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Hypercyclic vectors and algebras

Fernando Vieira Costa Junior

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ÉCOLE DOCTORALE DES SCIENCES FONDAMENTALES
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Fernando VIEIRA COSTA JÚNIOR

Algèbres et vecteurs hypercycliques

Hypercyclic vectors and algebras

Sous la direction de **Frédéric BAYART**

Soutenance le 27/05/2021 à 14h30 devant le jury composé de :

Karl-G. Grosse-Erdmann , Université de Mons	Président du jury / examinateur
Evgueni Abakoumov , Université Gustave Eiffel	Rapporteur
Juan Bès , Bowling Green State University	Rapporteur
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Christoph Kriegler , Université Clermont Auvergne	Examineur interne

Je dédie cette thèse à ma femme Wiliane,
for having left so many things behind,
para continuar sempre ao meu lado.

A man is not where he lives, but where he loves.
– *old proverb*

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Abstract

This work contributes to the theory of hypercyclicity and related concepts. We are mainly focused on algebras of hypercyclic vectors for operators acting on Fréchet sequence algebras, although some contributions were made concerning the existence of single vectors. We have improved a Baire criterion for the existence of hypercyclic algebras which now grants algebras of hypercyclic vectors that are not finitely generated. Variations of this main criterion apply to some related concepts such as upper frequent, disjoint and common hypercyclicity. We provide criteria for the existence of algebras of vectors satisfying these properties (except for the 0 vector) for weighted backward shifts on classical sequence spaces equipped either with the Cauchy product or with the coordinatewise product. In particular for common hypercyclicity, we provide criteria for the existence of common hypercyclic vectors and algebras when the parameter set is not one-dimensional. We also explore the problem of closed hypercyclic algebras and provide counterproofs even for classical operators and spaces.

Keywords: hypercyclicity, hypercyclic algebras, linear dynamics, chaos, frequent hypercyclicity, upper frequent hypercyclicity, disjoint hypercyclicity, common hypercyclicity.

Résumé

Ce travail contribue à la théorie de l'hypercyclicité et à des concepts liés. Nous nous sommes principalement intéressés à des algèbres de vecteurs hypercycliques pour opérateurs agissant sur une algèbre de Fréchet de suites, bien que quelques contributions portent sur l'existence d'un seul vecteur. Nous avons amélioré un critère de Baire pour l'existence d'une algèbre hypercyclique, ce qui permet maintenant de considérer des algèbres de vecteurs hypercycliques qui ne sont pas finiment générées. Des variantes de ce critère s'appliquent à d'autres concepts relatifs comme l'hypercyclicité supérieurement fréquente, disjointe et simultanée. Nous fournissons de nouveaux critères d'existence d'algèbres de vecteurs satisfaisant ces propriétés (sauf le vecteur 0) pour les opérateurs de décalage à gauche pondérés agissant sur des espaces de suites classiques munis soit du produit de convolution, soit du produit coordonnée par coordonnée. En particulier pour l'hypercyclicité simultanée, nous fournissons des critères d'existence de vecteurs et d'algèbres hypercycliques communs dans le cas où l'ensemble de paramètres n'est pas unidimensionnel. Nous explorons aussi le problème d'existence des algèbres hypercycliques fermées dont nous avons montré l'inexistence même pour des opérateurs et des espaces classiques.

Mots-clés : hypercyclicité, algèbres hypercycliques, dynamique linéaire, chaos, hypercyclicité fréquente, hypercyclicité supérieurement fréquente, hypercyclicité simultanée.

Resumo

Este trabalho é uma contribuição à teoria de hiperciclicidade e conceitos relacionados. Nos interessamos principalmente em álgebras de vetores hipercíclicos para operadores agindo sobre uma álgebra de Fréchet de sequências, apesar de que algumas contribuições tratam da existência de vetores simplesmente. Nós melhoramos um critério de Baire que garante a existência de álgebras hipercíclicas, o qual agora fornece álgebras hipercíclicas que não são finitamente geradas. Variações do novo critério se aplicam à diversos conceitos relacionados à hiperciclicidade, como por exemplo à hiperciclicidade superiormente frequente, disjunta e simultânea. Nos provemos critérios de existência de álgebras de vetores satisfazendo essas propriedades (com exceção do vetor 0) para operadores de deslocamento à esquerda (ou “shift para trás”) agindo sobre espaços de sequência clássicos equipados ou com o produto de convolução, ou com o produto coordenada por coordenada. Em particular para o conceito de hiperciclicidade simultânea, nós fornecemos critérios de existência de vetores e álgebras hipercíclicas mesmo no caso em que o conjunto de parâmetros não é unidimensional. Nos ainda exploramos o problema de existência de uma álgebra hipercíclica fechada e encontramos respostas negativas mesmo para operadores e espaços clássicos.

Palavras-chave: hiperciclicidade, álgebras hipercíclicas, dinâmica linear, caos, hiperciclicidade frequente, hiperciclicidade superiormente frequente, hiperciclicidade disjunta, hiperciclicidade simultânea.

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Preface

Many works in mathematics consist of proving expected results, although a solution can be very difficult, even for simple problems. On the flip side, many interesting problems come from strange behaviors, at least at a first glance. In this work we studied some very specific problems in hypercyclicity, which is a strange behavior at first sight but can be well understood in numerous contexts. Nowadays the theory of linear dynamical systems is well developed and one can expect the outcome multiple kinds of related problems, but even though many tools have been used to successfully attack problems in the theory, his strange nature sometimes demand some creativity in order to either adapt previous techniques to new contexts or find counterexamples for the result in question. Moreover, at some situations we can even prove that the techniques fail but the result itself remain a mystery, sometimes we are so uncertain that we can't even have a clear filling of what to expect. We have faced all of these situations during the period of my doctorate and a large number of problems and possibilities remained open for future research. Of course not all open question were written in this document, we have rather focused ourselves on some natural problems following the actual discussions.

This is a "thesis by published works", although not all of our articles are published. The document is divided in three parts. The first is an introduction (also available in French), where we discuss topic by topic some of the ideas behind our results. It can either be read in parallel with part two or it can be used as reference for key points of the techniques present within our articles. The second part consists of a compilation of four articles, written during the preparation of the thesis, two of them are in prepublication. The third part is a conclusion where, once more, we discuss the remaining open problems in general lines and we make question there are not present in the articles of part two.

We hope the introduction will be useful for a better understanding of our work and he hope the reader will also find the open questions from part three as interesting as we do.

Fernando V. Costa Júnior,
Aubière, June 8, 2021

Part I

Introduction in english

The study of hypercyclicity is one of the most important subjects in Linear Dynamics as it constitutes the main ingredient of the Devaney's definition of Chaos from 1989 (see [18]). If X is a topological vector space and $T : X \rightarrow X$ is a bounded linear operator on X , we say that a vector $x \in X$ is *hypercyclic* for the *dynamical system* (X, T) when its *orbit under T* , defined by $\text{orb}(x, T) = \{x, T(x), T^2(x), \dots\}$, is a dense subset of X . We denote by $HC(T)$ the set of hypercyclic vectors of T . When $HC(T) \neq \emptyset$, we say that (X, T) is a *hypercyclic dynamical system* or that T is a *hypercyclic operator* on X . In other words, the iterates sequence of T on a single point x approaches any point of the space as much as we want. However, the first example of a system satisfying this peculiar property is much older than Devaney. In 1929, Birkhoff [15] has shown that the operator of translation by 1 on the space of entire functions, that is, $\tau_1 : H(\mathbb{C}) \rightarrow H(\mathbb{C})$ given by $\tau_1 f(x) = f(x + 1)$ for all $f \in H(\mathbb{C})$, admits a hypercyclic function $g \in H(\mathbb{C})$. In 1952, MacLane [22] proved the hypercyclicity of the operator of complex derivation of an entire function $D : H(\mathbb{C}) \rightarrow H(\mathbb{C})$, given by $D(f) = f'$, $f \in H(\mathbb{C})$. In 1969, Rolewicz [23] proved the same result for all multiples λB , $\lambda > 1$, of the backward shift operator $B : \ell_2(\mathbb{N}) \rightarrow \ell_2(\mathbb{N})$ given by $B(x_0, x_1, x_2, \dots) = (x_1, x_2, x_3, \dots)$, $(x_n)_{n \geq 0} \in \ell_2(\mathbb{N})$. Today these three operators are known by the respective names of these mathematicians. Nevertheless, the research topic now known as Linear Dynamics has begun with the thesis of Kitai [21], where the first systematic study on orbits of bounded linear operators was developed.

This thesis is a contribution to this research domain. In this introduction, we will describe its main results.

1 Hypercyclic algebras

It is known that $HC(T) \cup \{0\}$ contains a dense vector subspace of X whenever T is hypercyclic. That is to say, we can always find a structure composed by hypercyclic vectors (but 0) "compatible" with the space. What we call *hypercyclic subspace* comes from a more interesting problem (commonly concerning Banach spaces): in the case where T is hypercyclic, is there a closed and infinite dimensional vector subspace of X within $HC(T) \cup \{0\}$? Once more, one looks for interesting structures of hypercyclic vectors. Now, we can make a similar question on the algebraic setting: when T is hypercyclic and X has the structure of algebra, is it possible to find a subalgebra of X withing $HC(T) \cup \{0\}$? The search for such a structure, which is called a *hypercyclic algebra*, motivates this work.

The general framework of our results is that of separable and commutative F -algebras X , induced by a sequence of seminorms $(\|\cdot\|_\rho)_{\rho \in \mathbb{N}}$, such that its product $\cdot : X \times X \rightarrow X$ satisfies, for all $x, y \in X$ and $\rho \in \mathbb{N}$,

$$\|x \cdot y\|_\rho \leq \|x\|_\rho \times \|y\|_\rho.$$

Moreover, most of our results concern sequence F -algebras, what means that X is a subspace of the space of all complex sequences $\omega = \mathbb{C}^{\mathbb{N}}$. In this contexts, there are two main products to consider: the coordinatewise product and the Cauchy (or convolution) product. As examples, the spaces $\ell_p(\mathbb{N})$, $1 \leq p < +\infty$, as well as $c_0(\mathbb{N})$ become Fréchet sequence algebras with the coordinatewise product and only $\ell_1(\mathbb{N})$ is a Fréchet algebra with the Cauchy product. We can

also look at $H(\mathbb{C})$ as a sequence space. In this case, it is a Fréchet algebra with both products. The coordinatewise product on $H(\mathbb{C})$ is also known as Hadamard product. Later on we will discuss each of these products.

The first result on the existence of hypercyclic algebras dates from 2007, when Aron et. al. [2] have shown that no translation operator has a hypercyclic algebra for the convolution product. Two years later, Shkarin [25] as well as Bayart and Matheron [7] showed (independently and by different methods) that the MacLane operator have a hypercyclic algebra for the convolution product. The argument used by the latter will prove to be quite fruitful in the subsequent years with the work of Bès, Conejero and Papathanasiou [9, 10].

2 The method of Bayart and Matheron

Contrary to the constructive method employed by Shkarin, the method of Bayart and Matheron is a kind of Birkhoff transitivity property for algebras (see [7, Remark 2.28]) and, as such, relies on the application of the Baire theorem.

The concept of topological transitivity, created in 1920 by Birkhoff [14], is one of the most important tools in the theory. We say that a dynamical system (X, T) is *topologically transitive* when, for any pair of non-empty open sets (U, V) in X , one can find $n \in \mathbb{N}$ such that $T^n(U) \cap V \neq \emptyset$. Its importance comes from the fact that, over complete and separable metric spaces, topological transitivity and hypercyclicity are equivalent. The link is clear when we state this fact in the following way.

Theorem 2.1 (Birkhoff transitivity) *Let T be a bounded linear operator on a separable F -space X . Suppose that, for each pair of nonempty open sets (U, V) of X , there exist a power $N \in \mathbb{N}$ and a vector $u \in U$ such that $T^N(u) \in V$. Then $HC(T)$ is a dense G_δ subset of X .*

Of course the conclusion that $HC(T)$ is G_δ dense comes from the application of the Baire theorem. The idea of this statement is that any pair of open sets can be connected by some iterate T^N of T . This way, it is enough to find one vector u in U together with an iterate T^N which will send u into V . When it comes to algebras, the criterion [7, Remark 2.28] aims to find a vector in U such that any combination of the form $\alpha_1 u + \alpha_2 u^2 + \dots + \alpha_m u^m$ can be send into V by some iterate T^N of T . This is simplified by showing that there exists an iterate T^N sending every power u^n with $n < m$ to 0 and sending the « main power » u^m into V . These ideas turn into the following result.

Theorem 2.2 (Bayart and Matheron [7]) *Let T be a continuous operator on a separable F -algebra. Suppose that, for all $m \in \mathbb{N}$ and all nonempty open sets $U, V, W \subset X$, with $0 \in W$, there exist $u \in U$ and $N \in \mathbb{N}$ such that $T^N(u^n) \in W$ for all $n < m$ and $T^N(u^m) \in V$. Then, the set of vectors generating a hypercyclic algebra for T is G_δ dense in X .*

These authors manage to apply this method for $D : H(\mathbb{C}) \rightarrow H(\mathbb{C})$ by making use of its shift-like behavior. Under the same approach, Bès, Conejero and Papathanasiou have shown more generally that $P(D)$ admits a hypercyclic algebra whenever P is a polynomial satisfying $P(0) = 0$ (see [9]). Later on, the same authors used a different method (by using eigenvalues rather than the

shift-like behavior) to find hypercyclic algebras for many others convolution operators (see [10]). While these approaches change, the result ultimately relies on an application of Theorem 2.2. In the next section we will discuss convolution operators and a generalization of this result, which will lead later on to many other applications.

3 Convolution operators and criteria for the existence of hypercyclic algebras

By *convolution operator* we mean operators $\phi(D) : H(\mathbb{C}) \rightarrow H(\mathbb{C})$ induced by an entire function $\phi \in H(\mathbb{C})$ of *finite exponential type*, where $D : H(\mathbb{C}) \rightarrow H(\mathbb{C})$ is the operator of complex derivation. More precisely, to be of finite exponential type means that one can find constants $A, B > 0$ such that $\phi(z) = \sum_{n=0}^{\infty} a_n z^n \in H(\mathbb{C})$ satisfies $|\phi(z)| \leq A \exp(B|z|)$ for all $z \in \mathbb{C}$. The operator $\phi(D)$ is then defined by $\phi(D)(f) = \sum_{n=0}^{\infty} a_n f^{(n)}$. In this section, $H(\mathbb{C})$ is equipped with its natural operation, the product pointwise multiplication of two entire functions $f(z) = \sum a_n z^n, g(z) = \sum b_n z^n$, which correspond to the convolution product of their coefficients. As we have said, D and more generally $P(D)$, for all polynomial P with $P(0) = 0$, admit a hypercyclic algebra. In article [10], the authors have shown in a interesting way that the convolution operator $\phi(D)$ have a hypercyclic algebra in the cases $\phi(z) = \cos(z)$, $\phi(z) = ze^z$ and $\phi(z) = e^z - a$ with $0 < a \leq 1$. These functions are clearly not polynomials. Their result can be applied as well to the case $\phi(z) = (a_0 + a_1 z^n)^k$ with $|a_0| \leq 1$ and $a_1 \neq 0$, which is a polynomial that doesn't satisfy $P(0) = 0$. These applications come from the following result.

Theorem 3.1 (Bès et. al. [10]) *Let $\phi \in H(\mathbb{C})$ be of finite exponential type satisfying that its level set $\{z \in \mathbb{C} : |\phi(z)| = 1\}$ contains a non-trivial strictly convex compact arc Γ_1 such that*

$$\text{conv}(\Gamma_1 \cup \{0\}) \setminus (\Gamma_1 \cup \{0\}) \subset \phi^{-1}(\mathbb{D}).$$

Then, the set of functions generating a hypercyclic algebra for the convolution operator $\phi(D)$ is a dense G_δ subset of $H(\mathbb{C})$.

It is remarkable that $|\phi(0)| \leq 1$ for all examples found, hence some natural question arise. When it comes to polynomials, can we remove the condition $P(0) = 0$? Could $|P(z)| \leq 1$, or even $|P(z)| < 1$, be enough? Also, the method employed by these authors, by using eigenvectors, seems to be promising as it can be applied for convolution operators not induced by polynomials.

