Heterogeneity in scalar conservation laws: approximation and applications

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- Hein ? Une thèse ? Sur quoi ?
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- Ok."

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RÉSUMÉ

Dans cette thèse, on traite la prise en compte de l'hétérogénéité dans les lois de conservation scalaires, c'est-à-dire les lois de conservation non invariantes par translation en espace. Ces équations apparaissent notamment dans les modèles de trafic. Par exemple, les mécanismes suivants introduisent de l'hétérogénéité : la présence de feux de circulation, des portions de route où la vitesse maximale est limitée, la variabilité de l'état de la route, *etc*... La prise en compte de l'hétérogénéité permet d'enrichir les modèles de trafic. On aborde trois classes de problèmes inhomogènes pour lesquelles on complète et approfondit le cadre mathématique pour l'analyse théorique et l'approximation numérique.
Nous explorons en détail le cadre où l'hétérogénéité est matérialisée par l'ajout d'une ou plusieurs interfaces mobiles. Le long des interfaces, on impose une condition de majoration sur le flux de la loi de conservation. Cette classe de modèles permet de tenir compte de la présence d'un petit nombre de véhicules encombrants et lents (ou alors, de véhicules autonomes qui ont pour rôle la régulation du trafic). Dans ce cadre, l'évolution des interfaces et des contraintes est couplée de façon non locale à l'état du trafic et/ou à des paramètres spécifiant l'état du véhicule ou du conducteur. En outre, nous élaborons une description de l'hétérogénéité du trafic résultant des variations du degré d'organisation des conducteurs, dans le cadre des modèles dits "du second ordre". L'aspect numérique est prépondérant pour les modèles de trafic que nous étudions. On construit des schémas numériques robustes et on élabore des techniques de compacité spécifiques. La convergence de ces schémas conduit à des résultats d'existence.
Enfin, en lien avec le modèle décrivant l'évolution d'une densité de véhicules sur une route hétérogène, on étudie théoriquement une loi de conservation dans laquelle la dépendance spatiale du flux est explicite. Des résultats classiques sur le caractère bien posé ou la correspondance avec l'équation de Hamilton-Jacobi associée sont obtenus sous des hypothèses plus en adéquation avec la modélisation que celles rencontrées dans la littérature. Les applications allant au-delà de la description du trafic, on se donne pour objectif l'analyse approfondie des problèmes d'identification de données initiales.

Mots clés : Lois de conservation hétérogènes ; Modèles de trafic ; Interfaces mobiles ; Schéma volumes finis ; Inverse design

ABSTRACT

This thesis is devoted to the treatment of heterogeneity in scalar conservation laws. We call heterogeneous a conservation law which is not invariant by space translation. These equations arise for instance in traffic flow dynamics modeling. The presence of traffic lights or roads that have a variable maximum speed limit are examples of mechanisms which lead to heterogeneous conservation laws. Considering such equations is a way to expand macroscopic traffic flow models. We tackle three classes of inhomogeneous problems for which we extend the mathematical framework for both the theoretical analysis and the numerical approximation.

We fully investigate the treatment of heterogeneity when one or several moving interfaces are added in the classic LWR model for traffic flow. Flux constraints are attached to each interfaces. The resulting class of models can be used to take into account the presence of slow moving vehicles that reduce the road capacity and thus generates moving bottlenecks for the surrounding traffic flow. They can also describe the regulating effect of autonomous vehicles. In this framework, the interfaces and the constraints are linked in a nonlocal way to the traffic density and/or to an orderliness marker describing the state of the drivers. The description of the heterogeneity caused by the variations in the drivers' organization leads to the analysis of a so-called second order model. The numerical aspect plays a central role in the analysis of these traffic flow models. We construct robust numerical schemes and establish specific techniques to obtain compactness of the approximate solutions. Proving the convergence of these schemes lead to existence results.

Finally, with the space-dependent LWR traffic flow model in mind, we theoretically analyze a class of scalar conservation laws with explicit space dependency. Classical results such as well-posedness or the link to the associated Hamilton-Jacobi equation are obtained under a set of assumptions more fitting with the modeling hypothesis. With applications that go beyond traffic modeling in mind, we aim to tackle initial data identification problems.

Keywords: Heterogenous conservation laws; Traffic flow models; Moving interfaces; Finite volume scheme; Inverse design

CONTENTS

INTRODUCTION

0.1 Introduction générale

0.1.1 Lois de conservation scalaires

Dans le cas le plus simple, une loi de conservation scalaire en dimension 1 est une équation aux dérivées partielles (EDP) de la forme suivante :

$$\frac{\partial \rho}{\partial t}(x,t) + \frac{\partial (f(\rho))}{\partial x}(x,t) = 0, \quad (x,t) \in \Omega = \mathbb{R} \times (0, +\infty). \tag{0.1.1}$$

Ce genre d'équations décrit le fait que la quantité $\rho = \rho(x,t)$ (désigne en général une densité) est conservée et précise son évolution *via* la fonction f, appelée *flux/diagramme fondamental*. Dans les applications, t désigne la variable temporelle, x la position spatiale et il arrive souvent que la fonction flux soit également une fonction du temps et de l'espace. Par ailleurs, ce sont surtout des *systèmes* de lois de conservation qui apparaissent naturellement dans la description de certains phénomènes physiques comme les écoulements en eau peu profonde, la propagation des ondes ou la dynamique des gaz (équations d'Euler pour des gaz compressibles, p-système). Par ailleurs, la dynamique des gaz a fortement guidé les travaux liés aux lois de conservation afin d'en donner une interprétation physique. Dans cette thèse, les lois de conservation considérées sont toutes scalaires et unidimensionnelles. Pour l'instant, restons dans le cadre d'une seule loi de conservation avec un flux indépendant du temps et de l'espace. Remarquons que si on intègre (formellement) (0.1.1) entre deux points $a, b \in \mathbb{R}$ ($a < b$), on obtient :

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_a^b \rho(x,t)\,\mathrm{d}x\right) = f(\rho(a,t)) - f(\rho(b,t)).$$

Cette égalité exprime le fait que la variation de ρ entre deux points a et b est égale à la différence du flux en ces deux points. Autrement dit, la quantité ρ n'est ni consommée, ni créée au cours du temps. Sous des hypothèses d'intégrabilité, on est conduit à

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\mathbb{R}} \rho(x,t)\,\mathrm{d}x\right) = 0.$$

Autrement dit, la masse totale de ρ ne varie pas au cours du temps. Une loi de conservation apparaît donc dès qu'on veut modéliser l'évolution d'une quantité qui se conserve : conservation de la masse, de la quantité de mouvement (moment), de l'énergie...

Une autre façon de voir les choses est de réécrire (0.1.1) sous forme *quasi-linéaire* (linéaire par rapport aux dérivées) :

$$\partial_t \rho + f'(\rho(x,t))\partial_x \rho = 0, \tag{0.1.2}$$

de sorte que la loi de conservation apparaît comme une équation de transport. On voit donc que lorsque f n'est pas linéaire, la quantité ρ est transportée à vitesse $f'(\rho)$, autrement dit, la vitesse de propagation de la solution dépend elle-même de la solution.

Exemple 0.1.1 (Équation de Burgers non visqueuse/Équation de Hopf)**.** C'est sans doute la plus simple des lois de conservation non linéaires. Elle s'écrit :

$$\partial_t u + \partial_x \left(\frac{u^2}{2} \right) = 0. \tag{0.1.3}$$

Cette équation peut par exemple décrire la vitesse $u = u(x,t)$ d'un champ de particules qui n'interagissent pas entre elles (milieu isolé, unidimensionnel). On gardera cette équation comme fil rouge tout au long de cette section afin d'illustrer les concepts introduits.

Exemple 0.1.2 (Modèle de trafic routier)**.**

$$\partial_t \rho + \partial_x \left(\rho v(\rho) \right) = 0, \tag{0.1.4}$$

où v est une fonction donnée. Sous de bonnes hypothèses sur v, entre d'autres décroissance et positivité, cette loi de conservation peut décrire le déplacement de la quantité $\rho = \rho(x,t) \in [0,1]$ qui représente une densité de voitures (par exemple) sur une route unidimensionnelle. Ici, le trafic se déplace à vitesse $v(\rho(x,t))$ de sorte que plus le trafic est dense, moins la vitesse de déplacement est importante. Le cas particulier $v(\rho) = V_{\max}(1-\rho)$ ($V_{\max} > 0$) conduit au célèbre modèle LWR, voir la section 0.1.2 pour plus de détails.

Penchons nous maintenant sur la résolution de (0.1.1), et plus précisément sur la résolution du problème de Cauchy associé :

$$\begin{cases} \partial_t \rho + \partial_x (f(\rho)) = 0 \\ \qquad\qquad \rho(\cdot, 0) = \rho_0, \end{cases} \tag{0.1.5}$$

ρ_0 étant une fonction donnée.

Solution classique et méthode des caractéristiques. On suppose ici que les fonctions f et ρ_0 sont aussi régulières que l'on veut. Assez naturellement, on commence par chercher des *solutions classiques* de (0.1.5), c'est-à-dire des fonctions $\rho \in \mathbf{C}^1(\Omega) \cap \mathbf{C}(\overline{\Omega})$ vérifiant l'équation en tout en point de Ω et la condition initiale sur $\mathbb{R} \times \{0\}$. Pour ce faire, on applique la *méthode des caractéristiques*, voir [101, 81, 144, 98] pour une approche complète et rigoureuse. L'idée est la suivante : on s'intéresse à l'évolution d'une solution de l'équation (0.1.5) le long d'une

courbe $x = x(t) \in \mathbf{C}^1((0,+\infty))$, qu'on appelle ici caractéristique. Dans le cas présent, on obtient :

$$\forall t > 0, \quad \frac{\mathrm{d}}{\mathrm{d}t}\left(\rho(x(t),t)\right) = \partial_t \rho(x(t),t) + x'(t)\partial_x \rho(x(t),t).$$

Mettant ceci en parallèle avec (0.1.2) (qui est bien équivalente à (0.1.1) puisque les fonctions sont régulières), on se rend compte qu'une solution classique de (0.1.5) est constante le long des courbes vérifiant $x'(t) = f'(\rho(x(t),t))$. Si x est une telle courbe, en notant $x_0 = x(0)$, alors pour tout $t > 0$, on obtient :

$$x'(t) = f'(\rho(x(t),t)) = f'(\rho_0(x_0)),$$

donc les caractéristiques de (0.1.5) sont des droites. Ceci nous permet *a priori* de définir la solution ρ de la façon suivante :

— on se fixe un point $(\xi,\tau) \in \Omega$;
— on résout l'équation

$$\xi = y + \tau f'(\rho_0(y)), \quad y \in \mathbb{R}; \tag{0.1.6}$$

— on pose $\rho(\xi,\tau) = \rho_0(y)$.

Contrairement au cas linéaire (*i.e.* f linéaire) où l'équation (0.1.6) a toujours une unique solution, la non-linéarité de f est source d'apparition de nouveaux phénomènes. Entre autres, si f n'est pas linéaire, alors il se peut que l'équation (0.1.6) admette plusieurs solutions, rendant alors impossible de définir ρ de manière continue.
Réciproquement, si ρ est définie de la manière suivante, alors on se rend compte que ρ vérifie l'équation implicite :

$$\forall (\xi,\tau) \in \Omega, \quad \rho(\xi,\tau) = \rho_0\left(\xi - \tau f'(\rho(\xi,\tau))\right). \tag{0.1.7}$$

Cette dernière égalité signifie que la solution est obtenue en propageant les valeurs de la donnée initiale. Dans le cas linéaire, toutes les valeurs sont propagées à la même vitesse, donc le graphe de la solution à un temps donné est obtenu en translatant le graphe de la donnée initiale. Dans le cas non-linéaire, toutes les valeurs de la donnée initiale ne se propagent pas à la même vitesse et donc la solution se déforme au cours du temps, voir la figure 2 ci-dessous. Mentionnons enfin que dans les conditions d'application du théorème des fonctions implicites, l'égalité (0.1.7) nous fournit la régularité de ρ et on vérifie ensuite qu'elle est effectivement une solution classique du problème de Cauchy (0.1.5), voir par exemple [33].

Exemple 0.1.3 (Équation de Burgers II)**.** Pour l'équation de Burgers (0.1.3), considérons les problèmes de Cauchy associées aux données initiales

$$u_0(x) = \arctan(x) \quad \text{et} \quad v_0 = -2\arctan(x).$$

Avec u_0, l'équation (0.1.6) est réduite à

$$\xi = y + \tau \arctan(y), \quad y \in \mathbb{R},$$

et une rapide étude de fonction montre que pour tout $(\xi,\tau) \in \Omega$, cette équation admet une unique solution. Cela se traduit par le fait que les caractéristiques ne s'intersectent pas dans

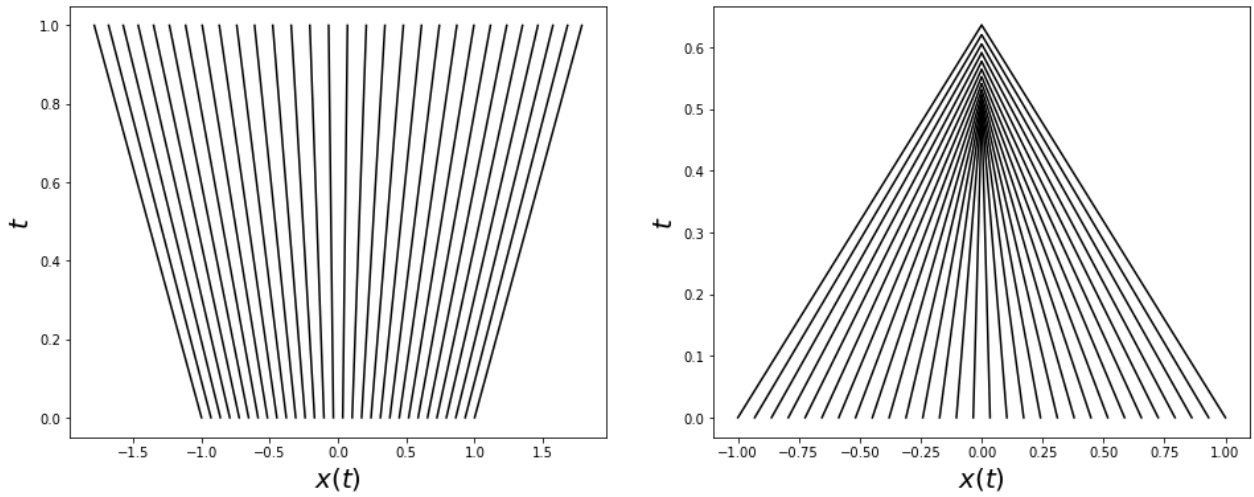

FIGURE 1 – Caractéristiques de l'équation de Burgers/Hopf avec donnée initiale u_0 (à gauche) et v_0 (à droite).

Ω comme on peut le voir sur la figure 1 (gauche). Ainsi, le problème de Cauchy (0.1.5) avec donnée initiale u_0 admet une solution classique définie sur Ω.
Avec la donnée initiale v_0, l'équation (0.1.6) devient

$$\xi = y - 2\tau \arctan(y), \quad y \in \mathbb{R}.$$

En notant $\phi_\tau(y) = y - 2\tau \arctan(y)$, on peut montrer que :

- si $\tau \leq \frac{1}{2}$, alors ϕ_τ est strictement croissante et pour tout $\xi \in \mathbb{R}$, (0.1.6) admet une unique solution. Les caractéristiques ne se croisent pas dans $\mathbb{R} \times (0, \tau)$ (voir la figure 1, à droite) et on peut donc définir une solution classique dans $\mathbb{R} \times (0, \tau)$;
- si $\tau > \frac{1}{2}$, alors pour tout $\xi \in [-\phi_\tau(\sqrt{2\tau - 1}), \phi_\tau(\sqrt{2\tau - 1})] \backslash \{0\}$, l'équation (0.1.6) admet exactement deux solutions (et trois solutions si $\xi = 0$). Ceci est confirmé par la figure 1, (à droite) où on peut voir les caractéristiques se croiser.

Pour illustrer le sens de l'égalité (0.1.7), on a représenté dans la figure 2 ci-dessous les deux solutions des problèmes de Cauchy associées aux données initiales u_0 et v_0. Comme on peut à droite, au temps $t = 0.5$ (premier temps d'intersection des caractéristiques), v présente une discontinuité en 0 (là où les caractéristiques se croisent).

Lorsque des caractéristiques se coupent en un point $(\xi, \tau) \in \Omega$, il n'est plus possible de définir une solution classique en (ξ, τ) puisque chaque caractéristique qui passe par (ξ, τ) transporte une valeur. Comme on le voit sur la Figure 2 (droite), la solution devient alors *multivaluée* (on parle de choc).
Dans certains cas, il est même possible de déterminer le temps d'existence d'une solution classique c'est-à-dire le plus petit temps à partir duquel des caractéristiques se croisent. Comme on a pu le voir dans l'exemple 0.1.3, ce temps sera lié aux variations de $f' \circ \rho_0$. Dans

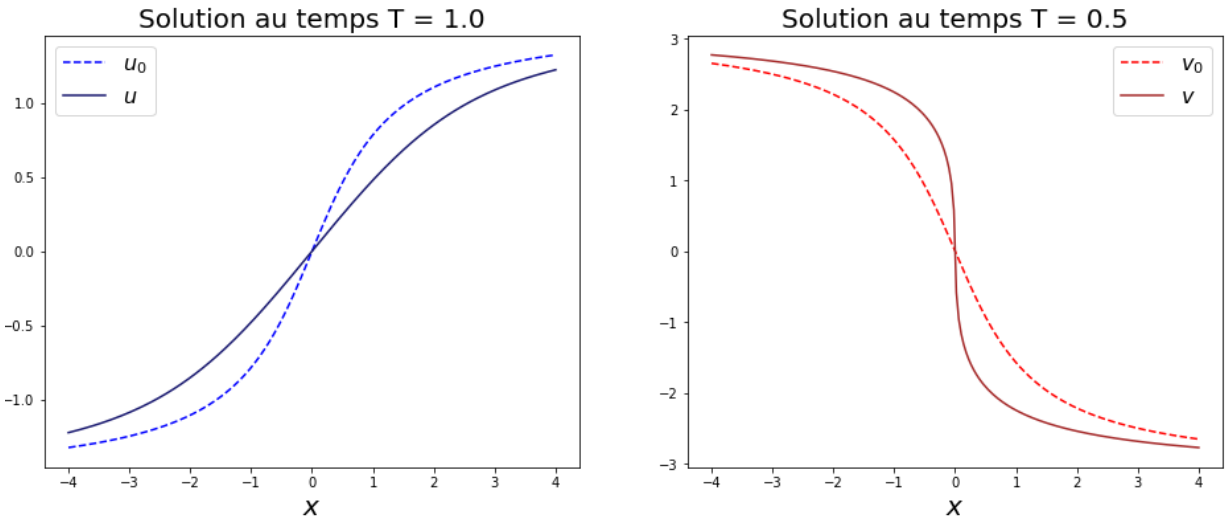

FIGURE 2 – Illustration de l'évolution des graphes des données initiales ; pour une évolution dynamique des solutions, voir : https://utbox.univ-tours.fr/s/iaQKcgooRNcWG3o

le cas où $\rho_0 \in \mathbf{C}^1(\mathbb{R}) \cap \mathbf{W}^{1,\infty}(\mathbb{R})$, on peut montrer qu'en définissant

$$T = \begin{cases} +\infty & \text{si } f' \circ \rho_0 \text{ est croissante} \\ -\dfrac{1}{\inf\,(f' \circ \rho_0)'} & \text{sinon,} \end{cases}$$

alors le problème de Cauchy (0.1.5) admet une unique solution classique définie dans $\mathbb{R} \times (0, T)$, voir [144, Proposition 2.1.1] par exemple.

Solution faible, condition de Rankine-Hugoniot, non-unicité. Ainsi, lorsque la fonction flux f n'est pas linéaire, peu importe la régularité de la donnée initiale, on ne peut pas s'attendre à pouvoir définir globalement (c'est-à-dire sur Ω) une solution classique au problème de Cauchy (0.1.5). Par exemple, pour l'équation de Burgers, seule une donnée initiale croissante conduit à l'existence d'une solution classique dans Ω. L'approche standard est alors d'étendre la notion de solutions à des fonctions possiblement discontinues, par exemple aux fonctions essentiellement bornées. Une *solution faible* de (0.1.1) est alors définie comme une fonction $\rho \in \mathbf{L}^\infty(\Omega)$, ou plus généralement $\rho \in \mathbf{L}^1_{\mathbf{loc}}(\Omega)$, vérifiant pour toute fonction test $\phi \in \mathbf{C}^\infty_{\mathbf{c}}(\Omega)$:

$$\int_0^{+\infty} \int_{\mathbb{R}} (\rho \partial_t \phi + f(\rho) \partial_x \phi)\, \mathrm{d}x\, \mathrm{d}t = 0. \tag{0.1.8}$$

Par ailleurs, si on veut prendre en compte une donnée initiale $\rho_0 \in \mathbf{L}^\infty(\mathbb{R})$, on dit que $\rho \in \mathbf{L}^\infty(\Omega)$ est une solution faible du problème de Cauchy (0.1.5) si pour toute fonction test $\phi \in \mathbf{C}^\infty_{\mathbf{c}}(\overline{\Omega})$, on a

$$\int_0^{+\infty} \int_{\mathbb{R}} (\rho \partial_t \phi + f(\rho) \partial_x \phi)\, \mathrm{d}x\, \mathrm{d}t + \int_{\mathbb{R}} \rho_0(x) \phi(x, 0)\, \mathrm{d}x = 0. \tag{0.1.9}$$

Pour arriver à (0.1.8), on a multiplié l'EDP (0.1.1) par ϕ puis intégré par parties, faisant ainsi porter les dérivées sur la fonction régulière. Clairement, toute solution classique de (0.1.1) en est une solution faible et réciproquement, toute fonction assez régulière vérifiant (0.1.8) est solution classique d'après le lemme de Du Bois-Reymond.

Puisque les formulations intégrales (0.1.8)-(0.1.9) ont été introduites pour prendre en compte les fonctions discontinues, on peut se demander ce qu'impose (0.1.8) aux discontinuités. Soient $\gamma = \gamma(t) \in \mathbf{C}^1((0,+\infty))$ une courbe de Ω et $\rho \in \mathbf{L}^1_{\mathbf{loc}}(\Omega)$. On suppose que ρ est de la forme :

$$\rho(x,t) = \begin{cases} \rho_l(x,t) & \text{si } x < \gamma(t) \\ \rho_r(x,t) & \text{si } x > \gamma(t), \end{cases} \tag{0.1.10}$$

où ρ_l (respectivement ρ_r) est une solution classique de la loi de conservation (0.1.1) dans l'ouvert $\{(x,t) \in \Omega \mid x < \gamma(t)\}$ (respectivement $\{(x,t) \in \Omega \mid x > \gamma(t)\}$). Alors ρ est solution faible de (0.1.1) si et seulement si :

$$\forall t > 0, \quad f(\rho_l(\gamma(t),t)) - f(\rho_r(\gamma(t),t)) = \gamma'(t) \times (\rho_l(\gamma(t),t) - \rho_r(\gamma(t),t)), \tag{0.1.11}$$

Cette égalité est appelée *condition de Rankine-Hugoniot*, voir [144, 98] pour une preuve. Elle exprime la conservation de ρ à travers une discontinuité et fournit une équation différentielle vérifiée par la courbe de discontinuité. Elle nous permet donc de construire facilement des solutions faibles constantes par morceaux, par exemple, à une loi de conservation dont les discontinuités sont séparées par des droites.

Exemple 0.1.4 (Équation de Burgers III)**.** Poursuivons avec l'équation de Burgers (0.1.3). Fixons $\nu > 0$ et considérons les fonctions :

$$\rho(x,t) = \begin{cases} 0 & \text{si} \quad x < 0 \\ \dfrac{x}{t} & \text{si} \quad 0 \le x < t \\ 1 & \text{si} \quad t \le x; \end{cases} \qquad \rho_\nu(x,t) = \begin{cases} 0 & \text{si} \quad x < \dfrac{\nu t}{2} \\ \nu & \text{si} \quad \dfrac{\nu t}{2} \le x < \dfrac{(1+\nu)t}{2} \\ 1 & \text{si} \quad \dfrac{(1+\nu)t}{2} < x, \end{cases} \tag{0.1.12}$$

représentées dans la figure 3. On vérifie rapidement en utilisant la caractérisation (0.1.11) que les $(\rho_\nu)_\nu$ sont toutes solution faible de (0.1.5) avec donnée initiale

$$\rho_0(x) = \begin{cases} 0 & \text{si } x < 0 \\ 1 & \text{si } x > 0. \end{cases}$$

La fonction ρ est continue dans Ω, de classe $\mathbf{C}^1$ dans l'ouvert $U = \{(x,t) \in \Omega \mid 0 < x < t\}$ et pour tout $(x,t) \in U$,

$$f'(\rho(x,t))\partial_x \rho(x,t) = \rho(x,t) \times \frac{1}{t} = \frac{x}{t^2} = -\partial_t \rho(x,t).$$

Puisque $\rho(\cdot,t) \underset{t\to 0^+}{\longrightarrow} \rho_0$ dans $\mathbf{L}^1_{\mathbf{loc}}(\mathbb{R})$, on en déduit que ρ est également solution faible de (0.1.5). On vient donc de construire une infinité de solutions faibles au même problème de Cauchy.

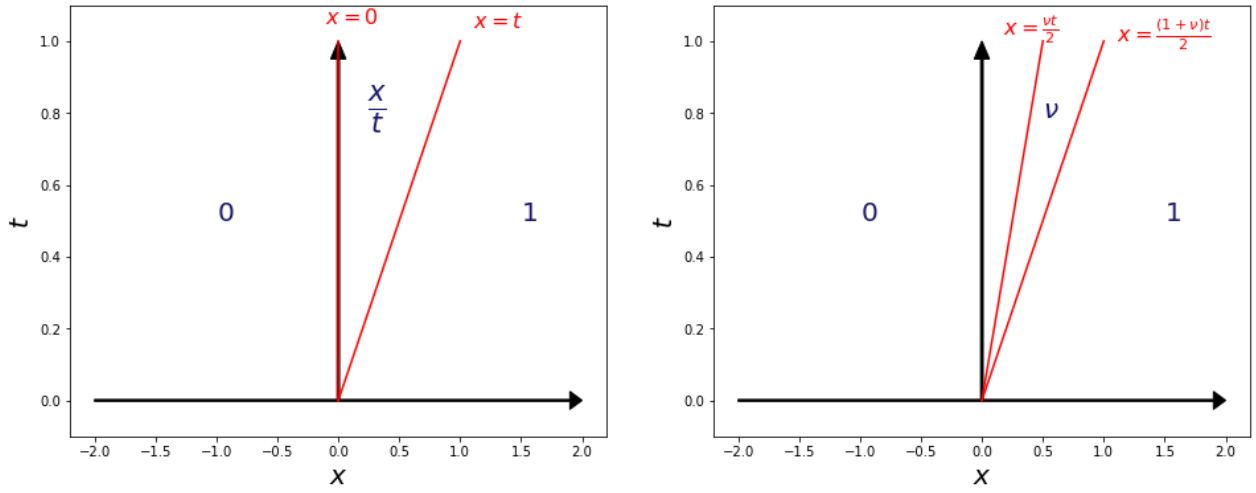

FIGURE 3 – Construction d'une infinité de solutions faibles (à droite).

Viscosité évanescente, solution entropique. Les deux paragraphes précédents ont mis en lumière les deux phénomènes suivants pour la loi de conservation (0.1.1) :

— non-existence (en général) d'une solution classique après un certain temps $T > 0$: les caractéristiques se croisent ;
— non-unicité des solutions faibles : la condition de Rankine-Hugoniot n'est pas assez restrictive sur les discontinuités.

Il est donc nécessaire d'imposer un critère supplémentaire qui permette de sélectionner la bonne solution. Dans le cadre des lois de conservation scalaires, il y a plusieurs moyens d'exprimer ce critère, chaque critère se basant sur le concept *d'entropie*. Un de ces critères consiste à voir la loi de conservation (0.1.5) comme limite du problème parabolique

$$\begin{cases} \partial_t \rho + \partial_x (f(\rho)) = \varepsilon \partial^2_{xx} \rho, \quad \varepsilon > 0 \\ \rho(\cdot, 0) = \rho_0. \end{cases} \tag{0.1.13}$$

Le terme $\varepsilon \partial^2_{xx} \rho$ est un terme de viscosité/diffusion artificielle. L'idée derrière l'introduction de l'équation (0.1.13) (héritée encore une fois de la dynamique des gaz) est que le problème physique modélisé par la loi de conservation (0.1.5) est visqueux, et que l'EDP représente le modèle limite quand la diffusion tend vers 0.

On peut montrer sous diverses hypothèses sur f et ρ_0, voir par exemple [114, 127, 101, 98], que pour tout $\varepsilon > 0$, (0.1.13) admet une solution régulière $\rho_\varepsilon \in \mathbf{C}^\infty(\Omega)$, et qu'on peut extraire de la suite $(\rho_\varepsilon)_\varepsilon$ une sous-suite qui converge presque partout sur Ω vers une fonction $\rho \in \mathbf{L}^\infty(\Omega)$. On s'attend alors à ce que la fonction limite ρ, qui est solution faible, possède des propriétés supplémentaires par rapport aux autres solutions. Fixons $\eta \in \mathbf{C}^2(\mathbb{R})$ une fonction convexe (appelée entropie) et notons Φ une primitive de $\eta' f'$. On multiplie l'EDP (0.1.13) par $\eta'(\rho_\varepsilon)$

et on utilise la convexité de η :

$$\begin{aligned}\partial_t\left(\eta(\rho_\varepsilon)\right)+\partial_x\left(\Phi(\rho_\varepsilon)\right)=\varepsilon\eta'(\rho_\varepsilon)\partial^2_{xx}\rho_\varepsilon&=\varepsilon\partial^2_{xx}\left(\eta(\rho_\varepsilon)\right)-\varepsilon\eta''(\rho_\varepsilon)(\partial_x\rho_\varepsilon)^2\\&\leq\varepsilon\partial^2_{xx}\left(\eta(\rho_\varepsilon)\right).\end{aligned}\tag{0.1.14}$$

Remarque 0.1.1 (Dissipation d'entropie)**.** L'inégalité (0.1.14) a une conséquence intéressante. Sous de bonnes hypothèses d'intégrabilité, on obtient que

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\mathbb{R}}\eta(\rho(x,t))\,\mathrm{d}x\right)\leq 0.$$

Cette inégalité traduit le caractère irréversible des phénomènes modélisés par une loi de conservation : au cours du temps l'entropie totale est dissipée, on perd de l'information. Notons au passage que si ρ est une solution classique de la loi de conservation sans viscosité ($\varepsilon=0$), alors en procédant aux mêmes manipulations que dans (0.1.14), on trouve que

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\mathbb{R}}\eta(\rho(x,t))\,\mathrm{d}x\right)=0.$$

Les solutions classiques sont, elles, *isentropiques*.

On déduit de (0.1.14) que ρ doit vérifier pour toute fonction test $\varphi\in\mathbf{C}^\infty(\Omega)$, $\varphi\geq 0$ et pour toute fonction convexe $\eta\in\mathbf{C}^2(\mathbb{R})$, l'inégalité intégrale :

$$\int_0^{+\infty}\int_{\mathbb{R}}\left(\eta(\rho)\partial_t\varphi+\Phi(\rho)\partial_x\varphi\right)\mathrm{d}x\,\mathrm{d}t\geq 0.$$

Par des arguments classiques d'approximation, on peut étendre cette analyse pour les fonctions convexes $\eta_\kappa(\rho)=|\rho-\kappa|$ ($\kappa\in\mathbb{R}$). En tenant compte de la donnée initiale, on est conduit à la notion suivante de *solution entropique*, due à Kruzhkov : une fonction $\rho\in\mathbf{L}^\infty(\Omega)$ est une solution entropique de (0.1.5) si pour tout $\kappa\in\mathbb{R}$ et pour toute fonction test $\varphi\in\mathbf{C}^\infty_{\mathbf{c}}(\overline{\Omega})$, $\varphi\geq 0$:

$$\int_0^{+\infty}\int_{\mathbb{R}}\left(|\rho-\kappa|\partial_t\varphi+\Phi(\rho,\kappa)\partial_x\varphi\right)\mathrm{d}x\,\mathrm{d}t+\int_{\mathbb{R}}|\rho_0(x)-\kappa|\varphi(x,0)\,\mathrm{d}x\geq 0,\tag{0.1.15}$$

où Φ, appelé *flux entropique*, est donné par

$$\Phi(\rho,\kappa)=\mathrm{sgn}(\rho-\kappa)(f(\rho)-f(\kappa)).$$

En considérant $\kappa\in\mathbb{R}$, $|\kappa|\geq\|\rho\|_{\mathbf{L}^\infty}$ dans (0.1.15), on se rend compte que ρ vérifie la formulation faible (0.1.9) pour toute fonction test $\varphi\in\mathbf{C}^\infty_{\mathbf{c}}(\overline{\Omega})$, $\varphi\geq 0$. Si φ n'est pas positive, on la décompose en partie positive et négative : $\varphi=\varphi^+-\varphi^-$, et la linéarité (en φ) de (0.1.9) assure que toute solution entropique de (0.1.5) est aussi solution faible.
Insistons sur le fait que (0.1.15) est en fait une famille d'inégalités. Contrairement à la définition de solution faible (0.1.9) où on ne pouvait jouer que sur la fonction test, la formulation de Kruzhkov permet également de jouer sur la paramètre κ, imposant naturellement des contraintes supplémentaires sur une solution entropique.

Reprenons le raisonnement développé dans la section 0.1.1. Soient donc $\gamma = \gamma(t) \in \mathbf{C}^1((0,+\infty))$ une courbe de Ω et $\rho \in \mathbf{L}^1_{\mathbf{loc}}(\Omega)$ de la forme (0.1.10). Alors ρ est solution entropique de (0.1.5) si et seulement si pour tout $\kappa \in \mathbb{R}$ et pour tout $t > 0$,

$$\Phi(\rho_l(\gamma(t),t)) - \Phi(\rho_r(\gamma(t),t)) \geq \gamma'(t) \times (|\rho_l(\gamma(t),t) - \kappa| - |\rho_r(\gamma(t),t) - \kappa|)\,, \tag{0.1.16}$$

voir par exemple [98] pour une preuve. Si on suppose par exemple qu'en un point $\tau > 0$, on a $\rho_l(\tau) < \rho_r(\tau)$ (on a omis la dépendance en γ pour clarifier la suite), alors la condition d'entropie (0.1.16) se réécrit :

$$\forall \kappa \in]\rho_l(\tau), \rho_r(\tau)[, \quad f(\kappa) \geq f(\rho_l(\tau)) + \frac{f(\rho_r(\tau)) - f(\rho_l(\tau))}{\rho_r(\tau) - \rho_l(\tau)}(\kappa - \rho_l(\tau)). \tag{0.1.17}$$

Autrement dit, sur $]\rho_l(\tau), \rho_r(\tau)[$, le graphe de f est au dessus de sa corde. On montrerait de la même façon que si $\rho_l(\tau) > \rho_r(\tau)$, alors le graphe de f doit être en dessous de sa corde.

Exemple 0.1.5 (Condition de Lax, équation de Burgers IV)**.** Dans le cas où la fonction est par exemple strictement convexe, la condition d'entropie (0.1.17) est équivalente à la *condition d'admissibilité de Lax* :

$$\forall t > 0, \quad f'(\rho_l(t)) > \frac{f(\rho_r(t)) - f(\rho_l(t))}{\rho_r(t) - \rho_l(t)} > f'(\rho_r(t)), \tag{0.1.18}$$

initialement formulée dans [117]. Cette inégalité a une interprétation géométrique illustrée en figure 4 : lorsque la condition de Lax est violée, des caractéristiques sortent de la discontinuité (on dit parfois que de l'information est créée). On voit également que dans le cas convexe, une solution entropique ne peut avoir que des *discontinuités décroissantes*. Par conséquent, aucune des solutions faibles $(\rho_\nu)_\nu$ définies par (0.1.12) n'est solution entropique. Dans la figure ci-dessous, on illustre géométriquement la condition d'admissibilité de Lax en dessinant les caractéristiques pour les données initiales

$$\rho_0^1(x) = \begin{cases} 1 & \text{si} \quad x < 0 \\ 0 & \text{si} \quad x > 0 \end{cases} \quad \text{et} \quad \rho_0^2(x) = \begin{cases} 0 & \text{si} \quad x < 0 \\ 1 & \text{si} \quad x > 0. \end{cases}$$

Entre

$$\rho^1(x,t) = \begin{cases} 1 & \text{si} \quad x < \dfrac{t}{2} \\ 0 & \text{si} \quad x > \dfrac{t}{2} \end{cases} \quad \text{et} \quad \rho^2(x,t) = \begin{cases} 0 & \text{si} \quad x < \dfrac{t}{2} \\ 1 & \text{si} \quad x > \dfrac{t}{2}. \end{cases}$$

seule ρ^1 est donc solution entropique.

Dans son papier [114], Kruzhkov a réglé de nombreux problèmes théoriques en montrant le caractère bien posé du problème de Cauchy dans le cadre des solutions entropiques, unifiant différents points de vue de ses prédécesseurs [142, 100, 131, 117, 155]. Sa preuve d'unicité utilise une méthode de dédoublement de variables et conduit à une contraction dans $\mathbf{C}((0,+\infty);\mathbf{L}^1(\mathbb{R}))$. Plus précisément si ρ et σ sont deux solutions entropiques de (0.1.5) associées aux données initiales ρ_0, σ_0, alors pour tout $t > 0$,

$$\|\rho(\cdot,t) - \sigma(\cdot,t)\|_{\mathbf{L}^1} \leq \|\rho_0 - \sigma_0\|_{\mathbf{L}^1}. \tag{0.1.19}$$

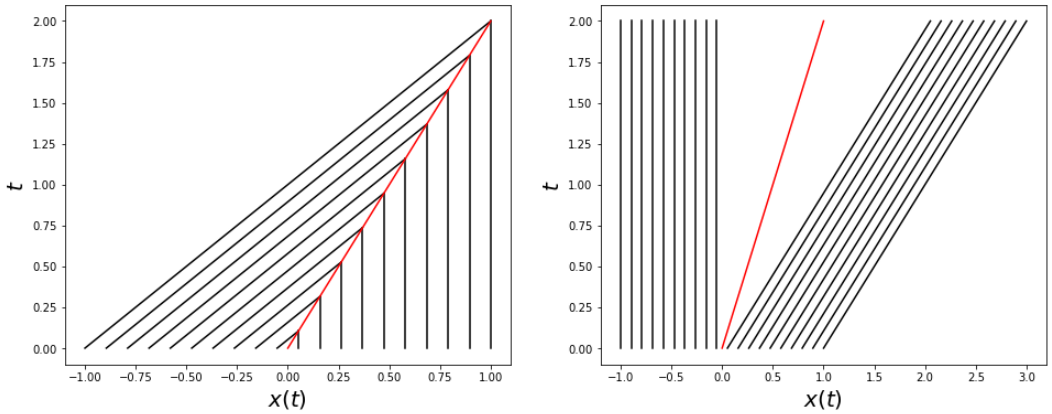

FIGURE 4 – Choc vérifiant la condition de Lax à gauche, et la violant à droite.

Concernant l'existence, Kruzhkov a basé sa preuve sur la méthode de la viscosité évanescente rapidement évoquée plus haut. En laissant de côté les problèmes dus à la régularité insuffisante de la fonction flux, le résultat de Kruzhkov ne pourrait être plus général dans le sens où il englobe le cas où $f = f(x,t,\rho)$ et où (0.1.1) possède un terme source $s = s(x,t,\rho)$, en dimension $d \in \mathbb{N}^*$ d'espace, pour les équations scalaires.
Dans le sillage de Kruzhkov, d'autres auteurs ont par la suite chercher à exprimer d'autres inégalités de stabilité du type (0.1.19) par rapport au flux, au terme source *etc...*, voir par exemple [32, 59, 120, 140]. Enfin, parmi les autres moyens de prouver l'existence des solutions entropiques mentionnons les méthodes d'approximation :

— par "front tracking", voir par exemple [33, 98, 68] ;
— par volumes/différences finis, voir par exemple [121, 48, 49, 83, 122, 128, 148].

Les schémas aux volumes finis, en particulier, sont utilisés de manière récurrente dans les chapitres de cette thèse.

0.1.2 Cadre LWR pour la modélisation du trafic

Bien avant que le papier de Kruzhkov apporte une réponse définitive quant au caractère bien posé des lois de conservation scalaires, ces dernières équations, et même des systèmes de lois de conservation, étaient déjà largement utilisés pour décrire des phénomènes de dynamiques de trafic routier/piétonnier. Parmi les différentes approches, deux visions se sont démarquées : la description *microscopique* et la description *macroscopique*.

Du point de vue microscopique, la trajectoire de chaque agent (piétons, voitures) est décrite par une équation différentielle ordinaire. Dans les cas les plus simples de description du trafic routier, voir [62, 93, 99], les modèles sont basés sur les hypothèses suivantes :

— la route est unidimensionnelle (pas de dépassement)
— la vitesse de chaque véhicule (à part du premier qui se déplace à vitesse maximale) ne dépend que de la distance au véhicule qui le précède.

Les modèles dérivant de ces hypothèses ont reçu la dénomination de modèles "Follow-the-Leader".

Du point de vue macroscopique, le trafic est vu comme un fluide/milieu continu et sa dynamique n'est pas plus décrite par la trajectoire de ses agents, mais par la densité d'agents. Assez naturellement, ce ne sont plus plusieurs équations différentielles qui décrivent le trafic mais une seule équation aux dérivées partielles. Ces modèles sont apparus pour la première au milieu du XX^{e} siècle quand Lighthill, Whitham [125] et Richards [141] indépendamment, ont introduit l'idée d'utiliser les lois de conservation pour décrire le trafic. Poussant un peu plus l'analogie entre le flux de trafic et la dynamique des fluides, leur modèle, appelé *LWR* depuis, décrit l'évolution de la densité de voitures $\rho = \rho(x,t) \in [0,1]$ sur une route "infinie" et est donné par une loi de conservation exprimant la conservation de la masse :

$$\partial_t \rho + \partial_x \left(\rho v(\rho)\right) = 0. \tag{0.1.20}$$

En mettant cette EDP en parallèle avec (0.1.1), la fonction v apparaît donc comme la vitesse du trafic. Dans le cadre LWR, il est supposé que $v = v(\rho)$ est une fonction positive, décroissante de la densité, et vérifiant $v(0) = 1$, $v(1) = 0$ ce qui modélise le fait qu'une faible densité de voitures se déplace plus rapidement qu'une plus importante densité. Le choix le plus simple est de choisir $v(\rho) = 1 - \rho$ ce qui nous ramène à l'équation présentée dans l'exemple 0.1.4. On a représenté en figure 5 des exemples de fonctions flux $f(\rho) = \rho v(\rho)$ résultant de ces hypothèses sur la vitesse. Cette classe de flux est qualifiée de *flux en cloche*.

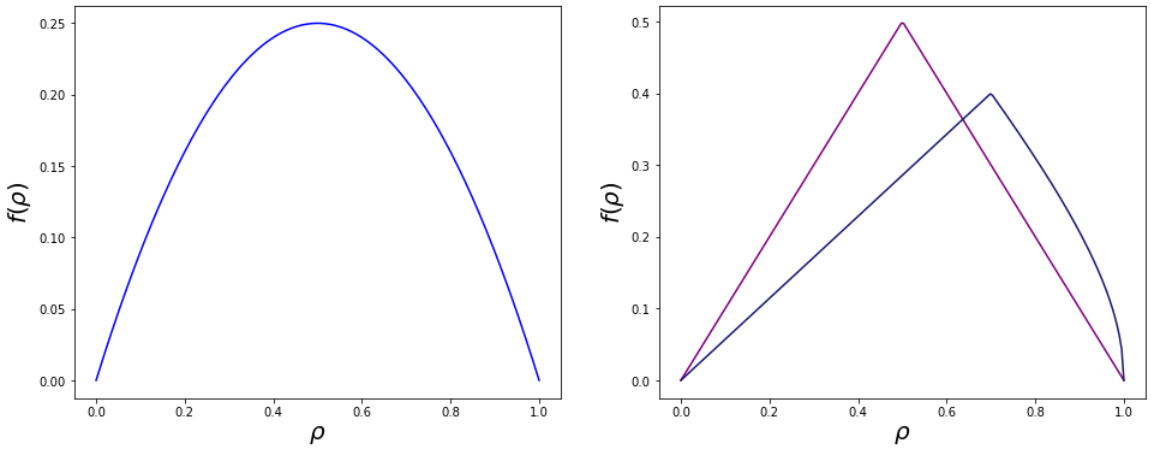

FIGURE 5 – Exemples de flux en cloche.

Malgré sa simplicité (une seule équation scalaire en dimension 1 d'espace), le cadre LWR (0.1.20) constitue une bonne base pour la description macroscopique du trafic. Dans la première partie de cette thèse, on propose quelques ajustements au cadre LWR afin de décrire des situations que le cadre LWR standard ne peut reproduire, et on étudie à la fois théoriquement et numériquement ces extensions.

0.1.3 Modèles de trafic du second ordre

Dans le contexte de dynamique de trafic, on désigne par modèle du second ordre un modèle décrivant l'évolution des variables d'état au moyen d'un système de deux EDP. Ces modèles

ont été remis au gout du jour par Aw, Rascle [25] et Zhang [157] (ARZ dans la suite) au début du XXI^{e} siècle et s'inspirent du système

$$\begin{cases} \partial_t \rho + \partial_x (\rho v) = 0 \\ \partial_t (\rho(v + p(\rho))) + \partial_x (\rho v(v + p(\rho))) = 0, \end{cases} \tag{0.1.21}$$

voir [97, 96, 12]. Dans le système ARZ, la première équation traduit toujours la conservation de la masse alors que la seconde exprime le transport, le long des trajectoires des véhicules, d'un "marqueur lagrangien" $w = v + p(\rho)$ où la fonction p, typiquement de la forme $p(\rho) = \rho^\gamma$, est parfois dite "pression" par analogie avec la dynamique des gaz. La complexité mathématique des modèles du second ordre, voir [144, 33, 98, 68, 148] pour l'étude théorique de ces systèmes, est compensée par une meilleure description du trafic. Dans la classe des modèles du second ordre, on s'intéressera plus particulièrement aux modèles de transition de phase. Dans la seconde partie du manuscrit, on introduira et étudiera un tel modèle. Lors de l'étude théorique et numérique, plutôt que d'adopter "le point de vue système" (hyperbolicité, valeurs propres, problèmes de Riemann), on verra plutôt le modèle comme deux lois de conservation scalaires *dé-couplables*.

0.2 Contribution et organisation du manuscrit

Le manuscrit est divisé en trois parties.

0.2.1 Description du trafic dans le cadre LWR et théorie du flux discontinu

Dans la partie I, on propose deux extensions du cadre LWR pour donner une description un peu plus sophistiquée de certains phénomènes du trafic. La première extension, qui est présente dans les deux premiers modèles, est liée aux contraintes ponctuelles. Ces contraintes ponctuelles sont apparues pour la première fois dans [60, 56], où les auteurs cherchaient à incorporer dans le modèle LWR un mécanisme qui tiendrait compte de phénomènes, localisés en espace, arrivant autour de sorties, de péages ou encore de feux de circulation, et qui agiraient comme des obstacles. Ce mécanisme est matérialisé par une inégalité sur le flux ajoutée à (0.1.20) et localisée en espace (position de l'obstacle), en général en $x = 0$ pour fixer les idées. Le modèle résultant est alors

$$\begin{cases} \partial_t \rho + \partial_x (f(\rho)) = 0 \\ \qquad f(\rho)|_{x=0} \leq q(t) \end{cases} \tag{0.2.1}$$

L'effet de l'obstacle est modélisé par la fonction q (contrainte dans la suite). On profite du chapitre 1 pour faire des rappels sur l'approximation par volumes finis des lois de conservation scalaires (0.1.1) et on traite également le cas du système (0.2.1). Les notations introduites dans ce chapitre seront reprises dans les différentes sections d'analyse numérique du manuscrit.

Dans le chapitre 2, on se base sur le système (0.2.1) et on propose un modèle décrivant la dynamique du trafic à un goulot d'étranglement. On ajoute à (0.2.1) un mécanisme mesurant le degré d'organisation/de désorganisation du trafic *via* une fonction $\omega = \omega(t)$ vérifiant une équation différentielle impliquant ρ. La valeur de ω au temps $t > 0$ influence la valeur de la contrainte.

Dans le chapitre 3, on décrit l'influence d'un véhicule lent sur le trafic qui l'entoure. Dans ce second modèle, qui présente également une contrainte ponctuelle, la contrainte n'est plus localisée en $x = 0$ mais est mobile et suit la trajectoire du véhicule lent :

$$\begin{cases} \partial_t \rho + \partial_x \left(f(\rho) \right) = 0 \\ f(\rho) - \dot{y}(t)\rho|_{x=y(t)} \leq Q(\dot{y}(t)) \\ \dot{y}(t) = \mathcal{M}[\rho], \end{cases} \tag{0.2.2}$$

y étant la trajectoire du véhicule lent et où on a noté $\mathcal{M}[\cdot]$ un opérateur non-local ; en pratique, $\mathcal{M}[\cdot]$ est un opérateur de moyennisation de ρ. Le système (0.2.2) présente plusieurs degrés de difficulté supplémentaires par rapport à (0.2.1). D'une part, la contrainte n'est plus fixe, mais ceci n'est pas un problème : on se ramène à un problème avec contrainte fixe *via* le changement de coordonnées $X = x - y(t)$, mais en contrepartie, le nouveau flux dépendra du temps *via* $\dot{y}$. Par ailleurs, la section 1.6.1 du chapitre 1 étend les résultats établis au début du chapitre 1 dans le cadre des systèmes de la forme (0.2.1) avec un flux dépendant du temps. Précisons que la variante locale de (0.2.2), c'est-à-dire avec $\mathcal{M}[\rho] = \rho(y(t)+, t)$ a été étudiée par les auteurs de [70], et insistons sur la difficulté d'analyse de leur modèle, en particulier sur la question de l'unicité, voir aussi [72, 123]. La difficulté principale pour l'étude du système (0.2.2) réside dans le couplage qui existe entre ρ et y : le trafic ambiant influence la vitesse du véhicule *via* $\mathcal{M}$, et le véhicule lent influence le trafic qui l'entoure *via* la contrainte $Q(\dot{y})$. Pour ces deux modèles, on prouve le caractère bien posé, et on construit un schéma numérique pour lequel on prouve la convergence. Des simulations numériques sont également réalisées.

Dans le chapitre 4, on se propose d'étendre l'analyse théorique de systèmes (0.2.2) au cas où un nombre fini de trajectoires/contraintes $(y_i, q_i)_i$ est donné. Contrairement au chapitre 3, on ne *rectifie* pas les trajectoires dans le sens où on travaille dans le repère d'origine, sans faire de changement de coordonnées (bien que possible, mais laborieux, voir par exemple [18]). La contribution majeure de ce chapitre est l'introduction d'un langage adapté à la géométrie définie par les trajectoires $(y_i)_i$ (*interfaces*). Dans la preuve d'existence basée sur la construction d'un schéma numérique, on adapte localement le maillage du schéma à la géométrie proposée. Le formalisme et les techniques dévelopées dans ce chapitre allant au delà des problèmes à flux contraints, on exploite le cadre fixé dans le chapitre 4 pour l'étude de la loi de conservation

$$\partial_t \rho(x,t) + \partial_x \left(\mathbf{f}(x,t,\rho(x,t)) \right) = 0, \tag{0.2.3}$$

où la fonction $\mathbf{f}$ est de classe $\mathbf{C}^1$ en ρ et présente des discontinuités en (x,t), l'exemple le plus

simple étant :

$$\mathbf{f}(x,t,\rho) = \begin{cases} f(\rho) & \text{si } x < 0 \\ g(\rho) & \text{si } x > 0, \end{cases} \tag{0.2.4}$$

avec $f, g \in \mathbf{C}^1(\mathbb{R})$. L'objectif du chapitre 5 est de fournir une approche systématique basée sur les volumes finis pour construire des solutions de (0.2.3) avec un couplage général à l'interface.

Papiers soumis/publiés liés à ces deux modèles : [23, 150, 149]. Précisons que dans le chapitre 2, du contenu a été ajouté par rapport à [23], notamment dans la section 2.2. Rajoutons également que le chapitre 5 fera l'objet d'un preprint qui est en cours de finalisation.

0.2.2 Étude théorique et numérique d'un modèle du second ordre

On proposera ensuite dans la partie II un modèle du second ordre constitué de deux équations : une loi de conservation (avec flux dépendant du temps et de l'espace) sur la densité ρ traduisant la conservation de la masse, et une loi de conservation sur un paramètre $\mathbf{w}$ traduisant le degré d'organisation du trafic qui évolue le long de la trajectoire des véhicules et influence de manière non-locale le diagramme fondamental du flux du trafic :

$$\begin{cases} \partial_t \rho + \partial_x(f(x,t,\rho)) = 0 \\ f(x,t,\rho) = (1-\omega(x,t))f_{\min}(\rho) + \omega(x,t) f_{\max}(\rho) \\ \omega = \mathcal{M}[\mathbf{w}] \\ \partial_t(\rho \mathbf{w}) + \partial_x(f(x,t,\rho)\mathbf{w}) = \rho \mathbf{K}[\rho]\mathbf{w}(1-\mathbf{w}), \end{cases} \tag{0.2.5}$$

les fonctions $f_{\min} \leq f_{\max}$ traduisant les deux niveaux extrêmes de desorganisation du trafic.

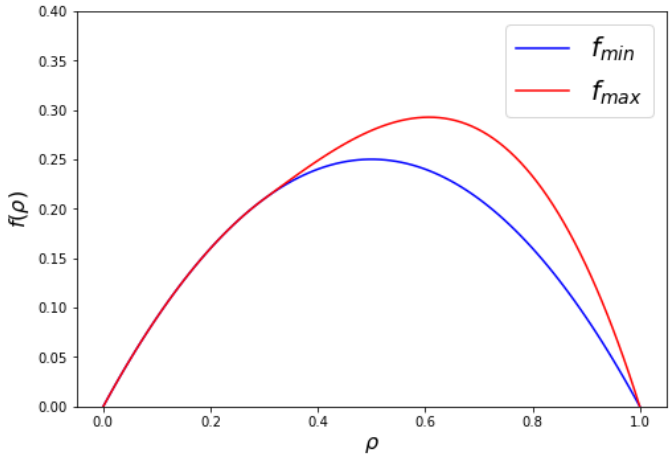

FIGURE 6 – Illustration des deux niveaux de désorganisation.

Les facteurs $\mathcal{M}[\mathbf{w}]$ et $\mathbf{K}[\rho]$ dans (0.2.5) introduisent un couplage non-local entre ρ et $\mathbf{w}$, ce qui rend l'analyse du système (0.2.5) similaire à celle du système classique de Keyfitz-Kanzer. L'étude du système s'organise de la façon suivante.

Après avoir étendu dans le chapitre 6 la théorie de Panov [134] sur les *solutions renormalisées* d'équations de transport linéaires, on profite pleinement de cette théorie dans le chapitre 7 où on prouve l'existence de solutions au système (0.2.5). Notre définition de solution est basée sur une combinaison des solutions entropiques de Kruzhkov pour ρ et des solutions renormalisées de Panov pour $\mathbf{w}$, voir la section 7.3. On propose également un schéma numérique pour (0.2.5) et on montre sa convergence dans le cadre de données initiales **BV** sous l'hypothèse que la densité ρ est séparée de 0. Ce schéma, qui est une combinaison entre la théorie classique des volumes finis (pour ρ) et la traduction numérique de la propagation de $\mathbf{w}$ le long des caractéristiques, mène toutefois à un schéma conservatif pour la quantité $\rho\mathbf{w}$.

Pré-publication liée à ce modèle : [22].

0.2.3 Lois de conservation et équations d'Hamilton-Jacobi hétérogènes

Finalement, dans la partie III, notre propos réside dans l'étude théorique des lois de conservation avec dépendance spatiale du flux :

$$\begin{cases} \partial_t u + \partial_x(f(x,u)) = 0 \\ u(\cdot,0) = u_0. \end{cases} \tag{0.2.6}$$

La première partie de l'étude consiste à montrer le caractère bien posé de ce problème de Cauchy pour des données initiales bornées $u_0 \in \mathbf{L}^\infty(\mathbb{R})$. Comme évoqué précédemment, voir le paragraphe 0.1.1, il est maintenant bien connu que (0.2.6) admet une unique solution entropique si $f \in \mathbf{W}^{1,\infty}(\mathbb{R}^2)$, voir par exemple [114, 88, 68, 140]. Cependant, cette hypothèse est assez restrictive puiqu'elle exclut les fonctions à croissance sur-linéaires en u par exemple.

Plutôt que de considérer des flux globalement Lipschitz, on choisira f dans la classe des fonctions $\mathbf{C}^2(\mathbb{R}^2)$ dont la dépendance spatiale est localisée dans un compact :

$$\exists X > 0,\ \forall x,u \in \mathbb{R}, \quad |x| \geq X \implies \frac{\partial f}{\partial x}(x,u) = 0. \tag{0.2.7}$$

Cette hypothèse, qui n'impose pas de condition de croissance de f en u, permettra par exemple de considérer des fonctions f *fortement convexes* en u, fonctions inatteignables sous l'hypothèse $f \in \mathbf{W}^{1,\infty}(\mathbb{R}^2)$. Pour faire le lien avec la modélisation du trafic, une fonction f rentrant dans notre cadre ($-f$ en fait) est

$$f(x,u) = \theta(x)u(1-u),$$

où l'allure de la fonction $\theta \in \mathbf{C}^2(\mathbb{R})$ est représentée en figure 7, à gauche. La loi de conservation associée à ce flux modélise par exemple l'évolution d'une densité de voitures se déplaçant sur une route où la limitation de vitesse change de manière continue.
Sur le chemin de la preuve d'existence de solutions entropiques de (0.2.6), on fera le lien entre les solutions (entropiques) de (0.2.6) et les solutions de viscosité (voir [126, 28]) de l'équation de Hamilton-Jacobi :

$$\begin{cases} \partial_t U + f(x,\partial_x U) = 0 \\ U(\cdot,0) = U_0. \end{cases} \tag{0.2.8}$$

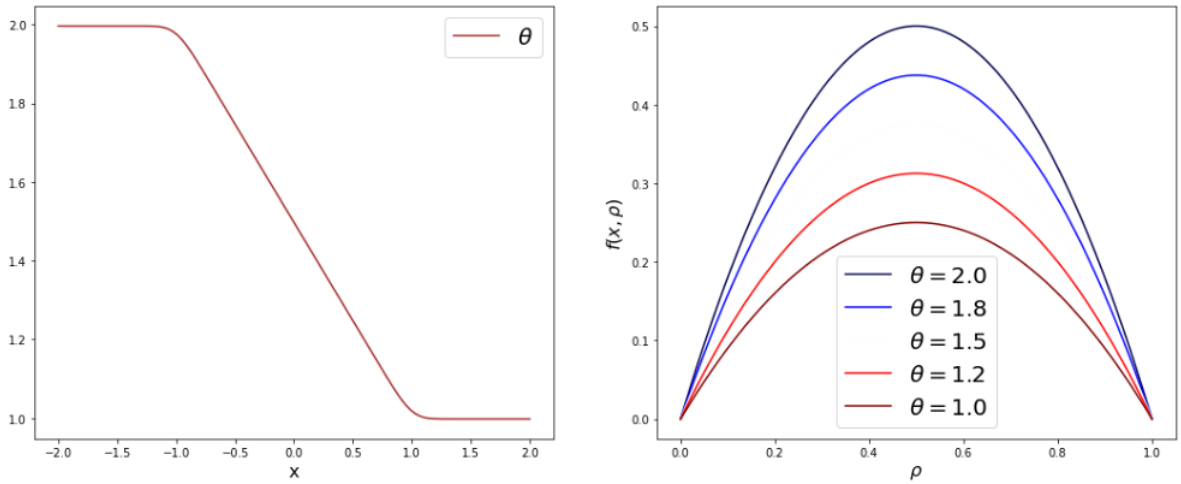

FIGURE 7 – Example d'un flux satisfaisant l'hypothèse (0.2.7).

Dans le cas homogène, *i.e.* quand $f = f(u) \in \mathbf{C}^1(\mathbb{R})$, il est connu que :

- si f est globalement Lipschitzienne, alors (0.2.6) admet une unique solution entropique, pour toute donnée initiale bornée, voir [114] ou [127, Chapter 2] ;
- si f est globalement Lipschitzienne ou super-linéaire, alors (0.2.8) admet une unique solution de viscosité pour toute donnée initiale Lipschitz, voir [81, Chapter 10] ou [27, Chapter 2] ;
- les solutions entropiques de (0.2.6) sont obtenus en dérivant par rapport à x les solutions de viscosité de (0.2.8), voir [105, 61].

Dans le cas hétérogène, sous l'hypothèse (0.2.7) (en plus de super-linéarité en u), ces résultats ne sont, *a priori*, pas connus. Outre l'extension de ces résultats dans notre cadre, on fait également le lien avec le problème de calcul des variations associé à (0.2.8) :

$$\inf_{\substack{y\in\mathbf{W}^{1,1}((0,t))\\ y(t)=x}} \left(\int_0^t f^*(y(s),\dot{y}(s))\,\mathrm{d}s + U_0(y(0)) \right), \quad (x,t) \in \Omega, \tag{0.2.9}$$

où f^* est la transformée de Legendre de f en la seconde variable.

Ces différents points de vue sont cruciaux dans la discussion reliée à la deuxième problématique de cette partie, à savoir celle *d'identification inverse*. Plus précisément, on se fixe un temps final $T > 0$ et un profil $w \in \mathbf{L}^\infty(\mathbb{R})$, et on s'interroge sur l'ensemble des données initiales u_0 tel que la solution entropique u de (0.2.6) vérifie $u(\cdot, T) = w$. Comme les solutions de (0.2.6) présentent des discontinuités qui causent une perte d'information, ce problème d'identification n'est pas trivial. Précisons que les auteurs de [61] ont fourni une description complète de l'ensemble des données initiales évoluant en un profil donné dans le cas où $f = f(u)$ était uniformément convexe. La partie III étend leurs résultats dans le cas de flux fortement convexe en u avec dépendance spatiale compacte. Mettons en lumière le fait que l'extension au cas hétérogène de ces problèmes *d'inverse design* n'est pas qu'une simple adaptation des résultats prouvés par les auteurs de [61], comme en atteste l'exemple de la section 8.4.2.

Part I

Heterogeneity by flux constraints

CHAPTER 1

PRELIMINARIES: FINITE VOLUME APPROXIMATION OF A BASIC PROBLEM WITH POINT CONSTRAINT

This first chapter is the occasion to introduce the notations we will adopt in the sequel regarding the numerical approximations of our different problems. It also serves to elaborate a handful of ingredients and techniques we will use.

Being given a Lipschitz concave flux $f : [0,1] \to \mathbb{R}$:

$$f(\rho) \geq 0,\ f(0) = f(1) = 0; \quad \exists!\, \overline{\rho} \in (0,1),\ \text{for a.e. } \rho \in (0,1), \quad f'(\rho)(\overline{\rho} - \rho) > 0, \tag{1.0.1}$$

and a constraint function $q \in \mathbf{L}^\infty((0,+\infty)), q \geq 0$, we tackle the following problem in $\Omega = \mathbb{R} \times (0,+\infty)$:

$$\begin{cases} \partial_t \rho + \partial_x \left(f(\rho)\right) & = 0 \\ \rho(\cdot,0) & = \rho_0 \\ f(\rho)|_{x=0} & \leq q(t). \end{cases} \tag{1.0.2}$$

This class of problems has been greatly investigated in the past few decades. Motivated by the modeling of tollgates and traffic lights for instance, the authors of [56] proved a well-posedness result for (1.0.2) in the **BV** framework (*i.e.* with both q and ρ_0 with bounded variation). The authors of [14] then extended the well-posedness in the $\mathbf{L}^\infty$ framework. In this part, we are interested in the numerical approximation of Problem (1.0.2) by the means of a finite volume scheme. Let us recall the notion of solution for (1.0.2).

Definition 1.0.1. A function $\rho \in \mathbf{L}^\infty(\Omega)$ is an admissible entropy solution to (1.0.2) with initial data $\rho_0 \in \mathbf{L}^\infty(\mathbb{R})$ if

(i) for all test functions $\varphi \in \mathbf{C}_\mathbf{c}^\infty(\overline{\Omega}), \varphi \geq 0$ and $\kappa \in [0,1]$, the following entropy inequalities

are verified:

$$\int_0^{+\infty}\int_{\mathbb{R}}\Big(|\rho-\kappa|\partial_t\varphi+\Phi(\rho,\kappa)\partial_x\varphi\Big)\,\mathrm{d}x\,\mathrm{d}t+\int_{\mathbb{R}}|\rho_0(x)-\kappa|\varphi(x,0)\,\mathrm{d}x \\ +2\int_0^{+\infty}\mathcal{R}(\kappa,q(t))\varphi(0,t)\,\mathrm{d}t\geq 0, \tag{1.0.3}$$

where

$$\mathcal{R}(\kappa,q(t))=f(\kappa)-\min\left\{f(\kappa),q(t)\right\};$$

(ii) for all test functions $\varphi\in\mathbf{C}_{\mathbf{c}}^{\infty}(\Omega),\varphi\geq 0$ the following constraint inequalities are verified:

$$-\int_0^{+\infty}\int_{\mathbb{R}^+}\Big(\rho\partial_t\varphi+f(\rho)\partial_x\varphi\Big)\,\mathrm{d}x\,\mathrm{d}t\leq\int_0^{+\infty}q(t)\varphi(0,t)\,\mathrm{d}t\,. \tag{1.0.4}$$

In [56, 14], the authors proved the well-posedness with a slightly different notion of solution. The one we consider here is equivalent (see all the details in Chapter 4) and is more adapted for passage to the limit of a.e. convergent sequences of approximate solutions, and thus for the proof of existence while the other one yields uniqueness more easily.

1.1 Constrained finite volume scheme

The idea behind the construction of a finite volume scheme for a partial differential equation in general (see [83]) is to decompose the domain $\overline{\Omega}$ into small *control volumes* and integrate the PDE on each of those control volumes. Throughout the manuscript (except in Chapters 4-5 where we will locally modify the mesh), we will always consider rectangle control volumes. For a fixed spatial mesh size $\Delta x>0$ and time mesh size $\Delta t>0$, let $x_j=j\Delta x$ $(j\in\mathbb{Z})$ and $t^n=n\Delta t$ $(n\in\mathbb{N})$. We define the cell grids:

$$\overline{\Omega}=\bigcup_{n\in\mathbb{N}}\bigcup_{j\in\mathbb{Z}}\mathcal{P}_{j+1/2}^n,\quad \mathcal{P}_{j+1/2}^n=(x_j,x_{j+1})\times[t^n,t^{n+1}).$$

The control volumes are the rectangles $\left(\mathcal{P}_{j+1/2}^n\right)_{n,j}$, see Figure 1.1, left. Integrating the PDE of (1.0.2) on $\mathcal{P}_{j+1/2}^n$ $(n\in\mathbb{N},j\in\mathbb{Z})$, we obtain:

$$\int_{x_j}^{x_{j+1}}\left(\rho(x,t^{n+1})-\rho(x,t^n)\right)\mathrm{d}x+\int_{t^n}^{t^{n+1}}\left(f(\rho(x_{j+1},t))-f(\rho(x_j,t))\right)\mathrm{d}t=0.$$

For $n\in\mathbb{N}$ and $j\in\mathbb{Z}$, let us denote $\rho_{j+1/2}^n$ an approximation of the solution ρ on $\mathcal{P}_{j+1/2}^n$, and f_j^n an approximation of $f(\rho)$ on $[t^n,t^{n+1})$ at $x=x_j$, see Figure 1.1, right. This leads to the scheme:

$$\rho_{j+1/2}^{n+1}=\rho_{j+1/2}^n-\frac{\Delta t}{\Delta x}(f_{j+1}^n-f_j^n). \tag{1.1.1}$$

We immediately see that if we can express the $(f_j^n)_j$ only with $(\rho_{j+1/2}^n)_j$, then by induction, we can deduce all the values $\left(\rho_{j+1/2}^n\right)_{n,j}$ from $\left(\rho_{j+1/2}^0\right)_j$ only, *i.e.* from approximate values of ρ_0. If $\rho_0\in\mathbf{L}^{\infty}(\mathbb{R};[0,1])$, a simple choice is:

$$\forall j\in\mathbb{Z},\quad \rho_{j+1/2}^0=\frac{1}{\Delta x}\int_{x_j}^{x_{j+1}}\rho_0(x)\,\mathrm{d}x\,.$$

Remark 1.1.1. Others choice could be made, for instance in the case $\rho_0 \in \mathbf{C}(\mathbb{R})$ such that $\lim_{|x|\to+\infty} \rho_0(x)$ exists, the values $\rho^0_{j+1/2} = \rho_0\left(\frac{x_j+x_{j+1}}{2}\right)$ can be used. The only requirements are

$$\forall j \in \mathbb{Z},\ \rho^0_{j+1/2} \in [0,1] \quad \text{and} \quad \rho^0_\Delta = \sum_{j\in\mathbb{Z}} \rho^0_{j+1/2} \mathbb{1}_{(x_j,x_{j+1})} \underset{\Delta x\to 0}{\longrightarrow} \rho_0 \text{ in } \mathbf{L}^1_{\mathbf{loc}}(\mathbb{R}).$$

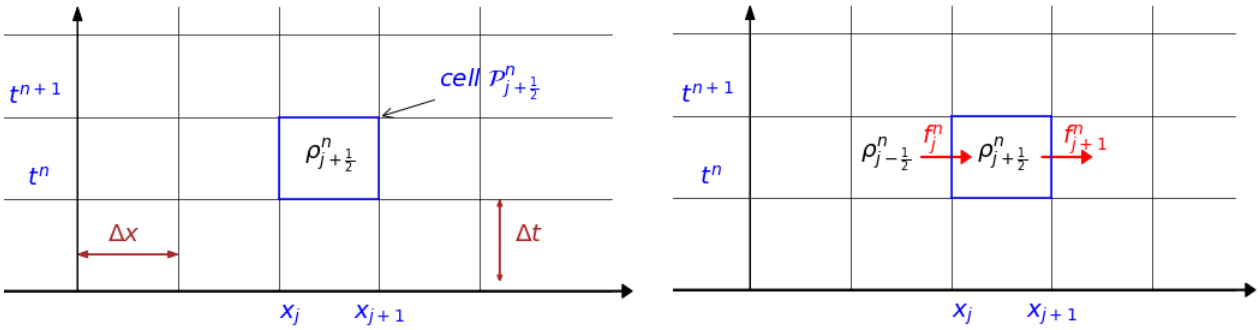

Figure 1.1 – Discretization of $\mathbb{R}\times\mathbb{R}^+$ and approximations of ρ and $f(\rho)$.

We can see that the remaining question is to find how to compute the approximation $(f^n_j)_j$ of $f(\rho)$. In first approach, it seems reasonable to define f^n_j as a function of $\rho^n_{j-1/2}$ and $\rho^n_{j+1/2}$: $f^n_j = \mathbf{F}(\rho^n_{j-1/2}, \rho^n_{j+1/2})$, see Figure 1.1, right. We choose the function $\mathbf{F}$ in the class of *monotone numerical flux*.

Definition 1.1.1. A monotone numerical flux associated to f is a function $\mathbf{F} : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ which satisfies:

(i) consistency: $\forall \kappa \in [0,1]$, $\mathbf{F}(\kappa,\kappa) = f(\kappa)$;

(ii) $\mathbf{F}$ is Lipschitz continuous on $[0,1]^2$, nondecreasing with respect to the first variable and nonincreasing with respect to the second variable.

Remark 1.1.2. The required properties of $\mathbf{F}$ are necessary only on $[0,1]$ because as we will see later (*cf.* Theorem 1.2.1), under a suitable assumption on Δt and Δx (CFL condition (1.1.5)), the values $(\rho^n_{j+1/2})_{n,j}$ computed by the scheme (1.1.1) will belong to $[0,1]$.

Example 1.1.1. Here are the most classical and most commonly used monotone numerical fluxes. These are the fluxes we will use and will refer to, throughout the manuscript.

Rusanov flux:

$$\forall a,b \in [0,1], \quad \mathbf{Rus}(a,b) = \frac{f(a)+f(b)}{2} + \frac{\Delta}{2}(a-b), \quad \Delta \geq \|f'\|_{\mathbf{L}^\infty}.$$

A special case of the Rusanov flux is the **Lax-Friedrichs** flux, obtained with $\Delta = \dfrac{\Delta x}{\Delta t}$, under the CFL condition (1.1.5).

Godunov flux:

$$\forall a,b\in[0,1],\quad \mathbf{God}(a,b)=\begin{cases} f(a) & \text{if } a=b\\ \min\limits_{u\in[a,b]} f(u) & \text{if } a<b\\ \max\limits_{u\in[b,a]} f(u) & \text{if } a>b. \end{cases}$$

Engquish-Osher flux:

$$\forall a,b\in[0,1],\quad \mathbf{EO}(a,b)=\frac{f(a)+f(b)}{2}-\frac{1}{2}\int_a^b |f'(u)|\,\mathrm{d}u\,.$$

More background as well as other examples of monotone numerical fluxes are given in [121, 122, 83]. Remark that for the ones we presented here, we have:

$$\left\|\frac{\partial \mathbf{F}}{\partial a}\right\|_{\mathbf{L}^\infty},\ \left\|\frac{\partial \mathbf{F}}{\partial b}\right\|_{\mathbf{L}^\infty}\leq \|f'\|_{\mathbf{L}^\infty}.$$

Throughout this chapter, if not explicitly stated otherwise, $\mathbf{F}=\mathbf{F}(a,b)$ will denote any monotone numerical flux associated to f. Once a monotone numerical flux is chosen, the scheme for the PDE in (1.0.2) is complete. However, we also have to take into account the constraint inequality in (1.0.2). To doing so, we first define a suitable approximation $(q^n)_n$ of q, for instance its mean value on $[t^n,t^{n+1})$, then introduce the constrained numerical flux:

$$\mathbf{F}_j^n(a,b)=\begin{cases} \mathbf{F}(a,b) & \text{if } j\neq 0\\ \min\{\mathbf{F}(a,b),q^n\} & \text{if } j=0, \end{cases} \tag{1.1.2}$$

and replace f_j^n by $\mathbf{F}_j^n(\rho_{j-1/2}^n,\rho_{j+1/2}^n)$ in (1.1.1). The resulting marching formula is:

$$\forall j\in\mathbb{Z},\quad \rho_{j+1/2}^{n+1}=\begin{cases} \rho_{j+1/2}^n-\dfrac{\Delta t}{\Delta x}(f_{j+1}^n-f_j^n) & \text{if } j\notin\{-1,0\}\\ \rho_{-1/2}^n-\dfrac{\Delta t}{\Delta x}\left(\min\{f_0^n,q^n\}-f_{-1}^n\right) & \text{if } j=-1\\ \rho_{1/2}^n-\dfrac{\Delta t}{\Delta x}(f_1^n-\min\{f_0^n,q^n\}) & \text{if } j=0. \end{cases} \tag{1.1.3}$$

Remark 1.1.3. The only necessary requirements on $(q^n)_n$ are

$$\forall n\in\mathbb{N},\ q^n\geq 0\quad\text{and}\quad q_\Delta=\sum_{n\in\mathbb{N}} q^n\mathbb{1}_{[t^n,t^{n+1})}\underset{\Delta t\to 0}{\longrightarrow} q \text{ in } \mathbf{L}^1_{\mathbf{loc}}((0,+\infty)).$$

Notice how in (1.1.3), we only modify two values of the approximate solution; the two values associated to the cells nearby the *interface* $x=0$. It is usual to rewrite (1.1.3) as

$$\forall j\in\mathbb{Z},\quad \rho_{j+1/2}^{n+1}=\mathbf{H}_j^n\left(\rho_{j-1/2}^n,\rho_{j+1/2}^n,\rho_{j+3/2}^n\right), \tag{1.1.4}$$

where $\mathbf{H}_j^n = \mathbf{H}_j^n(u,v,w)$ is given by the right-hand side of (1.1.3). Finally, the approximate solution ρ_Δ is defined almost everywhere on $\overline{\Omega}$:

$$\rho_\Delta = \sum_{n\in\mathbb{N}}\sum_{j\in\mathbb{Z}} \rho_{j+1/2}^n \mathbb{1}_{\mathcal{P}_{j+1/2}^n}.$$

Let $\Delta = (\Delta x, \Delta t)$. For our convergence analysis, we will assume that $\Delta \to 0$, with $\lambda = \Delta t/\Delta x$ verifying the *CFL condition*

$$\lambda \mathbf{L} \leq 1, \quad \mathbf{L} = \left\| \frac{\partial \mathbf{F}}{\partial a} \right\|_{\mathbf{L}^\infty} + \left\| \frac{\partial \mathbf{F}}{\partial b} \right\|_{\mathbf{L}^\infty}. \tag{1.1.5}$$

Remark 1.1.4. When considering the monotone numerical fluxes presented in Example 1.1.1, the CFL condition (1.1.5) simply reduces to

$$2\lambda \|f'\|_{\mathbf{L}^\infty} \leq 1.$$

The reasoning is as follows:
- proving $\mathbf{L}^\infty$ stability of the scheme and discrete entropy inequalities;
- deriving continuous entropy/constraint inequalities for the approximate solution similar to (1.0.3)-(1.0.4);
- proving sufficient compactness for the sequence $(\rho_\Delta)_\Delta$;
- passing to the limit in the continuous entropy/constraint inequalities.

In Chapters 2 – 4 and Chapter 7, even if the models considered are different, the numerical approximation sections will all have this skeleton.

1.2 Stability and discrete entropy inequalities

In one space dimension, the monotonicity plays a huge role in the stability and consistency of the scheme. In this section, we detail how much this is true.

Theorem 1.2.1 ($\mathbf{L}^\infty$ stability)**.** *Under the CFL condition* (1.1.5)*, the scheme* (1.1.4) *is*

(i) monotone: for all $n \in \mathbb{N}$ and $j \in \mathbb{Z}$, $\mathbf{H}_j^n$ is nondecreasing with respect to its three arguments;
(ii) stable:

$$\forall n \in \mathbb{N},\ \forall j \in \mathbb{Z}, \quad \rho_{j+1/2}^n \in [0,1]. \tag{1.2.1}$$

Proof. *(i)* Fix $n \in \mathbb{N}$ and $j \in \mathbb{Z}$. To prove that $\mathbf{H}_j^n$ is nondecreasing, we differentiate it (licit since it is a Lipschitz function) and verifies that the CFL condition ensures the nonnegativity of its partial derivatives.
Suppose first that $j \notin \{-1, 0\}$ *i.e.* the interface $x = 0$ does not enter the calculations in the scheme (1.1.3). The monotonicity of $\mathbf{F}$ implies that for a.e. $u, v, w \in [0,1]$,

$$\frac{\partial \mathbf{H}_j^n}{\partial u}(u,v,w) = \lambda \frac{\partial \mathbf{F}}{\partial a}(u,v) \geq 0, \quad \frac{\partial \mathbf{H}_j^n}{\partial w}(u,v,w) = -\lambda \frac{\partial \mathbf{F}}{\partial b}(v,w) \geq 0,$$

Using now the CFL condition, we obtain:

$$\frac{\partial \mathbf{H}_j^n}{\partial v}(u,v,w) = 1 - \lambda\left(\frac{\partial \mathbf{F}}{\partial a}(v,w) - \frac{\partial \mathbf{F}}{\partial b}(u,v)\right) \geq 1 - \lambda \mathbf{L} \geq 0,$$

proving the statement in this case.
Suppose now that $j = -1$ for instance. Let us prove that the modification in (1.1.3) does not affect the monotonicity of the scheme. In the present case, $\mathbf{H}_{-1}^n$ takes the form:

$$\mathbf{H}_{-1}^n(u,v,w) = v - \lambda\left(\min\{\mathbf{F}(v,w), q^n\} - \mathbf{F}(u,v)\right).$$

Clearly, $\mathbf{H}_{-1}^n$ is still nondecreasing with respect to the u variable thanks to the previous calculations. Using the classical formula

$$\min\{u,v\} = \frac{(u+v) - |u-v|}{2},$$

we obtain:

$$\frac{\partial \mathbf{H}_{-1}^n}{\partial v}(u,v,w) = -\frac{\lambda}{2}\frac{\partial \mathbf{F}}{\partial b}(v,w)\underbrace{\left(1 - \operatorname{sgn}\left(\mathbf{F}(v,w) - q^n\right)\right)}_{\geq 0} \geq 0,$$

because of the monotonicity of $\mathbf{F}$. Now, with the CFL condition, we get:

$$\begin{aligned}\frac{\partial \mathbf{H}_j^n}{\partial v}(u,v,w) &= 1 - \lambda\left(\frac{1}{2}\frac{\partial \mathbf{F}}{\partial a}(v,w)\left(1 - \operatorname{sgn}\left(\mathbf{F}(v,w) - q^n\right)\right) - \frac{\partial \mathbf{F}}{\partial b}(u,v)\right)\\ &\geq 1 - \lambda\left(\frac{\partial \mathbf{F}}{\partial a}(v,w) - \frac{\partial \mathbf{F}}{\partial b}(u,v)\right) \geq 1 - \lambda \mathbf{L} \geq 0,\end{aligned}$$

concluding the proof of the statement in this case as well. The case $j = 0$ is similar so we omit the details of the proof for that case.

(ii) We prove (1.2.1) by induction on n. If $n = 0$, it is verified by our choice of the discretization $\left(\rho_{j+1/2}^0\right)_j$, see Remark 1.1.1. Suppose now that (1.2.1) is true for some integer $n \geq 0$ and let us show that it still holds for $n+1$. Remark that 0 and 1 are stationary solutions to the scheme. Indeed, for all $j \in \mathbb{Z}$, if $j \notin \{-1, 0\}$ we have

$$\mathbf{H}_j^n(0,0,0) = 0 \quad \text{and} \quad \mathbf{H}_j^n(1,1,1) = 1,$$

and if $j = 0$ for instance, then using Remark 1.1.3,

$$\begin{aligned}\mathbf{H}_0^n(0,0,0) &= -\lambda(f(0) - \min\{f(0), q^n\}) = 0 \quad \text{and}\\ \mathbf{H}_0^n(1,1,1) &= 1 - \lambda(f(1) - \min\{f(1), q^n\}) = 1,\end{aligned}$$

and it is also the case for $\mathbf{H}_{-1}$. Using now the monotonicity of $\mathbf{H}_j^n$ and the induction property, we deduce that for all $j \in \mathbb{Z}$, we have

$$\begin{aligned}0 = \mathbf{H}_j^n(0,0,0) &\leq \mathbf{H}_j^n(\rho_{j-1/2}^n, \rho_{j+1/2}^n, \rho_{j+3/2}^n)\\ &= \rho_{j+1/2}^{n+1}\\ &= \mathbf{H}_j^n(\rho_{j-1/2}^n, \rho_{j+1/2}^n, \rho_{j+3/2}^n) \leq \mathbf{H}_j^n(1,1,1) = 1,\end{aligned}$$

which concludes the induction argument. □

Remark 1.2.1. The stability estimate (1.2.1) immediately implies:

$$\forall \Delta, \quad 0 \leq \rho_\Delta \leq 1.$$

Remark 1.2.2. In the context of traffic flow dynamics, the fact that the constants 0 and 1 are stationary solutions to the scheme makes sense: vacuum ($\rho \equiv 0$) and bumper-to-bumper ($\rho \equiv 1$) are obvious equilibrium of the model.

In order to show that the limit of $(\rho_\Delta)_\Delta$, when compactness is proved, is the admissible entropy solution to (1.0.2), we derive discrete entropy inequalities.

Corollary 1.2.2 (Discrete entropy inequalities)**.** *The numerical scheme* (1.1.4) *fulfills the following discrete entropy inequalities for all* $n \in \mathbb{N}$, $j \in \mathbb{Z}$ *and* $\kappa \in [0,1]$*:*

$$\begin{aligned} &\left(|\rho_{j+1/2}^{n+1} - \kappa| - |\rho_{j+1/2}^{n} - \kappa|\right) \Delta x \\ &\leq \begin{cases} -\left(\Phi_{j+1}^n - \Phi_j^n\right) \Delta t & \text{if} \quad j \notin \{-1, 0\} \\ -\left(\Phi_{int}^n - \Phi_{-1}^n\right) \Delta t + \mathcal{R}(\kappa, q^n) \Delta t & \text{if} \quad j = -1 \\ -\left(\Phi_1^n - \Phi_{int}^n\right) \Delta t + \mathcal{R}(\kappa, q^n) \Delta t & \text{if} \quad j = 0, \end{cases} \end{aligned} \tag{1.2.2}$$

where $\mathcal{R}(\kappa, q^n)$ *was defined in Definition 1.0.1, and* Φ_j^n *and* Φ_{int}^n *are the numerical entropy fluxes:*

$$\begin{aligned} \Phi_j^n &= \mathbf{F}(\rho_{j-1/2}^n \vee \kappa, \rho_{j+1/2}^n \vee \kappa) - \mathbf{F}(\rho_{j-1/2}^n \wedge \kappa, \rho_{j+1/2}^n \wedge \kappa); \\ \Phi_{int}^n &= \min\{\mathbf{F}(\rho_{-1/2}^n \vee \kappa, \rho_{1/2}^n \vee \kappa), q^n\} - \min\{\mathbf{F}(\rho_{-1/2}^n \wedge \kappa, \rho_{1/2}^n \wedge \kappa), q^n\}. \end{aligned}$$

Proof. This is mostly a consequence of the scheme monotonicity. Fix $n \in \mathbb{N}$ and $j \in \mathbb{Z}$. Suppose first that $j \notin \{-1, 0\}$. In this case, all the constant states $\kappa \in [0,1]$ are stationary solutions of the scheme. Consequently,

$$\begin{aligned} |\rho_{j+1/2}^{n+1} - \kappa| &= \rho_{j+1/2}^{n+1} \vee \kappa - \rho_{j+1/2}^{n+1} \wedge \kappa \\ &= \mathbf{H}_j^n\left(\rho_{j-1/2}^n, \rho_{j+1/2}^n, \rho_{j+3/2}^n\right) \vee \mathbf{H}_j^n(\kappa, \kappa, \kappa) - \mathbf{H}_j^n\left(\rho_{j-1/2}^n, \rho_{j+1/2}^n, \rho_{j+3/2}^n\right) \wedge \mathbf{H}_j^n(\kappa, \kappa, \kappa) \\ &\leq \mathbf{H}_j^n\left(\rho_{j-1/2}^n \vee \kappa, \rho_{j+1/2}^n \vee \kappa, \rho_{j+3/2}^n \vee \kappa\right) - \mathbf{H}_j^n\left(\rho_{j-1/2}^n \wedge \kappa, \rho_{j+1/2}^n \wedge \kappa, \rho_{j+3/2}^n \wedge \kappa\right) \\ &= |\rho_{j+1/2}^{n+1} - \kappa| - \lambda(\Phi_{j+1}^n - \Phi_j^n), \end{aligned}$$

which is exactly (1.2.2) in the case $j \notin \{-1, 0\}$.
Suppose now that $j = 0$ for instance. The previous observation regarding the constants $\kappa \in [0,1]$ is not longer valid in this case since

$$\mathbf{H}_0^n(\kappa, \kappa, \kappa) = \kappa - \lambda \mathcal{R}(\kappa, q^n) \quad \textit{i.e. } \kappa \begin{cases} \leq \mathbf{H}_0^n(\rho_{-1/2}^n \vee \kappa, \rho_{1/2}^n \vee \kappa, \rho_{3/2}^n \vee \kappa) + \lambda \mathcal{R}(\kappa, q^n) \\ \geq \mathbf{H}_0^n(\rho_{-1/2}^n \wedge \kappa, \rho_{1/2}^n \wedge \kappa, \rho_{3/2}^n \wedge \kappa). \end{cases}$$

We deduce:

$$
\begin{aligned}
|\rho_{1/2}^{n+1}-\kappa| &= \rho_{1/2}^{n+1}\vee\kappa-\rho_{1/2}^{n+1}\wedge\kappa\\
&= \mathbf{H}_0^n\left(\rho_{-1/2}^n,\rho_{1/2}^n,\rho_{3/2}^n\right)\vee\kappa-\mathbf{H}_0^n\left(\rho_{-1/2}^n,\rho_{1/2}^n,\rho_{3/2}^n\right)\wedge\kappa\\
&\le \mathbf{H}_0^n\left(\rho_{-1/2}^n\vee\kappa,\rho_{1/2}^n\vee\kappa,\rho_{3/2}^n\vee\kappa\right)\vee\kappa-\mathbf{H}_0^n\left(\rho_{-1/2}^n\wedge\kappa,\rho_{1/2}^n\wedge\kappa,\rho_{3/2}^n\wedge\kappa\right)\wedge\kappa\\
&\le \mathbf{H}_0^n\left(\rho_{-1/2}^n\vee\kappa,\rho_{1/2}^n\vee\kappa,\rho_{3/2}^n\vee\kappa\right)\vee\left(\mathbf{H}_0^n(\rho_{-1/2}^n\vee\kappa,\rho_{1/2}^n\vee\kappa,\rho_{3/2}^n\vee\kappa)+\lambda\mathcal{R}(\kappa,q^n)\right)\\
&-\mathbf{H}_0^n\left(\rho_{-1/2}^n\wedge\kappa,\rho_{1/2}^n\wedge\kappa,\rho_{3/2}^n\wedge\kappa\right)\wedge\mathbf{H}_0^n(\rho_{-1/2}^n\wedge\kappa,\rho_{1/2}^n\wedge\kappa,\rho_{3/2}^n\wedge\kappa)\\
&\le \mathbf{H}_0^n(\rho_{-1/2}^n\vee\kappa,\rho_{1/2}^n\vee\kappa,\rho_{3/2}^n\vee\kappa)+\lambda\mathcal{R}(\kappa,q^n)-\mathbf{H}_0^n(\rho_{-1/2}^n\wedge\kappa,\rho_{1/2}^n\wedge\kappa,\rho_{3/2}^n\wedge\kappa)\\
&= |\rho_{1/2}^n-\kappa|-\lambda\left(\Phi_1^n-\Phi_{int}^n\right)+\lambda\mathcal{R}(\kappa,q^n),
\end{aligned}
$$

which is (1.2.2) in the case $j=0$. The obtaining of (1.2.2) in the case $j=-1$ is similar so we omit the details of the proof for this case. □

1.3 Continuous inequalities for the approximate solution

In this section, if $\varphi\in\mathbf{C}_{\mathbf{c}}^{\infty}(\overline{\Omega}),\varphi\ge 0$ is a test function, we define:

$$
\forall n\in\mathbb{N},\ \forall j\in\mathbb{Z},\quad \varphi_{j+1/2}^n=\frac{1}{\Delta x}\int_{x_j}^{x_{j+1}}\varphi(x,t^n)\,\mathrm{d}x\,.
$$

1.3.1 Approximate entropy inequalities

We start by deriving continuous entropy inequalities similar to (1.0.3) verified by ρ_Δ. Let us define the approximate entropy flux:

$$
\Phi_\Delta(\rho_\Delta,\kappa)=\sum_{n\in\mathbb{N}}\sum_{j\in\mathbb{Z}}\Phi_j^n\mathbb{1}_{\mathcal{P}_{j+1/2}^n}.
$$

Proposition 1.3.1 (Approximate entropy inequalities)**.** *Fix $n\in\mathbb{N}$ and $\kappa\in[0,1]$. Then as $\Delta\to 0$, we have*

$$
\begin{aligned}
&\int_{t^n}^{t^{n+1}}\int_{\mathbb{R}}\Big(|\rho_\Delta-\kappa|\partial_t\varphi+\Phi_\Delta\left(\rho_\Delta,\kappa\right)\partial_x\varphi\Big)\,\mathrm{d}x\,\mathrm{d}t\\
&+\int_{\mathbb{R}}|\rho_\Delta(x,t^n)-\kappa|\varphi(x,t^n)\,\mathrm{d}x-\int_{\mathbb{R}}|\rho_\Delta(x,t^{n+1})-\kappa|\varphi(x,t^{n+1})\,\mathrm{d}x\\
&+2\int_{t^n}^{t^{n+1}}\mathcal{R}(\kappa,q_\Delta(t))\varphi(0,t)\,\mathrm{d}t\ge O(\Delta x\Delta t)+O\big(\Delta t^2\big)\,.
\end{aligned}
\tag{1.3.1}
$$

Proof. Let us multiply the discrete entropy inequalities (1.2.2) by $\varphi_{j+1/2}^{n+1}\Delta x$ and sum over $j \in \mathbb{Z}$:

$$\begin{aligned}\sum_{j\in\mathbb{Z}} \left|\rho_{j+1/2}^{n+1} - \kappa\right| \varphi_{j+1/2}^{n+1}\Delta x \leq & \sum_{\substack{j\in\mathbb{Z}\\ j\notin\{-1,0\}}} \left(\left|\rho_{j+1/2}^{n} - \kappa\right| \Delta x - (\Phi_{j+1}^n - \Phi_j)^n \Delta t\right) \varphi_{j+1/2}^{n+1} \\ & + |\rho_{-1/2}^n - \kappa|\varphi_{-1/2}^{n+1}\Delta x - \left(\Phi_{int}^n - \Phi_{-1}^n\right)\varphi_{-1/2}^{n+1}\Delta t + \mathcal{R}(\kappa, q^n)\varphi_{-1/2}^{n+1}\Delta t \\ & + |\rho_{1/2}^n - \kappa|\varphi_{1/2}^{n+1}\Delta x - \left(\Phi_1^n - \Phi_{int}^n\right)\varphi_{1/2}^{n+1}\Delta t + \mathcal{R}(\kappa, q^n)\varphi_{1/2}^{n+1}\Delta t.\end{aligned}$$

We now proceed to the Abel's transformation as well as adding some quantities and their opposites to obtain:

$$\begin{aligned}&\sum_{j\in\mathbb{Z}} \left|\rho_{j+1/2}^{n+1} - \kappa\right| \varphi_{j+1/2}^{n+1}\Delta x - \sum_{j\in\mathbb{Z}} \left|\rho_{j+1/2}^{n} - \kappa\right| \varphi_{j+1/2}^{n}\Delta x - \sum_{j\in\mathbb{Z}} \left|\rho_{j+1/2}^{n} - \kappa\right| \left(\varphi_{j+1/2}^{n+1} - \varphi_{j+1/2}^{n}\right)\Delta x \\ &\leq \underbrace{\sum_{j\in\mathbb{Z}} \Phi_j^n \left(\varphi_{j+1/2}^{n+1} - \varphi_{j-1/2}^{n+1}\right)\Delta t}_{A} + \underbrace{\mathcal{R}(\kappa, q^n)(\varphi_{-1/2}^{n+1} + \varphi_{1/2}^{n+1})\Delta t}_{B} + \underbrace{(\Phi_{int}^n - \Phi_0^n)(\varphi_{1/2}^{n+1} - \varphi_{-1/2}^{n+1})\Delta t}_{C}.\end{aligned}$$

The left-hand side of this inequality is equal to

$$\int_{\mathbb{R}} |\rho_\Delta(x, t^{n+1}) - \kappa|\varphi(x, t^{n+1})\,\mathrm{d}x - \int_{\mathbb{R}} |\rho_\Delta(x, t^n) - \kappa|\varphi(x, t^n)\,\mathrm{d}x - \int_{t^n}^{t^{n+1}} \int_{\mathbb{R}} |\rho_\Delta - \kappa|\partial_t\varphi\,\mathrm{d}x\,\mathrm{d}t\,.$$

We now estimate the members of the right-hand side.

Estimating A**.** We write:

$$\begin{aligned}A &= \Delta t \int_{\mathbb{R}} \Phi_\Delta(\rho_\Delta, \kappa)\partial_x\varphi(x, t^{n+1})\,\mathrm{d}x + \underbrace{\lambda \sum_{j\in\mathbb{Z}} \int_{x_j}^{x_{j+1}} \int_{x-\Delta x}^{x} \int_x^y \Phi_j^n \partial_{xx}^2\varphi(z, t^n)\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x}_{A_1} \\ &= \int_{t^n}^{t^{n+1}} \int_{\mathbb{R}} \Phi_\Delta(\rho_\Delta, \kappa)\partial_x\varphi\,\mathrm{d}x\,\mathrm{d}t + A_1 + \underbrace{\int_{t^n}^{t^{n+1}} \int_{\mathbb{R}} \int_t^{t^{n+1}} \Phi_\Delta(\rho_\Delta, \kappa)\partial_{tx}^2\varphi(x, \tau)\,\mathrm{d}\tau\,\mathrm{d}x\,\mathrm{d}t}_{A_2},\end{aligned}$$

and we have the estimations:

$$|A_1| \leq 4\mathbf{L} \sup_{t\geq 0} \|\partial_{xx}^2\varphi(\cdot, t)\|_{\mathbf{L}^1}\Delta x\Delta t; \quad |A_2| \leq \mathbf{L} \sup_{t\geq 0} \|\partial_{tx}^2\varphi(\cdot, t)\|_{\mathbf{L}^1}\Delta t^2.$$

Estimating B. We have

$$\begin{aligned} B = \mathcal{R}(\kappa, q^n)\Bigg(2\varphi(0,t^{n+1}) + \underbrace{\frac{1}{\Delta x}\int_0^{\Delta x}(\varphi(x,t^{n+1}) - \varphi(0,t^{n+1}))\,\mathrm{d}x}_{\leq \|\partial_x\varphi\|_{\mathbf{L}^\infty}\Delta x} \\ + \underbrace{\frac{1}{\Delta x}\int_{-\Delta x}^{0}(\varphi(x,t^{n+1}) - \varphi(0,t^{n+1}))\,\mathrm{d}x}_{\leq \|\partial_x\varphi\|_{\mathbf{L}^\infty}\Delta x}\Bigg)\Delta t \\ = 2\mathcal{R}(\kappa,q^n)\varphi(0,t^{n+1})\Delta t + O(\Delta x\Delta t) \\ = 2\int_{t^n}^{t^{n+1}}\mathcal{R}(\kappa,q_\Delta(t))\varphi(0,t)\,\mathrm{d}t + \underbrace{2\int_{t^n}^{t^{n+1}}\mathcal{R}(\kappa,q_\Delta(t))\left(\varphi(0,t^{n+1}) - \varphi(0,t)\right)\mathrm{d}t}_{\leq 2\|q\|_{\mathbf{L}^\infty}\|\partial_t\varphi\|_{\mathbf{L}^\infty}\Delta t^2} + O(\Delta x\Delta t) \\ = 2\int_{t^n}^{t^{n+1}}\mathcal{R}(\kappa,q_\Delta(t))\varphi(0,t)\,\mathrm{d}t + O(\Delta x\Delta t) + O\left(\Delta t^2\right). \end{aligned}$$

Estimating C. Finally,

$$|C| \leq (2\mathbf{L})\Delta x\Delta t,$$

concluding the proof of the statement. □

Remark 1.3.1. If φ is supported in time in some $[0,T]$ ($T>0$), with $T \in [t^N, t^{N+1})$, then by summing (1.3.1) over $n \in \{0,\dots,N+1\}$, we obtain:

$$\begin{aligned} \int_0^T\int_{\mathbb{R}}\Big(|\rho_\Delta - \kappa|\partial_t\varphi + \Phi_\Delta\left(\rho_\Delta,\kappa\right)\partial_x\varphi\Big)\,\mathrm{d}x\,\mathrm{d}t + \int_{\mathbb{R}}|\rho_\Delta^0 - \kappa|\varphi(x,0)\,\mathrm{d}x \\ + 2\int_0^T\mathcal{R}(\kappa,q_\Delta(t))\varphi(0,t)\,\mathrm{d}t \geq O(\Delta x) + O(\Delta t). \end{aligned} \tag{1.3.2}$$

1.3.2 Approximate constraint inequalities

We now turn to the proof of an approximate version of the constraint inequalities, one similar to (1.0.4). In that optic, let us define the approximate flux function:

$$\mathbf{F}_\Delta\left(\rho_\Delta\right) = \sum_{n\in\mathbb{N}}\sum_{j\in\mathbb{Z}} f_j^n \mathbb{1}_{\mathcal{P}_{j+1/2}^n}.$$

Proposition 1.3.2 (Approximate constraint inequalities)**.** *For all $n \in \mathbb{N}$, as $\Delta \to 0$, we have*

$$\begin{aligned} \int_{\mathbb{R}^+}\rho_\Delta(x,t^{n+1})\varphi(x,t^{n+1})\,\mathrm{d}x - \int_{\mathbb{R}^+}\rho_\Delta(x,t^n)\varphi(x,t^n)\,\mathrm{d}x \\ - \int_{t^n}^{t^{n+1}}\int_{\mathbb{R}^+}\Big(\rho_\Delta\partial_t\varphi + \mathbf{F}_\Delta(\rho_\Delta)\partial_x\varphi\Big)\,\mathrm{d}x\,\mathrm{d}t \leq \int_{t^n}^{t^{n+1}} q_\Delta(t)\varphi(0,t)\,\mathrm{d}t + O(\Delta x\Delta t) + O\left(\Delta t^2\right). \end{aligned} \tag{1.3.3}$$

Proof. This proof follows the same step as the proof of proposition 1.3.1. Let us multiply the scheme (1.1.3) by $\varphi_{j+1/2}^{n+1}\Delta x$ and sum over $j \geq 0$:

$$\sum_{j\geq 0} \rho_{j+1/2}^{n+1}\varphi_{j+1/2}^{n+1}\Delta x = \sum_{j\geq 1} \rho_{j+1/2}^{n}\varphi_{j+1/2}^{n+1}\Delta x - \sum_{j\geq 1}(f_{j+1}^n - f_j^n)\varphi_{j+1/2}^{n+1}\Delta t + \rho_{1/2}^n\varphi_{1/2}^{n+1}\Delta x - (f_1^n - \min\{f_0^n, q^n\})\varphi_{1/2}^{n+1}\Delta t.$$

Like before, we proceed to the Abel/"summation-by-parts" technique. After reorganization of the terms, we have:

$$\begin{aligned}
&\sum_{j\geq 0} \rho_{j+1/2}^{n+1}\varphi_{j+1/2}^{n+1}\Delta x - \sum_{j\in\mathbb{Z}} \rho_{j+1/2}^{n}\varphi_{j+1/2}^{n}\Delta x - \sum_{j\in\mathbb{Z}} \rho_{j+1/2}^{n}\left(\varphi_{j+1/2}^{n+1} - \varphi_{j+1/2}^{n}\right)\Delta x \\
&= \sum_{j\geq 1} f_j^n\left(\varphi_{j+1/2}^{n+1} - \varphi_{j-1/2}^{n+1}\right)\Delta t + \min\{f_0^n, q^n\})\varphi_{1/2}^{n+1}\Delta t \\
&\leq \underbrace{\sum_{j\geq 1} f_j^n\left(\varphi_{j+1/2}^{n+1} - \varphi_{j-1/2}^{n+1}\right)\Delta t}_{A} + \underbrace{q^n\varphi_{1/2}^{n+1}\Delta t}_{B}.
\end{aligned}$$

Inequality (1.3.3) follows from the estimates

$$\left|A - \int_{t^n}^{t^{n+1}}\int_{\mathbb{R}^+} \mathbf{F}_\Delta(\rho_\Delta)\partial_x\varphi \,\mathrm{d}x\,\mathrm{d}t\right| \leq 4\mathbf{L}\sup_{t\geq 0}\|\partial_{xx}^2\varphi(\cdot,t)\|_{\mathbf{L}^1}\Delta x\Delta t + \mathbf{L}\sup_{t\geq 0}\|\partial_{tx}^2\varphi(\cdot,t)\|_{\mathbf{L}^1}\Delta t^2$$

and

$$\left|B - \int_{t^n}^{t^{n+1}} q_\Delta(t)\varphi(0,t)\,\mathrm{d}t\right| \leq \|q\|_{\mathbf{L}^\infty}\|\partial_t\varphi\|_{\mathbf{L}^\infty}\Delta t^2.$$

□

Remark 1.3.2. If φ is supported in time in some $(0,T)$ $(T>0)$, with $T \in [t^N, t^{N+1})$, then by summing (1.3.3) over $n \in \{0,\dots,N+1\}$, we obtain:

$$\int_0^T\int_{\mathbb{R}^+}\left(\rho_\Delta\partial_t\varphi + \mathbf{F}_\Delta(\rho_\Delta)\partial_x\varphi\right)\mathrm{d}x\,\mathrm{d}t \leq \int_0^T q_\Delta(t)\varphi(0,t)\,\mathrm{d}t + O(\Delta x) + O(\Delta t)\,. \tag{1.3.4}$$

1.4 Compactness and convergence

We now wish to pass to the limit in inequalities (1.3.2) and (1.3.4). The only missing ingredient is the compactness of the sequence $(\rho_\Delta)_\Delta$. Of course, the uniform $\mathbf{L}^\infty$ bound (1.2.1) ensures that a subsequence of $(\rho_\Delta)_\Delta$ converges in the $\mathbf{L}^\infty$ weak* sense. However, since $(\rho_\Delta)_\Delta$ appears in (1.3.2) and (1.3.4) in a nonlinear way, this convergence does not allow us to the pass to the limit directly. For example, the authors of [14] were able to pass to the limit by deriving the so-called weak **BV** estimates and using convergence of Young's measures, see [76, 138] for instance. The advantage of this approach is that it can be used

to treat multi-dimensional conservation laws, where monotonicity is not as restrictive as in the one-dimensional case, see [82, 48, 83, 49, 156].
In this section, we present two methods to obtain a.e. convergence (called strong compacteness from here on out) of a subsequence of $(\rho_\Delta)_\Delta$. The key point is to derive **BV** bounds, whether they are global (Section 1.4.1) or localized (Section 1.4.2). Let us also point out that in the case where the problem is invariant by time translation (this is the case for (1.0.2)), Bürger and al, see [38, Lemma 4.2] and [41, Lemmas 5.3, 5.4], were able to provide localized **BV** estimates in a different setting that the one presented in Section 1.4.7.

1.4.1 Compactness *via* global BV bounds

It is well known that the entropy solution to

$$\begin{cases} \partial_t \rho + \partial_x \left(f(\rho) \right) &= 0 \\ \rho(\cdot, 0) &= \rho_0 \end{cases}$$

belongs to the space $\mathbf{L}^\infty((0,+\infty); \mathbf{BV}(\mathbb{R}))$ if ρ_0 has bounded variation. More precisely, the solution ρ verifies:

$$\forall t \geq 0, \quad \mathbf{TV}(\rho(\cdot, t)) \leq \mathbf{TV}(\rho_0).$$

This is a consequence of the $\mathbf{L}^1$ contraction and of the fact that the problem is invariant under spatial translations, see [114, 98, 59] for instance. This property is also true for numerical approximations constructed with monotone numerical fluxes: we say that the scheme (1.1.1) is *total variation diminishing* (TVD).

Lemma 1.4.1. *Suppose that $\rho_0 \in \mathbf{BV}(\mathbb{R})$. Then the scheme (1.1.1) verifies:*

$$\forall n \in \mathbb{N}, \quad \mathbf{TV}(\rho_\Delta(\cdot, t^n)) \leq \mathbf{TV}(\rho_0). \tag{1.4.1}$$

Proof. Fix $n \in \mathbb{N}$ and $j \in \mathbb{Z}$. We start by writing the scheme (1.1.1) under the form:

$$\rho_{j+1/2}^{n+1} = \rho_{j+1/2}^n - \underbrace{\lambda \left(\frac{\mathbf{F}\left(\rho_{j+1/2}^n, \rho_{j+3/2}^n\right) - \mathbf{F}\left(\rho_{j+1/2}^n, \rho_{j+1/2}^n\right)}{\rho_{j+3/2}^n - \rho_{j+1/2}^n} \right)}_{-B_{j+1}} \left(\rho_{j+3/2}^n - \rho_{j+1/2}^n\right)$$

$$- \underbrace{\lambda \left(\frac{\mathbf{F}\left(\rho_{j+1/2}^n, \rho_{j+1/2}^n\right) - \mathbf{F}\left(\rho_{j-1/2}^n, \rho_{j+1/2}^n\right)}{\rho_{j+1/2}^n - \rho_{j-1/2}^n} \right)}_{A_j} \left(\rho_{j+1/2}^n - \rho_{j-1/2}^n\right).$$

The monotonicity of $\mathbf{F}$ ensures that $A_j, B_{j+1} \geq 0$. We deduce that

$$\begin{aligned} \rho_{j+1/2}^{n+1} - \rho_{j-1/2}^{n+1} = &\left(1 - A_j - B_j\right)\left(\rho_{j+1/2}^n - \rho_{j-1/2}^n\right) \\ &+ A_{j-1}\left(\rho_{j-1/2}^n - \rho_{j-3/2}^n\right) + B_{j+1}\left(\rho_{j+3/2}^n - \rho_{j+1/2}^n\right). \end{aligned}$$

Making use of the CFL condition (1.1.5), we have $|A_j| + |B_j| \leq \lambda \mathbf{L} \leq 1$, hence:

$$\begin{aligned}\sum_{j\in\mathbb{Z}} \left|\rho^{n+1}_{j+1/2} - \rho^{n+1}_{j-1/2}\right| &\leq \sum_{j\in\mathbb{Z}} (1 - A_j - B_j) \left|\rho^{n}_{j+1/2} - \rho^{n}_{j-1/2}\right| \\ &\quad + \sum_{j\in\mathbb{Z}} A_{j-1} \left|\rho^{n}_{j-1/2} - \rho^{n}_{j-3/2}\right| + \sum_{j\in\mathbb{Z}} B_{j+1} \left|\rho^{n}_{j+3/2} - \rho^{n}_{j+1/2}\right| \\ &= \sum_{j\in\mathbb{Z}} \left|\rho^{n}_{j+1/2} - \rho^{n}_{j-1/2}\right|\end{aligned}$$

Inequality (1.4.1) follows by immediate induction on n. □

The TVD property (1.4.1) does not hold anymore when the flux is space dependent or in the context of flux constraint problems. The total variation of the solution to (1.0.2) may well increase as the following example highlights.

Example 1.4.1. The exact solution to (1.0.2) with data

$$f(\rho) = \rho(1-\rho); \quad \rho_0(x) = \begin{cases} 0.6 & \text{if } x < 0 \\ 0.2 & \text{if } x > 0 \end{cases}; \quad q(t) = 0.2$$

is represented at time $T = 1$ in Figure 1.2, see [56] for the construction of the constrained Riemann solver. As we can see, $\mathbf{TV}(\rho(\cdot, 1)) > \mathbf{TV}(\rho_0)$.

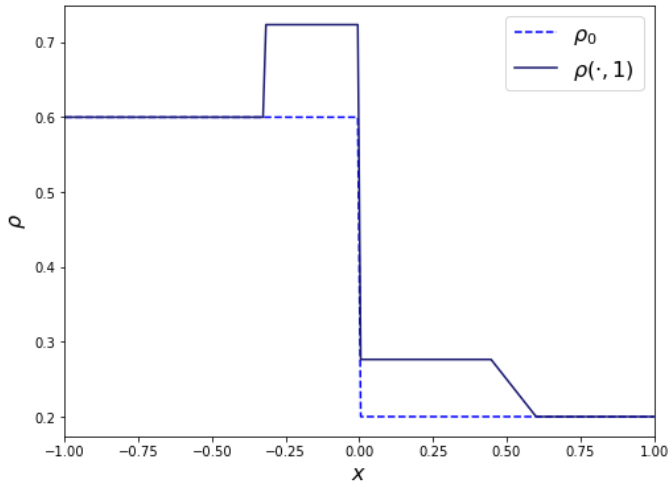

Figure 1.2 – Illustration of the loss of TVD property.

Though not nonincreasing, we will show in this section that if the initial data has bounded variation and if the constraint level does not reach the maximum level, see Assumption (1.4.2) below, then we can derive global **BV** bounds for the sequence $(\rho_\Delta)_\Delta$.

Lemma 1.4.2. *Let us assume that $f \in \mathbf{C}^1([0,1]\backslash\{\overline{\rho}\})$, that $\rho_0 \in \mathbf{BV}(\mathbb{R})$ and that in (1.1.2), we use the Godunov flux when $j = 0$. Then if q verifies the assumption*

$$\exists \varepsilon > 0,\ \forall t > 0, \quad q(t) \leq \max_{\rho\in[0,1]} f(\rho) - \varepsilon := q_\varepsilon, \tag{1.4.2}$$

there exists a constant $\mathbf{C}_\varepsilon > 0$ *such that for all* $n \in \mathbb{N}$,

$$\mathbf{TV}(\rho_\Delta(\cdot, t^{n+1})) \leq \mathbf{TV}(\rho_0) + 4 + \mathbf{C}_\varepsilon \sum_{k=0}^{n} |q^{k+1} - q^k|. \tag{1.4.3}$$

Proof. Note that by assumption, any suitable approximation $(q^n)_n$ of q (mean value, middle point...) will verify:

$$\forall n \in \mathbb{N}, \quad q^n \leq q_\varepsilon.$$

Fix $n \in \mathbb{N}$. With th present set up, we can follow the proofs of [45, Section 2] to obtain the following estimate:

$$\sum_{j\in\mathbb{Z}} |\rho^{n+1}_{j+1/2} - \rho^{n+1}_{j-1/2}| \leq \mathbf{TV}(\rho_0) + 4 + 2\sum_{k=0}^{n} \left| \left(\widehat{\rho}_{q^{k+1}} - \widehat{\rho}_{q^k}\right) - \left(\check{\rho}_{q^{k+1}} - \check{\rho}_{q^k}\right) \right|,$$

where for all $k \in \{0, \dots, n\}$, the couple $\left(\widehat{\rho}_{q^k}, \check{\rho}_{q^k}\right) \in [0,1]^2$ is uniquely defined by the conditions

$$f\left(\check{\rho}_{q^k}\right) = f\left(\widehat{\rho}_{q^k}\right) = q^k \quad \text{and} \quad \check{\rho}_{q^k} < \widehat{\rho}_{q^k}.$$

By assumption, the continuous function $|f'|$ is positive on the compact subset $[0,1]\backslash(\check{\rho}_{q_\varepsilon}, \widehat{\rho}_{q_\varepsilon})$. Hence, it attains its maximal value C_0. Consequently, if one denotes by $I : [0, \check{\rho}_{q_\varepsilon}] \to [0, q_\varepsilon]$ the increasing part of f, this function carries out a $\mathbf{C}^1$-diffeomorphism. Moreover,

$$\forall q \in [0, q_\varepsilon], \quad \left|(I^{-1})'(q)\right| \leq \frac{1}{\mathrm{C}_0}.$$

Consequently, for all $k \in \{0, \dots, n\}$,

$$\left|\check{\rho}_{q^{k+1}} - \check{\rho}_{q^k}\right| = \left|(I^{-1})(q^{k+1}) - (I^{-1})(q^k)\right| \leq \frac{1}{\mathrm{C}_0}|q^{k+1} - q^k|.$$

Using the same techniques, one can show that the same inequality holds for $\left|\widehat{\rho}_{q^{k+1}} - \widehat{\rho}_{q^k}\right|$. Therefore, inequality (1.4.3) follows with $\mathbf{C}_\varepsilon = \dfrac{4}{\mathrm{C}_0}$. □

Corollary 1.4.3. *Let us assume that* $f \in \mathbf{C}^1([0,1]\backslash\{\overline{\rho}\})$ *and that* $\rho_0 \in \mathbf{BV}(\mathbb{R}; [0,1])$. *Suppose also that* $q \in \mathbf{BV}((0, +\infty)), q \geq 0$ *verifies* (1.4.2) *and that in* (1.1.2), *we use the Godunov flux when* $j = 0$ *and any other monotone numerical flux when* $j \neq 0$. *Then there exists* $\rho \in \mathbf{L}^\infty(\Omega) \cap \mathbf{C}([0, +\infty); \mathbf{L}^1_{\mathbf{loc}}(\mathbb{R}))$, *such that along a subsequence,* $\rho_\Delta \to \rho$ *a.e. on* Ω.

Proof. From (1.4.3), we obtain

$$\forall n \in \mathbb{N}, \quad \mathbf{TV}(\rho_\Delta(\cdot, t^n)) \leq \mathbf{TV}(\rho_0) + 4 + \mathbf{C}_\varepsilon \mathbf{TV}(q).$$

Then using the marching formula (1.1.3), for all $n \in \mathbb{N}$,

$$\begin{aligned}
\sum_{j\in\mathbb{Z}} |\rho^{n+1}_{j+1/2} - \rho^n_{j+1/2}|\Delta x &\leq \sum_{j\in\mathbb{Z}} \left|\mathbf{F}^n_{j+1}\left(\rho^n_{j+1/2}, \rho^n_{j+3/2}\right) - \mathbf{F}^n_j\left(\rho^n_{j-1/2}, \rho^n_{j+1/2}\right)\right| \Delta t \\
&\leq \mathbf{L} \sum_{j\in\mathbb{Z}} |\rho^n_{j+1/2} - \rho^n_{j-1/2}|\Delta t \\
&= \mathbf{L} \times \mathbf{TV}(\rho_\Delta(\cdot, t^n))\Delta t.
\end{aligned}$$

The compactness comes from the result of [98, Appendix A] using the two previous bounds and the uniform $\mathbf{L}^\infty$ estimate (1.2.1). □

Theorem 1.4.4. *Fix* $\rho_0 \in \mathbf{BV}(\mathbb{R};[0,1])$, $f \in \mathbf{C}^1([0,1]\backslash\{\overline{\rho}\})$ *verifying* (1.0.1) *and* $q \in \mathbf{BV}((0,+\infty)), q \geq 0$. *Suppose that in* (1.1.2), *we use the Godunov flux when* $j = 0$ *and any other monotone numerical flux associated with* f *when* $j \neq 0$. *Finally, suppose that* q *satisfies* (1.4.2). *Then under the CFL condition* (1.1.5), *the scheme* (1.1.4) *converges to the unique admissible entropy solution* $\rho \in \mathbf{L}^\infty(\Omega) \cap \mathbf{C}([0,+\infty);\mathbf{L}^1_{\mathbf{loc}}(\mathbb{R}))$ *of* (1.0.2) *in the sense of Definition 1.0.1. Moreover, there exists a constant* $\mathbf{C}_\varepsilon > 0$ *such that*

$$\forall t > 0, \quad \mathbf{TV}(\rho(\cdot,t)) \leq \mathbf{TV}(\rho_0) + 4 + \mathbf{C}_\varepsilon \mathbf{TV}(q). \tag{1.4.4}$$

Proof. We have proved in Corollary 1.4.3 that a subsequence of $(\rho_\Delta)_\Delta$ converges a.e. on Ω to some $\rho \in \mathbf{L}^\infty(\Omega) \cap \mathbf{C}([0,+\infty);\mathbf{L}^1_{\mathbf{loc}}(\mathbb{R}))$. Passing to the limit in (1.3.4) and (1.3.4), we obtain that ρ verifies (1.0.3) and (1.0.4). This shows that ρ is an admissible entropy solution to (1.0.2). By uniqueness, the whole sequence $(\rho_\Delta)_\Delta$ converges to ρ. Finally, from (1.4.3), the lower semi-continuity of the $\mathbf{BV}$ semi-norm ensures that $\rho \in \mathbf{L}^\infty((0,+\infty);\mathbf{BV}(\mathbb{R}))$ and verifies (1.4.4), concluding the proof. □

1.4.2 Compactness *via* one-sided Lipschitz condition technique

We now adapt techniques and results put forward by Towers in [152]. We no longer suppose that $\rho_0 \in \mathbf{BV}(\mathbb{R})$ but in compensation, we assume that $f \in \mathbf{C}^2([0,1])$ is strictly concave (still verifying (1.0.1)). By continuity,

$$\exists \alpha > 0,\ \forall \rho \in [0,1], \quad f''(\rho) \leq -\alpha. \tag{1.4.5}$$

We will also assume, following [152], that

$$\begin{array}{c} \text{the numerical flux chosen in (1.1.1) is either} \\ \text{the Engquist-Osher one or the Godunov one.} \end{array} \tag{1.4.6}$$

To be precise, the choice made for the numerical flux at the interface – *i.e.* when $j = 0$ in (1.1.2) – does not play any role. What is important is that away from the interface, one chooses either the Engquist-Osher flux or the Godunov one. We denote for all $n \in \mathbb{N}$ and $j \in \mathbb{Z}$,

$$\mathbf{D}^n_j = \max\left\{\rho^n_{j-1/2} - \rho^n_{j+1/2}, 0\right\}.$$

We will also use the notation

$$\hat{\mathbb{Z}} = \mathbb{Z}\backslash\{-1,0,1\}.$$

In [152], the author dealt with a discontinuous in both time and space flux and the specific "vanishing viscosity" coupling at the interface. The discontinuity in space was localized along the curve $\{x = 0\}$. The applicability of the technique of [152] for our case (and in Chapters 3-4 as well with moving interface and flux-constrained interface coupling) relies on the fact that one can derive a bound on $\mathbf{D}^n_j$ as long as the interface does not enter the calculations for $\mathbf{D}^n_j$ *i.e.* $j \in \hat{\mathbb{Z}}$. This is what the following lemma points out under Assumptions (1.4.5)-(1.4.6). For readers' convenience and in order to highlight the generality of the technique of Towers [152], let us provide the key elements of the argumentation leading to compactness.

Lemma 1.4.5. *Let $n \in \mathbb{N}$ and $j \in \hat{\mathbb{Z}}$. Then if $a = \dfrac{\lambda\alpha}{4}$, we have*

$$\mathbf{D}_j^{n+1} \leq \max\left\{\mathbf{D}_{j-1}^n, \mathbf{D}_j^n, \mathbf{D}_{j+1}^n\right\} - a\left(\max\left\{\mathbf{D}_{j-1}^n, \mathbf{D}_j^n, \mathbf{D}_{j+1}^n\right\}\right)^2 \tag{1.4.7}$$

and

$$\mathbf{D}_j^{n+1} \leq \frac{1}{\min\{|j|-1, n+1\}a}. \tag{1.4.8}$$

Proof. *(Sketched, see also Chapter 4)* Inequality (1.4.8) is an immediate consequence of inequality (1.4.7), see [152, Lemma 4.3]. Obtaining inequality (1.4.7) however, is less immediate. Let us give some details of the proof.
First, note that by introducing the function $\psi : z \mapsto z - az^2$, inequality (1.4.7) can be stated as:

$$\mathbf{D}_j^{n+1} \leq \psi\left(\max\left\{\mathbf{D}_{j-1}^n, \mathbf{D}_j^n, \mathbf{D}_{j+1}^n\right\}\right). \tag{1.4.9}$$

Then, one can show, only using the monotonicity of both the scheme and of the function ψ, that under the assumption

$$\text{inequality (1.4.9) holds when } (\rho_{j+3/2}^n - \rho_{j+1/2}^n),\ (\rho_{j-1/2}^n - \rho_{j-3/2}^n) \leq 0, \tag{1.4.10}$$

it follows that inequality (1.4.9) holds for all cases. And finally in [152, Page 23], the author proves that if the flux considered is either the Engquist-Osher flux or the Godunov flux, then (1.4.10) holds. □

The following lemma is an immediate consequence of inequality (1.4.8).

Lemma 1.4.6. *Fix $0 < \varepsilon < X$. Let $i, J \in \mathbb{N}^*$ such that $\varepsilon \in (x_i, x_{i+1})$ and $X \in (x_{J-1}, x_J)$. Then if $\Delta x/\varepsilon$ is sufficiently small, there exists a constant $\Lambda = \Lambda\left(X, \dfrac{1}{a}, \dfrac{1}{\varepsilon}\right)$, nondecreasing with respect to its arguments, such that for all $n \geq i$,*

$$\sum_{j=i+1}^{J-1} |\rho_{j+1/2}^n - \rho_{j-1/2}^n|, \sum_{j=-J+1}^{-i-1} |\rho_{j+1/2}^n - \rho_{j-1/2}^n| \leq \Lambda \tag{1.4.11}$$

and

$$\sum_{j=i+1}^{J-2} |\rho_{j+1/2}^{n+1} - \rho_{j+1/2}^n|\Delta x, \sum_{j=-J+1}^{-i-2} |\rho_{j+1/2}^{n+1} - \rho_{j+1/2}^n|\Delta x \leq \mathbf{L}\Lambda\Delta t. \tag{1.4.12}$$

Proof. We have:

$$\begin{aligned}\sum_{j=i+1}^{J-1} |\rho_{j+1/2}^n - \rho_{j-1/2}^n| &= 2\sum_{j=i+1}^{J-1} \mathbf{D}_j^n - \sum_{j=i+1}^{J-1} (\rho_{j+1/2}^n - \rho_{j-1/2}^n) \\ &= 2\sum_{j=i+1}^{J-1} \mathbf{D}_j^n - (\rho_{J-1/2}^n - \rho_{i+1/2}^n) \leq 1 + 2\sum_{j=i+1}^{J-1} \mathbf{D}_j^n.\end{aligned}$$

Now, since

$$i+1=\frac{(i+1)\Delta x}{\Delta x}\geq\frac{\varepsilon}{\Delta x},$$

we deduce that if $\dfrac{\Delta x}{\varepsilon}\leq\dfrac{1}{2}$, then $j\geq i+1\implies j\in\widehat{\mathbb{Z}}$. Lemma 1.4.5 ensures that

$$\sum_{j=i+1}^{J-1}|\rho^n_{j+1/2}-\rho^n_{j-1/2}|\leq 1+2\sum_{j=i+1}^{J-1}\mathbf{D}^n_j\leq 1+\frac{2}{a}\sum_{j=i+1}^{J-1}\frac{1}{\min\{n,j-1\}}.$$

However, since $n\geq i$, for all $j\geq i+1$, we have $\min\{n,j-1\}\geq i$, hence:

$$\sum_{j=i+1}^{J-1}|\rho^n_{j+1/2}-\rho^n_{j-1/2}|\leq 1+\frac{2}{a}\times\left(\frac{J-i-1}{i}\right)\leq 1+\frac{4X}{a\varepsilon}:=\Lambda.$$

Then,

$$\sum_{j=i+1}^{J-2}|\rho^{n+1}_{j+1/2}-\rho^n_{j+1/2}|\Delta x=\sum_{j=i+1}^{J-2}\left|\mathbf{F}(\rho^n_{j+1/2},\rho^n_{j+3/2})-\mathbf{F}(\rho^n_{j-1/2},\rho^n_{j+1/2})\right|\Delta t\leq\mathbf{L}\Lambda\Delta t.$$

□

Corollary 1.4.7. *Let us assume that $f\in\mathbf{C}^2([0,1]$ is strictly concave and that $\rho_0\in\mathbf{L}^\infty(\mathbb{R};[0,1])$. Suppose that in (1.1.2), we use either the Godunov flux or the Engquist-Osher flux when $j\neq 0$. Then there exists $\rho\in\mathbf{L}^\infty(\Omega)\cap\mathbf{C}([0,+\infty);\mathbf{L}^1_{\mathbf{loc}}(\mathbb{R}))$, such that along a subsequence, $\rho_\Delta\to\rho$ a.e. on Ω.*

Proof. Fix $0<\varepsilon<X$ and $t>\lambda\varepsilon$. Denote by $\Omega(X,\varepsilon)=(-X,-\varepsilon)\cup(\varepsilon,X)$. Introduce $i,J,n\in\mathbb{N}$ such that $\varepsilon\in(x_i,x_{i+1})$, $X\in(x_{J-1},x_J)$ and $t\in[t^n,t^{n+1})$. Remark that

$$(n+1)\Delta t>t>\lambda\varepsilon\geq\lambda(i\Delta x)=i\Delta t,$$

i.e. $n\geq i-1$. Then if we suppose that $\Delta x/\varepsilon$ is sufficiently small, we can use Lemma 1.4.6. From (1.4.11), we get

$$\mathbf{TV}(\rho_\Delta(\cdot,t)_{|\Omega(X,\varepsilon)})\leq 2\Lambda \tag{1.4.13}$$

and from (1.4.12), we deduce

$$\int_{\Omega(X,\varepsilon)}|\rho_\Delta(x,t+\Delta t)-\rho_\Delta(x,t)|\,\mathrm{d}x\leq 2\mathbf{L}\Lambda\Delta t. \tag{1.4.14}$$

Combining (1.4.13)-(1.4.14) and the $\mathbf{L}^\infty$ bound (1.2.1), like in the proof of Corollary 1.4.3, [98, Theorem A.8] provides the compactness statement. □

Theorem 1.4.8. *Fix $\rho_0 \in \mathbf{L}^\infty(\mathbb{R};[0,1])$ and $f \in \mathbf{C}^2([0,1]$ strictly concave. Suppose that in (1.1.2), we use either the Godunov flux or the Engquist-Osher flux when $j \neq 0$ and any other monotone numerical flux associated with f when $j = 0$. Then under the CFL condition (1.1.5), the scheme (1.1.4) converges to the unique admissible entropy solution $\rho \in \mathbf{L}^\infty(\Omega) \cap \mathbf{C}([0,+\infty);\mathbf{L}^1_{\mathbf{loc}}(\mathbb{R}))$ of (1.0.2) in the sense of Definition 1.0.1. Moreover, for all $t > 0$, $\rho(\cdot,t) \in \mathbf{BV}_{\mathbf{loc}}(\mathbb{R}^*)$.*

Proof. We have proved in Corollary 1.4.7 that a subsequence of $(\rho_\Delta)_\Delta$ converges a.e. on Ω to some $\rho \in \mathbf{L}^\infty(\Omega) \cap \mathbf{C}([0,+\infty);\mathbf{L}^1_{\mathbf{loc}}(\mathbb{R}))$. Passing to the limit in (1.3.4) and (1.3.4), we obtain that ρ verifies (1.0.3) and (1.0.4). This shows that ρ is a solution to (1.0.2). Note also that since such a solution is unique, the whole sequence $(\rho_\Delta)_\Delta$ converges to ρ. Finally, from (1.4.13), the lower semi-continuity of the $\mathbf{BV}$ semi-norm provides the last part of the statement. This concludes the proof. □

Remark 1.4.1. Note the complementarity of the hypotheses made in the above theorem with the ones of Theorem 1.4.4. Recall that in Theorem 1.4.4, we needed the Godunov flux only at the interface.

