\Omega}(x) & x \in \Omega, \\ -\delta_{\partial\Omega}(x) & x \in M \setminus \Omega, \end{cases} \quad (4.28)$$

where  $\delta_{\partial\Omega}: M \rightarrow [0, +\infty)$  denotes the usual distance function from  $\partial\Omega$ . Let us introduce the following notation: for any  $a, b \in \mathbb{R}$ , with  $a < b$ , we set

$$\Omega_a^b = \{x \in M \mid a < \delta(x) < b\}, \quad (4.29)$$

with the understanding that if  $b$  (or  $a$ ) is omitted, it is assumed to be  $+\infty$  (or  $-\infty$ ), for example<sup>4</sup>

$$\Omega_r = \Omega_r^{+\infty} = \{x \in M \mid r < \delta(x)\}.$$

In the non-characteristic case, the existence of a tubular neighborhood, cf. Theorem 2.4.2, can be extended without difficulties to the signed distance function. A proof of this result can be found in Chapter 5, cf. Theorem 5.2.6.

**Theorem 4.4.1** (Double-sided tubular neighborhood). *Let  $M$  be a sub-Riemannian manifold and let  $\Omega \subset M$  be an open relatively compact subset of  $M$  whose boundary is smooth and has no characteristic points. Let  $\delta: M \rightarrow \mathbb{R}$  be the signed distance function from  $\partial\Omega$ . Then, we have:*

- i)  $\delta$  is Lipschitz with respect to the sub-Riemannian distance and  $\|\nabla\delta\|_g \leq 1$  a.e.;
- ii) there exists  $r_0 > 0$  such that  $\delta: \Omega_{-r_0}^{r_0} \rightarrow \mathbb{R}$  is smooth;
- iii) there exists a smooth diffeomorphism  $G: (-r_0, r_0) \times \partial\Omega \rightarrow \Omega_{-r_0}^{r_0}$ , such that

$$\delta(G(t, y)) = t \quad \text{and} \quad G_*\partial_t = \nabla\delta, \quad \forall (t, y) \in (-r_0, r_0) \times \partial\Omega.$$

Moreover,  $\|\nabla\delta\|_g \equiv 1$  on  $\Omega_{-r_0}^{r_0}$ .

In particular, the following co-area formula for the signed distance function holds

$$\int_{\Omega_0^r} v(x) d\omega(x) = \int_0^r \int_{\partial\Omega_s} v(s, y) d\sigma(y) ds, \quad \forall r \geq 0, \quad (4.30)$$

from which we deduce the sub-Riemannian mean value lemma, see Theorem 2.4.1 for a proof.

**Proposition 4.4.2.** *Let  $M$  be a compact sub-Riemannian manifold, equipped with a smooth measure  $\omega$ , let  $\Omega \subset M$  be an open subset of  $M$  with smooth boundary and no*

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<sup>4</sup>Notice that the set  $\Omega_{-\infty}^{+\infty} = M$ , thus omitting both indexes can create confusion. We will never do that and  $\Omega$  will always denote the starting subset of  $M$ .

characteristic points and let  $\delta: M \rightarrow \mathbb{R}$  be the signed distance function from  $\partial\Omega$ . Fix a smooth function  $v \in C^\infty(M)$  and define

$$F(r) = \int_{\Omega_r} v(x) d\omega(x), \quad \forall r \geq 0. \quad (4.31)$$

Then there exists  $r_0 > 0$  such that the function  $F$  is smooth on  $[0, r_0)$  and, for  $0 \leq r < r_0$ :

$$F''(r) = \int_{\Omega_r} \Delta v(x) d\omega(x) + \int_{\partial\Omega_r} v(y) \operatorname{div}_\omega(\nu_r(y)) d\sigma(y),$$

where  $\nu_r$  is the outward-pointing unit normal to  $\Omega_r$ ,  $\sigma$  is the induced measure on  $\partial\Omega_r$ .

**Remark 4.4.3.** If  $v \in C_c^\infty(M)$ , then neither  $M$  nor  $\Omega$  is required to be compact for Proposition 4.4.2 to be true, indeed its proof relies on (4.30), which continues to hold, and the divergence theorem (1.13), which applies if  $\operatorname{supp}(v)$  is compact. Moreover, we remark that  $\nu_r$  is equal to  $\nabla\delta$  up to sign. We prefer to keep  $\nu_r$  in (4.31), since we are going to apply it when the integral is performed over  $\Omega_r$  or its complement.

If we choose the function  $v$  in the definition of  $F$  to be  $1 - u(t, x)$ , where  $u(t, \cdot) = e^{t\Delta}\mathbf{1}_\Omega$ , then,  $F$  satisfies a non-homogeneous one-dimensional heat equation.

**Corollary 4.4.4.** *Under the hypotheses of Proposition 4.4.2, the function*

$$F(t, r) = \int_{\Omega_r} (1 - u(t, x)) d\omega(x), \quad \forall t > 0, \quad r \geq 0, \quad (4.32)$$

where  $u(t, x) = e^{t\Delta}\mathbf{1}_\Omega(x)$ , satisfies the following non-homogeneous one-dimensional heat equation:

$$(\partial_t - \partial_r^2)F(t, r) = - \int_{\partial\Omega_r} (1 - u(t, \cdot)) \operatorname{div}_\omega(\nu_r) d\sigma, \quad t > 0, \quad r \in [0, r_0), \quad (4.33)$$

where  $\nu_r$  is the outward-pointing unit normal to  $\Omega_r$ ,  $\sigma$  is the induced measure on  $\partial\Omega_r$ .

Corollary 4.4.4 holds only for  $r \leq r_0$ , however we would like to extend it to the whole positive half-line, in order to apply a Duhamel's principle. This can be done up to an error which is exponentially small.

#### 4.4.2 Localization principle

**Proposition 4.4.5.** *Let  $M$  be a compact sub-Riemannian manifold, equipped with a smooth measure  $\omega$ , and let  $\Omega \subset M$  be an open subset of  $M$ , with smooth boundary. Moreover, let  $K \subset M$  be a closed set such that  $K \cap \partial\Omega = \emptyset$ . Then*

$$\mathbf{1}_\Omega(x) - u(t, x) = O(t^\infty), \quad \text{uniformly for } x \in K,$$

where  $u(t, x) = e^{t\Delta}\mathbf{1}_\Omega(x)$ .

*Proof.* The statement is a direct consequence of the off-diagonal estimate for the heat kernel in compact sub-Riemannian manifold (see [JSC86, Prop. 3]):

$$p_t(x, y) \leq C_a e^{-c_a/t}, \quad \forall x, y \text{ with } d(x, y) \geq a, \quad t < 1, \quad (4.34)$$

for suitable constants  $C_a, c_a > 0$ , depending only on  $a$ . Now, since  $K \cap \partial\Omega = \emptyset$ , we can write  $K$  as a disjoint union

$$K = K_1 \sqcup K_2 \quad \text{with } K_1 \subset \Omega, \quad K_2 \subset M \setminus \Omega.$$

At this point, for  $i = 1, 2$ , set  $a_i = d_{\text{SR}}(K_i, \partial\Omega) > 0$  by hypothesis, and let  $x \in K_1$ . Then, using the stochastic completeness of  $M$ , we have:

$$|\mathbb{1}_\Omega(x) - u(t, x)| = 1 - u(t, x) = \int_{M \setminus \Omega} p_t(x, y) d\omega(y) \leq C_1 e^{-c_1/t} \omega(M \setminus \Omega), \quad (4.35)$$

which is exponentially decaying, uniformly in  $K_1$ . Analogously, if  $x \in K_2$ , we have

$$|\mathbb{1}_\Omega(x) - u(t, x)| = u(t, x) = \int_M p_t(x, y) \mathbb{1}_\Omega(y) d\omega(y) = \int_\Omega p_t(x, y) d\omega(y) \leq C_2 e^{-c_2/t} \omega(\Omega),$$

uniformly in  $K_2$ .  $\square$

*Remark 4.4.6.* In the non-compact case, Proposition 4.4.5 may fail. Indeed, on the one hand the off-diagonal estimate (4.34) is not always available, on the other hand the measure of  $M \setminus \Omega$  appearing in (4.35) is infinite. Under additional assumption on  $M$ , we are able to recover a localization principle, see Section 4.7.

Let  $M$  be compact. Thanks to Proposition 4.4.5, we can extend the function  $F$  defined in (4.32), to a solution to a non-homogeneous heat equation such as (4.33) on the whole half-line. More precisely, let  $\phi, \eta \in C_c^\infty(M)$  such that

$$\phi + \eta \equiv 1, \quad \text{supp}(\phi) \subset \Omega_{-r_0}^{r_0}, \quad \text{supp}(\eta) \subset \Omega^{-r_0/2} \cup \Omega_{r_0/2}, \quad (4.36)$$

where  $r_0$  is defined in Proposition 4.4.2. We have then, for  $r \in [0, r_0)$ ,

$$\begin{aligned} F(t, r) &= \int_{\Omega_r} (1 - u(t, x)) \phi(x) d\omega(x) + \int_{\Omega_r} (1 - u(t, x)) \eta(x) d\omega(x) \\ &= \int_{\Omega_r} (1 - u(t, x)) \phi(x) d\omega(x) + \int_{\text{supp}(\eta) \cap \Omega_r} (1 - u(t, x)) \eta(x) d\omega(x) \\ &= \int_{\Omega_r} (1 - u(t, x)) \phi(x) d\omega(x) + O(t^\infty), \end{aligned} \quad (4.37)$$

where we used Proposition 4.4.5 to deal with the second term, having set  $K = \text{supp}(\eta) \cap \Omega_r$ . For this reason, we may focus on the first term in (4.37).

**Definition 4.4.7.** For all  $t > 0$  and  $r \geq 0$ , we define the one-parameter families of operators  $I_\Omega$  and  $\Lambda_\Omega$ , associated with  $\Omega$  and acting on the space  $C_c^\infty(\Omega_{-r_0}^{r_0})$ , by

$$\begin{aligned} I_\Omega \phi(t, r) &= \int_{\Omega_r} (1 - u(t, x)) \phi(x) d\omega(x), \\ \Lambda_\Omega \phi(t, r) &= -\partial_r I_\Omega \phi(t, r) = -\int_{\partial\Omega_r} (1 - u(t, y)) \phi(y) d\sigma(y), \end{aligned}$$

for any  $\phi \in C_c^\infty(\Omega_{-r_0}^{r_0})$ , and where  $\sigma$  denotes the induced measure on  $\partial\Omega_r$  and  $u(t, \cdot) = e^{t\Delta} \mathbb{1}_\Omega(\cdot)$ .

**Lemma 4.4.8.** Let  $L = \partial_t - \partial_r^2$  be the one-dimensional heat operator. Then, for any  $\phi \in C_c^\infty(\Omega_{-r_0}^{r_0})$ ,

$$L(I_\Omega \phi(t, r)) = I_\Omega \Delta \phi(t, r) - \Lambda_\Omega N_r \phi(t, r), \quad \forall t > 0, \quad r \geq 0,$$

where  $N_r$  is the operator defined by:

$$N_r \phi = 2g(\nabla \phi, \nu_r) + \phi \text{div}_\omega(\nu_r), \quad \forall \phi \in C_c^\infty(M), \quad (4.38)$$

and  $\nu_r$  is the outward-pointing unit normal to  $\Omega_r$ .

Thanks to the localization principle, we can improve Corollary 4.4.4, obtaining the following result for  $I\phi(t, r)$ .

**Corollary 4.4.9.** For any  $\phi \in C_c^\infty(\Omega_{-r_0}^{r_0})$ , the function  $I_\Omega \phi(t, r)$ , cf. Definition 4.4.7, satisfies the non-homogeneous one-dimensional heat equation on the half-line:

$$(\partial_t - \partial_r^2) I_\Omega \phi(t, r) = I_\Omega \Delta \phi(t, r) - \Lambda_\Omega N_r \phi(t, r), \quad \forall t > 0, \quad r \geq 0.$$

### 4.4.3 Duhamel's principle for $I_\Omega\phi$

We recall for the convenience of the reader a one-dimensional version of the Duhamel's principle, see Lemma 2.5.4 for a proof.

**Lemma 4.4.10** (Duhamel's principle). *Let  $f \in C((0, \infty) \times [0, \infty))$ ,  $v_0, v_1 \in C([0, \infty))$ , such that  $f(t, \cdot)$  and  $v_0$  are compactly supported and assume that*

$$\exists \lim_{t \rightarrow 0} f(t, r), \quad \forall r \geq 0.$$

Consider the non-homogeneous heat equation on the half-line:

$$\begin{aligned} Lv(t, r) &= f(t, r), & \text{for } t > 0, r > 0, \\ v(0, r) &= v_0(r), & \text{for } r > 0, \\ \partial_r v(t, 0) &= v_1(t), & \text{for } t > 0, \end{aligned} \tag{4.39}$$

where  $L = \partial_t - \partial_r^2$ . Then, for  $t > 0$ , the solution to (4.39) is given by

$$\begin{aligned} v(t, r) &= \int_0^\infty e(t, r, s)v_0(s)ds + \int_0^t \int_0^\infty e(t - \tau, r, s)f(\tau, s)dsd\tau \\ &\quad - \int_0^t e(t - \tau, r, 0)v_1(\tau)d\tau, \end{aligned} \tag{4.40}$$

where  $e(t, r, s)$  is the Neumann heat kernel on the half-line, that is

$$e(t, r, s) = \frac{1}{\sqrt{4\pi t}} \left( e^{-(r-s)^2/4t} + e^{-(r+s)^2/4t} \right).$$

Finally, we apply Lemma 4.4.10 to obtain an asymptotic equality for  $I_\Omega\phi$ . The main difference with Theorem 2.5.6 is that the former will not be a true first-order asymptotic expansion.

**Corollary 4.4.11.** *Let  $M$  be a compact sub-Riemannian manifold, equipped with a smooth measure  $\omega$ , and let  $\Omega \subset M$  be an open subset whose boundary is smooth and has no characteristic points. Then, for any function  $\phi \in C_c^\infty(\Omega_{-r_0}^0)$ ,*

$$I_\Omega\phi(t, 0) = \frac{1}{\sqrt{\pi}} \int_0^t \int_{\partial\Omega} (1 - u(\tau, y)) \phi(y) d\sigma(y) (t - \tau)^{-1/2} d\tau + O(t),$$

as  $t \rightarrow 0$ , where  $u(t, \cdot) = e^{t\Delta}\mathbf{1}_\Omega(\cdot)$ .

*Proof.* By Corollary 4.4.9, the function  $I_\Omega\phi(t, r)$  satisfies the following Neumann problem on the half-line:

$$\begin{aligned} LI_\Omega\phi(t, r) &= f(t, r), & \text{for } t > 0, r > 0, \\ I_\Omega\phi(0, r) &= 0, & \text{for } r > 0, \\ \partial_r I_\Omega\phi(t, 0) &= -\Lambda_\Omega\phi(t, 0), & \text{for } t > 0, \end{aligned}$$

where the source is given by Lemma 4.4.8, i.e.  $f(t, r) = I_\Omega\Delta\phi(t, r) - \Lambda_\Omega N_r\phi(t, r)$ . Thus, applying Duhamel's formula (4.40), we have:

$$I_\Omega\phi(t, 0) = \int_0^t \int_0^{+\infty} e(t - \tau, 0, s)f(\tau, s)dsd\tau + \frac{1}{\sqrt{\pi}} \int_0^t \frac{1}{\sqrt{t - \tau}} \Lambda_\Omega\phi(t, 0)d\tau.$$

Since the source is uniformly bounded by the weak maximum principle (4.15), the first integral is a remainder of order  $t$ , as  $t \rightarrow 0$ , concluding the proof.  $\square$

*Remark 4.4.12.* We mention that a relevant role in the sequel will be played by the operators  $I_\Omega$ , cf. Definition 4.4.7, associated with either  $\Omega$  or its complement  $\Omega^c$ .

#### 4.4.4 First-order asymptotics

In this section we prove the first-order asymptotic expansion of  $H_\Omega(t)$ , cf. Theorem 4.1.1 at order 1. We will use Corollary 4.4.11, for the *inside contribution*:

$$I\phi(t, r) = \int_{\Omega_r} (1 - u(t, x)) \phi(x) d\omega(x), \quad \forall t > 0, \quad r \geq 0, \quad (4.41)$$

for any  $\phi \in C_c^\infty(\Omega_{-r_0}^{r_0})$ , and where  $\sigma$  denotes the induced measure on  $\partial\Omega_r$  and  $u(t, \cdot) = e^{t\Delta} \mathbb{1}_\Omega(\cdot)$  is the solution to (4.14). The quantity (4.41) is just Definition 4.4.7, applied with base set  $\Omega \subset M$ .

**Theorem 4.4.13.** *Let  $M$  be a compact sub-Riemannian manifold, equipped with a smooth measure  $\omega$ , and let  $\Omega \subset M$  be an open subset whose boundary is smooth and has no characteristic points. Then,*

$$H_\Omega(t) = \omega(\Omega) - \frac{1}{\sqrt{\pi}} \sigma(\partial\Omega) t^{1/2} + O(t), \quad \text{as } t \rightarrow 0.$$

*Proof.* Let  $\phi \in C_c^\infty(\Omega_{-r_0}^{r_0})$  be as in (4.36), namely

$$0 \leq \phi \leq 1, \quad \text{and} \quad \phi \equiv 1 \quad \text{in} \quad \Omega_{-r_0/2}^{r_0/2}.$$

Then, by the localization principle, cf. (4.37), we have that

$$\omega(\Omega) - H_\Omega(t) = I\phi(t, 0) + O(t^\infty), \quad \text{as } t \rightarrow 0. \quad (4.42)$$

Applying Corollary 4.4.11, we have:

$$I\phi(t, 0) = \frac{1}{\sqrt{\pi}} \int_0^t \int_{\partial\Omega} (1 - u(\tau, y)) \phi(y) d\sigma(y) (t - \tau)^{-1/2} d\tau + O(t), \quad \text{as } t \rightarrow 0. \quad (4.43)$$

Thus, to infer the first-order term of the asymptotic expansion we have to compute the following limit:

$$\lim_{t \rightarrow 0} \frac{I\phi(t, 0)}{t^{1/2}} = \lim_{t \rightarrow 0} \frac{1}{t^{1/2} \sqrt{\pi}} \int_0^t \int_{\partial\Omega} (1 - u(\tau, y)) \phi(y) d\sigma(y) (t - \tau)^{-1/2} d\tau. \quad (4.44)$$

Firstly, by the change of variable in the integral  $\tau \mapsto t\tau$ , we rewrite the argument of the limit (4.44) as

$$\frac{1}{\sqrt{\pi}} \int_0^1 \int_{\partial\Omega} (1 - u(t\tau, y)) \phi(y) d\sigma(y) (1 - \tau)^{-1/2} d\tau.$$

Secondly, we apply the dominated convergence theorem. Indeed, on the one hand, by Theorem 4.3.1 we have point-wise convergence

$$(1 - u(t\tau, y)) \phi(y) \xrightarrow{t \rightarrow 0} \frac{1}{2} \phi(y), \quad \forall y \in \partial\Omega, \quad \tau \in (0, 1),$$

and on the other hand, by the maximum principle

$$\left| \int_{\partial\Omega} (1 - u(t\tau, y)) \phi(y) d\sigma(y) (1 - \tau)^{-1/2} \right| \leq \int_{\partial\Omega} |\phi| d\sigma (1 - \tau)^{-1/2} \in L^1(0, 1),$$

for any  $t > 0$ . Therefore, we finally obtain that:

$$I\phi(t, 0) = \sqrt{\frac{t}{\pi}} \int_{\partial\Omega} \phi(y) d\sigma(y) + o(t^{1/2}), \quad \text{as } t \rightarrow 0.$$

Recalling that  $\phi|_{\partial\Omega} \equiv 1$ , we conclude the proof.  $\square$

*Remark 4.4.14.* The above technique used to evaluate the first-order coefficient causes a loss of precision in the remainder, with respect to the expression (4.43), where the remainder is  $O(t)$ . This loss comes from the application of Theorem 4.3.1, which does not contain any remainder estimate.

## 4.5 Higher-order asymptotic expansion of $H_\Omega(t)$

We iterate Duhamel's formula (4.40) for the inside contribution to study the higher-order asymptotics of  $H_\Omega(t)$ . We obtain the following expression for  $I\phi$ , at order 3:

$$\begin{aligned} I\phi(t, 0) &= \frac{1}{\sqrt{\pi}} \int_0^t \int_{\partial\Omega} (1 - u(\tau, \cdot)) \phi d\sigma (t - \tau)^{-1/2} d\tau \\ &\quad + \frac{1}{2\pi} \int_0^t \int_0^\tau \int_{\partial\Omega} (1 - u(\hat{\tau}, \cdot)) N\phi d\sigma ((\tau - \hat{\tau})(t - \tau))^{-1/2} d\hat{\tau} d\tau + O(t^{3/2}), \end{aligned} \quad (4.45)$$

where  $u(t, \cdot) = e^{t\Delta} \mathbb{1}_\Omega(\cdot)$  denotes the solution to (4.14) and  $N$  is the operator acting on smooth functions defined by

$$N\phi = 2g(\nabla\phi, \nabla\delta) + \phi\Delta\delta, \quad \forall \phi \in C^\infty(M), \quad (4.46)$$

with  $\delta: M \rightarrow \mathbb{R}$  the signed distance function from  $\partial\Omega$ . The computations for deriving (4.45) are technical. We refer to Appendix E for further details, and in particular to Lemma B.0.6.

Motivated by (4.45), we introduce the following functional.

**Definition 4.5.1.** Let  $M$  be a sub-Riemannian manifold, equipped with a smooth measure  $\omega$ , let  $\Omega \subset M$  be a relatively compact subset with smooth boundary and let  $v \in C^\infty((0, +\infty) \times M)$ . Define the functional  $\mathcal{G}_v$ , for any  $\phi \in C_c^\infty(\Omega_{-r_0}^{r_0})$  as:

$$\mathcal{G}_v[\phi](t) = \frac{1}{2\sqrt{\pi}} \int_0^t \int_{\partial\Omega} v(\tau, \cdot) \phi d\sigma (t - \tau)^{-1/2} d\tau, \quad \forall t \geq 0, \quad (4.47)$$

where  $\sigma$  is the sub-Riemannian induced measure on  $\partial\Omega$ .

Notice that the functional  $\mathcal{G}_v$  is linear with respect to the subscript function  $v$ , by linearity of the integral. Moreover, when the function  $v$  is chosen to be the solution to (4.14), we easily obtain the following corollary of Theorem 4.3.1, which is just a rewording of (4.44).

**Corollary 4.5.2.** *Let  $M$  be a compact sub-Riemannian manifold, equipped with a smooth measure  $\omega$ , and let  $\Omega \subset M$  be an open subset whose boundary is smooth and has no characteristic points. Let  $\phi \in C_c^\infty(\Omega_{-r_0}^{r_0})$ , then,*

$$\mathcal{G}_u[\phi](t) = \frac{1}{2\sqrt{\pi}} \int_{\partial\Omega} \phi(y) d\sigma(y) t^{1/2} + o(t^{1/2}), \quad \text{as } t \rightarrow 0.$$

Then, we can rewrite (4.45) in a compact notation:

$$I\phi(t, 0) = 2\mathcal{G}_{1-u}[\phi](t) + \frac{1}{\sqrt{\pi}} \int_0^t \mathcal{G}_{1-u}[N\phi](t) d\sigma (t - \tau)^{-1/2} d\tau + O(t^{3/2}). \quad (4.48)$$

However, on the one hand, the application of Corollary 4.5.2 to (4.45) does not give any new information on the asymptotics of  $H_\Omega(t)$ , as the first term produces an error of  $o(t^{1/2})$ . On the other hand, it is clear the an asymptotic series of  $\mathcal{G}_u$  is enough to deduce the small-time expansion of  $H_\Omega(t)$ .



### 4.5.1 The outside contribution and an asymptotic series for $\mathcal{G}_u[\phi]$

In this section, we deduce an asymptotic series, at order 3, of  $\mathcal{G}_u[\phi](t)$  as  $t \rightarrow 0$ . This is done exploiting the fact that the diffusion of heat is not confined in  $\Omega$ , and as a result we can define an *outside contribution*, namely the quantity obtained from Definition 4.4.7, applied with base set  $\Omega^c \subset M$ :

$$I^c\phi(t, r) = \int_{(\Omega^c)_r} (1 - u^c(t, x)) \phi(x) d\omega(x), \quad \forall t > 0, \quad r \geq 0, \quad (4.49)$$

for any  $\phi \in C_c^\infty(\Omega_{-r_0}^{r_0})$ , and where  $\sigma$  denotes the induced measure on  $\partial(\Omega^c)_r$  and  $u^c(t, x) = e^{t\Delta} \mathbf{1}_{\Omega^c}(x)$ . We remark that, since  $\Omega$  and its complement share the boundary, then  $(\Omega^c)_{-r_0}^{r_0} = \Omega_{-r_0}^{r_0}$ .

It is convenient to introduce (4.49), because it turns out that the quantity  $I\phi - I^c\phi$ , where  $I\phi$  is the inside contribution (4.41), has an explicit asymptotic series in integer powers of  $t$ .

**Proposition 4.5.3.** *Let  $M$  be a compact sub-Riemannian manifold, equipped with a smooth measure  $\omega$ , and let  $\Omega \subset M$  be an open subset with smooth boundary. Let  $\phi \in C_c^\infty(\Omega_{-r_0}^{r_0})$ , then, for any  $k \in \mathbb{N}$ ,*

$$I\phi(t, 0) - I^c\phi(t, 0) = \sum_{i=1}^k a_i(\phi) t^i + O(t^{k+1}), \quad \text{as } t \rightarrow 0, \quad (4.50)$$

where

$$a_i(\phi) = \int_{\partial\Omega} g(\nabla(\Delta^{i-1}\phi), \nabla\delta) d\sigma, \quad \text{for } i \geq 1.$$

*Proof.* Recall that in the definition of the outside contribution (4.49), the integrand function involves  $u^c(t, x) = e^{t\Delta} \mathbf{1}_{\Omega^c}(x)$ . Since  $M$  is compact, and hence stochastically complete, we have:

$$1 - u^c(t, x) = e^{t\Delta} \mathbf{1}(x) - e^{t\Delta} \mathbf{1}_{\Omega^c}(x) = u(t, x), \quad \forall t > 0, \quad x \in M,$$

having used the point-wise equality  $1 - \mathbf{1}_{\Omega^c} = \mathbf{1}_\Omega$  in  $M \setminus \partial\Omega$ . Therefore, we can write the difference  $I\phi(t, 0) - I^c\phi(t, 0)$  as follows:

$$\begin{aligned} I\phi(t, 0) - I^c\phi(t, 0) &= \int_{\Omega} (1 - u(t, \cdot)) \phi d\omega - \int_{\Omega^c} (1 - u^c(t, \cdot)) \phi d\omega \\ &= \int_{\Omega} (1 - u(t, \cdot)) \phi d\omega - \int_{\Omega^c} u(t, \cdot) \phi d\omega \\ &= \int_{\Omega} \phi(x) d\omega(x) - \int_M u(t, x) \phi(x) d\omega(x). \end{aligned} \quad (4.51)$$

Since  $u(t, x)$  is the solution to (4.14), the function (4.51) is smooth as  $t \in [0, \infty)$ . Indeed, the smoothness in the open interval is guaranteed by hypoellipticity of the sub-Laplacian. At  $t = 0$ , the divergence theorem, together with the fact that  $\phi$  has compact support in  $M$ , implies that

$$\begin{aligned} \partial_t^i \left( \int_M u(t, x) \phi(x) d\omega(x) \right) &= \int_M \partial_t^i (u(t, x) \phi(x)) d\omega(x) = \int_M \Delta^i u(t, x) \phi(x) d\omega(x) \\ &= \int_M u(t, x) \Delta^i \phi(x) d\omega(x) \xrightarrow{t \rightarrow 0} \int_{\Omega} \Delta^i \phi(x) d\omega(x). \end{aligned}$$

The previous limit shows that (4.51) is smooth at  $t = 0$ , and also that its asymptotic expansion at order  $k$ , as  $t \rightarrow 0$ , coincides with its  $k$ -th Taylor polynomial at  $t = 0$ . Finally, we recover (4.50), applying once again the divergence theorem:

$$\int_{\Omega} \Delta^i \phi d\omega = \int_{\partial\Omega} g(\nabla(\Delta^{i-1}\phi), \nu) d\sigma = - \int_{\partial\Omega} g(\nabla(\Delta^{i-1}\phi), \nabla\delta) d\sigma,$$

where we recall that  $\nu = -\nabla\delta$  is the outward-pointing normal vector to  $\Omega$  at its boundary.  $\square$

Applying the (iterated) Duhamel's principle (4.40) to the difference  $I\phi - I^c\phi$ , we are able to obtain relevant information on the functional  $\mathcal{G}_u$ .

**Theorem 4.5.4.** *Let  $M$  be a compact sub-Riemannian manifold, equipped with a smooth measure  $\omega$ , and let  $\Omega \subset M$  be an open subset whose boundary is smooth and has no characteristic points. Then, for any  $\phi \in C_c^\infty(\Omega_{-r_0}^0)$ ,*

$$\mathcal{G}_u[\phi](t) = \frac{1}{2\sqrt{\pi}} \int_{\partial\Omega} \phi d\sigma t^{1/2} + \frac{1}{8} \int_{\partial\Omega} \phi \Delta \delta d\sigma + o(t^{3/2}), \quad (4.52)$$

as  $t \rightarrow 0$ .

*Proof.* Let us study the difference of the inside and outside contributions  $I\phi(t, 0) - I^c\phi(t, 0)$ : on the one hand, we have an iterated Duhamel's principle, cf. Lemma E.0.6, which we report here:

$$\begin{aligned} (I\phi - I^c\phi)(t, 0) &= 2\mathcal{G}_{1-2u}[\phi](t) + \frac{1}{2} \int_{\partial\Omega} N\phi d\sigma t \\ &\quad + \frac{1}{2\pi} \int_0^t \int_0^\tau \mathcal{G}_{1-2u}[N^2\phi](\hat{\tau}) ((\tau - \hat{\tau})(t - \tau))^{-1/2} d\hat{\tau} d\tau \\ &\quad + \frac{1}{4\sqrt{\pi}} \int_0^t \int_{\partial\Omega} (1 - 2u(\tau, \cdot)) (4\Delta - N^2)\phi d\sigma (t - \tau)^{1/2} d\tau + O(t^2), \end{aligned} \quad (4.53)$$

where we recall that  $N$  is the operator acting on smooth functions defined by

$$N\phi = 2g(\nabla\phi, \nabla\delta) + \phi\Delta\delta, \quad \forall \phi \in C^\infty(M).$$

Using Corollary 4.5.2 and the linearity of  $\mathcal{G}_v$  with respect to  $v$ , we know that

$$\mathcal{G}_{1-2u}[\phi](t) = o(t^{1/2}), \quad \text{as } t \rightarrow 0, \quad \forall \phi \in C_c^\infty(\Omega_{-r_0}^0). \quad (4.54)$$

In addition, an application of Theorem 4.3.1 and dominated convergence theorem implies that

$$\int_0^t \int_{\partial\Omega} (1 - 2u(\tau, \cdot)) (4\Delta - N^2)\phi d\sigma (t - \tau)^{1/2} d\tau = o(t^{3/2}) \quad \text{as } t \rightarrow 0. \quad (4.55)$$

Thus, using (4.54) and (4.55), we can improve (4.53), obtaining

$$I\phi(t, 0) - I^c\phi(t, 0) = 2\mathcal{G}_{1-2u}[\phi](t) + \frac{1}{2} \int_{\partial\Omega} N\phi d\sigma t + o(t^{3/2}). \quad (4.56)$$

On the other hand, the quantity  $I\phi(t, 0) - I^c\phi(t, 0)$  has a complete asymptotic series by Proposition 4.5.3, which at order 3 becomes:

$$I\phi(t, 0) - I^c\phi(t, 0) = \int_{\partial\Omega} g(\nabla\phi, \nabla\delta) d\sigma t + o(t^{3/2}), \quad \text{as } t \rightarrow 0. \quad (4.57)$$

Comparing (4.56) and (4.57), we deduce that, as  $t \rightarrow 0$ ,

$$\begin{aligned} 2\mathcal{G}_{1-2u}[\phi](t) &= -\frac{1}{2} \int_{\partial\Omega} N\phi d\sigma t + o(t^{3/2}) + \int_{\partial\Omega} g(\nabla\phi, \nabla\delta) d\sigma t + o(t^{3/2}) \\ &= -\frac{1}{2} \int_{\partial\Omega} \phi \Delta \delta d\sigma t + o(t^{3/2}). \end{aligned}$$

Finally, using the linearity of the functional  $\mathcal{G}_v[\phi]$  with respect to  $v$ , we conclude the proof.  $\square$

*Remark 4.5.5.* The asymptotics (4.52) for the functional  $\mathcal{G}_u[\phi](t)$  is the best result that we are able to achieve. In the expression (4.53), the problematic term is given by (4.55), i.e.

$$\int_0^t \int_{\partial\Omega} (1 - 2u(\tau, \cdot)) (4\Delta - N^2) \phi d\sigma (t - \tau)^{1/2} d\tau,$$

which can not be expressed in terms of  $\mathcal{G}_u$ , hence the only relevant information is given by Theorem 4.3.1. In conclusion, we can not repeat the strategy of the proof of Theorem 4.5.4, replacing the series of  $\mathcal{G}_u$  at order 3 in (4.53) to deduce the higher-order terms.

## 4.5.2 Fourth-order asymptotics

In this section we prove Theorem 4.1.1. We recall here the statement.

**Theorem 4.5.6.** *Let  $M$  be a compact sub-Riemannian manifold, equipped with a smooth measure  $\omega$ , and let  $\Omega \subset M$  be an open subset whose boundary is smooth and has no characteristic points. Then, as  $t \rightarrow 0$ ,*

$$H_\Omega(t) = \omega(\Omega) - \frac{1}{\sqrt{\pi}} \sigma(\partial\Omega) t^{1/2} - \frac{1}{12\sqrt{\pi}} \int_{\partial\Omega} \left( 2g(\nabla\delta, \nabla(\Delta\delta)) - (\Delta\delta)^2 \right) d\sigma t^{3/2} + o(t^2).$$

Before giving the proof of the theorem, let us comment on its strategy. Recall that, on the one hand, for a cutoff function  $\phi \in C_c^\infty(\Omega_{-r_0}^{r_0})$  which is identically 1 close to  $\partial\Omega$ , cf. (4.36), the localization principle (4.42) holds, namely

$$\omega(\Omega) - H_\Omega(t) = I\phi(t, 0) + O(t^\infty), \quad \text{as } t \rightarrow 0. \quad (4.58)$$

Moreover, by the iterated Duhamel's principle for  $I\phi(t, 0)$ , cf. Lemma B.0.6, we can deduce expression (4.48), namely

$$I\phi(t, 0) = 2\mathcal{G}_{1-u}[\phi](t) + \frac{1}{\sqrt{\pi}} \int_0^t \mathcal{G}_{1-u}[N\phi](t) d\sigma (t - \tau)^{-1/2} d\tau + O(t^{3/2}). \quad (4.59)$$

On the other hand, we have an asymptotic series of the functional  $\mathcal{G}_u$  at order 3, cf. Theorem 4.3.1. Therefore, if we naively insert this series in (4.59), we can obtain, at most, a third-order asymptotic expansion of the relative heat content  $H_\Omega(t)$ , whereas we are interested in the fourth-order expansion.

Using the outside contribution, we are able to overcome this difficulty. In particular, applying Proposition 4.5.3, for a function  $\phi \in C_c^\infty(\Omega_{-r_0}^{r_0})$  which is identically 1 close to  $\partial\Omega$ , we have the following asymptotic relation:

$$I\phi(t, 0) = I^c\phi(t, 0) + O(t^\infty), \quad \text{as } t \rightarrow 0. \quad (4.60)$$

Notice that (4.60) is a direct consequence of Proposition 4.5.3 since all the coefficients of the expansion vanish. Therefore, thanks to (4.60), we can rephrase (4.58) as follows:

$$\omega(\Omega) - H_\Omega(t) = \frac{1}{2} (I\phi(t, 0) + I^c\phi(t, 0)) + O(t^\infty), \quad \text{as } t \rightarrow 0. \quad (4.61)$$

The advantage of (4.61) is that we can now apply the iterated Dirichlet principle for the sum  $I\phi + I^c\phi$ , cf. Lemma E.0.7. Already at order 3, we obtain

$$(I\phi + I^c\phi)(t, 0) = \frac{2}{\sqrt{\pi}} \int_{\partial\Omega} \phi d\sigma t^{1/2} + \frac{1}{\sqrt{\pi}} \int_0^t \mathcal{G}_{1-2u}[N\phi](\tau) (t - \tau)^{-1/2} d\tau + O(t^{3/2}), \quad (4.62)$$

where  $N$  is the operator defined in (4.46). As we can see, in (4.62), the functional  $\mathcal{G}_u$  occurs for the first time in the second iteration of the Duhamel's principle, as opposed to the expansion for  $I\phi$ , where it appeared already in the first application, cf. (4.59). Hence we gain an order with respect to the asymptotic series of  $\mathcal{G}_u$ . More generally, if we were able to develop the  $k$ -th order asymptotics for  $\mathcal{G}_u$ , this would imply the  $(k + 1)$ -th order expansion for  $H_\Omega(t)$ .

*Proof of Theorem 4.5.6.* Following the discussion above, it is enough to expand the sum  $I\phi + I^c\phi$ , with  $\phi \in C_c^\infty(\Omega_{-r_0}^{r_0})$ . For this quantity, Lemma E.0.7 holds, namely we have the following iterated version of Duhamel's principle:

$$\begin{aligned}
 (I\phi + I^c\phi)(t, 0) &= \frac{2}{\sqrt{\pi}} \int_{\partial\Omega} \phi d\sigma t^{1/2} + \frac{1}{\sqrt{\pi}} \int_0^t \mathcal{G}_{1-2u}[N\phi](\tau)(t-\tau)^{-1/2} d\tau \\
 &\quad + \frac{1}{6\sqrt{\pi}} \int_{\partial\Omega} (4\Delta + N^2)\phi d\sigma t^{3/2} \\
 &\quad + \frac{1}{4\pi^{3/2}} \int_0^t \int_0^\tau \int_0^{\hat{\tau}} \mathcal{G}_{1-2u}[N^3\phi](s) ((\hat{\tau}-s)(\tau-\hat{\tau})(t-\tau))^{-1/2} ds d\hat{\tau} d\tau \\
 &\quad + \frac{1}{4\sqrt{\pi}} \int_0^t \mathcal{G}_{1-2u}[(6N\Delta - N^3 - 2\Delta N)\phi](\tau)(t-\tau)^{1/2} d\tau + O(t^{5/2}),
 \end{aligned} \tag{4.63}$$

where  $N$  is defined in (4.46). Moreover, recall that by Theorem 4.5.4, the functional  $\mathcal{G}_{1-2u}[\phi]$  has the following expansion for any  $\phi \in C_c^\infty(\Omega_{-r_0}^{r_0})$ :

$$\mathcal{G}_{1-2u}[\phi](t) = -\frac{1}{4} \int_{\partial\Omega} \phi \Delta \delta d\sigma t + o(t^{3/2}), \quad \text{as } t \rightarrow 0.$$

Thus, replacing the above expansion in (4.63), we obtain for any  $\phi \in C_c^\infty(\Omega_{-r_0}^{r_0})$ ,

$$\begin{aligned}
 I\phi(t, 0) + I^c\phi(t, 0) &= \frac{2}{\sqrt{\pi}} \int_{\partial\Omega} \phi d\sigma t^{1/2} - \frac{1}{3\sqrt{\pi}} \left( \int_{\partial\Omega} N\phi \Delta \delta d\sigma \right) t^{3/2} \\
 &\quad + \frac{1}{6\sqrt{\pi}} \int_{\partial\Omega} (4\Delta + N^2)\phi d\sigma t^{3/2} + o(t^2).
 \end{aligned} \tag{4.64}$$

In particular, if we choose  $\phi \in C_c^\infty(\Omega_{-r_0}^{r_0})$  such that  $\phi \equiv 1$  close to  $\partial\Omega$ , then on the one hand, from (4.64), we obtain, as  $t \rightarrow 0$ ,

$$\begin{aligned}
 I\phi(t, 0) + I^c\phi(t, 0) &= \frac{2}{\sqrt{\pi}} \sigma(\partial\Omega) t^{1/2} \\
 &\quad + \frac{1}{6\sqrt{\pi}} \int_{\partial\Omega} \left( 2g(\nabla\delta, \nabla(\Delta\delta)) - (\Delta\delta)^2 \right) d\sigma t^{3/2} + o(t^2).
 \end{aligned}$$

On the other hand, the asymptotic relation (4.61) holds. This concludes the proof.  $\square$

**Third-order vs fourth-order asymptotics.** We stress that we could have obtained the third-order asymptotic expansion of  $H_\Omega(t)$  *without* introducing the sum of the inside and outside contributions  $I\phi + I^c\phi$ , and only using the Duhamel's principle for  $I\phi$ , cf. Lemma B.0.6, and the asymptotic series for  $\mathcal{G}_u$ , cf. Theorem 4.5.4. However, for the improvement to the fourth-order asymptotics, the argument of the sum of contributions seems necessary.

