Interpolation inequalities and spectral geometry on sub-Riemannian structures

Luca Rizzi

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CNRS and Institut Fourier, Université Grenoble Alpes

Mémoire d'Habilitation à Diriger des Recherches

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Gérard Besson	Président du jury
Dario Cordero-Erausquin	Rapporteur
Michel Ledoux	Examinateur
Ludovic Rifford	Rapporteur
Séverine Rigot	Examinatrice
Anton Thalmaier	Rapporteur

Contents

1. Summary

This mémoire presents a selection of the results I obtained after my Ph.D. in 2014, carried out first during my postdoc at the CMAP, École Polytechnique (Paris, June 2014 – Sept. 2016), and then as a Chargé de Recherche at the Institut Fourier (Grenoble, Oct. 2016 – onwards). None of the result presented here is taken from my Ph.D. thesis [R13], defended in SISSA (Trieste, May 2014), and the associated publications [R14, R15, R16, R17], even though some of the work discussed here is connected to that previous research.

The mémoire is divided in two parts. Part I concerns geometric inequalities on sub-Riemannian structures, while Part II focuses on spectral and heat asymptotics for singular structures. The two parts are independent but connected, as the typical singularity treated in Part II naturally arises in the study of sub-Riemannian manifolds of Part I.

Part I consists of four chapters:

- Chapter 2, containing the results of [R1] in collaboration with D. Barilari about certain interpolation inequalities for optimal transport on ideal sub-Riemannian structures. We will also describe briefly the related results of [R2, R3, R4];
- Chapter 3 focuses on branching geodesics, which are the subject of the article [R5], in collaboration with T. Mietton (Ph.D. student at the Institut Fourier, that I have been co-supervising with H. Pajot since October 2019);
- Chapter 4 can be seen as a follow-up of Chapter 2, and reports on a comparison theory adapted to sub-Riemannian structures developed in [R6], in collaboration with D. Barilari. Some application of these results follow from the article [R7], with P. Silveira (Ph. D. student at SISSA, that I co-supervised with A. Agrachev in 2014–15);
- Chapter 5 focuses on a different comparison theory, for special sub-Riemannian structures called *H-type foliations*, which is the subject of [R11], with F. Baudoin, E. Grong and G. Molino. These structures are classified in [R25], with the same co-authors, even though these results are not described in this mémoire.

Part II consists of two chapters:

- Chapter 6 deals with spectral properties of singular Riemannian structures. The precise definition of singularity varies slightly across the chapter, but it always corresponds to an explosion of all geometrical invariants (curvature, volume, ...). Section 6.2 contains the results about essential self-adjointess obtained in [R8, R9] with D. Prandi, V. Franceschi and M. Seri, while Section 6.3 reports on the Weyl's law for singular structures [R12], obtained with Y. Chitour and D. Prandi;

- Chapter 7 summarizes the results on the small-times asymptotics of the sub-Riemannian heat content, obtained in [R10] with T. Rossi (Ph.D. student in a joint program SISSA/Institut Fourier, that I have been co-supervising with A. Agrachev and G. Charlot since October 2018).

Each chapter begins with a self-contained introduction, and highlights a list of questions which are addressed in the rest of the chapter. At the end of each chapter it can be found a list of further research directions.

The following articles have been left out from this mémoire: [R25] on the classification of H-type foliations, [R18] on the geodesic counting on contact structures, [R19] on the cut-locus of some Carnot groups, [R20] dealing with an approach to curvature for contact structures, [R21] on a Santaló-type integral formula on sub-Riemannian manifolds, [R22, R23] on intrinsic random walks on sub-Riemannian structures, [R24] on a canonical connection in sub-Riemannian geometry, the proceedings [R26, R27], the contents of the Ph.D. thesis [R13] and the associated articles [R14, R15, R16, R17].

1.1. Works presented in this mémoire

Articles

[R1] D. Barilari and **L. Rizzi**. Sub-Riemannian interpolation inequalities. *Invent. Math.*, 215(3):977–1038, 2019.

[R2] **L. Rizzi**. Measure contraction properties of Carnot groups. *Calc. Var. Partial Differential Equations*, 55(3):Art. 60, 20, 2016.

[R3] D. Barilari and **L. Rizzi**. Sharp measure contraction property for generalized H-type Carnot groups. *Commun. Contemp. Math.*, 20(6):1750081, 24, 2018.

[R4] **L. Rizzi**. A counterexample to gluing theorems for MCP metric measure spaces. *Bull. Lond. Math. Soc.*, 50(5):781–790, 2018.

[R5] T. Mietton and **L. Rizzi**. Branching geodesics in sub-Riemannian geometry. *Geom. Funct. Anal.*, 30(4):1139–1151, 2020.

[R6] D. Barilari and **L. Rizzi**. Bakry-Émery curvature and model spaces in sub-Riemannian geometry. *Math. Ann.*, 377(1-2):435–482, 2020.

[R7] **L. Rizzi** and P. Silveira. Sub-Riemannian Ricci curvatures and universal diameter bounds for 3-Sasakian manifolds. *J. Inst. Math. Jussieu*, 18(4):783–827, 2019.

[R8] D. Prandi, **L. Rizzi**, and M. Seri. Quantum confinement on non-complete Riemannian manifolds. *J. Spectr. Theory*, 8(4):1221–1280, 2018.

[R9] V. Franceschi, D. Prandi, and **L. Rizzi**. On the essential self-adjointness of singular sub-Laplacians. *Potential Anal.*, 53(1):89–112, 2020.

[R10] **L. Rizzi** and T. Rossi. Heat content asymptotics for sub-Riemannian manifolds. *J. Math. Pures Appl. (9)*, 148:267–307, 2021.

Preprints

[R11] F. Baudoin, E. Grong, G. Molino, and **L. Rizzi**. Comparison theorems on H-type sub-Riemannian manifolds. *arXiv e-prints*, Sept. 2019. arXiv: 1812.02563.

[R12] Y. Chitour, D. Prandi, and **L. Rizzi**. Weyl's law for singular Riemannian manifolds. *arXiv e-prints*, 2019. arXiv: 1903.05639.

1.2. Works not presented in this mémoire

Ph.D. Thesis

[R13] **L. Rizzi**. *The curvature of optimal control problems with applications to sub-Riemannian geometry*. Ph.D. Thesis, SISSA, 2014.

Articles included in Ph.D. Thesis

[R14] A. Agrachev, D. Barilari, and **L. Rizzi**. Curvature: a variational approach. *Mem. Amer. Math. Soc.*, 256(1225):v+142, 2018.

[R15] D. Barilari and **L. Rizzi**. Comparison theorems for conjugate points in sub-Riemannian geometry. *ESAIM Control Optim. Calc. Var.*, 22(2):439–472, 2016.

[R16] A. Agrachev, **L. Rizzi**, and P. Silveira. On conjugate times of LQ optimal control problems. *J. Dyn. Control Syst.*, 21(4):625–641, 2015.

[R17] D. Barilari and **L. Rizzi**. A formula for Popp's volume in sub-Riemannian geometry. *Anal. Geom. Metr. Spaces*, 1:42–57, 2013.

Articles

[R18] A. Lerario and **L. Rizzi**. How many geodesics join two points on a contact sub-Riemannian manifold? *J. Symplectic Geom.*, 15(1):247–305, 2017.

[R19] **L. Rizzi** and U. Serres. On the cut locus of free, step two Carnot groups. *Proc. Amer. Math. Soc.*, 145(12):5341–5357, 2017.

[R20] A. Agrachev, D. Barilari, and **L. Rizzi**. Sub-Riemannian curvature in contact geometry. *J. Geom. Anal.*, 27(1):366–408, 2017.

[R21] D. Prandi, **L. Rizzi**, and M. Seri. A sub-Riemannian Santaló formula with applications to isoperimetric inequalities and first Dirichlet eigenvalue of hypoelliptic operators. *J. Differential Geom.*, 111(2):339–379, 2019.

[R22] U. Boscain, R. Neel, and **L. Rizzi**. Intrinsic random walks and sub-Laplacians in sub-Riemannian geometry. *Adv. Math.*, 314:124–184, 2017.

[R23] A. Agrachev, U. Boscain, R. Neel, and **L. Rizzi**. Intrinsic random walks in Riemannian and sub-Riemannian geometry via volume sampling. *ESAIM Control Optim. Calc. Var.*, 24(3):1075–1105, 2018.

[R24] D. Barilari and **L. Rizzi**. On Jacobi fields and a canonical connection in sub-Riemannian geometry. *Arch. Math. (Brno)*, 53(2):77–92, 2017.

Preprints

[R25] F. Baudoin, E. Grong, G. Molino, and **L. Rizzi**. H-type foliations. *arXiv e-prints*, Dec. 2018. arXiv: 1812.02563.

Proceedings

[R26] **L. Rizzi**. An introduction to Sub-Laplacians and their self-adjointness properties. In *Mini-Workshop: Self-adjoint Extensions in New Settings*, volume 16 of number 4 in *Oberwolfach Rep.* Pages 2917–2921, 2020.

[R27] D. Prandi, V. Franceschi, and **L. Rizzi**. Recent results on the essential self-adjointness of sub-Laplacians, with some remarks on the presence of characteristic points. *Séminaire de Théorie spectrale et géométrie (Grenoble)*, 33:1–15, 2015–2016.

1.3. Remerciements

Je remercie vivement Dario Cordero-Erausquin, Ludovic Rifford et Anton Thalmaier d'avoir accepté d'être les rapporteurs de mon mémoire. Je remercie également les autres membres du jury : Gérard Besson, Michel Ledoux et Séverine Rigot, qui ont accepté d'être présents à cette soutenance. Je tiens à remercier aussi Grégoire Charlot, qui a été un garant efficace.

Part I.

Interpolation inequalities and comparison results on sub-Riemannian manifolds

2. Interpolation inequalities

2.1. Introduction

The simplest example of interpolation inequalities we are dealing with is the Brunn-Minkowski inequality on the Euclidean space. In one of its equivalent formulations, it corresponds to a lower bound on the N-dimensional Lebesgue measure $|\cdot|$ of the linear interpolation between two non-empty Borel sets $A, B \subset \mathbb{R}^N$, of the form:

$$|tA + (1-t)B|^{1/N} \geq t|A|^{1/N} + (1-t)|B|^{1/N}, \qquad \forall t \in [0,1]. \tag{2.1}$$

In (2.1), the set on the left hand side, $tA + (1-t)B$, is the linear interpolation between B and A, obtained by taking the points at distance ratio t of segments with endpoints in B and A, respectively. It is well-known that (2.1) is part of a hierarchy of geometric an analytic inequalities, with applications to several domains of mathematics [47].

The starting point of this chapter is the seminal paper [41], where it is proved that some natural interpolation inequalities, ultimately related with (2.1), can be extended from the Euclidean to the Riemannian setting, taking into account the ambient geometry.

The main results of [41] can be stated in terms of optimal transport with quadratic cost on a Riemannian manifold (M, g), which is the following problem: given two probability measures μ_0, μ_1 defined on M, find a measurable map $T : M \to M$ with

$$T_\sharp \mu_0 = \mu_1, \qquad (i.e.\ \mu_1(A) = \mu_0(T^{-1}(A)) \text{ for all measurable } A \subset M), \tag{2.2}$$

such that T is optimal with respect to the quadratic cost, that is:

$$\int_M d^2(x, T(x)) d\mu_0(x) = \min_{S_\sharp \mu_0 = \mu_1} \int_M d^2(x, S(x)) d\mu_0(x), \tag{2.3}$$

where d is the Riemannian distance. This problem is well-understood in the Riemannian setting, thanks to the works of McCann [62], who adapted the Euclidean theory of Brenier [34]. We only give here the following well-posedness result.

Theorem 2.1 (McCann, Brenier)**.** *Assume that μ_0, μ_1 are compactly supported, and absolutely continuous with respect to the Riemannian measure m_g. Then, there exists a unique solution $T : M \to M$ to the problem (2.3) (uniqueness is meant up to changing T on a null set). Furthermore, for μ_0-a.e. $x \in M$, there exists a unique constant-speed geodesic $t \mapsto T_t(x)$, with $0 \leq t \leq 1$, such that $T_0(x) = x$ and $T_1(x) = T(x)$.*

In particular, the map $T_t : M \to M$ defines a curve $\mu_t := (T_t)_\sharp \mu_0$ in the space of probability measures on M, interpolating between the two marginal measures μ_0, μ_1.

Roughly speaking, if we think at μ_0 and μ_1 as the initial and final states of a distribution of mass, then μ_t represents the evolution at time t of the transport process that moves, in an optimal way, μ_0 to μ_1. (In more technical terms, the curve $(\mu_t)_{0\leq t\leq 1}$ is the unique Wasserstein geodesic between μ_0 and μ_1 in the space of probability measures on M, but we will not need this terminology).

One of the fundamental results of [41] is that μ_t satisfies the following estimate.

Theorem 2.2 (Cordero-Erausquin, McCann, Schmuckenschläger)**.** *Let (M, g) be a complete Riemannian manifold, and let μ_0, μ_1 be two probability measures, absolutely continuous with respect to m_g (shortly $\mu_0, \mu_1 \ll \mathsf{m}_g$). Let $T : M \to M$ be the unique optimal transport map between μ_0, μ_1. Then, for all times $t \in [0, 1]$ we have $\mu_t \ll \mathsf{m}_g$ and, letting $\mu_t = \rho_t \mathsf{m}_g$ for $\rho_t \in L^1(M, \mathsf{m}_g)$, the following interpolation inequality holds:*

$$\frac{1}{\rho_t(T_t(x))^{1/n}} \geq \frac{\beta_{1-t}(T(x), x)^{1/n}}{\rho_0(x)^{1/n}} + \frac{\beta_t(x, T(x))^{1/n}}{\rho_1(T(x))^{1/n}}, \qquad \mu_0 - \text{a.e. } x \in M, \tag{2.4}$$

where $\beta_t(x, y)$, for $t \in [0, 1]$, are distortion coefficients, defined in Definition 2.15, which depend only the geometry of the underlying Riemannian manifold, and not on μ_0, μ_1.

The strength of (2.4) is that it isolates the contribution of the geometry in the transportation problem, via the *distortion coefficients* β_t. More can be said about them if the Ricci curvature of M is bounded from below; we have the following comparison result.

Theorem 2.3. *Let (M, g) be a n-dimensional Riemannian manifold. Assume that there exists $\kappa \in \mathbb{R}$ such that $\mathrm{Ric}_g \geq \kappa g$. Then:*

$$\beta_t(x, y) \geq \beta_t^{(\kappa,n)}(d(x, y)), \tag{2.5}$$

where $\beta_t^{(\kappa,n)}$ are the distortion coefficients for the simply connected space form of dimension n and constant Ricci curvature equal to κ:

$$\beta_t^{(\kappa,n)}(\theta) := \begin{cases} t\left(\frac{\sin(t\sqrt{\kappa/(n-1)}\theta)}{\sin(\sqrt{\kappa/(n-1)}\theta)}\right)^{n-1} & \text{if } \kappa > 0, \\ t^n & \text{if } \kappa = 0, \\ t\left(\frac{\sinh(t\sqrt{|\kappa|/(n-1)}\theta)}{\sinh(\sqrt{|\kappa|/(n-1)}\theta)}\right)^{n-1} & \text{if } \kappa < 0. \end{cases} \tag{2.6}$$

In particular, combining (2.5) with (2.4), one can prove a geodesic generalization of the Brunn-Minkowski inequality (2.1) on Riemannian manifold with Ricci curvature bounded from below, of the form:

$$\mathsf{m}_g(Z_t(A, B))^{1/n} \geq \beta_{1-t}^{(\kappa,n)}(\Theta)^{1/n}\mathsf{m}_g(A)^{1/n} + \beta_t^{(\kappa,n)}(\Theta)^{1/n}\mathsf{m}_g(B)^{1/n}, \quad \forall t \in [0, 1], \tag{2.7}$$

where $Z_t(A, B)$ is the set of t-midpoints between two Borel sets A and B (cf. Definition 2.14), and where the worst possible distortion is encoded by the parameter:

$$\Theta = \begin{cases} \inf\{d(x, y) \mid (x, y) \in A \times B\}, & \text{if } \kappa > 0, \\ \sup\{d(x, y) \mid (x, y) \in A \times B\}, & \text{if } \kappa < 0. \end{cases} \tag{2.8}$$

Inequality (2.7) makes sense on general geodesic metric space, equipped with a measure (metric measure spaces) and it is morally one of the equivalent forms of the so-called curvature-dimension inequality ($\mathrm{CD}(K,N)$, or simply CD in the rest of the mémoire). An important fact in the theory is that the $\mathrm{CD}(K,N)$ conditions enjoy pre-compactness and stability properties with respect to the measured-Gromov-Hausdorff topology of metric spaces. This observation paved the way to the so-called synthetic approach to Ricci curvature lower bounds for metric measure spaces initiated by Lott-Villani and Sturm [61, 81, 82] and extensively developed in the last decade.

The main tools in the proofs of the Riemannian Theorems 2.2 and 2.3 are:

- the well-posedness of the optimal transport problem with quadratic cost in Riemannian spaces;
- the properties of the Riemannian cut-locus, and in particular its relation with the regularity properties of the distance;
- the classical theory of Jacobi fields, the second variation of the energy, their relation with the Riemannian curvature, all of which is handled via Levi-Civita connection;

All these ingredients change dramatically – or do not even exist – in the sub-Riemannian world, which is the main setting of this mémoire (see Section 2.2 for definitions).

Towards sub-Riemannian interpolation inequalities

It is nowadays well-known that no sub-Riemannian structure (which is not Riemannian) can fit into the theory of curvature-dimension bounds.

The first result in this direction was obtained in the influential work of Juillet [53]. He proved that the three-dimensional Heisenberg group equipped with its left-invariant measure, which is the simplest sub-Riemannian structure, does not satisfy any form of geodesic Brunn-Minkowski inequality as in (2.7). In particular, no curvature-dimension inequality $\mathrm{CD}(K,N)$ à la Lott-Sturm-Villani can be satisfied (this result was recently extended to almost all sub-Riemannian structures in [54], and to the case $N=\infty$ for non-commutative Carnot groups in [10]).

The next breakthrough was the one by Balogh, Kristály and Sipos in [17], who proved that the Heisenberg group supports interpolation inequalities for optimal transport similar to (2.4), but with quite different distortion coefficients with respect to the Riemannian ones. The authors of [17] employ a one-parameter family of Riemannian extension g_ε of the Heisenberg structure, converging to the latter as $\varepsilon \to 0$ (the so-called *canonical variation*, see also Chapter 5). Starting from the Riemannian interpolation inequalities for the structure g_ε, and since the Ricci curvature of the Riemannian extensions is unbounded from below as $\varepsilon \to 0$, a fine analysis is required to study the limit behaviour.

The results of [17] for the three-dimensional Heisenberg group, and their extension to case of corank 1 Carnot groups in [18], supported the existence of a sub-Riemannian theory of interpolation inequalities, and led to the following questions.

Do general sub-Riemannian manifolds support weighted interpolation inequalities à la [41]? What are the correct geometrical weights and what are their properties?

We gave a answer to these questions, at least for the so-called *ideal* structures (which is generically the case, see Proposition 2.6). In the way we proved regularity properties of the sub-Riemannian cut-locus, and geometric inequalities for several classes of sub-Riemannian structures. The objective of this chapter is to describe these results, which are mainly contained in:

- D. Barilari and **L. Rizzi**. Sub-Riemannian interpolation inequalities. *Invent. Math.*, 215(3):977–1038, 2019

We also report, without fully focusing on them, results obtained in:

- **L. Rizzi**. Measure contraction properties of Carnot groups. *Calc. Var. Partial Differential Equations*, 55(3):Art. 60, 20, 2016
- D. Barilari and **L. Rizzi**. Sharp measure contraction property for generalized H-type Carnot groups. *Commun. Contemp. Math.*, 20(6):1750081, 24, 2018
- **L. Rizzi**. A counterexample to gluing theorems for MCP metric measure spaces. *Bull. Lond. Math. Soc.*, 50(5):781–790, 2018

Some of the results discussed in this chapter have roots in the previous works:

- A. Agrachev, D. Barilari, and **L. Rizzi**. Curvature: a variational approach. *Mem. Amer. Math. Soc.*, 256(1225):v+142, 2018
- A. Agrachev, D. Barilari, and **L. Rizzi**. Sub-Riemannian curvature in contact geometry. *J. Geom. Anal.*, 27(1):366–408, 2017

which were part of my Ph.D. thesis.

2.2. Preliminaries in sub-Riemannian geometry

We recall some facts in sub-Riemannian geometry, which will be used for the rest of the mémoire. For comprehensive references, see [4, 74, 65]. Experts may want to quickly skim through this section to adjust to our notation, or skip it.

We work with the most general definition of sub-Riemannian structure, in which the a non-holonomic distribution can have possibly non-constant rank. This slightly more technical viewpoint allowed us to consider, in an unified setting, several interesting structures which would be otherwise excluded (for example the Grushin plane, important for applications).

A sub-Riemannian structure on a smooth, connected n-dimensional manifold M, where $n \geq 2$, is defined by a set of m global smooth vector fields $X_1, \dots, X_m$, called

a *generating frame*. The *distribution* is the family of subspaces of the tangent spaces spanned by the vector fields at each point

$$\mathcal{D}_x := \operatorname{span}\{X_1(x), \dots, X_m(x)\} \subseteq T_xM, \qquad \forall\, x \in M. \tag{2.9}$$

The generating frame induces an inner product g_x on $\mathcal{D}_x$ as follows: given $v, w \in \mathcal{D}_x$ the inner product $g_x(v, w)$ is defined by the polarization formula, letting

$$g_x(v, v) := \min\left\{ \sum_{i=1}^m u_i^2 \mid \sum_{i=1}^m u_i X_i(x) = v,\ u_i \in \mathbb{R} \right\}. \tag{2.10}$$

We assume that the distribution is *bracket-generating*, i.e., the tangent space T_xM is spanned by the vector fields $X_1, \dots, X_m$ and their iterated Lie brackets evaluated at x (this is also called *Hörmander condition*). A *horizontal curve* $\gamma : [0, 1] \to M$ is an absolutely continuous path such that there exists $u \in L^2([0, 1], \mathbb{R}^m)$ satisfying

$$\dot\gamma(t) = \sum_{i=1}^m u_i(t) X_i(\gamma(t)), \qquad \text{a.e. } t \in [0, 1]. \tag{2.11}$$

This implies that $\dot\gamma(t) \in \mathcal{D}_{\gamma(t)}$ for almost every t. If γ is horizontal, the map $t \mapsto \sqrt{g(\dot\gamma(t), \dot\gamma(t))}$ is integrable on $[0, 1]$, and we define its *length* as:

$$\ell(\gamma) := \int_0^1 \sqrt{g(\dot\gamma(t), \dot\gamma(t))}dt. \tag{2.12}$$

The *sub-Riemannian distance* is defined by:

$$d_{SR}(x, y) := \inf\{\ell(\gamma) \mid \gamma(0) = x,\ \gamma(1) = y,\ \gamma \text{ horizontal}\}. \tag{2.13}$$

Remark. When $\dim \mathcal{D}_x$ is constant, then $\mathcal{D}$ is a vector distribution in the classical sense, and g is a smooth scalar product on $\mathcal{D}$. If furthermore $m \le n$ and the vector fields $X_1, \dots, X_m$ are linearly independent, then they are orthonormal with respect to g.

By the Chow-Rashevskii theorem, and thanks to the bracket-generating condition, $d_{SR} : M \times M \to \mathbb{R}$ is finite, continuous and the metric topology coincides with the manifold one. In place of the length ℓ, it is convenient to consider the *energy functional*

$$J(\gamma) := \frac{1}{2} \int_0^1 g(\dot\gamma(t), \dot\gamma(t))dt. \tag{2.14}$$

On the set of horizontal curves defined on $[0, 1]$ and with fixed endpoints, the minimizers of J coincide with the minimizers of ℓ parametrized with constant speed. Since ℓ is invariant by reparametrization, and every horizontal curve is the reparametrization of a constant-speed one, we define *geodesics* as horizontal curves that locally minimize the energy between their endpoints (while *minimizing geodesics* are those that globally minimize the energy between their endpoints).