1.5 Numerical simulations

In all the simulations, we choose the uniformly concave flux $f(\rho) = \rho(1-\rho)$. Since the initial data take values in $[0,1]$, the CFL condition reduces to $2\dfrac{\Delta t}{\Delta x} \leq 1$.

1.5.1 Classical case (without constraint)

We take advantage of this section to illustrate some known results about scalar conservation related in particular to convergence orders or numerical diffusion. We will consider the following initial data:

$$\rho_0^1(x) = \begin{cases} 0.4 & \text{if} \quad x < 0 \\ 0.8 & \text{if} \quad x > 0 \end{cases} \quad \text{and} \quad \rho_0^2(x) = \begin{cases} 0.6 & \text{if} \quad x < 0 \\ 0.2 & \text{if} \quad x > 0. \end{cases}$$

The solution ρ^1 associated with initial data ρ_0^1 is a *shock wave i.e.* a discontinuity traveling at a velocity given by the Rankine-Hugoniot condition while the values of the initial condition ρ_0^2 create a rarefaction wave ρ^2 of speeds $f'(0.6)$ and $f'(0.2)$:

$$\rho^1(x,t) = \begin{cases} 0.4 & \text{if} \quad \dfrac{x}{t} < -0.2 \\ 0.8 & \text{if} \quad \dfrac{x}{t} > -0.2, \end{cases} \qquad \rho^2(x,t) = \begin{cases} 0.6 & \text{if} \quad \dfrac{x}{t} < -0.2 \\ \dfrac{t-x}{2t} & \text{if} \quad -0.2 \leq \dfrac{x}{t} \leq 0.6 \\ 0.2 & \text{if} \quad \dfrac{x}{t} > 0.6. \end{cases}$$

With a domain of computation equal to $[-1,1]$ and time horizon $T = 1$, the numerical solutions computed with the Godunov flux and the Rusanov flux are presented in Figure 1.3.

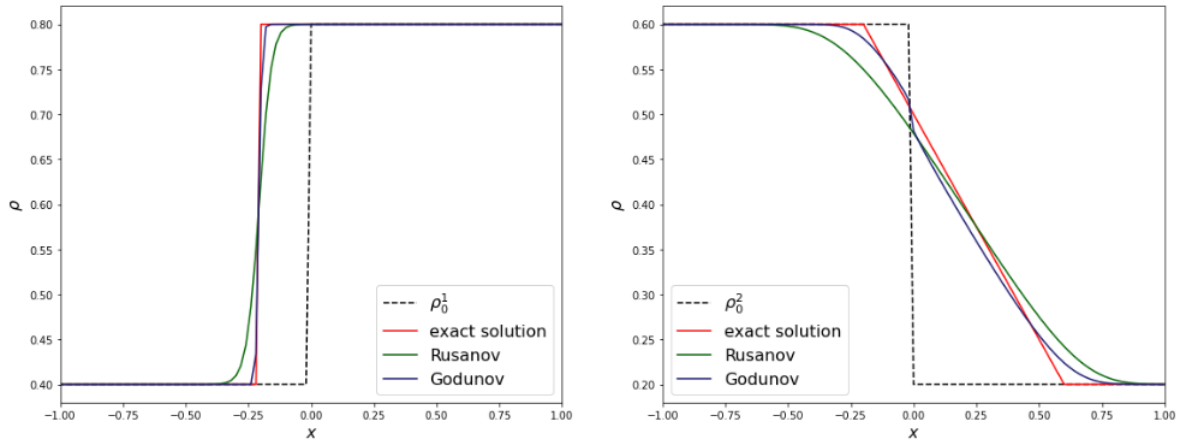

Figure 1.3 – Comparisons between the Godunov flux and the Rusanov flux with $\Delta x = 0.01$.

These figures show that the Godunov scheme is slightly better than the Rusanov flux in the sense that it is less diffusive. For these two Riemann problems, the $\mathbf{L}^\infty$ stability and the TVD property are observable under the CFL condition.
A convergence analysis is also performed. Since we can easily compute the exact solutions of these two Riemann problems, in Figure 1.4, we present the computed $\mathbf{L}^1$ error at time T:

$$\mathbf{E}_\Delta = \|\rho_\Delta - \rho\|_{\mathbf{L}^1} = \int_0^T \int_{-1}^1 |\rho_\Delta(x,t) - \rho(x,t)| \, \mathrm{d}x \, \mathrm{d}t \,,$$

and see that these errors converge with convergence order approximately equal to 1. In the litterature, convergence of order 1/4 or 1/2 is proved, see [48, 49] for instance, but usually one can observe convergence of order 1 like in the present case.

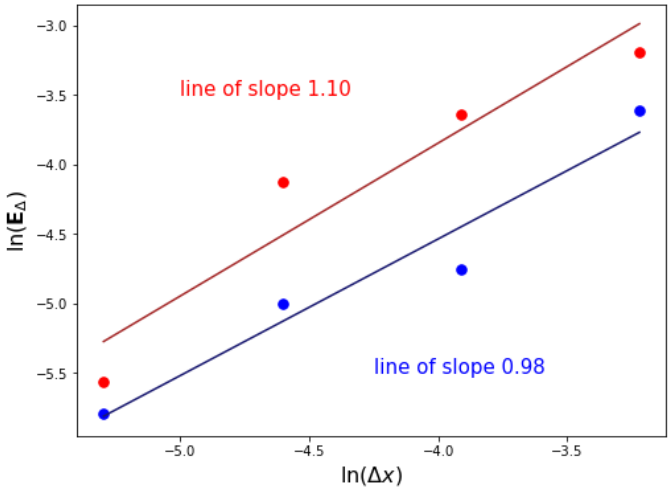

Figure 1.4 – Rates of convergence for the shock (blue) and the rarefaction (red) at time $T = 1.0$.

1.5.2 With local point constraint

We now turn to numerical simulations for the constrained problem (1.0.2).

Four Riemann problems

We consider the initial data

$$\rho_0^1(x) = \begin{cases} 0.8 & \text{if} \quad x < 0 \\ 0.2 & \text{if} \quad x > 0; \end{cases} \qquad \rho_0^2(x) = \begin{cases} 0.6 & \text{if} \quad x < 0 \\ 0.2 & \text{if} \quad x > 0; \end{cases}$$

$$\rho_0^3(x) = \begin{cases} 0.8 & \text{if} \quad x < 0 \\ 0.4 & \text{if} \quad x > 0; \end{cases} \qquad \rho_0^4(x) = \begin{cases} 0.6 & \text{if} \quad x < 0 \\ 0.4 & \text{if} \quad x > 0. \end{cases}$$

We choose a constant constraint function $q \equiv 0.2$ so that for each initial data ρ_0^i $(1 \leq i \leq 4)$, the exact solution is composed of rarefaction/shock wave followed by a nonclassical shock satisfying the constraint and again followed by a rarefaction/shock wave. Let us recall that the authors of [56] gave a full construction of a contrained Riemann solver for (1.0.2). Following the assumptions of Theorem 1.4.8, Figure 1.5 presents the numerical solutions computed with the Godunov flux at the interface and the Rusanov flux away from the interface.

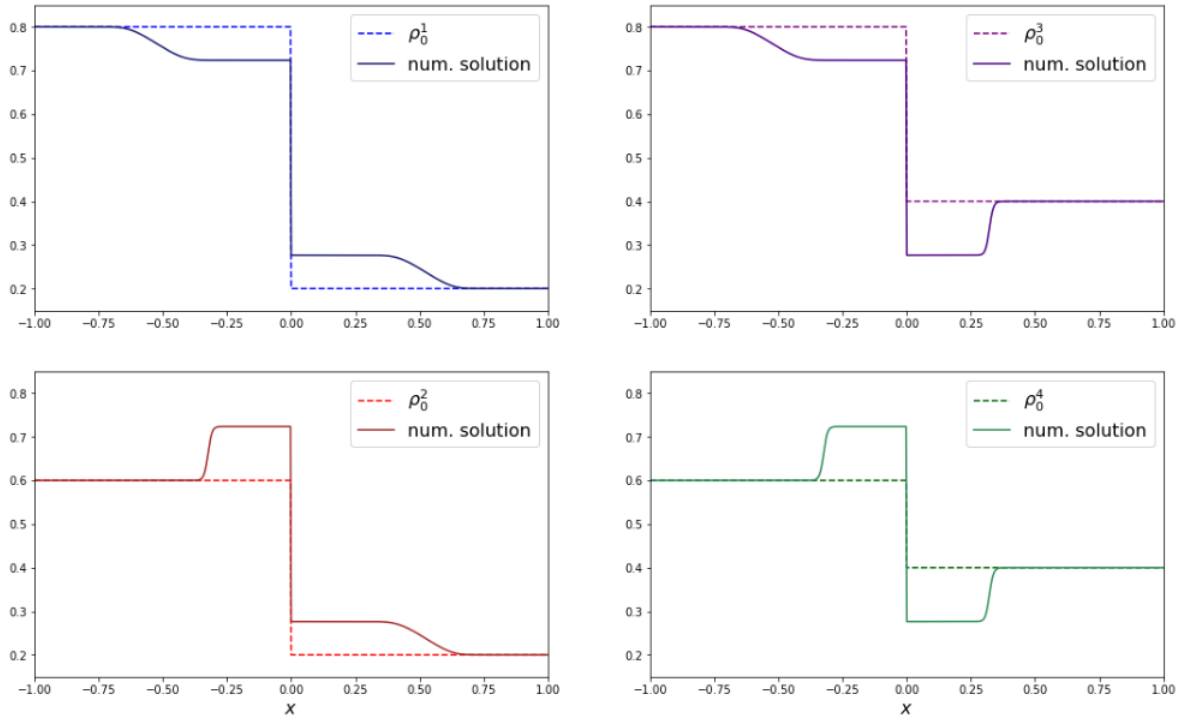

Figure 1.5 – Numerical solutions of four constrained Riemann problems at time $T = 1.0$, with $\Delta x = 0.001$.

The loss of the total variation diminishing property is highlighted by the numerical solutions ρ^2, ρ^3 and ρ^4. Like in the previous section, we computed the $\mathbf{L}^1$ errors $\mathbf{E}_\Delta^i$ $(1 \leq i \leq 4)$. The convergence of these ratio are presented in Figure 1.6.

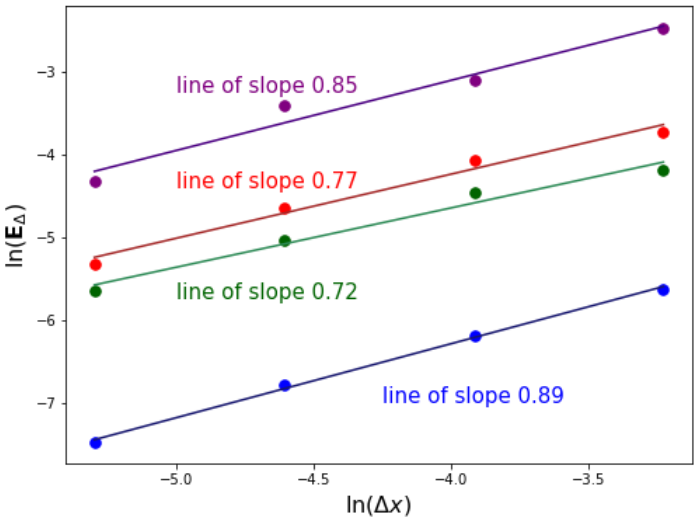

Figure 1.6 – Rates of convergence for four constrained Riemann problems at time $T = 1.0$.

1.6 On global BV bounds for a time-dependent limited flux model

We now focus on the study of the following class of models:

$$\begin{cases} \partial_t \rho + \partial_x \left(F(s(t), \rho) \right) = 0 \\ \rho(\cdot, 0) = \rho_0 \\ F(s(t), \rho)|_{x=0} \leq q(t), \end{cases} \tag{1.6.1}$$

where $s \in \mathbf{BV}((0, +\infty); [0, \Sigma])$ for some $\Sigma > 0$ and $q \in \mathbf{BV}((0, +\infty)), q \geq 0$. We suppose that $F \in \mathbf{C}^1([0, \Sigma] \times [0, 1])$ and that for all $s \in [0, \Sigma]$, $F(s, \cdot)$ verifies *i.e.*

$$\begin{aligned} &\forall s \in [0, \Sigma],\ F(s, 0) = 0,\ F(s, 1) \leq 0 \quad \text{and} \\ &\exists!\ \overline{\rho_s} \in (0, 1),\ \partial_\rho F(s, \rho)\,(\overline{\rho_s} - \rho) > 0 \text{ for a.e. } \rho \in (0, 1). \end{aligned} \tag{1.6.2}$$

This framework covers the particular case when F takes the form:

$$F(s(t), \rho) = f(\rho) - s(t)\rho,$$

with concave $f : [0, 1] \to \mathbb{R}^+$ verifying (1.0.1), which our model in Chapter 3 is based on. Remark that (1.6.1) reduces to (1.0.2) when $s \equiv 0$. In this section, we establish in passing the well-posedness of Problem (1.6.1), but our main interest lies in the **BV** in space regularity of the solutions. More precisely, we aim at obtaining a bound on the total variation of the solutions to (1.6.1), using a finite volume approximation which allows for sharp control of the variation at the constraint. We adapt the techniques and results we developed in the previous sections. Note that the alternative offered by wave-front tracking would be cumbersome because of the explicit time-dependency in (1.6.1). As we previously mentionned and emphasized in Sections 1.4.1-1.5, entropy solutions to limited flux problems like (1.6.1)

do not belong to $\mathbf{L}^\infty((0,+\infty);\mathbf{BV}(\mathbb{R}))$. However, by following the same ideas as Section 1.4.1, we will show that it is the case under a mild assumption on the constraint function q – see Assumption (1.6.10) below – and provided that $\rho_0 \in \mathbf{BV}(\mathbb{R};[0,1])$. Throughout this section, for all $s \in [0,\Sigma]$ and $a,b \in [0,1]$, we denote by

$$\Phi_s(a,b) = \operatorname{sgn}(a-b)(F(s,a)-F(s,b))$$

the classical Kruzhkov entropy flux associated with the Kruzhkov entropy $\rho \mapsto |\rho-\kappa|$, for all $\kappa \in [0,1]$, see [114].

1.6.1 Equivalent definitions of solution and uniqueness

Let us give the following definition of solution for (1.6.1), which is an adapatation of Definition 1.0.1.

Definition 1.6.1. A function $\rho \in \mathbf{L}^\infty(\Omega)$ is an admissible weak solution to (1.6.1) with initial data $\rho_0 \in \mathbf{L}^\infty(\mathbb{R})$ if

(i) for all test functions $\varphi \in \mathbf{C}^\infty_{\mathbf{c}}(\overline{\Omega}), \varphi \geq 0$ and $\kappa \in [0,1]$, the following entropy inequalities are verified:

$$\begin{aligned}\int_0^{+\infty}\int_{\mathbb{R}}\Big(|\rho-\kappa|\partial_t\varphi + \Phi_{s(t)}(\rho,\kappa)\partial_x\varphi\Big)\,\mathrm{d}x\,\mathrm{d}t + \int_{\mathbb{R}}|\rho_0(x)-\kappa|\varphi(x,0)\,\mathrm{d}x \\ + 2\int_0^{+\infty}\mathcal{R}_{s(t)}(\kappa,q(t))\varphi(0,t)\,\mathrm{d}t \geq 0,\end{aligned} \tag{1.6.3}$$

where

$$\mathcal{R}_{s(t)}(\kappa,q(t)) = F(s(t),\kappa) - \min\left\{F(s(t),\kappa),q(t)\right\};$$

(ii) for all test functions $\varphi \in \mathbf{C}^\infty_{\mathbf{c}}(\Omega), \varphi \geq 0$ the following constraint inequalities are verified:

$$-\int_0^{+\infty}\int_{\mathbb{R}^+}\Big(\rho\partial_t\varphi + F(s(t),\rho)\partial_x\varphi\Big)\,\mathrm{d}x\,\mathrm{d}t \leq \int_0^{+\infty} q(t)\varphi(0,t)\,\mathrm{d}t\,. \tag{1.6.4}$$

Definition 1.6.2. Any admissible weak solution belonging to $\rho \in \mathbf{L}^\infty((0,+\infty);\mathbf{BV}(\mathbb{R}))$ will be called $\mathbf{BV}$-regular solution.

As we pointed out after Definition 1.0.1, this notion of solution is well suited for passage to the limit of a.e. convergent sequences of exact or approximate solutions. However, it is not so well-adapted to prove uniqueness. An equivalent notion of solution, based on explicit treatment of traces of ρ at the constraint, was introduced by the authors of [15] (see also Chapter 4). This notion of solution leads to the following stability estimate.

Theorem 1.6.3. *Fix* $s^1, s^2 \in \mathbf{BV}((0,+\infty);[0,\Sigma])$, $\rho_0^1, \rho_0^2 \in \mathbf{BV}(\mathbb{R};[0,1])$ *and* $q^1, q^2 \in \mathbf{BV}((0,+\infty);\mathbb{R}^+)$. *Denote by* ρ^1 *a* $\mathbf{BV}$*-regular solution to* (1.6.1) *with data* ρ_0^1, q^1, s^1 *and*

ρ^2 an admissible weak solution to (1.6.1) with data ρ_0^2, q^2, s^2. Suppose that the flux functions $(t,\rho) \mapsto F(s^1(t),\rho), F(s^2(t),\rho)$ satisfy (1.6.2). Then for all $t>0$, we have:

$$\begin{aligned}\|\rho^1(\cdot,t)-\rho^2(\cdot,t)\|_{\mathbf{L}^1} \leq \|\rho_0^1-\rho_0^2\|_{\mathbf{L}^1} &+ 2\int_0^t |q^1(\tau)-q^2(\tau)|\,\mathrm{d}\tau \\ &+ 2\int_0^t \|F(s^1(\tau),\cdot)-F(s^2(\tau),\cdot)\|_{\mathbf{L}^\infty}\,\mathrm{d}\tau \\ &+ \int_0^t \left|\left|\partial_\rho F(s^1(\tau),\cdot)-\partial_\rho F(s^2(\tau),\cdot)\right|\right|_{\mathbf{L}^\infty} \mathbf{TV}(\rho^1(\cdot,\tau))\,\mathrm{d}\tau\,.\end{aligned} \tag{1.6.5}$$

In particular, Problem (1.6.1) admits at most one **BV***-regular solution.*

Proof. Since our interest to details lies rather on the numerical approximation point of view, we do not fully prove this statement but we give the essential steps leading to this stability result. A complete proof is given in a more general setting in Chapter 4.

— *Definition of solution.* First, the authors of [15] introduce a subset of $\mathbb{R}^2$ called *germ*, which can be seen as the set of all the possible traces of a solution to (1.6.1). Then, they say that ρ is a solution to (1.6.1) if it satisfies entropy inequalities away from the interface – *i.e.* with $\varphi \in \mathbf{C}_\mathbf{c}^\infty(\mathbb{R}^*\times\mathbb{R}^+)$ in the entropy inequalities – and if the couple constituted of left-side and the right-side traces of ρ belongs to this so-called germ.

— *Equivalence of the two definitions.* The next step is to prove that this latter definition of solution is equivalent to Definition 1.6.1. This part is done using good choices of test functions, see [15, Theorem 3.18], [14, Proposition 2.5, Theorem 2.9].

— *First stability estimate.* One first shows that if $s^1=s^2$, then for all $t>0$, one has

$$\|\rho^1(\cdot,t)-\rho^2(\cdot,t)\|_{\mathbf{L}^1} \leq \|\rho_0^1-\rho_0^2\|_{\mathbf{L}^1} + 2\int_0^t |q^1(\tau)-q^2(\tau)|\,\mathrm{d}\tau\,. \tag{1.6.6}$$

The proof starts with the classical doubling of variables method of Kruzhkov [114, Theorem 1] and then uses the germ structure, what the authors of [15] called $\mathbf{L}^1$-*dissipativity*, see [15, Definition 3.1] and [14, Lemma 2.7].

— *Proof of estimate (1.6.5).* The proof is based upon estimate (1.6.6) and elements borrowed from [32, 59]. Most details can be found in the proof of [72, Theorem 2.1]. □

Remark 1.6.1. Though the definition of solutions with the germ explicitly involves the traces of ρ, we did not discuss the existence of such traces. A first way to ensure such existence is to deal with **BV**-regular solutions. That way, traces do exist and are to be understood in the sense of **BV** functions. Outside the **BV** framework, existence of strong traces for solutions to (1.6.1) is ensured provided an non-degeneracy assumption on the fundamental diagram like (3.2.5), see [3, 129]. Finally, if one does not want to impose such a condition on the flux, one can follow what the authors of [15] proposed (in Section 2) and consider the "singular mapping traces."

1.6.2 Existence of BV-regular solutions

In this section, we prove the existence of **BV**-regular solutions by the means of a finite volume scheme. The reasoning is a direct adaptation of what we proposed in Sections 1.1-1.4. Problem (1.6.1) falls into that framework with the exception that here the flux is time dependent, but this is barely an inconvenience. Most of the results from Sections 1.1-1.4 still hold here, with slight modifications we are going to make precise. Of course, we keep the notations introduced back then.

Fix $\rho_0 \in \mathbf{L}^\infty(\mathbb{R};[0,1])$. First, let us discretize the initial data ρ_0 and the functions s, q with $\left(\rho^0_{j+1/2}\right)_j$, $(s^n)_n$ and $(q^n)_n$ where for all $j \in \mathbb{Z}$ and $n \in \mathbb{N}$, $\rho^0_{j+1/2}$, s^n and q^n are their mean values on each cell (x_j, x_{j+1}) and $[t^n, t^{n+1})$. Following [14] and Section 1.1, the marching formula of the scheme reads for all $n \in \mathbb{N}$ and $j \in \mathbb{Z}$:

$$\rho^{n+1}_{j+1/2} = \rho^n_{j+1/2} - \frac{\Delta t}{\Delta x}\left(\mathbf{F}^n_{j+1}(\rho^n_{j+1/2}, \rho^n_{j+3/2}) - \mathbf{F}^n_j(\rho^n_{j-1/2}, \rho^n_{j+1/2})\right), \tag{1.6.7}$$

where

$$\mathbf{F}^n_j(a,b) = \begin{cases} \mathbf{F}^n(a,b) & \text{if } j \neq 0 \\ \min\left\{\mathbf{F}^n(a,b), q^n\right\} & \text{if } j = 0, \end{cases} \tag{1.6.8}$$

$\mathbf{F}^n$ being a monotone numerical flux associated to $F(s^n, \cdot)$. We then define

$$\rho_\Delta(x,t) = \rho^n_{j+1/2} \text{ if } (x,t) \in \mathcal{P}^n_{j+1/2} \quad \text{and} \quad s_\Delta(t), q_\Delta(t) = s^n, q^n \text{ if } t \in [t^n, t^{n+1}).$$

In the present framework, the CFL condition reads:

$$\underbrace{\lambda \sup_{s\in[0,\Sigma]} \left(\left\| \frac{\partial \mathbf{F}^s}{\partial a} \right\|_{\mathbf{L}^\infty} + \left\| \frac{\partial \mathbf{F}^s}{\partial b} \right\|_{\mathbf{L}^\infty} \right)}_{\mathbf{L}} \leq 1, \tag{1.6.9}$$

where $\mathbf{F}^s = \mathbf{F}^s(a,b)$ is the monotone numerical flux, associated to $F(s,\cdot)$, we use in the scheme (1.6.7). From now, the analysis of the scheme follows the same path as in Sections 1.1-1.4: monotonicity/$\mathbf{L}^\infty$ stability, discrete entropy inequalities, approximate entropy/constraint inequalities and compactness. Only the compactness for $(\rho_\Delta)_\Delta$ is left to obtain since the $\mathbf{L}^1_{\mathbf{loc}}$ compactness for the sequences $(s_\Delta)_\Delta$ and $(q_\Delta)_\Delta$ is clear.

Lemma 1.6.4. *We suppose that $\rho_0 \in \mathbf{BV}(\mathbb{R};[0,1])$ and that $q \in \mathbf{BV}((0,+\infty)), q \geq 0$ verifies the assumption*

$$\exists \varepsilon > 0,\ \forall t > 0,\ \forall s \in [0,\Sigma], \quad q(t) \leq \max_{\rho\in[0,1]} F(s,\rho) - \varepsilon := q_\varepsilon(s). \tag{1.6.10}$$

Then there exists a constant $\mathbf{C}_\varepsilon = \mathbf{C}_\varepsilon(\|\partial_s F\|_{\mathbf{L}^\infty})$ nondecreasing with respect to its argument such that for all $n \in \mathbb{N}$,

$$\mathbf{TV}(\rho_\Delta(\cdot, t^{n+1})) \leq \mathbf{TV}(\rho_0) + 4 + \mathbf{C}_\varepsilon\left(\sum_{k=0}^n |q^{k+1} - q^k| + \sum_{k=0}^n |s^{k+1} - s^k|\right), \tag{1.6.11}$$

where $\rho_\Delta = \left(\rho^n_{j+1/2}\right)_{n,j}$ is the finite volume approximation constructed with the scheme (1.6.7)-(1.6.8), using the Godunov flux when $j = 0$ in (1.6.8).

Proof. Fix $n \in \mathbb{N}$. With this set up we can follow the proofs of [45, Section 2] to obtain the following estimate:

$$\sum_{j\in\mathbb{Z}} |\rho^{n+1}_{j+1/2} - \rho^{n+1}_{j-1/2}| \leq \mathbf{TV}(\rho_0) + 4 + 2\sum_{k=0}^{n} \left| \left(\widehat{\rho}_{s^{k+1}}(q^{k+1}) - \widehat{\rho}_{s^k}(q^k)\right) - \left(\check{\rho}_{s^{k+1}}(q^{k+1}) - \check{\rho}_{s^k}(q^k)\right)\right|,$$

where for all $k \in \{0,\dots,n\}$, the couple $\left(\widehat{\rho}_{s^k}(q^k), \check{\rho}_{s^k}(q^k)\right) \in [0,1]^2$ is uniquely defined by the conditions

$$F(s^k, \widehat{\rho}_{s^k}(q^k)) = F(s^k, \check{\rho}_{s^k}(q^k)) = q^k \quad \text{and} \quad \widehat{\rho}_{s^k}(q^k) > \check{\rho}_{s^k}(q^k).$$

Denote by $\Omega(\varepsilon)$ the open subset

$$\Omega(\varepsilon) = \bigcup_{s\in[0,\Sigma]} \Omega_s(\varepsilon)$$

where for all $s \in [0,\Sigma]$, $\Omega_s(\varepsilon) = (\check{\rho}_s(q_\varepsilon(s)), \widehat{\rho}_s(q_\varepsilon(s)))$. By Assumption (1.6.10), the continuous function $(s,\rho) \mapsto |\partial_\rho F(s,\rho)|$ is positive on the compact subset $[0,\Sigma]\times[0,1]\backslash\Omega(\varepsilon)$. Hence, it attains its minimal value $\mathrm{C}_0 > 0$. Consequently, for all $s \in [0,\Sigma]$, if one denotes by $I_s : [0, \check{\rho}_s(q_\varepsilon(s))] \to [0, q_\varepsilon(s)]$ the increasing part of $F(s,\cdot)$, this function carries out a $\mathbf{C}^1$-diffeomorphism. Moreover,

$$\forall q \in [0, q_\varepsilon(s)],\ \left|(I_s^{-1})'(q)\right| \leq \frac{1}{\mathrm{C}_0}.$$

Then, for all $k \in \{0,\dots,n\}$,

$$\begin{aligned}
\left|\check{\rho}_{s^{k+1}}(q^{k+1}) - \check{\rho}_{s^k}(q^k)\right| &= \left|(I^{-1}_{s^{k+1}})(q^{k+1}) - \check{\rho}_{s^k}(q^k)\right| \\
&\leq \frac{1}{\mathrm{C}_0}|q^{k+1} - q^k| + \left|(I^{-1}_{s^{k+1}})(q^k) - \check{\rho}_{s^k}(q^k)\right| \\
&= \frac{1}{\mathrm{C}_0}|q^{k+1} - q^k| + \left|(I^{-1}_{s^{k+1}})(q^k) - (I^{-1}_{s^{k+1}}) \circ I_{s^{k+1}}\left(\check{\rho}_{s^k}(q^k)\right)\right| \\
&\leq \frac{1}{\mathrm{C}_0}\left(|q^{k+1} - q^k| + \left|q^k - I_{s^{k+1}}\left(\check{\rho}_{s^k}(q^k)\right)\right|\right) \\
&= \frac{1}{\mathrm{C}_0}\left(|q^{k+1} - q^k| + \left|F\left(s^k, \check{\rho}_{s^k}(q^k)\right) - F\left(s^{k+1}, \check{\rho}_{s^k}(q^k)\right)\right|\right) \\
&\leq \frac{1}{\mathrm{C}_0}\left(|q^{k+1} - q^k| + \|\partial_s F\|_{\mathbf{L}^\infty}|s^{k+1} - s^k|\right) \\
&\leq \frac{1 + \|\partial_s F\|_{\mathbf{L}^\infty}}{\mathrm{C}_0}\left(|q^{k+1} - q^k| + |s^{k+1} - s^k|\right).
\end{aligned}$$

The same inequality holds when considering $\left|\widehat{\rho}_{s^{k+1}}(q^{k+1}) - \widehat{\rho}_{s^k}(q^k)\right|$. Therefore, inequality (1.6.11) follows with

$$\mathbf{C}_\varepsilon = 4 \times \left(\frac{1 + \|\partial_s F\|_{\mathbf{L}^\infty}}{\mathrm{C}_0}\right).$$

□

Remark 1.6.2. Note how the proof is a direct adaptation of the proof of Lemma 1.4.2.

Remark 1.6.3. Recall we suppose that $F : [0,\Sigma] \times [0,1]$ is continuously differentiable, but if we look in the details of the proof above, we actually need $F = F(s,\rho)$ to be continuously differentiable with respect to s and

$$\forall s \in [0,\Sigma],\ F(s,\cdot) \in \mathbf{C}^1([0,1]\backslash\{\overline{\rho_s}\}), \quad \overline{\rho_s} = \underset{\rho\in[0,1]}{\operatorname{argmax}}\ F(s,\rho).$$

Corollary 1.6.5. *Fix $\rho_0 \in \mathbf{BV}(\mathbb{R};[0,1])$, $s \in \mathbf{BV}((0,+\infty);[0,\Sigma])$ and $q \in \mathbf{BV}((0,+\infty)), q \geq 0$. Suppose that q verifies Assumption (1.6.10). Let $\rho_\Delta = \left(\rho^n_{j+1/2}\right)_{n,j}$ be the finite volume approximate solution constructed with the scheme (1.6.7)-(1.6.8), using the Godunov flux when $j = 0$ in (1.6.8), and any other monotone numerical flux when $j \neq 0$. Then there exists $\rho \in \mathbf{C}([0,+\infty);\mathbf{L}^1_{\mathbf{loc}}(\mathbb{R}))$ such that along a subsequence, $\rho_\Delta \to \rho$ a.e. on Ω.*

Proof. Since s and q have bounded variation, inequality (1.6.11) leads to an uniform in time $\mathbf{BV}$ bound for the sequence $(\rho_\Delta)_\Delta$. Then the result from [78, Appendix] or [98, Theorem A.8] establishes the compactness statement. □

Theorem 1.6.6. *Fix $\rho_0 \in \mathbf{BV}(\mathbb{R};[0,1])$, $s \in \mathbf{BV}((0,+\infty);[0,\Sigma])$, $F \in \mathbf{C}^1([0,\Sigma]\times[0,1])$ verifying (1.6.2) and $q \in \mathbf{BV}((0,+\infty)), q \geq 0$. Suppose that in (1.6.8), we use the Godunov flux when $j = 0$ and any other monotone numerical flux when $j \neq 0$. Finally, suppose that q satisfies (1.6.10). Then under the CFL condition (1.6.9), the scheme (1.6.7)-(1.6.8) converges to an admissible weak solution ρ to (1.6.1), which is also $\mathbf{BV}$-regular. More precisely, there exists a constant $\mathbf{C}_\varepsilon = \mathbf{C}_\varepsilon(\|\partial_s F\|_{\mathbf{L}^\infty})$ nondecreasing with respect to its argument such that*

$$\forall t > 0, \quad \mathbf{TV}(\rho(\cdot,t)) \leq \mathbf{TV}(\rho_0) + 4 + \mathbf{C}_\varepsilon\left(\mathbf{TV}(q) + \mathbf{TV}(s)\right). \tag{1.6.12}$$

Proof. From the scheme (1.6.7), one can derive approximate entropy/constraint inequalities analogous to (1.3.2)-(1.3.4). Let ρ be the limit (of a subsequence) to $(\rho_\Delta)_\Delta$, the compactness coming from the last corollary. We already know that $\rho \in \mathbf{C}([0,+\infty);\mathbf{L}^1_{\mathbf{loc}}(\mathbb{R}))$. By passing to the limit in the approximate entropy/constraint inequalities verified by $(\rho_\Delta)_\Delta$ we get that ρ satisfies the entropy/constraint inequalities of Definition 1.6.1. This shows that ρ is an admissible weak solution to Problem (1.6.1). By uniqueness of such a solution (*cf.* Theorem 1.6.3), the whole sequence converges to ρ. Finally, from (1.6.11), the lower semi-continuity of the $\mathbf{BV}$ semi-norm ensures that $\rho \in \mathbf{L}^\infty((0,+\infty);\mathbf{BV}(\mathbb{R}))$ and verifies (1.6.12). This concludes the proof. □

Corollary 1.6.7. *Fix $\rho_0 \in \mathbf{BV}(\mathbb{R};[0,1])$, $s \in \mathbf{BV}((0,+\infty);[0,\Sigma])$, $F \in \mathbf{C}^1([0,\Sigma]\times[0,1])$ verifying (1.6.2) and $q \in \mathbf{BV}((0,+\infty)), q \geq 0$. Suppose that q satisfies Assumption (1.6.10). Then Problem (1.6.1) admits a unique $\mathbf{BV}$-regular solution ρ. Moreover, ρ satisfies the bound (1.6.12).*

Proof. Uniqueness comes from Theorem 1.6.3, the existence and the $\mathbf{BV}$ bound come from Theorem 1.6.6. □

Remark 1.6.4. Under the hypotheses of Corollary 1.6.7, if we prove the existence of an other admissible weak solution $\widetilde{\rho}$ to (1.6.1) (by another method, splitting for instance like in Chapter 3), then Theorem 1.6.3 ensures that $\widetilde{\rho} = \rho$.

CHAPTER 2

TRAFFIC DYNAMICS AT BOTTLENECKS

After the preliminaries of Chapter 1, we carry on in this chapter with a macroscopic model to describe traffic dynamics at bottlenecks. As previously evoked, the LWR framework is the simplest one that can be used to describe macroscopically pedestrian/road traffic in a corridor or on a road. It takes the form

$$\partial_t \rho + \partial_x(f(\rho)) = 0,$$

where $\rho = \rho(x,t) \in [0,1]$ is the density of pedestrians/cars at $(x,t) \in \Omega = \mathbb{R} \times (0,+\infty)$ and $f : [0,1] \to \mathbb{R}$ is the flux function, assumed concave while verifying (1.0.1). Point constraints were introduced in [60, 56] in the LWR model in order to account for localized in space phenomena that may occur at exits, such as traffic lights or tollgates in the context of road traffic, and which act as obstacles. To do so, one can impose a localized constraint on the flux such as

$$f(\rho)|_{x=0} \leq q(t).$$

One of the typical features of both vehicle and pedestrian flows is self-organization (or orderliness and cooperation), see [109, 66, 47] for empirical data that put in evidence this phenomenon. Here, we focus on orderliness/cooperation near exits. We do not intend to model the different mechanisms behind self-organization, but only to reproduce its phenomenology. In [10] the authors attempted to reproduce self-organization with a model based on the LWR-flux constraint framework:

$$\begin{cases} \partial_t \rho + \partial_x(f(\rho)) = 0 \\ \qquad f(\rho)|_{x=0} \leq p\left(\displaystyle\int_{\mathbb{R}} \rho(x,t)\mu(x)\,\mathrm{d}x\right). \end{cases} \tag{2.0.1}$$

Above, μ is a weight function, supported in a compact neighborhood upstream the exit, used to average the density around the exit. Hence, the quantity

$$\xi(t) = \int_{\mathbb{R}} \rho(x,t)\mu(x)\,\mathrm{d}x\,,$$

called *subjective density* in the sequel is the average density upstream the exit. In (2.0.1), the function $p : [0,1] \to \mathbb{R}^+$ is nonincreasing and Lipschitz, and models the exit efficiency. The idea behind this choice is the following modeling assumption: in an evacuation, the exit outflow is a nonincreasing function of the upstream density; the more crowded the exit, the slower the evacuation. Problem (2.0.1) belongs to a class of models which has has been tremendously studied in the last decades, see [56, 14, 51, 11, 70] for instance. In particular, the authors of [9, 10] were able to reproduce the main effects linked to the *capacity drop* that are the Braess paradox and the "Faster Is Slower" effect, (see Sections 2.3 for related numerical simulations) but not so much the self-organization. Our first goal is to further advance in this direction. We introduce a model which interpolates between two states of the traffic (organized and disorganized) which we represent by the presence of two levels of constraints and by an organization parameter which evolves through an ODE. This model admits a natural and efficient approximation strategy, relying on a combination of splitting, explicit Euler time integration and of a monotone finite volume scheme for LWR. In passing, we prove well-posedness for our model in Sections 2.1-2.2, but our main interest lies in Sections 2.3-2.4, where we perform a test to validate and discuss the model.

2.1 Notion of solution and uniqueness

Our starting point is the model (2.0.1) proposed by the authors of [11], see also [9, 10]. To go further, we introduce two levels of exit efficiencies $p_{\min} \leq p_{\max}$, both are required to be Lipschitz continuous nonincreasing functions, and set

$$q(t) = (1-\omega(t))p_{\min}(\xi(t)) + \omega(t)p_{\max}(\xi(t)) := p_{\min}(\xi(t)) + \omega(t)\delta p(\xi(t)), \tag{2.1.1}$$

where as before, ξ denotes the subjective density:

$$\forall t \geq 0, \quad \xi(t) = \int_{\mathbb{R}} \rho(x,t)\mu(x)\,\mathrm{d}x\,,$$

with weight function $\mu \in \mathbf{C}^2_{\mathbf{c}}(\mathbb{R}^-)$. Above, $\omega(t) \in [0,1]$ is an organization parameter which describes the state of the traffic and evolves through the ODE

$$\dot{\omega}(t) = \mathbf{K}\left(\xi(t), \dot{\xi}(t)\right)\omega(t)(1-\omega(t)). \tag{2.1.2}$$

Mathematically speaking, we only suppose that $\mathbf{K} \in \mathbf{Lip_{loc}}(\mathbb{R}^2)$. The idea behind phenomenologically relevant choices of $\mathbf{K}$, see Figure 2.1(right), is to allow for progressive organization of traffic with time, while keeping the possibility of return to disorganization when sudden and strong variations of the traffic occur; see Section 2.4. For the sake of being definite, in simulations we will choose $\mathbf{K}$ under the form

$$\mathbf{K}(\xi,\chi) = C\left(\frac{\xi}{\xi_{\mathrm{c}}} - 1\right)^+ \left(1 - \frac{\chi^+}{D_+} - \frac{\chi^-}{D_-}\right), \tag{2.1.3}$$

with some positive parameters $\xi_{\mathrm{c}}, C, D_+, D_-$ and the notations $z^+ = \max(z,0)$, $z^- = |z| - z^+$. This choice will be discussed later. We have the following coupled PDE-ODE system to study

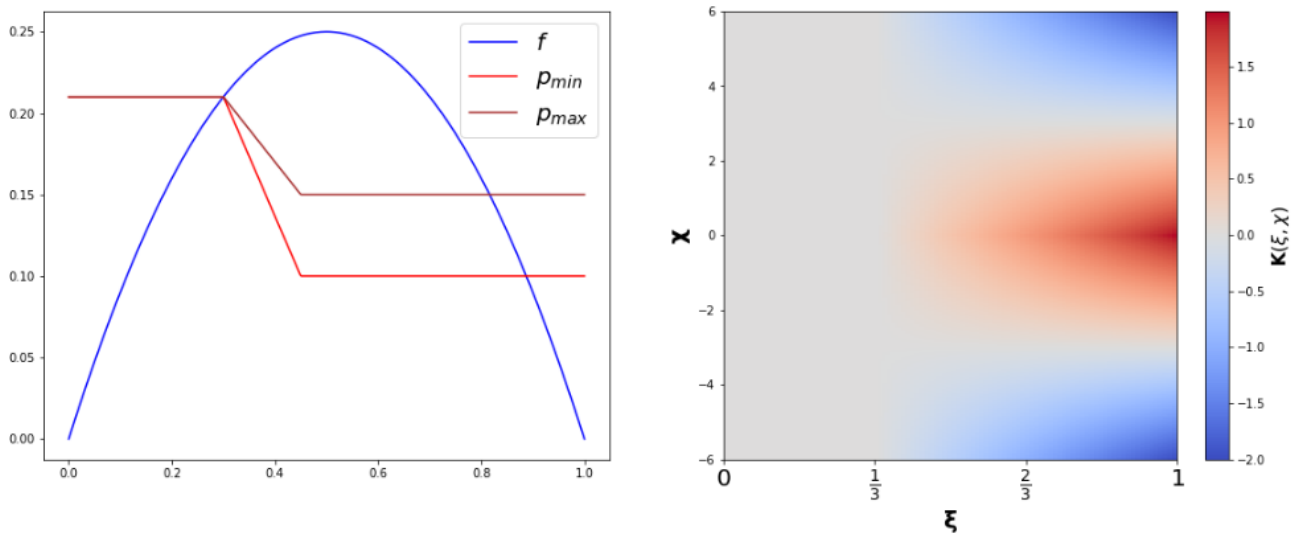

Figure 2.1 – Typical behavior of exit efficiencies $p_{\min}, p_{\max}$ (left) and organization-driving function $\mathbf{K}$ in (2.1.2) (right).

in $\mathbb{R} \times (0,T)$ $(T>0)$:

$$\begin{cases} \partial_t \rho + \partial_x(f(\rho)) = 0 \\ \qquad f(\rho)|_{x=0} \leq q(t) \\ \qquad\quad q(t) = (1-\omega(t))p_{\min}(\xi(t)) + \omega(t)p_{\max}(\xi(t)) \\ \qquad\quad \dot{\omega}(t) = \mathbf{K}\left(\xi(t), \dot{\xi}(t)\right)\omega(t)(1-\omega(t)). \end{cases} \tag{2.1.4}$$

The notion of solution for Problem (2.1.4) is based on Definition 1.0.1 in which we simply take into account the coupling between ρ and q *via* ω and add a weak ODE formulation that ω has to verify.

Definition 2.1.1. A couple (ρ, ω) with $\rho \in \mathbf{L}^\infty(\mathbb{R} \times (0,T))$ and $\omega \in \mathbf{W}^{1,\infty}((0,T))$ is an admissible weak solution to (2.1.4) with initial data $\rho_0 \in \mathbf{L}^\infty(\mathbb{R})$ and $\omega_0 \in (0,1)$ if

(i) for all test functions $\varphi \in \mathbf{C}_\mathbf{c}^\infty(\mathbb{R} \times [0,T)), \varphi \geq 0$ and $\kappa \in [0,1]$, the following entropy inequalities are verified:

$$\begin{aligned} \int_0^T \int_{\mathbb{R}} \Big(|\rho - \kappa| \partial_t \varphi + \Phi(\rho, \kappa) \partial_x \varphi \Big) \,\mathrm{d}x \,\mathrm{d}t + \int_{\mathbb{R}} |\rho_0(x) - \kappa| \varphi(x,0) \,\mathrm{d}x \\ + 2 \int_0^T \mathcal{R}(\kappa, q(t)) \varphi(0,t) \,\mathrm{d}t \geq 0, \end{aligned} \tag{2.1.5}$$

where for all $t \in [0,T]$,

$$\mathcal{R}(\kappa, q(t)) = f(\kappa) - \min\{f(\kappa), q(t)\}; \quad q(t) = (1-\omega(t))p_{\min}(\xi(t)) + \omega(t)p_{\max}(\xi(t));$$

(ii) for all test functions $\varphi \in \mathbf{C}_{\mathbf{c}}^{\infty}(\mathbb{R} \times (0,T)), \varphi \geq 0$ the following constraint inequalities are verified:

$$-\int_0^T \int_{\mathbb{R}^+} \Big(\rho \partial_t \varphi + f(\rho) \partial_x \varphi \Big) \,\mathrm{d}x \,\mathrm{d}t \leq \int_0^{+\infty} q(t) \varphi(0,t) \,\mathrm{d}t \,; \tag{2.1.6}$$

(iii) the following weak ODE formulation is verified for all $t \in [0,T]$:

$$\omega(t) = \omega_0 + \int_0^t \mathbf{K}\Big(\xi(s), \dot{\xi}(s)\Big)\, \omega(s)(1-\omega(s)) \,\mathrm{d}s \,. \tag{2.1.7}$$

Remark 2.1.1. Suppose that (ρ, ω) is an admissible weak solution to (2.1.4). In particular, ρ is an entropy solution to $\partial_t \rho + \partial_x(f(\rho)) = 0$ in $\mathbb{R}^{+*} \times (0,T)$ and $\mathbb{R}^{-*} \times (0,T)$. Consequently, it is also a weak/distributional solution to the PDE in $\mathbb{R} \times (0,T)$. Using $\mu\psi$, $\psi \in \mathbf{C}_{\mathbf{c}}^{\infty}((0,T))$, $\psi \geq 0$ as a test function in the weak formulation, we obtain that

$$\int_0^T \xi(t) \dot{\psi}(t) \,\mathrm{d}t = -\int_0^T \left(\int_{\mathbb{R}} f(\rho(x,t)) \mu'(x) \,\mathrm{d}x \right) \psi(t) \,\mathrm{d}t \,,$$

which proves that ξ is differentiable in the distributional sense and that for a.e. $t \in (0,T)$,

$$\dot{\xi}(t) = \int_{\mathbb{R}} f(\rho(x,t)) \mu'(x) \,\mathrm{d}x \,.$$

In particular, if $\rho(x,t) \in [0,1]$ a.e. on $\mathbb{R} \times (0,T)$ (which will be the case, see Theorem 2.2.1), then for a.e. $t \in (0,T)$,

$$\left| \dot{\xi}(t) \right| \leq \|f\|_{\mathbf{L}^\infty} \|\mu'\|_{\mathbf{L}^1}.$$

Consequently, $\xi \in \mathbf{W}^{1,\infty}((0,T))$.

Before we prove stability with respect to initial data and uniqueness for admissible weak solutions to the system (2.1.4), let us note that we can directly integrate the ODE in (2.1.4). This feature is not crucial nor essential but it will be useful in the proof of Theorem 2.1.3.

Lemma 2.1.2. *Let (ρ, ω) be an admissible weak solution to the system (2.1.4) associated with initial data $\rho_0 \in \mathbf{L}^\infty(\mathbb{R})$ and $\omega_0 \in (0,1)$. Then for all $t \in [0,T]$,*

$$\omega(t) = \frac{\exp(W(t))}{1+\exp(W(t))}; \quad W(t) = \ln\left(\frac{\omega_0}{1-\omega_0}\right) + \int_0^t \mathbf{K}\Big(\xi(s), \dot{\xi}(s)\Big) \,\mathrm{d}s \,.$$

Proof. The expression of ω follows from a classical integration of the ODE in (2.1.4). □

Introduce the notations:

$$\|\mathbf{K}\|_{\mathbf{L}^\infty} = \sup_{\substack{0 \leq \xi \leq 1 \\ |\chi| \leq 2\|f\|_{\mathbf{L}^\infty} \|\mu'\|_{\mathbf{L}^1}}} |\mathbf{K}(\xi,\chi)|; \quad \|\nabla \mathbf{K}\|_{\mathbf{L}^\infty} = \sup_{\substack{0 \leq \xi_1, \xi_2 \leq 1 \\ |\chi_1|, |\chi_2| \leq 2\|f\|_{\mathbf{L}^\infty} \|\mu'\|_{\mathbf{L}^1}}} |\mathbf{K}(\xi_1,\chi_1) - \mathbf{K}(\xi_2,\chi_2)|$$

and

$$\|\delta p\|_{\mathbf{L}^\infty} = \sup_{0 \leq \xi \leq 1} \delta p(\xi); \quad \|\delta p'\|_{\mathbf{L}^\infty} = \sup_{0 \leq \xi \leq 1} |\delta p'(\xi)|.$$

Theorem 2.1.3. *Fix $\rho_0^1, \rho_0^2 \in \mathbf{L}^\infty(\mathbb{R})$ and $\omega_0^1, \omega_0^2 \in (0,1)$. We denote by (ρ^1, ω^1) and (ρ^2, ω^2) two admissible weak solutions to the system (2.1.4) corresponding to the initial data (ρ_0^1, ω_0^1) and (ρ_0^2, ω_0^2), respectively. Then there exist $A, B, C \geq 0$ such that for all $t \in [0,T]$,*

$$\|\rho^1(\cdot,t) - \rho^2(\cdot,t)\|_{\mathbf{L}^1} \leq \|\rho_0^1 - \rho_0^2\|_{\mathbf{L}^1} G(t) + A|w_0^1 - w_0^2| \int_0^t G(s)\,\mathrm{d}s \tag{2.1.8}$$

and

$$|\omega^1(t) - \omega^2(t)| \leq \frac{|w_0^1 - w_0^2|}{4} + \frac{C}{2\|\delta p\|_{\mathbf{L}^\infty}} \int_0^t \left(A|w_0^1 - w_0^2|(t-s) + \|\rho_0^1 - \rho_0^2\|_{\mathbf{L}^1}\right) G(s)\,\mathrm{d}s\,, \tag{2.1.9}$$

where

$$G(t) = \exp\left(Bt + \frac{Ct^2}{2}\right); \quad w_0^i = \ln\left(\frac{\omega_0^i}{1-\omega_0^i}\right) \ i \in \{1,2\}.$$

In particular, the system (2.1.4) admits at most one admissible weak solution.

Proof. Fix $t \in [0,T]$. First, from [14, Proposition 2.10], Theorem 1.6.3 or Theorem (4.1.8), we get:

$$\|\rho^1(\cdot,t) - \rho^2(\cdot,t)\|_{\mathbf{L}^1} \leq \|\rho_0^1 - \rho_0^2\|_{\mathbf{L}^1} + 2\int_0^t |q^1(s) - q^1(s)|\,\mathrm{d}s\,.$$

This estimate is typical of constraint problems like (2.1.4) and can be seen as a Lipschitz continuous dependence $q \mapsto \rho$ for $q \in \mathbf{L}^1((0,T))$ and $\rho \in \mathbf{C}([0,T]; \mathbf{L}^1(\mathbb{R}))$. Then, using the expressions of q^i $(i \in \{1,2\})$, we obtain

$$\begin{aligned}
|q^1(t) - q^2(t)| &= \left|p_{\min}(\xi^1(t)) + \omega^1(t)\delta p(\xi^1(t)) - p_{\min}(\xi^2(t)) - \omega^2(t)\delta p(\xi^2(t))\right| \\
&\leq \left(\|p'_{\min}\|_{\mathbf{L}^\infty} + \|\delta p'\|_{\mathbf{L}^\infty}\right)|\xi^1(t) - \xi^2(t)| + \|\delta p\|_{\mathbf{L}^\infty}|\omega^1(t) - \omega^2(t)| \\
&\leq \left(\|p'_{\min}\|_{\mathbf{L}^\infty} + \|\delta p'\|_{\mathbf{L}^\infty}\right)\|\mu\|_{\mathbf{L}^\infty}\|\rho^1(\cdot,t) - \rho^2(\cdot,t)\|_{\mathbf{L}^1} + \|\delta p\|_{\mathbf{L}^\infty}|\omega^1(t) - \omega^2(t)|.
\end{aligned}$$

We now express the distance between the ω^i using their expression (see Lemma 2.1.2):

$$\begin{aligned}
|\omega^1(t) - \omega^2(t)| &\leq \frac{|w_0^1 - w_0^2|}{4} + \frac{\|\nabla \mathbf{K}\|_{\mathbf{L}^\infty}}{4}\int_0^t \left(|\xi^1(s) - \xi^2(s)| + |\dot{\xi}^1(s) - \dot{\xi}^2(s)|\right)\mathrm{d}s \\
&\leq \frac{|w_0^1 - w_0^2|}{4} + \frac{\|\nabla \mathbf{K}\|_{\mathbf{L}^\infty}(\|\mu\|_{\mathbf{L}^\infty} + \|f'\|_{\mathbf{L}^\infty}\|\mu'\|_{\mathbf{L}^\infty})}{4}\int_0^t \|\rho^1(\cdot,s) - \rho^2(\cdot,s)\|_{\mathbf{L}^1}\,\mathrm{d}s\,.
\end{aligned} \tag{2.1.10}$$

Putting these three estimates together, we have shown that:

$$\begin{aligned}
\|\rho^1(\cdot,t) - \rho^2(\cdot,t)\|_{\mathbf{L}^1} &\leq \|\rho_0^1 - \rho_0^2\|_{\mathbf{L}^1} + A|w_0^1 - w_0^2|t + B\int_0^t \|\rho^1(\cdot,s) - \rho^2(\cdot,s)\|_{\mathbf{L}^1}\,\mathrm{d}s \\
&\quad + C\int_0^t \left(\int_0^s \|\rho^1(\cdot,\tau) - \rho^2(\cdot,\tau)\|_{\mathbf{L}^1}\,\mathrm{d}\tau\right)\mathrm{d}s \\
&= \|\rho_0^1 - \rho_0^2\|_{\mathbf{L}^1} + A|w_0^1 - w_0^2|t + \int_0^t (B + C(t-s))\|\rho^1(\cdot,s) - \rho^2(\cdot,s)\|_{\mathbf{L}^1}\,\mathrm{d}s\,,
\end{aligned}$$

with

$$A = \frac{\|\delta p\|_{\mathbf{L}^\infty}}{2}; \quad B = 2\left(\|p'_{\min}\|_{\mathbf{L}^\infty} + \|\delta p'\|_{\mathbf{L}^\infty}\right)\|\mu\|_{\mathbf{L}^\infty};$$

$$C = \frac{\|\delta p\|_{\mathbf{L}^\infty}\|\nabla \mathbf{K}\|_{\mathbf{L}^\infty}(\|\mu\|_{\mathbf{L}^\infty} + \|f'\|_{\mathbf{L}^\infty}\|\mu'\|_{\mathbf{L}^\infty})}{2}.$$

An application of Gronwall's lemma leads to (2.1.8), and (2.1.9) follows by putting (2.1.8) in (2.1.10). □

2.2 Finite volume approximation of the model

Here, we prove the existence of admissible weak solutions to the system (2.1.4). To do that, we construct and prove the convergence of an explicit Euler in time scheme for the ODE (2.1.2) combined with a monotone finite volume scheme for the constrained LWR equation. The framework is almost identical to the one of Chapter 1, Sections 1.1-1.4 with the exception of the coupling between ρ and the constraint level q *via* ω, expressed by (2.1.1). However, this will not be an issue for the construction of the scheme. Indeed, the only modification is in the definition of the approximate constraint, which is done rather simply by (2.2.1). We keep the notations of Chapter 1.

Fix $\rho_0 \in \mathbf{L}^\infty(\mathbb{R}; [0,1])$ and $\omega_0 \in (0,1)$.

First, let us discretize the initial data ρ_0 and the weight function μ with $\left(\rho^0_{j+1/2}\right)_{j\in\mathbb{Z}}$ and $\left(\mu_{j+1/2}\right)_{j\in\mathbb{Z}}$ where for all $j \in \mathbb{Z}$, $\rho^0_{j+1/2}$ and $\mu_{j+1/2}$ are their mean values on the cell (x_j, x_{j+1}).

Initialization:

$$\xi^0 = \sum_{j\in\mathbb{Z}} \rho^0_{j+1/2}\mu_{j+1/2}\Delta x \quad \text{and} \quad w^0 = \omega_0.$$

Induction. Fix $n \in \mathbb{N}$.

At the time step t^n, we first define a constraint level q^n:

$$q^n = (1-w^n)p_{\min}(\xi^n) + w^n p_{\max}(\xi^n). \tag{2.2.1}$$

We use this value to update the approximate traffic density with the marching formula (for all $j \in \mathbb{Z}$):

$$\rho^{n+1}_{j+1/2} = \rho^n_{j+1/2} - \frac{\Delta t}{\Delta x}\left(\mathbf{F}^n_{j+1}(\rho^n_{j+1/2}, \rho^n_{j+3/2}) - \mathbf{F}^n_j(\rho^n_{j-1/2}, \rho^n_{j+1/2})\right), \tag{2.2.2}$$

where, following the recipe of [14, 51] or Chapter 1, Section 1.1,

$$\mathbf{F}^n_j(a,b) = \begin{cases} \mathbf{F}(a,b) & \text{if } j \neq 0 \\ \min\{\mathbf{F}(a,b), q^n\} & \text{if } j = 0, \end{cases} \tag{2.2.3}$$

$\mathbf{F} = \mathbf{F}(a,b)$ being a monotone numerical flux associated to f, see Definition 1.1.1 and Example 1.1.1. Then, setting

$$\xi^{n+1} = \sum_{j\in\mathbb{Z}} \rho^{n+1}_{j+1/2}\mu_{j+1/2}\Delta x,$$

we update the organization parameter

$$\chi^{n+1} = \frac{\xi^{n+1} - \xi^n}{\Delta t}, \quad \theta^{n+1} = \mathbf{K}\left(\xi^{n+1}, \chi^{n+1}\right) w^n(1-w^n), \quad w^{n+1} = w^n + \theta^{n+1}\Delta t. \tag{2.2.4}$$

Conclusion. Define the functions

- $\rho_\Delta = \sum_{n\in\mathbb{N}} \sum_{j\in\mathbb{Z}} \rho^n_{j+1/2} \mathbb{1}_{\mathcal{P}^n_{j+1/2}}$
- $q_\Delta(t), \chi_\Delta(t), \theta_\Delta(t) = \sum_{n\in\mathbb{N}} \left(q^n, \chi^{n+1}, \theta^{n+1}\right) \mathbb{1}_{[t^n, t^{n+1})}$
- $\forall t > 0, \quad \xi_\Delta(t) = \xi^0 + \int_0^t \chi_\Delta(s)\,\mathrm{d}s\,; \quad \omega_\Delta(t) = w^0 + \int_0^t \theta_\Delta(s)\,\mathrm{d}s\,.$

As always, the CFL condition reads ($\lambda = \Delta t/\Delta x$):

$$\lambda \mathbf{L} \le 1, \quad \mathbf{L} = \left(\left\| \frac{\partial \mathbf{F}}{\partial a} \right\|_{\mathbf{L}^\infty} + \left\| \frac{\partial \mathbf{F}}{\partial b} \right\|_{\mathbf{L}^\infty} \right). \tag{2.2.5}$$

2.2.1 $\mathbf{L}^\infty$ stability and approximate inequalities

The analysis of the scheme follows the skeleton developed in Chapter 1. Even better, most of the results proved through Sections 1.2-1.4 remain exactly the same in the present framework. More precisely, any statement in Chapter 1 which does not depend on the way the sequence $(q^n)_n$ is constructed still holds here.

Proposition 2.2.1 ($\mathbf{L}^\infty$ stability)**.** *Given q^n to define the constrained flux in (2.2.3), the scheme (2.2.2) is stable:*

$$\forall n \in \mathbb{N},\ \forall j \in \mathbb{Z}, \quad \rho^n_{j+1/2} \in [0,1]. \tag{2.2.6}$$

Proof. *cf.* the proof of Theorem 1.2.1. □

Remark 2.2.1. From (2.2.6) and the fact that μ is a weight function, we immediately obtain:

$$\forall n \in \mathbb{N}, \quad \xi^n = \int_{\mathbb{R}} \rho_\Delta(x, t^n)\mu(x)\,\mathrm{d}x \in [0,1].$$

We can also derive a bound for $(\chi^{n+1})_n$. Indeed, for all $n \in \mathbb{N}$,

$$\begin{aligned}
\chi^{n+1} &= \frac{1}{\lambda} \sum_{j\in\mathbb{Z}} (\rho^{n+1}_{j+1/2} - \rho^n_{j+1/2})\mu_{j+1/2} \\
&= -\sum_{j\in\mathbb{Z}} \left(\mathbf{F}^n_{j+1}(\rho^n_{j+1/2}, \rho^n_{j+3/2}) - \mathbf{F}^n_j(\rho^n_{j-1/2}, \rho^n_{j+1/2}) \right) \mu_{j+1/2} \\
&= \sum_{j\in\mathbb{Z}} \mathbf{F}^n_j(\rho^n_{j-1/2}, \rho^n_{j+1/2})(\mu_{j+1/2} - \mu_{j-1/2}) \\
&= \frac{1}{\Delta x} \sum_{j\in\mathbb{Z}} \int_{x_j}^{x_{j+1}} \int_{x-\Delta x}^{x} \mathbf{F}^n_j(\rho^n_{j-1/2}, \rho^n_{j+1/2})\mu'(z)\,\mathrm{d}z\,\mathrm{d}x\,,
\end{aligned}$$

from which we deduce:

$$\forall n \in \mathbb{N}, \quad \left|\chi^{n+1}\right| \leq 2\mathbf{L}\|\mu'\|_{\mathbf{L}^1}.$$

Corollary 2.2.2 (Discrete entropy inequalities)**.** *The numerical scheme* (2.2.2) *fulfills the following discrete entropy inequalities for all* $n \in \mathbb{N}$, $j \in \mathbb{Z}$ *and* $\kappa \in [0,1]$*:*

$$|\rho^{n+1}_{j+1/2} - \kappa| \leq \begin{cases} |\rho^n_{j+1/2} - \kappa| - \lambda\left(\Phi^n_{j+1} - \Phi^n_j\right) & \text{if} \quad j \notin \{-1, 0\} \\ |\rho^n_{-1/2} - \kappa| - \lambda\left(\Phi^n_{int} - \Phi^n_{-1}\right)\Delta t + \lambda\mathcal{R}(\kappa, q^n) & \text{if} \quad j = -1 \\ |\rho^n_{1/2} - \kappa| - \lambda\left(\Phi^n_1 - \Phi^n_{int}\right)\Delta t + \lambda\mathcal{R}(\kappa, q^n) & \text{if} \quad j = 0, \end{cases} \tag{2.2.7}$$

where $\mathcal{R}(\kappa, q^n)$ *was defined in Definition 2.1.1, and* Φ^n_j *and* Φ^n_{int} *are the numerical entropy fluxes:*

$$\Phi^n_j = \mathbf{F}(\rho^n_{j-1/2} \vee \kappa, \rho^n_{j+1/2} \vee \kappa) - \mathbf{F}(\rho^n_{j-1/2} \wedge \kappa, \rho^n_{j+1/2} \wedge \kappa);$$
$$\Phi^n_{int} = \min\{\mathbf{F}(\rho^n_{-1/2} \vee \kappa, \rho^n_{1/2} \vee \kappa), q^n\} - \min\{\mathbf{F}(\rho^n_{-1/2} \wedge \kappa, \rho^n_{1/2} \wedge \kappa), q^n\}.$$

Proof. *cf.* the proof of Corollary 1.2.2. □

Proposition 2.2.3 (Approximate entropy/constraint inequalities)**.** *Fix* $\varphi \in \mathbf{C}^\infty_{\mathbf{c}}(\mathbb{R} \times [0,T))$, $\varphi \geq 0$ *and* $\kappa \in [0,1]$*. Then as* $\Delta x, \Delta t \to 0$*, we have*

$$\begin{aligned} \int_0^T \int_{\mathbb{R}} \Big(|\rho_\Delta - \kappa| \partial_t \varphi + \Phi_\Delta\left(\rho_\Delta, \kappa\right) \partial_x \varphi \Big) \,\mathrm{d}x \,\mathrm{d}t + \int_{\mathbb{R}} |\rho^0_\Delta - \kappa| \varphi(x,0) \,\mathrm{d}x \\ + 2\int_0^T \mathcal{R}(\kappa, q_\Delta(t)) \varphi(0,t) \,\mathrm{d}t \geq O(\Delta x) + O(\Delta t)\,, \end{aligned} \tag{2.2.8}$$

and if $\varphi \in \mathbf{C}^\infty_{\mathbf{c}}(\mathbb{R} \times (0,T))$*, then*

$$\int_0^T \int_{\mathbb{R}^+} \Big(\rho_\Delta \partial_t \varphi + \mathbf{F}_\Delta(\rho_\Delta) \partial_x \varphi \Big) \,\mathrm{d}x \,\mathrm{d}t \leq \int_0^T q_\Delta(t) \varphi(0,t) \,\mathrm{d}t + O(\Delta x) + O(\Delta t)\,, \tag{2.2.9}$$

where

$$\Phi_\Delta(\rho_\Delta, \kappa) = \sum_{n \in \mathbb{N}} \sum_{j \in \mathbb{Z}} \Phi^n_j \mathbb{1}_{\mathcal{P}^n_{j+1/2}}; \quad \mathbf{F}_\Delta\left(\rho_\Delta\right) = \sum_{n \in \mathbb{N}} \sum_{j \in \mathbb{Z}} \mathbf{F}(\rho^n_{j-1/2}, \rho^n_{j+1/2}) \mathbb{1}_{\mathcal{P}^n_{j+1/2}}.$$

Proof. *cf.* the proofs of Propositions 1.3.1-1.3.2. □

We now turn to the study of $(w^n)_n$.

Proposition 2.2.4 ($\mathbf{L}^\infty$ stability)**.** *Under the additional assumption*

$$\Delta t \|\mathbf{K}\|_{\mathbf{L}^\infty} < 1, \tag{2.2.10}$$

we have

$$\forall n \in \mathbb{N}, \quad w^n \in (0,1). \tag{2.2.11}$$

Proof. We prove the result by induction on n. The result is clearly true for $n = 0$ since $\omega \in (0,1)$. Suppose now that (2.1.9) holds for some $n \in \mathbb{N}$. Introduce the function

$$g : w \mapsto w + \mathbf{K}\left(\xi^{n+1}, \chi^{n+1}\right) w(1-w)\Delta t,$$

so that $w^{n+1} = g(w^n)$. Using (2.2.10), we obtain that for all $w \in [0,1]$,

$$g'(w) = 1 + \mathbf{K}\left(\xi^{n+1}, \chi^{n+1}\right)(1-2w)\Delta t \geq 1 - \left|\mathbf{K}\left(\xi^{n+1}, \chi^{n+1}\right)\right| \Delta t > 0.$$

Since $g(0) = 0$ and $g(1) = 1$, the monotonicity of g implies that $w^{n+1} = g(w^n) \in (0,1)$, which completes the induction argument. □

We now prove that ω_Δ satisfies an approximate version of (2.1.7).

Proposition 2.2.5 (Approximate weak ODE formulation)**.** *There exists a constant $D \geq 0$ such that*

$$\forall t > 0, \quad \left|\omega_\Delta(t) - \left(\omega_0 + \int_0^t \mathbf{K}\left(\xi_\Delta(s), \chi_\Delta(s)\right) \omega_\Delta(s)(1-\omega_\Delta(s))\,\mathrm{d}s\right)\right| \leq (D\Delta t)t. \quad (2.2.12)$$

Proof. Fix $|\chi| \leq 2\mathbf{L}\|\mu'\|_{\mathbf{L}^1}$, $n \in \mathbb{N}$ and $t \in [t^n, t^{n+1})$. We have

$$\begin{aligned}
&\left|\mathbf{K}\left(\xi^{n+1}, \chi\right) w^n(1-w^n) - \mathbf{K}\left(\xi_\Delta(t), \chi\right) \omega_\Delta(t)(1-\omega_\Delta(t))\right| \\
&\leq \frac{\|\mathbf{K}\|_{\mathbf{L}^\infty}}{4} \left|w^n - \omega_\Delta(t)\right| + \frac{\|\nabla \mathbf{K}\|_{\mathbf{L}^\infty}}{4} \left|\xi^{n+1} - \xi_\Delta(t)\right|
\end{aligned}$$

Note also that

$$|w^n - \omega_\Delta(t)| \leq \frac{\|\mathbf{K}\|_{\mathbf{L}^\infty}}{4}\Delta t; \quad \left|\xi^{n+1} - \xi_\Delta(t)\right| \leq 2\mathbf{L}\|\mu'\|_{\mathbf{L}^1}\Delta t,$$

hence (2.2.12) follows with

$$D = \frac{1}{4}\left(\frac{\|\mathbf{K}\|_{\mathbf{L}^\infty}^2}{4} + 2\mathbf{L}\|\mu'\|_{\mathbf{L}^1}\|\nabla \mathbf{K}\|_{\mathbf{L}^\infty}\right).$$

□

To conclude this section, we make precise the link between $(q_\Delta)_\Delta$, $(\omega_\Delta)_\Delta$ and $(\xi_\Delta)_\Delta$.

Proposition 2.2.6 (Approximate weak ODE formulation)**.** *There exists a constant $E \geq 0$ such that*

$$\forall t > 0, \quad \left|q_\Delta(t) - \left((1-\omega_\Delta(t))p_{\min}(\xi_\Delta(t)) + \omega_\Delta(t)p_{\max}(\xi_\Delta(t))\right)\right| \leq E\Delta t. \quad (2.2.13)$$

Proof. Fix $n \in \mathbb{N}$ and $t \in [t^n, t^{n+1})$. We have

$$\begin{aligned}
&\left|q^n - \left((1-\omega_\Delta(t))p_{\min}(\xi_\Delta(t)) + \omega_\Delta(t)p_{\max}(\xi_\Delta(t))\right)\right| \\
&\leq \left(\|p'_{\min}\|_{\mathbf{L}^\infty} + \|p'_{\max}\|_{\mathbf{L}^\infty}\right)\left|\xi^n - \xi_\Delta(t)\right| + \left(\|p_{\min}\|_{\mathbf{L}^\infty} + \|p_{\max}\|_{\mathbf{L}^\infty}\right)\left|\omega^n - \omega_\Delta(t)\right| \\
&\leq \underbrace{\left(2\mathbf{L}\|\mu'\|_{\mathbf{L}^1}\left(\|p_{\min'}\|_{\mathbf{L}^\infty} + \|p'_{\max}\|_{\mathbf{L}^\infty}\right) + \frac{\|\mathbf{K}\|_{\mathbf{L}^\infty}\left(\|p_{\min'}\|_{\mathbf{L}^\infty} + \|p'_{\max}\|_{\mathbf{L}^\infty}\right)}{4}\right)}_{E}\Delta t,
\end{aligned}$$

which is exactly (2.2.13). □

2.2.2 Compactness and convergence

The only thing left to do is to pass the limit in (2.2.8)-(2.2.9) and (2.2.12)-(2.2.13), and to do so, we need sufficient compactness for all the sequences involved. First, note that Remark 2.2.1 coupled with the compact embedding $\mathbf{W}^{1,\infty}((0,T)) \subset \mathbf{C}([0,T])$ ensure that there exists $\xi \in \mathbf{C}([0,T])$ such that along a subsequence, $\xi_\Delta \to \xi$ uniformly on $[0,T]$. The same way, the stability (2.2.11) and the bound

$$\forall n \in \mathbb{N}, \quad \left|\theta^{n+1}\right| \leq \frac{\|\mathbf{K}\|_{\mathbf{L}^\infty}}{4}$$

provide the existence of $\omega \in \mathbf{C}([0,T])$ such that along a subsequence, $\omega_\Delta \to \omega$ uniformly on $[0,T]$. Then, from

$$\begin{aligned}\forall n \in \mathbb{N}, \quad |q^{n+1} - q^n| &\leq \left(\|p'_{\min}\|_{\mathbf{L}^\infty} + \|\delta p'\|_{\mathbf{L}^\infty}\right)|\xi^{n+1} - \xi^n| + \|\delta p\|_{\mathbf{L}^\infty}|w^{n+1} - w^n| \\ &= \left(\|p'_{\min}\|_{\mathbf{L}^\infty} + \|\delta p'\|_{\mathbf{L}^\infty}\right)|\chi^{n+1}|\Delta t + \|\delta p\|_{\mathbf{L}^\infty}|\theta^{n+1}|\Delta t \\ &\leq \left(2\mathbf{L}\|\mu'\|_{\mathbf{L}^1}\left(\|p'_{\min}\|_{\mathbf{L}^\infty} + \|\delta p'\|_{\mathbf{L}^\infty}\right) + \frac{\|\delta p\|_{\mathbf{L}^\infty}\|\mathbf{K}\|_{\mathbf{L}^\infty}}{4}\right)\Delta t.\end{aligned}$$

and the obvious $\mathbf{L}^\infty$ bound:

$$\forall n \in \mathbb{N}, \quad |q^n| \leq \|p_{\max}\|_{\mathbf{L}^\infty},$$

Helly's theorem yields the existence of $q \in \mathbf{BV}([0,T])$ such that along a subsequence, $q_\Delta \to q$ a.e. on $(0,T)$. Combining this with (2.2.13), we establish the link between q, ω and ξ: for a.e. $t \in (0,T)$,

$$q(t) = (1 - \omega(t))p_{\min}(\xi(t)) + \omega(t)p_{\max}(\xi(t))$$

Regarding the sequence $(\rho_\Delta)_\Delta$, in Chapter 1, Section 1.4, we discussed in great lengths diverse ways to obtain a.e. convergence and focused on two of them which involve $\mathbf{BV}$ bounds.

— We saw that global $\mathbf{BV}$ bounds are available if the data (initial data and constraint) are $\mathbf{BV}$ and if the constraint does not reach the maximum level, under the additional assumption that $f \in \mathbf{C}^1([0,1]\backslash\{\overline{\rho}\})$, see Lemma 1.4.2. To apply this result, we would need to prove that the sequence $(q^n)_n$ is bounded in $\mathbf{BV}([0,T])$ and verifies Assumption 1.4.2. This last assumption will be fulfilled by supposing that

$$\exists \varepsilon > 0, \quad \forall \xi \in [0,1], \quad p_{\max}(\xi) \leq \max_{0 \leq \rho \leq 1} f(\rho) - \varepsilon. \tag{2.2.14}$$

For the $\mathbf{BV}$ regularity of $(q^n)_n$, we simply write that for all $n \in \mathbb{N}$,

$$\begin{aligned}|q^{n+1} - q^n| &\leq \left(\|p'_{\min}\|_{\mathbf{L}^\infty} + \|\delta p'\|_{\mathbf{L}^\infty}\right)|\xi^{n+1} - \xi^n| + \|\delta p\|_{\mathbf{L}^\infty}|w^{n+1} - w^n| \\ &= \left(\|p'_{\min}\|_{\mathbf{L}^\infty} + \|\delta p'\|_{\mathbf{L}^\infty}\right)|\chi^{n+1}|\Delta t + \|\delta p\|_{\mathbf{L}^\infty}|\theta^{n+1}|\Delta t \\ &\leq \left(2\mathbf{L}\|\mu'\|_{\mathbf{L}^1}\left(\|p'_{\min}\|_{\mathbf{L}^\infty} + \|\delta p'\|_{\mathbf{L}^\infty}\right) + \frac{\|\delta p\|_{\mathbf{L}^\infty}\|\mathbf{K}\|_{\mathbf{L}^\infty}}{4}\right)\Delta t.\end{aligned}$$

In short, if $\rho_0 \in \mathbf{BV}(\mathbb{R})$, under Assumption (2.2.14), there exists $\rho \in \mathbf{C}([0,T];\mathbf{L}^1_{\mathbf{loc}}(\mathbb{R}))$ such that along a subsequence, $\rho_\Delta \to \rho$ a.e. on $\mathbb{R} \times (0,T)$.

— We also saw that local $\mathbf{BV}$ bounds are available if $f \in \mathbf{C}^2([0,1])$ is strictly concave, without any additional assumption on the data, see Lemmas 1.4.5-1.4.6. We could directly apply this result since the reasoning behind the proof of these local $\mathbf{BV}$ bounds did not involve the sequence $(q^n)_n$.

Theorem 2.2.7. *Fix $\rho_0 \in \mathbf{L}^\infty(\mathbb{R};[0,1])$ and $\omega_0 \in (0,1)$ and f a concave flux verifying (1.0.1).*

(i) Suppose that $\rho_0 \in \mathbf{BV}(\mathbb{R})$, $f \in \mathbf{C}^1([0,1]\backslash\{\overline{\rho}\})$ and that (2.2.14) holds. Finally, suppose that in (2.2.3), we use the Godunov flux when $j=0$ and any other monotone numerical flux associated with f when $j \neq 0$. Then under the CFL conditions (2.2.5)-(2.2.10), the scheme (2.2.1)-(2.2.2)-(2.2.3) converges to the unique admissible weak solution to Problem (2.1.4).

(ii) Suppose that $f \in \mathbf{C}^2([0,1])$ is strictly concave and that in (2.2.3), we use either the Godunov flux or the Engquist-Osher flux when $j \neq 0$ and any other monotone numerical flux associated with f when $j=0$. Then the conclusion of (i) holds.

Proof. The difference between these sets of assumptions is on how they provide strong compacteness for $(\rho_\Delta)_\Delta$. Once said compactness is obtained, the proofs of convergence are identical. We show that the couple (ρ,ω) constructed above is an admissible weak solution in the sense of Definition 2.1.1.

First, we pass to the limit in (2.2.8)-(2.2.9) and (2.2.13) which proves that (i)-(ii) of Definition 2.1.1 are satisfied. Moreover, from

$$\forall n \in \mathbb{N},\ \forall t \in [t^n, t^{n+1}), \quad \left|\xi_\Delta(t) - \int_{\mathbb{R}} \rho_\Delta(x,t)\mu(x)\,\mathrm{d}x\right| \leq \left|\chi^{n+1}\right|(t-t^n) \leq 2\mathbf{L}\|\mu'\|_{\mathbf{L}^1}\Delta t,$$

we deduce that for a.e. $t \in (0,T)$,

$$\xi(t) = \int_{\mathbb{R}} \rho(x,t)\mu(x)\,\mathrm{d}x\,.$$

Since these two functions are continuous, the equality holds for all $t \in [0,T]$. Moreover, since ρ is a distributional solution to the PDE in (2.1.4), we know that $\xi \in \mathbf{W}^{1,\infty}((0,T))$ and for a.e. $t \in (0,T)$,

$$\dot{\xi}(t) = \int_{\mathbb{R}} f(\rho(x,t))\mu'(x)\,\mathrm{d}x\,,$$

see Remark 2.1.1. The only thing left to do is to prove that ω verifies the weak ODE formulation (2.1.7). To this end, we want to pass to the limit in (2.2.12) and to do so, we are going to prove that $(\chi_\Delta)_\Delta$ converges to $\dot{\xi}$ a.e. on $(0,T)$. This is here, and only here that the hypothesis $\mu \in \mathbf{C}^2_{\mathbf{c}}(\mathbb{R}^-)$ is used. Note that until now, we only used the $\mathbf{C}^1_{\mathbf{c}}$ regularity of μ. For all $n \in \mathbb{N}$ and $t \in [t^n, t^{n+1})$, we have

$$\begin{aligned}
\left|\chi_\Delta(t) - \int_{\mathbb{R}} \mathbf{F}_\Delta(\rho_\Delta)\mu'(x)\,\mathrm{d}x\right| &= \frac{1}{\Delta x}\left|\sum_{j\in\mathbb{Z}} \int_{x_j}^{x_{j+1}} \int_{x-\Delta x}^{x} \int_x^y \mathbf{F}(\rho^n_{j-1/2}, \rho^n_{j+1/2})\mu''(z)\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x\right| \\
&\leq \frac{2\mathbf{L}}{\Delta x}\sum_{j\in\mathbb{Z}} \int_{x_j}^{x_{j+1}} \int_{x_{j-1}}^{x_{j+1}} \int_{x_{j-1}}^{x_{j+1}} |\mu''(z)|\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x \\
&= 8\mathbf{L}\|\mu''\|_{\mathbf{L}^1}\Delta x,
\end{aligned}$$

proving the statement. Having all the sufficient convergences, we let $\Delta \to 0$ in (2.2.12), proving that ω verifies the weak ODE formulation (2.1.7) and concluding the proof of the theorem. □

Corollary 2.2.8. *Fix $\rho_0 \in \mathbf{L}^\infty(\mathbb{R};[0,1])$ and $\omega_0 \in (0,1)$.*

(i) Suppose that $\rho_0 \in \mathbf{BV}(\mathbb{R})$, $f \in \mathbf{C}^1([0,1]\backslash\{\overline{\rho}\})$ and that (2.2.14) holds. Then Problem (2.1.4) admits a unique admissible weak solution.

(ii) Suppose that $f \in \mathbf{C}^2([0,1])$ is strictly concave. Then Problem (2.1.4) admits a unique admissible weak solution.

Proof. Uniqueness comes from Theorem 2.1.3 while existence was proved in Theorem 2.2.7. □

Remark 2.2.2. Adopting the formalism proposed in [10], one could also prove well-posedness with fixed point arguments.

2.3 Numerical simulations

We report on numerical experiments with the scheme described in Section 2.2. In all the simulations, we take the normalized uniformly concave flux $f(\rho) = \rho(1-\rho)$. Following the hypothesis of Theorem 2.2.7, we choose to use the Godunov flux at the interface ($j = 0$ in (2.2.3)) and the Rusanov flux away from the interface ($j \neq 0$ in (2.2.3)).

Following [10, Section 7], the setup for our simulation is as follows. We consider the domain of computation $[-5,1]$, the initial data $\rho_0(x) = \mathbb{1}_{[-4,-2]}(x)$, $\omega_0 = 0.2$ and the efficiencies of the exit $p_{\min}$, $p_{\max}$ are represented in Figure 2.1(left). For the simulations, we have fixed a locally Lipschitz prefactor $\mathbf{K}$ in (2.1.2) with behaviour is depicted in Figure 2.1(right) and parameters $\xi_c = 1/3$, $C = 2/3$, $D_+ = 1/10$ and $D_- = D_+/2$. The phenomenological features encoded in this choice will be addressed in Section 2.4. A mild regularisation of the function

$$x \mapsto 2n\left(x + \frac{1}{n}\right)\mathbb{1}_{\left[-\frac{1}{n},0\right]}(x),$$

(with $n = 3$) is issued as weight function.
Let us comment on qualitative features of the simulated traffic flow and provide its interpretation in terms of agents' behaviors. First, as we can see in Figure 2.2, the introduction of the organization parameter favors the evacuation time. Figure 2.3 highlights the fact that the model reproduces some features expected from self-organization. At first, the exit flux increases until it reaches the maximum level of the exit efficiency. As traffic densifies, the exit flux falls down to the lowest value of this efficiency, which reflects rapid disorganization, *i.e.* , predominance of agents' individualistic strategies over the rational collective behavior. Then, in the time interval $[6,16]$, the elevated density upstream has very small variations which leads to the emergence of a coherent collective behavior of the agents. This is witnessed through the increase of both the organization marker and the exit flux. We stress out

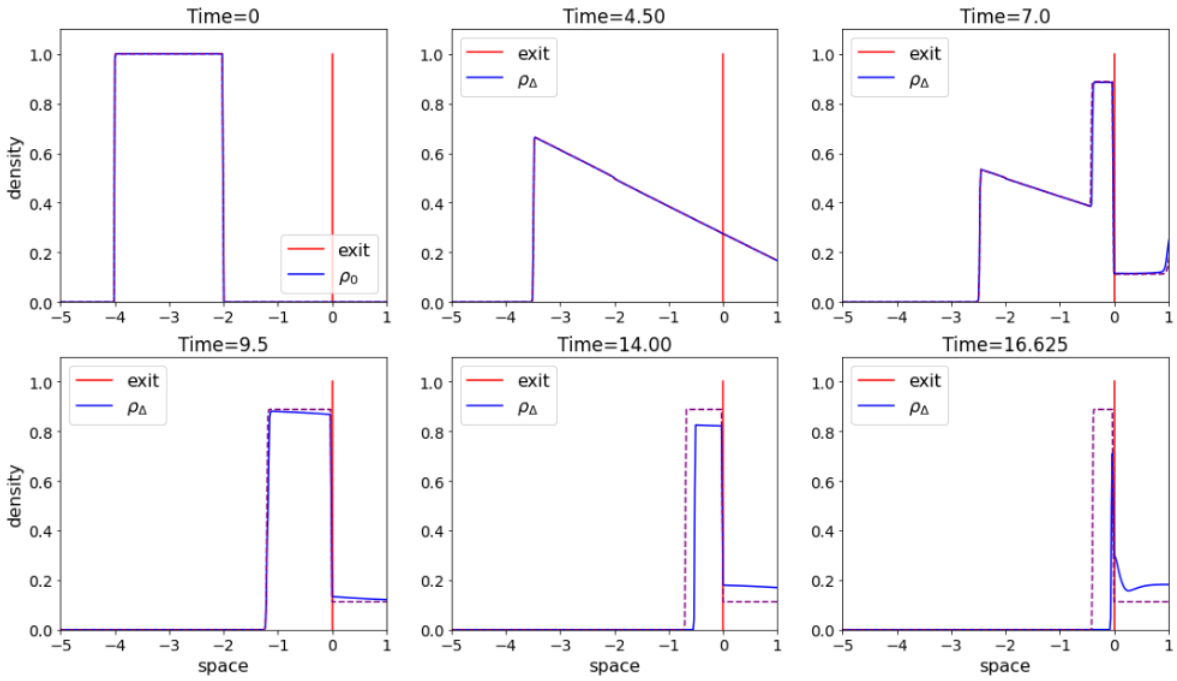

Figure 2.2 – The numerically computed solution $x \mapsto \rho_\Delta(x,t)$ at different fixed times t; dashed lines correspond to the reference solution in absence of orderliness $\omega = 0$ in (2.1.4).

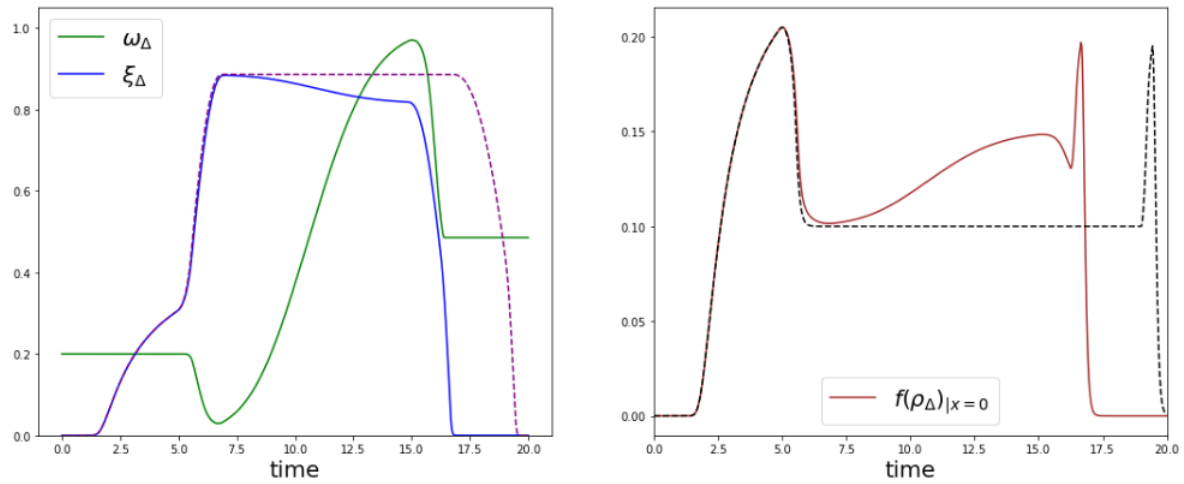

Figure 2.3 – Left: computed subjective density ξ_Δ and organization marker ω_Δ. Right: computed exit flux $f(\rho_\Delta)|_{x=0^-}$; dashed lines correspond to the reference solution in absence of orderliness $\omega = 0$ in (2.1.4), with $\Delta x = 0.015$.

that without orderliness, the exit flux keeps its minimal value in this time interval. Then a notable phenomenon seems to take place. In the time interval $[15.5, 16.3]$, the jam upstream the exit starts to resorb, and the exit efficiency (which is monitored by the exit flux) slightly falls down while the organization level regresses significantly. In other words, the agents abandon collective strategies in rapidly evolving environments, but this does not affect the traffic dramatically because densities are also strongly decreased.

2.4 Conclusions and perspectives

The model we propose here permits a rigorous analysis of well-posedness as well as a robust and simple numerical approximation. It enriches the qualitative behavior of the simple LWR-based models for bottlenecks ([56, 11, 10]), due to its ability to reproduce a few self-organization features. Let us deeper discuss the model construction, in particular the role of the function $\mathbf{K}$ whose behavior is depicted in Figure 2.1(right). Its key features are as follows:

- invariance of the organization marker ω in the region of low densities;
- rapid decrease of ω for moderate and particularly for high densities, under strong density variations;
- progressive increase of ω in dense and very dense traffic with small density variations.

The idea behind these features is: rapidly changing traffic conditions, at considerable densities, promote individual behavior and rapidly lead to a somewhat chaotic interactions among agents, thus lowering the exit efficiency; while persistent coercive traffic conditions, such as a jam, help to emerge and promote a collective behavior like formation of well-organized queues, the alternate in the order of passage through the bottleneck, and a higher degree of mutual courtesy among agents; thus the exit efficiency improves accordingly, which enhances the jam evacuation. The form (2.1.3) provides a simple example of such behavior, which is confirmed by the simulations of Section 2.3. The parameter ξ_c has the meaning of activation threshold for organization/disorganization of the traffic at bottleneck; D_+, D_- indicate thresholds of transition from cooperative (low variations of ξ) to individualistic (higher ones) dynamics of agents.

One way to improve this model would be to take into account unexpected/rash behavior of certain agents. Let us recall that unlike fluid mechanics models, traffic models deal with a relatively small number of agents. In consequence, we would expect the dynamics to be greatly impacted by the behavior of a few agents. An idea to model such rash behaviors is to introduce a stochastic term in the definition of the prefactor $\mathbf{K}$, for example

$$\mathbf{K}(t,\xi,\chi) = C\left(\frac{\xi}{\xi_c} - 1\right)^+ \left(1 - \frac{\chi^+}{D_+} - \frac{\chi^-}{D_-} - X(t)\right),$$

where X is a stochastic process modeling the harmful impact of a random number of mindless agents on the collective dynamics. We plan to study numerically this variant of the model and provide indications concerning the impact of undisciplined agents on the evacuation time.

In Part II, we will take inspiration from second-order macroscopic models of traffic [25, 157] to model self-organization globally on the road; note that bottlenecks can be as well modelled with non-local point constraints within such models, see, *e.g.*, [12]. Mimicking the key elements (2.1.1)-(2.1.2) of the model we addressed in the present note, we will introduce two fundamental graphs $f_{\min} \leq f_{\max}$ to describe the two states of the traffic and make the space-and-time dependent organization parameter act both on the constraint levels (2.1.1) and on the fundamental graphs. We will then have to study a variant of nonlocal LWR model, *cf.* [31, 95] for related mathematical and numerical issues.

CHAPTER 3

INFLUENCE OF A SLOW MOVING VEHICLE ON TRAFFIC

Delle Monache and Goatin developed in [70] a macroscopic model aiming at describing the situation in which a slow moving large vehicle – a bus for instance – reduces the road capacity and thus generates a moving bottleneck for the surrounding traffic flow. Their model is given by a Cauchy problem for Lightwill-Whitham-Richards scalar conservation law in one space dimension with local point constraint (1.0.2). Unlike in (1.0.2), the constraint is prescribed along the slow vehicle trajectory $(y(t), t)$, the unknown y being coupled to the unknown ρ of the constrained LWR equation. Point constraints were introduced in [60, 56] to account for localized in space phenomena that may occur at exits and which act as obstacles. The constraint in the model of [70] depends upon the slow vehicle speed $\dot{y}$, where its position y verifies the following ODE

$$\dot{y}(t) = \omega\left(\rho(y(t)+, t)\right). \tag{A}$$

Above, $\rho = \rho(x,t) \in [0,1]$ is the traffic density and $\omega : [0,1] \to \mathbb{R}^+$ is a nonincreasing Lipschitz continuous function which links the traffic density to the slow vehicle velocity. Delle Monache and Goatin proved an existence result for their model in [70] with a wave-front tracking approach in the **BV** framework. Adjustments to the result were recently brought by Liard and Piccoli in [124]. Despite the step forward made in [72], the uniqueness issue remained open for a time. Indeed, the appearance of the trace $\rho(y(t)+, t)$ makes it fairly difficult to get a Lipschitz continuous dependency of the trajectory $y = y(t)$ from the solution $\rho = \rho(x,t)$. Nonetheless, a highly nontrivial uniqueness result was achieved by Liard and Piccoli in [123]. To describe the influence of a single vehicle on the traffic flow, the authors of [115] proposed a PDE-ODE coupled model without constraint on the flux for which they proposed in [35] two convergent schemes. In the present chapter, we consider a modified model where the point constraint becomes nonlocal, making the velocity of the slow vehicle depend on the mean density evaluated in a small vicinity ahead the driver. More precisely, instead of A, we consider the relation

$$\dot{y}(t) = \omega\left(\int_{\mathbb{R}} \rho(x + y(t), t)\mu(x)\,\mathrm{d}x\right), \tag{B}$$

where $\mu \in \mathbf{BV}(\mathbb{R};\mathbb{R}^+)$ is a weight function used to average the density. From the mathematical point of view, this choice makes the study of the new model easier. Indeed, the authors of [11, 9, 10] put forward techniques for full well-posedness analysis of similar models with nonlocal point constraints. From the modeling point of view, considering B makes sense for several reasons outlined in Section 3.2.5.
The chapter is organized as follows. Sections 3.1 and 3.2 are devoted to the proof of the well-posedness of the model. In Section 3.3 we introduce the numerical finite volume scheme and prove its convergence. An important step of the reasoning is to prove a **BV** regularity for the approximate solutions. It serves both in the existence proof and it is central in the uniqueness argument. With this in mind, the results proved in Section 1.6 will be essential. Indeed, this section was devoted to the proof of a **BV** regularity for entropy solutions to a large class of limited flux models and our model (3.1.1) falls directly in that framework. In the numerical section 3.4, first we perform numerical simulations to validate our model. Then we investigate both qualitatively and quantitatively the proximity between our model – in which we considered B – as $\delta \to \mu_{0^+}$ and the model of [70] in which the authors considered A.

3.1 Model, notion of solution and uniqueness

3.1.1 Model in the bus frame

Note that we find it convenient to study the problem in the bus frame, which means setting $X = x - y(t)$ in the model of Delle Monache and Goatin in [70]. Keeping in mind what we said above about the nonlocal constraint, the problem we consider, set up in $\mathbb{R} \times (0,T)$ $(T > 0)$, takes the following form:

$$\begin{cases} \partial_t \rho + \partial_x \left(F(\dot{y}(t), \rho) \right) = 0 \\ \rho(\cdot, 0) = \rho_0(\cdot + y_0) \\ F(\dot{y}(t), \rho)|_{x=0} \leq Q(\dot{y}(t)) \\ \dot{y}(t) = \omega \left(\displaystyle\int_{\mathbb{R}} \rho(x,t)\mu(x)\,\mathrm{d}x \right) \\ y(0) = y_0. \end{cases} \tag{3.1.1}$$

Above, $\rho = \rho(x,t) \in [0,1]$ denotes the traffic density and

$$F(\dot{y}(t), \rho) = f(\rho) - \dot{y}(t)\rho$$

denotes the normal flux through the curve $x = y(t)$. We assume that the flux function $f : [0,1] \to \mathbb{R}$ is Lipschitz continuous, concave and satisfies (1.0.1). In [70], the authors chose the function $Q(s) = \alpha \times \left(\dfrac{1-s}{2} \right)^2$ to prescribe the maximal flow allowed through a bottleneck located at $x = 0$. The parameter $\alpha \in (0,1)$ was giving the reduction rate of the road capacity due to the presence of the slow vehicle. We use the s variable to stress that

the value of the constraint is a function of the speed of the slow vehicle. In the sequel the s variable will refer to quantities related to the slow vehicle velocity. Regarding the function Q, we can allow for more general choices. Specifically,

$$Q : [0, \|\omega\|_{\mathbf{L}^\infty}] \to \mathbb{R}^+$$

can be any Lipschitz continuous function. It is a well known fact that in general, the total variation of an entropy solution to a constraint Cauchy problem may increase (see Sections 1.4.1 and 1.5 for examples). However, as we proved in Section 1.4.1 or Section 1.6, this increase can be controlled if the constraint level does not reach the maximum level. A mild assumption on Q – see Assumption (3.2.7) below which is the analogue to (1.4.2) and (1.6.10) – will guarantee availability of $\mathbf{BV}$ bounds, provided we suppose that $\rho_0 \in \mathbf{BV}(\mathbb{R}; [0,1])$.

3.1.2 Notion of solution

Throughout the chapter, we denote by

$$\Phi(a,b) = \operatorname{sgn}(a-b)(f(a)-f(b)) \quad \text{and} \quad \forall s \in \mathbb{R},\ \Phi_s(a,b) = \Phi(a,b) - s \times |a-b|$$

the entropy fluxes associated with the Kruzhkov entropy $\rho \mapsto |\rho - \kappa|$, for all $\kappa \in [0,1]$, see [114]. Following [70, 56, 14, 51], we give the following definition of solution for Problem (3.1.1).

Definition 3.1.1. A couple (ρ, y) with $\rho \in \mathbf{L}^\infty(\mathbb{R} \times (0,T))$ and $y \in \mathbf{W}^{1,\infty}((0,T))$ is an admissible weak solution to (3.1.1) with initial data $\rho_0 \in \mathbf{L}^\infty(\mathbb{R})$ and $y_0 \in \mathbb{R}$ if

(i) the following regularity is fulfilled:

$$\rho \in \mathbf{C}([0,T]; \mathbf{L}^1_{\mathbf{loc}}(\mathbb{R})), \tag{3.1.2}$$

and $\rho(\cdot, 0) = \rho_0(\cdot + y_0)$ in $\mathbf{L}^1_{\mathbf{loc}}(\mathbb{R})$;

(ii) for all test functions $\varphi \in \mathbf{C}^\infty_{\mathbf{c}}(\mathbb{R} \times [0,T)), \varphi \geq 0$ and $\kappa \in [0,1]$, the following entropy inequalities are verified for all $0 \leq \tau < \tau' \leq T$:

$$\begin{aligned} &\int_{\mathbb{R}} |\rho(x,\tau) - \kappa| \varphi(x,\tau)\,\mathrm{d}x - \int_{\mathbb{R}} |\rho(x,\tau') - \kappa| \varphi(x,\tau')\,\mathrm{d}x \\ &+ \int_\tau^{\tau'} \int_{\mathbb{R}} \Big(|\rho - \kappa| \partial_t \varphi + \Phi_{\dot y(t)}(\rho,\kappa) \partial_x \varphi \Big)\,\mathrm{d}x\,\mathrm{d}t + 2 \int_\tau^{\tau'} \mathcal{R}_{\dot y(t)}(\kappa, q(t)) \varphi(0,t)\,\mathrm{d}t \geq 0, \end{aligned} \tag{3.1.3}$$

where

$$\mathcal{R}_{\dot y(t)}(\kappa, q(t)) = F(\dot y(t), \kappa) - \min\{F(\dot y(t), \kappa), q(t)\} \quad \text{and} \quad q(t) = Q(\dot y(t));$$

(iii) for all test functions $\varphi \in \mathbf{C}^\infty_{\mathbf{c}}(\mathbb{R} \times (0,T)), \varphi \geq 0$, the following weak constraint inequalities are verified for all $0 \leq \tau < \tau' \leq T$:

$$\begin{aligned} &\int_{\mathbb{R}^+} \rho(x,\tau') \varphi(x,\tau')\,\mathrm{d}x - \int_{\mathbb{R}^+} \rho(x,\tau) \varphi(x,\tau)\,\mathrm{d}x \\ &- \int_\tau^{\tau'} \int_{\mathbb{R}^+} \Big(\rho \partial_t \varphi + F(\dot y(t), \rho) \partial_x \varphi \Big)\,\mathrm{d}x\,\mathrm{d}t \leq \int_\tau^{\tau'} q(t) \varphi(0,t)\,\mathrm{d}t\,; \end{aligned} \tag{3.1.4}$$

(iv) the following weak ODE formulation is verified for all $t \in [0,T]$:

$$y(t) = y_0 + \int_0^t \omega\left(\int_{\mathbb{R}} \rho(x,u)\mu(x)\,\mathrm{d}x\right)\mathrm{d}u\,. \tag{3.1.5}$$

Definition 3.1.2. We will call **BV**-regular solution any admissible weak solution (ρ, y) to the Problem (3.1.1) which also verifies

$$\rho \in \mathbf{L}^\infty((0,T);\mathbf{BV}(\mathbb{R})).$$

Remark 3.1.1. It is more usual to formulate (3.1.3)-(3.1.4) with $\tau = 0$ and $\tau' = T$, like we did in Chapters 1-2. The equivalence between the two formulations is due to the regularity (3.1.2). As it happens, this time-continuity regularity is actually a consequence of inequalities (3.1.3). Indeed, we will use the result [44, Theorem 1.2] (or [114], [59]) which states that if U is an open subset of $\mathbb{R}$ and if for all test functions $\varphi \in \mathbf{C}_\mathbf{c}^\infty(U\times[0,T)), \varphi \geq 0$ and $\kappa \in [0,1]$, ρ satisfies the following entropy inequalities:

$$\int_0^T \int_U \left(|\rho-\kappa|\partial_t\varphi + \Phi_{\dot y(t)}(\rho,\kappa)\partial_x\varphi\right)\mathrm{d}x\,\mathrm{d}t + \int_U |\rho_0(x)-\kappa|\varphi(x,0)\,\mathrm{d}x \geq 0,$$

then $\rho \in \mathbf{C}([0,T];\mathbf{L}^1_{\mathbf{loc}}(U))$. Moreover, since ρ is bounded and $\overline{U}\backslash U$ has a Lebesgue measure 0, $\rho \in \mathbf{C}([0,T];\mathbf{L}^1_{\mathbf{loc}}(\overline{U}))$. We will use this remark several times in the sequel of the chapter, with $U = \mathbb{R}^*$.

The interest of weak formulations (3.1.4)-(3.1.5) for the flux constraint and for the ODE governing the slow vehicle lies in their stability with respect to ρ. Formulation (3.1.3) – (3.1.5) is well suited for passage to the limit of a.e. convergent sequences of exact or approximate solutions.

3.1.3 Uniqueness of the BV-regular solution

In this section, we prove stability with respect to the initial data and uniqueness for **BV**-regular solutions to Problem (3.1.1). We start with the

Lemma 3.1.3. *If (ρ, y) is an admissible weak solution to Problem (3.1.1), then $\dot y \in \mathbf{W}^{1,\infty}((0,T))$. In particular, $\dot y \in \mathbf{BV}([0,T])$.*

Proof. Denote for all $t \in [0,T]$,

$$s(t) = \omega\left(\int_{\mathbb{R}} \rho(x,t)\mu(x)\,\mathrm{d}x\right).$$

Since $\mu \in \mathbf{L}^1(\mathbb{R}) \cap \mathbf{L}^\infty(\mathbb{R})$ and $\rho \in \mathbf{C}([0,T];\mathbf{L}^1_{\mathbf{loc}}(\mathbb{R}))$, s is continuous on $[0,T]$. By definition, y satisfies the weak ODE formulation (3.1.5). Consequently, for a.e. $t \in (0,T)$, $\dot y(t) = s(t)$. We are going to prove that s is Lipschitz continuous on $[0,T]$, which will ensure that $\dot y \in \mathbf{W}^{1,\infty}((0,T))$. Since $\mu \in \mathbf{BV}(\mathbb{R})$, there exists a sequence $(\mu_n)_{n\in\mathbb{N}} \subset \mathbf{BV}(\mathbb{R}) \cap \mathbf{C}_\mathbf{c}^\infty(\mathbb{R})$ such that:

$$\|\mu_n - \mu\|_{\mathbf{L}^1} \underset{n\to+\infty}{\longrightarrow} 0 \quad \text{and} \quad \mathbf{TV}(\mu_n) \underset{n\to+\infty}{\longrightarrow} \mathbf{TV}(\mu).$$

Introduce for all $n \in \mathbb{N}$ and $t \in [0,T]$, the function

$$\xi_n(t) = \int_{\mathbb{R}} \rho(x,t)\mu_n(x)\,\mathrm{d}x\,.$$

Fix $\psi \in \mathbf{C}^\infty_{\mathbf{c}}((0,T))$. Since ρ is a distributional solution to the conservation law in (3.1.1), we have for all $n \in \mathbb{N}$,

$$\begin{aligned}\int_0^T \xi_n(t)\dot{\psi}(t)\,\mathrm{d}t &= \int_0^T \int_{\mathbb{R}} \rho\partial_t(\psi\mu_n)\,\mathrm{d}x\,\mathrm{d}t \\ &= -\int_0^T \int_{\mathbb{R}} F(\dot{y}(t),\rho)\partial_x(\psi\mu_n)\,\mathrm{d}x\,\mathrm{d}t \\ &= -\int_0^T \left(\int_{\mathbb{R}} F(\dot{y}(t),\rho)\mu_n'(x)\,\mathrm{d}x\right)\psi(t)\,\mathrm{d}t\,,\end{aligned}$$

which means that for all $n \in \mathbb{N}$, ξ_n is differentiable in the weak sense, and that for a.e. $t \in (0,T)$,

$$\dot{\xi}_n(t) = \int_{\mathbb{R}} F(\dot{y}(t),\rho)\mu_n'(x)\,\mathrm{d}x\,.$$

In particular, since both the sequences $(\|\mu_n\|_{\mathbf{L}^1})_n$ and $(\mathbf{TV}(\mu_n))_n$ are bounded – say by $C > 0$ – we also have for all $n \in \mathbb{N}$,

$$\|\xi_n\|_{\mathbf{L}^\infty} \leq C \quad \text{and} \quad \|\dot{\xi}_n\|_{\mathbf{L}^\infty} \leq C(\|f\|_{\mathbf{L}^\infty} + \|\omega\|_{\mathbf{L}^\infty}).$$

Therefore, the sequence $(\xi_n)_n$ is bounded in $\mathbf{W}^{1,\infty}((0,T))$. Now, for all $t,\tau \in [0,T]$ and $n \in \mathbb{N}$, triangle inequality yields:

$$\begin{aligned}|s(t)-s(\tau)| &\leq 2\|\omega'\|_{\mathbf{L}^\infty}\|\mu_n-\mu\|_{\mathbf{L}^1} + \|\omega'\|_{\mathbf{L}^\infty}\left|\int_{\mathbb{R}}(\rho(x,t)-\rho(x,\tau))\mu_n(x)\,\mathrm{d}x\right| \\ &= 2\|\omega'\|_{\mathbf{L}^\infty}\|\mu_n-\mu\|_{\mathbf{L}^1} + \|\omega'\|_{\mathbf{L}^\infty}|\xi_n(t)-\xi_n(\tau)| \\ &\leq 2\|\omega'\|_{\mathbf{L}^\infty}\|\mu_n-\mu\|_{\mathbf{L}^1} + \underbrace{C\|\omega'\|_{\mathbf{L}^\infty}(\|f\|_{\mathbf{L}^\infty}+\|\omega\|_{\mathbf{L}^\infty})}_{K}|t-\tau|.\end{aligned}$$

Letting $n \to +\infty$, we get that for all $t,\tau \in [0,T]$, $|s(t)-s(\tau)| \leq K|t-\tau|$, which proves that s is Lipschitz continuous on $[0,T]$. The proof of the statement is completed. □

Before stating the uniqueness result, we make the following additional assumption:

$$\forall s \in [0,\|\omega\|_{\mathbf{L}^\infty}],\ \operatorname*{argmax}_{\rho\in[0,1]} F(s,\rho) > 0. \tag{3.1.6}$$

This ensures that for all $s \in [0,\|\omega\|_{\mathbf{L}^\infty}]$, the function $F(s,\cdot)$ verifies the assumptions (1.6.2). For example, when considering the flux $f(\rho) = \rho(1-\rho)$, (3.1.6) reduces to $\|\omega\|_{\mathbf{L}^\infty} < 1$, which only means that the maximum velocity of the slow vehicle is smaller than the maximum velocity of the surrounding cars.

Theorem 3.1.4. *Suppose that f is Lipschitz, concave and satisfies (1.0.1)-(3.1.6). Fix $\rho_0^1, \rho_0^2 \in \mathbf{BV}(\mathbb{R}; [0,1])$ and $y_0^1, y_0^2 \in \mathbb{R}$. We denote by (ρ^1, y^1) a $\mathbf{BV}$-regular solution to Problem (3.1.1) corresponding to initial data (ρ_0^1, y_0^1), and by (ρ^2, y^2) an admissible weak solution with initial data (ρ_0^2, y_0^2). Then there exist constants $\alpha, \beta, \gamma > 0$ such that for all $t \in [0,T]$,*

$$\|\rho^1(\cdot,t) - \rho^2(\cdot,t)\|_{\mathbf{L}^1} \leq \left(|y_0^1 - y_0^2| \mathbf{TV}(\rho_0^1) + \|\rho_0^1 - \rho_0^2\|_{\mathbf{L}^1}\right) \exp(\alpha t) \tag{3.1.7}$$

and

$$|y^1(t) - y^2(t)| \leq |y_0^1 - y_0^2| + (\beta |y_0^1 - y_0^2| + \gamma \|\rho_0^1 - \rho_0^2\|_{\mathbf{L}^1})(\exp(\alpha t) - 1). \tag{3.1.8}$$

In particular, Problem (3.1.1) admits at most one $\mathbf{BV}$-regular solution.

Proof. Since (ρ^1, y^1) is a $\mathbf{BV}$-regular solution to Problem (3.1.1), there exists $C \geq 0$ such that

$$\forall t \in [0,T],\ \mathbf{TV}(\rho^1(\cdot,t)) \leq C.$$

Lemma 3.1.3 ensures that $\dot{y}^1, \dot{y}^2 \in \mathbf{BV}([0,T]; \mathbb{R}^+)$. We can use result Theorem (1.6.3) to obtain that for all $t \in [0,T]$,

$$\begin{aligned} \|\rho^1(t) - \rho^2(t)\|_{\mathbf{L}^1} \leq\ & |y_0^1 - y_0^2| \mathbf{TV}(\rho_0^1) + \|\rho_0^1 - \rho_0^2\|_{\mathbf{L}^1} \\ & + (2\|Q'\|_{\mathbf{L}^\infty} + 2 + C) \int_0^t |\dot{y}^1(\tau) - \dot{y}^2(\tau)| \, \mathrm{d}\tau \,. \end{aligned} \tag{3.1.9}$$

Moreover, since for a.e. $t \in (0,T)$,

$$|\dot{y}^1(t) - \dot{y}^2(t)| \leq \|\omega'\|_{\mathbf{L}^\infty} \|\mu\|_{\mathbf{L}^\infty} \|\rho^1(\cdot,t) - \rho^2(\cdot,t)\|_{\mathbf{L}^1},$$

Gronwall's lemma yields (3.1.7) with $\alpha = (2\|Q'\|_{\mathbf{L}^\infty} + 2 + C)\, \|\omega'\|_{\mathbf{L}^\infty} \|\mu\|_{\mathbf{L}^\infty}$. Then,

$$\begin{aligned} \forall t \in [0,T], \quad |y^1(t) - y^2(t)| &\leq |y_0^1 - y_0^2| + \int_0^t |\dot{y}^1(s) - \dot{y}^2(s)| \, \mathrm{d}s \\ &\leq |y_0^1 - y_0^2| + \|\omega'\|_{\mathbf{L}^\infty} \|\mu\|_{\mathbf{L}^\infty} \int_0^t \|\rho^1(\cdot,s) - \rho^2(\cdot,s)\|_{\mathbf{L}^1} \, \mathrm{d}s \\ &\leq |y_0^1 - y_0^2| + (\beta |y_0^1 - y_0^2| + \gamma \|\rho_0^1 - \rho_0^2\|_{\mathbf{L}^1})(\exp(\alpha t) - 1), \end{aligned}$$

where

$$\beta = \frac{\mathbf{TV}(\rho_0^1)}{2\|Q'\|_{\mathbf{L}^\infty} + 2 + C} \quad \text{and} \quad \gamma = \frac{1}{2\|Q'\|_{\mathbf{L}^\infty} + 2 + C}.$$

The uniqueness of a $\mathbf{BV}$-regular solution is then clear. □

Remark 3.1.2. Up to inequality (3.1.9), our proof was very much following the one of [72, Theorem 2.1]. However, the authors of [72] faced an issue to derive a Lipschitz stability estimate between the car densities and the slow vehicle velocities starting from

$$|\omega\left(\rho^1(0+,t)\right)-\omega\left(\rho^2(0+,t)\right)|.$$

For us, due to the nonlocality of our problem, it was straightforward to obtain the bound

$$\left|\omega\left(\int_{\mathbb{R}}\rho^1(x,t)\mu(x)\,\mathrm{d}x\right)-\omega\left(\int_{\mathbb{R}}\rho^2(x,t)\mu(x)\,\mathrm{d}x\right)\right|\leq\|\omega'\|_{\mathbf{L}^\infty}\|\mu\|_{\mathbf{L}^\infty}\|\rho^1(\cdot,t)-\rho^2(\cdot,t)\|_{\mathbf{L}^1}.$$

Remark 3.1.3. A noteworthy consequence of Theorem 3.1.4 is that existence of a **BV**-regular solution will ensure uniqueness of an admissible weak one.

3.2 Two existence results

3.2.1 Time-splitting technique

In [70], to prove existence for their problem, the authors took a wave-front tracking approach. We choose here to use a time-splitting technique. The main advantage of this technique is that it relies on a ready-to-use theory. More precisely, at each time step, we will deal with exact solutions to a conservation law with a flux constraint, which have now become standard, see [56, 14, 51].

Fix $\rho_0\in\mathbf{L}^\infty(\mathbb{R};[0,1])$ and $y_0\in\mathbb{R}$. Let $\nu>0$ be a time step, $N\in\mathbb{N}$ such that $T\in[N\nu,(N+1)\nu)$ and denote for all $n\in\{0,\dots,N+1\}$, $t^n=n\nu$. We initialize with

$$\forall t\in\mathbb{R},\ \rho^0(t)=\rho_0(\cdot+y_0)\quad\text{and}\quad\forall t\in[0,T],\ y^0(t)=y_0.$$

Fix $n\in\{1,\dots,N+1\}$. First, we define for all $t\in(t^{n-1},t^n]$,

$$\sigma^n(t)=\omega\left(\int_{\mathbb{R}}\rho^{n-1}(x,t-\nu)\mu(x)\,\mathrm{d}x\right),\ s^n=\sigma^n(t^n)\text{ and }q^n=Q(s^n).$$

Since both q^n and $\rho^{n-1}(\cdot,t^{n-1})$ are bounded, [14, Theorem 2.11] ensures the existence and uniqueness of a solution $\rho^n\in\mathbf{L}^\infty(\mathbb{R}\times(t^{n-1},t^n))$ to

$$\begin{cases}\partial_t\rho+\partial_x\left(F(s^n,\rho)\right)=0\\ \qquad\qquad\rho(\cdot,t^{n-1})=\rho^{n-1}(\cdot,t^{n-1})\\ \qquad F(s^n,\rho)|_{x=0}\leq q^n,\end{cases}$$

in the sense of Definition 3.1.1 (i)-(ii)-(iii) with suitable flux/constraint function and initial data. We then define the following functions:

- $\rho_\nu(t)=\rho^0\mathbb{1}_{\mathbb{R}^-}(t)+\displaystyle\sum_{n=1}^{N+1}\rho^n(t)\mathbb{1}_{(t^{n-1},t^n]}(t)$
- $\sigma_\nu(t),q_\nu(t),s_\nu(t)=\sigma^n(t),q^n,s^n$ if $t\in(t^{n-1},t^n]$
- $y_\nu(t)=y_0+\displaystyle\int_0^t\sigma_\nu(\tau)\,\mathrm{d}\tau\,.$

First, let us prove that (ρ_ν, y_ν) solves an approximate version of Problem (3.1.1).

Proposition 3.2.1. *The couple (ρ_ν, y_ν) is an admissible weak solution, in $\mathbb{R}\times(0,T)$, to*

$$\begin{cases} \partial_t \rho_\nu + \partial_x \left(F(s_\nu(t), \rho_\nu)\right) = 0 \\ \rho_\nu(\cdot, 0) = \rho_0(\cdot + y_0) \\ F(s_\nu(t), \rho_\nu)|_{x=0} \leq q_\nu(t) \\ \dot{y}_\nu(t) = \omega\left(\displaystyle\int_{\mathbb{R}} \rho_\nu(x, t-\nu)\mu(x)\,\mathrm{d}x\right) \\ y_\nu(0) = y_0. \end{cases} \tag{3.2.1}$$

in the sense that $\rho_\nu \in \mathbf{C}([0,T]; \mathbf{L}^1_{\mathbf{loc}}(\mathbb{R}))$, $\rho_\nu(\cdot,0) = \rho_0(\cdot + y_0)$ in $\mathbf{L}^1_{\mathbf{loc}}(\mathbb{R})$ and satisfies entropy/constraint inequalities analogous to (3.1.3)-(3.1.4) with flux $F(s_\nu(\cdot),\cdot)$ and constraint q_ν; and y_ν satisfies, instead of (3.1.5), the following weak ODE formulation:

$$\forall t \in [0,T], \quad y_\nu(t) = y_0 + \int_0^t \omega\left(\int_{\mathbb{R}} \rho_\nu(x, \tau-\nu)\mu(x)\,\mathrm{d}x\right)\mathrm{d}\tau\,. \tag{3.2.2}$$

Proof. By construction, for all $n \in \{1,\dots,N+1\}$, $\rho^n \in \mathbf{C}([t^{n-1}, t^n]; \mathbf{L}^1_{\mathbf{loc}}(\mathbb{R}))$. Combining this with the "stop-and-restart" conditions $\rho^n(\cdot, t^{n-1}) = \rho^{n-1}(\cdot, t^{n-1})$, one ensures that $\rho_\nu \in \mathbf{C}([0,T]; \mathbf{L}^1_{\mathbf{loc}}(\mathbb{R}))$. Let $t \in [0,T]$ and $n \in \{1,\dots,N+1\}$ such that $t \in [t^{n-1}, t^n)$. Then,

$$\begin{aligned} y_\nu(t) - y_0 &= \sum_{k=1}^{n-1} \int_{t^{k-1}}^{t^k} \sigma^k(\tau)\,\mathrm{d}\tau + \int_{t^{n-1}}^{t} \sigma^n(\tau)\,\mathrm{d}\tau \\ &= \sum_{k=1}^{n-1} \int_{t^{k-1}}^{t^k} \omega\left(\int_{\mathbb{R}} \underbrace{\rho^{k-1}(x, \tau-\nu)}_{\rho_\nu(x,\tau-\nu)}\mu(x)\,\mathrm{d}x\right)\mathrm{d}\tau + \int_{t^{n-1}}^{t} \omega\left(\int_{\mathbb{R}} \underbrace{\rho^{n-1}(x, \tau-\nu)}_{\rho_\nu(x,\tau-\nu)}\mu(x)\,\mathrm{d}x\right)\mathrm{d}\tau \\ &= \int_0^t \omega\left(\int_{\mathbb{R}} \rho_\nu(x, \tau-\nu)\mu(x)\,\mathrm{d}x\right)\mathrm{d}\tau\,, \end{aligned}$$

which is exactly (3.2.2). Fix now $\varphi \in \mathbf{C}^\infty_{\mathbf{c}}(\mathbb{R}\times[0,T)), \varphi \geq 0$ and $\kappa \in [0,1]$. By construction of $((\rho^k, y^k))_k$, we have for all $n, m \in \{0,\dots,N+1\}$ $(n < m)$,

$$\begin{aligned} &\int_{t^n}^{t^m} \int_{\mathbb{R}} \Big(|\rho_\nu - \kappa| \partial_t \varphi + \Phi_{s_\nu(t)}(\rho_\nu, \kappa)\partial_x \varphi \Big)\,\mathrm{d}x\,\mathrm{d}t \\ &= \sum_{k=n+1}^{m} \int_{t^{k-1}}^{t^k} \int_{\mathbb{R}} \Big(|\rho^k - \kappa| \partial_t \varphi + \Phi_{s^k}(\rho^k, \kappa)\partial_x \varphi \Big)\,\mathrm{d}x\,\mathrm{d}t \\ &\geq \sum_{k=n+1}^{m} \left\{ \int_{\mathbb{R}} |\rho^k(x,t^k) - \kappa| \varphi(x,t^k)\,\mathrm{d}x - \int_{\mathbb{R}} |\underbrace{\rho^k(x,t^{k-1})}_{\rho^{k-1}(x,t^{k-1})} - \kappa| \varphi(x,t^{k-1})\,\mathrm{d}x - 2\int_{t^{k-1}}^{t^k} \mathcal{R}_{s^k}(\kappa, q^k)\varphi(0,t)\,\mathrm{d}t \right\} \\ &= \int_{\mathbb{R}} |\rho_\nu(x,t^m) - \kappa| \varphi(x,t^m)\,\mathrm{d}x - \int_{\mathbb{R}} |\rho_\nu(x,t^n) - \kappa| \varphi(x,t^n)\,\mathrm{d}x - 2\int_{t^n}^{t^m} \mathcal{R}_{s_\nu(t)}(\kappa, q_\nu(t))\varphi(0,t)\,\mathrm{d}t\,. \end{aligned}$$

It is then straightforward to prove that for all $0 \leq \tau < \tau' \leq T$,

$$\int_{\mathbb{R}} |\rho_\nu(x,\tau) - \kappa|\varphi(x,\tau)\,\mathrm{d}x - \int_{\mathbb{R}} |\rho_\nu(x,\tau') - \kappa|\varphi(x,\tau')\,\mathrm{d}x$$
$$+ \int_\tau^{\tau'} \int_{\mathbb{R}} \Big(|\rho_\nu - \kappa|\partial_t\varphi + \Phi_{s_\nu(t)}(\rho_\nu,\kappa)\partial_x\varphi \Big)\,\mathrm{d}x\,\mathrm{d}t + 2\int_\tau^{\tau'} \mathcal{R}_{s_\nu(t)}(\kappa, q_\nu(t))\varphi(0,t)\,\mathrm{d}t \geq 0. \tag{3.2.3}$$

Proving that ρ_ν satisfies constraint inequalities is very similar so we omit the details. One has to start from

$$-\int_{t^n}^{t^m} \Big(\int_{\mathbb{R}^+} \rho_\nu \partial_t\varphi + F(s_\nu(t),\rho_\nu)\partial_x\varphi \Big)\,\mathrm{d}x\,\mathrm{d}t$$

and make use once again of the construction of the sequence $((\rho^k, y^k))_k$ to obtain

$$\begin{aligned} &\int_{\mathbb{R}^+} \rho_\nu(x,\tau')\varphi(x,\tau')\,\mathrm{d}x - \int_{\mathbb{R}^+} \rho_\nu(x,\tau)\varphi(x,\tau)\,\mathrm{d}x \\ &- \int_\tau^{\tau'} \int_{\mathbb{R}^+} \Big(\rho_\nu\partial_t\varphi + F(s_\nu(t),\rho_\nu)\partial_x\varphi \Big)\,\mathrm{d}x\,\mathrm{d}t \leq \int_\tau^{\tau'} q_\nu(t)\varphi(0,t)\,\mathrm{d}t\,. \end{aligned} \tag{3.2.4}$$

This concludes the proof. □

Remark 3.2.1. Remark that we have for all $\nu > 0$,

$$\|\sigma_\nu\|_{\mathbf{L}^\infty} \leq \|\omega\|_{\mathbf{L}^\infty} \quad \text{and} \quad \|y_\nu\|_{\mathbf{L}^\infty} \leq |y_0| + T\|\omega\|_{\mathbf{L}^\infty}.$$

This means that the sequence $(y_\nu)_\nu$ is bounded in $\mathbf{W}^{1,\infty}((0,T))$. Then the compact embedding of $\mathbf{W}^{1,\infty}((0,T))$ in $\mathbf{C}([0,T])$ yields a subsequence of $(y_\nu)_\nu$, which we do not relabel, which converges uniformly on $[0,T]$ to some $y \in \mathbf{C}([0,T])$.

At this point, we propose two ways to obtain compactness for the sequence $(\rho_\nu)_\nu$, which will lead to two existence results.