### 4.5.3 The weighted relative heat content

Theorem 4.5.6 holds for any function  $\phi \in C_c^\infty(\Omega_{-r_0}^{r_0})$  regardless of its value at the boundary of  $\partial\Omega$ . Indeed, we can prove a slightly more general result which we state here for completeness.

**Proposition 4.5.7.** *Let  $M$  be a compact sub-Riemannian manifold, equipped with a smooth measure  $\omega$ , and let  $\Omega \subset M$  be an open subset whose boundary is smooth and has no characteristic points. Let  $\chi \in C_c^\infty(M)$  and define the weighted relative heat content*

$$H_\Omega^\chi(t) = \int_\Omega u(t, x) \chi(x) d\omega(x), \quad \forall t > 0.$$

Then, as  $t \rightarrow 0$ ,

$$\begin{aligned} H_{\Omega}^{\chi}(t) &= \int_{\Omega} \chi d\omega - \frac{1}{\sqrt{\pi}} \int_{\partial\Omega} \chi d\sigma t^{1/2} - \frac{1}{2} \int_{\partial\Omega} g(\nabla\chi, \nabla\delta) d\sigma t \\ &\quad - \left( \frac{1}{12\sqrt{\pi}} \int_{\partial\Omega} (4\Delta + N^2)\chi d\sigma - \frac{1}{6\sqrt{\pi}} \int_{\partial\Omega} (N\chi)\Delta\delta d\sigma \right) t^{3/2} \\ &\quad - \frac{1}{2} \int_{\partial\Omega} g(\nabla(\Delta\chi), \nabla\delta) d\sigma t^2 + o(t^2). \end{aligned}$$

*Proof.* Let us consider a cutoff function  $\phi$  as in (4.36). Then, applying the usual localization argument, cf. (4.37), we have:

$$\int_{\Omega} \chi(x) d\omega(x) - H_{\Omega}^{\chi}(t) = I[\phi\chi](t, 0) + O(t^{\infty}), \quad \text{as } t \rightarrow 0,$$

where now the function  $\phi\chi \in C_c^{\infty}(\Omega_{-r_0}^{r_0})$  and  $\phi\chi = \chi$  close to  $\partial\Omega$ .

As we did in the proof of Theorem 4.5.6, we relate  $H_{\Omega}^{\chi}(t)$  with the sum of contributions. Applying Proposition 4.5.3, we have the following asymptotic relation at order 4:

$$I[\phi\chi](t, 0) - I^c[\phi\chi](t, 0) = \int_{\partial\Omega} g(\nabla\chi, \nabla\delta) d\sigma t + \int_{\partial\Omega} g(\nabla(\Delta\chi), \nabla\delta) d\sigma t^2 + o(t^2),$$

as  $t \rightarrow 0$ , having used the fact that  $\phi\chi \equiv \chi$  close to  $\partial\Omega$ . Notice that this relation coincides with (4.60) when  $\chi \equiv 1$  close to  $\partial\Omega$ . Thus, we obtain

$$\begin{aligned} \int_{\Omega} \chi(x) d\omega(x) - H_{\Omega}^{\chi}(t) &= \frac{1}{2} (I[\phi\chi](t, 0) + I^c[\phi\chi](t, 0)) \\ &\quad + \int_{\partial\Omega} g(\nabla\chi, \nabla\delta) d\sigma t + \int_{\partial\Omega} g(\nabla(\Delta\chi), \nabla\delta) d\sigma t^2 + o(t^2), \quad \text{as } t \rightarrow 0. \end{aligned}$$

Finally, applying (4.64) for  $I[\phi\chi](t, 0) + I^c[\phi\chi](t, 0)$ , we conclude.  $\square$

*Remark 4.5.8.* We compare the coefficients of the expansions of  $H_{\Omega}(t)$  and  $Q_{\Omega}(t)$ , defined in (4.6), respectively. On the one hand, by the results obtained in Chapter 2, and in particular Proposition 2.1.2, the  $k$ -th coefficient of the expansion of  $Q_{\Omega}(t)$  is of the form

$$- \int_{\partial\Omega} D_k(\chi) d\sigma, \quad \forall \chi \in C_c^{\infty}(M),$$

where  $D_k$  is a differential operator acting on  $C_c^{\infty}(M)$  and belonging to  $\text{span}_{\mathbb{R}}\{\Delta, N\}$  as algebra of operators. On the other hand, Proposition 4.5.7 shows that this is no longer true for the third coefficient of the expansion of  $H_{\Omega}(t)$ , as we need to add the operator multiplication by  $\Delta\delta$ .

## 4.6 An alternative approach using the heat kernel asymptotics

As we can see by a first application of Duhamel's principle, cf. (4.10), and its iterations, the small-time asymptotics of  $u(t, \cdot)|_{\partial\Omega}$ , together with uniform estimates on the remainder with respect to  $x \in \partial\Omega$ , would be enough to determine the asymptotic expansion of the relative heat content, at any order.

In Theorem 4.3.1, we studied the zero-order asymptotics of  $u(t, \cdot)|_{\partial\Omega}$ . The technique used for its proof does not work at higher-order, since the exponential remainder term in (4.26) would be unbounded as  $t \rightarrow 0$ . In this section, we comment how such a higher-order asymptotics of  $u(t, \cdot)|_{\partial\Omega}$  can be obtained exploiting the asymptotic formula for the heat kernel proved in [CdVHT20, Thm. A].

Let  $M$  be a compact sub-Riemannian manifold and  $\Omega \subset M$  an open subset with smooth boundary. For  $x \in \partial\Omega$ , let us consider  $\psi = (z_1, \dots, z_n): U \rightarrow V$  a chart of privileged coordinates centered at  $x$ , with  $U$  a relatively compact set. Since the heat kernel is exponentially decaying outside the diagonal, cf. (4.17),

$$\begin{aligned} u(t, x) &= \int_{\Omega} p_t(x, y) d\omega(y) = \int_{\Omega \cap U} p_t(x, y) d\omega(y) + O(t^\infty) \\ &= \int_{V_1} p_t(0, z) d\omega(z) + O(t^\infty), \end{aligned} \quad (4.65)$$

where  $V_1 = \psi(U \cap \Omega)$ , and we denote with the same symbols  $\omega$  and  $p_t(0, z)$  the coordinate expression of the measure and heat kernel, respectively. For example, if  $x \in \partial\Omega$  is non-characteristic, we may choose  $\psi$  as in (4.16), then  $V_1 = V \cap \{z_1 > 0\}$ . Recall the asymptotic expansion of the heat kernel of Theorem 1.4.5, evaluated in  $(0, z)$ : for any  $m \in \mathbb{N}$  and compact set  $K \subset (0, \infty) \times V$ ,

$$|\varepsilon|^{\mathcal{Q}} p_{\varepsilon^2\tau}(0, \delta_\varepsilon(z)) = \hat{p}_\tau(0, z) + \sum_{i=0}^m \varepsilon^i f_i(\tau, 0, z) + o(|\varepsilon|^m), \quad \text{as } \varepsilon \rightarrow 0, \quad (4.66)$$

uniformly as  $(\tau, z) \in K$ , where  $\mathcal{Q}$ ,  $\hat{p}$  and  $f_i$ 's are defined in Section 1.4. We will omit the dependance on the center of the privileged coordinates  $x$ , it being fixed for the moment. At this point, we would like to integrate (4.66) to get information of  $u(t, x)$  as  $t \rightarrow 0$ . Proceeding formally, let us choose the parameters  $\varepsilon, \tau$  in (4.66) such that:

$$\varepsilon^2\tau = t, \quad \varepsilon = t^{\frac{\alpha}{2\alpha+1}}, \quad \tau = t^{\frac{1}{2\alpha+1}}, \quad (4.67)$$

for some  $\alpha > 0$  to be fixed. For convenience of notation, set

$$V_s = \delta_s(V_1) \quad \forall s \in [-1, 1],$$

then, split the integral over  $V_1$  in (4.65) in two, so that the first one is computed on  $V_\varepsilon$  and the second one is computed on its complement in  $V_1$ , i.e.  $V_1 \setminus V_\varepsilon$ . Notice that, by usual off-diagonal estimates, see [JSC86, Prop. 3] and our choice of the parameter  $\varepsilon$  as in (4.67), the following is a remainder term, independently of the value of  $\alpha$ :

$$\int_{V_1 \setminus V_\varepsilon} p_t(0, z) d\omega(z) = O\left(e^{-\beta \frac{\varepsilon^2}{t}}\right) = O(t^\infty), \quad \text{as } t \rightarrow 0.$$

Thus, writing the measure in coordinates  $d\omega(z) = \omega(z) dz$  with  $\omega(\cdot) \in C^\infty(V_1)$ , we have, as  $t \rightarrow 0$ ,

$$\begin{aligned} u(t, x) &= \int_{V_\varepsilon} p_t(0, z) \omega(z) dz + O(t^\infty) = \int_{V_1} \varepsilon^{\mathcal{Q}} p_{\varepsilon^2\tau}(0, \delta_\varepsilon(z)) \omega(\delta_\varepsilon(z)) dz + O(t^\infty) \\ &= \int_{V_1} \left( \hat{p}_\tau(0, z) + \sum_{i=0}^{m-1} \varepsilon^i f_i(\tau, 0, z) + \varepsilon^m R_m(\varepsilon, \tau, z) \right) \omega(\delta_\varepsilon(z)) dz + O(t^\infty), \end{aligned} \quad (4.68)$$

where  $R_m$  is a smooth function on  $[-1, 1] \times (0, \infty) \times \mathbb{R}^n$ , such that

$$\sup_{\substack{\varepsilon \in [-1, 1] \\ z \in K}} |R_m(\varepsilon, \tau, z)| \leq C_m(\tau, K), \quad (4.69)$$

for any compact set  $K \subset \mathbb{R}^n$ , according to (4.66). Up to restricting the domain of privileged coordinates  $U$ , we can assume that (4.69) holds on  $\bar{V}$ . By our choices (4.67), we would like the following term

$$t^{\frac{m\alpha}{2\alpha+1}} \int_{V_1} \left| R_m \left( t^{\frac{\alpha}{2\alpha+1}}, t^{\frac{1}{2\alpha+1}}, z \right) \right| \omega(\delta_{t^{\alpha/(2\alpha+1)}}(z)) dz \quad (4.70)$$

to be an error term of order greater than  $\frac{m-1}{2}$ , as  $t \rightarrow 0$ . Thus, assume for the moment that  $\forall K \subset V$  compact and  $\forall m \in \mathbb{N}$ ,  $\exists \ell = \ell(m, K) \in \mathbb{N}$  and  $C_m(K) > 0$  such that

$$\sup_{\substack{\varepsilon \in [-1, 1] \\ z \in K}} |R_m(\varepsilon, \tau, z)| \leq \frac{C_m(K)}{\tau^\ell}, \quad \forall \tau \in (0, 1). \quad (\mathbf{H})$$

Thanks to assumption **(H)**, choosing  $\alpha$  large enough, we see that (4.70) is a  $o(t^{\frac{m-1}{2}})$ . Performing the change of variables  $z \mapsto \delta_{1/\sqrt{\tau}}(z)$  in (4.68), and exploiting the homogeneity properties of  $\hat{p}$  and  $f_i$ , namely (1.31), we finally obtain the following expression for  $u$  as  $t \rightarrow 0$ :

$$u(t, x) = \int_{V_{t^{-1/(2(2\alpha+1))}}} \left( \hat{p}_1(0, z) + \sum_{i=0}^{m-1} t^{i/2} a_i(z) \right) \omega(\delta_{\sqrt{t}}(z)) dz + o(t^{\frac{m-1}{2}}), \quad (4.71)$$

having set  $a_i(z) = f_i(1, 0, z)$ , for all  $i \in \mathbb{N}$ . Therefore, we find an asymptotic expansion of  $u(t, x)$  under assumption **(H)**, which is crucial to overcome the fact that (4.66) is formulated on an asymptotic neighborhood of the diagonal, and not uniformly as  $\tau \rightarrow 0$ . It is likely<sup>5</sup> that **(H)** can be proven in the nilpotent case, and more generally when the ambient manifold is  $M = \mathbb{R}^n$  and the generating family of the sub-Riemannian structure,  $\{X_1, \dots, X_N\}$  satisfies the uniform Hörmander polynomial condition, see [CdVHT20, App. B] for details. Although this strategy could be used to prove the existence of an asymptotic expansion of  $H_\Omega(t)$ , we refrain to go in this directions since two technical difficulties would arise nonetheless:

- *Uniformity of the expansion of  $u(t, x)$  with respect to  $x \in \partial\Omega$ .* In the non-equiregular case, cf. Section 1.4 for details, the expansion (4.66) is not uniform as  $x$  varies in compact subsets of  $M$ , hence the same would be true for the expansion (4.71).
- *Computations of the coefficients.* The coefficients appearing in (4.71) depend on the nilpotent approximation at  $x \in \partial\Omega$  and are not clearly related to the invariants of  $\partial\Omega$ .

Our strategy avoids almost completely the knowledge of the small-time asymptotics of  $u(t, \cdot)_{\partial\Omega}$ , it being based on an asymptotic series of the auxiliary functional  $\mathcal{G}_u$ . Moreover, we stress that our method to prove the asymptotics of  $H_\Omega(t)$  up to order 4, cf. Theorem 4.1.1, holds for any sub-Riemannian manifold, including also the non-equiregular ones.

*Remark 4.6.1.* In order to pass from (4.71) to the asymptotic expansion of  $H_\Omega(t)$ , we would use Duhamel's formula, which holds under the non-characteristic assumption. This means that, even though (4.66) of course is true even in presence of characteristic points, we can't say much about the asymptotics of  $H_\Omega(t)$  in the general case.

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<sup>5</sup>Personal communication by Yves Colin de Verdière, Luc Hillairet and Emmanuel Trélat.



## 4.7 The non-compact case

In the non-compact case, we have the following difficulties:

- The localization principle, cf. Proposition 4.4.5, may fail.
- Set  $u(t, x) = e^{t\Delta}\mathbf{1}_\Omega(x)$  and  $u^c(t, x) = e^{t\Delta}\mathbf{1}_{\Omega^c}(x)$ . If the manifold is not stochastically complete, the relation  $u(t, x) + u^c(t, x) = 1$  does not hold.
- The Gaussian bounds for the heat kernel and its time-derivatives, à la Jerison and Sanchez-Calle [JSC86, Thm. 3], may not hold, thus Lemma E.0.2 may not be true.

**Definition 4.7.1.** Let  $M$  be a sub-Riemannian manifold, equipped with a smooth measure  $\omega$ . We say that  $(M, \omega)$  is (globally) doubling if there exist constants  $C_D > 0$  such that:

$$V(x, 2\rho) \leq C_D V(x, \rho), \quad \forall \rho > 0, x \in M,$$

where  $V(x, \rho) = \omega(B_\rho(x))$ . We say that  $(M, \omega)$  satisfies a (global) weak Poincaré inequality, if there exist constants  $C_P > 0$  such that,

$$\int_{B_\rho(x)} |f - f_{x,\rho}|^2 d\omega \leq C_P \rho^2 \int_{B_{2\rho}(x)} \|\nabla f\|^2 d\omega, \quad \rho > 0, x \in M,$$

for any smooth function  $f \in C^\infty(M)$ . Here  $f_{x,\rho} = \frac{1}{V(x,\rho)} \int_{B_\rho(x)} f d\omega$ . We refer to these properties as local whenever they hold for any  $\rho < \rho_0$ .

*Remark 4.7.2.* If  $M$  is a sub-Riemannian manifold, equipped with a smooth globally doubling measure  $\omega$ , then it is stochastically complete, namely

$$\int_M p_t(x, y) d\omega(y) = 1, \quad \forall t > 0, x \in M.$$

This is a straightforward consequence of the characterization given by [Stu96, Thm. 4] on the volume growth of balls.

**Theorem 4.7.3.** Let  $M$  be a complete sub-Riemannian manifold, equipped with a smooth measure  $\omega$ . Assume that  $(M, \omega)$  is globally doubling and satisfies a global weak Poincaré inequality. Then, there exist constants  $C_k, c_k > 0$ , for any integer  $k \geq 0$ , depending only on  $C_D, C_P$ , such that, for any  $x, y \in M$  and  $t > 0$ ,

$$|\partial_t^k p_t(x, y)| \leq \frac{C_k t^{-k}}{V(x, \sqrt{t})} \exp\left(-\frac{d_{\text{SR}}^2(x, y)}{c_k t}\right), \quad (4.72)$$

where we recall  $V(x, \sqrt{t}) = \omega(B_{\sqrt{t}}(x))$ .

In addition, there exists constants  $C_\ell, c_\ell > 0$ , depending only on  $C_D, C_P$ , such that, for any  $x, y \in M$  and  $t > 0$ ,

$$p_t(x, y) \geq \frac{C_\ell}{V(x, \sqrt{t})} \exp\left(-\frac{d_{\text{SR}}^2(x, y)}{c_\ell t}\right). \quad (4.73)$$

*Proof.* Following the notations of [Stu96], one can easily verify that

$$\mathcal{E}(u, v) = \int_M 2H(du, dv) d\omega, \quad \forall u, v \in C_c^\infty(M),$$

where  $H$  is the sub-Riemannian Hamiltonian (1.8) viewed as a bilinear form on fibers, defines a strongly local Dirichlet form with domain  $\text{dom}(\mathcal{E}) = C_c^\infty(M)$ . Notice that the Friedrichs extension of  $\mathcal{E}$  is exactly the sub-Laplacian, moreover, the intrinsic metric

$$d_I(x, y) = \sup\{|u(x) - u(y)| \text{ s.t. } u \in C_c^\infty(M), |2H(du, du)| \leq 1\}, \quad \forall x, y \in M.$$



coincides with the usual sub-Riemannian distance, as  $|2H(du, du)| = \|\nabla u\|^2$ , cf. [BBS16, Ch. 2, Prop. 12.4]. Thus,  $\mathcal{E}$  is also strongly regular and, by our assumptions on  $(M, \omega)$ , [SC92, Thm. 4.3] holds true, proving (4.72). For the Gaussian lower bound (4.73), it is enough to apply [Stu96, Cor. 4.10], cf. also [SC92, Thm. 4.2]. This concludes the proof.  $\square$

*Remark 4.7.4.* Theorem 4.7.3 ensures that the time-derivatives of the heat kernel satisfy Gaussian bounds, which are sufficient to prove Lemma E.0.2 in the non-compact case. This lemma is crucial to obtain the asymptotics expansion of  $H_\Omega(t)$  at order *strictly greater* than 1.

We prove now the non-compact analogue of Proposition 4.4.5.

**Corollary 4.7.5.** *Under the assumptions of Theorem 4.7.3, let  $\Omega \subset M$  be an open and bounded subset with smooth boundary. Then, for any  $K \subset M$  compact subset of  $M$  such that  $K \cap \partial\Omega = \emptyset$ , we have:*

$$\mathbb{1}_\Omega(x) - u(t, x) = O(t^\infty), \quad \text{as } t \rightarrow 0, \quad \text{uniformly for } x \in K,$$

where  $u(t, x) = e^{t\Delta}\mathbb{1}_\Omega(x)$  is the solution to (4.14).

*Proof.* Let us assume that  $K \subset \Omega$  such that  $K \cap \partial\Omega = \emptyset$ . The other part of the statement can be done similarly.

Since  $M$  is stochastically complete, cf. Remark 4.7.2, for any  $x \in K$ , we can write:

$$\mathbb{1}_\Omega(x) - u(t, x) = 1 - e^{t\Delta}\mathbb{1}_\Omega(x) = e^{t\Delta}\mathbb{1}(x) - e^{t\Delta}\mathbb{1}_\Omega(x) = \int_{M \setminus \Omega} p_t(x, y) d\omega(y).$$

Thanks to Theorem 4.7.3, we can apply (4.72) for  $k = 0$  obtaining

$$\int_{M \setminus \Omega} p_t(x, y) d\omega(y) \leq \int_{M \setminus \Omega} \frac{C_0}{V(x, \sqrt{t})} \exp\left(-\frac{d_{\text{SR}}^2(x, y)}{c_0 t}\right),$$

for suitable constants  $C_0, c_0 > 0$  not depending on  $x, y \in M, t > 0$ . Now, fix  $L > 1$ : since  $K \subset \Omega$  and is well-separated from  $\partial\Omega$ , we deduce there exists  $a = a(K) > 0$  such that  $d_{\text{SR}}(x, y) > a$  for any  $x \in K, y \in M \setminus \Omega$ , and so

$$\begin{aligned} \int_{M \setminus \Omega} p_t(x, y) d\omega(y) &\leq \int_{M \setminus \Omega} \frac{C_0}{V(x, \sqrt{t})} \exp\left(-\frac{d_{\text{SR}}^2(x, y)}{c_0 t}\right) d\omega(y) \\ &\leq \exp\left(-\frac{C(a, L)}{c_0 t}\right) \int_{M \setminus \Omega} \frac{C_0}{V(x, \sqrt{t})} \exp\left(-\frac{d_{\text{SR}}^2(x, y)}{2^L c_0 t}\right) d\omega(y), \end{aligned} \quad (4.74)$$

where  $C(a, L) = \frac{a^2(2^L - 1)}{2^L} > 0$ . Thus, if we prove that the integral in (4.74) is finite, we conclude. Firstly, recall the Gaussian lower bound (4.73), which holds thanks to Theorem 4.7.3:

$$p_t(x, y) \geq \frac{C_\ell}{V(x, \sqrt{t})} \exp\left(-\frac{d_{\text{SR}}^2(x, y)}{c_\ell t}\right). \quad (4.75)$$

for suitable constants constants  $C_\ell, c_\ell > 0$ , not depending on  $x, y \in M, t > 0$ . Secondly, by the doubling property of  $\omega$ , it is well-known that there exists  $C'_D, s > 0$  depending only on  $C_D$  such that

$$V(x, R) \leq C'_D \left(\frac{R}{\rho}\right)^s V(x, \rho), \quad \forall \rho \leq R. \quad (4.76)$$

Therefore, choosing  $L > 1$  so big that  $\tilde{c}^2 = (2^L c_0)/c_\ell > 1$  and applying (4.76) for  $\rho = \sqrt{\tilde{t}}$  and  $R = \tilde{c}\sqrt{\tilde{t}}$ , we have  $R > \rho$  and

$$V(x, \tilde{c}\sqrt{\tilde{t}}) \leq \tilde{C}V(x, \sqrt{\tilde{t}}), \quad \forall t > 0, \quad (4.77)$$

having denoted by  $\tilde{C} = C'_D \tilde{c}^s > 0$ . Finally, using (4.77) and the Gaussian lower bound (4.75), we can estimate the integral in (4.74) as follows:

$$\begin{aligned} & \int_{M \setminus \Omega} \frac{1}{V(x, \sqrt{\tilde{t}})} \exp\left(-\frac{d_{\text{SR}}^2(x, y)}{2^L c_0 t}\right) d\omega(y) \\ & \leq \int_M \frac{\tilde{C}}{V(x, \tilde{c}\sqrt{\tilde{t}})} \exp\left(-\frac{d_{\text{SR}}^2(x, y)}{c_\ell \tilde{c} t}\right) d\omega(y) \leq \frac{\tilde{C}}{C_\ell} \int_M p_{\tilde{t}}(x, y) d\omega(y) \leq \frac{\tilde{C}}{C_\ell}, \end{aligned}$$

where  $\tilde{t} = \tilde{c}t$ . Since the resulting constant does not depend on  $x \in K$ , we conclude the proof.  $\square$

Using Corollary 4.7.5 and adopting the same strategy of the compact case, one can finally prove the following result.

**Theorem 4.7.6.** *Let  $M$  be a complete sub-Riemannian manifold, equipped with a smooth measure  $\omega$ . Assume that  $(M, \omega)$  is globally doubling and satisfies a global weak Poincaré inequality. Let  $\Omega \subset M$  be an open and bounded subset whose boundary is smooth and has no characteristic points. Then, as  $t \rightarrow 0$ ,*

$$H_\Omega(t) = \omega(\Omega) - \frac{1}{\sqrt{\pi}} \sigma(\partial\Omega) t^{1/2} - \frac{1}{12\sqrt{\pi}} \int_{\partial\Omega} \left(2g(\nabla\delta, \nabla(\Delta\delta)) - (\Delta\delta)^2\right) d\sigma t^{3/2} + o(t^2).$$

*Remark 4.7.7.* Theorem 4.7.6 holds true also for the weighted relative heat content, cf. Section 4.5.3. In both cases, we do not know whether its assumptions are sharp in the non-compact case.

#### 4.7.1 Notable examples

We list here some notable examples of sub-Riemannian manifolds satisfying the assumptions of Theorem 4.7.3. For these examples Theorem 4.7.6 is valid.

- $M$  is a Lie group with polynomial volume growth, the distribution is generated by a family of left-invariant vector fields satisfying the Hörmander condition and  $\omega$  is the Haar measure. This family includes also Carnot groups. See for example [Var96, SC92, GS12].
- $M = \mathbb{R}^n$ , equipped with a sub-Riemannian structure induced by a family of vector fields  $\{Y_1, \dots, Y_N\}$  with bounded coefficients together with their derivatives, and satisfying the Hörmander condition. Under these assumptions, the Lebesgue measure is doubling, cf. [NSW85, Thm. 1], and the Poincaré inequality is verified in [Jer86]. We remark that these works provide the local properties of Definition 4.7.1, with constants depending only on the  $C^k$ -norms of the vector fields  $Y_i$ , for  $i = 1, \dots, N$ . Thus, if the  $C_k$ -norms are globally bounded, we obtain the corresponding global properties.
- $M$  is a complete Riemannian manifold with metric  $g$ , equipped with the Riemannian measure, and with non-negative Ricci curvature.

We mention that a Riemannian manifold  $M$  with Ricci curvature bounded below by a negative constant satisfies only locally Definition 4.7.1, i.e. for some  $\rho_0 < \infty$ , depending on the Ricci bound. Nevertheless, we can prove Corollary 4.7.5 in this case, as Li and Yau provides an upper Gaussian bound, see [LY86, Cor. 3.1], and a lower bound as (4.73) holds, cf. [BQ99, Cor. 2]. Thus, the first-order asymptotic expansion of  $H_\Omega(t)$ , cf. Theorem 4.4.13, is valid in this setting.



# 5

## Relative heat content for submanifolds

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*We prove the existence of a tubular neighborhood for submanifolds of any codimension, assuming they do not have characteristic points. Next, we propose a definition of relative heat content for submanifolds of codimension  $k \geq 1$  and we build a possible approximation of this quantity, via tubular neighborhoods. However, we show that this approximation fails to recover the asymptotic expansion of the relative heat content of the submanifold, by comparing the two approaches in an example.*

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### 5.1 Introduction

In this chapter, we study the relative heat content for submanifolds. Up to our knowledge, this quantity has never been systematically studied, not even in Riemannian geometry, so we propose the following definition. Let  $M$  be a sub-Riemannian manifold, equipped with a smooth measure  $\omega$ , and let  $S \subset M$  be a smooth, compact submanifold of codimension  $k \geq 0$ . Fix a smooth measure  $\mu$  on  $S$  as initial datum for the heat equation

in the sense of distributions, and study the associated Cauchy problem:

$$\begin{aligned} (\partial_t - \Delta)u(t, x) &= 0, & \forall (t, x) \in (0, \infty) \times M, \\ u(t, \cdot) &\xrightarrow{t \rightarrow 0} \mu, & \text{in } \mathcal{D}'(M), \end{aligned}$$

where  $\Delta = \operatorname{div}_\omega \circ \nabla$  is the usual sub-Laplacian associated with  $\omega$ . A solution to this problem, in the sense of distribution, is given by

$$u(t, x) = \int_S p_t(x, y) d\mu(y), \quad \forall (t, x) \in (0, \infty) \times M,$$

where  $p_t(x, y)$  is the usual heat kernel associated with  $\Delta$  and  $\omega$ . We define the *relative heat content for a submanifold  $S$*  as:

$$H_S(t) = \int_S \int_S p_t(x, y) d\mu(x) d\mu(y), \quad \forall t > 0. \quad (5.1)$$

Notice that, on the one hand if  $S = \{x_0\}$  and  $\mu = \delta_{x_0}$ , we obtain the trace heat kernel  $p_t(x_0, x_0)$ . On the other hand, if  $S$  is a 0-codimensional submanifold, i.e.  $S$  is an open, relatively compact and smooth set, and we choose as measure  $\mu = \mathbb{1}_\Omega \omega$ , then (5.1) coincides with the usual relative heat content as defined in Chapter 4. Therefore, a small-time asymptotic expansion of (5.1) would include many cases of interest, ranging from the trace heat kernel asymptotics to the results of Chapter 4.

Our attempt to compute the asymptotics of (5.1) consists in building a suitable approximation of it, using tubular neighborhoods of  $S$ . Firstly, we introduce the general definition of non-characteristic submanifold, which applies for any codimension  $k \geq 0$ , cf. Definition 5.2.1, and we prove the following crucial result.

**Theorem 5.1.1.** *Let  $M$  be a sub-Riemannian manifold and  $S \subset M$  be a compact smooth non-characteristic submanifold of codimension  $k \geq 1$ . Then, there exists  $r_0 > 0$  such that, denoting by  $S_{r_0} = \{p \in M \mid 0 < \delta_S < r_0\}$ , the following conditions hold:*

- i)  $\delta_S: S_{r_0} \rightarrow [0, \infty)$  is smooth and such that  $\|\nabla \delta_S\|_g = 1$ ;
- ii) there exists a diffeomorphism  $G: (0, r_0) \times \{\delta_S = r_0\} \rightarrow S_{r_0}$  such that:

$$\delta_S(G(r, p)) = r \quad \text{and} \quad G_* \partial_r = \nabla \delta_S.$$

Theorem 5.1.1 generalizes the analogous result in [FPR20, Prop. 3.1], and [PRS18, Lem. 23], to submanifolds of any codimension. Its proof follows the blueprint of those previous works.

Secondly, for any  $\varepsilon \leq r_0$ , sufficiently small, we consider the (rescaled) relative heat content associated with  $S_\varepsilon$ , namely the tubular neighborhood of  $S$  of radius  $\varepsilon$ :

$$H_S^\varepsilon(t) = \frac{1}{\omega(S_\varepsilon)^2} \int_{S_\varepsilon} \int_{S_\varepsilon} p_t(x, y) d\omega(x) d\omega(y), \quad \forall t > 0. \quad (5.2)$$

We show that  $H_S^\varepsilon$  converges point-wise to  $H_S$ , as  $\varepsilon \rightarrow 0$ , cf. Proposition 5.4.1. However, this convergence is too weak to recover any information regarding the small-time asymptotic expansion of the limit. Actually, we show that (5.2) fails to recover the asymptotic expansion of the relative heat content of the submanifold, by studying an example in which is possible to perform explicit computations.

**Structure of the chapter.** In Section 5.2, we give the definition of non-characteristic submanifold and we prove Theorem 5.1.1. In Section 5.3, we build a specific smooth measure on  $S$ , which is induced by the outer measure  $\omega$ . In Section 5.4, we introduce the definition of relative heat content for a submanifold  $S$ , we build its approximation and we prove its point-wise convergence. Finally, in Section 5.5, we show how the approximation fails to recover the asymptotic expansion of the relative heat content of  $S$ .

## 5.2 Tubular neighborhood for submanifolds

Let  $M$  be a sub-Riemannian manifold and let  $S \subset M$  be a smooth submanifold of codimension  $k \geq 0$ . Consider the sub-Riemannian distance function from  $S$ , defined by:

$$\delta_S(p) = \inf\{d_{\text{SR}}(q, p), q \in S\}, \quad \forall p \in M.$$

We refer to Chapter 1 for the relevant definitions in sub-Riemannian geometry.

**Definition 5.2.1** (Non-characteristic submanifolds). Let  $M$  be a sub-Riemannian manifold and let  $S \subset M$  be a smooth submanifold of codimension  $k \geq 0$ . We say that a point  $q \in S$  is *non-characteristic* if

$$\mathcal{D}_q + T_q S = T_q M. \quad (5.3)$$

We say that  $S$  is a *non-characteristic submanifold* if (5.3) holds for any point  $q \in S$ .

Notice that Definition (5.2.1) includes the one we gave for 0-codimensional submanifolds, cf. Definition 2.2.2. Under the assumption of non-characteristic submanifold, we can extend the regularity properties of the distance function to this situation.

**Theorem 5.2.2.** *Let  $M$  be a sub-Riemannian manifold and  $S \subset M$  be a compact smooth non-characteristic submanifold of codimension  $k \geq 1$ . Then, there exists  $r_0 > 0$  such that, denoting by  $S_{r_0} = \{p \in M \mid 0 < \delta_S < r_0\}$ , the following conditions hold:*

- i)  $\delta_S: S_{r_0} \rightarrow [0, \infty)$  is smooth and such that  $\|\nabla \delta_S\|_g = 1$ ;
- ii) there exists a diffeomorphism  $G: (0, r_0) \times \{\delta_S = r_0\} \rightarrow S_{r_0}$  such that:

$$\delta_S(G(r, p)) = r \quad \text{and} \quad G_* \partial_r = \nabla \delta_S.$$

Before giving the proof of the theorem, we need a preliminary lemma, which can be regarded as a partial generalization of Proposition 1.2.3.

**Lemma 5.2.3.** *Let  $M$  be a sub-Riemannian manifold and  $S \subset M$  be a smooth submanifold of codimension  $k \geq 1$ . Let  $\gamma: [0, 1] \rightarrow M$  be a minimizing geodesic such that*

$$\gamma(0) \in S, \quad \gamma(1) = p \in M \setminus S, \quad \delta_S(p) = \ell(\gamma).$$

*If  $\gamma$  is an abnormal geodesic, then  $\gamma(0)$  is a characteristic point of  $S$ .*

*Proof.* Let  $\lambda: [0, 1] \rightarrow T^*M$  be an abnormal lift of  $\gamma$ : this means in particular that  $\pi(\lambda(t)) = \gamma(t)$  and

$$\langle \lambda(0), \mathcal{D}_{\gamma(0)} \rangle = 0, \quad \text{with } \lambda(0) \neq 0, \quad (5.4)$$

where  $\langle \cdot, \cdot \rangle$  denotes the dual coupling. Moreover, since  $\gamma$  is a minimizing geodesic, any lift must necessarily satisfy the following transversality condition, cf. [AS04, Thm. 12.4],

$$\langle \lambda(0), T_{\gamma(0)} S \rangle = 0. \quad (5.5)$$

Thus, since  $\lambda(0) \neq 0$ , conditions (5.4), (5.5) imply that (5.3) fails at  $q = \gamma(0)$ .  $\square$

*Proof of Theorem 5.2.2.* Let us consider the annihilator bundle of  $S$ ,  $\mathcal{A}S$ , namely the vector bundle of rank  $k$ , whose fibers are given by

$$\mathcal{A}_q S = \{\lambda \in T_q^* M \mid \langle \lambda, T_q S \rangle = 0\}, \quad \forall q \in S.$$

At a point  $q \in S$ , let us fix a basis of the fiber  $\mathcal{A}_q S$ , say  $\{\lambda_1, \dots, \lambda_k\}$  and define, for any  $j = 1, \dots, k$ , the element  $v_j \in \mathcal{D}_q$  dual to  $\lambda_j$  via the Hamiltonian  $H$ , i.e.

$$v_j = \pi_* \vec{H}(\lambda_j) = \sum_{i=1}^N \langle \lambda_j, X_i(q) \rangle X_i(q) \quad j = 1, \dots, k. \quad (5.6)$$

(Step 1) If  $q$  is non-characteristic, then the set  $\{v_1, \dots, v_k\}$  is linearly independent. Indeed assume there exists constants  $\alpha_i$  for  $i = 1, \dots, k$ , such that  $\sum_{i=1}^k \alpha_i v_i = 0$ . Then,

$$0 = \sum_{j=1}^k \alpha_j v_j = \sum_{j=1}^k \alpha_j \sum_{i=1}^N \langle \lambda_j, X_i(q) \rangle X_i(q) = \sum_{i=1}^N \left\langle \sum_{j=1}^k \alpha_j \lambda_j, X_i(q) \right\rangle X_i(q) = \pi_* \vec{H}(\lambda), \quad (5.7)$$

having set  $\lambda = \sum_{j=1}^k \alpha_j \lambda_j \in \mathcal{A}_q S$ . Notice that, by the Lagrange multiplier rule, denoting by  $v_\lambda = \pi_* \vec{H}(\lambda)$ , for any  $\lambda \in T_q^* M$ , we have

$$\|v_\lambda\|_g^2 = \inf \left\{ \sum_{i=1}^N u_i^2 \mid v_\lambda = \sum_{i=1}^N u_i X_i(q) \right\} = \sum_{i=1}^N \langle \lambda, X_i(q) \rangle^2 = 2H(\lambda). \quad (5.8)$$

Therefore, (5.7) implies that  $\|\pi_* \vec{H}(\lambda)\|_g^2 = 2H(\lambda) = 0$ , or equivalently:

$$\langle \lambda, \mathcal{D}_q \rangle = 0. \quad (5.9)$$

Since  $\lambda \in \mathcal{A}_q S$  and  $q$  is non-characteristic, by (5.9), we deduce that  $\lambda = 0$ . Thus:

$$0 = \lambda = \sum_{j=1}^k \alpha_j \lambda_j = 0 \quad \Rightarrow \quad \alpha_j = 0, \quad \text{for any } j = 1, \dots, k,$$

since  $\{\lambda_1, \dots, \lambda_k\}$  was a basis of the fiber of  $\mathcal{A}S$ . This concludes the proof of the first step. Define now the sub-Riemannian exponential map from  $S$ , i.e. the map

$$E: D \cap \mathcal{A}S \rightarrow M; \quad E(\lambda) = \pi \circ e^{\vec{H}}(\lambda),$$

where  $D \subset T^*M$  is the open set where the flow of  $\vec{H}$  is defined up to time 1. Consider also the zero section of the annihilator bundle, namely

$$i: S \rightarrow \mathcal{A}S; \quad i(q) = (q, 0) \in \mathcal{A}_q S.$$

(Step 2)  $E$  is a local diffeomorphism at points of  $i(S)$ .

To prove the claim, we consider a point  $(q, 0) \in i(S)$  and verify that  $d_{(q,0)}E$  is invertible. Identifying  $T_{(q,0)}(D \cap \mathcal{A}S) \cong T_q S \oplus \mathcal{A}_q S$ , we have, on the one hand  $E \circ i = Id_S$ , therefore for a vector  $v = (v, 0) \in T_q S \oplus \mathcal{A}_q S$ ,

$$d_{(q,0)}E(v) = \left. \frac{d}{dt} \right|_{t=0} E(\lambda(t)) = \left. \frac{d}{dt} \right|_{t=0} E \circ i(\gamma(t)) = \left. \frac{d}{dt} \right|_{t=0} \gamma(t) = v,$$

since  $\lambda(t) = (\gamma(t), 0)$ , with  $\gamma: (-\varepsilon, \varepsilon) \rightarrow S$ , such that  $\gamma(0) = q$  and  $\dot{\gamma}(0) = v$ . On the other hand, take an element  $\lambda = (0, \lambda) \in T_q S \oplus \mathcal{A}_q S$ , then by definition, we obtain

$$d_{(q,0)}E(\lambda) = \left. \frac{d}{dt} \right|_{t=0} E(q, t\lambda) = \left. \frac{d}{dt} \right|_{t=0} E(q, t\lambda) = \left. \frac{d}{dt} \right|_{t=0} \pi \circ e^{t\vec{H}}(\lambda) = \pi_* \vec{H}(\lambda) = v_\lambda.$$

Thus, choosing any basis for  $T_q S$  and the basis  $\{\lambda_1, \dots, \lambda_k\}$  for  $\mathcal{A}_q S$ , as before, we may write the  $n \times n$  matrix representing the differential of  $E$  as

$$d_{(q,0)}E = \left( \begin{array}{c|c} Id_{n-k} & \\ \hline & v_1, \dots, v_k \end{array} \right),$$



where the vectors  $v_j$  are defined as in (5.6). Since, by the previous step, the set  $\{v_1, \dots, v_k\}$  consists of linearly independent vectors of  $\mathcal{D}_q$ , we conclude that  $dE$  is invertible at  $i(S)$ . (Step 3) There exists  $U \subset D \cap \mathcal{AS}$ , such that  $E|_U$  is a diffeomorphism on its image. Moreover,  $U$  can be chosen of the form:

$$U = \{\lambda \in \mathcal{AS} \mid \sqrt{2H(\lambda)} < r_0\}, \quad \text{for some } r_0 > 0. \quad (5.10)$$

The proof of this step follows verbatim what has been done in [FPR20, Prop. 3.1], cf. also [PRS18, Lem. 23], once we have verified that  $\sqrt{2H(\cdot)}$  is a fiber-wise norm on the annihilator bundle. Since  $H$  is quadratic on fibers, it immediately follows that  $\sqrt{2H(\cdot)}$  is positive, 1-homogeneous and sub-additive. We are left to prove that, for  $\lambda \in \mathcal{A}_q S$ ,

$$\sqrt{2H(\lambda)} = 0 \quad \Leftrightarrow \quad \lambda = 0.$$

As already remarked in (5.9), an element  $\lambda \in \mathcal{A}_q S$ , such that  $\sqrt{2H(\overline{(q, \lambda)})} = 0$ , annihilates both the distribution and  $T_q S$ , thus, being  $q$  non-characteristic,  $\lambda = 0$ .