2.2.1. End-point map and Lagrange multipliers

Let $\gamma_u : [0,1] \to M$ be an horizontal curve, where $u \in L^2([0,1], \mathbb{R}^m)$ is such that

$$\dot{\gamma}_u(t) = \sum_{i=1}^{m} u_i(t) X_i(\gamma_u(t)), \qquad \text{a.e. } t \in [0,1]. \tag{2.15}$$

Let $x = \gamma(0)$, and let $\mathcal{U} \subset L^2([0,1], \mathbb{R}^m)$ be the neighborhood of u such that, for $v \in \mathcal{U}$, the Cauchy problem

$$\dot{\gamma}_v(t) = \sum_{i=1}^{m} v_i(t) X_i(\gamma_v(t)), \qquad \gamma_v(0) = x, \tag{2.16}$$

has a well defined solution for a.e. $t \in [0,1]$. We define the *end-point map* with base point x as the smooth (in the Fréchet sense) map $E_x : \mathcal{U} \to M$, which sends v to $\gamma_v(1)$.

We can consider $J : \mathcal{U} \to \mathbb{R}$ as a smooth functional on $\mathcal{U}$. Let γ_u be a minimizing geodesic between $x, y \in M$, that u is a solution of the constrained minimum problem

$$\min\{J(v) \mid v \in \mathcal{U},\, E_x(v) = y\}. \tag{2.17}$$

By the Lagrange multipliers rule, there exists a non-trivial pair (λ_1, ν), such that

$$\lambda_1 \circ D_u E_x = \nu D_u J, \qquad \lambda_1 \in T_y^* M, \qquad \nu \in \{0,1\}, \tag{2.18}$$

where $\circ$ denotes the composition of linear maps and D the (Fréchet) differential. More generally, if $\gamma_u : [0,1] \to M$ with control $u \in \mathcal{U}$ is an horizontal curve (not necessarily a minimizing geodesic), we say that a non-zero pair $(\lambda_1, \nu) \in T_y^* M \times \{0,1\}$ is a *Lagrange multiplier* for γ_u if (2.18) is satisfied. The multiplier (λ_1, ν) and the associated curve γ_u are called *normal* if $\nu = 1$ and *abnormal* if $\nu = 0$. Observe that Lagrange multipliers are not unique, and a horizontal curve may be both normal *and* abnormal. Observe also that γ_u is an abnormal curve if and only if u is a critical point for E_x.

We give first order necessary conditions for minimizing geodesics. First, define the *Hamiltonian* of the sub-Riemannian structure as the function $H : T^*M \to \mathbb{R}$ given by

$$H(\lambda) := \frac{1}{2} \sum_{i=1}^{m} \langle \lambda, X_i \rangle^2, \qquad \lambda \in T^*M, \tag{2.19}$$

where $X_1, \ldots, X_m$ is the generating frame. Here $\langle \lambda, \cdot \rangle$ denotes the dual action of covectors on vectors. One can check that the definition is well-posed and H does not change if we choose an equivalent generating frame, that is one yielding the same distribution and scalar product at all points). Then we have the following characterization.

Theorem 2.4. *Let $\gamma_u : [0,1] \to M$ be an horizontal curve joining x with y. A non-zero pair $(\lambda_1, \nu) \in T_y^* M \times \{0,1\}$ is a Lagrange multiplier for γ_u if and only if there exists a Lipschitz curve $\lambda(t) \in T^*_{\gamma_u(t)} M$ with $\lambda(1) = \lambda_1$, such that*

(N) if $\nu = 1$ then $\lambda(t)$ is a solution of the Hamilton's equation:

$$\dot{\lambda}(t) = \vec{H}(\lambda(t)), \tag{2.20}$$

($\vec{H}$ is the vector field on T^*M such that $\sigma(\cdot, \vec{H}) = dH$, σ is the symplectic form).

(A) if $\nu = 0$ then $\lambda(t)$ satisfies the following identity:

$$\sigma(\dot{\lambda}(t), T_{\lambda(t)}\mathcal{D}^\perp) = 0, \tag{2.21}$$

where $\mathcal{D}^\perp \subset T^*M$ is the set of covectors that annihilate the distribution.[1]

In the first (resp. second) case, $\lambda(t)$ is called a *normal* (resp. *abnormal*) *extremal*. Normal extremals are integral curves $\lambda(t)$ of $\vec{H}$. As such, they are smooth, and uniquely determined by their *initial covector* $\lambda = \lambda(0)$. A geodesic is normal (resp. abnormal) if admits a normal (resp. abnormal) extremal. It is well-known that the projection $\gamma_\lambda(t) = \pi(\lambda(t))$ of a normal extremal is locally minimizing, hence it is a normal geodesic.

The following terminology was introduced by Rifford [73, 74].

Definition 2.5. A sub-Riemannian structure $(\mathcal{D}, g)$ on M is *ideal* if the metric space (M, d_{SR}) is complete and there exists no non-trivial abnormal minimizing geodesics (i.e., the only possible abnormal geodesics are constant curves).

The ideal assumption removes all non-trivial abnormal minimizing geodesics, but the trivial ones are always present as soon as the structure is truly sub-Riemannian (i.e. as soon as $\dim(\mathcal{D}_x) < n$, then the curve $\gamma(t) = x$ is always an abnormal geodesic). Their presence is the source of several technical complications in the theory.

Generic sub-Riemannian structures of rank ≥ 3 are ideal. This is made precise in the following result due to Chitour, Jean and Trélat.

Proposition 2.6 ([38, Thm. 2.8])**.** *Let $k \geq 3$ be a positive integer, and $\mathcal{G}_k$ be the set of constant-rank sub-Riemannian structures $(\mathcal{D}, g)$ on M with $\operatorname{rank}\mathcal{D} = k$, endowed with the Whitney C^∞ topology. There exists an open dense subset W_k of $\mathcal{G}_k$ such that every element of W_k does not admit non-trivial abnormal minimizers.*

2.2.2. Regularity of the sub-Riemannian distance and the cut locus

The next definition is given in the case in which (M, d_{SR}) is complete, in this case solutions of (2.20) are defined for all times.

Definition 2.7. The *exponential map* at $x \in M$ is the map $\exp_x : T^*_xM \to M$, which assigns to $\lambda \in T^*_xM$ the final point $\pi(\lambda(1))$ of the corresponding normal geodesic.

The curve $\gamma_\lambda(t) = \exp_x(t\lambda)$, for $t \in [0, 1]$, is the normal geodesic corresponding to λ, which has constant speed $\|\dot{\gamma}_\lambda(t)\| = \sqrt{2H(\lambda)}$ and length $\ell(\gamma|_{[t_1,t_2]}) = \sqrt{2H(\lambda)}(t_2 - t_1)$.

Next, we recall the definition of conjugate points.

[1] If the rank of $\mathcal{D}$ not locally constant, then $\mathcal{D}^\perp$ is not a smooth manifold. In this case $T_\lambda\mathcal{D}^\perp \subset T_\lambda(T^*M)$ is meant as the intersection of the kernels $\cap_{i=1}^m \ker d_\lambda h_i$, where $h_i(\lambda) = \langle \lambda, X_i \rangle$ for $i = 1, \dots, m$.

Definition 2.8. Let $\gamma : [0,1] \to M$ be a normal geodesic with initial covector $\lambda \in T_x^*M$, that is $\gamma(t) = \exp_x(t\lambda)$. We say that $y = \exp_x(\bar{t}\lambda)$ is a *conjugate point* to x along γ if $\bar{t}\lambda$ is a critical point for $\exp_x$.

In the Riemannian setting, conjugate points on a geodesic are isolated, and geodesics cease to be minimizers after the first conjugate point. In the general sub-Riemannian setting, the picture is more complicated. In fact, any normal geodesic can contain segments made of conjugate points, and this happens precisely in presence of abnormal segments. This fact is also related with the branching of geodesics, see Chapter 3.

An important fact in the theory is that, barring the aforementioned exception, the behaviour of conjugate points is similar to the more familiar Riemannian one.

Theorem 2.9. *Let $\gamma : [0,1] \to M$ be a minimizing geodesic which does not contain abnormal segments (i.e. if for all $0 \leq s_1 < s_2 \leq 1$ the restriction $\gamma|_{[s_1,s_2]}$ is not abnormal.). Then $\gamma(s)$ is not conjugate to $\gamma(s')$ along γ for all $s, s' \in [0,1]$ with $|s - s'| < 1$.*

The proof of the above theorem in the Riemannian setting is an easy consequence of the second variation formula for the energy. In the sub-Riemannian setting, such a formula is not available, do to the lack of a canonical connection describing geodesics. Therefore its proof is more complex and makes use of a control-theoretical form of second variation (cf. e.g. [77]). A self-contained proof can be found in [R1, Appendix B].

We recall now some regularity properties of the sub-Riemannian distance.

Definition 2.10 (Smooth points & cut locus)**.** Let $(\mathcal{D}, g)$ be an ideal sub-Riemannian structure on M, and let $x \in M$. We say that $y \in M$ is a *smooth point* (with respect to x) if there exists a unique minimizing geodesic joining x with y, which is not abnormal, and the two points are not conjugate along such a curve. The *cut locus* $\mathrm{Cut}(x)$ is the complement of the set of smooth points with respect to x. The *global cut locus* of M is

$$\mathrm{Cut}(M) := \{(x, y) \in M \times M \mid y \in \mathrm{Cut}(x)\}. \tag{2.22}$$

We have the following result, due to Agrachev, Rifford and Trelat.

Theorem 2.11 ([1, 72])**.** *The set of smooth points is open and dense in M, and the squared sub-Riemannian distance is smooth on $M \times M \setminus \mathrm{Cut}(M)$.*

In general case it is not known whether $\mathrm{Cut}(x)$ has zero measure (cf. Sard's conjecture [65]). The proof of Theorem 2.11 in the ideal case is easier, thanks to the absence of non-trivial abnormal geodesics. Furthermore, in that case, for all $x \in M$ the set $\mathrm{Cut}(x)$ has zero Lebesgue measure and its complement is geodesically star-shaped.

2.2.3. Optimal transport

The study of the Monge optimal transportation problem in sub-Riemannian geometry has been initiated in [8, 45] for the Heisenberg group and subsequently developed in [7, 46, 56] for more general structures. The well-posedness of the optimal transport problem in the general case is still open, but the ideal case is now quite well-understood.

Unlike the Riemannian case, there is no intrinsic smooth measure induced by the metric structure (see [R17]). Thus, let us fix a general smooth measure m on M. The space of compactly supported probability measures on M is denoted by $\mathcal{P}_c(M)$, while $\mathcal{P}_c^{ac}(M)$ is the subset of the absolutely continuous ones w.r.t. m. We denote by $\pi_i : M \times M \to M$, for $i = 1, 2$, the projection on the i-th factor. Furthermore, let $D = \{(x, y) \in M \times M \mid x = y\}$ be the diagonal.

We take from [46] the main results about well-posedness of the quadratic Monge problem in the ideal setting, in a simplified form.

Theorem 2.12 (Well-posedness of Monge problem)**.** *Let $(\mathcal{D}, g)$ be an ideal sub-Riemannian structure on M, and $\mu_0 \in \mathcal{P}_c^{ac}(M)$, $\mu_1 \in \mathcal{P}_c(M)$. There exists a unique transport map $T : M \to M$ such that $T_\sharp \mu_0 = \mu_1$, optimal w.r.t. the quadratic cost, that is*

$$\int_M d_{SR}^2(x, T(x)) d\mu_0(x) = \min_{S_\sharp \mu_0 = \mu_1} \int_M d_{SR}^2(x, S(x)) d\mu_0(x). \tag{2.23}$$

The map T is characterized as follows. There exist a closed set $\mathcal{S}^\psi$ (the static set), an open set $\mathcal{M}^\psi = M \setminus \mathcal{S}^\psi$ (the moving set), and a function $\psi : M \to \mathbb{R}$ (the Kantorovich potential) locally semiconvex in a neighborhood of $\mathcal{M}^\psi \cap \operatorname{supp}(\mu_0)$ (and hence locally Lipschitz in charts), such that, letting

$$T_t(x) := \begin{cases} \exp_x(t d_x \psi) & x \in \mathcal{M}^\psi \cap \operatorname{supp}(\mu_0), \\ x & x \in \mathcal{S}^\psi \cap \operatorname{supp}(\mu_0), \end{cases} \qquad t \in [0, 1]. \tag{2.24}$$

the optimal transport map is given by $T = T_1$. Furthermore, for μ_0-a.e. $x \in M$ there exists a unique minimizing geodesic between x and $T(x)$ given by $t \mapsto T_t(x)$.

Theorem 2.13 (Absolute continuity of Wasserstein geodesic)**.** *Under the same assumptions of Theorem 2.12, there exists a unique Wasserstein geodesic joining μ_0 with μ_1, given by $\mu_t = (T_t)_\sharp \mu_0$, for $t \in [0, 1]$. Moreover, $\mu_t \ll \mathsf{m}$ for all $t \in [0, 1]$, so that there exists $\rho_t \in L^1(M, \mathsf{m})$ such that $\mu_t = \rho_t \mathsf{m}$.*

Notice that we can recover the regularity result $\mu_t \ll \mathsf{m}$ in an independent way from [46], as a consequence of our main Jacobian estimate presented below (Theorem 2.17).

2.3. Sub-Riemannian interpolation inequalities

Let $(\mathcal{D}, g)$ be a (possibly rank-varying) sub-Riemannian structure on a smooth manifold M, and fix a smooth (outer) measure m as explained in Section 2.2.3.

Definition 2.14. Let $A, B \subset M$ be non-empty sets, and $t \in [0, 1]$. The set $Z_t(A, B)$ of *t-intermediate points* is the set of all points $\gamma(t)$, where $\gamma : [0, 1] \to M$ is a minimizing geodesic such that $\gamma(0) \in A$ and $\gamma(1) \in B$.

Let $\mathcal{B}_r(x)$ denote the sub-Riemannian ball of center $x \in M$ and radius $r > 0$.

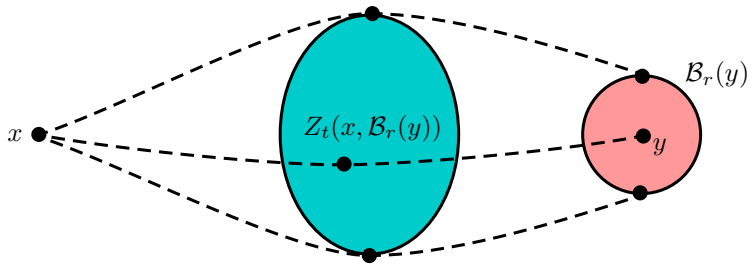


Figure 2.1.: The distortion coefficient $\beta_t(x, y)$.

Definition 2.15. Let $x, y \in M$. The *distortion coefficient* from x to y is

$$\beta_t(x, y) := \limsup_{r \downarrow 0} \frac{\mathsf{m}(Z_t(x, \mathcal{B}_r(y)))}{\mathsf{m}(\mathcal{B}_r(y))}, \qquad t \in [0, 1]. \tag{2.25}$$

Notice that $\beta_0(x, y) = 0$ and $\beta_1(x, y) = 1$.

Our first main result is the extension of (2.4) to the ideal sub-Riemannian setting.

Theorem 2.16 (Interpolation inequality)**.** *Let $(\mathcal{D}, g)$ be an ideal sub-Riemannian structure on M, and $\mu_0, \mu_1 \in \mathcal{P}_c^{ac}(M)$. Let $\rho_s = d\mu_s/d\mathsf{m}$. For all $t \in [0, 1]$, it holds*

$$\frac{1}{\rho_t(T_t(x))^{1/n}} \geq \frac{\beta_{1-t}(T(x), x)^{1/n}}{\rho_0(x)^{1/n}} + \frac{\beta_t(x, T(x))^{1/n}}{\rho_1(T(x))^{1/n}}, \qquad \mu_0 - \text{a.e. } x \in M. \tag{2.26}$$

If μ_1 is not absolutely continuous, an analogous result holds, provided that $t \in [0, 1)$, and that in (2.26) the second term on the right hand side is omitted.

With respect to the Riemannian case proved in [41], several technical complications arise in the sub-Riemannian setting, and in particular: (i) the lack of Levi-Civita connection, standard Jacobi fields, and their comparison theory, (ii) the lack of positive definiteness of sub-Riemannian Hamiltonian, (iii) the lack of semi-concavity of the sub-Riemannian distance on the diagonal.

At the root of the Theorem 2.16 and its proof lies a Jacobian estimate for the sub-Riemannian exponential map which is the true fundamental – albeit technical – result. We sketch in the next section the proof of Theorem 2.16, which will give us the occasion to illustrate the aforementioned Jacobian estimate and some of its consequences.

2.3.1. Sketch of the proof of Theorem 2.16 and Jacobian estimate

Fix $\mu_0, \mu_1 \in \mathcal{P}_c^{ac}(M)$. By the theory of optimal transport on ideal structures (cf. Theorems 2.12 and 2.13 in Section 2.2.3), there is a unique Wasserstein geodesic $\mu_t = (T_t)_\sharp \mu_0$ joining the two extremal measures, and T_t is characterized explicitly as in Theorem 2.12 in terms of the Kantorovich potential ψ.

The behaviour of $\mu_t = (T_t)_\sharp \mu_0$ is hence controlled by the Jacobian determinant of T_t (defined in a suitable sense to take into account the general lack of smoothness of

the optimal transport map). We remind that the transport map is defined differently depending on whether x is in the moving set or the static set (cf. Theorem 2.12). This distinction is absent in the Riemannian case, and comes from the fact that the squared distance d^2_{SR} loses regularity (in particular, it loses semiconcavity in charts) at the diagonal $D \subset M \times M$, due to the presence of the trivial abnormal geodesic.

Static points. First, we deal with the case of points $x \in \mathcal{S}^\psi \cap \mathrm{supp}(\mu_0)$, that is points which are not moved by the transport map. They cannot be treated in an unified way as it happens in the Riemannian setting. For these points the estimate (2.26) is proved using local volume estimates coming from the sub-Riemannian Ball-Box theorem and (uniform) privileged coordinates around x, which are technical tools in sub-Riemannian geometry [25, 50]. In particular, (2.26) follow from suitable estimates of $\beta_t(x, x)$.

Moving points. We can pass to the case $x \in \mathcal{M}^\psi \cap \mathrm{supp}(\mu_0)$. In [46, Thm. 3.7], Figalli and Rifford obtained a formula for the differential of the transport map in this case, akin the classical one of [41], in terms of the Hessian of the distance, under additional hypothesis on the sub-Riemannian cut locus (i.e., essentially, that there are no conjugate cut points). To avoid this restrictive assumption, we adopted a more intrinsic approach exploiting the symplectic structure of T^*M, and using directly the regularity properties of the Kantorovich potential. In fact, in the ideal setting, ψ is locally semiconvex in charts on $\mathcal{M}^\psi \cap \mathrm{supp}(\mu_0)$. In particular, it is twice differentiable almost everywhere.

An intrinsic approach to second differentials. Recall that ψ is locally Lipschitz on $\mathcal{M}^\psi \cap \mathrm{supp}(\mu_0)$, and hence also a.e. differentiable. We look at its differential as a (a.e. defined) map $d\phi : M \to T^*M$. We define the second differential at a point of twice-differentiability of ψ as the differential of this map, namely $d^2_x\psi := d_x(d\psi) : T_xM \to T_{d_x\psi}(T^*M)$. Hence, since $T_t(x) = \exp_x(td_x\psi) = \pi \circ e^{t\vec{H}}(d_x\psi)$, we have

$$d_xT_t = \pi_* \circ e^{t\vec{H}}_* \circ d^2_x\psi, \tag{2.27}$$

where the star denotes the push-forward of a smooth map. We stress that all of this makes sense at all points where ψ is twice-differentiable which, by Alexandrov theorem on locally semiconvex functions, is true for a.e. point on M.

Main Jacobian estimate. Lacking a Riemannian metric on the whole tangent space, to compute the Jacobian determinant $\det(d_xT_t)$ one has to fix a basis on the initial and final tangent spaces. More precisely, fix a smoothly moving frame $X_1(t), \dots, X_n(t)$ along the geodesic $t \mapsto T_t(x)$. The determinant of the linear map $d_xT_t : T_xM \to T_{T_t(x)}M$ is computed with respect to the given frame, that is

$$d_xT_t(X_i(0)) = \sum_{j=1}^n N_{ij}(t)X_j(t), \qquad \det(d_xT_t) := \det N(t). \tag{2.28}$$

In the Riemannian case, this can be naturally done by choosing a parallel orthonormal frame (this also enables the use of classical comparison theory and estimates for Jacobi

fields). This is not possible in the sub-Riemannian case, and thus we must work with a general moving frame as described above. One can now prove the following key estimate.

Theorem 2.17 (Main Jacobian estimate)**.** *Denote by $\gamma(t) = \exp_x(td_x\psi)$, with $t \in [0,1]$, the unique minimizing curve joining x with $T(x)$, which does not contain non-trivial abnormal segments. Then, the linear maps*

$$d_xT_t : T_xM \to T_{\gamma(t)}M, \qquad d_xT_t := \pi_* \circ e_*^{t\vec{H}} \circ d_x^2\psi, \tag{2.29}$$

satisfy the following estimate, for all fixed $s \in (0,1]$:

$$\det(d_xT_t)^{1/n} \geq f(s,t) + g(s,t)\det(d_xT_s)^{1/n}, \qquad \forall\, t \in [0,s], \tag{2.30}$$

where $(t,s) \mapsto f(s,t), g(s,t)$ are explicit functions expressed in terms of the sub-Riemannian Hamiltonian flow $e^{t\vec{H}}$ and its derivatives (cf. [R1, Section 3.3]).

We stress that, at a first step, one can only prove the above inequality for all $s < 1$ (this is due to the fact that the final point can be a priori conjugate). A key role in the proof is played by a positivity lemma (cf. [R1, Lemma 29]) inspired by [90, Ch. 14, Appendix: Jacobi fields forever], which allows to overcome the non positive definiteness of the sub-Riemannian Hamiltonian. As a consequence of this fact, and Theorem 2.9, both terms in the right hand side of (2.30) are non-negative for $t \in [0,s)$ and the first one is actually positive. In particular $\det(d_xT_t) > 0$ for all $t \in [0,s)$.

Transport avoids cut locus. Thanks to the aforementioned positivity of the right hand side of (2.30), one can show that if the final point $T_1(x)$ were conjugate with x, then $\det(d_xT_t) = +\infty$, contradicting the differentiability of T_t at x (and thus the twice-differentiability of ψ at x). This fact not only improves (2.30) from $s \in (0,1)$ to $s \in (0,1]$, but it also has the following nice corollary.

Proposition 2.18. *Under the same assumptions of Theorem 2.12, for μ_0-a.e. $x \in M$ and all $t \in (0,1]$, either $T_t(x) = x$ or $T_t(x) \notin \mathrm{Cut}(x)$.*[2]

As a consequence of Theorem 2.17, and thus of the fact that $\det(d_xT_t) > 0$ for μ_0-a.e. x, and all $t \in [0,1)$, we obtain an independent proof of [46, Thm. 3.5], about the absolute continuity of the Wasserstein geodesic μ_t between μ_0 and μ_1.

Theorem 2.19 (Jacobian identity)**.** *Under the assumptions of Theorem 2.12 we have that $\mu_t \in \mathcal{P}_c^{ac}(M)$ for all $t \in [0,1]$. Letting $\rho_t \in L^1(M,\mathsf{m})$ such that $\mu_t = \rho_t\mathsf{m}$ for all $t \in [0,1]$ we have*

$$\frac{\rho_0(x)}{\rho_t(T_t(x))} = \det(d_xT_t)\frac{\mathsf{m}(X_1(t),\dots,X_n(t))}{\mathsf{m}(X_1(0),\dots,X_n(0))} > 0, \qquad \mu_0 - a.e.\ x \in \mathcal{M}^\psi \cap \mathrm{supp}(\mu_0), \tag{2.31}$$

where $X_1(t),\dots,X_n(t)$ is a smooth moving frame along the geodesic $t \mapsto T_t(x)$, and the determinant is computed with respect to that frame as in (2.28).

[2] Recall that, as soon as the structure is truly sub-Riemannian at x, that is $\dim \mathcal{D}_x < n$, then $x \in \mathrm{Cut}(x)$.