3.2.2 The case of a nondegenerately nonlinear flux

Theorem 3.2.2. *Fix $\rho_0 \in \mathbf{L}^\infty(\mathbb{R};[0;1])$ and $y_0 \in \mathbb{R}$. Suppose that f is Lipschitz, concave, satisfies (1.0.1)-(3.1.6) and the following nondegeneracy assumption*

$$\text{for a.e. } s \in (0, \|\omega\|_{\mathbf{L}^\infty}), \quad \mathrm{mes}\{\rho \in [0,1] \mid f'(\rho) - s = 0\} = 0. \tag{3.2.5}$$

Then Problem (3.1.1) admits at least one admissible weak solution.

Proof. Condition (3.2.5) combined with the obvious uniform $\mathbf{L}^\infty$ bound

$$\forall \nu > 0,\ \forall (x,t) \in \mathbb{R}\times[0,T], \quad \rho_\nu(x,t) \in [0,1],$$

and the results proved by Panov in [135, 136] ensure the existence of a subsequence – which we do not relabel – that converges in $\mathbf{L}^1_{\mathbf{loc}}(\mathbb{R}^*\times(0,T))$ to some $\rho \in \mathbf{L}^1_{\mathbf{loc}}(\mathbb{R}^*\times(0,T))$; and

a further extraction yields the almost everywhere convergence on $\mathbb{R}\times(0,T)$. We now show that the couple (ρ,y) constructed above is an admissible weak solution to (3.1.1) in the sense of Definition 3.1.1.

For all $\nu>0$ and $t\in[0,T]$,

$$\begin{aligned} y_\nu(t)-y_0 &= \int_0^t \omega\left(\int_{\mathbb{R}}\rho_\nu(x,\tau-\nu)\mu(x)\,\mathrm{d}x\right)\mathrm{d}\tau \\ &= \int_{-\nu}^{t-\nu} \omega\left(\int_{\mathbb{R}}\rho_\nu(x,\tau)\mu(x)\,\mathrm{d}x\right)\mathrm{d}\tau \\ &= \int_0^t \omega\left(\int_{\mathbb{R}}\rho_\nu(x,\tau)\mu(x)\,\mathrm{d}x\right)\mathrm{d}\tau + \left(\int_{-\nu}^0 - \int_{t-\nu}^t\right)\omega\left(\int_{\mathbb{R}}\rho_\nu(x,\tau)\mu(x)\,\mathrm{d}x\right)\mathrm{d}\tau\,. \end{aligned}$$

The last term vanishes as $\nu\to 0$ since ω is bounded. Then, Lebesgue theorem combined with the continuity of ω gives, for all $t\in[0,T]$,

$$y_\nu(t)\underset{\nu\to 0}{\longrightarrow} y_0+\int_0^t\omega\left(\int_{\mathbb{R}}\rho(x,\tau)\mu(x)\,\mathrm{d}x\right)\mathrm{d}\tau\,.$$

This last quantity is also equal to $y(t)$ due to the uniform convergence of $(y_\nu)_\nu$ to y. This proves that y verifies (3.1.5). Now, we aim at passing to the limit in (3.2.3) and (3.2.4). With this in mind, we prove the a.e. convergence of the sequence $(\sigma_\nu)_\nu$ towards $\dot y$. Since $\mu\in\mathbf{BV}(\mathbb{R})$, there exists a sequence of smooth functions $(\mu_n)_{n\in\mathbb{N}}\subset\mathbf{BV}(\mathbb{R})\cap\mathbf{C}^\infty_{\mathbf{c}}(\mathbb{R})$ such that:

$$\|\mu_n-\mu\|_{\mathbf{L}^1}\underset{n\to+\infty}{\longrightarrow}0\quad\text{and}\quad\mathbf{TV}(\mu_n)\underset{n\to+\infty}{\longrightarrow}\mathbf{TV}(\mu).$$

Introduce for every $\nu>0$ and $n\in\mathbb{N}$, the function

$$\xi_\nu^n(t)=\int_{\mathbb{R}}\rho_\nu(x,t)\mu_n(x)\,\mathrm{d}x\,.$$

Since for all $\nu>0$, ρ_ν is a distributional solution to the conservation law in (3.2.1), one can show – following the proof of Lemma 3.1.3 for instance – that for every $n\in\mathbb{N}$, $\xi_\nu^n\in\mathbf{W}^{1,\infty}((0,T))$, and that for a.e. $t\in(0,T)$,

$$\dot\xi_\nu^n(t)=\int_{\mathbb{R}}F(s_\nu(t),\rho_\nu)\mu_n'(x)\,\mathrm{d}x\,.$$

Moreover, since both the sequences $\left(\|\mu_n\|_{\mathbf{L}^1}\right)_n$ and $\left(\mathbf{TV}(\mu_n)\right)_n$ are bounded, it is clear that $\left(\xi_\nu^n\right)_{\nu,n}$ is uniformly bounded in $\mathbf{W}^{1,\infty}((0,T))$, therefore so is $\left(\omega(\xi_\nu^n)\right)_{\nu,n}$. Consequently, for all $n\in\mathbb{N}$, $\nu>0$ and almost every $t\in(0,T)$, the triangle inequality yields:

$$\begin{aligned}\left|\sigma_\nu(t)-\omega\left(\int_{\mathbb{R}}\rho(x,t)\mu(x)\,\mathrm{d}x\right)\right| &\le 2\|\omega'\|_{\mathbf{L}^\infty}\|\mu_n-\mu\|_{\mathbf{L}^1}+\nu\sup_{n\in\mathbb{N}}\|\omega(\xi_\nu^n)\|_{\mathbf{W}^{1,\infty}} \\ &\quad+\|\omega'\|_{\mathbf{L}^\infty}\left|\int_{\mathbb{R}}(\rho_\nu(x,t)-\rho(x,t))\mu(x)\,\mathrm{d}x\right|\underset{\substack{\nu\to0\\ n\to+\infty}}{\longrightarrow}0,\end{aligned}$$

which proves that $(\sigma_\nu)_\nu$ converges a.e. on $(0,T)$ to $\dot{y}$. To prove the time-continuity regularity, we first apply inequality (3.2.3) with $\tau = 0$, $\tau' = T$ (which is licit since ρ_ν is continuous in time), $\varphi \in \mathbf{C}^\infty_{\mathbf{c}}(\mathbb{R}^* \times [0,T)), \varphi \geq 0$ and $\kappa \in [0,1]$:

$$\int_0^T \int_{\mathbb{R}} \Big(|\rho_\nu - \kappa| \partial_t \varphi + \Phi_{\sigma_\nu(t)}(\rho_\nu, \kappa) \partial_x \varphi \Big) \,\mathrm{d}x \,\mathrm{d}t + \int_{\mathbb{R}} |\rho_0(x+y_0) - \kappa| \varphi(x,0) \,\mathrm{d}x \geq 0.$$

Then, we let $\nu \to 0$ to get

$$\int_0^T \int_{\mathbb{R}} \Big(|\rho - \kappa| \partial_t \varphi + \Phi_{\dot{y}(t)}(\rho, \kappa) \partial_x \varphi \Big) \,\mathrm{d}x \,\mathrm{d}t + \int_{\mathbb{R}} |\rho_0(x+y_0) - \kappa| \varphi(x,0) \,\mathrm{d}x \geq 0.$$

Consequently, $\rho \in \mathbf{C}([0,T]; \mathbf{L}^1_{\mathbf{loc}}(\mathbb{R}))$, see Remark 3.1.1. Finally, the a.e. convergences of $(\sigma_\nu)_\nu$ and $(\rho_\nu)_\nu$ to $\dot{y}$ and ρ, respectively, are enough to pass to the limit in (3.2.3). This ensures that for all test functions $\varphi \in \mathbf{C}^\infty_{\mathbf{c}}(\mathbb{R} \times [0,T)), \varphi \geq 0$ and $\kappa \in [0,1]$, the following inequalities hold for a.e. $0 \leq \tau < \tau' \leq T$:

$$\begin{aligned} &\int_{\mathbb{R}} |\rho(x,\tau) - \kappa| \varphi(x,\tau) \,\mathrm{d}x - \int_{\mathbb{R}} |\rho(x,\tau') - \kappa| \varphi(x,\tau') \,\mathrm{d}x \\ &+ \int_\tau^{\tau'} \int_{\mathbb{R}} \Big(|\rho - \kappa| \partial_t \varphi + \Phi_{\dot{y}(t)}(\rho, \kappa) \partial_x \varphi \Big) \,\mathrm{d}x \,\mathrm{d}t + 2 \int_\tau^{\tau'} \mathcal{R}_{\dot{y}(t)}(\kappa, q(t)) \varphi(0,t) \,\mathrm{d}t \geq 0. \end{aligned}$$

Observe that the expression in the left-hand side of the previous inequality is a continuous function of (τ, τ') which is almost everywhere greater than the continuous function 0. By continuity, this expression is everywhere greater than 0, which proves that ρ satisfies the entropy inequalities (3.1.3). Using similar arguments, we show that ρ satisfies the constraint inequalities (3.1.4). This proves that the couple (ρ, y) is an admissible weak solution to Problem (3.1.1), concluding the proof. □

In this section, we proved an existence result for $\mathbf{L}^\infty$ initial data, but we have no guarantee of uniqueness since *a priori* we have no information regarding the **BV** regularity of such solutions.

Assumption (3.2.5) ensures the compactness for sequences of entropy solutions to conservation laws with flux function F. However, it prevents us from using flux functions with linear parts, like in Figure 5 (right) – which corresponds to constant traffic velocity for small densities – whereas such fundamental diagrams are often used in traffic modeling. The results of the next section will extend to this interesting case, under the extra **BV** assumption on the data.

3.2.3 Well-posedness for BV data

To obtain compactness for $(\rho_\nu)_\nu$, an alternative to the setting of Section 3.2.2 is to derive uniform **BV** bounds.

Theorem 3.2.3. *Fix $\rho_0 \in \mathbf{BV}(\mathbb{R}; [0,1])$ and $y_0 \in \mathbb{R}$. Suppose that f is Lipschitz, concave and satisfies (1.0.1)-(3.1.6). Suppose also that*

$$\forall s \in [0, \|\omega\|_{\mathbf{L}^\infty}], \quad F(s,\cdot) \in \mathbf{C}^1([0,1] \backslash \{\overline{\rho_s}\}), \tag{3.2.6}$$

where $\overline{\rho_s} = \underset{\rho\in[0,1]}{\operatorname{argmax}} F(s,\rho)$. *Finally assume that* Q *satisfies the condition*

$$\exists \varepsilon > 0,\ \forall s \in [0, \|\omega\|_{\mathbf{L}^\infty}], \quad Q(s) \leq \max_{\rho\in[0,1]} F(s,\rho) - \varepsilon. \tag{3.2.7}$$

Then Problem (3.1.1) *admits a unique admissible weak solution, which is also* $\mathbf{BV}$*-regular.*

Proof. Fix $\nu > 0$. Recall that (ρ_ν, y_ν) is an admissible weak solution to (3.2.1). In particular, ρ_ν is an admissible weak solution to the constrained conservation law in (3.2.1) in the sense of Definition 3.1.1 (i)-(ii)-(iii). It is clear from the splitting construction that for a.e. $t \in (0,T)$,

$$\sigma_\nu(t) = \omega\left(\int_{\mathbb{R}} \rho_\nu(x, t-\nu)\mu(x)\,\mathrm{d}x\right).$$

Following the steps of the proof of Lemma 3.1.3, we can show that for all $\nu > 0$, $\sigma_\nu \in \mathbf{BV}([0,T];\mathbb{R}^+)$. Even more than that, by doing so we show that the sequence $\left(\mathbf{TV}(\sigma_\nu)\right)_\nu$ is bounded. Therefore, the sequence $\left(\mathbf{TV}(s_\nu)\right)_\nu$ is bounded as well. Moreover, since Q verifies (3.2.7), all the hypotheses of Corollary 1.6.7 are fulfilled: there exists a constant $\mathbf{C}_\varepsilon > 0$ such that for all $t \in [0,T]$,

$$\begin{aligned}\mathbf{TV}(\rho_\nu(\cdot,t)) &\leq \mathbf{TV}(\rho_0) + 4 + \mathbf{C}_\varepsilon\left(\mathbf{TV}(q_\nu) + \mathbf{TV}(s_\nu)\right)\\ &\leq \mathbf{TV}(\rho_0) + 4 + \mathbf{C}_\varepsilon(1 + \|Q'\|_{\mathbf{L}^\infty})\mathbf{TV}(s_\nu).\end{aligned} \tag{3.2.8}$$

Consequently, the sequence $(\rho_\nu)_\nu$ is bounded in $\mathbf{L}^\infty((0,T);\mathbf{BV}(\mathbb{R}))$. A classical analysis argument, see [78, Appendix] or [98, Theorem A.8], ensures the existence of $\rho \in \mathbf{C}([0,T];\mathbf{L}^1_{\mathbf{loc}}(\mathbb{R}))$ such that along a subsequence, $\rho_\nu \to \rho$ a.e. on $\mathbb{R}\times(0,T)$. With this convergence, we can follow the proof of Theorem 3.2.2 to show that (ρ, y) is an admissible weak solution to (3.1.1). Then, when passing to the limit in (3.2.8), the lower semi-continuity of the $\mathbf{BV}$ semi-norm ensures that (ρ, y) is also $\mathbf{BV}$-regular. By Remark 3.1.3, it ensures uniqueness and concludes the proof. □

3.2.4 Stability with respect to the weight function

To end this section, we now study the stability of Problem (3.1.1) with respect to the weight function μ. More precisely, let $\left(\mu^\ell\right)_\ell \subset \mathbf{BV}(\mathbb{R};\mathbb{R}^+)$ be a sequence of weight functions that converges to μ in the weak $\mathbf{L}^1$ sense:

$$\forall g \in \mathbf{L}^\infty(\mathbb{R}),\ \int_{\mathbb{R}} g(x)\mu^\ell(x)\,\mathrm{d}x \underset{\ell\to+\infty}{\longrightarrow} \int_{\mathbb{R}} g(x)\mu(x)\,\mathrm{d}x\,. \tag{3.2.9}$$

Let $(y_0^\ell)_\ell \subset \mathbb{R}$ be a sequence of real numbers that converges to some y_0 and let $(\rho_0^\ell)_\ell \subset \mathbf{L}^1(\mathbb{R};[0,1])$ be a sequence of initial data that converges to ρ_0 in the strong $\mathbf{L}^1$ sense. We suppose that the flux function f is Lipschitz, concave and satisfies Assumptions (1.0.1)-(3.1.6)-(3.2.5). Theorem 3.2.2 allows us to define for all $\ell \in \mathbb{N}$, the couple (ρ^ℓ, y^ℓ) as an

admissible weak solution to the following problem, set up in $\mathbb{R}\times(0,T)$:

$$\begin{cases} \partial_t\rho^\ell + \partial_x\left(F(\dot{y}^\ell(t),\rho^\ell)\right) = 0 \\ \qquad\qquad \rho^\ell(\cdot,0) = \rho_0^\ell(\cdot + y_0^\ell) \\ \qquad F(\dot{y}^\ell(t),\rho^\ell)\big|_{x=0} \leq Q(\dot{y}^\ell(t)) \\ \qquad\qquad \dot{y}^\ell(t) = \omega\left(\displaystyle\int_{\mathbb{R}} \rho^\ell(x,t)\mu^\ell(x)\,\mathrm{d}x\right) \\ \qquad\qquad y^\ell(0) = y_0^\ell. \end{cases}$$

Remark 3.2.2. Using the same arguments as in Remark 3.2.1 and as in the proof of Theorem 3.2.2, we get that up to the extraction of a subsequence, $(y^\ell)_\ell$ converges uniformly on $[0,T]$ to some $y\in\mathbf{C}([0,T])$ and $(\rho^\ell)_\ell$ converges a.e. on $\mathbb{R}\times(0,T)$ to some $\rho\in\mathbf{L}^\infty(\mathbb{R}\times(0,T))$.

Theorem 3.2.4. *The couple (ρ,y) constructed above is an admissible weak solution to Problem (3.1.1).*

Proof. The sequence $(\mu^\ell)_\ell$ converges in the weak $\mathbf{L}^1$ sense and is bounded in $\mathbf{L}^1(\mathbb{R})$; by the Dunford-Pettis theorem, this sequence is equi-integrable:

$$\forall\varepsilon>0,\ \exists\alpha>0,\ \forall A\in\mathcal{B}(\mathbb{R}),\ \mathrm{mes}(A)<\alpha \implies \forall\ell\in\mathbb{N},\ \int_A \mu^\ell(x)\,\mathrm{d}x\leq\varepsilon \tag{3.2.10}$$

and

$$\forall\varepsilon>0,\ \exists X>0,\ \forall\ell\in\mathbb{N},\ \int_{|x|\geq X}\mu^\ell(x)\,\mathrm{d}x\leq\varepsilon. \tag{3.2.11}$$

Fix $t\in(0,T)$ and $\varepsilon>0$. Fix $\alpha,X>0$ given by (3.2.10) and (3.2.11). Egoroff theorem yields the existence of a measurable subset $E_t\subset[-X,X]$ such that

$$\mathrm{mes}([-X,X]\backslash E_t)<\alpha \text{ and } \rho^\ell(\cdot,t)\longrightarrow\rho(\cdot,t) \text{ uniformly on } E_t.$$

For a sufficiently large $\ell\in\mathbb{N}$,

$$\begin{aligned} &\left|\int_{\mathbb{R}}\rho^\ell(x,t)\mu^\ell(x)\,\mathrm{d}x-\int_{\mathbb{R}}\rho(x,t)\mu(x)\,\mathrm{d}x\right| \\ &\leq\int_{|x|\geq X}|\rho^\ell-\rho|\mu^\ell\,\mathrm{d}x+\left|\int_{E_t}(\rho^\ell-\rho)\mu^\ell\,\mathrm{d}x\right|+\left|\int_{[-X,X]\backslash E_t}(\rho^\ell-\rho)\mu^\ell\,\mathrm{d}x\right|+\left|\int_{\mathbb{R}}\rho\mu^\ell\,\mathrm{d}x-\int_{\mathbb{R}}\rho\mu\,\mathrm{d}x\right| \\ &\leq\varepsilon+\|\rho^\ell-\rho\|_{\mathbf{L}^\infty(E_t)}\int_{E_t}\mu^\ell(x)\,\mathrm{d}x+\int_{[-X,X]\backslash E_t}\mu^\ell(x)\,\mathrm{d}x+\varepsilon\leq 4\varepsilon, \end{aligned}$$

which proves that for a.e. $t\in(0,T)$,

$$\int_{\mathbb{R}}\rho^\ell(x,t)\mu^\ell(x)\,\mathrm{d}x\underset{\ell\to+\infty}{\longrightarrow}\int_{\mathbb{R}}\rho(x,t)\mu(x)\,\mathrm{d}x\,. \tag{3.2.12}$$

We get that y verifies the weak ODE formulation (3.1.5) by passing to the limit in

$$y^\ell(t) = y_0^\ell + \int_0^t \omega\left(\int_{\mathbb{R}} \rho^\ell(x,u)\mu^\ell(x)\,\mathrm{d}x\right)\mathrm{d}u\,.$$

By definition, for all $\ell \in \mathbb{N}$, the couple (ρ^ℓ, y^ℓ) satisfies the analogue of entropy/constraint inequalities (3.1.3)-(3.1.4) with suitable flux/constraint functions. Applying these inequalities with $\tau = 0$, $\tau' = T$, $\varphi \in \mathbf{C}_{\mathbf{c}}^\infty(\mathbb{R}^* \times [0,T)), \varphi \geq 0$ and $\kappa \in [0,1]$, we get

$$\int_0^T \int_{\mathbb{R}} \left(|\rho^\ell - \kappa|\partial_t\varphi + \Phi_{\dot{y}^\ell(t)}(\rho^\ell,\kappa)\partial_x\varphi\right)\mathrm{d}x\,\mathrm{d}t + \int_{\mathbb{R}} |\rho_0^\ell(x+y_0^\ell) - \kappa|\varphi(x,0)\,\mathrm{d}x \geq 0.$$

The continuity of ω and the convergence (3.2.12) ensure that $(\dot{y}^\ell)_\ell$ converges a.e. to $\dot{y}$. This, combined with the a.e. convergence of $(\rho^\ell)_\ell$ to ρ and Riesz-Frechet-Kolmogorov theorem – $\left(\rho_0^\ell\right)_\ell$ being strongly compact in $\mathbf{L}^1(\mathbb{R})$ – is enough to show that when letting $\ell \to +\infty$ in the inequality above, we get, up to the extraction of a subsequence, that

$$\int_0^T \int_{\mathbb{R}} \left(|\rho - \kappa|\partial_t\varphi + \Phi_{\dot{y}(t)}(\rho,\kappa)\partial_x\varphi\right)\mathrm{d}x\,\mathrm{d}t + \int_{\mathbb{R}} |\rho_0(x+y_0) - \kappa|\varphi(x,0)\,\mathrm{d}x \geq 0.$$

Consequently $\rho \in \mathbf{C}([0,T];\mathbf{L}^1_{\mathbf{loc}}(\mathbb{R}))$, see Remark 3.1.1. Finally, the combined a.e. convergences of $(\dot{y}^\ell)_\ell$ and $(\rho^\ell)_\ell$ to $\dot{y}$ and ρ, respectively, guarantee that (ρ, y) verifies inequalities (3.1.3)-(3.1.4) for almost every $0 \leq \tau < \tau' \leq T$. The same continuity argument we used in the proof Theorem 3.2.2 holds here to ensure that (ρ, y) actually satisfies the inequalities for all $0 \leq \tau < \tau' \leq T$. This concludes the proof of our stability claim. □

3.2.5 Discussion

The last section concludes the theoretical analysis of Problem (3.1.1). The nonlocality in space of the constraint delivers an easy proof of stability with respect to the initial data in the **BV** framework. Although a proof of existence using a fixed point theorem was possible (*cf.* [10]), we chose to propose a proof based on a time-splitting technique. The stability with respect to μ is a noteworthy feature, which shows a certain sturdiness of the model. However, the case we had in mind – namely $\mu \to \delta_{0+}$ – is not reachable with the assumptions we used to prove the stability, especially (3.2.9). We will explore this singular limit numerically, after having built a robust convergent numerical scheme for Problem (3.1.1). Let us also underline that unlike in [123, 124] where the authors required a particular form for the function ω to prove well-posedness for their model, our result holds as long as ω is Lipschitz continuous.

As evoked earlier, the nonlocality in space of the constraint makes the mathematical study of the model easier. But in the modeling point of view, this choice also makes sense for several reasons. First of all, one can think that the velocity $\dot{y}$ of the slow moving vehicle – unlike its acceleration – is a rather continuous value. Even if the driver of the slow vehicle suddenly applies the brakes, the vehicle will not decelerate instantaneously. Note that the LWR model allows for discontinuous averaged velocity of the agents, however while modeling the slow vehicle we are concerned with an individual agent and can model its behavior more precisely.

Moreover, considering the mean value of the traffic density in a vicinity ahead of the driver could be seen at taking into account both the driver anticipation and a psychological effect. For example, if the driver sees – several dozens of meters ahead of him/her – a speed reduction on traffic, he/she will start to slow down. This observation can be related to the fact that, compared to the fluid mechanics models where the typical number of agents is governed by the Avogadro constant, in traffic models the number of agents is at least 10^{20} times less. Therefore, a mild nonlocality (evaluation of the downstream traffic flow via averaging over a handful of preceding cars) is a reasonable assumption in the macroscopic traffic models inspired by fluid mechanics. This point of view is exploited in the model of [52]. Note that it is feasible to substitute the basic LWR equation on ρ by the nonlocal LWR introduced in [52] in our nonlocal model for the slow vehicle. Such mildly nonlocal model remains close to the basic local model of [70]. It can be studied combining the techniques of [52] and the ones we developed in this section.

3.3 Numerical approximation of the model

In this section, we aim at constructing a finite volume scheme and at proving its convergence toward the **BV**-regular solution to (3.1.1). The reasoning is a direct adaptation of what we proposed in Chapter 1, from where we keep the notations.

Fix $\rho_0 \in \mathbf{L}^\infty(\mathbb{R};[0,1])$ and $y_0 \in \mathbb{R}$.

3.3.1 Finite volume scheme in the bus frame

The change of variables $X = x - y(t)$ transforms the problem into a classical fixed interface point constraint problem; one that falls into the framework covered in Chapter 1, Section 1.6, where we precisely dealt with a time dependent flux.

First, let us discretize the initial data $\rho_0(\cdot + y_0)$ and the weight function μ with $\left(\rho^0_{j+1/2}\right)_{j\in\mathbb{Z}}$ and $\left(\mu_{j+1/2}\right)_{j\in\mathbb{Z}}$ where for all $j \in \mathbb{Z}$, $\rho^0_{j+1/2}$ and $\mu_{j+1/2}$ are their mean values on the cell (x_j, x_{j+1}).

Fix $n \in \mathbb{N}$. At the time step t^n, we first define an approximate velocity of the slow vehicle s^n and a constraint level q^n:

$$s^n = \omega\left(\sum_{j\in\mathbb{Z}} \rho^n_{j+1/2}\mu_{j+1/2}\Delta x\right); \quad q^n = Q\left(s^n\right). \tag{3.3.1}$$

With these values, we update the approximate traffic density with the marching formula for all $j \in \mathbb{Z}$:

$$\rho^{n+1}_{j+1/2} = \rho^n_{j+1/2} - \frac{\Delta t}{\Delta x}\left(\mathbf{F}^n_{j+1}(\rho^n_{j+1/2}, \rho^n_{j+3/2}) - \mathbf{F}^n_j(\rho^n_{j-1/2}, \rho^n_{j+1/2})\right), \tag{3.3.2}$$

where, following the recipe of [14, 51] and Section 1.6,

$$\mathbf{F}^n_j(a,b) = \begin{cases} \mathbf{F}^n(a,b) & \text{if } j \neq 0 \\ \min\left\{\mathbf{F}^n(a,b), q^n\right\} & \text{if } j = 0, \end{cases} \tag{3.3.3}$$

$\mathbf{F}^n = \mathbf{F}^n(a,b)$ being a monotone numerical flux associated to $F(s^n,\cdot)$, see Definition 1.1.1 and Example 1.1.1. The conservative form of the scheme reads:

$$\rho_{j+1/2}^{n+1} = \mathbf{H}_j^n(\rho_{j-1/2}^n, \rho_{j+1/2}^n, \rho_{j+3/2}^n), \tag{3.3.4}$$

where $\mathbf{H}_j^n$ is given by the expression in the right-hand side of (3.3.2). We then define the functions

- $\rho_\Delta(x,t) = \sum_{n\in\mathbb{N}}\sum_{j\in\mathbb{Z}} \rho_{j+1/2}^n \mathbb{1}_{\mathcal{P}_{j+1/2}^n}(x,t)$
- $s_\Delta(t), q_\Delta(t) = s^n, q^n$ if $t \in [t^n, t^{n+1})$
- $y_\Delta(t) = y_0 + \int_0^t s_\Delta(u)\,\mathrm{d}u\,.$

In the present framework, the CFL condition reads ($\lambda = \Delta t/\Delta x$):

$$\lambda \underbrace{\sup_{s\in[0,\|\omega\|_{\mathbf{L}^\infty}]} \left(\left\| \frac{\partial \mathbf{F}^s}{\partial a} \right\|_{\mathbf{L}^\infty} + \left\| \frac{\partial \mathbf{F}^s}{\partial b} \right\|_{\mathbf{L}^\infty} \right)}_{\mathbf{L}} \leq 1, \tag{3.3.5}$$

where $\mathbf{F}^s = \mathbf{F}^s(a,b)$ is the monotone numerical flux associated to $F(s,\cdot)$ we use in (3.3.2).

Remark 3.3.1. When considering one of the monotone numerical fluxes we introduced in Example 1.1.1, the CFL condition can be reduced to:

$$2\lambda(\|f'\|_{\mathbf{L}^\infty} + \|\omega\|_{\mathbf{L}^\infty}) \leq 1.$$

3.3.2 $\mathbf{L}^\infty$ stability and approximate inequalities

The results are stated without proof, we refer to Chapter 1, Sections 1.2-1.3 and 1.6 for all the details.

Proposition 3.3.1 ($\mathbf{L}^\infty$ stability)**.** *The scheme (3.3.4) is monotone and stable:*

$$\forall n \in \mathbb{N},\ \forall j \in \mathbb{Z}, \quad \rho_{j+1/2}^n \in [0,1]. \tag{3.3.6}$$

Corollary 3.3.2 (Discrete entropy inequalities)**.** *The numerical scheme (3.3.4) fulfills the following inequalities for all $n \in \mathbb{N}$, $j \in \mathbb{Z}$ and $\kappa \in [0,1]$:*

$$|\rho_{j+1/2}^{n+1} - \kappa| \leq \begin{cases} |\rho_{j+1/2}^n - \kappa| - \lambda\left(\Phi_{j+1}^n - \Phi_j^n\right) & \text{if} \quad j \notin \{-1,0\} \\ |\rho_{-1/2}^n - \kappa| - \lambda\left(\Phi_{int}^n - \Phi_{-1}^n\right)\Delta t + \lambda\mathcal{R}(\kappa, q^n) & \text{if} \quad j = -1 \\ |\rho_{1/2}^n - \kappa| - \lambda\left(\Phi_1^n - \Phi_{int}^n\right)\Delta t + \lambda\mathcal{R}(\kappa, q^n) & \text{if} \quad j = 0, \end{cases} \tag{3.3.7}$$

where $\mathcal{R}(\kappa, q^n)$ was defined in Definition 1.0.1, and Φ_j^n and Φ_{int}^n are the numerical entropy fluxes:

$$\Phi_j^n = \mathbf{F}(\rho_{j-1/2}^n \vee \kappa, \rho_{j+1/2}^n \vee \kappa) - \mathbf{F}(\rho_{j-1/2}^n \wedge \kappa, \rho_{j+1/2}^n \wedge \kappa);$$

$$\Phi_{int}^n = \min\{\mathbf{F}(\rho_{-1/2}^n \vee \kappa, \rho_{1/2}^n \vee \kappa), q^n\} - \min\{\mathbf{F}(\rho_{-1/2}^n \wedge \kappa, \rho_{1/2}^n \wedge \kappa), q^n\}.$$

where $\mathcal{R}_{s^n}(\kappa, q^n) = F(s^n, \kappa) - \min\{F(s^n, \kappa), q^n\}$, *and* Φ_j^n, Φ_{int}^n *denote the numerical entropy fluxes:*

$$\Phi_j^n = \mathbf{F}^n(\rho_{j-1/2}^n \vee \kappa, \rho_{j+1/2}^n \vee \kappa) - \mathbf{F}^n(\rho_{j-1/2}^n \wedge \kappa, \rho_{j+1/2}^n \wedge \kappa),$$
$$\Phi_{int}^n = \min\{\mathbf{F}^n(\rho_{-1/2}^n \vee \kappa, \rho_{1/2}^n \vee \kappa), q^n\} - \min\{\mathbf{F}^n(\rho_{-1/2}^n \wedge \kappa, \rho_{1/2}^n \wedge \kappa), q^n\}.$$

As in Propositions 1.3.1-1.3.2, we define the approximate entropy flux and the approximate flux function:

$$\Phi_\Delta(\rho_\Delta, \kappa) = \sum_{n\in\mathbb{N}} \sum_{j\in\mathbb{Z}} \Phi_j^n \mathbb{1}_{\mathcal{P}_{j+1/2}^n}; \quad \mathbf{F}_\Delta(s_\Delta, \rho_\Delta) = \sum_{n\in\mathbb{N}} \sum_{j\in\mathbb{Z}} \mathbf{F}^n(\rho_{j-1/2}^n, \rho_{j+1/2}^n) \mathbb{1}_{\mathcal{P}_{j+1/2}^n}.$$

Proposition 3.3.3 (Approximate entropy/constraint inequalities)**.** *Fix* $\varphi \in \mathbf{C}_\mathbf{c}^\infty(\mathbb{R}\times[0,T)), \varphi \geq 0$, $\kappa \in [0,1]$ *and* $0 \leq \tau < \tau' \leq T$. *Then as* $\Delta x, \Delta t \to 0$, *we have*

$$\begin{aligned}
&\int_{\mathbb{R}} |\rho_\Delta(x,\tau) - \kappa| \varphi(x,\tau)\,\mathrm{d}x - \int_{\mathbb{R}} |\rho_\Delta(x,\tau') - \kappa| \varphi(x,\tau')\,\mathrm{d}x \\
&+ \int_\tau^{\tau'} \int_{\mathbb{R}} \Big(|\rho_\Delta - \kappa| \partial_t \varphi + \Phi_\Delta(\rho_\Delta, \kappa) \partial_x \varphi \Big)\,\mathrm{d}x\,\mathrm{d}t + 2\int_\tau^{\tau'} \mathcal{R}_{s_\Delta(t)}(\kappa, q_\Delta(t)) \varphi(0,t)\,\mathrm{d}t \qquad (3.3.8) \\
&\geq O(\Delta x) + O(\Delta t)\,.
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\mathbb{R}^+} \rho(x,\tau') \varphi(x,\tau')\,\mathrm{d}x - \int_{\mathbb{R}^+} \rho(x,\tau) \varphi(x,\tau)\,\mathrm{d}x \\
&- \int_\tau^{\tau'} \int_{\mathbb{R}^+} \Big(\rho_\Delta \partial_t \varphi + \mathbf{F}_\Delta(s_\Delta, \rho_\Delta) \partial_x \varphi \Big)\,\mathrm{d}x\,\mathrm{d}t \leq \int_\tau^{\tau'} q_\Delta(t) \varphi(0,t)\,\mathrm{d}t + O(\Delta x) + O(\Delta t)\,.
\end{aligned} \qquad (3.3.9)$$

3.3.3 Compactness

The final step is to obtain compactness for the sequences $(\rho_\Delta)_\Delta$ and $(y_\Delta)_\Delta$ in order to pass to the limit in (3.3.8)-(3.3.9). We start with $(y_\Delta)_\Delta$.

Proposition 3.3.4. *For all* $t \in [0,T]$,

$$y_\Delta(t) = y_0 + \int_0^t \omega \left(\int_{\mathbb{R}} \rho_\Delta(x,u) \mu(x)\,\mathrm{d}x \right) \mathrm{d}u\,. \qquad (3.3.10)$$

Consequently, there exists $y \in \mathbf{C}([0,T])$ *such that along a subsequence,* $y_\Delta \to y$ *uniformly on* $[0,T]$.

Proof. For all $t \in [0,T]$, if $t \in [t^n, t^{n+1})$ for some $n \in \{0,\dots,N\}$, then we can write

$$\begin{aligned} y_\Delta(t) - y_0 &= \sum_{k=0}^{n-1} \int_{t^k}^{t^{k+1}} s^{k+1}\,\mathrm{d}u + \int_{t^n}^{t} s^n\,\mathrm{d}u \\ &= \sum_{k=0}^{n-1} \int_{t^k}^{t^{k+1}} \omega\left(\sum_{j\in\mathbb{Z}} \int_{\mathbb{R}} \rho^k_{j+1/2}\mu_{j+1/2}\Delta x\right)\mathrm{d}u + \int_{t^n}^{t} \omega\left(\sum_{j\in\mathbb{Z}} \int_{\mathbb{R}} \rho^n_{j+1/2}\mu_{j+1/2}\Delta x\right)\mathrm{d}u \\ &= \int_0^t \omega\left(\int_{\mathbb{R}} \rho_\Delta(x,u)\mu(x)\,\mathrm{d}x\right)\mathrm{d}u\,. \end{aligned}$$

Let us also point out that from (3.3.1), we get that for all Δ and almost every $t \in (0,T)$,

$$s_\Delta(t) = \omega\left(\int_{\mathbb{R}} \rho_\Delta(x,t)\mu(x)\,\mathrm{d}x\right). \tag{3.3.11}$$

Combining (3.3.10) and (3.3.11), we obtain that for all Δ,

$$\|\dot{y}_\Delta\|_{\mathbf{L}^\infty} = \|s_\Delta\|_{\mathbf{L}^\infty} \le \|\omega\|_{\mathbf{L}^\infty} \quad \text{and} \quad \|y_\Delta\|_{\mathbf{L}^\infty} \le |y_0| + T\|\omega\|_{\mathbf{L}^\infty}.$$

The sequence $(y_\Delta)_\Delta$ is therefore bounded in $\mathbf{W}^{1,\infty}((0,T))$. Making use of the compact embedding of $\mathbf{W}^{1,\infty}((0,T))$ in $\mathbf{C}([0,T])$, we get the existence of $y \in \mathbf{C}([0,T])$ such that up to the extraction of subsequence, $(y_\Delta)_\Delta$ converges uniformly to y on $[0,T]$. □

We now turn to $(\rho_\Delta)_\Delta$.

Global BV bounds

The following result is the discrete version of Lemma 3.1.3 so it is consistent that the proof uses the discrete analogous arguments of the ones we used in the proof of Lemma 3.1.3.

Lemma 3.3.5. *Introduce for all $\Delta > 0$ the function ξ_Δ defined for all $t \in [0,T]$ by*

$$\xi_\Delta(t) = \int_{\mathbb{R}} \rho_\Delta(x,t)\mu(x)\,\mathrm{d}x\,.$$

Then ξ_Δ has bounded variation and consequently, so does s_Δ.

Proof. Since $\mu \in \mathbf{BV}(\mathbb{R})$, there exists a sequence of smooth functions $(\mu_\ell)_{\ell\in\mathbb{N}} \subset \mathbf{BV}(\mathbb{R}) \cap \mathbf{C}^\infty_{\mathbf{c}}(\mathbb{R})$ such that

$$\|\mu_\ell - \mu\|_{\mathbf{L}^1} \underset{\ell\to+\infty}{\longrightarrow} 0 \quad \text{and} \quad \mathbf{TV}(\mu_\ell) \underset{\ell\to+\infty}{\longrightarrow} \mathbf{TV}(\mu).$$

Introduce for all $\ell \in \mathbb{N}$ and $t \in [0,T]$, the function $\xi_{\Delta,\ell}(t) = \displaystyle\int_{\mathbb{R}} \rho_\Delta(x,t)\mu_\ell(x)\,\mathrm{d}x$ and let $K > 0$ such that

$$\forall \ell \in \mathbb{N}, \quad \|\mu_\ell\|_{\mathbf{L}^1}, \mathbf{TV}(\mu_\ell) \le K.$$

For all $\ell \in \mathbb{N}$ and $t, \tau \in [0,T]$, if $t \in [t^p, t^{p+1})$ and $\tau \in [t^q, t^{q+1})$, for some $p, q \in \mathbb{N}$, then we have

$$\begin{aligned}
|\xi_{\Delta,\ell}(t) - \xi_{\Delta,\ell}(\tau)| &= |\xi_{\Delta,\ell}(t^p) - \xi_{\Delta,\ell}(t^q)| \\
&= \left| \int_{\mathbb{R}} \rho_\Delta(x,t^p)\mu_\ell(x)\,\mathrm{d}x - \int_{\mathbb{R}} \rho_\Delta(x,t^q)\mu_\ell(x)\,\mathrm{d}x \right| \\
&= \left| \sum_{j\in\mathbb{Z}} (\rho^p_{j+1/2} - \rho^q_{j+1/2})\mu^\ell_{j+1/2}\Delta x \right|, \quad \mu^\ell_{j+1/2} = \frac{1}{\Delta x}\int_{x_j}^{x_{j+1}} \mu_\ell(x)\,\mathrm{d}x \\
&= \left| \sum_{j\in\mathbb{Z}} \sum_{k=q}^{p-1} (\rho^{k+1}_{j+1/2} - \rho^k_{j+1/2})\mu^\ell_{j+1/2}\Delta x \right| \\
&= \left| \sum_{k=q}^{p-1} \sum_{j\in\mathbb{Z}} \left(\mathbf{F}^k_j(\rho^k_{j-1/2}, \rho^k_{j+1/2}) - \mathbf{F}^k_{j+1}(\rho^k_{j+1/2}, \rho^k_{j+3/2})\right) \mu^\ell_{j+1/2}\Delta t \right| \\
&= \left| \sum_{k=q}^{p-1} \sum_{j\in\mathbb{Z}} \mathbf{F}^k_{j+1}(\rho^k_{j+1/2}, \rho^k_{j+3/2})(\mu^\ell_{j+3/2} - \mu^\ell_{j+1/2})\Delta t \right| \\
&\leq \mathbf{L} \sum_{k=q}^{p-1} \mathbf{TV}(\mu_\ell)\Delta t \leq \mathbf{L}K(|t-\tau| + 2\Delta t).
\end{aligned}$$

Consequently, for all $\ell \in \mathbb{N}$, $\Delta > 0$ and $t, \tau \in [0,T]$, the triangle inequality yields:

$$|\xi_\Delta(t) - \xi_\Delta(\tau)| \leq 2\|\mu - \mu_\ell\|_{\mathbf{L}^1} + \mathbf{L}K(|t-\tau| + 2\Delta t).$$

Letting $\ell \to +\infty$, we get that for all $\Delta > 0$ and $t, \tau \in [0,T]$,

$$|\xi_\Delta(t) - \xi_\Delta(\tau)| \leq \mathbf{L}K(|t-\tau| + 2\Delta t),$$

which leads to

$$\mathbf{TV}(\xi_\Delta) = \sum_{k=0}^{N} \left|\xi_\Delta(t^{k+1}) - \xi_\Delta(t^k)\right| \leq 3\mathbf{L}K(T + \Delta t).$$

This proves that $\xi_\Delta \in \mathbf{BV}([0,T])$. Since ω is Lipschitz continuous, s_Δ also has bounded variation. □

Theorem 3.3.6. *Fix $\rho_0 \in \mathbf{BV}(\mathbb{R};[0,1])$ and $y_0 \in \mathbb{R}$. Suppose that f is concave and satisfies (1.0.1)-(3.1.6)-(3.2.6) and that Q satisfies (3.2.7). Suppose also that in (3.3.3), we use the Godunov flux when $j = 0$ and any other monotone numerical flux when $j \neq 0$. Then there exists $\rho \in \mathbf{L}^\infty(\mathbb{R}\times(0,T)) \cap \mathbf{C}([0,T];\mathbf{L}^1_{\mathbf{loc}}(\mathbb{R}))$, such that along a subsequence, $\rho_\Delta \to \rho$ a.e. on $\mathbb{R}\times(0,T)$.*

Proof. All the hypotheses of Lemma 1.6.4 are fulfilled. Consequently, there exists a constant $\mathbf{C}_\varepsilon > 0$ such that for all $n \in \{0,\dots,N-1\}$,

$$\begin{aligned}
\mathbf{TV}\left(\rho_\Delta(\cdot,t^{n+1})\right) &\leq \mathbf{TV}(\rho_0) + 4 + \mathbf{C}_\varepsilon \left(\sum_{k=0}^{n} \left|q^{k+1} - q^k\right| + \sum_{k=0}^{n} \left|s^{k+1} - s^k\right| \right) \\
&\leq \mathbf{TV}(\rho_0) + 4 + \mathbf{C}_\varepsilon(1 + \|Q'\|_{\mathbf{L}^\infty}) \sum_{k=0}^{n} \left|s^{k+1} - s^k\right|.
\end{aligned} \tag{3.3.12}$$

Making use of Lemma 3.3.5, we obtain that for all $n \in \{0, \dots, N\}$,

$$\sum_{k=0}^{n} |s^{k+1} - s^k| = \sum_{k=0}^{n} |s_\Delta(t^{k+1}) - s_\Delta(t^k)| \leq \|\omega\|_{\mathbf{L}^\infty} \sum_{k=0}^{n} |\xi_\Delta(t^{k+1}) - \xi_\Delta(t^k)| \leq 3\mathbf{L}K\|\omega\|_{\mathbf{L}^\infty}(T + \Delta t).$$

where the constant K was introduced in the proof of Lemma 3.3.5. The two last inequalities imply that for all $t \in [0, T]$, we have

$$\forall t \in [0,T], \quad \mathbf{TV}(\rho_\Delta(\cdot, t)) \leq \mathbf{TV}(\rho_0) + 4 + 3\mathbf{C}_\varepsilon(1 + \|Q'\|_{\mathbf{L}^\infty})\|\omega\|_{\mathbf{L}^\infty}\mathbf{L}K(T + \Delta t). \quad (3.3.13)$$

Therefore, the sequence $(\rho_\Delta)_\Delta$ is bounded in $\mathbf{L}^\infty((0,T); \mathbf{BV}(\mathbb{R}))$; [78, Appendix] then provides the compactness statement. □

OSLC technique

We state without proof the following compacteness result. The result comes from the study done in Sections 1.4.2-1.6.

Theorem 3.3.7. *Fix $\rho_0 \in \mathbf{L}^\infty(\mathbb{R}; [0,1])$ and $y_0 \in \mathbb{R}$. Let us assume that $f \in \mathbf{C}^2([0,1]$ is strictly concave. Suppose also that in (3.3.3), we use either the Godunov flux or the Engquist-Osher flux when $j \neq 0$ and any other monotone numerical flux when $j = 0$. Then there exists $\rho \in \mathbf{L}^\infty(\mathbb{R} \times (0,T)) \cap \mathbf{C}([0,T]; \mathbf{L}^1_{\mathbf{loc}}(\mathbb{R}))$, such that along a subsequence, $\rho_\Delta \to \rho$ a.e. on $\mathbb{R} \times (0,T)$.*

3.3.4 Convergence and existence results

We were able to get compacteness using two different methods, and under two different sets of assumptions. This will to two convergence/existence results. However, note that the proofs of convergence only differ in the obtaining of compacteness.

Theorem 3.3.8. *Fix $\rho_0 \in \mathbf{BV}(\mathbb{R}; [0,1])$ and $y_0 \in \mathbb{R}$. Suppose that f is concave and satisfies (1.0.1)-(3.1.6)-(3.2.6) and that Q satisfies (3.2.7). Suppose also that in (3.3.3), we use the Godunov flux when $j = 0$ and any other monotone numerical flux when $j \neq 0$. Then under the CFL condition (3.3.5), the scheme (3.3.1) – (3.3.3) converges to a $\mathbf{BV}$-regular solution ρ to Problem (3.1.1). Moreover, there exists a constant $\mathbf{C}_\varepsilon > 0$ such that*

$$\forall t \in [0,T], \quad \mathbf{TV}(\rho(\cdot, t)) \leq \mathbf{TV}(\rho_0) + 4 + 3\mathbf{C}_\varepsilon(1 + \|Q'\|_{\mathbf{L}^\infty})\|\omega\|_{\mathbf{L}^\infty}\mathbf{L}KT, \quad (3.3.14)$$

where K was defined in Lemma 3.3.5.

Proof. We have shown that – up to the extraction of a subsequence – y_Δ converges uniformly on $[0,T]$ to some $y \in \mathbf{C}([0,T])$ and that ρ_Δ converges a.e. on $\mathbb{R} \times (0,T)$ to some $\rho \in \mathbf{L}^\infty(\mathbb{R} \times (0,T)) \cap \mathbf{C}([0,T]; \mathbf{L}^1_{\mathbf{loc}}(\mathbb{R}))$. We now prove that this couple (ρ, y) is a $\mathbf{BV}$-regular solution to Problem (3.1.1) in the sense of Definition 3.1.1.

Recall that for all Δ and $t \in [0,T]$,

$$y_\Delta(t) = y_0 + \int_0^t \omega \left(\int_{\mathbb{R}} \rho_\Delta(x,u)\mu(x)\,\mathrm{d}x \right) \mathrm{d}u\,.$$

Now, we pass to the limit in (3.3.8) and (3.3.9) using the a.e. convergence of $(s_\Delta)_\Delta$ to $\dot{y}$ and of $(\rho_\Delta)_\Delta$ to ρ as well as the continuity of Q and ω. Consequently, for all test functions $\varphi \in \mathbf{C}_\mathbf{c}^\infty(\mathbb{R}\times[0,T)), \varphi \geq 0$ and $\kappa \in [0,1]$, the following inequalities hold for almost every $0 \leq \tau < \tau' \leq T$:

$$\int_{\mathbb{R}} |\rho(x,\tau)-\kappa|\varphi(x,\tau)\,\mathrm{d}x - \int_{\mathbb{R}} |\rho(x,\tau')-\kappa|\varphi(x,\tau')\,\mathrm{d}x$$
$$+ \int_\tau^{\tau'} \int_{\mathbb{R}} \Big(|\rho-\kappa|\partial_t\varphi + \Phi_{\dot{y}(t)}(\rho,\kappa)\partial_x\varphi \Big)\,\mathrm{d}x\,\mathrm{d}t + 2\int_\tau^{\tau'} \mathcal{R}_{\dot{y}(t)}(\kappa,q(t))\varphi(0,t)\,\mathrm{d}t \geq 0.$$

To conclude, note that the expression in the left-hand side of the previous inequality is a continuous function of (τ,τ') which is almost everywhere greater than the continuous function 0. By continuity, this expression is everywhere greater than 0, which proves that ρ satisfies the entropy inequalities (3.1.3). Using similar arguments, one shows that ρ also satisfies the constraint inequalities (3.1.4). This shows that the couple (ρ, y) is an admissible weak solution to (3.1.1). Finally, estimate (3.3.14) comes from (3.3.13) and the lower semi-continuity property of the **BV** semi-norm. □

Theorem 3.3.9. *Fix $\rho_0 \in \mathbf{L}^\infty(\mathbb{R};[0,1])$ and $y_0 \in \mathbb{R}$. Let us assume that $f \in \mathbf{C}^2([0,1]$ is strictly concave. Suppose also that in (3.3.3), we use either the Godunov flux or the Engquist-Osher flux when $j \neq 0$ and any other monotone numerical flux when $j = 0$. Then under the CFL condition (3.3.5), the scheme (3.3.1) – (3.3.3) converges to an admissible weak solution to Problem (3.1.1).*

Proof. The proof is identical to the one of Theorem 3.3.8. □

We proved that in the $\mathbf{L}^\infty$ framework, the scheme converges to an admissible weak solution, but note that there is no guarantee of uniqueness in this construction. Also stress that we cannot extend this result to general monotone numerical fluxes beyond the Engquist-Osher flux or the Godunov flux.

3.4 Numerical simulations

In this section we present some numerical tests performed with the scheme analyzed in Section 3.3. In all the simulations we take the uniformly concave flux $f(\rho) = \rho(1-\rho)$ (the maximal car velocity and the maximal density are assumed to be equal to one). Following the hypotheses of Theorem 3.3.8, we choose the Godunov flux at the interface, and the Rusanov one away from the interface. We will use weight functions of the kind

$$\mu_k(x) = 2^k \mathbb{1}_{[0;\frac{1}{2^k}]}(x),$$

for one (in Section 3.4.1) or several (in Section 3.4.2) values of $k \in \mathbb{N}^*$.

3.4.1 Validation of the scheme

In this section, consider a two-lane road on which a bus travels with a speed given by the function

$$\omega(\rho) = \begin{cases} \dfrac{\alpha}{(\beta+\rho)^2} & \text{if} \quad 0 \le \rho \le 0.6 \\ 1-\rho & \text{if} \quad 0.6 \le \rho \le 1, \end{cases}$$

where α and β are chosen so that $\omega(0) = 0.7$ and $\omega(0.6) = 0.4$, as illustrated in Figure 3.1 (left). The set-up of the experiment is the following. Consider a domain of computation $[0, 11]$, the weight function μ_4 and the following data:

$$\rho_0(x) = 0.5\mathbb{1}_{[0.5;1]}(x), \quad y_0 = 1.5, \quad Q(s) = 0.75 \times \left(\frac{1-s}{2}\right)^2.$$

The idea behind the choice of Q is that in average (between the two lanes), the presence of the slow vehicle reduces by 25% the maximum traffic flow. As we can see in Figure 3.1 (right), the slow vehicle nearly always travels at maximum velocity. It makes sense because even though we can see that cars are overtaking it (Figure 3.1, right and Figure 3.2), the density ξ ahead of it is never sufficiently important to make it go slower.

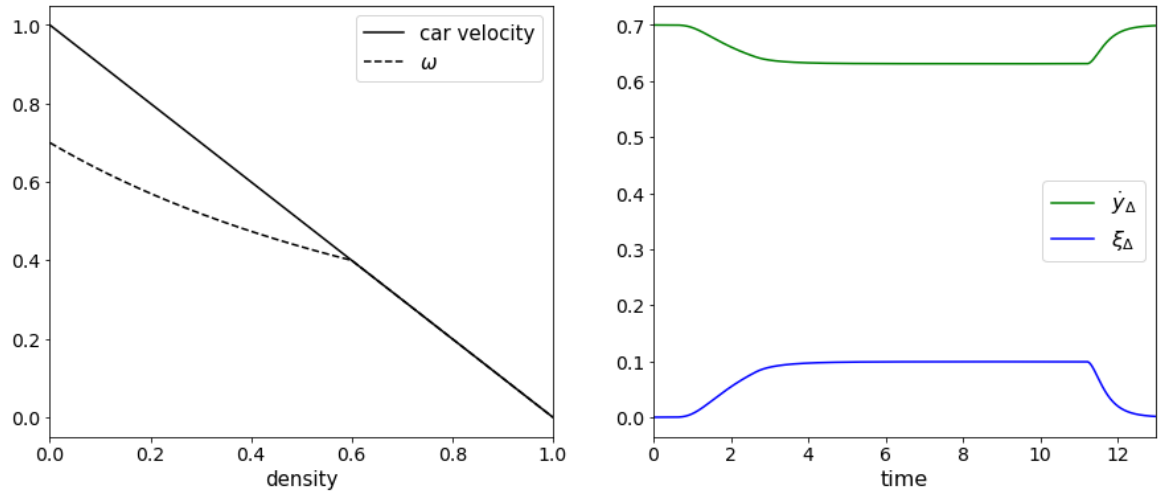

Figure 3.1 – Evolution in time of the bus velocity $\dot{y}_\Delta$ and of the subjective density ξ_Δ, with $\Delta x = 0.01$.

Remark 3.4.1. The function ω we chose above is not of the form as required in [123, 124]. Once again, let us stress that the particular form $\omega(\rho) = \min\{V_{\text{bus}}; 1-\rho\}$, where V_{bus} is the maximum bus velocity, is crucial for the well-posedness result of [123, 124] to hold. Indeed, it is essential in the analysis of [123, 124] that the velocity of the bus be constant (equal to V_{bus}) across the nonclassical shocks. Our nonlocal model is not bound to this restriction.

We also perform a convergence analysis for this test. In Table 3.1, we computed the relative errors

$$\mathbf{E}_{\rho,\Delta} = \|\rho_\Delta - \rho_{\Delta/2}\|_{\mathbf{L}^1((0,T);\mathbf{L}^1(\mathbb{R}))} \quad \text{and} \quad \mathbf{E}_{y,\Delta} = \|y_\Delta - y_{\Delta/2}\|_{\mathbf{L}^\infty},$$

for different number of space cells at the final time $T = 13$. We see (Figure 3.3) that those ratio converge with convergence orders approximately equal to 0.76 for the car density and approximately equal to 1.1 for the slow moving vehicle position.

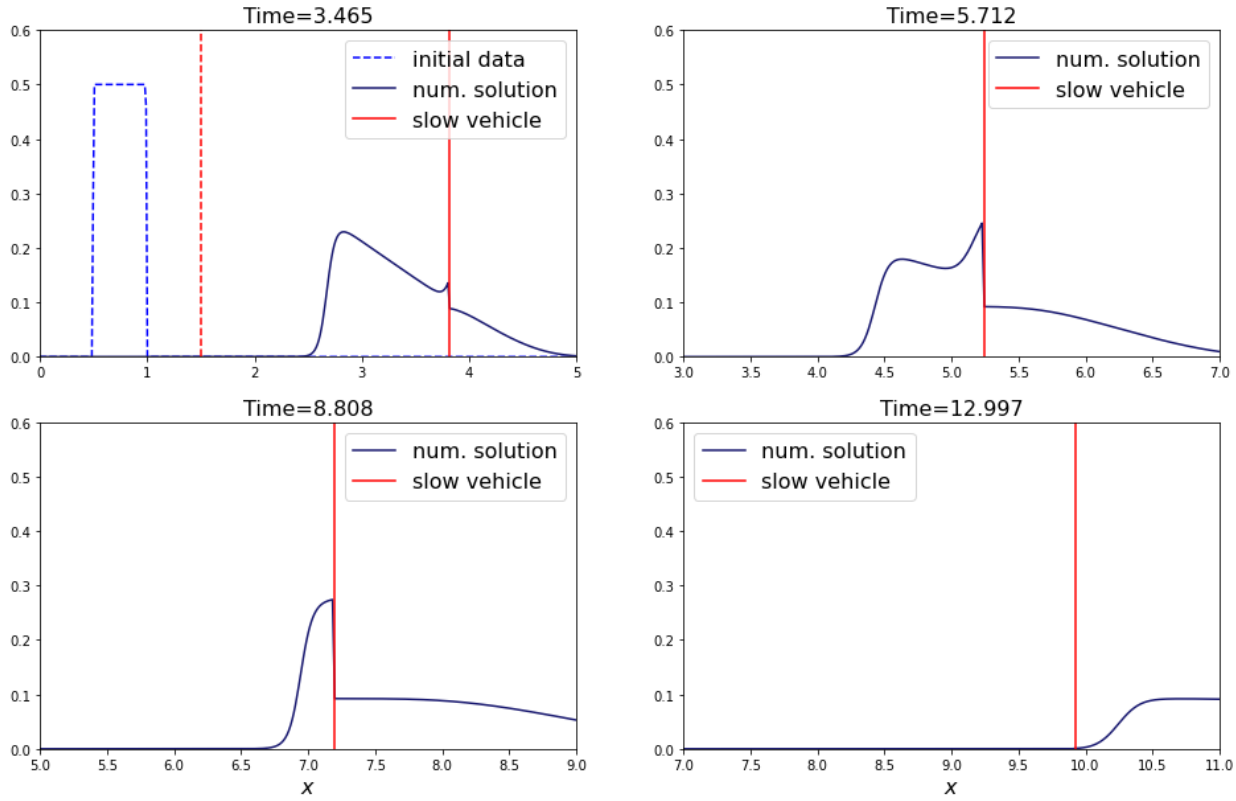

Figure 3.2 – The numerical solution at different fixed times, red dashed lines correspond to the slow vehicle initial position; for an animated representation of the solution, see https://utbox.univ-tours.fr/s/BoTnsEqmrjndy66

3.4.2 Comparisons with experiments on the local model

Now we confront the numerical tests performed with our model with the tests done by the authors in [50] approximating the original problem of [70]. We deal with a road of length 1 parametrized by the interval $[0, 1]$ and choose the weight function μ_3. Moreover,

$$\omega(\rho) = \min\{0.3;\ 1-\rho\} \quad \text{and} \quad Q(s) = 0.6 \times \left(\frac{1-s}{2}\right)^2.$$

First, consider the initial datum

$$\rho_0(x) = \begin{cases} 0.4 & \text{if} \quad x < 0.5 \\ 0.5 & \text{if} \quad x > 0.5 \end{cases} ; \quad y_0 = 0.5. \tag{3.4.1}$$

Number of cells	$\mathbf{E}_{\rho,\Delta}$ $(\times 10^{-2})$	$\mathbf{E}_{y,\Delta}$ $(\times 10^{-3})$
160	24.053	48.0643
320	15.731	15.939
640	9.647	7.698
1280	6.197	3.715
2560	3.226	1.777
5120	1.936	0.889
10240	1.055	0.443

Table 3.1 – Measured errors ($T = 13$).

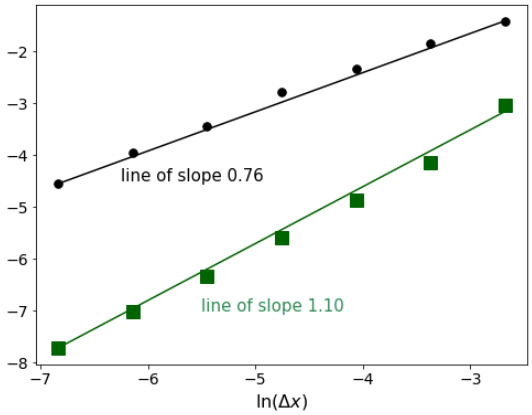

Figure 3.3 – Rates of convergence for ρ_Δ (in black) and y_Δ (in green), with $T = 13$.

The numerical solution is composed of two classical shocks separated by a nonclassical discontinuity, as illustrated in Figure 3.4 (left). Next, we choose

$$\rho_0(x) = \begin{cases} 0.8 & \text{if} \quad x < 0.5 \\ 0.5 & \text{if} \quad x > 0.5 \end{cases} \quad ; \quad y_0 = 0.5. \tag{3.4.2}$$

The values of the initial condition create a rarefaction wave followed by a nonclassical and classical shocks, as illustrated in Figure 3.4 (right).

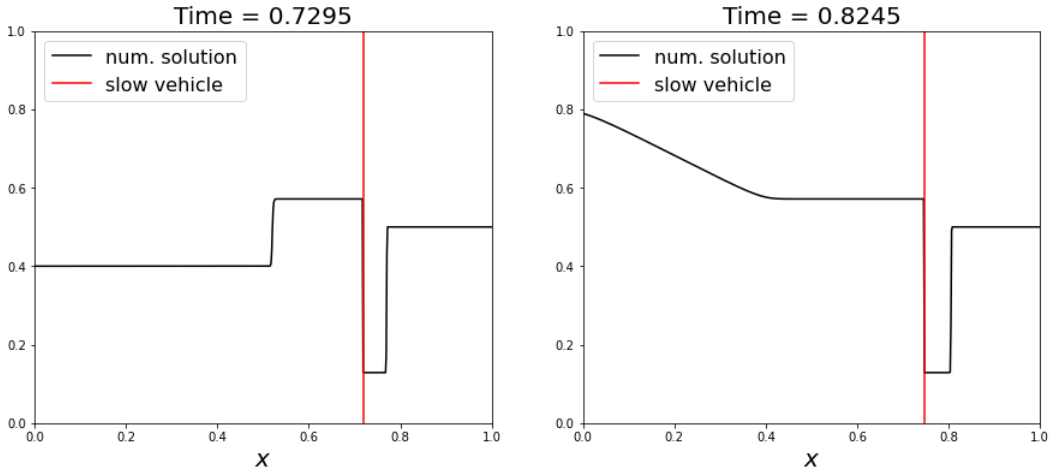

Figure 3.4 – Evolution in time of the numerical density corresponding to initial data (3.4.1) (left) and (3.4.2) (right), with $\Delta x = 0.001$.

Finally, still following [50], we consider

$$\rho_0(x) = \begin{cases} 0.8 & \text{if} \quad x < 0.5 \\ 0.4 & \text{if} \quad x > 0.5 \end{cases} \quad ; \quad y_0 = 0.4. \tag{3.4.3}$$

Here the solution is composed of a rarefaction wave followed by nonclassical and classical shocks on the density that are created when the slow vehicle approaches the rarefaction and initiates a moving bottleneck, as illustrated in Figure 3.5.

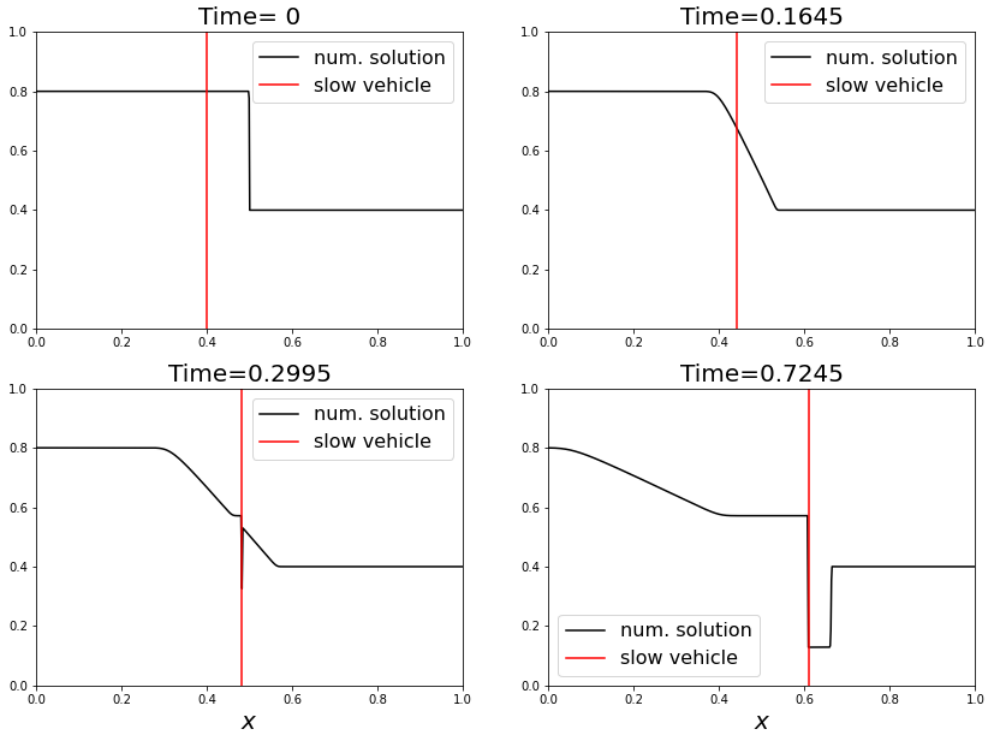

Figure 3.5 – Evolution in time of the numerical density corresponding to initial data (3.4.3), with $\Delta x = 0.001$.

With these three tests, we can already see, in a qualitative way, the resemblance between the numerical approximations to the solutions to our model and the numerical approximations of [50]. One way to quantify their proximity is for example to evaluate the $\mathbf{L}^1$ error between the car densities and the $\mathbf{L}^\infty$ error between the bus positions. More precisely, denote by (ρ_Δ, y_Δ) the approximation of the $\mathbf{BV}$-regular solution to (3.1.1) obtained with the scheme (3.3.1) – (3.3.3), and denote by $(\overline{\rho_\Delta}, \overline{y_\Delta})$ the couple obtained with this same scheme but

$$\text{replacing} \quad s^n = \omega\left(\sum_{j\in\mathbb{Z}} \rho^n_{j+1/2}\mu_{j+1/2}\Delta x\right) \quad \text{by} \quad s^n = \omega\left(\rho^n_{1/2}\right).$$

Let us make precise that this is not the scheme the authors of [50] proposed. However, this scheme is consistent with the problem

$$\begin{cases} \partial_t \rho + \partial_x \left(F(\dot{y}(t), \rho)\right) = 0 \\ F(\dot{y}(t), \rho)|_{x=0} \leq Q(\dot{y}(t)) \\ \dot{y}(t) = \omega\left(\rho(0+, t)\right) \end{cases} \tag{3.4.4}$$

and behaves in a stable way in the calculations we performed. Therefore, the couple $(\overline{\rho_\Delta}, \overline{y_\Delta})$ is expected to give a reasonable approximation of the solution to (3.4.4). With this in mind,

for the case (3.4.3) and still with the weight function μ_3, we computed in Table 3.2 the measured errors

$$\mathbf{E}^1_\Delta = \|\rho_\Delta - \overline{\rho_\Delta}\|_{\mathbf{L}^1((0,T);\mathbf{L}^1(\mathbb{R}))} \quad \text{and} \quad \mathbf{E}^\infty_\Delta = \|y_\Delta - \overline{y_\Delta}\|_{\mathbf{L}^\infty}.$$

Number of cells	$\mathbf{E}^1_\Delta$ $(\times 10^{-4})$	$\mathbf{E}^\infty_\Delta$ $(\times 10^{-3})$
160	32.672	18.519
320	14.236	7.341
640	5.837	3.701
1280	3.833	4.879
2560	3.207	6.405
5120	2.922	7.144
10240	2.776	7.501
20480	2.698	7.674
40960	2.658	7.759

Table 3.2 – Measured errors at time $T = 0.7245$.

These calculations indicate that for a sufficiently large number of cells $J \geq 40960$,

$$\mathbf{E}^1_\Delta \simeq 2.7 \times 10^{-4} \quad \text{and} \quad \mathbf{E}^\infty_\Delta \simeq 7.6 \times 10^{-3}.$$

This indicates the discrepancy between our nonlocal model and the local model (3.4.4) of [70]. The idea is now to fix the number of cells $J = 40960$ and to make the length of the weight function support go to zero. In Table 3.3, we have computed, for different weight functions, the error between the approximations of the two models. This error corresponds, as in the above calculation, to the residual error observed starting from a sufficiently small Δx.

weight function	$\mathbf{E}^1_\Delta$	$\mathbf{E}^\infty_\Delta$
μ_1	6.810×10^{-3}	5.489×10^{-2}
μ_2	1.105×10^{-3}	1.972×10^{-2}
μ_3	2.658×10^{-4}	7.759×10^{-3}
μ_4	9.232×10^{-5}	2.913×10^{-3}
μ_5	6.190×10^{-5}	9.110×10^{-4}

Table 3.3 – Measured errors at time $T = 0.7245$

Remark 3.4.2. The previous simulations show a closeness between our model as $\mu \to \delta_{0^+}$ and (3.4.4). Let us however point that the nonlocality in space for the slow vehicle introduces an undesirable artefact into the model. In the rarefaction regime one may observe that the large vehicle may move a bit faster that the surrounding flow. The situation where this effect becomes truly perceptible is when considering initial data of the type

$$\rho_0(x) = \begin{cases} 1 & \text{if} \quad x < x_b \\ 0 & \text{if} \quad x > x_b. \end{cases} \tag{3.4.5}$$

Indeed, for such data, there exists a small time interval $[0, \nu]$ ($\nu > 0$) in which $\dot{y}(t) > v(\rho(y(t)^+)) = 0$, which would suggest that the slow vehicle moves forward while the cars in front of it do not. This time interval is in fact quite small due to the narrowness of the support of the weight function. The local model does not develop such phenomena. This qualitative artifact precludes us from giving a microscopic interpretation to the model, which main output is the global influence of the slow vehicle on the flow; however, let us stress that the phenomenon becomes quantitatively negligible for larger times. Indeed, Oleĭnik estimate on decay of positive waves ensures that data of the type (3.4.5) evolve into rarefaction waves and do not appear while driving: the classical LWR model precludes the formation of rarefaction waves focused at positive time. The modification of the classical LWR brought by the constraint may produce nonclassical waves at positive times; while these waves are downward jumps in density like in (3.4.5), they are situated precisely at the location of the constraint and not slightly behind it, like in (3.4.5).

Even if we are unable, at this time, to rigorously link our problem (3.1.1) with $\mu \to \delta_{0^+}$ and the original problem (3.4.4) of the authors in [70], this last experiment corroborates the conjecture that the local model (3.4.4) is the singular limit of our model in the case ω is of the form $\omega(\rho) = \min\{V_{\text{bus}}; 1 - \rho\}$. The other interesting question is whether the local model is well posed beyond this particular choice of ω.

CHAPTER 4

A LWR MODEL WITH CONSTRAINTS AT MOVING INTERFACES

Being given a regular concave flux $f \in \mathbf{C}^2([0,1])$ verifying

$$f(\rho) \geq 0,\ f(0) = f(1) = 0; \quad \exists!\, \overline{\rho} \in (0,1),\ \text{for a.e. } \rho \in (0,1), \quad f'(\rho)(\overline{\rho} - \rho) > 0, \tag{4.0.1}$$

and a finite family of trajectories $(y_i)_{i\in[\![1;J]\!]}$ and constraints $(q_i)_{i\in[\![1;J]\!]}$ defined on (s_i, T_i) $(0 \leq s_i < T_i)$, we tackle the following problem:

$$\begin{cases} \partial_t \rho(x,t) + \partial_x \left(f(\rho(x,t))\right) = 0 & (x,t) \in \mathbb{R} \times (0,+\infty) = \Omega \\ \rho(x,0) = \rho_0(x) & x \in \mathbb{R} \\ \forall i \in [\![1;J]\!], \quad \left(f(\rho) - \dot{y}_i(t)\rho\right)\big|_{x=y_i(t)} \leq q_i(t) & t \in (s_i, T_i). \end{cases} \tag{4.0.2}$$

Systems of the type (4.0.2) have naturally arisen in the recent years. Let us give a non-exhaustive review on how our Problem (4.0.2) relates to the existing literature.

— The authors of [73, 89] considered a model very similar to (4.0.2). In their framework, $(y_i)_i$ represented the trajectories of autonomous vehicles, and the authors aimed at modeling the regulation impact on a few autonomous vehicles on the traffic flow. In the same framework but with different applications in mind, the model of [116] accounts for the boundedness of traffic acceleration. Note that in each of these models, the trajectories of the moving interfaces $(y_i)_i$ were not given *a priori*, but rather obtained as solutions to an ODE involving the density of traffic, a mechanism reminiscent of [14, 70, 150] for instance. Let us also mention the work of [92] where the authors studied a different model for the situation of several moving bottlenecks.

— The numerical aspect of (4.0.2) was treated in [50] (for one trajectory) and [71] (for multiple trajectories), where the authors modeled the moving bottlenecks created by buses on a road.

— In a class of problems close to (4.0.2), *i.e.* without constraint on the flux, but still with coupling interfaces/density, the authors of [84] described the interaction between a platoon of vehicles and the surrounding traffic flow on a highway.

— Problem (4.0.2) can be seen as a conservation law with discontinuous flux and special treatments at the interfaces. In that directions, the authors of [107, 18, 5, 34, 153] studied such problems but with the classical vanishing viscosity coupling at the interfaces.

In several of these works [89, 116], the existence issue is tackled using the wave-front tracking procedure which is very sensible to the details of the model. On the other hand, when numerical schemes are considered, see [71, 50], the numerical analysis is usually left out.

The contribution of this chapter is to provide a robust mathematical setting both in the theoretical and numerical aspects of (4.0.2). The proof of uniqueness is based upon a combination of Kruzhkov classical method of doubling variables and the theory of dissipative germs in the framework of discontinuous flux [15] and it is analogous to the one of [18]. To prove existence, we build a finite volume scheme with a grid that adapts locally to the trajectories $(y_i)_i$ and to their crossing points, but remains a simple cartesian grid away from the interfaces. Our work can serve as a basis for constructing solutions to more involved models, *e.g.* *via* the splitting approach. As an example of application, we can point out the variant of our recent work [150] with multiple slow vehicles involved; this is a mildly non-local analogue of the problem considered numerically in [71].

As the fundamental ingredient of the well-posedness proof and numerical approximation of (4.0.2), we will first tackle the one trajectory/one constraint problem:

$$\begin{cases} \partial_t \rho + \partial_x \left(f(\rho) \right) = 0 \\ \rho(\cdot, 0) = \rho_0 \\ \left(f(\rho) - \dot{y}(t)\rho \right)|_{x=y(t)} \leq q(t) \quad t > 0, \end{cases} \tag{4.0.3}$$

with $y \in \mathbf{W}^{1,\infty}_{\mathbf{loc}}((0,+\infty))$ and $q \in \mathbf{L}^{\infty}_{\mathbf{loc}}((0,+\infty))$. Recall that in Chapter 1, we studied (4.0.3) in the case $y \equiv 0$.

The reduction of (4.0.2) to localized problem (4.0.3) requires the construction of a finite volume scheme in the original coordinates (x,t), while the treatment of (4.0.3) in the literature is most often based upon the rectification of the interface *via* a variable change, see [70, 72] and Chapter 3. For (4.0.2), this approach leads to a cumbersome and singular construction, see [18]. In our well-posedness analysis and approximation of (4.0.3), having in mind (4.0.2), we will not change the coordinate system.

Let us detail how the chapter is organized. Sections 4.1-4.2 are devoted to Problem (4.0.3). We start by giving two definitions of solutions. One, most frequently used in traffic dynamics (see [56, 23]), is composed of classical Kruzhkov entropy inequalities with reminder term taking into account the constraint and of a weak formulation for the constraint, see Definition 4.1.1. The second definition emanates from the theory of conservation laws with dissipative interface coupling (see [15, 5]). It consists of Kruzhkov entropy inequalities with test functions that vanish along the interface $\{x = y(t)\}$ and of an explicit treatment of the traces of the solution along the interface, see Definition 4.1.4. Before tackling the well-posedness issue, we

prove that these two definitions are equivalent, see Propositions 4.1.6-4.1.6, similarly to what the authors of [14] did. Uniqueness follows from the stability obtained in Section 4.1, see Theorem 4.1.13. In Section 4.2, we construct a finite volume scheme for (4.0.3) and prove of its convergence. In the construction, we do not rectify the trajectory but instead we locally modify the mesh to mold the trajectory. Moreover, we fully make use of techniques and results put forward by the author of [152] to derive localized **BV** estimates away from the interface, essential to obtain strong compactness for the approximate solutions created by the scheme, see Corollary 4.2.7. This is a way to highlight the generality of the compactness technique of [152].
In Section 4.3, we get back to the original problem (4.0.2). Our strategy is to *assemble* the study of (4.0.2) from several local studies of (4.0.3) with the help of a partition of unity argument. This concerns, in particular, the convergence of finite volume approximation of (4.0.2) which is addressed *via* a localization argument. However, the scheme needs to be defined globally, which makes it impossible to use the rectification strategy as soon as the interfaces have crossing points, *cf.* [18] for a singular rectification strategy.

4.1 Uniqueness and stability for the single trajectory problem

The content of this section is not original in the sense that it is a rigorous adaptation and assembling of existing techniques reminiscent of [155, 114, 56, 14, 15].

4.1.1 Equivalent definitions of solutions

Throughout the chapter, for all $s \in \mathbb{R}$, we denote by

$$\forall \rho \in [0,1],\ F_s(\rho) = f(\rho) - s\rho \quad \text{and} \quad \forall a,b \in [0,1],\ \Phi_s(a,b) = \operatorname{sgn}(a-b)(F_s(a) - F_s(b))$$

the normal flux through $\{x = x_0 + st\}$ ($x_0 \in \mathbb{R}$) and its entropy flux associated with the Kruzhkov entropy $\rho \mapsto |\rho - \kappa|$, for all $\kappa \in [0,1]$, see [114]. Let us also denote by Γ the trajectory:

$$\Gamma = \{(x,t) \in \overline{\Omega} \mid x = y(t)\}.$$

Definition 4.1.1. A function $\rho \in \mathbf{L}^\infty(\Omega;[0,1])$ is an admissible entropy solution to (4.0.3) with initial data $\rho_0 \in \mathbf{L}^\infty(\mathbb{R};[0,1])$ if
(i) for all test functions $\varphi \in \mathbf{C}_{\mathbf{c}}^\infty(\overline{\Omega}), \varphi \geq 0$ and $\kappa \in [0,1]$, the following entropy inequalities are verified:

$$\begin{aligned}\int_0^{+\infty} \int_{\mathbb{R}} \Big(|\rho - \kappa| \partial_t \varphi + \Phi(\rho,\kappa) \partial_x \varphi \Big) \,\mathrm{d}x \,\mathrm{d}t + \int_{\mathbb{R}} |\rho_0(x) - \kappa| \varphi(x,0) \,\mathrm{d}x \\ + \int_0^{+\infty} \mathcal{R}_{\dot{y}(t)}(\kappa, q(t)) \varphi(y(t),t) \,\mathrm{d}t \geq 0,\end{aligned} \tag{4.1.1}$$

where

$$\mathcal{R}_{\dot{y}(t)}(\kappa, q(t)) = 2\left(F_{\dot{y}(t)}(\kappa) - \min\left\{F_{\dot{y}(t)}(\kappa), q(t)\right\}\right);$$

(ii) for all test functions $\varphi \in \mathbf{C}_{\mathbf{c}}^{\infty}(\Omega), \varphi \geq 0$ the following constraint inequalities are verified:

$$-\iint_{\Omega^+} \Big(\rho \partial_t \varphi + f(\rho) \partial_x \varphi \Big) \,\mathrm{d}x \,\mathrm{d}t \leq \int_0^{+\infty} q(t) \varphi(y(t), t) \,\mathrm{d}t \,, \tag{4.1.2}$$

where $\Omega^+ = \{(x,t) \in \Omega \mid x > y(t)\}$.

Remark 4.1.1. Taking $\kappa = 0$, then $\kappa = 1$ in (4.1.1), from the condition $\rho(x,t) \in [0,1]$ a.e. we deduce that any admissible weak solution to Problem (4.0.3) is also a distributional solution to the conservation law $\partial_t \rho + \partial_x f(\rho) = 0$. If ρ is a regular enough solution, then for all test functions $\varphi \in \mathbf{C}_{\mathbf{c}}^{\infty}(\Omega), \varphi \geq 0$, we have

$$\begin{aligned}
0 &= \iint_{\Omega^+} \mathrm{div}_{(x,t)} \begin{pmatrix} f(\rho) \\ \rho \end{pmatrix} \varphi \;\, \mathrm{d}x \,\mathrm{d}t \\
&= \int_{\partial\Omega^+} \begin{pmatrix} f(\rho)\varphi \\ \rho\varphi \end{pmatrix} \cdot \begin{pmatrix} -1 \\ \dot{y}(t) \end{pmatrix} \mathrm{d}t - \iint_{\Omega^+} \begin{pmatrix} f(\rho) \\ \rho \end{pmatrix} \cdot \nabla_{x,t} \varphi \;\, \mathrm{d}x \,\mathrm{d}t \\
&= -\int_0^{+\infty} \Big((f(\rho) - \dot{y}(t)\rho)_{|x=y(t)} \Big) \varphi(y(t), t) \,\mathrm{d}t - \iint_{\Omega^+} \Big(\rho \partial_t \varphi + f(\rho) \partial_x \varphi \Big) \,\mathrm{d}x \,\mathrm{d}t \,.
\end{aligned}$$

Moreover, if ρ satisfies the flux inequality of (4.0.3) a.e. on $(0, +\infty)$, then the previous computations lead to

$$-\iint_{\Omega^+} \Big(\rho \partial_t \varphi + f(\rho) \partial_x \varphi \Big) \,\mathrm{d}x \,\mathrm{d}t \leq \int_0^{+\infty} q(t) \varphi(y(t), t) \,\mathrm{d}t \,;$$

this is where inequalities (4.1.2) come from. Note how they make sense irrespective of the regularity of ρ. Integrating on $\Omega^- = \{(x,t) \in \Omega \mid x < y(t)\}$ would lead to similar and equivalent inequalities.

Definition 4.1.1 is well suited for passage to the limit of a.e. convergent sequences of exact or approximate solutions. However, we cannot derive uniqueness by the standard arguments like in the classical case of Kruzhkov. Using an equivalent notion of solution, which we adapt from [15], based on explicit treatment of traces of ρ on Γ, we rather combine the arguments of [114] and [155]. In this definition a couple plays a major role, the one which realizes the equality in the flux constraint in (4.0.3). More precisely, fix first $s \geq 0$. By (4.0.1) and concavity of f, for all $q \in [0, \max F_s)$, the equation $F_s(\rho) = q$ admits exactly two solutions in $[0,1]$, see Figure 4.1, left. The same way, if $s \leq 0$, then for all $q \in [-\dot{s}, \max F_s)$, the equation still admits two solutions in $[0,1]$. The couple formed by these two solutions, denoted by $(\widehat{\rho}_s(q), \check{\rho}_s(q))$ in Definition 4.1.2 below, will serve both in the prove of uniqueness and existence.

Following the previous discussion, in the sequel, we will assume that q verifies the following assumption:

$$\text{for a.e. } t > 0, \quad q(t) \in [0, \max F_{\dot{y}(t)}) \text{ if } \dot{y}(t) \geq 0 \;\text{ and }\; q(t) \in [-\dot{y}(t), \max F_{\dot{y}(t)}) \text{ if } \dot{y}(t) < 0. \tag{4.1.3}$$

In particular, remark that

$$\text{for a.e. } t > 0, \quad \dot{y}(t) + q(t) \geq 0. \tag{4.1.4}$$

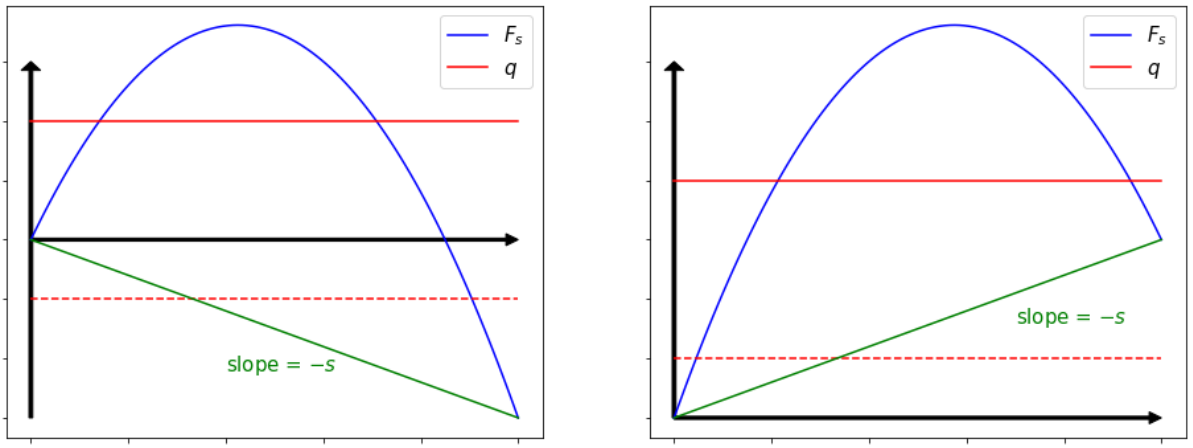

Figure 4.1 – Illustration of Assumption (4.1.3)

Definition 4.1.2. Let $s \in \mathbb{R}^+$ and $q \in [0, \max F_s)$, or $s \in \mathbb{R}^-$ and $q \in [-s, \max F_s)$. The admissibility germ for the conservation law in (4.0.3) associated with the constraint $F_s(\rho)_{|x=st} \leq q$ is the subset $\mathcal{G}_s(q) \subset [0,1]^2$ defined as the union:

$$\mathcal{G}_s(q) = \underbrace{(\widehat{\rho}_s(q), \check{\rho}_s(q))}_{\mathcal{G}_s^1(q)} \bigcup \underbrace{\{(\kappa,\kappa) \mid F_s(\kappa) \leq q\}}_{\mathcal{G}_s^2(q)} \bigcup \underbrace{\{(k_l,k_r) \mid k_l < k_r \text{ and } F_s(k_l) = F_s(k_r) \leq q\}}_{\mathcal{G}_s^3(q)},$$

where, due to the profile of F_s, the couple $(\widehat{\rho}_s(q), \check{\rho}_s(q))$ is uniquely defined by the conditions

$$F_s(\widehat{\rho}_s(q)) = F_s(\check{\rho}_s(q)) = q \quad \text{and} \quad \widehat{\rho}_s(q) > \check{\rho}_s(q).$$

Lemma 4.1.3. *For all $s \in \mathbb{R}^+$ and $q \in [0, \max F_s)$, and for all $s \in \mathbb{R}^-$ and $q \in [-s, \max F_s)$, the admissibility germ $\mathcal{G}_s(q)$ is $\mathbf{L}^1$-dissipative in the sense that:*

(i) for all $(k_l, k_r) \in \mathcal{G}_s(q)$, $F_s(k_l) = F_s(k_r)$ (Rankine-Hugoniot condition);

(ii) for all $(k_l, k_r), (c_l, c_r) \in \mathcal{G}_s(q)$,

$$\Phi_s(k_l, c_l) \geq \Phi_s(k_r, c_r). \tag{4.1.5}$$

Proof. The point *(i)* is obvious from the definition. Let us prove the dissipative feature (4.1.5). The following table summarizes which values can take the difference $\Delta = \Phi_s(k_l, c_l) - \Phi_s(k_r, c_r)$ according with which parts of the germ the couples $(k_l, k_r), (c_l, c_r) \in \mathcal{G}_s(q)$ belong to.

(c_l, c_r) \ (k_l, k_r)	$\in \mathcal{G}_s^1(q)$	$\in \mathcal{G}_s^2(q)$	$\in \mathcal{G}_s^3(q)$
$\in \mathcal{G}_s^1(q)$	0	0	0 or $2(q - F_s(k_l))$
$\in \mathcal{G}_s^2(q)$	0	0	0 or $2\lvert F_s(c) - F_s(k_l)\rvert$
$\in \mathcal{G}_s^3(q)$	0 or $2(q - F_s(c_l))$	0 or $2\lvert F_s(c_l) - F_s(k)\rvert$	0 or $2\lvert F_s(c_l) - F_s(k_l)\rvert$

Having in mind the definition of $\mathcal{G}_s^3(q)$, we can conclude that $\Delta \geq 0$. □

Definition 4.1.4. A function $\rho \in \mathbf{L}^\infty(\Omega;[0,1])$ is a $\mathcal{G}_{\dot y}(q)$-entropy solution to (4.0.3) with initial data $\rho_0 \in \mathbf{L}^\infty(\mathbb{R};[0,1])$ if:
(i) for all test functions $\varphi \in \mathbf{C}^\infty_{\mathbf{c}}(\overline{\Omega}\backslash\Gamma), \varphi \geq 0$ and $\kappa \in [0,1]$, the following entropy inequalities are verified:

$$\int_0^{+\infty}\int_{\mathbb{R}}\Big(|\rho-\kappa|\partial_t\varphi + \Phi(\rho,\kappa)\partial_x\varphi\Big)\,\mathrm{d}x\,\mathrm{d}t + \int_{\mathbb{R}}|\rho_0(x)-\kappa|\varphi(x,0)\,\mathrm{d}x \geq 0; \tag{4.1.6}$$

(ii) for a.e. $t>0$,

$$(\rho(y(t)-,t),\rho(y(t)+,t)) \in \mathcal{G}_{\dot y(t)}(q(t)). \tag{4.1.7}$$

Remark 4.1.2. Condition (4.1.7) is to be understood in the sense of strong traces along Γ. An important fact we stress is that it is not restrictive to assume that entropy solutions, *i.e.* bounded functions verifying (4.1.6), admit strong traces. Usually, it is ensured provided a nondegeneracy assumption on the flux function:

$$\text{for any nonempty interval } (a,b)\subset(0,1), \quad f_{|(a,b)} \text{ is not constant.} \tag{4.1.8}$$

In the context of traffic flow, however, we sometimes consider fluxes which do not verify (4.1.8). Such fluxes, which have linear parts, usually model constant traffic velocity for small densities. In those situations, and when $y\equiv 0$, one can prove that under a mild assumption on the constraint, if the initial data has bounded variation, then solutions to (4.0.3) are in $\mathbf{L}^\infty((0,T);\mathbf{BV}(\mathbb{R}))$, and traces are then to be understood in the sense of $\mathbf{BV}(\mathbb{R})$ functions, see [150, Theorem 3.2]. Also note that the germ formalism can be adapted to the situations where the flux is degenerate and no variation bound is assumed, see [15, Remarks 2.2, 2.3].

We now prove that Definitions 4.1.1 and 4.1.4 are equivalent.

Proposition 4.1.5. *Any admissible entropy solution to (4.0.3) is a $\mathcal{G}_{\dot y}(q)$-entropy solution.*

Proof. Fix $\rho \in \mathbf{L}^\infty(\Omega)$ an admissible entropy solution to (4.0.3), $\varphi \in \mathbf{C}^\infty_{\mathbf{c}}(\overline{\Omega})$, $\varphi\geq 0$ and $\kappa\in[0,1]$. If φ vanishes along Γ, then (4.1.1) becomes (4.1.6). Moreover, it is known that the Rankine-Hugoniot condition is contained in (4.1.1). Combining it with (4.1.2) gives us:

$$\text{for a.e. } t>0, \quad F_{\dot y(t)}(\rho(y(t)-,t)) = F_{\dot y(t)}(\rho(y(t)+,t)) \leq q(t). \tag{4.1.9}$$

Let us show that for a.e. $t>0$, $(\rho(y(t)-,t),\rho(y(t)+,t)) \in \mathcal{G}_{\dot y(t)}(q(t))$.

Case 1: $\rho(y(t)-,t)\leq\rho(y(t)+,t)$. Condition (4.1.9) implies that $(\rho(y(t)-,t),\rho(y(t)+,t)) \in \mathcal{G}^2_{\dot y(t)}(q(t))\cup\mathcal{G}^3_{\dot y(t)}(q(t))$.

Case 2: $\rho(y(t)-,t)>\rho(y(t)+,t)$. Suppose now that $\varphi\in\mathbf{C}^\infty_{\mathbf{c}}(\Omega)$ and fix $n\in\mathbb{N}^*$. By a standard approximation argument, we can apply (4.1.1) with the Lipschitz test function $\xi_n\varphi$, where ξ_n is the cut-off function:

$$\xi_n(x,t) = \begin{cases} 1 & \text{if} \quad |x-y(t)| < \dfrac{1}{n} \\ 2-n|x-y(t)| & \text{if} \quad \dfrac{1}{n}\leq|x-y(t)|\leq\dfrac{2}{n} \\ 0 & \text{if} \quad |x-y(t)|>\dfrac{2}{n}. \end{cases}$$

This yields:

$$\int_0^{+\infty}\int_{\mathbb{R}} |\rho-\kappa|\left(\xi_n\partial_t\varphi + n\dot{y}(t)\,\mathrm{sgn}(x-y(t))\mathbb{1}_{\left\{\frac{1}{n}<|x-y(t)|<\frac{2}{n}\right\}}\varphi\right)\mathrm{d}x\,\mathrm{d}t$$
$$+\int_0^{+\infty}\int_{\mathbb{R}} \Phi(\rho,\kappa)\left(\xi_n\partial_x\varphi - n\,\mathrm{sgn}(x-y(t)\mathbb{1}_{\left\{\frac{1}{n}<|x-y(t)|<\frac{2}{n}\right\}}\varphi\right)\mathrm{d}x\,\mathrm{d}t$$
$$+\int_0^{+\infty}\mathcal{R}_{\dot{y}(t)}(\kappa,q(t))\varphi(y(t),t)\,\mathrm{d}t \geq 0.$$

Taking the limit when $n\to+\infty$, we obtain:

$$\int_0^{+\infty}\left(\Phi_{\dot{y}(t)}\left(\rho(y(t)-,t),\kappa\right)-\Phi_{\dot{y}(t)}\left(\rho(y(t)+,t),\kappa\right)+\mathcal{R}_{\dot{y}(t)}(\kappa,q(t))\right)\varphi(y(t),t)\,\mathrm{d}t\geq 0$$

which implies that for a.e. $t>0$ and for all $\kappa\in[0,1]$,

$$\Phi_{\dot{y}(t)}\left(\rho(y(t)-,t),\kappa\right)-\Phi_{\dot{y}(t)}\left(\rho(y(t)+,t),\kappa\right)+\mathcal{R}_{\dot{y}(t)}(\kappa,q(t))\geq 0.$$

Taking in particular $\kappa=\mathrm{argmax}(F_{\dot{y}(t)})$, we get:

$$\Phi_{\dot{y}(t)}\left(\rho(y(t)-,t),\kappa\right)-\Phi_{\dot{y}(t)}\left(\rho(y(t)+,t),\kappa\right)+2(F_{\dot{y}(t)}(\kappa)-q(t))\geq 0. \tag{4.1.10}$$

Since $\rho(y(t)-,t)>\rho(y(t)+,t)$, (4.1.10) leads to $F_{\dot{y}(t)}(\rho(y(t)-,t))\geq q(t)$, which combined with (4.1.9), implies $F_{\dot{y}(t)}(\rho(y(t)-,t))=F_{\dot{y}(t)}(\rho(y(t)+,t))=q(t)$. We deduce that $(\rho(y(t)-,t),\rho(y(t)+,t))\in\mathcal{G}^1_{\dot{y}(t)}(q(t))$, which completes the proof. □

Proposition 4.1.6. *Any $\mathcal{G}_{\dot{y}}(q)$-entropy solution to (4.0.3) is an admissible entropy solution.*

Proof. Fix $\rho\in\mathbf{L}^\infty(\Omega)$ a $\mathcal{G}_{\dot{y}}(q)$-entropy solution to (4.0.3), $\varphi\in\mathbf{C}^\infty_{\mathbf{c}}(\Omega)$, $\varphi\geq 0$, $\kappa\in[0,1]$ and $n\in\mathbb{N}^*$. We still denote by ξ_n the cut-off function from the last proof. We write

$\varphi = (1-\xi_n)\varphi + \xi_n\varphi$. Since $\phi_n = (1-\xi_n)\varphi$ vanishes along Γ, we have

$$\begin{aligned}
\mathbf{I} &= \int_0^{+\infty}\int_{\mathbb{R}}\Big(|\rho-\kappa|\partial_t\varphi + \Phi(\rho,\kappa)\partial_x\varphi\Big)\,\mathrm{d}x\,\mathrm{d}t + \int_{\mathbb{R}}|\rho_0(x)-\kappa|\varphi(x,0)\,\mathrm{d}x \\
&\quad + \int_0^{+\infty}\mathcal{R}_{\dot y(t)}(\kappa,q(t))\varphi(y(t),t)\,\mathrm{d}t \\
&= \underbrace{\int_0^{+\infty}\int_{\mathbb{R}}\Big(|\rho-\kappa|\partial_t\phi_n + \Phi(\rho,\kappa)\partial_x\phi_n\Big)\,\mathrm{d}x\,\mathrm{d}t + \int_{\mathbb{R}}|\rho_0(x)-\kappa|\phi_n(x,0)\,\mathrm{d}x}_{\geq 0} \\
&\quad + \int_0^{+\infty}\int_{\mathbb{R}}\Big(|\rho-\kappa|\partial_t(\xi_n\varphi) + \Phi(\rho,\kappa)\partial_x(\xi_n\varphi)\Big)\,\mathrm{d}x\,\mathrm{d}t + \int_{\mathbb{R}}|\rho_0(x)-\kappa|\xi_n(x,0)\varphi(x,0)\,\mathrm{d}x \\
&\quad + \int_0^{+\infty}\mathcal{R}_{\dot y(t)}(\kappa,q(t))\varphi(y(t),t)\,\mathrm{d}t \\
&\geq \int_0^{+\infty}\int_{\mathbb{R}}|\rho-\kappa|\left(\xi_n\partial_t\varphi + n\dot y(t)\operatorname{sgn}(x-y(t))\mathbb{1}_{\left\{\frac{1}{n}<|x-y(t)|<\frac{2}{n}\right\}}\varphi\right)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \int_0^{+\infty}\int_{\mathbb{R}}\Phi(\rho,\kappa)\left(\xi_n\partial_x\varphi - n\operatorname{sgn}(x-y(t)\mathbb{1}_{\left\{\frac{1}{n}<|x-y(t)|<\frac{2}{n}\right\}}\varphi\right)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \int_{\mathbb{R}}|\rho_0(x)-\kappa|\xi_n(x,0)\varphi(x,0)\,\mathrm{d}x + \int_0^{+\infty}\mathcal{R}_{\dot y(t)}(\kappa,q(t))\varphi(y(t),t)\,\mathrm{d}t\,.
\end{aligned}$$

Taking the limit when $n\to+\infty$, we obtain:

$$\mathbf{I} \geq \int_0^{+\infty}\Big(\underbrace{\Phi_{\dot y(t)}\left(\rho(y(t)-,t),\kappa\right) - \Phi_{\dot y(t)}\left(\rho(y(t)+,t),\kappa\right) + \mathcal{R}_{\dot y(t)}(\kappa,q(t))}_{\Delta(t,\kappa)}\Big)\varphi(y(t),t)\,\mathrm{d}t\,.$$

To conclude, we are going to prove that for a.e. $t>0$ and for all $\kappa\in[0,1]$, $\Delta(t,\kappa)\geq 0$. Remember that by assumption, for a.e. $t>0$, $(\rho(y(t)-,t),\rho(y(t)+,t))\in\mathcal{G}_{\dot y(t)}(q(t))$. The following table, in which we dropped the $\dot y(t)/q(t)$-indexing, summarizes which values can take the difference $\Delta(t,\kappa)$ according to the position of κ with respect to the couple $(\rho(y(t)-,t),\rho(y(t)+,t))$, which is simply denoted by (ρ_l,ρ_r). Note that the case marked by $\times$ is impossible.

κ \ (ρ_l,ρ_r)	$\in\mathcal{G}^1$	$\in\mathcal{G}^2$	$\in\mathcal{G}^3$
$\kappa<\min\{\rho_l,\rho_r\}$	0	$\mathcal{R}(\kappa,q(t))$	0
$\kappa>\max\{\rho_l,\rho_r\}$	0	$\mathcal{R}(\kappa,q(t))$	0
κ between ρ_l and ρ_r	0	$\times$	$2(F(\kappa)-F(\rho_l))+\mathcal{R}(\kappa,q(t))$

Clearly, $\Delta(t,\kappa) \geq 0$, which proves that $\mathbf{I} \geq 0$, hence ρ satisfies (4.1.1). Moreover, by assumption, for a.e. $t>0$, $(\rho(y(t)-,t),\rho(y(t)+,t)) \in \mathcal{G}_{\dot{y}(t)}(q(t))$. This implies, in particular, that ρ satisfies the flux constraint inequality $(f(\rho)-\dot{y}(t)\rho)_{|x=y(t)} \leq q(t)$ in the a.e. sense. By Remark 4.1.1, ρ satisfies (4.1.2) as well *i.e.* ρ is an admissible entropy solution to (4.0.3). □

4.1.2 Uniqueness of $\mathcal{G}$-entropy solutions

We now prove uniqueness using Definition 4.1.4.

Lemma 4.1.7 (Kato inequality)**.** *Fix $\rho_0,\sigma_0 \in \mathbf{L}^\infty(\mathbb{R};[0,1])$, $y \in \mathbf{W}^{1,\infty}_{\mathbf{loc}}((0,+\infty))$ and $q,r \in \mathbf{L}^\infty_{\mathbf{loc}}((0,+\infty))$. We denote by ρ (resp. σ) a $\mathcal{G}_{\dot{y}}(q)$-entropy solution (resp. $\mathcal{G}_{\dot{y}}(r)$-entropy solution) to Problem (4.0.3) corresponding to initial data ρ_0 (resp. σ_0). We suppose that q,r satisfy (4.1.3). Then for all test functions $\varphi \in \mathbf{C}^\infty_{\mathbf{c}}(\overline{\Omega}), \varphi \geq 0$, we have*

$$\int_0^{+\infty}\int_{\mathbb{R}}\Big(|\rho-\sigma|\partial_t\varphi+\Phi(\rho,\sigma)\partial_x\varphi\Big)\,\mathrm{d}x\,\mathrm{d}t+\int_{\mathbb{R}}|\rho_0(x)-\sigma_0(x)|\varphi(x,0)\,\mathrm{d}x$$
$$+\int_0^{+\infty}\Big(\Phi_{\dot{y}(t)}\left(\rho(y(t)+,t),\sigma(y(t)+,t)\right)-\Phi_{\dot{y}(t)}\left(\rho(y(t)-,t),\sigma(y(t)-,t)\right)\Big)\varphi(y(t),t)\,\mathrm{d}t\geq 0. \tag{4.1.11}$$

Proof. Take $\phi=\phi(x,t,\chi,\tau)\in\mathbf{C}^\infty_{\mathbf{c}}(\overline{\Omega}^2), \phi\geq 0$ with support contained in the set $\left(\overline{\Omega}\backslash\Gamma\right)^2$. The classical method of doubling variables leads us to:

$$\iiiint |\rho(x,t)-\sigma(\chi,\tau)|(\partial_t\phi+\partial_\tau\phi)+\Phi(\rho(x,t),\sigma(\chi,\tau))(\partial_x\phi+\partial_\chi\phi)\;\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}\chi\,\mathrm{d}\tau$$
$$+\iiint|\rho_0(x)-\sigma(\chi,\tau)|\phi(x,0,\chi,\tau)\,\mathrm{d}x\,\mathrm{d}\chi\,\mathrm{d}\tau+\iiint|\rho(x,t)-\sigma_0(\chi)|\phi(x,t,\chi,0)\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}\chi\geq 0. \tag{4.1.12}$$

Again, a standard approximation argument allows us to apply (4.1.12) with the Lipschitz function

$$\phi_n(x,t,\chi,\tau)=\gamma_n(x,t)\varphi\left(\frac{x+\chi}{2},\frac{t+\tau}{2}\right)\delta_n\left(\frac{x-\chi}{2}\right)\delta_n\left(\frac{t-\tau}{2}\right)$$

where $\varphi=\varphi(X,T)\in\mathbf{C}^\infty_{\mathbf{c}}(\overline{\Omega})$ is a nonnegative test function, $(\delta_n)_n$ is a smooth approximation of the Dirac mass at the origin, and

$$\gamma_n(x,t)=\begin{cases}0 & \text{if} \quad |x-y(t)|<\dfrac{1}{n}\\ n\left(|x-y(t)|-\dfrac{1}{n}\right) & \text{if} \quad \dfrac{1}{n}\leq|x-y(t)|\leq\dfrac{2}{n}\\ 1 & \text{if} \quad |x-y(t)|>\dfrac{2}{n}.\end{cases}$$

Using the fact that for a.e. $t>0$,

$$\begin{aligned}\partial_t\phi_n+\partial_\tau\phi_n&=-n\dot y(t)\operatorname{sgn}(x-y(t))\mathbb{1}_{\left\{\frac1n<|x-y(t)|<\frac2n\right\}}\varphi\left(\frac{x+\chi}{2},\frac{t+\tau}{2}\right)\delta_n\left(\frac{x-\chi}{2}\right)\delta_n\left(\frac{t-\tau}{2}\right)\\&\quad+\gamma_n(x,t)\partial_T\varphi\left(\frac{x+\chi}{2},\frac{t+\tau}{2}\right)\delta_n\left(\frac{x-\chi}{2}\right)\delta_n\left(\frac{t-\tau}{2}\right)\\\partial_x\phi_n+\partial_\chi\phi_n&=n\operatorname{sgn}(x-y(t))\mathbb{1}_{\left\{\frac1n<|x-y(t)|<\frac2n\right\}}\varphi\left(\frac{x+\chi}{2},\frac{t+\tau}{2}\right)\delta_n\left(\frac{x-\chi}{2}\right)\delta_n\left(\frac{t-\tau}{2}\right)\\&\quad+\gamma_n(x,t)\partial_X\varphi\left(\frac{x+\chi}{2},\frac{t+\tau}{2}\right)\delta_n\left(\frac{x-\chi}{2}\right)\delta_n\left(\frac{t-\tau}{2}\right),\end{aligned}$$

we obtain:

$$\begin{aligned}&\iiiint|\rho(x,t)-\sigma(\chi,\tau)|(\partial_t\phi_n+\partial_\tau\phi_n)\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}\chi\,\mathrm{d}\tau\\&\underset{n\to+\infty}{\longrightarrow}-\int_0^{+\infty}\dot y(t)\Big(|\rho(y(t)+,t)-\sigma(y(t)+,t)|-|\rho(y(t)-,t)-\sigma(y(t)-,t)|\Big)\varphi(y(t),t)\,\mathrm{d}t\\&+\int_0^{+\infty}\int_{\mathbb{R}}|\rho(x,t)-\sigma(x,t)|\partial_T\varphi(x,t)\,\mathrm{d}x\,\mathrm{d}t\end{aligned}$$

and

$$\begin{aligned}&\iiiint\Phi(\rho(x,t),\sigma(\chi,\tau))(\partial_x\phi_n+\partial_\chi\phi_n)\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}\chi\,\mathrm{d}\tau\\&\underset{n\to+\infty}{\longrightarrow}\int_0^{+\infty}\Big(\Phi(y(t)+,t),\sigma(y(t)+,t)-\Phi(\rho(y(t)-,t),\sigma(y(t)-,t))\Big)\varphi(y(t),t)\,\mathrm{d}t\\&+\int_0^{+\infty}\int_{\mathbb{R}}\Phi(\rho(x,t),\sigma(x,t))\partial_X\varphi(x,t)\,\mathrm{d}x\,\mathrm{d}t\,.\end{aligned}$$

Finally, since

$$\iiint|\rho_0(x)-\sigma(\chi,\tau)|\phi_n(x,0,\chi,\tau)\,\mathrm{d}x\,\mathrm{d}\chi\,\mathrm{d}\tau\quad\text{and}\quad\iiint|\rho(x,t)-\sigma_0(\chi)|\phi_n(x,t,\chi,0)\,\mathrm{d}x\,\mathrm{d}\chi\,\mathrm{d}t$$

both converge to $\dfrac12\displaystyle\int_{\mathbb{R}}|\rho_0(x)-\sigma_0(x)|\varphi(x,0)\,\mathrm{d}x\,,$

we get (4.1.11) by assembling the above ingredients together. □

Theorem 4.1.8. *Fix $\rho_0,\sigma_0\in\mathbf{L}^\infty(\mathbb{R};[0,1])$, $y\in\mathbf{W}^{1,\infty}_{\mathbf{loc}}((0,+\infty))$ and $q,r\in\mathbf{L}^\infty_{\mathbf{loc}}((0,+\infty))$. We denote by ρ (resp. σ) a $\mathcal{G}_{\dot y}(q)$-entropy solution (resp. $\mathcal{G}_{\dot y}(r)$-entropy solution) to Problem (4.0.3) corresponding to initial data ρ_0 (resp. σ_0). We suppose that q,r satisfy (4.1.3). Then for all $T>0$, we have*

$$\|\rho(\cdot,T)-\sigma(\cdot,T)\|_{\mathbf{L}^1}\le\|\rho_0-\sigma_0\|_{\mathbf{L}^1}+2\int_0^T|q(t)-r(t)|\,\mathrm{d}t\,.\tag{4.1.13}$$

In particular, Problem (4.0.3) admits at most one solution.

Proof. Fix $T > 0$, $R \geq \|y\|_{\mathbf{L}^\infty((0,T))}$ and set $\mathbf{L} = \|f'\|_{\mathbf{L}^\infty} + \|\dot{y}\|_{\mathbf{L}^\infty((0,T))}$. Consider for all $n \in \mathbb{N}$ the function:

$$\varphi_n(x,t) = \frac{1}{4}\left(1 - \xi_n(t-T)\right)\left(1 - \xi_n\left(|x| - R + \mathbf{L}(t-T)\right)\right),$$

where $(\xi_n)_n$ is a smooth approximation of the sign function. The sequence $(\varphi_n)_n$ is a smooth approximation of the characteristic function of the trapezoid

$$\mathcal{T} = \left\{(x,t) \in \overline{\Omega} \mid t \in [0,T] \text{ and } |x| \leq R - \mathbf{L}(t-T)\right\} \supset \left\{(x,t) \in \overline{\Omega} \mid t \in [0,T] \text{ and } x = y(t)\right\}.$$

Let us apply Kato inequality (4.1.11) with $(\varphi_n)_n$. For all $n \in \mathbb{N}$, we have

$$\begin{aligned}\int_0^{+\infty}\int_{\mathbb{R}} |\rho-\sigma|\partial_t\varphi_n \,\mathrm{d}x\,\mathrm{d}t &= -\frac{1}{4}\int_0^{+\infty}\int_{\mathbb{R}} |\rho-\sigma|\xi_n'(t-T)\left(1-\xi_n\left(|x|-R+\mathbf{L}(t-T)\right)\right)\mathrm{d}x\,\mathrm{d}t \\ &\quad - \frac{\mathbf{L}}{4}\int_0^{+\infty}\int_{\mathbb{R}} |\rho-\sigma|\left(1-\xi_n(t-T)\right)\xi_n'\left(|x|-R+\mathbf{L}(t-T)\right)\mathrm{d}x\,\mathrm{d}t \\ &\underset{n\to+\infty}{\longrightarrow} -\int_{|x|\leq R}|\rho(x,T)-\sigma(x,T)|\,\mathrm{d}x - \mathbf{L}\int_0^T\int_{|x|=R-\mathbf{L}(t-T)}|\rho-\sigma|\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

Then,

$$\begin{aligned}\int_0^{+\infty}\int_{\mathbb{R}} \Phi(\rho,\sigma)\partial_x\varphi_n \,\mathrm{d}x\,\mathrm{d}t &= -\frac{1}{4}\int_0^{+\infty}\int_{\mathbb{R}} \Phi(\rho,\sigma)\left(1-\xi_n(t-T)\right)\mathrm{sgn}(x)\xi_n'\left(|x|-R+\mathbf{L}(t-T)\right)\mathrm{d}x\,\mathrm{d}t \\ &\underset{n\to+\infty}{\longrightarrow} -\int_0^T\int_{|x|=R-\mathbf{L}(t-T)}\Phi(\rho,\sigma)\,\mathrm{sgn}(x)\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

Finally, we have

$$\int_{\mathbb{R}}|\rho_0(x)-\sigma_0(x)|\varphi_n(x,0)\,\mathrm{d}x \underset{n\to+\infty}{\longrightarrow} \int_{|x|\leq R+\mathbf{L}T}|\rho_0(x)-\sigma_0(x)|\,\mathrm{d}x.$$

Remark also that the choices of R and $\mathbf{L}$ imply that for all $t > 0$,

$$\varphi_n(y(t),t) \underset{n\to+\infty}{\longrightarrow} 1.$$

Assembling the previous limits together, we get:

$$\begin{aligned}&-\int_{|x|\leq R}|\rho(x,T)-\sigma(x,T)|\,\mathrm{d}x + \int_{|x|\leq R+\mathbf{L}T}|\rho_0(x)-\sigma_0(x)|\,\mathrm{d}x \\ &-\int_0^T\int_{|x|=R-\mathbf{L}(t-T)}\left(\mathbf{L}|\rho-\sigma| + \Phi(\rho,\sigma)\,\mathrm{sgn}(x)\right)\mathrm{d}x\,\mathrm{d}t \\ &+\int_0^T\left(\Phi_{\dot{y}(t)}\left(\rho(y(t)+,t),\sigma(y(t)+,t)\right) - \Phi_{\dot{y}(t)}\left(\rho(y(t)-,t),\sigma(y(t)-,t)\right)\right)\mathrm{d}t \geq 0.\end{aligned}$$

Note that for all $\rho, \sigma \in [0,1]$ and for all $x \in \mathbb{R}$,

$$\mathbf{L}|\rho-\sigma| + \Phi(\rho,\sigma)\operatorname{sgn}(x) \geq \mathbf{L}|\rho-\sigma| - |f(\rho)-f(\sigma)| \geq (\mathbf{L} - \|f'\|_{\mathbf{L}^\infty})|\rho-\sigma| \geq 0.$$

Consequently, we have shown that

$$\int_{|x|\leq R} |\rho(x,T)-\sigma(x,T)|\,\mathrm{d}x \leq \int_{|x|\leq R+\mathbf{L}T} |\rho_0(x)-\sigma_0(x)|\,\mathrm{d}x + \int_0^T \Big(\underbrace{\Phi_{\dot y(t)}\left(\rho(y(t)+,t),\sigma(y(t)+,t)\right) - \Phi_{\dot y(t)}\left(\rho(y(t)-,t),\sigma(y(t)-,t)\right)}_{\Delta(t)} \Big)\,\mathrm{d}t\,.$$

What is left to do is to take the limit when $R \to +\infty$ and to estimate the last two terms of the right-hand side of the previous inequality. The following table, in which we dropped the t-indexing, summarizes which values can take the difference $\Delta(t)$ according to which parts of their respective germs the couples $(\rho(y(t)-,t),\rho(y(t)+,t))$ and $(\sigma(y(t)-,t),\sigma(y(t)+,t))$, respectively denoted by (ρ_l,ρ_r) and (σ_l,σ_r) belong to.

(σ_l,σ_r) \ (ρ_l,ρ_r)	$\in \mathcal{G}^1_{\dot y}(q)$	$\in \mathcal{G}^2_{\dot y}(q)$	$\in \mathcal{G}^3_{\dot y}(q)$
$\in \mathcal{G}^1_{\dot y}(r)$	$2(q-r)$	0 or $2(F_{\dot y}(\rho_l)-r)$	$2(F_{\dot y}(\rho_l)-r)$
$\in \mathcal{G}^2_{\dot y}(r)$	0	0	≤ 0
$\in \mathcal{G}^3_{\dot y}(r)$	$2(F_{\dot y}(\sigma_l)-q)$	≤ 0	≤ 0

We clearly see the bound $\Delta(t) \leq 2|q(t)-r(t)|$, which leads us to (4.1.13), which clearly implies uniqueness. This concludes the proof. □

4.2 Existence for the single trajectory problem

We build a simple finite volume scheme and prove its convergence to an admissible entropy solution to (4.0.3). From now on, we denote by

$$a \vee b = \max\{a,b\} \quad \text{and} \quad a \wedge b = \min\{a,b\}.$$

Fix $\rho_0 \in \mathbf{L}^\infty(\mathbb{R};[0,1])$.

4.2.1 Adapted mesh and definition of the scheme

We start by defining the sequence of approximate slopes:

$$\forall n \in \mathbb{N},\ s^n = \frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} \dot y(t)\,\mathrm{d}t\,; \quad \forall t \geq 0,\ s_\Delta(t) = \sum_{n\in\mathbb{N}} s^n \mathbb{1}_{[t^n,t^{n+1})}(t).$$

Remark 4.2.1. Fix $T > 0$. Let us prove that $(s_\Delta)_\Delta$ converges to $\dot{y}$ in $\mathbf{L}^1((0,T))$. Fix $\varepsilon > 0$. Since $\dot{y} \in \mathbf{L}^1((0,2T))$, by density, there exists a continuous function $\gamma \in \mathbf{C}([0,2T])$ such that$\|\dot{y} - \gamma\|_{\mathbf{L}^1((0,2T))} \leq \varepsilon$. Heine theorem ensures that γ is uniformly continuous, hence:

$$\exists \alpha > 0,\ \forall t, \tau \in [0,2T], \quad |t-\tau| \leq \alpha \implies |\gamma(t) - \gamma(\tau)| \leq \frac{\varepsilon}{2T}.$$

Suppose now that $\Delta t \leq \alpha$ and let $N \in \mathbb{N}^*$ such that $T \in [t^N, t^{N+1})$. We have:

$$\begin{aligned}
\|\dot{y} - s_\Delta\|_{\mathbf{L}^1((0,T))} &\leq \|\dot{y} - \gamma\|_{\mathbf{L}^1((0,T))} + \|\gamma - s_\Delta\|_{\mathbf{L}^1((0,T))} \\
&\leq \|\dot{y} - \gamma\|_{\mathbf{L}^1((0,2T))} + \|\gamma - s_\Delta\|_{\mathbf{L}^1((0,t^{N+1}))} \\
&\leq \varepsilon + \sum_{n=0}^{N} \int_{t^n}^{t^{n+1}} |\gamma(t) - s^n| \,\mathrm{d}t \\
&\leq \varepsilon + \sum_{n=0}^{N} \int_{t^n}^{t^{n+1}} \int_{t^n}^{t^{n+1}} \left| \frac{\gamma(t) - \dot{y}(\tau)}{\Delta t} \right| \mathrm{d}\tau \,\mathrm{d}t \\
&\leq \varepsilon + \sum_{n=0}^{N} \int_{t^n}^{t^{n+1}} \int_{t^n}^{t^{n+1}} \underbrace{\left| \frac{\gamma(t) - \gamma(\tau)}{\Delta t} \right|}_{\leq \frac{\varepsilon}{2T\Delta t}} \mathrm{d}\tau \,\mathrm{d}t + \sum_{n=0}^{N} \int_{t^n}^{t^{n+1}} \int_{t^n}^{t^{n+1}} \left| \frac{\gamma(\tau) - \dot{y}(\tau)}{\Delta t} \right| \mathrm{d}\tau \,\mathrm{d}t \\
&\leq \varepsilon + \frac{(N+1)\Delta t}{2T}\varepsilon + \|\gamma - \dot{y}\|_{\mathbf{L}^1((0,t^{N+1}))} \leq 3\varepsilon.
\end{aligned}$$

Then, we define the sequence of approximate trajectories:

$$\forall t \geq 0,\ y_\Delta(t) = y_0 + \int_0^t s_\Delta(\tau) \,\mathrm{d}\tau\,; \quad \forall n \in \mathbb{N},\ y^n = y_\Delta(t^n),$$

which converges to y in $\mathbf{L}^\infty_{\mathbf{loc}}((0,+\infty))$ since $(s_\Delta)_\Delta$ converges to $\dot{y}$ in $\mathbf{L}^1_{\mathbf{loc}}((0,+\infty))$. We also define $(q_\Delta)_\Delta$, the sequence of approximate constraints:

$$q_\Delta(t) = \sum_{n \in \mathbb{N}} q^n \mathbb{1}_{[t^n, t^{n+1})}(t); \quad q^n = \frac{1}{\Delta t} \int_{t^n}^{t^{n+1}} q(t) \,\mathrm{d}t\,.$$

Since $q \in \mathbf{L}^\infty_{\mathbf{loc}}((0,+\infty))$, we can show, as we did in Remark 4.2.1, that $(q_\Delta)_\Delta$ converges to q in $\mathbf{L}^1_{\mathbf{loc}}((0,+\infty))$.

Remark 4.2.2. Remark that with our choices, from (4.1.4), we deduce that

$$\forall n \in \mathbb{N}, \quad s^n + q^n = \frac{1}{\Delta t} \int_{t^n}^{t^{n+1}} (\dot{y}(t) + q(t)) \,\mathrm{d}t \geq 0. \tag{4.2.1}$$

This fact will come in handy in the proof of stability for the scheme.

Fix now $T > 0$ and a spatial mesh size $\Delta x > 0$ with $\lambda = \Delta t/\Delta x$ fixed, verifying the CFL condition

$$2\left(\underbrace{\|f'\|_{\mathbf{L}^\infty} + \|\dot{y}\|_{\mathbf{L}^\infty((0,T))}}_{\mathbf{L}}\right)\lambda \leq 1. \tag{4.2.2}$$

For all $n \in \mathbb{N}$, there exists a unique index $j_n \in \mathbb{Z}$ such that $y^n \in (x_{j_n}, x_{j_n+1})$, see Figure 4.2. Introduce the sequence $(\chi_j^n)_{j\in\mathbb{Z}}$ defined by

$$\chi_j^n = \begin{cases} x_j & \text{if} \quad j \leq j_n - 1 \\ y^n & \text{if} \quad j = j_n \\ x_{j+1} & \text{if} \quad j \geq j_n + 1. \end{cases}$$

We define the cell grids:

$$\overline{\Omega} = \bigcup_{n\in\mathbb{N}} \bigcup_{j\in\mathbb{Z}} \mathcal{P}_{j+1/2}^n,$$

where for all $n \in \mathbb{N}$ and $j \in \mathbb{Z}$, $\mathcal{P}_{j+1/2}^n$ is the rectangle $(\chi_j^n, \chi_{j+1}^n) \times [t^n, t^{n+1})$ if $j \leq j_n - 2$, one of the parallelograms represented in Figure 4.2 if $j \in \{j_n - 1, j_n\}$ and the rectangle $(\chi_{j+1}^n, \chi_{j+2}^n) \times [t^n, t^{n+1})$ if $j \geq j_n + 1$.

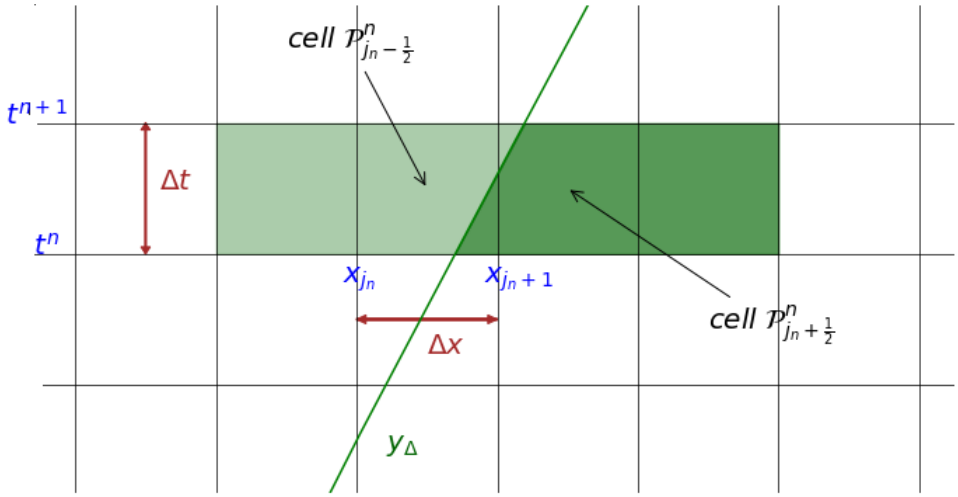

Figure 4.2 – Illustration of the modification to the mesh.

We start by discretizing the initial data ρ_0 with $\left(\rho_{j+1/2}^0\right)_j$ where for all $j \in \mathbb{Z}$, $\rho_{j+1/2}^0$ is its mean value on the cell (χ_j^0, χ_{j+1}^0). Clearly, for this choice, we have:

$$\rho_{j+1/2}^0 \in [0,1] \quad \text{and} \quad \rho_\Delta^0 = \sum_{j\in\mathbb{Z}} \rho_{j+1/2}^0 \mathbb{1}_{(\chi_j^0, \chi_{j+1}^0)} \underset{\Delta x\to 0}{\longrightarrow} \rho_0 \text{ in } \mathbf{L}^1_{\mathbf{loc}}(\mathbb{R}).$$

Let us denote by $\mathbf{EO} = \mathbf{EO}(a,b)$ the Engquist-Osher numerical flux associated with f and for all $s \in \mathbb{R}$, $\mathbf{God}^s = \mathbf{God}^s(u,v)$ be the Godunov flux associated with $\rho \mapsto f(\rho) - s\rho$, see

Definition 1.1.1 and Example 1.1.1.

Fix $n \in \mathbb{N}$. To simplify the reading, we introduce the notations:

$$\forall j \in \mathbb{Z}, \quad f_j^n = \mathbf{EO}\left(\rho_{j-1/2}^n, \rho_{j+1/2}^n\right) \quad \text{and} \quad f_{int}^n = \mathbf{God}^{s^n}\left(\rho_{j_n-1/2}^n, \rho_{j_n+1/2}^n\right) \wedge q^n. \tag{4.2.3}$$

We now proceed to the definition of the scheme. It comes from a discretization of the conservation law written in each volume control $\mathcal{P}_{j+1/2}^n$ ($n \in \mathbb{N}$, $j \in \mathbb{Z}$). Away from the trajectory/constraint, it is the standard 3-point marching formula and when $j \in \{j_n - 1, j_n\}$, we have to deal with both the constraint and the interface which is not vertical. Three cases have to be considered when describing the marching formula of the scheme, but we really give the details for only one of them.