(Step 4)  $E(U) = \{p \in M \mid \delta_S(p) < r_0\} = S_{r_0} \cup S$  and, for elements  $(q, \lambda) \in U$  we have  $\delta_S(E(q, \lambda)) = \sqrt{2H(\lambda)}$ . In particular,  $\delta_S \in C^\infty(S_{r_0})$ .

Firstly, we recall that, for an element  $\lambda \in U$ , the length of the curve

$$[0, 1] \ni t \mapsto \pi \circ e^{t\tilde{H}}(\lambda) \in M$$

is equal to  $\sqrt{2H(\lambda)} < r_0$ , as one can check using (5.8). Thus,  $E(U) \subset S_{r_0} \cup S$ . Secondly, we prove the opposite inclusion: up to restricting  $r_0$ , we may assume that  $S_{r_0} \subset K$ , for a compact set  $K \subset M$ . Therefore, for an element  $p \in S_{r_0}$ , there exists a minimizing geodesic  $\gamma: [0, 1] \rightarrow M$  such that

$$\gamma(0) = q \in S, \quad \gamma(1) = p \quad \text{and} \quad \ell(\gamma) = \delta_S(p).$$

Applying Lemma 5.2.3, we deduce that  $\gamma$  is not an abnormal geodesic, meaning that there exists a unique normal lift for  $\gamma$ , with initial covector given by  $\lambda \in T_q^* M$ , which implies that

$$\gamma(t) = \pi \circ e^{t\tilde{H}}(\lambda),$$

and in particular,  $E(q, \lambda) = p$ . Moreover,  $\lambda \in U$  as, by optimality, it satisfies the transversality condition (5.5), and also

$$\ell(\gamma) = \sqrt{2H(\lambda)} < r_0, \quad (5.11)$$

being  $p \in S_{r_0}$ . Finally, we conclude that  $p \in E(U)$  and  $\delta_S(E(q, \lambda)) = \sqrt{2H(\lambda)}$ , by (5.11). Since  $\sqrt{2H(\cdot)}$  is smooth, as long as  $H(\lambda) \neq 0$ , we also have that  $\delta_S$  is smooth on the set  $E(U \setminus i(S)) = S_{r_0}$ .

(Step 5) There exists a diffeomorphism  $G: (0, r_0) \times \{\delta_S = r_0\} \rightarrow S_{r_0}$  satisfying item (ii) of the statement. Moreover,  $\|\nabla \delta_S\|_g = 1$  in  $S_{r_0}$ .

Once again, this part of the proof follows verbatim [FPR20, Prop. 3.1].  $\square$

*Remark 5.2.4.* Consider the set  $U = \mathcal{AS} \cap \{\sqrt{2H(\cdot)} < r_0\}$  defined in (5.10). What we proved in the previous Theorem is that  $E$  defines a diffeomorphism between  $U$  and  $S_{r_0} \cup S$ . In particular, choosing a local trivialization of the annihilator bundle, this means that

$$S_{r_0} \cup S \cong \mathcal{AS} \cap \left\{ \sqrt{2H(\cdot)} < r_0 \right\} \underset{\text{locally}}{\cong} S \times B_{r_0}^H(0), \quad (5.12)$$

where  $B_{r_0}^H(0)$  denotes the ball of radius  $r_0$ , centered at the origin of the Euclidean space  $(\mathbb{R}^k, \sqrt{2H(\cdot)})$ . Of course, in general, the annihilator bundle will not be globally trivializable, however, this is the case when  $S$  is the boundary of an open set and we are able to extend (5.12) to the whole submanifold.

Whenever  $S$  is a boundary of an open set, we can refine Theorem 5.2.2 building a double-sided tubular neighborhood of  $S$ , in which we are able to distinguish the inside and the outside of the open set. This is done using the signed distance function (4.28). We recall here its definition.

**Definition 5.2.5** (Signed distance). Let  $M$  be a sub-Riemannian manifold and  $\Omega \subset M$  be an open subset. Define  $\delta: M \rightarrow \mathbb{R}$  to be the *signed distance function* from  $\partial\Omega$ , i.e.

$$\delta(p) = \begin{cases} \delta_{\partial\Omega}(p) & p \in \Omega, \\ -\delta_{\partial\Omega}(p) & p \in M \setminus \Omega, \end{cases}$$

where  $\delta_{\partial\Omega}: M \rightarrow [0, +\infty)$  denotes the usual distance function from the boundary of  $\Omega$ .

**Theorem 5.2.6** (Double-sided tubular neighborhood). *Let  $M$  be a sub-Riemannian manifold and  $\Omega \subset M$  be an open, relatively compact subset, whose boundary is smooth and has no characteristic points. Denote by  $\Omega_{-r_0}^{r_0} = \{p \in M \mid -r_0 < \delta < r_0\}$ , consistently with (4.29). Then, there exists  $r_0 > 0$  such that, the following conditions hold:*

- i)  $\delta: \Omega_{-r_0}^{r_0} \rightarrow \mathbb{R}$  is smooth and such that  $\|\nabla\delta\|_g = 1$ ;*
- ii) there exists a diffeomorphism  $G: (-r_0, r_0) \times \partial\Omega \rightarrow \Omega_{-r_0}^{r_0}$  such that:*

$$\delta(G(t, p)) = t \quad \text{and} \quad G_*\partial_t = \nabla\delta.$$

*Remark 5.2.7.* The main differences with respect to Theorem 5.2.2 are that  $\delta$  is smooth up to the boundary of  $\Omega$ , and the diffeomorphism is built starting from  $\partial\Omega$ .

*Proof of Theorem 5.2.6.* By Theorem 5.2.2, applied with  $S = \partial\Omega$ , the sub-Riemannian exponential map from  $\partial\Omega$  is a diffeomorphism for small covectors, namely there exists  $r_0 > 0$ , such that:

$$E: \mathcal{A}(\partial\Omega) \cap \left\{ \sqrt{2H(\lambda)} < r_0 \right\} \xrightarrow{\cong} \Omega_{-r_0}^{r_0}$$

and  $|\delta(E(q, \lambda))| = \sqrt{2H(\lambda)}$ . Now, since  $\Omega$  is an open set with smooth boundary,  $\mathcal{A}(\partial\Omega)$  is trivializable, i.e. there exists a never-vanishing and inward-pointing smooth section

$$\lambda^+: \partial\Omega \rightarrow \mathcal{A}(\partial\Omega); \quad q \mapsto \lambda_q^+.$$

Furthermore, by non-characteristic assumption,  $\sqrt{2H(\cdot)}$  is a fiber-wise norm on the annihilator bundle, hence we may assume without loss of generality that

$$\sqrt{2H(\lambda_q^+)} = 1, \quad \forall q \in S.$$

Thus, we find a unique smooth function  $\xi(\lambda) \in C^\infty(\mathcal{A}(\partial\Omega))$  such that

$$\lambda = \xi(\lambda)\lambda_q^+, \quad \lambda \in \mathcal{A}_q(\partial\Omega).$$

Hence, the annihilator bundle is trivializable via the map  $\xi$ , i.e.

$$F: \mathcal{A}(\partial\Omega) \xrightarrow{\cong} \partial\Omega \times \mathbb{R}; \quad F(\lambda) = (\pi(\lambda), \xi(\lambda)).$$

Notice that, by definition,  $|\xi(\lambda)| = \sqrt{2H(\lambda)}$ . Moreover,  $\xi(\lambda) > 0$ , whenever  $E(q, \lambda) \in \Omega$ , by definition of  $\lambda^+$ ,  $\xi(0) = 0$  and negative otherwise. Therefore, having defined the signed distance such that it is positive inside of  $\Omega$ , we obtain that

$$\delta(E(q, \lambda)) = \xi(\lambda), \quad \forall \lambda \in \left\{ \sqrt{2H(\lambda)} < r_0 \right\},$$

proving the smoothness of  $\delta$  on the set  $\Omega_{-r_0}^{r_0}$ . Finally, define  $G$  as the composition of  $E \circ F^{-1}$  restricted to the set  $(-r_0, r_0) \times \partial\Omega$ . Since  $E$  and  $F$  are diffeomorphisms, also  $G$  is and moreover,

$$G(t, q) = E(q, t\lambda_q^+) \quad \forall (t, q) \in (-r_0, r_0) \times \partial\Omega,$$

therefore  $\delta(G(t, q)) = \delta(E(q, t\lambda_q^+)) = \xi(t\lambda_q^+) = t$ . This concludes the proof.  $\square$

### 5.3 Induced measure on $S$

Let  $\omega$  be a smooth measure on  $M$ . We define a measure on  $S$  induced by  $\omega$ , assigning a tensor density. This construction specializes to the sub-Riemannian perimeter measure, when  $S$  is the boundary of an open set. Recall that, by Theorem 5.2.2, there exists  $r_0 > 0$  such that (5.12) holds locally and define  $\text{vol}_H$  as the Riemannian measure associated with  $(\mathbb{R}^k, \|\cdot\|_\perp)$ , where  $\|\cdot\|_\perp$  is a shorthand notation for  $\sqrt{2H(\cdot)}|_{\mathcal{A}_q S}$ . In particular,  $\text{vol}_H$  is well-defined since  $\|\cdot\|_\perp$  is induced by the fiber-wise bilinear form

$$(\lambda_1, \lambda_2)_\perp = \sum_{i=1}^N \langle \lambda_1, X_i \rangle \langle \lambda_2, X_i \rangle, \quad \forall \lambda_1, \lambda_2 \in \mathcal{A}_q S, \quad q \in S, \quad (5.13)$$

where  $\langle \cdot, \cdot \rangle$  denotes the dual coupling.

**Lemma 5.3.1.** *Let  $M$  be a sub-Riemannian manifold and  $S \subset M$  be a compact smooth non-characteristic submanifold of codimension  $k \geq 1$ . Then, there exists a unique smooth probability measure  $\mu_S$  on  $S$ , such that,*

$$\int_M h(p) \omega^\varepsilon(p) \xrightarrow{\varepsilon \rightarrow 0} \int_S h(q) \mu_S(q), \quad (5.14)$$

for any  $h \in C_c(M)$ , where,

$$\omega^\varepsilon = \frac{1}{\omega(S_\varepsilon)} \mathbf{1}_{S_\varepsilon}, \quad \forall \varepsilon > 0,$$

*Proof.* Proceeding with hindsight, we are going to define explicitly the measure  $\mu_S$  and then prove the convergence. We may define  $\mu_S$  locally, hence, fix an open coordinate chart  $V \subset S$  for  $S$  and a local trivialization of  $\mathcal{A}S$  over  $V$ , so that

$$\mathcal{A}S|_V \cong V \times \mathbb{R}^k.$$

By Theorem 5.2.2, we have that, denoting by  $V_{r_0} = E(\mathcal{A}S|_V \cap \{\sqrt{2H(\cdot)} < r_0\})$ ,

$$V_{r_0} \cong \mathcal{A}S|_V \cap \left\{ \sqrt{2H(\cdot)} < r_0 \right\} \cong V \times B_{r_0}^H(0).$$

Consider on  $V_{r_0}$ , coordinates  $(x, z)$  where  $(x_1, \dots, x_{n-k})$  are coordinates on  $V$  and  $S \cap V_{r_0} = \{(x, z) \mid z = 0\}$ . Thus, since  $\omega$  is smooth, we have

$$d\omega(x, z) = \omega(x, z) dx dz, \quad \text{with } \omega(\cdot) \in C^\infty(V_{r_0}),$$

where  $dx$  and  $dz$  are the Lebesgue measures in coordinates. Moreover, since (5.13) is a metric along the fibers, we can define canonically a volume associated with  $H$ , which in coordinates is given by

$$d\text{vol}_H(z) = \sqrt{\det H_q(z)} dz, \quad \forall q \in S,$$

with never-vanishing density. Therefore, we may rewrite  $\omega$  in terms of  $\text{vol}_H$ , obtaining

$$d\omega(x, z) = \omega(x, z) dx dz = \frac{\omega(x, z)}{\sqrt{\det H_q(z)}} dx d\text{vol}_H(z) \quad (5.15)$$

Finally, on the fiber, we can choose an orthonormal (w.r.t.  $\sqrt{2H(\cdot)}$ ) basis of smooth local sections  $\{\lambda_1, \dots, \lambda_k\}$ , so that  $\text{vol}_H(\lambda_1, \dots, \lambda_k) = 1$ , and define  $\tilde{\mu}_S$  in coordinates  $(x, z)$ , to be the contraction of (5.15) along these covectors, restricted to  $S$ , namely

$$\tilde{\mu}_S = \frac{\omega(x, 0)}{\sqrt{\det H_q(0)}} dx d\text{vol}_H(\lambda_1, \dots, \lambda_k) = \frac{\omega(x, 0)}{\sqrt{\det H_q(0)}} dx.$$

One can check that this procedure defines a smooth measure on  $S$ , independently on the choice of the coordinates. We can now verify the convergence, using a partition of the unity argument. Fix a covering of  $S$  with a finite number of open charts  $\{V_i\}_{i=1}^N$  and consider the associated covering  $\{V_{r_0}^i\}$  of  $S \cup S_{r_0}$ , defined by

$$V_{r_0}^i = E \left( \mathcal{A}S|_{V_i} \cap \{ \sqrt{2H(\cdot)} < r_0 \} \right), \quad \forall i = 1, \dots, N.$$

Then, consider  $\{\rho_i\}_{i=1}^N$  to be a partition of unity subordinate to the covering  $\{V_{r_0}^i\}$  of  $S \cup S_{r_0}$ . Exploiting the coordinate expression of  $\tilde{\mu}_S$ , we have, for any  $\varepsilon \leq r_0$ :

$$\begin{aligned} \omega(S_\varepsilon) &= \int_{S_\varepsilon} d\omega = \int_{S_\varepsilon} \sum_{i=1}^N \rho_i(q) d\omega(q) = \sum_{i=1}^N \int_{V_i} \int_{B_\varepsilon^H(x)} \rho_i(x, z) \frac{\omega(x, z)}{\sqrt{\det H_q(z)}} d\text{vol}_H(z) dx \\ &= \varepsilon^k \sum_{i=1}^N \int_{V_i} \int_0^1 \int_{\mathbb{S}^{k-1}} \rho_i(x, \varepsilon r, \theta) \frac{\omega(x, \varepsilon r, \theta)}{\sqrt{\det H_q(\varepsilon r, \theta)}} r^{k-1} d\theta dr dx, \end{aligned}$$

having expressed the volume  $\text{vol}_H$  in polar coordinates  $r^{k-1} d\theta dr$ . Therefore, up to a factor  $\varepsilon^k$ , we see that

$$\frac{\omega(S_\varepsilon)}{\varepsilon^k} \xrightarrow{\varepsilon \rightarrow 0} \varpi_k \sum_{i=1}^N \int_{V_i} \rho_i(x, 0) d\tilde{\mu}_S(x) = \varpi_k \int_S d\tilde{\mu}_S$$

where  $\varpi_k$  is the volume of the standard unit ball in  $\mathbb{R}^k$ . Finally, reasoning as above, since for any  $h \in C_c(M)$ , we are able to extract a factor  $\varepsilon^k$  from the integral of  $h$  over  $S_\varepsilon$ , we obtain the convergence in the weak-star topology (5.14), having normalized  $\tilde{\mu}_S$  to obtain a probability measure  $\mu_S$ .  $\square$

*Remark 5.3.2.* Since  $\omega^\varepsilon$ , for any  $\varepsilon \leq r_0$ , has compact support which is contained in  $S_{r_0}$ , we can extend the convergence (5.14) to any continuous function on  $M$ .

## 5.4 Heat content for submanifolds

Let  $M$  be a sub-Riemannian manifold, equipped with a smooth measure  $\omega$ , and let  $S \subset M$  be a smooth, compact submanifold of codimension  $k \geq 1$ . We may consider  $\mu$  a smooth probability measure on  $S$  as initial datum for the heat equation, in the sense of distributions, and study the associated Cauchy problem:

$$\begin{aligned} (\partial_t - \Delta)u(t, x) &= 0, & \forall (t, x) \in (0, \infty) \times M, \\ u(t, \cdot) &\xrightarrow{t \rightarrow 0} \mu, & \text{in } \mathcal{D}'(M). \end{aligned}$$

A solution to this problem, in the sense of distribution, is given by

$$u(t, x) = \int_S p_t(x, y) d\mu(x), \quad \forall (t, x) \in (0, \infty) \times M,$$

which, by hypoellipticity, is a smooth function for positive times. Recall that, by definition of the relative heat content associated with an open set  $\Omega \subset M$ , we have

$$H_\Omega(t) = \int_\Omega \int_\Omega p_t(x, y) d\omega(x) d\omega(y), \quad \forall t > 0. \quad (5.16)$$

Thus, a suitable generalization of (5.16) for a submanifold  $S$  seems to be:

$$H_S(t) = \int_S \int_S p_t(x, y) d\mu(x) d\mu(y), \quad \forall t > 0. \quad (5.17)$$

Moreover, when  $S$  is a non-characteristic submanifold, Lemma 5.3.1 provides with a natural probability measure on  $S$ , induced by  $\omega$ , i.e.  $\mu_S$ . Henceforth, we shall assume  $S$  non-characteristic and fix  $\mu = \mu_S$ . In this setting, we can hope to obtain an asymptotic expansion of (5.17).

**Proposition 5.4.1.** *Let  $M$  be a sub-Riemannian manifold, equipped with a smooth measure  $\omega$ , let  $S \subset M$  be a smooth, compact and non-characteristic submanifold of codimension  $k \geq 1$  and fix the probability measure  $\mu_S$  on  $S$ . Define, for any  $\varepsilon \leq r_0$ ,*

$$H_S^\varepsilon(t) = \int_M \int_M p_t(x, y) d\omega^\varepsilon(x) d\omega^\varepsilon(y), \quad \forall t > 0. \quad (5.18)$$

Then, for any  $t > 0$ ,

$$H_S^\varepsilon(t) \xrightarrow{\varepsilon \rightarrow 0} H_S(t). \quad (5.19)$$

*Proof.* Firstly, notice that, applying Lemma 5.3.1, we have that

$$u^\varepsilon(t, x) = \int_M p_t(x, y) d\omega^\varepsilon(y) = \langle \omega^\varepsilon, p_t(x, \cdot) \rangle \xrightarrow{\varepsilon \rightarrow 0} \langle \mu_S, p_t(x, \cdot) \rangle, \quad (5.20)$$

for any  $t > 0$  and  $y \in M$ . Secondly, since the heat kernel  $p_t$  is smooth on  $M \times M$ , there exists a constant  $C(t) > 0$  depending on  $t$ , such that:

$$\|p_t(\cdot, \cdot)\|_{L^\infty_{\text{loc}}(M \times M)} \leq C(t), \quad (5.21)$$

and we remark that the constant  $C(t)$  explodes as  $t \rightarrow 0$ . Therefore, the convergence (5.20) is locally uniform with respect to  $x \in M$ . In conclusion,

$$\begin{aligned} |H_S^\varepsilon(t) - H_S(t)| &= |\langle \omega^\varepsilon, u^\varepsilon(t, \cdot) \rangle - \langle \mu_S, u(t, \cdot) \rangle| \\ &\leq |\langle \omega^\varepsilon, u^\varepsilon(t, \cdot) - u(t, \cdot) \rangle| + |\langle \omega^\varepsilon, u(t, \cdot) \rangle - \langle \mu_S, u(t, \cdot) \rangle| \\ &\leq \|u^\varepsilon(t, \cdot) - u(t, \cdot)\|_{L^\infty(S_{r_0})} |\langle \omega^\varepsilon, 1 \rangle| + |\langle \omega^\varepsilon - \mu_S, u(t, \cdot) \rangle|, \end{aligned}$$

and taking the limit as  $\varepsilon \rightarrow 0$  in the last line proves the desired result.  $\square$

*Remark 5.4.2.* The convergence (5.19) is never uniform as  $t \rightarrow 0$ , being the constant  $C(t)$  in (5.21) not bounded as  $t \rightarrow 0$ . This suggests that, while  $H_S^\varepsilon$  seems to be the best possible approximation of the heat content associated with  $S$ , using such a strategy to deduce the asymptotics of  $H_S(t)$  is not correct. Indeed, we can show that the coefficients of the expansion of  $H_S^\varepsilon$  can not approximate those of  $H_S(t)$ , in general.

## 5.5 An example: closed simple curve in $\mathbb{R}^3$

In  $\mathbb{R}^3$  equipped with the Euclidean scalar product and the Lebesgue measure, let us consider a biregular closed simple curve, parametrized by arc-length,  $\gamma: [0, \ell] \rightarrow \mathbb{R}^3$ , where  $\ell$  denotes the length of  $\gamma$ . Recall that a smooth curve  $\gamma: I \rightarrow \mathbb{R}^3$  is biregular if

$$\dot{\gamma}(s) \wedge \ddot{\gamma}(s) \neq 0, \quad \forall s \in I,$$

where  $\wedge$  denotes the cross product in  $\mathbb{R}^3$ . In this setting, define  $S = \gamma([0, \ell]) \subset \mathbb{R}^3$ , submanifold of codimension 2. The tubular neighborhood of  $S$  given by Theorem 5.2.2 coincides with the usual Euclidean tubular neighborhood, which can be conveniently described by the Frenet-Serret moving frame along  $\gamma$ , it being biregular. In particular, denoting by  $\{T(s), N(s), B(s)\}$  the Frenet-Serret frame for  $s \in [0, \ell]$ , we have, for any  $\varepsilon \leq r_0$

$$S_\varepsilon = \{\gamma(s) + r(\cos \theta N(s) + \sin \theta B(s)) \mid s \in [0, \ell], \theta \in (0, 2\pi], r \in (0, \varepsilon)\}.$$

Thus, in coordinates  $(s, r, \theta)$ , the Lebesgue measure is

$$dx dy dz = r(1 - rk(s) \cos \theta) ds dr d\theta, \quad (5.22)$$

where  $k(s) = \|\ddot{\gamma}(s)\|$  is the curvature of  $\gamma$ , and the procedure of Lemma 5.3.1 gives the probability measure  $\mu_S = ds/\ell$ . Following the discussion of Section 5.4, we define the heat content associated with  $S$  as

$$H_S(t) = \frac{1}{\ell^2} \frac{1}{(4\pi t)^{3/2}} \int_0^\ell \int_0^\ell e^{-\frac{|\gamma(s) - \gamma(\tau)|^2}{4t}} ds d\tau, \quad \forall t > 0.$$

### 5.5.1 Asymptotic expansion of $H_S(t)$

Denote by  $\phi_\tau(s) = |\gamma(s) - \gamma(\tau)|^2$ , then we can explicitly compute the asymptotic expansion of  $H_S$  applying the Laplace method to the integral

$$I_\tau(\lambda) = \int_0^\ell e^{-\phi_\tau(s)\lambda} ds, \quad \text{as } \lambda \rightarrow +\infty. \quad (5.23)$$

In particular, since  $\gamma$  is a simple curve, the phase  $\phi_\tau(s)$  has a strict minimum at  $s = \tau$ , thus there exists  $\varepsilon = \varepsilon(\tau)$  such that  $\phi'_\tau(s) > 0$  for any  $s \in [\tau - \varepsilon, \tau + \varepsilon] \setminus \{\tau\}$ . A direct computation, building upon  $\|\dot{\gamma}\| = 1$ , yields:

$$\begin{aligned} \phi'_\tau(\tau) &= 0, & \phi''_\tau(\tau) &= 2, & \phi_\tau^{(5)}(\tau) &= -5\partial_\tau(k(\tau)^2), \\ \phi_\tau^{(3)}(\tau) &= 0, & \phi_\tau^{(4)}(\tau) &= -2k(\tau)^2, & \phi_\tau^{(6)}(\tau) &= -9\partial_\tau^2(k(\tau)^2) + 2\|\ddot{\gamma}(\tau)\|^2. \end{aligned}$$

Therefore the phase has a Taylor expansion at its minimum and we can apply the Laplace method, which gives a full asymptotic expansion, cf. [Olv97, Thm. 8.1],

$$I_\tau(\lambda) \sim e^{-\lambda\phi_\tau(\tau)} \sum_{i=0}^{\infty} \Gamma\left(\frac{i+1}{2}\right) \frac{a_i(\tau)}{\lambda^{\frac{i+1}{2}}} \quad \text{as } \lambda \rightarrow \infty,$$

where  $\Gamma$  is the Euler Gamma function, and the  $a_i(\tau)$  are given by explicit formulas in terms of the derivatives of  $\phi_\tau$  at its minimum. Moreover, since, for any  $\tau \in [0, \ell]$ , the phase has an interior minimum, the odd coefficients of the expansion vanish. For the even-order coefficients, we have

$$a_0 = 1, \quad a_2 = \frac{1}{8}k(\tau)^2, \quad a_4 = \frac{1}{1152} \left( 36\partial_\tau^2(k(\tau)^2) + 35k(\tau)^2 - 8\|\ddot{\gamma}(\tau)\|^2 \right). \quad (5.24)$$

To conclude, we have to integrate with respect to  $\tau$  the asymptotic expansion of (5.23). In general, the expansion may not be uniform in  $\tau \in (0, \ell)$ , however, since  $\gamma$  is uniformly continuous on  $[0, \ell]$  and can be extended by periodicity on the whole real line, the choice of  $\varepsilon > 0$  such that  $\phi'_\tau(s) > 0$  for any  $s \in [\tau - \varepsilon, \tau + \varepsilon] \setminus \{\tau\}$  can be made uniform, providing uniform estimates of the remainder. In particular, we have

$$H_S(t) \sim \frac{1}{\ell^2} \frac{1}{(4\pi t)^{3/2}} \int_0^\ell \int_{\tau-\varepsilon}^{\tau+\varepsilon} e^{-\frac{\phi_\tau(s)}{4t}} ds d\tau, \quad \text{as } t \rightarrow 0,$$

where  $\varepsilon > 0$  is chosen uniformly with respect to  $\tau$ . Hence, we conclude that:

$$\begin{aligned} H_S(t) &\sim \frac{1}{\ell^2} \frac{1}{(4\pi t)^{3/2}} \int_0^\ell \sum_{i=0}^{\infty} \Gamma\left(\frac{2i+1}{2}\right) a_{2i}(\tau) (4t)^{\frac{2i+1}{2}} d\tau \\ &= \frac{1}{\ell^2} \frac{1}{4\pi t} \sum_{i=0}^{\infty} \alpha_i t^i, \end{aligned} \quad (5.25)$$

as  $t \rightarrow 0$ , where the coefficients  $\alpha_i$ 's are defined by

$$\alpha_i = 2^{2i-1} (2i+1) \int_0^\ell a_{2i}(\tau) d\tau, \quad \forall i \geq 0,$$

and are given explicitly by (5.24), up to  $i = 2$ .

### 5.5.2 Approximation via tubular neighborhoods

We compute the asymptotic expansion of the approximation defined in Proposition 5.4.1 for a closed simple curve, and we compare its coefficients with those obtained in (5.25). Recall the definition (5.18), when  $S = \gamma([0, \ell])$ :

$$H_S^\varepsilon(t) = \frac{1}{|S_\varepsilon|^2} \int_{S_\varepsilon} \int_{S_\varepsilon} \frac{1}{(4\pi t)^{3/2}} e^{-\frac{|x-y|^2}{4t}} dx dy, \quad \forall t > 0.$$

By Theorem 4.1.1, we know there exists an asymptotic expansion up to order 4 in  $\sqrt{t}$ , of the form

$$H_S^\varepsilon(t) = \frac{1}{|S_\varepsilon|^2} \left( \alpha_0^\varepsilon + \alpha_1^\varepsilon t^{1/2} + \alpha_3^\varepsilon t^{3/2} + o(t^2) \right), \quad (5.26)$$

where  $\alpha_0^\varepsilon = |S_\varepsilon|$  and

$$\alpha_1^\varepsilon = -\frac{1}{\sqrt{\pi}} \sigma^\varepsilon (\partial S_\varepsilon), \quad \alpha_3^\varepsilon = -\frac{1}{12\sqrt{\pi}} \int_{\partial S_\varepsilon} \left( 2\nabla \delta_S \cdot \nabla (\Delta \delta_S) - (\Delta \delta_S)^2 \right) d\sigma^\varepsilon. \quad (5.27)$$

Notice that, in tubular coordinates  $(s, r, \theta)$ ,  $\nabla \delta_S = \partial_r$ , and, since (5.22) holds,

$$d\sigma^r = r(1 - rk(s) \cos \theta) ds d\theta, \quad \Delta \delta_S = \frac{1}{r} + \partial_r (\log(1 - rk(s) \cos \theta)).$$

Thus, we can explicitly compute the coefficients (5.27):

$$\begin{aligned} \alpha_1^\varepsilon &= -\frac{\varepsilon}{\sqrt{\pi}} \int_0^\ell \int_0^{2\pi} (1 - \varepsilon k(s) \cos \theta) ds d\theta = -2\varepsilon \ell \sqrt{\pi} \\ \alpha_3^\varepsilon &= -\frac{\varepsilon}{12\sqrt{\pi}} \int_0^\ell \int_0^{2\pi} \left( -\frac{3}{\varepsilon^2} + \frac{A_1(s, \varepsilon, \theta)}{\varepsilon} + A_0(s, \varepsilon, \theta) \right) (1 - \varepsilon k(s) \cos \theta) ds d\theta, \end{aligned} \quad (5.28)$$

where  $A_0, A_1$  are smooth functions defined by

$$\begin{aligned} A_0(s, r, \theta) &= 2\partial_r^2 (\log(1 - rk(s) \cos \theta)) - (\partial_r (\log(1 - rk(s) \cos \theta)))^2, \\ A_1(s, r, \theta) &= -2\partial_r (\log(1 - rk(s) \cos \theta)). \end{aligned}$$

### 5.5.3 Comparison between the two approaches

Let us compare the asymptotic expansion of the two quantities in exam: for a fixed  $\varepsilon > 0$  and as  $t \rightarrow 0$ , we have from (5.25) and (5.26)

$$H_S^\varepsilon(t) = \frac{1}{|S_\varepsilon|^2} \left( \alpha_0^\varepsilon + \alpha_1^\varepsilon t^{1/2} + \alpha_3^\varepsilon t^{3/2} + o(t^2) \right),$$
$$H_S(t) = \frac{1}{\ell^2} \frac{1}{4\pi t} \left( \alpha_0 + \alpha_1 t + \alpha_3 t^2 + o(t^2) \right),$$

where the coefficients are given by (5.28) and (5.24), respectively. At this stage, on the one hand, we notice that the order in  $t$  of the expansions doesn't agree. On the other hand, the coefficients  $\alpha_1^\varepsilon, \alpha_3^\varepsilon$  do not contain fine geometrical information of  $S$ , indeed, the functions  $A_0, A_1$  depend only the curvature of  $\gamma$ , as opposed to (5.24), where derivatives of  $k(s)$  appear. Moreover, at a formal level,  $\alpha_1^\varepsilon \rightarrow 0$  as  $\varepsilon \rightarrow 0$ , whereas  $\alpha_3^\varepsilon$  explodes: it is possible to give meaning to these limits, taking into account the parabolic scaling between the space and time variables of the heat equation and formally replacing  $t \mapsto \varepsilon^2 t$ , in (5.26), however, we do not recover any geometrical meaning.

*Remark 5.5.1.* The relative heat content is too coarse a tool to detect the geometry of a submanifold of high codimension and a different strategy is needed. Inspired by the construction of the tubular neighborhood for  $S$ , we may study the asymptotic expansion of  $H_S(t)$  using a perturbative approach similarly to what has been done in [Bar13], presenting the sub-Laplacian of  $S$  as a perturbation of a simpler operator. This will be object of future research.





# A

## Convergence of the heat semi-group of the Riemannian variation

---

*We recall some general facts about sequences of unbounded operator and heat semi-groups and we give an alternative proof of the localization result, i.e. Theorem 2.3.1, which does not make use of the hypoelliptic version of Kac's principle.*

---

Throughout the appendix,  $\mathcal{H}$  denotes a Hilbert space and any unbounded linear operator defined on  $\mathcal{H}$  is tacitly assumed to be densely defined.

**Definition A.0.1.** Assume that  $A$  is a closed (possibly unbounded) operator on  $\mathcal{H}$ . A linear dense subset  $D \subset \text{dom}(A)$  is called a *core* for  $A$  if  $\overline{A|_D} = A$ .

We state here Trotter-Kato's Theorem, useful to deduce convergence of heat semi-groups (see implication (a)  $\Rightarrow$  (d) in [EN00, Thm. 4.8, Ch. III]).

**Theorem A.0.2.** Let  $A_\varepsilon, A$  be self-adjoint (possibly unbounded) operators on  $\mathcal{H}$ , for any  $\varepsilon \in \mathbb{R}$ , such that their spectra are uniformly bounded, i.e. there exists  $\gamma \in \mathbb{R}$  such that  $\lambda \leq \gamma, \forall \lambda \in \sigma(A) \cup \sigma(A_\varepsilon)$ . Assume that  $D$  is a core for  $A$ , such that the following properties hold:

- i)  $D \subset \text{dom}(A_\varepsilon)$ , for any  $\varepsilon \leq \varepsilon_0$
- ii)  $A_\varepsilon \varphi \rightarrow A\varphi$  in  $\mathcal{H}$ , as  $\varepsilon \rightarrow 0$ , for each  $\varphi \in D$ .

Then, for any  $t \geq 0$ ,  $e^{tA_\varepsilon} \rightarrow e^{tA}$  as  $\varepsilon \rightarrow 0$ , strongly in  $\mathcal{H}$ . Moreover, for any  $T > 0$ :

$$e^{tA_\varepsilon} \varphi \xrightarrow{\varepsilon \rightarrow 0} e^{tA} \varphi, \quad \forall \varphi \in \mathcal{H}, \text{ uniformly on } [0, T].$$

**Corollary A.0.3.** Let  $(\mathcal{D}, g)$  be a sub-Riemannian structure on  $M$  and  $\{(M, g_\varepsilon)\}_{\varepsilon \in \mathbb{R}}$  be a Riemannian variation. Fix a smooth volume  $\omega$  on  $M$  and let  $\Omega \subseteq M$  be an open, relatively compact set. Let  $\Delta_\varepsilon, \Delta$  be the weighted Dirichlet  $\varepsilon$ -Riemannian and sub-Riemannian Laplacian, defined in (2.9) and (1.14) respectively. Then, for any  $T > 0$

$$e^{t\Delta_\varepsilon} \varphi \xrightarrow{\varepsilon \rightarrow 0} e^{t\Delta} \varphi, \quad \forall \varphi \in L^2(\Omega), \text{ uniformly on } [0, T]. \quad (\text{A.1})$$

*Proof.* Set here  $\mathcal{H} = L^2(\Omega, \omega)$ , while  $A_\varepsilon = \Delta_\varepsilon$ , for any  $\varepsilon \in \mathbb{R}$  and  $A = \Delta$  are the Dirichlet Laplacians of  $(M, g_\varepsilon)$  and the sub-Laplacian, i.e. the Friedrichs extensions of the Dirichlet energy functionals, defined respectively by:

$$\mathcal{E}_\varepsilon(f) = \int_\Omega 2H_\varepsilon(df)d\omega, \quad \mathcal{E}(f) = \int_\Omega 2H(df)d\omega, \quad \forall f \in C_c^\infty(\Omega),$$

where  $H_\varepsilon, H$  are respectively the  $\varepsilon$ -Riemannian and the sub-Riemannian Hamiltonian. It is enough to check that these operators satisfy the hypotheses of Theorem A.0.2. First of all, both  $A$  and  $A_\varepsilon$  have negative spectrum, so that their spectra is uniformly bounded above by 0. Second of all, a core for  $A$  is the set

$$D = \{f \in C^\infty(\bar{\Omega}) \mid f|_{\partial\Omega} = 0\}.$$

This is a well-known fact in the Euclidean case, but the proof (e.g. see [RS78, Prop. 1, p. 264]) holds unchanged in the (sub-)Riemannian case, by the spectral theorem. Hence, we also have  $D \subset \text{dom}(A_\varepsilon)$ , for any  $\varepsilon > 0$ . Using the explicit expressions of  $\Delta$  and  $\Delta_\varepsilon$  on smooth functions, in terms of a generating frame for the Riemannian variation (2.9), we obtain:

$$A_\varepsilon\varphi \xrightarrow{\varepsilon \rightarrow 0} A\varphi, \quad \forall \varphi \in D,$$

uniformly on  $\Omega$ , and hence also on  $\mathcal{H}$ . Consequently, applying Theorem A.0.2, we finally get the convergence (A.1).  $\square$

## A.1 An alternative proof of Theorem 2.3.1

We prove now a weaker version of Theorem 2.3.1, which is enough to deduce the heat content asymptotics, cf. Section 2.5. Here, we use the shorthand notation  $L^2$  for the space  $L^2(\Omega, \omega)$ .

**Theorem A.1.1.** *Let  $M$  be a sub-Riemannian manifold, equipped with a smooth measure  $\omega$ , and let  $\Omega \subset M$  be an open relatively compact subset with smooth boundary. Let  $\eta \in C_c^\infty(\Omega)$ , with support separated from  $\partial\Omega$ . Then, for all  $m \geq 1$ ,*

$$\int_\Omega (1 - u(t, x))\eta(x)d\omega(x) = O(t^m), \quad \text{as } t \rightarrow 0, \quad (\text{A.2})$$

where  $u(t, x)$  denotes the solution to (2.3).

To prove the theorem, we need the following technical lemma.

**Lemma A.1.2.** *Under the hypotheses of Theorem A.1.1 fix  $\{(M, g_\varepsilon)\}_{\varepsilon \in \mathbb{R}}$ , a Riemannian variation of  $M$ . Then, for any  $T > 0$  and any integer  $k \geq 1$ , we have:*

$$\partial_t^k e^{t\Delta_\varepsilon} \eta \xrightarrow[\|\cdot\|_{L^2}]{\varepsilon \rightarrow 0} \partial_t^k e^{t\Delta} \eta, \quad \text{uniformly w.r.t. } t \in (0, T). \quad (\text{A.3})$$

*Proof.* First of all, let  $\{\psi_\varepsilon\}_{\varepsilon > 0} \subset L^2$  be a convergent sequence to  $\psi \in L^2$ , then, for any  $\rho > 0$ , choosing  $\varepsilon$  suitably small, we have:

$$\begin{aligned} \|e^{t\Delta_\varepsilon}\psi_\varepsilon - e^{t\Delta}\psi\|_{L^2} &\leq \|e^{t\Delta_\varepsilon}\psi_\varepsilon - e^{t\Delta_\varepsilon}\psi\|_{L^2} + \|e^{t\Delta_\varepsilon}\psi - e^{t\Delta}\psi\|_{L^2} \\ &\leq \|e^{t\Delta_\varepsilon}\|_{\mathcal{B}(L^2)} \|\psi_\varepsilon - \psi\|_{L^2} + \|e^{t\Delta_\varepsilon}\psi - e^{t\Delta}\psi\|_{L^2} \\ &\leq \|\psi_\varepsilon - \psi\|_{L^2} + \|e^{t\Delta_\varepsilon}\psi - e^{t\Delta}\psi\|_{L^2} \leq \rho, \end{aligned} \quad (\text{A.4})$$

where, in the last inequality, we have used the fact the heat semi-group is a contraction semi-group for any  $\varepsilon$ , therefore its operator norm is smaller than 1, and Corollary A.0.3. We remark that, since the estimate (A.4) is uniform with respect to the time variable, also the convergence remains uniform on  $[0, T]$ .