In the Riemannian case, when $\mathsf{m} = \mathsf{m}_g$ is the Riemannian volume, one can compute the determinant in (2.31) with respect to orthonormal frames, eliminating any dependence on the frame and obtaining the classical Monge-Ampère equation for the transport map.

Conclusion. Using (2.31), the main Jacobian estimate (2.30), and a local computation of the distortion coefficients $\beta_t(x, y)$ for $y \notin \mathrm{Cut}(x)$, one concludes the proof of Theorem 2.16. The case in which μ_1 is not absolutely continuous with respect to m is handled easily by dropping all singular terms.

2.4. Regularity of distance

The proof of Theorem 2.16 is also related with the structure of the cut locus. We say that a continuous function $f : M \to \mathbb{R}$ *fails to be semiconvex* at $x \in M$ if, in any set of local coordinates around x, we have

$$\inf_{0<|v|<1} \frac{f(x+v) + f(x-v) - 2f(x)}{|v|^2} = -\infty. \tag{2.32}$$

We remark only that the negation of the failure of semiconvexity (2.32) does not mean that f is locally semiconvex! See [36] for background on locally semiconvex functions.

In Riemannian geometry, it is well-known that for almost every geodesic γ involved in the transport, $\gamma(1) \notin \mathrm{Cut}(\gamma(0))$. In particular, this implies (in a non-trivial way), that the cut locus, which is defined as the set of points where the squared distance is not smooth, can be characterized actually as the set of points where the squared distance fails to be semiconvex [41]. The validity of this characterization was questioned by Figalli and Rifford in [46, Open problems in Sec. 5.8]. Another main consequence of our Jacobian estimates is the affirmative answer to this question in the ideal setting.

Theorem 2.20 (Failure of semiconvexity at the cut locus)**.** *Let $(\mathcal{D}, g)$ be an ideal sub-Riemannian structure on M. Let $y \neq x$. Then $x \in \mathrm{Cut}(y)$ if and only if the squared sub-Riemannian distance from y fails to be semiconvex at x, that is, in local coordinates around x, we have*

$$\inf_{0<|v|<1} \frac{d^2_{SR}(x+v, y) + d^2_{SR}(x-v, y) - 2d^2_{SR}(x, y)}{|v|^2} = -\infty. \tag{2.33}$$

The characterization of Theorem 2.20 is false in the non-ideal case, so its statement is essentially optimal. To see that, consider the standard left-invariant sub-Riemannian structure on the product $\mathbb{H} \times \mathbb{R}$ of the three-dimensional Heisenberg group and the Euclidean line. Denoting points $x = (q, s) \in \mathbb{H} \times \mathbb{R}$, one has

$$d^2_{SR}((q, s), (q', s')) = d^2_{\mathbb{H}}(q, q') + |s - s'|^2. \tag{2.34}$$

Without loss of generality, fix $(q', s') = (0, 0)$. The set of points reached by abnormal minimizers is $\{(0, s) \mid s \in \mathbb{R}\}$. Here, the squared distance $(q, s) \mapsto d^2_{SR}((q, s), (0, 0))$ is not smooth, but the infimum in (2.32) is finite. The loss of smoothness is rather due to the failure of semiconcavity on the diagonal for $d^2_{\mathbb{H}}$.

2.5. Properties of the distortion coefficients

In this section, we discuss some of the general properties of distortion coefficients in the sub-Riemannian setting.

2.5.1. Small times asymptotics

The small-time asymptotics of the distortion coefficients $\beta_t(x,y)$ allows us to introduce the concept of *geodesic dimension*. Its original definition was given in [R14] (containing part of the work done in my Ph.D. thesis), and later extended in [R2] for metric measure spaces. The following theorem, stated in [R1] and proved using techniques developed in [R14], allows to give a technicalities-free definition of the geodesic dimension through the distortion coefficients.

Theorem 2.21 (Asymptotics and geodesic dimension)**.** *Let $(\mathcal{D},g)$ be a sub-Riemannian structure on M, not necessarily ideal. Let $x\in M$ and $y\notin \mathrm{Cut}(x)$. Then, there exists $\mathcal{N}(x,y)\in\mathbb{N}\cup\{+\infty\}$ and a constant $C(x,y)>0$ such that*[3]

$$\beta_t(x,y)\sim C(x,y)t^{\mathcal{N}(x,y)},\qquad \text{for } t\to 0^+. \tag{2.35}$$

Furthermore, for a.e. $y\notin\mathrm{Cut}(x)$, the exponent $\mathcal{N}(x,y)$ attains its minimal value

$$\mathcal{N}(x):=\min\{\mathcal{N}(x,z)\mid z\notin\mathrm{Cut}(x)\}. \tag{2.36}$$

The number $\mathcal{N}(x)$ is called the geodesic dimension *of the sub-Riemannian structure at x. Finally, the following inequality holds*

$$\mathcal{N}(x)\geq \dim_{Haus}(M)\geq \dim(M), \tag{2.37}$$

with equality if and only if the structure is Riemannian at x, that is $\mathcal{D}_x=T_xM$.

We mention that there is an explicit formula for the geodesic dimension of a sub-Riemannian manifold of the form

$$\mathcal{N}(x)=\sum_{i=1}^{m}(2i-1)(\dim\mathcal{F}_x^i-\dim\mathcal{F}_x^{i-1}), \tag{2.38}$$

where $\mathcal{F}_x^1\subset\cdots\subset\mathcal{F}_x^m=T_xM$ is a flag of subspaces associated to generic geodesics. This formula is reminiscent of Mitchell's one [64] for the Hausdorff dimension:

$$Q=\sum_{j=1}^{r}j(\dim\mathcal{D}_x^j-\dim\mathcal{D}_x^{j-1}), \tag{2.39}$$

where $\mathcal{D}_x^1\subset\cdots\subset\mathcal{D}_x^r=T_xM$ is the classical flag of the distribution. We stress that the two flags are different, in general. We refer to [R2] for details.

[3] The case $\mathcal{N}(x,y)=+\infty$ means that $\beta_t(x,y)=o(t^N)$ as $t\to0^+$ for all $N\in\mathbb{N}$.

2.5.2. Dependence on the distance

Let $(\mathcal{D}, g)$ be an ideal sub-Riemannian structure on M, and let $x, y \in M$, with $y \notin \mathrm{Cut}(x)$. We know that there exists a unique $\lambda \in T_x^*M$ such that $y = \exp_x(\lambda)$. One can regard the sub-Riemannian distortion coefficient $\beta_t(x, y)$ as a one-parameter family of functions depending on the initial covector $\lambda \in T^*M$. Loosely speaking:

$$\beta_t(x, \exp_x(\lambda)) = f_t(\lambda). \tag{2.40}$$

The basic Riemannian examples where the β_t's are explicit are space forms, where they depend on λ *only* through its (dual) norm $\|\lambda\| = d(x, y)$ see (2.5).

In the simplest sub-Riemannian structure, the three-dimensional Heisenberg group, the dependence on λ is fundamentally more complicated, and the distortion coefficients, seen as a function of λ as in (2.40), are *not* a function of $d(x, y) = \sqrt{2H(\lambda)}$ only. This fact is the result of a direct computation, which was originally obtained by Balogh, Kristály and Sipos in [17]. Thanks to the homogeneous structure of the Heisenberg group, it is sufficient to consider distortion coefficients $\beta_t(0, q)$, where 0 is the identity of the group.

Proposition 2.22 (Heisenberg distortion coefficient)**.** *Let $q \notin \mathrm{Cut}(0)$. Let $\lambda_0 = u_0 dx + v_0 dy + w_0 dz$ be the initial covector of the unique geodesic joining 0 with q, that is $\exp_0(\lambda_0) = q$. Then*

$$\beta_t(0, q) = t \frac{\sin\left(\frac{tw_0}{2}\right)}{\sin\left(\frac{w_0}{2}\right)} \frac{\sin\left(\frac{tw_0}{2}\right) - \frac{tw_0}{2}\cos\left(\frac{tw_0}{2}\right)}{\sin\left(\frac{w_0}{2}\right) - \frac{w_0}{2}\cos\left(\frac{w_0}{2}\right)}, \qquad \forall t \in [0, 1], \tag{2.41}$$

while $d_{SR}(0, q) = \sqrt{u_0^2 + v_0^2}$.

The Heisenberg distortion coefficient *does not* depend on $d_{SR}(0, q)$, but rather on the "vertical component" w_0 of $\lambda_0 \in T_0^*M$.

A similar phenomenon occurs in the case of the Grushin plane, generalized H-type structures, Sasakian and 3-Sasakian structures (cf. [R1] and reference within).

2.6. Geometric inequalities

A classical consequence of Theorem 2.16 are geometric inequalities on M. In [R1, Section 6] we discuss the the p-mean and the Borell-Brascamp-Lieb inequalities, following [41]. Here we focus on the Brunn-Minkowski inequality.

To introduce the geodesic version of the Brunn-Minkowski inequality, we define for any pair of Borel non-empty subsets $A, B \subset M$ the following quantity:

$$\beta_t(A, B) := \inf\left\{\beta_t(x, y) \mid (x, y) \in (A \times B) \setminus \mathrm{Cut}(M)\right\}, \tag{2.42}$$

with the convention that $\inf \emptyset = 0$. Notice that $0 \leq \beta_t(A, B) < +\infty$.

Theorem 2.23 (Sub-Riemannian Brunn-Minkowski inequality)**.** *Let $(\mathcal{D}, g)$ be an ideal sub-Riemannian structure on a n-dimensional manifold M, equipped with a smooth measure m. Let $A, B \subset M$ be non-empty Borel subsets. Then we have*

$$\mathsf{m}(Z_t(A, B))^{1/n} \geq \beta_{1-t}(B, A)^{1/n}\mathsf{m}(A)^{1/n} + \beta_t(A, B)^{1/n}\mathsf{m}(B)^{1/n}. \tag{2.43}$$

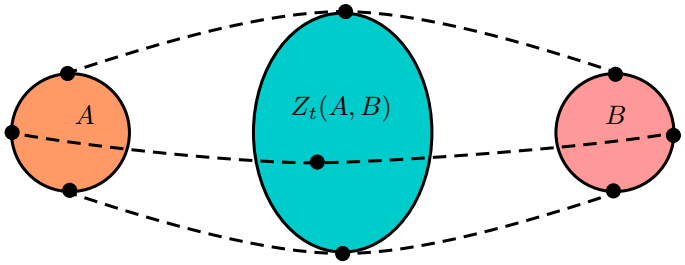


Figure 2.2.: The set $Z_t(A,B)$.

Multiplicative versions of the Brunn-Minkowski inequalities (on Carnot groups) are not the subject of this mémoire, and we refer to [58, 67] for some results on that topic.

A special role in Theorem 2.23 is played by structures where $\beta_t(x,y) \geq t^N$ for some $N \in \mathbb{N}$, for all $t \in [0,1]$ and $(x,y) \notin \mathrm{Cut}(M)$. By Theorem 2.23, this implies the so-called *measure contraction property* $\mathrm{MCP}(0,N)$, first introduced in [69] (see also [82] for a similar formulation). The validity of the MCP was first investigated in Carnot groups in [53, 73], and it is an active subject of research in sub-Riemannian geometry. In our setting, we prove the following equivalence result.

Theorem 2.24 (Equivalence of inequalities)**.** *Let $(\mathcal{D},g)$ be an ideal sub-Riemannian structure on a n-dimensional manifold M, equipped with a smooth measure m. Let $N \geq 1$. Then, the following properties are equivalent:*

(i) $\beta_t(x,y) \geq t^N$, *for all* $(x,y) \notin \mathrm{Cut}(M)$ *and* $t \in [0,1]$;

(ii) the Brunn-Minkowski inequality holds: for all non-empty Borel sets A, B

$$\mathsf{m}(Z_t(A,B))^{1/n} \geq (1-t)^{N/n}\mathsf{m}(A)^{1/n} + t^{N/n}\mathsf{m}(B)^{1/n}, \qquad \forall\, t \in [0,1]; \tag{2.44}$$

(iii) the following measure contraction property $\mathrm{MCP}(0,N)$ *is satisfied: for all non-empty Borel sets* B *and* $x \in M$

$$\mathsf{m}(Z_t(x,B)) \geq t^N \mathsf{m}(B), \qquad \forall\, t \in [0,1]. \tag{2.45}$$

Theorem 2.24 clarifies the fact that an Euclidean Brunn-Minkowski inequality with linear weights (that is (2.44) with $N = n$), is not suitable for genuine sub-Riemannian situations, as well as the classical curvature-dimension condition. In fact, on a n-dimensional sub-Riemannian manifold that is not Riemannian, the $\mathrm{MCP}(0,n)$ is never satisfied (see [R2, Thm. 6]). Theorem 2.24 also shows the interplay between the best N for which (i) holds and the topological dimension n, in the Brunn-Minkowski-type inequality (2.44).

We remark that Theorem 2.24 was one of the starting points of a recent extension of the curvature-dimension condition, due to E. Milman, suitable for several sub-Riemannian structures. This condition is called *quasi curvature-dimension*, or QCD, see [63]. It is stronger than the MCP, but weaker then the CD, and it has several nice applications, such as the determination of explicit constants for Poincaré inequalities on domains for Carnot groups.

2.7. Old and new examples

As a consequence of previous works on the geometry of several sub-Riemannian structures, that we carried on in [R2, R3, R20, R7] with different co-authors, we were able to exhibit several new classes of structures with explicit distortion coefficients (and thus, explicit versions of Brunn-Minkowski inequalities and related ones). Particular emphasis has been given to the structures for which the $\mathrm{MCP}(0,N)$ holds, that is point (iii) of Theorem 2.24. In particular we recall, without giving details:

The Heisenberg group. In this case we recover the seminal results of [17].

Generalized H-type groups. This is a class of Carnot groups of arbitrary large corank, that we introduced in [R3], and which extends the class of Kaplan H-type groups. For *ideal* generalized H-type groups we obtain sharp interpolations inequalities for general measures. In the non-ideal case, we prove sharp Brunn-Minkowski inequalities and measure contraction properties. The proof for non-ideal generalized H-type Carnot groups is reduced to the ideal case thanks to recent tensorization results of Ritoré [75]. This result is interesting since it shows, analogously to [18], that the ideal assumption may be weakened, at least for Brunn-Minkowski-type inequalities.

Grushin plane. Our techniques work also for sub-Riemannian distributions $\mathcal{D}$ whose rank is not constant. We were able to obtain for the first time, in this setting, sharp Brunn-Minkowski inequalities and measure-contraction properties. In particular, consider the important example of the Grushin plane, which can be seen as the singular Riemannian structure on $\mathbb{R}^2$ given by the Riemannian metric

$$g = dx^2 + \frac{1}{x^2}dy^2. \tag{2.46}$$

This structure is a special case of (rank-varying) sub-Riemannian structures with generating frame $X_1 = \partial_x$ and $X_2 = x\partial_y$ (cf. Section 2.2). An interesting by-product of this analysis is the following theorem, proved in [R4].

Theorem 2.25. *Let $\mathbb{G}$ be the Grushin plane, equipped with its sub-Riemannian distance and the Lebesgue measure. Let also $\mathbb{G}_\pm$ be the left/right Grushin plane, which are geodesically convex subsets of $\mathbb{G}$. Then*

- *$\mathbb{G}$ satisfies the $\mathrm{MCP}(0,N)$ if and only if $N \geq 5$;*
- *$\mathbb{G}_\pm$ satisfies the $\mathrm{MCP}(0,N)$ if and only if $N \geq 4$;*

Since the full Grushin plane is the metric double of the half Grushin plane, Theorem 2.25 yields a counter-example of Perelman's doubling theorem in the setting of metric measure spaces satisfying weak Ricci curvature lower bounds. We do not describe further this result here, referring to [R4] for details.

Sasakian structures. Sasakian manifolds are a particular class of contact sub-Riemannian structures. When endowed with their canonical volume, Sasakian manifolds satisfy a measure contraction property under suitable curvature lower bounds. Combining these results with Theorem 2.24, we get a sharp Brunn-Minkowski inequality in this setting.

We do not go into details, referring to [R1, Sec. 7] for further details and description of the above structures. Here, we only mention that in all the above cases, we are able to prove that the distortion coefficients satisfies

$$\beta_t(x,y) \geq t^{\mathcal{N}}, \qquad \forall (x,y) \notin \mathrm{Cut}(M),\ \forall t \in [0,1], \tag{2.47}$$

for some minimal $\mathcal{N}$, given by the geodesic dimension of the sub-Riemannian structure (cf. Theorem 2.21).

2.8. Further developments

Extension to non-ideal case

A challenging problem is to understand how to include abnormal minimizers in the framework. In general, an organic theory of transport and Jacobi fields along abnormal geodesics is still lacking and this will be the subject of further studies. Abnormal geodesics, as [18] suggests for the case of corank 1 Carnot groups, do not preclude interpolation inequalities for optimal transport as in Theorem 2.16.

In a ongoing project, T. Mietton (Ph.D. student at the Institut Fourier, co-advised with H. Pajot), was able to extend the theory of interpolation inequalities to metric products of ideal structures, which are non-ideal as soon as one of the factors is not Riemannian (this work, in particular, contains the results in [18]).

In general, there are two main obstacles. Firstly, abnormal minimizers might not follow the classical Hamiltonian dynamics. Secondly, the well-posedness and regularity of the Monge transportation problem, fundamental for the formulation of interpolation inequalities, still needs to be fully developed in the non-ideal case. Another disruptive phenomenon occurring in non-ideal structures is the one of branching geodesics, which will be discussed in Chapter 3.

Regularity at the cut locus

One of our main results in [R1] was the characterization of the sub-Riemannian cut locus as the set of points where the squared distance loses local semiconvexity (cf. Theorem 2.20), extending the Riemannian result of [41]. This characterization is false in the non-ideal case (so that our result is optimal). We conjectured in [R1, Sec. 4.2, see also Addendum on my website] that abnormal minimizing geodesics are related with loss of local semiconcavity, while loss of local semiconvexity is associated with loss of minimality of geodesics (i.e. the classical concept of cut-length in Riemannian geometry). We do not detail here these conjectures. We only remark that, at least for a sub-class of step 2 Carnot group, some of these conjectures have been proved to be true, in [59].

Measure contraction properties of Carnot groups

One major open problem in sub-Riemannian geometry is related with the measure contraction properties of Carnot groups. In [73], Rifford proved that for any ideal Carnot group there exists $N \in \mathbb{N}$ such that the $\mathrm{MCP}(0, N)$ property is satisfied (and so, by Theorem 2.24, this implies a weighted Brunn-Minkowski inequality). In [14], Badreddine and Rifford extended this result to the so-called *Lipschitz* Carnot groups (i.e. Carnot groups where the sub-Riemannian distance is locally Lipschitz outside of the diagonal, which are not necessarily ideal). For this latter class, Theorem 2.24 does not necessarily hold. Hence, we have the following questions: is it true that, for Lipschitz Carnot groups, the Brunn-Minkowski type inequality (2.44) holds for some $N \in \mathbb{N}$? Of course, if such N exists, then it is greater or equal than the geodesic dimension $\mathcal{N}$ of the Carnot group, as a consequence of the asymptotics of Theorem 2.21 as $t \to 0$. Hence, is the optimal N such that (2.44) holds equal to the geodesic dimension? The answer to those questions will require also a deeper understanding of optimal transport for non-ideal structures.

3. Branching geodesics

3.1. Introduction

A key fact in our work about interpolation inequalities (cf. Chapter 2), is that in the ideal setting sub-Riemannian geodesics lose minimality after the first conjugate point.

However, if a normal geodesic $\gamma : [0,1] \to M$ contains a non-trivial abnormal segment $\gamma|_{[t,s]}$, with $t < s$, then the whole segment is made of conjugate points. Furthermore, this can happen even if γ is a minimizing geodesic, so that minimality is not lost after the first conjugate point! However, no examples of this type were known, and the following question was open.

Can a strictly normal geodesic contain a non-trivial abnormal segment?

This chapter reports on the affirmative answer to the above question, contained in:

- T. Mietton and **L. Rizzi**. Branching geodesics in sub-Riemannian geometry. *Geom. Funct. Anal.*, 30(4):1139–1151, 2020

and obtained in collaboration with my Ph.D. student T. Mietton (Institut Fourier, co-advised with H. Pajot, 2019 – onwards). There, we proved not only that the answer to the above question is affirmative, but that it is related with branching geodesics.

Common examples of branching geodesics are found in Finsler geometry, or on graphs (but not on Riemannian manifolds). In sub-Riemannian geometry, even though normal geodesics are obtained through the action of a Hamiltonian flow, and are therefore smooth, they are not uniquely characterized by their jet at some point! To our best knowledge, it is the first time that this fact is observed. Our contribution adds this phenomenon to the list of remarkable features of sub-Riemannian geometry. We refer to Section 2.2 for preliminaries about normal/abnormal geodesics, which will be important in this chapter.

3.2. Branching

First let us define precisely what we will call branching here.

Definition 3.1. A normal geodesic γ is *branching* at time $t \in (0,1)$ if there exists a normal geodesic γ' such that $\gamma|_{[0,t]} = \gamma'|_{[0,t]}$ and $\gamma|_{[0,t+\varepsilon]} \neq \gamma'|_{[0,t+\varepsilon]}$ for all $\varepsilon > 0$.

Consider now a strictly normal geodesic γ, that is a geodesic that does not admit any abnormal lift. The following result is a simplified version of what proved in [R5].

Theorem 3.2. *A strictly normal geodesic γ is branching for some time $t \in (0,1)$ if and only if the restriction $\gamma|_{[0,t]}$ is abnormal. In particular if t is the last branching time, $\gamma|_{[0,t]}$ is the maximal abnormal segment starting from 0.*

Let us stress that normal geodesics cannot branch in real-analytic sub-Riemannian structures, that is when the corresponding Hamiltonian function is real-analytic. In fact in this case, by the Cauchy-Kowalevski theorem, normal geodesics, which are projections of the solutions of the Hamiltonian equation, are real-analytic paths. Thus, two distinct real-analytic paths cannot be equal on a segment. That is, the following fact holds:

Proposition 3.3. *If the sub-Riemannian Hamiltonian H is a real-analytic function, then normal geodesic cannot branch.*

Thus, to build an example of branching, we need to find a smooth, but non real-analytic structure, in which an abnormal geodesic, when prolonged, becomes strictly normal. A natural idea is to start from a structure admitting non-trivial abnormal geodesics (such as the flat Martinet structure in $\mathbb{R}^3$) and "glue" it in a non-real-analytic way to a structure that does not admit non-trivial abnormal paths, like the Heisenberg one. By the above discussion, branching must occur when an abnormal Martinet geodesic enters the Heisenberg region. This idea is implemented explicitly in [R5]. In this particular structure on $\mathbb{R}^3$, any abnormal geodesics starting from the Martinet region branches in a one-parameter family of normal geodesics γ_α upon entering the gluing region, accumulating on a strictly abnormal geodesic, see Figure 3.1.

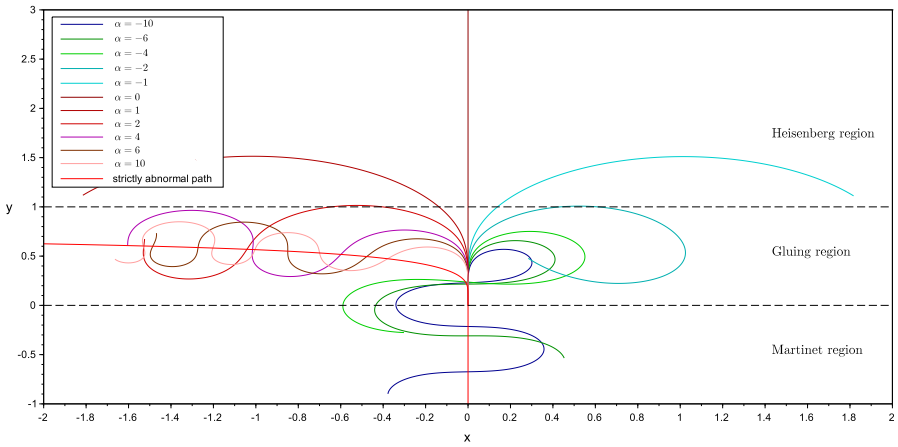


Figure 3.1.: Numerical plot of the branching geodesics γ_α, projected on the xy plane. Notice that the abnormal path lies in the Martinet surface, which must bend in order to avoid the Heisenberg region.