This technique makes a step forward in the article [4]. By using eigenvalues in a different way, F. Bayart characterizes the convolution operators having a common hypercyclic algebra in the case $|\phi(0)| < 1$ as well as establishes sufficient condition for the existence of such algebras in the case $|\phi(0)| = 1$.

Theorem 3.2 (Bayart [4]) *Let $\phi \in H(\mathbb{C})$ be a non constant entire function with exponential type.*

(1) *Suppose that $|\phi(0)| < 1$. Then the following assertions are equivalent:*

- (i) $\phi(D)$ admits a hypercyclic algebra;
- (ii) ϕ is not a multiple of an exponential function.

(2) Suppose that $|\phi(0)| = 1$ and ϕ is of subexponential growth. If either $\phi'(0) \neq 0$ or ϕ has order less than $1/2$, then $\phi(D)$ admits a hypercyclic algebra.

Regarding the fact that ϕ is not multiple of an exponential, one could not get better. In fact, convolution operators induced by functions of the form $\phi(x) = \lambda \exp(ax)$ are precisely multiple of a translation operator (and we know they do not have hypercyclic algebras). What Theorem 3.2 show is that, in the case $|\phi(0)| < 1$, these are the only operators not having a hypercyclic algebra. Later on, Bès, Ernst and Prieto [11] have shown that, in the case $|\phi(0)| = 1$, neither conditions on the derivative of ϕ nor on its order are necessary.

Theorem 3.3 (Bès, Ernst and Prieto [11]) *Let $\phi \in H(\mathbb{C})$ of finite exponential type such that $|\phi(0)| = 1$. If ϕ is of subexponential growth then $\phi(D)$ admits a hypercyclic algebra.*

They have also established other criteria for the existence (and even the nonexistence) of hypercyclic algebra (see their article for a detailed discussion).

One can ask what happens when $|\phi(0)| > 1$. Is this case naturally pathogenic? Could one use this condition to produce a hypercyclic algebra somehow? Still in [4], we can find some clues towards an answer to this question.

The first part of Theorem 3.2 comes from a more general criterion which largely increases the class of convolution operators supporting a hypercyclic algebra. In fact, it can be applied for example to the case $\phi(z) = \cos(z)$ and $\phi(z) = e^z - 2$, which satisfy $|\phi(0)| = 1$ (see [4, Examples 2.6 and 2.7], but (surprisingly enough) it also applies to the case $\phi(z) = 2e^{-z} + \sin(z)$, which in its turn satisfies $|\phi(z)| > 1$ (see [4, Example 2.8]). We state it here for the sake completeness.

Theorem 3.4 (Bayart [4]) *Let $\phi \in H(\mathbb{C})$ of finite exponential type. Suppose that:*

- (1) $\phi(D)$ is not multiple of an exponential function;
- (2) for all $m \in \mathbb{N}$, there exist $a, b \in \mathbb{C}$ such that $|\phi(mb)| > 1$ and, for all $n \in \{1, \dots, m\}$ and $d \in \{0, \dots, n\}$ with $(n, d) \neq (m, m)$, $|\phi(nb + (n - d)a)| < 1$.

Then $\phi(D)$ admit a hypercyclic algebra.

What happens in condition (2) is represented in the Figure 1. Returning to the discussion about Theorem 2.2, the condition in the theorem above allow us to choose a candidate $u \in U$ and a power N so that we can make T^N send u^n to 0 when $n < m$ (by using $|\phi(nb + (n - d)a)| < 1$) and send the “main power” u^m into V (by using $|\phi(mb)| > 1$). The key point which makes $\phi(z) = 2e^{-z} + \sin(z)$ work is the oscillation of this function, what allows us to choose two points $a, b \in \mathbb{R}$ such that the set $\{nb + (n - d)a : n = 1, \dots, m, d = 0, \dots, n, (n, d) \neq (m, m)\}$ falls into $\phi^{-1}(\mathbb{D}) \cap \mathbb{R}$ and, at the same time, b satisfies $|\phi(mb)| > 1$.

Now, if we look at the case $|\phi(0)| < 1$ in view of Theorem 3.2, we see that all points x sufficiently near the origin will have modulus $|\phi(x)|$ less than 1 and, at the same time, since

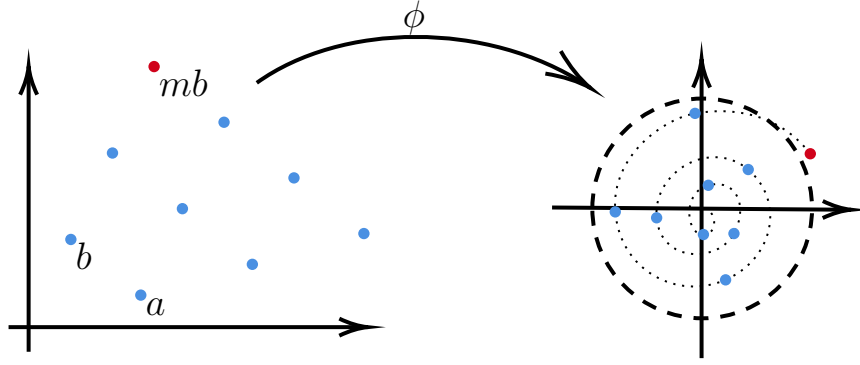


Fig. 1 – Example with $m = 3$.

$\overline{\phi(\mathbb{C})} = \mathbb{C}$, is it easy to find points y with modulus $|\phi(y)|$ bigger than 1. Hence, one only need to pick a direction $w \in \mathbb{C}$ such that there are $t_0, t_1 > 0$ satisfying

$$\begin{cases} |\phi(tw)| < 1 \text{ si } t \in [0, t_0] \\ |\phi(tw)| > 1 \text{ si } t \in (t_0, t_1] \end{cases}$$

and then choose $a = \varepsilon w$ and $b = \frac{t_0 + \varepsilon}{m} w$ with $\varepsilon > 0$ very small. We see that $mb = (t_0 + \varepsilon)w$, where $(t_0 + \varepsilon) \in (t_0, t_1]$, and $db + (n - d)a = (d\frac{t_0 + \varepsilon}{m} + (n - d)\varepsilon)w$, where

$$0 \leq d\frac{t_0 + \varepsilon}{m} + (n - d)\varepsilon \leq \frac{m - 1}{m}(t_0 + \varepsilon) + m\varepsilon \xrightarrow{\varepsilon \rightarrow 0} \frac{m - 1}{m}t_0 \in (0, t_0).$$

What can we say about $|\phi(0)| > 1$? In this case, the points x near the origin will have modulus $|\phi(x)|$ bigger than 1, that's why we link the “main power” to the combination $db + (n - d)a$ nearest to the origin. If we assume that $|\phi|$ vanishes in on direction w , that is $\lim_{t \rightarrow +\infty} |\phi(tw)| = 0$, then we can try to choose a, b in order to have one element of the combinations $db + (n - d)a$ near the origin and the other ones far away. Everything happens in the same direction $\{tw : t \geq 0\}$ as soon as we choose $b = t_0 w$ and $a = t_1 w$ with t_1 much bigger than t_0 . By doing that, for all $m \in \mathbb{N}$, the combination $db + (n - d)a$ nearest the origin if simply b (which corresponds to $n = d = 1$) and all the other combinations are far away due to a . Of course we will need another version of Theorem 2.2 changing the “main power” to the minimum.

Theorem 3.5 (false) *Let T be a bounded operator on a separable F -algebra X . Suppose that, for all $m \geq 1$ and all open sets U, V, W in X , with $0 \in W$, there exist $u \in U$ and $N \in \mathbb{N}$ such that $T^N(u^n) \in W$ for all $1 < n \leq m$ and $T^N(u) \in V$. Then the set of points generating a hypercyclic algebra for T is G_δ dense in X .*

It turns out that this result is in fact impossible. Indeed, the term u not always appear in the elements of the algebra $A(u)$ generated by u . For example, $u^2 + 2u^3 \in A(u)$ but the coefficient of u in this combination is $\alpha_1 = 0$, whence T^N can't send $\alpha_1 u$ anywhere! What we can have, for each element of $A(u)$, is the presence of a lower and a higher power, say u^{m_0} and u^{m_1} where m_0 is the smallest and m_1 is the biggest. This way we can send u^{m_0} to V and all the other powers $m_0 < n < m_1$ to 0. For example again with the element $u^2 + 2u^3$, the operator T^N will send u^2 into V and u^3 into 0. We have got the following (true) version.

Theorem 3.6 (true) *Let T be a bounded operator on a separable F -algebra X . Suppose that, for all $1 \leq m_0 \leq m_1$ and all nonempty open sets U, V, W in X , with $0 \in W$, there exist $u \in U$ and $N \in \mathbb{N}$ such that $T^N(u^n) \in W$ for all $m_0 < n \leq m_1$ and $T^N(u^{m_0}) \in V$. Then the set of points generating a hypercyclic algebra for T is G_δ dense in X .*

Hence, we have found a criterion that can be applied for the case $|\phi(0)| > 1$ when ϕ is not a multiple of an exponential and there exists a direction $w \in \mathbb{C}$ such that $\lim_{t \rightarrow +\infty} |\phi(tw)| = 0$. We will find more examples in the section 3 of Article A.

Theorem 3.6 was the first version of the more general Theorem 2.1 of Article A, where we consider an arbitrary finite subset $I \subset \mathbb{N}$ and we choose as the “main power” some element $m_0 \in I$. Then Theorem 2.2 corresponds to the case $m_0 = \max(I)$ and Theorem 3.6 corresponds to the case $m_0 = \min(I)$. This possibility of choosing the “main power” will be useful to get hypercyclic algebras for weighted backward shift operators on Fréchet sequence algebras. We will always consider $m_0 = \min(I)$ when we work with the coordinatewise product and we will always choose $m_0 = \max(I)$ when we work with the convolution product.

Later on this Baire argument for algebras will be adapted to other different kinds of hypercyclicity, such as common, upper frequently and disjoint (see sections 5, 6 and 7).

4 Cauchy and coordinatewise products: first results on Fréchet sequence algebras

From now on, we will turn ourselves to the F -algebras of sequences. In this context, as one might expect, the coordinatewise product is easier to work with in comparison with the Cauchy product. Although the treatment of these two settings are naturally different, knowing to work with the simpler case helps to understand the more difficult one. In this section we will discuss their differences and some of the first results for weighted backward shift operators. As far as the literature goes, the best results were from [19], where the authors obtain in a constructive way some quite general criteria on weighted shifts under the hypothesis that they are mixing. Here, we make use of some of their ideas but insisting on the use of a Baire argument under the form of Theorems 2.2 and 3.6 and we also obtain in a quite general results without the hypothesis that the shifts are mixing.

In order to simplify the discussion, we will call a weighted backward shift operator simply by “weighted shift”. It is defined as

$$B_w(x_0, x_1, \dots) = (w_1x_1, w_2x_2, \dots),$$

where $x = (x_n)_n$ and $w = (w_n)_n$ is its “weight sequence”. We say that a weight w is “admissible” in a sequence space X when the weighted shift B_w is continuous on X .

4.1 Coordinatewise product

In linear dynamics, when we work on an F -space X is common to assume this space have a continuous norm (as one does when the subject is hypercyclic subspaces). The space ω is the

canonical case where this property is not satisfied, although this space is easier to work with due to its permissive topology. In fact, all backward shift operators B_w on ω admit a hypercyclic algebra for the coordinatewise product (see Theorem 4.8 of Article A). We will then suppose that X has a continuous norm. The main consequence of this hypothesis is that the basis $(e_n)_n$ is bounded away from 0. We will equally assume that $(e_n)_n$ is a Schauder basis of X , what makes possible multiple criteria for the existence of hypercyclic vectors, for which we will furnish their algebraic counterpart with the coordinatewise product. Aiming to clarify the discussion, is it important to define what we understand as the “support” of a vector $x \in X$. This element can be uniquely written as $x = \sum x_n e_n$. Its *support* is defined by $\text{supp}(x) = \{n \in \mathbb{N} : x_n \neq 0\}$.

The main property satisfied by the coordinatewise product is the following: for all element $x \in X$ and all $m \geq 1$,

$$x^m = \left(\sum_{n \geq 0} x_n e_n \right)^m = \sum_{n \geq 0} x_n^m e_n. \quad (1)$$

This means that, taking powers on vectors doesn't change their supports. This is the fundamental difference with the Cauchy product. This also means that, when two vectors $x, y \in X$ have disjoint support, then

$$(x + y)^m = x^m + y^m. \quad (2)$$

The property (1) allows us to easily define m th roots of vectors (for all complex z , we denote by $z^{1/m}$ one of its m th roots):

$$x^{1/m} = \sum_{n \geq 0} x_n^{1/m} e_n.$$

These properties let us apply Theorem 3.6. Let us describe informally how do we do this.

Let B_w be a weighted shift induced by the weight sequence $w = (w_n)$. Then B_w admits a right inverse $F_{w^{-1}}$ (which is the weighted forward shift induced by the weight sequence $w^{-1} := (w_n^{-1})$). Suppose that U, V, W are non-empty open sets of X with $0 \in W$ and let $0 \leq m_0 \leq m_1$. We fix two “targets” $x \in U$ and $y \in V$ with support in $\{0, \dots, p\}$ for some $p \in \mathbb{N}$ (what is possible as $(e_n)_n$ is a Schauder basis). Now, define $S := F_{w^{-1/m_0}}$ and consider

$$u = x + S^N y^{1/m_0}, \quad (3)$$

where $N > p$ sufficiently big (conditions later). All these definition are quite natural. Effectively, the condition $N > p$ separates the supports of x and $S^N y^{1/m_0}$ (so we can apply property (2)) and makes x fall in the kernel of B_w^N (what allows us to get rid of some parcels). Looking at the “main power” u^{m_0} , we define S and the candidate u in order to have

$$u^{m_0} = x^{m_0} + F_{w^{-1}}^N y.$$

Since the support of x^{m_0} didn't move, we still have $B^N(x^{m_0}) = 0$ whence $B^N(u^{m_0}) = y \in V$. Finally, there are two questions to answer: (1) under which condition the vector u will be an element of U ? (2) what happens with the remaining terms $B_w^N u^n$ where $m_0 < n \leq m_1$?

For the first question, by looking at (3) we see that it is enough to impose $\lim_{N \rightarrow +\infty} S^N y^{1/m_0} = 0$ and we can take N big enough so that $u \in U$. If we write $y = \sum_{l=0}^p y_l e_l$, we get

$$S^N y^{1/m_0} = \sum_{l=0}^p \frac{1}{(w_{l+1} \cdots w_{N+l})^{1/m_0}} y_l^{1/m_0} e_{l+N}.$$

Hence, what we can put as a condition is

$$\lim_{N \rightarrow +\infty} \frac{1}{(w_{l+1} \cdots w_{N+l})^{1/m}} e_{l+N} = 0, \quad \forall m \geq 1, \forall l \geq 0. \quad (4)$$

For the second question we need to look at $u^n = x^n + (S^N y^{1/m_0})^n$. Since the support of x^n hasn't moved, we have $B_w^N x^n = 0$. Now, using that

$$(S^N y^{1/m_0})^n = \sum_{l=0}^p \frac{1}{(w_{l+1} \cdots w_{N+l})^{n/m_0}} y_l^{n/m_0} e_{l+N},$$

we find

$$B_w^N u^n = \sum_{l=0}^p \frac{w_{l+1} \cdots w_{N+l}}{(w_{l+1} \cdots w_{N+l})^{n/m_0}} y_l^{n/m_0} e_l = \sum_{l=0}^p \frac{1}{(w_{l+1} \cdots w_{N+l})^{n/m_0 - 1}} e_l.$$

Since $n/m_0 - 1 > 0$, for $B_w^N u^n$ to be small, it is enough to impose

$$\lim_{N \rightarrow +\infty} w_{l+1} \cdots w_{N+l} = +\infty.$$

But this condition is a consequence of (4) as $(e_n)_n$ is bounded below! In conclusion, the only condition that we need is (4). If we repeat this argument but replacing N by a sequence $(n_k)_k$, we obtain the following characterization.