Let now  $\eta \in C_c^\infty(\Omega)$  be as in the statement. Since  $\eta$  is supported away from  $\partial\Omega$ ,  $\Delta_\varepsilon^k \eta \in C_c^\infty(\Omega) \subset \text{dom}(\Delta_\varepsilon)$  for all  $k \geq 1$  and  $\varepsilon \in \mathbb{R}$ . Since on  $\text{dom}(\Delta_\varepsilon)$  the heat semi-group commutes with  $\Delta_\varepsilon$ , it holds:

$$\partial_t^k e^{t\Delta_\varepsilon} \eta = \partial_t^{k-1} \Delta_\varepsilon e^{t\Delta_\varepsilon} \eta = \partial_t^{k-1} e^{t\Delta_\varepsilon} \Delta_\varepsilon \eta = \dots = e^{t\Delta_\varepsilon} \Delta_\varepsilon^k \eta = e^{t\Delta_\varepsilon} \eta_\varepsilon, \quad \forall t > 0,$$

where we defined  $\eta_\varepsilon = \Delta_\varepsilon^k \eta \in C_c^\infty(\Omega)$ . It follows directly from the expression (2.9) that:

$$\eta_\varepsilon = \Delta_\varepsilon^k \eta \xrightarrow{\varepsilon \rightarrow 0} \Delta^k \eta,$$

uniformly on  $\Omega$ , and in particular also in  $L^2$ . Therefore, using (A.4):

$$e^{t\Delta_\varepsilon} \eta_\varepsilon \xrightarrow[\|\cdot\|_{L^2}]{\varepsilon \rightarrow 0} e^{t\Delta} \Delta^k \eta = \partial_t^k e^{t\Delta} \eta, \quad \text{uniformly w.r.t. } t \in (0, T),$$

concluding the proof.  $\square$

*Proof of Theorem A.1.1.* First of all, we can interpret the integral appearing in (A.2), i.e.

$$\int_\Omega u(t, x) \eta(x) d\omega(x),$$

as the heat content associated with the problem (2.3), with initial datum  $\eta$ , instead of 1, indeed, denoting with  $p_t^\Omega(x, y)$  the sub-Riemannian Dirichlet heat kernel, we have:

$$\int_\Omega u(t, x) \eta(x) d\omega(x) = \int_\Omega \int_\Omega p_t^\Omega(x, z) \eta(x) d\omega(z) d\omega(x) = \int_\Omega u^\eta(t, z) d\omega(z), \quad (\text{A.5})$$

where  $u^\eta(t, x)$  is a shorthand notation for the solution to (2.3) with initial datum  $\eta$ . The existence of the heat kernel is guaranteed by the Schwartz kernel theorem. Second of all, let us consider the Riemannian variation of  $M$ , i.e. a sequence of Riemannian manifolds  $\{(M, g_\varepsilon)\}_{\varepsilon \in \mathbb{R}}$ , defined as in (2.2.2). Notice that formula (A.5) holds without any changes in the Riemannian case, since it relies only on the existence of the heat kernel. Hence, using (A.5), we have:

$$\begin{aligned} \int_\Omega (1 - u(t, x)) \eta(x) d\omega(x) &= \int_\Omega (1 - u_\varepsilon(t, x)) \eta(x) d\omega(x) + \int_\Omega (u_\varepsilon(t, x) - u(t, x)) \eta(x) d\omega(x) \\ &= \int_\Omega (1 - u_\varepsilon(t, x)) \eta(x) d\omega(x) + \int_\Omega (u_\varepsilon^\eta(t, x) - u^\eta(t, x)) d\omega(x), \end{aligned}$$

where  $\varepsilon \in \mathbb{R}$  has to be suitably chosen. The first term can be estimated using the Riemannian localization principle, see [Sav98, Prop. 5.1]. In details, for the fixed  $\varepsilon$ -Riemannian structure, we may choose  $a < \min\{R_{\text{inj}}^\varepsilon, d_\varepsilon(\text{supp}(\eta), \partial\Omega)\}$ , where  $R_{\text{inj}}^\varepsilon$  is the  $\varepsilon$ -injectivity radius of  $\partial\Omega$  in  $\Omega$ . Notice that  $a$  can be chosen to be positive, indeed the injectivity radius  $R_{\text{inj}}^\varepsilon$  is strictly positive, being  $\partial\Omega$  smooth, while  $d_\varepsilon(\text{supp}(\eta), \partial\Omega) > 0$  since  $\text{supp}(\eta)$  is separated from the boundary. Then, for any  $x \in \Omega$ , such that  $d_\varepsilon(x, \partial\Omega) > a$ , we have:

$$|1 - u_\varepsilon(t, x)| \leq C_{m, \varepsilon} t^m, \quad \forall m \geq 1, \quad \text{and } t \leq 1, \quad (\text{A.6})$$

where the constant  $C_{m, \varepsilon}$  depends only on  $\Omega$  and the parameter  $a$ , which, in turn, depends only on  $\eta$  and the Riemannian approximant  $(M, g_\varepsilon)$ , but not on  $m$ . We remark that the

parameter  $\varepsilon$  of the Riemannian approximant is fixed. To deal with the second term, for  $t \leq 1$ , write:

$$\begin{aligned} & \left| \int_{\Omega} (u_{\varepsilon}^{\eta}(t, x) - u^{\eta}(t, x)) d\omega(x) \right| \\ &= \left| \int_{\Omega} \int_0^t \int_0^{t_2} \dots \int_0^{t_m} (\partial_{\tau}^m u_{\varepsilon}^{\eta}(\tau, x) - \partial_{\tau}^m u^{\eta}(\tau, x)) d\tau dt_m \dots dt_2 d\omega(x) \right|. \end{aligned} \quad (\text{A.7})$$

This can be done, since for any integer  $m \geq 1$ , we have:

$$\begin{aligned} \int_{\Omega} \partial_{\tau}^m u_{\varepsilon}^{\eta}(0, x) d\omega(x) &= \int_{\Omega} \partial_{\tau}^m e^{\tau \Delta_{\varepsilon}} \eta(x) |_{\tau=0} d\omega(x) = \int_{\Omega} e^{\tau \Delta_{\varepsilon}} \Delta_{\varepsilon}^m \eta(x) |_{\tau=0} d\omega(x) \\ &= \int_{\Omega} \Delta_{\varepsilon}^m \eta(x) d\omega(x) = 0, \end{aligned} \quad (\text{A.8})$$

by the divergence formula (1.13). Notice that, in the second equality in (A.8), we used the fact that the heat semi-group commutes with the Laplacian on  $\text{dom}(\Delta_{\varepsilon})$  and, for any  $k$ ,  $\Delta_{\varepsilon}^k \eta \in C_c^{\infty}(\Omega) \subset \text{dom}(\Delta_{\varepsilon})$ . Clearly, formula (A.8) holds when we replace  $u_{\varepsilon}^{\eta}$  with  $u^{\eta}$ . We may estimate (A.7), using Lemma A.1.2, indeed:

$$\begin{aligned} & \left| \int_{\Omega} (u_{\varepsilon}^{\eta}(t, x) - u^{\eta}(t, x)) d\omega(x) \right| \\ &= \left| \int_{\Omega} \int_0^t \dots \int_0^{t_m} \partial_{\tau}^m (u_{\varepsilon}^{\eta}(\tau, \cdot) - u^{\eta}(\tau, \cdot)) d\tau dt_m \dots dt_2 d\omega(x) \right| \\ &\leq \int_0^t \dots \int_0^{t_m} \int_{\Omega} |\partial_{\tau}^m (u_{\varepsilon}^{\eta}(\tau, \cdot) - u^{\eta}(\tau, \cdot))| d\omega d\tau dt_m \dots dt_2 \\ &\leq \omega(\Omega)^{1/2} \int_0^t \dots \int_0^{t_m} \|\partial_{\tau}^m (u_{\varepsilon}^{\eta}(\tau, \cdot) - u^{\eta}(\tau, \cdot))\|_{L^2} d\tau dt_m \dots dt_2 \\ &\leq \rho t^m, \end{aligned} \quad (\text{A.9})$$

having chosen  $\varepsilon$  sufficiently small, such that:

$$\frac{\omega(\Omega)^{1/2}}{m!} \|\partial_{\tau}^m (u_{\varepsilon}^{\eta}(\tau, \cdot) - u^{\eta}(\tau, \cdot))\|_{L^2} \leq \rho, \quad \forall \tau \in (0, 1).$$

We mention that the choice of  $\varepsilon$  does not depend on  $t$ , as showed in (A.3), but only on the function  $\eta$  and on the parameter  $\rho > 0$ . Finally, putting together (A.6) and (A.9),

$$\left| \int_{\Omega} (1 - u(t, x)) \eta(x) d\omega(x) \right| \leq (C_{m, \varepsilon} + \rho) t^m = C_m t^m, \quad \forall t \leq 1,$$

concluding the proof. □

# B

## Asymptotic expansion of $I\phi(t, 0)$

---

We show how to prove Theorem B.0.7, following closely [Sav98]. The proof consists in the application of an iterated version of the Duhamel's formula.

---

First of all, let us prove a general iterated Duhamel's principle, that is an higher-order version of Lemma 2.5.4.

**Proposition B.0.1.** *Let  $F \in C^\infty((0, \infty) \times [0, \infty))$  be a smooth function compactly supported in the second variable. Assume that the following conditions hold:*

- (i)  $L^k F(0, r) = \lim_{t \rightarrow 0} L^k F(t, r)$  exists in the sense of distributions<sup>1</sup> for any  $k \geq 0$ ;
- (ii)  $L^k F(t, 0)$  and  $\partial_r L^k F(t, 0)$  converge to a finite limit as  $t \rightarrow 0$ , for any  $k \geq 0$ .

Then, for all  $m \in \mathbb{N}$  and  $t > 0$ , we have

$$F(t, 0) = \sum_{k=0}^m \left( \frac{t^k}{k!} \int_0^\infty e(t, r, 0) L^k F(0, r) dr - \frac{1}{\sqrt{\pi k!}} \int_0^t \partial_r L^k F(\tau, 0) (t - \tau)^{k-1/2} d\tau \right) + \frac{1}{m!} \int_0^t \int_0^\infty e(t - \tau, r, 0) L^{m+1} F(\tau, r) (t - \tau)^m dr d\tau, \quad (\text{B.1})$$

where  $e(t, r, s)$  is the Neumann heat kernel on the half-line, cf. (2.23).

*Proof.* Define, for all  $\epsilon > 0$ , the function  $v(t, r) = F(t + \epsilon, r)$ , for  $t > 0$  and  $r \geq 0$ . We stress that  $v(t, r)$  is smooth on the closed set  $[0, \infty) \times [0, \infty)$ . In particular all the assumptions of Lemma 2.5.4 are verified with for the function  $v$ , and it holds

$$F(t + \epsilon, s) = \int_0^\infty e(t, s, r) F(\epsilon, r) dr - \int_0^t e(t - \tau, 0, s) \partial_r F(\tau + \epsilon, 0) d\tau + \int_0^t \int_0^\infty e(t - \tau, s, r) L F(\tau + \epsilon, r) dr d\tau,$$

---

<sup>1</sup>Namely, for any  $\psi \in C^\infty([0, \infty))$ , there exists finite  $\lim_{t \rightarrow 0} \int_0^\infty f(t, r) \psi(r) dr$ . With a slight abuse of notation, we define  $\int_0^\infty f(0, r) \psi(r) dr = \lim_{t \rightarrow 0} \int_0^\infty f(t, r) \psi(r) dr$ .

for all  $s \geq 0$  and  $t > 0$ . Iterating the above formula, and using the semi-group property, we obtain for all  $m \in \mathbb{N}$  and  $\epsilon > 0$

$$\begin{aligned} F(t + \epsilon, s) &= \sum_{k=0}^m \left( \frac{t^k}{k!} \int_0^\infty e(t, s, r) L^k F(\epsilon, r) dr \right. \\ &\quad \left. - \frac{1}{k!} \int_0^t e(t - \tau, 0, s) \partial_r L^k F(\tau + \epsilon, 0) (t - \tau)^k d\tau \right) \\ &\quad + \frac{1}{m!} \int_0^t \int_0^\infty e(t - \tau, r, s) L^{m+1} F(\tau + \epsilon, r) (t - \tau)^m dr d\tau. \end{aligned}$$

Then we send  $\epsilon \rightarrow 0$  checking that, under the assumptions (i) and (ii), all terms on the right hand side is well-defined. We set finally  $s = 0$ .  $\square$

Since we want to apply Proposition B.0.1 to the function  $I\phi(t, 0)$ , first, we study in detail the operators  $L^k I$ , for any  $k \geq 1$ . Recall the definition of the family of operators  $R_{kj}, S_{kj}$  in (2.25)–(2.26):

$$R_{00} = \text{Id}, \quad S_{00} = 0,$$

while, for all  $k \geq 1$ , and  $0 \leq j \leq k$ , define

$$\begin{aligned} R_{kj} &= -(N^2 - \Delta)R_{k-1,j} + NS_{k-1,j}, \\ S_{kj} &= NR_{k-1,j-1} - \Delta NR_{k-1,j} + \Delta S_{k-1,j}, \end{aligned}$$

and  $R_{kj} = S_{kj} = 0$ , for all other values of the indices, i.e.  $k < 0$ ,  $j < 0$  or  $k < j$ . In addition, we define a second family, still lying in the algebra of operators generated by  $N$  and  $\Delta$ . We set first

$$P_{00} = 0, \quad Q_{00} = \text{Id},$$

while, for all  $k \geq 1$ , and  $0 \leq j \leq k$ , we define

$$P_{kj} = -(N^2 - \Delta)P_{k-1,j} + NQ_{k-1,j}, \quad (\text{B.2})$$

$$Q_{kj} = NP_{k-1,j-1} - \Delta NP_{k-1,j} + \Delta Q_{k-1,j}, \quad (\text{B.3})$$

and  $P_{kj} = Q_{kj} = 0$ , for all other values of the indices, i.e.  $k < 0$ ,  $j < 0$  or  $k < j$ . These operators, which may seem obscure, arise naturally in the iterative application of the one-dimensional heat operator  $L$  to  $I\phi$  and  $\Lambda\phi$ , as we show in the next lemma. Recall that  $\Omega'(r_0) = \bar{\Omega} \setminus \Omega(r_0) = \{x \in \bar{\Omega} \mid \delta(x) \leq r_0\}$ .

**Lemma B.0.2.** *Let  $M$  be a sub-Riemannian manifold, equipped with a smooth measure  $\omega$ , and let  $\Omega \subset M$  be an open relatively compact subset whose boundary is smooth and has not characteristic points. Denote by  $\delta : \bar{\Omega} \rightarrow [0, \infty)$  the sub-Riemannian distance function from  $\partial\Omega$ . Then, as operators on  $C_c^\infty(\Omega'(r_0))$ , we have:*

$$(i) \quad LI = \Lambda N + I\Delta;$$

$$(ii) \quad L\Lambda = \Lambda(-N^2 + \Delta) + \partial_t IN - I\Delta N;$$

(iii) For any  $k \in \mathbb{N}$ ,

$$L^k I = \sum_{j=0}^k \frac{\partial^j}{\partial t^j} (\Lambda P_{kj} + I Q_{kj}) \quad \text{and} \quad L^k \Lambda = \sum_{j=0}^k \frac{\partial^j}{\partial t^j} (\Lambda R_{kj} + I S_{kj}).$$

*Remark B.0.3.* The time derivatives make sense, since, for any  $\phi \in C_c^\infty(\Omega'(r_0))$ , the functions  $I\phi, \Lambda\phi$  are defined on  $(t, r) \in (0, \infty) \times [0, r_0)$ . Notice that only the operators  $\Lambda$  and  $I$  are time-dependent, while the families  $P, Q, R, S$  do not depend on time.

*Proof.* Recall that  $L = \partial_t - \partial_r^2$ , hence, we use formula (2.15) to compute

$$\partial_r^2 I\phi(t, r) = \int_{\Omega(r)} \Delta((1 - u(t, x))\phi(x)) d\omega(x) - \int_{\partial\Omega(r)} (1 - u(t, y))\phi(y)\Delta\delta(y) d\sigma(y). \quad (\text{B.4})$$

Now,  $\Delta((1 - u)\phi) = -\phi\Delta u - 2g(\nabla u, \nabla\phi) + (1 - u)\Delta\phi$ , therefore the first term in (B.4) becomes

$$\begin{aligned} \int_{\Omega(r)} \Delta((1 - u)\phi) d\omega &= \int_{\Omega(r)} (-\phi\Delta u - 2g(\nabla u, \nabla\phi) + (1 - u)\Delta\phi) d\omega \\ &= - \int_{\Omega(r)} (\phi\Delta u + (1 - u)\Delta\phi) d\omega + 2 \int_{\partial\Omega(r)} (1 - u)g(\nabla\phi, \nu) d\sigma, \end{aligned}$$

where  $\nu = -\nabla\delta$  is the outward-pointing unit normal and where we used the divergence formula (1.13). Finally, (B.4) becomes

$$\begin{aligned} -\partial_r^2 I\phi &= \int_{\Omega(r)} (\phi\Delta u + (1 - u)\Delta\phi) d\omega + \int_{\partial\Omega(r)} (1 - u)(-2g(\nabla\phi, \nu) + \phi\Delta\delta) d\sigma \\ &= \int_{\Omega(r)} \phi \partial_t u d\omega + I\Delta\phi + \Lambda N\phi = - \int_{\Omega(r)} \partial_t(1 - u)\phi d\omega + I\Delta\phi + \Lambda N\phi \\ &= -\partial_t I\phi + I\Delta\phi + \Lambda N\phi. \end{aligned} \quad (\text{B.5})$$

This concludes the proof of (i), recalling that  $L = \partial_t - \partial_r^2$ .

To prove (ii), we need to compute  $L\Lambda\phi$ . Since by definition  $\partial_r I = -\Lambda$  (cf. Definition 2.5.1), we rewrite the equality (B.5) as

$$\partial_r \Lambda\phi = -\partial_t I\phi + I\Delta\phi + \Lambda N\phi,$$

and we differentiate it with respect to  $r$ , obtaining

$$\begin{aligned} \partial_r^2 \Lambda\phi(t, r) &= \partial_t \Lambda\phi(t, r) - \Lambda\Delta\phi(t, r) + \partial_r \Lambda N\phi(t, r) \\ &= \partial_t \Lambda\phi(t, r) - \Lambda\Delta\phi(t, r) + (-\partial_t I + I\Delta + \Lambda N) N\phi(t, r), \end{aligned}$$

yielding the statement of (ii). Point (iii) follow easily by induction on  $k$ .  $\square$

As suggested by Theorem 2.5.6, we would like to apply Proposition B.0.1 to  $I\phi(t, r)$  for  $k \geq 2$ , in order to obtain higher-order asymptotics. However, by Lemma B.0.2, the terms  $L^k I\phi$  involve time derivatives of  $u(t, x)$  and these are not well-defined due to the lack of smoothness of  $u$  at the boundary, at  $t = 0$ . Therefore, we consider the following approximations of  $I\phi$  and  $\Lambda\phi$ :

$$\begin{aligned} I_\epsilon\phi(t, r) &= \int_{\Omega(r)} (1 - u_\epsilon(t, x))\phi(x) d\omega(x), \\ \Lambda_\epsilon\phi(t, r) &= -\partial_r I_\epsilon\phi(t, r) = \int_{\partial\Omega(r)} (1 - u_\epsilon(t, x))\phi(x) d\omega(x), \end{aligned} \quad (\text{B.6})$$

where  $u_\epsilon(t, x)$  denotes the solution to (2.3) with initial datum  $\varphi(x) = \mathbf{1}_{\Omega(\epsilon)}(x)$ , where, we recall  $\Omega(\epsilon) = \{x \in \Omega \mid \delta(x) > \epsilon\}$ . Notice that, by the dominated convergence theorem, we have

$$I_\epsilon\phi(t, 0) \xrightarrow{\epsilon \rightarrow 0} I\phi(t, 0),$$

uniformly on  $[0, T]$ . Moreover, Lemma B.0.2 holds unchanged for  $I_\epsilon$  and  $\Lambda_\epsilon$ , and both  $I_\epsilon\phi(t, r)$  and  $\Lambda_\epsilon\phi(t, r)$  are compactly supported in the  $r$ -variable.



---

**Lemma B.0.4.** *Under the same hypotheses of Lemma B.0.2, let  $\psi \in C^\infty([0, \infty))$ ,  $\epsilon \in (0, r_0)$  and define*

$$\psi^{(-1)}(r) = \int_0^r \psi(s) ds.$$

*Then, for any  $\phi \in C_c^\infty(\Omega'(r_0))$ , the following identities hold:*

$$(i) \lim_{t \rightarrow 0} \int_0^\infty \frac{\partial^j}{\partial t^j} \Lambda_\epsilon \phi(t, r) \psi(r) dr = \begin{cases} \int_{\Omega \setminus \Omega(\epsilon)} \phi(\psi \circ \delta) d\omega & \text{if } j = 0, \\ - \int_{\Omega(\epsilon)} \Delta^j (\phi(\psi \circ \delta)) d\omega & \text{if } j \geq 1; \end{cases}$$

$$(ii) \lim_{t \rightarrow 0} \int_0^\infty \frac{\partial^j}{\partial t^j} I_\epsilon \phi(t, r) \psi(r) dr = \begin{cases} \int_{\Omega \setminus \Omega(\epsilon)} \phi(\psi^{(-1)} \circ \delta) d\omega & \text{if } j = 0, \\ - \int_{\Omega(\epsilon)} \Delta^j (\phi(\psi^{(-1)} \circ \delta)) d\omega & \text{if } j \geq 1; \end{cases}$$

$$(iii) \forall t \geq 0, \quad \frac{\partial^j}{\partial t^j} \Lambda_\epsilon \phi(t, 0) = \begin{cases} \int_{\partial\Omega} \phi d\sigma & \text{if } j = 0, \\ 0 & \text{if } j \geq 1; \end{cases}$$

$$(iv) \frac{\partial^j}{\partial t^j} I_\epsilon \phi(0, 0) = \begin{cases} \int_{\Omega \setminus \Omega(\epsilon)} \phi d\omega & \text{if } j = 0, \\ - \int_{\Omega(\epsilon)} \Delta^j \phi d\omega & \text{if } j \geq 1; \end{cases}$$

where, we recall,  $\Omega(\epsilon) = \{x \in \Omega \mid \delta(x) > \epsilon\}$ .

*Proof.* The idea of the proof is to use the divergence formula (1.13) and the fact that  $u_\epsilon$ , with all its derivative, converges to zero, as  $t \rightarrow 0$ , uniformly on  $\partial\Omega$ . The proof is identical to the Riemannian case and we omit it. See [Sav98, Lem. 5.6], for details.  $\square$

The next step is to obtain a small-time asymptotic expansion for the function  $I_\epsilon \phi(t, 0)$ , cf. (B.6), for fixed  $\epsilon > 0$ . Then, passing to the limit as  $\epsilon \rightarrow 0$ , one obtains the complete asymptotic series of  $I\phi(t, 0)$ .

**Lemma B.0.5.** *Under the same hypotheses of Lemma B.0.2, let  $\phi \in C_c^\infty(\Omega'(r_0))$ . Then, for any  $m \in \mathbb{N}$ , we have*

$$I\phi(t, 0) = Z^{(m)}(t) + \frac{1}{\sqrt{\pi}} B^{(m)}(t) + O(t^{(m+1)/2}), \quad \text{as } t \rightarrow 0,$$

with

$$Z^{(m)}(t) = \lim_{\epsilon \rightarrow 0} \sum_{k=0}^m \frac{t^k}{k!} \int_0^\infty e(t, r, 0) L^k I_\epsilon \phi(0, r) dr,$$

$$B^{(m)}(t) = \lim_{\epsilon \rightarrow 0} \sum_{k=0}^m \frac{1}{k!} \int_0^t L^k \Lambda_\epsilon \phi(\tau, 0) (t - \tau)^{k-1/2} d\tau,$$

where  $L^k I_\epsilon \phi(0, r) = \lim_{t \rightarrow 0} L^k I_\epsilon \phi(t, r)$  in the sense of distributions, cf. footnote in Proposition B.0.1.

*Proof.* First of all, we should check that the function  $I_\epsilon \phi(t, r)$  satisfies the hypotheses of Proposition B.0.1. This can be done employing the previous Lemmas. Indeed, thanks to Lemma B.0.2, it is enough to check the hypotheses for the time derivatives of  $I_\epsilon \phi$  and  $\Lambda_\epsilon \phi$ ,

and Lemma B.0.4 explicitly verifies them. Hence, we may apply the iterated Duhamel's principle. It remains to ensure that the last term in (B.1) is a remainder of order  $(m+1)/2$ , as  $\epsilon \rightarrow 0$ , i.e.

$$\lim_{\epsilon \rightarrow 0} \int_0^t \int_0^\infty e(t-\tau, r, 0) L^{m+1} I_\epsilon \phi(\tau, r) (t-\tau)^m dr d\tau = O(t^{(m+1)/2}),$$

as  $t \rightarrow 0$ . This can be done using again Lemma B.0.4 and integration by parts.  $\square$

**Lemma B.0.6.** *Under the same hypotheses of Lemma B.0.2, let  $\phi \in C_c^\infty(\Omega'(r_0))$ . Then, for any  $m \in \mathbb{N}$ , as  $t \rightarrow 0$ , we have*

$$\begin{aligned} I\phi(t, 0) &= \frac{1}{\sqrt{\pi}} \sum_{k=1}^{\lfloor (m+1)/2 \rfloor} \int_{\partial\Omega} Z_k \phi(y) d\sigma(y) t^{k-1/2} \\ &\quad + \frac{1}{\sqrt{\pi}} \sum_{k=0}^{\lfloor (m-1)/2 \rfloor} \int_0^t I A_k \phi(\tau, 0) (t-\tau)^{k-1/2} d\tau + O(t^{(m+1)/2}), \end{aligned} \quad (\text{B.7})$$

where,  $Z_k, A_k$  are the operators (defined by compositions of  $\Delta$  and  $N$ ) defined in (2.27).

We omit the proof, since it is a long computation formally identical to the Riemannian case as done in [Sav98, Lem. 5.8]. The key idea is to express the operators  $Z^{(m)}(t), B^{(m)}(t)$  of Lemma B.0.5 in terms of the operators  $A_k, Z_k$ .

**Theorem B.0.7.** *Under the same hypotheses of Lemma B.0.2, let  $\phi \in C_c^\infty(\Omega'(r_0))$ . Then, for any  $m \in \mathbb{N}$*

$$I\phi(t, 0) = \sum_{k=1}^m \left( \int_{\partial\Omega} D_k \phi(y) d\sigma(y) \right) t^{k/2} + O(t^{(m+1)/2}), \quad \text{as } t \rightarrow 0. \quad (\text{B.8})$$

where the operators  $D_k$  are given in (2.28)–(2.30).

*Proof.* Let us denote by  $\beta_k(\phi)$  the  $k$ -th coefficient in (B.8). We proceed by induction on  $m \in \mathbb{N}$ . The case  $m = 1$  is given in formula (2.24), and the first coefficient is

$$\beta_1(\phi) = \sqrt{\frac{4}{\pi}} \int_{\partial\Omega} \phi d\sigma = \int_{\partial\Omega} D_1(\phi) d\sigma.$$

Assume that (B.8) holds for  $m-1$  and for any  $\phi \in C_c^\infty(\Omega'(r_0))$ . Using Lemma B.0.6, we recognize that the only term that we should discuss in the equality (B.7) is

$$\sum_{k=0}^{\lfloor (m-1)/2 \rfloor} \int_0^t I A_k \phi(\tau, 0) (t-\tau)^{k-1/2} d\tau.$$

By the induction hypothesis, we obtain an asymptotic expansion of  $I A_k \phi$ , for any  $k = 0, \dots, \lfloor (m-1)/2 \rfloor$ , up to order  $m/2$ :

$$I A_k \phi(\tau, 0) = \sum_{j=1}^{m-1} \beta_j(A_k \phi) \tau^{j/2} + O(\tau^{m/2}).$$

Inserting this expression in (B.7) and integrating with respect to  $\tau$ , we obtain

$$\begin{aligned} I\phi(t, 0) &= \frac{1}{\sqrt{\pi}} \sum_{k=1}^{\lfloor (m+1)/2 \rfloor} \int_{\partial\Omega} Z_k \phi(y) d\sigma(y) t^{k-1/2} \\ &\quad + \frac{1}{\sqrt{\pi}} \sum_{k=0}^{\lfloor (m-1)/2 \rfloor} \sum_{j=1}^{m-1} \frac{\Gamma(j/2 + 1) \Gamma(k + 1/2)}{\Gamma(k + (j+3)/2)} \beta_j(A_k \phi) t^{k+(j+1)/2} + O(t^{(m+1)/2}). \end{aligned} \quad (\text{B.9})$$

---

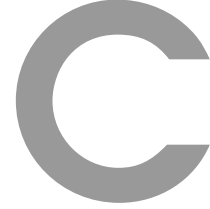
Since, by induction hypothesis, we know that  $I\phi(t, 0)$  admits an asymptotic expansion up to order  $m/2$ , (B.9) already shows that an asymptotic expansion up to order  $(m+1)/2$  exists, moreover, it provides an explicit expression for the  $\frac{m}{2}$ -th coefficient. If  $m = 2n$  is even, then the first sum does not give any contribution, since the highest power of  $t$  is  $(m-1)/2$ , while in the second sum, we have to consider only those indexes,  $k, j$ , such that the power of  $t$ ,  $k + (j+1)/2 = n$ , thus  $j$  must be odd, i.e.  $j = 2i - 1$ , with  $i = 1, \dots, n$ , and  $k = n - i$ . Therefore, we get

$$\beta_{2n}(\phi) = \frac{1}{\sqrt{\pi}} \sum_{i=1}^n \frac{\Gamma(i+1/2)\Gamma(n-i+1/2)}{n!} \beta_{2i-1}(A_{n-i}\phi).$$

Recalling the recursive definition of  $D_{2n}$  in (2.29) we obtain the result for even order. On the other hand, if  $m = 2n + 1$  is odd, then the first sum gives a contribution, whereas in the second sum  $j$  must be even, i.e.  $j = 2i$ , with  $i = 1, \dots, n$ , and  $k = n - i$ . Hence, we obtain

$$\beta_{2n+1}(\phi) = \frac{1}{\sqrt{\pi}} \int_{\partial\Omega} Z_{n+1}\phi + \frac{1}{\sqrt{\pi}} \sum_{i=1}^n \frac{\Gamma(i+1)\Gamma(n-i+1/2)}{\Gamma(n+3/2)} \beta_{2i}(A_{n-i}\phi).$$

Recalling the recursive definition of  $D_{2n+1}$  in (2.30) we obtain the result for odd order, concluding the proof.  $\square$



## Algorithm for computing the coefficients of Theorem 2.1.1

---

*In Theorem 2.1.1, we establish the existence of the full asymptotic series for the classical heat content  $Q_\Omega(t)$ . In this appendix, we describe an algorithm for computing the coefficients of the expansion at any order.*

---

In Chapter 2, we study the small-time asymptotics of the heat content of smooth non-characteristic domains of a general rank-varying sub-Riemannian structure, equipped with an arbitrary smooth measure. In Theorem 2.1.1, we establish the existence of the full asymptotic series:

$$Q_\Omega(t) \sim \sum_{i=1}^{\infty} a_k t^{k/2}, \quad \text{as } t \rightarrow 0.$$

As explained in Proposition 2.1.2, the coefficients  $a_k$  can be computed explicitly via an iterative procedure. In particular, for any  $k \geq 1$ , there exists an operator  $D_k$  acting on the set  $C^\infty(\bar{\Omega})$ , such that

$$a_k = - \int_{\partial\Omega} D_k(1) d\sigma, \tag{C.1}$$

where the operators  $D_k$  are defined recursively in (2.28)–(2.30). We describe a *Mathematica* implementation of the algorithm for determining the operators  $D_k$  and the coefficients  $a_k$ . See [Ros20].

### C.1 Construction of the operators $D_k$

In the following script, we will set a non commutative algebra of operators. To properly deal with non commutative formal operators, we will need the NCAAlgebra package (for download and further information, see the following [link](#), Copyright (c) Helton and de Oliveira 2017, All rights reserved).

We are going to follow the procedure explained in Section 2.5.2. First of all, recall the family of operators (2.25) and (2.26), (B.2) and (B.3). Those families can be compactly

described introducing the following matrix formalism: for any  $k, j \in \mathbb{Z}$ , set

$$M_{kj} = \begin{pmatrix} Q_{kj} & S_{kj} \\ P_{kj} & R_{kj} \end{pmatrix}.$$

Then, letting the 1-dimensional heat operator  $L = \partial_t - \partial_r^2$  act on the vector  $v = (I, \Lambda)^T$ , where  $I, \Lambda$  are as in Definition 2.5.1, we obtain the initial conditions for  $\{M_{kj}\}$ , cf. Lemma B.0.2, items (i) and (ii):

$$M_{10} = \begin{pmatrix} Q_{10} & S_{10} \\ P_{10} & R_{10} \end{pmatrix} = \begin{pmatrix} \Delta & -\Delta N \\ N & \Delta - N^2 \end{pmatrix}, \quad M_{11} = \begin{pmatrix} Q_{11} & S_{11} \\ P_{11} & R_{11} \end{pmatrix} = \begin{pmatrix} 0 & N \\ 0 & 0 \end{pmatrix}.$$

Studying the iterations  $L^k v$ , we obtain the following recursive formula for the family  $\{M_{kj}\}$ : for any  $j, k$  such that  $0 \leq j \leq k$ ,

$$M_{kj} = M_{10}M_{k-1,j} + M_{11}M_{k-1,j-1}$$

and  $M_{kj} = 0$  for any other couple of indexes. Having built the matrices  $\{M_{kj}\}$ , we can extract the families of operators  $\{S_{kj}\}$  and  $\{R_{kj}\}$ , which allow us to define  $A_k$  and  $Z_k$  as in (2.27). Finally, using (2.28)–(2.30) we find the operators  $\{D_k\}_{k \geq 1}$ .

## C.2 The algorithm

We present a working code for the implementation of the algorithm above in *Mathematica*. Once we have determined the operators  $D_k$ , we formally evaluate  $D_k(1)$  to obtain the integrand of the  $k$ -th coefficients, see (C.1), according to the following calculus rules:

$$\Delta(1) = 0, \quad N(1) = \Delta\delta = -\mathcal{H}, \quad (\text{C.2})$$

where  $\delta: \Omega \rightarrow [0, \infty)$  is the distance function from  $\partial\Omega$  and  $\mathcal{H}$  is its horizontal mean curvature. Note that (C.2) can be implemented treating 1 and  $\mathcal{H}$  as non-commutative symbols.

```
(*Load NC packages*)
<< NC';
<< NCAlgebra';
<< NCTex';
NCSetOutput[NonCommutativeMultiply -> True];

(*Set formal symbols to be non commutative*)
ClearAll[M, N, Δ, 1, δ, A, aux];
SetNonCommutative[Δ, N, 1, δ, H];

(*Iterative definition of the matrices Mkj*)
M[0, 0] = IdentityMatrix[2];
M[1, 0] = {{Δ, -Δ**N}, {N, -(N2 - Δ)}};
M[1, 1] = {{0, N}, {0, 0}};
M[k_, j_] :=
  Which[k >= 1 && 0 <= j <= k, NCDot[M[1, 0], M[k-1, j]] + NCDot[M[1, 1], M[k-1, j-1]],
    True, M[k, j] = {{0, 0}, {0, 0}}
  ];
```

(\*Extraction of the operators  $S_{kj}$  and  $R_{kj}$ \*)

$$\begin{aligned} S[k\_ , j\_ ] &:= \mathbf{M}[k, j][[1, 2]]; \\ R[k\_ , j\_ ] &:= \mathbf{M}[k, j][[2, 2]]; \end{aligned}$$

(\*Definition of auxiliary pairings  $\{\cdot, \cdot\}_1, \{\cdot, \cdot\}_2, \{\cdot, \cdot\}_3$ \*)

$$\begin{aligned} p1[k\_ , j\_ ] &:= \mathbf{Gamma}[k+j+1/2]/((k+j)! \mathbf{Gamma}[k+1/2]); \\ p2[k\_ , j\_ ] &:= (\mathbf{Gamma}[j+1/2] \mathbf{Gamma}[k-j+1/2])/(k!); \\ p3[k\_ , j\_ ] &:= (\mathbf{Gamma}[j+1] \mathbf{Gamma}[k-j+1/2])/\mathbf{Gamma}[k+3/2]; \end{aligned}$$

(\*Definition of  $A_k$  and  $Z_k$ \*)

$$\begin{aligned} A[k\_ ] &:= \sum_{j=0}^{k+1} (p1[k, j] S[k+j, j]); \\ Z[k\_ ] &:= \sum_{j=0}^{k-1} (p1[k, j-1] R[k+j-1, j]); \end{aligned}$$

(\*Iterative construction of the operators  $D_k$ \*)

$$\begin{aligned} \text{OpD}[1] &= 2/\mathbf{Sqrt}[\mathbf{Pi}]; \\ \text{OpD}[k\_ ] &:= \\ &\quad \mathbf{Which}[k \leq 0, \text{OpD}[k]=0, \\ &\quad \quad \mathbf{EvenQ}[k], 1/\mathbf{Sqrt}[\mathbf{Pi}] \sum_{i=1}^{k/2} (p2[k/2, i] \text{OpD}[2i-1] ** A[k/2-i]), \\ &\quad \quad \mathbf{True}, 1/\mathbf{Sqrt}[\mathbf{Pi}] \left( Z[(k+1)/2] \right. \\ &\quad \quad \quad \left. + \sum_{i=1}^{(k-1)/2} (p3[(k-1)/2, i] \text{OpD}[2i] ** A[(k-1)/2-i]) \right) \\ &\quad ]; \end{aligned}$$

(\*Formal evaluation of the operators on the function 1\*)

$$\begin{aligned} \text{aux}[k\_ ] &:= -\text{OpD}[k] ** 1 // \mathbf{NCSimplifyRational}; \\ A[k\_ ] &:= \mathbf{NCReplaceRepeated}[\text{aux}[k], \{\mathbf{N} ** 1 \rightarrow -\mathcal{H}, \Delta ** 1 \rightarrow 0\}]; \end{aligned}$$

Finally the function  $A[k\_ ]$  reproduce the integrand function of  $a_k$  for any  $k \geq 1$ .

# D

## An alternative proof of Theorem 4.3.1

---

We give an alternative proof of Theorem 4.3.1 which holds also in the non-compact case, without any additional assumption on lower Gaussian estimates.

---

Let us recall the statement of Theorem 4.3.1.

**Theorem D.0.1.** *Let  $M$  be a sub-Riemannian manifold, equipped with a smooth measure  $\omega$  and let  $\Omega \subset M$  be an open and relatively compact subset, whose boundary is smooth and has no characteristic points. Let  $x \in \partial\Omega$  and consider a chart of privileged coordinates  $\psi: U \rightarrow V \subset \mathbb{R}^n$  centered at  $x$ , such that  $\psi(U \cap \Omega) = V \cap \{z_1 > 0\}$ . Then,*

$$\lim_{t \rightarrow 0} u(t, x) = \int_{\{z_1 > 0\}} \hat{p}_1^x(0, z) d\hat{\omega}^x(z) = \frac{1}{2}, \quad \forall x \in \partial\Omega,$$

where  $\hat{\omega}^x$  denotes the nilpotentization of  $\omega$  at  $x$  and  $\hat{p}_t^x$  denotes the heat kernel associated with the nilpotent approximation of  $M$  at  $x$  and measure  $\hat{\omega}^x$ .

*Proof of Theorem 4.3.1.* Let  $p_t(x, y)$  be the heat kernel of  $M$  defined in (1.17), then we may write

$$u(t, x) = e^{t\Delta} \mathbf{1}_\Omega(x) = \int_\Omega p_t(x, y) d\omega(y), \quad \forall x \in M.$$

For a fixed  $x \in M$ , denoting by  $U$  any relatively compact neighborhood of  $x$ , we have

$$\begin{aligned} u(t, x) &= \int_{U \cap \Omega} p_t(x, y) d\omega(y) + \int_{\Omega \setminus U} p_t(x, y) d\omega(y) \\ &= \int_{U \cap \Omega} p_t(x, y) d\omega(y) + o(1), \end{aligned} \tag{D.1}$$

as  $t \rightarrow 0$ . Indeed, since  $x \notin \Omega \setminus U$ , we have

$$\int_{\Omega \setminus U} p_t(x, y) d\omega(y) = e^{t\Delta} \mathbf{1}_{\Omega \setminus U}(x) \xrightarrow{t \rightarrow 0} 0,$$

having used the fact that  $x \in U$  is a point of continuity of  $\mathbf{1}_{\Omega \setminus U}$ . Now, for  $x \in \partial\Omega$ , fix the set of privileged coordinates  $\psi: U \rightarrow V \subset \mathbb{R}^n$ , defined in (4.16). Following the

notation of Section 1.4, let  $W_1, W_2 \subset \mathbb{R}^n$  open and relatively compact subsets of  $\mathbb{R}^n$  such that  $\overline{W_1} \subset W_2 \subset \overline{W_2} \subset V$  and  $\delta_\varepsilon(W_i) \subset W_i$ , for  $i = 1, 2$  and for every  $\varepsilon \leq \varepsilon_0$ . Consider  $\chi \in C_c^\infty(\mathbb{R}^n)$  such that  $\chi \equiv 1$  on  $W_1$  and  $\text{supp}(\chi) \subset W_2$ , as in (1.24). Define  $Y_i \in \Gamma(\mathbb{R}^n)$

$$Y_i = \chi(\psi_* X_i), \quad \forall i = 1, \dots, N,$$

where  $\{X_1, \dots, X_N\}$  is a generating frame (1.1) for the sub-Riemannian structure at the point  $x$ . Moreover, consider also the coordinate representation of  $\omega$ , i.e. the smooth measure  $\mu$  on  $\mathbb{R}^n$  as in (1.26).