3.3. Branching and magnetic fields

We describe succinctly a class of examples including the one we just described. We will exploit the connection between the equations of motion of a particle in a magnetic field

and sub-Riemannian geodesics, pointed out by Montgomery in his original construction of abnormal minimizers [66].

Take the distribution on $\mathbb{R}^3$ defined by the kernel of a one-form $\omega = dz - A(x,y)dy$, and consider the sub-Riemannian metric given by the restriction of $dx^2 + dy^2$. Let $B = \partial_x A$ be the magnetic field associated with ω, that is $\omega \wedge d\omega = -B(x,y)dx \wedge dy \wedge dz$. Notice that, without changing B, we can alter A in such a way that $A(0,y) = 0$ (gauge freedom), and thus the straight line $\gamma_0(t) = (0,t,0)$ for $t \in \mathbb{R}$ is horizontal.

It is well-known that abnormal paths are the horizontal curves contained in the zero-locus of B, while normal geodesics are those whose projection on the xy-plane satisfies

$$\kappa(t) = \lambda B(x(t), y(t)), \tag{3.1}$$

for some $\lambda \in \mathbb{R}$, and where $\kappa(t)$ is the curvature of the projection $t \mapsto (x(t), y(t))$, which we assume to be parametrized with constant speed. The ODE corresponding to (3.1) describes the motion of a particle with charge λ under the action of the magnetic field $B(x,y)$ normal to the plane.

Choose a smooth potential $A(x,y)$ such that $B(x,y) = x$ for $y < 0$ and $B(0,y) > 0$ when $y > 0$. The zero-locus of B coincides in this case with $x = 0$ when $y < 0$. In particular, $\gamma_0(t) = (0,t,0)$, for $t < 0$, is an abnormal geodesic. Since $B(0,y) = 0$ for $y < 0$, the curve $\gamma_0(t)$ for $t < 0$ satisfies also (3.1) for any $\lambda \in \mathbb{R}$, so it is also normal. We can now extend such a curve to a normal geodesic $\gamma_\lambda(t)$ for $t \in \mathbb{R}$, by solving (3.1) for different values of $\lambda \in \mathbb{R}$. Of course, γ_0 corresponds to the straight line but, from the fact that $B(0,y) > 0$ for $y > 0$, the curve γ_λ must have non-vanishing curvature for small non-zero λ, hence a branching phenomenon occurs at the origin. Moreover, one can prove that the projection on the xy-plane of the trajectories γ_λ contains an open neighborhood of the positive y-axis and those trajectories are all distinct for λ sufficiently small. From the physical viewpoint, this phenomenon corresponds to particles having different charges, which "spray out" following different trajectories under the influence of the magnetic field.

3.4. Further developments

In this chapter we only considered the branching of normal geodesics, and we do not cover the case of strictly abnormal ones. It is easy to produce sub-Riemannian structures with branching abnormal paths. For example, consider a degenerate Martinet-type structure in a three-dimensional space, whose Martinet surface itself branches. Such a structure cannot verify the usual non-degeneracy condition, cf. [65, Sec. 3.2]. The Liu-Sussmann local minimality result for abnormal paths does not apply [60], and we are not able to prove that these paths are length-minimizing. Thus, an interesting problem would be to find an example of branching strictly abnormal length-minimizing curves.

4. Bakry-Émery comparison theory

4.1. Introduction

In Chapter 2, we discussed interpolation inequalities for optimal transport, where distortion coefficients of the underlying structure appeared naturally. The subject of this chapter is the relation between these coefficients and Ricci-type curvature lower bounds.

In the Riemannian setting, a lower bound on the Ricci curvature implies a lower bound on the distortion coefficients (cf. Theorem 2.3). Let now (M, g) be a Riemannian manifold, but equipped with a general[1] smooth measure $\mathsf{m} = e^{-\psi}\mathsf{m}_g$, for some smooth function $\psi : M \to \mathbb{R}$. In this setting, the relevant curvature-type quantity is the so-called *Bakry-Émery Ricci tensor* of parameter $N > n$, defined for every unit $v \in TM$ by:

$$\mathrm{Ric}_{\mathsf{m}}^N(v) := \mathrm{Ric}_g(v) + \nabla^2\psi(v,v) - \frac{g(\nabla\psi, v)^2}{N-n}, \tag{4.1}$$

where $\nabla^2\psi$ denotes the Riemannian Hessian of ψ. One has then the following classical result (see e.g. [91, Appendix A] for an equivalent statement).

Theorem 4.1. *Let (M, g) be a n-dimensional Riemannian manifold, equipped with a smooth volume m. Assume that there exists $K \in \mathbb{R}$ and $N > n$ such that $\mathrm{Ric}_{\mathsf{m}}^N(v) \geq K$ for every unit vector $v \in TM$. Then for all $t \in [0,1]$ we have*

$$\beta_t(x,y) \geq \beta_t^{(K,N)}(d(x,y)), \tag{4.2}$$

where $\beta_t^{(K,N)}$ are the model distortion coefficients for the simply connected space form of dimension N and constant curvature $K/(N-1)$ (and thus Ricci curvature K):

$$\beta_t^{(K,N)}(\theta) = \begin{cases} t\left(\frac{\sin(t\sqrt{K/(N-1)}\theta)}{\sin(\sqrt{K/(N-1)}\theta)}\right)^{N-1} & \text{if } K > 0, \\ t^N & \text{if } K = 0, \\ t\left(\frac{\sinh(t\sqrt{|K|/(N-1)}\theta)}{\sinh(\sqrt{|K|/(N-1)}\theta)}\right)^{N-1} & \text{if } K < 0. \end{cases} \tag{4.3}$$

The natural next step in the development of the theory is to establish a comparison theory for distortion coefficients suitable for sub-Riemannian spaces.

What is a suitable concept of Ricci curvature in sub-Riemannian geometry? Is there an associated comparison theory? What are corresponding models?

[1] This is important for sub-Riemannian generalizations, as there is no canonical measure [R17, 65].

Our answer to the above question comes in the form of an intrinsic comparison theory, and it is inspired by the theory of optimal control. Our analysis suggests that, in this context, model spaces are microlocal, i.e. associated to a fixed geodesic. Our models are not sub-Riemannian manifolds. Rather, they belong to a more general class of variational problems, called Linear Quadratic optimal control problems. This is a significant difference with respect to the Riemannian case, but at the same time it is the crucial idea that allowed the development of this theory.

The results reported in this chapter are contained in:

- D. Barilari and **L. Rizzi**. Bakry-Émery curvature and model spaces in sub-Riemannian geometry. *Math. Ann.*, 377(1-2):435–482, 2020

The applications to 3-Sasakian manifolds (Theorem 4.20) is based on the results in

- **L. Rizzi** and P. Silveira. Sub-Riemannian Ricci curvatures and universal diameter bounds for 3-Sasakian manifolds. *J. Inst. Math. Jussieu*, 18(4):783–827, 2019

The comparison theory for distortion coefficient that we present here can be paired with the results of Chapter 2, yielding explicit interpolation inequalities under suitable sub-Riemannian curvature bounds.

4.2. Sub-Riemannian curvature

The notion of sub-Riemannian curvature we use has its roots in the works of Agrachev-Zelenko [2, 3] and Zelenko-Li [92] and was then subsequently developed by us [R14, R15, R24]. Applications of this theory, such as Bonnet-Myers theorems and measure contraction properties, have been given in the recent years in [6, 57, R15, R7, R20, 20]. The work [R6], whose results we present here, is at the same time an extension (in the Bakry-Émery sense) and a synthesis of these works. For simplicity, assume in this chapter that the sub-Riemannian distribution $\mathcal{D}$ has constant rank.

Definition 4.2. Let γ be a smooth horizontal curve and let T be a smooth horizontal vector field such that $\mathsf{T}|_{\gamma(t)} = \dot\gamma(t)$ for all t. For $i \geq 1$, let

$$\mathcal{F}^i_{\gamma(t)} := \operatorname{span}\{(\operatorname{ad}\mathsf{T})^j Y|_{\gamma(t)} \mid Y \in \Gamma(\mathcal{D}),\ j \leq i-1\} \subseteq T_{\gamma(t)}M, \tag{4.4}$$

where $(\operatorname{ad} X)Y = [X, Y]$. The *growth vector* of the curve is the sequence

$$\mathcal{G}_{\gamma(t)} := \{\dim \mathcal{F}^1_{\gamma(t)}, \dim \mathcal{F}^2_{\gamma(t)}, \dots\}. \tag{4.5}$$

Definition 4.3. We say that the curve γ with growth vector $\mathcal{G}_{\gamma(t)}$ is:

(a) *equiregular* if $\dim \mathcal{F}^i_{\gamma(t)}$ does not depend on t for all $i \geq 1$,

(b) *ample* if for all t there exists $m \geq 1$ such that $\dim \mathcal{F}^m_{\gamma(t)} = \dim T_{\gamma(t)}M$.

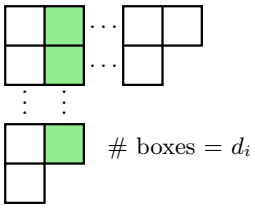
Assume from now on that γ is ample and equiregular. The smallest integer $m \geq 1$ such that $\dim \mathcal{F}^m_{\gamma(t)} = \dim T_{\gamma(t)}M$ is called *step* of γ. Let

$$d_i := \dim \mathcal{F}^i_{\gamma(t)} - \dim \mathcal{F}^{i-1}_{\gamma(t)}, \qquad i \geq 1, \tag{4.6}$$

the increase in dimension at each step, with the convention that $\dim \mathcal{F}^0_{\gamma(t)} = 0$. It is easy to show that $d_1 \geq d_2 \geq \ldots \geq d_m$.

4.2.1. Young diagram and canonical curvature

We can associate with γ a Young diagram D, which encodes (4.5). The Young diagram D has, by construction, m columns, each one of length d_i, for $i = 1, \ldots, m$, as follows:



The total number of boxes in D is equal to the dimension of the manifold $\sum_{i=1}^m d_i = n$.

Remark 4.4. In the Riemannian case, any curve has the same Young diagram with only one column and n boxes.

For the sake of brevity and clarity, the precise definition of the forthcoming objects (which is quite technical) is omitted. We refer to [R6, Appendix A] for a minimal overview, or to the original work [92]. Let γ be an ample and equiregular geodesic. Then the following objects are well defined:

- a quadratic form $\mathfrak{R}_\gamma(t) : T_{\gamma(t)}M \times T_{\gamma(t)}M \to \mathbb{R}$ along γ (*canonical curvature*);
- a scalar product $\langle \cdot | \cdot \rangle_{\gamma(t)}$, on $T_{\gamma(t)}M$, extending g along γ (*canonical extension*);
- a *canonical moving frame* $X_1(t), \ldots, X_n(t)$ along γ, orthonormal with respect to $\langle \cdot | \cdot \rangle_{\gamma(t)}$, and adapted to the flag $\mathcal{F}_{\gamma(t)}$. The canonical moving frame is a generalization of the concept of parallel transported frame. It is uniquely defined up to constant orthogonal transformations respecting the structure of the flag $\mathcal{F}^i_{\gamma(t)}$.

All above objects, at any point $\gamma(t)$, depend in general on the choice of γ.

Remark 4.5. Every Riemannian geodesic is ample and equiregular, and letting R_g be the Riemannian curvature tensor it holds $\mathfrak{R}_\gamma(t)(v, v) = R_g(v, \dot\gamma(t), \dot\gamma(t), v)$. Furthermore, in the Riemannian case, $\mathfrak{R}_\gamma(t)$ is quadratic also with respect to $\dot\gamma(t)$.

Definition 4.6. Given a smooth measure m and an ample and equiregular geodesic γ, we define the *geodesic volume derivative* along γ as the function

$$\rho_{\mathsf{m},\gamma}(t) = \frac{d}{dt} \log \mathsf{m}_{\gamma(t)}(X_1(t), \ldots, X_n(t)), \tag{4.7}$$

where $X_1, \ldots, X_n$ is a canonical moving frame along γ.

Remark 4.7. In the Riemannian case $X_1, \dots, X_n$ is parallel and orthonormal, hence if $\mathsf{m} = e^{-\psi}\mathsf{m}_g$, then $\rho_{\mathsf{m},\gamma}(t) = -g(\dot\gamma(t), \nabla\psi)$. In particular $\rho_{\mathsf{m}_g,\gamma}(t) = 0$ for any geodesic.

The scalar product $\langle\cdot|\cdot\rangle_{\gamma(t)}$ induces a quadratic form

$$\mathfrak{B}_\gamma(t) : T_{\gamma(t)}M \times T_{\gamma(t)}M \to \mathbb{R}, \tag{4.8}$$

where $\mathfrak{B}_\gamma(t)(v,w)$ is the scalar product of the orthogonal projections of v, w on $\mathcal{D}_{\gamma(t)}$ with respect to $\langle\cdot|\cdot\rangle_{\gamma(t)}$.

Definition 4.8 (Bakry-Émery curvature)**.** Let $(M, \mathcal{D}, g)$ be a sub-Riemannian manifold of dimension n. The *Bakry-Émery curvature* along an ample and equiregular geodesic γ is the family of quadratic forms $\mathfrak{R}^N_{\mathsf{m},\gamma}(t) : T_{\gamma(t)}M \times T_{\gamma(t)}M \to \mathbb{R}$ defined by

$$\mathfrak{R}^N_{\mathsf{m},\gamma}(t) := \mathfrak{R}_\gamma(t) - \left(\frac{\dot\rho_{\mathsf{m},\gamma}(t)}{k} + \frac{n}{N-n}\frac{\rho^2_{\mathsf{m},\gamma}(t)}{k^2}\right)\mathfrak{B}_\gamma(t), \tag{4.9}$$

where $N > n$ is a real parameter, and $k = \operatorname{rank}\mathcal{D}_{\gamma(t)}$. If $\rho_{\mathsf{m},\gamma} = 0$ along the given geodesic, one can take $N = n$ in the above expression, with the convention that $0/\infty = 0$.

Remark 4.9. In the Riemannian case $k = n$ and $\langle\cdot|\cdot\rangle_{\gamma(t)} = g_{\gamma(t)}$, so that $\mathfrak{B}_\gamma(t)$ coincides with the Riemannian metric on $T_{\gamma(t)}M$. Hence, taking the trace we have

$$\operatorname{Tr}\mathfrak{R}^N_{\mathsf{m},\gamma}(t) = \operatorname{Tr}\mathfrak{R}_\gamma(t) - \dot\rho_{\mathsf{m},\gamma}(t) - \frac{1}{N-n}\rho^2_{\mathsf{m},\gamma}(t). \tag{4.10}$$

Letting $\mathsf{m} = e^{-\psi}\mathrm{vol}_g$, we have $\rho_{\mathsf{m},\gamma}(t) = -g(\nabla\psi, \dot\gamma(t))$ and therefore (4.10) reduces to the classical Bakry-Émery Ricci curvature defined in (4.1). In particular, in the Riemannian setting, we can regard $\mathfrak{R}^N_{\mathsf{m},\gamma}(t)$ as a sectional version of the Bakry-Émery Ricci curvature.

4.3. Model spaces

We introduce our models, associated to a fixed ample and equiregular geodesic $\gamma : [0,1] \to M$. Let A, B be $n \times n$ matrices, with $B \geq 0$ and symmetric. Their explicit formula is not important right now, but we only stress that A, B are determined uniquely by the Young diagram D of γ. Letting $k \leq n$ be the rank of B, there exist vectors $b_1, \dots, b_k \in \mathbb{R}^n$, unique up to an orthogonal transformation, such that $B = \sum_{i=1}^k b_i b_i^*$.

Let Q be a symmetric $n \times n$ matrix (playing the role of the curvature parameter). We consider a variational problem on $\mathbb{R}^n$, that consists in minimizing the functional

$$\frac{1}{2}\int_0^1 (u^*u - x_u^* Q x_u)\, dt, \tag{4.11}$$

among all trajectories $x_u : [0,1] \to \mathbb{R}^n$ with fixed endpoint satisfying

$$\dot x_u = Ax_u + \sum_{i=1}^k u_i b_i, \tag{4.12}$$

for some control $u \in L^2([0,1],\mathbb{R}^k)$. These models are called *Linear Quadratic optimal control problems* in control theory (LQ problems).

The functional (4.11), when minimized over all trajectories x_u with fixed endpoints, does not define a metric spaces structure on $\mathbb{R}^n$, in general. However one can define the set $Z_t(\Omega_0,\Omega_1)$ of t-intermediate points between two non-empty sets $\Omega_0,\Omega_1 \subset \mathbb{R}^n$ as the set of all points $x(t)$, where $x:[0,1]\to\mathbb{R}^n$ is a minimizer for the problem (4.11)-(4.12) such that $x(0)\in\Omega_0$ and $x(1)\in\Omega_1$. Then we define the model distortion coefficient as

$$\beta_t^{D,Q} := \limsup_{r\to 0} \frac{|Z_t(x,B_r(y))|}{|B_r(y)|}, \qquad t\in[0,1], \tag{4.13}$$

where $x,y\in\mathbb{R}^n$, $B_r(y)$ denotes the Euclidean ball with center y and radius $r>0$, and $|\cdot|$ denotes the Lebesgue measure of $\mathbb{R}^n$. An important fact, which we proved in [R6], is that the right hand side of (4.13) is independent on the choice of $x,y\in\mathbb{R}^n$, and so the definition is well posed, in the sense that $\beta_t^{D,Q}$ depends only on the Young diagram D (via the matrices A,B) and Q.

Remark 4.10. If D is the Young diagram of a geodesic on a n-dimensional Riemannian manifold, then $A=A(D)=\mathbb{0}_n$, $B=B(D)=\mathbb{1}_n$. If we choose $Q=\frac{\kappa}{n}\mathbb{1}_n$, we obtain the homogeneous distortion coefficient

$$\beta_t^{D,Q} = \begin{cases} \left(\frac{\sin(t\sqrt{\kappa})}{\sin(\sqrt{\kappa})}\right)^n & \kappa>0,\\ t^n & \kappa=0,\\ \left(\frac{\sinh(t\sqrt{|\kappa|})}{\sinh(\sqrt{|\kappa|})}\right)^n & \kappa<0. \end{cases} \tag{4.14}$$

One can recover the sharp Riemannian model coefficient $\beta_t^{(\kappa,n)}$ of Theorem 4.1 by choosing, instead, the $n\times n$ matrix $Q=\frac{\kappa}{n-1}d^2(x,y)\,\mathrm{diag}(1,\dots,1,0)$. With hindsight, this is natural: since the potential Q mimics the effect of (sectional) curvature, this choice correctly takes into account that, for a space form with Ricci curvature κ and dimension n, the sectional curvature $R_g(\cdot,\dot\gamma,\dot\gamma,\cdot)$ has $n-1$ eigenvalues equal to $\kappa\|\dot\gamma\|^2=\kappa d^2(x,y)$ and a vanishing eigenvalue corresponding to the direction of the motion (here γ is a minimizing geodesic joining x with y).

4.4. Sectional-type comparison results

We now state the first pair of results of [R6]. Theorem 4.11 requires separate assumptions on the curvature and on the volume derivative. Theorem 4.13 unifies both assumptions in a single Bakry-Émery-type lower bound.

Theorem 4.11. *Let $(x,y)\notin \mathrm{Cut}(M)$ and assume that the unique geodesic γ joining x and y is ample and equiregular, with Young diagram D. Assume that the geodesic volume derivative satisfies $\rho_{\mathsf{m},\gamma}(t)\le 0$ along γ, and that there exists a symmetric $n\times n$ matrix Q such that $\mathfrak{R}_\gamma(t)\ge Q$ for every $t\in[0,1]$. Then*

$$\frac{\beta_t(x,y)}{\beta_t^{D,Q}} \text{ is a non-increasing function of } t\in(0,1]. \tag{4.15}$$

In particular we have

$$\beta_t(x,y) \geq \beta_t^{D,Q}, \qquad \forall t \in [0,1]. \tag{4.16}$$

If, instead, $\mathfrak{R}_\gamma(t) \leq Q$ *and* $\rho_{\mathsf{m},\gamma}(t) \geq 0$ *along* γ *for every* $t \in [0,1]$*, then the function in* (4.15) *is non-decreasing and* (4.16) *holds with the opposite inequality.*

The inequality $\mathfrak{R}_\gamma(t) \geq Q$ is understood by identifying the quadratic form $\mathfrak{R}_\gamma(t)$ with a $n \times n$ matrix using a canonical frame $X_1(t), \dots, X_n(t)$.

Remark 4.12. In Theorem 4.11 the assumption $\rho_{\mathsf{m},\gamma}(t) \leq 0$ (resp. ≥ 0) can be weakened to $\rho_{\mathsf{m},\gamma}(t) \leq c$ for some $c \in \mathbb{R}$ with the following modifications in the conclusion:

$$\frac{\beta_t(x,y)}{\beta_t^{D,Q}} e^{-ct} \text{ is a non-increasing function of } t \in (0,1]. \tag{4.17}$$

In particular we have

$$\beta_t(x,y) \geq \beta_t^{D,Q} e^{c(t-1)}, \qquad \forall t \in [0,1], \tag{4.18}$$

and similarly with reversed inequalities if $\mathfrak{R}_\gamma(t) \leq Q$ and $\rho_{\mathsf{m},\gamma}(t) \geq c$ for $t \in [0,1]$.

Theorem 4.13. *Let* $(x,y) \notin \mathrm{Cut}(M)$ *and assume that the unique geodesic* γ *joining* x *and* y *is ample and equiregular, with Young diagram* D*. Assume that there exists* $N > n$ *and a symmetric* $n \times n$ *matrix* Q *such that* $\frac{1}{N}\mathfrak{R}^N_{\mathsf{m},\gamma}(t) \geq \frac{1}{n}Q$ *for every* $t \in [0,1]$*. Then*

$$\frac{\beta_t(x,y)^{1/N}}{(\beta_t^{D,Q})^{1/n}} \text{ is a non-increasing function of } t \in (0,1]. \tag{4.19}$$

In particular we have

$$\beta_t(x,y)^{1/N} \geq (\beta_t^{D,Q})^{1/n}, \qquad \forall t \in [0,1]. \tag{4.20}$$

Equation (4.20) gives an interpretation of the parameter N as an *effective dimension*.

All those results are obtained by exploiting a suitable Hamiltonian version of the Jacobi equation, without making use of any Levi-Civita connection, and employing instead the canonical frame of Zelenko-Li. The comparison theory then boils down to a delicate study of a matrix Riccati-type equation, which is substantially more complex (and rich of additional structure) than its Riemannian counterpart.

4.5. Ricci-type comparison results

The sectional-type curvature bounds in the assumptions of Theorems 4.11 and 4.13 can be weakened to Ricci-type bounds. In the Riemannian case, this is done by tracing the equation describing the evolution of Jacobi fields, and turning it into a simple scalar inequality (see [90, Ch. 14]). In the sub-Riemannian case, the process of "taking the trace" is more delicate. Due to the anisotropy of the structure, it only makes sense to take partial traces, leading to a number of Ricci curvatures (each one obtained as a partial trace on an invariant subspace). This technique was introduced by us in [R24].

4.5.1. Sub-Riemannian Bakry-Émery Ricci curvatures

In order to state our main results, we need to introduce some terminology related to the boxes of a Young diagram D associated with an ample and equiregular geodesic. We refer to Figure 4.1.

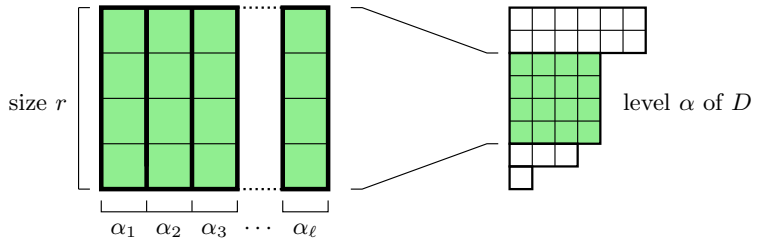


Figure 4.1.: Level α and superboxes α_i of a Young diagram.