Case 1: $j_{n+1} = j_n + 1$. This means that the line joining (y^n, t^n) and (y^{n+1}, t^{n+1}) crosses the line $x = x_{j_n+1}$, see Figure 4.2. If $j \notin \{j_n - 1, j_n\}$, the conservation written in the rectangle $\mathcal{P}_{j+1/2}^n$ is given by the standard equation:

$$\left(\rho_{j+1/2}^{n+1} - \rho_{j+1/2}^n\right) \Delta x + (f_{j+1}^n - f_j^n)\Delta t = 0. \tag{4.2.4}$$

From the conservation in the cell $\mathcal{P}_{j_n-1/2}^n$, we set:

$$\rho_{j_{n+1}-1/2}^{n+1}\left(y^{n+1} - \chi_{j_{n+1}-2}^{n+1}\right) - \rho_{j_n-1/2}^n\left(y^n - \chi_{j_n-1}^n\right) + (f_{int}^n - f_{j_n-1}^n)\Delta t = 0. \tag{4.2.5}$$

This formula corresponds to the choice of putting the same value for ρ_Δ on $(\chi_{j_{n+1}-2}^{n+1}, \chi_{j_{n+1}-1}^{n+1})$ and on $(\chi_{j_{n+1}-1}^{n+1}, y^{n+1})$ at time $t = t^{n+1}$, *i.e.* $\rho_{j_{n+1}-3/2}^{n+1} = \rho_{j_{n+1}-1/2}^{n+1}$. In the cell $\mathcal{P}_{j_n+1/2}^n$, the conservation takes the form:

$$\rho_{j_{n+1}+1/2}^{n+1}\left(\chi_{j_{n+1}+1}^{n+1} - y^{n+1}\right) - \rho_{j_n+1/2}^n\left(\chi_{j_n+1}^n - y^n\right) - \rho_{j_n+3/2}^n \Delta x + (f_{j_n+2}^n - f_{int}^n)\Delta t = 0. \tag{4.2.6}$$

Let us introduce the two functions

$$\mathbf{H}_{j_n-1}^n(u, v, w) = \frac{v(y^n - \chi_{j_n-1}^n) - \left(\mathbf{God}^{s^n}(v, w) \wedge q^n - \mathbf{EO}(u, v)\right)\Delta t}{y^{n+1} - \chi_{j_{n+1}-2}^{n+1}}$$

and

$$\mathbf{H}_{j_n}^n(u, v, w, z) = \frac{v(\chi_{j_n+1}^n - y^n) + w\Delta x - \left(\mathbf{EO}(w, z) - \mathbf{God}^{s^n}(u, v) \wedge q^n\right)\Delta t}{\chi_{j_{n+1}+1}^{n+1} - y^{n+1}}$$

so that

$$\begin{cases} \rho_{j_{n+1}-1/2}^{n+1} = \mathbf{H}_{j_n-1}^n(\rho_{j_n-3/2}^n, \rho_{j_n-1/2}^n, \rho_{j_n+1/2}^n) \\ \rho_{j_{n+1}+1/2}^{n+1} = \mathbf{H}_{j_n}^n(\rho_{j_n-1/2}^n, \rho_{j_n+1/2}^n, \rho_{j_n+3/2}^n, \rho_{j_n+5/2}^n). \end{cases} \tag{4.2.7}$$

The key point in the proofs of the next section (stability and discrete entropy inequalities) is that the functions $\mathbf{H}_{j_n-1}$ and $\mathbf{H}_{j_n}$ are nondecreasing with respect to their arguments *i.e.* the modification in (4.2.3) did not affect the monotonicity of the resulting scheme (4.2.4) – (4.2.6).

Finally, the approximate solution ρ_Δ is defined almost everywhere on $\overline{\Omega}$:

$$\rho_\Delta = \sum_{n\in\mathbb{N}} \left(\sum_{j\le j_n} \rho^n_{j+1/2} \mathbb{1}_{\mathcal{P}^n_{j+1/2}} + \sum_{j\ge j_n+1} \rho^n_{j+3/2} \mathbb{1}_{\mathcal{P}^n_{j+1/2}} \right).$$

The other cases ($j_{n+1} = j_n$ or $j_{n+1} = j_n - 1$) follow from similar geometric considerations. Note that in the context of traffic dynamics, y would be the trajectory of a stationary or a forward moving obstacle and therefore, we should have $\dot{y} \ge 0$. This implies that for all $n \in \mathbb{N}$, either $j_{n+1} = j_n$ or $j_{n+1} = j_n + 1$. This is why we will focus on the case presented in Figure 4.2.

4.2.2 Stability and discrete entropy inequalities

Proposition 4.2.1 ($\mathbf{L}^\infty$ stability)**.** *Under the CFL condition* (4.2.2)*, the scheme* (4.2.4) – (4.2.6) *is stable:*

$$\forall n \in \mathbb{N},\ \forall j \in \mathbb{Z}, \quad \rho^n_{j+1/2} \in [0,1]. \tag{4.2.8}$$

Proof. *Monotonicity.* Fix $n \in \mathbb{N}$. Clearly, the expression (4.2.4) allows to express ρ^{n+1} as a function of three values of ρ^n in an nondrecreasing way, see the [83, Chapter 5] for instance. We now verify that the functions $\mathbf{H}^n_{j_n-1}$ and $\mathbf{H}^n_{j_n}$ are also nondecreasing. Let us detail the proof for $\mathbf{H}^n_{j_n}$. Recall that $\mathbf{H}^n_{j_n}$ is Lipschitz continuous by construction, therefore we can study its monotonicity in terms of its a.e. derivatives. Making use of both the CFL condition (4.2.2) and of the monotonicity of **EO** and $\mathbf{God}^{s^n}$, for a.e. $u, v, w, z \in [0,1]$, we have

$$\begin{aligned}
\frac{\partial \mathbf{H}^n_{j_n}}{\partial u}(u,v,w,z) &= \frac{1}{2}\frac{\Delta t}{\chi^{n+1}_{j_{n+1}+1} - y^{n+1}} \frac{\partial \mathbf{God}^{s^n}}{\partial a}(u,v)\left(1 - \operatorname{sgn}(\mathbf{God}^{s^n}(u,v) - q^n)\right) \ge 0,\\
\frac{\partial \mathbf{H}^n_{j_n}}{\partial v}(u,v,w,z) &= \frac{\chi^n_{j_n+1} - y^n}{\chi^{n+1}_{j_{n+1}+1} - y^{n+1}} + \frac{\Delta t}{\chi^{n+1}_{j_{n+1}+1} - y^{n+1}} \frac{\partial \mathbf{God}^{s^n}}{\partial b}(u,v) \frac{\left(1 - \operatorname{sgn}(\mathbf{God}^{s^n}(u,v) - q^n)\right)}{2}\\
&\ge \frac{\chi^n_{j_n+1} - (y^n + \mathbf{L}\Delta t)}{\chi^{n+1}_{j_{n+1}+1} - y^{n+1}} \ge \frac{\chi^n_{j_n+1} - \left(y^n + \frac{\Delta x}{2}\right)}{\chi^{n+1}_{j_{n+1}+1} - y^{n+1}} \ge 0,\\
\frac{\partial \mathbf{H}^n_{j_n}}{\partial w}(u,v,w,z) &= \frac{\Delta x}{\chi^{n+1}_{j_{n+1}+1} - y^{n+1}} - \frac{\Delta t}{\chi^{n+1}_{j_{n+1}+1} - y^{n+1}} \frac{\partial \mathbf{EO}}{\partial a}(w,z)\\
&\ge \frac{\Delta x - \mathbf{L}\Delta t}{\chi^{n+1}_{j_{n+1}+1} - y^{n+1}} \ge \frac{\Delta x - \Delta x/2}{\chi^{n+1}_{j_{n+1}+1} - y^{n+1}} \ge 0,\\
\frac{\partial \mathbf{H}^n_{j_n}}{\partial z}(u,v,w,z) &= -\frac{\Delta t}{\chi^{n+1}_{j_{n+1}+1} - y^{n+1}} \frac{\partial \mathbf{EO}}{\partial b}(w,z) \ge 0,
\end{aligned}$$

proving the monotonicity of $\mathbf{H}^n_{j_n}$. Similar computations show that $\mathbf{H}^n_{j_n-1}$ is nondecreasing with respect to its arguments as well.

Stability. We now turn to the proof of (4.2.8), which is done by induction on n. If $n = 0$, it

is verified by definition of $\left(\rho^0_{j+1/2}\right)_j$. Suppose now that (4.2.8) holds for some integer $n \geq 0$ and let us show that it still holds for $n+1$. Remark that 0 and 1 are stationary solutions to the scheme. It is obviously true in the case (4.2.4). The definitions of $\mathbf{H}^n_{j_n-1}$ and $\mathbf{H}^n_{j_n}$ do not change this fact. For instance, $\mathbf{H}^n_{j_n-1}(0,0,0) = 0$ since $q^n \geq 0$ and because of (4.2.1), we also have:

$$\mathbf{H}^n_{j_n-1}(1,1,1) = \frac{(y^n - \chi^n_{j_n-1}) - ((-s^n) \wedge q^n)\,\Delta t}{y^{n+1} - \chi^{n+1}_{j_{n+1}-2}} = \frac{(y^n - \chi^n_{j_n-1}) + s^n \Delta t}{y^{n+1} - \chi^{n+1}_{j_{n+1}-2}} = 1.$$

Similar computations would ensure that it holds also for $\mathbf{H}^n_{j_n}$. Using now the monotonicity of $\mathbf{H}^n_{j_n-1}$ for instance, we deduce that

$$\begin{aligned} 0 = \mathbf{H}^n_{j_n-1}(0,0,0) &\leq \mathbf{H}^n_{j_n-1}(\rho^n_{j_n-3/2}, \rho^n_{j_n-1/2}, \rho^n_{j_n+1/2}) \\ &= \rho^{n+1}_{j_{n+1}-1/2} \\ &= \mathbf{H}^n_{j_n-1}(\rho^n_{j_n-3/2}, \rho^n_{j_n-1/2}, \rho^n_{j_n+1/2}) \leq \mathbf{H}^n_{j_n-1}(1,1,1) = 1, \end{aligned}$$

which concludes the induction argument. The remaining cases follow from similar computations. □

Corollary 4.2.2 (Discrete entropy inequalities). *Fix $n \in \mathbb{N}$, $j \in \mathbb{Z}\backslash\{j_{n+1}-2\}$ and $\kappa \in [0,1]$. Then the numerical scheme (4.2.4) – (4.2.6) fulfills the following discrete entropy inequalities:*

$$\begin{aligned} &|\rho^{n+1}_{j+1/2} - \kappa|(\chi^{n+1}_{j+1} - \chi^{n+1}_j) \\ &\leq \begin{cases} |\rho^n_{j+1/2} - \kappa|(\chi^n_{j+1} - \chi^n_j) - \left(\Phi^n_{j+1} - \Phi^n_j\right)\Delta t & \text{if } j \notin \{j_{n+1}-1, j_{n+1}\} \\ -|\rho^{n+1}_{j_{n+1}-1/2} - \kappa|\Delta x + |\rho^n_{j_n-1/2} - \kappa|(\chi^n_{j_n} - \chi^n_{j_n-1}) & \\ \quad - \left(\Phi^n_{int} - \Phi^n_{j_n-1}\right)\Delta t + \frac{1}{2}\mathcal{R}_{s^n}(\kappa, q^n)\Delta t & \text{if } j = j_{n+1}-1 \\ |\rho^n_{j_n+1/2} - \kappa|(\chi^n_{j_n+1} - \chi^n_{j_n}) + |\rho^n_{j_n+3/2} - \kappa|\Delta x & \\ \quad - \left(\Phi^n_{j_n+2} - \Phi^n_{int}\right)\Delta t + \frac{1}{2}\mathcal{R}_{s^n}(\kappa, q^n)\Delta t & \text{if } j = j_{n+1}, \end{cases} \end{aligned} \tag{4.2.9}$$

where Φ^n_j and Φ^n_{int} denote the numerical entropy fluxes:

$$\Phi^n_j = \mathbf{EO}(\rho^n_{j-1/2} \vee \kappa, \rho^n_{j+1/2} \vee \kappa) - \mathbf{EO}(\rho^n_{j-1/2} \wedge \kappa, \rho^n_{j+1/2} \wedge \kappa);$$

$$\Phi^n_{int} = \min\{\mathbf{God}^{s^n}(\rho^n_{j_n-1/2} \vee \kappa, \rho^n_{j_n+1/2} \vee \kappa), q^n\} - \min\{\mathbf{God}^{s^n}(\rho^n_{j_n-1/2} \wedge \kappa, \rho^n_{j_n+1/2} \wedge \kappa), q^n\}$$

Proof. This result is mostly a consequence of the scheme monotonicity. When the interface/constraint does not enter the calculations *i.e.* when $j \notin \{j_{n+1}-1, j_{n+1}\}$, the proof follows [83, Lemma 5.4]. The key point is not only the monotonicity, but also the fact that in the classical case, all the constants states $\kappa \in [0,1]$ are stationary solutions of the scheme. This observation does not hold when the constraint enters the calculations. Suppose for example

that $j = j_{n+1}$ (which corresponds to the function $\mathbf{H}^n_{j_n}$). Here, we have

$$\begin{aligned}\mathbf{H}^n_{j_n}(\kappa,\kappa,\kappa,\kappa) &= \frac{\kappa(\chi^n_{j_n+1}-y^n)+\kappa\Delta x-(f(\kappa)-(f(\kappa)-s^n\kappa)\wedge q^n)\,\Delta t}{\chi^{n+1}_{j_{n+1}+1}-y^{n+1}}\\ &= \frac{(\chi^n_{j_n+2}-y^n-s^n\Delta t)\kappa}{\chi^{n+1}_{j_{n+1}+1}-y^{n+1}}-\frac{\Delta t}{2(\chi^{n+1}_{j_{n+1}+1}-y^{n+1})}\mathcal{R}_{s^n}(\kappa,q^n)\\ &= \kappa-\frac{\Delta t}{2(\chi^{n+1}_{j_{n+1}+1}-y^{n+1})}\mathcal{R}_{s^n}(\kappa,q^n),\end{aligned}$$

and it implies:

$$\begin{aligned}&\mathbf{H}^n_{j_n}(\rho^n_{j_n-1/2}\wedge\kappa,\rho^n_{j_n+1/2}\wedge\kappa,\rho^n_{j_n+3/2}\wedge\kappa,\rho^n_{j_n+5/2}\wedge\kappa)\\ &\le \rho^{n+1}_{j_{n+1}+1/2}\wedge\kappa,\ \ \rho^{n+1}_{j_{n+1}+1/2}\vee\kappa\\ &\le \mathbf{H}^n_{j_n}(\rho^n_{j_n-1/2}\vee\kappa,\rho^n_{j_n+1/2}\vee\kappa,\rho^n_{j_n+3/2}\vee\kappa,\rho^n_{j_n+5/2}\vee\kappa)+\frac{\Delta t}{2(\chi^{n+1}_{j_{n+1}+1}-y^{n+1})}\mathcal{R}_{s^n}(\kappa,q^n).\end{aligned}$$

We deduce:

$$\begin{aligned}|\rho^{n+1}_{j_{n+1}+1/2}-\kappa| &= \rho^{n+1}_{j_{n+1}+1/2}\vee\kappa-\rho^{n+1}_{j_{n+1}+1/2}\wedge\kappa\\ &\le \mathbf{H}^n_{j_n}(\rho^n_{j_n-1/2}\vee\kappa,\rho^n_{j_n+1/2}\vee\kappa,\rho^n_{j_n+3/2}\vee\kappa,\rho^n_{j_n+5/2}\vee\kappa)\\ &\quad-\mathbf{H}^n_{j_n}(\rho^n_{j_n-1/2}\wedge\kappa,\rho^n_{j_n+1/2}\wedge\kappa,\rho^n_{j_n+3/2}\wedge\kappa,\rho^n_{j_n+5/2}\wedge\kappa)+\frac{\Delta t}{2(\chi^{n+1}_{j_{n+1}+1}-y^{n+1})}\mathcal{R}_{s^n}(\kappa,q^n)\\ &= \frac{\chi^n_{j_n+1}-y^n}{\chi^{n+1}_{j_{n+1}+1}-y^{n+1}}|\rho^n_{j_n+1/2}-\kappa|+\frac{\Delta x}{\chi^{n+1}_{j_{n+1}+1}-y^{n+1}}|\rho^n_{j_n+3/2}-\kappa|\\ &\quad-\frac{\Delta t}{\chi^{n+1}_{j_{n+1}+1}-y^{n+1}}\left(\Phi^n_{j_n+2}-\Phi^n_{int}\right)+\frac{\Delta t}{2(\chi^{n+1}_{j_{n+1}+1}-y^{n+1})}\mathcal{R}_{s^n}(\kappa,q^n),\end{aligned}$$

which is exactly (4.2.9) in the case $j = j_{n+1}$. The obtaining of (4.2.9) in the case $j = j_{n+1}-1$ is similar so we omit the details of the proof for this case. □

4.2.3 Continuous inequalities for the approximate solution

The next step of the reasoning is to derive continuous inequalities, analogous to (4.1.1)-(4.1.2), verified by the approximate solution ρ_Δ, starting from the discrete entropy inequalities (4.2.9) and the marching formula (4.2.4) – (4.2.6).

In this section, we fix a test function $\varphi \in \mathbf{C}^\infty_{\mathbf{c}}(\overline{\Omega}), \varphi \ge 0$ and define:

$$\forall n\in\mathbb{N},\ \forall j\in\mathbb{Z},\quad \varphi^n_{j+1/2} = \frac{1}{\chi^n_{j+1}-\chi^n_j}\int_{\chi^n_j}^{\chi^n_{j+1}}\varphi(x,t^n)\,\mathrm{d}x = \fint_{\chi^n_j}^{\chi^n_{j+1}}\varphi(x,t^n)\,\mathrm{d}x\,.$$

We start by deriving continuous entropy inequalities verified by ρ_Δ. Let us define the approximate entropy flux:

$$\Phi_\Delta(\rho_\Delta,\kappa)=\sum_{n\in\mathbb{N}}\left(\sum_{j\le j_n}\Phi_j^n\mathbb{1}_{\mathcal{P}^n_{j+1/2}}+\sum_{j\ge j_n+1}\Phi_{j+1}^n\mathbb{1}_{\mathcal{P}^n_{j+1/2}}\right).$$

Proposition 4.2.3 (Approximate entropy inequalities)**.** *Fix $n\in\mathbb{N}$ and $\kappa\in[0,1]$. Then we have*

$$\begin{aligned}&\int_{t^n}^{t^{n+1}}\int_{\mathbb{R}}\Big(|\rho_\Delta-\kappa|\partial_t\varphi+\Phi_\Delta\left(\rho_\Delta,\kappa\right)\partial_x\varphi\Big)\,\mathrm{d}x\,\mathrm{d}t\\&+\int_{\mathbb{R}}|\rho_\Delta(x,t^n)-\kappa|\varphi(x,t^n)\,\mathrm{d}x-\int_{\mathbb{R}}|\rho_\Delta(x,t^{n+1})-\kappa|\varphi(x,t^{n+1})\,\mathrm{d}x\\&+\int_{t^n}^{t^{n+1}}\mathcal{R}_{s_\Delta(t)}(\kappa,q_\Delta(t))\varphi(y_\Delta(t),t)\,\mathrm{d}t\ge O\big(\Delta x^2\big)+O(\Delta x\Delta t)+O\big(\Delta t^2\big).\end{aligned}\tag{4.2.10}$$

Proof. For all $j\in\mathbb{Z}\backslash\{j_{n+1}-2\}$, we multiply the discrete entropy inequalities (4.2.9) by $\varphi^{n+1}_{j+1/2}$ and take the sum to obtain:

$$\begin{aligned}&\sum_{j\neq j_{n+1}-2}\left|\rho^{n+1}_{j+1/2}-\kappa\right|\varphi^{n+1}_{j+1/2}(\chi^{n+1}_{j+1}-\chi^{n+1}_j)\\&\le\sum_{j\notin\{j_{n+1}-2,j_{n+1}-1,j_{n+1}\}}\left(\left|\rho^n_{j+1/2}-\kappa\right|(\chi^n_{j+1}-\chi^n_j)-(\Phi^n_{j+1}-\Phi^n_j)\Delta t\right)\varphi^{n+1}_{j+1/2}\\&+|\rho^n_{j_n-1/2}-\kappa|\varphi^{n+1}_{j_{n+1}-1/2}(\chi^n_{j_n}-\chi^n_{j_n-1})-|\rho^{n+1}_{j_{n+1}-1/2}-\kappa|\varphi^{n+1}_{j_{n+1}-1/2}\Delta x-\left(\Phi^n_{int}-\Phi^n_{j_n-1}\right)\varphi^{n+1}_{j_{n+1}-1/2}\Delta t\\&+|\rho^n_{j_n+1/2}-\kappa|\varphi^{n+1}_{j_{n+1}+1/2}(\chi^n_{j_n+1}-\chi^n_{j_n})+|\rho^n_{j_n+3/2}-\kappa|\varphi^{n+1}_{j_{n+1}+1/2}\Delta x-\left(\Phi^n_{j_n+2}-\Phi^n_{int}\right)\varphi^{n+1}_{j_{n+1}+1/2}\Delta t\\&+\frac{1}{2}\mathcal{R}_{s^n}(\kappa,q^n)(\varphi^{n+1}_{j_{n+1}-1/2}+\varphi^{n+1}_{j_{n+1}+1/2})\Delta t.\end{aligned}$$

This inequality can be rewritten as

$$\begin{aligned}&\sum_{j\in\mathbb{Z}}\left|\rho^{n+1}_{j+1/2}-\kappa\right|\varphi^{n+1}_{j+1/2}(\chi^{n+1}_{j+1}-\chi^{n+1}_j)-\sum_{j\in\mathbb{Z}}\left|\rho^n_{j+1/2}-\kappa\right|\varphi^{n+1}_{j+1/2}(\chi^n_{j+1}-\chi^n_j)\\&\le-\underbrace{\left|\rho^{n+1}_{j_{n+1}-1/2}-\kappa\right|\left(\varphi^{n+1}_{j_{n+1}-1/2}-\varphi^{n+1}_{j_{n+1}-3/2}\right)\Delta x}_{\varepsilon_1}+\underbrace{\left|\rho^n_{j_n-1/2}-\kappa\right|\left(\varphi^{n+1}_{j_{n+1}-1/2}-\varphi^{n+1}_{j_{n+1}-3/2}\right)(\chi^n_{j_n}-\chi^n_{j_n-1})}_{\varepsilon_2}\\&+\underbrace{\left|\rho^n_{j_n+1/2}-\kappa\right|\left(\varphi^{n+1}_{j_{n+1}+1/2}-\varphi^{n+1}_{j_{n+1}-1/2}\right)(\chi^n_{j_n+1}-\chi^n_{j_n})}_{\varepsilon_3}\\&-\sum_{j\notin\{j_{n+1}-2,j_{n+1}-1,j_{n+1}\}}(\Phi^n_{j+1}-\Phi^n_j)\varphi^{n+1}_{j+1/2}\Delta t-\left(\Phi^n_{int}-\Phi^n_{j_n-1}\right)\varphi^{n+1}_{j_{n+1}-1/2}\Delta t-\left(\Phi^n_{j_n+2}-\Phi^n_{int}\right)\varphi^{n+1}_{j_{n+1}+1/2}\Delta t\\&+\frac{1}{2}\mathcal{R}_{s^n}(\kappa,q^n)(\varphi^{n+1}_{j_{n+1}-1/2}+\varphi^{n+1}_{j_{n+1}+1/2})\Delta t,\end{aligned}$$

with

$$\forall i \in \{1,2,3\}, \quad |\varepsilon_i| \leq 8\|\partial_x\varphi\|_{\mathbf{L}^\infty}\Delta x^2.$$

We now proceed to the Abel's transformation and reorganize the terms of the inequality. This leads us to:

$$\underbrace{\sum_{j\in\mathbb{Z}} \left|\rho^{n+1}_{j+1/2}-\kappa\right| \varphi^{n+1}_{j+1/2}(\chi^{n+1}_{j+1}-\chi^{n+1}_j) - \sum_{j\in\mathbb{Z}} \left|\rho^{n}_{j+1/2}-\kappa\right| \varphi^{n}_{j+1/2}(\chi^{n}_{j+1}-\chi^{n}_j)}_{A}$$

$$\underbrace{-\sum_{j\in\mathbb{Z}} \left|\rho^{n}_{j+1/2}-\kappa\right| \left(\varphi^{n+1}_{j+1/2}-\varphi^{n}_{j+1/2}\right)(\chi^{n}_{j+1}-\chi^{n}_j)}_{B} + \underbrace{\sum_{j\notin\{j_{n+1}-2,j_{n+1}-1\}} \Phi^n_j\left(\varphi^{n+1}_{j+1/2}-\varphi^{n+1}_{j-1/2}\right)\Delta t}_{C}$$

$$\leq \underbrace{\frac{1}{2}\mathcal{R}_{s^n}(\kappa,q^n)(\varphi^{n+1}_{j_{n+1}-1/2}+\varphi^{n+1}_{j_{n+1}+1/2})\Delta t}_{D} + \sum_{i=1}^{5}\varepsilon_i,$$

with

$$\forall i \in \{4,5\}, \quad |\varepsilon_i| \leq 4\|f\|_{\mathbf{L}^\infty}\|\partial_x\varphi\|_{\mathbf{L}^\infty}\Delta x\Delta t.$$

We immediately see that

$$A = \int_{\mathbb{R}} \left|\rho_\Delta(x,t^{n+1})-\kappa\right| \varphi(x,t^{n+1})\,\mathrm{d}x - \int_{\mathbb{R}} \left|\rho_\Delta(x,t^{n})-\kappa\right| \varphi(x,t^{n})\,\mathrm{d}x\,.$$

We conclude this proof by estimating the remaining terms of the inequality.

Estimating B. First, note that

$$\begin{aligned} B = {} & \sum_{j\leq j_n-2} \iint_{\mathcal{P}^n_{j+1/2}} |\rho_\Delta-\kappa|\,\partial_t\varphi\,\mathrm{d}x\,\mathrm{d}t + \sum_{j\geq j_n+1} \iint_{\mathcal{P}^n_{j+1/2}} |\rho_\Delta-\kappa|\,\partial_t\varphi\,\mathrm{d}x\,\mathrm{d}t \\ & \underbrace{+ \left|\rho^n_{j_n-1/2}-\kappa\right| \left(\fint_{\chi^{n+1}_{j_n-1}}^{\chi^{n+1}_{j_n+1}} \varphi(x,t^{n+1})\,\mathrm{d}x - \fint_{\chi^{n}_{j_n-1}}^{y^n} \varphi(x,t^{n})\,\mathrm{d}x\right)(y^n-\chi^n_{j_n-1})}_{B_1} \\ & \underbrace{+ \left|\rho^n_{j_n+1/2}-\kappa\right| \left(\fint_{\chi^{n+1}_{j_n}}^{y^{n+1}} \varphi(x,t^{n+1})\,\mathrm{d}x - \fint_{y^{n}}^{\chi^n_{j_n+1}} \varphi(x,t^{n})\,\mathrm{d}x\right)(\chi^n_{j_n+1}-y^n)}_{B_2} \\ & \underbrace{+ \left|\rho^n_{j_n+3/2}-\kappa\right| \left(\fint_{y^{n+1}}^{\chi^{n+1}_{j_n+2}} \varphi(x,t^{n+1})\,\mathrm{d}x - \fint_{\chi^{n}_{j_n+1}}^{\chi^n_{j_n+2}} \varphi(x,t^{n})\,\mathrm{d}x\right)\Delta x}_{B_3}\,. \end{aligned}$$

Since

$$\begin{aligned}
&\iint_{\mathcal{P}^n_{j_n-1/2}} |\rho_\Delta - \kappa|\, \partial_t\varphi \,\mathrm{d}x\,\mathrm{d}t \\
&= \left|\rho^n_{j_n-1/2} - \kappa\right| \left(\int_{\chi^{n+1}_{j_n-1}}^{y^{n+1}} \varphi(x,t^{n+1})\,\mathrm{d}x - \int_{\chi^n_{j_n-1}}^{y^n} \varphi(x,t^n)\,\mathrm{d}x - s^n \int_{t^n}^{t^{n+1}} \varphi(y^n + s^n(t-t^n),t)\,\mathrm{d}t \right) \\
&= \left|\rho^n_{j_n-1/2} - \kappa\right| \left(\frac{y^{n+1} - \chi^{n+1}_{j_n-1}}{y^n - \chi^n_{j_n-1}} \fint_{\chi^{n+1}_{j_n-1}}^{y^{n+1}} \varphi(x,t^{n+1})\,\mathrm{d}x - \fint_{\chi^n_{j_n-1}}^{y^n} \varphi(x,t^n)\,\mathrm{d}x \right. \\
&\left. + \frac{y^n - y^{n+1}}{y^n - \chi^n_{j_n-1}} \fint_{t^n}^{t^{n+1}} \varphi(y^n + s^n(t-t^n),t)\,\mathrm{d}t \right) (y^n - \chi^n_{j_n-1}),
\end{aligned}$$

we deduce the bound:

$$\begin{aligned}
&\left| B_1 - \iint_{\mathcal{P}^n_{j_n-1/2}} |\rho_\Delta - \kappa|\, \partial_t\varphi \,\mathrm{d}x\,\mathrm{d}t \right| \\
&= \left|\rho^n_{j_n-1/2} - \kappa\right| (y^{n+1} - y^n) \left| \fint_{\chi^{n+1}_{j_n-1}}^{y^{n+1}} \varphi(x,t^{n+1})\,\mathrm{d}x - \fint_{t^n}^{t^{n+1}} \varphi(y^n + s^n(t-t^n),t)\,\mathrm{d}t \right| \\
&\leq \|\dot y\|_{\mathbf{L}^\infty} \left(3\|\partial_x\varphi\|_{\mathbf{L}^\infty}\Delta x + \|\partial_t\varphi\|_{\mathbf{L}^\infty}\Delta t + 2\|\dot y\|_{\mathbf{L}^\infty}\|\partial_x\varphi\|_{\mathbf{L}^\infty}\Delta t \right) \Delta t.
\end{aligned}$$

The same way, we would derive the estimation:

$$\begin{aligned}
&\left| B_2 + B_3 - \iint_{\mathcal{P}^n_{j_n+1/2}} |\rho_\Delta - \kappa|\, \partial_t\varphi \,\mathrm{d}x\,\mathrm{d}t \right| \\
&\leq 6\|\partial_x\varphi\|_{\mathbf{L}^\infty}\Delta x^2 + \|\dot y\|_{\mathbf{L}^\infty} \left(2\|\partial_x\varphi\|_{\mathbf{L}^\infty}\Delta x + \|\partial_t\varphi\|_{\mathbf{L}^\infty}\Delta t + 2\|\dot y\|_{\mathbf{L}^\infty}\|\partial_x\varphi\|_{\mathbf{L}^\infty}\Delta t \right) \Delta t.
\end{aligned}$$

Estimating C. We write:

$$\begin{aligned}
C &= \lambda \sum_{j\notin\{j_{n+1}-2, j_{n+1}-1, j_{n+1}\}} \int_{\chi^n_j}^{\chi^n_{j+1}} \int_{x-\Delta x}^{x} \Phi^n_j \partial_x\varphi(y,t^{n+1})\,\mathrm{d}y\,\mathrm{d}x + \underbrace{\Phi^n_{j_{n+1}} \left(\varphi^{n+1}_{j_{n+1}+1/2} - \varphi^{n+1}_{j_{n+1}-1/2} \right) \Delta t}_{\varepsilon_6} \\
&= \int_{t^n}^{t^{n+1}} \int_{\mathbb{R}} \Phi_\Delta(\rho_\Delta,\kappa)\partial_x\varphi\,\mathrm{d}x\,\mathrm{d}t + \varepsilon_6 - \underbrace{\sum_{j_{n+1}-2\leq j\leq j_{n+1}-1} \iint_{\mathcal{P}^n_{j+1/2}} \Phi_\Delta(\rho_\Delta,\kappa)\partial_x\varphi\,\mathrm{d}x\,\mathrm{d}t}_{\varepsilon_7} \\
&+ \underbrace{\sum_{j\notin\{j_{n+1}-2, j_{n+1}-1, j_{n+1}\}} \left(\lambda \int_{\chi^n_j}^{\chi^n_{j+1}} \int_{x-\Delta x}^{x} \Phi^n_j \partial_x\varphi(y,t^{n+1})\,\mathrm{d}y\,\mathrm{d}x \right) - \int_{t^n}^{t^{n+1}} \int_{\mathbb{R}} \Phi_\Delta(\rho_\Delta,\kappa)\partial_x\varphi\,\mathrm{d}x\,\mathrm{d}t}_{\varepsilon_8},
\end{aligned}$$

with

$$|\varepsilon_6| + |\varepsilon_7| \leq 8\|f\|_{\mathbf{L}^\infty}\|\partial_x\varphi\|_{\mathbf{L}^\infty}\Delta x\Delta t$$

and

$$|\varepsilon_8| \leq \|f\|_{\mathbf{L}^\infty}\left(4\sup_{t\geq 0}\|\partial^2_{xx}\varphi(\cdot,t)\|_{\mathbf{L}^1}\Delta x + \sup_{t\geq 0}\|\partial^2_{tx}\varphi(\cdot,t)\|_{\mathbf{L}^1}\Delta t\right)\Delta t.$$

Estimating D. Finally, we have

$$\begin{aligned}
D &= \mathcal{R}_{s^n}(\kappa, q^n)\varphi(y^{n+1}, t^{n+1})\Delta t + \underbrace{\frac{1}{y^{n+1} - \chi_{j_{n+1}-1}}\int_{\chi^{n+1}_{j_{n+1}-1}}^{y^{n+1}} (\varphi(x, t^{n+1}) - \varphi(y^{n+1}, t^{n+1}))\Delta t}_{\varepsilon_9} \\
&\quad + \underbrace{\frac{1}{\chi_{j_{n+1}+1} - y^{n+1}}\int_{y^{n+1}}^{\chi^{n+1}_{j_{n+1}+1}} (\varphi(x, t^{n+1}) - \varphi(y^{n+1}, t^{n+1}))\Delta t}_{\varepsilon_{10}} \\
&= \int_{t^n}^{t^{n+1}} \mathcal{R}_{s_\Delta(t)}(\kappa, q_\Delta(t))\varphi(y_\Delta(t), t)\,\mathrm{d}t + \varepsilon_9 + \varepsilon_{10} \\
&\quad + \underbrace{\int_{t^n}^{t^{n+1}} \mathcal{R}_{s_\Delta(t)}(\kappa, q_\Delta(t))(\varphi(y^{n+1}, t^{n+1}) - \varphi(y_\Delta(t), t))\,\mathrm{d}t}_{\varepsilon_{11}},
\end{aligned}$$

with

$$|\varepsilon_9| + |\varepsilon_{10}| + |\varepsilon_{11}| \leq 2\|f\|_{\mathbf{L}^\infty}\left(2\|\partial_x\varphi\|_{\mathbf{L}^\infty}\Delta x + \|\dot{y}\|_{\mathbf{L}^\infty}\|\partial_x\varphi\|_{\mathbf{L}^\infty}\Delta t + \|\partial_t\varphi\|_{\mathbf{L}^\infty}\Delta t\right)\Delta t$$

□

Note that if φ is supported in time in $[0, T]$, with $T \in [t^N, t^{N+1})$, then by summing (4.2.10) over $n \in \{0, \ldots, N+1\}$, we obtain (recall that λ is fixed):

$$\begin{aligned}
&\int_0^T \int_{\mathbb{R}} \left(|\rho_\Delta - \kappa|\partial_t\varphi + \Phi_\Delta(\rho_\Delta, \kappa)\,\partial_x\varphi\right)\mathrm{d}x\,\mathrm{d}t + \int_{\mathbb{R}} |\rho^0_\Delta - \kappa|\varphi(x, 0)\,\mathrm{d}x \\
&\qquad + \int_0^T \mathcal{R}_{s_\Delta(t)}(\kappa, q_\Delta(t))\varphi(y_\Delta(t), t)\,\mathrm{d}t \geq O(\Delta x) + O(\Delta t)\,.
\end{aligned} \tag{4.2.11}$$

We now turn to the proof of an approximate version of (4.1.2). Let us define the approximate flux function:

$$\mathbf{F}_\Delta(\rho_\Delta) = \sum_{n\in\mathbb{N}}\left(\sum_{j\leq j_n} f^n_j \mathbb{1}_{\mathcal{P}^n_{j+1/2}} + \sum_{j\geq j_n+1} f^n_{j+1}\mathbb{1}_{\mathcal{P}^n_{j+1/2}}\right).$$

Proposition 4.2.4 (Approximate constraint inequalities)**.** *Fix $n \in \mathbb{N}$ and $\kappa \in [0,1]$. Then we have*

$$\begin{aligned}&\int_{y^n}^{+\infty} \rho_\Delta(x,t^n)\varphi(x,t^n)\,\mathrm{d}x - \int_{y^{n+1}}^{+\infty} \rho_\Delta(x,t^{n+1})\varphi(x,t^{n+1})\,\mathrm{d}x\\ &-\int_{t^n}^{t^{n+1}}\int_{\mathbb{R}}\Big(\rho_\Delta\partial_t\varphi + \mathbf{F}_\Delta\left(\rho_\Delta\right)\partial_x\varphi\Big)\,\mathrm{d}x\,\mathrm{d}t \;\le \int_{t^n}^{t^{n+1}} q_\Delta(t)\varphi(y_\Delta(t),t)\,\mathrm{d}t\\ &\qquad\qquad + O(\Delta x^2) + O(\Delta x\Delta t) + O(\Delta t^2)\,.\end{aligned} \tag{4.2.12}$$

Proof. Following the steps of the proof of Proposition 4.2.3, we first multiply the scheme (4.2.4)-(4.2.6) by $\varphi^{n+1}_{j+1/2}$, sum over $j \ge j_{n+1}$ and then apply the summation by parts procedure. This time, we obtain:

$$\underbrace{\sum_{j\ge j_{n+1}} \rho^{n+1}_{j+1/2}\varphi^{n+1}_{j+1/2}(\chi^{n+1}_{j+1}-\chi^{n+1}_j) - \sum_{j\ge j_n}\rho^n_{j+1/2}\varphi^n_{j+1/2}(\chi^n_{j+1}-\chi^n_j)}_{A}$$

$$\underbrace{-\sum_{j\ge j_n}\rho^n_{j+1/2}\left(\varphi^{n+1}_{j+1/2}-\varphi^n_{j+1/2}\right)(\chi^n_{j+1}-\chi^n_j)}_{B} + \underbrace{\sum_{j\ge j_n+2} f^n_j\left(\varphi^{n+1}_{j+1/2}-\varphi^{n+1}_{j-1/2}\right)\Delta t}_{C} \le \underbrace{q^n\varphi^{n+1}_{j_{n+1}+1/2}\Delta t}_{D} + \varepsilon,$$

with $\varepsilon \le 8\|\partial_x\varphi\|_{\mathbf{L}^\infty}\Delta x^2$. Clearly,

$$A = \int_{y^{n+1}}^{+\infty}\rho_\Delta(x,t^{n+1})\varphi(x,t^{n+1})\,\mathrm{d}x - \int_{y^n}^{+\infty}\rho_\Delta(x,t^n)\varphi(x,t^n)\,\mathrm{d}x\,,$$

and estimate (4.2.12) follows from the bounds:

$$\left|B - \int_{t^n}^{t^{n+1}}\int_{\mathbb{R}}\rho_\Delta\partial_t\varphi\,\mathrm{d}x\,\mathrm{d}t\right|$$

$$\le (3\|\partial_x\varphi\|_{\mathbf{L}^\infty}\Delta x + \|\partial_t\varphi\|_{\mathbf{L}^\infty}\Delta t)\Delta t + \|\dot y\|_{\mathbf{L}^\infty}\Big(2\|\partial_x\varphi\|_{\mathbf{L}^\infty}\Delta x + 2\|\dot y\|_{\mathbf{L}^\infty}\|\partial_x\varphi\|_{\mathbf{L}^\infty}\Delta t + \|\partial_t\varphi\|_{\mathbf{L}^\infty}\Delta t\Big)\Delta t$$

$$\left|C - \int_{t^n}^{t^{n+1}}\int_{\mathbb{R}}\mathbf{F}_\Delta\left(\rho_\Delta\right)\partial_x\varphi\,\mathrm{d}x\,\mathrm{d}t\right| \le \|f\|_{\mathbf{L}^\infty}\Big(6\|\partial_x\varphi\|_{\mathbf{L}^\infty} + 4\sup_{t\ge0}\|\partial^2_{xx}\varphi(\cdot,t)\|_{\mathbf{L}^1} + \sup_{t\ge0}\|\partial^2_{tx}\varphi(\cdot,t)\|_{\mathbf{L}^1}\Big)\Delta x\Delta t$$

$$\left|D - \int_{t^n}^{t^{n+1}} q_\Delta(t)\varphi(y_\Delta(t),t)\,\mathrm{d}t\right| \le \|q\|_{\mathbf{L}^\infty}\Big(2\|\partial_x\varphi\|_{\mathbf{L}^\infty}\Delta x + \|\partial_t\varphi\|_{\mathbf{L}^\infty}\Delta t + \|\dot y\|_{\mathbf{L}^\infty}\|\partial_x\varphi\|_{\mathbf{L}^\infty}\Delta t\Big)\Delta t.$$

□

If φ is supported in time in $(0,T)$, with $T \in [t^N, t^{N+1})$, then by summing (4.2.10) over $n \in \{0,\dots,N+1\}$, we obtain:

$$-\int_0^T\int_{\mathbb{R}}\Big(\rho_\Delta\partial_t\varphi + \mathbf{F}_\Delta\left(\rho_\Delta\right)\partial_x\varphi\Big)\,\mathrm{d}x\,\mathrm{d}t \le \int_0^T q_\Delta(t)\varphi(y_\Delta(t),t)\,\mathrm{d}t + O(\Delta x) + O(\Delta t)\,. \tag{4.2.13}$$

4.2.4 Compactness and convergence

The remaining part of the reasoning consists in obtaining sufficient compactness for the sequence $(\rho_\Delta)_\Delta$ in order to pass to the limit in (4.2.11)-(4.2.13). To doing so, we adapt techniques and results put forward by Towers in [152]. With this in mind, we suppose in this section that the flux function, still satisfying (4.0.1), is also strictly concave. By continuity,

$$\exists \mu > 0,\ \forall \rho \in [0,1], \quad f''(\rho) \leq -\mu. \tag{4.2.14}$$

We denote for all $n \in \mathbb{N}$ and $j \in \mathbb{Z}$,

$$\mathbf{D}_j^n = \max\left\{\rho_{j-1/2}^n - \rho_{j+1/2}^n, 0\right\}.$$

We will also use the notation

$$\forall n \in \mathbb{N}, \quad \widehat{\mathbb{Z}}_{n+1} = \mathbb{Z}\backslash\{j_{n+1}-2, j_{n+1}-1, j_{n+1}, j_{n+1}+1\}.$$

In [152], the author dealt with a discontinuous in both time and space flux and the specific "vanishing viscosity" coupling at the interface. The discontinuity in space was localized along the curve $\{x = 0\}$. Here, we deal with a smooth flux but we have a flux constraint along the curve $\{x = y(t)\}$. The applicability of the technique of [152] for our case with moving interface and flux-constrained interface coupling relies on the fact that one can derive a bound on $\mathbf{D}_j^{n+1}$ as long as the interface does not enter the calculations for $\mathbf{D}_j^{n+1}$ *i.e.* as long as $j \in \widehat{\mathbb{Z}}_{n+1}$ in the case $j_{n+1} = j_n + 1$.

Lemma 4.2.5. *Let $n \in \mathbb{N}$, $j \in \widehat{\mathbb{Z}}_{n+1}$, $a = \mu\dfrac{\Delta t}{4\Delta x}$ and $\psi(x) = x - ax^2$. Then*

$$\mathbf{D}_j^{n+1} \leq \psi\left(\max\left\{\mathbf{D}_{j-1}^n, \mathbf{D}_j^n, \mathbf{D}_{j+1}^n\right\}\right). \tag{4.2.15}$$

Proof. A complete proof, which is largely inspired by [152], can be found in [149, Appendix]. □

Remark 4.2.3. Fix $n \in \mathbb{N}$ and $j \in \widehat{\mathbb{Z}}_{n+1}$. Remark that if $\mathbf{D}_j^n > 0$, then we can write that for some $\nu(j) \in \{j-1, j, j+1\}$, we have

$$\begin{aligned}\mathbf{D}_j^{n+1} &\leq \mathbf{D}_{\nu(j)}^n - a\left(\mathbf{D}_{\nu(j)}^n\right)^2 \\ &= \mathbf{D}_{\nu(j)}^n\left(1 - a\mathbf{D}_{\nu(j)}^n\right) = \mathbf{D}_{\nu(j)}^n \frac{1 - a^2\left(\mathbf{D}_{\nu(j)}^n\right)^2}{1 + a\mathbf{D}_{\nu(j)}^n} \leq \frac{\mathbf{D}_{\nu(j)}^n}{1 + a\mathbf{D}_{\nu(j)}^n} = \frac{1}{\frac{1}{\mathbf{D}_{\nu(j)}^n} + a}.\end{aligned}$$

Corollary 4.2.6. *Let $n \in \mathbb{N}$. Then the scheme (4.2.4) – (4.2.6) verifies the following one-sided Lipschitz condition (OSLC):*

$$\mathbf{D}_j^{n+1} \leq \begin{cases} \dfrac{1}{(n+1)a} & \text{if} \quad j \leq j_{n+1} - 3 - n \\ \dfrac{1}{((j_{n+1}-2)-j)a} & \text{if} \quad j_{n+1} - 3 - n \leq j \leq j_{n+1} - 3 \\ \dfrac{1}{(j-(j_{n+1}+1))a} & \text{if} \quad j_{n+1} + 2 \leq j \leq j_{n+1} + 2 + n \\ \dfrac{1}{(n+1)a} & \text{if} \quad j \geq j_{n+1} + 2 + n. \end{cases} \tag{4.2.16}$$

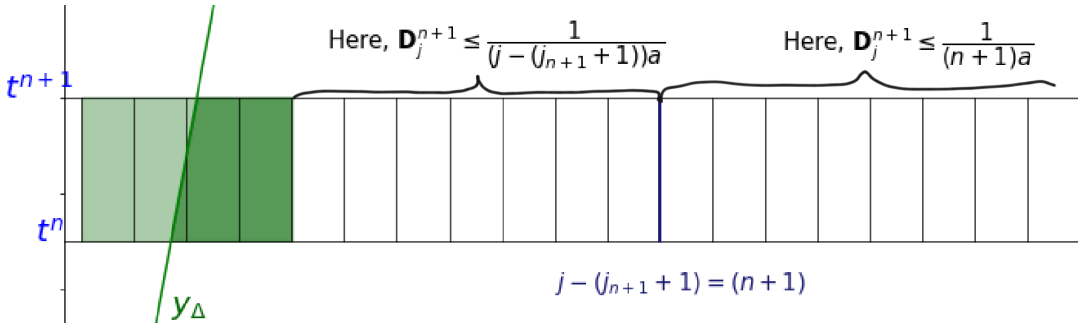

Figure 4.3 – Illustration of the OSL bound (4.2.16).

Proof. Fix $n \in \mathbb{N}$. We only prove (4.2.16) in the cases $j \ge j_{n+1}+2$. The reasoning for the cases $j \le j_0 - 3$ is very similar. Let us first prove by induction on $k \in \mathbb{N}^*$ that

$$\forall k \in \mathbb{N}^*,\ \forall j \in \mathbb{Z}, \quad \min\{n+1, j-(j_{n+1}+1)\} \ge k \implies \mathbf{D}_j^{n+1} \le \frac{1}{ka}. \tag{4.2.17}$$

Inequality (4.2.17) holds if $k = 1$. Indeed, if $k = 1$, then $j \ge j_{n+1}+2$ *i.e.* $j \in \widehat{\mathbb{Z}}_{n+1}$. By (4.2.15),

$$\exists \nu_j \in \{j-1, j, j+1\}, \quad \mathbf{D}_j^{n+1} \le \mathbf{D}_{\nu_j}^n - a\left(\mathbf{D}_{\nu_j}^n\right)^2.$$

If $\mathbf{D}_{\nu_j}^n = 0$, then $\mathbf{D}_j^{n+1} = 0 \le 1/a$. Otherwise, we can write:

$$\mathbf{D}_j^{n+1} \le \frac{1}{\frac{1}{\mathbf{D}_{\nu_j}^n} + a} \le \frac{1}{a} = \frac{1}{ka}.$$

Now, let us assume that (4.2.17) holds for some integer $k \in \mathbb{N}^*$ and suppose that $\min\{n+1, j-(j_{n+1}+1)\} \ge k+1$. Again, by (4.2.15),

$$\exists \nu_j \in \{j-1, j, j+1\}, \quad \mathbf{D}_j^{n+1} \le \mathbf{D}_{\nu_j}^n - a\left(\mathbf{D}_{\nu_j}^n\right)^2.$$

Since

$$n \ge k \quad \text{and} \quad \nu_j - (j_n+1) \ge (j-1) - (j_{n+1}+1) = j - (j_{n+1}+1) - 1 \ge k,$$

we deduce that $\min\{n, j-(j_n+1)\} \ge k$, hence, using the induction property:

$$\mathbf{D}_j^{n+1} \le \frac{1}{\frac{1}{\mathbf{D}_{\nu_j}^n} + a} \le \frac{1}{(k+1)a},$$

which concludes the induction argument. Estimates (4.2.16) in the cases $j \ge j_{n+1}+2$ follow for suitable choices of k in (4.2.17). □

Corollary 4.2.7 (Localized **BV** estimates)**.** *Fix $0 < \varepsilon < X$ and suppose that $3\Delta x \le \varepsilon$ and that $t^{n+1} \ge \dfrac{\varepsilon}{2\mathbf{L}}$. Then there exists a constant $\Lambda = \Lambda\left(\dfrac{1}{\varepsilon}, X\right)$, nondecreasing with respect to its arguments such that*

$$\mathbf{TV}\left(\rho_\Delta(\cdot, t^{n+1})_{|(y^{n+1}+\varepsilon, y^{n+1}+X)}\right) \le \Lambda \tag{4.2.18}$$

and

$$\int_{y^{n+1}+\varepsilon}^{y^{n+1}+X} \left|\rho_\Delta(x, t^{n+2}) - \rho_\Delta(x, t^{n+1})\right| \mathrm{d}x \le 2\Delta x + \mathbf{L}\,(2\Lambda + 1)\,\Delta t. \tag{4.2.19}$$

Note that we have the same bounds for the quantities:

$$\mathbf{TV}\left(\rho_\Delta(\cdot, t^{n+1})_{|(y^{n+1}-X, y^{n+1}-\varepsilon)}\right) \quad \text{and} \quad \int_{y^{n+1}-X}^{y^{n+1}-\varepsilon} \left|\rho_\Delta(x, t^{n+2}) - \rho_\Delta(x, t^{n+1})\right| \mathrm{d}x\,.$$

Proof. Let $k_{n+1}, J_{n+1} \in \mathbb{Z}$ such that $y^{n+1} + \varepsilon \in (\chi^{n+1}_{k_{n+1}}, \chi^{n+1}_{k_{n+1}} + \Delta x)$ and $y^{n+1} + X \in (\chi^{n+1}_{J_{n+1}}, \chi^{n+1}_{J_{n+1}} + \Delta x)$. We have:

$$\begin{aligned}
\mathbf{TV}(\rho_\Delta(\cdot, t^{n+1})_{|(y^{n+1}+\varepsilon, y^{n+1}+X)}) & \sum_{j=k_{n+1}+1}^{J_{n+1}} |\rho^{n+1}_{j+1/2} - \rho^{n+1}_{j-1/2}| \\
&= 2 \sum_{j=k_{n+1}+1}^{J_{n+1}} \mathbf{D}^{n+1}_j - \sum_{j=k_{n+1}+1}^{J_{n+1}} (\rho^{n+1}_{j+1/2} - \rho^{n+1}_{j-1/2}) \\
&= 2 \sum_{j=k_{n+1}+1}^{J_{n+1}} \mathbf{D}^{n+1}_j - (\rho^{n+1}_{J_{n+1}-1/2} - \rho^{n+1}_{k_{n+1}+1/2}) \le 1 + 2 \sum_{j=k_{n+1}+1}^{J_{n+1}} \mathbf{D}^{n+1}_j.
\end{aligned}$$

Now, for all $j \ge k_{n+1} + 1$, we have

$$\begin{aligned}
j - (j_{n+1} + 1) \ge \frac{(k_{n+1} + 1) - (j_{n+1} + 1))\Delta x}{\Delta x} &= \frac{(\chi^{n+1}_{k_{n+1}} + \Delta x) - \chi^{n+1}_{j_{n+1}}}{^\Delta x} \\
&\ge \frac{(y^{n+1} + \varepsilon) - (y^{n+1} + 2\Delta x)}{\Delta x} = \frac{\varepsilon}{\Delta x} - 2 \ge 1.
\end{aligned}$$

Lemma 4.2.16 ensures that

$$\mathbf{TV}(\rho_\Delta(\cdot, t^{n+1})_{|(y^{n+1}+\varepsilon, y^{n+1}+X)}) \le 1 + \frac{2}{a} \sum_{j=k_{n+1}+1}^{J_{n+1}} \frac{1}{\min\{n+1, j - (j_{n+1}+1)\}}.$$

However, we also have:

$$n+1 = \frac{t^{n+1}}{\Delta t} \ge \frac{\varepsilon}{2\mathbf{L}\Delta t} \ge \frac{\varepsilon}{\Delta x} = \frac{(y^{n+1} + \varepsilon) - y^{n+1}}{\Delta x} \ge \frac{\chi^{n+1}_{k_{n+1}} - (\chi^{n+1}_{j_{n+1}} + \Delta x)}{\Delta x} = k_{n+1} - (j_{n+1}+1).$$

We deduce that for all $j \in \{k_{n+1}+1, \dots, J_{n+1}\}$, $\min\{n+1, j-(j_{n+1}+1)\} \geq k_{n+1}-(j_{n+1}+1)$; hence:

$$\begin{aligned}\sum_{j=k_{n+1}+1}^{J_{n+1}} |\rho_{j+1/2}^{n+1} - \rho_{j-1/2}^{n+1}| &\leq 1 + \frac{2}{a} \times \left(\frac{J_{n+1} - k_{n+1}}{k_{n+1} - (j_{n+1}+1)} \right) \\ &\leq 1 + \frac{2}{a} \times \left(\frac{X - \varepsilon + \Delta x}{\varepsilon - 2\Delta x} \right) \\ &\leq \Lambda, \quad \Lambda := 1 + \frac{6X}{a\varepsilon},\end{aligned}$$

which is exactly (4.2.18). Then,

$$\begin{aligned}&\int_{y^{n+1}+\varepsilon}^{y^{n+1}+X} \left| \rho_\Delta(x, t^{n+2}) - \rho_\Delta(x, t^{n+1}) \right| \mathrm{d}x \\ &\leq 2\Delta x + \sum_{j=k_{n+1}+1}^{J_{n+1}} |\rho_{j+1/2}^{n+2} - \rho_{j+1/2}^{n+1}| \Delta x \\ &\leq 2\Delta x + \|f'\|_{\mathbf{L}^\infty} \left(\sum_{j=k_{n+1}+1}^{J_{n+1}} |\rho_{j+3/2}^{n+1} - \rho_{j+1/2}^{n+1}| + \sum_{j=k_{n+1}+1}^{J_{n+1}} |\rho_{j+1/2}^{n+1} - \rho_{j-1/2}^{n+1}| \right) \Delta t \\ &\leq 2\Delta x + \mathbf{L}\,(2\Lambda + 1)\,\Delta t,\end{aligned}$$

concluding the proof. □

Theorem 4.2.8. *Fix $\rho_0 \in \mathbf{L}^\infty(\mathbb{R}; [0,1])$, $y \in \mathbf{W}^{1,\infty}_{\mathbf{loc}}((0,+\infty))$, $\dot{y} \geq 0$ and $q \in \mathbf{L}^\infty_{\mathbf{loc}}((0,+\infty))$, $q \geq 0$. Suppose that $f \in \mathbf{C}^2([0,1])$ satisfies (4.0.1)-(4.2.14). Then as $\Delta \to 0$ while satisfying the CFL condition (4.2.2), $(\rho_\Delta)_\Delta$ converges a.e. on Ω to the admissible entropy solution to (4.0.3).*

Proof. Fix $n \in \mathbb{N}^*$. The uniform convergence of $(y_\Delta)_\Delta$ to y, coupled with the **BV** bounds (4.2.18)-(4.2.19) and the uniform $\mathbf{L}^\infty$ bound (4.2.8) provide (up to a subsequence) a.e. convergence for the sequence $(\rho_\Delta)_\Delta$ in any rectangular bounded domains of the open subset

$$O_n = \{(x,t) \in \Omega \mid |x - y(t)| > 1/n\},$$

see [98, Appendix A]. The a.e. convergence on any compact subsets of Ω_n follows by a classical covering argument. Then a diagonal procedure provides the a.e. convergence on any compact subsets of $O = \{(x,t) \in \Omega \mid x \neq y(t)\}$. A further extraction yields the a.e. convergence on Ω. Equipped with the convergence of $(\rho_\Delta)_\Delta$ to ρ, we let $\Delta \to 0$ in (4.2.11) and (4.2.13) to establish that ρ is an admissible entropy solution to (4.0.3). By uniqueness, the whole sequence converges to ρ, which proves the theorem. □

Corollary 4.2.9. *Fix $\rho_0 \in \mathbf{L}^\infty(\mathbb{R}; [0,1])$, $y \in \mathbf{W}^{1,\infty}_{\mathbf{loc}}((0,+\infty))$, $\dot{y} \geq 0$ and $q \in \mathbf{L}^\infty_{\mathbf{loc}}((0,+\infty))$, $q \geq 0$. Suppose that $f \in \mathbf{C}^2([0,1])$ satisfies (4.0.1)-(4.2.14). Then Problem (4.0.3) admits a unique admissible entropy solution.*

Proof. Existence comes from Theorem 4.2.8 while uniqueness was established by Theorem 4.1.8. □

4.3 Well-posedness for the multiple trajectory problem

We now get back to the original problem (4.0.2). Let us detail the organization of this section. First, we construct a partition of the unity to reduce the study of (4.0.2) to an assembling of several local studies of (4.0.3), see Section 4.3.1. Using the definition based on germs, analogous to Definition 4.1.4, we will prove a stability estimate, leading to uniqueness, see Theorem 4.3.3. Then in Section 4.3.3, we construct a finite volume scheme in which we fully use the precise study of Section 4.2. A special treatment of the crossing points is described, see Section 4.3.3.

Let us recall that we are given a finite (or more generally locally finite) family of trajectories and constraints $(y_i, q_i)_{i\in[\![1;J]\!]}$ defined on (s_i, T_i) $(0 \le s_i < T_i)$. Introduce the notations:

$$\forall i \in [\![1; J]\!], \quad \Gamma_i = \{(x,t) \in \overline{\Omega} \mid t \in [s_i, T_i] \text{ and } x = y_i(t)\}.$$

We suppose that for all $i \in [\![1; J]\!]$, $y_i \in \mathbf{W}^{1,\infty}((s_i, T_i))$ and $q_i \in \mathbf{L}^\infty((s_i, T_i); \mathbb{R}^+)$. This notation means that what can be seen as crossing points between interfaces will be considered as endpoints of the interfaces; for instance, given two crossing lines, we split them into four interfaces having a common endpoint. We denote by $(\mathcal{C}_m)_{1\le m\le M}$ the set of all endpoints of the interfaces Γ_i, $i \in [\![1; J]\!]$.

4.3.1 Reduction to a single interface

Fix $\varphi \in \mathbf{C}^\infty_{\mathbf{c}}(\overline{\Omega}\setminus \cup_{m=1}^M \mathcal{C}_m)$. Let us denote by K the compact support of φ.

Step 1. For all $i \in [\![1; J]\!]$, $K \cap \Gamma_i$ is a compact subset (maybe empty) of $\overline{\Omega}$, and the family $(K \cap \Gamma_i)_i$ is pairwise disjoint. By compactness,

$$\exists \delta > 0,\ \forall i, j \in [\![1; J]\!], \quad i \neq j \implies \operatorname{dist}(K \cap \Gamma_i, K \cap \Gamma_j) \ge 2\delta.$$

Step 2. For all $i \in [\![1; J]\!]$, set

$$\Omega_i = \bigcup_{(x,t)\in K\cap\Gamma_i} \mathbf{B}((x,t), \delta),$$

where $\mathbf{B}((x,t),\delta)$ denotes the $\mathbb{R}^2$-euclidean open ball centered on (x,t) and of radius δ. Clearly, Ω_i is an open subset of $\overline{\Omega}$ containing Γ_i. Moreover, the family $(\Omega_i)_i$ is pairwise disjoint. Indeed, suppose instead that for some $i, j \in [\![1; J]\!]$ $(i \neq j)$, we have

$$\Omega_i \cap \Omega_j \neq \emptyset,$$

and fix $(x,t) \in \Omega_i \cap \Omega_j$. By definition, there exists $(x_i, t_i) \in K \cap \Gamma_i$ and $(x_j, t_j) \in K \cap \Gamma_j$ such that

$$(x,t) \in \mathbf{B}((x_i,t_i),\delta) \cap \mathbf{B}((x_j,t_j),\delta).$$

Using the triangle inequality, we deduce that

$$\operatorname{dist}(K \cap \Gamma_i, K \cap \Gamma_j) \le \operatorname{dist}((x_i,t_i),(x_j,t_j)) \le \operatorname{dist}((x_i,t_i),(x,t)) + \operatorname{dist}((x,t),(x_j,t_j)) < 2\delta,$$

yielding the contradiction.

Step 3. Define the open subset (finite intersection of open subsets):

$$\Omega_0 = \left\{ (x,t) \in \overline{\Omega} \;\middle|\; \forall i \in [\![1;J]\!],\ \mathrm{dist}((x,t), K \cap \Gamma_i) \geq \frac{\delta}{2} \right\}.$$

The family $(\Omega_i)_{i\in[\![0;J]\!]}$ is an open cover of $\mathbb{R}\times\mathbb{R}^+$. Consequently, there exists a partition of the unity $(\theta_i)_{i\in[\![0;J]\!]}$ associated with this cover:

$$\forall i \in [\![0;J]\!], \quad \theta_i \geq 0;\ \theta_i \in \mathbf{C}^\infty_{\mathbf{c}}(\Omega_i); \quad \forall (x,t) \in \mathbb{R}\times\mathbb{R}^+,\ \sum_{i=0}^{J} \theta_i(x,t) = 1.$$

Step 4. We write the function φ in the following manner:

$$\varphi = \sum_{i=0}^{J} (\varphi\theta_i) = \varphi_0 + \sum_{i=1}^{J} \varphi_i. \tag{4.3.1}$$

Note that:

1. φ_0 vanishes along all the interfaces;
2. for all $i \in [\![1;J]\!]$, φ_i vanishes along all the interfaces but Γ_i.

4.3.2 Definition of solutions and uniqueness

Following Section 4.1 and Definition 4.1.4, we give the following definition of solution.

Definition 4.3.1. A function $\rho \in \mathbf{L}^\infty(\Omega;[0,1])$ is a $\mathcal{G}$-entropy solution to (4.0.2) with initial data $\rho_0 \in \mathbf{L}^\infty(\mathbb{R})$ if:
(i) for all test functions $\varphi \in \mathbf{C}^\infty_{\mathbf{c}}(\overline{\Omega}\setminus \cup_{i=1}^J \Gamma_i), \varphi \geq 0$ and $\kappa \in [0,1]$, the following entropy inequalities are verified:

$$\int_0^{+\infty}\int_{\mathbb{R}} \Big(|\rho-\kappa|\partial_t\varphi + \Phi(\rho,\kappa)\partial_x\varphi \Big)\,\mathrm{d}x\,\mathrm{d}t + \int_{\mathbb{R}} |\rho_0(x)-\kappa|\varphi(x,0)\,\mathrm{d}x \geq 0; \tag{4.3.2}$$

(ii) for all $i \in [\![1;J]\!]$ and for a.e. $t \in (s_i,T_i)$,

$$(\rho(y_i(t)-,t), \rho(y_i(t)+,t)) \in \mathcal{G}_{\dot{y}_i(t)}(q_i(t)), \tag{4.3.3}$$

where the admissibility germ $\mathcal{G}_{\dot{y}_i}(q_i)$ was defined in Definition 4.1.2.

Lemma 4.3.2 (Kato inequality)**.** *Fix $\rho_0, \sigma_0 \in \mathbf{L}^\infty(\mathbb{R};[0,1])$. Let $(q_i)_{i\in[\![1;J]\!]}$ and $(\widetilde{q}_i)_{i\in[\![1;J]\!]}$ be two family of constraints, where for all $i \in [\![1;J]\!]$, $q_i,\ \widetilde{q}_i \in \mathbf{L}^\infty((s_i,T_i))$. We denote by ρ (resp. σ) a $\mathcal{G}$-entropy solution to Problem (4.0.2) corresponding to initial data ρ_0 (resp. σ_0) and constraints $(q_i)_{i\in[\![1;J]\!]}$ (resp. $(\widetilde{q}_i)_{i\in[\![1;J]\!]}$). Then for all test functions $\varphi \in \mathbf{C}^\infty_{\mathbf{c}}(\overline{\Omega}), \varphi \geq 0$, we have*

$$\begin{aligned}
&\int_0^{+\infty}\int_{\mathbb{R}} \Big(|\rho-\sigma|\partial_t\varphi + \Phi(\rho,\sigma)\partial_x\varphi \Big)\,\mathrm{d}x\,\mathrm{d}t + \int_{\mathbb{R}} |\rho_0(x)-\sigma_0(x)|\varphi(x,0)\,\mathrm{d}x \\
&+ \sum_{i=1}^{J} \int_{s_i}^{T_i} \Big(\Phi_{\dot{y}_i(t)}\left(\rho(y_i(t)+,t), \sigma(y_i(t)+,t)\right) - \Phi_{\dot{y}_i(t)}\left(\rho(y_i(t)-,t), \sigma(y_i(t)-,t)\right) \Big)\varphi(y_i(t),t)\,\mathrm{d}t \geq 0.
\end{aligned} \tag{4.3.4}$$

Proof. We split the reasoning in two steps.
Step 1. Suppose first that $\varphi \in \mathbf{C}_{\mathbf{c}}^{\infty}(\overline{\Omega}\backslash \cup_{m=1}^{M} \mathcal{C}_m)$. In this case, we write φ using the partition of unity (4.3.1). Fix $i \in [\![1;J]\!]$. Following the computations of Lemma 4.1.7, we obtain:

$$\iint_{\Omega_i} \Big(|\rho - \sigma| \partial_t \varphi_i + \Phi(\rho,\sigma) \partial_x \varphi_i \Big) \,\mathrm{d}x \,\mathrm{d}t + \int_{\{x \in \mathbb{R} \,|\, (x,0) \in \Omega_i\}} |\rho_0(x) - \sigma_0(x)| \varphi_i(x,0) \,\mathrm{d}x$$
$$+ \int_{s_i}^{T_i} \Big(\Phi_{\dot{y}_i(t)} \left(\rho(y_i(t)+,t), \sigma(y_i(t)+,t)\right) - \Phi_{\dot{y}_i(t)} \left(\rho(y_i(t)-,t), \sigma(y_i(t)-,t)\right) \Big) \varphi_i(y_i(t),t) \,\mathrm{d}t \geq 0. \tag{4.3.5}$$

Now, since φ_0 vanishes along all the interfaces, standard computations lead to

$$\iint_{\Omega_0} \Big(|\rho - \sigma| \partial_t \varphi_0 + \Phi(\rho,\sigma) \partial_x \varphi_0 \Big) \,\mathrm{d}x \,\mathrm{d}t + \int_{\{x \in \mathbb{R} \,|\, (x,0) \in \Omega_0\}} |\rho_0(x) - \sigma_0(x)| \varphi_0(x,0) \,\mathrm{d}x \geq 0. \tag{4.3.6}$$

We now sum (4.3.5) ($i \in [\![1;J]\!]$) and (4.3.6) to obtain (4.3.4). This inequality is the analogous of (4.1.11).

Step 2. Consider now $\varphi \in \mathbf{C}_{\mathbf{c}}^{\infty}(\overline{\Omega})$. Fix $n \in \mathbb{N}^*$. From the first step, a classical approximation argument allows us to apply (4.3.4) with the Lipschitz test function

$$\psi_n(x,t) = \left(\sum_{m=1}^{M} \delta_{m,n}(x,t) \right) \varphi(x,t),$$

where for all $m \in [\![1;M]\!]$,

$$\delta_{m,n}(x,t) = \begin{cases} 0 & \text{if} \quad \mathrm{dist}_1((x,t),\mathcal{C}_m) < \dfrac{1}{n} \\ n \left(\mathrm{dist}_1((x,t),\mathcal{C}_m) - \dfrac{1}{n} \right) & \text{if} \quad \dfrac{1}{n} \leq \mathrm{dist}_1((x,t),\mathcal{C}_m) \leq \dfrac{2}{n} \\ 1 & \text{if} \quad \mathrm{dist}_1((x,t),\mathcal{C}_m) > \dfrac{2}{n}, \end{cases}$$

where, by analogy with the proof of Lemma 4.1.7, dist_1 denotes the $\mathbb{R}^2$ distance associated with the norm $\|\cdot\|_1$. We let $n \to +\infty$, keeping in mind that:

$$\left\| \left(\sum_{m=1}^{M} \delta_{m,n} \right) \varphi - \varphi \right\|_{\mathbf{L}^1(\Omega)} \underset{n \to +\infty}{\longrightarrow} 0; \quad \forall m \in [\![1;M]\!], \ \|\nabla \delta_{m,n}\|_{\mathbf{L}^1(\Omega)} = O\left(\frac{1}{n}\right).$$

Straightforward computations lead to (4.3.4) with $\varphi \in \mathbf{C}_{\mathbf{c}}^{\infty}(\overline{\Omega})$, concluding the proof. □

Theorem 4.3.3. *Fix $\rho_0, \sigma_0 \in \mathbf{L}^{\infty}(\mathbb{R};[0,1])$. Let $(q_i)_{i \in [\![1;J]\!]}$ and $(\widetilde{q}_i)_{i \in [\![1;J]\!]}$ be two family of constraints, where for all $i \in [\![1;J]\!]$, q_i, $\widetilde{q}_i \in \mathbf{L}^{\infty}((s_i,T_i))$. We denote by ρ (resp. σ) a $\mathcal{G}$-entropy solution to Problem (4.0.2) corresponding to initial data ρ_0 (resp. σ_0) and constraints $(q_i)_{i \in [\![1;J]\!]}$ (resp. $(\widetilde{q}_i)_{i \in [\![1;J]\!]}$). Then for all $T > 0$, we have*

$$\|\rho(\cdot,T) - \sigma(\cdot,T)\|_{\mathbf{L}^1} \leq \|\rho_0 - \sigma_0\|_{\mathbf{L}^1} + \sum_{i=1}^{J} 2 \int_{s_i}^{T_i} \left| q_i(t) - \widetilde{q}_i(t) \right| \mathrm{d}t \,. \tag{4.3.7}$$

In particular, Problem (4.0.2) admits at most one $\mathcal{G}$-entropy solution.

Proof. Estimate (4.3.7) follows from Kato inequality (4.3.4) with a suitable choice of test function and in light of the inequality:

$$\forall i \in [\![1;J]\!], \text{ for a.e. } t \in (s_i, T_i),$$

$$\Phi_{\dot{y}_i(t)}\left(\rho(y_i(t)+,t),\sigma(y_i(t)+,t)\right) - \Phi_{\dot{y}_i(t)}\left(\rho(y_i(t)-,t),\sigma(y_i(t)-,t)\right) \leq 2|q_i(t) - \widetilde{q}_i(t)|,$$

see Theorem 4.1.8. □

4.3.3 Proof of existence

Following the reasoning of Sections 4.1-4.2, we introduce a second definition of solutions, more suitable to prove existence.

Definition 4.3.4. A function $\rho \in \mathbf{L}^\infty(\Omega;[0,1])$ is an admissible entropy solution to (4.0.2) with initial data $\rho_0 \in \mathbf{L}^\infty(\mathbb{R})$ if
(i) for all test functions $\varphi \in \mathbf{C}^\infty_{\mathbf{c}}(\overline{\Omega}), \varphi \geq 0$ and $\kappa \in [0,1]$, the following entropy inequalities are verified:

$$\begin{aligned}\int_0^{+\infty} \int_{\mathbb{R}} \Big(|\rho - \kappa| \partial_t \varphi + \Phi(\rho,\kappa) \partial_x \varphi \Big) \,\mathrm{d}x \,\mathrm{d}t + \int_{\mathbb{R}} |\rho_0(x) - \kappa| \varphi(x,0) \,\mathrm{d}x \\ + \sum_{i=1}^{J} \int_{s_i}^{T_i} \mathcal{R}_{\dot{y}_i(t)}(\kappa, q_i(t)) \varphi(y_i(t),t) \,\mathrm{d}t \geq 0,\end{aligned} \tag{4.3.8}$$

where $\mathcal{R}_{\dot{y}_i}(\kappa, q_i)$ was defined in Definition 4.1.1;

(ii) for all test functions $\varphi \in \mathbf{C}^\infty_{\mathbf{c}}(\Omega \backslash \cup_{m=1}^M \mathcal{C}_m), \varphi \geq 0$, written under the form (4.3.1), the following constraint inequalities are verified for all $i \in [\![1;J]\!]$:

$$-\iint_{\Omega_i^+} \Big(\rho \partial_t \varphi + f(\rho) \partial_x \varphi \Big) \,\mathrm{d}x \,\mathrm{d}t \leq \int_{s_i}^{T_i} q_i(t) \varphi_i(y_i(t),t) \,\mathrm{d}t\,, \tag{4.3.9}$$

where $\Omega_i^+ = \{(x,t) \in \Omega_i \mid x > y_i(t)\}$.

Proposition 4.3.5. *Definition 4.3.1 and Definition 4.3.4 are equivalent. Moreover, in Definition 4.3.4 (i), it is equivalent that (4.3.8) holds with $\varphi \in \mathbf{C}^\infty_{\mathbf{c}}(\overline{\Omega} \backslash \cup_{m=1}^M \mathcal{C}_m)$.*

Proof. The proof of the equivalence of Definitions 4.3.1 and 4.3.4 is a straightforward adaptation of the proofs of Propositions 4.1.5-4.1.6. The last part of the statement follows using the same approximation argument described at the end of the proof of Lemma 4.3.2. □

We now turn to the proof of existence for admissible entropy solutions of (4.0.2). We make use of the precise study of Section 4.2 in the case of a single trajectory and build a finite volume scheme. We keep the notations of Section 4.2 when there is no ambiguity.

Construction of the mesh, definition of the scheme

For the sake of clarity, suppose that we only have two trajectories/constraints (y_i, q_i) $(1 \leq i \leq 2)$ defined on $[0, \tau]$, which cross at time τ. We denote by $\mathcal{C}$ this crossing point. Suppose also that this crossing point results in two additional trajectories/constraints (y_i, q_i) $(3 \leq i \leq 4)$ defined on $[\tau, T]$, and which do not cross, as represented in Figure 4.4.

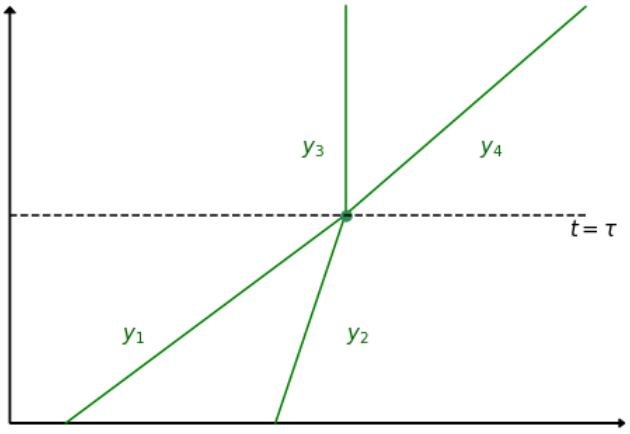

Figure 4.4 – Illustration of the configuration.

Let us fully make explicit the steps of the reasoning leading to the construction of our scheme in that situation. Suppose that $\lambda = \Delta t/\Delta x$ is fixed and verifies the CFL condition

$$2\left(\underbrace{\|f'\|_{\mathbf{L}^\infty} + \max_{1\leq i\leq 4} \|\dot{y}_i\|_{\mathbf{L}^\infty((0,T))}}_{\mathbf{L}}\right)\lambda \leq 1. \tag{4.3.10}$$

Set $N \in \mathbb{N}$ such that $\tau \in [t^N, t^{N+1})$. We divide the discussion in four parts.

Part 1. Introduce the number

$$N_1 = \inf\left\{n \in \mathbb{N}, \quad |y^1_\Delta(t^n) - y^2_\Delta(t^n)| \leq 4\Delta x\right\}.$$

The definition of N_1 ensures that for all $n \in \{0, \dots, N_1 - 1\}$, we can independently modify the mesh near the two trajectories y^1_Δ and y^2_Δ, as presented in Figure 4.5. Consequently, we can simply define the approximate solution ρ_Δ on $\mathbb{R} \times [0, t^{N_1-1}]$ as the finite volume approximation of a conservation law, with initial data ρ_0, with flux constraints on two non-interacting trajectories, using the recipe of Section 4.2 for each trajectory/constraint.

Part 2. Fix now $n \in \{N_1, \dots, N\}$. In these time intervals, since the two trajectories are too close to each other, one cannot modify the mesh in the neighbourhood of one of them without affecting the other. However, the scheme has to be defined globally so we proceed as described below.

— First, introduce the mean trajectory and the new constraint:

$$\forall t \in [0, \tau], \quad y_{12}(t) = \frac{y_1(t) + y_2(t)}{2}; \quad q_{12}(t) = \min\{q_1(t), q_2(t)\},$$

represented in purple in Figure 4.5, before the crossing point (in red). The choice of taking the minimal level of constraint in the definition of q_{12} stems from the nature of the constrained problem; see however Remark 4.3.1 below.

— Then, define ρ_Δ on $\mathbb{R} \times [t^{N_1}, t^N]$ as the finite volume approximation of the one trajectory/one constraint problem:

$$\begin{cases} \partial_t \rho + \partial_x \left(f(\rho) \right) = 0 \\ \rho(\cdot, t^{N_1}) = \rho_\Delta(\cdot, t^{N_1 - 1}) \\ \left(f(\rho) - \dot{y}_{12}(t) \rho \right)|_{x = y_{12}(t)} \leq q_{12}(t) \qquad t \in (t^{N_1}, t^N), \end{cases}$$

using exactly the recipe of Section 4.2.1.

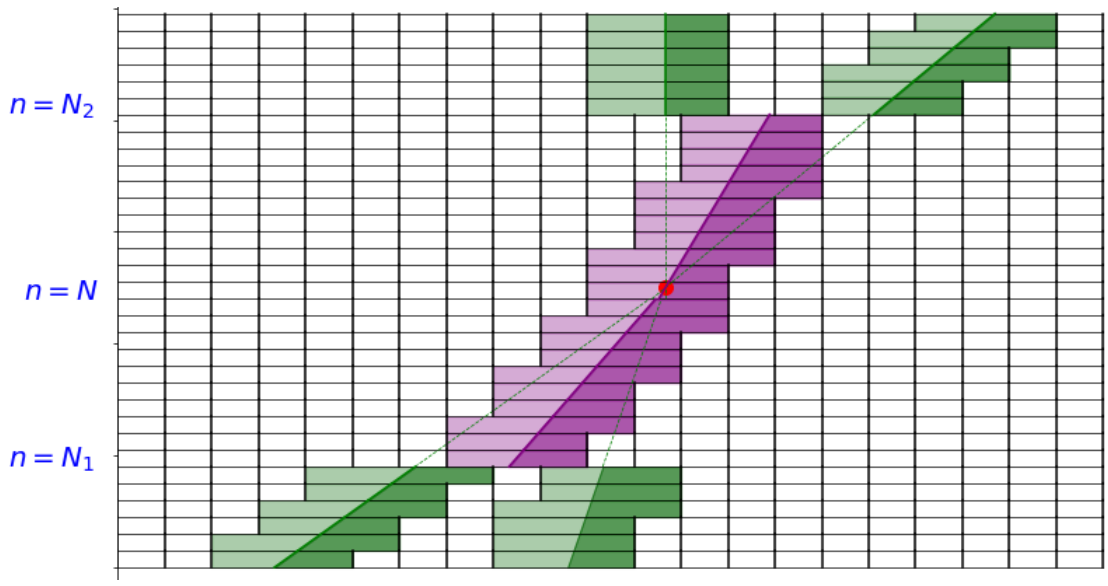

Figure 4.5 – Illustration of the local modifications of the mesh.

Part 3. Introduce the number:

$$N_2 = \inf \left\{ n > N, \quad |y^3_\Delta(t^n) - y^4_\Delta(t^n)| \geq 4\Delta x \right\}.$$

For $n \in \{N, \dots, N_2\}$, we are in the same situation as Part 2. We proceed to the same construction, *mutatis mutandis.*

— As in Part 2, define the mean trajectory and the new constraint:

$$\forall t \in [\tau, T], \quad y_{34}(t) = \frac{y_3(t) + y_4(t)}{2}; \quad q_{34}(t) = \min\{q_3(t), q_4(t)\},$$

represented in purple in Figure 4.5, after the crossing point.

— Define ρ_Δ on $\mathbb{R} \times [t^N, t^{N_2}]$ as the finite volume approximation of the one trajectory/one constraint problem:

$$\begin{cases} \partial_t \rho + \partial_x \left(f(\rho) \right) = 0 \\ \rho(\cdot, t^N) = \rho_\Delta(\cdot, t^N) \\ \left(f(\rho) - \dot{y}_{34}(t)\rho \right)\big|_{x = y_{34}(t)} \leq q_{34}(t) \qquad t \in (t^N, t^{N_2}). \end{cases}$$

Part 4. Finally, ρ_Δ is defined on $\mathbb{R} \times [t^{N_2}, T]$ like in Part 1 with $y_3, q_3, \rho_\Delta(\cdot, t^{N_2})$ (resp. y_4, q_4) playing the role of y_1, q_1, ρ_0 (resp. of y_2, q_2).

Remark 4.3.1. Let us stress out that the details of the treatment done in Parts 2-3 do not play any significant role in the convergence proof below thanks to the choice of test functions vanishing at neighbourhood of the crossing points, see Proposition 4.3.5. Consequently, taking the mean trajectory and the minimum of the constraint is merely an example aiming at preserving some consistency while keeping the scheme simple to understand and implement.

The general case of a finite number of interfaces (locally finite number can be easily included) is treated in the same way, leading to a pattern with the uniform rectangular mesh adapted to each of the interfaces Γ_i, $i \in [\![1; J]\!]$ except for small (in terms of the number of impacted mesh cells) neighbourhoods of the crossing points $\mathcal{C}_m$, $m \in [\![1; M]\!]$.

Proof of convergence

Theorem 4.3.6. *Fix $T > 0$, $f \in \mathbf{C}^2([0,1])$ satisfying (4.0.1)-(4.2.14) and $\rho_0 \in \mathbf{L}^\infty(\mathbb{R}; [0,1])$. Let $(y_i, q_i)_{i \in [\![1;J]\!]}$ be a finite family of trajectories and constraints defined on (s_i, T_i) ($0 \leq s_i < T_i$). We suppose that for all $i \in [\![1; J]\!]$, $y_i \in \mathbf{W}^{1,\infty}((s_i, T_i))$ and $q_i \in \mathbf{L}^\infty((s_i, T_i); \mathbb{R}^+)$. Suppose also that the interfaces $(\Gamma_i)_i$ defined by the trajectories $(y_i)_i$ have a finite number of crossing points. Then as $\Delta \to 0$ while satisfying the CFL condition*

$$2 \left(\underbrace{\|f'\|_{\mathbf{L}^\infty} + \max_{1 \leq i \leq J} \|\dot{y}_i\|_{\mathbf{L}^\infty((0,T))}}_{\mathbf{L}} \right) \lambda \leq 1,$$

the sequence $(\rho_\Delta)_\Delta$ constructed by the procedure of Section 4.3.3 converges a.e. on Ω to the admissible entropy solution to (4.0.2).

Proof. We make use of the fact that in Definition 4.3.4, we only need to consider test functions that vanish at a neighbourhood of the crossing points (this is the key observation leading to Remark 4.3.1 hereabove).

(i) Proof of the entropy inequalities. Fix $\varphi \in \mathbf{C}_{\mathbf{c}}^{\infty}(\overline{\Omega}\backslash\cup_{m=1}^{M}\mathcal{C}_m), \varphi \geq 0$, written as $\varphi = \varphi_0 + \sum_{i=1}^{J}\varphi_i$, using the appropriate partition of unity, see Section 4.3.1. Since φ_0 vanishes along all the interfaces, ρ_Δ verifies inequality (4.2.11) with $\mathcal{R} \equiv 0$ on the domain Ω_0 and with test function φ_0. Indeed, for a sufficiently small $\Delta x > 0$, the scheme we constructed in the previous section reduces to a standard finite volume in Ω_0. Fix now $i \in [\![1; J]\!]$. Since φ_i vanishes along all the interfaces but Γ_i, ρ_Δ verifies inequality (4.2.11) with reminder term $\mathcal{R}_{s_\Delta^i}(\kappa, q_\Delta^i)$ along the trajectory y_Δ^i on the domain Ω_i and with test function φ_i, due to the analysis of Section 4.2; indeed, in the support of the test function, our scheme for the multi-interface problem reduces to the scheme for the single-interface problem. By summing these previous inequalities, we obtain an approximate version of (4.3.8) verified by ρ_Δ:

$$\begin{aligned}\int_0^{+\infty}\int_{\mathbb{R}}\Big(|\rho_\Delta - \kappa|\partial_t\varphi + \Phi_\Delta(\rho_\Delta,\kappa)\partial_x\varphi\Big)\,\mathrm{d}x\,\mathrm{d}t + \int_{\mathbb{R}}|\rho_\Delta^0(x) - \kappa|\varphi(x,0)\,\mathrm{d}x \\ + \sum_{i=1}^{J}\int_{s_i}^{T_i}\mathcal{R}_{s_\Delta^i(t)}(\kappa, q_\Delta^i(t))\varphi(y_\Delta^i(t),t)\,\mathrm{d}t \geq O(\Delta x) + O(\Delta t)\,.\end{aligned} \tag{4.3.11}$$

(ii) Proof of the weak constraint inequalities. Let $\varphi \in \mathbf{C}_{\mathbf{c}}^{\infty}(\Omega\backslash\cup_{m=1}^{M}\mathcal{C}_m), \varphi \geq 0$, written under the form (4.3.1). Fix $i \in [\![1; J]\!]$. Since φ_i vanishes along all the interfaces but Γ_i, for a sufficiently small Δx, ρ_Δ verifies inequality (4.2.13) with constraint q_Δ^i along the trajectory y_Δ^i on the domain Ω_i^+ and with test function φ_i. We obtain an approximate version of (4.3.12) verified by ρ_Δ:

$$-\iint_{\Omega_i^+}\Big(\rho_\Delta\partial_t\varphi + \mathbf{F}_\Delta(\rho_\Delta)\partial_x\varphi\Big)\,\mathrm{d}x\,\mathrm{d}t \leq \int_{s_i}^{T_i} q_\Delta^i(t)\varphi_i(y_\Delta^i(t),t)\,\mathrm{d}t + O(\Delta x) + O(\Delta t)\,. \tag{4.3.12}$$

(iii) Compactness and convergence. Compactness of the sequence $(\rho_\Delta)_\Delta$ follows directly from the study of Section 4.2.4 where we derived local $\mathbf{BV}$ bounds for $(\rho_\Delta)_\Delta$ under the assumption (4.2.14). Indeed, these local bounds lead to compactness in the domain complementary to the interfaces, we only use the fact that the interfaces together with the crossing points form a closed subset of Ω with zero Lebesgue measure. Once the a.e. convergence (up to a subsequence) on Ω to some $\rho \in \mathbf{L}^\infty(\Omega;[0,1])$ obtained, we simply pass to the limit in (4.3.11)-(4.3.12). This proves that ρ is an admissible solution to (4.0.2). By the uniqueness of Theorem 4.3.3, the whole sequence converges to ρ. This concludes the proof. □

Corollary 4.3.7. *Fix $T > 0$, $f \in \mathbf{C}^2([0,1])$ satisfying (4.0.1)-(4.2.14) and $\rho_0 \in \mathbf{L}^\infty(\mathbb{R};[0,1])$. Let $(y_i, q_i)_{i\in[\![1;J]\!]}$ be a finite family of trajectories and constraints defined on (s_i, T_i) $(0 \leq s_i < T_i)$. We suppose that for all $i \in [\![1; J]\!]$, $y_i \in \mathbf{W}^{1,\infty}((s_i,T_i))$ and $q_i \in \mathbf{L}^\infty((s_i,T_i);\mathbb{R}^+)$. Finally, suppose that the interfaces $(\Gamma_i)_i$ defined by the trajectories $(y_i)_i$ have a finite number of crossing points. Then Problem (4.0.2) admits a unique admissible entropy solution.*

Proof. Existence comes from Theorem 4.3.6 while uniqueness was established by Theorem 4.3.3. □

4.4 Numerical experiment with crossing trajectories

In this section, we perform a numerical test to illustrate the scheme analyzed in Section 4.2 and Section 4.3.3. We take the GNL flux $f(\rho) = \rho(1-\rho)$.

We model the following situation. A vehicle breaks down on a road and reduces by half the surrounding traffic flow, which initial state is given by $\rho_0 = 0.8 \times \mathbb{1}_{[1,3]}$. At some point, a tow truck comes to move the immobile vehicle. We summarized this situation in Figure 4.6. Notice the time interval in which $q_3 \equiv 0.1$. This corresponds to the time needed for the tow truck to move the vehicle. Remark also that the value of the constraint on this time interval is smaller than the one when only the broken down vehicle was reducing the traffic flow.

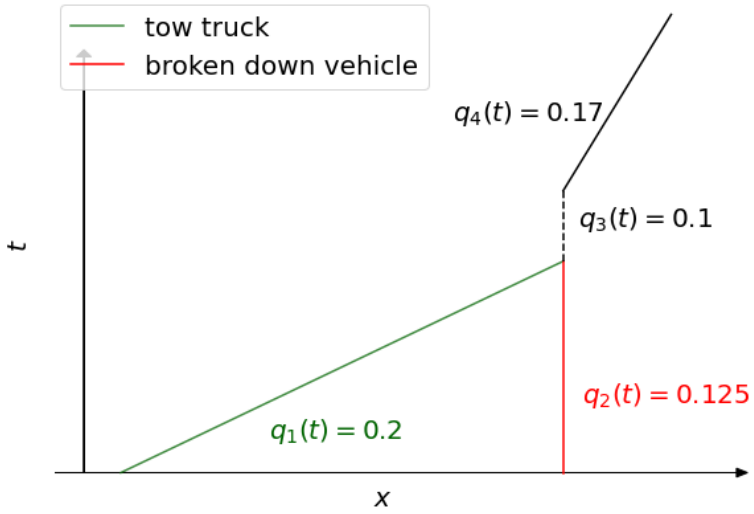

Figure 4.6 – A tow truck comes moving an immobile vehicle.

The evolution of the numerical solution is represented in Figure 4.7. Let us comment on the profile of the numerical solution.

- At first ($0 \leq t \leq 5.80$), the solution is composed of traveling waves separated by a stationary nonclassical shock located at the immobile vehicle position.
- When the tow truck catches up with the vehicle ($6.30 \leq t \leq 8.0$), the profile of the numerical solution is the same, but the greater value of the constraint in this time interval changes the magnitude of the nonclassical shock; at this point the combined presence of both the tow truck and the immobile vehicle clogs the traffic flow even more.
- Finally, once the tow truck starts again ($t > 8.0$), the traffic congestion is reduced.

Notice at time $t = 7.44$ the small artefact (circled in red in Figure 4.7) created by Parts 2-3 in the construction of the approximate solution and reproduced by the scheme. This highlights the fact that even if the treatment of the crossing points brings inconsistencies or artefacts to the numerical solution, these undesired effects are not amplified by the scheme, and become negligible when one refines the mesh.

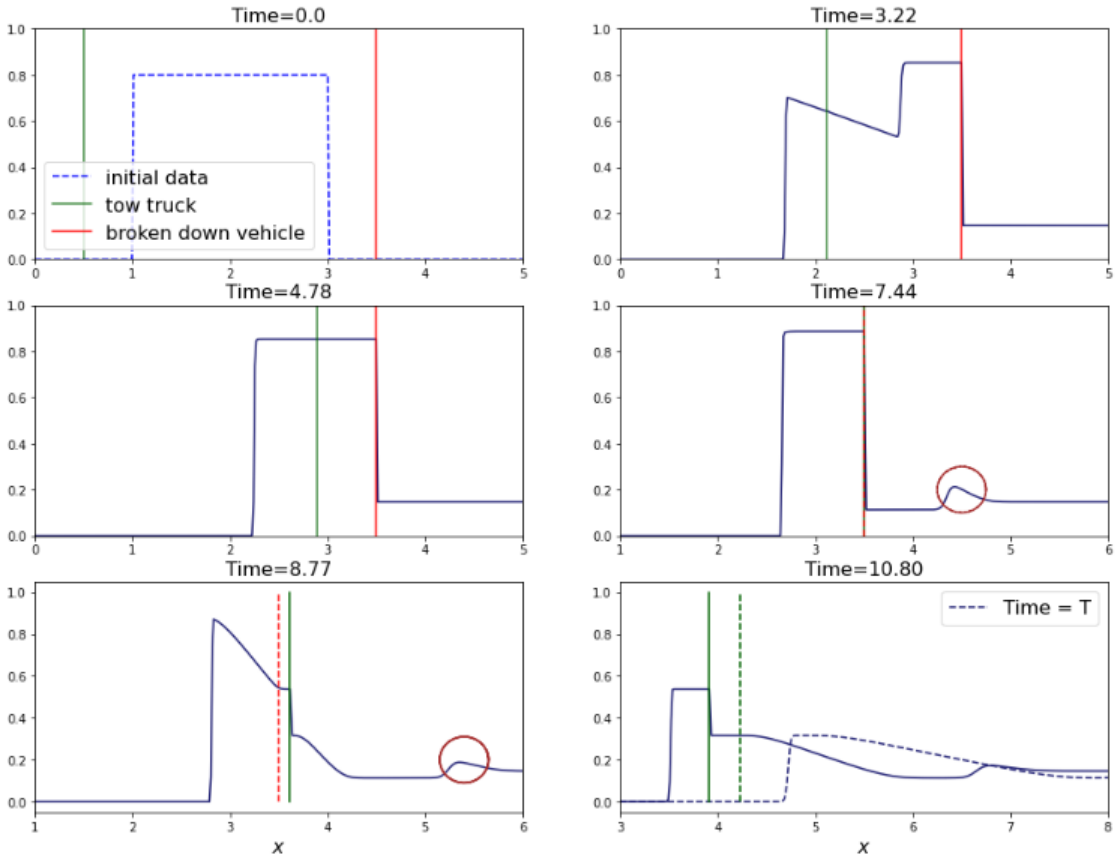

Figure 4.7 – The numerical solution at different fixed times; for an animated evolution of the solution, follow: https://utbox.univ-tours.fr/s/Zn6oF3ts4B2smAH

CHAPTER 5

TOWARDS WELL-POSEDNESS OF (X,T)-DISCONTINUOUS FLUX CONSERVATION LAWS UNDER ABSTRACT COUPLING CONDITIONS AT INTERFACES

5.1 Introduction

In the domain $\Omega = \mathbb{R} \times (0, +\infty)$ consider the formal Cauchy problem

$$\begin{cases} \partial_t \rho(x,t) + \partial_x \left(\mathbf{f}(x,t,\rho(x,t)) \right) &= 0 \quad \text{for } (x,t) \in \Omega \\ \rho(\cdot,0) &= \rho_0 \qquad\qquad \text{on } \mathbb{R} \end{cases} \tag{5.1.1}$$

where $\rho_0 \in \mathbf{L}^\infty(\mathbb{R})$ and $\mathbf{f} : \Omega \times \mathbb{R} \mapsto \mathbb{R}$ is a Carathéodory function, as specified below. The physical range of values of the state variable may be restricted to a subset $[a,b] \subset \mathbb{R}$ in many applications.

In the case where $\mathbf{f}$ extends to a locally Lipschitz continuous on $\overline{\Omega} \times \mathbb{R}$ function, and under some restrictions on the growth of $\mathbf{f}$ in ρ to ensure global in time existence (see in particular Part III), the definition of entropy solutions in the sense of Kruzhkov [114] is the cornerstone of the well-posedness theory. Discontinuous-flux problems and related problems of conservation laws with embedded interfaces appear in applications such as sedimentation, porous media, road traffic, *etc*... By "discontinuous-flux" we mean only problems where the flux is continuous (and even locally Lipschitz or even $\mathbf{C}^1$) in the state variable ρ while it may present discontinuities in the (x,t)-plane. Sense can be given to (5.1.1) (in particular, a weak formulation can be written) in the general situation where $\mathbf{f}$ is a Carathéodory function (measurable in (x,t), continuous in ρ); to the best of the authors' knowledge, no consistent theory has been constructed yet in this very general setting. The most general setting where

partial existence and uniqueness results were established concerns fluxes $\mathbf{f}$ that are at least $\mathbf{BV_{loc}}$ in (x,t), for all $\rho \in \mathbb{R}$, see in particular [107, 136]. To the authors' knowledge, the existence results under the $\mathbf{BV}$ kind assumption on the flux concern exclusively solutions constructed by the classical vanishing viscosity strategy going back to Rayleigh, Hopf and Kruzhkov. The early concepts of admissibility such as the minimal jump condition ([94]) or the Γ-condition ([75], see also [74]) at interfaces select the vanishing viscosity solution, *cf.* [15]; they were elaborated in the context of sedimentation applications. However, vanishing viscosity limits are not always appropriate from the modeling perspective. Optimal entropy solutions (*i.e.* solutions which maximize the flow across interfaces) were a second class of solutions identified as relevant in applications in porous media, see [104, 2, 42]; they also appear in the context of road traffic with varying road conditions (see [20] and references therein). It has been realized that the optimal solutions may be different from the vanishing viscosity limits (see, in particular, [107, Sect. 7]). More generally, co-existence of infinitely many different kinds of solutions, equally consistent from the purely mathematical standpoint, was explicitly pointed out in [2]. Later on, the different solution notions ((A,B)-connections) put forward in [2] were linked to different vanishing capillarity limits in the porous medium context [43, 6]. Moreover, fully analogous situation arose from modeling of road traffic by a classical, continuous-flux Lighthill-Whitham-Richards equation with point constraints on the flux ([56, 14]) where different solvers stem from different levels of constraint. Thus, different notions of solution correspond to different modeling assumptions at the interface, quite analogously to what happens when one prescribes different boundary conditions to a given PDE (we refer to [5] for the viewpoint of "Interface Coupling Conditions" and highlighted analogies with nonlinear boundary conditions for scalar conservation laws [21]). For this reason, writing (5.1.1) is formal, even having in mind a Kruzhkov-like entropy formulation: indeed, one needs to specify the expected (expected, given the underlying modeling context) behavior of solutions at interfaces which are the jump sets of $(x,t) \mapsto \mathbf{f}(x,t,\rho)$. From this perspective, these jump sets should be common for all values of ρ. A rather general set of assumptions on $\mathbf{f}$ that allows to interpret (5.1.1) as a discontinuous-flux conservation law is elaborated in [65]. Coupling at interfaces and the associated uniqueness analysis for the reference setting of [65] (see also [107, 18] for simpler but still rather complex variants) are mimicked from the model case, which we now discuss.

The model problem (5.1.1) features the flux function $\mathbf{f}$ discontinuous with respect to the space variable across the interface $\{x=0\}$. The expression for such flux reads as follows:

$$\mathbf{f}(x,t,\rho) = \begin{cases} f(\rho) & \text{if } x<0 \\ g(\rho) & \text{if } x>0. \end{cases} \tag{5.1.2}$$

Problem (5.1.1),(5.1.2) has been the main playground for understanding the issue of admissibility of solutions and of their uniqueness (see [24, 137] for a different line of research on this topic which applies to particular flux configurations, and where discontinuities need not to arrange along interfaces). General structure of interface coupling leading to $\mathbf{L}^1$-contractive solution semigroup for (5.1.1), (5.1.2) has been described in [15] in terms of "$\mathbf{L}^1 D$ germs" recalled below. It gave a common framework to a number of uniqueness arguments developed in the literature ([94, 75, 151, 1, 107, 2, 41]). It has been exploited for the sake of uniqueness and stability analysis of road traffic models with point constraints [14] and of a

non-conservative fluid-particle interaction model [17]. The very general uniqueness result of [65] highlights the fact that abstract "germ formulations" readily lead to uniqueness of the associated solutions, far beyond the model problem case (in this respect, let us underline that [65] deals with the multi-dimensional analogue of (5.1.1)). For the sake of completeness, let us point out a very different uniqueness result of [34], which requires much weaker than **BV** regularity assumptions on the (x,t)-dependence of $\mathbf{f}$ but is only applicable to vanishing viscosity solutions.

In what concerns existence for (5.1.1) - for a given choice of interface coupling conditions, as highlighted above, - the situation is far less explored. We refer in particular to [65, Remark 2.10]. Typically, to prove existence for (5.1.1) having in mind a specific interface coupling, it is required to construct sequences of approximate solutions, pass to the limit using the appropriate compactness structures, and obtain at the limit entropy formulations encoding, in particular, the coupling expected at the interfaces. To the best of the authors' knowledge only one situation was explored systematically beyond the model flux (5.1.2) case: this is the vanishing viscosity interface coupling. The existence results of [151, 106, 107, 54, 16, 18, 108, 152, 34] are based either on a vanishing viscosity approximation (which can be a very tricky one, see [18]) or on a numerical finite volume approximation which enforces, at the numerical level, the continuity of the state variable ρ at interfaces. Actually, the notion of vanishing viscosity solution corresponds to an implicit assumption of continuity - up to an interface layer, like for the case of Bardos-LeRoux-Nédélec boundary-value problems - of solution ρ at interfaces ([18, 5, 7]), and such continuity is explicitly or implicitly incorporated into the above mentioned approximation schemes. Another natural construction procedure which is the smoothing of the discontinuities of $\mathbf{f}$ (see, *e.g.*, [26] in the model case, see also [147] for a more elaborate situation coming from vanishing viscosity approximation of a triangular system of conservation laws): it may produce relevant solutions in particular situations, but it cannot be used to produce solutions for any kind of coupling. The adapted viscosity procedure of [15] was a rather artificial attempt to produce solutions with more general interface coupling, but it has not been extended beyond the model case. Note in passing that the above viscosity, smoothing and discretization methods are applicable for multi-dimensional generalization of (5.1.1), and some of the above references deal with multiple space dimensions. In one space dimension, wave-front tracking approximations were used for constructing solutions ([39] for the vanishing viscosity case, [90] for the general setting fully comparable to [15]) but beyond the model situations, their use for problems of the kind (5.1.1) was mainly restricted to road traffic models in concrete situations. Now, road traffic with point flux limitations at interfaces is another context where solutions to (5.1.1) were constructed for slanted, curved and possibly crossing interfaces either through Wave-Front Tracking or through Finite Volume approximation. The latter is addressed in detail in Chapter 4; we refer to its introduction for a set of references to related works.

The goal of this Chapter is to provide a systematic Finite Volume approach to construction of solutions to (5.1.1) with piecewise $\mathbf{C}^1$ dependence on (x,t) and with general interface coupling. To this end, we exploit the constructions put forward in Chapter 4 (the definition of the scheme, the treatment of interfaces and of interface crossings). Naturally, the numerical scheme consistent with the desired interface coupling should use specific fluxes at the locations of the interfaces; for the sake of maximal generality, we use the Godunov flux associated with

the underlying germ (*cf.* [15]). Since the germ depends on the inclination $\dot{y}(t)$ of the interface $x = y(t)$ and the interfaces are approximated, we require a mild restriction on the family $\mathcal{G}_{\dot{y}(t)}$ prescribing the interface coupling. The essential tool of our analysis is the adapted entropy formulation incorporating remainder terms, as suggested in [15]. The cornerstone of our contribution is the identification of the suitable form of the remainder term, compatible with the Godunov numerical approximation at interfaces.

The Chapter is organized as follows. In Section 5.2 we make precise the assumptions on $\mathbf{f}$ in (5.1.1) and on the coupling enforced on each of the interfaces associated with the (x,t)-discontinuity of $\mathbf{f}$. We state the definition of solution for which uniqueness follows from [65] (but we also rely upon the analysis of Chapter 4 for a technically simpler proof, since our geometrical assumptions on the structure of interfaces are much stronger that those of [65]), and reformulate the notion of solution in terms of adapted entropy inequalities. At this point, a specific Carathéodory structure of the family of interface couplings at hand comes into play. We illustrate the resulting setting with an example (another example being explored in detail in Chapter 4). To conclude Section 5.2, we state the main result of existence *via* convergence of a Finite volume scheme that is described in subsequent sections. We also provide comments about the compactness assumptions we take and on the convergence of other approximation procedures. In Section 5.3, we coin the key tools of our study in the model case (5.1.2) with t-dependent coupling at the interface $\{(x,t) \,:\, x = 0, t \in (0,+\infty)\}$. We link our adapted entropy formulation to the Godunov numerical approach at the interface, formulate the numerical scheme and sketch the compactness and convergence analysis, treating in detail the interface terms. In Section 5.4 we briefly develop the adaptations needed in the scheme to take into account slanted interfaces and interface crossings, and perform the reduction of the general case with multiple, possibly crossing interfaces to the case on an isolated interface; we then call upon the convergence analysis of the model case, showing that it also applies to the general setting. Finally, in Section 5.5 we present some conclusions and comment on extensions of our existence result to weaker genuine nonlinearity assumptions on $\mathbf{f}$ and to the multi-dimensional case.

5.2 Flux and interface coupling structure. Notion of solution, uniqueness and existence result

For the sake of clarity and in order to avoid technical details related to t and x dependence of $\mathbf{f}$ in regions between interfaces (note that Chapter 3 and Part III for the relevant adaptations), we will assume that the space-time heterogeneity of the flux is reduced to the presence of sharp interfaces; in simple words, we assume that $\mathbf{f}$ is piecewise constant with respect to (x,t). We refer to Remark 5.2.1 and Remark 5.2.4 for a discussion on feasible generalizations to flux heterogeneous between interfaces.

5.2.1 Piecewise constant flux

We assume we are given a finite family of interfaces $(y_i)_{i\in[\![1;J]\!]}$ defined on (s_i,T_i) $(0\le s_i<T_i\le+\infty)$. Introduce the notations:

$$\forall i\in[\![1;J]\!],\quad \Gamma_i=\{(x,t)\in\overline{\Omega}\mid t\in(s_i,T_i)\text{ and }x=y_i(t)\}. \tag{5.2.1}$$

Note that the extension to locally finite number of interfaces in straightforward. We suppose that for all $i\in[\![1;J]\!]$, $y_i\in\mathbf{W}^{1,\infty}((s_i,T_i))$. As explained in Section 4.3, this notation means that what can be seen as crossing points between interfaces will be considered as endpoints of the interfaces. We denote by $(\mathcal{C}_m)_{1\le m\le M}$ the set of all endpoints of the interfaces Γ_i, $i\in[\![1;J]\!]$.
As suggested hereabove, we assume that in each of the regions of the (x,t)-plane delimited by the interfaces, $\mathbf{f}$ depends on ρ only (*i.e.* the flux is homogeneous in each such region).

5.2.2 Germs and Godunov fluxes for interface coupling

Further, to each interface we attach a two-parameter family of subsets of $\mathbb{R}^2$ denoted by $\mathcal{G}_s^i(t)$, $t\in(s_i,T_i)$, $s\in\mathbb{R}$. In the terminology of [15, 5] recalled in Section 5.3 below, $\mathcal{G}_s^i(t)$ is assumed to be a complete $\mathbf{L}^1D$ germ for the couple of fluxes

$$g_s^i(\rho)=\mathbf{f}(t,y_i(t)-,\rho)-s\rho,\quad f_s^i(\rho)=\mathbf{f}(t,y_i(t)+,\rho)-s\rho\ \ (\forall t\in(s_i,T_i)); \tag{5.2.2}$$

note that due to the piecewise constant assumption on $\mathbf{f}$, the expressions of g_s^i, f_s^i are actually t-independent. We will need $g_{\dot y(t)}^i$, $f_{\dot y(t)}^i$ and $\mathcal{G}_{\dot y(t)}^i(t)$ to define the interface coupling conditions associated with the interfaces in the formal problem (5.1.1), see Definition 5.2.2 below; but we exploit $\mathcal{G}_s^i(t)$, for s in a vicinity of $\dot y(t)$, in order to construct approximate solutions *via* a numerical scheme. The assumption we impose on the family $\{\mathcal{G}_s^i(t)\}_{t\in(s_i,T_i),s\in\mathbb{R}}$ is a Carathéodory-kind assumption: the family should be continuous in s and measurable in t. To this end, we need to define a topology on the set of $\mathbf{L}^1D$ germs; the one we take is inferred from our numerical approach and it is described in terms of the Godunov flux associated to the Riemann solver determined by the germ at hand. Note that definitions of a neighbourhood of a germ and associated measurability properties were elaborated and discussed in [15, 5], but we take a different (more practical) viewpoint here. For given $s\in\mathbb{R}$ and a fixed $t_0\in(s_i,T_i)$, consider the flux (5.1.2) with $g=g_s^i$, $f=f_s^i$, with interface coupling prescribed (in the sense of [15], see also Section 5.3 below) by the maximal $\mathbf{L}^1D$ germ $\mathcal{G}_s^i(t_0)$ and with Riemann initial data

$$\rho_0(x)=\begin{cases}\kappa_L & \text{if } x<0\\ \kappa_R & \text{if } x>0.\end{cases}$$

This problem admits a unique solution which, we denote $\mathcal{RS}_s^i(\kappa_L,\kappa_R,t_0)$. Since a maximal $\mathbf{L}^1D$ germ is also complete (see [15, 5]), this solution is self-similar, therefore the Godunov flux

$$\mathbf{F}_s^{i,int}(\cdot,\cdot,t_0):(\kappa_L,\kappa_R)\ \mapsto\ g_s^i\big(\mathcal{RS}_s^i(\kappa_L,\kappa_R,t_0)\big)|_{x=0^-}\equiv f_s^i\big(\mathcal{RS}_s^i(\kappa_L,\kappa_R,t_0)\big)|_{x=0^+} \tag{5.2.3}$$

is well defined (in the right-hand side, we have a constant in t function that we see as a real value; in other words, we have $\mathbf{F}_s^{i,int}(\cdot,\cdot,t_0) : \mathbb{R}^2 \to \mathbb{R}$). Note that we could also use the formalism of [90], which directly prescribes the interface coupling in terms of a Riemann solver defined at the interface.
We are now in a position to define the Carathéodory structure on families of germs used in this chapter.

Definition 5.2.1. Given $i \in [\![1; J]\!]$ and a family $\{\mathcal{G}_s^i(t)\}_{t\in(s_i,T_i),s\in\mathbb{R}}$ of maximal $\mathbf{L}^1 D$ germs associated with fluxes g_s^i, f_s^i, we say that the family is Carathéodory (measurable in $t \in (s_i, T_i)$, continuous in $s \in \mathbb{R}$) if for every $(\kappa_L, \kappa_R) \in \mathbb{R}^2$ the associated function

$$\mathbb{R} \times (s_i, T_i) \to \mathbb{R},\ (s,t) \mapsto \mathbf{F}_s^{i,int}(\kappa_L, \kappa_R, t)$$

is a Carathéodory function.

Recall that s will stand for $\dot{y}(t)$ (the slope of the interface) or for its approximations, while t replaces the fixed value t_0 in the definition of the Godunov fluxes $\mathbf{F}_s^{i,int}(\cdot,\cdot,t)$ associated with the family of germs.

5.2.3 Notion of solution and uniqueness

We are now in a position to define solutions; the definition readily leads to uniqueness. For $\kappa \in \mathbb{R}$, denote by $\Phi(x,t,\rho,\kappa)$ the Kruzhkov entropy flux corresponding to $\mathbf{f}(x,t,\rho)$, *i.e.*

$$\Phi(x,t,\rho,\kappa) = \operatorname{sign}(\rho-\kappa)(\mathbf{f}(x,t,\rho) - \mathbf{f}(x,t,\kappa)).$$

Definition 5.2.2. Consider a piecewise constant flux $\mathbf{f} : \Omega \times \mathbb{R} \to \mathbb{R}$ with a set of interfaces of the form (5.2.1). Assume that for each $i \in [\![1; J]\!]$ we are given a family $\{\mathcal{G}_s^i(t)\}_{t\in(s_i,T_i),s\in\mathbb{R}}$ of maximal $\mathbf{L}^1 D$ germs associated with fluxes g_s^i, f_s^i in (5.2.2); assume that this family is Carathéodory, in the sense of Definition 5.2.1.
Consider a function $\rho \in \mathbf{L}^\infty(\Omega)$ such that ρ has strong one-sided traces in the sense of [154, 133] on interfaces Γ_i; we denote them $\gamma_L^i\rho, \gamma_R^i\rho : (s_i, T_i) \to \mathbb{R}$.
We say that such a function ρ is a $\mathcal{G}_s^i(t)$-entropy solution to (5.1.1) with initial data $\rho_0 \in \mathbf{L}^\infty(\mathbb{R})$ if for all test functions $\varphi \in \mathbf{C}_\mathbf{c}^\infty(\overline{\Omega} \setminus \cup_{i=1}^J \Gamma_i)$, $\varphi \geq 0$ and for all $\kappa \in \mathbb{R}$, the classical Kruzhkov entropy inequalities are satisfied:

$$\int_0^{+\infty} \int_{\mathbb{R}} \Big(|\rho-\kappa| \partial_t \varphi + \Phi(x,t,\rho,\kappa) \partial_x \varphi \Big) \,\mathrm{d}x\,\mathrm{d}t + \int_{\mathbb{R}} |\rho_0(x) - \kappa| \varphi(x,0)\,\mathrm{d}x \geq 0 \tag{5.2.4}$$

and moreover, for all $i \in [\![1; J]\!]$, for a.e. $t \in (s_i, T_i)$ there holds

$$\Big((\gamma_L^i \rho)(t)\,,\, (\gamma_R^i \rho)(t) \Big) \in \mathcal{G}_{\dot{y}_i(t)}^i(t). \tag{5.2.5}$$

Note that it is easy to assess that a solution ρ in the above sense belongs to $\mathbf{C}(\mathbb{R}^+; \mathbf{L}^1_\mathbf{loc}(\mathbb{R}))$ in the sense that it is time-continuous taking values in the $\mathbf{L}^1_\mathbf{loc}$ space of functions of the space variable; see, *e.g.*, Chapter 3.

Remark 5.2.1. Regarding the assumption of existence of strong one-sided traces for ρ on interfaces Γ_i, we have two important comments. First, our existence result will require uniform convexity or uniform concavity in ρ of the flux $\mathbf{f}$ on each of the regions of the (x,t)-plane delimited by the interfaces. Under this assumption which implies the genuine nonlinearity of the flux, existence of strong traces is well known since [154]. Second, using the machinery of [133], one can circumvent the genuine nonlinearity assumption with the help of a kind of singular mappings which allow to to define the relevant traces and with the help of "reduced germs" to replace (5.2.5) (see [15, Rem. 2.2, Def. 3.5]). For these two reasons, we consider that the trace assumption is non-restrictive in practice.

Remark 5.2.2. For the sake of conciseness, we have chosen to formulate Definition 5.2.2 in the context of piecewise constant $\mathbf{f}$. Extension of this notion of solution to piecewise regular fluxes, heterogeneous but $\mathbf{C}^1$ (or even merely Lipschitz continuous) in (x,t) in regions delimited by the interfaces, is straightforward. Note that also the existence of strong one-sided interface traces extends to this framework under mild assumptions, see in particular [3, 65, 130].

The uniqueness proof under the assumption of a finite number of interfaces is standard (see, in particular, [15, 18]; see also Chapter 4); moreover, even in the much more general situation of **SBV** fluxes uniqueness is proved in [65]. We state the corresponding claim for the sake of completeness, along with the $\mathbf{L}^1$ contraction result.

Theorem 5.2.3. *In the situation of Definiton 5.2.2, there exists at most one $\mathcal{G}_s^i(t)$-entropy solution for every initial datum $\rho_0 \in \mathbf{L}^\infty(\mathbb{R})$, moreover, if $\hat\rho_0 \in \mathbf{L}^\infty(\mathbb{R})$ with $\hat\rho_0 - \rho_0 \in \mathbf{L}^1(\mathbb{R})$ and $\hat\rho$ is the associated $\mathcal{G}_s^i(t)$-entropy solution, then for all $t>0$ there holds*

$$\|\hat\rho(\cdot,t) - \rho(\cdot,t)\|_{\mathbf{L}^1(\mathbb{R})} \leq \|\hat\rho_0 - \rho_0\|_{\mathbf{L}^1(\mathbb{R})}.$$

Note that also the continuous dependence on interface coupling conditions can be obtained along the same line of argumentation, see [15, Prop. 3.21] for a prototype statement.
Finally, note that the Carathéodory assumption on the family $\{\mathcal{G}_s^i(t)\}_{t\in(s_i,T_i),s\in\mathbb{R}}$ plays no role in the uniqueness proof; actually, we will exploit it for proving existence of solutions. Moreover, we guess that it is important for stability of solutions under perturbation of interface locations. To sum up, we believe that this assumption is an important one on the way to a consistent theory of problem (5.1.1) and therefore, we have included it into Definition 5.2.2.

5.2.4 Adapted entropy inequalities and existence result

Definition 5.2.2 is particularly well suited for uniqueness proof, but it cannot be used directly to establish existence of solutions. It became standard in the literature to use different kinds of "adapted entropy inequalities" in order to describe the interface coupling, in the place of (5.2.5). In the case of traffic models with flux limitation, since [56] one uses entropy inequalities of Kruzhkov (with a constant value κ) with a remainder term $\mathcal{R}(\kappa)$ supported by the interface; see Chapters 1 – 4. The choice of a constant κ can be done for the vanishing viscosity interface coupling, see [106, 107]. The corresponding remainder term $\mathcal{R}(\kappa)$ appears quite naturally even in a very general context, see [136], but this natural formulation leads to uniqueness only if the so-called crossing condition is fulfilled (see [15]). The case without the

crossing condition is significantly more delicate but it can be handled as well by introducing a singular form of remainder term $\mathcal{R}(\kappa)$, see [18, 64] (see also [108]).
However, the presence of remainder terms supported on the interfaces is not the main adaptation of the classical Kruzhkov entropy conditions. Adapted entropy inequalities with κ that may jump across interface offer more flexibility; *e.g.*, in many situations including the road traffic with limited flux, only one entropy inequality (with a special choice of the piecewise constant function κ) is needed, see [41, 14, 6] and [15, Sect. 4.8,4.9]. However, beyond the homogeneous situation with constant in time choice of coupling across interfaces, adapted entropy inequalities should be written for arbitrary choice of the piecewise constant κ, which jumps across the interfaces of $\mathbf{f}$, see in particular [16, 18]; these inequalities incorporate a remainder term that depends both upon the piecewise constant κ and on the prescribed coupling (*i.e.* on the given family of germs along interfaces).
We start by reformulating the definition of $\mathcal{G}_s^i(t)$-entropy solution under the form of adapted entropy inequalities which incorporate a remainder term $\mathcal{R}^i_{\dot{y}_i(t)}(\kappa_L, \kappa_R, t)$ different from those previously proposed in [15, 5] but satisfying the key structural properties that ensure the equivalence of definitions.

Proposition 5.2.4. *Consider a piecewise constant flux $\mathbf{f} : \Omega \times \mathbb{R} \to \mathbb{R}$ with a set of interfaces of the form (5.2.1). Assume that for each $i \in [\![1; J]\!]$ we are given a family $\{\mathcal{G}_s^i(t)\}_{t\in(s_i,T_i),s\in\mathbb{R}}$ of maximal $\mathbf{L}^1D$ germs associated with fluxes g_s^i, f_s^i in (5.2.2); assume that this family is Carathéodory, in the sense of Definition 5.2.1. Denote by $\mathcal{K}$ the set of all the functions, piecewise on Ω, that share the same interfaces as $\mathbf{f}$. For a function $\kappa \in \mathcal{K}$ and $i \in [\![1; J]\!]$, we denote by k_L^i, k_R^i the one-sided limits of κ on Γ_i (being $\kappa_L = \kappa(y_i(t) - 0, t)$ and $\kappa_R = \kappa(y_i(t) + 0, t)$ for all $t \in (s_i, T_i)$).*
Define for all $i \in [\![1; J]\!]$, $t \in (s_i, T_i)$, $s \in \mathbb{R}$ and $(\kappa_L, \kappa_R) \in \mathbb{R}^2$ the "remainder term"

$$\mathcal{R}_s^i(\kappa_L, \kappa_R, t) := \left|f(\kappa_L) - \mathbf{F}_s^{i,int}(\kappa_L, \kappa_R, t)\right| + \left|\mathbf{F}_s^{i,int}(\kappa_L, \kappa_R, t) - g(\kappa_R)\right|, \tag{5.2.6}$$

where $\mathbf{F}_s^{i,int}(\cdot, \cdot, t)$ is the Godunov flux defined in (5.2.3) associated with the germ family $\{\mathcal{G}_s^i(t)\}_{t\in(s_i,T_i),s\in\mathbb{R}}$.

Assume that $\rho \in \mathbf{L}^\infty(\Omega)$ satisfies, for all test functions $\varphi \in \mathbf{C}_\mathbf{c}^\infty(\overline{\Omega})$, $\varphi \geq 0$ and for all $\kappa \in \mathcal{K}$, the following adapted entropy inequalities:

$$\int_0^{+\infty} \int_{\mathbb{R}} \Big(|\rho - \kappa| \partial_t \varphi + \Phi(x, t, \rho, \kappa) \partial_x \varphi \Big) \, \mathrm{d}x \, \mathrm{d}t + \int_{\mathbb{R}} |\rho_0(x) - \kappa| \varphi(x, 0) \, \mathrm{d}x \\ \geq - \sum_{i=1}^{J} \int_{s_i}^{T_i} \mathcal{R}^i_{\dot{y}_i(t)}(\kappa_L, \kappa_R, t) \varphi(y_i(t), t) \, \mathrm{d}t \,. \tag{5.2.7}$$

Then ρ is the $\mathcal{G}_s^i(t)$-entropy solution to (5.1.1) with initial data $\rho_0 \in \mathbf{L}^\infty(\mathbb{R})$.

As a consequence of the uniqueness of $\mathcal{G}_s^i(t)$-entropy solution (Theorem 5.2.3), the reciprocal implication of Proposition 5.2.4 can be proved as soon one can ensure the existence of solutions in the sense of inequality (5.2.7), see Remark 5.3.3.

Remark 5.2.3. As demonstrated in Chapter 4, in the context of formulations of the kind (5.2.7) we can replace the set $\mathbf{C}_{\mathbf{c}}^{\infty}(\overline{\Omega})$ for the test functions by $\mathbf{C}_{\mathbf{c}}^{\infty}(\overline{\Omega} \setminus \cup_{m=1}^{M} \mathcal{C}_m)$. This permits to relax, in a small vicinity of the cross-points $\mathcal{C}_m$, the consistency constraints on the approximation procedure used to construct solutions, see Section 5.4.2.

With this reformulation of the notion of solution we are concerned with, we are in a position to state the main result of the Chapter.

Theorem 5.2.5. *Consider a piecewise constant flux $\mathbf{f} : \Omega \times \mathbb{R} \to \mathbb{R}$ with a set of interfaces of the form (5.2.1). Assume that for each $i \in [\![1; J]\!]$ we are given a family $\{\mathcal{G}_s^i(t)\}_{t\in(s_i,T_i),s\in\mathbb{R}}$ of maximal $\mathbf{L}^1 D$ germs associated with fluxes g_s^i, f_s^i in (5.2.2); assume that this family is Carathéodory, in the sense of Definition 5.2.1.*
Assume moreover that the confinement assumption holds:

$$\exists a, b \in \mathbb{R} \ : \ \ a < b \text{ and for a.e. } (x,t) \in \Omega \ \ \mathbf{f}(x,t,a) = a, \ \mathbf{f}(x,t,b) = b. \tag{5.2.8}$$

Assume moreover $\mathbf{f}$ is $\mathbf{C}^2$ in the state variable $\rho \in [a,b]$ for a.e. $(x,t) \in \Omega$ and the uniform convexity/concavity assumption holds:

$$\exists c > 0 \ \ \forall \rho \in [a,b] \ \ |\partial_\rho^2 \mathbf{f}(x,t,\rho)| \geq c. \tag{5.2.9}$$

Finally, suppose that for each $i \in [\![1; J]\!]$,

$$(\kappa_L, \kappa_R) \mapsto \mathbf{F}_s^i(\kappa_L, \kappa_R, \cdot) \tag{5.2.10}$$

is Lipschitz continuous on $[a,b]^2$ uniformly in $t \in (s_i, T_i)$ and locally uniformly in $s \in \mathbb{R}$.

Then for any initial datum $\rho_0 \in \mathbf{L}^\infty(\mathbb{R})$ taking values in $[a,b]$, there exists a $\mathcal{G}_s^i(t)$-entropy solution (which is unique, due to Theorem 5.2.3) of the discontinuous-flux conservation law (5.1.1). Moreover, it can be obtained as the limit of a sequence of approximate solutions generated by a finite volume numerical scheme described in Section 5.4.

Let us stress that, due to the piecewise constant in (x,t) structure of $\mathbf{f}$, assumption (5.2.9) is a concise way to state that, in each of the regions delimited by the interfaces Γ_i, f is either subject to the uniform convexity or to the uniform concavity assumption. Observe that the change of convexity across the interface is relevant, *e.g.*, in the Hughes model of pedestrian evacuation [4]. Further, assumption (5.2.8) is a standard way to ensure uniform $\mathbf{L}^\infty$ bounds on the approximate solutions (see, *e.g.*, [107]); for instance in traffic and porous media models, it is satisfied with $[a,b] = [0,1]$, the physical range of the state variable.
Next, we guess that assumption (5.2.10) is verified in all practical cases. In particular, for the interface coupling based on transmission maps (see Section 5.2.5 below) the Lipschitz property is checked in [7], while for the flux-limitation coupling of Chapters 1-4, this property is obvious from the definition of the Godunov flux (see in particular [45] for the definition of the Godunov flux in this case).
We stress that the convergence and existence result of Theorem 5.2.5 requires local strong compactness of the sequence of approximate solutions. Our choice of compactness argument is the one-sided Lipschitz (OSL) bound, which imposes the restriction (5.2.9) of uniform convexity or uniform concavity of $\mathbf{f}$ with respect to ρ in each of the subdomains separated by the interfaces Γ_i. Three remarks are in order.

Remark 5.2.4. Theorem 5.2.5, stated for piecewise constant in (x,t) flux $\mathbf{f}$ for the sake of readability, extends to t-dependent and to x-dependent flux in regions between the interfaces. This can be seen from the proofs of Chapter 3, under the assumption of the uniform in t or in x convexity or concavity of the flux with respect to the state variable (unpublished). Indeed, in this chapter the Finite Volume scheme is constructed taking into account the t-dependence and the local OSL-based compactness argument for this case is written in detail. We guess that the extension to the (x,t)-dependent flux under the uniform convexity/concavity assumption can be obtained in a straightforward way.

Remark 5.2.5. Different compactness tools, such as the compensated compactness, would permit to justify the existence and convergence result of Theorem 5.2.5 under weaker assumptions on the genuine nonlinearity of $\mathbf{f}$ with respect to ρ in each of the subdomains separated by interfaces. We refer to [107] for such arguments, in the setting of a different scheme; we guess that these arguments can be applied in our framework as well. This direction is left for future work.

Remark 5.2.6. As soon as the existence of a solution is established for a dense set of initial data, the weak convergence methods (see, *e.g.*, the "entropy process" framework put forward in [83]) can be applied to justify convergence of other numerical schemes that are consistent with the weak formulation and the adapted entropy inequalities. We refer to [15, Thm. 3.28,Thm. 6.5] and [14] for this line of argumentation.