Then, denoting by  $\tilde{\Delta}$  the sub-Laplacian associated with the family  $\{Y_1, \dots, Y_N\}$  and with the measure  $\mu$ , and by  $\tilde{p}_t$  its heat kernel, we have

$$p_t(z, z') = \tilde{p}_t(z, z') + O(t^\infty), \quad \text{as } t \rightarrow 0,$$

uniformly on the compact subsets of  $W_1$ , cf. [CdVHT20, Thm. 3.1]. Therefore, using again (D.1), we may restrict the domain of integration to  $W \subset W_1$ , a relatively compact neighborhood of  $0 \in \mathbb{R}^n$ , obtaining

$$\lim_{t \rightarrow 0} \int_{V \cap \{z_1 > 0\}} p_t(0, z) d\omega(z) = \lim_{t \rightarrow 0} \int_{W \cap \{z_1 > 0\}} \tilde{p}_t(0, z) d\mu(z). \quad (\text{D.2})$$

Starting from the family  $\{Y_i\}$  and the measure  $\mu$ , for any  $|\varepsilon| \leq \varepsilon_0$ , define the associated perturbed family and measure:

$$\mu^\varepsilon = \frac{1}{|\varepsilon|^\mathcal{Q}} \delta_\varepsilon^* \mu, \quad Y_i^\varepsilon = \varepsilon(\delta_{\frac{1}{\varepsilon}})_* Y_i, \quad \forall i = 1, \dots, N,$$

where  $\mathcal{Q}$  is the homogeneous dimension of  $M$  at  $x$ , defined in (1.20) and  $\delta_\varepsilon$  is the dilation (1.23). The family  $\{Y_i^\varepsilon\}$  generates a sub-Riemannian structure on  $\delta_{1/\varepsilon}(W_1)$  and, denoting by  $p_t^\varepsilon(z, z')$  its heat kernel computed with respect to the measure  $\mu^\varepsilon$ , we have the following homogeneity property:

$$p_t^\varepsilon(z, z') = \varepsilon^\mathcal{Q} \tilde{p}_{\varepsilon^2 t}(\delta_\varepsilon(z), \delta_\varepsilon(z')), \quad \forall (t, z, z') \in (0, \infty) \times W_1 \times W_1. \quad (\text{D.3})$$

Furthermore, notice that, by definition of perturbed family and measure, we can rewrite (1.25) and (1.27) as follows: for any  $i = 1, \dots, N$ ,

$$\widehat{X}_i^x = \lim_{\varepsilon \rightarrow 0} Y_i^\varepsilon \quad \text{and} \quad \widehat{\omega}^x = \lim_{\varepsilon \rightarrow 0} \mu^\varepsilon, \quad (\text{D.4})$$

in the  $C^\infty$  topology of  $\mathbb{R}^n$  and the weak\* topology of  $C_c'(\mathbb{R}^n)$ , respectively. Now, to evaluate the limit (D.2), introduce the change of variables  $z \mapsto \delta_{1/\sqrt{t}}(z)$ , for  $t > 0$ . Thus, since the set  $\{z_1 > 0\}$  remains unchanged under anisotropic dilations,

$$\begin{aligned} \int_{W \cap \{z_1 > 0\}} \tilde{p}_t(0, z) d\mu(z) &= \int_{\delta_{\frac{1}{\sqrt{t}}}(W) \cap \{z_1 > 0\}} \tilde{p}_t(0, \delta_{\sqrt{t}}(z)) d(\delta_{\sqrt{t}}^* \mu)(z) \\ &= \int_{\delta_{\frac{1}{\sqrt{t}}}(W) \cap \{z_1 > 0\}} t^{\mathcal{Q}/2} \tilde{p}_t(0, \delta_{\sqrt{t}}(z)) d\mu^{\sqrt{t}}(z) \\ &= \int_{\delta_{\frac{1}{\sqrt{t}}}(W) \cap \{z_1 > 0\}} p_1^{\sqrt{t}}(0, z) d\mu^{\sqrt{t}}(z), \end{aligned} \quad (\text{D.5})$$

where, in the last passage, we have used the homogeneity property (D.3). Let us evaluate the limit:

$$\lim_{t \rightarrow 0} \int_{\delta_{\frac{1}{\sqrt{t}}}(W) \cap \{z_1 > 0\}} p_1^{\sqrt{t}}(0, z) d\mu^{\sqrt{t}}(z).$$



First of all, by (D.4), the sequence of smooth measures  $\{\mu^\varepsilon\}_{\varepsilon>0}$  converges, in the weak\* topology of  $C'_c(\mathbb{R}^n)$ , to the smooth measure  $\hat{\omega}$ . Moreover, at level of densities, this convergence is stronger indeed, since  $\mu^\varepsilon, \hat{\omega} \ll dz$  (here  $dz$  is the Lebesgue measure) and are smooth measures, there exist smooth positive functions  $f_\varepsilon, \hat{f}$  such that

$$\hat{\omega} = \hat{f}dz, \quad \mu_\varepsilon = f_\varepsilon dz,$$

for any  $|\varepsilon| \leq \varepsilon_0$ . The sequence of smooth densities  $\{f_\varepsilon\}$  converges to  $\hat{f}$ , uniformly on the compact subsets of  $\mathbb{R}^n$ , with all the derivatives. Second of all, since the heat kernel  $p_1^\varepsilon(0, z)$  is associated with the family  $\{Y_i^\varepsilon\}$ , we have the following point-wise convergence,  $\forall z \in \mathbb{R}^n$ :

$$\mathbb{1}_{\delta_{\frac{1}{\varepsilon}}(W) \cap \{z_1 > 0\}} p_1^\varepsilon(0, z) f_\varepsilon(z) \xrightarrow{\varepsilon \rightarrow 0} \mathbb{1}_{\{z_1 > 0\}} \hat{p}_1(0, z) \hat{f}(z), \quad (\text{D.6})$$

where  $\hat{p}$ , defined in (1.29), is the heat kernel of  $\widehat{M}^x$  the nilpotent approximation of  $M$  at  $x$ . This convergence follows from an application of [CdVHT20, Thm. 3.2] and it is only point-wise, since, for any  $z \in \mathbb{R}^n$  we have to fix  $\varepsilon'_0$  so small that  $z \in \delta_{1/\varepsilon'_0}(W)$  and then apply the result. Finally, for a sequence of times  $t_k \rightarrow 0$ , define the following sequence of positive functions:

$$h_k(z) = \mathbb{1}_{\delta_{\frac{1}{\sqrt{t_k}}}(W) \cap \{z_1 > 0\}} p_1^{\sqrt{t_k}}(0, z) f_{\sqrt{t_k}}(z), \quad \forall z \in \mathbb{R}^n$$

whose point-wise limit  $h(z)$  is given in (D.6), then, we can apply Fatou's Lemma, obtaining:

$$\begin{aligned} \int_{\{z_1 > 0\}} \hat{p}_1(0, z) d\hat{\omega}(z) &\leq \liminf_{k \rightarrow \infty} \int_{\delta_{\frac{1}{\sqrt{t_k}}}(W) \cap \{z_1 > 0\}} p_1^{\sqrt{t_k}}(0, z) d\mu^{\sqrt{t_k}}(z) \\ &= \liminf_{k \rightarrow \infty} \int_{U \cap \Omega} p_{t_k}(x, y) d\omega(y), \end{aligned} \quad (\text{D.7})$$

having used the homogeneity property (D.5) and relation (D.2) in the above equality. Now we can exchange the role of  $\Omega$  and its complement: in particular, by construction of the set of privileged coordinates, we have

$$\psi(U \cap \Omega^c) = V \cap \{z_1 \leq 0\},$$

thus, repeating the above argument, we can prove (D.7) for  $U \cap \Omega^c$ , i.e.

$$\begin{aligned} \int_{\{z_1 \leq 0\}} \hat{p}_1(0, z) d\hat{\omega}(z) &\leq \liminf_{k \rightarrow \infty} \int_{\delta_{\frac{1}{\sqrt{t_k}}}(W) \cap \{z_1 \leq 0\}} p_1^{\sqrt{t_k}}(0, z) d\mu^{\sqrt{t_k}}(z) \\ &= \liminf_{k \rightarrow \infty} \int_{U \cap \Omega^c} p_{t_k}(x, y) d\omega(y) \\ &= \liminf_{k \rightarrow \infty} \left( \int_U p_{t_k}(x, y) d\omega(y) - \int_{U \cap \Omega} p_{t_k}(x, y) d\omega(y) \right) \\ &\leq \liminf_{k \rightarrow \infty} \left( 1 - \int_{U \cap \Omega} p_{t_k}(x, y) d\omega(y) \right), \end{aligned}$$

having used the maximum principle in last inequality. Therefore, we obtain:

$$\int_{\{z_1 \leq 0\}} \hat{p}_1(0, z) d\hat{\omega}(z) \leq \liminf_{k \rightarrow \infty} \left( 1 - \int_{U \cap \Omega} p_{t_k}(x, y) d\omega(y) \right). \quad (\text{D.8})$$

Putting together estimates (D.7) and (D.8), and using the stochastic completeness of the nilpotent approximation, we conclude that

$$\begin{aligned}
 \int_{\{z_1 > 0\}} \hat{p}_1(0, z) d\hat{\omega}(z) &\leq \liminf_{k \rightarrow \infty} \int_{U \cap \Omega} p_{t_k}(x, y) d\omega(y) \leq \limsup_{k \rightarrow \infty} \int_{U \cap \Omega} p_{t_k}(x, y) d\omega(y) \\
 &\leq 1 - \int_{\{z_1 \leq 0\}} \hat{p}_1(0, z) d\hat{\omega}(z) \\
 &= \int_{\{z_1 > 0\}} \hat{p}_1(0, z) d\hat{\omega}(z).
 \end{aligned} \tag{D.9}$$

Finally, we can evaluate the quantity (D.9), using the homogeneity properties (1.31) of  $\hat{p}$ , as we did in the original proof.  $\square$

# E

## Iterated Duhamel's principle for $I_\Omega\phi(t, 0)$

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We study the iterated Duhamel's principle for the  $I_\Omega\phi$ , cf. Definition 4.4.7. The main result is Lemma B.0.6, which will imply formulas (4.45), (4.53) and (4.63).

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We want to apply Proposition B.0.1 to the function  $I_\Omega\phi(t, 0)$ , thus, we study in detail the operators  $L^k I_\Omega$ , for any  $k \geq 1$ . Define iteratively the family of matrices of operators, acting on smooth functions:

$$M_{kj} = \begin{pmatrix} Q_{kj} & S_{kj} \\ P_{kj} & R_{kj} \end{pmatrix},$$

as follows. Set

$$M_{10} = \begin{pmatrix} \Delta & \Delta N_r \\ -N_r & -N_r^2 + \Delta \end{pmatrix} \quad \text{and} \quad M_{11} = \begin{pmatrix} 0 & -N_r \\ 0 & 0 \end{pmatrix},$$

and, for all  $k \geq 1$  and  $0 \leq j \leq k$ , set

$$M_{kj} = M_{10}M_{k-1,j} + M_{11}M_{k-1,j-1}, \tag{E.1}$$

while  $M_{kj} = 0$ , for all other values of the indices, i.e.  $k < 0$ ,  $j < 0$  or  $k < j$ . Here  $N_r$  is the operator defined in (4.38), namely

$$N_r\phi = 2g(\nabla\phi, \nu_r) + \phi \operatorname{div}_\omega(\nu_r), \quad \forall \phi \in C_c^\infty(M), \tag{E.2}$$

where  $\nu_r$  is the outward-pointing normal from  $\Omega_r$ .

Recall the definition of  $I_\Omega$  and  $\Lambda_\Omega$ : for any  $\phi \in C_c^\infty(\Omega_{-r_0}^{r_0})$  and for all  $t > 0$ ,  $r \geq 0$ ,

$$I_\Omega\phi(t, r) = \int_{\Omega_r} (1 - u(t, x)) \phi(x) d\omega(x),$$

$$\Lambda_\Omega\phi(t, r) = -\partial_r I_\Omega\phi(t, r) = - \int_{\partial\Omega_r} (1 - u(t, y)) \phi(y) d\sigma(y),$$

where  $u(t, \cdot) = e^{t\Delta}\mathbf{1}_\Omega$ . Iterations of  $L^k I_\Omega\phi$  satisfy the following lemma.

**Lemma E.0.1.** *Let  $M$  be a sub-Riemannian manifold, equipped with a smooth measure  $\omega$ , and let  $\Omega \subset M$  be an open relatively compact subset whose boundary is smooth and has no characteristic points. Then, as operators on  $C_c^\infty(\Omega_{-r_0}^{r_0})$ , we have:*

$$(i) \quad LI_\Omega = I_\Omega\Delta - \Lambda_\Omega N_r;$$

$$(ii) \quad L\Lambda_\Omega = \Lambda_\Omega(-N_r^2 + \Delta) - \partial_t I_\Omega N_r + I_\Omega\Delta N_r;$$

(iii) For any  $k \in \mathbb{N}$ ,

$$L^k I_\Omega = \sum_{j=0}^k \frac{\partial^j}{\partial t^j} (\Lambda_\Omega P_{kj} + I_\Omega Q_{kj}) \quad \text{and} \quad L^k \Lambda_\Omega = \sum_{j=0}^k \frac{\partial^j}{\partial t^j} (\Lambda_\Omega R_{kj} + I_\Omega S_{kj}).$$

Here we mean that, for any  $\phi \in C_c^\infty(\Omega_{-r_0}^{r_0})$ , the operator  $L^k$  acts on the functions  $I_\Omega\phi(t, r)$ ,  $\Lambda_\Omega\phi(t, r)$ . Analogously the right-hand side when evaluated in  $\phi$  is a function of  $(t, r)$ .

*Proof.* The proof of items (i) and (ii) follows from Proposition 4.4.2 and the divergence theorem, see Lemma B.0.2 for a proof. We show how to recover the iterative law (E.1).

Consider the vector  $V = (I_\Omega, \Lambda_\Omega)$ , then by items (i) and (ii), we have

$$LV = (LI_\Omega, L\Lambda_\Omega) = VM_{10} + \partial_t VM_{11}. \quad (\text{E.3})$$

Notice that the operator  $L^k$  contains at most  $k$  derivatives with respect to  $t$ , therefore we have

$$L^k V = \sum_{j=0}^k \partial_t^j (VM_{kj}), \quad \forall k \geq 0,$$

On the other hand, we can evaluate  $L^k V$ , using (E.3):

$$\begin{aligned} L^k V &= L(L^{k-1}V) = \sum_{j=0}^{k-1} L\partial_t^j (VM_{k-1,j}) = \sum_{j=0}^{k-1} \partial_t^j (LVM_{k-1,j}) \\ &= \sum_{j=0}^{k-1} \partial_t^j VM_{10}M_{k-1,j} + \sum_{j=0}^{k-1} \partial_t^{j+1} VM_{11}M_{k-1,j}. \end{aligned}$$

Reorganizing the sum, we find (E.1), concluding the proof.  $\square$

We want to apply Proposition B.0.1 to  $I_\Omega\phi(t, r)$  for  $k \geq 2$ , in order to obtain higher-order asymptotics. However, Lemma E.0.1 shows that  $L^k I_\Omega$ , for  $k \geq 2$ , involves time derivatives of  $u(t, x)$  which are not well-defined at  $\partial\Omega$  as  $t \rightarrow 0$ . Therefore, we consider the following approximation of  $I_\Omega\phi$  and  $\Lambda_\Omega\phi$ , respectively: fix  $\epsilon > 0$  and define, for any  $t > 0, r \geq 0$ ,

$$\begin{aligned} I_\epsilon\phi(t, r) &= \int_{\Omega_r} (1 - u_\epsilon(t, x)) \phi(x) d\omega(x), \\ \Lambda_\epsilon\phi(t, r) &= -\partial_r I_\epsilon\phi(t, r) = \int_{\partial\Omega_r} (1 - u_\epsilon(t, x)) \phi(y) d\sigma(y), \end{aligned}$$

where  $u_\epsilon(t, x) = e^{t\Delta}\mathbf{1}_{\Omega_\epsilon}(x)$ . We recall that, for any  $a \in \mathbb{R}$ ,  $\Omega_a = \{x \in M \mid \delta(x) > a\}$ . Notice that, by the dominated convergence theorem, we have

$$I_\epsilon\phi(t, 0) \xrightarrow{\epsilon \rightarrow 0} I_\Omega\phi(t, 0), \quad \text{uniformly on } [0, T],$$

and, in addition, Lemma E.0.1 holds unchanged also for  $I_\epsilon$  and  $\Lambda_\epsilon$ .

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**Lemma E.0.2.** *Let  $M$  be compact a sub-Riemannian manifold, equipped with a smooth measure  $\omega$ , and let  $\Omega \subset M$  be an open subset whose boundary is smooth and has no characteristic points. Let  $\psi \in C^\infty([0, \infty))$ ,  $\epsilon \in (0, r_0)$  and define*

$$\psi^{(-1)}(r) = \int_0^r \psi(s) ds, \quad \forall r \geq 0.$$

*Then, for any  $\phi \in C_c^\infty(\Omega_{-r_0}^{r_0})$ , the following identities hold:*

$$(i) \lim_{t \rightarrow 0} \int_0^\infty \frac{\partial^j}{\partial t^j} \Lambda_\epsilon \phi(t, r) \psi(r) dr = \begin{cases} \int_{\Omega_\epsilon^0} \phi(\psi \circ \delta) d\omega & \text{if } j = 0, \\ - \int_{\Omega_\epsilon} \Delta^j(\phi(\psi \circ \delta)) d\omega & \text{if } j \geq 1; \end{cases}$$

$$(ii) \lim_{t \rightarrow 0} \int_0^\infty \frac{\partial^j}{\partial t^j} I_\epsilon \phi(t, r) \psi(r) dr = \begin{cases} \int_{\Omega_\epsilon^0} \phi(\psi^{(-1)} \circ \delta) d\omega & \text{if } j = 0, \\ - \int_{\Omega_\epsilon} \Delta^j(\phi(\psi^{(-1)} \circ \delta)) d\omega & \text{if } j \geq 1; \end{cases}$$

$$(iii) \frac{\partial^j}{\partial t^j} \Lambda_\epsilon \phi(0, 0) = \begin{cases} \int_{\partial\Omega} \phi d\sigma & \text{if } j = 0, \\ 0 & \text{if } j \geq 1; \end{cases}$$

$$(iv) \frac{\partial^j}{\partial t^j} I_\epsilon \phi(0, 0) = \begin{cases} \int_{\Omega_\epsilon^0} \phi d\omega & \text{if } j = 0, \\ - \int_{\Omega_\epsilon} \Delta^j \phi d\omega & \text{if } j \geq 1; \end{cases}$$

where, we recall,  $\Omega_\epsilon = \{x \in M \mid \delta(x) > \epsilon\}$  and  $\Omega_\epsilon^0 = \Omega \setminus \Omega_\epsilon$ .

*Remark E.0.3.* The only difference with respect to Lemma B.0.4 is item (iii), which now holds only as  $t \rightarrow 0$  and not for all positive times.

*Proof of Lemma E.0.2.* We claim that, for any  $j \geq 1$ ,

$$\lim_{t \rightarrow 0} \int_\Omega \phi(x) \Delta^j u_\epsilon(t, x) d\omega(x) = \int_{\Omega_\epsilon} \Delta^j \phi(x) d\omega(x). \quad (\text{E.4})$$

Let us prove it by induction: for  $j = 1$ , applying the divergence theorem, we have

$$\int_\Omega \phi \Delta u_\epsilon d\omega = - \int_{\partial\Omega} \phi g(\nabla u_\epsilon, \nabla \delta) d\sigma + \int_{\partial\Omega} u_\epsilon g(\nabla \phi, \nabla \delta) d\sigma + \int_\Omega u_\epsilon \Delta \phi d\omega. \quad (\text{E.5})$$

Let us discuss the first term in (E.5): by divergence theorem applied with respect to the set  $\Omega^c$ , we have

$$\int_{\partial\Omega} \phi g(\nabla u_\epsilon, \nabla \delta) d\sigma = \int_{\Omega^c} \phi \Delta u_\epsilon d\omega + \int_{\partial\Omega} u_\epsilon g(\nabla \phi, \nabla \delta) d\sigma - \int_{\Omega^c} u_\epsilon \Delta \phi d\omega, \quad (\text{E.6})$$

then, using [JSC86, Thm. 3] and noticing that  $d_{\text{SR}}(x, y) \geq \epsilon$ , for any  $x \in \Omega_\epsilon$  and  $y \in \Omega^c$ , we conclude that in the limit as  $t \rightarrow 0$ , (E.6) converges to 0. This proves (E.4), for  $j = 1$ . For  $j > 1$ , proceeding by induction, we conclude. Finally, using the co-area formula (4.30), we complete the proof of the statement as in the usual argument of [Sav98, Lem. 5.6].  $\square$

*Remark E.0.4.* In the non-compact case, under the assumption of Theorem 4.7.3, the above lemma holds. In particular, on the one hand, the divergence theorem holds since  $\phi$  has compact support. On the other hand, notice that the time derivative estimates (4.72) are enough to ensure that (E.6) converges to 0 as  $t \rightarrow 0$ , regardless of the compactness of the set of integration. The same is true for  $j > 1$ , where higher-order time derivatives appear.

The next step is to finally apply Proposition B.0.1 to  $I_\epsilon$ , which now satisfies its assumptions, then, pass to the limit as  $\epsilon \rightarrow 0$ . The computations are long but straightforward: we report here the result at order  $t^{5/2}$ .

**Lemma E.0.5.** *Under the same assumptions of Lemma E.0.2, let  $\phi \in C_c^\infty(\Omega_{-r_0}^{r_0})$ . Then, as  $t \rightarrow 0$ , we have:*

$$\begin{aligned}
 I_\Omega\phi(t, 0) &= 2\mathcal{G}_{1-u}[\phi](t) - \frac{1}{\sqrt{\pi}} \int_0^t \mathcal{G}_{1-u}[N_0\phi](\tau)(t-\tau)^{-1/2} d\tau \\
 &\quad + \frac{1}{2\pi} \int_0^t \int_0^\tau \mathcal{G}_{1-u}[N_0^2\phi](\hat{\tau})((\tau-\hat{\tau})(t-\tau))^{-1/2} d\hat{\tau} d\tau \\
 &\quad - \frac{1}{4\pi^{3/2}} \int_0^t \int_0^\tau \int_0^{\hat{\tau}} \mathcal{G}_{1-u}[N_0^3\phi](s)((\hat{\tau}-s)(\tau-\hat{\tau})(t-\tau))^{-1/2} ds d\hat{\tau} d\tau \\
 &\quad + \frac{1}{4\sqrt{\pi}} \int_0^t \int_{\partial\Omega} (1-u(\tau, \cdot))(4\Delta - N_0^2)\phi d\sigma (t-\tau)^{1/2} d\tau \\
 &\quad - \frac{1}{4\sqrt{\pi}} \int_0^t \mathcal{G}_{1-u}[(6N_0\Delta - N_0^3 - 2\Delta N_0)\phi](\tau)(t-\tau)^{1/2} d\tau + O(t^{5/2}),
 \end{aligned} \tag{E.7}$$

where  $u(t, \cdot) = e^{t\Delta}\mathbf{1}_\Omega$  and  $\mathcal{G}_u[\phi]$  is the functional defined in (4.47). We recall that  $N_0$  is the operator defined in (E.2), associated with  $\nu_0$  the outward-pointing normal to  $\Omega$ , namely

$$N_0\phi = 2g(\nabla\phi, \nu_0) + \phi \operatorname{div}_\omega(\nu_0), \quad \forall \phi \in C^\infty(M). \tag{E.8}$$

The expression (4.45) is a direct consequence of B.0.6. Moreover, we can apply it, when the base set is chosen to be  $\Omega^c$ . Then, evaluating the difference between  $I_\Omega\phi(t, 0)$  and  $I_{\Omega^c}\phi(t, 0)$  we obtain (4.53). We use the shorthands  $I, I^c$  for  $I_\Omega$  and  $I_{\Omega^c}$  respectively.

**Lemma E.0.6.** *Under the same assumptions of Lemma E.0.2, let  $\phi \in C_c^\infty(\Omega_{-r_0}^{r_0})$ . Then, as  $t \rightarrow 0$ , we have:*

$$\begin{aligned}
 (I\phi - I^c\phi)(t, 0) &= 2\mathcal{G}_{1-2u}[\phi](t) + \frac{1}{2} \int_{\partial\Omega} N\phi d\sigma \\
 &\quad + \frac{1}{2\pi} \int_0^t \int_0^\tau \mathcal{G}_{1-2u}[N^2\phi](\hat{\tau})((\tau-\hat{\tau})(t-\tau))^{-1/2} d\hat{\tau} d\tau \\
 &\quad + \frac{1}{4\sqrt{\pi}} \int_0^t \int_{\partial\Omega} (1-2u(\tau, \cdot))(4\Delta - N^2)\phi d\sigma (t-\tau)^{1/2} d\tau + O(t^2),
 \end{aligned}$$

where  $N$  is the operator given by

$$N\phi = 2g(\nabla\phi, \nabla\delta) + \phi\Delta\delta, \quad \forall \phi \in C^\infty(M),$$

with  $\delta: M \rightarrow \mathbb{R}$  the signed distance function from  $\partial\Omega$ .

*Proof.* Firstly, we apply Lemma B.0.6 to  $I\phi$ : we obtain exactly the expression (E.7), with the operator  $N_0$  given by  $-N$ , since the outward-pointing normal to  $\Omega$  is  $-\nabla\delta$ . Secondly, for the outside contribution, recall that we have the following equality of smooth functions:

$$1 - u^c(t, x) = 1 - e^{t\Delta}\mathbf{1}_{\Omega^c}(x) = e^{t\Delta}\mathbf{1}_\Omega(x) = u(t, x), \quad \forall t > 0, x \in M.$$

Therefore, when we apply Lemma B.0.6 to  $I^c\phi$ , we replace  $1 - u^c(t, \cdot) = 1 - e^{t\Delta}\mathbf{1}_{\Omega^c}$  with the function  $u(t, \cdot) = e^{t\Delta}\mathbf{1}_\Omega(x)$ . Moreover, the operator  $N_0$  defined in (E.8) is equal to  $N$ , since the outward-pointing normal to  $\Omega^c$  is  $\nabla\delta$ . Therefore, writing the difference of the

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two contributions, and noticing that  $\Omega$  and its complement share the boundary, we have:

$$\begin{aligned}
(I\phi - I^c\phi)(t, 0) &= 2\mathcal{G}_{1-2u}[\phi](t) + \frac{1}{\sqrt{\pi}} \int_0^t \mathcal{G}_1[N\phi](\tau)(t - \tau)^{-1/2} d\tau \\
&+ \frac{1}{2\pi} \int_0^t \int_0^\tau \mathcal{G}_{1-2u}[N^2\phi](\hat{\tau})((\tau - \hat{\tau})(t - \tau))^{-1/2} d\hat{\tau} d\tau \\
&+ \frac{1}{4\pi^{3/2}} \int_0^t \int_0^\tau \int_0^{\hat{\tau}} \mathcal{G}_1[N^3\phi](s)((\hat{\tau} - s)(\tau - \hat{\tau})(t - \tau))^{-1/2} ds d\hat{\tau} d\tau \text{ (E.9)} \\
&+ \frac{1}{4\sqrt{\pi}} \int_0^t \int_{\partial\Omega} (1 - 2u(\tau, \cdot)) (4\Delta - N^2)\phi d\sigma (t - \tau)^{1/2} d\tau \\
&+ \frac{1}{4\sqrt{\pi}} \int_0^t \mathcal{G}_1[(6N\Delta - N^3 - 2\Delta N)\phi](\tau)(t - \tau)^{1/2} d\tau + O(t^{5/2}). \text{ (E.10)}
\end{aligned}$$

To conclude, it is enough to notice that the functional  $\mathcal{G}_1$  can be explicitly computed:

$$\mathcal{G}_1[\phi](t) = \frac{1}{\sqrt{\pi}} \int_{\partial\Omega} \phi d\sigma t^{1/2}, \quad \forall \phi \in C_c^\infty(\Omega_{-r_0}^{r_0}).$$

Thus, the terms in (E.9) and (E.10) are remainder of order  $O(t^2)$ .  $\square$

Applying Lemma B.0.6 to the sum of  $I_\Omega\phi(t, 0)$  and  $I_{\Omega^c}\phi(t, 0)$  instead, we obtain (4.63). We state here the result and we omit the proof, it being equal to the one of Lemma E.0.6.

**Lemma E.0.7.** *Under the same assumptions of Lemma E.0.2, let  $\phi \in C_c^\infty(\Omega_{-r_0}^{r_0})$ . Then, as  $t \rightarrow 0$ , we have:*

$$\begin{aligned}
(I\phi + I^c\phi)(t, 0) &= \frac{2}{\sqrt{\pi}} \int_{\partial\Omega} \phi d\sigma t^{1/2} + \frac{1}{\sqrt{\pi}} \int_0^t \mathcal{G}_{1-2u}[N\phi](\tau)(t - \tau)^{-1/2} d\tau \\
&+ \frac{1}{6\sqrt{\pi}} \int_{\partial\Omega} (4\Delta + N^2)\phi d\sigma t^{3/2} \\
&+ \frac{1}{4\pi^{3/2}} \int_0^t \int_0^\tau \int_0^{\hat{\tau}} \mathcal{G}_{1-2u}[N^3\phi](s)((\hat{\tau} - s)(\tau - \hat{\tau})(t - \tau))^{-1/2} ds d\hat{\tau} d\tau \\
&+ \frac{1}{4\sqrt{\pi}} \int_0^t \mathcal{G}_{1-2u}[(6N\Delta - N^3 - 2\Delta N)\phi](\tau)(t - \tau)^{1/2} d\tau + O(t^{5/2}),
\end{aligned}$$

where  $N$  is the operator given by

$$N\phi = 2g(\nabla\phi, \nabla\delta) + \phi\Delta\delta, \quad \forall \phi \in C^\infty(M).$$





# Résumé détaillé de la thèse en français

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## R.1 Introduction

Ces dernières années, la géométrie sous-riemannienne a connu un regain d'intérêt, caractérisé par une grande variété de sujets et de techniques. D'un point de vue théorique, on peut trouver des exemples notables de variétés sous-riemanniennes comme limites Gromov-Hausdorff de séquences de variétés riemanniennes, comme géométries sous-jacentes de la

diffusion hypoelliptique ou encore comme fibrés  $G$ -principaux avec une connexion horizontale. En mécanique, les variétés sous-riemanniennes modélisent des contraintes non-holonomes fournissant une bonne description de divers systèmes mécaniques, tels que des corps qui roulent sans glisser ou des systèmes à moment angulaire conservé. En outre, en modélisation mathématique, la géométrie sous-riemannienne s'avère être un outil extrêmement flexible et efficace. Par exemple, elle a été employée dans le problème de la reconstruction d'images partiellement endommagés, ou pour décrire un modèle biologique de la vision qui explique la reconnaissance des formes dans le cortex visuel.

Notre principal intérêt pour la géométrie sous-riemannienne est lié à l'analyse asymptotique des phénomènes de diffusion hypoelliptique. Tout d'abord, on montre, dans le cadre riemannien, comment l'analyse asymptotique des équations aux dérivées partielles peut être un outil puissant pour étudier les propriétés géométriques. Soit  $(M, g)$  une variété riemannienne complète, et on considère le problème classique de Cauchy pour l'équation de la chaleur sur  $M$  :

$$\begin{aligned} (\partial_t - \Delta_g)u(t, x) &= 0, & \forall (t, x) \in (0, \infty) \times M, \\ u(0, \cdot) &= f, & \text{dans } L^2(M, \mu_g), \end{aligned} \tag{R.11}$$

où  $\Delta_g, \mu_g$  désignent respectivement l'opérateur de Laplace-Beltrami et le volume riemannien. Il est bien connu qu'il existe une solution unique  $u(t, x)$  au problème (R.11), qui peut être représenté comme suit :

$$u(t, x) = \int_M p_t(x, y) f(y) d\mu_g(y), \quad \forall (t, x) \in (0, \infty) \times M,$$

où  $p_t(x, y)$  est une fonction positive lisse sur  $(0, +\infty) \times M \times M$  appelée noyau de la chaleur, voir [BGM71, Ros97] pour une introduction complète sur le sujet. Dans le même esprit que la définition de Riemann de la courbure d'une variété riemannienne, les asymptotiques en temps petit du noyau de la chaleur encodent des informations géométriques de  $(M, g)$ , dévoilant une relation profonde entre la diffusion de la chaleur et la géométrie de la structure sous-jacente. Parmi les nombreux résultats intéressants concernant l'analyse asymptotique des quantités liées à la diffusion de la chaleur, voir par exemple [Kac66, Pat71, Cha84], nous mentionnons ici le résultat séminal de Minakshisundaram et Pleijel [MP49] : pour chaque  $x \in M$ , il existe une expansion asymptotique complète

$$p_t(x, x) = \frac{1}{t^{n/2}} \left( a_0(x) + a_1(x)t + a_2(x)t^2 + \dots \right) \quad \text{as } t \rightarrow 0, \tag{R.12}$$

où  $a_j(x)$  sont des fonctions lisses sur  $M$ , et sont déterminées par le tenseur de courbure riemannienne et ses dérivées covariantes en  $x$ , donc elles reflètent la géométrie locale de  $M$  près du point  $x$ . Par exemple, le coefficient d'ordre zéro est juste une constante, mais déjà le coefficient d'ordre 1 en  $t$ , c'est-à-dire  $a_1(x)$ , est proportionnel à la courbure scalaire de  $M$  au point  $x$ .

D'un point de vue physique, l'équation de la chaleur (R.11) modélise une diffusion isotrope sur la variété  $M$ . On peut donc obtenir une généralisation naturelle de ce modèle en imposant des *contraintes non-holonomes* sur la variété, décrivant à son tour un phénomène de diffusion anisotrope. Cela revient à remplacer l'opérateur de Laplace-Beltrami  $\Delta_g$  dans (R.11) par un opérateur sous-elliptique  $\mathcal{L}$  à la Hörmander [Hör67, Str86], généré par un ensemble de contraintes non-holonomes  $\{X_1, \dots, X_m\} \subset \Gamma(TM)$ . Plus précisément,  $\mathcal{L}$  est un opérateur différentiel du second ordre défini comme la somme des carrés des champs de vecteurs  $X_1, \dots, X_m$ , plus éventuellement un terme de dérive  $X_0 \in \Gamma(TM)$ . On remarque qu'en général,  $X_1, \dots, X_m$  n'ont pas besoin d'être linéairement indépendants en tout point, ainsi le symbole de  $\mathcal{L}$  peut être dégénéré, ce qui le rend remarquablement

différent de  $\Delta_g$ , qui est au contraire un opérateur elliptique. Néanmoins, un résultat fondamental obtenu par Hörmander donne une condition suffisante garantissant l'hypoellipticité de  $\mathcal{L}$ , c'est-à-dire la *condition de Hörmander* : l'algèbre de Lie engendrée par la famille  $\{X_1, \dots, X_m\}$  en tout point  $x \in M$  doit coïncider avec l'espace tangent entier  $T_x M$ . On dit aussi que  $\{X_1, \dots, X_m\}$  est une famille de champs de vecteurs *génératrice par crochets de Lie*. D'un point de vue géométrique, contrairement à ce qui se passe pour l'opérateur de Laplace-Beltrami, la structure géométrique sous-jacente d'un tel opérateur  $\mathcal{L}$  n'est plus riemannienne. En effet, il est possible de donner une notion de longueur des vecteurs tangents mais uniquement le long des directions admissibles dictées par la famille  $\{X_1, \dots, X_m\}$ . La structure résultante est ce que l'on appelle une variété sous-riemannienne.

Les variétés sous-riemanniennes constituent une classe riche d'espaces métriques, généralisant le concept classique de variété riemannienne. Une structure sous-riemannienne sur  $M$  est définie par un couple  $(\mathcal{D}, g)$ , où  $g$  est un produit scalaire lisse défini sur un sous-ensemble de directions admissibles :

$$\mathcal{D}_x = \text{span}\{X_1(x), \dots, X_m(x)\} \subset T_x M, \quad \forall x \in M,$$

qui est appelée *distribution* et doit satisfaire la condition de Hörmander, voir les monographies [Mon02, ABB20]. Puisque  $g$  n'est défini que sur un sous-ensemble de directions privilégiées, toutes les courbes lisses n'ont pas une longueur finie dans une variété sous-riemannienne. Par conséquent, on doit se restreindre à l'ensemble des *courbes horizontales*, à savoir les courbes qui sont tangentes à la distribution et de longueur finie. Le théorème de connectivité de Chow-Rashevsky garantit que deux points quelconques (dans la même composante connexe de  $M$ ) peuvent être reliés par une courbe horizontale. Ainsi, la fonction de distance sous-riemannienne  $d_{SR}$  entre deux points quelconques, obtenue comme l'infimum de la longueur, parmi toutes les courbes horizontales les reliant, est finie. L'espace métrique résultant  $(M, d_{SR})$  présente des phénomènes remarquablement différents par rapport au cas riemannien : premièrement, il existe des points où il n'est pas bi-lipschitzien à un espace euclidien, et par conséquent sa dimension de Hausdorff est strictement supérieure à la dimension topologique dans un voisinage de ces points. Deuxièmement,  $(M, d_{SR})$  est un espace métrique géodésique comme dans le cas riemannien, cependant toutes les géodésiques ne satisfont pas une équation de Hamilton, ce qui conduit à l'existence de *géodésiques anormales*, qui sont liés à plusieurs problèmes ouverts dans le domaine. Troisièmement, il n'y a pas de manière canonique de définir une mesure intrinsèque associée à une variété sous-riemannienne, en général, puisque  $g$  n'est pas défini sur l'espace tangent entier, donc la procédure riemannienne ne produit pas une forme de volume bien définie.

Par conséquent, pour un opérateur hypoelliptique  $\mathcal{L}$  donné sur une variété lisse  $M$ , on considère la structure sous-riemannienne sous-jacente et on étudie sa diffusion : une fois qu'une mesure extérieure lisse  $\omega$  est fixée, on peut considérer l'équation de la chaleur associée à  $\mathcal{L}$ ,

$$\begin{aligned} (\partial_t - \mathcal{L})u(t, x) &= 0, & \forall (t, x) \in (0, \infty) \times M, \\ u(0, \cdot) &= f, & \text{dans } L^2(M, \omega). \end{aligned} \tag{R.13}$$

L'hypoellipticité de  $\mathcal{L}$  assure l'existence d'un noyau de la chaleur hypoelliptique, que l'on désigne toujours par  $p_t(x, y)$ , cependant, l'interaction entre la géométrie de la structure sous-riemannienne sur  $M$  et la diffusion régie par  $\mathcal{L}$  est moins bien comprise que dans le cas riemannien. En premier lieu, la diffusion est affectée par le choix de la mesure  $\omega$ , car  $\mathcal{L}$  peut ne pas être symétrique par rapport au produit scalaire  $L^2(M, \omega)$  : on sera intéressé en particulier à la diffusion hypoelliptique régie par un opérateur hypoelliptique spécifique, appelé *sous-laplacien* et noté  $\Delta$ , qui est en effet symétrique dans  $L^2(M, \omega)$  et généralise directement l'opérateur de Laplace-Beltrami, lorsque  $M$  est riemannien et  $\omega = \mu_g$ . Deuxièmement, dans le cadre sous-riemannien général, il n'existe pas d'analogue

du tenseur de courbure riemannienne, ce qui rend le problème d'intérêt plus difficile à résoudre. Une définition de la courbure dans le cadre sous-riemannien a été récemment proposée dans [ABR18] : l'approche utilisée par les auteurs rappelle celle de Riemann, et donc il est directement liée à l'asymptotique de la distance sous-riemannienne. L'inconvénient de cette approche est que les calculs explicites sont beaucoup plus difficiles par rapport au cas riemannien.

Un grand effort dans l'étude des propriétés analytiques de la diffusion de la chaleur (R.13) a été réalisé par plusieurs auteurs et une liste non exhaustive est [BA88, BA89, Léa92, Nee07, BM09, BBN12, Bar13, BBN19, CdVHT20]. Par exemple, pour ce qui concerne la trace du noyau de la chaleur, l'existence d'une expansion asymptotique complète est prouvée dans [BA89], cependant l'interprétation géométrique des coefficients fait toujours défaut. À cet égard, un résultat notable est contenu dans [Bar13], où l'auteur est capable de calculer explicitement les deux premiers coefficients de l'asymptotique de la trace du noyau de la chaleur, dans les variétés sous-riemanniennes contact 3D :

$$p_t(x, x) = \frac{1}{16t^2} (1 + \kappa(x)t + o(t)), \quad \text{as } t \rightarrow 0, \forall x \in M, \quad (\text{R.14})$$

où  $\kappa(x)$  est un invariant de type "courbure" de la structure sous-riemannienne. D'une part, (R.14) est singulier en  $t$  d'ordre  $Q/2$ , où  $Q$  est la dimension de Hausdorff de la variété et, dans le cas du contact 3D, est égal à 4 et supérieur à la dimension topologique. D'autre part, en comparant (R.14) avec le résultat riemannien correspondant (R.12), on peut s'attendre à un invariant qui ressemble à la courbure scalaire d'une variété riemannienne, on peut donc *définir*  $\kappa(x)$  comme la courbure sous-riemannienne scalaire de  $M$  au point  $x$ . De plus, on mentionne également les développements récents de [CdVHT20], qui concerne l'existence d'une expansion asymptotique en temps petit pour les noyaux de la chaleur hypoelliptiques, et la description de ses coefficients en termes d'approximation nilpotente de la structure sous-riemannienne sous-jacente.

Dans cette optique, notre objectif principal est de développer la relation entre la diffusion hypoelliptique de la chaleur et les invariants géométriques de la structure sous-riemannienne, en mettant l'accent sur les propriétés géométriques des hypersurfaces plongées.

## R.2 Préliminaires de géométrie sous-riemannienne

On rappelle quelques faits essentiels de géométrie sous-riemannienne. Pour plus de détails, nous renvoyons le lecteur à [ABB20].