A *level* is the collection of all the rows of the Young diagram with the same length. A *superbox* is the collection of all boxes of the Young diagram in a given level, belonging to the same column. The *size* of a level or a superbox is the number r of boxes in each of its columns. For a given level α of the Young diagram, of length ℓ, we denote its superboxes as $\alpha_1, \dots, \alpha_\ell$. Every superbox α_i is associated with an invariant subspace $S^{\alpha_i}_{\gamma(t)} \subseteq T_{\gamma(t)}M$, of dimension equal to its size.

Definition 4.14. For each superbox α_i, we define a sub-Riemannian Ricci curvature (resp. Bakry-Émery Ricci) denoted $\mathfrak{Ric}^{\alpha_i}_\gamma(t)$ (resp. $\mathfrak{Ric}^{N,\alpha_i}_{\mathsf{m},\gamma}(t)$) for $i = 1, \dots, \ell$,

$$\mathfrak{Ric}^{\alpha_i}_\gamma(t) := \operatorname{Tr}\left(\mathfrak{R}_\gamma(t)|_{S^{\alpha_i}_{\gamma(t)}}\right), \qquad \mathfrak{Ric}^{N,\alpha_i}_{\mathsf{m},\gamma}(t) := \operatorname{Tr}\left(\mathfrak{R}^N_{\mathsf{m},\gamma}(t)|_{S^{\alpha_i}_{\gamma(t)}}\right). \tag{4.21}$$

The total number of Ricci curvatures is equatl to the number of superboxes in D.

In the Riemannian case, the Young diagram has a single column with $n = \dim M$ boxes. Thus there is only one superbox, and one Ricci curvature, corresponding to the full trace of $\mathfrak{R}_\gamma(t) = R_g(\dot\gamma(t), \cdot, \cdot, \dot\gamma(t))$ (or its Bakry-Émery generalization).

In the following theorems, Υ denotes the set of levels of the Young diagram.

Theorem 4.15. *Let $(x, y) \notin \operatorname{Cut}(M)$ and assume that the unique geodesic joining x and y is ample and equiregular, with Young diagram D.*

Assume that $\rho_{\mathsf{m},\gamma}(t) \le 0$ along γ and that for every level α of size r_α and length ℓ_α of D there exist $\kappa_{\alpha_i} \in \mathbb{R}$, for $i = 1, \dots, \ell_\alpha$, such that for every superbox α_i

$$\frac{1}{r_\alpha}\,\mathfrak{Ric}^{\alpha_i}_\gamma(t) \ge \kappa_{\alpha_i}, \qquad \forall t \in [0,1]. \tag{4.22}$$

Then

$$\frac{\beta_t(x,y)}{\prod_{\alpha\in\Upsilon}\left(\beta_t^{D_\alpha,Q_\alpha}\right)^{r_\alpha}} \text{ is a non-increasing function of } t \in (0,1], \tag{4.23}$$

where D_α is the Young diagram composed by a single row of length ℓ, and $Q_\alpha = \mathrm{diag}(\kappa_{\alpha_1}, \dots, \kappa_{\alpha_{\ell_\alpha}})$. In particular

$$\beta_t(x,y) \geq \prod_{\alpha \in \Upsilon} \left(\beta_t^{D_\alpha, Q_\alpha}\right)^{r_\alpha}, \qquad \forall t \in [0,1]. \tag{4.24}$$

A similar conclusion can be obtained if, in Theorem 4.15, one assumes $\rho_{\mathsf{m},\gamma}(t) \leq c$ for some $c \in \mathbb{R}$ along γ (cf. Remark 4.12).

Theorem 4.16. *Let $(x,y) \notin \mathrm{Cut}(M)$ and assume that the unique geodesic joining x and y is ample and equiregular, with Young diagram D.*

Assume that there exists $N > n$ such that for every level α of size r_α and length ℓ_α of D there exist $\kappa_{\alpha_i} \in \mathbb{R}$, for $i = 1, \dots, \ell_\alpha$, such that for every superbox α_i

$$\frac{1}{r_\alpha} \mathfrak{Ric}_{\mathsf{m},\gamma}^{N,\alpha_i}(t) \geq \frac{N}{n} \kappa_{\alpha_i}, \qquad \forall t \in [0,1]. \tag{4.25}$$

Then

$$\frac{\beta_t(x,y)^{1/N}}{\displaystyle\prod_{\alpha \in \Upsilon} \left(\beta_t^{D_\alpha, Q_\alpha}\right)^{r_\alpha/n}} \text{ is a non-increasing function of } t \in (0,1], \tag{4.26}$$

where D_α is the Young diagram composed by a single row of length ℓ, and $Q_\alpha = \mathrm{diag}(\kappa_{\alpha_1}, \dots, \kappa_{\alpha_{\ell_\alpha}})$. In particular

$$\beta_t(x,y)^{1/N} \geq \prod_{\alpha \in \Upsilon} \left(\beta_t^{D_\alpha, Q_\alpha}\right)^{r_\alpha/n}, \qquad \forall t \in [0,1]. \tag{4.27}$$

4.6. Removing the direction of motion

Theorems 4.15 and 4.16 do not take into account the fact that distances are not distorted in the direction of a geodesic. This is well known in Riemannian geometry (see e.g. the discussion in [90, p. 384]). This fact remains true in sub-Riemannian geometry as a consequence of the homogeneity of the Hamiltonian.

At a technical level, the distortion coefficient can always be factorized as

$$\beta_t(x,y) = t\beta_t^\perp(x,y), \qquad \forall\, (x,y) \notin \mathrm{Cut}(M), \tag{4.28}$$

where $\beta_t^\perp(x,y)$ is, roughly speaking, the distortion felt in the transverse directions to the geodesic joining x with y. In all proofs, the direction of the motion can be factored out, proving comparison results for $\beta_t^\perp(x,y)$. In the Young diagram, the direction of the motion corresponds to a block situated in the bottom level, the only one of length 1. Thus, factoring out the direction of the motion amounts to removing such a block from the Young diagram D. We omit the details, recording only the following sharper version of Theorem 4.16.

Theorem 4.17. *Let $(x,y) \notin \mathrm{Cut}(M)$ and assume that the unique geodesic joining x and y is ample and equiregular, with Young diagram D.*

Assume that there exists $N > n$ such that for every level α of size r_α and length ℓ_α there exist $\kappa_{\alpha_i} \in \mathbb{R}$, for $i = 1, \dots, \ell_\alpha$, such that for every superbox α_i

$$\frac{1}{r_\alpha}\mathfrak{Ric}^{N,\alpha_i}_{\mathsf{m},\gamma}(t) \geq \frac{N-1}{n-1}\kappa_{\alpha_i}, \qquad \forall t \in [0,1], \tag{4.29}$$

with the convention that if α is the level of length 1 then r_α is replaced by $r_\alpha - 1$, and if $r_\alpha = 0$ then this level is omitted. Then, with the same convention, we have

$$\frac{\beta_t^{\perp}(x,y)^{1/(N-1)}}{\prod_{\alpha\in\Upsilon}\left(\beta_t^{D_\alpha,Q_\alpha}\right)^{r_\alpha/(n-1)}} \text{ is a non-increasing function of } t \in (0,1], \tag{4.30}$$

where D_α is the Young diagram composed by a single row of length ℓ, and $Q_\alpha = \mathrm{diag}(\kappa_{\alpha_1}, \dots, \kappa_{\alpha_{\ell_\alpha}})$. In particular

$$\beta_t(x,y)^{1/(N-1)} \geq t^{1/(N-1)} \prod_{\alpha\in\Upsilon}\left(\beta_t^{D_\alpha,Q_\alpha}\right)^{r_\alpha/(n-1)}, \qquad \forall t \in [0,1]. \tag{4.31}$$

Remark 4.18. If $\rho_{\mathsf{m},\gamma} \leq 0$ along the geodesic joining x with y, then one can take formally $N = n$ in the previous theorem, and obtain a version of Theorem 4.15 with the direction of the motion taken out. This recovers the sharp statement of Theorem 4.1.

4.7. The two columns case

As a consequence of Theorem 4.17, and non-trivial inequalities for the model distortion coefficients, we obtain polynomial bounds for β_t. We only give a statement for $\rho_{\mathsf{m},\gamma} \leq 0$, in which case the Bakry-Émery curvature is not necessary (formally $N = n$ in Theorem 4.17). We adopt an ad-hoc labeling notation for the superboxes of a two-columns Young diagram and the corresponding Ricci curvatures, as in Figure 4.2.

Theorem 4.19. *Let $(x,y) \notin \mathrm{Cut}(M)$ and assume that the unique geodesic joining x and y is ample and equiregular, with Young diagram D as in Figure 4.2. Assume that for all $t \in [0,1]$ we have $\rho_{\mathsf{m},\gamma}(\gamma(t)) \leq 0$ and*

$$\mathfrak{Ric}^a_{\gamma(t)} \geq (n-k)\kappa_a, \tag{4.32}$$

$$\mathfrak{Ric}^b_{\gamma(t)} \geq (n-k)\kappa_b, \tag{4.33}$$

$$\mathfrak{Ric}^c_{\gamma(t)} \geq (2k-n-1)\kappa_c, \tag{4.34}$$

for some $\kappa_a, \kappa_b, \kappa_c \in \mathbb{R}$ satisfying

$$4\kappa_a + \kappa_b^2 \geq 0, \qquad \kappa_b \geq 0, \qquad \kappa_c \geq 0. \tag{4.35}$$

Then $\beta_t(x,y)/t^{k+3(n-k)}$ is a non-increasing of $t \in (0,1]$, and hence

$$\beta_t(x,y) \geq t^{k+3(n-k)}. \tag{4.36}$$

The exponent $k + 3(n-k)$ is optimal, i.e. the smallest one such that (4.36) holds true.

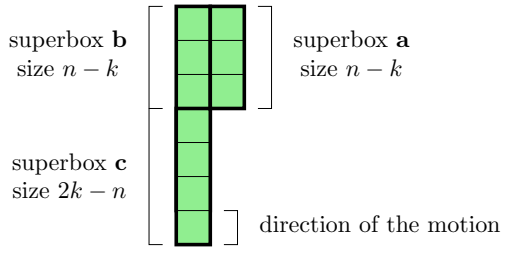


Figure 4.2.: Young diagram with two columns. Here, k is the rank of the sub-Riemannian distribution, while n is the dimension of the manifold. The first level, of size $n-k$ and length 2, is composed by the superboxes denoted, respectively, **b** and **a**. The second level, of size $2k-n$ and length 1, is the superbox **c**. This is, for example, the Young diagram of any non-trivial geodesic for a contact sub-Riemannian structure (where $k = 2d$ and $n = 2d+1$ for some $d \in \mathbb{N}$).

As an application of these comparison results, combined with the ones in [R7], we obtained new explicit inequalities for 3-Sasakian structures. These are co-rank 3 sub-Riemannian structures, where the distribution is given by the kernel of three distinct contact forms, satisfying some compatibility conditions, see [R6, Sec. 6.3] or [R7] for precise definitions. These structures are equipped with a standard Riemannian extension, and thus an auxiliary familiar curvature tensor that can be used to state the result.

Theorem 4.20. *Let $(M, \mathcal{D}, g)$ be a 3-Sasakian manifold of dimension $4d+3$, equipped with its canonical measure. Assume that, for every non-zero $X \in \mathcal{D}$*

$$\mathrm{Sec}(X \wedge Y) \geq K \geq -9, \qquad \forall Y \in \mathrm{span}\{\phi_I X, \phi_J X, \phi_K X\}, \tag{4.37}$$

where Sec *is the Riemannian sectional curvature of the standard Riemannian extension of the 3-Sasakian structure. Then $\beta_t(x,y)/t^{4d+9}$ is a non-increasing of $t \in (0,1]$ and $(x,y) \notin \mathrm{Cut}(M)$. In particular*

$$\beta_t(x,y) \geq t^{4d+9}, \qquad \forall t \in [0,1], \tag{4.38}$$

and the exponent is optimal.

By Theorem 2.24, the bound (4.38) adds 3-Sasakian structures to the list of sub-Riemannian manifolds satisfying a measure contraction properties, with an explicit and sharp dimensional parameter [73, 14, R2, R3, 6, 57].

4.8. Equivalence to sub-Laplacian comparison

Comparison results for distortion coefficients are equivalent to comparison theorem for the sub-Laplacian of the sub-Riemannian distance. In what follows Δ_{m} denotes the sub-Laplacian on M associated with the measure m, i.e. the generator of the Dirichlet form $\int_M \|\nabla u\|^2 d\mathsf{m}$, where ∇u is the sub-Riemannian gradient of u, defined pointwise as the horizontal vector such that $g(\nabla u, \cdot) = du(\cdot)$.

Theorem 4.21. *Let $(M, \mathcal{D}, g)$ be a sub-Riemannian manifold. Let $y \notin \mathrm{Cut}(x)$, and let $\gamma : [0,1] \to M$ be the unique geodesic joining x with y. Assume that $\gamma(t) \notin \mathrm{Cut}(x)$ for all $t \in (0,1]$ (the latter requirement is automatically satisfied if γ does not contain non-trivial abnormal segments, and in particular on ideal structures). Then, letting $\mathsf{f}_x(\cdot) = \frac{1}{2} d^2_{SR}(x, \cdot)$, we have*

$$\Delta_{\mathsf{m}} \mathsf{f}_x(\gamma(t)) = t \frac{d}{dt} \log \beta_t(x, y), \qquad \forall t \in (0,1]. \tag{4.39}$$

Hence, for any smooth $h : (0,1] \to \mathbb{R}_+$, with $h(1) = 1$, the following are equivalent:

- *the function $t \mapsto \beta_t(x,y)/h(t)$ is non-increasing on $(0,1]$;*
- *$\Delta_{\mathsf{m}} \mathsf{f}_x(\gamma(t)) \le t \frac{d}{dt} \log h(t)$ for all $t \in (0,1]$.*

In particular, both statements imply that $\beta_t(x,y) \ge h(t)$ for all $t \in [0,1]$.

This fact is not new, but important in our opinion. The first formula is a consequence of the fact that $\dot\gamma(t) = \nabla \mathsf{f}_x(\gamma(t))$, and of the definition of divergence. See for example [26, Prop. B.1], or also the proof in [14, Prop. 9]. The remaining implications are obvious.

4.9. Maximal length bounds

From the proof of the above theorems, one can recover a bound for the maximal length of minimizing geodesics, that we had already obtained in [R15].

In the following statement, recall the LQ problem introduced in Section 4.3 as models for our comparison theory. The first conjugate time t_c of the LQ problem is an invariant related with existence and uniqueness of its solutions (cf. [R16]).

Theorem 4.22. *Let $\gamma : [0,T] \to M$ be a length-parametrized minimizing geodesic, ample and equiregular, with Young diagram D. Assume that there exists a level α with size r_α and length ℓ_α, and $\kappa_i \in \mathbb{R}$, for $i = 1, \dots, \ell_\alpha$, such that*

$$\frac{1}{r_\alpha} \mathfrak{Ric}^{\alpha_i}_\gamma(t) \ge \kappa_i, \qquad \forall t \in [0,T], \tag{4.40}$$

with the convention that, if α is the level of length 1, then r_α is replaced by $r_\alpha - 1$. Then $\ell(\gamma) \le t_c(\kappa_1, \dots, \kappa_{\ell_\alpha})$, where the latter is the first conjugate time of the LQ problem whose Young diagram has a single row of length ℓ and $Q = \mathrm{diag}(\kappa_1, \dots, \kappa_{\ell_\alpha})$.

Of course, in general there is no explicit expression for $t_c(\kappa_1, \dots, \kappa_{\ell_\alpha})$, and the latter can be infinite (in which case the above statement is vacuous). However, as a consequence of the study performed in [R16] (part of my Ph.D. thesis), one can find necessary and sufficient conditions on $\kappa_1, \dots, \kappa_{\ell_\alpha}$ such that $t_c(\kappa_1, \dots, \kappa_{\ell_\alpha}) < +\infty$, by analyzing the algebraic and geometric multiplicity of the eigenvalues of the associated Hamiltonian system. We refer to [R16] for a general formulation, we just give two examples.

If α is a level of length $\ell_\alpha = 1$, then the finiteness condition is $\kappa_1 > 0$, in which case

$$t_c(\kappa_1) = \frac{\pi}{\sqrt{\kappa_1}}. \tag{4.41}$$

This bound, in particular, recovers the classical Bonnet-Myers diameter bound as a special case of Theorem 4.22.

If α is a level of length $\ell_\alpha = 2$, the conditions for the finiteness of $t_c(\kappa_1, \kappa_2)$ are

$$\begin{cases} \kappa_1 > 0, \\ \kappa_1^2 + 4\kappa_2 > 0, \end{cases} \qquad \text{or} \qquad \begin{cases} \kappa_1 \leq 0, \\ \kappa_2 > 0, \end{cases} \tag{4.42}$$

in which case

$$t_c(\kappa_1, \kappa_2) \leq \frac{2\pi}{\mathrm{Re}(\sqrt{x+y} - \sqrt{x-y})}, \qquad x = \frac{\kappa_1}{2}, \quad y = \frac{\sqrt{\kappa_1^2 + 4\kappa_2}}{2}. \tag{4.43}$$

These results yield the sharp diameter of the standard sub-Riemannian structure on the Hopf fibrations $\mathbb{S}^1 \hookrightarrow \mathbb{S}^{2d+1} \to \mathbb{CP}^d$ and the quaternionic Hopf fibrations $\mathbb{S}^3 \hookrightarrow \mathbb{S}^{2d+3} \to \mathbb{HP}^d$, cf. [R7] and references therein.

4.10. Further developments

Eulerian vs Lagrangian approaches

We note that a different approach to sub-Riemannian curvature, based on the extension of Bochner-type formulas and curvature-dimension inequalities, has been proposed by Baudoin, Garofalo and collaborators (see [22, 21, 24] and references therein). The latter is an instance of the so-called *Eulerian approach* to curvature bounds, alternative to the *Lagrangian approach* we adopted. In the Riemannian setting both approaches are equivalent. It is presently unknown whether our approach to curvature has an Eulerian counterpart. In my opinion this is a very interesting question for future research.

Unification

In [R1] and [R6] we developed a satisfying theory for interpolation inequalities in sub-Riemannian geometry, coupled with an intrinsic comparison theory. As we have seen, the natural reference spaces *do not belong* to the category of sub-Riemannian structures, but rather are optimal control problems (LQ problems). This class of variational problems is large enough to include infinitesimal models for all of the three great classes of geometries: Riemannian, sub-Riemannian, Finsler. This can be seen as a first step of the "great unification" auspicated by Villani (cf. [89, Sec. 9]). In a work in progress with Barilari and Mondino we are developing such a unified theory, extending these insight to the non-smooth setting, creating a bridge between sub-Riemannian geometry and the theory of curvature bounds on metric spaces.

5. Comparison theory via canonical variation for H-type foliations

5.1. Introduction

The subject of this chapter is a different approach to comparison, alternative to the one presented in Chapter 4. To introduce it, let (M, g) be a Riemannian manifold, equipped with an orthogonal splitting of the tangent bundle

$$TM = \mathcal{H} \oplus \mathcal{V}, \tag{5.1}$$

called the *horizontal* and *vertical bundles*, respectively. The *canonical variation* of the metric (cf. [27, Ch. 9]) is the one-parameter family of Riemannian metrics defined by

$$g_\varepsilon = g_{\mathcal{H}} \oplus \frac{1}{\varepsilon} g_{\mathcal{V}}, \qquad \forall \varepsilon > 0. \tag{5.2}$$

where $g_{\mathcal{H}}$ and $g_{\mathcal{V}}$ denote the restriction of g to the corresponding bundle.

Assuming that $\mathcal{H}$ is bracket-generating, the sequence of Riemannian distances d_ε converges, as $\varepsilon \to 0$, to the metric d_0 of the sub-Riemannian manifold $(M, \mathcal{H}, g_{\mathcal{H}})$. Conversely, any sub-Riemannian metric can be obtained as a limit of Riemannian ones in this (non-unique) way. An important fact is that the Riemannian curvature of g_ε is unbounded below as $\varepsilon \to 0$, so that it is not clear how one can recover comparison results for the limit structure exploiting this approximation. In this chapter we aim to answer to the following questions.

> Can we recover sub-Riemannian comparison theorems by studying the canonical variation g_ε? Are there specific classes of sub-Riemannian structures which are better suited for this analysis?

As stated in Theorem 4.21, comparison theorems for distortion coefficients are equivalent to comparison theorems for the sub-Laplacian of the sub-Riemannian distance from a point. Thus, letting $r_\varepsilon(\cdot) := d_\varepsilon(x, \cdot)$ the distance from a given point for the Riemannian structure g_ε, we are interested in proving inequalities of the form

$$\Delta_{\mathcal{H}} r_\varepsilon \leq F_\varepsilon(r_\varepsilon), \qquad \forall \varepsilon > 0, \tag{5.3}$$

where $F_\varepsilon : (0, \infty) \to \mathbb{R}$ is a model function, depending on the particular class of structures under investigation and curvature-type assumptions, while $\Delta_{\mathcal{H}}$ denotes the sub-Laplacian (also called *horizontal Laplacian* in this chapter).

We require that the right hand side of (5.3) admits a limit as $\varepsilon \to 0$, yielding thus a truly sub-Riemannian comparison theorem. This singles out a special class of structures, that we called *H-type foliations*, which are our main focus.

The results reported in this chapter are taken from the pre-print:

- F. Baudoin, E. Grong, G. Molino, and **L. Rizzi**. Comparison theorems on H-type sub-Riemannian manifolds. *arXiv e-prints*, Sept. 2019. arXiv: 1812.02563

The former article focuses on comparison results for H-type foliations, and extends a theory whose development was initiated by Baudoin and collaborators in [23]. We did not include in this memoir the classification of H-type foliations, contained in [R25].

5.2. H-type sub-Riemannian manifolds

In this section, we fix a Riemannian manifold (M, g) equipped with an orthogonal splitting $TM = \mathcal{H} \oplus \mathcal{V}$. We assume that $\mathcal{H}$ is bracket generating so that the restriction of g to $\mathcal{H}$ yields a sub-Riemannian structure.

5.2.1. The sub-Riemannian limit

Let $\{g_\varepsilon\}_{\varepsilon>0}$ be the canonical variation (5.2). The Riemannian distance associated with g_ε will be denoted by d_ε, while the sub-Riemannian one, which depends only on the restriction $g_\mathcal{H}$ of g to $\mathcal{H}$, is d_0. In this section we detail how d_ε approximates d_0.

We assume that (M, d_1) is complete, and so the same necessarily holds for (M, d_ε) for all $\varepsilon \geq 0$ (the converse is not true in general). The cut locus $\mathrm{Cut}_\varepsilon(x)$ of $x \in M$, $\varepsilon \geq 0$ for the distance d_ε is defined as the complement of the set of points $y \in M$ such that there exists a unique minimizing normal geodesic joining x and y, and its endpoints are not conjugate (this definition is correct even in the sub-Riemannian case).

A result in [R11], of independent interest, is the following proposition, establishing the C^∞-convergence of d_ε to d_0 outside of the cut locus. We will use this result to obtain the sub-Riemannian limit of the uniform horizontal Laplacian comparison theorems.

Proposition 5.1. *Let $x, y \in M$ with $y \notin \mathrm{Cut}_0(x)$. Then there exists an open neighborhood V of y and $\varepsilon' > 0$ such that $V \cap \mathrm{Cut}_\varepsilon(x) = \emptyset$ for all $0 \leq \varepsilon < \varepsilon'$, and the map*

$$(\varepsilon, z) \mapsto r_\varepsilon(z) = d_\varepsilon(x, z) \tag{5.4}$$

is smooth for $(\varepsilon, z) \in [0, \varepsilon') \times V$. In particular, we have uniform convergence $r_\varepsilon \to r_0$ together with their derivatives of arbitrary order on compact subsets of $M \setminus \mathrm{Cut}_0(x)$.