Theorem 4.1 *Let X be a Fréchet sequence algebra for the coordinatewise product and with a continuous norm. Assume that (e_n) is a Schauder basis for X . Let also B_w be a bounded weighted shift on X . The following assumptions are equivalent.*

(i) B_w supports a dense and not finitely generated hypercyclic algebra.

(ii) There exists a sequence of integers (n_k) such that for all $\gamma > 0$, for all $l \in \mathbb{N}$, $((w_{l+1} \cdots w_{n_k+l})^{-\gamma} e_{n_k+l})$ tends to zero.

(The proof of the reciprocal mimics that of the necessary conditions for hypercyclicity). In the particular cases $X = \ell_p(\mathbb{N})$, $1 \leq p < \infty$, or $c_0(\mathbb{N})$, this condition is the same as the hypercyclicity criterion [7, Theorem 1.40].

It is clear in the previous discussion why we use Theorem 3.6 rather than Theorem 2.2: we need the “main power” m to satisfy $n/m - 1 > 0$. The exponent $n/m - 1$ appears in all results concerning hypercyclic algebras with the coordinatewise product. Even when the context is more complicated (like with upper frequent hypercyclicity), the idea behind is the same. This is how the coordinatewise product works.

4.2 Cauchy product and the separation term

As we have discussed for the coordinatewise product, when we look at the definition of the candidate $u = x + z \in U$, we see that the support of the “main” parcel z^{m_0} , which will be sent into V by B_w^N , doesn't change with the power m_0 , hence one need to know what happens with the coefficients for the remaining powers u^n of u . For the Cauchy product, however, the support moves to the right when we increase the powers (that is, $\max \text{supp}(u^{n_1}) < \max \text{supp}(u^{n_2})$ when $n_1 < n_2$) and, moreover, all the coefficients are mixed in the product.

In order to solve this problem, we are going to introduce in u a separating term by taking

$$u = x + z + \varepsilon e_\sigma, \tag{5}$$

where ε is small and $\sigma > \max(\text{supp}(z))$ in a way that εe_σ is placed far to the right. Some traces of this idea are present in the article [25], but it in the paper by Falcó and Grosse-Erdmann [19] where the application of this term is fully developed. The interest is that, when we take the m th power in (5), we can write

$$u^m = (x + z + \varepsilon e_\sigma)^m = P_0 + mz \cdot (\varepsilon e_\sigma)^{m-1} + (\varepsilon e_\sigma)^m,$$

where $\max(\text{supp}(P_0)) < \min(\text{supp}(z(e_\sigma)^{m-1}))$. This way, if ε, z, σ and $N > \max(\text{supp}(P_0))$ are well chosen, we can define u so that $B_w^N(u^m)$ belongs to V . The problem on the support is no more! Furthermore, if $m' < m$ and if z, σ, N are well chosen, we can have $\max(\text{supp}(u^{m'})) < N$ so that $B_w^N(u^{m'}) = 0$.

All these ideas bring us to the results in the Section 4.3 of Article A, where we have assumed a supplementary hypothesis on the space X (see the definition of “regular space” in Article A), which is satisfied by all of our favorite spaces $\ell_p, 1 \leq p < +\infty, c_0(\mathbb{N})$ and $H(\mathbb{C})$. We have obtained the following characterization.

Theorem 4.2 *Let X be a regular Fréchet sequence algebra for the Cauchy product and let B_w be a bounded weighted shift on X . The following assertions are equivalent.*

- (i) B_w is hypercyclic.
- (ii) B_w supports a dense and not finitely generated hypercyclic algebra.

As for the coordinatewise product, this discussion elucidates what happens when we deal with the Cauchy product. Again, even in more complex contexts, a separating term of the form εe_σ is always present.

5 Common hypercyclicity

When we consider a family of hypercyclic operators $(T_\lambda)_{\lambda \in \Lambda}$ acting on the same F -space X , it is natural to ask whether there exists a single vector x which is hypercyclic for each member T_λ of this family. In this section, we assume as always that the parameter set Λ is a σ -compact subset of $\mathbb{R}^d, d \geq 1$, and that $(\lambda, x) \mapsto T_\lambda(x)$ is a continuous application from $\Lambda \times X$ into X . An element $x \in \bigcap_{\lambda \in \Lambda} HC(T_\lambda)$ is called *common hypercyclic vector* for the *continuous family* $(T_\lambda)_{\lambda \in \Lambda}$, which is then called *common hypercyclic family*.

This question is interesting and non-trivial. On the one hand, many classic families of operators are common hypercyclic, the first example being given in 2003 by Abakumov and Gordon [1] for the family of Rolewicz operators $(\lambda B)_{\lambda > 1}$. On the other hand, as shows an example by A. Borichev (also cited in [1]), the family $(\lambda B \times \mu B)_{(\lambda, \mu) \in [2, 3]^2}$ doesn't have any common hypercyclic vector.

Example 5.1 (A. Borichev) For any $K \subset (1, +\infty) \times (1, +\infty)$ with positive Lebesgue measure, the family $(\lambda B \times \mu B)_{(\lambda, \mu) \in K}$ has no common hypercyclic vector on $\ell_2(\mathbb{N}) \times \ell_2(\mathbb{N})$.

Intuitively, the smaller the parameter set is, the bigger are our chances of finding a common hypercyclic vector. This way, the example above shows a possible link between the existence of such a vector and the dimension of the parameter set. We will return to this discussion later in Section 5.2. For now, let us assume $d = 1$.

5.1 Partitioning intervals

As it often happens, the first results on the existence of a vector satisfying some properties are complicated and constructive. The same happened with the common hypercyclicity, as the result of Abakumov and Gordon consists of a complex construction of the vector. Nowadays, the theory of common hypercyclic vectors is well developed and the powerful argument is also based on the Baire theorem (see [20, Theorem 11.5]).

Theorem 5.2 *Assume that, for all $K \subset \Lambda$ compact and all couples (U, V) of nonempty open sets of X , there exists $u \in U$ such that, for any $\lambda \in K$, one can find $N \in \mathbb{N}$ satisfying $T_\lambda^N(u) \in V$. Then the set of common hypercyclic vectors for $(T_\lambda)_{\lambda \in \Lambda}$ is a dense G_δ subset of X .*

In the particular case of weighted shifts, this criterion give many simple applications (see [7] and [20] for more details), which cover the classic families such as $(\lambda D)_{\lambda > 0}$ on $H(\mathbb{C})$ and $(\lambda B)_{\lambda > 1}$ on $\ell_p(\mathbb{N})$, $1 \leq p < +\infty$, or $c_0(\mathbb{N})$.

A successful application of Theorem 5.2 consists of considering K as a (possibly small) interval $[a, b]$ and partitioning it in the form $a = \lambda_0 < \lambda_1 < \dots < \lambda_q = b$. For each sub interval $\Lambda_i = [a_{i-1}, a_i]$ of this partition, we associate some $u \in U$ such that each $\lambda \in K$ falls into some Λ_i which verifies $T_{\lambda_i}^{N_i}(u) \in V$. The goal is to define a very fine partition so that λ will be so close to λ_i that we will also have $T_\lambda^{N_i}(u) \in V$. The important Costakis-Sambarino criterion allows us to construct such a partition in a natural way. Here, we assume that each T_λ , $\lambda \in \Lambda$ has a right inverse $S_\lambda : \mathcal{D} \rightarrow \mathcal{D}$, where \mathcal{D} is a dense subset of X .

Theorem 5.3 (Costakis-Sambarino [17]) *Let Λ be an interval of the real line. Assume that for every compact interval $K \subset \Lambda$, every $u \in \mathcal{D}$,*

(1) *there exist $m \geq 1$ and $(c_k)_{k \geq m}$ a sequence of positive real numbers with $\sum_{k \geq m} c_k < \infty$ such that*

- $\|T_\lambda^{n+k} S_\mu^n u\| \leq c_k$ for every $n \in \mathbb{N}$, $k \geq m$, $\mu \leq \lambda$, $\mu, \lambda \in K$;
- $\|T_\lambda^n S_\mu^{n+k} u\| \leq c_k$ for every $n \in \mathbb{N}$, $k \geq m$, $\mu \geq \lambda$, $\mu, \lambda \in K$;

(2) *for all $\eta > 0$, there exists $\tau > 0$ such that, for all $n \geq 1$,*

$$0 \leq \mu - \lambda \leq \frac{\tau}{n} \Rightarrow \|T_\lambda^n S_\mu^n(u) - u\| < \eta. \quad (6)$$

Then $\bigcap_{\lambda \in \Lambda} HC(T_\lambda)$ is a dense G_δ subset of X .

Condition (2) allows us to define a partition $\min(K) = a_0 < a_1 < \dots < a_q = \max(K)$ of the form

$$a_i = a_{i-1} + \frac{\tau}{iN}, \quad i = 1, \dots, q,$$

where N is suitably fixed. We then choose $N_i = iN$, $i = 1, \dots, q$. The divergence of $\sum_{i \geq 1} \frac{\tau}{iN}$ is important to the construction of the partition, as it guarantees the existence of q such that a_q reaches (or surpasses) $b = \max(K)$.

The partition introduced by Costakis and Sambarino motivates some generalizations of these conditions. Before discussing them in detail, let us see how condition (2) behaves in the context of weighted shifts $T_\lambda = B_{w(\lambda)}$. We assume that each weight $w(\lambda) = (w_n(\lambda))_{n \in \mathbb{N}}$ is such that $(\lambda, n) \in \Lambda \times \mathbb{N} \mapsto w_n(\lambda)$ is continuous. The right inverse operators are defined by $S_\mu = F_{w(\lambda)^{-1}}$. Let $u \in \text{span}(e_i)$ of the form $u = \sum_{l=0}^p u_l e_l$ for some $p \in \mathbb{N}$. We have

$$\begin{aligned} \|T_\lambda^n S_\mu^n(u) - u\| &= \|B_{w(\lambda)}^n F_{w(\mu)^{-1}}^n(u) - u\| \\ &= \left\| \sum_{l=0}^p \left(\frac{w_{l+1}(\lambda) \cdots w_{l+n}(\lambda)}{w_{l+1}(\mu) \cdots w_{l+n}(\mu)} - 1 \right) u_l e_l \right\|. \end{aligned} \quad (7)$$

Thus, it is important to ensure that the quotients $\frac{w_1(\lambda) \cdots w_n(\lambda)}{w_1(\mu) \cdots w_n(\mu)}$ get close to 1 when λ gets close to μ (the indices $l = 0, \dots, p$ will be included later on). By using logarithms, we see that it is enough to assume that there exists $N \geq 1$ such that all functions $\lambda \mapsto \log(w_i(\lambda))$, $i \in \mathbb{N}$, are N -Lipschitz. In fact, knowing that $|\exp(x) - 1| \leq 2|x|$ for all small x , we find

$$\begin{aligned} \left| \frac{w_1(\lambda) \cdots w_n(\lambda)}{w_1(\mu) \cdots w_n(\mu)} - 1 \right| &= \left| \exp \left(\log \left(\frac{w_1(\lambda) \cdots w_n(\lambda)}{w_1(\mu) \cdots w_n(\mu)} \right) \right) - 1 \right| \\ &= \left| \exp \left(\sum_{j=1}^n \left(\log(w_j(\lambda)) - \log(w_j(\mu)) \right) \right) - 1 \right| \\ &\leq 2 \left| \sum_{j=1}^n \left(\log(w_j(\lambda)) - \log(w_j(\mu)) \right) \right| \\ &\leq 2 \sum_{j=1}^n N |\lambda - \mu| \\ &= 2nN |\lambda - \mu|. \end{aligned}$$

Hence, condition(6) will be satisfied if we take $\tau \leq \frac{\eta}{2nC_u}$, for some constant C_u depending on u (and which will allow us to have the same conclusion for the remaining parcels $l = 0, \dots, p$). The most important step in the above calculations is the second inequality, where we use the N -Lipschitz property of the functions $\lambda \mapsto \log(w_i(\lambda))$. More precisely, it is sufficient if these functions are Lipschitz on compact sets with uniformly bounded Lipschitz constants, this idea appears in [6]. A first generalization of this condition appears in Article B, Theorem 3.4, where we ask for each function $\lambda \mapsto \log(w_i(\lambda))$ to be L_n -Lipschitz and we ask for the sequence (L_n) to satisfy some properties (see next section). A second generalization will appear later in Article C, where we rather look at the functions $\lambda \mapsto \sum_{k=1}^n \log(w_k(\lambda))$.

In the next sections, we will see how to generalize these ideas to produce algebras of common hypercyclic vectors.

5.1.1 Common hypercyclic algebras for the coordinatewise product

One of the most basic application of Theorem 5.2 is the Common Hypercyclicity Criterion (of which Costakis-Sambarino criterion is a consequence).

Theorem 5.4 (Basic Criterion [7]) *Let us suppose that, for all $K \subset \Lambda$ compact, for every couple $(u, v) \in \mathcal{D} \times \mathcal{D}$ and for all neighborhoods O of the origin 0 in X , one can find $q \in \mathbb{N}$, parameters $\lambda_1, \dots, \lambda_q \in \Lambda$, subsets $\Lambda_1, \dots, \Lambda_q \subset \Lambda$ with $\lambda_i \in \Lambda_i, i = 1, \dots, q$, and positive integers $n_1, \dots, n_q \in \mathbb{N}$ such that:*

- (i) $\bigcup_i \Lambda_i \supset \Lambda$;
- (ii) $\sum_{j=1}^q S_{\lambda_j}^{n_j}(v) \in O$;
- (iii) for all $i \in \{1, \dots, q\}$ and all $\lambda \in \Lambda_i$,
 - $\sum_{j \neq i} T_{\lambda}^{n_i} S_{\lambda_j}^{n_j}(u) \in O$;
 - $T_{\lambda_i}^{n_i}(u) \in O$;
 - $T_{\lambda}^{n_i} S_{\lambda_i}^{n_i}(v) - v \in O$.

Then $\bigcap_{\lambda \in \Lambda} HC(T_{\lambda})$ is G_{δ} dense in X .

These conditions are not very different than the ones previously adapted to the algebraic context for the coordinatewise product. Thus, one can expect to successfully adapt as we have done before, the first step being to reformulate Theorem 5.2 to produce algebras. We have found the following statement.

Theorem 5.5 *Let Λ be a σ -compact metric space and let $(T_{\lambda})_{\lambda \in \Lambda}$ be a continuous family of operators on an F -space X . Assume that, for all $K \subset \Lambda$ compact and all $I \in \mathcal{P}_f(\mathbb{N})$, there exists $m_0 \in I$ such that, for each couple (U, V) of nonempty open sets of X and for all neighborhoods W of 0 , one can find $u \in U$ such that, for each $\lambda \in K$ there is $N \in \mathbb{N}$ satisfying*

$$\begin{cases} T_{\lambda}^N(u^n) \in W \text{ pour } n \in I \setminus \{m_0\}, \\ T_{\lambda}^N u^{m_0} \in V. \end{cases}$$

Then, the set of vectors generating a common hypercyclic algebra for $(T_{\lambda})_{\lambda \in \Lambda}$ is G_{δ} dense in X .

This theorem allows us to state an algebraic version of the basic criterion for the coordinatewise product (see Article B, Lemma 3.2) and also to get practical applications, which derive from the result below.

Theorem 5.6 *Let $\Lambda = [a, b]$ be a compact interval and let $(w(\lambda))_{\lambda \in \Lambda}$ be a family of admissible weight sequences. Assume that*

- (a) (e_i) is an unconditional basis of X ;
- (b) for each n , the function $\log(w_n)$ is non-decreasing and Lipschitz on Λ , with Lipschitz constant less than or equal to $L_n > 0$.