### R.2.1 Variétés sous-riemanniennes

Soit  $M$  une variété lisse, connexe et de dimension finie. Une structure sous-riemannienne sur  $M$  est définie par un ensemble de  $N$  champs des vecteurs lisses globaux  $X_1, \dots, X_N$ , que on appelle *famille génératrice*. La famille génératrice définit une *distribution* de sous-espaces de l'espace tangent en chaque point  $x \in M$ , donnée par

$$\mathcal{D}_x = \text{span}\{X_1(x), \dots, X_N(x)\} \subseteq T_x M. \quad (\text{R.15})$$

On suppose que la distribution est *génératrice par crochets de Lie*, à savoir que l'algèbre de Lie des champs de vecteurs lisses engendrés par  $X_1, \dots, X_N$ , évalués au point  $x$ , coïncide avec  $T_x M$ , pour tout  $x \in M$ . On rappelle que, par définition, l'algèbre de Lie engendrée

par une famille de champs de vecteurs  $\{X_1, \dots, X_N\}$  est la plus petite sous-algèbre de  $\Gamma(TM)$  contenant la famille elle-même, soit

$$\text{Lie}\{X_1, \dots, X_N\} = \{[X_{j_1}, [\dots [X_{j_{k-1}}, X_{j_k}]]] \mid j_i \in \{1, \dots, N\}, k \in \mathbb{N}\}.$$

Donc, la distribution (R.15) est génératrice par crochets de Lie si

$$\text{Lie}_x(\mathcal{D}) = \{X(x) \mid X \in \text{Lie}(\mathcal{D})\} = T_x M, \quad \forall x \in M. \quad (\text{R.16})$$

On se réfère parfois à (R.16) comme *condition d'Hörmander*. La famille génératrice induit également une norme sur la distribution en  $x$ , à savoir

$$\|v\|_g = \inf \left\{ \sum_{i=1}^N u_i^2 \mid \sum_{i=1}^N u_i X_i(x) = v \right\}, \quad \forall v \in \mathcal{D}_x, \quad (\text{R.17})$$

qui, à son tour, définit un produit interne sur  $\mathcal{D}_x$  par polarisation, noté  $g_x(\cdot, \cdot)$ . On remarque que les champs de vecteurs dans la famille génératrice (R.15) n'ont pas besoin d'être linéairement indépendants en tout point, de plus, nous pouvons facilement observer que si  $\mathcal{D}_x = T_x M$ , pour tout  $x \in M$ , alors  $g$  défini par (R.17) est une métrique riemannienne sur  $M$ .

On dit que  $\gamma : [0, T] \rightarrow M$  est une *courbe horizontale*, si elle est absolument continue et

$$\dot{\gamma}(t) \in \mathcal{D}_{\gamma(t)}, \quad \text{pour p.t. } t \in [0, T].$$

Ceci implique qu'il existe une *fonction de contrôle*  $u : [0, T] \rightarrow \mathbb{R}^N$ , telle que

$$\dot{\gamma}(t) = \sum_{i=1}^N u_i(t) X_i(\gamma(t)), \quad \text{pour p.t. } t \in [0, T].$$

De plus, on exige que  $u \in L^2([0, T], \mathbb{R}^N)$ . On rappelle ici [ABB20, Lem. 3.12], qui assure l'intégrabilité de la fonction  $t \mapsto \|\dot{\gamma}(t)\|_g$ .

**Lemme R.2.1.** *Soit  $M$  une variété sous-riemannienne, et soit  $\gamma : [0, T] \rightarrow M$  une courbe horizontale. On définit le contrôle minimal de  $\gamma$  par*

$$u^*(t)^2 = \operatorname{argmin} \left\{ \sum_{i=1}^N u_i(t)^2 \mid \dot{\gamma}(t) = \sum_{i=1}^N u_i(t) X_i(\gamma(t)) \right\}, \quad \text{pour p.t. } t \in [0, T]. \quad (\text{R.18})$$

Alors, la fonction  $[0, T] \ni t \mapsto u^*(t)$  est mesurable et son carré est intégrable sur  $[0, T]$ .

On remarque que, par définition du contrôle minimal,  $\|\dot{\gamma}(t)\|_g = |u^*(t)|$  pour presque tout  $t \in [0, T]$ , donc la fonction  $t \mapsto \|\dot{\gamma}(t)\|_g$  est intégrable sur  $[0, T]$  et on peut définir la *longueur* d'une courbe horizontale comme suit :

$$\ell(\gamma) = \int_0^T \|\dot{\gamma}(t)\|_g dt.$$

Ensuite, la *distance sous-riemannienne* est définie par la procédure de minimisation de longueur : pour tout  $x, y \in M$ , on définit

$$d_{\text{SR}}(x, y) = \inf \{ \ell(\gamma) \mid \gamma : [0, T] \rightarrow M, \text{ courbe horizontale joignant } x \text{ et } y \}.$$

Par le théorème de Chow-Rashevsky, l'hypothèse sur la distribution d'être génératrice par crochets de Lie garantit que la distance  $d_{\text{SR}} : M \times M \rightarrow \mathbb{R}$  est finie, continue et qu'elle induit la même topologie que celle de la variété. De plus, cela implique que l'espace métrique résultant  $(M, d_{\text{SR}})$  est localement compact.

*Remarque R.2.2.* La définition ci-dessus inclut toutes les structures sous-riemanniennes classiques à rang constant comme dans [Mon02, Rif14] (où  $\mathcal{D}$  est défini un sous-fibré du fibré tangent et  $g$  un tenseur symétrique et positif sur  $\mathcal{D}$ ), mais aussi les structures sous-riemanniennes générales à rang variable. De plus, la même structure sous-riemannienne peut provenir de différentes familles génératrices.

## R.2.2 Géodésiques et flot hamiltonien

Nous rappelons quelques faits de base sur les courbes qui minimisent la longueur, et sur le formalisme hamiltonien.

Une *géodésique* est une courbe horizontale  $\gamma: [0, T] \rightarrow M$ , paramétrisée à vitesse constante, et telle que tout segment suffisamment court minimise la longueur. On peut formuler le problème de se trouver une courbe de longueur minimale joignant  $x, y \in M$  comme un problème de contrôle optimal de la forme :

$$\begin{cases} \dot{\gamma}(t) = \sum_{i=1}^N u_i(t) X_i(\gamma(t)), \\ \gamma(0) = x, \quad \gamma(T) = y, \\ \int_0^T \sqrt{\sum u_i^2(t)} dt \rightarrow \min. \end{cases} \quad (\text{R.19})$$

Le problème (R.19) consiste à trouver les minima locaux d'une fonction soumise à certaines contraintes, et donc on peut employer la méthode des multiplicateurs de Lagrange.

**Exemple R.2.3.** Soient  $f, f_0: \mathbb{R}^2 \rightarrow \mathbb{R}$  deux fonctions  $C^1$  et on considère le problème d'optimisation avec une contrainte

$$\begin{cases} f_0(x, y) \rightarrow \min, \\ f(x, y) = 0. \end{cases}$$

En appliquant la méthode des multiplicateurs de Lagrange, on introduit la fonction  $\Lambda = \lambda f + \lambda_0 f_0$ , puis on obtient les points extrêmes en résolvant l'équation suivante

$$\nabla_{(x,y),\lambda} \Lambda = 0. \quad (\text{R.20})$$

On peut donc distinguer deux cas :

- (i)  $\lambda_0 = 0$ , alors (R.20) se réduit à  $\lambda \nabla f = 0$ , avec  $\lambda \neq 0$ , ce qui implique  $\nabla f = 0$ . Ce sont des extrêmes *abnormal* ;
- (ii) Si l'on considère  $\lambda_0 \neq 0$ , on peut alors supposer que  $\lambda_0 = -1$ , d'où (R.20) devient  $\lambda \nabla f - \nabla f_0 = 0$ . Ce sont des extrêmes *normal*.

Ce type de différence entre les extrêmes d'un problème d'optimisation avec une contrainte est typique et apparaît également lors de l'étude d'un problème de dimension infinie, comme celui de la recherche des géodésiques joignant deux points fixés. La caractérisation des extrémals du problème (R.19) est donnée par le *principe du maximum de Pontryagin*, cf. [ABB20, Sec. 4.3].

Étant donné une structure sous-riemannienne sur  $M$  avec une famille génératrice  $\{X_1, \dots, X_N\}$ , on définit les fonctions  $h_i \in C^\infty(T^*M)$ , associées à ces champs de vecteurs et linéaires sur les fibres, tels que

$$h_i: T^*M \rightarrow \mathbb{R}; \quad h_i(\lambda) = \langle \lambda, X_i \rangle, \quad \forall i = 1, \dots, N,$$

où  $\langle \lambda, \cdot \rangle$  désigne l'action des covecteurs sur les vecteurs. Pour une fonction lisse  $h \in C^\infty(T^*M)$ , le *champ de vecteurs hamiltonien* associé à  $h$  est le champ de vecteurs  $\vec{h}$  sur  $T^*M$ , défini par  $\zeta(\cdot, \vec{h}) = dh$ , où  $\zeta \in \Lambda^2(T^*M)$  est la 2-forme symplectique canonique sur le fibré cotangent.

**Théorème R.2.1** (Principe du maximum de Pontryagin). *Soit  $M$  une variété sous-riemannienne, soit  $\gamma: [0, T] \rightarrow M$  une courbe horizontale paramétrisée à vitesse constante, minimisant la distance entre  $\gamma(0)$  et  $\gamma(T)$ , et soit  $u^*(t)$  le contrôle minimal correspondant. Alors il existe un relèvement de  $\gamma$  dans  $T^*M$ , c'est-à-dire une courbe absolument continue  $\lambda(t) \in T^*_{\gamma(t)}M$ , telle que,*

$$\dot{\lambda}(t) = \sum_{i=1}^N u_i^*(t) \vec{h}_i(\lambda(t)), \quad \text{pour p.t. } t \in [0, T], \quad (\text{R.21})$$

et l'une des conditions suivantes est vérifiée :

$$h_i(\lambda(t)) = u_i^*(t), \quad \forall t \in [0, T] \text{ et pour tout } i = 1, \dots, N, \text{ ou} \quad (\text{N})$$

$$h_i(\lambda(t)) = 0, \quad \forall t \in [0, T] \text{ et pour tout } i = 1, \dots, N. \quad (\text{A})$$

De plus, dans le cas (A),  $\lambda(t) \neq 0$ , pour tout  $t \in [0, T]$ .

Parmi les solutions de (R.21), on distingue les *extrémales normales*, satisfaisant (N), et les *extrémales anormales*, satisfaisant (A). En particulier, les extrémales normales sont les solutions d'un véritable système hamiltonien : on définit le *hamiltonien sous-riemannien* comme la fonction lisse  $H: T^*M \rightarrow \mathbb{R}$ , donnée par

$$H(\lambda) = \frac{1}{2} \sum_{i=1}^N \langle \lambda, X_i \rangle^2, \quad \lambda \in T^*M,$$

où  $\{X_1, \dots, X_N\}$  est une famille génératrice pour la structure sous-riemannienne. Ensuite, les solutions  $\lambda: [0, T] \rightarrow T^*M$  des *équations d'Hamilton*

$$\dot{\lambda}(t) = \vec{H}(\lambda(t)), \quad (\text{R.22})$$

sont des extrémales normales au sens du Théorème R.2.1. Par conséquent, les extrémales normales sont lisses et l'hamiltonien est constant sur les trajectoires (R.22). Leurs projections  $\gamma(t) = \pi(\lambda(t))$  sur  $M$ , où  $\pi: T^*M \rightarrow M$  est la projection canonique du fibré, sont des courbes horizontales localement minimisant, paramétrisées à vitesse constante, et sont appelées *géodésiques normales*. Si  $\gamma$  est une géodésique normale avec un relèvement extrémal normal  $\lambda$ , alors sa vitesse est donnée par  $\|\dot{\gamma}\|_g = \sqrt{2H(\lambda)}$ . Notamment

$$\ell(\gamma|_{[0,t]}) = t\sqrt{2H(\lambda(0))}, \quad \forall t \in [0, T].$$

L'exponentielle sous-riemannienne  $\exp_x: T_x^*M \rightarrow M$ , avec point de base  $x \in M$  est

$$\exp_x(\lambda) = \pi \circ e^{\vec{H}}(\lambda), \quad \lambda \in T_x^*M, \quad (\text{R.23})$$

où  $e^{\vec{H}}$  indique le flot de  $\vec{H}$  dans le fibré cotangent, et on peut supposer qu'il est défini jusqu'au temps 1. C'est le cas, par exemple, lorsque  $(M, d_{\text{SR}})$  est un espace métrique complet. On remarque aussi que l'exponentielle sous-riemannienne (R.23) est un difféomorphisme local à  $\lambda = 0$  si et seulement si  $\mathcal{D}_x = T_xM$ , donc dans une variété vraiment sous-riemannienne on trouve toujours des points dans lesquels la première propriété échoue.

En ce qui concerne les extrémales anormales  $\lambda: [0, T] \rightarrow T^*M$ , satisfaisant (R.21), ainsi que la condition (A), on définit *géodésiques anormales* leurs projections  $\gamma(t) = \pi(\lambda(t))$  sur  $M$ . À la différence des extrémales normales, la courbe  $\lambda(t)$  peut ne pas suivre la dynamique hamiltonienne (R.22), donc on ne peut en déduire aucune propriété de régularité. Ici, on observe seulement que la condition (A) pour une extrémale anormale  $\lambda(t) \in T^*M$  est équivalente à

$$\langle \lambda(t), \mathcal{D}_{\pi(\lambda(t))} \rangle = 0 \quad \text{et} \quad \lambda(t) \neq 0, \quad \forall t \in [0, T],$$

c'est-à-dire  $H(\lambda(t)) \equiv 0$ . On remarque qu'une géodésique peut être anormale et normale en même temps.



**Courbes minimisantes de la distance à une hypersurface.** On considère maintenant une sous-variété fermée et plongée  $S \subset M$  de codimension positive. Soit  $\gamma : [0, T] \rightarrow M$  une courbe horizontale, paramétrisée à vitesse constante, telle que  $\gamma(0) \in S$ ,  $\gamma(T) = x \in M \setminus S$ , et telle qu'elle minimise la distance à  $S$ , soit

$$\ell(\gamma) = \inf\{d_{\text{SR}}(z, x) \mid z \in S\}.$$

En particulier,  $\gamma$  est une géodésique. Tout relèvement normal ou anormal correspondant  $\lambda : [0, T] \rightarrow T^*M$ , doit satisfaire les conditions de transversalité [AS04, Thm. 12.4]

$$\langle \lambda(0), v \rangle = 0, \quad \forall v \in T_{\gamma(0)}S, \quad (\text{R.24})$$

en d'autres termes, le covecteur initial  $\lambda(0)$  doit appartenir au fibré annihilateur  $\mathcal{A}(S) = \{\lambda \in T^*M \mid \langle \lambda, T_{\pi(\lambda)}S \rangle = 0\}$  de  $S$ . Une conséquence immédiate pour le cas de codimension 1 est la suivante, cf. [FPR20, Prop. 2.7].

**Proposition R.2.4.** *On considère une structure sous-riemannienne sur une variété lisse  $M$ . Soit  $S \subset M$  une hypersurface plongée et soit  $\gamma : [0, T] \rightarrow M$  une courbe horizontale telle que  $\gamma(0) \in S$ ,  $\gamma(T) = p \in M \setminus S$ , et qui minimise la distance à  $S$ . Alors  $\gamma$  est anormal si et seulement si  $\gamma(0) \in S$  est un point caractéristique, c'est-à-dire*

$$\mathcal{D}_{\gamma(0)} \subset T_{\gamma(0)}S.$$

En particulier, la Proposition R.2.4 implique que, dès que  $S$  n'a pas de points caractéristiques, cf. définition (R.3.2), les courbes qui réalisent la distance à  $S$  sont toutes normales, c'est-à-dire  $\gamma(t) = \exp_x(t\lambda)$ , pour un certain  $\lambda$  unique (à reparamétrisation près) satisfaisant (R.24).

### R.2.3 Le sous-laplacien

Soit  $M$  une variété sous-riemannienne et soit  $\omega$  une mesure lisse sur  $M$ , définie par une densité tensorielle positive. La *divergence* d'un champ de vecteurs lisse est définie par

$$\text{div}_\omega(X)\omega = \mathcal{L}_X\omega, \quad \forall X \in \Gamma(TM),$$

où  $\mathcal{L}_X$  indique la dérivée de Lie dans la direction de  $X$ . Le *gradient horizontal* d'une fonction  $f \in C^\infty(M)$ , noté  $\nabla f$ , est défini comme le champ de vecteurs horizontal (c'est-à-dire tangent à la distribution en chaque point), tel que

$$g_x(\nabla f(x), v) = v(f)(x), \quad \forall v \in \mathcal{D}_x, \quad (\text{R.25})$$

où  $v$  agit sur  $f$  comme une dérivation. En termes de la famille génératrice (R.15), on a

$$\nabla f = \sum_{i=1}^N X_i(f)X_i, \quad \forall f \in C^\infty(M).$$

On rappelle le théorème de la divergence (on souligne que  $M$  n'est pas forcément orientable) : soit  $\Omega \subset M$  un sous-ensemble ouvert avec bord lisse, alors

$$\int_{\partial\Omega} fg(X, \nu) d\sigma = \int_{\Omega} (f \text{div}_\omega X + g(\nabla f, X)) d\omega, \quad (\text{R.26})$$

pour toute fonction lisse  $f$  et tout champ de vecteurs lisse  $X$ , de sorte que le champ de vecteurs  $fX$  ait support compact. Dans (R.26),  $\nu$  est le champ de vecteurs normaux pointant vers l'extérieur de  $\partial\Omega$  et  $\sigma$  est la mesure sous-riemannienne induite sur  $\partial\Omega$  (c'est-à-dire celle dont la densité est  $\sigma = |i_\nu\omega|_{\partial\Omega}$ ).



Le *sous-laplacien* est l'opérateur  $\Delta = \operatorname{div}_\omega \circ \nabla$ , agissant sur  $C^\infty(M)$ . De nouveau, on peut écrire son expression par rapport à la famille génératrice (R.15), en obtenant

$$\Delta f = \sum_{i=1}^N \left\{ X_i^2(f) + X_i(f) \operatorname{div}_\omega(X_i) \right\}, \quad \forall f \in C^\infty(M). \quad (\text{R.27})$$

Si la distribution  $\mathcal{D}$ , définissant la structure sous-riemannienne sur  $M$ , ne coïncide pas avec tout l'espace tangent, le sous-laplacien n'est pas un opérateur elliptique, donc la théorie classique de régularité n'est pas disponible. Néanmoins,  $\Delta$  est hypoelliptique, cf. [Hör67, Str86].

**Théorème R.2.2.** *Soit  $M$  une variété lisse et soit  $\{Y_0, \dots, Y_N\}$  un ensemble de champs de vecteurs sur  $M$  qui satisfait la condition d'Hörmander. Alors l'opérateur*

$$L = Y_0 + \sum_{i=1}^N Y_i^2$$

*est hypoelliptique, ce qui signifie que si  $\varphi$  est une distribution définie sur un ensemble ouvert  $\Omega \subset M$ , telle que  $L\varphi \in C^\infty(\Omega)$ , alors  $\varphi \in C^\infty(\Omega)$ .*

Dans une variété sous-riemannienne  $M$ , avec la famille génératrice (R.15) satisfaisant la condition d'Hörmander (R.16), on observe immédiatement que le sous-laplacien est hypoelliptique grâce à (R.27). De plus, le Théorème R.2.2 assure également que l'opérateur de la chaleur est hypoelliptique, en effet

$$L = \partial_t - \Delta = \partial_t - \sum_{i=1}^N \operatorname{div}_\omega(X_i)X_i - \sum_{i=1}^N X_i^2, \quad (\text{R.28})$$

donc en choisissant  $Y_0 = \partial_t - \sum_{i=1}^N \operatorname{div}_\omega(X_i)X_i$  et  $Y_i = X_i$ , pour tout  $i = 1, \dots, N$ , on conclut. En particulier, toute solution de l'équation de la chaleur

$$L\varphi = (\partial_t - \Delta)\varphi = 0, \quad \text{au sens des distributions sur } \Omega$$

est une fonction lisse dans  $\Omega$ , pour tout sous-ensemble ouvert  $\Omega \subset M$ .

**Le problème de Cauchy pour l'équation de la chaleur.** Soit  $M$  une variété sous-riemannienne, munie d'une mesure lisse  $\omega$ . On désigne par  $L^2(M, \omega)$  l'espace des fonctions réelles sur  $M$  dont le carré est intégrable par rapport à la mesure  $\omega$ . Soit  $f \in L^2(M, \omega)$  et on considère le *problème de Cauchy pour l'équation de la chaleur* :

$$\begin{aligned} (\partial_t - \Delta)u(t, x) &= 0, & \forall (t, x) \in (0, \infty) \times M, \\ u(0, \cdot) &= f, & \text{dans } L^2(M, \omega), \end{aligned} \quad (\text{R.29})$$

où  $u(0, \cdot)$  est une notation abrégée pour la limite  $L^2$  de  $u(t, x)$  lorsque  $t \rightarrow 0$  et  $\Delta$  est le sous-laplacien (R.27). On remarque que  $\Delta$  est symétrique par rapport au produit scalaire  $L^2$  et négatif, de plus, si  $(M, d_{\text{SR}})$  est complet en tant qu'espace métrique, il est essentiellement auto-adjoint, voir [Str86]. Ainsi, il existe une solution unique à (R.29), et elle peut être représentée par

$$u(t, x) = e^{t\Delta}f(x), \quad \forall x \in M, t > 0,$$

où  $e^{t\Delta}: L^2(M, \omega) \rightarrow L^2(M, \omega)$  désigne le semi-groupe de la chaleur, associé à  $\Delta$ . On remarque aussi que pour tout  $f \in L^2(M, \omega)$ , la fonction  $e^{t\Delta}f$  est lisse pour tout  $(t, x) \in (0, \infty) \times M$ , par hypoellipticité de l'opérateur de la chaleur, voir le Théorème 1.3.1, et il existe un noyau de la chaleur associé à (R.29), c'est-à-dire une fonction positive  $p_t(x, y) \in C^\infty((0, +\infty) \times M \times M)$ , telle que :

$$u(t, x) = \int_M p_t(x, y) f(y) d\omega(y), \quad \forall x \in M, t > 0.$$

## R.2.4 Approximation nilpotente de $M$

Nous introduisons ici la notion d'approximation nilpotente d'une variété sous-riemannienne. Voir [Jea14, Bel96] pour plus de détails.

**Drapeau sous-riemannien.** Soit  $M$  une variété sous-riemannienne de dimension  $n$ , avec distribution  $\mathcal{D}$ . On définit le *drapeau de  $\mathcal{D}$*  comme la suite de sous-ensembles  $\mathcal{D}^k \subset TM$  tels que

$$\mathcal{D}^1 = \mathcal{D}, \quad \mathcal{D}^{k+1} = \mathcal{D}^k + [\mathcal{D}, \mathcal{D}^k], \quad \forall k \geq 1,$$

avec la convention que  $\mathcal{D}^0 = \{0\}$ . Si  $\mathcal{D}$  est génératrice par crochets de Lie, le drapeau de la distribution définit un épuisement de  $T_x M$ , pour tout point  $x \in M$ , c'est-à-dire qu'il existe  $r(x) \in \mathbb{N}$  tel que :

$$\{0\} = \mathcal{D}_x^0 \subset \mathcal{D}_x^1 \subset \dots \subset \mathcal{D}_x^{r(x)-1} \subsetneq \mathcal{D}_x^{r(x)} = T_x M. \quad (\text{R.30})$$

Le nombre  $r(x)$  est appelé *degré de non-holonomie* en  $x$ . On pose  $n_k(x) = \dim \mathcal{D}_x^k$ , pour tout  $k \geq 0$ , alors la collection de  $r(x)$  entiers donnée par

$$(n_1(x), \dots, n_{r(x)}(x))$$

est appelé *vecteur de croissance* en  $x$ , et on a  $n_{r(x)}(x) = n = \dim M$ . Associé au vecteur de croissance, on peut définir les *poids sous-riemanniens*  $w_i(x)$  en  $x$ , en fixant pour tout  $i \in \{1, \dots, n\}$ ,

$$w_i(x) = j, \quad \text{si et seulement si} \quad n_{j-1}(x) + 1 \leq i \leq n_j(x). \quad (\text{R.31})$$

Un point  $x \in M$  est dit *régulier* si le vecteur de croissance est constant dans un voisinage de  $x$ , et *singulier* autrement. La structure sous-riemannienne sur  $M$  est dite *équirégulière* si tous les points de  $M$  sont réguliers. Dans ce cas, les poids sont constants sur  $M$ . Enfin, étant donné  $x \in M$ , on définit la *dimension homogène* de  $M$  en  $x$  comme

$$\mathcal{Q}(x) = \sum_{i=1}^{r(x)} i(n_i(x) - n_{i-1}(x)) = \sum_{i=1}^n w_i(x).$$

On rappelle que, si  $x$  est régulier, alors  $\mathcal{Q}(x)$  coïncide avec la dimension de Hausdorff de  $(M, d_{\text{SR}})$  en  $x$ , cf. [Mit85]. De plus,  $\mathcal{Q}(x) > n$ , pour tout  $x \in M$  tel que  $\mathcal{D}_x \subsetneq T_x M$ .

**Coordonnées privilégiées.** Soit  $M$  une variété sous-riemannienne avec famille génératrice (R.15) et soit  $f$  le germe d'une fonction lisse en  $x \in M$ . On appelle *dérivée non-holonyme* d'ordre  $k \in \mathbb{N}$  de  $f$ , la quantité

$$X_{j_1} \dots X_{j_k} f(x),$$

pour toute famille d'indices  $\{j_1, \dots, j_k\} \subset \{1, \dots, N\}$ . Alors, l'*ordre non-holonyme* de  $f$  au point  $x$  est

$$\text{ord}_x(f) = \min \{k \in \mathbb{N} \mid \exists \{j_1, \dots, j_k\} \subset \{1, \dots, N\} \text{ t.q. } X_{j_1} \dots X_{j_k} f(x) \neq 0\}.$$

**Définition R.2.5** (Coordonnées privilégiées). Soit  $M$  une variété sous-riemannienne de dimension  $n$  et soit  $x \in M$ . Un système de coordonnées locales  $(z_1, \dots, z_n)$ , centré en  $x$ , est dit *privilégié* en  $x$  si

$$\text{ord}_x(z_j) = w_j, \quad \forall j = 1, \dots, n.$$

On remarque que les coordonnées privilégiées  $(z_1, \dots, z_n)$  en  $x$  satisfont la propriété suivante

$$\partial_{z_i|x} \in \mathcal{D}_x^{w_i}, \quad \partial_{z_i|x} \notin (\mathcal{D}_x^{w_i-1}) = 0, \quad \forall i = 1, \dots, n. \quad (\text{R.32})$$

Une base locale de  $TM$  constitué de  $n$  champs de vecteurs  $\{Z_1, \dots, Z_n\}$  et satisfaisant (R.32) est dit *adapté* au drapeau (R.30) en  $x$ . Ainsi, les coordonnées privilégiées sont toujours adaptées au drapeau. De plus, étant donné une base locale adapté au drapeau sous-riemannien en  $x$ , notée  $\{Z_1, \dots, Z_n\}$ , on peut définir un ensemble de coordonnées privilégiées en  $x$ , à partir de celle-là :

$$\mathbb{R}^n \ni (z_1, \dots, z_n) \mapsto e^{z_1 Z_1} \circ \dots \circ e^{z_n Z_n}(x).$$

De plus, dans ces coordonnées, le champ de vecteurs  $Z_1$  est exactement  $\partial_{z_1}$ .

**Approximation nilpotente.** Soit  $M$  une variété sous-riemannienne et soit  $x \in M$  avec poids définis dans (R.31). On considère un ensemble de coordonnées privilégiées en  $x$ ,  $\psi = (z_1, \dots, z_n): U \rightarrow V$ , alors, pour tout  $\varepsilon \in \mathbb{R}$ , on peut définir la *dilatation* en  $x$  comme suit

$$\delta_\varepsilon: \mathbb{R}^n \rightarrow \mathbb{R}^n; \quad \delta_\varepsilon(z) = \left( \varepsilon^{w_1(x)} z_1, \dots, \varepsilon^{w_n(x)} z_n \right).$$

En utilisant ces dilatations, on obtient l'approximation nilpotente (ou du premier ordre) de la famille génératrice (R.15). En suivant la construction de [CdVHT20], soit  $W_1, W_2 \subset \mathbb{R}^n$  des sous-ensembles ouverts et relativement compacts de  $\mathbb{R}^n$  tels que  $\overline{W}_1 \subset W_2 \subset \overline{W}_2 \subset V$  et  $\delta_\varepsilon(W_i) \subset W_i$ , pour  $i = 1, 2$  et pour chaque  $|\varepsilon| \leq 1$ . On considère

$$\chi \in C_c^\infty(\mathbb{R}^n), \quad \text{tel que } \chi|_{W_1} \equiv 1 \text{ et } \text{supp}(\chi) \subset W_2, \quad (\text{R.33})$$

et on définit  $Y_i = \chi(\psi_* X_i)$ , pour tout  $i = 1, \dots, N$ , où  $X_i$  sont les champs de vecteurs de (R.15). Alors, on définit l'*approximation de premier ordre* de  $X_i$  en  $x$  comme suit

$$\widehat{X}_i^x = \lim_{\varepsilon \rightarrow 0} \varepsilon \delta_{\frac{1}{\varepsilon}*}(Y_i), \quad \forall i = 1 \dots, N, \quad (\text{R.34})$$

où la limite est prise dans la topologie  $C^\infty$  de  $\mathbb{R}^n$ .

**Théorème R.2.3.** *Soit  $M$  une variété sous-riemannienne de dimension  $n$  avec famille génératrice  $\{X_1, \dots, X_N\}$  et on considère son approximation du premier ordre en  $x$  comme dans (R.34). Alors, la famille  $\{\widehat{X}_1^x, \dots, \widehat{X}_N^x\}$  de champs de vecteurs sur  $\mathbb{R}^n$  engendre une algèbre de Lie nilpotente de pas  $r(x) = w_n(x)$  et satisfait la condition de Hörmander.*

La preuve de ce théorème peut être trouvée dans [Jea14]. On rappelle qu'une algèbre de Lie est dite nilpotente de pas  $s$  si  $s$  est le plus petit entier tel que toutes les parenthèses de longueur supérieure à  $s$  sont nulles.

**Définition R.2.6** (Approximation nilpotente). Soit  $M$  une variété sous-riemannienne et soit  $x \in M$ . Alors, le Théorème R.2.3 implique que la famille  $\{\widehat{X}_1^x, \dots, \widehat{X}_N^x\}$  est une famille génératrice pour une structure sous-riemannienne sur  $\mathbb{R}^n$  : on indique la variété sous-riemannienne résultante  $\widehat{M}^x$ . C'est ce qu'on appelle l'*approximation nilpotente* de  $M$  au point  $x$ .

*Remarque R.2.7.* À isométrie près, l'approximation nilpotente de  $M$  au point  $x$  coïncide avec l'espace tangent métrique à  $(M, d_{\text{SR}})$  au sens de Gromov-Hausdorff. De plus,  $\widehat{M}^x$  est isométrique à un quotient d'un groupe de Carnot. Voir [Gro96, Bel96, Mon02] pour plus de détails.

**Sous-laplacien nilpotentisé.** Soit  $M$  une variété sous-riemannienne, munie d'une mesure lisse  $\omega$ , et soit  $\psi = (z_1, \dots, z_n)$  un ensemble de coordonnées privilégiées à  $x \in M$ . On considère une fonction de troncature  $\chi \in C_c^\infty(\mathbb{R}^n)$  comme (R.33), et on définit une mesure lisse  $\mu$  sur  $\mathbb{R}^n$ .

$$\langle \mu, f \rangle = \langle \omega, \chi f \rangle, \quad \forall f \in C^0(\mathbb{R}^n).$$

La *nilpotentisation*  $\hat{\omega}^x$  de  $\omega$  en  $x$  est définie comme suit

$$\langle \hat{\omega}^x, f \rangle = \lim_{\varepsilon \rightarrow 0} \frac{1}{|\varepsilon|^{\mathcal{Q}(x)}} \langle \delta_\varepsilon^* \mu, f \rangle, \quad \forall f \in C_c^\infty(\mathbb{R}^n). \quad (\text{R.35})$$

On remarque que, en indiquant par  $dz = dz_1 \dots dz_n$  la mesure de Lebesgue sur  $\mathbb{R}^n$ , on a

$$\delta_\varepsilon^*(dz) = |\varepsilon|^{\mathcal{Q}(x)} dz, \quad \forall \varepsilon \neq 0,$$

donc, la limite en (R.35) existe. Finalement, on peut définir le *sous-laplacien nilpotentisé*, agissant sur  $C^\infty(\mathbb{R}^n)$ , comme suit :

$$\hat{\Delta}^x = \text{div}_{\hat{\omega}^x} \left( \hat{\nabla}^x \right) = \sum_{i=1}^N (\hat{X}_i^x)^2. \quad (\text{R.36})$$

On remarque que dans (R.36) il n'y a pas de terme de divergence, puisque

$$\text{div}_{\hat{\omega}^x}(\hat{X}_i^x) = 0 \quad \forall i \in \{1, \dots, N\}.$$

Comme dans le contexte sous-riemannien général, dans l'approximation nilpotente  $\hat{M}^x$ , on peut considérer le problème de la chaleur de Cauchy (R.29) en  $L^2(\mathbb{R}^n, \hat{\omega}^x)$ .

$$\begin{aligned} (\partial_t - \hat{\Delta}^x)u(t, x) &= 0, & \forall (t, x) \in (0, \infty) \times \mathbb{R}^n, \\ u(0, x) &= f, & \text{dans } L^2(\mathbb{R}^n, \hat{\omega}^x). \end{aligned}$$

On indique le noyau de la chaleur associé à ce problème comme suit

$$\hat{p}_t^x(z, z') \in C^\infty((0, +\infty) \times \mathbb{R}^n \times \mathbb{R}^n).$$

**Asymptotiques du noyau de la chaleur.** Soit  $M$  une variété sous-riemannienne, munie d'une mesure lisse  $\omega$  et on dénote par  $p_t(x, y)$  le noyau de la chaleur (R.54). On a le résultat suivant.

**Théorème R.2.4** ([CdVHT20, Thm. A]). *Soit  $M$  une variété sub-Riemannian et soit  $\psi: U \rightarrow V$  un ensemble de coordonnées privilégiées en  $x \in M$ . Alors, pour toute  $m \in \mathbb{N}$ ,*

$$|\varepsilon|^{\mathcal{Q}(x)} p_{\varepsilon^2 \tau}(\delta_\varepsilon(z), \delta_\varepsilon(z')) = \hat{p}_\tau^x(z, z') + \sum_{i=1}^m \varepsilon^i f_i^x(\tau, z, z') + o(|\varepsilon|^m), \quad \text{lorsque } \varepsilon \rightarrow 0, \quad (\text{R.37})$$

dans la topologie  $C^\infty$  de  $(0, \infty) \times V \times V$ , où les  $f_i^x$  sont des fonctions lisses satisfaisant la propriété d'homogénéité suivante : pour  $i = 0, \dots, m$ ,

$$\varepsilon^{\mathcal{Q}(x)} \varepsilon^{-i} f_i^x(\varepsilon^2 \tau, \delta_\varepsilon(z), \delta_\varepsilon(z')) = f_i^x(\tau, z, z'), \quad \forall (\tau, z, z') \in (0, \infty) \times \mathbb{R}^n \times \mathbb{R}^n,$$

où, pour  $i = 0$ , on fixe  $f_0^x(\tau, z, z') = \hat{p}_\tau^x(z, z')$ . Dans (R.37), on considère le noyau de la chaleur  $p_t$  en coordonnées, avec un petit abus de notation.

## R.3 Asymptotiques du contenu thermique en variétés sous-riemanniennes

### R.3.1 Introduction

Soit  $(M, g)$  une variété riemannienne complète, et  $\Omega \subset M$  un ensemble ouvert et relativement compact avec bord lisse. On considère la solution  $u(t, x)$  de l'équation de la chaleur avec des conditions aux limites de Dirichlet et une donnée initiale homogène :

$$\begin{aligned} (\partial_t - \Delta)u(t, x) &= 0, & \forall (t, x) \in (0, \infty) \times \Omega, \\ u(t, x) &= 0, & \forall (t, x) \in (0, \infty) \times \partial\Omega, \\ u(0, x) &= 1, & \forall x \in \Omega, \end{aligned}$$

où  $\Delta$  est l'opérateur de Laplace-Beltrami de  $(M, g)$ . Le contenu thermique riemannien de  $\Omega$  est la fonction

$$Q_\Omega(t) = \int_\Omega u(t, x) d\mu_g(x), \quad t \in [0, \infty),$$

où  $\mu_g$  est la mesure riemannienne. D'un point de vue physique,  $Q_\Omega(t)$  représente la chaleur totale contenue dans  $\Omega$  au temps  $t$ , correspondant à une distribution initiale uniforme de la température, et où la frontière  $\partial\Omega$  est maintenue à température nulle. On montre que  $Q_\Omega(t)$  admet un développement asymptotique en fonction de  $\sqrt{t}$  dont les coefficients encodent des informations géométriques sur  $\Omega$  et sa frontière.

Pour les domaines euclidiens  $\Omega \in \mathbb{R}^n$ , l'asymptotique de  $Q_\Omega(t)$  à l'ordre 1 a été calculée dans [vdBD89], et jusqu'à l'ordre 2 dans [vdBLG94], en utilisant des méthodes probabilistes. En particulier, sous la condition que  $\partial\Omega$  est de classe  $C^3$ , on montre que :

$$Q_\Omega(t) = |\Omega| - \sqrt{\frac{4t}{\pi}} \sigma(\partial\Omega) + \frac{t}{2} \int_{\partial\Omega} \mathcal{H} d\sigma + O(t^{3/2}), \quad (\text{R.38})$$

où  $|\cdot|$  désigne ici la mesure de Lebesgue,  $\sigma$  est la mesure de surface induite sur  $\partial\Omega$ , et  $\mathcal{H}$  est la courbure moyenne de  $\partial\Omega$ . Un premier cas non plat a été étudié dans [vdB91], où les auteurs ont calculé l'asymptotique du contenu thermique à l'ordre 2 pour l'hémisphère supérieur d'une sphère, en exploitant la connaissance explicite du noyau de la chaleur.

Pour les ensembles ouverts et lisses dans une variété riemannienne, l'existence d'une expansion asymptotique en  $\sqrt{t}$  à un ordre arbitraire a été établie dans [vdBG94], où les auteurs ont également calculé tous les coefficients jusqu'à l'ordre 4. Dans ce cas, le volume, le périmètre et la courbure moyenne apparaissant dans (R.38) sont remplacés par leurs homologues riemanniens, tandis que, dans les termes suivants, la deuxième forme fondamentale de  $\partial\Omega$  et le tenseur de courbure riemannienne interviennent. Il faut souligner ici que l'existence d'une série asymptotique complète n'est pas triviale, car le contenu thermique n'est pas une fonction lisse de  $\sqrt{t}$  en  $t = 0$  (on peut facilement vérifier ce fait en calculant le contenu thermique d'un segment euclidien). La méthode de Van den Berg et Gilkey dans le [vdBG94], qui exploite fortement les propriétés fonctorielles des coefficients et la théorie de l'invariance pour la courbure riemannienne, a été étendue pour calculer le contenu thermique asymptotique jusqu'à l'ordre 5 dans [vdBG99], et au cas des conditions aux limites de Neumann, voir [vdBDG93, DG94].

### Asymptotiques du contenu thermique sous-riemannien

Dans le Chapitre 2, nous étudions l'asymptotique du contenu thermique en géométrie sous-riemannienne. Cette étude est intéressante pour plusieurs raisons. Premièrement, il n'existe pas d'analogue de la connexion de Levi-Civita, de la courbure riemannienne, ou de la théorie de l'invariance pour une structure sous-riemannienne générale. Il s'agit d'outils

fondamentaux pour l'étude du problème riemannien réalisée par Van den Berg et Gilkey, et de nouvelles méthodes doivent donc être utilisées dans le cadre sous-riemannien. Deuxièmement, comme on l'a déjà mentionné, dans le cas général sous-riemannien, il n'existe pas de choix canonique de mesure. Pour cette raison, il faut fixer une mesure extérieure lisse  $\omega$ , qui est nécessaire pour la définition du sous-laplacien, voir la Section R.2.3. On remarque que, si  $M$  est une variété sous-riemannienne équirégulière, on peut définir la mesure canonique de Popp [Mon02, BR13], étendant le volume riemannien. Cette construction n'est pas possible pour les structures non équirégulières, par exemple les structures à rang variable. Troisièmement, l'étude du contenu thermique sous-riemannien peut améliorer notre compréhension de la géométrie intrinsèque des hypersurfaces, qui n'est bien développée que pour le cas du groupe de Heisenberg [Pau04, AF07, AF08, CDPT07, BFF<sup>+</sup>15a, BTV17] et des groupes de Carnot [DGN07a] (voir aussi [Vel20] pour un concept de courbure gaussienne pour les surfaces dans les structures de contact tridimensionnelles, généralisant [BTV17]). Enfin, un phénomène véritablement nouveau apparaît dans le cas sous-riemannien : les points caractéristiques, où la distribution est tangente à  $\partial\Omega$ , et dont la présence est source de plusieurs problèmes techniques.