5.2.2. Totally geodesic foliations

In the setting of the previous section, if $\mathcal{V}$ is integrable, then (M, g) is equipped with a foliation whose leaves are tangent to $\mathcal{V} = \mathcal{H}^\perp$. Furthermore, we say that g is *bundle-like* and *metric*, if the following two conditions are satisfied:

$$\mathcal{L}_\mathcal{V} g_\mathcal{H} = 0, \qquad \mathcal{L}_\mathcal{H} g_\mathcal{V} = 0. \tag{5.5}$$

The notation $\mathcal{L}_{\mathcal{H}}$ and $\mathcal{L}_{\mathcal{V}}$ denotes the Lie derivative along horizontal and vertical directions, respectively. The first condition, in particular, means that any vertical vector field is Killing for the horizontal part of the metric, while any horizontal vector field is killing for the horizontal part of the metric.

Definition 5.2. A *totally geodesic foliation* $(M, g, \mathcal{H})$ is a Riemannian structure (M, g), together with an orthogonal splitting $TM = \mathcal{H} \oplus \mathcal{V}$, such that $\mathcal{V} = \mathcal{H}^{\perp}$ is integrable, and g is bundle-like and metric. Furthermore, we assume that $\mathcal{H}$ is bracket-generating, so that the restriction of $g_{\mathcal{H}}$ defines a sub-Riemannian structure on M.

In this setting there exists a canonical g-metric connection ∇ preserving the splitting, called *Bott connection*, given in terms of the Levi-Civita one ∇^g by

$$\nabla_X Y = \begin{cases} \pi_{\mathcal{H}}(\nabla^g_X Y) & X, Y \in \Gamma(\mathcal{H}), \\ \pi_{\mathcal{H}}([X, Y]) & X \in \Gamma(\mathcal{V}), \quad Y \in \Gamma(\mathcal{H}), \\ \pi_{\mathcal{V}}([X, Y]) & X \in \Gamma(\mathcal{H}), \quad Y \in \Gamma(\mathcal{V}), \\ \pi_{\mathcal{V}}(\nabla^g_X Y) & X, Y \in \Gamma(\mathcal{V}). \end{cases} \tag{5.6}$$

The torsion T of the Bott connection ∇ is given by

$$T(X, Y) = \begin{cases} -\pi_{\mathcal{V}}([X, Y]) & X, Y \in \Gamma(\mathcal{H}), \\ 0 & \text{otherwise.} \end{cases} \tag{5.7}$$

The above formulas show that the Bott connection defined relative to the canonical variation g_ε in (5.2) does not depend on $\varepsilon > 0$. In particular ∇ is g_ε-metric for all $\varepsilon > 0$.

5.2.3. The comparison principle

The following construction is a specification to foliations of a more general comparison result proved in [R11, Appendix B], for general Riemannian structures (M, g) equipped with an orthogonal splitting.

Let $(M, g, \mathcal{H})$ a totally geodesic foliation, and let ∇ be the Bott connection. We define an associated metric connection:

$$\hat{\nabla}^\varepsilon_X Y = \nabla_X Y + J^\varepsilon_X Y, \qquad \forall X, Y \in \Gamma(TM), \tag{5.8}$$

where J and J^ε are defined as in (5.13) relative to g and g_ε respectively. The main property of connection 5.8 is that its adjoint connection[1] is also g_ε-metric, given by

$$\nabla^\varepsilon_X Y = \nabla_X Y - T(X, Y) + J^\varepsilon_Y X, \qquad \forall X, Y \in \Gamma(TM). \tag{5.9}$$

Let $\hat{R}^\varepsilon$ be the curvature of $\hat{\nabla}^\varepsilon$. The Jacobi equation for a vector field W along a g_ε-geodesic γ reads

$$\hat{\nabla}^\varepsilon_{\dot\gamma} \nabla^\varepsilon_{\dot\gamma} W + \hat{R}^\varepsilon(W, \dot\gamma)\dot\gamma = 0. \tag{5.10}$$

The following comparison principle is the key for our results.

[1] The adjoint of a connection D is $\hat{D}_X Y = D_X Y - \mathrm{Tor}^D(Y, X)$.

Theorem 5.3 (Comparison principle)**.** *Fix $\varepsilon > 0$ and let g_ε be the corresponding one-parameter family of Riemannian metrics as in (5.2). Choose $x \in M$ and $y \notin \mathrm{Cut}_\varepsilon(x)$. Let $\gamma : [0, r_\varepsilon] \to M$ be the unique g_ε-geodesic, parametrized with unit speed, joining x with y. For $\ell \in \mathbb{N}$, let $W_1, \dots, W_\ell$ be ℓ-tuple of vector fields along γ and g_ε-orthogonal to $\dot\gamma$ such that*

$$\sum_{i=1}^{\ell} \int_0^{r_\varepsilon} \langle \hat\nabla^\varepsilon_{\dot\gamma} \nabla^\varepsilon_{\dot\gamma} W_i + \hat R^\varepsilon(W_i, \dot\gamma)\dot\gamma, W_i \rangle_\varepsilon \, dt \geq 0. \tag{5.11}$$

Then, at $y = \gamma(r_\varepsilon)$, it holds

$$\sum_{i=1}^{\ell} \mathrm{Hess}^{\hat\nabla^\varepsilon}(r_\varepsilon)(W_i, W_i) \leq \sum_{i=1}^{\ell} \langle W_i(r_\varepsilon), \hat\nabla^\varepsilon_{\dot\gamma} W_i(r_\varepsilon) \rangle_\varepsilon, \tag{5.12}$$

and the equality holds if and only if $W_1, \dots, W_\ell$ are Jacobi fields for the metric g_ε.

Of course a similar statement holds in terms of the Levi-Civita connection of g (which is equal to its own adjoint, removing thus several complications), and is classical. The main advantage of Theorem 5.3 is that, in terms of the connection $\hat\nabla^\varepsilon$, it is easier to single out special subspaces of the horizontal bundle where the corresponding curvature $\hat R^\varepsilon$ does not explode as $\varepsilon \to 0$. This will be key for proving uniform comparison theorems for g_ε which remain well-defined in the limit.

We show two applications of Theorem 5.3: one horizontal Laplacian comparison theorem for the structures g_ε, uniform for all $\varepsilon \geq 0$ (Section 5.3), and one Bonnet-Myers type result for the sub-Riemannian limit structure (Section 5.4).

5.2.4. The H-type and the J^2 conditions

We introduce a condition that will play a prominent role. For any $Z \in TM$, let $J_Z : TM \to TM$ be defined by the identity:

$$\langle J_Z X, Y \rangle = \langle Z, T(X, Y) \rangle, \qquad \forall X, Y \in TM. \tag{5.13}$$

If $(M, g, \mathcal{H})$ is a totally geodesic foliation, it holds $J_{\mathcal{H}} = 0$, $J_{\mathcal{V}}\mathcal{V} = 0$, and $J_{\mathcal{V}}\mathcal{H} \subseteq \mathcal{H}$.

Definition 5.4. We say that the *H-type condition* is satisfied if

$$J_Z^2 = -\|Z\|^2 \mathbb{1}_{\mathcal{H}}, \qquad \forall Z \in \Gamma(\mathcal{V}). \tag{5.14}$$

Definition 5.5 ([42, 35])**.** We say that J^2 *condition* holds if for all $Z, Z' \in \Gamma(\mathcal{V})$, $X \in \Gamma(\mathcal{H})$ with $\langle Z, Z' \rangle = 0$ there exists $Z'' \in \Gamma(\mathcal{V})$ such that

$$J_Z J_{Z'} X = J_{Z''} X. \tag{5.15}$$

We stress that Z'' depends on Z, Z' but may also depend on X. This condition is true in particular if the vector space generated by $\mathbb{1}_{\mathcal{H}}$ and the J_Z, for $Z \in \Gamma(\mathcal{V})$ is a subalgebra of linear endomorphisms.

Complex Type
Sasakian manifolds
Quaternionic Type
3-Sasakian manifolds
Negative 3-Sasakian manifolds
Torus bundle over hyperkähler manifolds
Octonionic Type
Octonionic Heisenberg Group
Octonionic Hopf Fibration $\mathbb{S}^7 \hookrightarrow \mathbb{S}^{15} \to \mathbb{O}P^1$
Octonionic Anti de-Sitter Fibration $\mathbb{S}^7 \hookrightarrow \mathbf{AdS}^{15}(\mathbb{O}) \to \mathbb{O}H^1$

Table 5.1.: Examples of H-type foliations with parallel horizontal Clifford structure and satisfying the J^2 condition.

Definition 5.6. ([R25, R11]) A totally geodesic foliation $(M, \mathcal{H}, g)$ for which the H-type condition holds is called an *H-type foliation*. Moreover:

(i) If the torsion of the Bott connection is horizontally parallel, i.e. $\nabla_{\mathcal{H}} T = 0$, then we say that $(M, \mathcal{H}, g)$ is an H-type foliation with *horizontally parallel torsion*.

(ii) If the torsion of the Bott connection is completely parallel, i.e. $\nabla T = 0$, then we say that $(M, \mathcal{H}, g)$ is an H-type foliation with *parallel torsion*.

(iii) Let $(M, \mathcal{H}, g)$ be an H-type foliation with horizontally parallel torsion. We say that $(M, \mathcal{H}, g)$ is an H-type foliation with a *parallel horizontal Clifford structure* if there exists a constant $\kappa \in \mathbb{R}$ such that for every $Z_1, Z_2 \in \Gamma\mathcal{V}$

$$(\nabla_{Z_1} J)_{Z_2} = -\kappa \left(J_{Z_1} J_{Z_2} + \langle Z_1, Z_2 \rangle_{\mathcal{V}} \mathbb{1}_{\mathcal{H}} \right). \tag{5.16}$$

H-type foliations are ideal, that is, they do not contain non-trivial abnormal geodesics.

H-type foliations are classified in [R25], together with several examples. Since H-type foliations with a parallel horizontal Clifford structure and satisfying J^2 condition will play an important role in the next section, we point out some examples that satisfy these assumptions in Table 5.1 and refer to [R25] for further details.

5.3. A uniform comparison theorem

Throughout the section, we will denote $n = \operatorname{rank} \mathcal{H}$ and $m = \operatorname{rank} \mathcal{V}$ and make use of the following notation for the comparison functions:

$$F_{\mathrm{Rie}}(r,k) = \begin{cases} \sqrt{k} \cot \sqrt{k} r & \text{if } k > 0, \\ \frac{1}{r} & \text{if } k = 0, \\ \sqrt{|k|} \coth \sqrt{|k|} r & \text{if } k < 0, \end{cases} \tag{5.17}$$

and

$$F_{\mathrm{Sas}}(r,k)=\begin{cases}\frac{\sqrt{k}(\sin\sqrt{k}r-\sqrt{k}r\cos\sqrt{k}r)}{2-2\cos\sqrt{k}r-\sqrt{k}r\sin\sqrt{k}r} & \text{if } k>0,\\ \frac{4}{r} & \text{if } k=0,\\ \frac{\sqrt{|k|}(\sqrt{|k|}r\cosh\sqrt{|k|}r-\sinh\sqrt{|k|}r)}{2-2\cosh\sqrt{|k|}r+\sqrt{|k|}r\sinh\sqrt{|k|}r} & \text{if } k<0.\end{cases} \tag{5.18}$$

Let $(M,\mathcal{H},g)$ be a totally geodesic foliation. We recall that sub-Laplacian $\Delta_{\mathcal{H}}$ is defined as the second order differential operator associated with the Dirichlet form

$$\mathcal{E}(u,v)=\int_M\langle\nabla_{\mathcal{H}}u,\nabla_{\mathcal{H}}v\rangle d\mathsf{m}_g,\qquad u,v\in C_c^\infty(M), \tag{5.19}$$

where m_g is the Riemannian measure of (M,g) (which, for H-type foliation, coincides with the Popp measure of the sub-Riemannian structure, see [R17]).

One can check that, for a totally geodesic foliation, the sub-Laplacian coincides with the horizontal trace of the Hessian appearing in Theorem 5.3:

$$\Delta_{\mathcal{H}}u=\sum_{i=1}^n\mathrm{Hess}^{\hat\nabla^\varepsilon}(u)(X_i,X_i), \tag{5.20}$$

where $X_1,\dots,X_n\in\Gamma(\mathcal{H})$ is an orthonormal frame. We obtain then the following result.

Theorem 5.7. *Let $(M,\mathcal{H},g)$ be an H-type foliation with parallel horizontal Clifford structure, satisfying the J^2 condition. Let κ^2 be the curvature of the vertical leaves. Assume that there exists $\rho\in\mathbb{R}$ such that, for the Bott curvature, it holds:*

$$\mathrm{Sec}(X\wedge Y)\geq\rho,\qquad\forall X,Y\in\mathcal{H}. \tag{5.21}$$

Fix $\varepsilon>0$. Then, for all $y\notin\mathrm{Cut}_\varepsilon(x)$ with $\nabla_{\mathcal{H}}r_\varepsilon(y)\neq0$, it holds

$$\begin{aligned}\Delta_{\mathcal{H}}r_\varepsilon(y)\leq\frac{1-\|\nabla_{\mathcal{H}}r_\varepsilon\|^2}{r_\varepsilon}+(n-m-1)F_{\mathrm{Riem}}(r_\varepsilon,K)\\ +F_{\mathrm{Sas}}(r_\varepsilon,K_1)+(m-1)F_{\mathrm{Sas}}(r_\varepsilon,K_2),\end{aligned} \tag{5.22}$$

where

$$K=\rho\|\nabla_{\mathcal{H}}r_\varepsilon\|^2+\tfrac14\|\nabla_{\mathcal{V}}r_\varepsilon\|^2, \tag{5.23}$$

$$K_1=\rho\|\nabla_{\mathcal{H}}r_\varepsilon\|^2+\|\nabla_{\mathcal{V}}r_\varepsilon\|^2, \tag{5.24}$$

$$K_2=\rho\|\nabla_{\mathcal{H}}r_\varepsilon\|^2+(2-\kappa\varepsilon)(\kappa\varepsilon-1)\|\nabla_{\mathcal{V}}r_\varepsilon\|^2. \tag{5.25}$$

An immediate consequence of this result and Proposition 5.1, we have the following sub-Riemannian comparison theorem. We stress that the assumptions and the conclusion of the statement depend only on the restriction of g to $\mathcal{H}$, that is on the induced sub-Riemannian structure. In particular the curvature of the leaves κ plays no role.

Theorem 5.8. *Let $(M, \mathcal{H}, g)$ be an H-type foliation with parallel horizontal Clifford structure, satisfying the J^2 condition. Assume that there exists $\rho \in \mathbb{R}$ such that*

$$\mathrm{Sec}(X \wedge Y) \geq \rho, \qquad \forall X, Y \in \mathcal{H}, \tag{5.26}$$

where Sec *is the sectional curvature of Bott connection. Then for $y \notin \mathrm{Cut}_0(x)$ it holds*

$$\Delta_{\mathcal{H}} r_0(y) \leq (n-m-1) F_{\mathrm{Riem}}(r_0, K) + F_{\mathrm{Sas}}(r_0, K_1) + (m-1) F_{\mathrm{Sas}}(r_0, K_2), \tag{5.27}$$

where

$$K = \rho + \tfrac{1}{4}\|\nabla_{\mathcal{V}} r_0\|^2, \tag{5.28}$$

$$K_1 = \rho + \|\nabla_{\mathcal{V}} r_0\|^2, \tag{5.29}$$

$$K_2 = \rho - 2\|\nabla_{\mathcal{V}} r_0\|^2. \tag{5.30}$$

5.4. A sharp sub-Riemannian Bonnet-Myers theorem

In this section we show a sharp Bonnet-Myers type theorem for structures which are not necessarily foliations. This result is quite general, but it holds only in the sub-Riemannian limit (and not, as in the previous section, uniformly for all $\varepsilon \geq 0$).

Consider a general Riemannian manifold (M, g) together with a vector bundle orthogonal splitting $TM = \mathcal{H} \oplus \mathcal{V}$. The vertical bundle $\mathcal{V}$ is not required to be integrable nor metric. In this case, one can define a generalized Bott connection, due to Hladky [48], denoted again with ∇. As Bott's connection, Hladky's one is g_ε-metric for all $\varepsilon > 0$, and also its adjoint is metric, and this allows us to use a version of Theorem 5.3.

The idea is that the index form, when restricted to an appropriate subspace, behaves as in the Riemannian case, up to corrections of order ε. The appropriate subspace is defined as follows: for a g_ε-geodesic $\gamma : [0, r_\varepsilon] \to M$, consider the $(n-m-1)$-vector subspace of $\mathcal{H}$ along γ given by

$$\mathfrak{L}_J(\dot\gamma) := \mathrm{span}\{X \in \mathcal{H} \mid X \perp J_{\mathcal{V}}\dot\gamma,\ X \perp \dot\gamma\}, \tag{5.31}$$

where the operator J is defined as in 5.13 via the torsion of Hladky connection. We have then the following.

Theorem 5.9. *Let $(M, \mathcal{H}, g)$ be a complete sub-Riemannian structure with corank m and rank $n > m+1$, satisfying:*

(i) the H-type condition: $J_Z^2 = -\|Z\|^2 \mathbb{1}_{\mathcal{H}}$ for all $Z \in \Gamma(\mathcal{V})$;

(ii) the J^2 condition: for all $Z, Z' \in \Gamma(\mathcal{V})$, $X \in \Gamma(\mathcal{H})$, with $\langle Z, Z'\rangle = 0$ there exists $Z'' \in \Gamma(\mathcal{V})$ such that

$$J_Z J_{Z'} X = J_{Z''} X. \tag{5.32}$$

(iii) for all $Y \in \Gamma(\mathcal{H})$ *and* $Z \in \Gamma(\mathcal{V})$*, we have*

$$\sum_{i=1}^{n-m-1} \langle (\nabla_{X_i} J)_Z X_i, Y \rangle = 0, \tag{5.33}$$

where $X_1, \dots, X_{n-m-1}$ *is an orthonormal frame for* $\mathfrak{L}_J(Y)$*.*

Assume that there exists $\rho > 0$ *such that, for all unit* $X \in \mathcal{H}$ *it holds*

$$\begin{aligned} \mathrm{Ric}^{\nabla}_{\mathcal{H}}(X,X) - \sum_{\alpha=1}^{m} R^{\nabla}(X, J_{Z_\alpha}X, J_{Z_\alpha}X, X) \\ - \sum_{\alpha=1}^{m} \left(\|(\nabla_X J)_{Z_\alpha} X\|^2 - \sum_{\beta=1}^{m} \langle (\nabla_X J)_{Z_\alpha} X, J_{Z_\beta} X \rangle^2 \right) \geq (m-n-1)\rho, \end{aligned} \tag{5.34}$$

where $Z_1, \dots, Z_m \in \mathcal{V}$ *is any orthonormal frame. Then* M *is compact with sub-Riemannian diameter not greater than* $\pi/\sqrt{\rho}$*, and the fundamental group is finite.*

Assumptions (i)-(ii)-(iii) are verified for any contact and quaternionic contact structures, as the one we studied in [R20] and [20], respectively. We refer to these reference for more precise definitions, see also [R11, Remark 4.2].

Theorem 5.9 thus includes the contact Bonnet-Myers theorem [R20, Thm. 1.7], and the quaternionic contact one of [20]. This result is sharp since the bound is attained in the case of the standard sub-Riemannian structure on the Hopf fibrations or their quaternionic counterparts.

Part II.

Spectral and heat asymptotics on singular structures

6. Spectral properties of singular structures

6.1. Introduction

Let (M, g) be a Riemannian manifold, and let Δ_g be its Laplace-Beltrami operator, initially defined on the domain $C_c^\infty(M)$ (smooth functions with compact support). The operator Δ_g is symmetric and densely defined in $L^2(M)$, and plays a fundamental role in the description of the heat diffusion (via the heat equation) or the dynamics of quantum particles (via the Schrödinger equation) on (M, g).

The problem of finding the self-adjoint extensions of Δ_g has a long and venerable history, dating back to Weyl at the beginning of the 20th century. This problem is physically relevant as any self-adjoint extension of Δ_g generates, by spectral calculus, a heat semigroup or, by Stone's theorem, a unitary semigroup on $L^2(M)$. For example, when $M \subset \mathbb{R}^n$ is a bounded region of the Euclidean space, different self-adjoint extensions correspond to different boundary conditions, leading to different physical evolutions. On the other hand, when Δ_g is *essentially self-adjoint*, that is, it admits a unique self-adjoint extension, there is no need to fix any boundary condition to determine an extension. The physical interpretation of this fact is that the heat (or quantum particles), evolving according to the heat (or Schrödinger) equation, remain confined to M. For this reason the essential self-adjointness property is referred to as *quantum confinement*.

A well-known sufficient condition for essential self-adjointness of Δ_g is the metric completeness of (M, g). Furthermore, if M is compact, the spectrum of $-\Delta_g$ is discrete, and consists of positive eigenvalues $\{\lambda_i\}_{i\in\mathbb{N}}$ accumulating to infinity, and they satisfy the following asymptotic relation, known as *Weyl's law*:

$$\lim_{\lambda\to\infty} \frac{N(\lambda)}{\lambda^{n/2}} = \frac{\omega_n}{(2\pi)^n}\mathrm{vol}(M), \tag{6.1}$$

where $N(\lambda)$ is the number of eigenvalues smaller than λ, $\mathrm{vol}(M)$ stands for the Riemannian volume of M and ω_n is the volume of the n-dimensional Euclidean unit ball.

In this chapter we study how the above properties are affected by metric singularities.

6.1.1. The Grushin sphere

Let us present an example of the kind of singular structures we will study. Consider the two-dimensional sphere $\mathbb{S}^2 \subset \mathbb{R}^3$. Let X and Y be the generators of rotations around the x and y axis, respectively. We define a Riemannian metric g by declaring X and Y to be orthonormal. These vector fields are collinear on the equator $S = \{(x, y, z) \in \mathbb{S}^2 \mid z = 0\}$, and hence the metric we defined is singular on S (the coefficients of the metric explode).

We can view this structure as an example of rank-varying sub-Riemannian structure on $\mathbb{S}^2$, generated by the two vector fields X, Y (cf. Section 2.2). Since the metric is Riemannian outside of the equator S, this type of structure is also called *almost-Riemannian* [5, 31]. We focus on the Riemannian structure on the upper hemisphere

$$\mathbb{S}^2_+ := \{(x, y, z) \in \mathbb{S}^2 \mid z > 0\}.$$

In cylindrical coordinates (θ, z), the Laplace-Beltrami operator Δ_g is

$$-\Delta_g = \frac{\partial^2}{\partial z^2} + z^2 \frac{\partial^2}{\partial \theta^2} + \left(\frac{1}{z} - z\right) \frac{\partial}{\partial z}. \tag{6.2}$$

By construction, Δ_g is symmetric on $C^\infty_c(\mathbb{S}^2_+)$ in $L^2(\mathbb{S}^2_+) = L^2(\mathbb{S}^2_+, \mathsf{m}_g)$, where

$$\mathsf{m}_g = \frac{1}{|z|} d\theta \wedge dz. \tag{6.3}$$

One can check that $(\mathbb{S}^2_+, g)$ is not geodesically complete: this is a consequence of the fact that the rank-varying sub-Riemannian structure on the whole $\mathbb{S}^2$ is complete, and its geodesics (which are Riemannian outside of S) can cross the equator.

The following fact was proved in [31]: despite the geodesic incompleteness, the Laplace-Beltrami operator Δ_g, with domain $C^\infty_c(\mathbb{S}^2_+)$, is essentially self-adjoint. Physically, this means that sub-Riemannian classical particle can cross the singularity, whereas a quantum particle (or the heat) remains confined to $\mathbb{S}^2_+$ for all times.

Furthermore, the resolvent of $-\Delta_g$ is compact, and its spectrum is discrete. By (6.3), the Riemannian volume of $\mathbb{S}^2_+$ is infinite, so that the classical Weyl's law (6.1) does not make sense. However, the spectrum of $-\Delta_g$ can be computed explicitly [29], and it satisfies the following non-classical Weyl's asymptotics:

$$N(\lambda) \sim \frac{1}{4} \lambda \log \lambda, \qquad \lambda \to \infty. \tag{6.4}$$

Our type of singularity can be modeled as the metric boundary of a non-complete Riemannian manifold, as in the Grushin sphere describe above. Intrinsic quantities such as the curvature, the measure of balls, et cætera, can blow up when approaching the metric boundary, which we may imagine as a singularity.

The purpose of this chapter is to report our results about (i) quantum confinement and (ii) Weyl's asymptotics, on those singular structures, and in particular we aim to answer to the following questions.