(c) for all $p > 0$, for all $m_0 \geq 1$, there exists an increasing sequence of integers (n_k) such that

- the series $\sum_{k \geq 1} \frac{1}{(w_{l+1}(a) \cdots w_{l+n_k}(a))^{1/m_0}} e_{l+n_k}$ are unconditionally convergent for all $l = 0, \dots, p$;
- the series $\sum_{k \geq 1} \frac{1}{w_{l+1}(a) \cdots w_{l+n_k+i-n_i}(a)} e_{l+n_k+i-n_i}$, $i \geq 1$, $l = 0, \dots, p$ are uniformly unconditionally convergent;
- for all $k \geq 1$, $w_{l+1}(a) \cdots w_{l+n_k}(a) \geq 1$;
- $\sum_{i > 1} 1/C_{n_i} = +\infty$, where $C_n = \sum_{k=1}^n L_k$.

Then the set of points generating a common hypercyclic algebra for $(T_\lambda)_{\lambda \in \Lambda}$ is residual in X .

This theorem applies to the classic families $(\lambda B)_{\lambda > 1}$ on $\ell_p(\mathbb{N})$, $1 \leq p < +\infty$, or $c_0(\mathbb{N})$; also $(\lambda D)_{\lambda > 0}$ on $HC(\mathbb{C})$; and also $(B_{w(\lambda)})_{\lambda > 0}$, where $w_n(\lambda) = 1 + \frac{\lambda}{n}$, on $\ell_p(\mathbb{N})$, $1 \leq p < +\infty$, or $c_0(\mathbb{N})$. This last application is particularly interesting because the existence of a common hypercyclic vector for this family on $\ell_p(\mathbb{N})$ was only knew for $\lambda > 1/p$. Our method implies in particular the following result.

Corollaire 5.7 *Consider the family of weights $(w(\lambda))_{\lambda > 0}$ defined by $w_n(\lambda) = 1 + \frac{\lambda}{n}$, $n \geq 1$. Then $(B_{w(\lambda)})_{\lambda > 0}$ admits a common hypercyclic vector on $\ell_p(\mathbb{N})$, $1 \leq p < \infty$.*

5.1.2 Common hypercyclic algebras for the Cauchy product

Theorem 5.6 let us state some practical results for algebras with the coordinatewise product, similar to those found in the literature for common hypercyclic vectors. This is possible since the coordinatewise product behaves well as we have discussed before, what is not at all the case for the Cauchy product. The problem is even more complicated when we look for algebras as the vector u for which we apply Theorem 5.5 is naturally written in the form

$$u = x + z_1 + z_2 + \cdots + z_q,$$

where each z_i are associated with the elements of the partition. More terms are then susceptible of being mixed when we take powers u^n of u . However, by using again the separating term and Theorem 5.5, we get the following general result.

Theorem 5.8 *Let $\Lambda \subset \mathbb{R}$ be an interval, let X be a regular Fréchet sequence algebra under the Cauchy product and let $(w(\lambda))_{\lambda \in \Lambda}$ be a family of admissible weighted sequences such that all functions $\log(w_n)$ are non-decreasing and Lipschitz on compact sets with uniformly bounded Lipschitz constants. Suppose that (e_n) is an unconditional basis of X and that*

(a) for all $\lambda \in \Lambda$,

$$\sum_{n=1}^{\infty} \frac{1}{w_1(\lambda) \cdots w_n(\lambda)} e_n \in X;$$

(b) for all $m \in \mathbb{N}$ and all $[a_0, b_0] \subset \Lambda$ there exist $c \in (0, 1)$ and $\kappa_0 > 1$ such that

$$\lim_{N \rightarrow \infty} \sum_{n=cN}^N \frac{[w_1(\kappa_0 a) \cdots w_{mN}(\kappa_0 a)]^{\frac{m-1}{m}}}{w_1(a) \cdots w_{(m-1)N+n}(a)} e_n = 0, \text{ for all } a \in [a_0, b_0]. \quad (8)$$

Then $(T_\lambda)_{\lambda \in \Lambda}$ admits a common hypercyclic algebra.

This theorem furnishes a common hypercyclic algebra with the Cauchy product for all classic families of Rolewicz operators $(\lambda B)_{\lambda > 1}$ on $\ell_1(\mathbb{N})$ and MacLane operators $(\lambda D)_{\lambda > 0}$ on $H(\mathbb{C})$, but leaves open the example $(B_{w(\lambda)})_{\lambda > 0}$ where $w_n(\lambda) = 1 + \frac{\lambda}{n}$, what doesn't satisfy condition (b). This problem will be solved later on in Article D, where we will show that this family have a common hypercyclic algebra on a more general context. The proof is discussed in Section 5.3.

5.2 Partitioning squares

In this part we will discuss common hypercyclicity when the parameter set Λ is a subset of \mathbb{R}^d with $d > 1$. It is possible to approach this problem in different ways, as in [26] or [3], where we increase the set of parameters without changing the underlying space. Here, we are rather interested on operators as in Borichev's example: direct sums of backward shifts acting on the same product space. Its example shows that the parameter set can't be too big, but what shapes could it have? Bayart and Matheron showed in [6] that $(sB \times tB)_{(s,t) \in \Lambda}$ admits a common hypercyclic vector whenever Λ is an monotonic Lipschitz curve. In this example, a one-dimensional set of parameters allows the existence of a common hypercyclic vector in the bi-dimensional environment. More generally, the same authors showed that in the d -dimensional case $(t_1 B \times \dots \times t_d B)_{(t_1, \dots, t_d) \in \Lambda}$, there is a common hypercyclic vector when Λ is a monotonic Lipschitz surface, what is a $(d-1)$ -dimensional subset of \mathbb{R}^d . Nevertheless, whether the set $\bigcap_{s \in [2,3]} HC(sB \times (4-s)B)$ is empty or not has remained as an open question. We don't know either how to estimate the "maximal size" of Λ for the existence of a common hypercyclic vector to be possible. What we know is that its d -dimensional Lebesgue measure might be 0. Also, a treatment of weighted shifts was not explored yet.

The question that we have just discussed motivated the results of the Article C. On the one hand, we have looked for necessary conditions on $\Lambda \subset \mathbb{R}^d$ for $\bigcap_{\lambda \in \Lambda} HC(\lambda B \times \mu B)$ to be nonempty. The analysis of the proof of the Borichev's example 5.1 reveals that, what limits the size of the set Λ for the case $(\lambda B \times \mu B)_{(a,b) \in \Lambda}$ is the fact that condition (2) of the Costakis-Sambarino criterion is "saturated" for this family T_λ : for many vectors x ,

$$\|T_\lambda^n S_\mu^s(x) - x\| \geq cn|\lambda - \mu|.$$

This observation has lead us to obtain the following result.

Theorem 5.9 *Let $\Lambda \subset \mathbb{R}^d$, let $(T_\lambda)_{\lambda \in \Lambda}$ be a family of operators acting on the Banach space X . Assume that there exist $\alpha > 0$, $v \in X$, $\delta > 0$ and $C > 0$ such that, for all $\lambda, \mu \in \Lambda$, for all $n \in \mathbb{N}$ and all $u \in X$ satisfying*

$$\|T_\lambda^n u - v\| < \delta \text{ and } \|T_\mu^n u - v\| < \delta,$$

then

$$\|T_\lambda^n u - T_\mu^n u\| \geq Cn^\alpha \|\lambda - \mu\|.$$

If $\bigcap_{\lambda \in \Lambda} HC(T_\lambda) \neq \emptyset$, then $\dim_{\mathcal{H}}(\Lambda) \leq \frac{1}{\alpha}$.

On the other hand, we have looked for sufficient conditions on $\Lambda \subset \mathbb{R}^d$ for $\bigcap_{\lambda \in \Lambda} HC(T_\lambda)$ to be nonempty. The fundamental difference from the one-dimensional case is the absence of a natural order on \mathbb{R}^d in order to produce a covering by small and well placed parts. We have defined a dyadic covering by cubes of side $\frac{\tau}{n_i^\alpha}$ together with an associated sequence $(n_i)_i$ chosen by considering the best emplacement possible of the parts (see Section 4.1 of Article C for a comparison of three possible emplacements of these dyadic cubes)). In particular, we get the following theorem.

Theorem 5.10 *Let $d \geq 1$ and $\Lambda \subset \mathbb{R}^d$ compact. We assume that there exist $\alpha \in (0, 1/d)$, $\beta > \alpha d$ and $D > 0$ such that, for all $u \in \text{span}(e_i)$,*

(a) *there exist $C, N > 0$ such that, for all $\lambda, \mu \in \Lambda$ and all $n \geq 0, k \geq N$ satisfying $\|\lambda - \mu\|_\infty \leq D \frac{k^\alpha}{(n+k)^\alpha}$, we have*

$$\begin{aligned} \|T_\lambda^{n+k} S_\mu^n u\| &\leq \frac{C}{k^\beta} \\ \|T_\lambda^n S_\mu^{n+k} u\| &\leq \frac{C}{k^\beta}. \end{aligned}$$

(b) *for all $\varepsilon > 0$, there is $\tau > 0$ such that, for all $n \geq 1$ and for all $\lambda, \mu \in \Lambda$,*

$$\|\lambda - \mu\|_\infty \leq \frac{\tau}{n^\alpha} \implies \|T_\lambda^n S_\mu^n u - u\| < \varepsilon.$$

Then $\bigcap_{\lambda \in \Lambda} HC(T_\lambda)$ is a dense G_δ subset of X .

Since the corollaries of this result will be generalized to the context of algebras in Article D, we will discuss them in the next section. One of its consequences is the existence of a common hypercyclic vector for $(B_{w(\lambda_1)} \times \cdots \times B_{w(\lambda_d)})_{\lambda \in \Lambda}$ in the case $w_1(a) \cdots w_n(a) = \exp(an^\alpha)$ whenever $\alpha < 1/d$. However, Theorem 5.9 implies that, in the same case $w_1(a) \cdots w_n(a) = \exp(an^\alpha)$, the set $\bigcap_{\lambda \in \Lambda} (B_{w(\lambda_1)} \times \cdots \times B_{w(\lambda_d)})$ is empty whenever $\alpha > 1/d$. All in all, the only case for which we don't have an answer is $\alpha = 1/d$.

Still in Article C, we have obtained a characterization of products of weighted backward shifts admitting a common hypercyclic vector. This result answer the open question from [6] by showing that $\bigcap_{s \in [2,3]} HC(sB \times (4-s)B) \neq \emptyset$. Also in the next section, we will discuss an algebraic version of this result.

5.2.1 Products of weighted backward shifts admitting a common hypercyclic algebra for the coordinatewise product

The characterization established by Theorem 4.1 is a generalization of the hypercyclic criterion [20, Theorem 4.8(a)] for the context of algebras with the coordinatewise product. One could try to do the same for some existing characterization of common hypercyclicity, but there is no such result in the literature. We have then obtained the first characterization of common hypercyclicity for weighted shifts in Article C, Theorem 2.7, and we have brought this result to the context of algebras with the coordinatewise product in Article D under the following form.

Theorem 5.11 *Let X be a Fréchet sequence algebra for the coordinatewise product with a continuous norm and admitting (e_n) as an unconditional basis. Let $I \subset \mathbb{R}$ be a nonempty interval, let $d \geq 1$ and let $\Lambda \subset I^d$ be σ -compact. Let $(B_{w(a)})_{a \in I}$ be a family of weighted shifts on X and assume that $a \in I \mapsto w_n(a)$ is nondecreasing with $\inf_{n \in \mathbb{N}, a \in I} w_n(a) > 0$. Assume also that, there exist $F : \mathbb{N} \rightarrow \mathbb{R}_+$ and $c, C > 0$ such that, for all $n \geq 1$, denoting by $f_n(a) = \sum_{k=1}^n \log(w_k(a))$,*

$$\forall (a, b) \in I^2, cF(n)|a - b| \leq |f_n(a) - f_n(b)| \leq CF(n)|b - a|$$

$$\forall (a, b) \in I^2, \frac{w_n(a)}{w_n(b)} \geq c.$$

Then the following assertions are equivalent:

- (a) $(B_{w(\lambda(1))} \times \cdots \times B_{w(\lambda(d))})_{\lambda \in \Lambda}$ admits a common hypercyclic algebra in X ;
- (b) For all $m \geq 1$, there exist $u \in X^d$ such that u^m is a common hypercyclic vector for $(B_{w(\lambda(1))} \times \cdots \times B_{w(\lambda(d))})_{\lambda \in \Lambda}$;
- (c) For all $m \in \mathbb{N}$, $\tau > 0$, for all $N \geq 1$, for all $\varepsilon > 0$, for all $K \subset \Lambda$ compact, there exists $N \leq n_1 < n_1 + N \leq n_2 < \cdots < n_{q-1} + N \leq n_q$ and $(\lambda_k)_{k=1, \dots, q} \in I^d$ such that

$$(i) K \subset \bigcup_{k=1}^q \prod_{l=1}^d \left[\lambda_k(l) - \frac{\tau}{F(n_k)}, \lambda_k(l) \right]$$

(ii) For all $i = 1, \dots, d$,

$$\left\| \sum_{k=1}^q \frac{1}{[w_1(\lambda_k(i)) \cdots w_{n_k}(\lambda_k(i))]^{1/m}} e_{n_k} \right\| < \varepsilon.$$

(iii) For all $k = 1, \dots, q - 1$, for all $i = 1, \dots, d$ and for all $l = 0, \dots, N$,

$$\left\| \sum_{j=k+1}^q \frac{w_{n_j - n_k + l + 1}(\lambda_k(i)) \cdots w_{n_j + l}(\lambda_k(i))}{w_{l+1}(\lambda_j(i)) \cdots w_{n_j + l}(\lambda_j(i))} e_{n_j - n_k + l} \right\| < \varepsilon.$$

The criterion for vectors is obtained from the conditions above by considering only the case $m = 1$. The rather technique hypothesis of Theorem 5.11 become much simpler when we restrict ourselves to products of Rolewicz operators $(\lambda_1 B \times \cdots \times \lambda_d B)_{\lambda \in (1, +\infty)^d}$ on $\ell_p(\mathbb{N})$, $1 \leq p < +\infty$, or $c_0(\mathbb{N})$. In order to simplify the hypothesis, let us look at this family under the form $(e^{\lambda_1} B \times \cdots \times e^{\lambda_d} B)_{\lambda \in (0, +\infty)^d}$.

Theorem 5.12 *Let $d \geq 1$, $\Lambda \subset (0, +\infty)^d$ and $X = \ell_p(\mathbb{N})$, $p \in [1, +\infty)$, or $X = c_0(\mathbb{N})$. The following assertions are equivalent.*

1. $(e^{\lambda_1} B \times \cdots \times e^{\lambda_d} B)_{\lambda \in (1, +\infty)^d}$ has a common hypercyclic algebra.
2. For all $N \geq 1$ and $\tau, \varepsilon > 0$, for all $K \subset \Lambda$ compact, there exist $N \leq n_1 < n_1 + N \leq n_2 < \cdots < n_{q-1} + N \leq n_q$ as well as $(\lambda_k)_{k=1, \dots, q} \in (0, +\infty)^d$ such that

$$(i) K \subset \bigcup_{k=1}^q \prod_{l=1}^d \left[\lambda_k(l) - \frac{\tau}{n_k}, \lambda_k(l) \right],$$

(ii) for each $k = 1, \dots, q$ and $l = 1, \dots, d$,

$$\lambda_{k+1}(l)n_{k+1} - \lambda_k(l)n_k \geq N.$$

We note here that there is no more $m \geq 1$ in this last statement. In fact, the existence of a common hypercyclic vector for the coordinatewise product is equivalent to the existence of at least one common hypercyclic vector. Furthermore, this remains true for the more general case $w_1(a) \cdots w_n(a) = \exp(\lambda n^\alpha)$, as shown in Proposition 2.9 of Article D. One interesting consequence of this result, as shown in Theorem 1.3 of Article C, is that for any Lipschitz curve $\Lambda \subset (1, +\infty)^d$, the family $(\lambda_1 B \times \cdots \times \lambda_d B)_{\lambda \in \Lambda}$ admits a common hypercyclic algebra for the coordinatewise product.

Other results from Article C brought to the context of algebras in Article D are consequence of Theorem 5.10 (see Article C, Section 4.3). An unified version is presented under the following form.