5.2.5 The example of transmission map coupling

In Section 4.3, the example of flux limitation interface coupling was treated; here, let us observe that the associated family of germs is indeed Carathéodory, due to the continuity in s of the fluxes g_s^i, f_s^i and to the measurability of the constraints $t \mapsto q_i(t)$. Moreover, the remainder term denoted by $\mathcal{R}(\kappa, q_i(t))$ used in Chapters 1 – 4, for the choice of constant κ, is precisely $\mathcal{R}_s^i(\kappa,\kappa,t)$ in our framework.
In this section, let us provide another important example of interface coupling fulfilling the Carathéodory assumption, and give the explicit expression of the associated Godunov fluxes. This example is based upon [7], see also [5] for a summary. For the sake of conciseness, let us consider the case of a single interface defined for $t \in (0,+\infty)$, *i.e.* $J=1$; in this case, we drop the interface label i in the notation. Given fluxes g, f on $[a,b] \subset \mathbb{R}$ such that $g(a) = f(a)$ and $g(b) = f(b)$, set $g_s(\rho) = g(\rho) - s\rho$, $f_s(\rho) = f(\rho) - s\rho$ and denote by $\mathbf{God}g_s, \mathbf{God}f_s : [a,b]^2 \to \mathbb{R}$ the Godunov fluxes associated to g_s, f_s, respectively. The explicit formula for the Godunov fluxes shows that the dependence of $\mathbf{God}g_s(\kappa_L,\kappa_R), \mathbf{God}f_s(\kappa_L,\kappa_R)$ on s is continuous.
To define the interface coupling, given $\{\beta(t)\}_{t\in(0,+\infty)}$ a family of maximal monotone graphs in $\mathbb{R}\times\mathbb{R}$, we can define for all $s \in \mathbb{R}$ and $t \in (s_i, T_i)$ the maximal $\mathbf{L}^1 D$ germ

$$\mathcal{G}_s(t) = \Big\{ (\rho_L,\rho_R) \in [a,b]^2 \,,\, \exists (k_-,k_+) \in \beta(t) \text{ such that } \\ g_s(\rho_L) = (\mathbf{God}g_s)(\rho_L,k_-) = (\mathbf{God}f_s)(k_+,\rho_R) = f_s(\rho_R) \Big\}$$

and the associated interface Godunov flux

$$\mathbf{F}_s^{int}(\kappa_L,\kappa_R,t) = (\mathbf{God}g_s)(\kappa_L,k_-) = (\mathbf{God}f_s)(k_+,\kappa_R) \text{ where } (k_-,k_+) \in \beta(t), \tag{5.2.11}$$

being understood that the equality between $F_L := (\mathbf{God}g_s)(\kappa_L, k_-)$ and $F_R := (\mathbf{God}f_s)(k_+, \kappa_R)$ may not define uniquely the couple $(k_-, k_+) \in \beta(t)$ but the common value $F_L = F_R$ is defined uniquely. We refer to [7] for the justification of the above claims. Note that the case $\beta(t) = \mathrm{Id}$ corresponds to the fundamental case of vanishing viscosity interface coupling.
We point out that the family $\{\mathcal{G}_s^i(t)\}_{t\in(0,+\infty),s\in\mathbb{R}}$ is Carathéodory, in the sense of Definition 5.2.1, provided the family $\{\beta(t)\}_{t\in(0,+\infty)}$ is measurable in a natural sense. To this end, let us represent any maximal monotone graph β in $\mathbb{R}\times\mathbb{R}$ as $\beta = \{(\pi_L(p), \pi_R(p))\,,\, p \in \mathbb{R}\}$ where $\pi_{L,R} : \mathbb{R} \to \mathbb{R}$ are 1-Lipschitz non-decreasing functions verifying $\pi_L(p) + \pi_R(p) = p$ for all $p \in \mathbb{R}$. In other words, we parametrize β by the sum $p = \kappa_L + \kappa_R$ of the two components of a point $(\kappa_L, \kappa_R) \in \beta$. Then the natural way to impose measurability of a family $\{\beta(t)\}_t$ is to consider the corresponding parametrisations $\pi_{L,R}(t, \cdot)$ and require that these functions be Carathéodory. Then, in view of the construction (5.2.11), it is clear that, given (κ_L, κ_R), $(s, t) \mapsto \mathbf{F}_s^{int}(\kappa_L, \kappa_R, t)$ is continuous with respect to s (because the Godunov fluxes are continuous with respect to s) and measurable with respect to t (because they are obtained solving an equation of the form $F_s(t, p) = 0$ for a measurable in t, monotone and continuous in p function F_s).

5.3 The basic ingredients in the model case with variable interface coupling

Let us now provide the key tools to our study for the model case (5.1.1)-(5.1.2) with a time-dependent coupling at the interface $\{x = 0\}$. Therefore, in this section, we consider a flux $\mathbf{f} : \Omega \times \mathbb{R} \to \mathbb{R}$ given by (5.1.2) verifying the confinement assumption (5.2.8) and the the uniform convexity/concavity assumption (5.2.9) with $f, g \in \mathbf{C}^2([a, b])$. Throughout the section, we denote by Φ_f (resp. Φ_g) the classical Kruzhkov entropy flux associated with f (resp. g) so that:

$$\forall x \in \mathbb{R}^*,\ \rho, \kappa \in [a, b], \quad \Phi(x, \rho, \kappa) = \begin{cases} \Phi_f(\rho, \kappa) & \text{if} \quad x < 0 \\ \Phi_g(\rho, \kappa) & \text{if} \quad x > 0. \end{cases}$$

Suppose also that we are given a family $\{\mathcal{G}(t)\}_{t>0}$ of maximal $\mathbf{L}^1D$ germs associated with fluxes f and g. We suppose that this family is Carathéodory in the sense of Definition 5.2.1 which, in this context, means that for all $\kappa_L, \kappa_R \in [a, b]$, the associated function

$$\mathbb{R} \to \mathbb{R},\ t \mapsto \mathbf{F}^{int}(\kappa_L, \kappa_R, t)$$

is a Carathéodory function.

5.3.1 Stability and uniqueness

For the sake of completeness, we recall the abstract definition of $\mathbf{L}^1D$ germ, see [15, Definition 3.1] or [5, Definition 1.1]. We restrict ourselves to subsets of $[a, b]^2$ in view of the confinement assumption verified by f and g.

Definition 5.3.1. A family $\{\mathcal{G}(t)\}_{t>0}$ of subsets of $[a,b]^2$ is called $\mathbf{L}^1D$ germ associated with fluxes f,g if for all $t>0$:
(i) for all $(\kappa_L,\kappa_R)\in\mathcal{G}(t)$, $f(\kappa_L)=g(\kappa_R)$;
(ii) for all $(\kappa_L,\kappa_R),(c_L,c_R)\in\mathcal{G}(t)$,

$$\Phi_f(\kappa_L,c_L)-\Phi_g(\kappa_R,c_R)\geq 0. \tag{5.3.1}$$

We say that the family is maximal if for all $t>0$, $\mathcal{G}(t)$ is not a strict subset of some other $\mathbf{L}^1D$ germ.

Remark 5.3.1. With this convention, it is readily seen that any maximal $\mathbf{L}^1D$ germ $\mathcal{G}(t)$ associated with fluxes f,g verifying the confinement condition (5.2.8) contains the couples (a,a) and (b,b).

Let us give the arguments of the proof of Theorem 5.2.3 for this model case. Following the proof of Lemma 4.1.7, we derive the Kato inequality:

$$\int_0^{+\infty}\int_{\mathbb{R}}\Big(|\rho-\hat\rho|\partial_t\varphi+\Phi(x,\rho,\hat\rho)\partial_x\varphi\Big)\,\mathrm{d}x\,\mathrm{d}t+\int_{\mathbb{R}}|\rho_0(x)-\hat\rho_0(x)|\varphi(x,0)\,\mathrm{d}x$$
$$+\int_0^{+\infty}\Big(\Phi_g\left(\gamma_R\rho(t),\gamma_R\hat\rho(t)\right)-\Phi_f\left(\gamma_L\rho(t),\gamma_L\hat\rho(t)\right)\Big)\varphi(0,t)\,\mathrm{d}t\geq 0.$$

The $\mathbf{L}^1D$ dissipativity assumption on the family of germs $\{\mathcal{G}(t)\}_{t>0}$ (5.3.1) ensures that for a.e. $t>0$,

$$\Phi_g\left(\gamma_R\rho(t),\gamma_R\hat\rho(t)\right)-\Phi_f\left(\gamma_L\rho(t),\gamma_L\hat\rho(t)\right)\leq 0.$$

Upon a suitable choice of test function, see the proof of Theorem 4.1.8, the stability estimate of Theorem 5.2.3 follows. This leads to uniqueness.

5.3.2 Adapted entropy inequalities with the Godunov remainder terms

We now turn to the proof of Proposition 5.2.4 where the remainder term becomes:

$$\mathcal{R}(\kappa_L,\kappa_R,t)=\left|f(\kappa_L)-\mathbf{F}^{int}(\kappa_L,\kappa_R,t)\right|+\left|\mathbf{F}^{int}(\kappa_L,\kappa_R,t)-g(\kappa_R)\right|.$$

Recall that for $t>0$ and $\kappa_L,\kappa_R\in[a,b]$, $\mathbf{F}^{int}(\kappa_L,\kappa_R,t)$ denotes the Godunov flux associated with the family $\{\mathcal{G}(t)\}_{t>0}$. We start with the following lemma.

Lemma 5.3.2. *Let $(\kappa_L,\kappa_R)\in[a,b]$. Then*

$$(\kappa_L,\kappa_R)\in\{\mathcal{G}(t)\}_{t>0}\implies\forall t>0,\ \mathcal{R}(\kappa_L,\kappa_R,t)=0.$$

Proof. Indeed, saying that $(\kappa_L,\kappa_R)\in\{\mathcal{G}(t)\}_{t>0}$ means that the piecewise constant function

$$\kappa(x)=\begin{cases}\kappa_L & \text{if}\quad x<0\\ \kappa_R & \text{if}\quad x>0\end{cases}$$

is the unique solution to the problem (5.1.1) with initial data $\rho_0=\kappa$. Consequently, for all $t>0$,

$$f(\kappa_L)=\mathbf{F}^{int}(\kappa_L,\kappa_R,t)=g(\kappa_R)\quad\text{and}\quad\forall t>0,\ \mathcal{R}(\kappa_L,\kappa_R,t)=0,$$

concluding the proof. □

We now turn to the proof of Proposition 5.2.4.

Suppose that ρ verifies (5.2.7). Clearly, ρ satisfies (5.2.4) if $\varphi \in \mathbf{C}_{\mathbf{c}}^{\infty}(\overline{\Omega}), \varphi \geq 0$ vanishes along the interface $\{x = 0\}$. Following the proof of Proposition 4.1.5, we obtain:

$$\int_0^{+\infty} \Big(\Phi_f(\gamma_L \rho(t), \kappa_L) - \Phi_g(\gamma_R \rho(t), \kappa_R) + \mathcal{R}(\kappa_L, \kappa_R, t) \Big) \varphi(0,t) \, \mathrm{d}t \geq 0. \tag{5.3.2}$$

Since $t \mapsto \mathcal{R}(\kappa_L, \kappa_R, t)$ is measurable and bounded, the function $t \mapsto \mathcal{R}(\kappa_L, \kappa_R, t)$ is in $\mathbf{L}^1_{\mathbf{loc}}((0, +\infty))$. Consequently, a.e. $t > 0$ is Lebesgue point of this function. From (5.3.2), we deduce that for a.e. $t > 0$,

$$\Phi_f(\gamma_L \rho(t), \kappa_L) - \Phi_g(\gamma_R \rho(t), \kappa_R) + \mathcal{R}(\kappa_L, \kappa_R, t) \geq 0.$$

In particular, using Lemma 5.3.2,

$$\forall (\kappa_L, \kappa_R) \in \{\mathcal{G}(t)\}_{t>0}, \text{ for a.e. } t > 0, \quad \Phi_g(\gamma_R \rho(t), \kappa_R) \leq \Phi_f(\gamma_L \rho(t), \kappa_L). \tag{5.3.3}$$

This last inequality implies that for a.e. $t > 0$, $\Big((\gamma_L \rho)(t) \, , \, (\gamma_R \rho)(t) \Big) \in \mathcal{G}(t)$ by maximality (and therefore completeness, see [15, 5]) of the family of germs. We proved that ρ is a $\mathcal{G}_s^i(t)$-entropy solution to (5.1.1) with initial data $\rho_0 \in \mathbf{L}^{\infty}(\mathbb{R}; [a,b])$.

Remark 5.3.2. In the case of an interface coupling like the one in Chapter 1, a case by case study ensures that a $\mathcal{G}(t)$-entropy solution to (5.1.1) verifies inequality (5.2.7) as well, see the proof of Proposition 4.1.6.

5.3.3 Finite volume scheme, compactness and convergence

We turn to the proof of the existence statement (Theorem 5.2.5) with the construction of finite volume scheme for which we prove the convergence. Let us keep the notations introduced back in Section 1.1.

Let $\rho_0 \in \mathbf{L}^{\infty}(\mathbb{R}; [a,b])$, where $a, b \in \mathbb{R}$ are the numbers defined in the confinement assumption 5.2.8 verified by f and g. We denote by $\mathbf{F}_f = \mathbf{F}_f(u,v)$ (resp. $\mathbf{F}_g = \mathbf{F}_g(u,v)$) a monotone numerical flux associated with f (resp. g), see Definition 1.1.1 and Example 1.1.1. Following the assumptions of Theorem 5.2.5, we assume that $(\kappa_L, \kappa_R) \mapsto \mathbf{F}^{int}(\kappa_L, \kappa_R, \cdot)$ is Lipschitz continuous on $[a,b]^2$, uniformly on $t > 0$. This can be expressed as:

$$\exists \mathbf{L} > 0, \ \forall t > 0, \ \forall x, y, \chi, \gamma \in [a,b], \quad \left| \mathbf{F}^{int}(x,y,t) - \mathbf{F}^{int}(\chi, \gamma, t) \right| \leq \mathbf{L} \left(|x - \chi| + |y - \gamma| \right). \tag{5.3.4}$$

In the analysis below, the following properties of the Godunov flux $(t, \kappa_L, \kappa_R) \mapsto \mathbf{F}^{int}(\kappa_L, \kappa_R, t)$ are used in addition to (5.3.4).

Lemma 5.3.3. *The Godunov flux defined in Section 5.2.2 has the following properties:*
(i) for all $t > 0$, $\mathbf{F}^{int}(\cdot, \cdot, t)$ is nondecreasing with respect to its first argument and nonincreasing with respect to its second argument;
(ii) for all $t > 0$,

$$\forall t > 0, \quad f(a) = \mathbf{F}^{int}(a, a, t) = g(a); \quad f(b) = \mathbf{F}^{int}(b, b, t) = g(b). \tag{5.3.5}$$

Proof. Point *(i)* follows from the order-preservation property encoded in $\mathbf{L}^1 D$ germs *via* the Crandall-Tartar Lemma (see [8]). Point *(ii)* comes from the fact that the couples (a,a) and (b,b) belong to $\{\mathcal{G}(t)\}_{t>0}$ (Remark 5.3.1). Identites (5.3.5) follow as in the proof of Lemma 5.3.2. □

We now proceed to the definition of the scheme. Fix $n \in \mathbb{N}$ and $j \in \mathbb{Z}$. Away from the interface, *i.e.* for $j \notin \{-1, 0\}$, our scheme reduces to a classical three-point finite volume scheme on the uniform grid defined in Section 1.1. More precisely,

$$\forall j \leq -2, \quad \rho_{j+1/2}^{n+1} = \rho_{j+1/2}^n - \lambda(\mathbf{F}_f(\rho_{j+1/2}^n, \rho_{j+3/2}^n) - \mathbf{F}_f(\rho_{j-1/2}^n, \rho_{j+1/2}^n)) \tag{5.3.6}$$

and

$$\forall j \geq 1, \quad \rho_{j+1/2}^{n+1} = \rho_{j+1/2}^n - \lambda(\mathbf{F}_g(\rho_{j+1/2}^n, \rho_{j+3/2}^n) - \mathbf{F}_g(\rho_{j-1/2}^n, \rho_{j+1/2}^n)). \tag{5.3.7}$$

To handle the coupling at the interface, introduce the mean numerical flux

$$\mathbf{God}_{int}^n(\rho_{-1/2}^n, \rho_{1/2}^n) = \frac{1}{\Delta t} \int_{t^n}^{t^{n+1}} \mathbf{F}^{int}(\rho_{-1/2}^n, \rho_{1/2}^n, t)\, \mathrm{d}t \tag{5.3.8}$$

and use it to define the numerical solution in the remaining cells:

$$\rho_{-1/2}^{n+1} = \rho_{-1/2}^n - \lambda(\mathbf{God}_{int}^n(\rho_{-1/2}^n, \rho_{1/2}^n) - \mathbf{F}_f(\rho_{-3/2}^n, \rho_{-1/2}^n)). \tag{5.3.9}$$

and

$$\rho_{1/2}^{n+1} = \rho_{1/2}^n - \lambda(\mathbf{F}_g(\rho_{1/2}^n, \rho_{3/2}^n) - \mathbf{God}_{int}^n(\rho_{-1/2}^n, \rho_{1/2}^n)). \tag{5.3.10}$$

For the sake of simplicity, we choose $\mathbf{F}_f$ and $\mathbf{F}_g$ equal to one of the numerical fluxes introduced in Example 1.1.1. This way, the CFL condition reduces to

$$2\lambda \max\{\|f'\|_{\mathbf{L}^\infty}, \|g'\|_{\mathbf{L}^\infty}, \mathbf{L}\} \leq 1, \tag{5.3.11}$$

with $\mathbf{L}$ defined in (5.3.4).

Stability and discrete entropy inequalities

Lemma 5.3.4 ($\mathbf{L}^\infty$ stability)**.** *The scheme* (5.3.6)-(5.3.10) *is monotone and stable:*

$$\forall n \in \mathbb{N},\ \forall j \in \mathbb{Z}, \quad \rho_{j+1/2}^n \in [a, b]. \tag{5.3.12}$$

Proof. The monotonicity of the scheme follows from the arguments used in the proof of Theorem 1.2.1 for instance. Note in particular that the Godunov interface fluxes $\mathbf{F}^{int}$ are monotone (which implies the monotonicity of $\mathbf{God}_{int}^n$ due to (5.3.8). The stability estimate is proved by induction with the introduction, for all $n \in \mathbb{N}$ and $j \in \mathbb{Z}$, of the function $\mathbf{H}_j^n = \mathbf{H}_j^n(x, y, z)$ used to express $\rho_{j+1/2}^{n+1}$ from $\rho_{j-1/2}^n$, $\rho_{j+1/2}^n$ and $\rho_{j+3/2}^n$, like in (1.1.4). The key point of the proof is the fact that a and b are stationary states of the scheme. Indeed, as a consequence of Lemma 5.3.3 *(iii)*, we have:

$$\forall n \in \mathbb{N},\ \forall j \in \mathbb{Z}, \quad \mathbf{H}_j^n(a, a, a) = a \ \text{ and } \ \mathbf{H}_j^n(b, b, b) = b.$$

We refer to the proof of Theorem 1.2.1 for more precise details. □

Corollary 5.3.5 (Discrete entropy inequalities). *Fix $\kappa_L, \kappa_R \in [a,b]$ and define*

$$\forall j \in \mathbb{Z}, \quad \kappa_{j+1/2} = \kappa_L \mathbb{1}_{\{j \leq -1\}} + \kappa_R \mathbb{1}_{\{j \geq 0\}}.$$

Then the numerical scheme (5.3.6)-(5.3.10) fulfills the following discrete entropy inequalities for all $n \in \mathbb{N}$ and $j \in \mathbb{Z}$:

$$\begin{aligned} &\left(|\rho^{n+1}_{j+1/2} - \kappa_{j+1/2}| - |\rho^n_{j+1/2} - \kappa_{j+1/2}| \right) \Delta x \\ &\leq \begin{cases} -\left(\Phi^n_{j+1} - \Phi^n_j\right) \Delta t & \text{if } j \notin \{-1, 0\} \\ -\left(\Phi^n_{int} - \Phi^n_{-1}\right) \Delta t + |f(\kappa_L) - \mathbf{God}^n_{int}(\kappa_L, \kappa_R)| \, \Delta t & \text{if } j = -1 \\ -\left(\Phi^n_1 - \Phi^n_{int}\right) \Delta t + |\mathbf{God}^n_{int}(\kappa_L, \kappa_R) - g(\kappa_R)| \, \Delta t & \text{if } j = 0, \end{cases} \end{aligned} \tag{5.3.13}$$

where Φ^n_j and Φ^n_{int} are the numerical entropy fluxes:

$$\begin{aligned} \Phi^n_j &= \begin{cases} \mathbf{F}_f(\rho^n_{j-1/2} \vee \kappa_L, \rho^n_{j+1/2} \vee \kappa_L) - \mathbf{F}_f(\rho^n_{j-1/2} \wedge \kappa_L, \rho^n_{j+1/2} \wedge \kappa_L) \text{ if } j \leq -1 \\ \mathbf{F}_g(\rho^n_{j-1/2} \vee \kappa_R, \rho^n_{j+1/2} \vee \kappa_R) - \mathbf{F}_g(\rho^n_{j-1/2} \wedge \kappa_R, \rho^n_{j+1/2} \wedge \kappa_R) \text{ if } j \geq 1 \end{cases} \\ \Phi^n_{int} &= \mathbf{God}^n_{int}(\rho^n_{-1/2} \vee \kappa_L, \rho^n_{1/2} \vee \kappa_R) - \mathbf{God}^n_{int}(\rho^n_{-1/2} \wedge \kappa_L, \rho^n_{1/2} \wedge \kappa_R). \end{aligned}$$

Proof. Fix $n \in \mathbb{N}$. The obtaining of (5.3.13) in the case $j \notin \{-1, 0\}$ is standard, see the proof of Corollary 1.2.2. Suppose that $j = -1$. In that case, we have

$$\mathbf{H}^n_{-1}(\kappa_L, \kappa_L, \kappa_R) = \kappa_L - \lambda \left(\mathbf{God}^n_{int}(\kappa_L, \kappa_R) - f(\kappa_L) \right),$$

frow which we obtain:

$$\begin{aligned} &\mathbf{H}^n_{-1}\left(\rho^n_{-3/2} \wedge \kappa_L, \rho^n_{-1/2} \wedge \kappa_L, \rho^n_{3/2} \wedge \kappa_R\right) - \lambda \left(\mathbf{God}^n_{int}(\kappa_L, \kappa_R) - f(\kappa_L) \right)^- \\ &\leq \kappa_L \leq \mathbf{H}^n_{-1}\left(\rho^n_{-3/2} \vee \kappa_L, \rho^n_{-1/2} \vee \kappa_L, \rho^n_{3/2} \vee \kappa_R\right) + \lambda \left(\mathbf{God}^n_{int}(\kappa_L, \kappa_R) - f(\kappa_L) \right)^+, \end{aligned}$$

where we denoted by z^+ (resp. z^-) the positive part (resp. negative part) of the real number z. We deduce:

$$\begin{aligned} |\rho^{n+1}_{-1/2} - \kappa_L| &= \rho^{n+1}_{-3/2} \vee \kappa_L - \rho^{n+1}_{-3/2} \wedge \kappa_L \\ &\leq \mathbf{H}^n_{-1}\left(\rho^n_{-3/2} \vee \kappa_L, \rho^n_{-1/2} \vee \kappa_L, \rho^n_{1/2} \vee \kappa_R\right) + \lambda \left(\mathbf{God}^n_{int}(\kappa_L, \kappa_R) - f(\kappa_L) \right)^+ \\ &\quad - \mathbf{H}^n_{-1}\left(\rho^n_{-3/2} \wedge \kappa_L, \rho^n_{-1/2} \wedge \kappa_L, \rho^n_{1/2} \wedge \kappa_R\right) + \lambda \left(\mathbf{God}^n_{int}(\kappa_L, \kappa_R) - f(\kappa_L) \right)^- \\ &= |\rho^n_{-1/2} - \kappa_L| - \lambda \left(\Phi^n_{int} - \Phi^n_{-1} \right) + \lambda \left| \mathbf{God}^n_{int}(\kappa_L, \kappa_R) - f(\kappa_L) \right|, \end{aligned}$$

which is exactly (5.3.13) in the case $j = -1$. The obtaining of (5.3.13) in the case $j = 0$ is similar so we omit the details of the proof for this case. □

Proposition 5.3.6 (Approximate entropy inequalities). *Fix $\kappa_L, \kappa_R \in [a,b]$ and define*

$$\kappa = \kappa_L \mathbb{1}_{\{x<0\}} + \kappa_R \mathbb{1}_{\{x>0\}}.$$

Let $\varphi \in \mathbf{C}_{\mathbf{c}}^{\infty}(\overline{\Omega}), \varphi \geq 0$ supported in time in $[0,T]$ $(T>0)$. Then as $\Delta \to 0$, we have

$$\begin{aligned} \int_0^T \int_{\mathbb{R}} \Big(|\rho_\Delta - \kappa| \partial_t \varphi + \Phi_\Delta \left(x, \rho_\Delta, \kappa\right) \partial_x \varphi \Big) \,\mathrm{d}x \,\mathrm{d}t + \int_{\mathbb{R}} |\rho_\Delta^0 - \kappa| \varphi(x,0) \,\mathrm{d}x \\ + \int_0^T \mathcal{R}_\Delta(\kappa_L, \kappa_R, t) \varphi(0,t) \,\mathrm{d}t \geq O(\Delta x) + O(\Delta t)\,, \end{aligned} \tag{5.3.14}$$

where

$$\Phi_\Delta \left(x, \rho_\Delta, \kappa\right) = \sum_{n \in \mathbb{N}} \sum_{j \in \mathbb{Z}^*} \Phi_j^n \mathbb{1}_{(x_j, x_{j+1}) \times [t^n, t^{n+1})}.$$

and

$$\mathcal{R}_\Delta(\kappa_L, \kappa_R, t) = \sum_{n \in \mathbb{N}} \Big(|f(\kappa_L) - \mathbf{God}_{int}^n(\kappa_L, \kappa_R)| + |\mathbf{God}_{int}^n(\kappa_L, \kappa_R) - g(\kappa_R)| \Big) \mathbb{1}_{[t^n, t^{n+1})}.$$

Proof. The proof follows step by step the proof of Proposition 1.3.1: start from the discrete entropy inequalities (5.3.13); apply the Abel procedure and estimate the members of the resulting inequality. The remainder term that replaces

$$\mathcal{R}(\kappa, q^n)(\varphi_{-1/2}^{n+1} + \varphi_{1/2}^{n+1})\Delta t$$

from the proof of Proposition 1.3.1 becomes, in our context,

$$\Big(|f(\kappa_L) - \mathbf{God}_{int}^n(\kappa_L, \kappa_R)| \, \varphi_{-1/2}^{n+1} + |\mathbf{God}_{int}^n(\kappa_L, \kappa_R) - g(\kappa_R)| \, \varphi_{1/2}^{n+1} \Big) \Delta t.$$

□

Compactness and convergence

We now prove existence of solutions in the sense of inequality (5.2.7).
The compactness of the sequence $(\rho_\Delta)_\Delta$ is obtained by deriving local **BV** bounds. Since f and g are assumed to have uniform convexity/convexity, we can use the OSL technique put forward by [152] described in detail in Chapter 4 and [149, Appendix] and applied in Chapters 3-4. As highlighted in Chapter 4, this technique "does not see" the interface coupling, so the proofs contained in the previous chapters cover our present case as well. This provides the existence of $\rho \in \mathbf{L}^\infty(\Omega; [a,b])$ such that a subsequence of $(\rho_\Delta)_\Delta$ converges to ρ a.e. on Ω. To pass to the limit in (5.3.14) and conclude the analysis, we need to ensure that

$$\int_0^T \mathcal{R}_\Delta(\kappa_L, \kappa_R, t) \varphi(0,t) \,\mathrm{d}t \underset{\Delta \to 0}{\longrightarrow} \int_0^T \mathcal{R}(\kappa_L, \kappa_R, t) \varphi(0,t) \,\mathrm{d}t\,.$$

This convergence claim comes from the measurability and the boundedness of $t \mapsto \mathcal{R}(\kappa_L, \kappa_R, t)$. Since $t \mapsto \mathcal{R}_\Delta(\kappa_L, \kappa_R, t)$ is obtained by taking the mean values of $t \mapsto \mathcal{R}(\kappa_L, \kappa_R, t)$ on a uniform grid, we are ensured that for all Lebesgue points $t > 0$ of $\mathcal{R}(\kappa_L, \kappa_R, \cdot)$ (which are a.e. $t > 0$), $(\mathcal{R}_\Delta(\kappa_L, \kappa_R, t))_\Delta$ converges to $\mathcal{R}(\kappa_L, \kappa_R, t)$. This last ingredient implies that by letting $\Delta \to 0$ in (5.3.14), we obtain that ρ satisfies (5.2.7).

Remark 5.3.3. As mentioned after Proposition 5.2.4, now that we proved the existence of solutions in the sense of inequality (5.2.7), we can prove, under the assumptions of Theorem 5.2.5, the uniqueness for the adapted entropy formulation (5.2.7) and the reciprocal statement of Proposition 5.2.4.
First, let us observe that combining the last step of the convergence proof reasoning and Section 5.3.2, we proved a well-posedness result for solutions in the sense of inequality (5.2.7). Existence is obtained by the scheme. Moreover, two solutions in the sense of inequality (5.2.7) are also $\mathcal{G}(t)$-entropy solutions by Section 5.3.2; for those solutions we have uniqueness, see Theorem 5.3.1, leading to uniqueness for the adapted entropy formulation of Proposition 5.2.4.
Now suppose that ρ is a $\mathcal{G}(t)$-entropy solution to (5.1.1) with initial data $\rho_0 \in \mathbf{L}^\infty(\mathbb{R};[a,b])$. Suppose that ρ is not the solution in the sense of inequality (5.2.7). Let us then denote by $\sigma \neq \rho$ the solution in the sense of inequality (5.2.7). As mentioned before, σ is also a $\mathcal{G}(t)$-entropy solution, and therefore equal to ρ by uniqueness. This contradicts the fact that ρ is not the solution in the sense of inequality (5.2.7). We conclude that ρ verifies (5.2.7).

5.4 Multi-interface problem with general interface coupling

This section is very brief because borrows almost all of its contents to the corresponding section of Chapter 4. Indeed, the only changes are that we use Godunov interface flux in a more general situation leading to a more abstract form of the remainder term; and that we do not need any more the constraint inequalities required for sharp characterization of solutions for the adapted entropy formulation of Chapter 4. As in Chapter 4, we proceed in two steps. First the case of a single interface is dealt with calling upon the analysis of the previous section in what concerns the remainder term $\mathcal{R}_s$. Second, the accurate use of partitions of unity along with the choice of test functions vanishing near the cross-points permits to reduce the general case described in Section 5.2.1 to the case of a single slanted interface.

5.4.1 The case of a single slanted interface

This section builds on the work done in Sections 4.1-4.2. If y denotes the trajectory of the interface, the remainder term has now the form:

$$\mathcal{R}_{\dot{y}(t)}(\kappa_L,\kappa_R,t) = \left|f_{\dot{y}(t)}(\kappa_L) - \mathbf{F}^{int}_{\dot{y}(t)}(\kappa_L,\kappa_R,t)\right| + \left|\mathbf{F}^{int}_{\dot{y}(t)}(\kappa_L,\kappa_R,t) - g_{\dot{y}(t)}(\kappa_R)\right|.$$

Using Definition 5.2.2, the proof of uniqueness follows the ones of Lemma 4.1.7 and Theorem 4.1.8.

Regarding the existence, the only difference with the model case is that here we need to discretize the interface and to adapt the mesh in a neighbourhood of the discrete interface.

Like in Section 4.2, we define the sequence of approximate slopes:

$$\forall n \in \mathbb{N},\ s^n = \frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} \dot{y}(t)\,\mathrm{d}t\,; \quad \forall t \geq 0,\ s_\Delta(t) = \sum_{n\in\mathbb{N}} s^n \mathbb{1}_{[t^n,t^{n+1})}(t),$$

which converges to $\dot{y}$ in $\mathbf{L}^1_{\mathbf{loc}}((0,+\infty))$, see Remark 4.2.1. The mesh is produced in the same way as in Section 4.3.3, see in particular Figure 4.2. One only needs to make precise the numerical flux used at the slanted mesh boundaries. At time step t^n, the approximate coupling Godunov flux is defined by:

$$\mathbf{God}^{s^n}_{int}(\rho^n_{-1/2},\rho^n_{1/2}) = \frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} \mathbf{F}^{int}_{s^n}(\rho^n_{-1/2},\rho^n_{1/2},t)\,\mathrm{d}t\,. \tag{5.4.1}$$

Note that by assumption, the family of germs $\{\mathcal{G}_s(t)\}_{t>0,s\in\mathbb{R}}$ is Carathéodory which implies that for all $(\kappa_L,\kappa_R) \in [a,b]^2$, $s \mapsto \mathbf{God}^s_{int}(\kappa_L,\kappa_R)$ is continuous since for all $t>0$, $s \mapsto \mathbf{F}^{int}_s(\kappa_L,\kappa_R,t)$ is continuous.

The approximate remainder term that appears in the approximate adapted entropy inequalities is:

$$\mathcal{R}_{s_\Delta(t)}(\kappa_L,\kappa_R,t) = \sum_{n\in\mathbb{N}}\left(\left|f_{s^n}(\kappa_L) - \mathbf{God}^{s^n}_{int}(\kappa_L,\kappa_R)\right| + \left|\mathbf{God}^{s^n}_{int}(\kappa_L,\kappa_R) - g_{s^n}(\kappa_R)\right|\right)\mathbb{1}_{[t^n,t^{n+1})}.$$

The convergence

$$\int_0^{+\infty} \mathcal{R}_{s_\Delta(t)}(\kappa_L,\kappa_R,t)\varphi(y_\Delta(t),t)\,\mathrm{d}t \underset{\Delta\to 0}{\longrightarrow} \int_0^{+\infty} \mathcal{R}_{\dot{y}(t)}(\kappa_L,\kappa_R,t)\varphi(y(t),t)\,\mathrm{d}t$$

comes from:

— the measurability of $t \mapsto \mathcal{R}_{\dot{y}(t)}(\kappa_L,\kappa_R,t)$;

— the continuity of $s \mapsto t \mapsto \mathcal{R}_s(\kappa_L,\kappa_R,t)$ $(t>0)$ combined with the strong convergence of $(s_\Delta)_\Delta$ to $\dot{y}$.

Existence of solutions in the sense of inequality (5.2.7) follows and like in Section 5.3.3, we can prove the equivalence between Definition 5.2.2 and inequality (5.2.7).

5.4.2 Isolating interfaces and neglecting cross-points

The construction of the mesh, of the finite volume scheme and the convergence analysis by reduction of the initial configuration of Section 5.2 to the case of a single slanted interface, tackled in Section 5.4.1 is identical to the reasoning of Section 4.3. Let us only highlight the fact that the choice of the precise coupling at the interfaces near the cross-points (the violet parts in Figure 4.5) does not matter for the convergence of the scheme and its overall consistency. Indeed, the choice of the test functions in Remark 5.2.3 permits to disregard the discrete solution near cross-points. We only have to ensure that the discrete solutions take their values within $[a,b]$, for this sake we can use any numerical flux at the interfaces consistent with the stationary solutions a and b.

In this way, we complete the proof of Theorem 5.2.5.

Remark 5.4.1. In the practice of numerical approximation of problem (5.1.1) by the schemes we propose in this Chapter, any numerical flux which is monotone and whose Lipschitz constant is consistent with the CFL can be used on the interfaces in a small vicinity of the crossing points. This may lead to appearance of numerical artefacts (see Section 4.3.3 for an example). However, the latter remain "under control" since, according to our analysis, they do not affect the convergence of the scheme.

5.5 Conclusion and possible extensions

In this chapter, we provided an existence result for one-dimensional discontinuous-flux conservation law (5.1.1) for a wide choice of interface couplings. To this end, we constructed a Finite Volume numerical scheme where the uniform mesh, rectangular in (x,t), is adapted to the presence of interfaces and of their cross-points. The adaptation to interfaces is accurate and the Godunov flux consistent with the interface condition is used along the discretized interfaces.
The convergence result is based upon a new variant of adapted entropy inequality which naturally arises from Godunov discretization at interfaces; the Godunov flux should be compatible with the interface coupling imposed for (5.1.1). We stress that the entropy formulation verified by the limit of the scheme enjoys uniqueness, *i.e.* both the entropy admissibility conditions within the regions of homogeneous flux and full information on the interface coupling are captured. The existence result is based upon consistency with such adapted entropy formulation and on a local strong compactness property, which requires fine analysis of the scheme. We highlight the fact that, as soon as existence is justified, convergence of other approximation schemes can be achieved with weak compactness arguments (*cf.* [15, Sec. 3.4,6.4] and [14]) as soon as these approximation schemes are consistent - away from the cross-points - with the adapted entropy inequalities for the interface coupling at hand.
The local compactness technique we exploit systematically in this Thesis is based upon one-sided Lipschitz regularization, under the assumption of uniform convexity or concavity of the flux $\mathbf{f}$ with respect to the state variable. Due to the local nature of the compactness arguments, it does not depend on the choice of the interface coupling. Regular dependence of $\mathbf{f}$ on t in each of the subdomains separated by the interfaces can be handled within this compactness approach: the necessary adjustments are pointed out in Chapter 4 (for the time dependence). The case of a regular dependence of $\mathbf{f}$ on x is left for future works.

Our compactness approach can be replaced by the compensated compactness technique, under the weaker assumption of non-degenerately nonlinear flux (*cf.* [107] for a very similar result). We highlight the fact that the use of the compensated compactness technique would make our construction applicable to the multi-dimensional generalization of (5.1.1). Indeed, for instance in the case of a two-dimensional space (so, $(x,t) \in \mathbb{R}^3$), interfaces are surfaces in $\mathbb{R}^3$ and interface crossings are lower-dimensional sets (curves and their cross-points). The crossings can be neglected and the rectangular space-time mesh can be locally adapted to the shape of interfaces away from a vicinity of crossings. Like for the one-dimensional problem, adapted entropy inequalities involving Godunov fluxes in the normal direction to the interfaces can be written at the discrete level and inherited at the limit; like in the one-dimensional

case, the resulting formulation enjoys uniqueness (*cf.* [65, 5]).

Part II

Existence analysis and numerical approximation for a second order model of traffic with orderliness marker

We propose a toy model for self-organized road traffic taking into account the state of orderliness in drivers' behavior. The model is reminiscent of the wide family of generalized second-order models (GSOM) of road traffic. It can also be seen as a phase-transition model. The orderliness marker is evolved along vehicles' trajectories and it influences the fundamental diagram of the traffic flow. The coupling we have in mind is non-local, leading to a kind of "weak decoupling" of the resulting 2×2 system; this makes the mathematical analysis similar to the analysis of the classical Keyfitz-Kranzer system. Taking advantage of the theory of weak and renormalized solutions of one-dimensional transport equations [Panov, 2008], which we further develop on this occasion in the first chapter, we prove the existence of admissible solutions defined via a mixture of the Kruzhkov and the Panov approaches; note that this approach to admissibility does not rely upon the classical hyperbolic structure for 2×2 systems. First, approximate solutions are obtained *via* a splitting strategy; compactification effects proper to the notion of solution we rely upon are carefully exploited, under general assumptions on the data. Second, we also address fully discrete approximation of the system, constructing a **BV**-stable Finite Volume numerical scheme and proving its convergence under the no-vacuum assumption and for data of bounded variation. As a byproduct of our approach, an original treatment of local GSOM-like models in the **BV** setting is briefly discussed, in relation to discontinuous-flux LWR models.

CHAPTER 6

WELL-POSEDNESS AND COMPACTIFICATION OF RENORMALIZED SOLUTIONS TO A SEMILINEAR ONE-DIMENSIONAL TRANSPORT EQUATION

We first extend the results put forward by Panov in [134]. In this chapter and the next, we write Ω for $\mathbb{R}\times(0,T)$. Let us recall the working framework. Fix $\rho,\mathbf{v}\in\mathbf{L}^\infty(\Omega)$ such that

$$\rho\geq 0;\quad \partial_t\rho+\partial_x\left(\rho\mathbf{v}\right)=0 \text{ in } \mathcal{D}(\Omega). \tag{6.0.1}$$

Given a source term $\mathbf{S}\in\mathbf{L}^\infty(\Omega)$ and an initial datum $\mathbf{w}_0\in\mathbf{L}^\infty(\mathbb{R})$, introduce the transport equation formally written as $\partial_t\mathbf{w}+\mathbf{v}\partial_x\mathbf{w}=\mathbf{S}$, $\mathbf{w}(\cdot,0)=\mathbf{w}_0$ and reformulated as:

$$\begin{cases}\partial_t\left(\rho\mathbf{w}\right)+\partial_x\left(\rho\mathbf{v}\mathbf{w}\right)=\rho\mathbf{S}\\ \qquad\qquad \rho(\cdot,0)\mathbf{w}(\cdot,0)=\rho(\cdot,0)\mathbf{w}_0.\end{cases} \tag{6.0.2}$$

Following [134], we give the following notions of solution for Problem (6.0.2).

Definition 6.0.1. A function $\mathbf{w}\in\mathbf{L}^\infty(\Omega)$ is a weak solution to (6.0.2) with initial data $\mathbf{w}_0\in\mathbf{L}^\infty(\mathbb{R})$ if for all test functions $\phi\in\mathbf{C}^\infty_\mathbf{c}(\mathbb{R}\times[0,T))$, the following weak formulation is satisfied:

$$\int_0^T\int_{\mathbb{R}}\Big((\rho\mathbf{w})\partial_t\phi+(\rho\mathbf{v}\mathbf{w})\partial_x\phi+(\rho\mathbf{S})\phi\Big)\,\mathrm{d}x\,\mathrm{d}t+\int_{\mathbb{R}}\rho(x,0)\mathbf{w}_0(x)\phi(x,0)\,\mathrm{d}x=0. \tag{6.0.3}$$

Remark 6.0.1. Since ρ is a distributional solution to $\partial_t\rho+\partial_x(\rho\mathbf{v})=0$, we know (see [134, Lemma 1]) that $t\mapsto\rho(\cdot,t)$ is weakly* continuous in $\mathbf{L}^\infty(\mathbb{R})$, and the quantity $\rho(\cdot,0)$ has to

be understood as the weak* limit of $\rho(\cdot,t)$ as $t\to 0^+$. Further, applying [134, Lemma 1] to the field $(\tilde{A},B)$,

$$\tilde{A}:(x,t)\mapsto(\rho\mathbf{w})(x,t)-\int_0^t(\rho S\mathbf{w})(x,s)\,\mathrm{d}s\,,\quad B=\rho v\mathbf{w}$$

satisfying $\partial_t\tilde{A}+\partial_x B=0$, we see that $\tilde{A}\in\mathbf{C}([0,T];w^*\text{–}\mathbf{L}^\infty(\mathbb{R}))$ and since the integral term in the definition of $\tilde{A}$ is in $\mathbf{C}([0,T];\mathbf{L}^\infty(\mathbb{R}))$, we also have $\rho\mathbf{w}\in\mathbf{C}([0,T];w^*\text{–}\mathbf{L}^\infty(\mathbb{R}))$. In particular, $\rho\mathbf{w}$ assumes the initial datum $\rho(\cdot,0)\mathbf{w}_0$ in the sense of the weak* limit in $\mathbf{L}^\infty(\mathbb{R})$.

Definition 6.0.2. We say that a weak solution $\mathbf{w}\in\mathbf{L}^\infty(\Omega)$ to (6.0.2) with initial data $\mathbf{w}_0\in\mathbf{L}^\infty(\mathbb{R})$ verifies the renormalization property if for any function $p\in\mathbf{C}^1(\mathbb{R})$, $u=p(\mathbf{w})$ is a weak solution to

$$\begin{cases}\partial_t(\rho u)+\partial_x(\rho\mathbf{v}u)=\rho\mathbf{S}p'(\mathbf{w})\\ \qquad\rho(\cdot,0)u(\cdot,0)=\rho(\cdot,0)\,(p\circ\mathbf{w}_0)(\cdot).\end{cases}\tag{6.0.4}$$

Let us recall the following results, put forward in [134].

Theorem 6.0.3. *Let $\rho,\mathbf{v}\in\mathbf{L}^\infty(\Omega)$ satisfy (6.0.1) and let $\mathbf{S}\in\mathbf{L}^\infty(\Omega)$.*
(i) For any initial data $\mathbf{w}_0\in\mathbf{L}^\infty(\mathbb{R})$, the transport equation (6.0.2) admits a unique weak solution. Moreover, this weak solution verifies the renormalization property.
(ii) If $\mathbf{w}^1$ and $\mathbf{w}^2$ are two weak solutions to (6.0.2) associated with data $(\mathbf{w}_0^1,\mathbf{S}^1)$ and $(\mathbf{w}_0^2,\mathbf{S}^2)$, respectively, then the following stability estimate holds: for a.e. $t\in(0,T)$,

$$\|\mathbf{w}^1(\cdot,t)-\mathbf{w}^2(\cdot,t)\|_{\mathbf{L}^\infty}\le\|\mathbf{w}_0^1-\mathbf{w}_0^2\|_{\mathbf{L}^\infty}+\int_0^t\|\mathbf{S}^1(\cdot,s)-\mathbf{S}^2(\cdot,s)\|_{\mathbf{L}^\infty}\,\mathrm{d}s\,.\tag{6.0.5}$$

Remark 6.0.2. The author of [134] even extended these results with source terms:

$$\mathbf{S}(x,t)=g(x,t)\mathbf{w}(x,t)+h(x,t);\quad g,h\in\mathbf{L}^\infty(\Omega),$$

$\mathbf{w}$ being the unknown.

The contribution of this chapter is to prove an analogous to Theorem 6.0.3 when the source term of (6.0.2) takes the form

$$\mathbf{S}(x,t)=g(x,t)\mathbf{F}(\mathbf{w}(x,t));\quad g\in\mathbf{L}^\infty(\Omega).\tag{6.0.6}$$

Remark that when the function $\mathbf{F}$ is separated from zero in the sense described below, existence of a weak solution for a given initial datum follows from the renormalization property.

Lemma 6.0.4. *Suppose that $\mathbf{F}\in\mathbf{C}(\mathbb{R})$ and that there exists $\delta>0$ such that $\mathbf{F}\ge\delta$. Then for any initial data $\mathbf{w}_0\in\mathbf{L}^\infty(\mathbb{R})$, the transport equation (6.0.2) with source term $\mathbf{S}$ given by (6.0.6) admits at least a weak solution.*

Proof. Introduce the $\mathbf{C}^1$ function p defined by

$$\forall w\in\mathbb{R},\quad p(w)=\int_0^w\frac{\mathrm{d}y}{\mathbf{F}(y)}.$$

Note that the assumption on $\mathbf{F}$ implies that p is a $\mathbf{C}^1$-diffeomorphism on its image. From Theorem 6.0.3 *(i)*, we know that the transport equation

$$\begin{cases} \partial_t(\rho u) + \partial_x(\rho \mathbf{v} u) = \rho g \\ \qquad \rho(\cdot,0)u(\cdot,0) = \rho(\cdot,0)\,(p \circ \mathbf{w}_0)(\cdot). \end{cases}$$

admits a unique weak solution u. Since u verifies the renormalization property, by remarking that $(p^{-1})'(u) = \mathbf{F}(\mathbf{w})$, we deduce that $\mathbf{w} = p^{-1} \circ u$ is a weak solution to (6.0.2). □

Under the mere local Lipschitz assumption on $\mathbf{F}$, uniqueness for the transport equation with source terms of the form (6.0.6) follows.

Proposition 6.0.5. *Let $\rho, \mathbf{v} \in \mathbf{L}^\infty(\Omega)$ satisfy (6.0.1), $g \in \mathbf{L}^\infty(\Omega)$ and $\mathbf{F} \in \mathbf{Lip_{loc}}(\mathbb{R})$. Then for any initial data $\mathbf{w}_0 \in \mathbf{L}^\infty(\mathbb{R})$, the transport equation (6.0.2) with source term $\mathbf{S}$ given by (6.0.6) admits at most one weak solution.*

Proof. Let $\mathbf{w}_0^1, \mathbf{w}_0^2 \in \mathbf{L}^\infty(\mathbb{R})$. We denote by $\mathbf{w}^1$ (resp. $\mathbf{w}^2$) a weak solution to (6.0.2) associated with initial data $\mathbf{w}_0^1$ (resp. $\mathbf{w}_0^2$). Remark in the particular that $\mathbf{w}^1$ (resp. $\mathbf{w}^2$) is a weak solution to (6.0.2) with source term $\mathbf{S}^1 = g\mathbf{F}(\mathbf{w}^1)$ (resp. $\mathbf{S}^2 = g\mathbf{F}(\mathbf{w}^2)$). Using the stability estimate (6.0.5), we obtain that for a.e. $t \in (0,T)$,

$$\|\mathbf{w}^1(\cdot,t) - \mathbf{w}^2(\cdot,t)\|_{\mathbf{L}^\infty} \leq \|\mathbf{w}_0^1 - \mathbf{w}_0^2\|_{\mathbf{L}^\infty} + \|g\|_{\mathbf{L}^\infty}\|\mathbf{F}'\|_{\mathbf{L}^\infty} \int_0^t \|\mathbf{w}^1(\cdot,s) - \mathbf{w}^2(\cdot,s)\|_{\mathbf{L}^\infty}\,\mathrm{d}s\,.$$

Gronwall lemma yields a stability estimate and the uniqueness follows. □

We now prove the main result of compactness/stability regarding weak solutions verifying the renormalization property.

Theorem 6.0.6. *Let $\rho, \mathbf{v} \in \mathbf{L}^\infty(\Omega)$ satisfy (6.0.1), $g \in \mathbf{L}^\infty(\Omega)$, $\mathbf{F} \in \mathbf{Lip}(\mathbb{R})$ and $\mathbf{w}_0 \in \mathbf{L}^\infty(\mathbb{R})$. Let $(\rho_\nu)_\nu$, $(\mathbf{v}_\nu)_\nu$, $(g_\nu)_\nu$, $(\mathbf{w}_{0,\nu})_\nu$ be sequences of uniformly bounded functions such that:*

$$\forall \nu > 0,\ \rho_\nu \geq 0; \quad (\rho_\nu)_\nu, (\rho_\nu \mathbf{v}_\nu)_\nu, (g_\nu)_\nu \underset{\nu \to 0}{\longrightarrow} \rho, \rho\mathbf{v}, g \text{ a.e. on } \Omega.$$

Moreover, suppose that there exist $a, b \in \mathbb{R}$ such that $\mathbf{F}_{|(a,b)} > 0$ and

$$\forall \nu > 0,\ a \leq \mathbf{w}_{0,\nu} \leq b; \quad \mathbf{w}_{0,\nu} \underset{\nu \to 0}{\longrightarrow} \mathbf{w}_0 \text{ a.e. on } \mathbb{R}.$$

Suppose that $(\mathbf{w}_\nu)_\nu \subset \mathbf{L}^\infty(\Omega)$ is a sequence of weak solutions to

$$\begin{cases} \partial_t(\rho_\nu \mathbf{w}_\nu) + \partial_x(\rho_\nu \mathbf{v}_\nu \mathbf{w}_\nu) = \rho_\nu g_\nu \mathbf{F}(\mathbf{w}_\nu) \\ \qquad \rho_\nu(\cdot,0)\mathbf{w}_\nu(\cdot,0) = \rho_\nu(\cdot,0)\mathbf{w}_{0,\nu}, \end{cases} \tag{6.0.7}$$

verifying the renormalization property. Then:

1. *There exists $\mathbf{w} \in \mathbf{L}^\infty(\Omega)$ such that $(\mathbf{w}_\nu)_\nu \to \mathbf{w}$ a.e. on Ω.*
2. *The function $\mathbf{w}$ is a weak solution to the transport equation (6.0.2) with source term given by (6.0.6), and it verifies the renormalization property.*

Proof. *1.* We split the study into two steps.
Step 1. The uniform $\mathbf{L}^\infty$ bound of $(\mathbf{w}_\nu)_\nu$ provides the existence, up to the extraction of a subsequence (not relabeled), for a.e. $(x,t)\in\Omega$ of a Borel probability measure $m_{(x,t)}$ on $\mathbb{R}$ such that for each $\varphi\in\mathbf{C}(\mathbb{R})$, $(\varphi(\mathbf{w}_\nu))_\nu$ converges $\mathbf{L}^\infty$-weakly* to $\overline{\varphi}$ where for a.e. $(x,t)\in\Omega$:

$$\overline{\varphi}(x,t)=\int_{\mathbb{R}}\varphi(y)\,\mathrm{d}m_{(x,t)}\,(y),$$

see for example [76, 138]. Let us suppose that there exists $\varepsilon>0$ such that for all $\nu>0$, $a+\varepsilon\leq\mathbf{w}_\nu\leq b-\varepsilon$. Introduce the $\mathbf{C}^1([a+\varepsilon,b-\varepsilon])$ function

$$p(w)=\int_{(a+b)/2}^{w}\frac{\mathrm{d}y}{\mathbf{F}(y)}.$$

By the renormalization property, for all $\nu>0$, $u_\nu=p(\mathbf{w}_\nu)\in\mathbf{L}^\infty(\Omega)$ is a weak solution to

$$\begin{cases}\partial_t\left(\rho_\nu u_\nu\right)+\partial_x\left(\rho_\nu\mathbf{v}_\nu u_\nu\right)=\rho_\nu g_\nu\\ \qquad\qquad\rho_\nu(\cdot,0)u_\nu(\cdot,0)=\rho_\nu(\cdot,0)\;(p\circ\mathbf{w}_{0,\nu})(\cdot).\end{cases}\tag{6.0.8}$$

Note that the source term does not depend on u_ν; this is the reason behind the choice of p above. Moreover, Theorem 6.0.3 ensures that u_ν verifies the renormalization property. By definition, for all test functions $\phi\in\mathbf{C}^\infty_{\mathbf{c}}(\mathbb{R}\times[0,T))$, we have

$$\int_0^T\int_{\mathbb{R}}\Big((\rho_\nu u_\nu)\partial_t\phi+(\rho_\nu\mathbf{v}_\nu u_\nu)\partial_x\phi+(\rho_\nu g_\nu)\phi\Big)\,\mathrm{d}x\,\mathrm{d}t+\int_{\mathbb{R}}\rho_\nu(x,0)p(\mathbf{w}_{0,\nu}(x))\phi(x,0)\,\mathrm{d}x=0.\tag{6.0.9}$$

Now from this, we take two routes.

Route 1: limit first, renormalization second. We can safely pass to the limit in (6.0.9). This proves that $\overline{p}$ is a weak solution to

$$\begin{cases}\partial_t\left(\rho\overline{p}\right)+\partial_x\left(\rho\mathbf{v}\overline{p}\right)=\rho g\\ \qquad\rho_\nu(\cdot,0)\overline{p}(\cdot,0)=\rho_\nu(\cdot,0)\;(p\circ\mathbf{w}_0)(\cdot).\end{cases}$$

Since the source term of this last transport equation is of the form covered by Theorem 6.0.3 (Remark 6.0.2), we are assured that $\overline{p}$ verifies the renormalization property. Applying it with exp, we obtain that $\overline{u}=\exp(\overline{p})$ is a weak solution to

$$\begin{cases}\partial_t\left(\rho\overline{u}\right)+\partial_x\left(\rho\mathbf{v}\overline{u}\right)=\rho g\overline{u}\\ \qquad\rho_\nu(\cdot,0)\overline{u}(\cdot,0)=\rho_\nu(\cdot,0)\exp(p\circ\mathbf{w}_0).\end{cases}\tag{6.0.10}$$

Route 2: renormalization first, limit second. From (6.0.8), we apply the renormalization property to u_ν ($\nu>0$) with $p=\exp$. This ensures that $U_\nu=\exp(u_\nu)$ is a weak solution to

$$\begin{cases}\partial_t\left(\rho_\nu U_\nu\right)+\partial_x\left(\rho_\nu\mathbf{v}_\nu U_\nu\right)=\rho_\nu g_\nu U_\nu\\ \qquad\qquad\rho_\nu(\cdot,0)U_\nu(\cdot,0)=\rho_\nu(\cdot,0)\exp(p\circ\mathbf{w}_{0,\nu}),\end{cases}$$

i.e. for all test functions $\phi \in \mathbf{C}_{\mathbf{c}}^{\infty}(\mathbb{R} \times [0,T))$, we have

$$\int_0^T \int_{\mathbb{R}} \Big((\rho_\nu U_\nu)\partial_t\phi + (\rho_\nu \mathbf{v}_\nu U_\nu)\partial_x\phi + (\rho_\nu g_\nu U_\nu)\phi \Big)\,\mathrm{d}x\,\mathrm{d}t + \int_{\mathbb{R}} \rho_\nu(x,0)\exp(p(\mathbf{w}_{0,\nu}(x)))\phi(x,0)\,\mathrm{d}x = 0.$$

We now let $\nu \to 0$ in this formulation to obtain that $\overline{\exp \circ p}$ is a weak solution to (6.0.10). By uniqueness (see Theorem 6.0.3 and Remark 6.0.2), $\overline{\exp \circ p} = \exp \circ \overline{p}$ a.e. on Ω. Consequently, for a.e. $(x,t) \in \Omega$,

$$\begin{aligned}\exp\left(\overline{p}(x,t)\right) &= \exp\left(\int_{\mathbb{R}} p(y)\,\mathrm{d}m_{(x,t)}\,(y)\right) \\ &\leq \int_{\mathbb{R}} \exp\left(p(y)\right)\mathrm{d}m_{(x,t)}\,(y) = \overline{\exp(p)}(x,t) = \exp\left(\overline{p}(x,t)\right).\end{aligned}$$

Since exp is strictly convex, the function $y \mapsto p(y)$ is constant $m_{(x,t)}$-a.e. and consequently, for a.e. $(x,t) \in \Omega$, $m_{(x,t)} = m_{\alpha(x,t)}$ for some function $\alpha : \Omega \to \mathbb{R}$. Finally, for all $\nu > 0$, and for all bounded open subsets $U \subset \Omega$,

$$\begin{aligned}\|\mathbf{w}_\nu\|^2_{\mathbf{L}^2(U)} &= \int_0^T \int_{\mathbb{R}} \mathbf{w}_\nu^2 \mathbb{1}_U\,\mathrm{d}x\,\mathrm{d}t \\ &\underset{\nu\to 0}{\longrightarrow} \int_0^T \int_{\mathbb{R}} \left(\int_{\mathbb{R}} y^2\,\mathrm{d}m_{(x,t)}\,(y)\right)\mathbb{1}_U\,\mathrm{d}x\,\mathrm{d}t \\ &= \int_0^T \int_{\mathbb{R}} \alpha(x,t)^2 \mathbb{1}_U\,\mathrm{d}x\,\mathrm{d}t = \|\mathbf{w}\|^2_{\mathbf{L}^2(U)},\end{aligned}$$

which implies that $\mathbf{w}_\nu \to \mathbf{w}$ in $\mathbf{L}^2_{\mathbf{loc}}(\Omega)$. A standard diagonal process yields a subsequence of $(\mathbf{w}_\nu)_\nu$ that converges a.e. on Ω to $\mathbf{w}$.

Step 2. We now get back to the general case. Fix $\varepsilon > 0$ and consider the cut-off functions

$$\mathbf{F}_\varepsilon(r) = \max\{F(r), \varepsilon\}; \quad T_\varepsilon(w) = \min\{\min\{a+\varepsilon, w\}, b-\varepsilon\}.$$

Since $\mathbf{F}_\varepsilon \in \mathbf{C}(\mathbb{R})$ and $\mathbf{F}_\varepsilon \geq \varepsilon > 0$, Lemma 6.0.4 ensures that the transport equation

$$\begin{cases} \partial_t\left(\rho_\nu \mathbf{w}\right) + \partial_x\left(\rho_\nu \mathbf{v}_\nu \mathbf{w}\right) = \rho_\nu g_\nu \mathbf{F}_\varepsilon(\mathbf{w}) \\ \qquad\qquad \rho_\nu(\cdot,0)\mathbf{w}(\cdot,0) = \rho_\nu(\cdot,0) T_\varepsilon(\mathbf{w}_{0,\nu}) \end{cases}$$

admits a weak solution $\mathbf{w}_{\nu,\varepsilon}$. Note that from Proposition 6.0.5, for all $\nu, \varepsilon > 0$,

$$\begin{aligned}\|\mathbf{w}_{\nu,\varepsilon}(\cdot,t) - \mathbf{w}_\nu(\cdot,t)\|_{\mathbf{L}^\infty} &\leq \|\mathbf{w}_{\nu,\varepsilon}(\cdot,t) - \mathbf{w}_\nu(\cdot,t)\|_{\mathbf{L}^\infty} \\ &\quad + \sup_{\nu>0}\|g_\nu\|_{\mathbf{L}^\infty}\int_0^t \|\mathbf{F}(\mathbf{w}_\nu(\cdot,s)) - \mathbf{F}_\varepsilon(\mathbf{w}_{\nu,\varepsilon}(\cdot,s))\|_{\mathbf{L}^\infty}\,\mathrm{d}s \\ &\leq \varepsilon + \sup_{\nu>0}\|g_\nu\|_{\mathbf{L}^\infty}\left(\|\mathbf{F}'\|_{\mathbf{L}^\infty}\int_0^t \|\mathbf{w}_\nu(\cdot,s) - \mathbf{w}_{\nu,\varepsilon}(\cdot,s)\|_{\mathbf{L}^\infty}\,\mathrm{d}s + \varepsilon t\right),\end{aligned}$$

since $\|\mathbf{F}-\mathbf{F}_\varepsilon\|_{\mathbf{L}^\infty}\leq\varepsilon$. From this, we deduce with Gronwall lemma, that

$$\forall\nu,\varepsilon>0,\quad \|\mathbf{w}_{\nu,\varepsilon}-\mathbf{w}_\nu\|_{\mathbf{L}^\infty(\Omega)}\leq \underbrace{\varepsilon\,(1+\sup_{\nu>0}\|g_\nu\|_{\mathbf{L}^\infty}\|\mathbf{F}'\|_{\mathbf{L}^\infty}T)\exp(\sup_{\nu>0}\|g_\nu\|_{\mathbf{L}^\infty}\|\mathbf{F}'\|_{\mathbf{L}^\infty}T)}_{\mathbf{C}}\,. \tag{6.0.11}$$

Clearly, if $0<\varepsilon\leq 1$, inequality (6.0.11) establishes a uniform $\mathbf{L}^\infty$ bound for the sequence $(\mathbf{w}_{\nu,\varepsilon})_\nu$ since $(\mathbf{w}_\nu)_\nu$ is bounded in $\mathbf{L}^\infty$ by assumption. Consequently, since $\mathbf{F}_\varepsilon\geq\varepsilon>0$, Step 1 provides the existence of $\mathbf{w}_\varepsilon\in\mathbf{L}^\infty(\Omega)$ such that a subsequence of $(\mathbf{w}_{\nu,\varepsilon})_\nu$ converges a.e. on Ω to $\mathbf{w}_\varepsilon$. Now, by a standard topological argument we prove that (6.0.11) leads to strong compactness for the sequence $(\mathbf{w}_\nu)_\nu$. More precisely, we are to prove that $(\mathbf{w}_\nu)_\nu$ is relatively compact in $\mathbf{L}^1_{\mathbf{loc}}(\Omega)$. Fix $\mathbf{K}\subset\Omega$ a compact subset of Ω and fix $\delta>0$. Since for all $\varepsilon>0$, $(\mathbf{w}_{\nu,\varepsilon})_\nu$ converges a.e. on Ω and is uniformly bounded in $\mathbf{L}^\infty$, the sequence converges in $\mathbf{L}^1(\mathbf{K})$. Consequently, for all $\varepsilon>0$, $(\mathbf{w}_{\nu,\varepsilon})_\nu$ is relatively compact in $\mathbf{L}^1_{\mathbf{loc}}(\mathbf{K})$. Fix $\varepsilon>0$ such that, with $\mathbf{C}$ defined in (6.0.11),

$$\mathrm{mes}(\mathbf{K})\mathbf{C}\varepsilon\leq\frac{\delta}{2}.$$

Now use the precompactness of $(\mathbf{w}_{\nu,\varepsilon})_\nu$ to introduce a finite covering

$$\left\{\mathbf{B}_{\mathbf{L}^1}\left(u_i,\frac{\delta}{2}\right)\right\}_{1\leq i\leq J};\quad u_i\in\mathbf{L}^1(\mathbf{K}),\ J\in\mathbb{N}^*.$$

We conclude by verifying that $\{\mathbf{B}_{\mathbf{L}^1}(u_i,\delta)\}_{1\leq i\leq J}$ is a covering of $(\mathbf{w}_\nu)_\nu$. Fix $\nu>0$ and $i\in\{1,\dots,J\}$ such that $\mathbf{w}_{\nu,\varepsilon}\in\mathbf{B}_{\mathbf{L}^1}\left(u_i,\frac{\delta}{2}\right)$. Using the triangle inequality and the definition of ε, we obtain:

$$\begin{aligned}\|\mathbf{w}_\nu-u_i\|_{\mathbf{L}^1(\mathbf{K})}&\leq\|\mathbf{w}_\nu-\mathbf{w}_{\nu,\varepsilon}\|_{\mathbf{L}^1(\mathbf{K})}+\|\mathbf{w}_{\nu,\varepsilon}-u_i\|_{\mathbf{L}^1(\mathbf{K})}\\&\leq\mathrm{mes}(\mathbf{K})\|\mathbf{w}_\nu-\mathbf{w}_{\nu,\varepsilon}\|_{\mathbf{L}^\infty(\Omega)}+\frac{\delta}{2}\\&\leq\mathrm{mes}(\mathbf{K})\mathbf{C}\varepsilon+\frac{\delta}{2}\leq\delta.\end{aligned}$$

We can conclude that a subsequence of $(\mathbf{w}_\nu)_\nu$ converges in $\mathbf{L}^1_{\mathbf{loc}}(\Omega)$ to some $\mathbf{w}\in\mathbf{L}^\infty(\Omega)$. A further extraction establishes the a.e. convergence.

2. Passing to the limit in the weak formulation satisfied by $(\mathbf{w}_\nu)_\nu$, we obtain that $\mathbf{w}$ is a weak solution to (6.0.2) with source term given by (6.0.6). By uniqueness of such a weak solution, see Proposition 6.0.5, the whole sequence $(\mathbf{w}_\nu)_\nu$ converges to $\mathbf{w}$. Finally, Theorem 6.0.3 *(i)* applied with

$$\mathbf{S}(x,t)=g(x,t)\mathbf{F}(\mathbf{w}(x,t))$$

ensures that $\mathbf{w}$ satisfies the renormalization property, concluding the proof. □

We conclude the chapter with a well-posedness result for the transport equation (6.0.2) with source term (6.0.6) where we consider functions $\mathbf{F}$ which satisfy:

$$\exists a, b \in \mathbb{R}\ (a < b), \quad \mathbf{F} \in \mathbf{Lip}([a,b]), \ \ \mathbf{F}(a) = \mathbf{F}(b) = 0 \text{ and } \mathbf{F} > 0 \text{ on } (a,b). \tag{6.0.12}$$

Our study is motivated by the particular case $a, b = 0, 1$ and $\mathbf{F}(\mathbf{w}) = \mathbf{w}(1-\mathbf{w})$.

Theorem 6.0.7. *Let $\rho, \mathbf{v} \in \mathbf{L}^\infty(\Omega)$ satisfy (6.0.1), $g \in \mathbf{L}^\infty(\Omega)$, $\mathbf{F}$ satisfying (6.0.12) and $\mathbf{w}_0 \in \mathbf{L}^\infty(\mathbb{R}; [a,b])$. Then the transport equation (6.0.2) with source term given by (6.0.6) admits at least a weak solution. Moreover, this solution verifies the renormalization property.*

Proof. The idea is to construct sequences $(\rho_k)_k, (\mathbf{v}_k)_k, (g_k)_k$ satisfying the assumptions of Theorem 6.0.6. For the sake of consistency, let us extend $\mathbf{F}$ on $\mathbb{R}\backslash[a,b]$ so that $\mathbf{F} \in \mathbf{Lip}(\mathbb{R})$ and verifies the assumption of Theorem 6.0.6.
Fix $\varphi \in \mathbf{C}^\infty_\mathbf{c}(\mathbb{R}), \varphi \geq 0$ a test function of mass 1 and supported in $[-1, 0]$. For all $k \in \mathbb{N}^*$, consider the function

$$\theta_k(x,t) = \frac{\varphi(kx)\varphi(kt)}{k^2} \in \mathbf{C}^\infty_\mathbf{c}((\mathbb{R}^2; \mathbb{R}^+)).$$

We now introduce the smooth approximations of the coefficients:

$$\rho_k = \rho * \theta_k + \frac{1}{k}; \quad \mathbf{V}_k = (\rho\mathbf{v}) * \theta_k + \frac{1}{k}; \quad g_k = g * \theta_k.$$

The sequences $(\rho_k)_k, (\mathbf{V}_k)_k$ and $(g_k)_k$ are sequences of smooth functions that converge in $\mathbf{L}^1_\mathbf{loc}(\Omega)$ to $\rho, \rho\mathbf{v}$ and g, respectively, and even if it means taking subsequences, we can assume that the convergence is a.e. on Ω. Note also that since $\rho \geq 0$, then $\rho_k \geq \dfrac{1}{k} > 0$. Fix $\phi \in \mathbf{C}^\infty_\mathbf{c}(\Omega)$. It is readily checked that $\partial_t \rho_k + \partial_x \mathbf{V}_k = 0$ in $\mathcal{D}'(\Omega)$, and since ρ_k and $\mathbf{V}_k$ are smooth, the equality holds pointwise. Consider now $(\mathbf{w}_{0,k})_k \subset \mathbf{C}^1(\mathbb{R})$ such that

$$\forall k \in \mathbb{N}^*, \ a \leq \mathbf{w}_{0,k} \leq b \quad \text{and} \quad \mathbf{w}_{0,k} \underset{k \to +\infty}{\longrightarrow} \mathbf{w}_0 \text{ a.e. on } \mathbb{R}.$$

Since ρ_k does not vanish, the function $\mathbf{v}_k = \dfrac{\mathbf{V}_k}{\rho_k}$ is smooth, moreover, it verifies the uniform $\mathbf{L}^\infty$ bound:

$$\forall k \in \mathbb{N}^*, \quad |\mathbf{v}_k| = \frac{|\mathbf{V}_k|}{\rho_k} = \frac{|(\rho\mathbf{v}) * \theta_k + 1/k|}{\rho * \theta_k + 1/k} \leq \|\mathbf{v}\|_{\mathbf{L}^\infty} + 1.$$

We can define $\mathbf{w}_k \in \mathbf{Lip}(\Omega)$ as the classical solution to the following transport equation:

$$\begin{cases} \partial_t \mathbf{w}_k + \mathbf{v}_k \partial_x \mathbf{w}_k = g_k \mathbf{F}(\mathbf{w}_k) \\ \qquad\qquad \mathbf{w}_k(\cdot, 0) = \mathbf{w}_{0,k}. \end{cases} \tag{6.0.13}$$

Indeed, we can solve this PDE using the method of characteristics. More precisely, fix $(x,t) \in \Omega$. First, we solve the following system of ODEs $(0 < s < t)$:

$$\begin{cases} \dot{\xi}_k(s) &= \mathbf{v}_k(\xi_k(s), s) \\ \xi_k(t) &= x \end{cases} \qquad \begin{cases} \dot{u}_k(s) &= g_k(\xi_k(s), s)\mathbf{F}(u_k(s)) \\ u_k(0) &= \mathbf{w}_{0,k}(\xi_k(0)). \end{cases}$$

The first ODE admits a unique global solution since $\mathbf{v}_k$ is smooth and bounded. Moreover, since $(s,u) \mapsto g_k(\xi_k(s),s)\mathbf{F}(u)$ is continuous and Lipschitz continuous with respect to the u variable, the second ODE admits a unique solution. This defines $\mathbf{w}_k$ everywhere in Ω. Note that since $u_k(0) \in [a,b]$, Assumption (6.0.12) ensures that $u(s) \in [a,b]$ for all $s \in [0,t]$. Consequently, $\mathbf{w}_k(x,t) = \mathbf{w}_k(\xi(t),t) = u(t) \in [a,b]$. Hence:

$$\forall k \in \mathbb{N}^*,\ \forall (x,t) \in \Omega, \quad a \leq \mathbf{w}_k(x,t) \leq b. \tag{6.0.14}$$

It is classical that $\mathbf{w}_k$ defined that way is a classical solution to the PDE (6.0.13) and also to

$$\begin{cases} \partial_t(\rho_k \mathbf{w}_k) + \partial_x(\rho_k \mathbf{v}_k \mathbf{w}_k) = g_k \mathbf{F}(\mathbf{w}_k) \\ \qquad\qquad \rho_k(\cdot,0)\mathbf{w}_k(\cdot,0) = \rho_k(\cdot,0)\mathbf{w}_{0,k}. \end{cases} \tag{6.0.15}$$

since $\rho_k > 0$. Therefore $\mathbf{w}_k$ is also a weak solution to (6.0.15). Since we also have, for any $p \in \mathbf{C}^1(\mathbb{R})$,

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(p(u_k(s))\right) = g_k(\xi_k(s),s)\mathbf{F}(u_k(s))p'(u_k(s)),$$

we deduce the same way that $U_k = p(\mathbf{w}_k)$ is a weak solution to

$$\begin{cases} \partial_t(\rho_k U_k) + \partial_x(\rho_k \mathbf{v}_k U_k) = g_k \mathbf{F}(\mathbf{w}_k)p'(\mathbf{w}_k) \\ \qquad\qquad \rho_k(\cdot,0)U_k(\cdot,0) = \rho_k(\cdot,0)\ (p \circ \mathbf{w}_{0,k})(\cdot), \end{cases} \tag{6.0.16}$$

i.e. the sequence $(\mathbf{w}_k)_k$ is a sequence of weak solutions to (6.0.15) which satisfy the renormalization property.

All the hypotheses of Theorem 6.0.6 are fulfilled. Consequently, there exists $\mathbf{w} \in \mathbf{L}^\infty(\Omega;[a,b])$ such that $(\mathbf{w}_k)_k$ converges a.e. to $\mathbf{w}$, $\mathbf{w}$ is a weak solution to (6.0.2) and it verifies the renormalization property. □

Putting together Proposition 6.0.5 and Theorem 6.0.7, we proved:

Corollary 6.0.8. *Let $\rho, \mathbf{v} \in \mathbf{L}^\infty(\Omega)$ satisfy (6.0.1), $g \in \mathbf{L}^\infty(\Omega)$ and $\mathbf{F}$ satisfying (6.0.12). Then for any initial data $\mathbf{w}_0 \in \mathbf{L}^\infty(\mathbb{R};[a,b])$, the transport equation*

$$\begin{cases} \partial_t(\rho \mathbf{w}) + \partial_x(\rho \mathbf{v} \mathbf{w}) = \rho g \mathbf{F}(\mathbf{w}) \\ \qquad\qquad \rho(\cdot,0)\mathbf{w}(\cdot,0) = \rho(\cdot,0)\mathbf{w}_0. \end{cases}$$

admits a unique weak solution $\mathbf{w} \in \mathbf{L}^\infty(\Omega;[a,b])$. Moreover, $\mathbf{w}$ verifies the renormalization property.

CHAPTER 7

EXISTENCE ANALYSIS AND NUMERICAL APPROXIMATION FOR A SECOND ORDER MODEL OF TRAFFIC WITH ORDERLINESS MARKER

7.1 Introduction

This chapter is devoted to mathematical and numerical analysis of a 2×2 system of balance laws with non-local coupling. Our motivations come from macroscopic modeling of road traffic, and more specifically, from taking into account the distinction between ordered or disordered behaviors of drivers within the paradigm of the so-called Generalized Second-Order Models (GSOM).

7.1.1 Generalities on macroscopic PDE traffic models

Let us start by providing a brief account on advantages and drawbacks (in terms of modeling, but also in terms of completeness and flexibility of their mathematical and numerical analysis) of first-order and second-order hyperbolic models for road traffic, including phase transition models that combine both of the above. More information can be found, *e.g.*, in the surveys and monographs [30, 139, 143]. In Section 7.1.2, we will insert our work within this general picture and highlight the analytical purpose of our work that goes beyond its modeling purpose.

The fundamental flow equation

Although traffic description in terms of individual agents and their interactions is relevant, typically it leads to large ODE systems which mathematical analysis is cumbersome; moreover, they may encrypt the relevant traffic information (such as presence of shock waves) in

a non-obvious way. The influence of fluid mechanics and the well developed mathematical machinery of hyperbolic PDEs and their approximation made macroscopic models very popular, starting from the pioneering Lighthill-Whitham and Richards model. All these models are based on the fundamental flow equation

$$\partial_t \rho + \partial_x(\rho v) = 0 \tag{7.1.1}$$

with ρ representing the density of the flow, bounded by some maximum value, and v representing the velocity. Different models are built upon this equation by adding functional and/or differential relations linking the two state variables ρ and v (or ρ and ρv).

First-order models

These models use an explicit closure relation linking v to ρ by a functional dependence, such as $v(\rho) = V_{\max}(1 - \frac{\rho}{\rho_{\max}})$. The classical Lighthill-Whitham and Richards model [125, 141] (LWR, in the sequel) is the prototype of the whole class. We refer to [29] for a survey of first-order models. The major advantage of such models is the possibility of their complete mathematical analysis, rigorous assessment of several approximation strategies, proved relation to certain microscopic many-particle models. Their theory is firmly attached to the classical theory of Kruzhkov entropy solutions to scalar conservation laws [114]. The robustness of the theory facilitates the introduction, into the first-order models, of additional features such as delays, non-locality, point constraints, variation of the number of lines, *etc*; see [29], see also [40, 56, 11, 31] for a few more recent examples. The clear drawback of the first-order models is their inadequacy to experimental data which exhibit a functional dependence of ρv on ρ only for low enough densities, see, *e.g.*, the experimental fundamental diagram in [87, Fig.1].

Second-order models

In the context of traffic flows, the name "second-order" is given to models describing the joint evolution of the state variables (ρ, v) (or $(\rho, \rho v)$) by means of a 2×2 system of PDEs. After the controversy of [69], the second-order model of Aw-Rascle and Zhang [25, 157] (ARZ, in the sequel) became popular. In [119, 118] a wide family of generalized second-order models (GSOM, in the sequel) was described. The mathematical structure of these models is a 2×2 system of conservation or balance laws, strictly hyperbolic away from the vacuum $\rho = 0$, with one genuinely nonlinear and one linearly degenerate characteristic fields. Selection criteria in terms of Riemann solver can be reformulated under the form of entropy conditions (see, in particular, [12] for a Kruzhkov-like choice of entropies). Variants of ARZ with additional features, as for the variants of the LWR model mentioned here above, were proposed. Existence analysis with, sometimes, numerical analysis could be extended to some of these variants, see, *e.g.*, [12]. However, the mathematical analysis of GSOM is not complete at the present stage, except for the case of the Riemann problems [118]. The additional complexity of ARZ and more generally, of GSOM is compensated by a better description of some of the features of traffic, yet for low densities and especially for vacuum the LWR model may represent a simpler and more reliable model.

Phase transition models

Phase transitions between a "free" and a "congested" states of flow were identified in the engineering literature, see *e.g.*, [109, 111], as the crucial property of real traffic flows responsible for the self-organization patterns such as the stop-and-go waves. The two phases are associated with two different regions of the experimental fundamental diagrams, like [87, Fig.1]. Several two-phase mathematical models with phase transitions were proposed. In particular, the model of [55] is close to the GSOM family, see [119]. In principle, these models offer a better description of traffic, combining the advantages of the first-order and the second-order models (*e.g.*, [91, 57]) and the insight from the engineering literature. This comes at the price of a much heavier mathematical treatment. Indeed, typically the phase-transition models are posed in terms of the Riemann solver (which describes, among other, the phase transition behavior) and the wave-front tracking algorithm with delicate control of variation is used for the existence analysis. Even slight modifications of such models may result in heavy modifications of the analysis of front interactions. We refer to [13] for one recent example of phase transition model enriched with point constraints and for a brief survey of mathematical literature on phase transition models.

7.1.2 Analytical and modeling purposes of the present work

Our purpose is two-fold. Our primary goal is to contribute to mathematical analysis for some GSOM models based upon the robust theory of scalar conservation laws like for the first-order case and on the theory of renormalization for the kind of transport equations encountered in typical GSOM. This line is an alternative to the classical line based on the general theory of hyperbolic systems of conservation laws, and it may allow for more flexibility when variants of the model are considered.
Our secondary goal is to enrich the GSOM family of models with a variant built on taking into account the state of orderliness in drivers' behavior and its evolution along vehicles' trajectories. Our mathematical analysis is developed having in mind the key features of this non-local variant of GSOM, though it may have wider applications.

Contributions into analysis and approximation of GSOM-kind models and systems of the Keyfitz-Kranzer kind

We develop adequate analysis and approximation tools for an exemplary GSOM model featuring non-local coupling between the equation for the density ρ and the equation for the auxiliary marker $\mathbf{w}$. The non-locality has a regularizing effect that makes the system under study reminiscent, in terms of the analytical approach, of the classical Keyfitz-Kranzer system [112]. In this situation, the central role is played by the renormalization property for the component $\mathbf{w}$ of the solution which evolves along the trajectories of the flow. This surprising - in view of the involved and celebrated theory [77] of renormalized solutions - structural property was established in [134] for general weak solutions $\mathbf{w}$ of the transport equation $\partial_t \mathbf{w} + \partial_x(\mathbf{w}v) = 0$ with the velocity v involved in the continuity equation $\partial_t \rho + \partial_x(\rho v) = 0$ for the density ρ, having in mind application to the Keyfitz-Kranzer system. We further develop the tool of the weak/renormalized solution adding nonlinear source terms in the Panov

setting [134] and uncovering a "propagation of compactness" mechanism proper to this linear equation.
Indeed, the renormalization structure yields compactness - either through the total variation control, or through the analysis in terms of Young measures. This structure also guides us in developing an original numerical strategy which enters, in a non-obvious way, the standard framework of finite volume approximations. It turns out that this numerical strategy can be seen as a generalization of the specific discretization strategy developed for the Keyfitz-Kranzer system [113]. Note that the renormalization property was already identified in [12] as a key ingredient in the study of the Aw-Rascle and Zhang system (ARZ, the best known example of GSOM) with point constraints at bottlenecks, and it can be instrumental as well for studying boundary-value problems for ARZ.
Applicable to a wider class of GSOM with non-local coupling, our analysis does not rely on the standard hyperbolic structure of the system. Instead, it relies upon a sort of decoupling due to the non-local dependence on $\mathbf{w}$ of the fundamental diagram $\rho \mapsto v(x,t,\rho)$. Moreover, we briefly discuss the possibility of pursuing this line of analysis for more standard local GSOM models, linking the question to the need for a deeper understanding of discontinuous-flux scalar conservation laws with moderately or wildly discontinuous in space flux function. Rigorous application of this approach to local GSOM is postponed to future work. Note that also the discretization strategy we pursue is applicable to the local GSOM.

Contribution to traffic modeling with GSOM

We propose a prototype model able to take into account the state of orderliness of drivers' behavior. Roughly speaking, we represent the state of the traffic by a family of fundamental diagrams $\rho \mapsto \rho v$ that depend on the additional orderliness parameter ω and interpolate between fundamental diagrams $\rho \mapsto \rho V_{\min}(\rho)$ (corresponding to $\omega = 0$, fully disordered traffic) and $\rho \mapsto \rho V_{\max}(\rho)$ (corresponding to $\omega = 1$, fully ordered traffic).
This idea was put forward by the authors in [23] (see also Chapter 2) with the goal to model self-organization (and disorganization) of traffic at bottlenecks, in the frame of the basic LWR model adapted to the presence of bottlenecks [11, 9, 10]. In [23], ω is a time-dependent parameter attached to the bottleneck; the passing capacity of the bottleneck is a function of the orderliness parameter ω. The dynamics of ω is governed by an ODE of the logistic type. This ODE is driven by averaged values of the density in the upstream neighbourhood of the bottleneck: this offers a mechanism of progressive ordering of the traffic (self-organization) in stable traffic conditions, and of quick disordering in the situations with abruptly growing averaged density upstream the bottleneck.
In view of the extensive evidence of self-organization of traffic beyond bottlenecks [110], we transpose this idea towards taking into account the influence of orderliness in drivers' behavior on the fundamental diagram of the flow in the bulk (so we do not focus on bottlenecks any more, unlike in [23]).
Many attempts have been made to model the self-organization in traffic and its salient features like the stop-and-go waves. One important paradigm for these models is phase transitions, resulting in formulation of two-phase models [111, 58]. Some of two-phase models are close, in their structure, to the GSOM models [55, 119]. In the present chapter, we propose a toy model which can be situated at the crossroads of the above mentioned ideas. It has the structure

of GSOM with the Lagrangian marker interpreted as the orderliness parameter. It can be seen as a two-phase model, due to the fact that we take $V_{\min} \equiv V_{\max}$ for low densities. And it borrows from [23] the mechanism for the evolution of the orderliness marker $\mathbf{w}$ attached to individual vehicles. We define the orderliness parameter $\omega = \omega(x,t)$ of the fundamental diagram as the weighted average, over a small vicinity of every point (x,t), of the individual orderliness marker $\mathbf{w}$. The corresponding local model (with $\omega = \mathbf{w}$) makes sense and it is briefly discussed.
Any attempt to link the model we work with, or the values of the parameters of this model, to road traffic data is far beyond the scope of this chapter. As a matter of fact, we have in mind the whole class of systems of non-local GSOM kind of which our exemplary model is a particular instance. Indeed, the mathematical analysis we carry out is suitable for a wide family of non-local GSOM models, including source terms for evolution of the Lagrangian marker.

7.2 The GSOM-kind model with orderliness

Once and for all, fix a time horizon $T > 0$ and denote $\Omega = \mathbb{R} \times (0,T)$. We consider that the maximal density $\rho_{\max}$ on the road equals 1. In our new model, the first equation on $[0,1]$-valued density ρ,

$$\partial_t \rho + \partial_x \left(\rho \mathbf{v}(x,t,\rho)\right) = 0, \quad (x,t) \in \Omega, \tag{7.2.1}$$

expresses the conservation of mass and it is driven by a time and space dependent velocity $\mathbf{v}$. This dependency reads:

$$\mathbf{v}(x,t,\rho) = (1-\omega(x,t))V_{\min}(\rho) + \omega(x,t)V_{\max}(\rho). \tag{7.2.2}$$

In (7.2.2), $V_{\min}, V_{\max}$ are the two levels of traffic velocity; the one for the ordered regime of traffic and the other for the disordered regime. As usual, we require both of them to be nonincreasing and nonnegative Lipschitz continuous functions defined for $\rho \in [0,1]$: naturally, $V_{\max} \geq V_{\min}$. The actual velocity $\mathbf{v}$ in (7.2.2) is a convex combination of the two regimes' velocities with $\omega(x,t) \in [0,1]$ representing the state of orderliness of the traffic at time t and position x. We further consider the orderliness parameter $\mathbf{w}$ associated to individual vehicles, which is evolved according to the transport equation

$$\partial_t \left(\rho \mathbf{w}\right) + \partial_x \left(\rho \mathbf{w} \mathbf{v}(x,t,\rho)\right) = \rho s(x,t,\mathbf{w}). \tag{7.2.3}$$

For a regular velocity field, equation (7.2.3) corresponds to the evolution of $\mathbf{w}$ according to the ODE $\dot{\mathbf{w}}(X(t),t) = s\big(X(t),t,\mathbf{w}(X(t),t)\big)$ along the integral curves $x = X(t)$ of the velocity field $\mathbf{v}$. In absence of regularity of $\mathbf{v}$, the rigorous meaning to such evolution is provided by the weak formulation (7.2.3) which, moreover, automatically implies the renormalization property (see Chapter 6). The coupling of (7.2.1), (7.2.2) with (7.2.3) is provided by relations linking ω, s to $\mathbf{w}$, ρ.
First, we concentrate on the choice of the source term s in (7.2.3): it is directly inspired by our previous work [23] where self-organization at bottlenecks, governed by an analogous orderliness parameter ω, is considered. Let us take $s(x,t,\mathbf{w}) = \mathbf{K}\mathbf{w}(1-\mathbf{w})$ where $\mathbf{K}$, depending on ρ and $\partial_t \rho$ in a non-local way, reflects a mechanism of ordering/disordering subject to the

traffic conditions in a vicinity of each point (x,t). To this end, we introduce the subjective density

$$\xi(x,t) = \int_{\mathbb{R}} \rho(y,t)\mu(x-y)\,\mathrm{d}y\,, \tag{7.2.4}$$

where $\mu \geq 0$, $\displaystyle\int_{\mathbb{R}} \mu(x)\,\mathrm{d}x = 1$, is a smooth weight function used to average ρ, similarly to non-local models of [11, 31, 23]. Further, we make $\mathbf{K}$ depend on ρ through the subjective density ξ and its time variations $\partial_t\xi$. For future use, let us make precise that classical PDE computations using the weak formulation of (7.2.1) ensure that ξ admits a time derivative in the sense of the distributions and that for a.e. $(x,t) \in \Omega$,

$$\partial_t\xi(x,t) = -\int_{\mathbb{R}} \rho(y,t)\mathbf{v}(y,t,\rho)\mu'(x-y)\,\mathrm{d}y\,.$$

This comes from using $\varphi(y,t) = \mu(x-y)\psi(t)$ $(x \in \mathbb{R})$ as a test function in the weak formulation, see for instance Remark 2.1.1. To sum up, we take

$$s(x,t,\mathbf{w}) = \mathbf{K}(\xi,\partial_t\xi)\mathbf{w}(1-\mathbf{w}) \tag{7.2.5}$$

for some $\mathbf{K} : [0,1]\times\mathbb{R} \mapsto \mathbb{R}$. To fix the ideas, in the simulations we will take, following [23],

$$\mathbf{K}(\xi,\chi) = C\Big(\frac{\xi}{\xi_c} - 1\Big)^+\Big(1 - \frac{\chi^+}{D_+} - \frac{\chi^-}{D_-}\Big)$$

with some threshold $\xi_c \in (0,1)$ and constants $C > 0$, $D_+ \gg D_- > 0$ (see Figure 7.1). Mathematically speaking, we only suppose that $\mathbf{K} \in \mathbf{Lip_{loc}}([0,1]\times\mathbb{R})$. The idea behind the above choice of $\mathbf{K}$ is to allow for progressive ordering of the traffic with time when the traffic conditions are stable, and for a quick disordering when sudden and strong variations (especially in the case of densification) of the traffic occur. Note that random fluctuations of $\mathbf{w}$ could be considered, as a further step of modeling, but this is beyond the scope of our work.

The key features of the dynamics of $\mathbf{w}$ encoded in (7.2.3)–(7.2.5) with the above choice of $\mathbf{K}$ are as follows:

- conservation of the "momentum" quantity $\rho\mathbf{w}$ in the region of low densities, because $\mathbf{K}$ is zero for low densities;
- rapid decrease of $\rho\mathbf{w}$ for moderate and particularly for high densities, under strong density variations (disordering);
- progressive increase of $\rho\mathbf{w}$ in dense and very dense traffic with small density variations (ordering).

Finally, let us write the link between ω in (7.2.2) and the individual ordering markers $\mathbf{w}$ as

$$\omega = \mathcal{M}[\mathbf{w}] \tag{7.2.6}$$

where $\mathcal{M}$ is an operator on $\mathbf{L}^\infty(\Omega;[0,1])$. We have in mind the following three choices. For the simplest one, $\mathcal{M} = \mathrm{Id}$, *i.e.*, $\omega = \mathbf{w}$, (7.2.1) can be seen as an LWR equation with space-time discontinuous flux. Its mathematical study still requires deeper analysis, despite much progress made in this direction. We briefly discuss the issue in Section 7.5.2. Because traffic

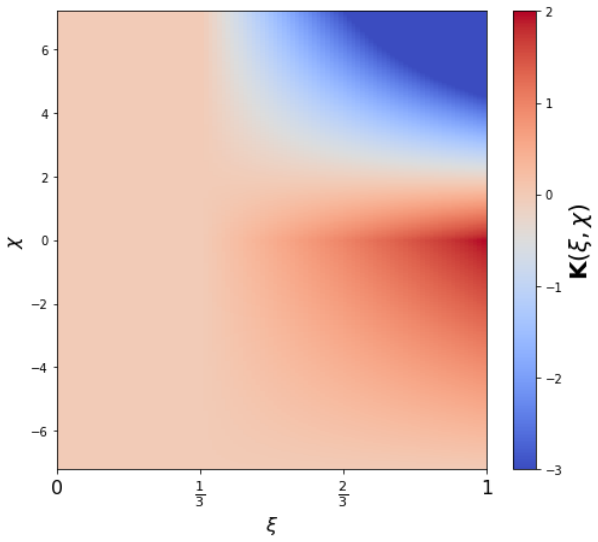

Figure 7.1 – Typical behavior of the orderliness-driving function $\mathbf{K}$.

involves only a limited number of agents in a neighbourhood of each point, in this chapter we focus on the non-local impact of the individual vehicle markers $\mathbf{w}$ on the global traffic orderliness ω. Two variants will be considered. In Section 7.4, the existence will be obtained with

$$\mathcal{M}[\mathbf{w}](x,t) = \int_{-\infty}^{t}\int_{\mathbb{R}} \mathbf{w}(y,s)\eta(x-y,t-s)\,\mathrm{d}y\,\mathrm{d}s\,. \tag{7.2.7}$$

In (7.2.7), the function η is a weight function of the form $\eta(x,t)=\eta_1(x)\eta_2(t)$ with $\eta_1 \in \mathbf{C}^1_{\mathbf{c}}(\mathbb{R})$ and $\eta_2 \in \mathbf{BV}(\mathbb{R})$ and supported in a compact subset of $[0,T)$. Note also that to make sense of (7.2.7), we will extend $\mathbf{w}$ by the initial data $\mathbf{w}_0$ for negative times. Note that the space averaging means that the perception, by the drivers, of the traffic conditions relies on their observations of their immediate neighbourhood (typically, several dozens of meters downstream the flow) and the time averaging means that the drivers' perception of the situation is not instantaneous. Remark that the non-locality in time only looks in the past. In Section 7.5.1 and throughout Section 7.6, we assume a stronger reactivity of the drivers to instantaneous traffic conditions in their immediate neighbourhood, and take the mere space averaging

$$\mathcal{M}[\mathbf{w}](x,t) = \int_{\mathbb{R}} \mathbf{w}(y,t)\eta(x-y)\,\mathrm{d}y\,. \tag{7.2.8}$$

with $\eta \equiv \eta_1$. For the mathematical analysis of the resulting system, the difference between (7.2.7) and (7.2.8) is that that the latter one requires the $\mathbf{BV}$ framework for existence analysis, while the first choice is regularizing enough to deal with mere $\mathbf{L}^\infty$ solutions and data.

Finally, we stress that we have in mind the situation where

$$\exists \rho_f \in (0,1),\ \forall \rho \in [0,\rho_f], \quad V_{\min}(\rho) = V_{\max}(\rho) \tag{7.2.9}$$

so that (7.2.1)–(7.2.6) exhibits a two-phase behavior with $\rho \in [0,\rho_f]$ corresponding to the free traffic flow phase while $\rho > \rho_f$ correspond to the congested traffic.

We are now in a position of presenting the outline of the chapter. In Section 7.3 we fix the mathematical framework of our work. The equation (7.2.1) is understood in the sense of Kruzhkov entropy solutions [114] of LWR models. The equation prescribing the evolution of the orderliness marker (7.2.3) is understood in the weak and renormalized sense of Panov [134] for one-dimensional transport equations driven by zero-divergence coefficients, with necessary adaptations. Indeed, an important ingredient of our analysis is the refinement of the theory of weak (and renormalized) solutions of transport PDEs of the kind (7.2.3) under the key assumptions that the coefficients form a zero-divergence field in Ω, and for a wide class of source fields with separation on (x,t) and $\mathbf{w}$ dependence. We gathered original results on this problem in Chapter 6 for this very purpose. Further, Section 7.4 is devoted to the proof of the existence of solutions of Problem (7.2.1) – (7.2.6) with the averaging choice (7.2.7). In Section 7.5 we discuss the extension of the existence analysis to other choices of $\mathcal{M}$ in (7.2.6). In Section 7.6 we build a numerical scheme adapted to the specific structure of the system at hand (LWR equation for ρ and a transport equation for $\mathbf{w}$). We make the simpler averaging choice (7.2.8) and prove that the scheme is $\mathbf{BV}$-stable and convergent. We point out structural similarities between our scheme and the scheme of the authors of [113] developed for the classical Keyfitz-Kranzer system. Finally, Section 7.7 is devoted to performing numerical simulations to illustrate our model.

7.3 Notion of solution

We denote by f the time and space dependent flux $f(x,t,\rho)=\rho\mathbf{v}(x,t,\rho)$ and Φ its Kruzhkov entropy flux (see [114]):

$$\forall \rho,\kappa\in[0,1],\ \forall (x,t)\in\Omega,\ \Phi(x,t,\rho,\kappa)=\operatorname{sgn}(\rho-\kappa)\left(f(x,t,\rho)-f(x,t,\kappa)\right).$$

Relying upon [114] for the PDE describing the evolution of ρ and upon [134] (see also Chapter 6) for the PDE describing the evolution of $\mathbf{w}$, we give the following definition of solution to Problem (7.2.1) – (7.2.6).

Definition 7.3.1. A couple $(\rho,\mathbf{w})\in\mathbf{L}^\infty(\Omega)^2$ is a solution to (7.2.1) – (7.2.6) with initial data $(\rho_0,\mathbf{w}_0)\in\mathbf{L}^\infty(\mathbb{R})^2$ if
(i) $\rho\in\mathbf{C}([0,T];\mathbf{L}^1_{\mathbf{loc}}(\mathbb{R};[0,1]))$ and $\rho\mathbf{w}\in\mathbf{C}([0,T];w^*\text{–}\mathbf{L}^\infty(\mathbb{R};[0,1]))$, where $w^*\text{–}\mathbf{L}^\infty$ means the space $\mathbf{L}^\infty$ endowed with its topology of weak-$*$ convergence;
(ii) ρ is an entropy solution to (7.2.1) with initial data ρ_0 in the following sense: $\rho(\cdot,0)=\rho_0$ in $\mathbf{L}^1_{\mathbf{loc}}(\mathbb{R})$; and for all test functions $\varphi\in\mathbf{C}^\infty_{\mathbf{c}}(\mathbb{R}\times\mathbb{R}^+),\varphi\geq 0$, for all $\kappa\in[0,1]$ and for all $\tau,s\in[0,T]$ $(s<\tau)$,

$$\begin{aligned}\int_s^\tau\int_{\mathbb{R}}\Big(|\rho-\kappa|\partial_t\varphi+\Phi(x,t,\rho,\kappa)\partial_x\varphi-\operatorname{sgn}(\rho-\kappa)\partial_x f(x,t,\kappa)\varphi\Big)\,\mathrm{d}x\,\mathrm{d}t\\ +\int_{\mathbb{R}}|\rho(x,s)-\kappa|\varphi(x,s)\,\mathrm{d}x-\int_{\mathbb{R}}|\rho(x,\tau)-\kappa|\varphi(x,\tau)\,\mathrm{d}x\geq 0;\end{aligned}\tag{7.3.1}$$

(iii) $\mathbf{w}$ is a weak solution to (7.2.3) with initial data $\mathbf{w}_0$ in the following sense: $\rho(\cdot,0)\mathbf{w}(\cdot,0)=\rho_0\mathbf{w}_0$ in $\mathbf{L}^\infty(\mathbb{R})$-weakly*; and for all test functions $\phi\in\mathbf{C}^\infty_{\mathbf{c}}(\mathbb{R}\times\mathbb{R}^+)$ and for all $\tau,s\in[0,T]$

$(s < \tau)$,

$$\int_s^\tau \int_{\mathbb{R}} \Big((\rho \mathbf{w}) \partial_t \phi + (\rho \mathbf{v} \mathbf{w}) \partial_x \phi + \rho \mathbf{K}\left(\xi, \partial_t \xi\right) \mathbf{w}(1-\mathbf{w}) \phi \Big) \, \mathrm{d}x \, \mathrm{d}t \\ + \int_{\mathbb{R}} \rho(x,s) \mathbf{w}(x,s) \phi(x,s) \, \mathrm{d}x - \int_{\mathbb{R}} \rho(x,\tau) \mathbf{w}(x,\tau) \phi(x,\tau) \, \mathrm{d}x = 0, \tag{7.3.2}$$

where ξ is linked to ρ by (7.2.4);

(iv) $\mathbf{v}$ and ω are linked by (7.2.2) and ω and $\mathbf{w}$ are linked by (7.2.6).

Remark 7.3.1. According to the result of Corollary 6.0.8 based upon the theory of [134], given $\rho, \mathbf{v}$ and setting $g = \mathbf{K}\left(\xi, \partial_t \xi\right)$ with ξ given by (7.2.4), the solution $\mathbf{w}$ in the sense (7.3.2) automatically verifies the renormalization property, *cf.* Definition 6.0.2. We will say, for short, that the weak solution in the sense (7.3.2) is also a renormalized solution, meaning that it fulfills this renormalization property. This aspect is essential for the compactness properties, and it also means that, in a sense, the solution is evolving as if characteristics could be defined (though the latter cannot be defined due to the possible irregularity of $\rho, \mathbf{v}$). The latter observation is the key to the construction of the the numerical scheme and it also ensures the propagation of the **BV** regularity, for **BV** initial data.

Remark 7.3.2 (On the time-continuity)**.** It is more usual to formulate (7.3.1)-(7.3.2) with $s = 0$, $\tau = T$ and $\varphi, \phi \in \mathbf{C}_{\mathbf{c}}^\infty(\mathbb{R} \times [0,T))$. Our present formulations are instrumental for the splitting argument we employ in our construction, see Section 7.4.1. The equivalence between the two formulations is due to the time-continuity of entropy solutions of LWR equation and of weak solutions of the transport equations at hand, see Definition 7.3.1(i).

In Section 7.4, we prove the following existence result.

Theorem 7.3.2. *Fix $\rho_0, \mathbf{w}_0 \in \mathbf{L}^\infty(\mathbb{R}; [0,1])$. Assume that $V_{\min} \leq V_{\max} \in \mathbf{C}^1([0,R])$ are nonnegative and that $V'_{\min}$ and $V'_{\max}$ do not vanish on any interval of $[0,1]$. Then Problem (7.2.1) – (7.2.6),(7.2.7) admits at least one solution.*

In Section 7.6, we obtain the following results of numerical approximation and existence for the time-local variant (7.2.8) of our model; note that (7.2.7) can also be considered in our numerical framework.

Theorem 7.3.3. *Suppose that $\mathbf{TV}(\rho_0) < +\infty$ and that $\mathbf{w}_0 \in \mathbf{L}^1(\mathbb{R}; [0,1]), \mathbf{TV}(\mathbf{w}_0) < +\infty$. Moreover suppose that ρ_0 is separated from the vacuum in the sense that*

$$\exists \varepsilon \in (0,1), \quad \varepsilon \leq \rho_0 \leq 1 \quad \textit{and} \quad V_{min}(\varepsilon) = V_{max}(\varepsilon). \tag{7.3.3}$$

Then up to a subsequence, the sequence of discrete solutions produced by the scheme of Section 7.6 converges to a solution of (7.2.1) – (7.2.6), (7.2.8).

Note that the second requirement in (7.3.3) follows from the assumption (7.2.9), while the first requirement in (7.3.3) is essential in order to define the CFL condition of the numerical scheme we develop.

Theorem 7.3.4. *Suppose that* $\mathbf{TV}(\rho_0) < +\infty$ *and* ρ_0 *satisfies* (7.3.3)*, and that* $\mathbf{w}_0 \in \mathbf{L}^1(\mathbb{R};[0,1])$*,* $\mathbf{TV}(\mathbf{w}_0) < +\infty$*. Then Problem* (7.2.1) – (7.2.6)*,* (7.2.8) *admits at least one solution.*

Let us make precise that the assumption (7.3.3) is only useful to construct and prove the convergence of the scheme developed in Section 7.6. The last existence result can be obtained without it, see the discussion in Section 7.5.1, by using the splitting construction borrowed the proof of Theorem 7.3.2 along with a **BV** stability argument ensuring compactness.

7.4 Existence of solutions *via* splitting

7.4.1 Time-splitting procedure and approximate solution

To prove existence of solutions to (7.2.1) – (7.2.6),(7.2.7), we use a time-splitting technique. This way, we split the model combining the notion of Kruzhkov entropy solution to LWR models with the notion of weak-and-renormalized solutions to transport equations under the specific form of Panov [134], extended in Chapter 6 in order to include the nonlinear source term.

Fix $\rho_0, \mathbf{w}_0 \in \mathbf{L}^\infty(\mathbb{R};[0,1])$. Let $\nu > 0$ be a time step, denote for all $n \in \mathbb{Z}$, $t^n = n\nu$ and let $N \in \mathbb{N}^*$ such that $T \in [t^N, t^{N+1})$.
Initialization. For all $t \in \mathbb{R}$,

$$\rho^0(\cdot,t) = \rho_0 \quad \text{and} \quad \forall n \in \mathbb{Z}^-,\ \mathbf{w}^n(\cdot,t) = \mathbf{w}_0.$$

Induction. Fix $n \in \{1,\dots,N+1\}$.
(1) First define the orderliness parameter: $\forall t \in [t^{n-1}, t^n),\ \forall x \in \mathbb{R}$,

$$\begin{aligned}\omega^n(x,t) = &\int_{t^{n-2}}^{t-\nu}\int_{\mathbb{R}} \mathbf{w}^{n-1}(y,s)\eta(x-y,t-s)\,\mathrm{d}y\,\mathrm{d}s \\ &+ \sum_{k\le n-2}\int_{t^{k-1}}^{t^k}\int_{\mathbb{R}} \mathbf{w}^{k}(y,s)\eta(x-y,t-s)\,\mathrm{d}y\,\mathrm{d}s\,.\end{aligned}$$

Remark that the values of ω^n only depend on the values of ρ and $\mathbf{w}$ before time t^{n-1}, which is the key to the splitting.
(2) We use ω^n to define the car velocity

$$\forall t \in [t^{n-1}, t^n),\ \forall x \in \mathbb{R},\ \mathbf{v}^n(x,t,\cdot) = (1-\omega^n(x,t))V_{\min}(\cdot) + \omega^n(x,t)V_{\max}(\cdot)$$

and the flux $f^n(x,t,\rho) = \rho\mathbf{v}^n(x,t,\rho)$.
(3) The flux function is smooth in x, Lipschitz in ρ and **BV** in t. Since $\rho^{n-1}(\cdot,t^{n-1})$ is bounded, we can define $\rho^n \in \mathbf{C}([t^{n-1},t^n];\mathbf{L}^1_{\mathbf{loc}}(\mathbb{R};[0,1]))$ as the unique entropy solution, in the sense of Definition 7.3.1 (i)-(ii), see [114, Theorem 1] and [59, Theorem 2.3], to

$$\begin{cases}\partial_t\rho^n + \partial_x\left(f^n(x,t,\rho^n)\right) = 0 \\ \qquad\qquad \rho^n(\cdot,t^{n-1}) = \rho^{n-1}(\cdot,t^{n-1}).\end{cases}$$

(4) Setting

$$\forall t \in [t^{n-1}, t^n),\ \forall x \in \mathbb{R},\ \xi^n(x,t) = \int_{\mathbb{R}} \rho^n(y,t)\mu(x-y)\,\mathrm{d}y\,,$$

and following Corollary 6.0.8, we can define $\mathbf{w}^n \in \mathbf{L}^\infty(\mathbb{R}\times(t^{n-1},t^n))$ as the unique weak solution to

$$\begin{cases} \partial_t\left(\rho^n\mathbf{w}^n\right) + \partial_x\left(f^n(x,t,\rho^n)\mathbf{w}^n\right) = \rho^n\mathbf{K}\left(\xi^n, \partial_t\xi^n\right)\mathbf{w}^n(1-\mathbf{w}^n) \\ \mathbf{w}^n(\cdot,t^{n-1}) = \mathbf{w}^{n-1}(\cdot,t^{n-1}). \end{cases}$$

Corollary 6.0.8 ensures that $\mathbf{w}^n$ verifies the renormalization property, see Definition 6.0.2; and Remark 6.0.1 based upon [134, Lemma 1] provides the required regularity in time: $\mathbf{w}^n \in \mathbf{C}([t^{n-1},t^n];w^*\text{–}\mathbf{L}^\infty(\mathbb{R}))$. Note that by construction, $\mathbf{w}$ takes values in $[0,1]$.

Conclusion. Define the following functions: for a.e. $(x,t)\in\Omega$,

$$\begin{aligned} (\rho_\nu(\cdot,t),\mathbf{w}_\nu(\cdot,t)) &= (\rho_0,\mathbf{w}_0)\,\mathbb{1}_{\mathbb{R}^-}(t) + \sum_{n=1}^{N+1}(\rho^n(\cdot,t),\mathbf{w}^n(\cdot,t))\,\mathbb{1}_{(t^{n-1},t^n]}(t); \\ (\mathbf{v}_\nu(x,t,\cdot),\omega_\nu(x,t),\xi_\nu(x,t)) &= \sum_{n=1}^{N+1}(\mathbf{v}^n(x,t,\cdot),\omega^n(x,t),\xi^n(x,t))\,\mathbb{1}_{[t^{n-1},t^n)}(t) \\ f_\nu(x,t,\cdot) &= \sum_{n=1}^{N+1} f^n(x,t,\cdot)\mathbb{1}_{[t^{n-1},t^n)}(t). \end{aligned}$$

Proposition 7.4.1. *The couple $(\rho_\nu,\mathbf{w}_\nu)$ constructed above is a solution in Ω to the following system:*

$$\begin{cases} \partial_t\rho_\nu + \partial_x\left(f_\nu(x,t,\rho_\nu)\right) = 0 \\ \mathbf{v}_\nu(x,t,\rho) = (1-\omega_\nu(x,t))V_{\min}(\rho) + \omega_\nu(x,t)V_{\max}(\rho) \\ \partial_t\left(\rho_\nu\mathbf{w}_\nu\right) + \partial_x\left(f_\nu(x,t,\rho_\nu)\mathbf{w}_\nu\right) = \rho_\nu\mathbf{K}\left(\xi_\nu,\partial_t\xi_\nu\right)\mathbf{w}_\nu(1-\mathbf{w}_\nu) \\ \omega_\nu(x,t) = \displaystyle\int_{-\infty}^{t-\nu}\int_{\mathbb{R}} \mathbf{w}_\nu(y,s)\eta(x-y,t-s)\,\mathrm{d}y\,\mathrm{d}s\,. \end{cases} \tag{7.4.1}$$

Proof. By construction, for all $n\in\{1,\dots,N+1\}$, $\rho^n\in\mathbf{C}([t^{n-1},t^n];\mathbf{L}^1_{\mathbf{loc}}(\mathbb{R}))$. Combining this with the *stop-and-restart* conditions $\rho^n(\cdot,t^{n-1}) = \rho^{n-1}(\cdot,t^{n-1})$, we ensure that $\rho_\nu\in\mathbf{C}([0,T];\mathbf{L}^1_{\mathbf{loc}}(\mathbb{R}))$. Using a similar reasoning, we obtain $\rho_\nu\mathbf{w}_\nu\in\mathbf{C}([0,T];w^*\text{–}\mathbf{L}^\infty(\mathbb{R}))$. Fix now $\varphi\in\mathbf{C}^\infty_{\mathbf{c}}(\mathbb{R}\times\mathbb{R}^+)$, $\varphi\geq 0$ and $\kappa\in[0,1]$. Let us denote by Φ_ν the Kruzhkov entropy flux

associated with f_ν. By construction, for every $n \in \{1, \dots, N+1\}$, we have

$$\begin{aligned}
&\int_{t^{n-1}}^{t^n} \int_{\mathbb{R}} \big|\rho_\nu - \kappa\big| \partial_t \varphi + \Phi_\nu(x,t,\rho_\nu,\kappa)\partial_x\varphi \,\mathrm{d}x\,\mathrm{d}t \\
&= \int_{t^{n-1}}^{t^n} \int_{\mathbb{R}} \big|\rho^n - \kappa\big| \partial_t \varphi + \operatorname{sgn}(\rho^n - \kappa)\Big(f^n(x,t,\rho^n) - f^n(x,t,\kappa)\Big)\partial_x\varphi \,\mathrm{d}x\,\mathrm{d}t \\
&\geq \int_{t^{n-1}}^{t^n} \int_{\mathbb{R}} \operatorname{sgn}(\rho^n - \kappa)\partial_x f^n(x,t,\kappa)\varphi \,\mathrm{d}x\,\mathrm{d}t \\
&- \int_{\mathbb{R}} \big|\rho^n(x,t^{n-1}) - \kappa\big| \,\varphi(x,t^{n-1})\,\mathrm{d}x + \int_{\mathbb{R}} |\rho^n(x,t^n) - \kappa| \,\varphi(x,t^n)\,\mathrm{d}x \\
&= \int_{t^{n-1}}^{t^n} \int_{\mathbb{R}} \operatorname{sgn}(\rho^n - \kappa)\partial_x f_\nu(x,t,\kappa)\varphi \,\mathrm{d}x\,\mathrm{d}t \\
&- \int_{\mathbb{R}} \big|\rho_\nu(x,t^{n-1}) - \kappa\big| \,\varphi(x,t^{n-1})\,\mathrm{d}x + \int_{\mathbb{R}} |\rho_\nu(x,t^n) - \kappa| \,\varphi(x,t^n)\,\mathrm{d}x\,.
\end{aligned}$$

From this inequality, it is straightforward to prove that for all $s, \tau \in [0,T]$ $(s < \tau)$, we have

$$\begin{aligned}
\int_s^\tau \int_{\mathbb{R}} \Big(|\rho_\nu - \kappa| \partial_t \varphi + \Phi_\nu(x,t,\rho_\nu,\kappa)\partial_x\varphi - \operatorname{sgn}(\rho_\nu - \kappa)\partial_x f_\nu(x,t,\kappa)\varphi \Big)\,\mathrm{d}x\,\mathrm{d}t \\
+ \int_{\mathbb{R}} |\rho_\nu(x,s) - \kappa|\varphi(x,s)\,\mathrm{d}x - \int_{\mathbb{R}} |\rho_\nu(x,\tau) - \kappa|\varphi(x,\tau)\,\mathrm{d}x \geq 0,
\end{aligned} \tag{7.4.2}$$

see Chapter 3, Section 3.2 for an analogous calculation. Let us make precise here the link between ρ_ν and ξ_ν. For all $t \in [0,T]$, if $t \in [t^{n-1}, t^n)$ for some $n \in \{1, \dots, N+1\}$, then for all $x \in \mathbb{R}$,

$$\xi_\nu(x,t) = \xi^n(x,t) = \int_{\mathbb{R}} \rho^n(y,t)\mu(x-y)\,\mathrm{d}y = \int_{\mathbb{R}} \rho_\nu(y,t)\mu(x-y)\,\mathrm{d}y\,.$$

We now turn to the obtaining of an approximate weak formulation similar to (7.3.2). Let $\phi \in \mathbf{C}_{\mathbf{c}}^\infty(\mathbb{R} \times \mathbb{R}^+)$. For every $n \in \{1, \dots, N+1\}$, we have

$$\begin{aligned}
&\int_{t^{n-1}}^{t^n} \int_{\mathbb{R}} \rho_\nu \mathbf{w}_\nu \partial_t \phi + f_\nu(x,t,\rho_\nu)\mathbf{w}_\nu \partial_x \phi \,\mathrm{d}x\,\mathrm{d}t \\
&= \int_{t^{n-1}}^{t^n} \int_{\mathbb{R}} \rho^n \mathbf{w}^n \partial_t \phi + f^n(x,t,\rho^n)\mathbf{w}^n \partial_x \phi \,\mathrm{d}x\,\mathrm{d}t \\
&= -\int_{t^{n-1}}^{t^n} \int_{\mathbb{R}} \rho^n \mathbf{K}\left(\xi^n, \partial_t \xi^n\right) \mathbf{w}^n (1 - \mathbf{w}^n)\phi \,\mathrm{d}x\,\mathrm{d}t \\
&- \int_{\mathbb{R}} \rho^n(x,t^{n-1})\mathbf{w}^n(x,t^{n-1})\phi(x,t^{n-1})\,\mathrm{d}x + \int_{\mathbb{R}} \rho^n(x,t^n)\mathbf{w}^n(x,t^n)\phi(x,t^n)\,\mathrm{d}x \\
&= -\int_{t^{n-1}}^{t^n} \int_{\mathbb{R}} \rho_\nu \mathbf{K}\left(\xi_\nu, \partial_t \xi_\nu\right) \mathbf{w}_\nu (1 - \mathbf{w}_\nu)\phi \,\mathrm{d}x\,\mathrm{d}t \\
&- \int_{\mathbb{R}} \rho_\nu(x,t^{n-1})\mathbf{w}_\nu(x,t^{n-1})\phi(x,t^{n-1})\,\mathrm{d}x + \int_{\mathbb{R}} \rho_\nu(x,t^n)\mathbf{w}_\nu(x,t^n)\phi(x,t^n)\,\mathrm{d}x\,,
\end{aligned}$$

and from this, once again, it is easy to prove that for all $s, \tau \in [0,T]$ $(s<\tau)$, we have

$$\begin{aligned}\int_s^\tau \int_{\mathbb{R}} \Big((\rho_\nu \mathbf{w}_\nu)\partial_t\phi + (\rho_\nu \mathbf{v}_\nu \mathbf{w}_\nu)\partial_x\phi + \rho_\nu \mathbf{K}\left(\xi_\nu, \partial_t\xi_\nu\right)\mathbf{w}_\nu(1-\mathbf{w}_\nu)\phi \Big)\,\mathrm{d}x\,\mathrm{d}t \\ + \int_{\mathbb{R}} \rho_\nu(x,s)\mathbf{w}_\nu(x,s)\phi(x,s)\,\mathrm{d}x - \int_{\mathbb{R}} \rho_\nu(x,\tau)\mathbf{w}_\nu(x,\tau)\phi(x,\tau)\,\mathrm{d}x = 0.\end{aligned} \tag{7.4.3}$$

By construction, $\mathbf{v}_\nu$ and ω_ν are linked by the second equality in (7.4.1). Finally, if $t \in [t^{n-1}, t^n)$ for some $n \in \{1, \dots, N+1\}$, then we have for all $x \in \mathbb{R}$,

$$\begin{aligned}\omega^n(x,t) &= \int_{t^{n-2}}^{t-\nu} \int_{\mathbb{R}} \mathbf{w}^{n-1}(y,s)\eta(x-y,t-s)\,\mathrm{d}y\,\mathrm{d}s + \sum_{k\le n-2} \int_{t^{k-1}}^{t^k} \int_{\mathbb{R}} \mathbf{w}^k(y,s)\eta(x-y,t-s)\,\mathrm{d}y\,\mathrm{d}s \\ &= \int_{-\infty}^{t-\nu} \int_{\mathbb{R}} \mathbf{w}_\nu(y,s)\eta(x-y,t-s)\,\mathrm{d}y\,\mathrm{d}s\,,\end{aligned}$$

i.e. ω_ν and $\mathbf{w}_\nu$ are linked by the last equality in (7.4.1). □

7.4.2 Compactness and convergence

We now want to pass to the limit in (7.4.2)-(7.4.3), and for that we need sufficient compactness of the sequences involved. The difficulty lies in the obtaining of strong compactness for the sequence $(\mathbf{w}_\nu)_\nu$. For this sake, we developed the "compactness from renormalization" argument for one-dimensional transport equations addressed in [134], see Theorem 6.0.6. To apply it, we need:

- uniform $\mathbf{L}^\infty$ bounds for the sequences $(\rho_\nu)_\nu$, $(\mathbf{v}_\nu)_\nu$, $(\mathbf{K}(\xi_\nu, \partial_t\xi_\nu))_\nu$ and $(\mathbf{w}_\nu)_\nu$;
- strong compactness for the sequences $(\rho_\nu)_\nu$, $(f_\nu(\cdot,\cdot,\rho_\nu))_\nu$, $(\mathbf{K}(\xi_\nu, \partial_t\xi_\nu))_\nu$;
- to prove that $(\mathbf{w}_\nu)_\nu$ is a sequence of weak solutions to the second PDE of (7.4.1), which implies that they verify the renormalization property, by virtue of Corollary 6.0.8.

Note that we proved the last point in the proof of Proposition 7.4.1. We now focus on the two other requirements. Let us start with the $\mathbf{L}^\infty$ bounds.

Lemma 7.4.2. *For all $\nu > 0$, we have the bounds:*

$$0 \le \rho_\nu, \mathbf{w}_\nu, \omega_\nu \le 1; \quad 0 \le \mathbf{v}_\nu \le V_{\max}; \quad \left|\mathbf{K}(\xi_\nu, \partial_t\xi_\nu)\right| \le \sup_{\substack{0\le\xi\le 1 \\ |\chi|\le V_{\max}\|\mu'\|_{\mathbf{L}^1}}} \left|\mathbf{K}(\xi,\chi)\right|.$$

Proof. The bounds for $(\rho_\nu)_\nu$ and $(\mathbf{w}_\nu)_\nu$ are clear. Since η is a weight function, for all $\nu > 0$, we have

$$\forall (x,t) \in \Omega, \quad 0 \le \omega_\nu(x,t) \le \int_0^T \int_{\mathbb{R}} \eta(y,s)\,\mathrm{d}y\,\mathrm{d}s = 1,$$

which implies the desired bounds for $(\mathbf{v}_\nu)_\nu$ since it is a convex combination of $V_{\min}$ and $V_{\max}$. Now, once we recall that for a.e. $(x,t) \in \Omega$,

$$\partial_t\xi_\nu(x,t) = -\int_{\mathbb{R}} \rho_\nu(y,t)\mathbf{v}_\nu(y,t,\rho_\nu)\mu'(x-y)\,\mathrm{d}y\,,$$

we immediately get the bound for $(\mathbf{K}(\xi_\nu, \partial_t\xi_\nu))_\nu$. □

We now turn to the strong compactness for the sequences $(\rho_\nu)_\nu$, $(f(\cdot,\cdot,\rho_\nu))_\nu$, $(\mathbf{K}(\xi_\nu,\partial_t\xi_\nu))_\nu$. Let us start with $(f(\cdot,\cdot,\rho_\nu))_\nu$.

Lemma 7.4.3. *There exists $\omega\in\mathbf{C}(\Omega)$ such that up to the extraction of a subsequence, $(\omega_\nu)_\nu$ converges uniformly on compact sets to ω. Moreover, for all $(x,t)\in\Omega$, $\omega(x,t)\in[0,1]$.*

Proof. We now prove that the sequence $(\omega_\nu)_\nu$ is bounded in $\mathbf{W}^{1,\infty}(\Omega)$. We already proved in Lemma 7.4.2 that $(\omega_\nu)_\nu$ is bounded in $\mathbf{L}^\infty(\Omega)$. Fix now $(x,t),(\xi,\tau)\in\Omega$. On the one hand, we have

$$\begin{aligned}|\omega_\nu(x,t)-\omega_\nu(\xi,t)|&\le\int_{-\infty}^{t-\nu}\int_{\mathbb{R}}|\eta(x-y,t-s)-\eta(\xi-y,t-s)|\,\mathrm{d}y\,\mathrm{d}s\\&\le|x-\xi|\int_{-\infty}^{t-\nu}\mathbf{TV}(\eta(\cdot,t-s))\,\mathrm{d}s\le\|\eta\|_{\mathbf{L}^1((0,T);\mathbf{BV})}|x-\xi|.\end{aligned}$$

On the other hand,

$$\begin{aligned}|\omega_\nu(x,t)-\omega_\nu(x,\tau)|&\le\int_{-\infty}^{t-\nu}\int_{\mathbb{R}}|\eta(x-y,t-s)-\eta(x-y,\tau-s)|\,\mathrm{d}y\,\mathrm{d}s\\&\quad+\left|\int_{t-\nu}^{\tau-\nu}\int_{\mathbb{R}}\eta(x-y,\tau-s)\,\mathrm{d}y\,\mathrm{d}s\right|\\&\le\left(\|\eta\|_{\mathbf{L}^1(\mathbb{R};\mathbf{BV})}+\|\eta\|_{\mathbf{L}^\infty((0,T);\mathbf{L}^1)}\right)|t-\tau|.\end{aligned}$$

The compactness result follows from the compact embedding $\mathbf{W}^{1,\infty}(\overset{\circ}{U})\subset\mathbf{C}(U)$ when $U\subset\Omega$ is a compact subset. A standard diagonal process ensures then the existence of subsequence of $(\omega_\nu)_\nu$ that converges to some $\omega\in\mathbf{C}(\Omega)$ on every compact subset of Ω. □

Corollary 7.4.4. *Define the velocity $\mathbf{v}(x,t,\rho)=(1-\omega(x,t))V_{\min}(\rho)+\omega(x,t)V_{\max}(\rho)$ and the flux $f(x,t,\rho)=\rho\mathbf{v}(x,t,\rho)$. Then, up to a subsequence, $(\mathbf{v}_\nu)_\nu$ and $(f_\nu)_\nu$ converge uniformly on compact subsets of $\Omega\times[0,1]$ to $\mathbf{v}$ and f, respectively.*

Proof. The claim is immediate because of the convergence of $(\omega_\nu)_\nu$. □

We see here the effect of the non-locality of $(\omega_\nu)_\nu$. To obtain strong compactness of $(\rho_\nu)_\nu$, we impose a non-degeneracy assumption on the flux.