On introduit maintenant la définition précise du contenu thermique en géométrie sous-riemannienne. Soit  $M$  une variété sous-riemannienne, munie d'une mesure lisse  $\omega$ , et soit  $\Omega \subset M$  un ensemble ouvert relativement compact avec bord lisse. Cela signifie que sa clôture  $\bar{\Omega}$  est une variété compacte avec bord. On considère le *problème de Dirichlet pour l'équation de la chaleur* sur  $\Omega$ , c'est-à-dire que on cherche des fonctions  $u$  telles que

$$\begin{aligned} (\partial_t - \Delta)u(t, x) &= 0, & \forall (t, x) \in (0, \infty) \times \Omega, \\ u(t, x) &= 0, & \forall (t, x) \in (0, \infty) \times \partial\Omega, \\ u(0, x) &= 1, & \forall x \in \Omega, \end{aligned} \tag{R.39}$$

où  $\Delta = \operatorname{div}_\omega \circ \nabla$  est le sous-laplacien défini dans (R.27). On désigne par  $L^2(\Omega, \omega)$  l'espace des fonctions réelles sur  $\Omega$  dont le carré est intégrable par rapport à la mesure  $\omega$ . Si  $f, h \in C^\infty(\bar{\Omega})$  satisfont la condition aux limites de Dirichlet, alors, par le théorème de la divergence (R.26), on obtient

$$(\Delta f, h)_{L^2} = (f, \Delta h)_{L^2},$$

où  $(\cdot, \cdot)_{L^2}$  désigne le produit scalaire dans  $L^2(\Omega, \omega)$ . Ceci implique que l'on peut représenter l'unique solution de (R.39), comme suit

$$u(t, \cdot) = e^{t\Delta}(1|_\Omega), \quad \forall t \geq 0,$$

où  $e^{t\Delta} : L^2(\Omega, \omega) \rightarrow L^2(\Omega, \omega)$  est le semi-groupe généré par l'extension auto-adjointe de Dirichlet du sous-laplacien sur  $\Omega$ . On remarque que pour tout  $\varphi \in L^2(\Omega, \omega)$ , la fonction  $e^{t\Delta}\varphi$  est lisse pour tout  $(t, x) \in (0, \infty) \times \Omega$ , par hypoellipticité de l'opérateur de la chaleur (R.28). De plus, les deux propriétés suivantes du semi-groupe de la chaleur sont vérifiées. Tout d'abord, la solution de (R.39) satisfait un principe de maximum faible, ce qui signifie que

$$0 \leq u(t, x) \leq 1, \quad \forall x \in \Omega, \quad \forall t > 0. \tag{R.40}$$

Deuxièmement, on a une propriété de monotonie du domaine. Soit  $\Omega' \subset M$  un ensemble ouvert et relativement compact, tel que  $\Omega \subset \Omega'$ . En indiquant par  $\Delta'$  le sous-laplacien de Dirichlet sur  $\Omega'$ , pour tout  $\varphi \in L^2(\Omega, \omega)$ , on a

$$e^{t\Delta}\varphi(x) \leq e^{t\Delta'}\varphi(x), \quad \forall x \in \Omega, \quad \forall t > 0, \tag{R.41}$$

où  $e^{t\Delta'}$  est le semi-groupe généré par  $\Delta'$  sur  $L^2(\Omega', \omega)$ . Les propriétés (R.40) et (R.41) peuvent être prouvées en suivant le schéma des preuves riemanniennes (voir [Gri09, Thm.

5.11, Thm. 5.23]). Une preuve alternative peut être donnée en utilisant des approximations riemanniennes (voir la Section R.3.2 pour plus de détails) en exploitant les propriétés respectives pour les semi-groupes de la chaleur riemanniens et en passant ensuite à la limite.

**Définition R.3.1** (Contenu thermique sous-riemannien). Soit  $u(t, x)$  la solution du problème (R.39). On définit le *contenu thermique sous-riemannien*, associé à  $\Omega$ , comme suit

$$Q_\Omega(t) = \int_\Omega u(t, x) d\omega(x), \quad \forall t > 0.$$

Afin d'obtenir l'existence d'un développement asymptotique du contenu thermique sous-riemannien associé à  $\Omega \subset M$ , on exige que l'ensemble soit non caractéristique.

**Définition R.3.2** (Point caractéristique). On dit que  $x \in \partial\Omega$  est un *point caractéristique*, ou point de tangence, si la distribution est tangente à  $\partial\Omega$  en  $x$ , c'est-à-dire :

$$\mathcal{D}_x \subseteq T_x(\partial\Omega).$$

Si  $\partial\Omega$  n'a aucun point caractéristique, on dit que  $\Omega$  est un *ensemble non-caractéristique*.

Si  $\Omega$  est un ensemble non-caractéristique, alors la solution  $e^{t\Delta}\phi$  du problème de Dirichlet, avec donnée initiale  $\phi \in C^\infty(\Omega)$ , existe, est unique, et est lisse sur  $(0, \infty) \times \bar{\Omega}$ , voir [GM18, Thm. 2.5]. Il s'agit d'une conséquence du résultat analogue pour le problème stationnaire considéré dans [KN65], tandis que dans le cas riemannien, il s'agit d'un résultat classique, voir [Eva10, Sec. 7, Thm. 7]. De plus, si  $\Omega$  est un ensemble non-caractéristique, on a également des propriétés de régularité de la fonction distance au son bord, où celle-ci est définie par  $\delta: M \rightarrow [0, \infty)$  tel que

$$\delta(x) = \inf\{d_{\text{SR}}(x, y) \mid y \in \partial\Omega\}, \quad \forall x \in M. \quad (\text{R.42})$$

Alors, il existe un voisinage de  $\partial\Omega$  dans  $\Omega$  où  $\delta$  est lisse. Précisément, on a le résultat suivant, cf. [FPR20, Prop. 3.1], et [PRS18, Lem. 23].

**Théorème R.3.1.** *Soit  $M$  une variété sous-riemannienne, munie d'une mesure lisse  $\omega$ , et soit  $\Omega \subset M$  un sous-ensemble ouvert et relativement compact de  $M$  dont le bord est lisse et n'a pas de points caractéristiques. Soit  $\delta: \bar{\Omega} \rightarrow [0, \infty)$  la fonction de distance à  $\partial\Omega$ . Pour tout  $r \geq 0$ , on définit*

$$\Omega(r) = \{x \in \bar{\Omega} \mid \delta(x) > r\}, \quad \Omega'(r) = \bar{\Omega} \setminus \Omega(r)$$

On a donc :

- i)  $\delta$  est lipschitzienne par rapport à la distance sous-riemannienne  $d_{\text{SR}}$  et  $\|\nabla\delta\|_g \leq 1$  presque partout ;
- ii) il existe  $r_0 > 0$  tel que  $\delta: \Omega'(r_0) \rightarrow [0, \infty)$  est lisse ;
- iii) il existe un difféomorphisme lisse  $G: [0, r_0) \times \partial\Omega \rightarrow \Omega'(r_0)$ , tel que

$$\delta(G(t, y)) = t \quad \text{et} \quad G_*\partial_t = \nabla\delta, \quad \forall (t, y) \in [0, r_0) \times \partial\Omega.$$

De plus,  $\|\nabla\delta\|_g \equiv 1$  sur  $\Omega'(r_0)$ .



On rappelle que pour un ensemble non-caractéristique  $\Omega$ , puisque  $\delta$  est lisse sur  $\partial\Omega$ , la normale horizontale pointant vers l'extérieur de  $\partial\Omega$  est donnée par  $\nu = -\nabla\delta$ . Dans ce cas, la mesure sous-riemannienne induite  $\sigma$  sur  $\partial\Omega$  est la mesure lisse et positive dont la densité est  $|i_\nu\omega|_{\partial\Omega}$ . De plus, on définit la courbure moyenne sous-riemannienne de  $\partial\Omega$  par

$$\mathcal{H} = \operatorname{div}_\omega(\nu)|_{\partial\Omega} = -\Delta\delta|_{\partial\Omega}. \quad (\text{R.43})$$

Nous pouvons maintenant énoncer nos principaux résultats concernant l'asymptotique du contenu thermique en géométrie sous-riemannienne.

**Théorème R.3.2.** *Soit  $M$  une variété sous-riemannienne, munie d'une mesure lisse  $\omega$ , et soit  $\Omega \subset M$  un sous-ensemble ouvert et relativement compact dont le bord est lisse et n'a pas de points caractéristiques. Alors, il existe  $a_k \in \mathbb{R}$  tel que, pour tout  $m \geq 4$ , on a*

$$Q_\Omega(t) = \omega(\Omega) - \sqrt{\frac{4t}{\pi}}\sigma(\partial\Omega) + \frac{t}{2} \int_{\partial\Omega} \mathcal{H}d\sigma + \sum_{k=3}^{m-1} a_k t^{k/2} + O(t^{m/2}), \quad \text{lorsque } t \rightarrow 0,$$

où  $\sigma$  est la mesure sous-riemannienne induite par  $\omega$  sur  $\partial\Omega$ , et  $\mathcal{H}$  est la courbure moyenne sous-riemannienne de  $\partial\Omega$ .

Afin de reporter les premiers coefficients, on introduit l'opérateur  $N$ , agissant sur les fonctions lisses définies sur un voisinage de  $\partial\Omega$ , donné par

$$N\phi = 2g(\nabla\phi, \nabla\delta) + \phi\Delta\delta. \quad (\text{R.44})$$

**Proposition R.3.3.** *Avec les hypothèses et notations du Théorème R.3.2, pour tout  $k \geq 1$ , on a  $a_k = -\int_{\partial\Omega} D_k(1)d\sigma$ , où  $D_k$  est un polynôme homogène de degré  $k-1$  dans les opérateurs  $\Delta$  et  $N$ . En particulier, on a*

$$\begin{aligned} a_0 &= \omega(\Omega), & a_1 &= -\sqrt{\frac{4}{\pi}}\sigma(\partial\Omega), \\ a_2 &= -\frac{1}{2} \int_{\partial\Omega} \Delta\delta d\sigma, & a_3 &= -\frac{1}{6\sqrt{\pi}} \int_{\partial\Omega} N\Delta\delta d\sigma, \\ a_4 &= -\frac{1}{16} \int_{\partial\Omega} \Delta^2\delta d\sigma, & a_5 &= \frac{1}{240\sqrt{\pi}} \int_{\partial\Omega} (N^3 - 8N\Delta)\Delta\delta d\sigma. \end{aligned}$$

Pour tout  $k \geq 1$ , les opérateurs  $D_k$  sont définis récursivement dans (2.28)–(2.30).

*Remarque R.3.4.* Les intégrandes de  $a_1$  et  $a_2$  ont une interprétation classique comme le périmètre et la courbure moyenne de  $\partial\Omega$ . On observe que l'intégrande de  $a_3$  est le soi-disant *potentiel effectif*, une quantité introduite dans [PRS18, FPR20] pour décrire les propriétés essentielles d'auto-adjonction des sous-laplaciens, en présence de mesures singulières.

*Remarque R.3.5.* Les opérateurs  $D_k$  appartiennent à l'algèbre engendrée par  $\Delta$  et  $N$ , et sont homogènes de degré  $k-1$  dans les générateurs, où  $N$  a degré 1 et  $\Delta$  a degré 2. Puisque  $\Delta(1) = 0$  et  $N(1) = \Delta\delta$ , pour  $k \geq 2$ , on a que chaque  $a_k = -\int_{\partial\Omega} \tilde{D}_k(\Delta\delta)d\sigma$ , où  $\tilde{D}_k$  est un polynôme homogène de degré  $k-2$  dans  $\Delta$  et  $N$ . En particulier, chaque  $a_k$  dépend de la courbure moyenne horizontale de  $\partial\Omega$  et de ses dérivées.

*Remarque R.3.6.* La construction itérative des opérateurs  $D_k$ , qui est assez complexe, a été implémentée dans le logiciel *Mathematica* en [Ros20], grâce auquel les coefficients  $a_k$  peuvent être immédiatement calculés. Voir l'Annexe C pour plus de détails.



### Stratégie de la preuve du Théorème R.3.2

L'étude de l'asymptotique du contenu thermique en temps petit dans le cadre sous-riemannien a été récemment initiée par Tyson et Wang, dans [TW18], où ils ont étudié le premier groupe de Heisenberg  $\mathbb{H}$ . Ils y ont établi l'existence d'une série asymptotique en temps petit jusqu'à l'ordre 2 dans  $\sqrt{t}$ , pour des ensembles non-caractéristiques. L'approche de [TW18] est probabiliste, basée sur l'interprétation de la solution du problème de Dirichlet en termes de temps de sortie du processus de Markov correspondant. Cette relation est valable si on produit une erreur d'ordre  $o(t)$ , cf. [TW18, Prop. 3.2], empêchant l'accès aux termes d'ordre supérieur dans l'asymptotique du contenu thermique.

Nous utilisons ici une méthode différente par rapport à celle de Tyson et Wang, en adaptant une technique développée dans le cas riemannien par Savo [Sav98, Sav01, Sav04]. Cette méthode nous permet de prouver l'existence d'un développement asymptotique à un ordre arbitraire, pour des ensembles non-caractéristiques de structures sous-riemanniennes générales (éventuellement à rang variable). L'idée principale est d'étudier la quantité

$$F(t, r) = \int_{\Omega(r)} u(t, x) d\omega(x),$$

où l'on se souvient que  $\Omega(r) = \{\delta > r\}$ . Après une localisation appropriée pour traiter le caractère non lisse de  $\Omega(r)$  pour de grands  $r$ , et en appliquant une version sous-riemannienne d'un lemme de valeur moyenne pour la fonction  $F(t, r)$ , on est capable de montrer qu'elle satisfait une équation de la chaleur unidimensionnelle non homogène sur la demi-droite  $[0, \infty)$ , avec une condition aux limites de Neumann à l'origine. Ensuite, l'asymptotique du Théorème R.3.2 et l'expression des coefficients sont obtenues en itérant la formule de Duhamel correspondante. Certaines modifications non triviales doivent être mises en œuvre pour adapter cette technique au cadre sous-riemannien. Par exemple, l'estimation de Li-Yau pour le noyau de la chaleur des variétés riemanniennes dont la courbure de Ricci est limitée par le bas n'est plus disponible (les variétés sous-riemanniennes ont, dans un certain sens, la courbure de Ricci non limitée par le bas). Un autre ingrédient important est la description des voisinages tubulaires de  $\partial\Omega$ , donnée par le Théorème R.3.1.

On remarque également que la même méthode peut être utilisée, sans modifications, pour étudier le contenu thermique associé à une condition initiale non uniforme  $\phi \in C^\infty(\Omega) \cap L^2(\Omega, \omega)$ . Dans ce cas, en utilisant la même notation de la Proposition R.3.3, on a  $a_k = -\int_{\partial\Omega} D_k(\phi) d\sigma$ .

### R.3.2 Approximation via des structures riemanniennes

Toute structure sous-riemannienne peut être obtenue comme une limite monotone de structures riemanniennes, au sens de Gromov-Hausdorff. Ce schéma d'approximation peut être facilement mis en œuvre pour les distributions à rangs constants. Dans ce cas, une séquence d'approximation naturelle est obtenue en prenant n'importe quelle métrique riemannienne  $g$  prolongeant la métrique sous-riemannienne, et en la multipliant par un facteur  $1/\varepsilon$  dans les directions transversales. Cette construction donne une famille à un paramètre de structures riemanniennes  $(M, g_\varepsilon)$ , dont la distance riemannienne associée  $d_\varepsilon$  converge, uniformément sur les ensembles compacts de  $M$ , vers la distance sous-riemannienne  $d_{\text{SR}}$ . En outre, en dehors du lieu de coupure sous-riemannien, on peut en fait prouver que  $d_\varepsilon \rightarrow d_{\text{SR}}$  dans la topologie  $C^\infty$ , voir par exemple [BGM19]. Pour les foliations totalement géodésiques, ce schéma est connu sous le nom de *variation canonique*, voir [Bes08]. On remarque cependant que la courbure riemannienne de la séquence d'approximation n'est pas bornée par le bas, ce qui pose des difficultés techniques lorsqu'on prend la limite.

Nous introduisons dans le Chapitre 2 une généralisation du schéma de variation canonique qui fonctionne pour des structures sous-riemanniennes générales à rang variable et

on construit une approximation des coefficients de l'asymptotique du contenu thermique de la Proposition [R.3.3](#).

Soit  $(\mathcal{D}, g)$  une structure sous-riemannienne sur  $M$ , avec famille génératrice [\(R.15\)](#). On considère une famille génératrice globale pour une structure riemannienne, c'est-à-dire un ensemble de  $L$  champs de vecteurs globaux  $\tilde{X}_1, \dots, \tilde{X}_L$  tels que

$$T_x M = \text{span}\{\tilde{X}_1(x), \dots, \tilde{X}_L(x)\}, \quad \forall x \in M. \quad (\text{R.45})$$

Pour tout  $\varepsilon \in \mathbb{R}$ , on considère la famille suivante de champs de vecteurs lisses globaux :

$$\{X_1, \dots, X_N, \varepsilon \tilde{X}_1, \dots, \varepsilon \tilde{X}_L\}. \quad (\text{R.46})$$

Il s'agit d'une famille génératrice globale pour tout l'espace tangent en chaque point, par conséquent, elle induit un produit scalaire sur  $T_x M$ , défini, comme dans le cas sous-riemannien, par la formule [\(R.17\)](#) et polarisation, que on désigne par  $g_\varepsilon$ . Puisque [\(R.45\)](#) est vérifiée,  $g_\varepsilon$  est une métrique riemannienne sur  $M$  et la distance riemannienne correspondante est notée  $d_\varepsilon$ . De plus, il est vrai que

$$d_\varepsilon \xrightarrow{\varepsilon \rightarrow 0} d_{\text{SR}}, \quad \text{uniformément sur les ensembles compacts de } M.$$

Une preuve explicite de ce fait dans le cas de variété sous-riemannienne à rang constant peut être trouvée dans [\[BGMR19, Lem. A.1\]](#). La même preuve est valable, mot pour mot, dans le cas général à rang variable, en remplaçant la base orthonormée locale par la famille génératrice [\(R.46\)](#), et en remplaçant les contrôles des géodésiques minimisantes par les contrôles minimaux, définis dans [\(R.18\)](#).

On appelle la famille d'espaces métriques à un paramètre  $\{(M, d_\varepsilon)\}_{\varepsilon \in \mathbb{R}}$  une *variation riemannienne* de  $(M, d_{\text{SR}})$ . Pour la mesure lisse que on a fixée  $\omega$  sur  $M$ , on définit le gradient horizontal et le laplacien correspondants, notés respectivement par  $\nabla_\varepsilon$  et  $\Delta_\varepsilon$ . L'expression de  $\nabla_\varepsilon$  agissant sur  $f \in C^\infty(M)$ , par rapport à la famille génératrice [\(R.46\)](#), est donnée par

$$\nabla_\varepsilon f = \sum_{i=1}^N X_i(f) X_i + \varepsilon^2 \sum_{\alpha=1}^L \tilde{X}_\alpha(f) \tilde{X}_\alpha = \nabla f + \varepsilon^2 \tilde{\nabla} f, \quad \forall f \in C^\infty(M),$$

tandis que l'expression de  $\Delta_\varepsilon$  est donnée par

$$\begin{aligned} \Delta_\varepsilon f &= \sum_{i=1}^N \{X_i^2(f) + X_i(f) \text{div}_\omega(X_i)\} + \varepsilon^2 \sum_{\alpha=1}^L \{\tilde{X}_\alpha^2(f) + \tilde{X}_\alpha(f) \text{div}_\omega(\tilde{X}_\alpha)\} \\ &= \Delta f + \varepsilon^2 \tilde{\Delta} f. \end{aligned}$$

Ici,  $\nabla$  et  $\Delta$  représentent respectivement le gradient et le sous-laplacien, tandis que  $\tilde{\nabla}$  et  $\tilde{\Delta}$  représentent respectivement le gradient et le laplacien, calculés par rapport à la structure riemannienne définie par la famille [\(R.45\)](#).

On fixe maintenant un sous-ensemble ouvert et relativement compact  $\Omega \subset M$  avec bord lisse. La séquence de structures riemanniennes donne naturellement une approximation de la solution  $u(t, x)$  du problème de Dirichlet [\(R.39\)](#) sur  $\Omega$ . En effet, en appliquant le théorème de Trotter-Kato (voir l'implication  $(a) \Rightarrow (d)$  dans [\[EN00, Thm. 4.8\]](#)), on a :

$$e^{t\Delta_\varepsilon} \varphi \xrightarrow[\|\cdot\|_{L^2(\Omega, \omega)}]{\varepsilon \rightarrow 0} e^{t\Delta} \varphi, \quad \forall \varphi \in L^2(\Omega, \omega), \quad \text{uniformément en } t \in [0, T], \quad (\text{R.47})$$

où  $e^{t\Delta_\varepsilon}$  et  $e^{t\Delta}$ , pour  $t > 0$ , désignent les semi-groupes associés à l'extension auto-adjointe de Dirichlet de  $\Delta_\varepsilon$  et  $\Delta$ , respectivement. En intégrant [\(R.47\)](#), avec  $\varphi = 1|_\Omega$ , on obtient un résultat analogue pour le contenu thermique :

$$Q_\Omega^\varepsilon(t) \xrightarrow{\varepsilon \rightarrow 0} Q_\Omega(t), \quad \text{uniformément sur } [0, T],$$

où  $Q_\Omega^\varepsilon(t)$  désigne le contenu thermique riemannien de  $\Omega$  pour la métrique  $g_\varepsilon$ .

## Approximations riemanniennes des coefficients

Notre deuxième résultat met en relation les coefficients de l'asymptotique en temps petit du contenu thermique riemannien  $Q_\Omega^\varepsilon(t)$  de l'approximation riemannienne avec les coefficients sous-riemanniens.

**Théorème R.3.3.** *Soit  $M$  une variété sous-riemannienne, munie d'une mesure lisse  $\omega$ , et soit  $\Omega \subset M$  un sous-ensemble ouvert et relativement compact dont le bord est lisse et n'a pas de points caractéristiques. Alors, il existe une famille de métriques riemanniennes  $g_\varepsilon$  telle que  $d_\varepsilon \rightarrow d_{\text{SR}}$  uniformément sur les ensembles compacts de  $M$ , et telle que*

$$\lim_{\varepsilon \rightarrow 0} a_k^\varepsilon = a_k, \quad \forall k \in \mathbb{N}, \quad (\text{R.48})$$

où  $a_k$  et  $a_k^\varepsilon$  désignent les coefficients de l'asymptotique sous-riemannienne du contenu thermique en temps petit, et les coefficients correspondants pour l'approximation riemannienne.

Même si  $Q_\Omega^\varepsilon(t) \rightarrow Q_\Omega(t)$  dans la topologie uniforme  $C^\infty$  sur les sous-ensembles compacts de  $(0, \infty)$ , ce fait seul n'implique pas (R.48). Une preuve directe du Théorème R.3.3 nécessiterait (i) une preuve a-priori de l'existence de l'asymptotique sous-riemannienne en temps petit pour  $Q_\Omega(t)$  et (ii) une inversion délicate de l'ordre des deux limites  $\varepsilon \rightarrow 0$  et  $t \rightarrow 0$ . Il est également important de souligner que le Théorème R.3.2 n'est pas prouvé en utilisant un schéma d'approximation et donc *il n'est pas* une conséquence du Théorème R.3.3. Ce dernier sera plutôt prouvé en utilisant la formule itérative explicite pour les coefficients des expansions du contenu thermique riemannien et sous-riemannien. En particulier, le Théorème R.3.3 sera une conséquence du résultat suivant reliant les fonctions de distance à  $\partial\Omega$  de la variété sous-riemannienne et de son approximation riemannienne, respectivement.

**Lemme R.3.7.** *Soit  $M$  une variété sous-riemannienne, et soit  $\Omega \subset M$  un sous-ensemble ouvert et relativement compact dont le bord est lisse et n'a aucun point caractéristique. Soient  $\delta, \delta_\varepsilon: \bar{\Omega} \rightarrow \mathbb{R}$  les distances à  $\partial\Omega$  sous-riemannienne et riemannienne de pas  $\varepsilon$ , respectivement et on fixe  $\bar{\varepsilon} > 0$ . Alors, il existe  $U \subset \bar{\Omega}$ , voisinage de  $\partial\Omega$ , tel que  $\delta, \delta_\varepsilon \in C^\infty(U)$  pour tout  $|\varepsilon| < \bar{\varepsilon}$  et  $\delta_\varepsilon \rightarrow \delta$ , lorsque  $\varepsilon \rightarrow 0$ , uniformément sur les ensembles compacts de  $U$ , avec toutes leurs dérivées.*

### R.3.3 Points caractéristiques

Une hypothèse cruciale dans tous nos résultats est que  $\partial\Omega$  ne contient pas de points caractéristiques. Ceci est assez restrictif pour le cas du groupe de Heisenberg, où les seuls ensembles non-caractéristiques sont homéomorphes à un tore. Plus généralement, pour toute variété sous-riemannienne de contact, l'hypothèse d'être non-caractéristique et la structure de contact impliquent que  $\partial\Omega$  doit avoir la caractéristique d'Euler évanouissante. D'autre part, l'hypothèse d'être non-caractéristique est moins restrictive pour les structures générales : il n'est pas difficile de prouver que pour toute variété lisse  $M$  de dimension  $n \geq 4$ , et tout ensemble ouvert et relativement compact  $\Omega$  avec bord lisse, il existe une structure sous-riemannienne, éventuellement à rang variable, sur  $M$  telle que  $\partial\Omega$  n'a pas de points caractéristiques.

Comme nous l'avons déjà mentionné, l'hypothèse d'être non-caractéristique est nécessaire pour la régularité de la distance à  $\partial\Omega$  et pour l'existence de voisinages tubulaires lisses, cf. le Théorème R.3.1. En outre, même si l'existence de solutions dans  $L^2(\Omega, \omega)$  à l'équation de la chaleur avec des conditions aux limites de Dirichlet est garantie par la théorie spectrale générale (et donc  $Q_\Omega(t)$  est bien défini), leur régularité jusqu'au bord

de l'ensemble peut échouer près des points caractéristiques [Jer81a, Jer81b]. Malgré toutes ces difficultés, on peut se demander si la formule asymptotique du contenu thermique en temps petit du Théorème R.3.2 a un sens, au moins formellement, pour les ensembles avec points caractéristiques. On note tout d'abord que si  $\Sigma$  est une hypersurface lisse plongée dans une variété sous-riemannienne  $M$ , alors l'ensemble des points caractéristiques a mesure nulle dans  $\Sigma$ , voir [Bal03]. Deuxièmement, tout choix de mesure lisse  $\omega$  sur  $M$  induit une mesure de surface lisse  $\sigma$  sur  $\Sigma$ , même en présence de points caractéristiques. Il suffit de considérer la contraction de  $\omega$  avec la normale horizontale unitaire à  $\Sigma$  (voir par exemple [DGN07a, Sec. 8] pour le cas des groupes de Carnot, où cette notion est liée à la mesure de périmètre horizontal). Troisièmement, la courbure moyenne sous-riemannienne est localement intégrable par rapport à  $\sigma$ , même en présence de points caractéristiques, cf. [DGN12, Ros21]. En conséquence de tous ces faits, tous les termes apparaissant dans la formule d'ordre 2 pour  $\mathbb{H}$  dans [TW18] sont bien définis également pour les ensembles avec des points caractéristiques. Cela semble suggérer que la même formule asymptotique en temps petit pourrait être valable aussi pour les ensembles avec des points caractéristiques. Notre analyse montre que cela ne peut pas être vrai à un ordre supérieur : on trouve en effet une surface explicite dans le groupe de Heisenberg où le cinquième coefficient  $a_5$ , défini dans la Proposition R.3.3, explose. Avant d'énoncer le théorème, on rappelle brièvement la définition du groupe de Heisenberg.

**Définition R.3.8** (Groupe de Heisenberg). Soit  $M = \mathbb{R}^3$  et on considère la famille génératrice globale donnée par

$$X_1 = \partial_x - \frac{y}{2}\partial_z, \quad X_2 = \partial_y + \frac{x}{2}\partial_z. \quad (\text{R.49})$$

On remarque que  $\{X_1, X_2\}$  satisfait la condition d'Hörmander. De plus, le produit scalaire induit par la famille est celui qui fait de  $\{X_1, X_2\}$  une base orthonormée. La variété sous-riemannienne résultante est le *premier groupe de Heisenberg*,  $\mathbb{H}$ . On le muni de la mesure de Lebesgue standard : dans ce cas, le sous-laplacien correspondant est  $\Delta = X_1^2 + X_2^2$ .

**Théorème R.3.4.** *Soit  $\mathbb{H}$  le premier groupe de Heisenberg, et on considère le plan  $\Sigma = \{z = 0\}$ . Tout d'abord, on observe que l'origine est un point caractéristique isolé. On désigne par  $\sigma$  la mesure de surface sous-riemannienne sur  $\Sigma$  induite par la mesure de Lebesgue sur  $\mathbb{H}$ . Alors l'intégrande du coefficient  $a_5$  de l'expansion du contenu thermique en temps petit n'est pas localement intégrable par rapport à  $\sigma$  autour du point caractéristique de  $\Sigma$ .*

Le Théorème R.3.4 montre que la formule asymptotique du Théorème R.3.2 est fautive à l'ordre  $k \geq 5$  pour les ensembles avec des points caractéristiques. Dans l'exemple du Théorème R.3.4, on montre que les intégrandes des coefficients  $a_3$  et  $a_4$  sont encore localement intégrables par rapport à la mesure de surface sous-riemannienne. On s'attend cependant à ce que l'on puisse construire un exemple moins symétrique où l'intégrande de  $a_4$  n'est pas non plus intégrable près d'un point caractéristique, cf. le Remark 2.7.7. D'autre part, le Théorème R.3.2 pourrait encore être vrai à un ordre inférieur (il a déjà été remarqué que les coefficients qui y apparaissent restent bien définis pour les ensembles avec des points caractéristiques dans  $\mathbb{H}$  jusqu'à  $k = 2$ ).

**problème ouvert.** *Est-il vrai que, pour des ensembles ouverts et lisses dans  $\mathbb{H}$  avec des points caractéristiques, le développement asymptotique du Théorème R.3.2 reste valide jusqu'à un certain ordre  $0 < k < 5$  ?*

Pour prouver le Théorème R.3.4, nous avons dérivé une formule exacte pour la distance sous-riemannienne du plan  $xy$  dans  $\mathbb{H}$ . À notre connaissance, c'est la première fois qu'une

telle formule globale explicite apparaît dans la littérature, et elle présente un intérêt indépendant. Par exemple, elle peut être utilisée pour étudier la perte de régularité de la distance à  $\Sigma$  aux points caractéristiques.

**Théorème R.3.5.** *La distance au plan  $xy$  dans le premier groupe de Heisenberg  $\mathbb{H}$ , pour tout  $p$  dans l'axe  $z$ , est donnée par*

$$\delta(p) = \sqrt{2\pi|z_p|},$$

tandis que pour tous les  $p$  n'appartenant pas à l'axe  $z$ , il est donné en coordonnées cylindriques par

$$\delta(p) = r_p \frac{4\xi_p + y_0(\xi_p)}{\sqrt{1 + y_0(\xi_p)^2}}, \quad \xi_p = \frac{|z_p|}{r_p^2},$$

où  $\xi \mapsto y_0(\xi)$  est la seule fonction lisse telle que

$$4\xi + y_0 + (1 + y_0^2) \arctan(y_0) = 0.$$

*Remarque R.3.9.* La preuve du Théorème R.3.5 consiste en une caractérisation non-triviale des géodésiques minimales qui sortent du plan  $xy$ , et le lieu de coupure correspondant. Les formules locales sont plus faciles à obtenir, cf. [BFF<sup>+</sup>15a, Ex. 5.1], en utilisant la propriété bien connue de minimalité des segments courts des géodésiques normales sous-riemanniennes. Cependant, ces formules locales et les estimations associées ne tiennent pas uniformément lorsqu'on s'approche des points caractéristiques. D'autre part, le Théorème R.3.5 est valable pour tous les  $p \in \mathbb{H}$ , même ceux situés arbitrairement près de l'origine. Cette caractéristique est essentielle pour la preuve du Théorème R.3.4.

## R.4 Intégrabilité de la courbure moyenne sous-riemannienne aux points caractéristiques dégénérés dans le groupe de Heisenberg

### R.4.1 Introduction

Soit  $M$  une variété sous-riemannienne, et soit  $\Sigma \subset M$  une hypersurface plongée. La courbure moyenne horizontale  $\mathcal{H} : \Sigma \rightarrow \mathbb{R}$  est un invariant géométrique qui apparaît naturellement dans différents domaines de l'analyse géométrique. Il apparaît dans la théorie des surfaces minimales [DGN07b, HP08, Mon15, Pau04, CHMY05], dans l'étude de l'asymptotique du contenu thermique dans les variétés sous-riemanniennes, voir [TW18] et [RR21], et dans les formules de type Steiner pour le volume des tubes autour des hypersurfaces [BFF<sup>+</sup>15b].

L'intégrabilité (locale) de  $\mathcal{H}$  est particulièrement importante dans toutes les applications susmentionnées, que ce soit par rapport à la mesure du périmètre horizontal ou à la mesure riemannienne, cf. la Définition R.4.3. Un fait important est que, même pour les hypersurfaces lisses  $\Sigma$ , la courbure moyenne horizontale explose aux points caractéristiques, cf. la Définition R.3.2, ce qui rend le problème de l'intégrabilité locale non trivial.

En ce qui concerne la mesure de périmètre sous-riemannienne  $\sigma_{\mathbb{H}}$ , comme cela a été remarqué pour la première fois dans [DGN12] pour le groupe de Heisenberg, la singularité de  $\mathcal{H}$  est compensée par le fait que  $\sigma_{\mathbb{H}}$  s'annule aux points caractéristiques, et donc  $\mathcal{H} \in L^1_{\text{loc}}(\Sigma, \sigma_{\mathbb{H}})$ . Cependant, cette compensation échoue si l'on remplace la mesure de périmètre sous-riemannienne  $\sigma_{\mathbb{H}}$  sur  $\Sigma$  par la mesure riemannienne  $\sigma_{\mathbb{R}}$ , et en général

$\mathcal{H} \notin L^1_{\text{loc}}(\Sigma, \sigma_R)$ . Dans tous les exemples connus, cependant, soit  $\Sigma$  n'est pas très lisse, soit l'ensemble des points caractéristiques a une dimension positive, cf. [DGN12]. D'autre part, si  $\Sigma$  est au moins  $C^2$  et que l'ensemble de points caractéristiques est discret, aucun contre-exemple d'intégrabilité locale n'est connu. En outre, une analyse approfondie de nombreux cas spécifiques a conduit les auteurs de l'article [DGN12] à conjecturer que, sous ces hypothèses,  $\mathcal{H} \in L^1_{\text{loc}}(\Sigma, \sigma_R)$ .

Dans le Chapitre 3, nous abordons la question de l'intégrabilité locale par rapport à la mesure de périmètre riemannienne dans le groupe de Heisenberg et dans le contexte plus général des variétés de contact tridimensionnelles.

### R.4.2 Courbure moyenne sous-riemannienne

Soit  $M$  une variété lisse de dimension 3. Soit  $\omega$  une 1-forme de contact, c'est-à-dire telle que  $\omega \wedge d\omega \neq 0$ . Alors, la distribution de contact est

$$\mathcal{D}_p = \ker(\omega_p) \subset T_p M, \quad \forall p \in M.$$

Puisque  $d\omega$  est non-dégénérée, la distribution  $\mathcal{D}$  est un sous-fibré de rang 2 du fibré tangent et est génératrice par crochets de Lie. Toute métrique sur  $\mathcal{D}$  définit une structure sous-riemannienne sur  $M$ . On appelle  $M$  une *variété sous-riemannienne de contact*.

*Remarque R.4.1.* Si  $M = \mathbb{R}^3$  et que l'on choisit comme forme de contact la suivant :

$$\omega = dz - \frac{1}{2}(xdy - ydx),$$

alors, une base globale pour la distribution des contacts est donné par  $\{X_1, X_2\}$ , où les champs de vecteurs  $X_1, X_2$  sont définis dans (R.49). Si l'on considère que  $\{X_1, X_2\}$  est une base orthonormée, alors la variété sous-riemannienne résultante est le groupe d'Heisenberg  $\mathbb{H}$ .

Soit  $M$  une variété sous-riemannienne de contact tridimensionnelle et soit  $\Sigma \subset M$  une surface plongée de classe  $C^2$  dans  $M$ . Puisque on s'intéresse aux propriétés locales de  $\Sigma$ , on peut toujours supposer, sans perdre de généralité, que  $\Sigma = \{u = 0\}$ , où

$$u \in C^2(M) \quad \text{et} \quad du|_{\Sigma} \neq 0.$$

Dans ce cas, on dit que  $p \in \Sigma$  est un point caractéristique et on écrit  $p \in C(\Sigma)$ , si

$$\mathcal{D}_p \subset T_p \Sigma \quad \text{ou de manière équivalente} \quad \nabla u(p) = 0.$$

**Définition R.4.2** (Courbure moyenne sous-riemannienne). Soit  $M$  une variété sous-riemannienne de contact tridimensionnelle et soit  $\Sigma \subset M$  une surface plongée de classe  $C^2$  dans  $M$ . La *courbure moyenne sous-riemannienne (ou horizontale)* en  $p \in \Sigma$  est définie par

$$\mathcal{H}(p) = -\operatorname{div}_{\mu} \left( \frac{\nabla u}{\|\nabla u\|} \right) \Big|_p. \quad (\text{R.50})$$

On remarque que, d'une part, la valeur de (R.50) en  $p$  ne dépend pas du choix de la fonction  $u$  qui définit  $\Sigma$ , et d'autre part, si  $p \in \Sigma \setminus C(\Sigma)$ , on retrouve la notion habituelle de courbure moyenne horizontale (R.43).

**Définition R.4.3** (Mesures induites). Soit  $M$  une variété sous-riemannienne de contact tridimensionnelle et soit  $\Sigma \subset M$  une surface plongée de classe  $C^2$  dans  $M$ . Soit  $\nu$  la normale horizontale unitaire à  $\Sigma$ , à savoir

$$\nu = -\frac{\nabla u}{\|\nabla u\|}, \quad (\text{R.51})$$



où  $\nabla$  est le gradient horizontal (R.25). Alors, la mesure sous-riemannienne  $\sigma_{\mathbb{H}}$  sur  $\Sigma$  est la mesure lisse positive de densité  $|i_{\nu}\mu|$ . De manière analogue, pour définir la mesure induite riemannienne  $\sigma_{\mathbb{R}}$ , on considère une extension riemannienne quelconque de la structure sous-riemannienne, puis on remplace  $\nu$  par la normale unitaire riemannienne, qui est donnée par (R.51) avec le gradient riemannien.

On remarque que  $\sigma_{\mathbb{R}}$  dépend du choix d'une extension riemannienne, mais que ce choix ne joue aucun rôle dans l'intégrabilité de la courbure moyenne horizontale.

**Définition R.4.4** (Hessien horizontal). . Soit  $M$  une variété sous-riemannienne de contact tridimensionnelle. On fixe une connexion affine  $\tilde{\nabla}$  sur la distribution  $\mathcal{D}$ . Alors, le *hessien horizontal* de  $u \in C^2(M)$  est le tenseur  $(0, 2)$  sur  $\mathcal{D}$ , défini par

$$\text{Hess}_{\mathbb{H}}(u)(V, W) = g(\tilde{\nabla}_V(\nabla u), W), \quad \forall V, W \in \Gamma(\mathcal{D}). \quad (\text{R.52})$$

Alors qu'en général la définition du hessien horizontal dépend du choix de la connexion, il est intrinsèque aux points caractéristiques. Plus précisément si  $\Sigma \subset M$  est une surface plongée de classe  $C^2$ , alors la valeur de (R.52) en  $p \in C(\Sigma)$  ne dépend ni du choix de  $\tilde{\nabla}$ , ni de  $u$ .

### R.4.3 Points caractéristiques non-dégénérés

On introduit maintenant une condition du second ordre sur les points caractéristiques, en utilisant le hessien horizontal.

**Définition R.4.5** (Point caractéristique non dégénéré). Soit  $M$  une variété sous-riemannienne de contact tridimensionnelle et soit  $\Sigma \subset M$  une surface plongée de classe  $C^2$  dans  $M$ . On dit que  $p \in C(\Sigma)$  est un *point caractéristique non-dégénéré* si

$$\det(\text{Hess}_{\mathbb{H}}(u)|_p) \neq 0.$$

Sinon, on dit que  $p \in C(\Sigma)$  est un *point caractéristique dégénéré*.

Le cas des points caractéristiques non-dégénérés et isolés est élémentaire, et on peut montrer le résultat suivant.

**Théorème R.4.1.** *Soit  $M$  une variété sous-riemannienne de contact tridimensionnelle, munie d'une mesure lisse  $\mu$  et soit  $\Sigma \subset M$  une surface plongée de classe  $C^2$ . On suppose que tous les points caractéristiques de  $\Sigma$  sont isolés et non-dégénérés. Alors ,*

$$\mathcal{H} \in L_{\text{loc}}^1(\Sigma, \sigma_{\mathbb{R}}),$$

où  $\sigma_{\mathbb{R}}$  désigne la mesure riemannienne induite par  $\mu$  sur  $\Sigma$ .

Le cas des points caractéristiques *dégénérés* est moins bien compris, même pour le cas des surfaces lisses dans le groupe de Heisenberg  $\mathbb{H}$ . Le résultat principal de ce chapitre est la définition d'un concept de point caractéristique *modérément dégénéré* pour les surfaces du groupe de Heisenberg, pour lequel nous sommes capables de prouver l'intégrabilité locale.