Theorem 5.13 *Let $\alpha \in (0, 1/d)$ and let $I \subset (0, +\infty)$ be σ -compact. Suppose that, for all $m_0 \in \mathbb{N}$ and all $I_0 \subset I$ compact, there exist $C_1, C_2 > 0$, $\beta > \alpha d$, $F : \mathbb{N} \rightarrow (0, +\infty)$ with $F(n) \leq C_1 n^\alpha$ and $N_0, M_0 > 0$ such that, for all $n \geq N_0$,*

(i) $a \in I_0 \mapsto \sum_{i=1}^n \log(w_i(a))$ is $F(n)$ -Lipschitz;

(ii) for all $a \in I_0$, $\inf_{a \in I_0} w_1(a) \cdots w_k(a) \rightarrow +\infty$ as $k \rightarrow +\infty$;

(iii) for all $a \in I_0$ and $k \geq N_0$,

$$\left\| \frac{\exp\left(C_2 \frac{F(n+k)}{(n+k)^\alpha} k^\alpha\right)}{w_1(a) \cdots w_k(a)} e_k \right\| \leq \frac{M_0}{k^\beta}, \quad (9)$$

$$\left\| \frac{1}{[w_1(a) \cdots w_k(a)]^{1/m_0}} e_k \right\| \leq \frac{M_0}{k^\beta}. \quad (10)$$

Then $(B_{w(\lambda_1)} \times \cdots \times B_{w(\lambda_d)})_{\lambda \in I^d}$ admits a common hypercyclic algebra on X^d for the coordinatewise product.

This result applies to products of multiples of the Rolewicz operators as well as for the family $(B_{w(\lambda_1)} \times \cdots \times B_{w(\lambda_d)})$ induced by the weighs $w_n(a) = 1 + \frac{a}{n^{1-\alpha}}$, $\lambda > 0$, for any $\alpha \in [0, 1/d)$, and acting on $\ell_p(\mathbb{N})$, $1 \leq p < +\infty$, or $c_0(\mathbb{N})$.

5.3 A common hypercyclic algebra for the Cauchy product for $(B_{w(\lambda_1)} \times \cdots \times B_{w(\lambda_d)})_{\lambda \in (0, +\infty)^d}$ where $w_n(a) = 1 + \frac{a}{n}$

In this last part of Section 5, we will discuss the answer given in Article D to the question left open in Article B on the existence of a common hypercyclic algebra for the family $(B_{w(\lambda)})_{\lambda > 0}$ induced by $w_n(\lambda) = 1 + \frac{\lambda}{n}$ and acting on $\ell_1(\mathbb{N})$ with the Cauchy product. In order to simplify

the discussion, when we say that a family of admissible weights $(w(a))_{a \in I}$ satisfy the $F(n)$ -Lipschitz property for some function $F : \mathbb{N} \rightarrow \mathbb{R}_+$, we mean that each function $a \in I \mapsto f_n(a) = \sum_{k=1}^n \log(w_n(a))$ is $F(n)$ -Lipschitz, $n \in \mathbb{N}$.

After having obtained many results on common hypercyclic algebras in higher dimensions, established mainly for the case n^α -Lipschitz, we have thought that we could apply the results developed in sections 4.4 and 4.5 of Article C to get an algebra in d -dimensions with the Cauchy product for the family of shifts induced by $w_n(a) = 1 + \frac{a}{n}$. However, we have faced technical complications related to the number of terms in the list n_1, \dots, n_q . These difficulties appear even in the one-dimensional case. The sequence of powers $(n_k)_k$ considered in the proof of Theorem 5.8 is linear on i and this linearity is hidden in condition (b). It seems that $n_i = iN$ works well in the case n -Lipschitz as the partition have parts of size $\frac{\tau}{iN}$, but that doesn't work for the case $\log(n)$ -Lipschitz if we maintain the old size of the parts. What we have done in Article D consists of choosing $(n_i)_i$ specially for the weights $w_n(a) = 1 + \frac{a}{n}$ by taking into account its $\log(n)$ -Lipschitz property. Even then, the dyadic covering used in Article C doesn't seem to be adapted to $w_n(a) = 1 + \frac{a}{n}$ on $\ell_1(\mathbb{N})^d$. In fact, it contains too much elements. Once more, the fact that this family satisfy the $\log(n)$ -Lipschitz condition (rather than n^α -Lipschitz) allows us to construct a covering adapted to this case. This led us to the following theorem, proved in Section 3 of Article D.

Theorem 5.14 *Let $(w(\lambda))_{\lambda > 0}$ be the family of weights defined by $w_n(a) = 1 + \frac{a}{n}, n \geq 1$. Then, $(B_{w(\lambda_1)} \times \dots \times B_{w(\lambda_d)})_{\lambda \in (0, +\infty)^d}$ admits a common hypercyclic algebra in $\ell_1(\mathbb{N})$ with the Cauchy product.*

6 Upper frequently hypercyclic algebras

Another concept related to hypercyclicity which was explored in this work is that of upper frequent hypercyclicity, introduced by S. Shkarin in [24] as a weaker version of frequent hypercyclicity (concept introduced by F. Bayart and S. Grivaux in [5]). Let us see the basic definitions.

The *upper* $\bar{d}(A)$ and *lower* $\underline{d}(A)$ densities of a subset $A \subset \mathbb{N}$ are defined by

$$\bar{d}(A) = \limsup_{n \rightarrow \infty} \frac{\#\{m \in A : m \leq n\}}{n} \quad \text{and} \quad \underline{d}(A) = \liminf_{n \rightarrow \infty} \frac{\#\{m \in A : m \leq n\}}{n},$$

where $\#E$ denotes the cardinal of E . Let X be an F -space and T a bounded linear operator on X . We say that $x \in X$ is a *frequently hypercyclic vector* for T when, for all $V \subset X$ open and nonempty, the lower density of the set $\{n \in \mathbb{N} : T^n(x) \in V\}$ is positive. In this case we say that T is a *frequently hypercyclic operator* and we denote by $FHC(T)$ the set of frequent hypercyclic vectors of T . If, instead, the upper density of $\{n \in \mathbb{N} : T^n(x) \in V\}$ is positive for all $V \subset X$ open and nonempty, we say that x is an *upper frequently hypercyclic vector* of T , which is then called an *upper frequently hypercyclic operator*. The set of all vectors satisfying this property is denoted by $UFHC(T)$. Although one can think that these two properties are slightly different, in practice they are quite distinct. In fact, as Bonnilla and Grosse-Erdmann have remarked in [16], we can get upper frequently hypercyclic vectors using the Baire theorem but the set of

frequently hypercyclic vectors is always meagre. We were able to adapt the previous techniques to get algebras of upper frequently hypercyclic vectors.

The first step, as always, consists of finding a Baire argument for upper frequently hypercyclic algebras. We have found the following formulation.

Proposition 6.1 *Let T be a continuous operator on an F -algebra X satisfying the following condition: for each $I \in \mathcal{P}_f(\mathbb{N}) \setminus \{\emptyset\}$, there exists $m_0 \in I$ such that, for each non-empty open subset V of X and each neighborhood W of the origin, there is $\delta > 0$ such that for each non-empty open subset U and each $N_0 \in \mathbb{N}$, there is $u \in U$ and $N \geq N_0$ satisfying*

$$\frac{\#\{p \leq N : T^p(u^m) \in W \text{ for } m \in I \setminus \{m_0\} \text{ and } T^p(u^{m_0}) \in V\}}{N+1} > \delta.$$

Then T admits an upper frequently hypercyclic algebra.

This same criterion will be applied for the two products: coordinatewise and Cauchy. Just as before, for the first we choose $m_0 = \min(I)$ and for the second we take $m_0 = \max(I)$.

6.1 Coordinatewise product

In the present case, we have found the following consequence of the previous proposition.

Theorem 6.2 *Let X be a Fréchet sequence algebra for the coordinatewise product and with a continuous norm. Assume that (e_n) spans a dense subspace of X . Let also B_w be a bounded weighted shift on X such that, for all $m \geq 1$, $\sum_{n \geq 1} (w_1 \cdots w_n)^{-1/m} e_n$ converges unconditionally. Then B_w admits an upper frequently hypercyclic algebra.*

Among its applications, we get quite easily upper frequently hypercyclic algebras for the classical operators λB , $\lambda > 1$, over $\ell_p(\mathbb{N})$, $1 \leq p < \infty$, or $c_0(\mathbb{N})$ with the coordinatewise product, as well as λD , $\lambda > 0$, on $H(\mathbb{C})$ with the Hadamard product. We found also that B_w , where $w_n = 1 + \frac{\lambda}{n}$, admits an upper frequently hypercyclic algebra on $c_0(\mathbb{N})$ for any $\lambda > 0$. However, on $\ell_p(\mathbb{N})$ the hypothesis are not satisfied for this weight since the series diverges when $p\lambda < m$. The problem of finding an upper frequently hypercyclic algebra in this case remains open.

6.2 Cauchy product

For the convolution product, we have found the following consequence of Proposition 6.1.

Theorem 6.3 *Let X be a regular Fréchet sequence algebra for the Cauchy product and let (w_n) be an admissible weight sequence. Assume that*

- (a) $\sum_{n \geq 1} (w_1 \cdots w_n)^{-1} e_n$ converges unconditionally.
- (b) for all $m \geq 2$, there exists $c \in (0, 1)$ such that

$$\lim_{\sigma \rightarrow +\infty} \sup_{z \in c_0 \cap B_{\ell_\infty}} \left\| \sum_{n \geq c\sigma} \frac{z_n (w_1 \cdots w_{m\sigma})^{(m-1)/m}}{w_1 \cdots w_{(m-1)\sigma+n}} e_n \right\| = 0.$$

Then B_w admits an upper frequently hypercyclic algebra.

It is remarkable the similarity between condition (b) above and condition (b) of Theorem 5.8 for common hypercyclic algebras, what corroborates the proximity of the methods. As for the coordinatewise product, Theorem 6.3 applies to tall classic operators $\lambda B, \lambda > 1$, over $\ell_1(\mathbb{N})$ as well as $\lambda D, \lambda > 0$, on $H(\mathbb{C})$, but not to B_w where $w_n = 1 + \frac{\lambda}{n}$. Again, the problem of finding an upper frequently hypercyclic algebra on $\ell_1(\mathbb{N})$ for this weighted shift remains open.

7 Disjoint hypercyclic algebras

The next topic explored is that of disjoint hypercyclicity. Defined for the first time independently by J. Bès and A. Peris [13] and L. Bernal-González [8], it is a sort of diagonal hypercyclicity. Let X be a Fréchet sequence space and let T_1 and T_2 be bounded linear operators on X . We say that T_1 and T_2 are *disjoint hypercyclic* when there exist $x \in X$ (called *disjoint hypercyclic vector* for T_1 and T_2) such that the sequence $(T_1^n(x), T_2^n(x))_{n \geq 0}$ is dense in X^2 . Many kinds of operators are investigated in the two initial articles of the theory, such as composition operators, differential operators and, what is particularly interesting for us, weighted shift operators. For simplicity, we decided to study the case of two operators, but the definition of disjoint hypercyclicity (and some of our results) works just fine for many operators. Moreover, we have concentrated ourselves on the case where T_1 and T_2 are power of two weighted shifts $T_1 = B_w^{r_1}$ and $T_2 = B_w^{r_2}$, say $0 < r_1 \leq r_2$. As observed in [13] and later confirmed in [12], the case $r_1 \neq r_2$ behaves fairly differently than $r_1 = r_2$. Here, we will discuss the case $r_1 < r_2$ only (although the case $r_1 = r_2$ is also explored in Article B).

We aim to bring to the context of algebras some known results. The definition of disjoint hypercyclic algebra is as natural as possible: when X is an F -algebra, we say that two bounded operators T_1 and T_2 on X admit a *disjoint hypercyclic algebra* when there exists a subalgebra A of X such that any nonzero vector $x \in A$ is disjoint hypercyclic for T_1 and T_2 . The first step, as before, consists of finding a Baire argument leading to disjoint hypercyclic algebras. We have found the following formulation.

Proposition 7.1 *Let T_1 and T_2 be two operators acting on the same F -algebra X . Assume that for each $I \in \mathcal{P}_f(\mathbb{N}) \setminus \{\emptyset\}$, there exists $m_0 \in I$ such that, for all U, V_1 and V_2 nonempty open subsets of X , for all W neighborhood of zero, there exist $u \in U$ and $N \in \mathbb{N}$ such that*

$$\begin{cases} T_1^N(u^n) \in W, T_2^N(u^n) \in W \text{ for all } n \in I, n \neq m_0 \\ T_1^N(u^{m_0}) \in V_1, T_2^N(u^{m_0}) \in V_2. \end{cases}$$

Then T_1 and T_2 admit a disjoint hypercyclic algebra.

This result will be applied for both coordinatewise (with $m_0 = \min(I)$) and Cauchy ($m_0 = \max(I)$) products. For the latter, a separating term is still present.

7.1 Coordinatewise product

For the coordinatewise product, by making use of some ideas similar to those applied to Theorem 4.1, we have characterized the powers B_w and B_w^2 of weighted shifts admitting a disjoint hypercyclic algebra.

Theorem 7.2 *Let X be a Fréchet sequence algebra for the coordinatewise product with a continuous norm and such that (e_n) is a Schauder basis. Let $B_1 = B_{w^{(1)}}$ and $B_2 = B_{w^{(2)}}$ be two bounded weighted shifts on X . The following assertions are equivalent.*

(i) B_1 and B_2^2 support a disjoint hypercyclic algebra.

(ii) There exists a sequence (n_k) such that, for all $l \geq 0$, for all $\gamma > 0$,

$$\begin{aligned} (w_{l+1}^{(1)} \cdots w_{n_k+l}^{(1)})^{-\gamma} e_{n_k+l} &\xrightarrow{k \rightarrow +\infty} 0, \\ (w_{l+1}^{(2)} \cdots w_{2n_k+l}^{(2)})^{-\gamma} e_{2n_k+l} &\xrightarrow{k \rightarrow +\infty} 0, \\ \frac{w_{n_k+l+1}^{(1)} \cdots w_{2n_k+l}^{(1)}}{w_{l+1}^{(2)} \cdots w_{2n_k+l}^{(2)}} e_{n_k+l} &\xrightarrow{k \rightarrow +\infty} 0. \end{aligned}$$

On spaces where $(e_n)_n$ is bounded, we can remove the terms e_{n_k+l} and $-\gamma$. The remaining conditions are sufficient for disjoint hypercyclicity (see [13]). This way, on spaces like $\ell_p(\mathbb{N}), 1 \leq p < \infty$, or $c_0(\mathbb{N})$, for example, the existence of a disjoint hypercyclic algebra is equivalent to the existence of a single disjoint hypercyclic vector. The result above also applies to $B_1 = B_{w^{(\lambda)}}$ and $B_2 = B_{w^{(\mu)}}$, with $w_n(a) := 1 + \frac{a}{n}$ and $\lambda, \mu > 0$, acting either on $\ell_p(\mathbb{N}), 1 \leq p < +\infty$, or $c_0(\mathbb{N})$ under the coordinatewise product. We can also apply it to get a disjoint hypercyclic algebra in $H(\mathbb{C})^2$ with the Hadamard product for any couple $\lambda D, \mu D^2$, $\lambda, \mu > 0$.

7.2 Cauchy product

As we have discussed before, the results for the Cauchy are more difficult to obtain. In our studies we didn't explored the general case of weighted shifts so, even for the example $B_1 = B_{w^{(\lambda)}}$ and $B_2 = B_{w^{(\mu)}}$, where $w_n(a) := 1 + \frac{a}{n}$ and $\lambda, \mu > 0$, we don't know if there exists a disjoint hypercyclic algebra in $\ell(\mathbb{N})$ with the Cauchy product. In the particular case of multiple of shifts $\lambda B, \mu B^2$, it is shown in [13] that this couple has a disjoint hypercyclic vector if and only if $1 < \lambda < \mu$. We managed to show that, under the same condition, this couple has also a disjoint hypercyclic algebra for the Cauchy product on $\ell_1(\mathbb{N})$. A similar result was also found for $\lambda D, \mu D^2$ on $H(\mathbb{C})$ for the convolution product, now with no constraints on $\lambda, \mu > 0$. Our discoveries are summarized in the following.