Lemma 7.4.5. *Suppose that $V'_{\min}$ and $V'_{\max}$ do not vanish on any interval of $[0,1]$. Then there exists a subsequence of $(\rho_\nu)_\nu$ which converges a.e. on Ω to some $\rho\in\mathbf{L}^\infty(\Omega)$. Moreover, for a.e. $(x,t)\in\Omega$, $\rho(x,t)\in[0,1]$.*

Proof. Fix U a bounded open subset of Ω, V a compact subset of Ω containing U and $\kappa\in[0,1]$. Using the formalism of [132, 134], we show that

$$\left(\mathrm{div}_{(t,x)}\begin{pmatrix}(\rho_\nu-\kappa)^+\\(\rho_\nu-\kappa)^+(f(x,t,\rho_\nu)-f(x,t,\kappa))\end{pmatrix}\right)_\nu\text{ is precompact in }\mathbf{H}^{-1}(U).$$

By construction, for all $\nu > 0$,

$$\begin{aligned} &2\partial_t(\rho_\nu - \kappa)^+ + 2(\rho_\nu - \kappa)^+(f(x,t,\rho_\nu) - f(x,t,\kappa)) \\ &= -\partial_x f(x,t,\kappa) + \partial_t|\rho_\nu - \kappa| + \partial_x \Phi(x,t,\rho_\nu,\kappa) \\ &+ \underbrace{\partial_x \left(f(x,t,\rho_\nu) - f_\nu(x,t,\rho_\nu)\right)}_{R_\nu(x,t)} \end{aligned} \tag{7.4.4}$$

For all $\varphi \in \mathbf{C}_\mathbf{c}^\infty(U)$, we have

$$\begin{aligned} \left| \iint_U R_\nu \varphi \,\mathrm{d}x\,\mathrm{d}t \right| &= \left| \iint_U \left(f(x,t,\rho_\nu) - f_\nu(x,t,\rho_\nu)\right) \partial_x \varphi \,\mathrm{d}x\,\mathrm{d}t \right| \\ &\leq \|f - f_\nu\|_{\mathbf{L}^\infty(V)} \mathrm{mes}(U)^{1/3} \|\partial_x \varphi\|_{\mathbf{L}^{3/2}(U)} \\ &\leq \sup_{\nu>0} \left(\|f - f_\nu\|_{\mathbf{L}^\infty(V)}\right) \mathrm{mes}(U)^{1/3} \|\varphi\|_{\mathbf{W}^{1,3/2}(U)}, \end{aligned}$$

which proves that the sequence $(R_\nu)_\nu$ is bounded in $\mathbf{W}^{-1,3}(U)$. Since $(R_\nu)_\nu$ is also clearly bounded in the space of finite signed Radon measures $\mathcal{M}_s(U)$, [80, Corollary 1.3.1] ensures that $(R_\nu)_\nu$ is precompact in $\mathbf{H}^{-1}(U)$. The same method applies to prove that the remainder of the right-hand side of (7.4.4) is precompact in $\mathbf{H}^{-1}(U)$. Hence,

$$\left(\mathrm{div}_{(t,x)} \begin{pmatrix} (\rho_\nu - \kappa)^+ \\ (\rho_\nu - \kappa)^+ (f(x,t,\rho_\nu) - f(x,t,\kappa)) \end{pmatrix} \right)_\nu \text{ is precompact in } \mathbf{H}^{-1}_{\mathbf{loc}}(\Omega).$$

Since $(\rho_\nu)_\nu \subset \mathbf{L}^\infty(\Omega)$ is bounded, for all $(x,t) \in \Omega$, the flux $f(x,t,\cdot)$ being non-degenerate in the sense required in [134] due to our assumption on $V_{\min}$, $V_{\max}$, [136, Corollary 2] yields a subsequence of $(\rho_\nu)_\nu$ that converges to some $\rho \in \mathbf{L}^\infty(\Omega)$ in $\mathbf{L}^1_{\mathbf{loc}}(\Omega)$. A further extraction yields the a.e. convergence on Ω. The fact that ρ takes values in $[0,1]$ comes from the $\mathbf{L}^\infty$ bound of Lemma 7.4.2. □

Corollary 7.4.6. *Define for all $(x,t) \in \Omega$,*

$$\xi(x,t) = \int_{\mathbb{R}} \rho(y,t)\mu(x-y)\,\mathrm{d}y\,; \quad \chi(x,t) = -\int_{\mathbb{R}} \rho(y,t)\mathbf{v}(y,t,\rho)\mu'(x-y)\,\mathrm{d}y\,.$$

Then, up to a subsequence, $(\xi_\nu)_\nu$, $(\partial_t \xi_\nu)_\nu$ and $(\mathbf{K}(\xi_\nu, \partial_t \xi_\nu))_\nu$ converge a.e. on Ω to ξ, χ and $\mathbf{K}(\xi,\chi)$, respectively.

Proof. The claim is immediate. □

We now assess the compactness of $(\mathbf{w}_\nu)_\nu$.

Corollary 7.4.7. *There exists $\mathbf{w} \in \mathbf{L}^\infty(\Omega;[0,1])$ such that $(\mathbf{w}_\nu)_\nu$ converges a.e. to $\mathbf{w}$ on Ω.*

Proof. Throughout this section, we ensured that all the hypotheses of Theorem 6.0.6 are fulfilled, yielding the desired compactness. □

With the established compactness, we can prove the

Theorem 7.4.8. *The couple* $(\rho, \mathbf{w})$ *constructed in Lemma 7.4.5 and Corollary 7.4.7 is a solution to Problem* (7.2.1) – (7.2.6),(7.2.7).

Proof. For all $\nu > 0$ and for all $(x,t) \in \Omega$, we have

$$\begin{aligned}\omega_\nu(x,t) &= \int_{-\infty}^{t-\nu}\int_{\mathbb{R}} \mathbf{w}_\nu(y,s)\eta(x-y,t-s)\,\mathrm{d}y\,\mathrm{d}s \\ &= -\int_{t-\nu}^{t}\int_{\mathbb{R}} \mathbf{w}_\nu(y,s)\eta(x-y,t-s)\,\mathrm{d}y\,\mathrm{d}s + \int_{-\infty}^{t}\int_{\mathbb{R}} \mathbf{w}_\nu(y,s)\eta(x-y,t-s)\,\mathrm{d}y\,\mathrm{d}s\,.\end{aligned}$$

The first term clearly vanishes as $\nu \to 0$, and since $\eta \in \mathbf{L}^1(\Omega)$, the second one converges to $\displaystyle\int_{-\infty}^{t}\int_{\mathbb{R}} \mathbf{w}(y,s)\eta(x-y,t-s)\,\mathrm{d}y\,\mathrm{d}s$ as $\nu \to 0$. Recall (*cf.* Lemma 7.4.3) that $(\omega_\nu)_\nu$ converges uniformly to ω on compact sets of Ω and we get:

$$\forall (x,t) \in \Omega,\ \omega(x,t) = \int_{-\infty}^{t}\int_{\mathbb{R}} \mathbf{w}(y,s)\eta(x-y,t-s)\,\mathrm{d}y\,\mathrm{d}s\,.$$

It is clear from this formula that $\omega \in \mathbf{W}^{1,\infty}(\Omega)$. Apply now (7.4.2) with $\varphi \in \mathbf{C}_\mathbf{c}^\infty(\mathbb{R}\times[0,T)), \varphi \geq 0$, $\kappa \in [0,1]$, $s = 0$ and $\tau = T$ and let $\nu \to 0$. We get:

$$\begin{aligned}\int_0^T\int_{\mathbb{R}} \Big(|\rho-\kappa|\partial_t\varphi + \Phi(x,t,\rho,\kappa)\partial_x\varphi - \operatorname{sgn}(\rho-\kappa)\partial_x f(x,t,\kappa)\varphi \Big)\,\mathrm{d}x\,\mathrm{d}t \\ + \int_{\mathbb{R}} |\rho_0(x)-\kappa|\varphi(x,0)\,\mathrm{d}x \geq 0.\end{aligned}$$

This proves that ρ is an entropy solution to (7.2.1). Therefore, $\rho \in \mathbf{C}([0,T];\mathbf{L}^1_{\mathbf{loc}}(\mathbb{R}))$, see [59]. Moreover, it implies that ξ defined in Lemma 7.4.6 verifies for all $x \in \mathbb{R}$, $\xi(x,\cdot) \in \mathbf{W}^{1,\infty}((0,T))$ and that for a.e $t \in (0,T)$,

$$\partial_t \xi(x,t) = \chi(x,t),$$

where χ was defined in 7.4.6 as well. Now the convergences we have proved for $(\rho_\nu)_\nu$ and $(f_\nu)_\nu$ ensure that for a.e. $\tau, s \in [0,T]$ $(s < \tau)$,

$$\begin{aligned}\int_s^\tau\int_{\mathbb{R}} \Big(|\rho-\kappa|\partial_t\varphi + \Phi(x,t,\rho,\kappa)\partial_x\varphi - \operatorname{sgn}(\rho-\kappa)\partial_x f(x,t,\kappa)\varphi\,\mathrm{d}x\,\mathrm{d}t \Big) \\ + \int_{\mathbb{R}} |\rho(x,s)-\kappa|\varphi(x,s)\,\mathrm{d}x - \int_{\mathbb{R}} |\rho(x,\tau)-\kappa|\varphi(x,\tau)\,\mathrm{d}x \geq 0.\end{aligned}$$

The expression in the left-hand side of the previous inequality is a continuous function of (s,τ) which is almost everywhere greater than the continuous function 0. By continuity, this expression is everywhere greater than 0, which proves that ρ satisfies the entropy inequalities (7.3.1). To conclude the proof of the statement, we have to prove that $\mathbf{w}$ is a weak solution

to (7.2.3). We apply (7.4.3) with $\phi \in \mathbf{C}_{\mathbf{c}}^{\infty}(\mathbb{R} \times [0,T))$, $s = 0$ and $\tau = T$, and we let $\nu \to 0$. The strong convergence of $(\mathbf{w}_\nu)_\nu$ and $(\mathbf{K}(\xi_\nu, \partial_t \xi_\nu)_\nu)$ are crucial here. We obtain:

$$\int_0^T \int_{\mathbb{R}} \Big((\rho \mathbf{w}) \partial_t \phi + (\rho \mathbf{v} \mathbf{w}) \partial_x \phi + \rho \mathbf{K}\left(\xi, \partial_t \xi\right) \mathbf{w}(1-\mathbf{w}) \phi \Big) \, \mathrm{d}x \, \mathrm{d}t + \int_{\mathbb{R}} \rho_0(x) \mathbf{w}_0(x) \phi(x,0) \, \mathrm{d}x = 0,$$

implying in particular that $\rho \mathbf{w} \in \mathbf{C}([0,T]; w^*\text{–}\mathbf{L}^\infty(\mathbb{R}))$. Therefore, we can conclude the same way we did for ρ that $\mathbf{w}$ satisfies the weak formulation (7.3.2), concluding the proof. □

Proof of Theorem 7.3.2. The existence claim readily follows from Theorem 7.4.8. □

7.5 Variants of the model

In the previous section, we conducted the existence analysis of Problem (7.2.1) – (7.2.6) with (7.2.7). The averaging in both space and time of the orderliness marker (7.2.6),(7.2.7) allowed for a strong decoupling of the system (7.2.1)–(7.2.3) and thus led us to a proof of existence *via* a time-splitting technique with merely bounded initial datum. Notice however that, while optimal results on scalar conservation laws feature merely $\mathbf{L}^\infty$ solutions ([114]), the assumption of bounded variation is typical in traffic modeling not only because of the numerous mathematical advantages it may offer and the consistency of the **BV**-based theory, but also because it is natural in the context due to the relative smallness of the number of traffic agents.

In this section, we will adopt the setting of densities with bounded variation; within the **BV** framework, we will consider two variants of the model (7.2.1) – (7.2.6). In Section 7.5.1 we replace (7.2.7) with (7.2.8) with only space averaging of the orderliness marker. Note that this will be the framework of our Section 7.6 devoted to numerical analysis of the model. The essential property that allows for analysis and numerical analysis of this variant is the propagation of the initial **BV** regularity of the orderliness marker ω uniformly with respect to the dynamics of ρ, which is the specific feature of solutions to (7.2.3) intimately related to the renormalization property of [134]. Further, in Section 7.5.2 we will briefly discuss the local variant of the model without averaging of the orderliness marker,*i.e.* , the variant where ω is taken equal to $\mathbf{w}$. Up to the source term in (7.2.3) that keeps non-local character, such model boils down to a system of conservation laws, thus falling within the class of so-called GSOM (generalized second-order) models put forward in [119, 118]. The unconditional **BV** regularity for $\mathbf{w}$ (provided initial data are **BV**) allows us to make a first step towards existence, however, we stress that mathematical tools for handling this situation are not ripe yet. Indeed, (7.2.1) becomes in this setting a conservation law with **BV** in space-time coefficients (see, *e.g.*, [136]) and one need to ensure that the candidate solutions fulfill selection criteria proper to the traffic context (see, *e.g.*, [19]) among infinitely many consistent selection criteria ([15]). The theory of (7.2.1),(7.2.2) is well understood for the case of isolated discontinuities in ω (*cf.* [107, 149]) but the case of interest, in the context of our model, requires much deeper investigation.

7.5.1 On the time-local model (7.2.1)–(7.2.6),(7.2.8)

Consider the variant of Problem (7.2.1) – (7.2.6) with averaging only in space of the orderliness marker (7.2.8). This simpler model keeps the non-local in space character reflecting the fact that, while the orderliness marker is attached to individual drivers, the impact (7.2.2),(7.2.6),(7.2.8) of the individual orderliness states on the fundamental diagram is taken in average.

The goal of this section is to sketch the existence theory, *via* convergence of the splitting approximations, based upon the propagation of the **BV** regularity of the initial datum $\mathbf{w}_0$. We do not expand this section, because the same problem is addressed in the setting of fully discrete numerical approximations in Section 7.6. We only point out the key arguments of the argumentation leading to convergence of the splitting approximations in this case.

To start with, we require $\rho_0, \mathbf{w}_0 \in \mathbf{BV}(\mathbb{R})$. The notion of solution is the one of Definition 7.3.1, with the necessary adjustment to replace (7.2.7) by (7.2.8); within the definition of solution, we can add the **BV** regularity of $\rho, \mathbf{w}$ since we achieve existence of such solutions. The splitting construction is unchanged. Our whole attention goes to the compactness issue, and at this point, we change the order of arguments and fully change the compactness analysis of $\mathbf{w}$. With **BV** datum ρ_0, compactness for $(\rho_\nu)_\nu$ is straightforward to obtain and it comes without the assumption on $V_{\min}, V_{\max}$ of Lemma 7.4.5. Indeed, due to the uniform space regularity of $(\omega_\nu)_\nu$ we can infer that $(\rho_\nu)_\nu$ is bounded in $\mathbf{L}^\infty([0,T];\mathbf{BV}(\mathbb{R}))$, see [59]. For $(\mathbf{w}_\nu)_\nu$, global **BV** bounds can be explained by the fact, highlighted in [134], that weak solutions to equations like (7.2.3) behave like if they were evolving along characteristics. In the basic sourceless case with piecewise constant data, this means that the solution at any time assumes the same states - and in the same order - as the initial datum, therefore its variation in space is controlled, for any time, by the variation of the initial data. For general **BV** datum and in presence of the source term, in order to infer this property one can rely upon the regularization approach of Chapter 6 and the renormalization property. We do not develop the argument here, but we stress that the numerical counterpart of the **BV** bound for $(\mathbf{w}_\nu)_\nu$ is assessed in detail in Section 7.6. While in Section 7.6 we require the restriction $\rho_0 \geq \varepsilon > 0$ in the appropriate area, see (7.3.3), let us stress here that this restriction is needed only to define the scheme and to guarantee the appropriate CFL condition. As far as the splitting procedure is considered, there is no need to introduce this restriction, as one can see it from the arguments of Chapter 6 where the case of $\rho \geq 0$ can be handled *via* a regularization procedure.

7.5.2 On the local model (7.2.1)–(7.2.6)

In this subsection, we discuss the purely local variant of our model, taking $\mathcal{M} = \mathrm{Id}$ in (7.2.6); in other words, we consider the situation where the 2×2 system on $\rho, \mathbf{w}$ and ω is closed by identifying ω with $\mathbf{w}$. The resulting model is a variant of GSOM (generalized second-order) models proposed in [119, 118], inspired by the already classical Aw-Rascle and Zhang model (ARZ). However, due to the choice (7.2.2) of the velocity, in our case the model need not reduce to a hyperbolic system with one genuinely nonlinear and one linearly degenerate field. Let us sketch a non-standard approach to this kind of GSOM models. First, as in Section 7.5.1, the dynamics of $\mathbf{w}$ ensures the propagation of **BV** regularity if we assume

$\mathbf{w}_0 \in \mathbf{BV}(\mathbb{R})$. For the sake of simplicity, consider first the case where $\mathbf{K} = 0$. Then it can be shown using the theory of [134] - due to the fact that the renormalization property is valid for general Borel functions - that piecewise constant $\mathbf{w}_0$ lead to piecewise constant $\mathbf{w}$ (*cf.* [118] for the analogous observation in the frame of GSOM). In this particular case equation (7.2.1) becomes a discontinuous-flux conservation law with separated interfaces. The theory (or, rather, multiple theories) of such equations were developed over more than 25 years, and we point out that it is possible to apply such theories in order to define the notion of solution to the model we are dealing with, and more generally, to GSOM models with or without the standard hyperbolicity structure. The key issue is to select the appropriate coupling conditions across discontinuities of $\omega \equiv \mathbf{w}$ (called interfaces), which is a clearly understood issue in the traffic context. According to phenomenological argumentation and to the numerical simulations involving the deterministic many-particle approximation (the so-called Follow-the-leader model), see [19], the coupling condition is the one maximizing the flux across interfaces. Either we do not pursue this line in the present chapter, let us point out that - for piecewise constant initial datum $\mathbf{w}_0$ of the orderliness marker - it is possible to define solutions (admissible in the sense of maximizing the flow across interfaces) for the splitting scheme we used in Section 7.4, and pass to the limit in the scheme. The compactness of $(\rho_\nu)_\nu$ can be assessed relying on the non-degeneracy of the flux [136]. The general setting with piecewise $\mathbf{C}^1$ or merely $\mathbf{BV}$ component $\mathbf{w}$ of the solution is a challenging issue for which some elements of analysis are ready, and others are lacking. Let us pinpoint the two main issues we leave for future work:

— One needs a plausible (on heuristic grounds, such as the uniqueness for Riemann problems) characterization of admissible solutions suitable for general $\omega \equiv \mathbf{w} \in \mathbf{BV}$.We stress that the one of [136], obtained in a very general setting, does not lead to uniqueness for general flux configurations but may be sufficient in the setting we are considering. In particular, due to the fact that $V_{\max} \geq V_{\min}$ in our model, fundamental diagrams for different values of $\omega \equiv \mathbf{w}$ do not cross, so that the crossing condition of [106] is automatically fulfilled. In this situation, the optimal-flux entropy solutions we are interested in coincide with the so-called vanishing viscosity solutions studied in [106, 18](see also [16] and in [64]). Note that a subtler characterization of admissible vanishing viscosity solutions is provided in [18] and [64]; the particularity of [64] is that the analysis extends to the general $\mathbf{BV}$ structure of the flux, which is what we have in mind.

— Being understood that the uniqueness of solutions for the system is probably beyond the reach of full analysis, it would be interesting to assess uniqueness of ρ, given $\omega \equiv \mathbf{w} \in \mathbf{BV}(\mathbb{R})$. Towards this goal, delicate refinements of techniques of [106, 18, 64] need to be elaborated.

To sum up, the present investigation of the non-local problem (7.2.1)–(7.2.6) highlights a novel approach to the definition of admissibility of solutionsof the local GSOM models, weakening at the same time the requirement on the hyperbolic structure of the system. Last but not least, the numerical strategy developed in Section 7.6 below for the spatially non-local problem of Section 7.5.1 is applicable also to the local problem of Section 7.5.2, provided consistent discretization of (7.2.1),(7.2.2) is used taking into account the possible sharp discontinuities in the expression of the flux function (*cf.* [149]).

7.6 Numerical approximation

In this section, we develop a finite volume numerical scheme for approximation of the model (7.2.1)–(7.2.6), with the averaging operator $\mathcal{M}$ in (7.2.6) given by (7.2.8). We analyze the **BV** stability and infer the convergence of the scheme. The approximation of the transport equation (7.2.3) is obtained exploiting the idea of propagation along characteristics; to state the idea clearly, we start with a simplified problem and expose the motivations behind the marching formula for the component $(\mathbf{w}^n_{j+1/2})_{j\in\mathbb{Z}}$ of the numerical solution. The scheme for the simplified problem turns out to be similar to the approximation of the Keyfitz-Kranzer [112] system put forward in [113], see Remark 7.6.1.

7.6.1 Motivation

We build a simple finite volume scheme and prove its convergence to a solution of (7.2.1) – (7.2.6) with (7.2.8) this time. Let us explain the ideas behind the construction of our scheme. For the sake of clarity, instead of (7.2.1) – (7.2.6), consider the problem

$$\begin{cases} \partial_t \rho + \partial_x \left(f(\rho)\right) &= 0 \\ \partial_t \left(\rho \mathbf{w}\right) + \partial_x \left(f(\rho)\mathbf{w}\right) &= \rho \mathbf{S}(x,t). \end{cases} \tag{7.6.1}$$

This system is a triangular one in the sense that we can solve the first equation and find ρ without $\mathbf{w}$, and then solve the second one. Numerically, this is what we do as well. The approximate density $\rho_\Delta = (\rho^n_{j+1/2})_{n,j}$ is constructed with a standard finite volume scheme:

$$(\rho^{n+1}_{j+1/2} - \rho^n_{j+1/2})\Delta x + (f^n_{j+1} - f^n_j)\Delta t = 0,$$

where f^n_j is a suitable approximation of the flux $f(\rho)$, see (7.6.2). We then use these values to construct $\mathbf{w}_\Delta$. The starting point is that if all the involved functions are smooth and if $\rho > 0$, the second PDE in (7.6.1) can be solved with the method of characteristics. More precisely, if $x \in \mathbf{C}^1((0,T))$ and $u(t) = \mathbf{w}(x(t),t)$, assuming in addition that $\rho > 0$ in Ω, the second equation in (7.6.1) can be solved by solving the family of ODE systems

$$\begin{cases} x'(t) = \mathbf{v}(\rho(x(t),t)) = \dfrac{f(\rho(x(t),t))}{\rho(x(t),t)} \\ \\ u'(t) = \mathbf{S}(x(t),t). \end{cases}$$

On each time step $[t^n, t^{n+1})$, for all $j \in \mathbb{Z}$, we draw characteristics starting from x_j with slope $s^n_j := \dfrac{f^n_j}{\rho^{n+1}_{j+1/2}}$, which is our choice for the approximation of $\dfrac{f(\rho(x(t),t))}{\rho(x(t),t)}$. At this point we need to know that $\rho^{n+1}_{j+1/2} \geq \varepsilon > 0$, in order to guarantee the existence of a CFL condition ensuring that at time t^{n+1}, the characteristics which started at x_j ends up at point $X^{n+1}_j \in (x_j, x_{j+1})$, see Figure 7.2.
Now, the ODE solved by $u(t) = \mathbf{w}(x(t),t)$ tells us that

$$u(t^{n+1}) = u(t^n) + \int_{t^n}^{t^{n+1}} \mathbf{S}(x(t),t)\,\mathrm{d}t \;\; \textit{i.e.} \;\; \mathbf{w}(X^{n+1}_j, t^{n+1}) \simeq \underbrace{\mathbf{w}(x_j,t^n) + \Delta t \mathbf{S}^{n+1}_{j+1/2}}_{\widetilde{\mathbf{w}}^n_{j+1/2}},$$

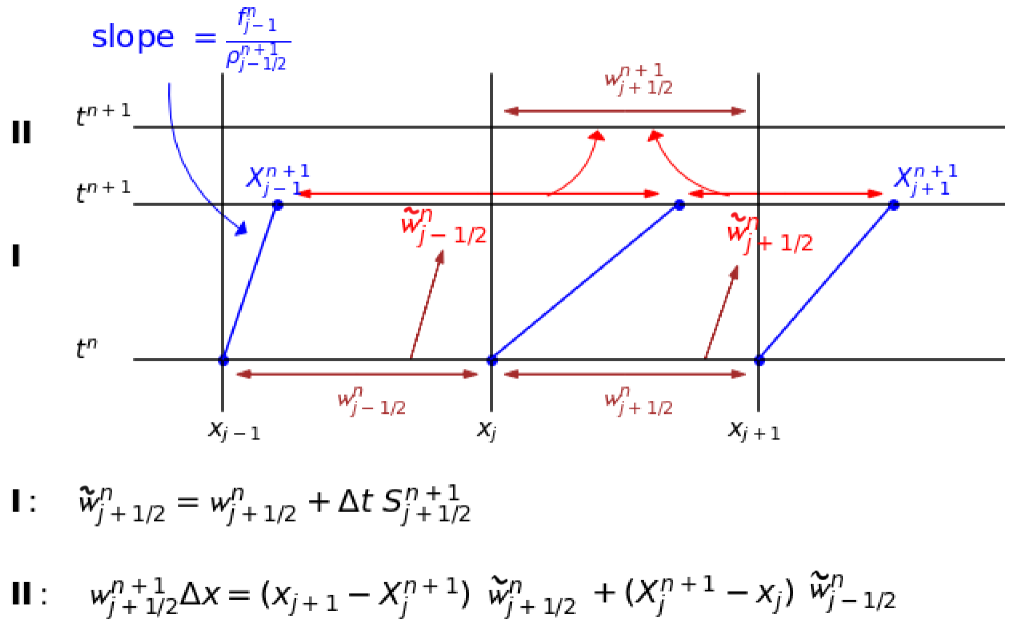

Figure 7.2 – Illustration of the two steps of the construction of the scheme.

$\mathbf{S}^{n+1}_{j+1/2}$ being a suitable approximation of the source term on the cell $(x_j, x_{j+1}) \times (t^n, t^{n+1}]$. At the numerical level, we are led to assign the value $\widetilde{\mathbf{w}}^n_{j+1/2}$ on (X^{n+1}_j, x_{j+1}). At this point we choose to define $\mathbf{w}_\Delta(\cdot, t^{n+1})$ on (x_j, x_{j+1}) by averaging the values $\widetilde{\mathbf{w}}^n_{j-1/2}$ and $\widetilde{\mathbf{w}}^n_{j+1/2}$ on (x_j, x_{j+1}). This is expressed as:

$$\begin{aligned}\mathbf{w}^{n+1}_{j+1/2}\Delta x &= (x_{j+1} - X^{n+1}_j)\widetilde{\mathbf{w}}^n_{j+1/2} + (X^{n+1}_j - x_j)\widetilde{\mathbf{w}}^n_{j-1/2} \\ &= \left(\left(1 - \frac{\Delta t}{\Delta x} s^n_j\right) \widetilde{\mathbf{w}}^n_{j+1/2} + \frac{\Delta t}{\Delta x} s^n_j \widetilde{\mathbf{w}}^n_{j-1/2}\right)\Delta x\end{aligned}$$

The above choices lead to a conservative scheme for $\rho\mathbf{w}$. Looking at the simplest case $\mathbf{S} = 0$ ($\widetilde{\mathbf{w}}^n_{j+1/2} = \mathbf{w}^n_{j+1/2}$), by multiplying the last expression by $\rho^{n+1}_{j+1/2}$, we find that:

$$\begin{aligned}&\left((\rho\mathbf{w})^{n+1}_{j+1/2} - (\rho\mathbf{w})^n_{j+1/2}\right)\Delta x \\ &= \rho^{n+1}_{j+1/2}\left(\mathbf{w}^{n+1}_{j+1/2} - \mathbf{w}^n_{j+1/2}\right)\Delta x + \left(\rho^{n+1}_{j+1/2} - \rho^n_{j+1/2}\right)\mathbf{w}^n_{j+1/2}\Delta x \\ &= -\rho^{n+1}_{j+1/2} s^n_j \left(\mathbf{w}^n_{j+1/2} - \mathbf{w}^n_{j-1/2}\right)\Delta t - (f^n_{j+1} - f^n_j)\mathbf{w}^n_{j+1/2}\Delta t \\ &= -f^n_j \left(\mathbf{w}^n_{j+1/2} - \mathbf{w}^n_{j-1/2}\right)\Delta t - (f^n_{j+1} - f^n_j)\mathbf{w}^n_{j+1/2}\Delta t \\ &= -\left(f^n_{j+1}\mathbf{w}^n_{j+1/2} - f^n_j\mathbf{w}^n_{j-1/2}\right)\Delta t,\end{aligned}$$

so that the numerical flux for $\rho\mathbf{w}$ turns out to be $f^n_j\mathbf{w}^n_{j-1/2}$. This observation is a cornerstone of our convergence proof.

Remark 7.6.1. In the case $\mathbf{S} \equiv 0$, system (7.6.1) has the same structure as the classical Keyfitz-Kranzer system [112] up to the properties of the flux function f, which is monotone

in the Keyfitz-Kranzer case and which is concave and bell-shaped (see (4.0.1)) in the case we are concerned with, see also [37]. Discretization of the Keyfitz-Kranzer system by finite difference schemes was addressed, in particular, in [113]. One of the schemes proposed in this reference (see [113, Section 5]) closely resembles our scheme. In the setting of [113] the flux has the form $f(\rho) = \rho\phi(\rho)$ but the assumptions on ϕ - different from our assumptions on $\mathbf{v}$ - ensure that f is increasing. Therefore the upwind choice is made for the numerical fluxes: $f_j^n = \rho_{j-1/2}^n \phi(\rho_{j-1/2}^n)$. The scheme of [113, Section 5] then reads:

$$\begin{aligned}
\rho_{j+1/2}^{n+1} &= \rho_{j+1/2}^n - \frac{\Delta t}{\Delta x}(f_{j+1}^n - f_j^n) \\
\mathbf{w}_{j+1/2}^{n+1} &= \left(1 - \frac{\Delta t}{\Delta x}\bar{s}_j^n\right)\mathbf{w}_{j+1/2}^n + \frac{\Delta t}{\Delta x}\bar{s}_j^n \mathbf{w}_{j-1/2}^n.
\end{aligned}$$

with $\bar{s}_j^n = \frac{f_j^n}{\rho_{j-1/2}^n} = \phi(\rho_{j-1/2}^n)$ due to the upwind choice for f_j^n. This choice of $\bar{s}_j^n$ differs slightly from our choice of s_j^n. It does not require the lower bound on $\rho_{j-1/2}^n$, but this is due to the monotonicity of f and cannot be mimicked in the setting of bell-shaped and concave f which is ours.

The ideas to deal with Problem (7.2.1) – (7.2.6) are the same as the ones we just develop. The difference is the presence of the coupling between ρ and $\mathbf{w}$. The coupling is taken care of in Step 1 below. Section 7.6.2 details the construction of the scheme for (7.2.1) – (7.2.6), following the ideas developed above.

7.6.2 Definition of the scheme

In what concerns the initial density, we assume that $\mathbf{TV}(\rho_0) < +\infty$ and that ρ_0 is separated from the vacuum in the sense stated in assumption (7.3.3); for the initial orderliness, we assume that $\mathbf{TV}(\mathbf{w}_0) < +\infty$ and $\mathbf{w}_0 \in \mathbf{L}^1(\mathbb{R}; [0,1])$.
For a fixed spatial mesh size $\Delta x > 0$ and time mesh size $\Delta t > 0$, let $x_j = j\Delta x$ $(j \in \mathbb{Z})$, $t^n = n\Delta t$ $(n \in \mathbb{N})$ and $N \in \mathbb{N}^*$ such that $T \in (t^N, t^{N+1}]$. We define the cell grids:

$$\mathbb{R} \times (0,T] \subset \bigcup_{n=0}^{N} \bigcup_{j\in\mathbb{Z}} \mathcal{P}_{j+1/2}^{n+1}, \quad \mathcal{P}_{j+1/2}^{n+1} = (x_j, x_{j+1}) \times (t^n, t^{n+1}].$$

We aim at constructing an approximate solution $(\rho_\Delta, \mathbf{w}_\Delta)$ defined almost everywhere on Ω:

$$\begin{cases}
\rho_\Delta = \rho_0 \mathbb{1}_{\{t\le 0\}} + \displaystyle\sum_{n=0}^{N}\sum_{j\in\mathbb{Z}} \rho_{j+1/2}^{n+1} \mathbb{1}_{\mathcal{P}_{j+1/2}^{n+1}} \\
\mathbf{w}_\Delta = \displaystyle\sum_{n=0}^{N}\sum_{j\in\mathbb{Z}} \mathbf{w}_{j+1/2}^{n} \mathbb{1}_{(x_j,x_{j+1})\times[t^n,t^{n+1})}.
\end{cases}$$

First, we discretize the initial data ρ_0 (resp. $\mathbf{w}_0$) with $\left(\rho_{j+1/2}^0\right)_j$, (resp. with $\left(\mathbf{w}_{j+1/2}^0\right)_j$) where for all $j \in \mathbb{Z}$, $\rho_{j+1/2}^0$ (resp. $\mathbf{w}_{j+1/2}^0$) is its mean value on the cell (x_j, x_{j+1}). Fix $n \in \{0, \dots, N\}$.

Step 1: Orderliness marker (mean value). For all $j \in \mathbb{Z}$, define

$$\omega_j^n = \int_{\mathbb{R}} \mathbf{w}_\Delta(x,t^n)\eta(x_j - y)\,\mathrm{d}y = \sum_{i\in\mathbb{Z}} \mathbf{w}_{i+1/2}^n \underbrace{\left(\int_{x_i}^{x_{i+1}} \eta(x_j - y)\,\mathrm{d}y\right)}_{\eta_{j-(i+1/2)}}.$$

Step 2: Finite volumes for the density. We use ω_j to define the velocity

$$\mathbf{v}_j^n(\rho) = (1-\omega_j^n)V_{\min}(\rho) + \omega_j^n V_{\max}(\rho)$$

and the flux $f_j^n(\rho) = \rho \mathbf{v}_j^n(\rho)$. Introduce the notations:

$$f_{\min,\max}(\rho) = \rho V_{\min,\max}(\rho);\ \delta f = f_{\max} - f_{\min}.$$

Let $\mathbf{F}_j^n = \mathbf{F}_j^n(u,v)$ be a monotone numerical flux associated with f_j^n, see Definition 1.1.1 and Example 1.1.1. For the sake of simplicity, we use the Rusanov flux, that is for all $u, v \in [0,1]$,

$$\mathbf{F}_j^n(u,v) = \frac{1}{2}\left(f_j^n(u) + f_j^n(v) + \mathbf{L}(u-v)\right),\ \mathbf{L} = \max\{\|f'_{\min}\|_{\mathbf{L}^\infty}, \|f'_{\max}\|_{\mathbf{L}^\infty}\}.$$

The conservation of ρ written in a cell $\mathcal{P}_{j+1/2}^{n+1}$ $(j \in \mathbb{Z})$ leads to the following marching formula:

$$\rho_{j+1/2}^{n+1} = \rho_{j+1/2}^n - \frac{\Delta t}{\Delta x}\left(\mathbf{F}_{j+1}^n(\rho_{j+1/2}^n, \rho_{j+3/2}^n) - \mathbf{F}_j^n(\rho_{j-1/2}^n, \rho_{j+1/2}^n)\right). \tag{7.6.2}$$

Eventually, it will be convenient to write the scheme under the form:

$$\rho_{j+1/2}^{n+1} = \mathbf{H}_j^n\left(\rho_{j-1/2}^n, \rho_{j+1/2}^n, \rho_{j+3/2}^n\right),$$

where $\mathbf{H}_j^n = \mathbf{H}_j^n(a,b,c)$ is given by the right-hand side of (7.6.2) with $\rho_{j-1/2}^n, \rho_{j+1/2}^n, \rho_{j+3/2}^n$ replaced by $a, b, c \in [0,1]$.

Step 3: Source term. For all $j \in \mathbb{Z}$, we set

$$\begin{cases} \xi_{j+1/2}^{n+1} = \displaystyle\sum_{i\in\mathbb{Z}} \rho_{i+1/2}^{n+1} \underbrace{\left(\int_{x_i}^{x_{i+1}} \mu(x_{j+1/2} - y)\,\mathrm{d}y\right)}_{\mu_{j+1/2-(i+1/2)}} \\ \chi_{j+1/2}^{n+1} = -\displaystyle\sum_{i\in\mathbb{Z}} \mathbf{F}_i^n(\rho_{i-1/2}^{n+1}, \rho_{i+1/2}^{n+1}) \underbrace{\left(\int_{x_i}^{x_{i+1}} \mu'(x_{j+1/2} - y)\,\mathrm{d}y\right)}_{\mathrm{d}\mu_{j+1/2-(i+1/2)}}. \end{cases}$$

Note that hereabove, we discretize the expression for $\chi = \partial_t \xi$ that is obtained combining the definition of ξ and the weak formulation of the mass conservation equation.

Then we define the source term by

$$\forall j \in \mathbb{Z}, \quad \mathbf{S}_{j+1/2}^{n+1} = \mathbf{K}\left(\xi_{j+1/2}^{n+1}, \chi_{j+1/2}^{n+1}\right)\mathbf{w}_{j+1/2}^n(1 - \mathbf{w}_{j+1/2}^n).$$

Step 4: Orderliness marker. Fix $j \in \mathbb{Z}$. Set

$$X_j^{n+1} = x_j + \Delta t \underbrace{\left(\frac{\mathbf{F}_j^n(\rho_{j-1/2}^n, \rho_{j+1/2}^n)}{\rho_{j+1/2}^{n+1}} \right)}_{s_j^n}.$$

We will prove that under (7.3.3) and a suitable CFL condition, see (7.6.4), the sequence $(X_j^{n+1})_j$ is well defined. Following the approach outlined in Section 7.6.1, we compute the updated orderliness marker as follows:

$$\begin{cases} \widetilde{\mathbf{w}}_{j+1/2}^n = \mathbf{w}_{j+1/2}^n + \Delta t \mathbf{S}_{j+1/2}^{n+1} \\ \mathbf{w}_{j+1/2}^{n+1} = \left(1 - \dfrac{\Delta t}{\Delta x} s_j^n\right) \widetilde{\mathbf{w}}_{j+1/2}^n + \dfrac{\Delta t}{\Delta x} s_j^n \widetilde{\mathbf{w}}_{j-1/2}^n. \end{cases} \tag{7.6.3}$$

We also define

$$(\xi_\Delta, \chi_\Delta, \mathbf{S}_\Delta) = \sum_{n=0}^{N} \sum_{j \in \mathbb{Z}} (\xi_{j+1/2}^{n+1}, \chi_{j+1/2}^{n+1}, \mathbf{S}_{j+1/2}^{n+1}) \mathbb{1}_{\mathcal{P}_{j+1/2}^{n+1}}$$

and

$$\omega_\Delta = \sum_{n=0}^{N} \sum_{j \in \mathbb{Z}} \omega_j^n \mathbb{1}_{(x_j, x_{j+1}) \times [t^n, t^{n+1})}.$$

For later use, introduce the notations:

$$\|\mathbf{K}\|_{\mathbf{L}^\infty} = \sup_{\substack{\varepsilon \le \xi \le 1 \\ |\chi| \le \mathbf{L} \times \mathbf{TV}(\mu)}} |\mathbf{K}(\xi, \chi)|; \quad \|\nabla \mathbf{K}\|_{\mathbf{L}^\infty} = \sup_{\substack{\varepsilon \le \xi_1, \xi_2 \le 1 \\ |\chi_1|, |\chi_2| \le \mathbf{L} \times \mathbf{TV}(\mu)}} |\mathbf{K}(\xi_1, \chi_1) - \mathbf{K}(\xi_2, \chi_2)|$$

and

$$\|\delta f\|_{\mathbf{L}^\infty} = \sup_{0 \le \rho \le 1} \delta f(\rho); \quad \|\delta f'\|_{\mathbf{L}^\infty} = \sup_{0 \le \rho \le 1} |\delta f'(\rho)|.$$

7.6.3 $\mathbf{L}^\infty$ stability *via* monotonicity

Proposition 7.6.1. *Under the conditions*

$$\lambda \max\left\{2, \frac{1}{\varepsilon}\right\} \mathbf{L} \le 1; \quad \lambda = \frac{\Delta t}{\Delta x} \tag{7.6.4}$$

and

$$\Delta t \|\mathbf{K}\|_{\mathbf{L}^\infty} \le 1, \tag{7.6.5}$$

the scheme (7.6.2)-(7.6.3) *is monotone and* $\mathbf{L}^\infty$ *stable. More precisely, for all* $n \in \{0, \dots, N+1\}$ *and* $j \in \mathbb{Z}$, *we have*

$$\varepsilon \le \rho_{j+1/2}^n \le 1 \quad \textit{and} \quad 0 \le \mathbf{w}_{j+1/2}^n \le 1. \tag{7.6.6}$$

Proof. We prove the result by induction on n.
The result is clearly true for $n = 0$ by definition of $\left(\rho^0_{j+1/2}\right)_j$ and $\left(\mathbf{w}^0_{j+1/2}\right)_j$. Suppose now that for some $n \in \{0, \dots, N\}$, (7.6.6) holds. Fix $j \in \mathbb{Z}$.
(i) Since $0 \le \mathbf{w}_\Delta(\cdot, t^n) \le 1$, we have

$$\omega^n_j = \int_{\mathbb{R}} \mathbf{w}_\Delta(y, t^n)\eta(x_j - y)\,\mathrm{d}y \in [0, 1],$$

from which we deduce that f^n_j is a convex combination of $f_{\min}$ and $f_{\max}$. Note also that

$$\begin{aligned}|\omega^n_{j+1} - \omega^n_j| &\le \sum_{i\in\mathbb{Z}} \int_{x_i}^{x_{i+1}} |\mathbf{w}^n_{j+1/2}| \cdot |\eta(x_{j+1} - y) - \eta(x_j - y)|\,\mathrm{d}y \\ &\le \int_{\mathbb{R}} |\eta(y - \Delta x) - \eta(y)|\,\mathrm{d}y \le \mathbf{TV}(\eta)\Delta x.\end{aligned}$$

(ii) Using the CFL condition, we can prove that the scheme (7.6.2) is monotone. More precisely, for a.e. $a, b, c \in [0, 1]$, we have:

$$\frac{\partial \mathbf{H}^n_j}{\partial a}(a, b, c) = \lambda \frac{\partial \mathbf{F}^n_j}{\partial u}(a, b) \ge 0; \quad \frac{\partial \mathbf{H}^n_j}{\partial c}(a, b, c) = -\lambda \frac{\partial \mathbf{F}^n_{j+1}}{\partial v}(b, c) \ge 0$$

and

$$\frac{\partial \mathbf{H}^n_j}{\partial b}(a, b, c) = 1 - \lambda \left(\frac{\partial \mathbf{F}^n_{j+1}}{\partial u}(b, c) - \frac{\partial \mathbf{F}^n_j}{\partial u}(a, b) \right) \ge 1 - 2\lambda \mathbf{L} \ge 0.$$

Using the monotonicity of the scheme and the induction property, we deduce that

$$\rho^{n+1}_{j+1/2} = \mathbf{H}^n_j \left(\rho^n_{j-1/2}, \rho^n_{j+1/2}, \rho^n_{j+3/2} \right) \le \mathbf{H}^n_j(1, 1, 1) = 1$$

and, since $\delta f(\varepsilon) = 0$ due to assumption (7.3.3),

$$\rho^{n+1}_{j+1/2} \ge \mathbf{H}^n_j(\varepsilon, \varepsilon, \varepsilon) = \varepsilon - \lambda(\omega^n_{j+1} - \omega^n_j)\delta f(\varepsilon) = \varepsilon.$$

(iii) Since $\varepsilon \le \rho_\Delta(\cdot, t^{n+1}) \le 1$, we have

$$\xi^{n+1}_{j+1/2} = \int_{\mathbb{R}} \rho_\Delta(y, t^{n+1})\mu(x_{j+1/2} - y)\,\mathrm{d}y \in [\varepsilon, 1],$$

and clearly,

$$|\chi^{n+1}_{j+1/2}| \le \mathbf{L} \times \mathbf{TV}(\mu).$$

(iv) Let us prove that $\widetilde{\mathbf{w}}^n_{j+1/2} \in [0, 1]$. Introduce the function

$$g : \mathbf{w} \mapsto \mathbf{w} + \Delta t \mathbf{K} \left(\xi^{n+1}_{j+1/2}, \chi^{n+1}_{j+1/2} \right) \mathbf{w}(1 - \mathbf{w}).$$

Using (7.6.5), we obtain that for all $\mathbf{w} \in [0, 1]$,

$$g'(\mathbf{w}) = 1 + \Delta t \mathbf{K} \left(\xi^{n+1}_{j+1/2}, \chi^{n+1}_{j+1/2} \right) (1 - 2\mathbf{w}) \ge 1 - \Delta t \left| \mathbf{K} \left(\xi^{n+1}_{j+1/2}, \chi^{n+1}_{j+1/2} \right) \right| \ge 0.$$

Since $g(0) = 0$ and $g(1) = 1$, the monotonicity of g implies that $\widetilde{\mathbf{w}}^n_{j+1/2} = g(\mathbf{w}^n_{j+1/2}) \in [0, 1]$. Due to the CFL condition, $\mathbf{w}^{n+1}_{j+1/2}$ is a convex combination of $\widetilde{\mathbf{w}}^n_{j+1/2}$ and $\widetilde{\mathbf{w}}^n_{j-1/2}$. This implies that $\mathbf{w}^{n+1}_{j+1/2} \in [0, 1]$, which completes the induction argument. □

Remark 7.6.2. The stability estimates (7.6.6) immediately imply:

$$\varepsilon \leq \rho_\Delta, \xi_\Delta \leq 1; \quad 0 \leq \mathbf{w}_\Delta, \omega_\Delta \leq 1; \quad |\chi_\Delta| \leq \mathbf{L} \times \mathbf{TV}(\mu); \quad |\mathbf{S}_\Delta| \leq \frac{\|\mathbf{K}\|_{\mathbf{L}^\infty}}{4}.$$

For all $a, b \in [0,1]$, set

$$a \wedge b = \min\{a, b\} \quad a \vee b = \max\{a, b\}.$$

Corollary 7.6.2 (Discrete entropy inequalities)**.** *The numerical scheme* (7.6.2) *fulfills the following discrete entropy inequalities for all* $n \in \{0, \dots, N\}$, $j \in \mathbb{Z}$ *and* $\kappa \in [0,1]$*:*

$$\begin{aligned} &\left(\left|\rho_{j+1/2}^{n+1} - \kappa\right| - \left|\rho_{j+1/2}^{n} - \kappa\right|\right)\Delta x + (\Phi_{j+1}^n - \Phi_j^n)\Delta t \\ &\leq -\operatorname{sgn}\left(\rho_{j+1/2}^{n+1} - \kappa\right) \times (f_{j+1}^n(\kappa) - f_j^n(\kappa))\Delta t, \end{aligned} \tag{7.6.7}$$

where Φ_j^n *denotes the numerical entropy flux:*

$$\Phi_j^n = \mathbf{F}_j^n\left(\rho_{j-1/2}^n \vee \kappa, \rho_{j+1/2}^n \vee \kappa\right) - \mathbf{F}_j^n\left(\rho_{j-1/2}^n \wedge \kappa, \rho_{j+1/2}^n \wedge \kappa\right).$$

Proof. This is mostly a consequence of the scheme monotonicity. Remark that

$$\forall j \in \mathbb{Z}, \quad \mathbf{H}_j^n(\kappa, \kappa, \kappa) = \kappa - \lambda(f_{j+1}^n(\kappa) - f_j^n(\kappa)).$$

We combine this with the convexity of the function $|\cdot - \kappa|$ to obtain:

$$\begin{aligned} &\left|\rho_{j+1/2}^{n+1} - \kappa\right| \\ &= \left|\mathbf{H}_j^n\left(\rho_{j-1/2}^n, \rho_{j+1/2}^n, \rho_{j+3/2}^n\right) - \kappa\right| \\ &\leq \left|\mathbf{H}_j^n\left(\rho_{j-1/2}^n, \rho_{j+1/2}^n, \rho_{j+3/2}^n\right) - \mathbf{H}_j^n(\kappa, \kappa, \kappa)\right| + \operatorname{sgn}\left(\rho_{j+1/2}^{n+1} - \kappa\right) \times \left(\mathbf{H}_j^n(\kappa, \kappa, \kappa) - \kappa\right) \\ &\leq \mathbf{H}_j^n\left(\rho_{j-1/2}^n \vee \kappa, \rho_{j+1/2}^n \vee \kappa, \rho_{j+3/2}^n \vee \kappa\right) - \mathbf{H}_j^n\left(\rho_{j-1/2}^n \wedge \kappa, \rho_{j+1/2}^n \wedge \kappa, \rho_{j+3/2}^n \wedge \kappa\right) \\ &- \lambda \operatorname{sgn}\left(\rho_{j+1/2}^{n+1} - \kappa\right) \times (f_{j+1}^n(\kappa) - f_j^n(\kappa)) \\ &= \left|\rho_{j+1/2}^n - \kappa\right| - \lambda(\Phi_{j+1}^n - \Phi_j^n) - \lambda \operatorname{sgn}\left(\rho_{j+1/2}^{n+1} - \kappa\right) \times (f_{j+1}^n(\kappa) - f_j^n(\kappa)). \end{aligned}$$

□

7.6.4 Compactness *via* BV stability

The key to obtain compactness is to derive global **BV** bounds for $(\rho_\Delta, \mathbf{w}_\Delta)_\Delta$.

Theorem 7.6.3. *There exists a constant* $c > 0$ *such that for all* $n \in \{1, \dots, N\}$*:*

$$\mathbf{TV}(\rho_\Delta(\cdot, t^n)) + \mathbf{TV}(\mathbf{w}_\Delta(\cdot, t^n)) \leq (\mathbf{TV}(\rho_0) + \mathbf{TV}(\mathbf{w}_0))e^{(2c+c^2\Delta t)t^n}. \tag{7.6.8}$$

Proof. Fix $n \in \{0, \dots, N\}$ and $j \in \mathbb{Z}$. For the sake of clarity, set

$$\mathcal{F}_j^n = \mathbf{F}_j^n \left(\rho_{j-1/2}^n, \rho_{j+1/2}^n\right).$$

We start by writing the scheme (7.6.2) under the form:

$$\rho_{j+1/2}^{n+1} = \rho_{j+1/2}^n - \underbrace{\lambda \left(\frac{\mathcal{F}_{j+1}^n - \mathbf{F}_{j+1}^n \left(\rho_{j+1/2}^n, \rho_{j+1/2}^n\right)}{\rho_{j+3/2}^n - \rho_{j+1/2}^n} \right)}_{-B_{j+1}} \left(\rho_{j+3/2}^n - \rho_{j+1/2}^n\right)$$

$$- \underbrace{\lambda \left(\frac{\mathbf{F}_j^n \left(\rho_{j+1/2}^n, \rho_{j+1/2}^n\right) - \mathcal{F}_j^n}{\rho_{j+1/2}^n - \rho_{j-1/2}^n} \right)}_{A_j} \left(\rho_{j+1/2}^n - \rho_{j-1/2}^n\right) - \lambda(\omega_{j+1}^n - \omega_j^n)\delta f(\rho_{j+1/2}^n).$$

The monotonicity of $\mathbf{F}_{j+1}^n$ and $\mathbf{F}_j^n$ ensures that $A_j, B_{j+1} \geq 0$. We deduce that

$$\begin{aligned}
\rho_{j+1/2}^{n+1} - \rho_{j-1/2}^{n+1} &= (1 - A_j - B_j)\left(\rho_{j+1/2}^n - \rho_{j-1/2}^n\right) \\
&\quad + A_{j-1}\left(\rho_{j-1/2}^n - \rho_{j-3/2}^n\right) + B_{j+1}\left(\rho_{j+3/2}^n - \rho_{j+1/2}^n\right) \\
&\quad - \lambda(\omega_{j+1}^n - \omega_j^n)\delta f(\rho_{j+1/2}^n) + \lambda(\omega_j^n - \omega_{j-1}^n)\delta f(\rho_{j-1/2}^n).
\end{aligned}$$

Making use of the CFL condition (7.6.4), we have

$$|A_j| + |B_j| \leq 2\lambda \mathbf{L} \leq 1,$$

hence:

$$\begin{aligned}
\sum_{j\in\mathbb{Z}} \left|\rho_{j+1/2}^{n+1} - \rho_{j-1/2}^{n+1}\right| &\leq \sum_{j\in\mathbb{Z}} (1 - A_j - B_j)\left|\rho_{j+1/2}^n - \rho_{j-1/2}^n\right| \\
&\quad + \sum_{j\in\mathbb{Z}} A_{j-1}\left|\rho_{j-1/2}^n - \rho_{j-3/2}^n\right| + \sum_{j\in\mathbb{Z}} B_{j+1}\left|\rho_{j+3/2}^n - \rho_{j+1/2}^n\right| \\
&\quad + \lambda \sum_{j\in\mathbb{Z}} \left|(\omega_{j+1}^n - 2\omega_j^n + \omega_{j-1}^n)\delta f(\rho_{j+1/2}^n)\right| \\
&\quad + \lambda \sum_{j\in\mathbb{Z}} \left|(\omega_j^n - \omega_{j-1}^n)\left(\delta f(\rho_{j+1/2}^n) - \delta f(\rho_{j-1/2}^n)\right)\right| \\
&\leq (1 + \Delta t \mathbf{TV}(\eta)\|\delta f'\|_{\mathbf{L}^\infty}) \sum_{j\in\mathbb{Z}} \left|\rho_{j+1/2}^n - \rho_{j-1/2}^n\right| \\
&\quad + \lambda \|\delta f\|_{\mathbf{L}^\infty} \sum_{j\in\mathbb{Z}} \left|\omega_{j+1}^n - 2\omega_j^n + \omega_{j-1}^n\right|.
\end{aligned}$$

We now rewrite the last term of the inequality using the Abel procedure. For all $j \in \mathbb{Z}$, we have

$$\begin{aligned}
\omega_{j+1}^n - 2\omega_j^n + \omega_{j-1}^n &= \sum_{i\in\mathbb{Z}} \mathbf{w}_{i+1/2}^n \Big(\left(\eta_{j-(i-1/2)} - \eta_{j-(i+1/2)}\right) - \left(\eta_{j-(i+1/2)} - \eta_{j-(i+3/2)}\right) \Big) \\
&= \sum_{i\in\mathbb{Z}} \left(\mathbf{w}_{i+1/2}^n - \mathbf{w}_{i-1/2}^n\right)\left(\eta_{j-(i-1/2)} - \eta_{j-(i+1/2)}\right),
\end{aligned}$$

from which we deduce:

$$\sum_{j\in\mathbb{Z}} |\omega_{j+1}^n - 2\omega_j^n + \omega_{j-1}^n| \leq \sum_{i\in\mathbb{Z}} |\mathbf{w}_{i+1/2}^n - \mathbf{w}_{i-1/2}^n| \left(\sum_{j\in\mathbb{Z}} |\eta_{j-(i-1/2)} - \eta_{j-(i+1/2)}| \right)$$
$$\leq \mathbf{TV}(\eta)\mathbf{TV}(\mathbf{w}_\Delta(\cdot, t^n))\Delta x.$$

We now derive a similar estimate for $(\mathbf{w}_\Delta)_\Delta$. We have

$$\mathbf{w}_{j+1/2}^{n+1} - \mathbf{w}_{j-1/2}^{n+1} = \left(1 - \lambda s_j^n\right)\left(\mathbf{w}_{j+1/2}^n - \mathbf{w}_{j-1/2}^n\right) + \lambda s_{j-1}^n \left(\mathbf{w}_{j-1/2}^n - \mathbf{w}_{j-3/2}^n\right)$$
$$+ \Delta t \left\{ \left(1 - \lambda s_j^n\right)\left(\mathbf{S}_{j+1/2}^{n+1} - \mathbf{S}_{j-1/2}^{n+1}\right) + \lambda s_{j-1}^n \left(\mathbf{S}_{j-1/2}^{n+1} - \mathbf{S}_{j-3/2}^{n+1}\right) \right\}.$$

Since $0 \leq \lambda s_j^n \leq 1$ due to the CFL condition, we obtain

$$\sum_{j\in\mathbb{Z}} \left| \mathbf{w}_{j+1/2}^{n+1} - \mathbf{w}_{j-1/2}^{n+1} \right| \leq \sum_{j\in\mathbb{Z}} \left| \mathbf{w}_{j+1/2}^n - \mathbf{w}_{j-1/2}^n \right| + \Delta t \sum_{j\in\mathbb{Z}} \left| \mathbf{S}_{j+1/2}^{n+1} - \mathbf{S}_{j-1/2}^{n+1} \right|.$$

But

$$\left| \mathbf{S}_{j+1/2}^{n+1} - \mathbf{S}_{j-1/2}^{n+1} \right| \leq \|\mathbf{K}\|_{\mathbf{L}^\infty} \left| \mathbf{w}_{j+1/2}^n - \mathbf{w}_{j-1/2}^n \right|$$
$$+ \frac{\|\nabla \mathbf{K}\|_{\mathbf{L}^\infty}}{4} \left(\left| \xi_{j+1/2}^{n+1} - \xi_{j-1/2}^{n+1} \right| + \left| \chi_{j+1/2}^{n+1} - \chi_{j-1/2}^{n+1} \right| \right),$$

so that from

$$\xi_{j+1/2}^{n+1} - \xi_{j-1/2}^{n+1} = \sum_{i\in\mathbb{Z}} \rho_{i+1/2}^{n+1} (\mu_{j+1/2-(i+1/2)} - \mu_{j-1/2-(i+1/2)})$$
$$= \sum_{i\in\mathbb{Z}} \rho_{i+1/2}^{n+1} (\mu_{j+1/2-(i+1/2)} - \mu_{j+1/2-(i-1/2)})$$
$$= \sum_{i\in\mathbb{Z}} (\rho_{i+1/2}^{n+1} - \rho_{i+3/2}^{n+1}) \mu_{j+1/2-(i+1/2)},$$

we deduce (remember that μ is a weight function):

$$\sum_{j\in\mathbb{Z}} \left| \xi_{j+1/2}^{n+1} - \xi_{j-1/2}^{n+1} \right| \leq \mathbf{TV}(\rho_\Delta(\cdot, t^{n+1})).$$

We prove in the same way that

$$\sum_{j\in\mathbb{Z}} \left| \chi_{j+1/2}^{n+1} - \chi_{j-1/2}^{n+1} \right| \leq 2\mathbf{L} \times \mathbf{TV}(\mu)\mathbf{TV}(\rho_\Delta(\cdot, t^{n+1})).$$

Finally, we proved that

$$\begin{cases} \mathbf{TV}(\rho_\Delta(\cdot, t^{n+1})) \leq (1 + \Delta t \mathbf{TV}(\eta)\|\delta f'\|_{\mathbf{L}^\infty})\, \mathbf{TV}(\rho_\Delta(\cdot, t^n)) \\ \qquad\qquad + \|\delta f\|_{\mathbf{L}^\infty} \mathbf{TV}(\eta)\Delta t \mathbf{TV}(\mathbf{w}_\Delta(\cdot, t^n)) \\ \mathbf{TV}(\mathbf{w}_\Delta(\cdot, t^{n+1})) \leq (1 + \Delta t \|\mathbf{K}\|_{\mathbf{L}^\infty})\, \mathbf{TV}(\mathbf{w}_\Delta(\cdot, t^n)) \\ \qquad\qquad + \Delta t \dfrac{\|\nabla \mathbf{K}\|_{\mathbf{L}^\infty}(1 + 2\mathbf{L} \times \mathbf{TV}(\mu))}{4} \mathbf{TV}(\rho_\Delta(\cdot, t^{n+1})), \end{cases} \tag{7.6.9}$$

i.e. by setting $u_n = \mathbf{TV}(\rho_\Delta(\cdot, t^n))$ and $v_n = \mathbf{TV}(\mathbf{w}_\Delta(\cdot, t^n))$,

$$\begin{cases} u_{n+1} \leq (1 + c_1\Delta t)u_n + c_2\Delta t v_n \\ v_{n+1} \leq (1 + c_3\Delta t + c_2c_4\Delta t^2)v_n + (1 + c_1\Delta t)c_4\Delta t u_n. \end{cases}$$

Putting the above inequalities into a matrix form, with standard linear algebra computations we are led to (7.6.8) with $c = \max_{1\leq i\leq 4} c_i$. □

Remark 7.6.3 ($\mathbf{L}^1$ stability)**.** Under the additional assumption that $\mathbf{w}_0 \in \mathbf{L}^1(\mathbb{R})$, the scheme (7.6.3) is $\mathbf{L}^1$ stable. Indeed, for all $n \in \{0, \dots, N-1\}$,

$$\begin{aligned} \|\mathbf{w}_\Delta(\cdot, t^{n+1})\|_{\mathbf{L}^1} &= \sum_{j\in\mathbb{Z}} \mathbf{w}^{n+1}_{j+1/2}\Delta x \\ &= \sum_{j\in\mathbb{Z}} \mathbf{w}^{n}_{j+1/2}\Delta x + \sum_{j\in\mathbb{Z}} \mathbf{S}^{n+1}_{j+1/2}\Delta x\Delta t \\ &+ \sum_{j\in\mathbb{Z}} \lambda s^n_j(\mathbf{w}^n_{j-1/2} - \mathbf{w}^n_{j+1/2}) + \sum_{j\in\mathbb{Z}} \lambda s^n_j(\mathbf{S}^{n+1}_{j-1/2} - \mathbf{S}^{n+1}_{j+1/2})\Delta t \\ &\leq (1 + \|\mathbf{K}\|_{\mathbf{L}^\infty}\Delta t)\|\mathbf{w}_\Delta(\cdot, t^n)\|_{\mathbf{L}^1} + \frac{\mathbf{L}}{\varepsilon} \times \mathbf{TV}(\mathbf{w}_\Delta(\cdot, t^n))\Delta t \\ &+ \frac{\mathbf{L}}{\varepsilon} \times \frac{\|\nabla\mathbf{K}\|_{\mathbf{L}^\infty}(1 + 2\mathbf{L} \times \mathbf{TV}(\mu))}{4}\mathbf{TV}(\rho_\Delta(\cdot, t^n))\Delta t \\ &\leq (1 + c\Delta t)\|\mathbf{w}_\Delta(\cdot, t^n)\|_{\mathbf{L}^1} + \frac{\mathbf{L}}{\varepsilon} \times \mathbf{TV}(\mathbf{w}_\Delta(\cdot, t^n))\Delta t \\ &+ \frac{\mathbf{L}}{\varepsilon} \times c\mathbf{TV}(\rho_\Delta(\cdot, t^n))\Delta t. \end{aligned}$$

Gronwall lemma yields $\sup_\Delta \|\mathbf{w}_\Delta\|_{\mathbf{L}^\infty((0,T);\mathbf{L}^1(\mathbb{R}))} < +\infty$.

Corollary 7.6.4. *We have:*

$$\begin{aligned} &\sum_{j\in\mathbb{Z}} |\rho^{n+1}_{j+1/2} - \rho^n_{j+1/2}|\Delta x \leq \left(2\mathbf{L} \times \mathbf{TV}(\rho_\Delta(\cdot, t^n)) + \|\delta f\|_{\mathbf{L}^\infty}\mathbf{TV}(\eta)\mathbf{TV}(\mathbf{w}_\Delta(\cdot, t^n))\right)\Delta t \\ &\sum_{j\in\mathbb{Z}} |\mathbf{w}^{n+1}_{j+1/2} - \mathbf{w}^n_{j+1/2}|\Delta x \leq \left(\frac{\mathbf{L}}{\varepsilon}\left(\mathbf{TV}(\mathbf{w}_\Delta(\cdot, t^n)) + c\mathbf{TV}(\rho_\Delta(\cdot, t^{n+1}))\right) + c\|\mathbf{w}_\Delta(\cdot, t^n)\|_{\mathbf{L}^1}\right)\Delta t \end{aligned} \tag{7.6.10}$$

Consequently, there exist $\rho, \mathbf{w} \in \mathbf{L}^\infty(\Omega) \cap \mathbf{C}([0,T]; \mathbf{L}^1_{\mathbf{loc}}(\mathbb{R}))$*, such that along a subsequence,* $(\rho_\Delta, \mathbf{w}_\Delta)_\Delta \to (\rho, \mathbf{w})$ *a.e. on* Ω*.*

Proof. Estimates (7.6.10) come from a combination of estimates (7.6.8) and the scheme (7.6.2)-(7.6.3). More precisely,

$$\begin{aligned} \sum_{j\in\mathbb{Z}} |\rho^{n+1}_{j+1/2} - \rho^n_{j+1/2}|\Delta x &\leq \sum_{j\in\mathbb{Z}} \left|\mathbf{F}^n_{j+1}\left(\rho^n_{j+1/2}, \rho^n_{j+3/2}\right) - \mathbf{F}^n_j\left(\rho^n_{j-1/2}, \rho^n_{j+1/2}\right)\right|\Delta t \\ &\leq 2\mathbf{L}\sum_{j\in\mathbb{Z}} |\rho^n_{j+1/2} - \rho^n_{j-1/2}|\Delta t + \sum_{j\in\mathbb{Z}} |f^n_{j+1}(\rho^n_{j+1/2}) - f^n_j(\rho^n_{j+1/2})|\Delta t \\ &\leq 2\mathbf{L} \times \mathbf{TV}(\rho_\Delta(\cdot, t^n))\Delta t + \|\delta f\|_{\mathbf{L}^\infty}\mathbf{TV}(\eta)\mathbf{TV}(\mathbf{w}_\Delta(\cdot, t^n))\Delta t. \end{aligned}$$

Regarding $(\mathbf{w}_\Delta)_\Delta$, we write

$$\sum_{j\in\mathbb{Z}}\left|\mathbf{w}_{j+1/2}^{n+1}-\mathbf{w}_{j+1/2}^{n}\right|\Delta x \le \frac{\mathbf{L}}{\varepsilon}\left(\mathbf{TV}(\mathbf{w}_\Delta(\cdot,t^n))+\sum_{j\in\mathbb{Z}}\left|\mathbf{S}_{j+1/2}^{n+1}-\mathbf{S}_{j-1/2}^{n+1}\right|\right)\Delta t+\sum_{j\in\mathbb{Z}}\left|\mathbf{S}_{j+1/2}^{n+1}\right|\Delta x\Delta t$$
$$\le \frac{\mathbf{L}}{\varepsilon}\left(\mathbf{TV}(\mathbf{w}_\Delta(\cdot,t^n))+c\mathbf{TV}(\rho_\Delta(\cdot,t^{n+1}))\right)\Delta t+c\|\mathbf{w}_\Delta(\cdot,t^n)\|_{\mathbf{L}^1}\Delta t.$$

The compactness comes from [98, Appendix A] since we have the bounds (7.6.6)-(7.6.8)-(7.6.10). □

7.6.5 Approximate entropy inequalities and weak formulation

We derive approximate entropy inequalities verified by ρ_Δ and an approximate version of the weak formulation (7.3.2) satisfied by $\mathbf{w}_\Delta$. We start with ρ_Δ. With Φ_j^n defined in Corollary 7.6.2, we define the approximate entropy flux and the $\mathbf{w}_\Delta$-related contribution:

$$\Phi_\Delta(\rho_\Delta,\kappa)=\sum_{n=0}^{N}\sum_{j\in\mathbb{Z}}\Phi_j^n\mathbb{1}_{\mathcal{P}_{j+1/2}^{n+1}};\quad \partial_\Delta f(x,t,\kappa)=\left(\int_{\mathbb{R}}\mathbf{w}_\Delta(y,t)\eta'(x-y)\,\mathrm{d}y\right)\delta f(\kappa). \tag{7.6.11}$$

Theorem 7.6.5 (Approximate entropy inequalities). *Fix $\varphi\in\mathbf{C}_\mathbf{c}^\infty(\mathbb{R}\times\mathbb{R}^+),\varphi\ge 0$, $\kappa\in[0,1]$ and $n\in\{0,\dots,N\}$. Then as $\Delta\to 0$, we have:*

$$\begin{aligned}&\int_{t^n}^{t^{n+1}}\int_{\mathbb{R}}\Big(|\rho_\Delta-\kappa|\partial_t\varphi+\Phi_\Delta(\rho_\Delta,\kappa)\partial_x\varphi-\operatorname{sgn}(\rho_\Delta-\kappa)\partial_\Delta f(x,t,\kappa)\varphi\Big)\,\mathrm{d}x\,\mathrm{d}t\\&+\int_{\mathbb{R}}|\rho_\Delta(x,t^n)-\kappa|\varphi(x,t^n)\,\mathrm{d}x-\int_{\mathbb{R}}|\rho_\Delta(x,t^{n+1})-\kappa|\varphi(x,t^{n+1})\,\mathrm{d}x\\&\ge O(\Delta x\Delta t)+O\big(\Delta t^2\big).\end{aligned}\tag{7.6.12}$$

Proof. Fix $n\in\{0,\dots,N\}$, $j\in\mathbb{Z}$, $\varphi\in\mathbf{C}_\mathbf{c}^\infty(\mathbb{R}\times\mathbb{R}^+),\varphi\ge 0$, $\kappa\in[0,1]$ and set

$$\varphi_{j+1/2}^n=\frac{1}{\Delta x}\int_{x_j}^{x_{j+1}}\varphi(x,t^n)\,\mathrm{d}x\,.$$

Multiply the discrete entropy inequalities (7.6.7) by $\varphi_{j+1/2}^n$ and take the sum over $j\in\mathbb{Z}$. Proceeding to the Abel summation, we obtain:

$$\begin{aligned}&\underbrace{\sum_{j\in\mathbb{Z}}|\rho_{j+1/2}^{n+1}-\kappa|\varphi_{j+1/2}^{n+1}\Delta x-\sum_{j\in\mathbb{Z}}|\rho_{j+1/2}^{n}-\kappa|\varphi_{j+1/2}^{n}\Delta x}_{A}\\&\underbrace{-\sum_{j\in\mathbb{Z}}|\rho_{j+1/2}^{n+1}-\kappa|\left(\varphi_{j+1/2}^{n+1}-\varphi_{j+1/2}^{n}\right)\Delta x}_{B}\underbrace{-\sum_{j\in\mathbb{Z}}\Phi_{j+1/2}^n\left(\varphi_{j+1/2}^n-\varphi_{j-1/2}^n\right)\Delta t}_{C}\\&\le\underbrace{-\sum_{j\in\mathbb{Z}}\operatorname{sgn}(\rho_{j+1/2}^{n+1}-\kappa)(f_{j+1}^n(\kappa)-f_j^n(\kappa))\varphi_{j+1/2}^n\Delta x\Delta t}_{D}\,.\end{aligned}$$

Remark that

$$A-B=\int_{\mathbb{R}}|\rho_\Delta(x,t^{n+1})-\kappa|\varphi(x,t^{n+1})\,\mathrm{d}x-\int_{\mathbb{R}}|\rho_\Delta(x,t^n)-\kappa|\varphi(x,t^n)\,\mathrm{d}x-\int_{t^n}^{t^{n+1}}\int_{\mathbb{R}}|\rho_\Delta-\kappa|\partial_t\varphi\,\mathrm{d}x\,\mathrm{d}t\,.$$

We now compare the other members of the inequality to their continuous counterparts.
Estimating C**.** We write:

$$\begin{aligned}C&=\int_{t^n}^{t^{n+1}}\int_{\mathbb{R}}\Phi_\Delta(x,\rho_\Delta,\kappa)\partial_x\varphi(x,t^n)\,\mathrm{d}x\,\mathrm{d}t+\underbrace{\lambda\sum_{j\in\mathbb{Z}}\int_{x_j}^{x_{j+1}}\int_{x-\Delta x}^{x}\int_x^y\Phi^n_{j+1/2}\partial_x\varphi(z,t^n)\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x}_{C_1}\\&=\int_{t^n}^{t^{n+1}}\int_{\mathbb{R}}\Phi_\Delta(x,\rho_\Delta,\kappa)\partial_x\varphi(x,t)\,\mathrm{d}x\,\mathrm{d}t+C_1+\underbrace{\int_{t^n}^{t^{n+1}}\int_{\mathbb{R}}\int_t^{t^n}\Phi_\Delta(x,\rho_\Delta,\kappa)\partial^2_{tx}\varphi(x,\tau)\,\mathrm{d}\tau\,\mathrm{d}x\,\mathrm{d}t}_{C_2},\end{aligned}$$

and we have the estimations:

$$|C_1|\le 4\mathbf{L}\sup_{t\ge0}\|\partial^2_{xx}\varphi(\cdot,t)\|_{\mathbf{L}^1}\Delta x\Delta t;\quad |C_2|\le\mathbf{L}\sup_{t\ge0}\|\partial^2_{tx}\varphi(\cdot,t)\|_{\mathbf{L}^1}\Delta t^2.$$

Estimating D**.** With the notation (7.6.11), we have

$$\begin{aligned}D&=\sum_{j\in\mathbb{Z}}\Delta t\int_{x_j}^{x_{j+1}}\mathrm{sgn}(\rho^{n+1}_{j+1/2}-\kappa)\partial_\Delta f(x,t^n,\kappa)\varphi(x,t^n)\,\mathrm{d}x\\&\quad+\underbrace{\sum_{j\in\mathbb{Z}}\lambda\int_{x_j}^{x_{j+1}}\int_{\mathbb{R}}\int_{x_j}^{x_{j+1}}\int_x^z\mathrm{sgn}(\rho^{n+1}_{j+1/2}-\kappa)\mathbf{w}_\Delta(y,t^n)\eta''(u-y)\delta f(\kappa)\varphi(x,t^n)\,\mathrm{d}u\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x}_{D_1}\\&=\int_{t^n}^{t^{n+1}}\int_{\mathbb{R}}\mathrm{sgn}(\rho_\Delta-\kappa)\partial_\Delta f(x,t,\kappa)\varphi(x,t)\,\mathrm{d}x+D_1\\&\quad+\underbrace{\int_{t^n}^{t^{n+1}}\int_{\mathbb{R}}\mathrm{sgn}(\rho_\Delta-\kappa)\partial_\Delta f(x,t^n,\kappa)(\varphi(x,t^n)-\varphi(x,t))\,\mathrm{d}x}_{D_2},\end{aligned}$$

which we combine with the bounds:

$$\begin{aligned}|D_1|&\le\|\eta''\|_{\mathbf{L}^1}\|\delta f\|_{\mathbf{L}^\infty}\sup_{t\ge0}\|\varphi(\cdot,t)\|_{\mathbf{L}^1}\Delta x\Delta t\\|D_2|&\le\|\eta'\|_{\mathbf{L}^1}\|\delta f\|_{\mathbf{L}^\infty}\sup_{t\ge0}\|\partial_t\varphi(\cdot,t)\|_{\mathbf{L}^1}\Delta t^2.\end{aligned}$$

□

We now turn to $\mathbf{w}_\Delta$. Let us define the approximate flux function:

$$f_\Delta(x,t,\rho)=(1-\omega_\Delta(x,t))f_{\min}(\rho)+\omega_\Delta(x,t)f_{\max}(\rho).$$

Theorem 7.6.6 (Approximate weak formulation). *Fix $\phi \in \mathbf{C}_{\mathbf{c}}^{\infty}(\mathbb{R}\times\mathbb{R}^+)$ and $n \in \{0,\dots,N\}$. Then as $\Delta \to 0$, we have:*

$$\begin{aligned}&\int_{t^n}^{t^{n+1}} \int_{\mathbb{R}} \Big((\rho_\Delta \mathbf{w}_\Delta)\partial_t\phi + (f_\Delta(x,t,\rho_\Delta)\mathbf{w}_\Delta)\partial_x\phi - \rho_\Delta \mathbf{S}_\Delta \phi \Big)\,\mathrm{d}x\,\mathrm{d}t \\ &+ \int_{\mathbb{R}} (\rho_\Delta \mathbf{w}_\Delta)(x,t^n)\phi(x,t^n)\,\mathrm{d}x - \int_{\mathbb{R}} (\rho_\Delta \mathbf{w}_\Delta)(x,t^{n+1})\phi(x,t^{n+1})\,\mathrm{d}x \\ &= O(\Delta x \Delta t) + O\big(\Delta t^2\big)\,.\end{aligned} \tag{7.6.13}$$

Proof. This proof follows the same steps as the one of Theorem 7.6.5.
Fix $n \in \{0,\dots,N\}$ and $j \in \mathbb{Z}$. Let us multiply (7.6.3) by $\rho_{j+1/2}^{n+1}$ and combine the result with (7.6.2). More precisely, we write:

$$\begin{aligned}&\Big((\rho\mathbf{w})_{j+1/2}^{n+1} - (\rho\mathbf{w})_{j+1/2}^{n} \Big)\,\Delta x \\ &= \rho_{j+1/2}^{n+1} \Big(\mathbf{w}_{j+1/2}^{n+1} - \mathbf{w}_{j+1/2}^{n} \Big)\,\Delta x + \Big(\rho_{j+1/2}^{n+1} - \rho_{j+1/2}^{n} \Big)\,\mathbf{w}_{j+1/2}^{n}\Delta x \\ &= \mathcal{F}_j^n \left(\mathbf{w}_{j-1/2}^{n} - \mathbf{w}_{j+1/2}^{n} \right)\Delta t + \rho_{j+1/2}^{n+1}\mathbf{S}_{j+1/2}^{n+1}\Delta x\Delta t + \mathcal{F}_j^n \times (\mathbf{S}_{j-1/2}^{n+1} - \mathbf{S}_{j+1/2}^{n+1})\Delta t^2 \\ &- \left(\mathcal{F}_{j+1}^n - \mathcal{F}_j^n \right)\mathbf{w}_{j+1/2}^{n}\Delta t \\ &= -\left(\mathcal{F}_{j+1}^n \mathbf{w}_{j+1/2}^{n} - \mathcal{F}_j^n \mathbf{w}_{j-1/2}^{n} \right)\Delta t + \rho_{j+1/2}^{n+1}\mathbf{S}_{j+1/2}^{n+1}\Delta x\Delta t + \mathcal{F}_j^n \times (\mathbf{S}_{j-1/2}^{n+1} - \mathbf{S}_{j+1/2}^{n+1})\Delta t^2.\end{aligned}$$

These computations are the analogous of the ones we did in Section 7.6.1. This last equality expresses the consistency of our scheme.
Fix now $\phi \in \mathbf{C}_{\mathbf{c}}^{\infty}(\mathbb{R}\times\mathbb{R}^+)$ and set

$$\phi_{j+1/2}^n = \frac{1}{\Delta x}\int_{x_j}^{x_{j+1}} \phi(x,t^n)\,\mathrm{d}x\,.$$

Multiply the previous equality by $\phi_{j+1/2}^{n+1}$ and take the sum over $j \in \mathbb{Z}$. Proceeding to the Abel summation, we obtain:

$$\begin{aligned}&\underbrace{\sum_{j\in\mathbb{Z}} (\rho\mathbf{w})_{j+1/2}^{n+1}\phi_{j+1/2}^{n+1}\Delta x - \sum_{j\in\mathbb{Z}} (\rho\mathbf{w})_{j+1/2}^{n}\phi_{j+1/2}^{n}\Delta x}_{A} - \underbrace{\sum_{j\in\mathbb{Z}} (\rho\mathbf{w})_{j+1/2}^{n} \left(\phi_{j+1/2}^{n+1} - \phi_{j+1/2}^{n} \right)\Delta x}_{B} \\ &- \underbrace{\sum_{j\in\mathbb{Z}} \mathcal{F}_{j+1}^n \mathbf{w}_{j+1/2}^{n} \left(\phi_{j+3/2}^{n+1} - \phi_{j+1/2}^{n+1} \right)\Delta t}_{C} - \underbrace{\sum_{j\in\mathbb{Z}} \rho_{j+1/2}^{n+1}\mathbf{S}_{j+1/2}^{n+1}\phi_{j+1/2}^{n+1}\Delta x\Delta t}_{D} \\ &- \underbrace{\sum_{j\in\mathbb{Z}} \mathcal{F}_j^n \times (\mathbf{S}_{j-1/2}^{n+1} - \mathbf{S}_{j+1/2}^{n+1})\phi_{j+1/2}^{n+1}\Delta t^2}_{E} = 0.\end{aligned}$$

The remaining part of the proof consists in estimating each member of this last equality, having in mind the previously established estimates such as (7.6.8). Like in the previous proof, we immediately see that:

$$A = \int_{\mathbb{R}} (\rho_\Delta \mathbf{w}_\Delta)(x,t^{n+1})\phi(x,t^{n+1})\,\mathrm{d}x - \int_{\mathbb{R}} (\rho_\Delta \mathbf{w}_\Delta)(x,t^n)\phi(x,t^n)\,\mathrm{d}x\,.$$

Estimating C:

$$
\begin{aligned}
C &= \lambda \sum_{j\in\mathbb{Z}} \int_{x_j}^{x_{j+1}} \int_{x}^{x+\Delta x} \mathcal{F}_{j+1}^n \mathbf{w}_{j+1/2}^n \partial_x \phi(y, t^{n+1}) \,\mathrm{d}y \,\mathrm{d}x \\
&= \lambda \sum_{j\in\mathbb{Z}} \int_{x_j}^{x_{j+1}} \int_{x}^{x+\Delta x} f_{j+1}^n(\rho_{j+1/2}^n) \mathbf{w}_{j+1/2}^n \partial_x \phi(y, t^{n+1}) \,\mathrm{d}y \,\mathrm{d}x \\
&\quad + \underbrace{\lambda \sum_{j\in\mathbb{Z}} \int_{x_j}^{x_{j+1}} \int_{x}^{x+\Delta x} (\mathcal{F}_{j+1}^n - f_{j+1}^n(\rho_{j+1/2}^n)) \mathbf{w}_{j+1/2}^n \partial_x \phi(y, t^{n+1}) \,\mathrm{d}y \,\mathrm{d}x}_{C_1} \\
&= \lambda \sum_{j\in\mathbb{Z}} \int_{x_j}^{x_{j+1}} \int_{x}^{x+\Delta x} f_{j}^n(\rho_{j+1/2}^n) \mathbf{w}_{j+1/2}^n \partial_x \phi(y, t^{n+1}) \,\mathrm{d}y \,\mathrm{d}x + C_1 \\
&\quad + \underbrace{\lambda \sum_{j\in\mathbb{Z}} \int_{x_j}^{x_{j+1}} \int_{x}^{x+\Delta x} (f_{j+1}^n(\rho_{j+1/2}^n) - f_{j}^n(\rho_{j+1/2}^n)) \mathbf{w}_{j+1/2}^n \partial_x \phi(y, t^{n+1}) \,\mathrm{d}y \,\mathrm{d}x}_{C_2} \\
&= \lambda \sum_{j\in\mathbb{Z}} \int_{x_j}^{x_{j+1}} \int_{x}^{x+\Delta x} f_{j}^n(\rho_{j+1/2}^{n+1}) \mathbf{w}_{j+1/2}^n \partial_x \phi(y, t^{n+1}) \,\mathrm{d}y \,\mathrm{d}x + C_1 + C_2 \\
&\quad + \underbrace{\lambda \sum_{j\in\mathbb{Z}} \int_{x_j}^{x_{j+1}} \int_{x}^{x+\Delta x} (f_{j}^n(\rho_{j+1/2}^n) - f_{j}^n(\rho_{j+1/2}^{n+1})) \mathbf{w}_{j+1/2}^n \partial_x \phi(y, t^{n+1}) \,\mathrm{d}y \,\mathrm{d}x}_{C_3} \\
&= \int_{t^n}^{t^{n+1}} \int_{\mathbb{R}} (f_\Delta(x, t, \rho_\Delta) \mathbf{w}_\Delta) \partial_x \phi(x, t^{n+1}) \,\mathrm{d}x \,\mathrm{d}t + C_1 + C_2 + C_3 \\
&\quad + \underbrace{\lambda \sum_{j\in\mathbb{Z}} \int_{x_j}^{x_{j+1}} \int_{x}^{x+\Delta x} f_{j}^n(\rho_{j+1/2}^{n+1}) \mathbf{w}_{j+1/2}^n (\partial_x \phi(y, t^{n+1}) - \partial_x \phi(x, t^{n+1})) \,\mathrm{d}y \,\mathrm{d}x}_{C_4} \\
&= \int_{t^n}^{t^{n+1}} \int_{\mathbb{R}} (f_\Delta(x, t, \rho_\Delta) \mathbf{w}_\Delta) \partial_x \phi(x, t) \,\mathrm{d}x \,\mathrm{d}t + C_1 + C_2 + C_3 + C_4 \\
&\quad + \underbrace{\int_{t^n}^{t^{n+1}} \int_{\mathbb{R}} (f_\Delta(x, t, \rho_\Delta) \mathbf{w}_\Delta) (\partial_x \phi(x, t^{n+1}) - \partial_x \phi(x, t)) \,\mathrm{d}x \,\mathrm{d}t}_{C_5},
\end{aligned}
$$

and we have the estimations:

$$
|C_1| \le 2\mathbf{L} \|\partial_x \phi\|_{\mathbf{L}^\infty} \mathbf{TV}(\rho_\Delta(\cdot, t^n)) \Delta x \Delta t; \quad |C_2| \le 2 \|\delta f\|_{\mathbf{L}^\infty} \mathbf{TV}(\eta) \sup_{t\ge 0} \|\partial_x \phi(\cdot, t)\|_{\mathbf{L}^1} \Delta x \Delta t;
$$

$$
|C_3| \le \mathbf{L} \left(\sum_{j\in\mathbb{Z}} |\rho_{j+1/2}^{n+1} - \rho_{j+1/2}^n| \Delta x \right) \|\partial_x \phi\|_{\mathbf{L}^\infty} \Delta t = O\big(\Delta t^2\big) \quad \text{due to Corollary 7.6.4;}
$$

$$
|C_4| \le 4\mathbf{L} \sup_{t\ge 0} \|\partial_{xx}^2 \phi(\cdot, t)\|_{\mathbf{L}^1} \Delta x \Delta t; \quad |C_5| \le \mathbf{L} \sup_{t\ge 0} \|\partial_{tx}^2 \phi(\cdot, t)\|_{\mathbf{L}^1} \Delta t^2.
$$

Moreover,

$$B = \int_{t^n}^{t^{n+1}} \int_{\mathbb{R}} (\rho_\Delta \mathbf{w}_\Delta) \partial_t \phi \, \mathrm{d}x \, \mathrm{d}t + \underbrace{\sum_{j\in\mathbb{Z}} (\rho^n_{j+1/2} - \rho^{n+1}_{j+1/2}) \mathbf{w}^n_{j+1/2} \left(\phi^{n+1}_{j+1/2} - \phi^n_{j+1/2} \right) \Delta x}_{B_1}$$

and, using Theorem 7.6.3 and Corollary 7.6.4, we have

$$|B_1| \leq \left(\sum_{j\in\mathbb{Z}} |\rho^{n+1}_{j+1/2} - \rho^n_{j+1/2}| \Delta x \right) \|\partial_t \phi\|_{\mathbf{L}^\infty} \Delta t = O\big(\Delta t^2\big) .$$

Estimating D**.** We write

$$\begin{aligned} D &= \int_{t^n}^{t^{n+1}} \int_{\mathbb{R}} \rho_\Delta(x,t) \mathbf{S}_\Delta(x,t) \phi(x, t^{n+1}) \, \mathrm{d}x \\ &= \int_{t^n}^{t^{n+1}} \int_{\mathbb{R}} \rho_\Delta(x,t) \mathbf{S}_\Delta(x,t) \phi(x,t) \, \mathrm{d}x \, \mathrm{d}t + \underbrace{\int_{t^n}^{t^{n+1}} \int_{\mathbb{R}} \rho_\Delta(x,t) \mathbf{S}_\Delta(x,t) (\phi(x, t^{n+1} - \phi(x,t)) \, \mathrm{d}x \, \mathrm{d}t}_{D_1}, \end{aligned}$$

and we have the bound:

$$|D_1| \leq \|\mathbf{S}_\Delta\|_{\mathbf{L}^\infty} \sup_{t\geq 0} \|\partial_t \phi(\cdot, t)\|_{\mathbf{L}^1} \Delta t^2.$$

To **estimate** E, we directly write:

$$|E| \leq c\mathbf{L} \|\phi\|_{\mathbf{L}^\infty} \left(\mathbf{TV}(\rho_\Delta(\cdot, t^n)) + \mathbf{TV}(\mathbf{w}_\Delta(\cdot, t^n)) \right) \Delta t^2,$$

concluding the proof. □

7.6.6 Convergence and existence statement

Before proving the convergence result, remark that the strong convergence of $(\rho_\Delta)_\Delta$ and $(\mathbf{w}_\Delta)_\Delta$ implies the strong convergence of $(\xi_\Delta)_\Delta$, $(\chi_\Delta)_\Delta$, $(\omega_\Delta)_\Delta$, $(f_\Delta(\cdot,\cdot,\rho_\Delta))_\Delta$ and $(\mathbf{S}_\Delta)_\Delta$. More precisely, fix $(x,t) \in \Omega$. Given Δ, let $n \in \{0, \dots, N\}, j \in \mathbb{Z}$ be such that $(x,t) \in \mathcal{P}^{n+1}_{j+1/2}$. We have:

$$\xi_\Delta(x,t) = \xi^{n+1}_{j+1/2} = \int_{\mathbb{R}} \rho_\Delta(y,t) \mu(x_{j+1/2} - y) \, \mathrm{d}y \underset{\Delta\to 0}{\longrightarrow} \int_{\mathbb{R}} \rho(y,t) \mu(x-y) \, \mathrm{d}y := \xi(x,t).$$

Moreover,

$$\omega_\Delta(x,t) = \omega^n_j = \int_{\mathbb{R}} \mathbf{w}_\Delta(y,t) \eta(x_j - y) \, \mathrm{d}y \underset{\Delta\to 0}{\longrightarrow} \int_{\mathbb{R}} \mathbf{w}(y,t) \eta(x-y) \, \mathrm{d}y := \omega(x,t).$$

Consequently,

$$f_\Delta(x,t,\rho_\Delta(x,t)) \underset{\Delta\to 0}{\longrightarrow} (1 - \omega(x,t)) f_{\min}(\rho(x,t)) + \omega(x,t) f_{\max}(\rho(x,t)) := f(x,t,\rho(x,t)),$$

from which we deduce:

$$\begin{aligned}\chi_\Delta(x,t) = \chi_{j+1/2}^{n+1} &= -\int_{\mathbb{R}} f_\Delta(y,t,\rho_\Delta)\mu'(x_{j+1/2}-y)\,\mathrm{d}y \\ &\quad \underbrace{-\sum_{i\in\mathbb{Z}}(\mathbf{F}_i^n(\rho_{i-1/2}^{n+1},\rho_{i+1/2}^{n+1}) - f_i^n(\rho_{i+1/2}^{n+1}))\int_{x_i}^{x_{i+1}}\mu'(x_{j+1/2}-y)\,\mathrm{d}y}_{=O(\Delta x)} \\ &\underset{\Delta\to 0}{\longrightarrow} -\int_{\mathbb{R}} f(y,t,\rho)\mu'(x-y)\,\mathrm{d}y := \chi(x,t).\end{aligned}$$

Also, by continuity of $\mathbf{K}$,

$$\begin{aligned}\mathbf{S}_\Delta(x,t) &= \mathbf{K}(\xi_\Delta(x,t),\chi_\Delta(x,t))\mathbf{w}_\Delta(x,t)(1-\mathbf{w}_\Delta(x,t)) \\ &\underset{\Delta\to 0}{\longrightarrow} \mathbf{K}(\xi(x,t),\chi(x,t))\mathbf{w}(x,t)(1-\mathbf{w}(x,t)) := \mathbf{S}(x,t).\end{aligned}$$

We now turn to the

Proof of Theorem 7.3.3. We verify that $(\rho,\mathbf{w})$ satisfies all the points of Definition 7.3.1. *(i)* Fix $\varphi\in\mathbf{C}_{\mathbf{c}}^\infty(\mathbb{R}\times[0,T))$, $\varphi\geq 0$, $\kappa\in[0,1]$ and $\tau,s\in[0,T]$ $(\tau<s)$. Being given $\Delta>0$, let $n,m\in\{0,\dots,N+1\}$ such that $\tau\in[t^n,t^{n+1})$ and $s\in[t^m,t^{m+1})$. By summing (7.6.12) over $k\in\{n,\dots,m-1\}$, we obtain:

$$\begin{aligned}&\int_s^\tau\int_{\mathbb{R}}\Big(|\rho_\Delta-\kappa|\partial_t\varphi+\Phi_\Delta(\rho_\Delta,\kappa)\partial_x\varphi-\mathrm{sgn}(\rho_\Delta-\kappa)\partial_\Delta f(x,t,\kappa)\varphi\Big)\,\mathrm{d}x\,\mathrm{d}t \\ &=-\int_{t^n}^s\int_{\mathbb{R}}\Big(|\rho_\Delta-\kappa|\partial_t\varphi+\Phi_\Delta(\rho_\Delta,\kappa)\partial_x\varphi-\mathrm{sgn}(\rho_\Delta-\kappa)\partial_\Delta f(x,t,\kappa)\varphi\Big)\,\mathrm{d}x\,\mathrm{d}t \\ &+\sum_{k=n}^{m-1}\int_{t^k}^{t^{k+1}}\int_{\mathbb{R}}\Big(|\rho_\Delta-\kappa|\partial_t\varphi+\Phi_\Delta(\rho_\Delta,\kappa)\partial_x\varphi-\mathrm{sgn}(\rho_\Delta-\kappa)\partial_\Delta f(x,t,\kappa)\varphi\Big)\,\mathrm{d}x\,\mathrm{d}t \\ &+\int_{t^m}^\tau\int_{\mathbb{R}}\Big(|\rho_\Delta-\kappa|\partial_t\varphi+\Phi_\Delta(\rho_\Delta,\kappa)\partial_x\varphi-\mathrm{sgn}(\rho_\Delta-\kappa)\partial_\Delta f(x,t,\kappa)\varphi\Big)\,\mathrm{d}x\,\mathrm{d}t\,.\end{aligned} \tag{7.6.14}$$

Using the uniform $\mathbf{L}^\infty$ bounds, we see that the first and last term of the right-hand side of

this equality can be written as $O(\Delta t)$. By (7.6.12),

$$\sum_{k=n}^{m-1}\int_{t^k}^{t^{k+1}}\int_{\mathbb{R}}\Big(|\rho_\Delta-\kappa|\partial_t\varphi+\Phi_\Delta(\rho_\Delta,\kappa)\partial_x\varphi-\operatorname{sgn}(\rho_\Delta-\kappa)\partial_\Delta f(x,t,\kappa)\varphi\Big)\,\mathrm{d}x\,\mathrm{d}t$$

$$\geq\int_{\mathbb{R}}|\rho_\Delta(x,t^m)-\kappa|\varphi(x,t^m)\,\mathrm{d}x-\int_{\mathbb{R}}|\rho_\Delta(x,t^n)-\kappa|\varphi(x,t^n)\,\mathrm{d}x+\sum_{k=n}^{m-1}\big(O(\Delta x\Delta t)+O\big(\Delta t^2\big)\big)$$

$$\geq\int_{\mathbb{R}}|\rho_\Delta(x,s)-\kappa|\varphi(x,s)\,\mathrm{d}x-\int_{\mathbb{R}}|\rho_\Delta(x,\tau)-\kappa|\varphi(x,\tau)\,\mathrm{d}x-T(\Delta x+\Delta t)$$

$$+\int_{\mathbb{R}}\left(|\rho_\Delta(x,t^m)-\kappa|\varphi(x,t^m)-|\rho_\Delta(x,s)-\kappa|\varphi(x,s)\right)\mathrm{d}x$$

$$-\int_{\mathbb{R}}\left(|\rho_\Delta(x,t^n)-\kappa|\varphi(x,t^n)-|\rho_\Delta(x,\tau)-\kappa|\varphi(x,\tau)\right)\mathrm{d}x\,.$$

Using the time **BV** estimate (7.6.10), we deduce that the last two members of this inequality can be written as $O(\Delta t)$ as well. Putting everything together, when letting $\Delta\to 0$ in (7.6.14), we obtain that ρ is an entropy solution to

$$\partial_t\rho+\partial_x\left(f(x,t,\rho)\right)=0.$$

(ii) From (7.6.13), and using the same ideas as in the previous reasoning, with in this case the second time **BV** estimate of Corollary 7.6.10, we easily obtain that for all $\phi\in\mathbf{C}_{\mathbf{c}}^{\infty}(\mathbb{R}\times[0,T))$ and $\tau,s\in[0,T]$ $(\tau<s)$, we have:

$$\int_s^\tau\int_{\mathbb{R}}\Big((\rho_\Delta\mathbf{w}_\Delta)\partial_t\phi+(f_\Delta(x,t,\rho_\Delta)\mathbf{w}_\Delta)\partial_x\phi-\rho_\Delta\mathbf{S}_\Delta\phi\Big)\,\mathrm{d}x\,\mathrm{d}t$$

$$+\int_{\mathbb{R}}\rho_\Delta(x,s)\mathbf{w}_\Delta(x,s)\phi(x,s)\,\mathrm{d}x-\int_{\mathbb{R}}\rho_\Delta(x,\tau)\mathbf{w}_\Delta(x,\tau)\phi(x,\tau)\,\mathrm{d}x=O(\Delta x)+O(\Delta t)\,,$$

which by taking the limit as $\Delta\to 0$ implies that $\mathbf{w}$ is a weak solution of

$$\partial_t\left(\rho\mathbf{w}\right)+\partial_x\left(f(x,t,\rho)\mathbf{w}\right)=\rho\mathbf{K}\left(\xi,\chi\right)\mathbf{w}(1-\mathbf{w}).$$

Finally, since ρ is a weak solution to $\partial_t\rho+\partial_x\left(f(x,t,\rho)\right)=0$, we deduce that ξ is differentiable with respect to t, with derivative χ.
(iii) We proved that $\mathbf{v}$ and ω are linked and that ω and $\mathbf{w}$ are linked by (7.2.6) at the beginning of the section. The proof is completed. □

Proof of Theorem 7.3.4. The existence claim readily follows from Theorem 7.3.3. □

7.7 Numerical simulation

In this section, we present a numerical test performed with the scheme analyzed in Section 7.6. For $f_{\min}$, we take the uniformly concave flux $f_{\min}(\rho)=\rho(1-\rho)$, and for $f_{\max}$, we take

$$f_{\max}(\rho)=\begin{cases}f_{\min}(\rho) & \text{if } 0\leq\rho\leq\rho_c\\ P(\rho) & \text{if } \rho_c<\rho\leq 1,\end{cases}$$

where ρ_c is some critical threshold and P is polynomial of degree 3 satisfying:

$$P \geq 0 \text{ on } [\rho_c, 1]; \quad P(\rho_c) = f_{\min}(\rho_c); \quad P'(\rho_c) = f'_{\min}(\rho_c); \quad P(1) = 0,$$

as depicted in Figure 7.3, left. For the sake of simplicity, we choose $\eta = \mu$, both equal to a suitable regularization of the triangle-shaped function $x \mapsto 2(1-2|x|)\mathbb{1}_{\{|x|\leq\frac{1}{2}\}}$. We deal with a road parametrized by the interval $[-2, 5]$ and time horizon $T = 6.0$. We choose initial data satisfying the hypotheses of Theorem 7.3.3:

$$\rho_0(x) = \begin{cases} 0.4 & \text{if} \quad -1 < x < 0 \\ 0.8 & \text{if} \quad 1 < x < 2 \\ 0.10 & \text{otherwise;} \end{cases} \qquad \mathbf{w}_0(x) = \begin{cases} 0.5 & \text{if} \quad |x| \leq 10 \\ 0 & \text{otherwise,} \end{cases}$$

as represented in Figure 7.3, right.

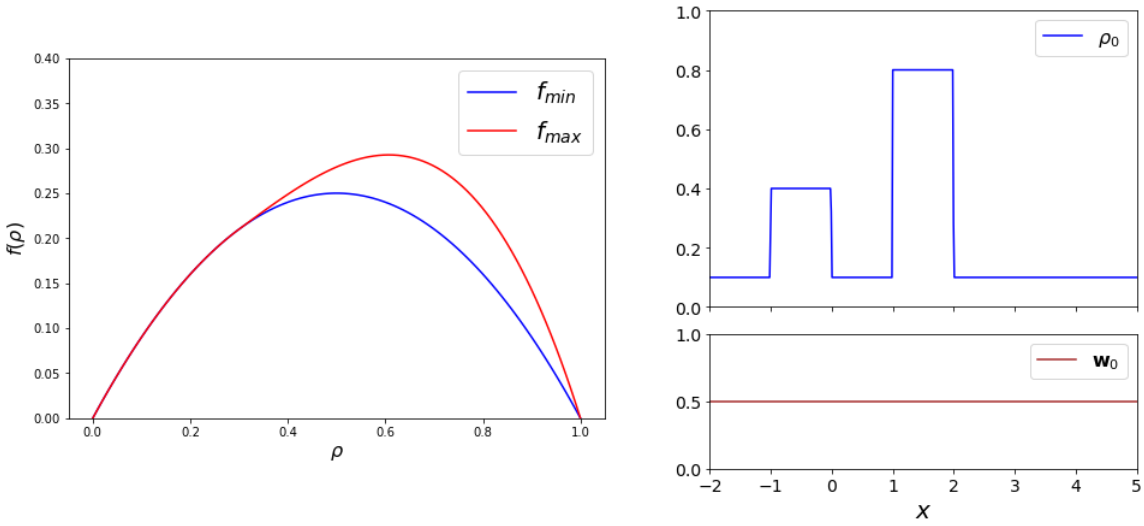

Figure 7.3 – Typical choice of fundamental diagrams and initial data.

Let us comment on the profile of the numerical solutions represented in Figure 7.4. Quite expectedly, as we can see from Figure 7.4 at time T, the introduction of the orderliness marker has favored the global velocity of the density. Now let us look more precisely at the different profiles of the numerical solution. We see that at times $t = 1.64$ and $t = 3.01$, the highest peaks of density correspond to the areas where the orderliness is the lowest. In the meantime, notice how this peak of the density is followed by an increase of the orderliness value, suggesting the emergence of an organizing pattern upstream the bottleneck. Finally, as incorporated in the model, everywhere the density is lesser than the threshold ρ_c, the value of $\mathbf{w}$ does not vary.

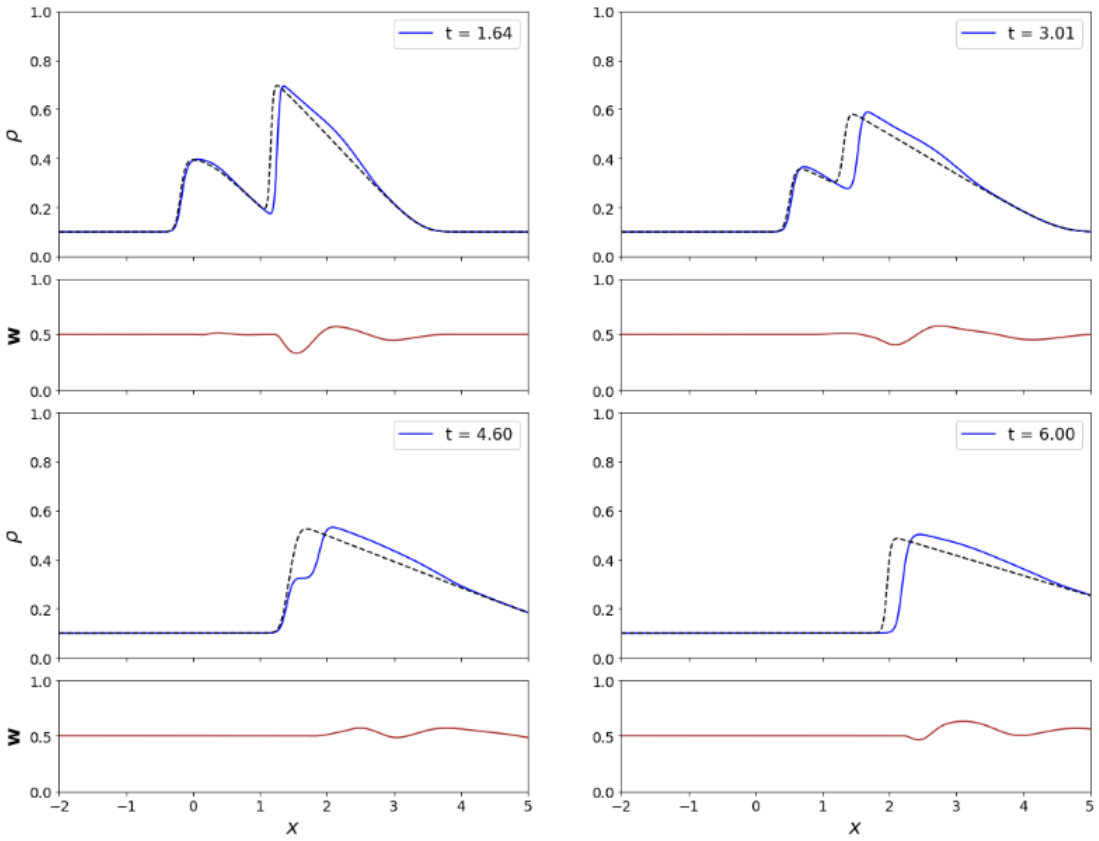

Figure 7.4 – The numerically computed solutions $\rho_\Delta(\cdot,t), \mathbf{w}_\Delta(\cdot,t)$ at different fixed times t; dashed lines correspond to the reference solution in absence of orderliness marker, *i.e.* for $\omega \equiv 0$ in (7.2.2); for an animated evolution of the numerical solution, follow: https://utbox.univ-tours.fr/s/AgDkkaQm9247ZEP.

Part III

Compactly non-homogeneous conservation laws and Hamilton-Jacobi equations

CHAPTER 8

CORRESPONDENCES AND THE INVERSE DESIGN PROBLEM

Fix a strictly positive time horizon $T > 0$ and set $\Omega = \mathbb{R} \times (0, T)$. For a flux $H : \mathbb{R}^2 \to \mathbb{R}$ [1] and an initial data $u_0 \in \mathbf{L}^\infty(\mathbb{R})$, consider the scalar conservation law in Ω:

$$\begin{cases} \partial_t u + \partial_x \left(H(x,u) \right) = 0 \\ \qquad\qquad u(\cdot, 0) = u_0. \end{cases} \tag{8.0.1}$$

It is known that solutions to these equations generally develop shocks which cause a loss of information. This expresses the irreversible nature of the phenomenon they describe. In [61], the authors considered equation (8.0.1) in the case when the flux has no space dependency *i.e.* $H(x,u) = H(u)$. They provided a full characterization of the set of initial data that evolve into a given profile. They also described some geometric and topological properties of this set. One of the crucial tools used by the authors of [61] is the connection between the solutions to the conservation law (8.0.1) and the solutions of the Hamilton-Jacobi equation

$$\begin{cases} \partial_t U + H \left(x, \partial_x U \right) = 0 \\ \qquad\qquad U(\cdot, 0) = U_0. \end{cases} \tag{8.0.2}$$

In the case where the flux H does not depend on the space variable, solutions of (8.0.1) are obtained by differentiating solutions of (8.0.2). The authors of [61] also clarified how to obtain solutions of (8.0.2) from solutions of (8.0.1), see [61, Proposition 2.5].

On an attempt to extend their studies to space-dependent flows, we first aim in this chapter to prove well-posedness of both problem (8.0.1) and problem (8.0.2), and the correspondence to their solutions for a class of $\mathbf{C}^2(\mathbb{R}^2)$ flux functions H which space dependency is localized in a compact subset of $\mathbb{R}$:

$$\exists X > 0,\ \forall x, u \in \mathbb{R}, \quad |x| \geq X \implies \frac{\partial H}{\partial x}(x,u) = 0. \tag{8.0.3}$$

1. We change the notation for the flux from f to H because we will extensively use the methods and viewpoints from the theory of the associated Hamilton-Jacobi equations, where the notation H is traditional.

In the literature, well-posedness for the conservation law (8.0.1) is usually obtained in the class of bounded functions with a flow H globally Lipschitz, see for instance [114, 88, 68, 140]. Regarding the Hamilton-Jacobi equation (8.0.2), the initial data is usually taken in the space $\mathbf{W}^{1,\infty}(\mathbb{R})$ and the flow H, convex with respect to the second variable, is sub-linear as $|u| \to +\infty$, see [81, Chapter 10] and [126, 79, 63, 28]. Note that this last assumption does not seem to have a counterpart in the usual assumptions on the flow of a conservation law.

To our knowledge, using assumption (8.0.3) is new. This assumption is worth considering for a few reasons:

— From the application point of view *i.e.* from the PDE point of view (Section 8.2), flows H which satisfy (8.0.3) naturally arise in the context of road traffic dynamics. Some simple examples would be flux functions:

$$H(x,u) = f(u) + g(x); \quad H(x,u) = \theta(x)f(u); \quad H(x,u) = f(u - \theta(x)),$$

for suitable choice of functions f, g and θ. Remark that assumption (8.0.3) allows for any growth of H in u. In particular, *strongly convex* (in u) Hamiltonians can be considered.

— From the calculus of variations viewpoint (Section 8.3), being able to consider coercive (in u) Hamiltonian will be essential to prove the existence of minimal arcs.

— From the ODE point of view (Section 8.1.2), assumption (8.0.3) will lead to global existence in time for the Hamiltonian rays and outside the compact $[-X, X]$ of space dependency, the Hamiltonian rays will be straight lines.

This chapter is organized as follows. We start in Section 8.1 by setting the framework and proving useful estimates and properties of the Hamiltonian H (and its Legendre transform) as well as ODE results and generalized characteristics. Then, Section 8.2 is devoted to the proof of correspondence between entropy solutions and viscosity solutions under the same assumptions on H. As a byproduct, we also obtain well-posedness results for (8.0.1) and (8.0.2). The result is extended for the Hamilton-Jacobi equation in Section 8.3 through the calculus of variations approach. More than the existence result, this section also provides the correspondence viscosity solutions/calculus of variations. We conclude in Section 8.4 by tackling the initial data identification problem for our class of space-dependent flows.

8.1 Assumptions and preliminary results

Recall that we suppose that $H \in \mathbf{C}^2(\mathbb{R}^2)$ satisfies (8.0.3). In particular, a feature often used in the sequel will be:

$$\forall R > 0, \quad \sup_{\substack{x\in\mathbb{R} \\ |u|\leq R}} H(x,u) = \sup_{\substack{|x|\leq X \\ |u|\leq R}} H(x,u). \tag{8.1.1}$$

We make the additional assumption that for all $x \in \mathbb{R}$, $H(x,\cdot)$ is *strongly convex*.

Definition 8.1.1. Let $f \in \mathbf{C}^2(\mathbb{R},\mathbb{R})$. We say that f is strongly convex if f' is an increasing diffeomorphism on $\mathbb{R}$.

The strong convexity of H translates as:

$$\forall x \in \mathbb{R}, \quad \frac{\partial^2 H}{\partial u^2}(x,\cdot) > 0 \quad \text{and} \quad \frac{\partial H}{\partial u}(x,\cdot) : \mathbb{R} \to \mathbb{R} \text{ is a bijection.} \tag{8.1.2}$$

Example 8.1.1 (Traffic flow I)**.** The Hamiltonian

$$H(x,u) = \theta(x)u(u-1),$$

where $\theta \in \mathbf{C}^2(\mathbb{R})$ has the form depicted in Figure 8.1, left, is in the class of flux functions we consider. Note that the conservation law (8.0.1) with flux $-H$ is a generalization of the LWR model, see Example 0.1.2. This heterogeneous extension describes the density of a flow of vehicles along a one-dimensional road with maximal speed smoothly varying from $\theta(-1)$ for $x \leq -1$, to $\theta(1)$ for $x \geq 1$.

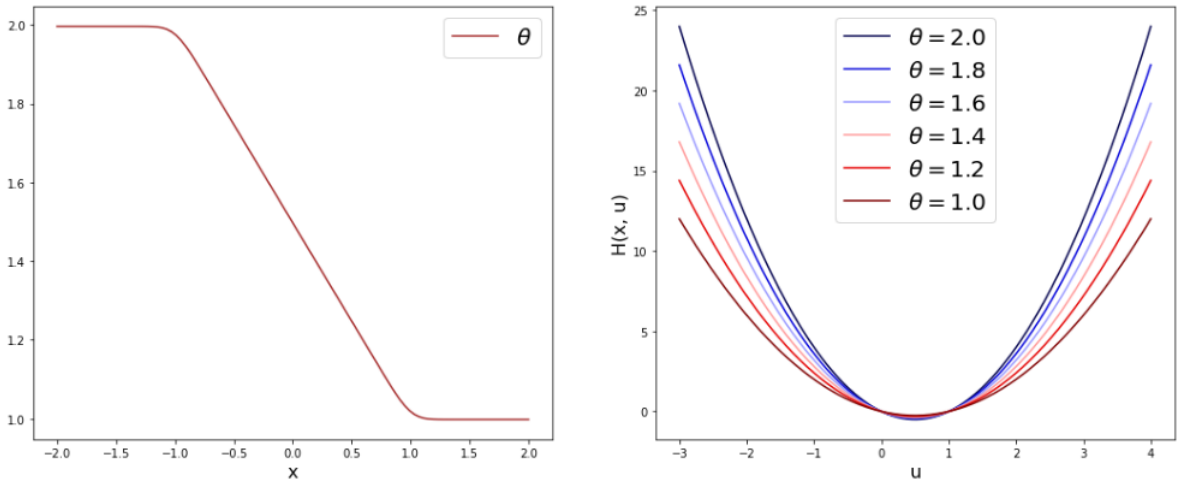

Figure 8.1 – Example of a strongly convex Hamiltonian.

8.1.1 Properties of the Legendre transform

Let us recall some properties of strongly convex functions we will use in the sequel.

Lemma 8.1.2 (On strongly convex functions)**.** *Let $f \in \mathbf{C}^2(\mathbb{R})$ be a strongly convex function. Then f has the following properties.*

(i) f is superlinear: $\lim_{|u|\to+\infty} \dfrac{f(u)}{|u|} = +\infty$.

(ii) Its Legendre transform f^*, given by the formula

$$f^*(v) = \sup_{u\in\mathbb{R}} \Big(uv - f(u) \Big),$$

is well-defined on $\mathbb{R}$. Moreover, f^* is $\mathbf{C}^2(\mathbb{R})$ and strongly convex as well.

(iii) For all $C \in \mathbb{R}$, the equation $f(u) = C$ admits at most two solutions.

Proof. Remark that f' is in particular a $\mathbf{C}^1$-diffeomorphism of $\mathbb{R}$.
(i) Fix $A > 0$. Since $f' : \mathbb{R} \to \mathbb{R}$ is a bijection,

$$\exists \beta \in \mathbb{R}, \quad f'(\beta) = 2A.$$

Moreover,

$$\exists U = U(A, \beta) > 0, \ \forall v \in \mathbb{R}, \quad v \geq U \implies \left| \frac{f(\beta) - 2A\beta}{v} \right| \leq A.$$

By convexity, for all $u \geq \max\{\beta, V\}$, we have

$$\frac{f(u)}{u} \geq \frac{f(\beta) + f'(\beta)(u - \beta)}{u} = \frac{f(\beta) + 2A(u - \beta)}{u} = 2A + \frac{f(\beta) - 2A\beta}{u} \geq A,$$

which proves that $\lim\limits_{u \to +\infty} \dfrac{f(u)}{u} = +\infty$. We would show in a similar fashion that $\lim\limits_{u \to -\infty} \dfrac{f(u)}{u} = -\infty$.
(ii) Fix $v \in \mathbb{R}$. The function $\phi : u \mapsto uv - f(u)$ is continuous on $\mathbb{R}$ and satisfies

$$\phi(u) \underset{|u| \to +\infty}{\longrightarrow} -\infty$$

because of *(i)*. We deduce that ϕ attains its maximum on $\mathbb{R}$; such a global maximum u_0 is a critical point of ϕ. We deduce:

$$\phi'(u_0) = 0 \iff v - f'(u_0) = 0 \iff u_0 = g(v), \quad g := (f')^{-1}.$$

Note that $g \in \mathbf{C}^1(\mathbb{R})$ as the reciprocal of f'. We deduce that f^* is given by:

$$\forall v \in \mathbb{R}, \quad f^*(v) = vg(v) - f(g(v)).$$

Since f and g are both continuously differentiable, f^* is continuously differentiable as well. Moreover, for all $v \in \mathbb{R}$,

$$(f^*)'(v) = g(v) - vg'(v) - g'(v)f'(v) = g(v).$$

Since g is continuously differentiable, $(f^*)'$ is continuously differentiable as well, *i.e.* $f^* \in \mathbf{C}^2(\mathbb{R})$. Finally, since f' is an increasing bijection, so is $(f^*)' = g = (f')^{-1}$. This proves that f^* is strongly convex.

(iii) Fix $C \in \mathbb{R}$. Property *(i)* ensures that f is bounded by below and attains its minimum value, say at point u_0. Note that $f'(u_0) = 0$. In particular, f is increasing on $[u_0, +\infty)$ and is decreasing on $(-\infty, u_0]$. The statement *(iii)* follows from a case by case study:

Case 1: $f(u_0) > C$. The equation does not have any solution.

Case 2: $f(u_0) = C$. The unique solution to the equation is u_0.

Case 3: $f(u_0) < C$. By monotonicity, the equation admits exactly two solutions. □

Notation 8.1.1. Let us introduce the Legendre transform of H, function we denote by L (sometimes referred to as Lagrangian) and which is defined by

$$\forall x, v \in \mathbb{R}, \quad L(x,v) = H^*(x,v) := \sup_{u\in\mathbb{R}}\Big(uv - H(x,u)\Big). \tag{8.1.3}$$

Example 8.1.2 (Traffic flow II)**.** The Legendre transform of the Hamiltonian from Example 8.1.1 is given by:

$$\forall x, v \in \mathbb{R}, \quad L(x,v) = \frac{(v+\theta(x))^2}{4\theta(x)}.$$

One remarkable feature is that L and H have the same properties, as highlighted by the following lemma.

Lemma 8.1.3. *Let $H \in \mathbf{C}^2(\mathbb{R}^2)$ verify (8.0.3)-(8.1.2). Then:*
(i) $L \in \mathbf{C}^2(\mathbb{R}^2)$ and verifies (8.0.3)-(8.1.2) as well;
(ii) we have:

$$\forall \lambda > 0,\ \forall x \in \mathbb{R},\ \forall v \in \mathbb{R}^*, \quad \frac{L(x,v)}{|v|} \geq \lambda - \frac{1}{|v|}\Big(\sup_{\substack{y\in\mathbb{R}\\ |u|\leq\lambda}} H(y,u)\Big) \tag{8.1.4}$$

and

$$\forall \lambda > 0,\ \forall x \in \mathbb{R},\ \forall u \in \mathbb{R}^*, \quad \frac{H(x,u)}{|u|} \geq \lambda - \frac{1}{|u|}\Big(\sup_{\substack{y\in\mathbb{R}\\ |v|\leq\lambda}} L(y,v)\Big). \tag{8.1.5}$$

Proof. Remark that the superlinear growth granted in *(i)* of Lemma 8.1.2 ensures that the supremum in the definition of L is a maximum, so that L is well defined on $\mathbb{R}^2$.
(i) By (8.1.2), for all $(x,v) \in \mathbb{R}^2$, there exists a unique $u \in \mathbb{R}$ such that $v = \dfrac{\partial H}{\partial u}(x,u)$. Since $\dfrac{\partial^2 H}{\partial u^2}(x,\cdot)$ does not vanish on $\mathbb{R}$, we can apply the implicit function theorem to inverse the relation:

$$v = \frac{\partial H}{\partial u}(x,u) \iff u = g(x,v),$$

with $g \in \mathbf{C}^1(\mathbb{R}^2)$ in a neighborhood of (x,v). Moreover, direct computations lead to

$$\frac{\partial L}{\partial v}(x,v) = g(x,v),$$

ensuring that $L \in \mathbf{C}^2(\mathbb{R}^2)$. Moreover, from (8.0.3), we deduce:

$$\forall (x,v) \in \mathbb{R}^2,\ |x| \geq X \implies L(x,v) = \sup_{u\in\mathbb{R}}\Big(uv - H(X,u)\Big),$$

and the space dependency of L is localized in the compact subset $[-X,X]$ as well. The fact that for all $x \in \mathbb{R}$, $L(x,\cdot)$ is strongly convex follows from Lemma 8.1.2.

(ii) We now turn to the proof of inequality (8.1.4). By specializing with $u = \dfrac{\lambda v}{|v|}$ in the definition (8.1.3) of L, we get:

$$L(x,v) \geq \lambda|v| - H\left(x, \frac{\lambda v}{|v|}\right) \geq \lambda|v| - \sup_{\substack{y\in\mathbb{R}\\ |u|\leq\lambda}} H(y,u).$$

From this last inequality, we deduce (8.1.4). Inequality (8.1.5) follows from the same computations, using the fact that the Legendre transform is an involution on convex functions. □

Note that choosing $H(x,u) = \dfrac{1}{4}u^4$ gives $L(x,v) = \dfrac{3}{4}v^{4/3}$, showing that no matter the regularity of H, as soon as $\partial^2_{uu}H(x,\cdot)$ vanishes, L may lack the $\mathbf{C}^2$ regularity, hence our assumption (8.1.2). We refer to [46, Appendix A.2] for more properties of the Legendre transform.

Corollary 8.1.4. *Both H and L have a Nagumo growth: there exists a continuous function $\phi : \mathbb{R}^+ \to \mathbb{R}$ which verifies $\dfrac{\phi(r)}{r} \underset{r\to+\infty}{\longrightarrow} +\infty$ such that*

$$\forall x, v \in \mathbb{R}, \quad H(x,v) \geq \phi(|v|). \tag{8.1.6}$$

Proof. We only give the details for H. Let us check that the function

$$\forall r \in \mathbb{R}^+, \quad \phi(r) := \inf_{x\in\mathbb{R}}\Big(\min\{H(x,r), H(x,-r)\}\Big)$$

suits the requirements. Remark that in light of (8.1.1),

$$\forall r \in \mathbb{R}^+, \quad \phi(r) = \inf_{|x|\leq X}\Big(\min\{H(x,r), H(x,-r)\}\Big).$$

Clearly, $\dfrac{\phi(r)}{r} \underset{r\to+\infty}{\longrightarrow} +\infty$ and ϕ satisfies (8.1.6); only the continuity is left to prove.
Let us introduce the continuous function F defined on $[-X,X]\times\mathbb{R}^+$ by

$$F(x,r) := \min\{H(x,r), H(x,-r)\}.$$

Fix $r_0 \geq 0$ and let $(r_n)_n$ be a sequence of $\mathbb{R}^+$ which converges to r. By compactness,

$$\forall n \in \mathbb{N},\ \exists x_n \in [-X,X], \quad \phi(r_n) = F(x_n, r_n)$$

and

$$\exists x_0 \in [-X,X], \quad \phi(r_0) = F(x_0, r_0).$$

Again, by compactness, up to the extraction of a subsequence, we can suppose that $(x_n)_n$ converges to some $\overline{x} \in [-X,X]$. The continuity of F gives us:

$$\phi(r_n) \underset{n\to+\infty}{\longrightarrow} F(\overline{x}, r_0).$$

We conclude the proof by showing that $F(\overline{x}, r_0) = F(x_0, r_0)$. By definition of the sequence $(x_n)_n$,

$$\forall n \in \mathbb{N}, \quad F(x_n, r_n) \leq F(x_0, r_n) \implies F(\overline{x}, r_0) \leq F(x_0, r_0).$$

Finally, by definition of x_0, we also have $F(\overline{x}, r_0) \geq F(x_0, r_0)$, hence the equality and the proof is complete. □

Remark 8.1.1. Condition (8.1.6) implies that L (resp. H) is bounded by below. Indeed, by assumption, there exists $R_1 > 0$ such that

$$\forall r \in \mathbb{R}^+, \quad r \geq R_1 \implies \phi(r) \geq r.$$

The function ϕ is continuous on the compact subset $[0, R_1]$, hence bounded on $[0, R_1]$. Consequently,

$$\forall x, v \in \mathbb{R}, \quad L(x, v) \geq \phi(|v|) \geq \min\left\{\min_{[0,R_1]} \phi; R_1\right\}.$$

8.1.2 ODE aspect

The characteristics system associated to (8.0.1) is:

$$\begin{cases} \dot{\xi}(t) = \dfrac{\partial H}{\partial u}(\xi(t), \nu(t)) \\ \dot{\nu}(t) = -\dfrac{\partial H}{\partial x}(\xi(t), \nu(t)); \quad \xi(0) = \xi_0, \ \nu(0) = \nu_0; \quad \xi_0, \nu_0 \in \mathbb{R}. \end{cases} \tag{8.1.7}$$

Remark that if (ξ, ν) is trajectory of (8.1.7) defined on (τ, σ), we have:

$$\forall t \in (\tau, \sigma), \quad \frac{\mathrm{d}}{\mathrm{d}t}\left(H(\xi(t), \nu(t))\right) = \dot{\xi}(t)\frac{\partial H}{\partial x}(\xi(t), \nu(t)) + \dot{\nu}(t)\frac{\partial H}{\partial u}(\xi(t), \nu(t)) = 0, \tag{8.1.8}$$

which means that H is constant along any trajectory $t \mapsto (\xi(t), \nu(t))$. Since $H \in \mathbf{C}^2(\mathbb{R}^2)$, Cauchy-Lipschitz theorem immediately ensures that for all $\xi_0, \nu_0 \in \mathbb{R}$, (8.1.7) admits a unique maximal solution. In this section we prove two principal results under assumptions (8.0.3)-(8.1.2): global existence of the maximal solutions and surjectivity of the shooting function.

Global existence

Lemma 8.1.5 (Projection of the level sets)**.** *Let $\xi_0, \nu_0 \in \mathbb{R}$. Set*

$$M := \sup_{\substack{x \in \mathbb{R} \\ |u| \leq |\nu_0|}} |H(x, u)| + \sup_{\substack{x \in \mathbb{R} \\ |v| \leq 1}} L(x, v).$$

Then the maximal solution $((\tau, \sigma), \xi, v)$ to (8.1.7) satisfies:

$$\forall t \in (\tau, \sigma), \quad |\nu(t)| \leq M. \tag{8.1.9}$$

Proof. This is a consequence of estimate (8.1.5) and assumption (8.0.3). Indeed, for all $t \in (\tau, \sigma)$, we have

$$\sup_{\substack{x \in \mathbb{R} \\ |u| \leq |\nu_0|}} |H(x, u)| \geq H(\xi_0, \nu_0) = H(\xi(t), \nu(t)) \geq |\nu(t)| - \sup_{\substack{x \in \mathbb{R} \\ |v| \leq 1}} L(x, v),$$

where we have used (8.1.5) with $\lambda = 1$ and the conservation of H along $t \mapsto (\xi(t), \nu(t))$, see (8.1.8). □

Lemma 8.1.6 (Global existence). *Fix $\xi_0, \nu_0 \in \mathbb{R}$ and let $((\tau,\sigma),\xi,\nu)$ be the maximal solution to (8.1.7). Then $(\tau,\sigma)=\mathbb{R}$.*

Proof. As in Lemma 8.1.5, set

$$M := \sup_{\substack{x\in\mathbb{R}\\ |u|\le|\nu_0|}} |H(x,u)| + \sup_{\substack{x\in\mathbb{R}\\ |v|\le 1}} L(x,v)$$

so that

$$\forall t\in(\tau,\sigma), \quad |\nu(t)|\le M.$$

Now, set

$$V := \sup_{\substack{x\in\mathbb{R}\\ |p|\le M}} \left|\frac{\partial H}{\partial u}(x,p)\right| < +\infty.$$

Remark that

$$\forall t\in(\tau,\sigma), \quad |\dot\xi(t)|\le V$$

and therefore,

$$\forall t,t'\in(\tau,\sigma), \quad |\xi(t)-\xi(t')|\le V|t-t'|,$$

and ξ does blow up in finite time. Since ν does not either, $(\tau,\sigma)=\mathbb{R}$. □

Surjectivity of the shooting function

Let us introduce the flow of (8.1.7):

$$\varphi : \left|\begin{array}{ccc} \mathbb{R}^3 & \longrightarrow & \mathbb{R}^2 \\ (t,\xi_0,\nu_0) & \longmapsto & (\xi(t),\nu(t)),\end{array}\right.$$

where (ξ,ν) is the maximal solution to (8.1.7) associated with initial data (ξ_0,ν_0). The differential system being autonomous, we know that φ has the same regularity as the derivatives of H, so here $\varphi\in\mathbf{C}^1(\mathbb{R}^3)$. Let us also denote by φ_1 and φ_2 the two projections of φ, that is for all $(t,\xi_0,\nu_0)\in\mathbb{R}^3$,

$$\begin{aligned}\varphi_1(t,\xi_0,\nu_0) &= \xi_0 + \int_0^t \frac{\partial H}{\partial u}(\varphi_1(s,\xi_0,\nu_0),\varphi_2(s,\xi_0,\nu_0))\,\mathrm{d}s \\ \varphi_2(t,\xi_0,\nu_0) &= \nu_0 - \int_0^t \frac{\partial H}{\partial x}(\varphi_1(s,\xi_0,\nu_0),\varphi_2(s,\xi_0,\nu_0))\,\mathrm{d}s\,.\end{aligned} \tag{8.1.10}$$

The principal of result of this section is Lemma 8.1.11. Before tackling the proof, we introduce some notations/definitions and prove intermediate results.
In view of (8.1.2),

$$\forall x\in\mathbb{R},\ \exists! u(x)\in\mathbb{R}, \quad \frac{\partial H}{\partial u}(x,u(x))=0. \tag{8.1.11}$$

Moreover, since $\dfrac{\partial^2 H}{\partial u^2}(x,\cdot)$ does not vanish on $\mathbb{R}$, an application of the implicit function theorem ensures that $u\in\mathbf{C}^1(\mathbb{R})$ and that

$$\forall x\in\mathbb{R},\ |x|\ge X \implies u'(x)=0,$$

see also the proof of Theorem 8.1.3. Therefore, u is bounded and attains both its minimum and its maximum, and the following quantities are well defined

$$\underline{u} := \min_{x\in\mathbb{R}} u(x), \quad \overline{u} := \max_{x\in\mathbb{R}} u(x), \quad \mathbf{K} := \sup_{x\in\mathbb{R}} H(x, u(x)). \tag{8.1.12}$$

Example 8.1.3 (Traffic flow III)**.** With the Hamiltonian

$$\forall x, p \in \mathbb{R}, \quad H(x,p) = \theta(x)p(p-1),$$

where $\theta \in \mathbf{C}^2(\mathbb{R})$ has the shape depicted in Figure 8.1, left, u is constant and we have:

$$\forall x \in \mathbb{R}, \quad u(x) = \frac{1}{2}; \quad \mathbf{K} = \sup_{x\in\mathbb{R}} \left(-\frac{\theta(x)}{4}\right) = -\frac{1}{4}.$$

Definition 8.1.7. For all $C > \mathbf{K}$, introduce the level sets

$$\Gamma_C^- := \left\{(x,p) \in \mathbb{R}^2 \mid H(x,p) = C \text{ and } p < u(x)\right\}$$
$$\Gamma_C^+ := \left\{(x,p) \in \mathbb{R}^2 \mid H(x,p) = C \text{ and } p > u(x)\right\}.$$

Proposition 8.1.8. *For all $C > \mathbf{K}$ and for all $x \in \mathbb{R}$,*

$$\forall C > \mathbf{K},\ \forall x \in \mathbb{R},\ \exists!(m(x,C), M(x,C)) \in \mathbb{R}^2, \quad (x, m(x,C)) \in \Gamma_C^-,\ (x, M(x,C)) \in \Gamma_C^+.$$

Moreover,

(i) $M, m \in \mathbf{C}^1(\mathbb{R} \times (\mathbf{K}, +\infty))$;

(ii) M and m have a compact space dependency:

$$\forall (x,C) \in \mathbb{R} \times (\mathbf{K}, +\infty),\ |x| \geq X \implies \frac{\partial M}{\partial x}(x,C) = 0;\ \frac{\partial m}{\partial x}(x,C) = 0; \tag{8.1.13}$$

(iii) for all $x \in \mathbb{R}$, $M(x,\cdot)$ is an increasing function and $m(x,\cdot)$ is a decreasing function.