### R.4.4 Points caractéristiques modérément dégénérés

On considère dans le groupe de Heisenberg  $\mathbb{H}$  une surface lisse  $\Sigma = \{u = 0\}$ , où  $u \in C^\infty(\mathbb{R}^3)$  avec  $du \neq 0$  sur  $\Sigma$ . On suppose que  $p \in C(\Sigma)$  soit un point caractéristique dégénéré, ce qui signifie que le hessien horizontal de  $u$  a déterminant nul en  $p$ . Ainsi, il a un noyau unidimensionnel en  $p$ , engendré par un vecteur unitaire que on désigne par

$N_p \in \mathcal{D}_p$ . Celui-ci est unique, à signe près, et donc on l'étend à un champ de vecteurs invariant à gauche  $N \in \Gamma(\mathcal{D})$ . En prenant un champ de vecteurs orthogonal à  $N$  dans  $\Gamma(\mathcal{D})$ , on obtient une base orthonormée  $\{N, T\}$  pour la distribution  $\mathcal{D}$ , co-orienté avec celui donné par (R.49).

**Définition R.4.6** (Courbe critique). Soit  $\Sigma \subset \mathbb{H}$  une surface lisse plongée et soit  $p \in C(\Sigma)$  dégénéré. La *courbe critique* de  $p$  est définie comme l'ensemble des points dans  $\Sigma$  où  $N$  est tangent à  $\Sigma$ , soit

$$\mathcal{C} = \{q \in \Sigma \mid N(q) \in T_q \Sigma\}.$$

La Définition R.4.6 est bien posée et nous permet d'introduire la notion de point caractéristique modérément dégénéré.

**Définition R.4.7** (Point caractéristique modérément dégénéré). Soit  $\Sigma \subset \mathbb{H}$  une surface lisse plongée, soit  $p \in C(\Sigma)$  un point caractéristique dégénéré et  $\mathcal{C}$  sa courbe critique. Soit  $\gamma: (-\varepsilon, \varepsilon) \rightarrow \mathcal{C}$  une paramétrisation régulière de  $\mathcal{C}$ , avec  $\gamma(0) = p$ . On dit que  $p$  est *modérément dégénéré* si la fonction

$$s \mapsto Tu(\gamma(s)),$$

a un zéro d'ordre fini en  $s = 0$ . On remarque que cette définition ne dépend pas ni du choix de  $u$ , ni de la paramétrisation régulière de  $\mathcal{C}$ .

Comme on peut le voir, le concept de point caractéristique modérément dégénéré est basé sur une condition d'ordre fini le long de la courbe intrinsèque  $\mathcal{C} \subset \Sigma$  émanant de  $p \in C(\Sigma)$  dégénéré. Pour les points caractéristiques isolés et modérément dégénérés, on montre le résultat d'intégrabilité suivant.

**Théorème R.4.2.** Soit  $\Sigma \subset \mathbb{H}$  une surface lisse plongée. On suppose que tous les points caractéristiques de  $\Sigma$  soient isolés et modérément dégénérés. Alors

$$\mathcal{H} \in L^1_{\text{loc}}(\Sigma, \sigma_{\mathbb{R}}),$$

où  $\sigma_{\mathbb{R}}$  désigne la mesure riemannienne induite sur  $\Sigma$ .

En particulier, si  $\Sigma$  est *analytique réel*, tous les points caractéristiques dégénérés sont modérément dégénérés. En conséquence, on obtient le corollaire suivant, qui répond affirmativement à la conjecture de [DGN12], au moins pour les surfaces analytiques réelles.

**Théorème R.4.3.** Soit  $\Sigma \subset \mathbb{H}$  une surface analytique réelle plongée. On suppose que tous les points caractéristiques de  $\Sigma$  soient isolés. Alors,

$$\mathcal{H} \in L^1_{\text{loc}}(\Sigma, \sigma_{\mathbb{R}}),$$

où  $\sigma_{\mathbb{R}}$  désigne la mesure riemannienne induite sur  $\Sigma$ .

## R.5 Contenu thermique relatif

### R.5.1 Introduction

Dans le Chapitre 4, nous étudions l'asymptotique du contenu thermique relatif en géométrie sous-riemannienne. Soit  $M$  une variété sous-riemannienne, munie d'une mesure lisse  $\omega$ . On rappelle la définition du sous-laplacien sur  $M$  :

$$\Delta f = \text{div}_{\omega} \circ \nabla f, \quad \forall f \in C^{\infty}(M).$$



On désigne par  $L^2(M, \omega)$  l'espace des fonctions réelles sur  $M$  dont le carré est intégrables par rapport à la mesure  $\omega$ . Soit  $\Omega \subset M$  un ensemble ouvert et relativement compact avec bord lisse et on considère  $u(t, x)$  comme la solution du problème de Cauchy pour l'équation de la chaleur sur  $\Omega$ , c'est-à-dire :

$$\begin{aligned} (\partial_t - \Delta) u(t, x) &= 0, & \forall (t, x) \in (0, \infty) \times M, \\ u(0, \cdot) &= \mathbf{1}_\Omega, & \text{dans } L^2(M, \omega), \end{aligned} \quad (\text{R.53})$$

où  $u(0, \cdot)$  est une notation abrégée pour la limite  $L^2$  de  $u(t, x)$  lorsque  $t \rightarrow 0$ . On rappelle que, par la théorie spectrale classique, cf. la Section R.2.3, il existe une solution unique au problème (4.14), et elle peut être représentée comme suit

$$u(t, x) = e^{t\Delta} \mathbf{1}_\Omega(x), \quad \forall x \in M, t > 0,$$

où  $e^{t\Delta} : L^2(M, \omega) \rightarrow L^2(M, \omega)$  désigne le semi-groupe de la chaleur, associé à  $\Delta$ . De plus, il existe un noyau de la chaleur associé à (4.14), c'est-à-dire une fonction positive  $p_t(x, y) \in C^\infty((0, +\infty) \times M \times M)$ , telle que :

$$u(t, x) = \int_M p_t(x, y) \mathbf{1}_\Omega(y) d\omega(y) = \int_\Omega p_t(x, y) d\omega(y). \quad (\text{R.54})$$

**Définition R.5.1** (Contenu thermique relatif). Soit  $u(t, x)$  la solution de (R.53). On définit le *contenu thermique relatif*, associé à  $\Omega$ , comme suit

$$H_\Omega(t) = \int_\Omega u(t, x) d\omega(x), \quad \forall t > 0.$$

Dans le cadre euclidien, le contenu thermique relatif a été étudié en relation avec les propriétés géométriques des sous-ensembles de  $\mathbb{R}^n$ , à partir des travaux fondamentaux de De Giorgi [DG54], où il a introduit la notion de périmètre d'un ensemble dans  $\mathbb{R}^n$  et il a prouvé une caractérisation des ensembles de périmètre fini en termes de noyau de la chaleur. Ensuite, son résultat a été affiné en utilisant les techniques des fonctions à variation bornée : il a été prouvé dans [Led94] pour les boules dans  $\mathbb{R}^n$ , et dans [MPPP07] pour les sous-ensembles quelconques de  $\mathbb{R}^n$ , qu'un ensemble borel  $\Omega \subset \mathbb{R}^n$  avec mesure de Lebesgue finie a un périmètre fini à la De Giorgi si et seulement si

$$\exists \lim_{t \rightarrow 0} \frac{\sqrt{\pi}}{\sqrt{t}} (|\Omega| - H_\Omega(t)) = P(\Omega), \quad (\text{R.55})$$

où  $|\cdot|$  est la mesure de Lebesgue et  $P$  est la mesure de périmètre dans  $\mathbb{R}^n$ . On remarque que (R.55) est équivalent à un développement asymptotique du premier ordre de  $H_\Omega(t)$ , lorsque  $t \rightarrow 0$ . Un développement supplémentaire dans cette direction a ensuite été obtenu dans [AMM13], où les auteurs ont étendu (R.55) à une asymptotique d'ordre 3 en  $\sqrt{t}$ , en supposant que le bord de  $\Omega \subset \mathbb{R}^n$  est un ensemble  $C^{1,1}$ . Pour simplifier, on énonce ici le résultat de [AMM13, Thm. 1.1] en supposant que  $\partial\Omega$  est lisse :

$$\begin{aligned} H_\Omega(t) &= |\Omega| - \frac{1}{\sqrt{\pi}} P(\Omega) t^{1/2} \\ &\quad + \frac{(n-1)^2}{12\sqrt{\pi}} \int_{\partial\Omega} \left( H_{\partial\Omega}^2(x) + \frac{2}{(n-1)^2} c_{\partial\Omega}(x) \right) d\mathcal{H}^{n-1}(x) t^{3/2} + o(t^{3/2}), \end{aligned} \quad (\text{R.56})$$

lorsque  $t \rightarrow 0$ , où  $\mathcal{H}^{n-1}$  est la mesure de Hausdorff et, en désignant par  $k_i^{\partial\Omega}$  les courbures principales de  $\partial\Omega$  au point  $x$ ,

$$H_{\partial\Omega}(x) = \frac{1}{n-1} \sum_{i=1}^{n-1} k_i^{\partial\Omega}(x), \quad c_{\partial\Omega}(x) = \sum_{i=1}^{n-1} k_i^{\partial\Omega}(x)^2,$$

Dans le cadre riemannien, Van den Berg et Gilkey ont prouvé dans [vdBG15] l'existence d'une expansion asymptotique complète pour  $H_\Omega(t)$ , généralisant (R.56), lorsque  $\partial\Omega$  est lisse. De plus, ils ont réussi à calculer explicitement les coefficients du développement jusqu'à l'ordre 4 en  $\sqrt{t}$ . Leurs techniques sont basées sur des méthodes d'opérateurs pseudo-différentiels, et ne peuvent pas être immédiatement adaptées au cadre sous-riemannien. En particulier, il manque une estimation de type paramétrice pour le noyau de la chaleur, cf. [vdBG15, Sec. 2.3] : pour tout  $k \in \mathbb{N}$ , il existe  $J_k, C_k > 0$  tels que

$$\left\| p_t(x, y) - \sum_{j=0}^{J_k} p_t^j(x, y) \right\|_{C^k(M \times M)} \leq C_k t^k, \quad \text{lorsque } t \rightarrow 0, \quad (\text{R.57})$$

où  $p_t^j(x, y)$  sont des fonctions lisses appropriées, données explicitement en termes de noyau de la chaleur euclidien et de convolutions itérées. L'estimation la plus proche analogue à (R.57) dans le cadre sous-riemannien est celle prouvée récemment dans [CdVHT20, Thm. A] (voir le Théorème R.2.4 pour l'énoncé précis), où les auteurs montrent une expansion asymptotique du noyau de chaleur dans un *voisinage asymptotique* de la diagonale, ce qui ne suffit pas à reproduire (R.57) et donc l'argument de Van den Berg et Gilkey. De plus, dans ce cas,  $p_t^j(x, y)$  est exprimé en termes de noyau de la chaleur de l'approximation nilpotente et des convolutions itérées, ce qui pose des difficultés techniques pour les calculs explicites des coefficients (qui ne seraient plus des intégrales "simples" de type gaussien).

Dans le Chapitre 4, sous l'hypothèse de l'absence de points caractéristiques, nous prouvons l'existence du développement asymptotique de  $H_\Omega(t)$ , jusqu'à l'ordre 4 en  $\sqrt{t}$ , lorsque  $t \rightarrow 0$ . Nous remarquons que notre preuve inclut également le cas de variété sous-riemannienne à rang variable. On rappelle la définition de l'opérateur auxiliaire (R.44), agissant sur  $C^\infty(M)$ ,

$$N\phi = 2g(\nabla\phi, \nabla\delta) + \phi\Delta\delta, \quad \forall \phi \in C^\infty(M),$$

où  $\delta: M \rightarrow \mathbb{R}$  désigne la fonction de distance sous-riemannienne à  $\partial\Omega$ , telle que définie dans (R.42). Notre résultat principal est donc le suivant.

**Théorème R.5.1.** *Soit  $M$  une variété sous-riemannienne compacte, munie d'une mesure lisse  $\omega$ , et soit  $\Omega \subset M$  un sous-ensemble ouvert dont le bord est lisse et n'a pas de points caractéristiques. Alors, lorsque  $t \rightarrow 0$ ,*

$$H_\Omega(t) = \omega(\Omega) - \frac{1}{\sqrt{\pi}}\sigma(\partial\Omega)t^{1/2} - \left( \frac{1}{12\sqrt{\pi}} \int_{\partial\Omega} N(\Delta\delta) d\sigma - \frac{1}{6\sqrt{\pi}} \int_{\partial\Omega} (\Delta\delta)^2 d\sigma \right) t^{3/2} + o(t^2), \quad (\text{R.58})$$

où  $\sigma$  représente la mesure de périmètre sous-riemannienne.

*Remarque R.5.2.* L'hypothèse de compacité du Théorème R.5.1 est technique et peut être relâchée en exigeant, à la place, un doublement global de la mesure et une inégalité de Poincaré globale, voir la Section 4.7 et en particulier le Théorème 4.7.3. Quelques exemples notables satisfaisant ses hypothèses sont :

- $M$  est un groupe de Lie à croissance de volume polynomiale, la distribution est engendrée par une famille de champs de vecteurs invariants à gauche satisfaisant la condition de Hörmander et  $\omega$  est la mesure de Haar. Cette famille comprend également les groupes de Carnot.
- $M = \mathbb{R}^n$ , munie d'une structure sous-riemannienne induite par une famille de champs de vecteurs  $\{Y_1, \dots, Y_N\}$  à coefficients bornés ainsi que leurs dérivées, et satisfaisant la condition de Hörmander.

- $M$  est une variété riemannienne complète, munie de la mesure riemannienne, et avec la courbure de Ricci non négative.

Voir la Section 4.7.1 pour plus de détails. Dans tous ces exemples, le Théorème R.5.1 est valable.

La stratégie de la preuve du Théorème R.5.1 suit une stratégie similaire de ce que nous avons fait dans le Chapitre 2, inspiré par la méthode introduite dans [Sav98]. Cependant, comme nous allons l'expliquer dans la Section R.5.2, de nouvelles difficultés techniques apparaissent, la principale étant liée au fait que  $u(t, \cdot)|_{\partial\Omega} \neq 0$ , où  $u$  désigne la solution à (R.53). À l'ordre zéro, on obtient le résultat suivant, voir la Section R.2 pour les définitions précises.

**Théorème R.5.2.** *Soit  $M$  une variété sous-riemannienne, munie d'une mesure lisse  $\omega$  et soit  $\Omega \subset M$  un sous-ensemble ouvert et relativement compact, dont le bord est lisse et n'a pas de points caractéristiques. Soit  $x \in \partial\Omega$  et on considère des coordonnées privilégiées  $\psi: U \rightarrow V \subset \mathbb{R}^n$  centré sur  $x$ , tel que  $\psi(U \cap \Omega) = V \cap \{z_1 > 0\}$ . Alors,*

$$\lim_{t \rightarrow 0} u(t, x) = \int_{\{z_1 > 0\}} \hat{p}_1^x(0, z) d\hat{\omega}^x(z) = \frac{1}{2}, \quad \forall x \in \partial\Omega,$$

où  $\hat{\omega}^x$  désigne la nilpotentisation de  $\omega$  à  $x$  et  $\hat{p}_1^x$  désigne le noyau de la chaleur associé à l'approximation nilpotente de  $M$  à  $x$  et à la mesure  $\hat{\omega}^x$ .

Ce résultat peut être considéré comme une généralisation partielle de [CCSGM13, Prop. 3], où les auteurs ont prouvé un développement asymptotique de  $u(t, x)$  jusqu'à l'ordre 1 en  $\sqrt{t}$  pour  $x \in \partial\Omega$ , pour une classe exceptionnelle d'ensembles non caractéristiques dans les groupes de Carnot.

*Remarque R.5.3.* Notre preuve du Théorème 4.1.3 ne donne pas une série asymptotique pour  $u(t, \cdot)|_{\partial\Omega}$  à un ordre supérieur à 0. En effet, une série asymptotique complète de cette quantité semble difficile à obtenir, cf. la Section 4.6.

## R.5.2 Stratégie de la preuve du Théorème R.5.1

Pour mieux comprendre les nouvelles difficultés techniques dans l'étude du contenu thermique relatif  $H_\Omega(t)$ , on le compare au contenu thermique classique  $Q_\Omega(t)$ , cf. le Chapitre 2, et illustrons la stratégie de la preuve du Théorème R.5.1.

**Le contenu thermique classique.** On met en évidence les différences entre le contenu thermique relatif  $H_\Omega(t)$  et le contenu thermique classique  $Q_\Omega(t)$  : soit  $\Omega \subset M$  un ensemble ouvert dans  $M$ , alors pour tout  $t > 0$ , on a

$$H_\Omega(t) = \int_\Omega u(t, x) d\omega(x), \quad Q_\Omega(t) = \int_\Omega u_0(t, x) d\omega(x), \quad (\text{R.59})$$

où  $u(t, x)$  est la solution de (R.53) et  $u_0(t, x)$  est la solution du problème de Dirichlet pour l'équation de la chaleur, associée à  $\Omega$ , c'est-à-dire

$$\begin{aligned} (\partial_t - \Delta)u_0(t, x) &= 0, & \forall (t, x) \in (0, \infty) \times \Omega, \\ u_0(t, x) &= 0, & \forall (t, x) \in (0, \infty) \times \partial\Omega, \\ u_0(0, x) &= 1, & \forall x \in \Omega, \end{aligned} \quad (\text{R.60})$$

La différence essentielle est que  $u_0(t, \cdot)|_{\partial\Omega} = 0$ , pour tout  $t > 0$ , alors que  $u(t, \cdot)|_{\partial\Omega} \neq 0$  en général. Il n'y a donc pas de relation a priori entre  $H_\Omega(t)$  et  $Q_\Omega(t)$  : la seule information pertinente est donnée par la monotonie du domaine (R.41), qui implique que :

$$Q_\Omega(t) \leq H_\Omega(t), \quad \forall t > 0,$$

et il est clair que cela ne donne pas l'asymptotique de cette dernière. Voir également [vdB13] pour d'autres résultats de comparaison dans le cadre euclidien.

**Echec du principe de Duhamel.** Dans le Chapitre 2, on établit un développement asymptotique complet de  $Q_\Omega(t)$ , lorsque  $t \rightarrow 0$ , à condition que  $\partial\Omega$  n'ait pas de points caractéristiques. La preuve de ce résultat reposait sur une application itérative du principe de Duhamel et sur le fait que  $u_0(t, x)|_{\partial\Omega} = 0$ . En suivant la même stratégie, on applique le principe de Duhamel à une version localisée de  $H_\Omega(t)$  : on fixe une fonction  $\phi \in C_c^\infty(M)$ , avec support compact dans un voisinage tubulaire autour de  $\partial\Omega$  et telle que  $0 \leq \phi \leq 1$  et  $\phi$  est identiquement 1, proche de  $\partial\Omega$ . Ensuite, en utilisant des estimations hors diagonale pour le noyau de la chaleur, on peut prouver que :

$$\omega(\Omega) - H_\Omega(t) = I\phi(t, 0) + O(t^\infty), \quad \text{lorsque } t \rightarrow 0, \quad (\text{R.61})$$

où  $I\phi(t, r)$  est défini pour  $t > 0$  et  $r \geq 0$  comme étant

$$I\phi(t, r) = \int_{\Omega(r)} (1 - u(t, x))\phi(x)d\omega(x), \quad (\text{R.62})$$

où  $\Omega(r) = \{x \in \Omega \mid \delta(x) > r\}$ , avec  $\delta: \Omega \rightarrow \mathbb{R}$  désignant la fonction de distance à  $\partial\Omega$ . Par conséquent, le comportement en temps petit de  $H_\Omega(t)$  est capturé par  $I\phi(t, 0)$ . Par le principe de Duhamel et le lemme de la valeur moyenne sous-riemannienne, on obtient ce qui suit :

$$I\phi(t, 0) = \frac{1}{\sqrt{\pi}} \int_0^t \int_{\partial\Omega} (1 - u(\tau, y)) \phi(y) d\sigma(y) (t - \tau)^{-1/2} d\tau + O(t), \quad (\text{R.63})$$

lorsque  $t \rightarrow 0$ . Pour le contenu thermique classique,  $u_0$  satisfait la condition aux limites de Dirichlet, cf. (R.60), donc (R.63) donnerait l'asymptotique du premier ordre (et on pourrait alors itérer). Au contraire, dans ce cas, on n'a pas de connaissance préalable de  $u(t, y)$  si  $y \in \partial\Omega$  et  $t \rightarrow 0$ . Ainsi, déjà pour l'asymptotique du premier ordre, le principe de Duhamel seul ne parvient pas à donner un développement précis, et nous avons besoin de quelques informations concernant le comportement asymptotique de  $u(t, \cdot)|_{\partial\Omega}$ .

**Asymptotique du premier ordre.** On étudie l'asymptotique de  $u(t, \cdot)|_{\partial\Omega}$ . En utilisant la notion d'approximation nilpotente d'une variété sous-riemannienne, cf. la Section R.2, on déduit l'expansion asymptotique d'ordre zéro de  $u(t, \cdot)|_{\partial\Omega}$  lorsque  $t \rightarrow 0$ , en prouvant le Théorème R.5.2. Ceci est suffisant pour déduire le développement du premier ordre de  $H_\Omega(t)$ , en utilisant (R.63). A ce stade, on itère le principe de Duhamel pour obtenir les termes d'ordre supérieur du développement de  $H_\Omega(t)$ . Cependant, dès la première itération, on obtient la formule suivante pour  $I\phi$  :

$$I\phi(t, 0) = \frac{1}{\sqrt{\pi}} \int_0^t \int_{\partial\Omega} (1 - u(\tau, \cdot))\phi d\sigma (t - \tau)^{-1/2} d\tau \\ + \frac{1}{2\pi} \int_0^t \int_0^\tau \int_{\partial\Omega} (1 - u(\hat{\tau}, \cdot)) N\phi d\sigma ((\tau - \hat{\tau})(t - \tau))^{-1/2} d\hat{\tau} d\tau + O(t^{3/2}), \quad (\text{R.64})$$

lorsque  $t \rightarrow 0$ . Par conséquent, le développement asymptotique d'ordre zéro de  $u(t, \cdot)|_{\partial\Omega}$  ne suffit plus pour obtenir l'asymptotique du second ordre de  $H_\Omega(t)$ .

**La contribution extérieure  $I^c\phi$ .** On a vu que la différence cruciale entre  $H_\Omega(t)$  et  $Q_\Omega(t)$  définis dans (R.59), est liée au fait que  $u(t, \cdot)|_{\partial\Omega} \neq 0$ , alors que  $u_0(t, \cdot)|_{\partial\Omega} = 0$ , pour tout  $t > 0$ . D'un point de vue physique, cette distinction vient du fait que, puisque le bord

$\partial\Omega$  n'est plus isolé, la chaleur régie par le problème de Cauchy  $u(t, x)$ , solution de (R.53), peut se propager également à l'extérieur de  $\Omega$ , tandis que  $u_0(t, x)$ , solution du problème de Dirichlet (R.60), est confinée dans  $\Omega$ , et la température extérieure est toujours 0. Par conséquent, on peut imaginer que l'expansion asymptotique de  $H_\Omega(t)$  est affectée par  $\partial\Omega$ , aussi bien de l'intérieur que de l'extérieur de  $\Omega$ .

En interprétant  $I\phi$  comme la *contribution intérieure* à l'asymptotique de  $H_\Omega$ , on formalise l'intuition physique d'un flot de la chaleur à l'extérieur de  $\Omega$ , en définissant une *contribution extérieure*,  $I^c\phi$  à l'asymptotique. L'observation de départ est la simple relation qui suit : en définissant

$$K_\Omega(t) = \int_{M \setminus \Omega} u(t, x) d\omega(x), \quad \forall t > 0,$$

on a, par le théorème de la divergence (R.26),

$$H_\Omega(t) + K_\Omega(t) = \omega(\Omega), \quad \forall t > 0. \quad (\text{R.65})$$

De manière similaire à (R.62), pour une fonction lisse appropriée  $\phi$ , on peut définir une version localisée de  $K_\Omega(t)$ , que on appelle  $I^c\phi(t, r)$ , de sorte que

$$K_\Omega(t) = I^c\phi(t, 0) + O(t^\infty), \quad \text{lorsque } t \rightarrow 0. \quad (\text{R.66})$$

En utilisant (R.61), (R.65) et (R.66), on montre la relation suivante :

$$I\phi(t, 0) - I^c\phi(t, 0) = O(t^\infty), \quad \text{lorsque } t \rightarrow 0,$$

pour une fonction lisse appropriée  $\phi$ . Par contre, pour la quantité localisée  $I\phi(t, 0) - I^c\phi(t, 0)$  on dispose d'un principe de Duhamel, grâce auquel on peut étudier le développement asymptotique, jusqu'à l'ordre 3, de l'intégrale de  $u(t, x)$  sur  $\partial\Omega$ , cf. le Théorème 4.5.4. La limitation à l'ordre 3 de l'asymptotique est technique et semble difficile à surmonter, cf. Remark 4.5.5. En insérant cette asymptotique dans (R.64), on obtient l'asymptotique jusqu'à l'ordre 3 du développement de  $H_\Omega(t)$ , lorsque  $t \rightarrow 0$ .

**Asymptotique d'ordre quatre.** Puisque nous n'avons à notre disposition que l'asymptotique de l'intégrale de  $u(t, x)$  sur  $\partial\Omega$ , jusqu'à l'ordre 3, il nous faut un argument plus fin pour obtenir l'asymptotique d'ordre quatre de  $H_\Omega(t)$ . La relation, simple mais contraignant, repose à nouveau sur (R.61), (R.65) et (R.66), grâce auxquels on peut écrire :

$$\omega(\Omega) - H_\Omega(t) = \frac{1}{2} (I\phi(t, 0) + I^c\phi(t, 0)) + O(t^\infty), \quad \text{as } t \rightarrow 0.$$

Maintenant pour la somme des contributions  $I\phi(t, 0) + I^c\phi(t, 0)$ , le principe de Duhamel implique ce qui suit :

$$\begin{aligned} I\phi(t, 0) + I^c\phi(t, 0) &= \frac{2}{\sqrt{\pi}} \sigma(\partial\Omega) t^{1/2} \\ &+ \frac{1}{2\pi} \int_0^t \int_0^\tau \int_{\partial\Omega} (1 - 2u(\hat{\tau}, x) N\phi(y)) d\sigma(y) ((\tau - \hat{\tau})(t - \tau))^{-1/2} d\hat{\tau} d\tau + o(t). \end{aligned}$$

Cette fois, on remarque comment l'intégrale de  $u(t, x)$  sur  $\partial\Omega$  apparaît dans un terme du premier ordre (contrairement à ce qui s'est passé dans (R.63) ou (R.64)), donc son développement asymptotique jusqu'à l'ordre 3 implique un développement du quatrième ordre pour  $H_\Omega(t)$ .

### R.5.3 De l'asymptotique du noyau de la chaleur à l'asymptotique du contenu thermique relatif

Dans [CdVHT20, Thm. A], les auteurs ont prouvé l'existence d'une asymptotique en temps petit du noyau de la chaleur hypoelliptique,  $p_t(x, y)$ , voir aussi le Théorème R.2.4 pour l'énoncé précis. Dans le Théorème R.5.2, on a été capable d'exploiter ce résultat pour obtenir l'asymptote d'ordre zéro de la fonction

$$u(t, x) = e^{t\Delta} \mathbf{1}_\Omega(x) = \int_\Omega p_t(x, y) d\omega(y), \quad \forall t > 0, \quad x \in \partial\Omega.$$

Cependant, nous ne sommes pas en mesure d'étendre le Théorème R.5.2 à une asymptotique d'ordre supérieur puisque, en gros, les termes de reste dans le Théorème R.2.4 ne sont pas uniformes lorsque  $t \rightarrow 0$ . Si nous avions un meilleur contrôle sur les restes, nous pourrions en effet intégrer (d'une manière appropriée) l'asymptotique du noyau de la chaleur en temps petit pour obtenir l'expansion correspondante pour  $u(t, x)$ . Enfin, à partir d'une telle expansion, l'asymptotique du contenu thermique relatif découlerait du principe de localisation (R.61) et du principe (itéré) de Duhamel (R.63). Ceci est traité dans la Section 4.6.

### R.5.4 Points caractéristiques

Afin de prouver nos principaux résultats, nous avons besoin de l'hypothèse de ne pas avoir points caractéristiques dans le bord  $\Omega$ , voir la Définition R.3.2. Comme c'était le cas pour le contenu thermique classique, cf. le Chapitre 2, cette hypothèse est cruciale pour suivre notre stratégie. Cependant, contrairement à ce qui se passe dans le cas du problème de Dirichlet, le noyau de la chaleur  $p_t(x, y)$  associé à (R.53) est lisse sur le bord de  $\Omega$ , pour temps positifs, même en présence de points caractéristiques. Ainsi, en principe, il n'y a pas d'obstacle à obtenir un développement asymptotique de  $H_\Omega(t)$  également dans ce cas. De plus, dans les groupes de Carnot de pas 2, un résultat similaire à (R.55) est valable, en remplaçant la mesure de Lebesgue et la mesure de périmètre par la mesure de Haar du groupe et la mesure de périmètre horizontale, respectivement, cf. [BMP12, GT20]. En particulier, la caractérisation des ensembles de périmètre horizontal fini dans les groupes de Carnot de pas 2 est indépendante de la présence de points caractéristiques, ce qui indique qu'une expansion asymptotique telle que (R.58) peut encore être valable, en abandonnant l'hypothèse de ne pas avoir points caractéristiques.

## R.6 Contenu thermique relatif associé à sous-variétés

Dans le Chapitre 5, nous étudions le contenu thermique relatif associé à une sous-variété. A notre connaissance, cette quantité n'a jamais été étudiée de manière systématique, même pas en géométrie riemannienne, aussi nous proposons la définition suivante. Soit  $M$  une variété sous-riemannienne, munie d'une mesure lisse  $\omega$ , et soit  $S \subset M$  une sous-variété lisse et compacte de codimension  $k \geq 0$ . On fixe une mesure lisse  $\mu$  sur  $S$  comme donnée initiale pour l'équation de la chaleur au sens des distributions, et on étudie le problème de Cauchy associé :

$$\begin{aligned} (\partial_t - \Delta)u(t, x) &= 0, & \forall (t, x) \in (0, \infty) \times M, \\ u(t, \cdot) &\xrightarrow{t \rightarrow 0} \mu, & \text{dans } \mathcal{D}'(M), \end{aligned}$$

où  $\Delta = \operatorname{div}_\omega \circ \nabla$  est le sous-laplacien habituel associé à  $\omega$ . Une solution à ce problème, dans le sens des distributions, est donnée par

$$u(t, x) = \int_S p_t(x, y) d\mu(x), \quad \forall (t, x) \in (0, \infty) \times M,$$

où  $p_t(x, y)$  est le noyau de la chaleur (R.54), associé à  $\Delta$  et à la mesure  $\omega$ . On définit le *contenu thermique relatif pour une sous-variété  $S$*  comme :

$$H_S(t) = \int_S \int_S p_t(x, y) d\mu(x) d\mu(y), \quad \forall t > 0. \quad (\text{R.67})$$

On remarque que, d'une part, si  $S = \{x_0\}$  et  $\mu = \delta_{x_0}$ , on obtient la trace du noyau de la chaleur  $p_t(x_0, x_0)$ . D'autre part, si  $S$  est une sous-variété de dimension 0, c'est-à-dire que  $S$  est un ensemble ouvert, relativement compact et lisse, et si on choisit comme mesure  $\mu = \mathbb{1}_\Omega \omega$ , alors (R.67) coïncide avec le contenu thermique relatif habituel comme dans la Définition R.5.1. Par conséquent, une expansion asymptotique en temps petit de (R.67) comprendrait de nombreux cas d'intérêt, allant de l'asymptotique de la trace du noyau de la chaleur aux résultats du Théorème R.5.1.

Notre tentative de calculer l'asymptotique de (R.67) consiste à en construire une approximation appropriée, en utilisant des voisinages tubulaires de  $S$ . Tout d'abord, nous introduisons la définition générale d'une sous-variété non-caractéristique, qui s'applique pour toute codimension  $k \geq 0$ .

**Définition R.6.1** (Sous-variété non-caractéristique). Soit  $M$  une variété sous-riemannienne et soit  $S \subset M$  une sous-variété plongée et lisse de codimension  $k \geq 0$ . On dit qu'un point  $q \in S$  est *non-caractéristique* si

$$\mathcal{D}_q + T_q S = T_q M.$$

On dit que  $S$  est un *sous-variété non caractéristique* si (5.3) vaut pour tout point  $q \in S$ .

On prouve le résultat suivant, au sujet de sous-variétés non-caractéristiques.

**Théorème R.6.1.** *Soit  $M$  une variété sous-riemannienne et  $S \subset M$  une sous-variété compacte, lisse et non-caractéristique de codimension  $k \geq 1$ . Alors, il existe  $r_0 > 0$  tel que, en désignant par  $S_{r_0} = \{p \in M \mid 0 < \delta_S < r_0\}$ , les conditions suivantes sont vérifiées :*

i)  $\delta_S : S_{r_0} \rightarrow [0, \infty)$  est lisse et tel que  $\|\nabla \delta_S\|_g = 1$  ;

ii) il existe un difféomorphisme  $G : (0, r_0) \times \{\delta_S = r_0\} \rightarrow S_{r_0}$  tel que :

$$\delta_S(G(r, p)) = r \quad \text{et} \quad G_* \partial_r = \nabla \delta_S.$$

Le Théorème R.6.1 généralise le résultat analogue dans [FPR20, Prop. 3.1], et [PRS18, Lem. 23], aux sous-variétés de toute codimension  $k \geq 1$ .

Deuxièmement, pour tout  $\varepsilon \leq r_0$ , on considère le contenu thermique relatif (renormalisé) associé à  $S_\varepsilon$ , qui désigne le voisinage tubulaire de  $S$  de rayon  $\varepsilon$  :

$$H_S^\varepsilon(t) = \frac{1}{\omega(S_\varepsilon)^2} \int_{S_\varepsilon} \int_{S_\varepsilon} p_t(x, y) d\omega(x) d\omega(y), \quad \forall t > 0. \quad (\text{R.68})$$

On montre que  $H_S^\varepsilon$  converge ponctuellement vers  $H_S$ , lorsque  $\varepsilon \rightarrow 0$ . En particulier, on prouve le résultat suivant.



**Proposition R.6.2.** *Soit  $M$  une variété sous-riemannienne, munie d'une mesure lisse  $\omega$ , soit  $S \subset M$  une sous-variété lisse, compacte et non-caractéristique de codimension  $k \geq 1$ . Alors, il existe une mesure de probabilité  $\mu_S$  sur  $S$  telle que pour tout  $t > 0$ ,*

$$H_S^\varepsilon(t) \xrightarrow{\varepsilon \rightarrow 0} H_S(t).$$

Cependant, cette convergence est trop faible pour récupérer quelque information concernant l'expansion asymptotique en temps petit de la limite. En fait, on montre que (R.68) ne parvient pas à récupérer l'expansion asymptotique du contenu thermique relatif du sous-variété, en étudiant un exemple dans lequel il est possible d'effectuer des calculs explicites.





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# Asymptotiques en temps petit du contenu thermique en géométrie sous-riemannienne

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## Résumé

La géométrie sous-riemannienne est une classe particulièrement riche de structures métriques, qui généralise la géométrie riemannienne. Une structure sous-riemannienne sur une variété lisse  $M$  est définie par un couple  $(D, g)$ , où  $g$  est un produit scalaire lisse défini sur un sous-ensemble de directions admissibles, qui est appelée distribution, et doit satisfaire la condition de Hörmander. Dans ce cas,  $M$  est connexe par des courbes horizontales, et la procédure habituelle de minimisation de la longueur donne une métrique bien définie. L'opérateur de Laplace-Beltrami est généralisé par le sous-Laplacien qui est sub-elliptique, mais possède néanmoins des propriétés de régularité appropriées (en particulier, il est hypoelliptique). Dans cette thèse, nous étudions l'asymptotique du contenu thermique et des sujets connexes en géométrie sous-riemannienne.

Pour un ensemble ouvert et relativement compact dans  $M$ , le contenu thermique est défini comme la quantité totale de chaleur contenue dans l'ensemble au temps  $t$ , en supposant que son bord est maintenu à température nulle. Dans le groupe de Heisenberg, qui est l'exemple le plus simple de structure sous-riemannienne, Tyson et Wang ont étudié le comportement à petit temps de cette quantité, prouvant l'existence d'une expansion asymptotique jusqu'à l'ordre 2 en racine carrée de  $t$ . Ici, en utilisant une approche différente, nous montrons l'existence d'une expansion asymptotique complète du contenu thermique dans toute variété sous-riemannienne, fournissant également un algorithme pour calculer ses coefficients à tout ordre. Comme dans le cas riemannien, le comportement en temps petit du contenu thermique contient des informations géométriques sur le bord de l'ensemble. Une hypothèse cruciale sur l'ensemble pour développer une expansion asymptotique complète du contenu thermique est l'absence de points caractéristiques. En gros, un point caractéristique est un point de la frontière du domaine (que l'on suppose être un sous-variété lisse de  $M$ ) où la fonction de distance de la frontière elle-même perd sa régularité. Nous montrons que, si des points caractéristiques sont présents, un nouveau phénomène se produit dans l'expansion asymptotique du contenu thermique. En particulier, l'asymptotique établie en absence de points caractéristiques ne peut plus être vraie à un ordre égal ou supérieur à 5, en général.

Ensuite, nous étudions la courbure moyenne horizontale du bord lorsque des points caractéristiques sont présents. Dans le cas du groupe de Heisenberg, nous introduisons la notion de point caractéristique modérément dégénéré, prouvant de nouveaux résultats d'intégrabilité pour la courbure moyenne horizontale des surfaces. Ce résultat, dans le cas de surfaces analytiques dans le groupe de Heisenberg, répond affirmativement à une conjecture formulée par Danielli, Garofalo et Nhieu.

Enfin, nous étudions le contenu thermique *relatif*. Pour un ensemble ouvert et relativement compact dans  $M$ , le contenu thermique relatif est défini comme la quantité totale de chaleur contenue dans l'ensemble au temps  $t$ , en permettant la chaleur de se propager à l'extérieur du domaine. Des difficultés importantes apparaissent par rapport au contenu thermique "classique", car cette fois-ci le comportement au bord de la fonction de température n'est plus connu. Nous utilisons un argument de symétrie "asymptotique" de la chaleur pour obtenir des informations sur le comportement en temps petit de la température à la frontière de l'ensemble et nous obtenons une expansion asymptotique à ordre 4 en racine carrée de  $t$ .

## Mots-clés

Géométrie sous-Riemannienne, contenu thermique, analyse asymptotique, points caractéristiques.



# Heat content asymptotics in sub-Riemannian geometry

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## Abstract

Sub-Riemannian geometry is a particularly rich class of metric structures, which generalizes Riemannian geometry, where a smoothly varying metric is defined only on a subset of preferred directions of the tangent space at each point of a smooth manifold  $M$  (called horizontal directions). Under the so-called Hörmander condition,  $M$  is horizontally-path connected, and the usual length-minimization procedure yields a well-defined metric. The Laplace-Beltrami operator is generalized by the sub-Laplacian which is sub-elliptic, but has nonetheless suitable regularity properties (in particular, it is hypoelliptic). In this thesis, we investigate the heat content asymptotics and related topics in sub-Riemannian geometry.

For a domain in  $M$ , the heat content is defined as the total amount of heat contained in the domain at time  $t$ , assuming that its boundary is kept at zero temperature for all times. In the Heisenberg group, which is the simplest example of sub-Riemannian structure, Tyson and Wang studied the small-time behavior of this quantity, proving the existence of an asymptotic expansion up to order 2 in square root of  $t$ . Here, using a different approach, we show the existence of a complete asymptotic expansion of the heat content in any sub-Riemannian manifold, also providing an algorithm for computing its coefficients at any order. As in the Riemannian case, the small-time behavior of the heat content contains geometrical information of the boundary of the domain.

A crucial assumption on the domain for developing a complete asymptotic expansion of the heat content is the absence of characteristic points. Roughly speaking, a characteristic point is a point of the boundary of the domain (which is assumed to be a smooth submanifold of  $M$ ) where the distance function from the boundary itself loses regularity. We show that, if characteristic points are present a new phenomenon occurs in the asymptotic expansion of the heat content. In particular, the latter can no longer be true to order equal to or greater than 5, in general.

In addition, we study the horizontal mean curvature of the boundary when characteristic points are present. In the particular case of the Heisenberg group, we introduce the notion of a mildly degenerate characteristic point, proving new integrability results for the horizontal mean curvature of surfaces. This result, in the case of analytic surfaces in the Heisenberg group, answers affirmatively to a conjecture formulated by Danielli, Garofalo and Nhieu.

Finally, we study a related, yet different, quantity called the *relative* heat content. For a domain in  $M$ , the relative heat content is defined as the total amount of heat contained in the domain at time  $t$ , allowing the heat to flow outside the domain. Significant difficulties emerge, as the boundary behavior of the temperature function is no longer known, as opposed to the case of the classical heat content. We use an “asymptotic” symmetry argument of the heat diffusion to obtain information on the small-time behavior of temperature at the boundary of the domain and we obtain a fourth-order asymptotic expansion in square root of  $t$ .

## Keywords

Sub-Riemannian geometry, heat content, asymptotic analysis, characteristic points