Theorem 7.3 *Let $\lambda, \mu > 0$.*

a) λB and μB^2 support a disjoint hypercyclic algebra on $\ell_1(\mathbb{N})$ (endowed with the convolution product) if and only if $1 < \lambda < \mu$.

b) λD and μD^2 support a disjoint hypercyclic algebra on $H(\mathbb{C})$.

An interesting fact in the proof of the item (1) of the theorem above is that the position of the separating term εe_σ has change according to the case within $1 < \lambda < \mu^{1/2}$: if $1 < \lambda < \mu^{1/2}$, we place εe_σ in the end of the definition of the candidate u just as before; and if $\mu^{1/2} \leq \lambda < \mu$, we look for a candidate of the form $u = x + \varepsilon e_\sigma + z$. Roughly speaking, this separation comes from $\mu B^2 = (\mu^{1/2} B)^2$ and the different emplacements have allowed us to successfully play with the supports.

8 Closed hypercyclic algebras

The last subject studied is the (non)existence of closed hypercyclic algebras (see Article A, Section 6.1). This question is the natural extension of the search for hypercyclic subspaces, namely closed infinite dimensional subspaces within $HC(T) \cup \{0\}$ (for some hypercyclic operator T). It is not clear how to adapt our methods in order to produce a closed algebra. Yet, we have found negative answers in some important cases.

The first result answer in the negative a question asked by Shkarin in [25] on the existence of a closed hypercyclic algebra for the operator of complex derivation D acting on $H(\mathbb{C})$. Not only this is not the case but, more generally, no convolution operator $P(D)$ induced by a polynomial P have a closed hypercyclic algebra in $H(\mathbb{C})$. However, the problem of finding an entire function ϕ such that $\phi(D)$ admits a closed hypercyclic algebra in $H(\mathbb{C})$ is still open.

When it comes to our favorite operators, that is to say weighted backward shifts acting on Fréchet sequence algebras, we have shown that the answer is still negative when we consider the coordinatewise product on $\ell_p(\mathbb{N})$, $1 \leq p < \infty$, on $c_0(\mathbb{N})$ or even on the more permissive space ω . In these cases, no weighted backward shift operator has a closed hypercyclic algebra. The problem remains open on $\ell_1(\mathbb{N})$ with the Cauchy product.

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Part II

Introduction en français

L'étude de l'hypercyclicité est l'un des plus importants thèmes en Dynamique Linéaire, alors que ce concept est l'ingrédient principal dans la définition du chaos de Devaney de 1989 (voir [44]). Si X est un espace topologique et $T : X \rightarrow X$ est un opérateur linéaire continu dans X , on dit qu'un vecteur $x \in X$ est *hypercyclique* pour le *système dynamique* (X, T) quand son *orbite sous T* , définie par $\text{orb}(x, T) = \{x, T(x), T^2(x), \dots\}$, est un sous-ensemble dense de X . On note $HC(T)$ l'ensemble des vecteurs hypercycliques de T . Lorsque $HC(T) \neq \emptyset$, on dit que (X, T) est un *système dynamique hypercyclique* ou que T est un *opérateur hypercyclique* sur X . Autrement dit, la suite d'itérées de T sur un seul point x se rapproche de chaque point de l'espace tant qu'on veut. Toutefois, le premier exemple d'un système satisfaisant cette propriété « étrange » est beaucoup plus vieux que Devaney. En 1929, Birkhoff [41] a montré que l'opérateur de translation par 1 sur l'espace des fonctions entières, c'est-à-dire, $\tau_1 : H(\mathbb{C}) \rightarrow H(\mathbb{C})$ donné par $\tau_1 f(x) = f(x + 1)$ pour tout $f \in H(\mathbb{C})$, admet une fonction hypercyclique $g \in H(\mathbb{C})$. En 1952, MacLane [48] a prouvé l'hypercyclicité de l'opérateur de dérivation d'une fonction entière $D : H(\mathbb{C}) \rightarrow H(\mathbb{C})$, donné par $D(f) = f'$, $f \in H(\mathbb{C})$. En 1969, Rolewicz [49] a fait pareil pour tous les multiples $\lambda B : \ell_2(\mathbb{N}) \rightarrow \ell_2(\mathbb{N})$, $\lambda > 1$, de l'opérateur de décalage à gauche B défini par $B(x_0, x_1, x_2, \dots) = (x_1, x_2, x_3, \dots)$, $(x_n)_{n \geq 0} \in \ell_2(\mathbb{N})$. Aujourd'hui ces trois opérateurs sont connus respectivement par les noms de ces mathématiciens. Néanmoins, le domaine de recherche aujourd'hui appelée Dynamique Linéaire a commencé avec la thèse de Kitai [47], où il a été fait la première étude systématique des orbites d'opérateurs bornées sur des espaces linéaires complets.

Cette thèse est une contribution à ce domaine de recherche. On va décrire maintenant ses principaux résultats.

9 Les algèbres hypercycliques

Il est connu que, pour tout opérateur hypercyclique T , $HC(T) \cup \{0\}$ contient un sous-espace linéaire dense. Alors, en général l'on peut trouver une structure de vecteurs hypercycliques « compatible » avec l'espace. Ce qu'on appelle *sous-espace hypercyclique* vient d'une question plus intéressante (qui concerne normalement les espaces de Banach) : dans le cas où T est hypercyclique, existe-t-il un sous-espace fermé de dimension infinie dans $HC(T) \cup \{0\}$? Encore une fois, on cherche des structures vectorielles intéressantes composées uniquement par des vecteurs hypercycliques (sauf 0). Maintenant on se pose la question: et si l'espace topologique X admet une structure d'algèbre ? Existe-t-il une sous-algèbre non-triviale de X dans $HC(T) \cup \{0\}$? La recherche d'une telle structure, appelée *algèbre hypercyclique* pour T , motive ce travail.

Le contexte le plus générique que nous avons considéré est celui d'une F -algèbre séparable et commutative, c'est-à-dire, un espace de Fréchet séparable X , induit par une suite croissante de semi-normes $(\|\cdot\|_\rho)_{\rho \in \mathbb{N}}$, où il y est défini un produit commutatif $\cdot : X \times X \rightarrow X$ satisfaisant, pour tout $x, y \in X$ et $\rho \in \mathbb{N}$,

$$\|x \cdot y\|_\rho \leq \|x\|_\rho \times \|y\|_\rho.$$

De plus, quasiment tous nos résultats portent sur les F -algèbres de suites, i.e., où X est un sous-espace de l'espace de toutes les suites $\omega = \mathbb{C}^{\mathbb{N}}$. Ici, on considère deux produits classiques:

le produit coordonnée-par-coordonnée (produit CPC) et le produit de Cauchy (ou produit de convolution). À titre d'exemple, les espaces $\ell_p(\mathbb{N})$, $1 \leq p < +\infty$ ainsi que $c_0(\mathbb{N})$ deviennent des algèbres de Fréchet avec le produit CPC et $\ell_1(\mathbb{N})$ devient une algèbre Fréchet avec le produit de Cauchy. On peut aussi regarder $H(\mathbb{C})$ comme un espace de séquence, ici les deux produits rendent cet espace une algèbre de Fréchet. Le produit CPC sur $H(\mathbb{C})$ est aussi appelé produit d'Hadamard. Nous discuterons chacun de ces produits plus tard.

Le premier résultat sur les algèbres hypercycliques date de 2007, où Aron et al. [28] montrent qu'aucun opérateur de Birkhoff n'admet une algèbre hypercyclique pour le produit de Cauchy. Deux ans plus tard (indépendamment et par des méthodes différentes), Shkarin [51] ainsi que Bayart et Matheron [33] ont montré que l'opérateur de MacLane admet une algèbre hypercyclique pour le produit de Cauchy. L'argument utilisé par ces derniers se montrera fructueux dans les années suivantes avec les travaux de Bès, Conejero et Papathanasiou [35, 36].

10 La méthode de Bayart et Matheron

Contrairement à la méthode constructive employée par Shkarin, la méthode de Bayart et Matheron est basée sur une adaptation (voir [33, Remark 2.28]) du théorème de transitivité de Birkhoff pour les algèbres.

Le concept de transitivité topologique, inventé en 1920 par Birkhoff [40], est l'un des plus importants outils dans toute la théorie. On dit qu'un système dynamique (X, T) est *topologiquement transitif* si, pour tous ouverts non vides U, V dans X , il existe $n \in \mathbb{N}$ tel que $T^n(U) \cap V \neq \emptyset$. L'importance de ce concept vient du fait que, sur les espaces métriques complets et séparables, transitivité topologique et hypercyclicité sont équivalentes. Pour mieux lier les points, on va énoncer ce théorème de la façon suivante.

Théorème 10.1 (de transitivité de Birkhoff) *Soit T un opérateur continu dans un F -espace séparable X . Supposons que, pour toute paire (U, V) d'ouverts non vides de X , il existe $u \in U$ et $N \in \mathbb{N}$ tels que $T^N(u) \in V$. Alors $HC(T)$ est un sous-ensemble comeagre de X .*

On observe que la conclusion que $HC(T)$ est G_δ dense vient de l'application du théorème de Baire. L'idée de cet énoncé est que n'importe quelle paire d'ouverts peuvent être connectés par un certain itéré T^N de T . Ainsi, pour démontrer que T est hypercyclique, on cherche pour toute paire d'ouverts (U, V) à trouver $u \in U$ et un itéré T^N de T tel que $T^N(u)$ appartienne à V . Dans le critère [33, Remark 2.28], les auteurs ont modifié les conditions pour qu'on cherche à trouver $u \in U$ tel qu'on puisse envoyer une combinaison arbitraire $\alpha_1 u + \alpha_2 u^2 + \dots + \alpha_m u^m$ dans V par un itéré T^N de T . Pour cela, on fait en sorte que T^N envoie toutes les puissances u, u^2, \dots, u^{m-1} vers 0 et la « puissance principale » $\alpha_m u^m$ dans V . Ils ont donc obtenu la formulation suivante.

Théorème 10.2 (Bayart et Matheron [33]) *Soit T un opérateur continu dans une F -algèbre séparable. Supposons que, pour tout $m \geq 1$, pour toute paire (U, V) d'ouverts non vides de X et pour tout voisinage W de 0, il existe $u \in U$ et $N \in \mathbb{N}$ tels que $T^N(u^n) \in W$ pour tous $n < m$ et $T^N(u^m) \in V$. Alors l'ensemble de points qui engendrent une algèbre hypercyclique pour T est G_δ dense dans X .*

Pour réussir à appliquer ce critère pour $D : H(\mathbb{C}) \rightarrow H(\mathbb{C})$, les auteurs profitent du « comportement de décalage » que D apporte. En utilisant la même approche, Bès, Conejero et Papatthanasiou ont montré que $P(D)$ admet une algèbre hypercyclique pour n'importe quel polynôme P qui satisfait $P(0) = 0$ (voir [35]), et avec une approche différente (en utilisant les valeurs propres de ces opérateurs), les mêmes auteurs ont trouvé une large classe d'opérateurs de convolution admettant une algèbre hypercyclique (voir [36]). Tandis que les approches changent, tous ces résultats reposent sur une application du Théorème 10.2. On discutera les opérateurs de convolution ainsi qu'une généralisation de ce résultat clé, ce qui va rendre possibles d'autres nombreuses applications.

11 Les opérateurs de convolution et les critères d'existence d'une algèbre hypercyclique

Par *opérateur de convolution* on désigne un opérateur $\phi(D) : H(\mathbb{C}) \rightarrow H(\mathbb{C})$ induit par une fonction entière $\phi \in H(\mathbb{C})$ de *type exponentiel*, où $D : H(\mathbb{C}) \rightarrow H(\mathbb{C})$ est l'opérateur de dérivation complexe. Plus précisément, être de type exponentiel signifie qu'il existe deux constantes $A, B > 0$ telles que $\phi(z) = \sum_{n=0}^{\infty} a_n z^n \in H(\mathbb{C})$ satisfait $|\phi(z)| \leq A \exp(B|z|)$ pour tout $z \in \mathbb{C}$. L'opérateur $\phi(D)$ est donc défini par $\phi(D)(f) = \sum_{n=0}^{\infty} a_n f^{(n)}$. Dans cette section, $H(\mathbb{C})$ est muni de son produit naturel, le produit ponctuel de deux fonctions, qui correspond au produit de convolution sur leurs coefficients. Nous avons déjà expliqué que D , ou plus généralement $P(D)$ pour tout polynôme P tel que $P(0) = 0$, admet une algèbre hypercyclique. Dans l'article [36], les auteurs ont montré d'une manière très intéressante que l'opérateur de convolution $\phi(D)$ admet une algèbre hypercyclique sur $H(\mathbb{C})$ dans les cas $\phi(z) = \cos(z)$, $\phi(z) = ze^z$ et $\phi(z) = e^z - a$ avec $0 < a \leq 1$. Évidemment ces fonctions ne sont pas des polynômes. Leur résultat s'applique aussi aux cas $\phi(z) = (a_0 + a_1 z^n)^k$ avec $|a_0| \leq 1$ et $a_1 \neq 0$, qui est un polynôme ne satisfaisant pas $P(0) = 0$. Ces applications viennent du résultat suivant.

Théorème 11.1 (Bès et. al. [36]) *Soit $\phi \in H(\mathbb{C})$ de type exponentiel telle que l'ensemble de niveau $\{z \in \mathbb{C} : |\phi(z)| = 1\}$ contient un arc compact strictement convexe et non trivial Γ_1 satisfaisant*

$$\text{conv}(\Gamma_1 \cup \{0\}) \setminus (\Gamma_1 \cup \{0\}) \subset \phi^{-1}(\mathbb{D}).$$

Alors, l'ensemble des fonctions entières engendrant une algèbre hypercyclique pour l'opérateur de convolution $\phi(D)$ est un sous-ensemble comaigne dans $H(\mathbb{C})$.

Il est remarquable que $|\phi(0)| \leq 1$ dans tous les exemples fournis. Quelques questions naturelles se posent. Au niveau des polynômes, peut-on enlever la condition $P(0) = 0$? Suffirait-il $|P(z)| \leq 1$, ou même $|P(z)| < 1$? Au niveau des fonctions entières, suffirait-il $|\phi(0)| \leq 1$? Ou peut-être $|\phi(0)| < 1$? La méthode employée par ces auteurs, en utilisant les valeurs propres des fonctions pour obtenir le résultat, paraissait suffisamment prometteuse pour qu'on puisse l'appliquer à d'autres fonctions de type exponentiel.

L'argument évolue encore une fois avec l'article [30]. En utilisant les valeurs propres d'une manière différente, F. Bayart caractérise les opérateurs de convolution satisfaisant $|\phi(0)| < 1$ qui admettent une algèbre hypercyclique, et il établit des conditions suffisantes pour que $\phi(D)$ supporte une telle algèbre dans le cas $|\phi(0)| = 1$.

Théorème 11.2 (Bayart [30]) *Soit $\phi \in H(\mathbb{C})$ de type exponentiel.*

(1) *Supposons que $|\phi(0)| < 1$. Les assertions suivantes sont équivalentes:*

- (i) *$\phi(D)$ admet une algèbre hypercyclique.*
- (ii) *ϕ n'est pas multiple d'une fonction exponentielle.*

(2) *Supposons que $|\phi(0)| = 1$ et ϕ a un taux de croissance sous-exponentiel. Si $\phi'(0) \neq 0$ ou si ϕ est d'ordre plus petit que $1/2$, l'opérateur $\phi(D)$ admet une algèbre hypercyclique.*

Par rapport au fait que ϕ n'est pas multiple d'une exponentielle, on ne pourrait pas espérer mieux. En effet, les opérateurs de convolution induits par les fonctions de la forme $\phi(x) = \lambda \exp(ax)$ correspondent aux multiples $\lambda \tau_a$ de l'opérateur de Birkhoff ! Ce que montre le Théorème 11.2 c'est que, dans le cas $|\phi(0)| < 1$, ce sont les seuls opérateurs qui n'ont pas d'algèbre hypercyclique. Postérieurement, Bès, Ernst et Prieto [37] ont montré que, dans le cas $|\phi(0)| = 1$, les conditions sur la dérivée de ϕ ou sur son ordre ne sont pas nécessaires pour l'existence d'une algèbre hypercyclique.