Under which conditions the Laplace-Beltrami operator of a non-complete Riemannian manifold is essentially self-adjoint? Is there a Weyl's-type law for its spectrum?

The related research is contained in the following articles:

- D. Prandi, **L. Rizzi**, and M. Seri. Quantum confinement on non-complete Riemannian manifolds. *J. Spectr. Theory*, 8(4):1221–1280, 2018

- V. Franceschi, D. Prandi, and **L. Rizzi**. On the essential self-adjointness of singular sub-Laplacians. *Potential Anal.*, 53(1):89–112, 2020
- Y. Chitour, D. Prandi, and **L. Rizzi**. Weyl's law for singular Riemannian manifolds. *arXiv e-prints*, 2019. arXiv: 1903.05639

For sake of simplicity and clarity of exposition, we consider only Laplace-Beltrami operators on non-complete Riemannian manifolds. All results about quantum completeness are valid in the more general setting of Schrödinger-type operators on rank-varying sub-Riemannian structure, that is operators of the form $-\Delta + V$, where $-\Delta$ is a sub-Laplacian, and V is a $L^2_{\text{loc}}(M)$ potential function. We do not report those results here, and we refer to [R8, R9] for details.

6.2. Quantum confinement

Let (M, g) be a smooth Riemannian manifold of dimension $n \geq 1$, equipped with a smooth measure m. We consider then the operator $\Delta_{\mathsf{m}} = \text{div}_{\mathsf{m}} \circ \nabla$, which is the symmetric operator associated with the quadratic form

$$\int_M \|\nabla u\|^2 d\mathsf{m}, \qquad u \in C^\infty_c(M). \tag{6.5}$$

In the following, unless otherwise specified, $L^2(M) = L^2(M, \mathsf{m})$. We remark that m is not necessarily the Riemannian measure. We assume that:

($\star$) the Riemannian manifold (M, g) is not complete, so that the distance function from the metric boundary, denoted by δ, is well-defined. Furthermore, there exists a neighborhood $M_\varepsilon = \{\delta < \varepsilon\}$ of the metric boundary such that δ is C^∞ on M_ε.

Assumption ($\star$) holds for different classes of non-complete structures, including those whose metric completion is not a smooth Riemannian manifold (such as the Grushin example of Section 6.1.1), or not even a topological manifold (such as cones). In this setting, we were able to apply and extend some Euclidean techniques inspired by [68, 40] to yield sufficient conditions for self-adjointness. A central role is played by the following intrinsic object which we introduced in [R8].

Definition 6.1. The *effective potential* $V_{\text{eff}} : M_\varepsilon \to \mathbb{R}$ is the function

$$V_{\text{eff}} := \left(\frac{\Delta_{\mathsf{m}}\delta}{2}\right)^2 + \left(\frac{\Delta_{\mathsf{m}}\delta}{2}\right)'. \tag{6.6}$$

where the symbol $'$ denotes the normal derivative with respect to the metric boundary, that is the derivative in the direction of $\nabla\delta$: $f' := df(\nabla\delta) = g(\nabla\delta, \nabla f)$.

The main result we proved in [R8] is the following essential self-adjointness criterion.

Theorem 6.2. *Let (M, g) be a Riemannian manifold satisfying $(\star)$. Assume that there exists $\kappa \geq 0$ such that, in a neighborhood of the metric boundary it holds*

$$V_{\text{eff}} \geq \frac{3}{4\delta^2} - \frac{\kappa}{\delta}. \tag{6.7}$$

Then, Δ_{m} with domain $C_c^\infty(M)$ is essentially self-adjoint in $L^2(M, \mathsf{m})$. Furthermore, if the metric completion is compact, the unique self-adjoint extension of Δ_{m} has compact resolvent, its spectrum is discrete and consists of eigenvalues with finite multiplicity.

Formula (6.7) provides a link between geometry and self-adjointness properties. In fact, by the generalized Bochner formula (see e.g. [90, Eqs. 14.28, 14.46]), we have

$$V_{\text{eff}} = \frac{1}{4}\left((\Delta_{\mathsf{m}}\delta)^2 - 2\|\operatorname{Hess}(\delta)\|^2 - 2\operatorname{Ric}_{\mathsf{m}}(\nabla\delta, \nabla\delta)\right), \tag{6.8}$$

where, if $\mathsf{m} = e^{-f}\mathsf{m}_g$, then $\operatorname{Ric}_{\mathsf{m}} := \operatorname{Ric} + \operatorname{Hess}(f)$ is the Bakry-Émery tensor.

6.2.1. Measure confinement

The condition of Theorem 6.2 can be expressed in terms of the degeneration (or explosion) of the measure m close to the metric boundary, and it is interesting to examine the case of measures with power-like behavior. Here, we identify $M_\varepsilon \simeq (0, \varepsilon] \times X_\varepsilon$, where X_ε is the level set $\{\delta = \varepsilon\}$, which is a well-defined, smooth hypersurface thanks to $(\star)$. Denote points of M as $p = (t, x)$, with $x \in X_\varepsilon$.

Corollary 6.3 (Pure measure confinement)**.** *Assume that the Riemannian manifold (M, g) satisfies $(\star)$ for $\varepsilon > 0$. Moreover, let m be a smooth measure such that there exists $a \in \mathbb{R}$ and a reference measure μ on X_ε for which*

$$d\mathsf{m}(t, x) = t^a\, dt\, d\mu(x), \qquad (t, x) \in (0, \varepsilon] \times X_\varepsilon. \tag{6.9}$$

Then, Δ_{m} with domain $C_c^\infty(M)$ is ess. self-adjoint in $L^2(M, \mathsf{m})$ if $a \geq 3$ or $a \leq -1$.

The preceding result can be directly applied to the Laplace-Beltrami operator (i.e. with $\mathsf{m} = \mathsf{m}_g$) of conical or anti-conical-type structures. These are Riemannian structures that satisfy $(\star)$ for some $\varepsilon > 0$ and such that their metric, under the identification $M_\varepsilon \simeq (0, \varepsilon] \times X_\varepsilon$, can be written as

$$g|_{M_\varepsilon} = dt \otimes dt + t^{2\alpha}\, h, \qquad \alpha \in \mathbb{R}, \tag{6.10}$$

where h is some Riemannian metric on X_ε (cf. Figure 6.1). We obtain thus the following.

Corollary 6.4. *Consider a conic or anti-conic-type structure as in (6.10). Then, the Laplace-Beltrami operator is essentially self-adjoint in $L^2(M)$ if $\alpha \geq \frac{3}{n-1}$ or $\alpha \leq -\frac{1}{n-1}$.*

Remark 6.5. The bounds of Corollaries 6.3 and Corollary 6.4 are sharp. Indeed, the Laplace-Beltrami operator $-\Delta_g$ on $M = (0, +\infty) \times \mathbb{S}^1$ given by the global metric

$$g = dt \otimes dt + t^{2\alpha} d\theta \otimes d\theta, \tag{6.11}$$

is essentially self-adjoint if and only if $\alpha \in (-\infty, -1] \cup [3, \infty)$. The proof of the "only if" part of this statement relies on the explicit knowledge of the symmetric solutions of $(-\Delta^* - \lambda)u = 0$ for this metric, and can be found, for example, in [32].

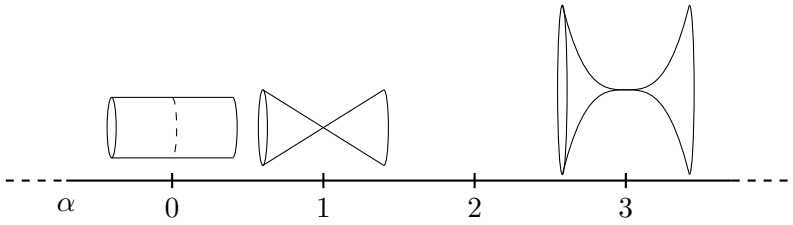


Figure 6.1.: Depiction of the embeddings in $\mathbb{R}^3$ of the 2-dimensional structures on $\mathbb{R}\times\mathbb{S}^1$ with metric $g = dt^2 + t^\alpha d\theta^2$, here only for the case $\alpha \geq 0$.

6.2.2. Curvature-based criteria for self-adjointness

In this section, we fix $\mathsf{m} = \mathsf{m}_g$, and investigate how the curvature of (M, g) is related with the essential self-adjointness of the Laplace-Beltrami operator $\Delta_g = \Delta_{\mathsf{m}_g}$. As we will see, curvature is not the only actor. In fact, consider for example the conic and anti-conic-type structures given by (6.10). In this case, for all planes σ containing $\nabla\delta$,

$$\mathrm{Sec}(\sigma) = -\frac{\alpha(\alpha-1)}{\delta^2}, \qquad \delta \leq \varepsilon, \tag{6.12}$$

and we have structures, with the same sectional curvature, whose Laplace-Beltrami operator can be essentially self-adjoint or not ($n = 2$ and $\alpha = -1$ or $\alpha = 2$, respectively).

The essential self-adjointness property of Δ_g is influenced also by the eigenvalues of the second fundamental form $H(t)$ of X_t, describing its extrinsic curvature:

$$H(t) := \mathrm{Hess}(\delta)|_{X_t}. \tag{6.13}$$

For example, for the conic and anti-conic-type structures (6.10), we have

$$H(t) = \frac{\alpha}{t} g, \tag{6.14}$$

which in particular allows to distinguish structures with the same sectional curvature (6.12), but different self-adjointntess behaviour.

In [R8, Section 6, Thms. 6.1 and 6.2], we prove two criteria for essential self-adjointness of the Laplace-Beltrami operator, under a control on the explosion of the sectional curvature and the principal curvatures of X_t close to the metric boundary. We present a simplified version of these results, which to our best knowledge are the first curvature-based criteria for essential self-adjointness.

Theorem 6.6. *Let (M, g) be a Riemannian manifold satisfying $(\star)$ for $\varepsilon > 0$. Assume that there exist $c_1 \geq c_2 \geq 0$ and $r \geq 2$ such that, for all planes σ containing $\nabla\delta$, one has*

$$-\frac{c_1}{\delta^r} \leq \mathrm{Sec}(\sigma) \leq -\frac{c_2}{\delta^r}, \qquad \delta \leq \varepsilon. \tag{6.15}$$

Then, there exist a region $\Sigma(n, r) \subset \mathbb{R}^2$, and a constant $h^*_\varepsilon(c_2, r) > 0$ such that, if $(c_1, c_2) \in \Sigma(n, r)$ and if the principal curvatures of the hypersurface $X_\varepsilon = \{\delta = \varepsilon\}$ satisfy

$$H(\varepsilon) < h^*_\varepsilon(c_2, r), \tag{6.16}$$

then the operator Δ_g with domain $C^\infty_c(M)$, is essentially self-adjoint in $L^2(M, \mathsf{m}_g)$.

For the explicit values of the constants $h^*_\varepsilon(c,r)$ and region $\Sigma(n,r)$ see [R8]. Here, we only observe that if we take $c_1 = c_2 = c$ in (6.15), then $(c_1, c_2) \in \Sigma(n,r)$ with $r > 2$ if $c > 0$, and $(c_1, c_2) \in \Sigma(n,2)$ if $c \geq n/(n-1)^2$.

6.2.3. Almost-Riemannian geometry

An almost-Riemannian structure (ARS in the following) is a rank-varying sub-Riemannian structure on a smooth manifold N, such that the rank of the distribution is maximal (equal to $n = \dim N$) on the complement of a closed region S, called singular locus, where the rank is strictly smaller than n. The restriction of the sub-Riemannian metric g to the regular region $M := N \setminus S$ is a well-defined Riemannian metric, which is not complete (if we assume that N, equipped with its sub-Riemannian distance, is a complete metric space). The typical example is the Grushin sphere of Section 6.1.1.

For the remainder of this section, we need the following local fact. For any $q \in S$ there exists a neighborhood U and a family of locally defined smooth vector fields

$$X_1, \dots, X_n, \tag{6.17}$$

orthonormal on $U \setminus S$, which are linearly dependent on S (and satisfy the Hörmander condition: $\mathrm{Lie}_q(X_1, \dots, X_n) = T_qU$, for all $q \in U$). We call $\{X_1, \dots, X_n\}$ a *local generating family* for the n-dimensional ARS.

The main motivation of our work comes from a conjecture on the essential self-adjointness of the Laplace-Beltrami operator of ARS, due to Boscain and Laurent, stemming from the following result they proved in [31] for two-dimensional ARS.

Theorem 6.7 (Boscain, Laurent)**.** *Let N be a 2-dimensional ARS on a compact orientable manifold, with smooth singular set S. Assume that, for every $q \in S$ and local generating family $\{X_1, X_2\}$, we have*

$$\mathrm{span}\{X_1, X_2, [X_1, X_2]\}_q = T_qN, \qquad (\textit{bracket-generating of step } 2). \tag{6.18}$$

Then, the Laplace-Beltrami operator $\Delta = \Delta_{\mathsf{m}_g}$, with domain $C^\infty_c(N \setminus S)$ is essentially self-adjoint in $L^2(N \setminus S)$ and its unique self-adjoint extension has compact resolvent.

Boscain and Laurent, motivated by Theorem 6.7 and other explicit examples, conjectured that the above result holds true in arbitrary dimension and step in the bracket-generating assumption (6.18).

The proof of Theorem 6.7 in [31] relies on the normal forms for the local family X_1, X_2 under the condition (6.18). Although normal forms for ARS are known also in dimension $n = 3$, [30], their complexity increases quickly with the number of degrees of freedom. This was the main obstacle to the proof of the conjecture. Applying our effective potential techniques, we were able to prove in [R8] the following extension of Theorem 6.7, which proves the Boscain-Laurent's conjecture under quite general assumptions.

Theorem 6.8. *Consider a regular ARS on a smooth manifold N with compact singular region S. Let $M = N \setminus S$, or one of its connected components. Then, the Laplace-Beltrami operator Δ_g with domain $C^\infty_c(M)$ is essentially self-adjoint in $L^2(M)$. Moreover, when $M \subset N$ is relatively compact, the unique self-adjoint extension of Δ_g has compact resolvent.*

The *regular* assumption in Theorem 6.8 consists in the following requirements:

- the singular set S is an embedded hypersurface without tangency points, that is, such that $\operatorname{span}\{X_1, \dots, X_n\} \pitchfork T_q S$ for all $q \in S$;
- in terms of the aforementioned local frame around S, it holds $\det(X_1, \dots, X_n) = \pm\psi^k$ for some $k \in \mathbb{N}$, where ψ is a local submersion defining S.

The regularity assumption does not depend on the choice of the local family $X_1, \dots, X_n$, and it is only a property of the rank-varying structure. The regularity condition implies that the Riemannian structure on $M = N \setminus S$ satisfies $(\star)$, allowing us to apply Theorem 6.2. Even in dimension 2, our result is stronger than Theorem 6.7, as it allows for higher step structures than the one prescribed by (6.18) (the step is the minimal numbers of Lie brackets needed to fulfill the bracket-generating condition).

Extensions to the sub-Riemannian case. In [R9], we extended these results to non-equiregular sub-Riemannian manifold in any dimension, satisfying a regularity assumption similar to the one introduced above (which is satisfied for example by the standard Martinet structure on $\mathbb{R}^3$). We do not describer further this generalization here.

6.3. Weyl's law

Let (M, g) be a non-complete Riemannian manifold. Intrinsic quantities such as the curvature, the measure of balls, et cætera, can blow up when approaching the metric boundary of M, which we imagine as a singularity.

Assumption A. Let δ be the distance from the metric boundary of M. Then, there exists a neighborhood $U = \{\delta < \varepsilon_0\}$ on which the following hold:

(a) regularity: δ is smooth;

(b) convexity: the level sets of δ are convex, i.e., $\operatorname{Hess}(\delta) \leq 0$;

(c) curvature: there exists $C > 0$ such that $|\operatorname{Sec}| \leq C\delta^{-2}$;

(d) injectivity radius: there exists $C > 0$ such that $\operatorname{inj} \geq C\delta$.

Assumption (a) is equivalent to assumption $(\star)$ of Section 6.2, but for consistency we decided to maintain the notation of the original references [R8, R12].

By (a), we identify $U \simeq (0, \varepsilon_0) \times Z$, for a fixed $(n-1)$-dimensional manifold Z without boundary. The metric on U has the form

$$g|_U = dx^2 + h(x), \tag{6.19}$$

where $h(x)$ is a smooth one-parameter family of Riemannian metrics on Z. In particular, it holds that $\delta(x, z) = x$ for $(x, z) \in U$. The convexity assumption (b) implies that, for any $V \in TZ$, the map $x \mapsto h(x)(V, V)$ is non-increasing.

Remark 6.9. Assumption (d) is implied by the others if the convexity in (b) is strict, or if the metric is of warped product type in a neighborhood of the singularity, cf. [R12]. We do not known whether (d) is independent from the other assumptions in general.

To quantify the rate of growth of the volume at the singularity, let M^∞_ε be the set at distance greater than $\varepsilon > 0$ from the metric boundary, and define the *volume function*:

$$v(\lambda) := \mathrm{vol}\left(M^\infty_{1/\sqrt{\lambda}}\right), \qquad \lambda > 0. \tag{6.20}$$

The first result of [R12] is the following rough Weyl-type asymptotics. We do not assume that the Laplace-Beltrami operator of (M, g), with domain $C^\infty_c(M)$, is essentially self-adjoint, and we choose the standard Friedrichs self-adjoint extension. We denote with the symbol $N(\lambda)$ the number of eigenvalues smaller than λ (*Weyl's counting function*):

$$N(\lambda) := \#\{\lambda_i \mid \lambda_i \text{ is an eigenvalue of } -\Delta_g \text{ and } \lambda_i \le \lambda\}. \tag{6.21}$$

Theorem 6.10 (Rough Weyl's asymptotics)**.** *Let M be a n-dimensional Riemannian manifold with compact metric completion and satisfying Assumption* **A**. *Then, there exist $C_\pm > 0$ and $\Lambda > 0$ such that*

$$C_- \le \frac{N(\lambda)}{\lambda^{n/2} v(\lambda)} \le C_+, \qquad \forall \lambda \ge \Lambda. \tag{6.22}$$

We do not know if the limit of (6.22) for $\lambda \to \infty$ exists under these general assumptions. The second main result in [R12], is a precise Weyl's law under an additional assumption on the volume function, ruling out rapid oscillations and growth. Recall that v is *slowly varying* at infinity if $v(a\lambda) \sim v(\lambda)$ as $\lambda \to \infty$ for all positive a, see [28]. Examples of slowly varying functions are logarithms and their iterations

$$\log \lambda, \qquad \log_k \lambda = \log_{k-1} \log \lambda, \qquad k = 2, 3, \dots, \tag{6.23}$$

and any rational function with positive coefficients formed with the above. This class also contains functions with non-logarithmic growth such as

$$\exp\left((\log \lambda)^{\alpha_1} \dots (\log_k \lambda)^{\alpha_k}\right), \qquad 0 < \alpha_i < 1. \tag{6.24}$$

Theorem 6.11 (Sharp Weyl's law)**.** *Let M be an n-dimensional Riemannian manifold with compact metric completion and satisfying Assumption* **A**. *Then, if v is slowly varying, we have*

$$\lim_{\lambda \to \infty} \frac{N(\lambda)}{\lambda^{n/2} v(\lambda)} = \frac{\omega_n}{(2\pi)^n}, \tag{6.25}$$

where ω_n is the volume of the n-dimensional Euclidean unit ball

The assumptions of Theorem 6.11 are verified for the Grushin sphere of Section 6.1.1, and more generally for generic 2-dimensional ARS [31]. We refer to [R12, Sec. 7] for details. In all these cases, $v(\lambda) = \sigma \log \lambda$ for some $\sigma > 0$ (depending on the structure).

6.3.1. Quantitative remainder for heat trace asymptotics

We sketch here the proof of Theorem 6.10, describing also a technical tool of independent interest. The proof consists in the simultaneous exploitation of Dirichlet-Neumann bracketing and Tauberian techniques, but the main difficulties are the precise heat trace remainder estimates required in presence of a singularity.

Step 1. We first split $M = M_0^\varepsilon \cup M_\varepsilon^\infty$ where (cf. Figure 6.2)

$$M_0^\varepsilon = \{0 < \delta \leq \varepsilon\}, \qquad M_\varepsilon^\infty = \{\varepsilon < \delta < \infty\}, \tag{6.26}$$

so that M_0^ε is adjacent to the metric boundary, while M_ε^∞ is separated from it.

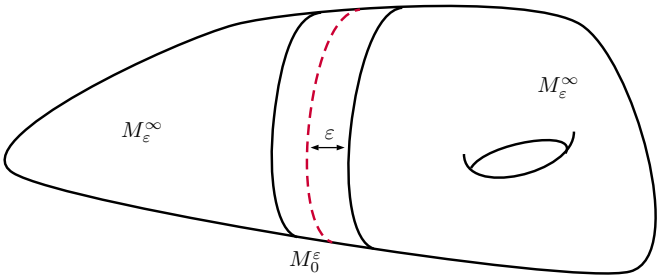


Figure 6.2.: Splitting of M. The dashed line represents the metric boundary.

The counting function $N(\lambda)$ is controlled by the counting functions for the Laplace-Beltrami operator on the two domains (6.26), with Neumann $(+)$ or Dirichlet $(-)$ boundary conditions, respectively (Dirichlet-Neumann bracketing, cf. [R12, Prop. 4.3]):

$$N^-_{[0,\varepsilon]}(\lambda) + N^-_{[\varepsilon,\infty]}(\lambda) \leq N(\lambda) \leq N^+_{[0,\varepsilon]}(\lambda) + N^+_{[\varepsilon,\infty]}(\lambda). \tag{6.27}$$

Step 2. Contribution of the singular region. Thanks to the convexity assumption, M_0^ε supports a Hardy-type inequality, cf. [R12, Prop. B.1], which yields a lower bound for the spectrum of M_0^ε. As a consequence of that, $N^\pm_{[0,\varepsilon(\lambda)]}(\lambda) = 0$, provided that $\varepsilon = \varepsilon(\lambda)$, for an explicit function $\varepsilon(\lambda)$, determined by the Hardy constant. In other words, this step links ε to λ in an explicit way, so that the contribution from $M_0^{\varepsilon(\lambda)}$ is negligible as $\lambda \to \infty$ (and $\varepsilon(\lambda) \to 0$ accordingly).

Step 3. Contribution of the regular region. In this regime (i.e. having chosen $\varepsilon = \varepsilon(\lambda)$ as explained above), the asymptotics of $N(\lambda)$ is controlled by the Weyl's function of the regular region $M^\infty_{\varepsilon(\lambda)}$. The latter is a Riemannian manifold with boundary and finite volume, which satisfies indeed the classical Weyl's law:

$$N^\pm_{[\varepsilon(\lambda),\infty]}(\lambda) = \frac{\omega_n}{(2\pi)^n} \mathrm{vol}\left(M^\infty_{\varepsilon(\lambda)}\right) \lambda^{n/2} + o_{\varepsilon(\lambda)}(\lambda^{n/2}). \tag{6.28}$$

Since the parameter ε is linked with λ in (6.28), we need more precise information on the remainder term in order to take the limit in (6.28) for $\lambda \to \infty$. The key ingredient

is the following heat-trace asymptotic formula with remainder, proved in [R12], valid for a general Riemannian manifold (P, g) with boundary. We use d_∂ and $\mathrm{inj}_\partial(P)$ to denote the Riemannian distance and the injectivity radius from ∂P, respectively. Furthermore, $\mathrm{inj}(P)$ denotes the injectivity radius of (P, g), suitably defined to take into account the presence of the boundary.