(iv) We have the limits:

$$\forall x \in \mathbb{R}, \quad M(x,C) \underset{C\to+\infty}{\longrightarrow} +\infty; \quad m(x,C) \underset{C\to+\infty}{\longrightarrow} -\infty.$$

Proof. We only prove the results for M, the details for m are similar.
(i)-(ii) The function M is defined by the relations

$$\forall (x,C) \in \mathbb{R} \times (\mathbf{K}, +\infty), \quad H(x, M(x,C)) = C, \quad M(x,C) > u(x).$$

Once again, the implicit function theorem provides the regularity. Then, the chain rule provides:

$$\frac{\partial H}{\partial x}(x, M(x,C)) + \frac{\partial M}{\partial x}(x,C)\frac{\partial H}{\partial u}(x, M(x,C)) = 0,$$

from which we deduce (8.1.13) when using (8.0.3).

(iii) By differentiating with respect to C the relation defining M, we obtain

$$\forall (x, C) \in \mathbb{R} \times (\mathbf{K}, +\infty), \quad \frac{\partial M}{\partial C}(x, C) \frac{\partial H}{\partial u}(x, M(x, C)) = 1. \tag{8.1.14}$$

However, recall that for all $(x, C) \in \mathbb{R} \times (\mathbf{K}, +\infty)$, $M(x, C) > u(x)$. By definition of u, this implies:

$$\forall (x, C) \in \mathbb{R} \times (\mathbf{K}, +\infty), \quad \frac{\partial H}{\partial u}(x, M(x, C)) > 0.$$

Combined with (8.1.14), it proves the statement.

(iv) In view of the monotonicity of M, suppose instead that for some $\overline{x} \in \mathbb{R}$, $M(\overline{x}, \cdot)$ is bounded by above. Therefore, $M(\overline{x}, \cdot)$ has a finite limit as $C \to +\infty$. This contradicts the equality

$$H(\overline{x}, M(\overline{x}, C)) = C.$$

□

Example 8.1.4 (Traffic flow IV)**.** With the H defined in Example 8.1.3, we can explicitly compute m and M. Recall that in this case, $\mathbf{K} = -1/4$.

$$\forall x \in \mathbb{R},\ \forall C > -\frac{1}{4}, \quad m(x, C) = \frac{1}{2} - \frac{1}{2}\sqrt{1 + \frac{4C}{\theta(x)}}; \quad M(x, C) = \frac{1}{2} + \frac{1}{2}\sqrt{1 + \frac{4C}{\theta(x)}}.$$

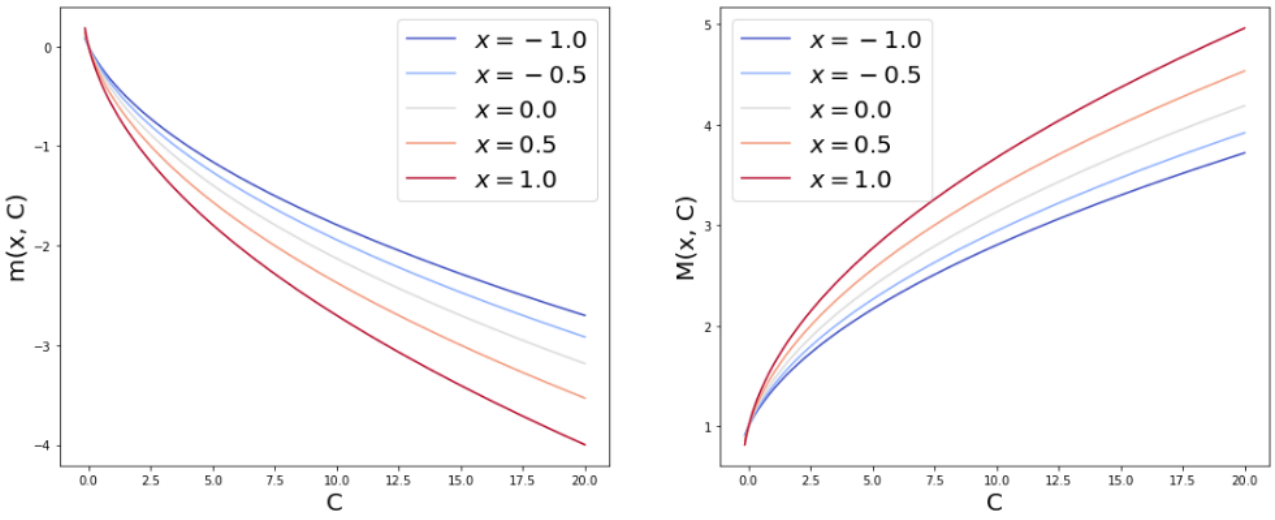

Figure 8.2 – Representation of the functions m and M.

Definition 8.1.9. For all $C > \mathbf{K}$, define the functions:

$$\forall C > \mathbf{K}, \quad V(C) := \inf_{x \in \mathbb{R}} \frac{\partial H}{\partial u}(x, M(x, C)); \quad v(C) := \sup_{x \in \mathbb{R}} \frac{\partial H}{\partial u}(x, m(x, C)).$$

Proposition 8.1.10. *(i) V is a nondecreasing function and v is a nonincreasing function;*
(ii) We have the limits:

$$V(C) \underset{C\to+\infty}{\longrightarrow} +\infty; \quad v(C) \underset{C\to+\infty}{\longrightarrow} -\infty.$$

Proof. Let us make precise that in view of assumption (8.1.1) and Proposition 8.1.8, V and v are well-defined. We now prove the statements and only give the details for V.

(i) This follows from the monotonicity of M and $\dfrac{\partial H}{\partial u}$ with respect to their second argument:

$$\forall x \in \mathbb{R},\ \forall C, \gamma > \mathbf{K}\ (C < \gamma), \quad \frac{\partial H}{\partial u}(x, M(x, C)) \leq \frac{\partial H}{\partial u}(x, M(x, \gamma)).$$

Then, take the infimum for $x \in \mathbb{R}$.

(ii) Suppose instead that it is not the case so that by monotonicity, $\overline{V} := \sup_{C>\mathbf{K}} V(C) < +\infty$. By definition,

$$\forall n \in \mathbb{N}\ (n \geq \mathbf{K}+1),\ \exists x_n \in [-X, X], \quad \frac{\partial H}{\partial u}(x_n, M(x_n, n)) \leq \overline{V}. \tag{8.1.15}$$

Up to the extraction of a subsequence, we can assume that $(x_n)_n$ converges to some $\overline{x} \in [-X, X]$. Making use of Proposition 8.1.8 *(iv)* and of the coercivity of H, we deduce:

$$\frac{\partial H}{\partial u}(\overline{x}, M(\overline{x}, n)) \underset{n\to+\infty}{\longrightarrow} +\infty.$$

Therefore,

$$\exists N \in \mathbb{N}\ (N \geq \mathbf{K}+1),\ \forall n \geq N, \quad \frac{\partial H}{\partial u}(\overline{x}, M(\overline{x}, n)) \geq 2\overline{V} > \overline{V}. \tag{8.1.16}$$

Now, the monotonicity of M combined with (8.1.15) results in:

$$\forall n \geq N, \quad \frac{\partial H}{\partial u}(x_n, M(x_n, N)) \leq \overline{V}.$$

Passing to the limit as $n \to +\infty$ contradicts (8.1.16), therefore proving the statement. □

We are now in position to prove:

Lemma 8.1.11. *Fix $T > 0$ and $\xi_0 \in \mathbb{R}$. Then*

$$\forall \xi_T \in \mathbb{R},\ \exists \nu_0 \in \mathbb{R}, \quad \varphi_1(T, \xi_0, \nu_0) = \xi_T. \tag{8.1.17}$$

Proof. Fix $\nu_0 > 0$ sufficiently large such that:

$$\nu_0 > u(\xi_0); \quad C_0 := H(\xi_0, \nu_0) > \mathbf{K}.$$

By (8.1.8) of H,

$$\forall t \in \mathbb{R}, \quad H(\varphi_1(t, \xi_0, \nu_0), \varphi_2(t, \xi_0, \nu_0)) = C_0 > \mathbf{K}.$$

By definition of $\mathbf{K}$,

$$\forall t \in \mathbb{R}, \quad \varphi_2(t, \xi_0, \nu_0) \neq u(\varphi_1(t, \xi_0, \nu_0)).$$

Using the continuity of φ_1 and φ_2 as well as the fact that $\nu_0 > u(\xi_0)$, we deduce that

$$\forall t \in \mathbb{R}, \quad \varphi_2(t, \xi_0, \nu_0) > u(\varphi_1(t, \xi_0, \nu_0)),$$

therefore, for all $t \in \mathbb{R}$, $(\varphi_1(t, \xi_0, \nu_0), \varphi_2(t, \xi_0, \nu_0)) \in \Gamma^+_{C_0}$. Consequently, by definition of V (Definition 8.1.9), we have:

$$\varphi_1(T, \xi_0, \nu_0) = \xi_0 + \int_0^T \dot{\xi}(s)\, \mathrm{d}s \geq \xi_0 + T \cdot V(C_0) \underset{\nu_0 \to +\infty}{\longrightarrow} +\infty$$

since $C_0 \underset{\nu_0 \to +\infty}{\longrightarrow} +\infty$. Using m and v, we would prove the same way that

$$\varphi_1(T, \xi_0, \nu_0) \underset{\nu_0 \to -\infty}{\longrightarrow} -\infty.$$

The continuity of φ_1 coupled with the intermediate value theorem concludes the proof. □

With similar ideas, we prove the variant:

Lemma 8.1.12. *Fix $T > 0$ and $\xi_T \in \mathbb{R}$. Then*

$$\forall \xi_0 \in \mathbb{R}, \ \exists \nu_T \in \mathbb{R}, \quad \varphi_1(0, \xi_T, \nu_T) = \xi_0.$$

8.1.3 Generalized characteristics

We now recall definitions and results from [67], which we mostly be used in Section 8.4. Note that we are indeed in the framework of [67] because of assumptions (8.1.2). Fix $T > 0$ and $u_0 \in \mathbf{L}^\infty(\mathbb{R})$. As we will prove in Section 8.2, (8.0.1) admits a unique entropy solution $u \in \mathbf{L}^\infty(\Omega)$, see Definition 8.2.1. Moreover, for a.e. $t \in (0, T)$, $u(\cdot, t)$ admits strong left-side and right-side traces at point x for all $x \in \mathbb{R}$, see Remark 8.2.1.

Definition 8.1.13. (i) A Lipschitz function $\gamma \in \mathbf{Lip}((0, T))$ is a generalized characteristics to (8.0.1) if for a.e. $t \in (0, T)$,

$$\frac{\partial H}{\partial u}(\gamma(t), u(\gamma(t)+, t) \leq \dot{\gamma}(t) \leq \frac{\partial H}{\partial u}(\gamma(t), u(\gamma(t)-, t)).$$

(ii) A generalized characteristics $\gamma \in \mathbf{Lip}((0, T))$ is said genuine on $(0, T)$ if for a.e. $t \in (0, T)$,

$$u(\gamma(t)-, t) = u(\gamma(t)+, t).$$

Remark 8.1.2. In Definition 8.1.13, the differential inclusion is to be understood in the classical sense of Filippov, see [85].

Now, let us recall the results from [67] that we will use.

Theorem 8.1.14. *(i) If $\gamma \in \mathbf{Lip}((0,T))$ is a genuine characteristics, there exists $v \in \mathbf{C}^1((0,T))$ such that:*

$$\limsup_{\varepsilon\to 0+} \frac{1}{\varepsilon}\int_{\gamma(0)-\varepsilon}^{\gamma(0)} u_0(x)\,\mathrm{d}x \le v(0) \le \liminf_{\varepsilon\to 0+} \frac{1}{\varepsilon}\int_{\gamma(0)}^{\gamma(0)+\varepsilon} u_0(x)\,\mathrm{d}x$$

$$\text{for a.e. } t\in(0,T),\quad u(\gamma(t)-,t) = v(t) = u(\gamma(t)+,t)$$

$$u(\gamma(T)+,T) \le v(T) \le u(\gamma(T)-,T).$$

Moreover, (γ, v) solves the ODE system (8.1.7).

(ii) Two genuine characteristics may intersect only at their endpoints.

(iii) For all $(x,t)\in\mathbb{R}\times(0,T]$, there exist two generalized characteristics ξ and ζ, called minimal and maximal, such that if $\gamma\in\mathbf{Lip}((0,T))$ is a generalized characteristics going through (x,t), then:

$$\forall s\in[0,t],\quad \xi(s)\le\gamma(s)\le\zeta(s)$$

$$\xi \text{ and } \zeta \text{ are genuine on } (0,t)$$

$$\xi(t) = u(x-,t);\quad \zeta(t) = u(x+,t).$$

8.2 Correspondence conservation law/Hamilton-Jacobi equation

We extend to our framework the correspondence between entropy solutions to (8.0.1) and viscosity solutions to (8.0.2) using the vanishing viscosity method. We fix $T>0$ and set $\Omega := \mathbb{R}\times(0,T)$.

8.2.1 Stability results

Let us introduce Φ, the space dependent classical entropy flux associated to H:

$$\forall x,u,\kappa\in\mathbb{R},\quad \Phi(x,u,\kappa) = \operatorname{sgn}(u-\kappa)(H(x,u)-H(x,\kappa)),$$

which we use for the notion of entropy solution in the sense of Kruzhkov [114, Definition 1].

Definition 8.2.1. A bounded function $u\in\mathbf{L}^\infty(\Omega)$ is an entropy solution to (8.0.1) with initial data $u_0\in\mathbf{L}^\infty(\mathbb{R})$ if for all test functions $\varphi\in\mathbf{C}^\infty_{\mathbf{c}}(\mathbb{R}\times\mathbb{R}^+), \varphi\ge 0$ and for all $\kappa\in\mathbb{R}$, the following entropy inequalities are verified:

$$\begin{aligned}\int_0^{+\infty}\int_{\mathbb{R}}\left(|u-\kappa|\partial_t\varphi + \Phi\left(x,u,\kappa\right)\partial_x\varphi - \operatorname{sgn}(u-\kappa)\frac{\partial H}{\partial x}(x,\kappa)\varphi\right)\mathrm{d}x\,\mathrm{d}t \\ + \int_{\mathbb{R}}|u_0(x)-\kappa|\varphi(x,0)\,\mathrm{d}x \ge 0.\end{aligned} \tag{8.2.1}$$

Remark 8.2.1. As we mentioned in Chapter 1 (see Remark 1.6.1), since the Hamiltonian is non-degenerate (see assumption (8.1.2)), entropy solutions to (8.0.1) admit strong traces for all positive times, see [3, 129].

Note that Lemma 8.1.5 coupled with the notion of generalized characteristics (see Section 8.1.3 provides an *a priori* bound for entropy solutions to (8.0.1).

Theorem 8.2.2. *Fix $u_0 \in \mathbf{L}^\infty(\mathbb{R}) \cap \mathbf{C}(\mathbb{R})$ and let u be an entropy solution to (8.0.1) associated with initial data u_0. Then we have:*

$$\|u\|_{\mathbf{L}^\infty(\Omega)} \leq \sup_{\substack{y\in\mathbb{R} \\ |p|\leq \|u_0\|_{\mathbf{L}^\infty}}} H(y,p) + \sup_{\substack{y\in\mathbb{R} \\ |v|\leq 1}} L(y,v). \tag{8.2.2}$$

Proof. Fix $(x,t) \in \Omega$ and denote by ξ (resp. ζ) the minimal (resp. maximal) backward generalized characteristics emanating from (x,t). Regarding ξ, it means that $\xi \in \mathbf{C}^1((0,t))$ and that there exists $\nu \in \mathbf{C}^1((0,t))$ such that ξ, ν solves (8.1.7) with final conditions $\xi(t) = x$ and $\nu(t) = u(x-,t)$, where we denoted by $u(x-,t)$ the left-side trace of u at point x and time t (see Remark 8.2.1 for the existence of such traces). Since u_0 is continuous, by conservation (see (8.1.8)) we have

$$H(\xi(t),\nu(t)) = H(\xi(0),\nu(0)) = H(\xi(0),u_0(\xi(0))).$$

Using now inequality (8.1.5) with $\lambda = 1$, we deduce:

$$\begin{aligned}
|u(x-,t)| &\leq H(x,u(x-,t)) + \sup_{\substack{y\in\mathbb{R} \\ |v|\leq 1}} L(y,v) \\
&= H(\xi(t),\nu(t)) + \sup_{\substack{y\in\mathbb{R} \\ |v|\leq 1}} L(y,v) \\
&= H(\xi(0),u_0(\xi(0))) + \sup_{\substack{y\in\mathbb{R} \\ |v|\leq 1}} L(y,v) \leq \sup_{\substack{y\in\mathbb{R} \\ |p|\leq \|u_0\|_{\mathbf{L}^\infty}}} H(y,p) + \sup_{\substack{y\in\mathbb{R} \\ |v|\leq 1}} L(y,v).
\end{aligned}$$

To obtain the same bound for $|u(x+,t)|$, we use ζ: there exists some function $\omega \in \mathbf{C}^1((0,t))$ such that ζ, ω solves (8.1.7) with final conditions $\zeta(t) = x$ and $\omega(t) = u(x+,t)$. We omit the details of the proof since they are similar to the ones for $u(x-,t)$. □

Remark 8.2.2. One can show that the bound (8.0.1) still holds for merely bounded initial data. However, Theorem 8.2.2 stated as it is will be sufficient for the application we have in mind, see Corollary 8.2.20.

The key argument to the proof of uniqueness for entropy solutions is Kato inequality, which proof can be found in [114, Theorem 1]. We provide here a simpler re-writing of the proof in our framework.

Lemma 8.2.3 (Kato inequality)**.** *Fix $u_0, v_0 \in \mathbf{L}^\infty(\mathbb{R})$. We denote by u (resp. v) an entropy solution to (8.0.1) corresponding to initial data u_0 (resp. v_0). Then for all test functions $\varphi \in \mathbf{C}_\mathbf{c}^\infty(\mathbb{R}\times\mathbb{R}^+)$, $\varphi \geq 0$, we have:*

$$\int_0^{+\infty} \int_{\mathbb{R}} \Big(|u-v| \partial_t \varphi + \Phi(x,u,v) \partial_x \varphi \, \mathrm{d}x \, \mathrm{d}t \Big) + \int_{\mathbb{R}} |u_0(x) - v_0(x)| \varphi(x,0) \, \mathrm{d}x \geq 0. \tag{8.2.3}$$

Proof. Take $\phi = \phi(x,t,y,s) \in \mathbf{C}_{\mathbf{c}}^{\infty}((\mathbb{R}\times\mathbb{R}^+)^2), \phi \geq 0$. The classical method of doubling variables (after adding/substracting identical functions and arranging terms) leads us to:

$$
\begin{aligned}
&\iiiint |u(x,t)-v(y,s)|(\partial_t\phi+\partial_s\phi)\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}y\,\mathrm{d}s\\
&+\iiiint \operatorname{sgn}(u(x,t)-v(y,s)\,(H(x,u(x,t))-H(y,v(y,s))\,(\partial_x\phi+\partial_y\phi)\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}y\,\mathrm{d}s\\
&+\iiiint \operatorname{sgn}(u(x,t)-v(y,s)\,(H(y,v(y,s))-H(x,v(y,s))\,\partial_x\phi\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}y\,\mathrm{d}s\\
&+\iiiint \operatorname{sgn}(u(x,t)-v(y,s)\,(H(y,u(x,t))-H(x,u(x,t))\,\partial_y\phi\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}y\,\mathrm{d}s\\
&-\iiiint \Big(\partial_x H(x,v(y,s))-\partial_x H(y,u(x,t))\Big)\operatorname{sgn}(u(x,t)-v(y,s))\phi\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}y\,\mathrm{d}s\\
&+\iiint |u_0(x)-v(y,s)|\phi(x,0,y,s)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s+\iiint |u(x,t)-v_0(y)|\phi(x,t,y,0)\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}y\geq 0.
\end{aligned}
\tag{8.2.4}
$$

We now apply (8.2.4) with

$$
\phi_n(x,t,y,s)=\varphi\left(\frac{x+y}{2},\frac{t+s}{2}\right)\delta_n\left(\frac{x-y}{2}\right)\delta_n\left(\frac{t-s}{2}\right),
$$

where $\varphi=\varphi(X,T)\in\mathbf{C}_{\mathbf{c}}^{\infty}(\mathbb{R}\times\mathbb{R}^+),\varphi\geq 0$ and $(\delta_n)_n$ is a smooth approximation (in the $\mathbf{C}_{\mathbf{c}}^{\infty}(\mathbb{R})'$ sense) of the Dirac mass at the origin. Using the fact that for all $n\in\mathbb{N}$,

$$
\begin{aligned}
\partial_t\phi_n+\partial_s\phi_n&=\partial_T\varphi\left(\frac{x+y}{2},\frac{t+s}{2}\right)\delta_n\left(\frac{x-y}{2}\right)\delta_n\left(\frac{t-s}{2}\right)\\
\partial_x\phi+\partial_y\phi&=\partial_X\varphi\left(\frac{x+y}{2},\frac{t+s}{2}\right)\delta_n\left(\frac{x-y}{2}\right)\delta_n\left(\frac{t-s}{2}\right)
\end{aligned}
$$

and

$$
\iiiint \Big(\partial_x H(x,v(y,s))-\partial_x H(y,u(x,t))\Big)\operatorname{sgn}(u(x,t)-v(y,s))\phi_n\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}y\,\mathrm{d}s\underset{n\to+\infty}{\longrightarrow}0,
$$

we obtain

$$
\iiiint |u(x,t)-v(y,s)|(\partial_t\phi_n+\partial_s\phi_n)\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}y\,\mathrm{d}s\underset{n\to+\infty}{\longrightarrow}\int_0^{+\infty}\int_{\mathbb{R}}|u(x,t)-v(x,t)|\partial_T\varphi(x,t)\,\mathrm{d}x\,\mathrm{d}t
$$

and

$$
\begin{aligned}
&\iiiint \operatorname{sgn}(u(x,t)-v(y,s)\,(H(x,u(x,t))-H(y,v(y,s))\,(\partial_x\phi_n+\partial_y\phi_n)\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}y\,\mathrm{d}s\\
&\underset{n\to+\infty}{\longrightarrow}\int_0^{+\infty}\int_{\mathbb{R}}\Phi(x,u(x,t),v(x,t))\partial_X\varphi(x,t)\,\mathrm{d}x\,\mathrm{d}t\,.
\end{aligned}
$$

Finally, since

$$\iiint |u_0(x) - v(y,s)|\phi_n(x,0,y,s)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s \text{ and } \iiint |u(x,t) - v_0(y)|\phi_n(x,t,y,0)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t$$

both converge to $\frac{1}{2}\int_{\mathbb{R}} |u_0(x) - v_0(x)|\varphi(x)\,\mathrm{d}x$, we get (8.2.3) by assembling the above ingredients together. □

Theorem 8.2.4. *Fix $u_0, v_0 \in \mathbf{L}^\infty(\mathbb{R})$. We denote by u (resp. v) an entropy solution to (8.0.1) corresponding to initial data u_0 (resp. v_0). Fix $R > 0$ and set*

$$\mathbf{L} = \sup_{\substack{x\in\mathbb{R} \\ |p|\le \|u\|_{\mathbf{L}^\infty} + \|v\|_{\mathbf{L}^\infty}}} \left|\frac{\partial H}{\partial u}(x,p)\right|.$$

Then for all $t \in (0,T)$,

$$\int_{|x|\le R} |u(x,t) - v(x,t)|\,\mathrm{d}x \le \int_{|x|\le R+\mathbf{L}t} |u_0(x) - v_0(x)|\,\mathrm{d}x\,. \tag{8.2.5}$$

In particular, (8.0.1) admits at most one entropy solution.

Proof. Fix $t \in (0,T)$. Consider for all $n \in \mathbb{N}$ the function:

$$\varphi_n(x,s) = \frac{1}{4}\left(1 - \xi_n(s-t)\right)\left(1 - \xi_n\left(|x| - R + \mathbf{L}(s-t)\right)\right),$$

where $(\xi_n)_n$ is a smooth approximation of the sign function. The sequence $(\varphi_n)_n$ is a smooth approximation of the characteristic function of the trapezoid

$$\mathcal{T} = \left\{(x,s) \in \mathbb{R}\times\mathbb{R}^+ \mid s \in [0,t] \text{ and } |x| \le R - \mathbf{L}(s-t)\right\}.$$

Let us apply Kato inequality (8.2.4) with $(\varphi_n)_n$. For all $n \in \mathbb{N}$, we have

$$\begin{aligned}
\int_0^{+\infty}\int_{\mathbb{R}} |u-v|\partial_t\varphi_n\,\mathrm{d}x\,\mathrm{d}s &= -\frac{1}{4}\int_0^{+\infty}\int_{\mathbb{R}} |u-v|\xi_n'(s-t)\left(1-\xi_n\left(|x|-R+\mathbf{L}(s-t)\right)\right)\mathrm{d}x\,\mathrm{d}s \\
&\quad - \frac{\mathbf{L}}{4}\int_0^{+\infty}\int_{\mathbb{R}} |u-v|\left(1-\xi_n(s-t)\right)\xi_n'\left(|x|-R+\mathbf{L}(s-t)\right)\mathrm{d}x\,\mathrm{d}s \\
&\underset{n\to+\infty}{\longrightarrow} -\int_{|x|\le R} |u(x,t) - v(x,t)|\,\mathrm{d}x - \mathbf{L}\int_0^t\int_{|x|=R-\mathbf{L}(s-t)} |u-v|\,\mathrm{d}x\,\mathrm{d}s\,.
\end{aligned}$$

Then,

$$\begin{aligned}
&\int_0^{+\infty}\int_{\mathbb{R}} \Phi(x,u,v)\partial_x\varphi_n\,\mathrm{d}x\,\mathrm{d}s \\
&= -\frac{1}{4}\int_0^{+\infty}\int_{\mathbb{R}} \Phi(x,u,v)\left(1-\xi_n(s-t)\right)\mathrm{sgn}(x)\xi_n'\left(|x|-R+\mathbf{L}(s-t)\right)\mathrm{d}x\,\mathrm{d}s \\
&\underset{n\to+\infty}{\longrightarrow} -\int_0^T\int_{|x|=R-\mathbf{L}(s-t)} \Phi(x,u,v)\,\mathrm{sgn}(x)\,\mathrm{d}x\,\mathrm{d}s\,.
\end{aligned}$$

Finally, we have

$$\int_{\mathbb{R}} |u_0(x) - v_0(x)| \varphi_n(x,0)\,\mathrm{d}x \underset{n\to+\infty}{\longrightarrow} \int_{|x|\le R+\mathbf{L}t} |u_0(x) - v_0(x)|\,\mathrm{d}x\,.$$

Assembling the previous limits together, we get:

$$-\int_{|x|\le R} |u(x,t) - v(x,t)|\,\mathrm{d}x + \int_{|x|\le R+\mathbf{L}t} |u_0(x) - v_0(x)|\,\mathrm{d}x$$
$$-\int_0^T \int_{|x|=R-\mathbf{L}(s-t)} (\mathbf{L}|u-v| + \Phi(x,u,v)\,\mathrm{sgn}(x))\,\mathrm{d}x\,\mathrm{d}s \ge 0.$$

Note that for all $s \in (0,t)$, if $x = R - \mathbf{L}(s-t)$, then

$$\mathbf{L}|u(x,t) - v(x,t)| + \Phi(x,u(x,t),v(x,t))\,\mathrm{sgn}(x)$$
$$\ge \mathbf{L}|u(x,t) - v(x,t)| - |H(x,u(x,t)) - H(x,v(x,t))| \ge 0.$$

We get (8.2.5) by assembling the above ingredients together. It clearly implies uniqueness and the proof is complete. □

Now, let us recall the standard Crandall-Lions definition of viscosity solution, see [126, 63, 105].

Definition 8.2.5. A continuous function $U \in \mathbf{C}(\Omega)$ is a viscosity solution to (8.0.2) with initial data $U_0 \in \mathbf{Lip}(\mathbb{R})$ if $U \in \mathbf{Lip}(\Omega)$; $U(\cdot,0) = U_0$ on $\mathbb{R}$ and if for all test functions $\varphi \in \mathbf{C}^\infty(\Omega)$ and $(x_0,t_0) \in \Omega$:

(i) if $U - \phi$ has a local maximum at (x_0,t_0), then:

$$\frac{\partial \varphi}{\partial t}(x_0,t_0) + H\left(x_0, \frac{\partial \varphi}{\partial x}(x_0,t_0)\right) \le 0, \tag{8.2.6}$$

(ii) if $U - \phi$ has a local minimum at (x_0,t_0), then:

$$\frac{\partial \varphi}{\partial t}(x_0,t_0) + H\left(x_0, \frac{\partial \varphi}{\partial x}(x_0,t_0)\right) \ge 0. \tag{8.2.7}$$

We call subsolution (resp. supsolution) to (8.0.2) a Lipschitz function $U \in \mathbf{Lip}(\Omega)$ which verifies $U(\cdot,0) = U_0$ on $\mathbb{R}$ and (8.2.6) (resp. (8.2.7)).

Remark 8.2.3. It is known that in Definition 8.2.5, we can allow $t = T$, see [81, Section 10.2].

Like for entropy solutions, uniqueness and stability of viscosity solutions is standard, see for instance [102], [28, Chapter 2], [46, Chapter 5] or [27, Chapter II]. However, these results do not apply in our framework, so we give the proof of the following statement.

Theorem 8.2.6. *Fix $U_0, V_0 \in \mathbf{Lip}(\mathbb{R})$. We denote by U (resp. V) a viscosity solution to (8.0.2) corresponding to initial data U_0 (resp. V_0). Then:*

$$\sup_{(x,t)\in\Omega} |U(x,t) - V(x,t)| \leq \sup_{x\in\mathbb{R}} |U_0(x) - V_0(x)|\,. \tag{8.2.8}$$

In particular, (8.0.2) admits at most one viscosity solution.

Proof. Let us assume that $M := \sup_{\mathbb{R}} |U_0 - V_0| < +\infty$.

Set $\mathbf{L} := \max\{\|\nabla U\|_{\mathbf{L}^\infty(\Omega)}, \|\nabla V\|_{\mathbf{L}^\infty(\Omega)}\}$. From a straightforward computation, we obtain that for all $(x,t),(y,s) \in \mathbb{R}\overline{\Omega}$,

$$|U(x,t) - V(y,s)| \leq \mathbf{L}(2T + |x-y|) + M. \tag{8.2.9}$$

Fix $A, \varepsilon, \nu, \eta > 0$ and consider, for all $(x,t),(y,s) \in \overline{\Omega}$,

$$\psi(x,y,t,s) := U(x,t) - V(y,s) - \frac{(x-y)^2}{2\varepsilon^2} - \frac{(t-s)^2}{2\nu^2} - A(x^2+y^2) - \eta(t+s).$$

In view of (8.2.9),

$$\psi(x,y,t,s) \underset{|x|,|y|\to+\infty}{\longrightarrow} -\infty,$$

uniformly on $t,s \in [0,T]$. Therefore, there exists $(\overline{x},\overline{y},\overline{t},\overline{s}) \in \overline{\Omega}^2$ such that

$$\sup_{(x,y,t,s)\in\overline{\Omega}^2} \psi(x,y,t,s) = \psi(\overline{x},\overline{y},\overline{t},\overline{s}). \tag{8.2.10}$$

(1) By definition of $(\overline{x},\overline{y},\overline{t},\overline{s})$,

$$\forall x \in \mathbb{R}, \quad \psi(x,\overline{y},\overline{t},\overline{s}) \leq \psi(\overline{x},\overline{y},\overline{t},\overline{s}),$$

which rewrites as:

$$\begin{aligned} \forall x \in \mathbb{R}, \quad \frac{(\overline{x}-\overline{y})^2}{2\varepsilon^2} - \frac{(x-\overline{y})^2}{2\varepsilon^2} - A(x^2-\overline{x}^2) &\leq U(x,\overline{t}) - U(\overline{x},\overline{t}) \\ &\leq \mathbf{L}|x-\overline{x}|. \end{aligned} \tag{8.2.11}$$

Apply (8.2.11) with $x > \overline{x}$ (resp. $x < \overline{x}$) and let $x \to \overline{x}+$ (resp. $x \to \overline{x}-$) to obtain:

$$\left|\frac{(\overline{x}-\overline{y})}{\varepsilon^2} + 2A\overline{x}\right| \leq \mathbf{L}. \tag{8.2.12}$$

With similar arguments, one can check that

$$\left|\frac{(\overline{x}-\overline{y})}{\varepsilon^2} + 2A\overline{y}\right| \leq \mathbf{L}. \tag{8.2.13}$$

(2) Once again, by definition of $(\overline{x},\overline{y},\overline{t},\overline{s})$,

$$\psi(0,0,\overline{t},\overline{s}) \leq \psi(\overline{x},\overline{y},\overline{t},\overline{s}),$$

therefore

$$A\left(\overline{x}^2+\overline{y}^2\right)+\frac{(\overline{x}-\overline{y})^2}{2\varepsilon^2}\leq \mathbf{L}\left(|\overline{x}|+|\overline{y}|\right).$$

Forgetting the second term in the left-hand side, we deduce

$$\left(A|\overline{x}|-\frac{\mathbf{L}}{2}\right)^2+\left(A|\overline{y}|-\frac{\mathbf{L}}{2}\right)^2\leq\frac{\mathbf{L}^2}{2},$$

and then:

$$\left|A|\overline{x}|-\frac{\mathbf{L}}{2}\right|\leq\frac{\mathbf{L}}{\sqrt{2}};\quad \left|A|\overline{y}|-\frac{\mathbf{L}}{2}\right|\leq\frac{\mathbf{L}}{\sqrt{2}}. \tag{8.2.14}$$

Combining this with (8.2.12) provides:

$$\frac{|\overline{x}-\overline{y}|}{\varepsilon^2}\leq\left|\frac{(\overline{x}-\overline{y})}{\varepsilon^2}+2A\overline{x}\right|+2A|\overline{x}|\leq 4\mathbf{L}. \tag{8.2.15}$$

(3) Also note that

$$\psi(\overline{x},\overline{y},\overline{t},\overline{t})\leq\psi(\overline{x},\overline{y},\overline{t},\overline{s})$$

implies

$$\frac{(\overline{t}-\overline{s})^2}{2\nu^2}\leq\mathbf{L}T+\eta(\overline{t}-\overline{s}). \tag{8.2.16}$$

(4) Using one more time the definition of $(\overline{x},\overline{y},\overline{t},\overline{s})$, we write

$$\forall(x,t)\in\mathbb{R}\times[0,T],\quad \psi(x,x,t,t)\leq\psi(\overline{x},\overline{y},\overline{t},\overline{s}).$$

This reads:

$$\begin{aligned}A(\overline{x}^2+\overline{y}^2)\leq{}&(U(\overline{x},\overline{t})-V(\overline{y},\overline{s}))-(U(x,t)-V(x,t))-\frac{(\overline{x}-\overline{y})^2}{2\varepsilon^2}-\frac{(\overline{t}-\overline{s})^2}{2\nu^2}\\&+2Ax^2+2\eta t-\eta(\overline{t}+\overline{s}).\end{aligned} \tag{8.2.17}$$

Now take the limit as $x\to 0$ and $t\to 0$:

$$\begin{aligned}A(\overline{x}^2+\overline{y}^2)&\leq(U(\overline{x},\overline{t})-V(\overline{y},\overline{s}))+M-\frac{(\overline{x}-\overline{y})^2}{2\varepsilon^2}-\frac{(\overline{t}-\overline{s})^2}{2\nu^2}\\&\leq\mathbf{L}(2T+|\overline{x}-\overline{y}|)+M-\frac{(\overline{x}-\overline{y})^2}{2\varepsilon^2}-\frac{(\overline{t}-\overline{s})^2}{2\nu^2}.\end{aligned}$$

In view of (8.2.15) and (8.2.16),

$$\exists\mathbf{C},\ \forall A,\varepsilon,\eta,\nu,\quad A(\overline{x}^2+\overline{y}^2)\leq\mathbf{C}. \tag{8.2.18}$$

(5) Suppose that $\overline{t}\neq 0$. In view of (8.2.16), we also have $\overline{s}\neq 0$.
The function

$$\varphi(x,t)=V(\overline{y},\overline{s})+\frac{(x-\overline{y})^2}{2\varepsilon^2}+\frac{(t-\overline{s})^2}{2\nu^2}+A(x^2+\overline{y}^2)+\eta(t+\overline{s})$$

is in $\mathbf{C}^1(\Omega)$ and $U-\varphi$ admits a local maximum at point $(\overline{x},\overline{t})$. Therefore, since U is a viscosity solution, we have:

$$\frac{\partial\varphi}{\partial t}(\overline{x},\overline{t})+H\left(\overline{x},\frac{\partial\varphi}{\partial x}(\overline{x},\overline{t})\right)\leq 0$$

which rewrites as:

$$\frac{(\overline{t}-\overline{s})}{\nu}+\eta+H\left(\overline{x},\frac{(\overline{x}-\overline{y})}{\varepsilon^2}+2A\overline{x}\right)\leq 0. \tag{8.2.19}$$

On the other hand, the function

$$\varphi(y,s)=U(\overline{x},\overline{t})-\frac{(\overline{x}-y)^2}{2\varepsilon^2}-\frac{(\overline{t}-s)^2}{2\nu^2}-A(\overline{x}^2+y^2)-\eta(\overline{t}+s)$$

is in $\mathbf{C}^1(\Omega)$ and $V-\varphi$ admits a local minimum at point $(\overline{y},\overline{s})$. Since V is a viscosity solution, we have:

$$\frac{\partial\varphi}{\partial t}(\overline{y},\overline{s})+H\left(\overline{y},\frac{\partial\varphi}{\partial x}(\overline{y},\overline{s})\right)\geq 0$$

which rewrites as:

$$\frac{(\overline{t}-\overline{s})}{\nu}-\eta+H\left(\overline{y},\frac{(\overline{x}-\overline{y})}{\varepsilon^2}-2A\overline{y}\right)\geq 0. \tag{8.2.20}$$

Substracting (8.2.19) by (8.2.20) yields:

$$2\eta+\underbrace{H\left(\overline{x},\frac{(\overline{x}-\overline{y})}{\varepsilon^2}+2A\overline{x}\right)-H\left(\overline{y},\frac{(\overline{x}-\overline{y})}{\varepsilon^2}-2A\overline{y}\right)}_{Q}\leq 0. \tag{8.2.21}$$

Note that in view of (8.2.12),

$$\begin{aligned}|Q|&\leq\left(\sup_{\substack{x\in\mathbb{R}\\|p|\leq 8\mathbf{L}}}\left|\frac{\partial H}{\partial x}(x,p)\right|\right)|\overline{x}-\overline{y}|+2\left(\sup_{\substack{x\in\mathbb{R}\\|p|\leq 8\mathbf{L}}}\left|\frac{\partial H}{\partial u}(x,p)\right|\right)A(|\overline{x}|+|\overline{y}|)\\&\leq 4\mathbf{L}\left(\sup_{\substack{x\in\mathbb{R}\\|p|\leq 8\mathbf{L}}}\left|\frac{\partial H}{\partial x}(x,p)\right|\right)\varepsilon^2+\left(4\sqrt{C}\sup_{\substack{x\in\mathbb{R}\\|p|\leq 8\mathbf{L}}}\left|\frac{\partial H}{\partial u}(x,p)\right|\right)\sqrt{A}.\end{aligned}$$

Now, let $A\to 0$ and $\varepsilon\to 0$ to obtain the contradiction $2\eta\leq 0$. Therefore, $\overline{t}=\overline{s}=0$.

(6) Now, for all $(x,t)\in\mathbb{R}\times[0,T]$,

$$\begin{aligned}U(x,t)-V(x,t)&=(U(x,t)-V(x,t)-2Ax^2-2\eta t)+2Ax^2+2\eta t\\&=\psi(x,x,t,t)+2Ax^2+2\eta t\\&\leq\psi(\overline{x},\overline{y},0,0)+2Ax^2+2\eta t\\&=U_0(\overline{x})-V_0(\overline{y})-\frac{(\overline{x}-\overline{y})^2}{2\varepsilon^2}-A(\overline{x}^2+\overline{y}^2)+2Ax^2+2\eta t\\&\leq\mathbf{L}|\overline{x}-\overline{y}|+M-\frac{(\overline{x}-\overline{y})^2}{2\varepsilon^2}-A(\overline{x}^2+\overline{y}^2)+2Ax^2+2\eta t.\end{aligned}$$

Take the limit as $\varepsilon, A, \eta \to 0$ to obtain:

$$\sup_{\mathbb{R}\times[0,T]} (U - V) \leq M = \sup_{\mathbb{R}} |U_0 - V_0|.$$

Then, exchange the roles played by U and V to obtain $\sup_{\mathbb{R}\times[0,T]} (V - U) \leq M = \sup_{\mathbb{R}} |U_0 - V_0|$.
□

8.2.2 Vanishing viscosity method

The ultimate goal of the next sections is to guarantee the correspondence between the entropy solutions of the conservation law and the viscosity solutions of the Hamilton-Jacobi equations, based on the construction of both kinds of solution as vanishing viscosity limits, that is, as limits of solutions to the viscous problems:

$$\begin{cases} \partial_t u + \partial_x \left(H(x,u)\right) = \varepsilon \partial^2_{xx} u \\ u(\cdot, 0) = u_0 \end{cases} \tag{8.2.22}$$

and

$$\begin{cases} \partial_t U + H(x, \partial_x U) = \varepsilon \partial^2_{xx} U \\ U(\cdot, 0) = U_0. \end{cases} \tag{8.2.23}$$

Notation 8.2.1. For a continuous function $u = u(x,t) \in \mathbf{C}(\Omega)$ and $k \in \mathbb{N}^*$, the notation $u \in \mathbf{C}^k_1(\Omega)$ will mean that u is continuously differentiable in Ω with respect to t and k times continuously differentiable in Ω with respect to x.

Let us make precise that the correspondence between solutions to (8.2.22) and (8.2.23) follows from simple computations when the solutions are sufficiently smooth, as outlined in the following lemma.

Lemma 8.2.7. *Fix $\varepsilon > 0$.*

(i) Fix $u_0 \in \mathbf{W}^{1,\infty}(\mathbb{R})$ and let $u_\varepsilon \in \mathbf{C}(\overline{\Omega}) \cap \mathbf{C}^2_1(\Omega)$ be a classical solution to (8.2.22) with initial data u_0. Define:

$$\forall (x,t) \in \overline{\Omega}, \quad U_\varepsilon(x,t) = \int_0^x u_\varepsilon(y,t)\,\mathrm{d}y - \int_0^t \Big(H(0, u_\varepsilon(0,s)) - \varepsilon \partial_x u_\varepsilon(0,s) \Big)\,\mathrm{d}s\,.$$

Then $U_\varepsilon \in \mathbf{C}(\overline{\Omega}) \cap \mathbf{C}^3_1(\Omega)$ is a classical solution to (8.2.23) with initial data $U_0(x) = \int_0^x u_0(y)\,\mathrm{d}y$.

(ii) Fix $U_0 \in \mathbf{C}^1(\mathbb{R})$ such that $U_0' \in \mathbf{W}^{1,\infty}(\mathbb{R})$ and let $U_\varepsilon \in \mathbf{C}(\overline{\Omega}) \cap \mathbf{C}^3_1(\Omega)$ be a classical solution to (8.2.23) with initial data U_0. Then $u_\varepsilon = \partial_x U_\varepsilon \in \mathbf{C}(\overline{\Omega}) \cap \mathbf{C}^2_1(\Omega)$ is a classical solution to (8.2.22) with initial data U_0'.

Proof. *(i)* Simple computations yield

$$\begin{aligned}
&\partial_t U_\varepsilon(x,t) + H(x, \partial_x U_\varepsilon(x,t)) \\
&= \int_0^x \partial_t u_\varepsilon(y,t)\,\mathrm{d}y - H(0, u_\varepsilon(0,t)) + \varepsilon \partial_x u_\varepsilon(0,t) + H(x, u_\varepsilon(x,t)) \\
&= \int_0^x \Big(-\partial_x \left(H(y, u_\varepsilon(y,t)) \right) + \varepsilon \partial_{xx} u_\varepsilon(y,t) \Big)\,\mathrm{d}y - H(0, u_\varepsilon(0,t)) + \varepsilon \partial_x u_\varepsilon(0,t) + H(x, u_\varepsilon(x,t)) \\
&= \varepsilon \partial_x u_\varepsilon(x,t) = \varepsilon \partial_{xx} U_\varepsilon(x,t).
\end{aligned}$$

(ii) This is obtained by differentiating (8.2.23) with respect to x. □

A priori bounds

Lemma 8.2.8. *Let $f \in \mathbf{C}^1(\mathbb{R})$, $R > 0$ and $\varepsilon > 0$. Denote by $\mathbf{U}$ the open subset*

$$\mathbf{U} = \{(x,t) \in \Omega \mid x > R\}.$$

Suppose that $u \in \mathbf{C}(\overline{\mathbf{U}}) \cap \mathbf{C}^2_1(\mathbf{U}) \cap \mathbf{L}^\infty(\mathbf{U})$ is a classical solution to

$$\partial_t u + \partial_x \left(f(u) \right) = \varepsilon \partial^2_{xx} u, \quad (x,t) \in \mathbf{U}.$$

Then u attains its maximum and minimum over the closure of $\mathbf{U}$, and both $\max u$ and $\min u$ are taken at a point of

$$\widetilde{\mathbf{U}} = \{(x,0) \mid x \geq R\} \cup \{(R,t) \mid 0 \leq t \leq T\}.$$

In particular,

$$\max_{(x,t)\in\overline{\mathbf{U}}} |u(x,t)| \leq \max \left\{ \sup_{x \geq R} |u(x,0)|, \sup_{0 \leq t \leq T} |u(R,t)| \right\}.$$

Proof. We take inspiration from [101, Chapter III] and the proof of [98, Theorem B.1]. For any $\eta \in (0,1)$, define the function

$$\forall (x,t) \in \overline{\mathbf{U}}, \quad v_\eta(x,t) = u(x,t) - \eta \left(2\varepsilon t + \frac{(\eta x)^2}{2} \right).$$

Fix $\eta \in (0,1)$. Since u is bounded, we have $v_\eta(x,t) \underset{x \to +\infty}{\longrightarrow} -\infty$, uniformly on $t \in [0,T]$. We deduce that v_η attains its maximum value at some point $(x_\eta, t_\eta) \in \overline{\mathbf{U}}$. Remark that

$$v_\eta(x_\eta, t_\eta) \geq v_\eta(R,0) \implies \frac{\eta^3 |x_\eta|^2}{2} \leq u(x_\eta, t_\eta) - u(R,0) - 2\varepsilon t_\eta + \frac{\eta^3 R^2}{2},$$

and since $\eta \in (0,1)$, we obtain:

$$\eta^{3/2} |x_\eta| \leq \sqrt{4\|u\|_{\mathbf{L}^\infty(\mathbf{U})} + R^2} := M. \tag{8.2.24}$$

We conclude the proof with a case by case study.
Case 1: $t_\eta = 0$ **and** $x_\eta \geq R$**.** In this case, for all $(x,t) \in \overline{\mathbf{U}}$, we have

$$v_\eta(x,t) \leq v_\eta(x_\eta,0) = u(x_\eta,0) - \frac{\eta^3|x_\eta|^2}{2} \leq \sup_{x\geq R} u(x,0).$$

We deduce that for all $(x,t) \in \overline{\mathbf{U}}$,

$$u(x,t) \leq v_\eta(x,t) + \eta\left(2\varepsilon t + \frac{(\eta x)^2}{2}\right) \leq \sup_{x\geq R} u(x,0) + \eta\left(2\varepsilon t + \frac{(\eta x)^2}{2}\right). \tag{8.2.25}$$

Case 2: $x_\eta = R$ **and** $0 \leq t_\eta \leq T$**.** In this case, for all $(x,t) \in \overline{\mathbf{U}}$, we have

$$v_\eta(x,t) \leq v_\eta(R,t_\eta) = u(R,t_\eta) - \eta\left(2\varepsilon t_\eta + \frac{(\eta R)^2}{2}\right) \leq \sup_{0\leq t\leq T} u(R,t).$$

We deduce that for all $(x,t) \in \overline{\mathbf{U}}$,

$$u(x,t) \leq v_\eta(x,t) + \eta\left(2\varepsilon t + \frac{(\eta x)^2}{2}\right) \leq \sup_{0\leq t\leq T} u(R,t) + \eta\left(2\varepsilon t + \frac{(\eta x)^2}{2}\right). \tag{8.2.26}$$

Case 3: $x_\eta > R$ **and** $t_\eta > 0$**.** At the maximum point, we have:

$$\partial_t v_\eta(x_\eta,t_\eta) \geq 0; \quad \partial_x v_\eta(x_\eta,t_\eta) = 0; \quad \partial_{xx} v_\eta(x_\eta,t_\eta) \leq 0,$$

which translates for u as:

$$\partial_t u(x_\eta,t_\eta) \geq 2\varepsilon\eta; \quad \partial_x u(x_\eta,t_\eta) = \eta^3 x_\eta; \quad \partial_{xx} u(x_\eta,t_\eta) \leq \eta^3.$$

Consequently,

$$\begin{aligned}
\Big(\partial_t u + \partial_x(f(u)) - \varepsilon\partial_{xx}u\Big)_{|(x_\eta,t_\eta)} &\geq 2\varepsilon\eta + \eta^3 x_\eta f'(u(x_\eta,t_\eta)) - \varepsilon\eta^3 \\
&\geq 2\varepsilon\eta - \eta^3|x_\eta| \sup_{|p|\leq\|u\|_{\mathbf{L}^\infty}} |f'(p)| - \varepsilon\eta^3 \\
&\geq 2\varepsilon\eta - \eta^{3/2}\left(M \sup_{|p|\leq\|u\|_{\mathbf{L}^\infty}} |f'(p)|\right) - \varepsilon\eta^3 \\
&= \underbrace{\left(2\varepsilon - \eta^{1/2}\left(M \sup_{|p|\leq\|u\|_{\mathbf{L}^\infty}} |f'(p)|\right) - \varepsilon\eta^2\right)}_{Q(\eta)}\eta,
\end{aligned}$$

where we have used the bound (8.2.24). Since $\eta Q(\eta) \underset{\eta\to 0}{\sim} 2\varepsilon\eta > 0$, we can choose $\eta \in (0,1)$ sufficiently small such that $Q(\eta) > 0$. This yields the contradiction

$$0 = \Big(\partial_t u + \partial_x(f(u)) - \varepsilon\partial_{xx}u\Big)_{|(x_\eta,t_\eta)} > 0.$$

Therefore, we are either in Case 1 or in Case 2. From (8.2.25)-(8.2.26), we deduce that for all $(x,t) \in \overline{\mathbf{U}}$,

$$u(x,t) \leq \max\left\{\sup_{x\geq R} u(x,0),\ \sup_{0\leq t\leq T} u(R,t)\right\} + \eta\left(2\varepsilon t + \frac{(\eta x)^2}{2}\right).$$

The proof is complete by passing to the limit as $\eta \to 0$. □

It is straightforward to check that a similar statement holds on the subset

$$\mathbf{U} = \{(x,t) \in \Omega \mid x < -R\}.$$

Corollary 8.2.9. *Let $u_\varepsilon \in \mathbf{C}(\overline{\Omega}) \cap \mathbf{C}_1^2(\Omega) \cap \mathbf{L}^\infty(\Omega)$ be a classical solution to (8.2.22) with initial data $u_0 \in \mathbf{C}(\mathbb{R})$. Then $|u_\varepsilon|$ attains its maximum over the closure of $\mathbf{U}$, which is taken at a point of*

$$\mathbf{K} = \{(x,0)\ :\ |x| \geq X\} \cup \{(x,t)\ :\ |x| \leq X \text{ and } t \in [0,T]\}.$$

In particular,

$$\max_{(x,t)\in\overline{\Omega}} |u_\varepsilon(x,t)| \leq \max\left\{\sup_{x\in\mathbb{R}} |u_0(x)|,\ \sup_{\substack{|x|\leq X\\ 0\leq t\leq T}} |u_\varepsilon(x,t)|\right\}. \tag{8.2.27}$$

Proof. Let us denote by $\mathbf{U}$ the subset

$$\mathbf{U} = (X, +\infty) \times (0,T)$$

and define $v_\varepsilon = u_{\varepsilon|\overline{\mathbf{U}}}$, the restriction of u_ε to $\overline{\mathbf{U}}$. By construction, $v_\varepsilon \in \mathbf{C}(\overline{\mathbf{U}}) \cap \mathbf{C}_1^2(\mathbf{U}) \cap \mathbf{L}^\infty(\mathbf{U})$ and v_ε is a classical solution to (8.2.22) in $\mathbf{U}$. Lemma 8.2.8 ensures that $|v_\varepsilon|$ attains its maximum, and that

$$\max_{(x,t)\in\overline{\mathbf{U}}} |v_\varepsilon(x,t)| \leq \max\left\{\sup_{x\geq X} |u_0(x)|,\ \sup_{0\leq t\leq T} |u_\varepsilon(X,t)|\right\}.$$

The same reasoning holds for the restriction $w_\varepsilon = u_{\varepsilon|(-\infty,-X]\times[0,T]}$. This leads to (8.2.27) and the proof is complete. □

Corollary 8.2.10. *Let $U_\varepsilon \in \mathbf{C}(\overline{\Omega}) \cap \mathbf{C}_1^3(\Omega)$ be a classical solution to (8.2.23) with initial data $U_0 \in \mathbf{C}^1(\mathbb{R})$. Suppose that $U_\varepsilon \in \mathbf{Lip}(\Omega)$. Then $|\partial_x U_\varepsilon|$ attains its maximum, which is taken at a point of*

$$\mathbf{K} = \{(x,0)\ :\ |x| \geq X\} \cup \{(x,t)\ :\ |x| \leq X \text{ and } t \in [0,T]\}.$$

In particular,

$$\max_{(x,t)\in\overline{\Omega}} |\partial_x U_\varepsilon(x,t)| \leq \max\left\{\sup_{x\in\mathbb{R}} |U_0'(x)|,\ \sup_{\substack{|x|\leq X\\ 0\leq t\leq T}} |\partial_x U_\varepsilon(x,t)|\right\}. \tag{8.2.28}$$

Proof. In light of Lemma 8.2.7 *(ii)*, apply Corollary 8.2.9 to $\partial_x U_\varepsilon$. □

Theorem 8.2.11. *Fix $\varepsilon \in (0,1)$ and let $U_\varepsilon \in \mathbf{C}(\overline{\Omega}) \cap \mathbf{C}_1^3(\Omega)$ be a classical solution to (8.2.23) with initial data $U_0 \in \mathbf{C}^1(\mathbb{R})$ such that $U_0' \in \mathbf{W}^{1,\infty}(\mathbb{R})$. Suppose that $U_\varepsilon \in \mathbf{Lip}(\Omega)$. Then there exists a constant $\mathbf{C}$, which does not depend on ε nor T such that*

$$\|\partial_t U_\varepsilon\|_{\mathbf{L}^\infty} + \|\partial_x U_\varepsilon\|_{\mathbf{L}^\infty} \leq \mathbf{C}. \tag{8.2.29}$$

Proof. We divide the proof in two steps.

Step 1. Bound for $\partial_t U_\varepsilon$. First, let us differentiate the PDE (8.2.23) with respect to t so that $\varphi_\varepsilon = \partial_t U_\varepsilon \in \mathbf{L}^\infty(\Omega)$ solves the linear parabolic equation:

$$\partial_t \varphi_\varepsilon + \partial_x \varphi_\varepsilon \frac{\partial H}{\partial u}(x, \partial_x U_\varepsilon) = \varepsilon \partial_{xx}^2 \varphi_\varepsilon.$$

Consequently, a comparison principle, ensures that

$$\inf_{x\in\mathbb{R}} \varphi_\varepsilon(x,0) \leq \varphi_\varepsilon \leq \sup_{x\in\mathbb{R}} \varphi_\varepsilon(x,0) \quad \textit{i.e.} \quad \inf_{x\in\mathbb{R}} \partial_t U_\varepsilon(x,0) \leq \partial_t U_\varepsilon \leq \sup_{x\in\mathbb{R}} \partial_t U_\varepsilon(x,0).$$

We now bound $\partial_t U_\varepsilon(\cdot,0)$ using super/subsolutions. Introduce the function ϕ defined by

$$\forall (x,t) \in \overline{\Omega}, \quad \phi(x,t) = U_0(x) + Mt, \quad M = \left(\sup_{\substack{\xi\in\mathbb{R} \\ |p|\leq \|U_0'\|_{\mathbf{L}^\infty}}} H(\xi,p) \right) + \|U_0''\|_{\mathbf{L}^\infty}. \tag{8.2.30}$$

Then $\phi(\cdot,0) = U_0$ and, recalling that $0 < \varepsilon \leq 1$, we have

$$\partial_t \phi + H(x, \partial_x \phi) - \varepsilon \partial_{xx} \phi = M + H(x, U_0'(x)) - \varepsilon U_0''(x) \geq M + H(x, U_0'(x)) - |U_0''(x)| \geq 0.$$

We deduce that $U_\varepsilon \leq \phi$, see for instance [86, Theorem 8.1]. It is straightforward to check that ψ defined by

$$\psi(x,t) = U_0(x) - Mt$$

is a subsolution to (8.2.23), implying that $U_\varepsilon \geq \psi$. We deduce that for all $(x,t) \in \Omega$,

$$-M \leq \frac{U_\varepsilon(x,t) - U_0(x)}{t} \leq M,$$

which leads to $\|\partial_t U_\varepsilon(\cdot,0)\|_{\mathbf{L}^\infty} \leq M$ by letting $t \to 0^+$. We just proved that

$$\|\partial_t U_\varepsilon\|_{\mathbf{L}^\infty} \leq M, \quad M := \left(\sup_{\substack{\xi\in\mathbb{R} \\ |p|\leq \|U_0'\|_{\mathbf{L}^\infty}}} H(\xi,p) \right) + \|U_0''\|_{\mathbf{L}^\infty}. \tag{8.2.31}$$

Step 2. Bound for $\partial_x U_\varepsilon$. We take inspiration from the computations done in [126, Appendix 1]. Let $\eta \in \mathbf{C}^2(\mathbb{R})$ be a function which verifies $\eta'' > 0$ on $\mathbb{R}$ and set $\omega_\varepsilon = \eta(\partial_x U_\varepsilon)$. We differentiate the PDE (8.2.23) with respect to x this time and multiply the resulting PDE by $\eta'(\partial_x U_\varepsilon)$. We write, recalling that $0 < \varepsilon < 1$,

$$\begin{aligned}
\partial_t \omega_\varepsilon + \frac{\partial H}{\partial u}(x, \partial_x U_\varepsilon) \partial_x \omega_\varepsilon - \varepsilon \partial_{xx} \omega_\varepsilon &= -\frac{\partial H}{\partial x}(x, \partial_x U_\varepsilon) \eta'(\partial_x U_\varepsilon) - \varepsilon (\partial_{xx} U_\varepsilon)^2 \eta''(\partial_x U_\varepsilon) \\
&\leq -\frac{\partial H}{\partial x}(x, \partial_x U_\varepsilon) \eta'(\partial_x U_\varepsilon) - (\varepsilon \partial_{xx} U_\varepsilon)^2 \eta''(\partial_x U_\varepsilon) \\
&= -\frac{\partial H}{\partial x}(x, \partial_x U_\varepsilon) \eta'(\partial_x U_\varepsilon) - (\partial_t U_\varepsilon + H(x, \partial_x U_\varepsilon))^2 \eta''(\partial_x U_\varepsilon) \\
&\leq -\frac{\partial H}{\partial x}(x, \partial_x U_\varepsilon) \eta'(\partial_x U_\varepsilon) - \left(\frac{1}{2} H(x, \partial_x U_\varepsilon)^2 - M^2 \right) \eta''(\partial_x U_\varepsilon),
\end{aligned}$$

where we have used the inequality

$$\forall a, b \in \mathbb{R}, \quad (a+b)^2 \geq \frac{a^2}{2} - b^2$$

and the bound (8.2.31). To summarize, ω_ε verifies:

$$\begin{aligned} &\left(\partial_t \omega_\varepsilon + \frac{\partial H}{\partial u}(x, \partial_x U_\varepsilon)\partial_x \omega_\varepsilon - \varepsilon \partial_{xx}\omega_\varepsilon\right) \\ &+ \eta''(\partial_x U_\varepsilon)\left(\frac{H(x, \partial_x U_\varepsilon)^2}{2} + \frac{\eta'(\partial_x U_\varepsilon)}{\eta''(\partial_x U_\varepsilon)}\frac{\partial H}{\partial x}(x, \partial_x U_\varepsilon) - M^2\right) \leq 0. \end{aligned} \tag{8.2.32}$$

Using Corollary 8.2.10, we know that $\partial_x U_\varepsilon$ and therefore ω_ε attains its maximum on $\overline{\Omega}$. Therefore, let $(x_0, t_0) \in \overline{\Omega}$ be a point of maximum of ω_ε. We conclude the proof by a case by case study.

Case 1: $t_0 = 0$**.** In that case, by convexity of η, we obtain:

$$\forall (x,t) \in \Omega, \quad \eta(\partial_x U_\varepsilon(x,t)) \leq \max\{\eta(\|U_0'\|_{\mathbf{L}^\infty}), \eta(-\|U_0'\|_{\mathbf{L}^\infty})\}. \tag{8.2.33}$$

Case 2: $t_0 > 0$**.** In that situation, we have:

$$\partial_t \omega_\varepsilon(x_0, t_0) \geq 0; \quad \partial_x \omega_\varepsilon(x_0, t_0) = 0; \quad \partial_{xx}\omega_\varepsilon(x_0, t_0) \leq 0.$$

Combining these inequalities with (8.2.32), we deduce that

$$\frac{H(x_0, \partial_x U_\varepsilon(x_0,t_0))^2}{2} + \frac{\eta'(\partial_x U_\varepsilon(x_0,t_0))}{\eta''(\partial_x U_\varepsilon(x_0,t_0))}\frac{\partial H}{\partial x}(x_0, \partial_x U_\varepsilon(x_0,t_0)) \leq M^2. \tag{8.2.34}$$

If $|x_0| \geq X$, then using assumption (8.0.3), we obtain:

$$|H(x_0, \partial_x U_\varepsilon(x_0,t_0))| \leq \sqrt{2}M,$$

and the growth condition (8.1.5) applied with $\lambda = 1$ yields:

$$|\partial_x U_\varepsilon(x_0,t_0)| \leq \sqrt{2}M + \sup_{\substack{x\in\mathbb{R} \\ |v|\leq 1}} L(x,v). \tag{8.2.35}$$

We conclude the proof by handling the case $|x_0| < X$. Let us exploit (8.2.34) with η of the form $\eta = \exp \circ \theta$, where $\theta \in \mathbf{C}^2(\mathbb{R})$ is a strictly convex function, to be determined. With that choice, (8.2.34) becomes

$$\frac{H(x_0, \partial_x U_\varepsilon(x_0,t_0))^2}{2} + \frac{\theta'(\partial_x U_\varepsilon(x_0,t_0))}{\theta''(\partial_x U_\varepsilon(x_0,t_0)) + (\theta'(\partial_x U_\varepsilon(x_0,t_0)))^2}\frac{\partial H}{\partial x}(x_0, \partial_x U_\varepsilon(x_0,t_0)) \leq M^2. \tag{8.2.36}$$

For the sake of clarity, introduce

$$\forall r \geq 0, \quad g(r) := \sup_{\substack{x\in\mathbb{R} \\ |p|\leq r}} \left|\frac{\partial H}{\partial x}(x,p)\right|.$$

The function g is continuous, nondecreasing and nonnegative on $\mathbb{R}^+$. We now define θ to control the contribution of g in (8.2.36). First, set

$$\forall r \geq 0, \quad A(r) := r + \int_r^{r+1} g(v)\,\mathrm{d}v\,.$$

Clearly, $A \in \mathbf{C}^1((0,+\infty))$ and $A' > 0$. Note that

$$\lim_{r\to 0+} A(r) = \int_0^1 g(v)\,\mathrm{d}v := M_1; \quad \lim_{r\to 0+} A'(r) = 1 + g(1) - g(0) := M_2.$$

Moreover, since g is nondecreasing, for all $r \geq 0$, we have

$$A(r) \geq r + g(r).$$

Now, set

$$\forall r < 0, \quad B(r) := 2M_1 - A(-r).$$

By definition, $B \in \mathbf{C}^1((-\infty,0))$, $B' > 0$, we have the limits

$$\lim_{r\to 0-} B(r) = M_1; \quad \lim_{r\to 0-} B'(r) = M_2,$$

and for all $r \leq -2M_1$,

$$B(r) \leq 2M_1 - (-r + g(-r)) = 2M_1 + r - g(-r) \leq -g(-r).$$

Therefore, the function θ' defined by

$$\forall r \in \mathbb{R}, \quad \theta'(r) = \begin{cases} A(r) & \text{if} \quad r \geq 0 \\ B(r) & \text{if} \quad r > 0 \end{cases}$$

verifies: $\theta' \in \mathbf{C}^1(\mathbb{R})$, $\theta'' > 0$ and

$$\forall r \in \mathbb{R}, \quad |r| \geq 2M_1 \implies |\theta'(r)| \geq g(|r|).$$

From (8.2.36), and the computation of the case $|x_0| \geq X$, we obtain:

$$|\partial_x U_\varepsilon(x_0,t_0)| \leq \max\left\{2M_1, \sqrt{2(M^2+1)} + \sup_{\substack{x\in\mathbb{R}\\ |v|\leq 1}} L(x,v)\right\}. \tag{8.2.37}$$

Combining (8.2.33)-(8.2.35)-(8.2.37), we obtain the desired bound. □

Corollary 8.2.12. *Fix $\varepsilon \in (0,1)$ and let $u_\varepsilon \in \mathbf{C}(\overline{\Omega}) \cap \mathbf{C}^2_1(\Omega) \cap \mathbf{L}^\infty(\Omega)$ be a classical solution to (8.2.22) with initial data $u_0 \in \mathbf{W}^{1,\infty}(\mathbb{R})$. Then there exists a constant $\mathbf{C}$, which does not depend on ε nor T such that*

$$\|u_\varepsilon\|_{\mathbf{L}^\infty} \leq \mathbf{C}. \tag{8.2.38}$$

Proof. In light of Lemma 8.2.7 *(i)*, apply Corollary 8.2.11 to

$$U_\varepsilon(x,t) = \int_0^x u_\varepsilon(y,t)\,\mathrm{d}y - \int_0^t \Big(H(0,u_\varepsilon(0,s)) - \varepsilon\partial_x u_\varepsilon(0,s)\Big)\,\mathrm{d}s\,.$$

□

Existence for the viscous equations

Fix $\varepsilon \in (0,1)$ and define the classical heat kernel:

$$\forall (x,t) \in \Omega, \quad \mathbf{K}_\varepsilon(x,t) = \frac{1}{\sqrt{4\pi\varepsilon t}} e^{-\frac{x^2}{4\varepsilon t}}.$$

Recall that

$$\forall t > 0, \quad \int_{\mathbb{R}} \mathbf{K}_\varepsilon(x,t)\,\mathrm{d}x = 1; \quad \int_{\mathbb{R}} \left| \frac{\partial \mathbf{K}_\varepsilon}{\partial x}(x,t) \right| \mathrm{d}x = \frac{1}{\sqrt{\pi\varepsilon t}},$$

see for instance [103, Chapter 7].

Theorem 8.2.13. *Fix $u_0 \in \mathbf{W}^{1,\infty}(\mathbb{R})$. Then (8.2.22) admits a classical solution $u_\varepsilon \in \mathbf{C}(\mathbb{R} \times [0,+\infty)) \cap \mathbf{C}^2_1(\mathbb{R} \times (0,+\infty))$.*

Proof. We split the study in two steps.

Step 1: Local existence. For a continuous function $v \in \mathbf{C}(\Omega)$, define

$$\forall (x,t) \in \Omega, \quad \mathcal{L}v(x,t) = \mathbf{K}_\varepsilon(\cdot,t) * u_0(x) - \int_0^t \int_{\mathbb{R}} \frac{\partial \mathbf{K}_\varepsilon}{\partial x}(x-y,t-s) H(y,v(y,s))\,\mathrm{d}y\,\mathrm{d}s$$

The idea is to prove that $\mathcal{L}$ has a fixed point in a suitable subset of $\mathbf{L}^\infty(\Omega)$.

Let $T^* > 0$ sufficiently small such that

$$\left(\frac{2}{\sqrt{\pi\varepsilon}} \sup_{\substack{x\in\mathbb{R} \\ |p| \le 1 + \|u_0\|_{\mathbf{L}^\infty}}} |H(x,p)| \right) \sqrt{T^*} \le 1; \quad \left(\frac{2}{\sqrt{\pi\varepsilon}} \sup_{\substack{x\in\mathbb{R} \\ |p| \le 1 + \|u_0\|_{\mathbf{L}^\infty}}} \left| \frac{\partial H}{\partial u}(x,p) \right| \right) \sqrt{T^*} \le \frac{1}{2} \tag{8.2.39}$$

and define

$$\mathbf{E} = \left\{ v \in \mathbf{C}(\Omega^*) \ : \ \|v - \mathbf{K}_\varepsilon(\cdot,t) * u_0\|_{\mathbf{L}^\infty(\Omega^*)} \le 1 \right\}; \quad \Omega^* = \mathbb{R} \times (0,T^*).$$

This subset $\mathbf{E}$ is closed in $\mathbf{L}^\infty(\Omega^*)$, therefore complete; $\mathbf{E}$ is invariant under $\mathcal{L}$ since for all $v \in \mathbf{E}$, $\mathcal{L}v \in \mathbf{C}(\Omega^*)$ and

$$\begin{aligned} \forall (x,t) \in \Omega^*, \quad |\mathcal{L}v(x,t) - \mathbf{K}_\varepsilon(\cdot,t) * u_0(x)| &\le \int_0^t \int_{\mathbb{R}} \left| \frac{\partial \mathbf{K}_\varepsilon}{\partial x}(x-y,t-s) H(y,v(y,s)) \right| \mathrm{d}y\,\mathrm{d}s \\ &\le \left(\frac{2}{\sqrt{\pi\varepsilon}} \sup_{\substack{\xi\in\mathbb{R} \\ |p| \le 1 + \|u_0\|_{\mathbf{L}^\infty}}} |H(\xi,p)| \right) \sqrt{T^*} \le 1, \end{aligned}$$

by definition of T^* and the fact that

$$\|v\|_{\mathbf{L}^\infty(\Omega^*)} \le \|v - \mathbf{K}_\varepsilon(\cdot,t) * u_0\|_{\mathbf{L}^\infty(\Omega^*)} + \|\mathbf{K}_\varepsilon(\cdot,t) * u_0\|_{\mathbf{L}^\infty(\Omega^*)} \le 1 + \|u_0\|_{\mathbf{L}^\infty}.$$

Moreover, for all $u, v \in \mathbf{E}$, and for all $(x,t) \in \Omega^*$,

$$\begin{aligned}|\mathcal{L}u(x,t) - \mathcal{L}v(x,t)| &\leq \int_0^t \int_{\mathbb{R}} \left| \frac{\partial \mathbf{K}_\varepsilon}{\partial x}(x-y,t-s)(H(y,u(y,s)) - H(y,v(y,s))) \right| \mathrm{d}y\,\mathrm{d}s \\ &\leq \left(\frac{2}{\sqrt{\pi\varepsilon}} \sup_{\substack{\xi \in \mathbb{R} \\ |p| \leq 1 + \|u_0\|_{\mathbf{L}^\infty}}} \left| \frac{\partial H}{\partial u}(\xi,p) \right| \right) \sqrt{T^*} \|u - v\|_{\mathbf{L}^\infty(\Omega^*)} \\ &\leq \frac{1}{2} \|u - v\|_{\mathbf{L}^\infty(\Omega^*)},\end{aligned}$$

by definition of T^*. Banach fixed point theorem ensures that $\mathcal{L}$ admits a fixed point $u_\varepsilon \in \mathbf{E}$. Consequently,

$$\forall (x,t) \in \Omega^*, \quad u_\varepsilon(x,t) = \mathbf{K}_\varepsilon(\cdot,t) * u_0(x) - \int_0^t \int_{\mathbb{R}} \frac{\partial \mathbf{K}_\varepsilon}{\partial x}(x-y,t-s) H(y,u_\varepsilon(y,s))\,\mathrm{d}y\,\mathrm{d}s\,.$$

Starting from the boundedness of u_ε, the regularity of the heat kernel ensures that $u_\varepsilon \in \mathbf{C}(\overline{\Omega^*}) \cap \mathbf{C}_1^2(\Omega^*)$ and is a classical solution to (8.2.22) with initial data u_0, see for instance [98, Lemma B.3]. Let us make precise that in this step of the reasoning, we only use the fact $u_0 \in \mathbf{C}(\mathbb{R}) \cap \mathbf{L}^\infty(\mathbb{R})$.

Step 2: Global existence. Now, introduce

$$T_m = \sup \left\{ \tau \geq 0 \mid (8.2.22) \text{ admits a classical solution } \mathbf{C}(\mathbb{R} \times [0,\tau]) \cap \mathbf{C}_1^2(\mathbb{R} \times (0,\tau)) \right\}.$$

From Step 1, we know that T_m is well-defined and that $T_m \geq T^*$. Let us prove that $T_m = +\infty$ by supposing that $T_m < +\infty$. Let $\mathbf{C}$ be the constant given by Corollary 8.2.12. Fix $\tau > 0$ such that

$$\left(\frac{2}{\sqrt{\pi\varepsilon}} \sup_{\substack{x \in \mathbb{R} \\ |p| \leq \mathbf{C}+1}} |H(x,p)| \right) \sqrt{\tau} \leq 1; \quad \left(\frac{2}{\sqrt{\pi\varepsilon}} \sup_{\substack{x \in \mathbb{R} \\ |p| \leq \mathbf{C}+1}} \left| \frac{\partial H}{\partial u}(x,p) \right| \right) \sqrt{\tau} \leq \frac{1}{2}. \tag{8.2.40}$$

Set $\tau_m = T_m - \tau/2$. By definition of T_m, we can find a function $u \in \mathbf{C}(\mathbb{R} \times [0,\tau_m]) \cap \mathbf{C}_1^2(\mathbb{R} \times (0,\tau_m))$ which is a classical solution of (8.2.22). From Corollary 8.2.12,

$$\|u\|_{\mathbf{L}^\infty(\mathbb{R} \times (0,\tau_m))} \leq \mathbf{C}.$$

Following the reasoning of Step 1, since $u(\cdot,\tau_m) \in \mathbf{C}(\mathbb{R}) \cap \mathbf{L}^\infty(\mathbb{R})$, we can construct a classical solution u_τ to

$$\begin{cases} \partial_t v + \partial_x (H(x,v)) = \varepsilon \partial_{xx}^2 v \\ v(\cdot,\tau_m) = u(\cdot,\tau_m)\,; \end{cases} \quad u_\tau \in \mathbf{C}(\mathbb{R} \times [\tau_m, \tau_m + \tau]) \cap \mathbf{C}_1^2(\mathbb{R} \times (\tau_m, \tau_m + \tau))\,.$$

The concatenation

$$v(\cdot,t) = \begin{cases} u(\cdot,t) & \text{if} \quad 0 \leq t \leq \tau_m \\ u_\tau(\cdot,t) & \text{if} \quad \tau_m < t \leq \tau_m + \tau \end{cases}$$

is a bounded function which verifies for all $(x,t) \in \mathbb{R} \times (0, \tau_m + \tau)$,

$$v(x,t) = \mathbf{K}_\varepsilon(\cdot,t) * u_0(x) - \int_0^t \int_{\mathbb{R}} \frac{\partial \mathbf{K}_\varepsilon}{\partial x}(x-y, t-s) H(y, v(y,s)) \, \mathrm{d}y \, \mathrm{d}s \, .$$

This implies, once again, that v is classical solution to

$$\begin{cases} \partial_t v + \partial_x \left(H(x,v) \right) = \varepsilon \partial^2_{xx} v \\ v(\cdot, 0) = u_0; \end{cases} ; \quad v \in \mathbf{C}\left(\mathbb{R} \times [0, \tau_m + \tau] \right) \cap \mathbf{C}^2_1 \left(\mathbb{R} \times (0, \tau_m + \tau) \right).$$

Therefore, by definition of T_m, we have $\tau_m \leq T_m$ *i.e.* $T_m + \tau/2 \leq T_m$. This contradiction proves that $T_m = +\infty$. □

Corollary 8.2.14. *Fix $U_0 \in \mathbf{C}^1(\mathbb{R})$ such that $U_0' \in \mathbf{W}^{1,\infty}(\mathbb{R})$. Then (8.2.23) admits a classical solution $U_\varepsilon \in \mathbf{C}(\mathbb{R} \times [0, +\infty)) \cap \mathbf{C}^3_1(\mathbb{R} \times (0, +\infty))$.*

Proof. Immediate in light of Lemma 8.2.7 *(i)*. □

Convergence and existence results

Theorem 8.2.15. *Fix $U_0 \in \mathbf{C}^1(\mathbb{R})$ such that $U_0' \in \mathbf{W}^{1,\infty}(\mathbb{R})$. Then:*
(i) for all $\varepsilon \in (0,1)$, the equation (8.2.23) admits a classical solution $U_\varepsilon \in \mathbf{C}(\overline{\Omega}) \cap \mathbf{C}^3_1(\Omega)$;

(ii) there is a subsequence of $(U_\varepsilon)_\varepsilon$ which converges uniformly on compact subsets of Ω to some $U \in \mathbf{C}(\Omega)$;

(iii) the function U is a viscosity solution to (8.0.2) with initial data U_0.

Proof. Point *(i)* comes from Corollary 8.2.14. Using the *a priori* bound of Theorem 8.2.11, Arzelà-Ascoli theorem (see [36, Theorem 4.25]) combined with a standard diagonal process yields the existence of a subsequence of $(U_\varepsilon)_\varepsilon$ which converges uniformly on compact subsets of Ω to some $U \in \mathbf{C}(\Omega)$. The fact that U is a viscosity solution to (8.0.2) is standard, see for instance [81, Chapter 10]. □

Theorem 8.2.16. *Suppose that $H \in \mathbf{C}^2(\mathbb{R}^2)$ verifies (8.0.3)-(8.1.2). Then for any initial data $U_0 \in \mathbf{C}^1(\mathbb{R})$ such that $U_0' \in \mathbf{W}^{1,\infty}(\mathbb{R})$, the Hamilton-Jacobi equation (8.0.2) admits a unique viscosity solution.*

Proof. Existence comes from Theorem 8.2.15 while uniqueness was proved in Theorem 8.2.6. □

Note that existence of a viscosity solution to (8.0.2) under assumptions (8.0.3)-(8.1.2) and with $U_0 \in \mathbf{Lip}(\mathbb{R})$ is obtained in Section 8.3 through the calculus of variations approach, see Theorem 8.3.12.

Theorem 8.2.17. *Fix $u_0 \in \mathbf{W}^{1,\infty}(\mathbb{R})$. Then:*
(i) for all $\varepsilon \in (0,1)$, the equation (8.2.22) admits a classical solution $u_\varepsilon \in \mathbf{C}(\overline{\Omega}) \cap \mathbf{C}^2_1(\Omega)$;

(ii) there is a subsequence of $(u_\varepsilon)_\varepsilon$ which converges a.e. on Ω to some $\overline{u} \in \mathbf{L}^\infty(\Omega)$;

(iii) the function $\overline{u}$ is an entropy solution to (8.0.1) with initial data u_0.

Proof. *(i)* It comes from Theorem 8.2.13.
(ii) The strong compactness of $(u_\varepsilon)_\varepsilon$ will follow from a compensated compactness reasoning, see [67, Chapter 16], [80, Chapter 5] or [145, Chapter 9]. The reasoning follows four steps.
Step 1. First, we prove that $(\sqrt{\varepsilon}|\partial_x u_\varepsilon|)_\varepsilon$ is bounded in $\mathbf{L}^2_{\mathbf{loc}}(\Omega)$.
For all $\varepsilon \in (0,1)$ and for all test functions $\varphi \in \mathbf{C}^1_{\mathbf{c}}(\Omega)$, we have for all $t \in (0,T)$,

$$\int_{\mathbb{R}} \partial_t u_\varepsilon(x,t)\varphi \,\mathrm{d}x + \varepsilon \int_{\mathbb{R}} \partial_x u_\varepsilon(x,t)\partial_x\varphi(x,t)\,\mathrm{d}x = -\int_{\mathbb{R}} \partial_x(H(x,u_\varepsilon(x,t)))\varphi \,\mathrm{d}x\,. \tag{8.2.41}$$

Fix $R>0$ and $\phi \in \mathbf{C}^\infty_{\mathbf{c}}(\mathbb{R}), \phi \geq 0$ such that $\phi \equiv 1$ on $[-R,R]$ and $\phi(x)=0$ if $|x| \geq R+1$. Let us apply (8.2.41) with $\varphi = u_\varepsilon \phi^2$. We obtain:

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\mathbb{R}} (u_\varepsilon\phi)^2\,\mathrm{d}x\right) + \int_{\mathbb{R}} |\sqrt{\varepsilon}\partial_x(u_\varepsilon\phi)|^2\,\mathrm{d}x \\ &= \varepsilon\int_{\mathbb{R}} |u_\varepsilon\phi'|^2\,\mathrm{d}x - \int_{\mathbb{R}} \partial_x(H(x,u_\varepsilon))u_\varepsilon\phi^2\,\mathrm{d}x\,. \end{aligned} \tag{8.2.42}$$

Now, we introduce the function

$$\forall (x,t)\in\Omega, \quad f_\varepsilon(x,t) = \int_0^{u_\varepsilon(x,t)} v\frac{\partial H}{\partial u}(x,v)\,\mathrm{d}v\,.$$

Note that $(f_\varepsilon)_\varepsilon$ is bounded in $\mathbf{L}^\infty(\Omega)$ and that

$$\begin{aligned} \partial_x f_\varepsilon(x,t) &= \partial_x u_\varepsilon(x,t)\frac{\partial H}{\partial u}(x,u_\varepsilon(x,t))u_\varepsilon(x,t) + \int_0^{u_\varepsilon(x,t)} v\frac{\partial^2 H}{\partial x\partial u}(x,v)\,\mathrm{d}v \\ &= \partial_x(H(x,u_\varepsilon(x,t)))u_\varepsilon(x,t) - \frac{\partial H}{\partial x}(x,u_\varepsilon(x,t))u_\varepsilon(x,t) + \int_0^{u_\varepsilon(x,t)} v\frac{\partial^2 H}{\partial x\partial u}(x,v)\,\mathrm{d}v\,. \end{aligned} \tag{8.2.43}$$

Integrating (8.2.42) on $t\in(0,T)$ yields:

$$\begin{aligned} \int_0^T\int_{\mathbb{R}} |\sqrt{\varepsilon}\partial_x(u_\varepsilon\phi)|^2\,\mathrm{d}x\,\mathrm{d}t \leq \frac{1}{2}\int_{\mathbb{R}} |u_0\phi|^2\,\mathrm{d}x + \varepsilon\int_0^T\int_{\mathbb{R}} |u_\varepsilon\phi'|^2\,\mathrm{d}x\,\mathrm{d}t \\ - \int_0^T\int_{\mathbb{R}}\left(\frac{\partial H}{\partial x}(x,u_\varepsilon)u_\varepsilon - \int_0^{u_\varepsilon(x,t)} v\frac{\partial^2 H}{\partial x\partial u}(x,v)\,\mathrm{d}v\right)\phi^2\,\mathrm{d}x\,\mathrm{d}t \\ - \int_0^T\int_{\mathbb{R}} \partial_x f_\varepsilon\phi^2\,\mathrm{d}x\,\mathrm{d}t\,. \end{aligned} \tag{8.2.44}$$

Using (8.2.43), an integration by parts gives us:

$$\left|\int_0^T\int_{\mathbb{R}} \partial_x f_\varepsilon(x,t)\phi^2(x)\,\mathrm{d}x\,\mathrm{d}t\right| \leq 2\sup_{\varepsilon>0}\|f_\varepsilon\|_{\mathbf{L}^\infty(\Omega)}\int_0^T\int_{\mathbb{R}} \phi(x)|\phi'(x)|\,\mathrm{d}x\,\mathrm{d}t\,.$$

From (8.2.43), this last estimate and the boundedness of $(u_\varepsilon)_\varepsilon$ in $\mathbf{L}^\infty(\Omega)$, we deduce that $(\sqrt{\varepsilon}|\partial_x u_\varepsilon|)_\varepsilon$ is bounded in $\mathbf{L}^2((-R,R)\times(0,T))$. The arbitrariness on $R>0$ ensures that

$(\sqrt{\varepsilon}|\partial_x u_\varepsilon|)_\varepsilon$ is bounded in $\mathbf{L}^2_{\mathbf{loc}}(\Omega)$.

Let us also make precise that for all $\eta \in \mathbf{C}^1(\mathbb{R})$, $(\varepsilon\partial_{xx}(\eta(u_\varepsilon)))_\varepsilon$ lies in a compact subset of $\mathbf{H}^{-1}_{\mathbf{loc}}(\Omega)$ for the strong topology since

$$\forall \varepsilon > 0, \quad |\varepsilon\partial_x(\eta(u_\varepsilon))| = \sqrt{\varepsilon}|\partial_x u_\varepsilon| \times \sqrt{\varepsilon}\eta'(u_\varepsilon) \underset{\varepsilon\to 0}{\longrightarrow} 0 \text{ in } \mathbf{L}^2_{\mathbf{loc}}(\Omega).$$

Step 2. Towards the divergence-curl lemma. Let us ensure that we are in position to apply the divergence-curl lemma. Fix $\eta \in \mathbf{C}^2(\mathbb{R})$ a convex function. Define its space-dependent entropy flux Φ:

$$\forall x, u \in \mathbb{R}, \quad \Phi(x,u) = \int_0^u \eta'(v)\frac{\partial H}{\partial u}(x,v)\,\mathrm{d}v\,.$$

By multiplying the PDE (8.2.22) by $\eta'(u_\varepsilon)$, standard computations lead to

$$\begin{aligned}\partial_t(\eta(u_\varepsilon)) + \partial_x\left(\Phi(x,u_\varepsilon)\right) = &\underbrace{\int_0^{u_\varepsilon} \eta'(v)\frac{\partial^2 H}{\partial x \partial u}(x,v)\,\mathrm{d}v - \eta'(u_\varepsilon)\frac{\partial H}{\partial x}(x,u_\varepsilon)}_{v_\varepsilon} \\ &+ \varepsilon\partial_{xx}\left(\eta(u_\varepsilon)\right) - \varepsilon\eta''(u_\varepsilon)(\partial_x u_\varepsilon)^2.\end{aligned} \tag{8.2.45}$$

— Since the sequences $(\eta(u_\varepsilon))_\varepsilon$ and $(\Phi(\cdot,u_\varepsilon))_\varepsilon$ are bounded in $\mathbf{L}^\infty(\Omega)$, we know that

$$(\partial_t(\eta(u_\varepsilon)) + \partial_x\left(\Phi(x,u_\varepsilon)\right))_\varepsilon \text{ is bounded in } \mathbf{W}^{-1,\infty}(\Omega).$$

— The sequence $(v_\varepsilon)_\varepsilon$ is bounded in $\mathbf{L}^1_{\mathbf{loc}}(\Omega)$ since $(u_\varepsilon)_\varepsilon$ is bounded in $\mathbf{L}^\infty(\Omega)$. Therefore, $(v_\varepsilon)_\varepsilon$ is bounded in the space of bounded measures, $\mathcal{M}(U)$ for every bounded open set U.

— Similarly, Step 1 ensures that $(\varepsilon|\partial_x u_\varepsilon|^2)_\varepsilon$ is bounded $\mathbf{L}^1_{\mathbf{loc}}(\Omega)$ and therefore, bounded in $\mathcal{M}(U)$ as well.

— Recall that $(\varepsilon\partial_{xx}(\eta(u_\varepsilon)))_\varepsilon$ lies in a compact subset of $\mathbf{H}^{-1}_{\mathbf{loc}}(\Omega)$.

To summarize, we wrote $(\partial_t(\eta(u_\varepsilon)) + \partial_x\left(\Phi(x,u_\varepsilon)\right))_\varepsilon$, a sequence bounded in $\mathbf{W}^{-1,\infty}(\Omega)$ as the sum of a sequence which is bounded in the space of measures $\mathcal{M}_{\mathbf{loc}}(\Omega)$ and of a sequence which is compact in $\mathbf{H}^{-1}_{\mathbf{loc}}(\Omega)$. From [145, Lemma 9.2.1], we deduce that $(\partial_t(\eta(u_\varepsilon)) + \partial_x\left(\Phi(x,u_\varepsilon)\right))_\varepsilon$ lies in a compact subset of $\mathbf{H}^{-1}_{\mathbf{loc}}(\Omega)$; and this holds for any convex function $\eta \in \mathbf{C}^2(\mathbb{R})$, with entropy flux Φ. A standard approximation argument ensures that we can choose $\eta \in \mathbf{C}(\mathbb{R})$, see the Remarks of [145, Proposition 9.2.2].

Step 3. Application of the divergence-curl lemma. Let us introduce the sequence $(\nu_{x,t})_{(x,t)\in\Omega}$ of Young measures associated to $(u_\varepsilon)_\varepsilon$, which comes from the uniform $\mathbf{L}^\infty$ bound, see Corollary 8.2.12. This means that for any continuous function $f \in \mathbf{C}(\mathbb{R})$,

$$f(u_\varepsilon) \rightharpoonup \overline{f} \text{ in } \mathbf{L}^\infty - w^*; \quad \overline{f}(x,t) = \int_{\mathbb{R}} f(y)\,\mathrm{d}\nu_{x,t}(y). \tag{8.2.46}$$

In particular, $(u_\varepsilon)_\varepsilon$ converges $\mathbf{L}^\infty - w^*$ to $\overline{u}$, where for a.e. $(x,t) \in \Omega$,

$$\overline{u}(x,t) = \int_{\mathbb{R}} y\,\mathrm{d}\nu_{x,t}(y).$$

Let us extend (8.2.46) for continuous functions $F \in \mathbf{C}(\mathbb{R}^2)$ having the property of compact space dependency, (8.0.3). Set $M := \sup_{\varepsilon>0} \|u_\varepsilon\|_{\mathbf{L}^\infty(\Omega)} < +\infty$ and fix $n \in \mathbb{N}^*$. Stone-Weierstrass theorem provides a number $N \in \mathbb{N}^*$ and sequences $(f_k)_{1\le k\le N}$, $(g_k)_{1\le k\le N}$ such that:

$$g_k \in \mathbf{C}([-X,X]); \quad f_k \in \mathbf{C}([-M,M]); \quad \left\| F - \sum_{k=1}^N g_k f_k \right\|_{\mathbf{L}^\infty([-X,X]\times[-M,M])} \le \frac{1}{n}.$$

For all $k \in \{1,\dots,N\}$, let us extend g_k by

$$\forall x \in \mathbb{R}, \quad G_k(x) = \begin{cases} g_k(-X) & \text{if} \quad x < -X \\ g_k(x) & \text{if} \quad |x| \le X \\ g_k(X) & \text{if} \quad x > X \end{cases}$$

so that

$$G_k \in \mathbf{C}(\mathbb{R}); \quad f_k \in \mathbf{C}([-M,M]); \quad \left\| F - \sum_{k=1}^N G_k f_k \right\|_{\mathbf{L}^\infty(\mathbb{R}\times[-M,M])} \le \frac{1}{n}. \tag{8.2.47}$$

Fix $\varphi \in \mathbf{L}^1(\Omega)$ and $k \in \mathbb{N}$. Since $G_k \in \mathbf{L}^\infty(\mathbb{R})$, $G_k\varphi \in \mathbf{L}^1(\Omega)$. We deduce that

$$\begin{aligned} \int_0^T \int_{\mathbb{R}} G_k(x) f_k(u_\varepsilon(x,t))\varphi(x,t)\,\mathrm{d}x\,\mathrm{d}t &= \int_0^T \int_{\mathbb{R}} f_k(u_\varepsilon(x,t)) G_k(x)\varphi(x,t)\,\mathrm{d}x\,\mathrm{d}t \\ &\underset{\varepsilon\to 0}{\longrightarrow} \int_0^T \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f_k(y)\,\mathrm{d}\nu_{x,t}(y) \right) G_k(x)\varphi(x,t)\,\mathrm{d}x\,\mathrm{d}t \\ &= \int_0^T \int_{\mathbb{R}} \left(\int_{\mathbb{R}} G_k(x) f_k(y)\,\mathrm{d}\nu_{x,t}(y) \right) \varphi(x,t)\,\mathrm{d}x\,\mathrm{d}t\,. \end{aligned}$$

By linearity,

$$\sum_{k=1}^N \int_0^T \int_{\mathbb{R}} G_k(x) f_k(u_\varepsilon(x,t))\varphi(x,t)\,\mathrm{d}x\,\mathrm{d}t \underset{\varepsilon\to 0}{\longrightarrow} \int_0^T \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \sum_{k=1}^N G_k(x) f_k(y)\,\mathrm{d}\nu_{x,t}(y) \right) \varphi(x,t)\,\mathrm{d}x\,\mathrm{d}t\,.$$

Using (8.2.47), straightforward computations ensure that for any continuous function $F \in \mathbf{C}(\mathbb{R}^2)$ verifying (8.0.3), the sequence $(x,t) \mapsto F(x,u_\varepsilon(x,t))$ converges in $\mathbf{L}^\infty - w^*$ to $\overline{F}$ where for a.e. $(x,t) \in \Omega$,

$$\overline{F}(x,t) = \int_{\mathbb{R}} F(x,y)\,\mathrm{d}\nu_{x,t}(y). \tag{8.2.48}$$

Now fix $\kappa \in \mathbb{R}$, $|\kappa| \le M$. Let us apply the divergence-curl lemma, see [80, Theorem 5.2.1] with $\eta_1(u) = u$, $\Phi_1(x,u) = H(x,u)$ and

$$\eta_2(u) = |u-\kappa|; \quad \Phi_2(x,u) = \operatorname{sgn}(u-\kappa)(H(x,u) - H(x,\kappa)).$$

Using (8.2.48), we obtain that for a.e. $(x,t)\in\Omega$,

$$\int_{\mathbb{R}} y\Phi_2(x,y)\,\mathrm{d}\nu_{x,t}(y) - \int_{\mathbb{R}} |y-\kappa|H(x,y)\,\mathrm{d}\nu_{x,t}(y)$$
$$= \overline{u}(x,t)\int_{\mathbb{R}}\int_{\mathbb{R}} \Phi_2(x,y)\,\mathrm{d}\nu_{x,t}(y) - \overline{H}(x,t)\int_{\mathbb{R}}\int_{\mathbb{R}} |y-\kappa|\,\mathrm{d}\nu_{x,t}(y).$$

Specializing with $\kappa=\overline{u}(x,t)$, we get

$$\left(\overline{H}(x,t) - H(x,\overline{u}(x,t))\right)\int_{\mathbb{R}} |y-\overline{u}(x,t)|\,\mathrm{d}\nu_{x,t}(y) = 0.$$

This equality implies that either $\overline{H}(x,t) = H(x,\overline{u}(x,t))$ or $\nu_{x,t}=\delta_{\overline{u}(x,t)}$; in any case, we proved that for a.e. $(x,t)\in\Omega$, $\overline{H}(x,t)=H(x,\overline{u}(x,t))$.

Step 4. Strong convergence. Let us conclude. For a.e. $(x,t)\in\Omega$, we have, using Jensen inequality,

$$H\left(x,\overline{u}(x,t)\right) = H\left(x,\int_{\mathbb{R}} y\,\mathrm{d}\nu_{x,t}(y)\right)$$
$$\leq \int_{\mathbb{R}} H(x,y)\,\mathrm{d}\nu_{x,t}(y) = \overline{H}(x,t) = H\left(x,\overline{u}(x,t)\right).$$

Since for all $x\in\mathbb{R}$, $H(x,\cdot)$ is strictly convex, the function $y\mapsto y$ is constant $\nu_{x,t}$-a.e. and consequently, for a.e. $(x,t)\in\Omega$, $\nu_{x,t}=\delta_{\alpha(x,t)}$ for some function $\alpha:\Omega\to\mathbb{R}$. Finally, for all $\varepsilon>0$, and for all bounded open subsets $U\subset\Omega$,

$$\|u_\varepsilon\|^2_{\mathbf{L}^2(U)} = \int_0^T\int_{\mathbb{R}} u_\varepsilon^2 \mathbb{1}_U\,\mathrm{d}x\,\mathrm{d}t$$
$$\underset{\varepsilon\to 0}{\longrightarrow} \int_0^T\int_{\mathbb{R}}\left(\int_{\mathbb{R}} y^2\,\mathrm{d}\nu_{x,t}(y)\right)\mathbb{1}_U\,\mathrm{d}x\,\mathrm{d}t$$
$$= \int_0^T\int_{\mathbb{R}} \alpha(x,t)^2\mathbb{1}_U\,\mathrm{d}x\,\mathrm{d}t = \|\overline{u}\|^2_{\mathbf{L}^2(U)},$$

which implies that $u_\varepsilon\to\overline{u}$ in $\mathbf{L}^2_{\mathbf{loc}}(\Omega)$, see [80, Theorem 1.1.1]. A standard diagonal process yields a subsequence of $(u_\varepsilon)_\varepsilon$ that converges a.e. on Ω to $\overline{u}$.

(iii) Fix $\eta\in\mathbf{C}^2(\mathbb{R})$ a convex function and $\Phi=\Phi(x,u)$ its entropy flux:

$$\forall x,u\in\mathbb{R},\quad \frac{\partial\Phi}{\partial u}(x,u) = \eta'(u)\frac{\partial H}{\partial u}(x,u).$$

Fix $\varepsilon>0$. Starting from (8.2.22), standard computations lead to:

$$\partial_t(\eta(u_\varepsilon)) + \partial_x(\Phi(x,u_\varepsilon)) + \eta'(u_\varepsilon)\partial_x H(x,u_\varepsilon) - \partial_x\Phi(x,u_\varepsilon) \leq \varepsilon\partial^2_{xx}\eta(u_\varepsilon). \tag{8.2.49}$$

Fix now a test function $\varphi\in\mathbf{C}^\infty_{\mathbf{c}}(\mathbb{R}\times\mathbb{R}^+)$, $\varphi\geq 0$. Multiply (8.2.49) by φ and integrate by

parts to obtain:

$$\begin{aligned}&\int_0^{+\infty}\int_{\mathbb{R}}\Big(\eta(u_\varepsilon)\partial_t\varphi+\Phi(x,u_\varepsilon)\partial_x\varphi\Big)\,\mathrm{d}x\,\mathrm{d}t\\&-\int_0^{+\infty}\int_{\mathbb{R}}\Big(\eta'(u_\varepsilon)\partial_x H(x,u_\varepsilon)-\partial_x\Phi(x,u_\varepsilon)\Big)\varphi\,\mathrm{d}x\,\mathrm{d}t\\&+\int_{\mathbb{R}}\eta(u_0(x))\varphi(x,0)\,\mathrm{d}x\geq 0.\end{aligned}\tag{8.2.50}$$

Now make use of the a.e. convergence of $(u_\varepsilon)_\varepsilon$ and the uniform $\mathbf{L}^\infty$ bound (8.2.38) to apply Lebesgue theorem to pass to the limit in (8.2.50):

$$\begin{aligned}&\int_0^{+\infty}\int_{\mathbb{R}}\Big(\eta(\overline{u})\partial_t\varphi+\Phi(x,\overline{u})\partial_x\varphi\Big)\,\mathrm{d}x\,\mathrm{d}t\\&-\int_0^{+\infty}\int_{\mathbb{R}}\Big(\eta'(\overline{u})\partial_x H(x,\overline{u})-\partial_x\Phi(x,\overline{u})\Big)\varphi\,\mathrm{d}x\,\mathrm{d}t\\&+\int_{\mathbb{R}}\eta(u_0(x))\varphi(x,0)\,\mathrm{d}x\geq 0.\end{aligned}\tag{8.2.51}$$

Inequality (8.2.51) is valid for any convex function $\eta\in\mathbf{C}^2(\mathbb{R})$. A standard approximation argument ensures that it holds for any convex function $\eta\in\mathbf{W}^{1,\infty}(\mathbb{R})$. In particular, with $\eta=|\cdot-\kappa|$ ($\kappa\in\mathbb{R}$), we obtain that $\overline{u}$ is an entropy solution to (8.0.1) with initial data u_0 in the sense of Definition 8.2.1. □

Remark that Theorems 8.2.4-8.2.17 provide a well-posedness result for (8.0.1).

Theorem 8.2.18. *Suppose that $H\in\mathbf{C}^2(\mathbb{R}^2)$ verifies (8.0.3)-(8.1.2). Then for any initial data $u_0\in\mathbf{W}^{1,\infty}(\mathbb{R})$, the conservation law (8.0.1) admits a unique entropy solution.*

8.2.3 Correspondence

Theorem 8.2.19. *Fix $u_0\in\mathbf{W}^{1,\infty}(\mathbb{R})$ and suppose that $H\in\mathbf{C}^2(\mathbb{R}^2)$ satisfies (8.0.3)-(8.1.2). Then:*

(i) the conservation law (8.0.1) admits a unique entropy solution with initial data u_0, denoted by u;

(ii) the Hamilton-Jacobi equation (8.0.2) admits a unique viscosity solution with initial data $U_0(x)=\displaystyle\int_0^x u_0(y)\,\mathrm{d}y$, denoted by U;

(iii) $\partial_x U=u$ in $\mathbf{L}^\infty(\Omega)$.

Proof. Points *(i)-(ii)* comes from Theorems 8.2.18-8.2.16.
(iii) Let $(U_\varepsilon)_\varepsilon$ be the sequence of classical solutions to (8.2.23) constructed in Corollary 8.2.14, which converges uniformly on compact subsets of Ω to U (see Theorem 8.2.15). Likewise, let $(u_\varepsilon)_\varepsilon$ the sequence of classical solutions to (8.2.22) constructed in Theorem 8.2.13, which converges a.e. on Ω to the entropy solution $u\in\mathbf{L}^\infty(\Omega)$ to (8.0.1). Note that for all $\varepsilon\in(0,1)$,

Lemma 8.2.7 ensures that we have $\partial_x U_\varepsilon = u_\varepsilon$. We now prove that $\partial_x U = u$. Fix $\phi \in \mathbf{C}^\infty_{\mathbf{c}}(\Omega)$. Then we have

$$\begin{aligned}\int_0^T \int_{\mathbb{R}} (u\phi)\,\mathrm{d}x\,\mathrm{d}t &= \lim_{\varepsilon\to 0} \int_0^T \int_{\mathbb{R}} (u_\varepsilon \phi)\,\mathrm{d}x\,\mathrm{d}t \\ &= \lim_{\varepsilon\to 0} \int_0^T \int_{\mathbb{R}} (\partial_x U_\varepsilon \phi)\,\mathrm{d}x\,\mathrm{d}t \\ &= -\lim_{\varepsilon\to 0} \int_0^T \int_{\mathbb{R}} (U_\varepsilon \partial_x \phi)\,\mathrm{d}x\,\mathrm{d}t \\ &= -\int_0^T \int_{\mathbb{R}} (U \partial_x \phi)\,\mathrm{d}x\,\mathrm{d}t = \int_0^T \int_{\mathbb{R}} (\partial_x U \phi)\,\mathrm{d}x\,\mathrm{d}t\,,\end{aligned}$$

which proves the statement. Notice that the convergence of $(u_\varepsilon)_\varepsilon$ to u in $\mathbf{L}^\infty(\Omega) - w^*$ would have suffice to make the previous computations. □

We conclude this section by extending the correspondence statement for less regular initial data.

Corollary 8.2.20. *Fix $u_0 \in \mathbf{L}^1(\mathbb{R}) \cap \mathbf{L}^\infty(\mathbb{R})$ and suppose that $H \in \mathbf{C}^2(\mathbb{R}^2)$ satisfies (8.0.3)-(8.1.2). Then the conclusions of Theorem 8.2.19 hold.*

Proof. Let $(u_0^{(k)})_{k\in\mathbb{N}} \subset \mathbf{C}^\infty_{\mathbf{c}}(\mathbb{R})$ such that:

$$u_0^{(k)} \underset{k\to+\infty}{\longrightarrow} u_0 \text{ in } \mathbf{L}^1(\mathbb{R}); \quad \underbrace{\sup_{k\in\mathbb{N}} \|u_0^{(k)}\|_{\mathbf{L}^\infty}}_{:=M} < +\infty.$$

Define

$$\forall k \in \mathbb{N},\ \forall x \in \mathbb{R}, \quad U_0^{(k)}(x) = \int_0^x u_0^{(k)}(y)\,\mathrm{d}y\,.$$

Remark that for all $k \in \mathbb{N}$, $u_0^{(k)} \in \mathbf{W}^{1,\infty}(\mathbb{R})$ and therefore, $U_0^{(k)} \in \mathbf{C}^1(\mathbb{R})$ and $(U_0^{(k)})' \in \mathbf{W}^{1,\infty}(\mathbb{R})$. Using Theorem 8.2.18 and Theorem 8.2.16, for all $k \in \mathbb{N}$, we denote by u_k (resp. U_k) the entropy solution to (8.0.1) (resp. the viscosity solution to (8.0.2)) with initial data $u_0^{(k)}$ (resp. $U_0^{(k)}$). Note that Theorem 8.2.19 ensures that for all $k \in \mathbb{N}$, $\partial_x U_k = u_k$ in $\mathbf{L}^\infty(\Omega)$. We will use the uniform bound obtained from Theorem 8.2.2. For all $k \in \mathbb{N}$, we have:

$$\|\partial_x U_k\|_{\mathbf{L}^\infty} = \|u_k\|_{\mathbf{L}^\infty} \leq \sup_{\substack{y\in\mathbb{R} \\ |p|\leq \|u_0^{(k)}\|_{\mathbf{L}^\infty}}} H(y,p) + \sup_{\substack{y\in\mathbb{R} \\ |v|\leq 1}} L(y,v) \leq \sup_{\substack{y\in\mathbb{R} \\ |p|\leq M}} H(y,p) + \sup_{\substack{y\in\mathbb{R} \\ |v|\leq 1}} L(y,v). \tag{8.2.52}$$

(i) From the stability result (see Theorem 8.2.4 with $R \to +\infty$), we deduce that

$$\forall p, q \in \mathbb{N}, \quad \sup_{t\in[0,T]} \|u_p(\cdot,t) - u_q(\cdot,t)\|_{\mathbf{L}^1} \leq \|u_0^{(p)} - u_0^{(q)}\|_{\mathbf{L}^1}.$$

This estimate yields the existence of $u \in \mathbf{L}^\infty((0,T);\mathbf{L}^1(\mathbb{R}))$ such that $(u_k)_k$ converges to u in $\mathbf{L}^\infty((0,T);\mathbf{L}^1(\mathbb{R}))$. Up to the extraction of a subsequence, we can assume that the

convergence is a.e. on Ω. Combined with (8.2.52), we deduce that u is an entropy solution to (8.0.1) with initial data u_0. Since $u_0 \in \mathbf{L}^\infty(\mathbb{R})$, u is the entropy solution to this Cauchy problem.

(ii) From the equality, licit for a.e. $x \in \mathbb{R}$,

$$\forall t, \tau \in [0,T]\ (\tau < t), \quad U_k(x,t) - U_k(x,\tau) = \int_\tau^t H(x, \partial_x U_k(x,s))\,\mathrm{d}s$$

and the bound (8.2.52), we deduce that $(U_k)_k$ is bounded in $\mathbf{W}^{1,\infty}_{\mathbf{loc}}(\Omega)$. The compact embedding of $\mathbf{W}^{1,\infty}(\mathbf{U})$ in $\mathbf{C}(\overline{\mathbf{U}})$ ($\mathbf{U} \subset \Omega$ a bounded open subset, see [36, Theorem 9.16]) combined with a standard diagonal process ensures the existence of subsequence of $(U_k)_k$ which converges uniformly on compact subsets of Ω to some $U \in \mathbf{Lip}(\Omega)$. U is a viscosity solution to (8.0.2) with initial data U_0, see [28, Chapter 2] or [81, Chapter 10].

(iii) With the convergences obtained above, we conclude that $\partial_x U = u$ with the same argument used in the proof of Theorem 8.2.19. □

Remark 8.2.4. The extension of Corollary 8.2.20 for merely bounded initial data $u_0 \in \mathbf{L}^\infty(\mathbb{R})$ follows by a standard approximation by $\mathbf{L}^1$ functions using the finite speed of propagation of (8.0.1), highlighted by estimate (8.2.5).

8.3 Calculus of variations

We focus on the calculus of variations problem associated to the Hamilton-Jacobi equation (8.0.2). For all $t \geq 0$, introduce the functional

$$\mathcal{J}_t : \left| \begin{array}{ccc} \mathbf{W}^{1,1}((0,t)) & \longrightarrow & \mathbb{R} \\ y & \longmapsto & \displaystyle\int_0^t L(y(s), \dot{y}(s))\,\mathrm{d}s + U_0(y(0)), \end{array} \right.$$

where $U_0 \in \mathbf{Lip}(\mathbb{R})$. In this section, we are interested in minimizing $\mathcal{J}_t$ and make the connection to the Hamilton-Jacobi equation (8.0.2). Let us also mention that even though we are not precisely in the framework covered by the authors of [46, Chapter 5], [81, Chapter 3] or [53, Part III], some ideas and techniques we use in the next sections are reminiscent of the ones of these authors.

8.3.1 Existence of a minimum arc

In light of Remark 8.1.1, L is bounded by below. In this section, and only in this section, we will assume that $L \geq 0$, only for the sake of clarity. This simply reduces to translate U_0 of a constant, which does not alter its Lipschitz continuity.

We start this section with a result on the functional $\mathcal{J}_t$ ($t > 0$) which resembles lower semicontinuity. Recall that functions of $\mathbf{W}^{1,1}((0,t))$ ($t > 0$) are called absolutely continuous, see [36, Remarks of Proposition 8.3].

Lemma 8.3.1. *Fix $t > 0$ and $U_0 \in \mathbf{Lip}(\mathbb{R})$. Let $(y_n)_n$ be a sequence of $\mathbf{W}^{1,1}((0,t))$ and $y \in \mathbf{W}^{1,1}((0,t))$ such that*

$$y_n \underset{n\to+\infty}{\longrightarrow} y \text{ uniformly on } [0,t] \quad \text{and} \quad \dot{y}_n \underset{n\to+\infty}{\longrightarrow} \dot{y} \text{ weakly in } \mathbf{L}^1((0,t)).$$

Then

$$\liminf_{n\to+\infty} \mathcal{J}_t(y_n) \geq \mathcal{J}_t(y).$$

Proof. This is a consequence of the convexity of L with respect to the second variable. For all $k \in \mathbb{N}$, introduce the subset

$$E_k = \{s \in (0,t) \ : \ |\dot{y}(s)| \leq k\}.$$

The idea is to first work on the $(E_k)_k$ subsets and then take the limit as $k \to +\infty$. For all $k, n \in \mathbb{N}$, we have

$$\begin{aligned}
\mathcal{J}_t(y_n) &= \int_0^t L(y_n(s), \dot{y}_n(s))\,\mathrm{d}s + U_0(y_n(0)) \\
&\geq \int_{E_k} L(y_n(s), \dot{y}_n(s))\,\mathrm{d}s + U_0(y_n(0)) \qquad (8.3.1)\\
&\geq \int_{E_k} L(y_n(s), \dot{y}(s))\,\mathrm{d}s + \int_{E_k} (\dot{y}_n(s) - \dot{y}(s)) \frac{\partial L}{\partial v}(y_n(s), \dot{y}(s))\,\mathrm{d}s + U_0(y_n(0)),
\end{aligned}$$

where the last inequality comes from the convexity of L with respect to v. Fix $k \in \mathbb{N}$. For all $s \in E_k$ we have $L(y_n(s), \dot{y}(s)) \underset{n\to+\infty}{\longrightarrow} L(y(s), \dot{y}(s))$ and the bound:

$$\forall n \in \mathbb{N}, \quad |L(y_n(s), \dot{y}(s))| \leq \sup_{\substack{x\in\mathbb{R} \\ |v|\leq k}} |L(x,v)| \in \mathbf{L}^1(E_k).$$

By dominated convergence,

$$\int_{E_k} L(y_n(s), \dot{y}(s))\,\mathrm{d}s \underset{n\to+\infty}{\longrightarrow} \int_{E_k} L(y(s), \dot{y}(s))\,\mathrm{d}s\,.$$

Then, for all $n \in \mathbb{N}$,

$$\begin{aligned}
&\int_{E_k} (\dot{y}_n(s) - \dot{y}(s)) \frac{\partial L}{\partial v}(y_n(s), \dot{y}(s))\,\mathrm{d}s \\
&= \int_{E_k} (\dot{y}_n(s) - \dot{y}(s)) \frac{\partial L}{\partial v}(y(s), \dot{y}(s))\,\mathrm{d}s + \int_{E_k} (\dot{y}_n(s) - \dot{y}(s)) \left(\frac{\partial L}{\partial v}(y_n(s), \dot{y}(s)) - \frac{\partial L}{\partial v}(y(s), \dot{y}(s)) \right) \mathrm{d}s\,.
\end{aligned}$$

As $n \to +\infty$, both terms of the last expression converge to 0. The first one does because $(\dot{y}_n)_n$ converges to $\dot{y}$ weakly in $\mathbf{L}^1((0,t))$ and $s \mapsto \frac{\partial L}{\partial v}(y(s), \dot{y}(s)) \in \mathbf{L}^\infty(E_k)$. The second one

does because of the bound:

$$\begin{aligned}\forall n \in \mathbb{N}, \quad & \left| \int_{E_k} (\dot{y}_n(s) - \dot{y}(s)) \left(\frac{\partial L}{\partial v}(y_n(s), \dot{y}(s)) - \frac{\partial L}{\partial v}(y(s), \dot{y}(s)) \right) \mathrm{d}s \right| \\ & \leq \|y_n - y\|_{\mathbf{L}^\infty(E_k)} \left(\sup_{n \in \mathbb{N}} \|\dot{y}_n\|_{\mathbf{L}^1(E_k)} + \|\dot{y}\|_{\mathbf{L}^1(E_k)} \right) \sup_{\substack{x \in \mathbb{R} \\ |v| \leq k}} \left| \frac{\partial^2 L}{\partial x \partial v}(x, v) \right| \\ & \underset{n \to +\infty}{\longrightarrow} 0.\end{aligned}$$

Recall that since $(\dot{y}_n)_n$ converges weakly in $\mathbf{L}^1((0,t))$, $\sup_{n \in \mathbb{N}} \|\dot{y}_n\|_{\mathbf{L}^1((0,t))} < +\infty$, see [80, Theorem 1.1.1]. Taking the limit in (8.3.1) when $n \to +\infty$ yields (remember that U_0 is continuous):

$$\forall k \in \mathbb{N}, \quad \liminf_{n \to +\infty} \mathcal{J}_t(y_n) \geq \int_0^t \underbrace{L(y(s), \dot{y}(s)) \mathbb{1}_{E_k}(s)}_{g_k(s)} \mathrm{d}s + U_0(y(0)). \tag{8.3.2}$$

The sequence $(g_k)_k$ is a sequence of measurable, nondecreasing, nonnegative functions. Since $\dot{y} \in \mathbf{L}^1((0,t))$, it is finite almost everywhere on $(0,t)$ which implies that $(g_k)_k$ converges to $s \mapsto L(y(s), \dot{y}(s))$ almost everywhere on $(0,t)$. By monotone convergence,

$$\int_0^t L(y(s), \dot{y}(s)) \mathbb{1}_{E_k}(s) \, \mathrm{d}s \underset{k \to +\infty}{\longrightarrow} \int_0^t L(y(s), \dot{y}(s)) \, \mathrm{d}s \,.$$

We obtain the announced result by taking the limit in (8.3.2) when $k \to +\infty$. □

Theorem 8.3.2 (Existence of a minimizer)**.** *Fix $t > 0$, $x \in \mathbb{R}$ and $U_0 \in \mathbf{Lip}(\mathbb{R})$. Then the functional $\mathcal{J}_t$ admits a minimizer in the subset $\mathcal{A}_{x,t} := \left\{ y \in \mathbf{W}^{1,1}((0,t)) \mid y(t) = x \right\}$.*

Proof. The proof mostly consists in proving sufficient compactness for a minimizing sequence and then invoke Lemma 8.3.1.
There exist arcs $y \in \mathcal{A}_{x,t}$ for which $\mathcal{J}_t(y)$ is finite, for example the constant arc:

$$\mathcal{J}_t(y \equiv x) = tL(x, 0) + U_0(x) \quad \text{(finite)}.$$

Accordingly, $\mathcal{J}_t$ admits a minimizing sequence $(y_n)_n$ in $\mathcal{A}_{x,t}$:

$$\lim_{n \to +\infty} \mathcal{J}_t(y_n) = \inf_{\mathcal{A}_{x,t}} \mathcal{J}_t \leq tL(x, 0) + U_0(x) < +\infty.$$

Compactness for $(y_n)_n$ with Arzelà-Ascoli theorem. Let $R > 0$ such that

$$\forall r \in \mathbb{R}^+, \quad r \geq R \implies \frac{\phi(r)}{r} \geq 1 + \|U_0'\|_{\mathbf{L}^\infty},$$

recall that $\phi \in \mathbf{C}(\mathbb{R}^+)$ is given by the Nagumo growth of L, see Corollary 8.1.4. Consequently, for all $n \in \mathbb{N}$, we have:

$$\begin{aligned}
\|\dot{y}_n\|_{\mathbf{L}^1} &= \int_{(0,t)\cap\{|\dot{y}_n|<R\}} |\dot{y}_n(s)|\,\mathrm{d}s + \int_{(0,t)\cap\{|\dot{y}_n|\geq R\}} |\dot{y}_n(s)|\,\mathrm{d}s \\
&\leq Rt + \frac{1}{1+\|U_0'\|_{\mathbf{L}^\infty}} \int_{(0,t)\cap\{|\dot{y}_n|\geq R\}} \phi(|\dot{y}_n(s)|)\,\mathrm{d}s \\
&\leq Rt + \frac{1}{1+\|U_0'\|_{\mathbf{L}^\infty}} \int_{(0,t)\cap\{|\dot{y}_n|\geq R\}} L(y_n(s),\dot{y}_n(s))\,\mathrm{d}s \\
&\leq Rt + \frac{1}{1+\|U_0'\|_{\mathbf{L}^\infty}} \int_0^t L(y_n(s),\dot{y}_n(s))\,\mathrm{d}s \\
&= Rt + \frac{1}{1+\|U_0'\|_{\mathbf{L}^\infty}} \left(\mathcal{J}_t(y_n) - U_0(y_n(0))\right) \\
&\leq Rt + \frac{1}{1+\|U_0'\|_{\mathbf{L}^\infty}} \left(\mathcal{J}_t(y_n) - U_0(x) + \|U_0'\|_{\mathbf{L}^\infty}\|\dot{y}_n\|_{\mathbf{L}^1}\right),
\end{aligned}$$

which can be rewritten as

$$\forall n \in \mathbb{N}, \quad \left(1 - \frac{\|U_0'\|_{\mathbf{L}^\infty}}{1+\|U_0'\|_{\mathbf{L}^\infty}}\right)\|\dot{y}_n\|_{\mathbf{L}^1} \leq Rt + \frac{1}{1+\|U_0'\|_{\mathbf{L}^\infty}}\left(\mathcal{J}_t(y_n) - U_0(x)\right).$$

The sequence $(\dot{y}_n)_n$ is bounded in $\mathbf{L}^1((0,t))$ which implies that the sequence $(y_n)_n$ is bounded in $\mathbf{C}([0,t])$:

$$\forall n \in \mathbb{N},\ \forall s \in [0,t], \quad |y_n(s)| \leq |x| + \int_s^t |\dot{y}_n(\tau)|\,\mathrm{d}\tau \leq |x| + \sup_{n\in\mathbb{N}} \|\dot{y}_n\|_{\mathbf{L}^1}.$$

We now prove that the sequence $(y_n)_n$ is equi-continuous.
First remark that

$$\forall n \in \mathbb{N}, \ \int_0^t \phi(|\dot{y}_n(s)|)\,\mathrm{d}s \leq \int_0^t L(y_n(s),\dot{y}_n(s))\,\mathrm{d}s \leq \mathcal{J}_t(y_n) - U_0(x) + \|U_0'\|_{\mathbf{L}^\infty}\|\dot{y}_n\|_{\mathbf{L}^1}$$

which implies that $M := \sup_{n\in\mathbb{N}} \int_0^t \phi(|\dot{y}_n(s)|)\,\mathrm{d}s < +\infty$.
Fix $\varepsilon > 0$ and $s \in [0,t]$. Let $R' > 0$ such that

$$\forall r \in \mathbb{R}^+, \quad r \geq R' \implies \frac{\phi(r)}{r} \geq \frac{2M}{\varepsilon}.$$

Suppose that $\tau \in [0,t]$ is such that $|\tau - s| \leq \dfrac{\varepsilon}{2R'}$. Then we have

$$\begin{aligned}\forall n \in \mathbb{N}, \quad |y_n(\tau) - y_n(s)| &\leq \int_{(\tau,s)\cap\{|\dot{y}_n|\geq R'\}} |\dot{y}_n(\sigma)|\,\mathrm{d}\sigma + \int_{(\tau,s)\cap\{|\dot{y}_n|<R'\}} |\dot{y}_n(\sigma)|\,\mathrm{d}\sigma \\ &\leq \frac{\varepsilon}{2M}\int_{(\tau,s)\cap\{|\dot{y}_n|\geq R'\}} \phi(|\dot{y}_n(\sigma)|)\,\mathrm{d}\sigma + R'|\tau - s| \\ &\leq \frac{\varepsilon}{2M}\int_{(\tau,s)\cap\{|\dot{y}_n|\geq R'\}} \phi(|\dot{y}_n(\sigma)|)\,\mathrm{d}\sigma + \frac{\varepsilon}{2} \\ &\leq \frac{\varepsilon}{2M}\int_0^t \phi(|\dot{y}_n(\sigma)|)\,\mathrm{d}\sigma + \frac{\varepsilon}{2} \leq \varepsilon.\end{aligned}$$

Arzelà-Ascoli theorem (see [36, Theorem 4.25]) ensures that there exist $y \in \mathbf{C}([0,t])$ and a subsequence $(n_k)_k$ such that $(y_{n_k})_k$ converges uniformly to y on $[0,t]$. In particular, $y(t) = \lim\limits_{k\to+\infty} y_{n_k}(t) = x$.

Compactness for $(\dot{y}_n)_n$ with Dunford-Pettis theorem. We now prove that the sequence $(\dot{y}_{n_k})_k$ is equi-integrable.
Let $A \subset (0,t)$ be a measurable set verifying $\mathrm{mes}(A) \leq \dfrac{\varepsilon}{2R'}$, then we have:

$$\begin{aligned}\forall k \in \mathbb{N}, \quad \int_A |\dot{y}_{n_k}(s)|\,\mathrm{d}s &= \int_{A\cap\{|\dot{y}_{n_k}|<R'\}} |\dot{y}_{n_k}(s)|\,\mathrm{d}s + \int_{A\cap\{|\dot{y}_{n_k}|>R'\}} |\dot{y}_{n_k}(s)|\,\mathrm{d}s \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2M}\int_{A\cap\{|\dot{y}_{n_k}|>R'\}} \phi(|\dot{y}_{n_k}(s)|)\,\mathrm{d}s \leq \varepsilon.\end{aligned}$$

Using this equi-integrability, the Dunford-Pettis theorem (see [36, Theorem 4.30]) ensures that there exist a further subsequence (which we do not relabel) and $z \in \mathbf{L}^1((0,t))$ such that $(\dot{y}_{n_k})_k$ converges to z weakly in $\mathbf{L}^1((0,t))$. By definition of the weak derivatives, we have

$$\forall k \in \mathbb{N},\ \forall \varphi \in \mathbf{C}_{\mathbf{c}}^{\infty}((0,t)), \quad \int_0^t y_{n_k}(s)\dot{\varphi}(s)\,\mathrm{d}s = -\int_0^t \dot{y}_{n_k}(s)\varphi(s)\,\mathrm{d}s\,.$$

Taking the limit as $k \to +\infty$ in the equality above, we get:

$$\forall \varphi \in \mathbf{C}_{\mathbf{c}}^{\infty}((0,t)), \quad \int_0^t y(s)\dot{\varphi}(s)\,\mathrm{d}s = -\int_0^t z(s)\varphi(s)\,\mathrm{d}s\,.$$

Since $z \in \mathbf{L}^1((0,t))$, this last equality ensures that $y \in \mathbf{W}^{1,1}((0,t))$ and that $\dot{y} = z$, hence $y \in \mathcal{A}_{x,t}$.