Théorème 11.3 (Bès, Ernst et Prieto [37]) *Soit $\phi \in H(\mathbb{C})$ de type exponentiel tel que $|\phi(0)| = 1$. Si ϕ a un taux de croissance sous-exponentiel, alors l'opérateur $\phi(D)$ admet une algèbre hypercyclique.*

Ils ont aussi établi d'autres critères pour l'existence, ou même l'absence, d'une algèbre hypercyclique (pour plus de détails, voir l'article).

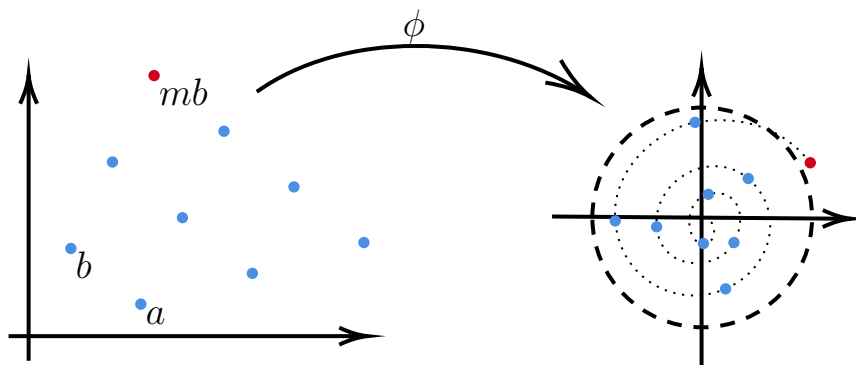
On se demande ce que se passe dans le cas $|\phi(0)| > 1$. Est-il naturellement pathogène ? La condition $|\phi(0)| > 1$ pourrait-elle être utilisée en quelque sorte pour produire une algèbre hypercyclique ? Encore dans [30], on trouve des pistes pour répondre à cette question.

La première partie du Théorème 11.2 vient d'un critère plus général qui élargit considérablement la classe des opérateurs de convolution avec une algèbre hypercyclique. En effet, il peut être appliqué par exemple aux cas $\phi(z) = \cos(z)$ et $\phi(z) = e^z - 2$, qui satisfont $|\phi(0)| = 1$ (voir [30, Exemples 2.6 et 2.7]). Cependant, il s'applique aussi au cas $\phi(z) = 2e^{-z} + \sin(z)$ qui satisfait $|\phi(z)| > 1$, ce qui est très surprenant (voir [30, Exemple 2.8]). On l'énonce ici par souci d'exhaustivité.

Théorème 11.4 (Bayart [30]) *Soit $\phi \in H(\mathbb{C})$ de type exponentiel. Supposons que:*

- (1) *$\phi(D)$ n'est pas multiple d'une fonction exponentielle;*
- (2) *pour tout $m \in \mathbb{N}$, il existe $a, b \in \mathbb{C}$ tels que $|\phi(mb)| > 1$ et, pour tous $n \in \{1, \dots, m\}$ et $d \in \{0, \dots, n\}$ avec $(n, d) \neq (m, m)$, $|\phi(nb + (n - d)a)| < 1$.*

Alors $\phi(D)$ admet une algèbre hypercyclique.


 Fig. 2 – Exemple avec $m = 3$.

Ce qui se passe dans la condition (2) est représenté dans la Figure 2. En revenant à la discussion du Théorème 10.2, ces conditions permettent de choisir un candidat $u \in U$ ainsi qu'une puissance N tels qu'on fasse T^N envoyer u^n vers 0 quand $n < m$ (en utilisant $|\phi(nb + (n-d)a)| < 1$) et envoyer la « puissance principale » u^m dans V (en utilisant $|\phi(mb)| > 1$). Le point-clé pour que l'exemple $\phi(z) = 2e^{-z} + \sin(z)$ fonctionne est l'oscillation de cette fonction ce qui fait qu'on peut choisir deux points convenables $a, b \in \mathbb{R}$ de façon à emboîter l'ensemble $\{nb + (n-d)a : n = 1, \dots, m, d = 0, \dots, n, (n, d) \neq (m, m)\}$ dans $\phi^{-1}(\mathbb{D}) \cap \mathbb{R}$ et de sorte que b satisfasse $|\phi(mb)| > 1$.

Maintenant, si l'on regarde le cas $|\phi(0)| < 1$ en vue du Théorème 11.2, on voit que les points suffisamment proches de l'origine auront des valeurs absolues inférieures à 1 et, en même temps, puisque $\overline{\phi(\mathbb{C})} = \mathbb{C}$, il est facile de trouver des points dont la valeur absolue est supérieure à 1. Alors, il faut juste choisir une direction w telle qu'il existe t_0, t_1 satisfaisant

$$\begin{cases} |\phi(tw)| < 1 \text{ si } t \in [0, t_0] \\ |\phi(tw)| > 1 \text{ si } t \in (t_0, t_1] \end{cases}$$

et puis choisir $a = \varepsilon w$ et $b = \frac{t_0 + \varepsilon}{m} w$ avec $\varepsilon > 0$ suffisamment petit. On voit que $mb = (t_0 + \varepsilon)w$, où $(t_0 + \varepsilon) \in (t_0, t_1]$, et $db + (n-d)a = (d\frac{t_0 + \varepsilon}{m} + (n-d)\varepsilon)w$, où

$$0 \leq d\frac{t_0 + \varepsilon}{m} + (n-d)\varepsilon \leq \frac{m-1}{m}(t_0 + \varepsilon) + m\varepsilon \xrightarrow{\varepsilon \rightarrow 0} \frac{m-1}{m}t_0 \in (0, t_0).$$

Que dire de $|\phi(0)| > 1$? Dans ce cas, les points proches de l'origine ont une norme supérieure à 1, d'où l'idée de lier la « puissance principale » à la combinaison $db + (n-d)a$ la plus proche de l'origine. Si l'on demande que $|\phi|$ soit petite dans une direction w , disons $\lim_{t \rightarrow +\infty} |\phi(tw)| = 0$, alors on pourrait essayer de faire que les autres combinaisons soient plus éloignées dans cette direction de sorte que la valeur de $|\phi|$ en ces points soit inférieure à 1. Maintenant, puisque cela se passera dans une même direction $\{tw : t \geq 0\}$, on peut se dire qu'on va choisir $b = t_0 w$ et $a = t_1 w$ avec t_1 beaucoup plus grand que t_0 . Comme cela on sait que, pour tout $m \geq 1$, la combinaison $db + (n-d)a$ plus proche de l'origine est b (obtenue en faisant $n = d = 1$) et toutes les autres combinaisons seront éloignées à cause de a . Autant dire que nous avons besoin d'une version du Théorème 10.2 comme la suivante.

Théorème 11.5 (faux) *Soit T un opérateur continu dans un F -algèbre séparable. Supposons que, pour tout $m \geq 1$, pour toute paire (U, V) d'ouverts non vides de X et pour tout voisinage W*

de 0, il existe $u \in U$ et $N \in \mathbb{N}$ tels que $T^N(u^n) \in W$ pour tous $1 < n \leq m$ et $T^N(u) \in V$. Alors l'ensemble de points qui engendrent une algèbre hypercyclique pour T est G_δ dense dans X .

Mais ce résultat est impossible. En effet, le terme u ne fait pas forcément partie de tous les éléments de l'algèbre $A(u)$ engendré par u . Par exemple $u^2 + 2u^3 \in A(u)$, mais le coefficient de u dans cette combinaison est $\alpha_1 = 0$ et donc T^N ne peut envoyer $\alpha_1 u$ nulle part ! Ce que l'on peut considérer pour chaque élément de $A(u)$ est la présence de termes u^{m_0} et u^{m_1} où m_0 est le plus petit et m_1 est le plus grand. Par exemple, pour l'élément $u^2 + 2u^3$, l'opérateur T^N enverra u^2 vers V et u^3 vers 0. On obtient donc la version (vraie) suivante.

Théorème 11.6 (vrai) *Soit T un opérateur continu dans un F -algèbre séparable. Supposons que, pour tout $1 \leq m_0 \leq m_1$, pour toute paire (U, V) d'ouverts non vides de X et pour tout voisinage W de 0, il existe $u \in U$ et $N \in \mathbb{N}$ tels que $T^N(u^n) \in W$ pour tous $m_0 < n \leq m$ et $T^N(u^{m_0}) \in V$. Alors l'ensemble de points qui engendrent une algèbre hypercyclique pour T est G_δ dense dans X .*

Par conséquent, on a trouvé un critère pour le cas $|\phi(0)| > 1$ et où ϕ n'est pas un multiple de la fonction exponentielle: s'il existe $w \in \mathbb{C}$ tel que $\lim_{t \rightarrow +\infty} |\phi(tw)| = 0$, alors $\phi(D)$ admet une algèbre hypercyclique ! On trouvera d'autres exemples dans le paragraphe 3 de l'Article A.

Le Théorème 11.6 a été la première version du critère général Théorème 2.1 de l'Article A, où on considère un sous-ensemble fini I quelconque de \mathbb{N}^* et on prend comme « puissance principale » un de ces éléments $m_0 \in I$. Le Théorème 10.2 est le cas où on choisit $m_0 = \max(I)$ et le Théorème 11.6 est le cas où on choisit $m_0 = \min(I)$. Cette possibilité de choisir la « puissance principale » va nous être très utile dans l'obtention d'algèbres hypercycliques pour les décalages à gauche. On considère toujours $m_0 = \min(I)$ si la F -algèbre de suites X est équipé avec le produit CPC et on prendra toujours $m_0 = \max(I)$ si X est muni du produit de Cauchy.

Dans un deuxième temps, cet argument de Baire pour l'obtention d'une algèbre va être adapté aux différents types d'hypercyclicité, comme l'hypercyclicité commune, supérieurement fréquente et disjointe (voir paragraphes 13, 14 et 15).

12 Les produits de Cauchy et CPC: premiers résultats sur les F -espaces de suites

À partir d'ici et jusqu'à la fin, nous nous concentrerons sur les F -algèbres de suites. Dans ce contexte, comme on s'y attendait, le produit CPC est plus facile à travailler que le produit de Cauchy. Le traitement de ces deux cas est naturellement différent, même si savoir travailler avec le produit CPC peut rendre plus simple le traitement du produit de Cauchy. Dans ce paragraphe, nous discuterons ces différences ainsi que parler des premiers résultats pour les opérateurs de décalage à gauche pondérés sur les F -espaces de suites. Au niveau de la littérature, antérieurement à nos résultats, les meilleurs résultats se trouvaient dans [45], où les auteurs obtiennent de façon constructive des critères très généraux pour les décalages à gauche pondérés sous l'hypothèse qu'ils soient mélangeants. Ici, nous allons utiliser certaines de leurs idées mais en insistant dans l'utilisation de l'argument de Baire sous la forme des Théorème 10.2 et Théorème

11.6 et nous obtenons de façon plus simple des résultats plus généraux sans avoir besoin de l'hypothèse qu'ils soient mélangeants.

Pour simplifier la discussion, les opérateurs de décalage à gauche vont être appelés simplement « décalages ». Ils sont définis par

$$B_w(x_0, x_1, \dots) = (w_1x_1, w_2x_2, \dots),$$

où $x = (x_n)_n$ et $w = (w_n)_n$ est son « poids ». En plus, on dit qu'un poids w est « admissible » dans un espace de suites X quand le décalage B_w induit par w est continu sur X .

12.1 Produit coordonnée par coordonnée

En Dynamique Linéaire, quand on travaille avec un F -espace X , il est fréquent qu'on suppose que cet espace admette une norme continue (pour l'existence des sous-espaces hypercycliques par exemple). L'espace ω est le cas canonique d'espace qui ne satisfait pas cette propriété, bien qu'il soit en général plus facile de travailler avec cet espace à cause de sa topologie plutôt permissive. En effet, tous les opérateurs de décalage à poids B_w sur ω admettent une algèbre hypercyclique pour le produit CPC (voir Théorème 4.8 de l'Article A). On supposera ici que X admet une norme continue. La conséquence importante de cette hypothèse est que la suite basique $(e_n)_n$ est bornée inférieurement. On va également supposer que $(e_n)_n$ est une base de Schauder de X , ce qui rend possible plusieurs critères d'existence de vecteurs hypercycliques, auxquels nous fournirons des versions pour les algèbres hypercycliques avec le produit CPC. Afin d'éclairer la discussion suivante, nous devons définir ce qu'on comprend par « support » d'un élément $x \in X$. Ce vecteur peut être écrit de façon unique comme $x = \sum x_n e_n$. Le *support* de x est défini par $\text{supp}(x) = \{n \in \mathbb{N} : x_n \neq 0\}$.

La principale propriété satisfaite par le produit CPC est la suivante: pour tout élément $x = (x_n) \in X$ et tout $m \geq 1$,

$$x^m = \left(\sum_{n \geq 0} x_n e_n \right)^m = \sum_{n \geq 0} x_n^m e_n. \quad (11)$$

Cela signifie que prendre des puissances sur les points ne change pas leurs supports, ce qui est la différence fondamentale par rapport au produit de Cauchy. Cela veut dire aussi que, si deux vecteurs $x, y \in X$ ont des supports disjoints, alors

$$(x + y)^m = x^m + y^m. \quad (12)$$

La propriété (11) permet aussi de définir facilement les racines m -ièmes des vecteurs de l'espace (pour tout complexe z , $z^{1/m}$ dénote l'une de ses racines m -ièmes):

$$x^{1/m} = \sum_{n \geq 0} x_n^{1/m} e_n.$$

Ces propriétés vont permettre d'appliquer le Théorème 11.6. Décrivons de façon informelle comment faire.

Soit B_w un décalage à poids induit par $w = (w_n)$. Alors B_w admet un inverse à droite $F_{w^{-1}}$ (opérateur de décalage à droite pondéré induit par le poids $w^{-1} := (w_n^{-1})$). Supposons que

U, V, W sont des ouverts dans X , avec $0 \in W$, et soit $0 \leq m_0 \leq m_1$. On fixe deux « cibles » $x \in U$ et $y \in V$ avec support dans $\{0, \dots, p\}$ pour un certain $p \in \mathbb{N}$ (ce qui est possible puisque $(e_n)_n$ est une base de Schauder). Maintenant, définissons $S := F_{w^{-1/m_0}}$ et considérons

$$u = x + S^N y^{1/m_0}, \quad (13)$$

où $N > p$ est un grand entier à définir. Toutes ces définitions sont très naturelles. Effectivement, la condition $N > p$ permet de séparer les supports de x et de $S^N y^{1/m_0}$ (pour qu'on puisse appliquer la propriété (12)) ainsi que d'avoir $x \in \ker(B_w^N)$. En vue de la « puissance principale » u^{m_0} , on a défini S ainsi que le candidat u afin d'obtenir

$$u^{m_0} = x^{m_0} + F_{w^{-1}}^N y.$$

Puisque le support dans x^{m_0} n'a pas bougé, $B^N x^{m_0} = 0$ et donc $B^N u^{m_0} = y \in V$. Pour conclure, il reste à répondre à deux questions: (1) sous quelles conditions l'élément u sera-t-il dans U ? (2) que se passe-t-il avec les autres puissances $B_w^N u^n$ où $m_0 < n \leq m_1$?