Theorem 6.12. *Let (P, g) be a compact n-dimensional Riemannian manifold with convex boundary ∂P. Let $K, H \geq 0$ such that $|\mathrm{Sec}(P)| \leq K$ and $|\mathrm{Hess}(d_\partial)| \leq H$ for $d_\partial < \mathrm{inj}_\partial(P)$. Then there exists a universal constant $c > 0$, depending only on n, such that the following estimate for the Dirichlet and Neumann heat kernels $E^\pm$ holds:*

$$\left| \frac{(4\pi t)^{n/2}}{\mathrm{vol}(P)} \int_M E^\pm(t, q, q) d\mathsf{m}_g(q) - 1 \right| \leq c \left(\frac{t}{t_0} \right)^{1/2}, \tag{6.29}$$

for all values of $t \in \mathbb{R}_+$ such that

$$\sqrt{t} \leq \sqrt{t_0} = \min \left\{ \mathrm{inj}(P), \frac{\mathrm{inj}_\partial(P)}{2}, \frac{\pi}{\sqrt{K}}, \frac{1}{H} \right\}. \tag{6.30}$$

It is well known that small-times heat trace asymptotics imply asymptotics for the Weyl's function, by means of Tauberian theorems. Thus, as a consequence of Theorem 6.12, and a Karamata-type theorem with remainder (due to Freud, [49, Thm. B]), we obtain an asymptotic formula for $N(\lambda)$ of (P, g) with remainder. The crucial fact is that the remainder is controlled in terms of a handful of geometric invariants of (P, g).

Step 4. Conclusion. Applying the above remainder formula to the case $P = M_\varepsilon^\infty$, we are able to keep under control the remainder term appearing in (6.28) as $\lambda \to \infty$. In fact, the quantitative explosion rates in Assumption **A** are precisely those that allow to keep under control the remainder term in the contribution $N^\pm_{[\varepsilon(\lambda), \infty]}(\lambda)$.

The proof of Theorem 6.11 (the precise Weyl's asymptotics for slowly varying volumes) is more delicate: one must consider a more refined three-parts splitting of M, and exploit the properties of slowly varying functions. See [R12, Sec. 5.1] for details.

6.3.2. Prescribing spectrum at infinity

We now turn to the inverse problem of building structures with prescribed large eigenvalues asymptotic. The next theorem can be seen as a counterpart at infinity of a celebrated result of Colin de Verdière [39] stating that, for any finite sequence of numbers $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_m$, one can find a compact Riemannian manifold such that these numbers are the first m eigenvalues.

Theorem 6.13. *Let N be an n-dimensional compact manifold, $S \subset N$ be a closed submanifold, and $\upsilon : \mathbb{R}_+ \to \mathbb{R}_+$ be a non-decreasing slowly varying function. Then, there exists a Riemannian structure on N, singular at S, such that*

$$\lim_{\lambda \to \infty} \frac{N(\lambda)}{\lambda^{n/2} \upsilon(\lambda)} = \frac{\omega_n}{(2\pi)^n}. \tag{6.31}$$

The proof of Theorem 6.13 uses Theorem 6.11. The idea is to build a non-complete Riemannian structure on $N \setminus S$, of warped-product type near S:

$$g|_U = dx^2 + f^2 \hat{g}, \tag{6.32}$$

where U is a neighborhood of S and the coordinate x represents the distance from S. The warping factor f has to be chosen so that $\mathrm{vol}(M_{1/\sqrt{\lambda}}) \sim \upsilon(\lambda)$, and (6.32) suggests the natural ansatz (assuming that υ is differentiable):

$$f(x)^{n-1} \propto \frac{\upsilon'(1/x^2)}{x^3}. \tag{6.33}$$

To apply Theorem 6.11, and thus obtain the desired Weyl's asymptotics (6.31), it remains only to check Assumption **A**. The latter, written in terms of f as in (6.33), amounts to growth conditions on the first and second derivatives of $f(x)$ as $x \to 0$ (and thus, second and third derivatives of $\upsilon(\lambda)$ as $\lambda \to \infty$). This cannot be expected for a general slowly varying function υ as in the statement of Theorem 6.13, even assuming that υ is a smooth function: this is not a simple regularity issue!

We were able to tackle this problem by exploiting the theory of regular variation [28] (more precisely, de Haan functions) to replace υ with a more tame slowly varying function with all desired asymptotics at infinity. This is a crucial but technical point which required a careful study. We refer to [R12, Section 5.2] for details.

Remark 6.14. As a consequence of the construction in the proof of Theorem 6.13, the corresponding Laplace-Beltrami operator is essentially self-adjoint, and this can be proved using the criteria described in Section 6.2, see [R12, Rmk. 5.1].

6.3.3. Concentration of eigenfunctions

Via a classical argument [71, Lemma 6.2], we also prove the concentration of eigenfunctions at the metric boundary, in presence of a non-classical Weyl's asymptotics. We recall that a subset $A \subseteq \mathbb{N}$ has density one if

$$\lim_{\ell \to \infty} \frac{1}{\ell} \sum_{k=0}^{\ell-1} \mathbb{1}_A(k) = 1. \tag{6.34}$$

Theorem 6.15. *Let M be an n-dimensional Riemannian manifold such that the Laplace-Beltrami operator $-\Delta_g$ has discrete spectrum, and*

$$\lim_{\lambda \to \infty} \frac{N(\lambda)}{\lambda^{n/2}} = \infty. \tag{6.35}$$

Let $\{\phi_i\}_{i \in \mathbb{N}}$, be a complete set of normalized eigenfunctions of $-\Delta_g$, associated with eigenvalues λ_i, arranged in non-decreasing order. Then, there exists a density one subset $A \subseteq \mathbb{N}$ such that for any compact U it holds

$$\lim_{\substack{i \to \infty \\ i \in A}} \int_U |\phi_i|^2 d\mathsf{m}_g = 0. \tag{6.36}$$

Theorem 6.15 applies to all Riemannian manifolds with compact metric completion, satisfying Assumption **A**, and having infinite volume (by Theorem 6.10).

6.4. Further developments

Quantum confinement

In [R8] and [R9] we proved the Boscain-Laurent's conjecture for regular ARS (cf. Section 6.2.3), but the non-regular case remains open. Notwithstanding, once a local generating family is given explicitly, it is easy to compute the associated effective potential V_{eff}, and apply the general criteria of [R9] (cf. Theorem 6.2). In this way, we were able to prove the essential self-adjointness property of the Laplace-Beltrami operator in several specific examples of non-regular ARS. We intend to return on this problem in the future.

Weyl's law

Concerning the Weyl's law for singular structures, several questions remained open after our investigation in [R12]. For example, is the slow variation of the volume function v really necessary for a sharp asymptotics, as in in Theorem 6.11? (We remind that, without that assumption, we were only able to prove the rough asymptotics in Theorem 6.10.) Furthermore, to what extent one can weaken the hypotheses in Assumption **A**? In particular, it would be interesting to study singular structures where (i) the distance from the singularity is not smooth, or (ii) where the geometrical invariants blow-up with a different growth rate from what is allowed in Assumption **A**. Finally, the classical Weyl's law has been recently established for non-smooth compact metric measure spaces satisfying a RCD condition in [9, 93]. It would be interesting to understand whether a singular version of these spaces admits a non-standard Weyl's law, analogue to the ones we described in Section 6.3. A proof of these facts in the non-smooth setting would be interesting also because it would give an answer to problem (i) mentioned above.

7. Heat content asymptotics for sub-Riemannian manifolds

7.1. Introduction

Let (M, g) be a complete Riemannian manifold, and $\Omega \subset M$ be a relatively compact open domain with smooth boundary. Consider the solution $u(t, x)$ of the heat equation with Dirichlet boundary conditions and homogeneous initial datum:

$$\begin{aligned} (\partial_t - \Delta)u(t,x) &= 0, && \forall (t,x) \in (0,\infty) \times \Omega, \\ u(t,x) &= 0, && \forall (t,x) \in (0,\infty) \times \partial\Omega, \\ u(0,x) &= 1, && \forall x \in \Omega. \end{aligned} \tag{7.1}$$

The Riemannian heat content of Ω is the function

$$Q_\Omega(t) := \int_\Omega u(t,x) d\mathsf{m}_g(x), \qquad t \in [0,\infty). \tag{7.2}$$

From a physical viewpoint, $Q_\Omega(t)$ represents the total heat contained in Ω at time t, corresponding to a uniform initial temperature distribution, and where the boundary $\partial\Omega$ is kept at zero temperature. It turns out that $Q_\Omega(t)$ admits an asymptotic expansion as a function of $\sqrt{t}$ whose coefficients encode geometrical information about Ω and its boundary. Early results can be found in [86, 88, 84]. For smooth domains in a Riemannian manifold, the existence of an asymptotic expansion in $\sqrt{t}$ was established by Van den Berg and Gilkey in [87], where the authors also computed all coefficients up to order 4. For our purposes, we recall the following order 2 expansion:

$$Q_\Omega(t) = \mathrm{vol}(\Omega) - \sqrt{\frac{4t}{\pi}}\sigma(\partial\Omega) + \frac{t}{2}\int_{\partial\Omega} H d\sigma + O(t^{3/2}), \tag{7.3}$$

where vol here denotes the Riemannian volume of Ω, σ is the corresponding surface measure on $\partial\Omega$, and H is the mean curvature of $\partial\Omega$. The subsequent terms involve the second fundamental form of $\partial\Omega$ and the Riemann curvature tensor. We stress that the existence of a full asymptotic series is non-trivial, as the heat content is not a smooth function of $\sqrt{t}$ around $t = 0$ (one can easily verify this fact by computing the heat content of a Euclidean segment).

In the sub-Riemannian setting, we have thus the following natural questions.

> Does the heat content of a general sub-Riemannian manifold admit an asymptotic series for small times? What is the geometric meaning of the coefficients? Is there a relation with the heat content of a Riemannian approximation?

In this chapter we report on our results obtained in the following paper:

- **L. Rizzi** and T. Rossi. Heat content asymptotics for sub-Riemannian manifolds. *J. Math. Pures Appl. (9)*, 148:267–307, 2021

in collaboration with T. Rossi (Ph.D. student in a joint program SISSA/Institut Fourier, that I have been co-supervising with A. Agrachev and G. Charlot since October 2018).

7.2. Sub-Riemannian heat content asymptotics

The study of the heat content asymptotics in the sub-Riemannian setting is interesting for several reasons. Firstly, there is no analogue of Levi-Civita connection, curvature, or invariance theory for a general sub-Riemannian structure. These were fundamental tools for the study of the Riemannian problem by Van den Berg and Gilkey, and hence new methods must be used in the sub-Riemannian setting. Secondly, in the general sub-Riemannian case, there is no canonical choice of measure. For this reason, we must work with a general smooth measure m, which is necessary for the definition of the sub-Laplacian. Thirdly, the study of the sub-Riemannian heat content can improve our understanding of the intrinsic geometry of hypersurfaces, which is well-developed only for the case of the Heisenberg group [70, 13, 37, 16, 19] and Carnot groups [44]. Lastly, a genuinely new phenomenon occurs in the sub-Riemannian case: characteristic points, where the distribution is tangent to $\partial\Omega$, which are source of subtle problems.

The study of the small-time heat content asymptotics in the sub-Riemannian setting was initiated by Tyson and Wang, in [83], where they studied the three-dimensional Heisenberg group $\mathbb{H}$. There, they established the existence of a small-time asymptotic series up to order 2 in $\sqrt{t}$, for non-characteristic domains. The approach in [83] is probabilistic, based on the interpretation of the solution of the Dirichlet problem in terms of the exit time of a Markov process. Their method introduces an error $o(t)$, cf. [83, Prop. 3.2], preventing the access to higher order terms.

In [R10], to attack the general problem and to higher order, we used a different method with respect to that of Tyson and Wang, by adapting a technique developed in the Riemannian case by Savo [80, 78, 79]. This method allows us to prove the existence of an asymptotic expansion at arbitrary order, for non-characteristic domains of general rank-varying sub-Riemannian structures. Before stating it, let us remind some facts about hypersurfaces in sub-Riemannian geometry.

Some facts about hypersurfaces of sub-Riemannian manifolds

Let $(\mathcal{D}, g)$ be a sub-Riemannian structure on a smooth manifold M, and let Σ be a smooth hypersurface (without boundary, for simplicity). We say that $p \in \Sigma$ is *non-characteristic* if

$$\mathcal{D}_p \pitchfork T_p\Sigma, \tag{7.4}$$

and p is *characteristic* if (7.4) is not true. The sub-Riemannian distance from Σ, denoted with δ, is well-defined and smooth in the neighborhood of any non-characteristic point.

Assume that Σ has no characteristic points, and let m be a smooth measure on M. Then we can induce a smooth surface measure on Σ via the tensor density

$$\sigma := |\iota_\nu \mathsf{m}|_\Sigma, \tag{7.5}$$

where $\nu = \nabla\delta|_\Sigma$ is the sub-Riemannian normal to Σ. Furthermore, we denote by

$$H := -\mathrm{div}_{\mathsf{m}}(\nabla\delta)|_\Sigma = -\Delta_{\mathsf{m}}\delta|_\Sigma \tag{7.6}$$

the sub-Riemannian mean curvature of Σ in M. Of course all these concepts reduce to their usual definition in the Riemannian setting.

Our first main result in [R10] is the following.

Theorem 7.1. *Let M be a sub-Riemannian manifold, equipped with a smooth measure m, and let $\Omega \subset M$ be an open relatively compact subset whose boundary is smooth and has no characteristic points. Then, the heat content of Ω admits an asymptotic expansion in $\sqrt{t}$ as $t \to 0$. In particular, we have, for all $m \geq 4$ and $a_k \in \mathbb{R}$:*

$$Q_\Omega(t) = \mathsf{m}(\Omega) - \sqrt{\frac{4t}{\pi}}\sigma(\partial\Omega) + \frac{t}{2}\int_{\partial\Omega} H d\sigma + \sum_{k=3}^{m-1} a_k t^{k/2} + O(t^{m/2}), \qquad \text{as } t \to 0, \tag{7.7}$$

where σ is the sub-Riemannian measure induced by m on $\partial\Omega$, and H is the sub-Riemannian mean curvature of $\partial\Omega$.

In order to report the first few coefficients, we introduce the operator N, acting on smooth functions in a neighborhood of $\partial\Omega$, given by

$$N\phi := 2g(\nabla\phi, \nabla\delta) + \phi\Delta\delta. \tag{7.8}$$

where ∇ is the sub-Riemannian gradient, $\Delta = \Delta_{\mathsf{m}}$, and δ is the distance from $\partial\Omega$.

Proposition 7.2. *With the assumptions and notations of Theorem 7.1, there exists operators D_k, which are homogeneous polynomial of degree $k-1$ in the operators $\Delta = \Delta_{\mathsf{m}}$ and N, such that for $k \geq 1$, we have $a_k = -\int_{\partial\Omega} D_k(1) d\sigma$. In particular, it holds*

$$a_0 = \mathsf{m}(\Omega), \qquad a_1 = -\sqrt{\frac{4}{\pi}}\sigma(\partial\Omega), \tag{7.9}$$

$$a_2 = -\frac{1}{2}\int_{\partial\Omega} \Delta\delta d\sigma, \qquad a_3 = -\frac{1}{6\sqrt{\pi}}\int_{\partial\Omega} N\Delta\delta d\sigma, \tag{7.10}$$

$$a_4 = -\frac{1}{16}\int_{\partial\Omega} \Delta^2\delta d\sigma, \qquad a_5 = \frac{1}{240\sqrt{\pi}}\int_{\partial\Omega} (N^3 - 8N\Delta)\Delta\delta d\sigma. \tag{7.11}$$

For the recursive definition of the operators D_k, see [R10]. Each a_k, for all $k \geq 1$, is the integrand over $\partial\Omega$ of a universal function of H and its normal derivatives.

Remark 7.3. The integrands of a_1 and a_2 have classical interpretation as the horizontal perimeter and horizontal mean curvature of $\partial\Omega$. We observe that the integrand of a_3 is the effective potential V_{eff} of Section 6.2. We do not know whether this is a coincidence or if there is a deeper relation between heat content asymptotics and quantum confinement.

Before presenting further results, let us comment the proof of Theorem 7.1. Savo's method amounts to study the quantity

$$F(t,r) = \int_{\Omega(r)} u(t,x) d\mathsf{m}(x), \tag{7.12}$$

where $\Omega(r) = \{\delta > r\}$. Upon appropriate localization to deal with the non-smoothness of $\Omega(r)$ for large r, it turns out that $F(t,r)$ satisfies a non-homogeneous one-dimensional heat equation on the half-line $[0,\infty)$, with Neumann boundary condition at the origin. Then, the whole asymptotics of Theorem 7.1 and the expression of the coefficients are obtained by iterating the corresponding Duhamel's formula. Some non-trivial modifications must be implemented to adapt this technique to the sub-Riemannian setting. For example, the Li-Yau estimate for the heat kernel of Riemannian manifolds with Ricci curvature bounded from below are no longer available (sub-Riemannian manifolds have, in a sense, Ricci curvature unbounded from below). Another important ingredient is the description of tubular neighborhoods of $\partial\Omega$. If, for the Heisenberg group, this can be achieved through the explicit formulas for geodesics as done in [12, 13], we must make in this case a better use of the Hamiltonian flow on the annihilator bundle of $\partial\Omega$.

7.3. Riemannian approximations

Any sub-Riemannian structure can be obtained as a monotonic limit of Riemannian ones. This approximation scheme can be easily implemented for constant-rank distributions. In this case, a natural approximating sequence is obtained by taking any Riemannian metric g extending the sub-Riemannian one, and rescaling it by a factor $1/\varepsilon$ in the transverse directions. This construction yields a one-parameter family of Riemannian structures g_ε. The associated Riemannian distance d_ε converges, uniformly on compact sets, to the sub-Riemannian one d_{SR}. Outside of the sub-Riemannian cut locus, one can actually prove that $d_\varepsilon \to d_{\mathrm{SR}}$ in the C^∞ topology, as we proved in [R11] (Proposition 5.1). We have already seen this approximation scheme in Chapter 5, for constant rank structures, under the name of canonical variation.

We introduce a generalization of the canonical variation scheme which works for general rank-varying sub-Riemannian structures. Our second result relates the coefficients of the small-time asymptotics of the Riemannian heat content $Q^\varepsilon_\Omega(t)$ of the approximating structure with the sub-Riemannian ones.

Theorem 7.4. *Let M be a sub-Riemannian manifold, equipped with a smooth measure m, and let $\Omega \subset M$ be an open relatively compact subset whose boundary is smooth and has no characteristic points. Then, there exists a family of Riemannian metrics g_ε such that $d_\varepsilon \to d_{\mathrm{SR}}$ uniformly on compact sets of M, and such that*

$$\lim_{\varepsilon\to 0} a_k^\varepsilon = a_k, \qquad \forall\, k \in \mathbb{N}, \tag{7.13}$$

where a_k and a_k^ε denote the coefficients of the sub-Riemannian small-time heat content asymptotics, and the corresponding ones for the Riemannian approximating structure.

Even though $Q^{\varepsilon}_{\Omega}(t) \to Q_{\Omega}(t)$ in the C^{∞} uniform topology on compact subsets of $(0,\infty)$, this fact alone does not imply (7.13). A direct proof of Theorem 7.4 would require (i) an a-priori proof of the existence of the small-time sub-Riemannian asymptotics for $Q_{\Omega}(t)$ and (ii) a delicate inversion of the order of the two limits $\varepsilon \to 0$ and $t \to 0$. It is also important to stress that Theorem 7.1 *is not* a consequence of Theorem 7.4. Actually, what happens is precisely the opposite: we prove Theorem 7.4 by combining 7.1, the explicit form the coefficients, and the smooth convergence of the distances $d_{\varepsilon} \to d_{SR}$.

7.4. Characteristic points

The main assumption in all our results is that $\partial\Omega$ does not contain characteristic points. This is quite restrictive for the case of Heisenberg group, where the only non-characteristic domains are homeomorphic to a torus. More generally, for any contact sub-Riemannian manifold, the non-characteristic assumption and the contact structure imply that $\partial\Omega$ must have vanishing Euler characteristic. On the other hand, the non-characteristic assumption is less restrictive for general structures: it is not hard to prove that for any smooth manifold M of dimension $n \geq 4$, and any smooth relatively compact domain Ω with smooth boundary $\partial\Omega$, there exists a possibly rank-varying sub-Riemannian structure on M such that $\partial\Omega$ has no characteristic points.

Around characteristic points, the sub-Riemannian distance from the boundary is no longer smooth. Furthermore, even if $Q_{\Omega}(t)$ remains well-defined by spectral theory, solutions to the heat equation with Dirichlet boundary conditions may fail to be smooth up to the boundary at characteristic points [51, 52].

Despite these difficulties, all terms appearing in the order 2 formula for $\mathbb{H}$ in [83] are well-defined also for characteristic domains. In fact:

(i) the set of characteristic points has zero Lebesgue measure in $\partial\Omega$, see [15]. Furthermore, the measure σ in (7.5) remains well-defined even at characteristic points (its density in coordinates tends to zero there). Thus, $a_1 \propto \int_{\partial\Omega} d\sigma$ is well-defined;

(ii) the sub-Riemannian mean curvature H is singular at characteristic points. However, it is well-known that this singularity is compensated by the degeneracy of σ, and as a result $H \in L^1_{\mathrm{loc}}(\partial\Omega,\sigma)$, see [43].[1] Hence, $a_2 \propto \int_{\partial\Omega} H d\sigma$ is well-defined.

This seems to suggest that the same small-time asymptotic formula holds also for characteristic domains. Our analysis shows that this cannot be true at higher order.

Theorem 7.5. *Let $\mathbb{H}$ be the Heisenberg group, and consider the plane $\Sigma = \{z = 0\}$. Observe that the origin is an isolated characteristic point. Denote with σ the sub-Riemannian surface measure on Σ induced by the Lebesgue measure on $\mathbb{H}$. Then the integrand of the coefficient a_5 of the small-time heat content expansion is not locally integrable with respect to σ around the characteristic point of Σ.*

[1] Recently, T. Rossi proved an analogue integrability result, but with respect to any smooth positive measure, for isolated *non-degenerate* characteristic points, see [76].

To prove Theorem 7.5 we derived a formula for the sub-Riemannian distance from the xy-plane in $\mathbb{H}$. This explicit global formula has independent interest, and it can be used to study the loss of regularity of the distance at characteristic points.

Theorem 7.6. *The distance from the xy-plane in the first Heisenberg group $\mathbb{H}$, for all $p \in z$-axis, is given by*

$$\delta(p) = \sqrt{2\pi|z_p|}, \tag{7.14}$$

while for all $p \notin z$-axis, it is given in cylindrical coordinates by

$$\delta(p) = r_p \frac{4\xi_p + y_0(\xi_p)}{\sqrt{1 + y_0(\xi_p)^2}}, \qquad \xi_p = \frac{|z_p|}{r_p^2}, \tag{7.15}$$

where $\xi \mapsto y_0(\xi)$ is the unique smooth function such that

$$4\xi + y_0 + (1 + y_0^2)\arctan(y_0) = 0. \tag{7.16}$$

7.5. Further developments

Relative heat content

In [R10] we focused on the heat content of a domain Ω of a sub-Riemannian manifold M. A related concept is that of *relative* heat content of a domain Ω in M, which is obtained by considering, instead of the Dirichlet problem (7.1), the solution to the heat equation on the whole manifold with initial condition $u(0, x) = \mathbb{1}_\Omega(x)$. In other words, in terms of the heat kernel $p_t^M(x, y)$ of M, the relative heat content is the function

$$H_\Omega(t) := \int_{\Omega\times\Omega} p_t^M(x, y) d\mathsf{m}(x) d\mathsf{m}(y), \qquad t > 0. \tag{7.17}$$

Despite earlier results concerning the relation between the small-time asymptotics of $H_\Omega(t)$ and the perimeter of Ω [55, 85, 11] in the Euclidean case, and in some Carnot groups [33], we are not aware of more general result. This is an interesting topic of investigation.

Characteristic points

Theorem 7.5 shows that the asymptotic formula of Theorem 7.1 is false at order $k \geq 5$ for domains with characteristic points. In the example of Theorem 7.5, it turns out that the integrands of the coefficients a_3 and a_4 are still locally integrable with respect to the sub-Riemannian surface measure. We expect, however, that one can build a less symmetric example where also the integrand of a_4 is not integrable close to a characteristic point. On the other hand, Theorem 7.1 might still be true at lower order (the coefficients appearing therein remain well-defined for characteristic domains in $\mathbb{H}$ up to $k = 2$). Thus, the following question is natural: is it true that, for smooth domains in $\mathbb{H}$ with characteristic points, the asymptotic expansion of Theorem 7.1 remains valid up to some intermediate order $0 < k < 5$? Is there a non-standard asymptotic expansion (perhaps with logarithmic terms in t) in presence of characteristic points?

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