Conclusion. Lemma 8.3.1 ensures that

$$\inf_{\mathcal{A}_{x,t}} \mathcal{J}_t = \lim_{k\to+\infty} \mathcal{J}_t(y_{n_k}) \geq \mathcal{J}_t(y) \geq \inf_{\mathcal{A}_{x,t}} \mathcal{J}_t,$$

which proves that y is a minimizer of $\mathcal{J}_t$. □

8.3.2 Regularity of minimizers

We now discuss the regularity of the minimizers of $\mathcal{J}_t$. We will use the following results.

Lemma 8.3.3. *Let $f \in \mathbf{C}^2(\mathbb{R})$ be a strongly convex function. Then for all $\xi \in \mathbb{R}$, the application*

$$g : \left| \begin{array}{ccc} [0,+\infty) & \longrightarrow & \mathbb{R} \\ x & \longmapsto & (1+x) f\left(\dfrac{\xi}{1+x}\right) \end{array} \right.$$

is convex. Moreover,

$$\forall x \in \mathbb{R}^+, \quad (1+x) f\left(\frac{\xi}{1+x}\right) - f(\xi) \leq -x f^*\left(f'\left(\frac{\xi}{1+x}\right)\right).$$

Proof. Since $g \in \mathbf{C}^2(\mathbb{R})$, the convexity of g follows from the computation:

$$\forall x \in \mathbb{R}^+, \quad g''(x) = \frac{\xi^2}{(1+x)^3} f''\left(\frac{\xi}{1+x}\right) \geq 0.$$

Then, using the convexity of g and classical Legendre transform properties, we can write that for all $x > 0$,

$$\frac{g(x) - g(0)}{x} \leq g'(x) = f\left(\frac{\xi}{1+x}\right) - \left(\frac{\xi}{1+x}\right) f'\left(\frac{\xi}{1+x}\right) = -f^*\left(f'\left(\frac{\xi}{1+x}\right)\right).$$

□

Lemma 8.3.4. *We have*

$$\forall x \in \mathbb{R},\ \forall v \in \mathbb{R}^*, \quad H\left(x, \frac{\partial L}{\partial v}(x,v)\right) \geq \frac{\phi(|v|)}{|v|} - \left(\sup_{\substack{y \in \mathbb{R} \\ |w| \leq 1}} L(y,w)\right) \cdot \left(1 + \frac{1}{|v|}\right), \tag{8.3.3}$$

Consequently, $H\left(x, \dfrac{\partial L}{\partial v}(x,v)\right) \underset{|v| \to +\infty}{\longrightarrow} +\infty$, uniformly in x.

Proof. Fix $v > 0$. Using the link between H and L:

$$H\left(x, \frac{\partial L}{\partial v}(x,v)\right) = \sup_{p \in \mathbb{R}} \left(p \frac{\partial L}{\partial v}(x,v) - L(x,p)\right)$$

with $p = 1$, we obtain:

$$H\left(x, \frac{\partial L}{\partial v}(x,v)\right) = \sup_{p \in \mathbb{R}} \left(p \frac{\partial L}{\partial v}(x,v) - L(x,p)\right) \geq \frac{\partial L}{\partial v}(x,v) - L(x,1) \geq \frac{\partial L}{\partial v}(x,v) - \left(\sup_{\substack{y \in \mathbb{R} \\ |w| \leq 1}} L(y,w)\right).$$

By convexity,

$$\frac{\partial L}{\partial v}(x,v) \geq \frac{L(x,v) - L(x,0)}{v} \geq \frac{\phi(|v|)}{|v|} - \frac{1}{|v|}\left(\sup_{\substack{y \in \mathbb{R} \\ |w| \leq 1}} L(y,w)\right),$$

which leads to (8.3.3). It is straightforward to prove the estimate in the case $v < 0$. The proof is complete. □

We are going to prove the following result.

Theorem 8.3.5 (Regularity of minimizers)**.** *Fix $t > 0$, $x \in \mathbb{R}$ and $U_0 \in \mathbf{Lip}(\mathbb{R})$. Let $y \in \mathcal{A}_{x,t}$ be a minimizer of $\mathcal{J}_t$. Then there exists a Lipschitz bijection $\Lambda \in \mathbf{Lip}((0,t))$ with $\Lambda^{-1} \in \mathbf{Lip}((0,t))$ such that $\gamma := y \circ \Lambda^{-1} \in \mathcal{A}_{x,t}$ is a Lipschitz minimizer of $\mathcal{J}_t$ in $\mathcal{A}_{x,t}$.*

Proof. The idea behind the construction of Λ is the following.
- On the subsets on which we can control $|\dot{y}|$ (E and O below), we make $\gamma \circ \Lambda$ travel faster (three times faster to be precise), but we have to pay a cost, see (8.3.5).
- On the other hand, if we cannot control $|\dot{y}|$ (subset U_N below), then we make $\gamma \circ \Lambda$ travel slower and adjust the cost (see (8.3.6)) so that γ is still a minimizer of $\mathcal{J}_t$ (Step 3 below).

Step 1: Setting. Set $m := \|y\|_{\mathbf{L}^\infty}$ and $M := \|\dot{y}\|_{\mathbf{L}^1}$. Introduce the subset

$$E := \left\{ s \in (0,t) \ : \ |\dot{y}(s)| \leq \frac{4M}{3t} \right\}.$$

Note that Markov inequality ensures that

$$\mathrm{mes}(E) \geq \frac{t}{4}. \tag{8.3.4}$$

Set

$$\delta := \frac{1}{3} \sup_{\substack{|y| \leq m \\ |w| \leq 4M/t}} |L(y,w)| - \inf_{\substack{|y| \leq m \\ |w| \leq 4M/3t}} L(y,w)$$

so that

$$\forall s \in E, \quad \frac{1}{3} L(y(s), 3\dot{y}(s)) - L(y(s), \dot{y}(s)) \leq \delta. \tag{8.3.5}$$

Lemma 8.3.4 yields the existence of $N_1 \in \mathbb{N}$ such that

$$\forall x, v \in \mathbb{R}, \quad \Big(|x| \leq m \text{ and } |v| \geq N_1 \Big) \implies H\left(x, \frac{\partial L}{\partial v}(x,v) \right) \geq \frac{3}{2}\delta.$$

Now for all $n \in \mathbb{N}^*$, introduce the subset

$$U_n := \{ s \in (0,t) \ : \ |\dot{y}(s)| > n \}.$$

Again, Markov inequality yields:

$$\forall n \in \mathbb{N}^*, \quad \mathrm{mes}(U_n) \leq \frac{M}{n}.$$

We also have

$$\forall n \in \mathbb{N}^*, \quad \underbrace{\int_{U_n} \left(\frac{|\dot{y}(s)|}{n} - 1 \right) \mathrm{d}s}_{\varepsilon_n} \leq \frac{M}{n}.$$

Set $N = \max\left\{\left\lfloor \frac{4M}{3t}\right\rfloor + 1, N_1, \left\lfloor \frac{12M}{t}\right\rfloor + 1\right\}$ so that

$$\begin{aligned} & E \cap U_N = \emptyset \\ & \forall x, v \in \mathbb{R}, \quad \left(|x| \le m \text{ and } |v| \ge N\right) \implies H\left(x, \frac{\partial L}{\partial v}(x,v)\right) \ge \frac{3}{2}\delta \\ & 0 \le \varepsilon_N \le \frac{t}{12}. \end{aligned} \tag{8.3.6}$$

Step 2: Construction of Λ. Let O be a measurable subset of E which has a Lebesgue measure equal to $\frac{3}{2}\varepsilon_N$. This is possible since, in light of (8.3.4) and(8.3.6), $\frac{3}{2}\varepsilon_N \le \frac{t}{8} \le \mathrm{mes}(E)$. Remark that with our choice of N, $E \cap U_N = \emptyset$ results in $O \cap U_N = \emptyset$. Define

$$\alpha : \left| \begin{array}{ccc} (0,t) & \longrightarrow & \mathbb{R} \\ s & \longmapsto & \left\{ \begin{array}{lll} \frac{1}{3} & \text{if} & s \in O \\ 1 + \left(\frac{|\dot y(s)|}{N} - 1\right) & \text{if} & s \in U_N \\ 1 & \text{otherwise.} & \end{array} \right. \end{array} \right.$$

Since

$$\int_0^t \alpha(s)\,\mathrm{d}s = \frac{1}{3}\mathrm{mes}(O) + \mathrm{mes}(U_N) + \varepsilon_N + (t - \mathrm{mes}(O) - \mathrm{mes}(U_N)) = t,$$

the relation

$$\Lambda(s) = \int_0^s \alpha(\tau)\,\mathrm{d}\tau$$

defines an increasing, absolutely continuous bijection from $[0,t]$ onto itself. Since $\Lambda' \ge \frac{1}{3}$ a.e. on $(0,t)$, we deduce that for all $p, q \in [0,t]$, written $p = \Lambda(P)$, $q = \Lambda(Q)$, $P, Q \in [0,t]$, we have

$$|\Lambda(P) - \Lambda(Q)| \ge \frac{1}{3}|P - Q| \implies |\Lambda^{-1}(p) - \Lambda^{-1}(q)| \le 3|p - q|,$$

i.e. Λ^{-1} is Lipschitz continuous with $\|(\Lambda^{-1})'\|_{\mathbf{L}^\infty} \le 3$. This ensures that both Λ and Λ^{-1} are absolutely continuous.

Step 3: $\gamma := y \circ \Lambda^{-1}$ is a minimizer of $\mathcal{J}_t$ on $\mathcal{A}_{x,t}$. By the change of variables formula (see [146, Theorem 3]), we obtain

$$\begin{aligned} \int_0^t L(\gamma(\tau), \dot\gamma(\tau))\,\mathrm{d}\tau &= \int_0^t L\left(y(s), \frac{\dot y(s)}{\alpha(s)}\right)\alpha(s)\,\mathrm{d}s \\ &= \int_O \frac{1}{3} L\left(y(s), 3\dot y(s)\right)\mathrm{d}s + \int_{U_N} \frac{|\dot y(s)|}{N} L\left(y(s), N\frac{\dot y(s)}{|\dot y(s)|}\right)\mathrm{d}s \\ &\quad + \int_{(0,t)\setminus O\cup U_N} L(y(s), \dot y(s))\,\mathrm{d}s\,. \end{aligned}$$

Since we chose $O \subset E$, we have, using (8.3.5):

$$\int_O \left(\frac{1}{3} L\left(y(s), 3\dot{y}(s)\right) - L\left(y(s), \dot{y}(s)\right) \right) \mathrm{d}s \leq \delta \ \mathrm{mes}(O) = \frac{3}{2} \varepsilon_N \delta.$$

Moreover, using Lemma 8.3.3 with $f = L$, we get that for all $s \in U_N$,

$$\begin{aligned} \frac{|\dot{y}(s)|}{N} L\left(y(s), N\frac{\dot{y}(s)}{|\dot{y}(s)|}\right) - L(y(s), \dot{y}(s)) &\leq -\left(\frac{|\dot{y}(s)|}{N} - 1\right) H\left(y(s), \frac{\partial L}{\partial v}\left(y(s), N\frac{\dot{y}(s)}{|\dot{y}(s)|}\right)\right) \\ &\leq -\frac{3}{2}\left(\frac{|\dot{y}(s)|}{N} - 1\right)\delta, \end{aligned}$$

where the last inequality comes from (8.3.6). Putting everything together, we get:

$$\mathcal{J}_t(\gamma) \leq \mathcal{J}_t(y) + \frac{3}{2}\varepsilon_N \delta - \int_{U_N} \frac{3}{2}\left(\frac{|\dot{y}(s)|}{N} - 1\right)\delta \, \mathrm{d}s = \mathcal{J}_t(y).$$

Step 4: γ is a Lipschitz arc. Now, using the chain rule [146, Corollary 2], we obtain for a.e. $s \in (0,t)$,

$$|\dot{\gamma}\left(\Lambda(s)\right)| = \frac{|\dot{y}(s)|}{\alpha(s)} \leq \begin{cases} \dfrac{4M}{t} & \text{if} \quad s \in O \\ N & \text{otherwise,} \end{cases}$$

hence the absolute value of $\dot{\gamma}$ is bounded at almost every point $\Lambda(s)$. Since $\Lambda : [0,t] \to [0,t]$ is one-to-one, almost every point of $[0,t]$ is of this form. □

We now prove that Lipschitz minimizers are actually in $\mathbf{W}^{2,\infty}((0,t))$. First let us recall the following technical result.

Lemma 8.3.6. *Fix $t > 0$, $\varphi \in \mathbf{C}^\infty((0,t))$ and $y \in \mathbf{W}^{1,\infty}((0,t))$. Define the function*

$$F : \left| \begin{array}{ccc} (-1,1) & \longrightarrow & \mathbb{R} \\ \eta & \longmapsto & \displaystyle\int_0^t L(y(s) + \eta\varphi(s), \dot{y}(s) + \eta\dot{\varphi}(s))\,\mathrm{d}s \end{array} \right.$$

Then $F \in \mathbf{C}^1((-1,1))$ and for all $\eta \in (-1,1)$,

$$\begin{aligned} F'(\eta) = \int_0^t \Bigg(& \varphi(s)\frac{\partial L}{\partial x}(y(s) + \eta\varphi(s), \dot{y}(s) + \eta\dot{\varphi}(s)) \\ & + \dot{\varphi}(s)\frac{\partial L}{\partial v}(y(s) + \eta\varphi(s), \dot{y}(s) + \eta\dot{\varphi}(s)) \Bigg) \mathrm{d}s\,. \end{aligned} \tag{8.3.7}$$

Proof. Define

$$\forall \eta \in (-1,1),\ \forall s \in (0,t), \quad f(\eta, s) := L(y(s) + \eta\varphi(s), \dot{y}(s) + \eta\dot{\varphi}(s)).$$

For all $\eta \in (-1,1)$, $f(\eta,\cdot) \in \mathbf{L}^1((0,t))$ since y is Lipschitz; for a.e. $s \in (0,t)$, $f(\cdot,s) \in \mathbf{C}^1((-1,1))$ and for all $\eta \in (-1,1)$ and for a.e. $s \in (0,t)$, we have the bound:

$$\left|\frac{\partial f}{\partial \eta}(\eta,s)\right| = \left|\varphi(s)\frac{\partial L}{\partial x}(y(s)+\eta\varphi(s),\dot{y}(s)+\eta\dot{\varphi}(s)) + \dot{\varphi}(s)\frac{\partial L}{\partial v}(y(s)+\eta\varphi(s),\dot{y}(s)+\eta\dot{\varphi}(s))\right|$$
$$\leq \|\varphi\|_{\mathbf{L}^\infty} \sup_{\substack{x\in\mathbb{R}\\ |v|\leq V}} \left|\frac{\partial L}{\partial x}(x,v)\right| + \|\dot{\varphi}\|_{\mathbf{L}^\infty} \sup_{\substack{x\in\mathbb{R}\\ |v|\leq V}} \left|\frac{\partial L}{\partial v}(x,v)\right|, \quad V := \|\dot{y}\|_{\mathbf{L}^\infty} + \|\dot{\varphi}\|_{\mathbf{L}^\infty},$$

which ensures that $F \in \mathbf{C}^1((-1,1))$ and that F' is given by (8.3.7). □

Corollary 8.3.7. *Fix $t>0$, $x \in \mathbb{R}$ and $U_0 \in \mathbf{Lip}(\mathbb{R})$. Denote by y a Lipschitz minimizer of $\mathcal{J}_t$ in $\mathcal{A}_{x,t}$. Then:*

(i) y satisfies the weak form of the Euler-Lagrange equations:

$$\forall \varphi \in \mathbf{C}^\infty_{\mathbf{c}}((0,t)), \quad \int_0^t \dot{\varphi}(s)\frac{\partial L}{\partial v}(y(s),\dot{y}(s))\,\mathrm{d}s = -\int_0^t \varphi(s)\frac{\partial L}{\partial x}(y(s),\dot{y}(s))\,\mathrm{d}s\,. \tag{8.3.8}$$

(ii) $y \in \mathbf{W}^{2,\infty}((0,t))$.

Proof. *(i)* Fix $\varphi \in \mathbf{C}^\infty_{\mathbf{c}}((0,t))$. Let F be the function of Lemma 8.3.6. The function

$$\eta \mapsto \mathcal{J}_t(y+\eta\varphi) = F(\eta) + U_0(y(0))$$

is in $\mathbf{C}^1((-1,1))$ and since y is a minimizer of $\mathcal{J}_t$, it admits a minimum at point $\eta = 0$. Its derivative at $s=0$ vanishes, which from (8.3.7), reads exactly as (8.3.8). Remark that since $s \mapsto \dfrac{\partial L}{\partial x}(y(s),\dot{y}(s)) \in \mathbf{L}^\infty((0,t))$, we have, by definition of weak derivatives,

$$s \mapsto \frac{\partial L}{\partial v}(y(s),\dot{y}(s)) \in \mathbf{W}^{1,\infty}((0,t)).$$

(ii) Since y is continuous, the strong convexity assumption of L (see Lemma 8.1.3) implies that

$$\exists \kappa > 0,\ \forall s \in [0,t],\ \forall v \in [-\|\dot{y}\|_{\mathbf{L}^\infty}, \|\dot{y}\|_{\mathbf{L}^\infty}], \quad \frac{\partial^2 L}{\partial v^2}(y(s),v) \geq \kappa.$$

Therefore, for a.e. $s,\tau \in (0,t)$, we have

$$\begin{aligned}
\kappa|\dot{y}(s)-\dot{y}(\tau)| &= \left|\int_{\dot{y}(\tau)}^{\dot{y}(s)} \kappa\,\mathrm{d}v\right| \\
&\leq \left|\int_{\dot{y}(\tau)}^{\dot{y}(s)} \frac{\partial^2 L}{\partial v^2}(y(s),v)\,\mathrm{d}v\right| \\
&= \left|\frac{\partial L}{\partial v}(y(s),\dot{y}(s)) - \frac{\partial L}{\partial v}(y(s),\dot{y}(\tau))\right| \\
&\leq \left|\frac{\partial L}{\partial v}(y(s),\dot{y}(s)) - \frac{\partial L}{\partial v}(y(\tau),\dot{y}(\tau))\right| + \left|\frac{\partial L}{\partial v}(y(\tau),\dot{y}(\tau)) - \frac{\partial L}{\partial v}(y(s),\dot{y}(\tau))\right|.
\end{aligned}$$

We proved in *(i)* that $s \mapsto \dfrac{\partial L}{\partial v}(y(s), \dot{y}(s))$ is Lipschitz; let's call $\Lambda \geq 0$ its Lipschitz constant. Moreover, we have

$$\left|\frac{\partial L}{\partial v}(y(\tau), \dot{y}(\tau)) - \frac{\partial L}{\partial v}(y(s), \dot{y}(\tau))\right| \leq |y(\tau) - y(s)| \sup_{\substack{x\in\mathbb{R} \\ |v|\leq \|\dot{y}\|_{\mathbf{L}^\infty}}} \left|\frac{\partial^2 L}{\partial x \partial v}(x, v)\right|$$

$$\leq \underbrace{\left(\|\dot{y}\|_{\mathbf{L}^\infty} \sup_{\substack{x\in\mathbb{R} \\ |v|\leq \|\dot{y}\|_{\mathbf{L}^\infty}}} \left|\frac{\partial^2 L}{\partial x \partial v}(x, v)\right| \right)}_{\Gamma} |\tau - s|.$$

We have shown that for a.e. $s, \tau \in (0, t)$,

$$|\dot{y}(s) - \dot{y}(\tau)| \leq \left(\frac{\Lambda + \Gamma}{\kappa}\right) |s - \tau|,$$

which proves that $y \in \mathbf{W}^{2,\infty}((0, t))$. □

Remark 8.3.1. Since $s \mapsto \dfrac{\partial L}{\partial v}(y(s), \dot{y}(s))$ is Lipschitz, it is differentiable a.e. on $(0, t)$ in the classical sense. Consequently, equality (8.3.8) implies that

$$\forall \varphi \in \mathbf{C}^\infty_{\mathbf{c}}((0, t)), \quad \int_0^t \varphi(s) \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\partial L}{\partial v}(y(s), \dot{y}(s))\right) \mathrm{d}s = \int_0^t \varphi(s) \frac{\partial L}{\partial x}(y(s), \dot{y}(s)) \, \mathrm{d}s \, .$$

Then, du Bois-Raymond lemma ensures that for a.e. $s \in (0, t)$:

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\partial L}{\partial v}(y(s), \dot{y}(s))\right) = \frac{\partial L}{\partial x}(y(s), \dot{y}(s)). \tag{8.3.9}$$

In short, any Lipschitz minimizer of $\mathcal{J}_t$ satisfies the a.e. form of the Euler-Lagrange equations.

Euler-Lagrange equations, combined with a variant of the transversality condition, will provide a uniform (w.r.t. x, t) Lipschitz bound for regular minimizers of $\mathcal{J}_t$. First:

Lemma 8.3.8. *Fix $t > 0$, $x \in \mathbb{R}$ and $U_0 \in \mathbf{Lip}(\mathbb{R})$. Denote by y a Lipschitz minimizer of $\mathcal{J}_t$ in $\mathcal{A}_{x,t}$. Then y satifies the condition*

$$\limsup_{\eta \to 0^-} \left(\frac{U_0(y(0) + \eta) - U_0(y(0))}{\eta}\right) \leq \frac{\partial L}{\partial v}(y(0), \dot{y}(0)) \leq \liminf_{\eta \to 0^+} \left(\frac{U_0(y(0) + \eta) - U_0(y(0))}{\eta}\right). \tag{8.3.10}$$

In particular,

$$\left|\frac{\partial L}{\partial v}(y(0), \dot{y}(0))\right| \leq \|U_0'\|_{\mathbf{L}^\infty} \tag{8.3.11}$$

Proof. Let us recall that Corollary 8.3.7 ensures that actually, $y \in \mathbf{W}^{2,\infty}((0, t))$.
Fix $\varphi \in \mathbf{C}^1_{\mathbf{c}}([0, t))$ such that $\varphi(0) = 1$. Let F be the function of Lemma 8.3.6 so that for all $\eta \in (-1, 1)$,

$$\mathcal{J}_t(y + \eta\varphi) = F(\eta) + U_0(y(0) + \eta).$$

From a Taylor expansion, we obtain, as $\eta \to 0$,

$$\mathcal{J}_t(y+\eta\varphi) = F(0) + \eta F'(0) + U_0(y(0)+\eta) + o(\eta)\,, \tag{8.3.12}$$

with, in light of Lemma 8.3.6 and the Euler-Lagrange equations (8.3.9),

$$F(0) = \mathcal{J}_t(y) - U_0(y(0)); \quad F'(0) = -\frac{\partial L}{\partial v}(y(0), \dot{y}(0)).$$

Since y is a minimizer, for all $\eta \in (-1,1)$, $\mathcal{J}_t(y+\eta\varphi) \geq \mathcal{J}_t(y)$. Therefore, (8.3.12) implies:

$$\forall \eta \in (-1,1), \quad \eta \frac{\partial L}{\partial v}(y(0), \dot{y}(0)) \leq U_0(y(0)+\eta) - U_0(y(0)) + o(\eta)\,. \tag{8.3.13}$$

Divide by $\eta > 0$ and take the lim inf as $\eta \to 0^+$ to obtain the right inequality of (8.3.10); then, divide by $\eta < 0$ and take the lim sup as $\eta \to 0^-$ to obtain the left inequality of (8.3.10). □

Fix now $U_0 \in \mathbf{Lip}(\mathbb{R})$ and let $\mathbf{C}_H > 0$ such that

$$\forall r \in \mathbb{R}^+,\ r \geq \mathbf{C}_H \implies \frac{\phi(r)}{1+r} > \left(\sup_{\substack{w\in\mathbb{R} \\ |u| \leq \|U_0'\|_{\mathbf{L}^\infty}}} |H(w,u)| + \sup_{\substack{w\in\mathbb{R} \\ |v|\leq 1}} |L(w,v)| \right), \tag{8.3.14}$$

where ϕ is given by the Nagumo growth of H, see Corollary 8.1.4.

Theorem 8.3.9. *For all $t > 0$, for all $x \in \mathbb{R}$, and for all all Lipschitz minimizers of $\mathcal{J}_t$ in $\mathcal{A}_{x,t}$, we have*

$$\|\dot{y}\|_{\mathbf{L}^\infty} \leq \mathbf{C}_H. \tag{8.3.15}$$

Proof. We adapt the reasoning put forward by the author of [53]. Fix $t > 0$, $x \in \mathbb{R}$ and let y be a Lipschitz minimizer of $\mathcal{J}_t$ in $\mathcal{A}_{x,t}$.

Using the transversality condition. Since $y \in \mathbf{W}^{2,\infty}((0,t))$ (see Corollary 8.3.7) and satisfies the a.e. form of the Euler-Lagrange equations, for a.e. $s \in (0,t)$, we have

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\dot{y}(s)\frac{\partial L}{\partial v}(y(s), \dot{y}(s)) - L(y(s), \dot{y}(s)) \right) = 0,$$

which implies that:

$$\exists \mathbf{C}_{x,t} \in \mathbb{R},\ \forall s \in [0,t], \quad \dot{y}(s)\frac{\partial L}{\partial v}(y(s), \dot{y}(s)) - L(y(s), \dot{y}(s)) = \mathbf{C}_{x,t}.$$

In particular,

$$\mathbf{C}_{x,t} = \dot{y}(0)\frac{\partial L}{\partial v}(y(0), \dot{y}(0)) - L(y(0), \dot{y}(0)) = H\left(y(0), \frac{\partial L}{\partial v}(y(0), \dot{y}(0)) \right).$$

Using the transversality condition (8.3.11), we deduce:

$$|\mathbf{C}_{x,t}| \leq \sup_{\substack{w\in\mathbb{R} \\ |u| \leq \|U_0'\|_{\mathbf{L}^\infty}}} |H(w,u)|.$$

Convexity and coercivity. By convexity, for all $s \in [0,t]$, we have

$$\begin{aligned} L\left(y(s), \frac{\dot{y}(s)}{1+|\dot{y}(s)|}\right) &\geq L(y(s),\dot{y}(s)) + \left(\frac{1}{1+|\dot{y}(s)|} - 1\right)\dot{y}(s)\frac{\partial L}{\partial v}(y(s),\dot{y}(s)) \\ &= L(y(s),\dot{y}(s)) + \left(\frac{1}{1+|\dot{y}(s)|} - 1\right)(\mathbf{C}_{x,t} + L(y(s),\dot{y}(s))) \\ &= \frac{L(y(s),\dot{y}(s))}{1+|\dot{y}(s)|} - \frac{|\dot{y}(s)|}{1+|\dot{y}(s)|}\mathbf{C}_{x,t} \\ &\geq \frac{\phi(|\dot{y}(s)|)}{1+|\dot{y}(s)|} - \frac{|\dot{y}(s)|}{1+|\dot{y}(s)|} \sup_{\substack{w\in\mathbb{R} \\ |u|\leq \|U_0'\|_{\mathbf{L}^\infty}}} |H(w,u)|, \end{aligned}$$

from which we deduce:

$$\forall s \in [0,t], \quad \frac{\phi(|\dot{y}(s)|)}{1+|\dot{y}(s)|} \leq \left(\sup_{\substack{w\in\mathbb{R} \\ |u|\leq \|U_0'\|_{\mathbf{L}^\infty}}} |H(w,u)| + \sup_{\substack{x\in\mathbb{R} \\ |v|\leq 1}} |L(x,v)| \right).$$

By (8.3.14), this ensures that for all $s \in [0,t]$, $|\dot{y}(s)| \leq \mathbf{C}_H$. □

8.3.3 Link to the Hamilton-Jacobi equation

We now link the minimization of $\mathcal{J}_t$ ($t > 0$) to the Hamilton-Jacobi equation (8.0.2). More precisely, fix $T > 0$ and consider the value function:

$$\forall (x,t) \in \Omega, \quad U(x,t) := \min_{y \in \mathcal{A}_{x,t}} \mathcal{J}_t(y); \quad \Omega = \mathbb{R} \times (0,T).$$

We prove that U is a viscosity solution to (8.0.2). Remark that by Theorems 8.3.2-8.3.5, an equivalent way to define U is with the following optimal control problem:

$$U(x,t) = \min_{\alpha \in \mathbf{L}^\infty((0,t))} \left(\int_0^t L(y(s),\alpha(s))\,\mathrm{d}s + U_0(y(0)) \right), \quad \begin{cases} \dot{y}(s) = \alpha(s) & 0 < s < t \\ y(t) = x. \end{cases} \tag{8.3.16}$$

Lemma 8.3.10 (Dynamic programming principle)**.** *Fix $(x,t) \in \Omega$. Then for all $h \in (0,t)$, we have*

$$U(x,t) = \inf_{\alpha \in \mathbf{L}^\infty((t-h,t))} \left(\int_{t-h}^t L(y(s),\alpha(s))\,\mathrm{d}s + U(y(t-h),t-h) \right), \tag{8.3.17}$$

where y is given by

$$\begin{cases} \dot{y}(s) = \alpha(s) & t-h < s < t \\ y(t) = x. \end{cases}$$

Proof. Fix $h \in (0,t)$ and let us call $V(x,t)$ the right-hand side of (8.3.17).

Claim: $\mathbf{U(x,t) \geq V(x,t)}$.
Let α be an optimal control for $U(x,t)$. If we call y its associated trajectory, then we have:

$$\begin{aligned} U(x,t) &= \int_0^t L(y(s),\alpha(s))\,\mathrm{d}s + U_0(y(0)) \\ &= \underbrace{\int_0^{t-h} L(y(s),\alpha(s))\,\mathrm{d}s + U_0(y(0))}_{\geq U(y(t-h),t-h)} + \int_{t-h}^t L(y(s),\alpha(s))\,\mathrm{d}s \\ &\geq \int_{t-h}^t L(y(s),\alpha(s))\,\mathrm{d}s + U(y(t-h),t-h) \geq V(x,t). \end{aligned}$$

Claim: $\mathbf{U(x,t) \leq V(x,t)}$.
Let $\alpha \in \mathbf{L}^\infty((t-h,t))$ be a control of associated trajectory y and β an optimal control (of trajectory z) for $U(y(t-h),t-h)$. Consider $\overline{\alpha} \in \mathbf{L}^\infty((0,t))$ the control defined by

$$\overline{\alpha}(s) = \begin{cases} \beta(s) & \text{if} \quad 0 < s < t-h \\ \alpha(s) & \text{if} \quad t-h < s < t. \end{cases}$$

Its trajectory $\overline{y}$ coincides with y on $[t-h,t]$ and with z on $[0,t-h]$. Consequently, we have

$$\begin{aligned} U(x,t) &\leq \int_0^t L(\overline{y}(s),\overline{\alpha}(s))\,\mathrm{d}s + U_0(\overline{y}(0)) \\ &= \int_0^{t-h} L(z(s),\beta(s))\,\mathrm{d}s + U_0(z(0)) + \int_{t-h}^t L(y(s),\alpha(s))\,\mathrm{d}s \\ &= U(y(t-h),t-h) + \int_{t-h}^t L(y(s),\alpha(s))\,\mathrm{d}s\,. \end{aligned}$$

Taking the infimum on all $\alpha \in \mathbf{L}^\infty((t-h,t))$, we obtain the second inequality, concluding the proof. □

Theorem 8.3.11 (Regularity of the value function)**.** *The function U is Lipschitz: $U \in$* $\mathbf{Lip}(\Omega)$.

Proof. **Step 1: Lipschitz continuity in space.**
Fix $(x,t),(\xi,t) \in \Omega$. Let z be a Lipschitz minimizer for $U(\xi,t)$ and introduce y defined by

$y(s) = z(s) + (x - \xi)$. Since $y \in \mathcal{A}_{x,t}$, by definition of U, we have

$$\begin{aligned}
U(x,t) &\leq \int_0^t L(y(s), \dot{y}(s))\,\mathrm{d}s + U_0(y(0)) \\
&= \int_0^t L(z(s) + x - \xi, \dot{z}(s))\,\mathrm{d}s + U_0(z(0) + x - \xi) \\
&= U(\xi,t) + \int_0^t (L(z(s) + x - \xi, \dot{z}(s)) - L(z(s), \dot{z}(s)))\,\mathrm{d}s + (U_0(z(0) + x - \xi) - U_0(z(0)) \\
&\leq U(\xi,t) + t\Bigg(\sup_{\substack{\chi \in \mathbb{R} \\ |v| \leq \mathbf{C}_H}} \left| \frac{\partial L}{\partial x}(\chi, v) \right| + \|U_0'\|_{\mathbf{L}^\infty} \Bigg) |x - \xi|,
\end{aligned}$$

where we used Theorem 8.3.9. It is straightforward to check that the reciprocal inequality holds in view of the symmetrical roles played by x and ξ.

Step 2: Lipschitz continuity in time.

Fix now $(x,t) \in \Omega$ and $h \in (0,t)$. First, by (8.3.17) (with $\alpha \equiv 0$), we have

$$U(x,t) \leq hL(x,0) + U(x,t-h) \implies U(x,t) - U(x,t-h) \leq h \sup_{\chi \in \mathbb{R}} |L(\chi,0)|.$$

Fix now an optimal Lipschitz trajectory y for $U(x,t)$. Define $z(s) = y(s+h)$ $(s \in [0, t-h])$. By definition, we have

$$\begin{aligned}
U(x,t-h) &\leq \int_0^{t-h} L(z(s), \dot{z}(s))\,\mathrm{d}s + U_0(z(0)) \\
&= \int_h^t L(y(s), \dot{y}(s))\,\mathrm{d}s + U_0(y(h)) \\
&= U(x,t) - \int_0^h L(y(s), \dot{y}(s))\,\mathrm{d}s + (U_0(y(h)) - U_0(y(0))) \\
&\leq U(x,t) + \Bigg(\sup_{\substack{\chi \in \mathbb{R} \\ |v| \leq \mathbf{C}_H}} |L(\chi, v)| + \|U_0'\|_{\mathbf{L}^\infty} \mathbf{C}_H \Bigg) h,
\end{aligned}$$

which concludes the proof. □

Theorem 8.3.12. *The function $U \in \mathbf{Lip}(\Omega)$ is a viscosity solution to the Hamilton-Jacobi equation (8.0.2) with initial data U_0. Moreover, for all $(x,t), (\xi,\tau) \in \Omega$,*

$$\begin{aligned}
|U(x,t) - U(\xi,\tau)| \leq T\Bigg(\sup_{\substack{y \in \mathbb{R} \\ |v| \leq \mathbf{C}_H}} \left| \frac{\partial L}{\partial x}(y, v) \right| + \|U_0'\|_{\mathbf{L}^\infty} \Bigg) |x - \xi| \\
+ \Bigg(\sup_{\substack{y \in \mathbb{R} \\ |v| \leq \mathbf{C}_H}} |L(y, v)| + \|U_0'\|_{\mathbf{L}^\infty} \mathbf{C}_H \Bigg) |t - \tau|.
\end{aligned} \tag{8.3.18}$$

Proof. We proved in Theorem 8.3.11 that U has the required regularity, and trivially, $U(\cdot, 0) = U_0$ since for all $x \in \mathbb{R}$, the subset $\mathcal{A}_{x,0}$ reduces to $\{x\}$.

Claim: U is a subsolution. Fix $\varphi \in \mathbf{C}^1(\Omega)$ and let (x_0, t_0) be a local point of maximum for $U - \varphi$. Even if it replacing $U - \varphi$ by $U - \varphi - (U(x_0, t_0) - \varphi(x_0, t_0))$, we can assume that $U(x_0, t_0) - \varphi(x_0, t_0) = 0$. By definition, there exists $r > 0$ such that for all $(x, t) \in \Omega$,

$$\max\{|x - x_0|, |t - t_0|\} \leq r \implies U(x, t) - \varphi(x, t) \leq U(x_0, t_0) - \varphi(x_0, t_0) = 0. \tag{8.3.19}$$

The function $v \mapsto v\dfrac{\partial \varphi}{\partial x}(x_0, t_0) - L(x_0, v) \underset{|v| \to +\infty}{\longrightarrow} -\infty$. By continuity, it attains its maximum value on some compact subset $[-R, R]$ $(R > 1)$. Fix $0 < h \leq \dfrac{r}{R}$. Consider $\alpha \in [-R, R]$ a constant control and y the trajectory defined by

$$\forall s \in [t_0 - h, t_0], \quad y(s) = x_0 + (s - t_0)\alpha.$$

The dynamic programming principle ensures that

$$U(x_0, t_0) \leq \int_{t_0 - h}^{t} L(y(s), \alpha)\, \mathrm{d}s + U(y(t_0 - h), t_0 - h).$$

Since

$$|y(t_0 - h) - x_0| \leq \int_{t_0 - h}^{t_0} \alpha\, \mathrm{d}s = \alpha h \leq r,$$

we can use (8.3.19) to obtain that

$$\varphi(x_0, t_0) = U(x_0, t_0) \leq \int_{t_0 - h}^{t} L(y(s), \alpha)\, \mathrm{d}s + U(y(t_0 - h), t_0 - h) \leq \int_{t_0 - h}^{t} L(y(s), \alpha)\, \mathrm{d}s + \varphi(y(t_0 - h), t_0 - h),$$

which leads us to

$$\frac{\varphi(x_0, t_0) - \varphi(y(t_0 - h), t_0 - h)}{h} \leq \frac{1}{h} \int_{t_0 - h}^{t_0} L(y(s), \alpha)\, \mathrm{d}s\,.$$

Taking the limit when $h \to 0$, we get:

$$\frac{\partial \varphi}{\partial t}(x_0, t_0) + \left(\alpha \frac{\partial \varphi}{\partial x}(x_0, t_0) - L(x_0, \alpha) \right) \leq 0.$$

Taking the supremum for all $\alpha \in [-R, R]$, we obtain:

$$\frac{\partial \varphi}{\partial t}(x_0, t_0) + H\left(x_0, \frac{\partial \varphi}{\partial x}(x_0, t_0) \right) \leq 0.$$

Claim: U is a supsolution. Fix $\varphi \in \mathbf{C}^1(\Omega)$ and let (x_0, t_0) be a local point of minimum for $U - \varphi$. Again, it is not restrictive to assume that $U(x_0, t_0) - \varphi(x_0, t_0) = 0$. There exists $r > 0$ such that for all $(x, t) \in \Omega$,

$$\max\{|x - x_0|, |t - t_0|\} \leq r \implies U(x, t) - \varphi(x, t) \geq U(x_0, t_0) - \varphi(x_0, t_0) = 0. \tag{8.3.20}$$

Fix $0 < h \leq \min\left\{\dfrac{r}{\mathbf{C}_H}, r\right\}$ and y an optimal Lipschitz trajectory for $U(x_0, t_0)$. Since y realizes the minimum for $U(x_0, t_0)$, we also have:

$$U(x_0, t_0) = \int_{t_0-h}^{t_0} L(y(s), \dot{y}(s))\,\mathrm{d}s + U(y(t_0-h), t_0-h).$$

Consequently, since by Theorem 8.3.9 we have $|y(t_0) - x_0| \leq h\mathbf{C}_H \leq r$, we can write:

$$\begin{aligned}\varphi(x_0, t_0) = U(x_0, t_0) &= \int_{t_0-h}^{t_0} L(y(s), \dot{y}(s))\,\mathrm{d}s + U(y(t_0-h), t_0-h)\\ &\geq \int_{t_0-h}^{t_0} L(y(s), \dot{y}(s))\,\mathrm{d}s + \varphi(y(t_0-h), t_0-h),\end{aligned}$$

and like in the first step, this leads to

$$\frac{\partial\varphi}{\partial t}(x_0, t_0) + \dot{y}(t_0)\frac{\partial\varphi}{\partial x}(x_0, t_0) - L(x_0, \dot{y}(t_0)) \geq 0$$

and

$$\begin{aligned}\frac{\partial\varphi}{\partial t}(x_0, t_0) + H\left(x_0, \frac{\partial\varphi}{\partial x}(x_0, t_0)\right) &= \frac{\partial\varphi}{\partial t}(x_0, t_0) + \sup_{v\in\mathbb{R}}\left(v\frac{\partial\varphi}{\partial x}(x_0, t_0) - L(x_0, v)\right)\\ &\geq \frac{\partial\varphi}{\partial t}(x_0, t_0) + \dot{y}(t_0)\frac{\partial\varphi}{\partial x}(x_0, t_0) - L(x_0, \dot{y}(t_0)) \geq 0,\end{aligned}$$

concluding the proof. □

8.3.4 Two explicit minimizers

We proved in Sections 8.3.1-8.3.2 the existence of $\mathbf{C}^1$ minimizers for $\mathcal{J}_t$ ($t \geq 0$). In this section, we establish that for all $(x, t) \in \mathbb{R} \times (0, T]$, the minimal and maximal backward characteristics emanating from (x, t) (see Section 8.1.3) are minimizers of $\mathcal{J}_t$ in $\mathcal{A}_{x,t}$. First, we prove the

Lemma 8.3.13. *Let $U \in \mathbf{Lip}(\Omega)$ be the viscosity solution to (8.0.2) with initial data $U_0 \in \mathbf{Lip}(\mathbb{R})$. Fix $\xi, \zeta \in \mathbf{Lip}((0,T))$, $\xi \leq \zeta$. Then for all $s, \tau \in [0,T]$ ($s < \tau$), we have:*

$$\begin{aligned}&\int_{\xi(\tau)}^{\zeta(\tau)} U(x,\tau)\,\mathrm{d}x - \int_{\xi(s)}^{\zeta(s)} U(x,s)\,\mathrm{d}x + \int_s^\tau \int_{\xi(t)}^{\zeta(t)} H(x, \partial_x U(x,t))\,\mathrm{d}x\,\mathrm{d}t\\ &= \int_s^\tau \Big(\dot{\zeta}(t)U(\zeta(t),t) - \dot{\xi}(t)U(\xi(t),t)\Big)\,\mathrm{d}t\,.\end{aligned} \tag{8.3.21}$$

Proof. The proof follows the one of [67, Lemma 3.2] and we only give the details for the sake of completeness. From the PDE (8.0.2), we deduce that for all test functions $\varphi \in \mathbf{C}^1_{\mathbf{c}}(\Omega)$,

$$\int_0^T \int_{\mathbb{R}} (U(x,t)\partial_t\varphi(x,t) - H(x, \partial_x U(x,t))\varphi(x,t))\,\mathrm{d}x\,\mathrm{d}t = 0. \tag{8.3.22}$$

A standard approximation argument allows us to choose $\varphi \in \mathbf{W}^{1,\infty}(\Omega)$, which we do now. Fix $\varepsilon > 0$. As [67], define for all $x \in \mathbb{R}$ and $t \in [0,T]$:

$$\chi_\varepsilon(t) = \begin{cases} 0 & \text{if} \quad 0 \le t < s \\ \dfrac{t-s}{\varepsilon} & \text{if} \quad s \le t < s+\varepsilon \\ 1 & \text{if} \quad s+\varepsilon \le t < \tau \\ 1 + \dfrac{\tau - t}{\varepsilon} & \text{if} \quad \tau \le t < \tau + \varepsilon \\ 0 & \text{if} \quad t \ge \tau + \varepsilon; \end{cases} \qquad \psi_\varepsilon(x,t) = \begin{cases} 0 & \text{if} \quad x < \xi(t) - \varepsilon \\ 1 + \dfrac{x - \xi(t)}{\varepsilon} & \text{if} \quad \xi(t) - \varepsilon \le x < \xi(t) \\ 1 & \text{if} \quad \xi(t) \le x < \zeta(t) \\ 1 + \dfrac{\zeta(t) - x}{\varepsilon} & \text{if} \quad \zeta(t) \le x < \zeta(t) + \varepsilon \\ 0 & \text{if} \quad x \ge \zeta(t) + \varepsilon. \end{cases}$$

Let us make precise that for a.e. $t \in (0,T)$,

$$\|\chi_\varepsilon - \mathbb{1}_{[s,\tau]}\|_{\mathbf{L}^1} \underset{\varepsilon \to 0}{\longrightarrow} 0; \quad \chi_\varepsilon' = \frac{1}{\varepsilon}\mathbb{1}_{(s,s+\varepsilon)} - \frac{1}{\varepsilon}\mathbb{1}_{(\tau,\tau+\varepsilon)},$$

and for a.e. $(x,t) \in \Omega$,

$$\|\psi_\varepsilon - \mathbb{1}_{\{\xi(t) \le x \le \zeta(t)\}}\|_{\mathbf{L}^1} \underset{\varepsilon \to 0}{\longrightarrow} 0; \quad \partial_t \chi_\varepsilon(x,t) = -\frac{\dot\xi(t)}{\varepsilon}\mathbb{1}_{(\xi(t)-\varepsilon,\xi(t))}(x,t) + \frac{\dot\zeta(t)}{\varepsilon}\mathbb{1}_{(\zeta(t),\zeta(t)+\varepsilon)}(x,t).$$

We apply (8.3.22) with the test function $\psi_\varepsilon \chi_\varepsilon$ and let $\varepsilon \to 0$. Since $U \in \mathbf{Lip}(\Omega)$, standard computations lead to (8.3.21). □

From Lemma 8.3.13, we deduce the announced statement.

Theorem 8.3.14. *Let $U \in \mathbf{Lip}(\Omega)$ be the viscosity solution to (8.0.2) with initial data $U_0 \in \mathbf{Lip}(\mathbb{R})$. Fix $(x,t) \in \mathbb{R} \times (0,T]$ and let ξ (resp. ζ) be the minimal (resp. maximal) characteristics emanating from (x,t). Then*

$$U(x,t) = \mathcal{J}_t(\xi) = \mathcal{J}_t(\zeta).$$

Proof. We only prove the result for ζ. The details of the proof for ξ are similar so we omit them.
Fix $\varepsilon > 0$. Let us apply (8.3.21) with ζ and $\zeta - \varepsilon$ on $(0,t)$. After dividing by ε, we obtain:

$$\begin{aligned} &\frac{1}{\varepsilon}\int_{\zeta(t)-\varepsilon}^{\zeta(t)} U(y,t)\,\mathrm{d}y - \frac{1}{\varepsilon}\int_{\zeta(0)-\varepsilon}^{\zeta(0)} U_0(y)\,\mathrm{d}y + \frac{1}{\varepsilon}\int_0^t \int_{\zeta(s)-\varepsilon}^{\zeta(s)} H(y, \partial_x U(y,s))\,\mathrm{d}y\,\mathrm{d}s \\ &= \frac{1}{\varepsilon}\int_0^t \dot\zeta(s)\Big(U(\zeta(s),s) - U(\zeta(s)-\varepsilon,s)\Big)\,\mathrm{d}s\,. \end{aligned} \tag{8.3.23}$$

We let $\varepsilon \to 0$ in (8.3.23) using the facts that:
(i) U and U_0 are continuous;
(ii) $\partial_x U$ is the entropy solution (8.0.1) with initial data U_0', see Corollary 8.2.20 and Remark 8.2.4.
(iii) Since ζ is genuine, for a.e. $s \in (0,t)$, $\partial_x U(\cdot,s)$ has left-side and right-side limit at $x = \zeta(s)$ that coincide, see [67, Theorem 3.3]. Since $U \in \mathbf{Lip}(\Omega)$, this implies that for a.e. $s \in (0,t)$,

$U(\cdot,s)$ is differentiable at point $x=\zeta(s)$ and that $\partial_x U(\zeta(s),s)=\lim_{h\to\pm 0}\partial_x U(\zeta(s)\pm h,s)$.

We obtain:

$$U(\zeta(t),t)-U_0(\zeta(0))=\int_0^t\Big(\dot\zeta(s)\partial_x U(\zeta(s),s)-H\left(\zeta(s),\partial_x U(\zeta(s),s)\right)\Big)\,\mathrm{d}s\,. \tag{8.3.24}$$

Since ζ is genuine, there exists a function $\omega\in\mathbf{C}^1((0,t))$ such that (ζ,ω) is a solution to the ODE system (8.1.7) with final conditions $\zeta(t)=x$ and $\omega(t)=\partial_x U(x^+,t)$. Moreover, for a.e. $s\in(0,t)$, $\omega(s)=\partial_x U(\zeta(s),s)$. Combining these details with (8.3.24), we obtain:

$$\begin{aligned}U(x,t)-U_0(\zeta(0))&=\int_0^t\Big(\dot\zeta(s)\omega(s)-H\left(\zeta(s),\omega(s)\right)\Big)\,\mathrm{d}s\\&=\int_0^t\left(\frac{\partial H}{\partial u}(\xi(s),\omega(s))\omega(s)-H\left(\zeta(s),\omega(s)\right)\right)\mathrm{d}s\\&=\int_0^t L\left(\zeta(s),\frac{\partial H}{\partial u}(\xi(s),\omega(s))\right)\mathrm{d}s\\&=\int_0^t L\left(\zeta(s),\dot\zeta(s)\right)\mathrm{d}s\,,\end{aligned}$$

concluding the proof. □

We deduce:

Corollary 8.3.15. *Fix $t>0$, $x\in\mathbb{R}$ and $U_0\in\mathbf{Lip}(\mathbb{R})$. Then*

$$\min_{y\in\mathcal{A}_{x,t}}\mathcal{J}_t(y)=\min_{\substack{y(t)=x\\ y\in\mathbf{Lip}((0,t))}}\mathcal{J}_t(y)=\min_{\substack{y(t)=x\\ y\in\mathbf{S}_t}}\mathcal{J}_t(y), \tag{8.3.25}$$

where

$$\mathcal{S}_t:=\left\{\gamma\in\mathbf{C}^1([0,t])\;\middle|\;\begin{array}{l}\exists\theta\in\mathbf{C}^1([0,t]),\ \forall s\in(0,t),\\ \dot\gamma(s)=\dfrac{\partial H}{\partial u}(\gamma(s),\theta(s))\ \ \textit{and}\ \ \dot\theta(s)=-\dfrac{\partial H}{\partial x}(\gamma(s),\theta(s))\end{array}\right\}.$$

Proof. Let us call A,B and C the three quantities in (8.3.25). Clearly, we have

$$A\geq B\geq C.$$

Theorem 8.3.5 ensures that $A=B$. Theorem 8.3.14 ensures that $A=C$. □

8.4 Initial data identification for space-dependent flows

We now get back to the original problem, that is the initial data identification for the conservation law (8.0.1) and the Hamilton-Jacobi equation (8.0.2). In [61], the authors considered

equation (8.0.1) with an homogeneous flow $H = H(u)$. A crucial tool used by the authors is triple correspondence entropy solutions/viscosity solutions and Lax-Hopf formula, see [81, Section 3.3, Theorem 4]. In our framework, *i.e.* with $H \in \mathbf{C}^2(\mathbb{R}^2)$ satisfying (8.0.3)-(8.1.2), we established the correspondence entropy solutions/viscosity solutions in Section 8.2. In the heterogeneous case, there is no Lax-Hopf formula. Its equivalent is given by the correspondence viscosity solutions/calculus of variations, established in Section 8.3.
With these tools, we aim at extending the results of [61]. The objective of this section is to adapt results put forward by the authors of [61].

Let us fix some notations. For any initial data $u_0 \in \mathbf{L}^\infty(\mathbb{R})$ and $U_0 \in \mathbf{Lip}(\mathbb{R})$, we denote by $(x,t) \mapsto \mathbf{S}_t^{CL} u_0(x)$ the entropy solution to (8.0.1) and by $(x,t) \mapsto \mathbf{S}_t^{HJ} U_0(x)$ the viscosity solution to (8.0.2). For given profiles $w \in \mathbf{L}^\infty(\mathbb{R})$ and $W \in \mathbf{Lip}(\mathbb{R})$, we aim at providing a characterization of the subsets:

$$\begin{aligned} \mathbf{I}_T^{CL}(w) &= \left\{ u_0 \in \mathbf{L}^\infty(\mathbb{R}) \mid \mathbf{S}_T^{CL} u_0 = w \right\} \\ \mathbf{I}_T^{HJ}(W) &= \left\{ U_0 \in \mathbf{Lip}(\mathbb{R}) \mid \mathbf{S}_T^{HJ} U_0 = W \right\}. \end{aligned} \tag{8.4.1}$$

Remark 8.4.1. In [61], the authors gave a necessary and sufficient condition on a given profile $w \in \mathbf{L}^\infty(\mathbb{R})$ to ensure that $\mathbf{I}_T^{CL}(w) \neq \emptyset$; they also provided a full characterization of the set of initial data that evolve into a given profile, and finally, they described some geometric and topological properties of $\mathbf{I}_T^{CL}(w)$ and $\mathbf{I}_T^{HJ}(W)$. We will follow the same path in this section.

In light of Remark 8.2.1, we have the following result.

Lemma 8.4.1. *Let $w \in \mathbf{L}^\infty(\mathbb{R})$ such that $\mathbf{I}_T^{CL}(w) \neq \emptyset$. Then for all $x \in \mathbb{R}$, w admits left-side and right-side traces at point x.*

In light of this regularity, in the sequel, if $w \in \mathbf{L}^\infty(\mathbb{R})$ is such that $\mathbf{I}_T^{CL}(w) \neq \emptyset$, then for all $x \in \mathbb{R}$, by $w(x)$, we will mean the left-side trace of w at point x. Note that Lemma 8.4.1 translates as:

Lemma 8.4.2. *Let $W \in \mathbf{Lip}(\mathbb{R})$ such that $\mathbf{I}_T^{HJ}(W) \neq \emptyset$. Then for all $x \in \mathbb{R}$, W' admits left-side and right-side traces at point x.*

8.4.1 Characterization of $\mathbf{I}_T^{CL}(w)$ and $\mathbf{I}_T^{HJ}(W)$

This section is organized as follows. First we give a necessary condition on a profile $w \in \mathbf{L}^\infty(\mathbb{R})$ to verify $\mathbf{I}_T^{CL}(w) \neq \emptyset$. Then, using the correspondence conservation laws/Hamilton Jacobi equations/calculus of variations (see Sections 8.2-8.3), we strengthen this condition to obtain a sufficient condition on a profile $W \in \mathbf{Lip}(\mathbb{R})$ to verify $\mathbf{I}_T^{HJ}(W) \neq \emptyset$. Finally, given $W \in \mathbf{Lip}(\mathbb{R})$ such that $\mathbf{I}_T^{HJ}(W) \neq \emptyset$, we characterize the initial data that belong to $\mathbf{I}_T^{HJ}(W)$.

Necessary condition for $\mathbf{I}_T^{CL}(w) \neq \emptyset$

Let $w \in \mathbf{L}^\infty(\mathbb{R})$ and define the function

$$p_w : \left| \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \longmapsto & \xi_x(0), \end{array} \right. \tag{8.4.2}$$

where for all $x \in \mathbb{R}$, $\xi_x, \nu_x \in \mathbf{C}^1((0,T))$ are such that:

$$\begin{cases} \dot{\xi}_x(t) = \dfrac{\partial H}{\partial u}(\xi_x(t), \nu_x(t)) \\ \dot{\nu}_x(t) = -\dfrac{\partial H}{\partial x}(\xi_x(t), \nu_x(t)); \quad \xi_x(T) = x,\ \nu_x(T) = w(x). \end{cases}$$

Remark 8.4.2. In the homogeneous case, p_w reduces to

$$\forall x \in \mathbb{R}, \quad p_w(x) = x - T \cdot H'(w(x)),$$

see [61, Section 2].

Proposition 8.4.3. *Let $w \in \mathbf{L}^\infty(\mathbb{R})$ such that $\mathbf{I}_T^{CL}(w) \neq \emptyset$. Then p_w is nondecreasing.*

Proof. In light of both Lemma 8.1.6 and Lemma 8.4.1, since $\mathbf{I}_T^{CL}(w) \neq 0$, p_w is well-defined on $\mathbb{R}$. Fix $x, y \in \mathbb{R}$ $(x < y)$. Since $\mathbf{I}_T^{CL}(w) \neq 0$, p_w assigns to x (resp. y) the value at time $t = 0$ of the minimal backward generalized characteristics emanating from (x,T) (resp. from (y,T)) see Section 8.1.3. By [67, Theorem 3.2], ξ_x and ξ_y are genuine, hence they do not intersect in $(0,T)$, see [67, Corollary 3.2]. This implies in particular that $\xi_x(0) \leq \xi_y(0)$ *i.e.* p_w is nondecreasing. □

Remark 8.4.3. In the homogeneous case, the monotonicity of p_w reads:

$$\forall x, y \in \mathbb{R}, \quad x < y \implies H'(w(y)) - H'(w(x)) \leq \frac{y-x}{T},$$

which is the classical Oleinik condition on the decay of positive waves, see [68, Chapter 6].

Sufficient and necessary condition for $\mathbf{I}_T^{HJ}(W) \neq \emptyset$

We now provide a sufficient and necessary condition on a profile $W \in \mathbf{Lip}(\mathbb{R})$ to ensure that $\mathbf{I}_T^{HJ}(W) \neq 0$.

Notation 8.4.1. For any $W \in \mathbf{Lip}(\mathbb{R})$, denote by $\mathbf{C}_{H,W} > 0$ a constant such that

$$\forall r \in \mathbb{R}^+,\ r \geq \mathbf{C}_{H,W} \implies \frac{\phi(r)}{1+r} > \left(\sup_{\substack{x \in \mathbb{R} \\ |u| \leq \|W'\|_{\mathbf{L}^\infty}}} |H(x,u)| + \sup_{\substack{x \in \mathbb{R} \\ |v| \leq 1}} |L(x,v)| \right), \tag{8.4.3}$$

where ϕ is given by the Nagumo growth of H, see Corollary 8.1.4.

Let us prove some intermediate results.

Lemma 8.4.4. *Define the subset*

$$\mathcal{S} := \left\{ \gamma \in \mathbf{C}^1([0,T]) \;\middle|\; \begin{array}{l} \exists \theta \in \mathbf{C}^1([0,T]),\ \forall t \in (0,T), \\ \dot{\gamma}(t) = \dfrac{\partial H}{\partial u}(\gamma(t), \theta(t)) \ \textit{and}\ \dot{\theta}(t) = -\dfrac{\partial H}{\partial x}(\gamma(t), \theta(t)) \end{array} \right\}. \tag{8.4.4}$$

Then

$$\forall (x_0, x_T) \in \mathbb{R}^2,\ \exists \gamma \in \mathcal{S}, \quad \gamma(0) = x_0 \ \textit{and}\ \gamma(T) = x_T.$$

Proof. This is a consequence of Lemmas 8.1.11-8.1.12. □

Lemma 8.4.5. *Let $W \in \mathbf{Lip}(\mathbb{R})$. Define*

$$\forall x_0 \in \mathbb{R}, \quad U_0^*(x_0) := \sup_{\substack{\gamma \in \mathcal{S} \\ \gamma(0)=x_0}} \left(W(\gamma(T)) - \int_0^T L(\gamma(s), \dot{\gamma}(s))\, \mathrm{d}s \right). \tag{8.4.5}$$

Then $U_0^* \in \mathbf{Lip}(\mathbb{R})$, and in its definition, the sup *is attained. More precisely,*

$$\|(U_0^*)'\|_{\mathbf{L}^\infty} \leq T\left(\sup_{\substack{x \in \mathbb{R} \\ |v| \leq \mathbf{C}_{H,W}}} \left| \frac{\partial L}{\partial x}(x, v) \right| + \|W'\|_{\mathbf{L}^\infty} \right). \tag{8.4.6}$$

Moreover, for any Lipschitz curves γ realizing the maximum in (8.4.5), we have:

$$\|\dot{\gamma}\|_{\mathbf{L}^\infty} \leq \mathbf{C}_{H,W}.$$

Proof. First, remark that

$$\forall x_0 \in \mathbb{R}, \quad U_0^*(x_0) = - \inf_{\substack{\gamma \in \mathcal{S} \\ \gamma(0)=x_0}} \left(\int_0^T L(\gamma(s), \dot{\gamma}(s))\, \mathrm{d}s - W(\gamma(T)) \right).$$

Now applying the change of variables $\tau = T - s$,

$$\forall x_0 \in \mathbb{R}, \quad U_0^*(x_0) = - \inf_{\substack{y \in \mathcal{S} \\ y(T)=x_0}} \left(\int_0^T L(y(T-\tau), \dot{y}(T-\tau))\, \mathrm{d}\tau - W(y(0)) \right).$$

Then, in view of Corollary 8.3.15,

$$\begin{aligned} \forall x_0 \in \mathbb{R}, \quad U_0^*(x_0) &= - \inf_{\substack{y \in \mathbf{Lip}((0,T)) \\ y(T)=x_0}} \left(\int_0^T L(y(T-s), \dot{y}(T-s))\, \mathrm{d}s - W(y(0)) \right) \\ &= - \inf_{\substack{y \in \mathbf{W}^{1,1}((0,T)) \\ y(T)=x_0}} \left(\int_0^T L(y(T-s), \dot{y}(T-s))\, \mathrm{d}s - W(y(0)) \right) \end{aligned}$$

which falls exactly in the framework developed in Section 8.3, and the results of the statement follow, in light of Theorem 8.3.9 and Theorem 8.3.12. □

The function U_0^* will play a significant role in the sequel. First, note that

Lemma 8.4.6. *Fix $W \in \mathbf{Lip}(\mathbb{R})$ such that $\mathbf{I}_T^{HJ}(W) \neq \emptyset$ and let U_0^* be defined as in Lemma 8.4.5. Then for all $U_0 \in \mathbf{I}_T^{HJ}(W)$, we have*

$$U_0 \geq U_0^*. \tag{8.4.7}$$

Proof. Fix $x_0 \in \mathbb{R}$ and $\gamma \in \mathcal{S}$ such that $\gamma(0) = x_0$. Since $U_0 \in \mathbf{I}_T^{HJ}(W)$, we have:

$$\begin{aligned} W(\gamma(T)) - \int_0^T L(\gamma(s), \dot{\gamma}(s))\, \mathrm{d}s &= \inf_{\substack{\zeta \in \mathcal{S} \\ \zeta(T)=\gamma(T)}} \left(\int_0^T L(\zeta(s), \dot{\zeta}(s))\, \mathrm{d}s + U_0(\zeta(0)) \right) - \int_0^T L(\gamma(s), \dot{\gamma}(s))\, \mathrm{d}s \\ &\underset{\zeta=\gamma}{\leq} U_0(\gamma(0)) = U_0(x_0). \end{aligned}$$

By taking the supremum on $\gamma \in \mathcal{S}$, we obtain $U_0^* \leq U_0$. □

Lemma 8.4.7. *Fix $W \in \mathbf{Lip}(\mathbb{R})$ and let U_0^* be defined as in Lemma 8.4.5. Define the subset:*

$$\mathcal{G} := \left\{ (x_0, x_T) \in \mathbb{R}^2 \;\middle|\; \exists \gamma \in \mathcal{S}, \begin{array}{l} (i)\ \gamma(0) = x_0,\ \gamma(T) = x_T \\ (ii)\ U_0^*(x_0) = W(x_T) - \displaystyle\int_0^T L(\gamma(s), \dot\gamma(s))\,\mathrm{d}s \end{array} \right\}. \tag{8.4.8}$$

Then $\mathcal{G}$ has the following properties:

(i) $\mathcal{G}$ is surjective in the following sense:

$$\forall x_0 \in \mathbb{R},\ \exists x_T \in \mathbb{R}, \quad (x_0, x_T) \in \mathcal{G}. \tag{8.4.9}$$

(ii) $\mathcal{G}$ is a closed subset of $\mathbb{R}^2$;

(iii) for all $(x_0, x_T) \in \mathcal{G}$, we have

$$|x_0 - x_T| \le T\mathbf{C}_{H,W}; \tag{8.4.10}$$

(iv) $\mathcal{G}$ is monotone in the following sense:

$$\forall (x_0, x_T), (y_0, y_T) \in \mathcal{G}, \quad \Big(x_0 < y_0 \implies x_T \le y_T\Big) \text{ and } \Big(x_T < y_T \implies x_0 \le y_0\Big). \tag{8.4.11}$$

Proof. *(i)* Property (8.4.9) comes from the definition of U_0^* and Lemma 8.4.5.

(ii) Let $(x_0^{(n)}, x_T^{(n)})_n$ be a sequence of $\mathcal{G}$ which converges to some $(x_0, x_T) \in \mathbb{R}^2$. By definition, for all $n \in \mathbb{N}$, there exists $\gamma^{(n)} \in \mathcal{S}$ such that

$$x_0^{(n)} = \gamma^{(n)}(0); \quad x_T^{(n)} = \gamma^{(n)}(T); \quad U_0^*(x_0^{(n)}) = W(x_T^{(n)}) - \int_0^T L(\gamma^{(n)}(s), \dot\gamma^{(n)}(s))\,\mathrm{d}s\,. \tag{8.4.12}$$

For all $n \in \mathbb{N}$, let us denote by $\theta^{(n)} \in \mathbf{C}^1((0,T))$ the curve associated with γ^n, given by $(x_0^{(n)}, x_T^{(n)}) \in \mathcal{G}$.
Lemma 8.4.5 ensures that

$$\forall n \in \mathbb{N}, \quad \|\dot\gamma^{(n)}\|_{\mathbf{L}^\infty} \le \mathbf{C}_{H,W}.$$

Note that for all $n \in \mathbb{N}$,

$$\left|\frac{\partial H}{\partial u}(\gamma^{(n)}(0), \theta^{(n)}(0))\right| = \left|\dot\gamma^{(n)}(0)\right| \le \mathbf{C}_{H,W},$$

which provides the boundedness of $(\theta^{(n)}(0))_n$. Therefore, up to a subsequence, we can assume that $(\theta^{(n)}(0), \gamma^{(n)}(0))_n$ converges to (θ_0, x_0) with $\theta_0 \in \mathbb{R}$. Using the continuity of the flow of the Hamiltonian system in the definition (8.4.4), we establish the existence of $\gamma, \theta \in \mathbf{C}([0,T])$ such that $(\gamma^{(n)})_n$ and $(\theta^{(n)})_n$ converge uniformly on $[0,T]$ to γ and θ, respectively.
From

$$\forall n \in \mathbb{N},\ \forall t \in [0,T], \quad \dot\gamma^{(n)}(t) = \frac{\partial H}{\partial u}(\gamma^{(n)}(t), \theta^{(n)}(t))$$

$$\dot\theta^{(n)}(t) = -\frac{\partial H}{\partial x}(\gamma^{(n)}(t), \theta^{(n)}(t)),$$

we deduce that $(\dot\gamma^{(n)})_n$ and $(\dot\theta^{(n)})_n$ converge uniformly on $[0,T]$ to $\dfrac{\partial H}{\partial u}(\gamma,\theta)$ and $-\dfrac{\partial H}{\partial x}(\gamma,\theta)$, respectively. Moreover, by passing to the limit in the equalities

$$\forall n\in\mathbb{N},\ \forall t\in[0,T],\quad \gamma^{(n)}(t)=x_0^{(n)}+\int_0^t\frac{\partial H}{\partial u}(\gamma^{(n)}(s),\theta^{(n)}(s))\,\mathrm{d}s$$
$$\theta^{(n)}(t)=\theta^{(n)}(0)-\int_0^t\frac{\partial H}{\partial x}(\gamma^{(n)}(s),\theta^{(n)}(s))\,\mathrm{d}s\,,$$

we deduce that:

— $\gamma,\theta\in\mathbf{C}^1((0,T))$ and $\dot\gamma=\dfrac{\partial H}{\partial u}(\gamma,\theta)$ and $\dot\theta-\dfrac{\partial H}{\partial x}(\gamma,\theta)$;

— $(\dot\gamma^{(n)})_n$ and $(\dot\theta^{(n)})_n$ converge uniformly on $[0,T]$ to $\dot\gamma$ and $\dot\theta$, respectively.

Finally, we let $n\to+\infty$ in (8.4.12) to conclude that first that $\gamma\in\mathcal{S}$ and then, that $(x_0,x_T)\in\mathcal{G}$.

(iii) Let $(x_0,x_T)\in\mathcal{G}$ and let $\gamma\in\mathcal{S}$ linking x_0 and x_T. Then in view of Lemma 8.4.5, we have

$$|x_0-x_T=|\gamma(0)-\gamma(T)|\le T\|\dot\gamma\|_{\mathbf{L}^\infty}\le T\mathbf{C}_{H,W}.$$

(iv) We only prove the first implication in (8.4.9), the details of the proof for the second one are similar so we omit them.

Let $\gamma,\zeta\in\mathcal{S}$ be two minimizers for $U_0^*(x_0)$ and $U_0^*(y_0)$, respectively. By assumption, we have $\gamma(0)<\zeta(0)$. We now check that γ and ζ do not intersect in $[0,T)$; this will prove the statement. Suppose instead that

$$\exists\tau\in(0,T),\quad \gamma(\tau)=\zeta(\tau).\tag{8.4.13}$$

Define the concatenation

$$\forall t\in[0,T],\quad \xi(t)=\begin{cases}\zeta(t) & \text{if}\quad 0\le t\le\tau\\ \gamma(t) & \text{if}\quad \tau<t\le T.\end{cases}$$

Clearly, $\xi\in\mathbf{Lip}(\mathbb{R})$ and $\xi(0)=y_0$. Remark that $\dot\gamma(\tau)\neq\dot\zeta(\tau)$. If it was the case, then by denoting $\theta,\omega\in\mathbf{C}^1((0,T))$ the curves associated with γ and ζ, respectively, given by $(x_0,x_T),(y_0,y_T)\in\mathcal{G}$, then we would have:

$$\dot\gamma(\tau)=\dot\zeta(\tau)\iff\frac{\partial H}{\partial u}(\gamma(\tau),\theta(\tau))=\frac{\partial H}{\partial u}(\zeta(\tau),\omega(\tau))$$
$$\iff\frac{\partial H}{\partial u}(\gamma(\tau),\theta(\tau))=\frac{\partial H}{\partial u}(\gamma(\tau),\omega(\tau))\iff\theta(\tau)=\omega(\tau),$$

since $u\mapsto\dfrac{\partial H}{\partial u}(\gamma(\tau),u)$ is bijection. However this would contradicts Cauchy Lipschitz theorem. In particular, ξ is not differentiable at point τ. Moreover, since γ and ζ are minimizers,

we have, in light of Corollary 8.3.15:

$$\begin{aligned}
U_0^*(y_0) - W(y_T) &= -\inf_{\substack{y\in\mathcal{S}\\ y(0)=y_0}} \left(\int_0^T L(y(s),\dot y(s))\,\mathrm{d}s\right)\\
&= -\inf_{\substack{y\in\mathbf{Lip}((0,T))\\ y(0)=y_0}} \left(\int_0^T L(y(s),\dot y(s))\,\mathrm{d}s\right)\\
&= -\inf_{\substack{y\in\mathbf{Lip}((0,\tau))\\ y(0)=y_0, y(\tau)=\xi(\tau)}} \left(\int_0^\tau L(y(s),\dot y(s))\,\mathrm{d}s\right) - \inf_{\substack{y\in\mathbf{Lip}((\tau,T))\\ y(\tau)=\xi(\tau), y(T)=y_T}} \left(\int_\tau^T L(y(s),\dot y(s))\,\mathrm{d}s\right)\\
&= -\int_0^\tau L(\zeta(s),\dot\zeta(s))\,\mathrm{d}s - \int_\tau^T L(\gamma(s),\dot\gamma(s))\,\mathrm{d}s\\
&= -\int_0^T L(\xi(s),\dot\xi(s))\,\mathrm{d}s\,.
\end{aligned}$$

This ensures that ξ is a Lipschitz minimizer for $U_0^*(y_0)$, therefore $\xi \in \mathbf{W}^{2,\infty}((0,T))$, see Corollary 8.3.7. However, this contradicts the fact that ξ is not differentiable at point $t=\tau$. We conclude that γ and ζ do not cross in $[0,T)$ which implies that $\gamma(T) \le \zeta(T)$ *i.e.* $x_T \le y_T$. □

Remark 8.4.4. Remark that in Lemma 8.4.7, Properties *(ii)-(iii)-(iv)* do not rely on the definition of U_0^*, only its Lipschitz continuity was required. In particular, for any Lipschitz function U_0, we can define its associated subset $\mathcal{G}_{U_0}$ by (8.4.8) (replacing U_0^* by U_0) and the subset $\mathcal{G}_{U_0}$ will also satisfy Properties *(ii)-(iii)-(iv)*. In contrast, Property (8.4.9) is intrinsic to U_0^*.

We are now in position to characterize the profiles which are reachable.

Theorem 8.4.8. *Fix $W \in \mathbf{Lip}(\mathbb{R})$. Let U_0^* and $\mathcal{G}$ be defined as in Lemmas 8.4.5-8.4.7. Then the following conditions are equivalent.*

1. *$U_0^* \in \mathbf{I}_T^{HJ}(W)$.*
2. *$\mathbf{I}_T^{HJ}(W) \neq \emptyset$.*
3. *(i) $p_{W'}$ is nondecreasing and (ii) $\mathcal{G}$ has the following maximal property:*

$$(x_0,x_T'),(x_0,x_T'') \in \mathcal{G} \implies \forall x_T \in [x_T',x_T''],\ (x_0,x_T) \in \mathcal{G}. \tag{8.4.14}$$

Proof. *1.* $\implies$ *2.* is clear.

2. $\implies$ *3.* Suppose that $\mathbf{I}_T^{HJ}(W) \neq \emptyset$. We have already established that *(i)* holds, see Proposition 8.4.3.
Fix now $U_0 \in \mathbf{I}_T^{HJ}(W)$, $(x_0,x_T'),(x_0,x_T'') \in \mathcal{G}$ $(x_T' < x_T'')$ and $x_T \in (x_T',x_T'')$.
Let $\gamma,\zeta \in \mathcal{S}$ two minimizers connecting (x_0,x_T') and (x_0,x_T''), respectively, and let ξ be the

minimal backward generalized characteristics emanating from (x_T, T), associated with the PDE (8.0.1) with initial data U_0'. Since ξ is genuine, $\xi \in \mathcal{S}$ and by Theorem 8.3.14,

$$W(x_T) = \int_0^T L(\xi(s), \dot{\xi}(s))\,\mathrm{d}s + U_0(\xi(0)) \geq \int_0^T L(\xi(s), \dot{\xi}(s))\,\mathrm{d}s + U_0^*(\xi(0)).$$

Above, we used the fact that $U_0 \geq U_0^*$, see Lemma 8.4.6. We deduce that

$$U_0^*(\xi(0)) \leq W(x_T) - \int_0^T L(\xi(s), \dot{\xi}(s))\,\mathrm{d}s\,.$$

By definition of U_0^*, we have equality above, and therefore ξ is a maximizer of U_0^*. We deduce that $(\xi(0), x_T) \in \mathcal{G}$. By (8.4.11),

$$x_T' < x_T \implies x_0 \leq \xi(0) \quad \text{and} \quad x_T'' > x_T \implies x_0 \geq \xi(0).$$

We deduce that $\xi(0) = x_0$ and therefore, $(x_0, x_T) \in \mathcal{G}$.

3. $\implies$ *1.* We now show that assumptions *(i)-(ii)* imply that $U_0^* \in \mathbf{I}_T^{HJ}(W)$. We first check that:

$$\forall x_T \in \mathbb{R},\ \exists x_0 \in \mathbb{R}, \quad (x_0, x_T) \in \mathcal{G}. \tag{8.4.15}$$

Fix $x_T \in \mathbb{R}$ and introduce the subset:

$$\mathbf{E} = \{x \in \mathbb{R} \mid \exists y < x_T,\ (x, y) \in \mathcal{G}\}\,.$$

Since $\mathbf{E}$ is nonempty and bounded by above since, as a consequence of (8.4.9) and (8.4.10),

$$(-\infty, -T\mathbf{C}_{H,W} + x_T) \subset \mathbf{E}$$

and

$$\forall x \in \mathbf{E}, \quad x \geq (x - y) + y \geq -|x - y| + x_T \geq -T\mathbf{C}_{H,W} + x_T.$$

Therefore, $\overline{x} = \sup \mathbf{E}$ is well-defined. Likewise, the subset

$$\mathbf{F} = \{x \in \mathbb{R} \mid \exists y > x_T,\ (x, y) \in \mathcal{G}\}\,.$$

is nonempty and bounded by below, therefore, $\underline{x} = \inf \mathbf{F}$ is well-defined. Clearly, we have $\overline{x} \leq \underline{x}$.

Let $(x_n)_n$ be a sequence of $\mathbf{E}$ which converges to $\overline{x}$. For all $n \in \mathbb{N}$, there exists $y_n < x_T$ such that (x_n, y_n). Since $(x_n)_n$ is bounded, $(y_n)_n$ is bounded as well, as a consequence of (8.4.10). Up to the extraction of a subsequence, we can assume that $(y_n)_n$ converges to some $\overline{y} \leq x_T$. Since $\mathcal{G}$ is closed, see Lemma 8.4.7 *(ii)*, $(\overline{x}, \overline{y}) \in \mathcal{G}$. The same way, there exists $\underline{y} \geq x_T$ such that $(\underline{x}, \underline{y}) \in \mathcal{G}$. Let us conclude the proof by a case by case study.

Case 1: $\overline{x} = \underline{x} := x_0$. Since $\overline{y} \leq x_T \leq \underline{y}$, by (8.4.14), we have:

$$(x_0, \overline{y}), (x_0, \underline{y}) \in \mathcal{G} \implies (x_0, x_T) \in \mathcal{G}.$$

Case 2: $\overline{x} < \underline{x}$. Fix $x_0 \in (\overline{x}, \underline{x})$. By Lemma 8.4.7, there exists $y \in \mathbb{R}$ such that $(x_0, y) \in \mathcal{G}$. However, by the definition of $\overline{x}$, we necessarily have $y \geq x_T$. The same way, the definition of

$\underline{x}$ ensures that $y \leq x_T$. We proved that $y = x_T$ and therefore, $(x_0, x_T) \in \mathcal{G}$ for any $x_0 \in (\overline{x}, \underline{x})$.

Equality (8.4.15) rewrites as:

$$\forall x \in \mathbb{R},\ \exists \gamma \in \mathcal{S},\ \gamma(T) = x, \quad W(x) = \int_0^T L(\gamma(s), \dot{\gamma}(s))\,\mathrm{d}s + U_0^*(\gamma(0)). \tag{8.4.16}$$

Moreover, the definition of U_0^* ensures that for all $x \in \mathbb{R}$ and for all curves $\gamma \in \mathcal{S}$, $\gamma(T) = x$, we have

$$\int_0^T L(\gamma(s), \dot{\gamma}(s))\,\mathrm{d}s + U_0^*(\gamma(0)) \geq W(\gamma(T)) = W(x). \tag{8.4.17}$$

Both (8.4.16) and (8.4.17) combined implies that

$$\begin{aligned} \forall x \in \mathbb{R}, \quad W(x) &= \inf_{\substack{\gamma \in \mathcal{S} \\ \gamma(T)=x}} \left(\int_0^T L(\gamma(s), \dot{\gamma}(s))\,\mathrm{d}s + U_0^*(\gamma(0)) \right) \\ &= \inf_{\substack{\gamma \in \mathbf{Lip}((0,T)) \\ \gamma(T)=x}} \left(\int_0^T L(\gamma(s), \dot{\gamma}(s))\,\mathrm{d}s + U_0^*(\gamma(0)) \right), \end{aligned}$$

in light of Corollary 8.3.15. This last equality exactly means that the viscosity solution U to (8.0.2) associated with initial data U_0^* verifies $U(\cdot, T) = W$, using the correspondence viscosity solution/calculus of variations, see Theorem 8.3.12. We proved that $U_0^* \in \mathbf{I}_T^{HJ}(W)$ and therefore $\mathbf{I}_T^{HJ}(W) \neq \emptyset$. □

Corollary 8.4.9. *Fix $w \in \mathbf{L}^\infty(\mathbb{R})$ and let W be a primitive of w. Let U_0^* and $\mathcal{G}$ be defined as in Lemmas 8.4.5-8.4.7. Then the following conditions are equivalent.*

1. $(U_0^*)' \in \mathbf{I}_T^{CL}(w)$.
2. $\mathbf{I}_T^{CL}(w) \neq \emptyset$.
3. *(i) p_w is nondecreasing and (ii) $\mathcal{G}$ has the property (8.4.14).*

Remark 8.4.5. Let us mention that in the homogeneous case, only assumption *(i)* was required.

Sufficient and necessary condition for $U_0 \in \mathbf{I}_T^{HJ}(W)$

Theorem 8.4.10. *Fix $W \in \mathbf{Lip}(\mathbb{R})$ such that $\mathbf{I}_T^{HJ}(W) \neq \emptyset$ and let U_0^* be defined as in Lemma 8.4.7. Fix $U_0 \in \mathbf{Lip}(\mathbb{R})$. Then $U_0 \in \mathbf{I}_T^{HJ}(W)$ if and only if:*

(i) $U_0 \geq U_0^*$;

(ii) $U_0 = U_0^*$ on $\overline{p_{W'}(\mathbb{R})}$.

Proof. $\Longrightarrow$ Point *(i)* comes from Lemma 8.4.6. Let us prove *(ii)*. Fix $x_0 \in p_{W'}(\mathbb{R})$. By definition,

$$\exists z_0 \in \mathbb{R}, \quad x_0 = p_{W'}(z_0).$$

This last equality simply means that x_0 is the value at time $t = 0$ of the minimal backward characteristics emanating from (z_0, T), denoted by ξ. Since $U_0 \in \mathbf{I}_T^{HJ}(W)$, Theorem 8.3.14 ensures that

$$W(z_0) = \int_0^T L(\xi(s), \dot{\xi}(s))\,\mathrm{d}s + U_0(x_0).$$

On the other hand,

$$U_0^*(x_0) = \sup_{\substack{\gamma \in \mathcal{S} \\ \gamma(0)=x_0}} \left(W(\gamma(T)) - \int_0^T L(\gamma(s), \dot{\gamma}(s)) \right) \mathrm{d}s$$

$$\geq_{\gamma=\xi} W(z_0) - \int_0^T L(\xi(s), \dot{\xi}(s))\,\mathrm{d}s = U_0(x_0).$$

We just proved that $U_0 = U_0^*$ on $p_{W'}(\mathbb{R})$. Since these two functions are continuous, they also coincide on $\overline{p_{W'}(\mathbb{R})}$.

$\Longleftarrow$ Let us make precise that since $\mathbf{I}_T^{HJ}(W) \neq \emptyset$, then Points *(i)-(ii)* of Theorem 8.4.8 hold and therefore, $U_0^* \in \mathbf{I}_T^{HJ}(W)$.
Fix $x \in \mathbb{R}$. Note that the assumption $U_0 \geq U_0^*$ immediately implies:

$$W(x) \leq \inf_{\substack{\gamma \in \mathcal{S} \\ \gamma(T)=x}} \left(\int_0^T L(\gamma(s), \dot{\gamma}(s))\,\mathrm{d}s + U_0(\gamma(0)) \right). \tag{8.4.18}$$

Denote by ξ the minimal backward characteristics emanating from (x, T). Using both the fact that $U_0^* \in \mathbf{I}_T^{HJ}(W)$ and that ξ is a minimizer, we have:

$$W(x) = \int_0^T L(\xi(s), \dot{\xi}(s))\,\mathrm{d}s + U_0^*(\xi(0)).$$

Clearly, $\xi(0) \in \overline{p_{W'}(\mathbb{R})}$ and therefore, we can replace $U_0^*(\xi(0))$ by $U_0(\xi(0))$ in the last equality. This ensures that we have equality in (8.4.18), which means that $U_0 \in \mathbf{I}_T^{HJ}(W)$. □

Corollary 8.4.11. *Fix $W \in \mathbf{Lip}(\mathbb{R})$ such that $\mathbf{I}_T^{HJ}(W) \neq \emptyset$. Then $\mathbf{I}_T^{HJ}(W)$ is a convex cone having at its vertex U_0^* defined in Lemma 8.4.5.*

Proof. Immediate using the characterization of Theorem 8.4.10. □

8.4.2 Illustration of differences with the homogeneous case

At this point, one could think that the inverse design problem for heterogeneous flow is a simple adapatation of the homogeneous case, especially in view of Theorem 8.4.10 and Corollary 8.4.11. Let us highlight in this section some subtleties. Let H be of the form

$$H(x, u) = \frac{u^2}{2} + g(x), \tag{8.4.19}$$

where g has the shape depicted in Figure 8.3, left. The main features of g are:

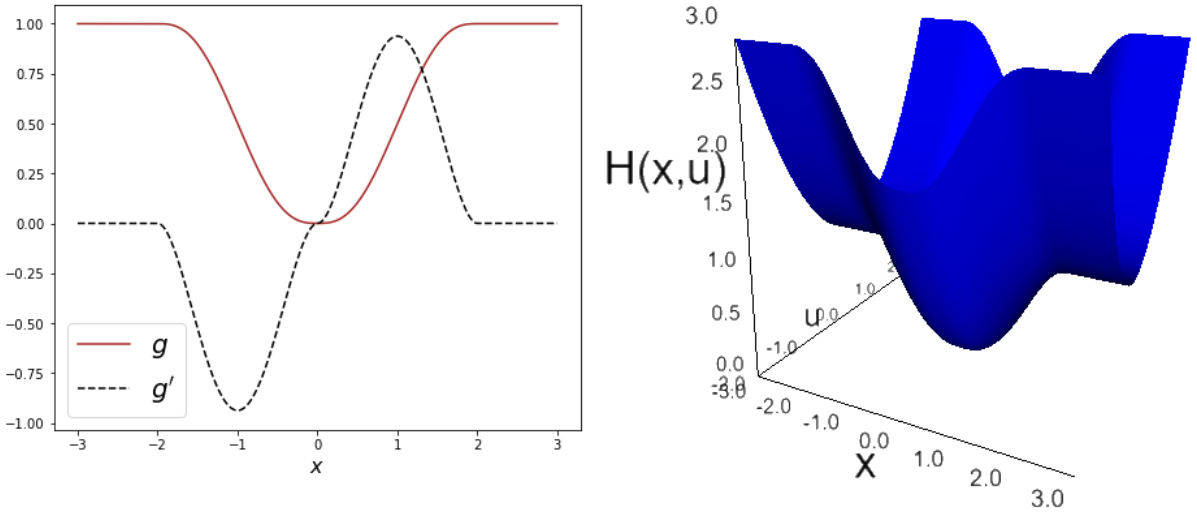

Figure 8.3 – Representation of the Hamiltonian.

— $g \in \mathbf{C}^2(\mathbb{R})$ is an even function;
— $g \equiv 1$ on $[2, +\infty)$;
— $g(0) = 0$ and $g' > 0$ on $(0, 2)$.

The resulting Hamiltonian is represented in Figure 8.3, right.
With this flow, the conservation law (8.0.1) reduces to the inviscid Burgers equation (0.1.3) with source term $-g'$. We attach to it the initial data

$$u_0(x) = \begin{cases} -2 & \text{if } x < 0 \\ 2 & \text{if } x > 0. \end{cases}$$

We compute the numerical solution with a standard finite volume scheme, that is replacing the marching formula (1.1.1) by

$$u^{n+1}_{j+1/2} = u^n_{j+1/2} - \frac{\Delta t}{\Delta x}(f^n_{j+1} - f^n_j) - \frac{\Delta t}{\Delta x}(g(x_{j+1}) - g(x_j)).$$

for instance. The evolution of the numerical solution, with time horizon $T = 3.5$ is represented in Figure 8.4. Remark, and this is intrinsic to the heterogeneous case, that the initial rarefaction profile evolves into a shock wave.
Let us denote by $w = u(\cdot, T)$ ($T = 3.5$) the final profile which is, by construction reachable. In Figure 8.5, we draw backward characteristics emanating from $(0, T)$.
Let us comment on the figure.
On the left, we drew the minimal (in brown) and maximal (in blue) backward characteristics emanating from $(0, T)$, denoted by γ and ζ, respectively. As expected, they do not cross in $(0, T)$, and recall that these curves are minimizers for the calculus of variations problem, see Theorem 8.3.14.

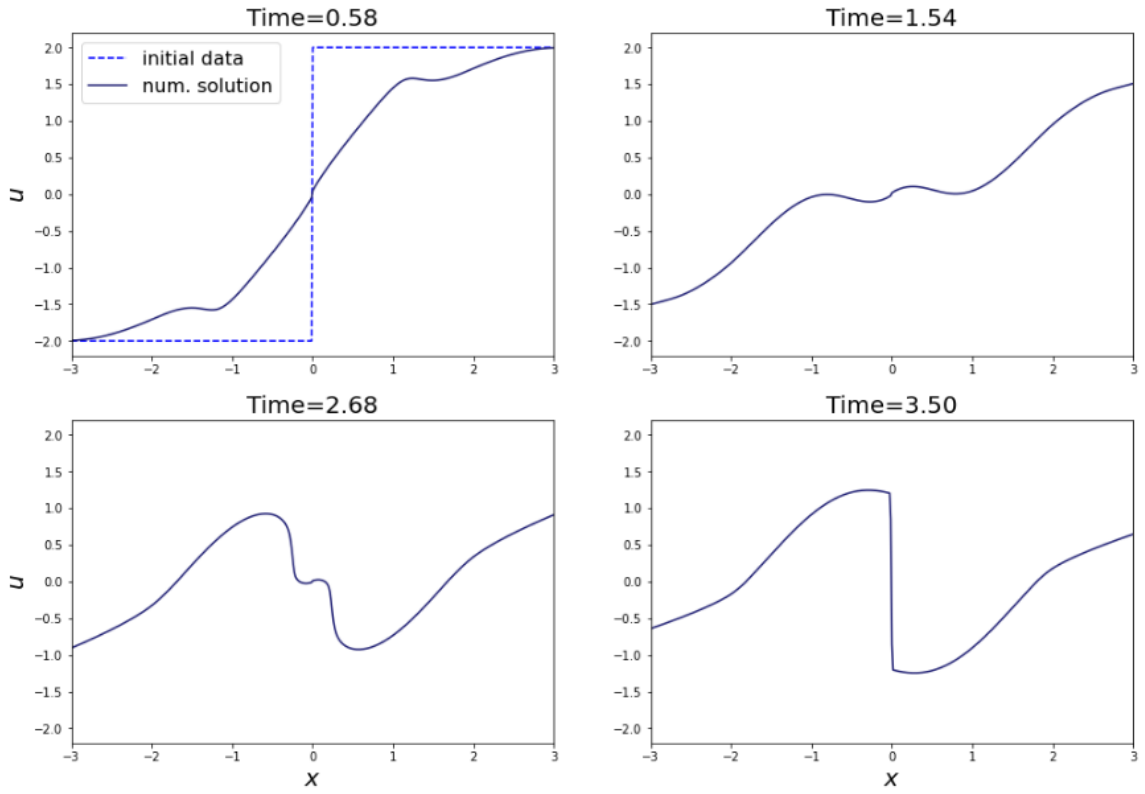

Figure 8.4 – The numerical solution $u_\Delta(\cdot, t)$ at time 3.5; for an animated evolution of the numerical solution, follow: https://utbox.univ-tours.fr/s/8gy5ByCw5gF5zb5.

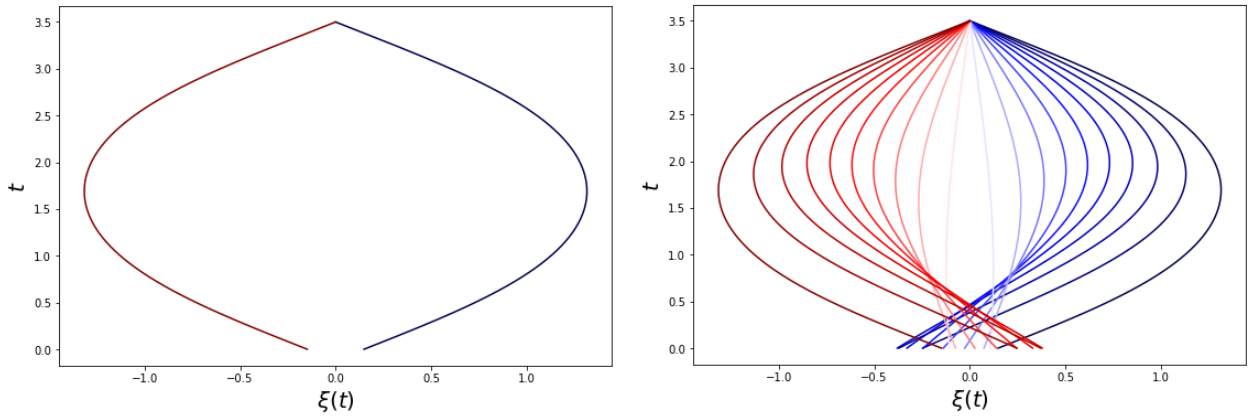

Figure 8.5 – Backward characteristics.

Then, on the right we drew Hamiltonian rays *i.e.* the projections ξ of the curves (ξ, ν) solving the system:

$$\begin{cases} \dot{\xi}(t) = \nu(t) \\ \dot{\nu}(t) = -g'(\xi(t)), \quad \xi(T) = 0,\ \nu(T) = \nu_T, \quad w(0+) \leq \nu_T \leq w(0-). \end{cases}$$

In view of the surjectivity of the shooting function, see Lemmas 8.1.11-8.1.12, we know that for any $x_0 \in [\gamma(0), \zeta(0)]$, we can connect $(0, T)$ and $(x_0, 0)$ by a Hamiltonian ray. Let us make precise that in the homogeneous case, there was a unique Hamiltonian ray joining $(0, T)$ and $(x_0, 0)$. Figure 8.5 highlights two facts.
First, the shooting function is not bijective anymore. And second, since w is reachable, we know that p_w is nondecreasing, see Theorem 8.4.8. However, this monotonicity does not imply maximality (in the sense of (8.4.14)) for the subset $\mathcal{G}_{u_0}$ associated with u_0, see (8.4.8), as we see Hamiltonian rays crossing γ and ζ.

BIBLIOGRAPHY

[1] Adimurthi and G. D. Veerappa Gowda. Conservation law with discontinuous flux. *J. Math. Kyoto Univ.*, **43**(1):27–70, 2003.

[2] Adimurthi, Siddhartha Mishra, and GD Veerappa Gowda. Optimal entropy solutions for conservation laws with discontinuous flux-functions. *Journal of Hyperbolic Differential Equations*, **2**(4):783–837, 2005.

[3] J. Aleksić and D. Mitrović. Strong traces for averaged solutions of heterogeneous ultra-parabolic transport equations. *Journal of Hyperbolic Differential Equations*, **10**(4):659–676, 2013.

[4] D. Amadori, P. Goatin, and M. D. Rosini. Existence results for Hughes' model for pedestrian flows. *J. Math. Anal. Appl.*, 420(1):387–406, 2014.

[5] B. Andreianov. New approaches to describing admissibility of solutions of scalar conservation laws with discontinuous flux. *ESAIM: Proceedings and Surveys*, **50**:40–65, 2015.

[6] B. Andreianov and C. Cancès. Vanishing capillarity solutions of Buckley-Leverett equation with gravity in two-rocks' medium. *Comput. Geosci.*, 17(3):551–572, 2013.

[7] B. Andreianov and C. Cancès. On interface transmission conditions for conservation laws with discontinuous flux of general shape. *J. Hyperbolic Differ. Equ.*, 12(2):343–384, 2015.

[8] B. Andreianov, G. M. Coclite, and C. Donadello. Well-posedness for vanishing viscosity solutions of scalar conservation laws on a network. *Discrete & Continuous Dynamical Systems*, **37**(11):5913–5942, 2017.

[9] B. Andreianov, C. Donadello, U. Razafison, and M. D. Rosini. Qualitative behaviour and numerical approximation of solutions to conservation laws with non-local point constraints on the flux and modeling of crowd dynamics at the bottlenecks. *ESAIM: M2AN*, **50**(5):1269–1287, 2016.

[10] B. Andreianov, C. Donadello, U. Razafison, and M. D. Rosini. Analysis and approximation of one-dimensional scalar conservation laws with general point constraints on the flux. *Journal de Mathématiques Pures et Appliquées*, **116**:309–346, 2018.

[11] B. Andreianov, C. Donadello, and M. D. Rosini. Crowd dynamics and conservation laws with nonlocal constraints and capacity drop. *Mathematical Models and Methods in Applied Sciences*, **24**(13):2685–2722, 2014.

[12] B. Andreianov, C. Donadello, and M. D. Rosini. A second-order model for vehicular traffics with local point constraints on the flow. *Mathematical Models and Methods in Applied Sciences*, **26**(04):751–802, 2016.

[13] B. Andreianov, C. Donadello, and M. D. Rosini. Entropy solutions for a two-phase transition model for vehicular traffic with metastable phase and time depending point constraint on the density flow. *NoDEA Nonlin. Differ. Equ. Appl.*, **28**(3):1–37, 2021.

[14] B. Andreianov, P. Goatin, and N. Seguin. Finite volume schemes for locally constrained conservation laws. *Numerische Mathematik*, **115**(4):609–645, 2010.

[15] B. Andreianov, K. Karlsen, and N. H. Risebro. A theory of L^1-dissipative solvers for scalar conservation laws with discontinuous flux. *Archive for Rational Mechanics and Analysis*, **201**(1):27–86, 2011.

[16] B. Andreianov, K. H. Karlsen, and N. H. Risebro. On vanishing viscosity approximation of conservation laws with discontinuous flux. *Netw. Heterog. Media*, **5**(3):617–633, 2010.

[17] B. Andreianov, F. Lagoutière, N. Seguin, and T. Takahashi. Well-posedness for a one-dimensional fluid-particle interaction model. *SIAM J. Math. Anal.*, **46**(2):1030–1052, 2014.

[18] B. Andreianov and D. Mitrović. Entropy conditions for scalar conservation laws with discontinuous flux revisited. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, **32**(6):1307–1335, 2015.

[19] B. Andreianov and M. D. Rosini. Microscopic selection of solutions to scalar conservation laws with discontinuous flux in the context of vehicular traffic. In *Conference on Semigroups of Operators: Theory and Applications*, pages 113–135. Springer, 2018.

[20] B. Andreianov and M. D. Rosini. Microscopic selection of solutions to scalar conservation laws with discontinuous flux in the context of vehicular traffic. In *Semigroups of operators – theory and applications. Selected papers based on the presentations at the conference, SOTA 2018, Kazimierz Dolny, Poland, September 30 – October 5, 2018. In honour of Jan Kisyński's 85th birthday*, pages 113–135. Cham: Springer, 2020.

[21] B. Andreianov and K. Sbihi. Well-posedness of general boundary-value problems for scalar conservation laws. *Trans. Amer. Math. Soc.*, 367(6):3763–3806, 2015.

[22] B. Andreianov and A. Sylla. Existence analysis and convergent approximations for a second order model of traffic with orderliness marker. hal-03214129.

[23] B. Andreianov and A. Sylla. A macroscopic model to reproduce self-organization at bottlenecks. In *International Conference on Finite Volumes for Complex Applications*, pages 243–254. Springer, 2020.

[24] E. Audusse and B. Perthame. Uniqueness for scalar conservation laws with discontinuous flux via adapted entropies. *Proc. Roy. Soc. Edinburgh Sect. A*, 135(2):253–265, 2005.

[25] A. Aw and M. Rascle. Resurrection of "second order" models of traffic flow. *SIAM Journal on Applied Mathematics*, **60**(3):916–938, 2000.

[26] F. Bachmann and J. Vovelle. Existence and uniqueness of entropy solution of scalar conservation laws with a flux function involving discontinuous coefficients. *Comm. Partial Differential Equations*, 31(1-3):371–395, 2006.

[27] M. Bardi and I. Capuzzo-Dolcetta. *Optimal control and viscosity solutions of Hamilton-Jacobi-Bellman equations*. Springer Science & Business Media, 2008.

[28] G. Barles. An introduction to the theory of viscosity solutions for first-order hamilton–jacobi equations and applications. In *Hamilton-Jacobi equations: approximations, numerical analysis and applications*, pages 49–109. Springer, 2013.

[29] N. Bellomo and V. Coscia. First-order models and closure of the mass conservation equation in the mathematical theory of vehicular traffic flow. *C. R., Méc., Acad. Sci. Paris*, **333**(11):843–851, 2005.

[30] N. Bellomo and C. Dogbe. On the modeling of traffic and crowds: a survey of models, speculations, and perspectives. *SIAM Rev.*, **53**(3):409–463, 2011.

[31] S. Blandin and P. Goatin. Well-posedness of a conservation law with non-local flux arising in traffic flow modeling. *Numer. Math.*, **132**(2):217–241, 2016.

[32] F. Bouchut and B. Perthame. Kruzhkov's estimates for scalar conservation laws revisited. *Transactions of the American Mathematical Society*, **350**(7):2847–2870, 1998.

[33] A. Bressan. *Hyperbolic systems of conservation laws: the one-dimensional Cauchy problem*, volume 20. Oxford University Press on Demand, 2000.

[34] A. Bressan, G. Guerra, and W. Shen. Vanishing viscosity solutions for conservation laws with regulated flux. *Journal of Differential Equations*, **266**(1):312–351, 2019.

[35] G. Bretti, E. Cristiani, C. Lattanzio, A. Maurizi, and B. Piccoli. Two algorithms for a fully coupled and consistently macroscopic pde-ode system modeling a moving bottleneck on a road. *Mathematics in Engineering*, **1**(1):55–83, 2018.

[36] H. Brezis. *Functional analysis, Sobolev spaces and partial differential equations*. Springer Science & Business Media, 2010.

[37] R. Bürger, C. Chalons, and L. M. Villada. Antidiffusive l agrangian-remap schemes for models of polydisperse sedimentation. *Numerical Methods for Partial Differential Equations*, **32**(4):1109–1136, 2016.

[38] R. Bürger, A. García, K. H. Karlsen, and J. D. Towers. A family of numerical schemes for kinematic flows with discontinuous flux. *Journal of Engineering Mathematics*, **60**:387–425, 2008.

[39] R. Bürger, K. H. Karlsen, C. Klingenberg, and N. H. Risebro. A front tracking approach to a model of continuous sedimentation in ideal clarifier-thickener units. *Nonlinear Anal. Real World Appl.*, **4**(3):457–481, 2003.

[40] R. Bürger, K. H. Karlsen, and J. D. Towers. A conservation law with discontinuous flux modelling traffic flow with abruptly changing road surface conditions. In *Hyperbolic problems: theory, numerics and applications*, volume 67 of *Proc. Sympos. Appl. Math.*, pages 455–464. Amer. Math. Soc., Providence, RI, 2009.

[41] R. Bürger, K. H. Karlsen, and J. D. Towers. An Engquist-Osher-type scheme for conservation laws with discontinuous flux adapted to flux connections. *SIAM Journal on Numerical Analysis*, **47**(3):1684–1712, 2009.

[42] C. Cancès. Asymptotic behavior of two-phase flows in heterogeneous porous media for capillarity depending only on space. I. Convergence to he optimal entropy solution. *SIAM J. Math. Anal.*, **42**(2):946–971, 2010.

[43] C. Cancès. On the effects of discontinuous capillarities for immiscible two-phase flows in porous media made of several rock-types. *Netw. Heterog. Media*, **5**(3):635–647, 2010.

[44] C. Cancès and T. Gallouët. On the time continuity of entropy solutions. *Journal of Evolution Equations*, **11**(1):43–55, 2011.

[45] C. Cancès and N. Seguin. Error estimate for Godunov approximation of locally constrained conservation laws. *SIAM Journal on Numerical Analysis*, **50**(6):3036–3060, 2012.

[46] P. Cannarsa and C. Sinestrari. *Semiconcave functions, Hamilton-Jacobi equations, and optimal control*, volume 58. Springer Science & Business Media, 2004.

[47] E. M. Cepolina. Phased evacuation: An optimisation model which takes into account the capacity drop phenomenon in pedestrian flows. *Fire Safety Journal*, **44**(4):532–544, 2009.

[48] C. Chainais-Hillairet. Finite volume schemes for a nonlinear hyperbolic equation. convergence towards the entropy solution and error estimate. *ESAIM: Mathematical Modelling and Numerical Analysis*, **33**(1):129–156, 1999.

[49] C. Chainais-Hillairet and S. Champier. Finite volume schemes for nonhomogeneous scalar conservation laws: error estimate. *Numerische Mathematik*, **88**(4):607–639, 2001.

[50] C. Chalons, M. L. Delle Monache, and P. Goatin. A conservative scheme for non-classical solutions to a strongly coupled PDE-ODE problem. *Interfaces and Free Boundaries*, **19**(4):553–570, 2017.

[51] C. Chalons, P. Goatin, and N. Seguin. General constrained conservation laws. application to pedestrian flow modeling. *Networks and Heterogeneous Media*, **2**(8):433–463, 2013.

[52] F. A. Chiarello, J. Friedrich, P. Goatin, S. Göttlich, and O. Kolb. A non-local traffic flow model for 1-to-1 junctions. *European Journal of Applied Mathematics*, pages 1–21, 2019.

[53] F. Clarke. *Functional Analysis, Calculus of Variations and Optimal Control*. Graduate Texts in Mathematics, Vol. 264. Springer, 2013.

[54] G. M. Coclite and N. H. Risebro. Conservation laws with time dependent discontinuous coefficients. *SIAM J. Math. Anal.*, 36(4):1293–1309, 2005.

[55] R. M. Colombo. Hyperbolic phase transitions in traffic flow. *SIAM J. Appl. Math.*, **63**(2):708–721, 2002.

[56] R. M. Colombo and P. Goatin. A well posed conservation law with a variable unilateral constraint. *Journal of Differential Equations*, **234**(2):654–675, 2007.

[57] R. M. Colombo, P. Goatin, and B. Piccoli. Road networks with phase transitions. *J. Hyperbolic Differ. Equ.*, **7**(1):85–106, 2010.

[58] R. M. Colombo, F. Marcellini, and M. Rascle. A 2-phase traffic model based on a speed bound. *SIAM Journal on Applied Mathematics*, **70**(7):2652–2666, 2010.

[59] R. M. Colombo, M. Mercier, and M. D. Rosini. Stability and total variation estimates on general scalar balance laws. *Commun. Math. Sci.*, **7**(1):37–65, 2009.

[60] R. M. Colombo and M. D. Rosini. Pedestrian flows and non-classical shocks. *Mathematical Methods in the Applied Sciences*, **28**(13):1553–1567, 2005.

[61] R. M. Colomobo and V. Perrollaz. Initial data identification in conservation laws and hamilton–jacobi equations. *Journal de Mathématiques Pures et Appliquées*, **138**:1–27, 2020.

[62] G. Costeseque, M. Herty, A. Seyfried, and A. Tordeux. From traffic and pedestrian follow-the-leader models with reaction time to first order convection-diffusion flow models. *SIAM Journal on Applied Mathematics*, **78**(1):63–79, 2018.

[63] M. G. Crandall and P.-L. Lions. Viscosity solutions of hamilton-jacobi equations. *Transactions of the American Mathematical Society*, **277**(1):1–42, 1983.

[64] G. Crasta, V. De Cicco, and G. De Philippis. Kinetic formulation and uniqueness for scalar conservation laws with discontinuous flux. *Comm. Partial Differential Equations*, **40**(4):694–726, 2015.

[65] G. Crasta, V. De Cicco, G. De Philippis, and F. Ghiraldin. Structure of solutions of multidimensional conservation laws with discontinuous flux and applications to uniqueness. *Arch. Ration. Mech. Anal.*, 221(2):961–985, 2016.

[66] E. Cristiani, B. Piccoli, and A. Tosin. How can macroscopic models reveal self-organization in traffic flow? In *2012 IEEE 51st IEEE Conference on Decision and Control (CDC)*, pages 6989–6994. IEEE, 2012.

[67] C. M. Dafermos. Generalized characteristics and the structure of solutions of hyperbolic conservation laws. *Indiana University Mathematics Journal*, **26**(6):1097–1119, 1977.

[68] C. M. Dafermos. *Hyperbolic conservation laws in continuum physics*, volume 3. Springer, 2005.

[69] C. F. Daganzo. Requiem for second order fluid approximations of traffic flow. *Transportation Research Part B: Methodological*, **29**:277–286, 1995.

[70] M. L. Delle Monache and P. Goatin. Scalar conservation laws with moving constraints arising in traffic flow modeling: an existence result. *Journal of Differential Equations*, **257**(11):4015––4029, 2014.

[71] M. L. Delle Monache and P. Goatin. A numerical scheme for moving bottlenecks in traffic flow. *Bulletin of the Brazilian Mathematical Society, New Series*, **47**(2):605–617, 2016.

[72] M. L. Delle Monache and P. Goatin. Stability estimates for scalar conservation laws with moving flux constraints. *Networks and Heterogeneous Media*, **12**(2):245–258, 2017.

[73] M. L. Delle Monache, T. Liard, B. Piccoli, R. Stern, and D. Work. Traffic reconstruction using autonomous vehicles. *SIAM Journal on Applied Mathematics*, **79**(5):1748–1767, 2019.

[74] S. Diehl. A uniqueness condition for nonlinear convection-diffusion equations with discontinuous coefficients. *J. Hyperbolic Differ. Equ.*, **6**(1):127–159, 2009.

[75] Stefan Diehl. On scalar conservation laws with point source and discontinuous flux function. *SIAM J. Math. Anal.*, **26**(6):1425–1451, 1995.

[76] R. J. DiPerna. Measure-valued solutions to conservation laws. *Archive for Rational Mechanics and Analysis*, **88**(3):223–270, 1985.

[77] R. J. DiPerna and P.-L. Lions. Ordinary differential equations, transport theory and Sobolev spaces. *Invent. Math.*, **98**(3):511–547, 1989.

[78] J. Droniou and R. Eymard. Uniform-in-time convergence result of numerical methods for non-linear parabolic equations. *Numerische Mathematik*, **132**(4):721–766, 2015.

[79] L. C. Evans. On solving certain nonlinear partial differential equations by accretive operator methods. *Israel Journal of Mathematics*, **36**(3-4):225–247, 1980.

[80] L. C. Evans. *Weak convergence methods for nonlinear partial differential equations.* CBMS Regional Conference Series in Mathematics, 74. American Mathematical Society, Providence, RI, 1990.

[81] Lawrence C. Evans. *Partial Differential Equations.* American Mathematical Society, Providence, R.I., 2010.

[82] R. Eymard, T. Gallouët, and R. Herbin. Existence and uniqueness of the entropy solution to a nonlinear hyperbolic equation. *Chin. Ann. Math.*, B**16**:1–14, 1995.

[83] R. Eymard, T. Gallouët, and R. Herbin. *Finite Volume Methods*, volume VII of *Handbook of Numerical Analysis.* North-Holland, Amsterdam, 2000.

[84] A. Ferrara, P. Goatin, and G. Piacentini. A macroscopic model for platooning in highway traffic. *SIAM Journal on Applied Mathematics*, **80**(1):639–656, 2020.

[85] A. F. Filippov. Differential equations with discontinuous right-hand side. *Matematicheskii sbornik*, **93**(1):99–128, 1960.

[86] W. H. Fleming and H. M. Soner. *Controlled Markov processes and viscosity solutions*, volume 25. Springer Science & Business Media, 2006.

[87] M. R. Flynn, A. R. Kasimov, R. R. Rosales, and B. Seibold. Constructing set-valued fundamental diagrams from jamiton solutions in second order traffic models. *Netw. Heterog. Media*, **8**(3):745–772, 2013.

[88] A. Friedman and E. Hopf. The cauchy problem for first order partial differential equations. *Indiana University Mathematics Journal*, **23**(1):27–40, 1973.

[89] M. Garavello, P. Goatin, T. Liard, and B. Piccoli. A multiscale model for traffic regulation via autonomous vehicles. *Journal of Differential Equations*, 269(7):6088–6124, 2020.

[90] M. Garavello, R. Natalini, B. Piccoli, and A. Terracina. Conservation laws with discontinuous flux. *Netw. Heterog. Media*, **2**(1):159–179, 2007.

[91] M. Garavello and B. Piccoli. Coupling of Lighthill-Whitham-Richards and phase transition models. *J. Hyperbolic Differ. Equ.*, **10**(3):577–636, 2013.

[92] I. Gasser, C. Lattanzio, and A. Maurizi. Vehicular traffic flow dynamics on a bus route. *Multiscale Modeling & Simulation*, **11**(3):925–942, 2013.

[93] D. C. Gazis, R. Herman, and R. W. Rothery. Nonlinear follow-the-leader models of traffic flow. *Operations Research*, **9**(4):545–567, 1961.

[94] T. Gimse and N. H. Risebro. Solution of the Cauchy problem for a conservation law with a discontinuous flux function. *SIAM J. Math. Anal.*, 23(3):635–648, 1992.

[95] P. Goatin and S. Scialanga. Well-posedness and finite volume approximations of the lwr traffic flow model with non-local velocity. *Networks and Heterogeneous Media*, **11**(1):107–121, 2016.

[96] H. Haj-Salem, J. P. Lebacque, X. Louis, S. Mammar, and B. Schnetzler. Modélisation du trafic autoroutier au second ordre. *C. R. Math. Acad. Sci. Paris*, **346**(21-22):1203–1206, 2008.

[97] H. Haj-Salem, J. P. Lebacque, S. Mammar, and B. Schnetzler. Generic second order traffic flow modelling. In *Transportation and Traffic Theory 2007*, pages 755–776. Elsevier, 2007.

[98] H. Holden and N. H. Risebro. *Front Tracking for Hyperbolic Conservation Laws*, volume 152 of *Applied Mathematical Sciences*. Springer-Verlag New York, 2002.

[99] H. Holden and N. H. Risebro. Follow-the-leader models can be viewed as a numerical approximation to the lighthill-whitham-richards model for traffic flow. *arXiv preprint arXiv:1702.01718*, 2017.

[100] E. Hopf. The partial differential equation ut+ uux= μxx. *Communications on Pure and Applied mathematics*, **3**(3):201–230, 1950.

[101] L. Hörmander. *Lectures on nonlinear hyperbolic differential equations*, volume 26. Springer Science & Business Media, 1997.

[102] H. Ishii. Uniqueness of unbounded viscosity solution of hamilton-jacobi equations. *Indiana University Mathematics Journal*, **33**(5):721–748, 1984.

[103] F. John. *Partial Differential Equations*. Applied Mathematical Sciences. Springer, 1982.

[104] E. F. Kaasschieter. Solving the Buckley-Leverett equation with gravity in a heterogeneous porous medium. *Comput. Geosci.*, **3**(1):23–48, 1999.

[105] K. H. Karlsen and N. H. Risebro. A note on front tracking and the equivalence between viscosity solutions of hamilton-jacobi equations and entropy solutions of scalar conservation laws. *Nonlinear Anal.*, **50** (4, Ser. A: Theory Methods):455–469, 2002.

[106] K. H. Karlsen, N. H. Risebro, and J. D. Towers. L^1 stability for entropy solutions of nonlinear degenerate parabolic convection-diffusion equations with discontinuous coefficients. *Skr. K. Nor. Vidensk. Selsk.*, (3):1–49, 2003.

[107] K. H. Karlsen and J. D. Towers. Convergence of the Lax-Friedrichs scheme and stability for conservation laws with a discontinous space-time dependent flux. *Chinese Ann. Math. Ser. B*, **25**(3):287–318, 2004.

[108] K. H. Karlsen and J. D. Towers. Convergence of a Godunov scheme for conservation laws with a discontinuous flux lacking the crossing condition. *J. Hyperbolic Differ. Equ.*, **14**(4):671–701, 2017.

[109] B. S. Kerner. Experimental features of self-organization in traffic flow. *Physical review letters*, **81**(17):3797–3800, 1998.

[110] B. S. Kerner. Theory of congested traffic flow: self-organization without bottlenecks. In *In Proceedings of the 14th International Symposium on Transportation and Traffic Theory, Jerusalem*, pages 147–171. Elsevier, 1999.

[111] B. S. Kerner. Phase transitions in traffic flow. In *Traffic and granular flow '99. Social traffic, and granular dynamics. 3rd conference, Stuttgart, Germany, September 1999*, pages 253–283. Berlin: Springer, 2000.

[112] B. L. Keyfitz and H. C. Kranzer. A system of non-strictly hyperbolic conservation laws arising in elasticity theory. *Archive for Rational Mechanics and Analysis*, **72**(3):219–241, 1980.

[113] U. Koley and N. H. Risebro. Finite difference schemes for the symmetric Keyfitz-Kranzer system. *Zeitschrift für angewandte Mathematik und Physik*, **64**(4):1057–1085, 2013.

[114] S. N. Kruzhkov. First order quasilinear equations with several independent variables. *Mathematics of the USSR-Sbornik*, **81**(123):228–255, 1970.

[115] C. Lattanzio, A. Maurizi, and B. Piccoli. Moving bottlenecks in car traffic flow: A PDE-ODE coupled model. *SIAM Journal on Mathematical Analysis*, **43**(1):50–67, 2011.

[116] N. Laurent-Brouty, G. Costeseque, and P. Goatin. A macroscopic traffic flow model accounting for bounded acceleration. *SIAM Journal on Applied Mathematics*, **81**(1):173–189, 2021.

[117] P. D. Lax. Hyperbolic systems of conservation laws ii. *Communications on pure and applied mathematics*, **10**(4):537–566, 1957.

[118] J. P. Lebacque, X. Louis, S. Mammar, B. Schnetzler, and H. Haj-Salem. Modélisation du trafic autoroutier au second ordre. *C. R. Math. Acad. Sci. Paris*, **346**(21-22):1203–1206, 2008.

[119] J. P. Lebacque, S. Mammar, B. Schnetzler, and H. Haj-Salem. Generic second order traffic flow modelling. In *Transportation and Traffic Theory 2007*, pages 755–776. Elsevier, 2007.

[120] M. Lécureux-Mercier. Improved stability estimates for general scalar conservation laws. *Journal of hyperbolic differential equations*, **8**(04):727–757, 2011.

[121] R. J. LeVeque. *Numerical methods for conservation laws*, volume 132. Springer, 1992.

[122] R. J. LeVeque. *Finite volume methods for hyperbolic problems*, volume 31. Cambridge university press, 2002.

[123] T. Liard and B. Piccoli. Well-posedness for scalar conservation laws with moving flux constraints. *SIAM Journal on Applied Mathematics*, **79**(2):641–667, 2018.

[124] T. Liard and B. Piccoli. On entropic solutions to conservation laws coupled with moving bottlenecks. hal-02149946, June 2019.

[125] M. J. Lighthill and G. B. Whitham. On kinematic waves II. a theory of traffic flow on long crowded roads. *Proceedings of the Royal Society of London. Series A. Mathematical and Physical Sciences*, **229**(1178):317–345, 1955.

[126] P.-L. Lions. *Generalized solutions of Hamilton-Jacobi equations*, volume 69. London Pitman, 1982.

[127] J. Málek, J. Necas, M. Rokyta, and M. Ruzicka. *Weak and measure-valued solutions to evolutionary PDEs*, volume 13. CRC Press, 1996.

[128] B. Merlet and J. Vovelle. Error estimate for finite volume scheme. *Numerische Mathematik*, **106**(1):129–155, 2007.

[129] W. Neves, E. Y. Panov, and J. Silva. Strong traces for conservation laws with general nonautonomous flux. *SIAM Journal on Mathematical Analysis*, **50**(6):6049–6081, 2018.

[130] W. Neves, E. Y. Panov, and J. Silva. Strong traces for conservation laws with general nonautonomous flux. *SIAM J. Math. Anal.*, 50(6):6049–6081, 2018.

[131] O. A. Oleinik. Discontinuous solutions of non-linear differential equations. *Uspekhi Matematicheskikh Nauk*, **12**(3):3–73, 1957.

[132] E. Y. Panov. On strong precompactness of bounded sets of measure-valued solutions of a first order quasilinear equation. *Sbornik: Mathematics*, **186**(5):729, 1995.

[133] E. Y. Panov. Existence of strong traces for quasi-solutions of multidimensional conservation laws. *J. Hyperbolic Differ. Equ.*, 4(4):729–770, 2007.

[134] E. Y. Panov. *Generalized solutions of the Cauchy problem for a transport equation with discontinuous coefficients*, pages 23–84. Springer New York, New York, NY, 2008.

[135] E. Y. Panov. On the strong pre-compactness property for entropy solutions of a degenerate elliptic equation with discontinuous flux. *Journal of Differential Equations*, **247**(10):2821–2870, 2009.

[136] E. Y. Panov. Existence and strong pre-compactness properties for entropy solutions of a first-order quasilinear equation with discontinuous flux. *Archive for Rational Mechanics and Analysis*, **195**(2):643–673, 2010.

[137] E. Yu. Panov. On existence and uniqueness of entropy solutions to the Cauchy problem for a conservation law with discontinuous flux. *J. Hyperbolic Differ. Equ.*, **6**(3):525–548, 2009.

[138] P. Pedregal. *Parametrized measures and variational principles*. Springer Science & Business Media, 1997.

[139] B. Piccoli and A. Tosin. Vehicular traffic: a review of continuum mathematical models. In *Mathematics of complexity and dynamical systems. Vols. 1–3*, pages 1748–1770. Springer, New York, 2012.

[140] N. Pogodaev. Estimates of the domain of dependence for scalar conservation laws. *Journal of Differential Equations*, **265**(4):1654–1677, 2018.

[141] P. I. Richards. Shock waves on the highway. *Operations Research*, **4**(1):42–51, 1956.

[142] B. Riemann. *Über die Fortpflanzung ebener Luftwellen von endlicher Schwingungsweite*. Dieterich, 1860.

[143] M. D. Rosini. *Macroscopic models for vehicular flows and crowd dynamics: theory and applications*. Understanding Complex Systems. Springer, Heidelberg, 2013.

[144] D. Serre. *Systems of Conservation Laws 1: Hyperbolicity, entropies, shock waves*. Cambridge University Press, 1999.

[145] D. Serre. *Systems of Conservation Laws 2: Geometric Structures, Oscillations, and Initial-Boundary Value Problems*, volume 2. Cambridge University Press, 1999.

[146] J. Serrin and D. E. Varberg. A general chain rule for derivatives and the change of variables formula for the lebesgue integral. *The American Mathematical Monthly*, **76**(5):514–520, 1969.

[147] W. Shen. On the uniqueness of vanishing viscosity solutions for Riemann problems for polymer flooding. *NoDEA Nonlinear Differential Equations Appl.*, **24**(4):Paper No. 37, 25, 2017.

[148] J. Smoller. *Shock waves and reaction—diffusion equations*, volume 258. Springer Science & Business Media, 2012.

[149] A. Sylla. A LWR model with constraints at moving interfaces. hal-03229291.

[150] A. Sylla. Influence of a slow moving vehicle on traffic: Well-posedness and approximation for a mildly nonlocal model. *Networks & Heterogeneous Media*, **16**(2):221–256, 2021.

[151] J. D. Towers. Convergence of a difference scheme for conservation laws with a discontinuous flux. *SIAM J. Numer. Anal.*, **38**(2):681–698, 2000.

[152] J. D. Towers. Convergence via OSLC of the godunov scheme for a scalar conservation law with time and space flux discontinuities. *Numerische Mathematik*, **139**(4):939–969, 2018.

[153] J. D. Towers. An existence result for conservation laws having BV spatial flux heterogeneities-without concavity. *Journal of Differential Equations*, **269**(7):5754–5764, 2020.

[154] A. Vasseur. Strong traces for solutions of multidimensional scalar conservation laws. *Arch. Ration. Mech. Anal.*, **160**(3):181–193, 2001.

[155] A. I. Vol'pert. The spaces BV and quasilinear equations. *Matematicheskii Sbornik*, **115**(2):255–302, 1967.

[156] J. Vovelle. Convergence of finite volume monotone schemes for scalar conservation laws on bounded domains. *Numerische Mathematik*, **90**(3):563–596, 2002.

[157] H. M. Zhang. A non-equilibrium traffic model devoid of gas-like behavior. *Transportation Research Part B: Methodological*, **36**(3):275–290, 2002.

Résumé :
Dans cette thèse, on traite la prise en compte de l'hétérogénéité dans les lois de conservation scalaires, c'est-à-dire les lois de conservation non invariantes par translation en espace. Ces équations apparaissent notamment dans les modèles de trafic. Par exemple, les mécanismes suivants introduisent de l'hétérogénéité : la présence de feux de circulation, des portions de route où la vitesse maximale est limitée, la variabilité de l'état de la route, *etc*... La prise en compte de l'hétérogénéité permet d'enrichir les modèles de trafic. On aborde trois classes de problèmes inhomogènes pour lesquelles on complète et approfondit le cadre mathématique pour l'analyse théorique et l'approximation numérique.
Nous explorons en détail le cadre où l'hétérogénéité est matérialisée par l'ajout d'une ou plusieurs interfaces mobiles. Le long des interfaces, on impose une condition de majoration sur le flux de la loi de conservation. Cette classe de modèles permet de tenir compte de la présence d'un petit nombre de véhicules encombrants et lents (ou alors, de véhicules autonomes qui ont pour rôle la régulation du trafic). Dans ce cadre, l'évolution des interfaces et des contraintes est couplée de façon non locale à l'état du trafic et/ou à des paramètres spécifiant l'état du véhicule ou du conducteur. En outre, nous élaborons une description de l'hétérogénéité du trafic résultant des variations du degré d'organisation des conducteurs, dans le cadre des modèles dits "du second ordre". L'aspect numérique est prépondérant pour les modèles de trafic que nous étudions. On construit des schémas numériques robustes et on élabore des techniques de compacité spécifiques. La convergence de ces schémas conduit à des résultats d'existence.
Enfin, en lien avec le modèle décrivant l'évolution d'une densité de véhicules sur une route hétérogène, on étudie théoriquement une loi de conservation dans laquelle la dépendance spatiale du flux est explicite. Des résultats classiques sur le caractère bien posé ou la correspondance avec l'équation de Hamilton-Jacobi associée sont obtenus sous des hypothèses plus en adéquation avec la modélisation que celles rencontrées dans la littérature. Les applications allant au-delà de la description du trafic, on se donne pour objectif l'analyse approfondie des problèmes d'identification de données initiales.

Mots clés : Lois de conservation hétérogènes ; Modèles de trafic ; Interfaces mobiles ; Schéma volumes finis ; Inverse design

Abstract:
This thesis is devoted to the treatment of heterogeneity in scalar conservation laws. We call heterogeneous a conservation law which is not invariant by space translation. These equations arise for instance in traffic flow dynamics modeling. The presence of traffic lights or roads that have a variable maximum speed limit are examples of mechanisms which lead to heterogeneous conservation laws. Considering such equations is a way to expand macroscopic traffic flow models. We tackle three classes of inhomogeneous problems for which we extend the mathematical framework for both the theoretical analysis and the numerical approximation.
We fully investigate the treatment of heterogeneity when one or several moving interfaces are added in the classic LWR model for traffic flow. Flux constraints are attached to each interfaces. The resulting class of models can be used to take into account the presence of slow moving vehicles that reduce the road capacity and thus generates moving bottlenecks for the surrounding traffic flow. They can also describe the regulating effect of autonomous vehicles. In this framework, the interfaces and the constraints are linked in a nonlocal way to the traffic density and/or to an orderliness marker describing the state of the drivers. The description of the heterogeneity caused by the variations in the drivers' organization leads to the analysis of a so-called second order model. The numerical aspect plays a central role in the analysis of these traffic flow models. We construct robust numerical schemes and establish specific techniques to obtain compactness of the approximate solutions. Proving the convergence of these schemes lead to existence results.
Finally, with the space-dependent LWR traffic flow model in mind, we theoretically analyze a class of scalar conservation laws with explicit space dependency. Classical results such as well-posedness or the link to the associated Hamilton-Jacobi equation are obtained under a set of assumptions more fitting with the modeling hypothesis. With applications that go beyond traffic modeling in mind, we aim to tackle initial data identification problems.

Keywords: Heterogenous conservation laws; Traffic flow models; Moving interfaces; Finite volume scheme; Inverse design