Pour la première question, en regardant (13) on voit qu'il suffit d'imposer que $\lim_{N \rightarrow +\infty} S^N y^{1/m_0} = 0$ pour qu'on puisse prendre N suffisamment grand et conclure que $u \in U$. Si l'on écrit $y = \sum_{l=0}^p y_l e_l$, on obtient

$$S^N y^{1/m_0} = \sum_{l=0}^p \frac{1}{(w_{l+1} \cdots w_{N+l})^{1/m_0}} y_l^{1/m_0} e_{l+N}.$$

Alors, ce que l'on pourrait admettre comme condition est

$$\lim_{N \rightarrow +\infty} \frac{1}{(w_{l+1} \cdots w_{N+l})^{1/m}} e_{l+N} = 0, \quad \forall m \geq 1, \forall l \geq 0. \quad (14)$$

Pour la deuxième question on regarde $u^n = x^n + (S^N y^{1/m_0})^n$. Puisque le support de x^n n'a pas bougé non plus, on a $B_w^N x^n = 0$ et, en utilisant

$$(S^N y^{1/m_0})^n = \sum_{l=0}^p \frac{1}{(w_{l+1} \cdots w_{N+l})^{n/m_0}} y_l^{n/m_0} e_{l+N},$$

on trouve que

$$B_w^N u^n = \sum_{l=0}^p \frac{w_{l+1} \cdots w_{N+l}}{(w_{l+1} \cdots w_{N+l})^{n/m_0}} y_l^{n/m_0} e_l = \sum_{l=0}^p \frac{1}{(w_{l+1} \cdots w_{N+l})^{n/m_0 - 1}} e_l.$$

Étant donné que $n/m_0 - 1 > 0$, pour que $B_w^N u^n$ soit petit, il suffit de poser la condition

$$\lim_{N \rightarrow +\infty} w_{l+1} \cdots w_{N+l} = +\infty.$$

Mais celle-ci est une conséquence de (14) puisque $(e_n)_n$ est bornée inférieurement! En conclusion, la seule condition dont l'on a besoin est (14). En considérant qu'il est possible de répéter l'argument en remplaçant N par une suite $(n_k)_k$, nous obtenons la caractérisation suivante.

Théorème 12.1 *Soit X une F -algèbre de suites munie du produit CPC et admettant une norme continue. Supposons que (e_n) est une base de Schauder de X . Soit B_w un opérateur de décalage à poids borné sur X . Les assertions suivantes sont équivalentes.*

(i) B_w admet une algèbre hypercyclique.

(ii) Il existe une suite (n_k) telle que, pour tout $\gamma > 0$ et pour tout $l \in \mathbb{N}$, $((w_{l+1} \cdots w_{n_k+l})^{-\gamma} e_{n_k+l})$ tend vers 0 avec k .

(La preuve de la réciproque imite la condition nécessaire pour l'hypercyclicité.) Dans les cas particuliers $X = \ell_p(\mathbb{N})$, $1 \leq p < \infty$, ou $c_0(\mathbb{N})$, ce résultat s'identifie avec le critère d'hypercyclicité [33, Theorem 1.40].

Il est clair dans la discussion précédente pourquoi l'on utilise le Théorème 11.6 plutôt que Théorème 10.2: on a besoin que la « puissance principale » m satisfasse $n/m - 1 > 0$. L'exposant $n/m - 1$ apparaît dans tous les résultats concernant les algèbres hypercycliques avec le produit CPC. Bien que le contexte peut être plus compliqué (comme pour les algèbres supérieurement hypercycliques), l'idée derrière est toujours celle qu'on vient de discuter. C'est ainsi que le produit CPC fonctionne.

12.2 Produit de Cauchy et le terme distingué

Pour le produit CPC, si l'on regarde les termes dans la définition $u = x + z \in U$, l'on voit que le support de la parcelle « principale » z^{m_0} qui va être envoyé vers V par B_w^N ne bouge pas avec m_0 , donc on a juste besoin de regarder ce que se passe avec ses coefficients quand on considère les puissances u^n de u . Au contraire, dans le cas du produit de Cauchy, le support se décale vers la droite quand on considère des plus grandes puissances (c'est-à-dire, $\max \text{supp}(u^{n_1}) < \max \text{supp}(u^{n_2})$ quand $n_1 < n_2$) et, en plus, tous les coefficients des parcelles se mélangent dans le produit. Pour résoudre ce problème, nous allons introduire dans u un terme de décalage en posant

$$u = x + z + \varepsilon e_\sigma, \tag{15}$$

où ε est petit et $\sigma > \max(\text{supp}(z))$ de sorte que εe_σ est bien décalé à droite. Les prémices de cette idée sont déjà présentes dans l'article [51] mais elle est beaucoup plus développée dans [45]. L'intérêt de ce terme de décalage est que si l'on élève (15) à la puissance m , on obtient

$$u^m = (x + z + \varepsilon e_\sigma)^m = P_0 + mz \cdot (\varepsilon e_\sigma)^{m-1} + (\varepsilon e_\sigma)^m,$$

où $\max(\text{supp}(P_0)) < \min(\text{supp}(z(e_\sigma)^{m-1}))$. On peut ainsi, en bien choisissant ε, z, σ et $N > \max(\text{supp}(P_0))$, espérer que $B_w^N(u^m)$ soit dans V , et on n'a plus de problème de mélange de support. De plus, si $m' < m$ et si z, σ, N sont bien choisis, on peut espérer avoir $\max(\text{supp}(u^{m'})) < N$ et donc $B_w^N(u^{m'}) = 0$.

Cela amène aux résultats de la partie 4.3 de l'Article A, où nous utilisons une hypothèse supplémentaire sur l'espace X (voir définition de « régularité » dans l'Article A) et qui est satisfaite par tous les espaces classiques ℓ_p , $1 \leq p < +\infty$, $c_0(\mathbb{N})$ et $H(\mathbb{C})$. Nous avons obtenus la caractérisation suivante.

Théorème 12.2 *Soit X un F -algèbre de suites régulière munie du produit de Cauchy et soit B_w un décalage à poids borné sur X . Les assertions suivantes sont équivalentes:*

(i) B_w est hypercyclique.

(ii) B_w admet une algèbre hypercyclique.

Comme pour le produit CPC, cette discussion élucide ce qui se passe toujours avec le produit de Cauchy. Même dans des contextes plus compliqués, un terme distingué du type εe_σ est toujours présent.

13 L'hypercyclicité commune

Quand on considère une famille d'opérateurs hypercycliques $(T_\lambda)_{\lambda \in \Lambda}$ sur un même F -espace X , il est naturel de se demander sous quelles conditions existe-t-il un même vecteur x hypercyclique pour tous les membres T_λ de cette famille ? Dans ce paragraphe on suppose de manière standard que l'ensemble de paramètres Λ est un sous-ensemble σ -compact (c'est-à-dire, une union dénombrable de sous-ensembles compacts) de \mathbb{R}^d , $d \geq 1$, et que $(\lambda, x) \mapsto T_\lambda(x)$ est une application continue de $\Lambda \times X$ dans X . Un élément $x \in \bigcap_{\lambda \in \Lambda} HC(T_\lambda)$ est appelé *vecteur hypercyclique commun* (ou *vecteur hypercyclique simultané*) pour la *famille continue* $(T_\lambda)_{\lambda \in \Lambda}$. Celle-ci est alors appelée *famille d'opérateurs simultanément hypercycliques*.

Cette question est intéressante et non triviale. D'un côté, plusieurs familles classiques d'opérateurs sont simultanément hypercycliques, le premier exemple ayant été donné en 2003 par Abakumov et Gordon dans [27] pour la famille d'opérateurs de Rolewicz $(\lambda B)_{\lambda > 1}$. D'un autre côté, comme le montre un exemple dû à A. Borichev (cité aussi dans [27]), la famille $(\lambda B \times \mu B)_{(\lambda, \mu) \in [2, 3]^2}$ n'admet pas de vecteur hypercyclique commun.

Exemple 13.1 (A. Borichev) Pour n'importe quel $K \subset (1, +\infty) \times (1, +\infty)$ dont la mesure de Lebesgue est positive, la famille $(\lambda B \times \mu B)_{(\lambda, \mu) \in K}$ n'admet aucun vecteur hypercyclique commun sur $\ell_2(\mathbb{N}) \times \ell_2(\mathbb{N})$.

Intuitivement, plus l'ensemble de paramètres est petit, plus de chances l'on a d'y trouver un vecteur hypercyclique commun. L'exemple ci-dessus montre ainsi un possible lien entre l'existence d'un vecteur hypercyclique commun et la dimension de l'ensemble de paramètres. On reviendra à cette discussion dans la Section 13.2. Pour l'instant, concentrons-nous sur le cas $d = 1$.

13.1 Partitions d'intervalles

Comme souvent, les premiers résultats d'existence d'un vecteur satisfaisant certaines propriétés sont compliqués et constructifs. L'hypercyclicité commune n'a pas échappé à la règle: le résultat d'Abakumov et Gordon est une construction complexe du vecteur. Aujourd'hui, la théorie des vecteurs hypercycliques communs est bien développée et l'argument le plus puissant est encore basé sur le Théorème de Baire (voir [46, Theorem 11.5]).

Théorème 13.1 *Supposons que, pour tout compact $K \subset \Lambda$, pour toute paire (U, V) d'ouverts non vides de X , il existe $u \in U$ tel que, pour tout $\lambda \in K$, on peut trouver $N \in \mathbb{N}$ tel que $T_\lambda^N(u) \in V$. L'ensemble de vecteurs hypercycliques communs pour $(T_\lambda)_{\lambda \in \Lambda}$ est alors comeagre dans X .*

Dans le cas particulier des opérateurs de décalage pondérés, ce critère amène à plusieurs applications très simples (voir [33] ou [46] pour plus de détails) et couvre des familles classiques d'opérateurs comme $(\lambda D)_{\lambda>0}$ sur $H(\mathbb{C})$ et $(\lambda B)_{\lambda>1}$ sur $\ell_p(\mathbb{N})$, $1 \leq p < +\infty$, ou $c_0(\mathbb{N})$.

Les applications réussies du Théorème 13.1 consistent à considérer K comme un petit intervalle $[a, b]$ et à le partitionner par $a = \lambda_0 < \lambda_1 < \dots < \lambda_q = b$. À chaque sous-intervalle $\Lambda_i = [a_{i-1}, a_i]$ de la partition, on associe une puissance N_i , $i = 1, \dots, q$. Tout est fait de façon à pouvoir définir $u \in U$ tel que chaque $\lambda \in K$ tombe dans un certain Λ_i qui vérifie $T_{\lambda_i}^{N_i}(u) \in V$. L'objectif est donc de définir une partition suffisamment fine pour que λ soit assez proche de λ_i de manière que $T_{\lambda}^{N_i}(u) \in V$. L'important critère de Costakis-Sambarino permet de construire cette partition d'une manière très simple. Ici, on demande que chaque T_{λ} , $\lambda \in \Lambda$, ait un inverse à droite $S_{\lambda} : \mathcal{D} \rightarrow \mathcal{D}$, où \mathcal{D} est un sous-ensemble dense de X .

Théorème 13.2 (Costakis-Sambarino [43]) *On suppose que, pour tout $u \in \mathcal{D}$ et pour tout intervalle compact $K \subset \Lambda$, les propriétés suivantes sont vérifiées, où tous les paramètres α, λ, μ appartiennent à K .*

(1) *Il existe $\kappa \in \mathbb{N}$ et une suite de nombres positifs $(c_k)_{k \geq \kappa}$ tel que*

- $\sum_{k=\kappa}^{+\infty} c_k < +\infty$;
- $\|T_{\lambda}^{n+k} S_{\alpha}^n(u)\| \leq c_k$ pour tous $n \in \mathbb{N}$, $k \geq \kappa$ et $\alpha \leq \lambda$;
- $\|T_{\lambda}^n S_{\alpha}^{n+k}(u)\| \leq c_k$ pour tous $n \in \mathbb{N}$, $k \geq \kappa$ et $\lambda \leq \alpha$.

(2) *Étant donné $\eta > 0$, on peut trouver $\tau > 0$ tel que, pour tout $n \geq 1$,*

$$0 \leq \mu - \lambda \leq \frac{\tau}{n} \implies \|T_{\lambda}^n S_{\alpha}^n(u) - u\| \leq \eta. \quad (16)$$

Alors $\bigcap_{\lambda \in \Lambda} HC(T_{\lambda})$ est comeigre dans X .

La condition (2) permet de définir une partition $\min(K) = a_0 < a_1 < \dots < a_q = \max(K)$ de la forme

$$a_i = a_{i-1} + \frac{\tau}{iN}, \quad i = 1, \dots, q,$$

où N est convenablement fixé. On choisit donc $N_i = iN$, $i = 1, \dots, q$. La divergence de $\sum_{i \geq 1} \frac{\tau}{iN}$ est importante pour que la partition soit possible, c'est-à-dire, pour qu'il existe q tel que a_q atteigne (ou dépasse) $b = \max(K)$.

La partition introduite par Costakis et Sambarino a motivé plusieurs généralisations de ces conditions. Avant de les discuter plus en détails, voyons comment la condition (2) se comporte dans le contexte des opérateurs de décalage à poids $T_{\lambda} = B_{w(\lambda)}$. On suppose que chaque poids $w(\lambda) = (w_n(\lambda))_{n \in \mathbb{N}}$ est tel que $(\lambda, n) \in \Lambda \times \mathbb{N} \mapsto w_n(\lambda)$ est continue. Les opérateurs inverses à droite sont défini par $S_{\mu} = F_{w(\lambda)^{-1}}$. Soit $u \in \text{span}(e_i)$ de la forme $u = \sum_{l=0}^p w_l e_l$ pour un certain $p \in \mathbb{N}$. On trouve

$$\begin{aligned} \|T_{\lambda}^n S_{\mu}^n(u) - u\| &= \|B_{w(\lambda)}^n F_{w(\mu)^{-1}}^n(u) - u\| \\ &= \left\| \sum_{l=0}^p \left(\frac{w_{l+1}(\lambda) \cdots w_{l+n}(\lambda)}{w_{l+1}(\mu) \cdots w_{l+n}(\mu)} - 1 \right) w_l e_l \right\|. \end{aligned} \quad (17)$$

Ainsi, il est important de pouvoir s'assurer que les quotients $\frac{w_1(\lambda)\cdots w_n(\lambda)}{w_1(\mu)\cdots w_n(\mu)}$ se rapprochent suffisamment de 1 quand λ se rapproche de μ (les indices $l = 0, \dots, p$ vont être inclus plus tard). En employant \log dans cette tâche, on voit qu'il suffit de demander qu'il existe $N \geq 1$ tel que toutes les fonctions $\lambda \mapsto \log(w_i(\lambda))$, $i \in \mathbb{N}$, soient N -Lipschitz. En effet, en sachant que $|\exp(x) - 1| \leq 2|x|$ pour tout x petit, on trouve

$$\begin{aligned} \left| \frac{w_1(\lambda)\cdots w_n(\lambda)}{w_1(\mu)\cdots w_n(\mu)} - 1 \right| &= \left| \exp\left(\log\left(\frac{w_1(\lambda)\cdots w_n(\lambda)}{w_1(\mu)\cdots w_n(\mu)}\right)\right) - 1 \right| \\ &= \left| \exp\left(\sum_{j=1}^n \left(\log(w_j(\lambda)) - \log(w_j(\mu))\right)\right) - 1 \right| \\ &\leq 2 \left| \sum_{j=1}^n \left(\log(w_j(\lambda)) - \log(w_j(\mu))\right) \right| \\ &\leq 2 \sum_{j=1}^n N|\lambda - \mu| \\ &= 2nN|\lambda - \mu|. \end{aligned}$$

Par conséquent, la condition (16) va être satisfaite si l'on prend $\tau \leq \frac{\eta}{2nC_u}$, où C_u est une constante qui dépend de u (et qui va permettre de gérer les $p+1$ termes $l = 0, \dots, p$). Le passage le plus important dans les calculs ci-dessus est la deuxième inégalité, où on utilise la propriété N -Lipschitz des fonctions $\lambda \mapsto \log(w_i(\lambda))$. Plus précisément, il suffit que ces fonctions soient Lipschitz sur les ensembles compacts avec des constantes de Lipschitz uniformément bornées, idée qui apparaît dans [32]. Une première généralisation de cette condition apparaît dans l'Article B, Théorème 3.4, où nous demandons que les fonctions $\lambda \mapsto \log(w_i(\lambda))$ soient L_n -Lipschitz pour une suite (L_n) satisfaisant certaines propriétés (voir le prochain paragraphe). Une deuxième sera faite plus tard dans l'Article C, en regardant plutôt les fonctions $\lambda \mapsto \sum_{k=1}^n \log(w_k(\lambda))$.

Dans les prochains paragraphes, nous verrons comment généraliser ces idées afin de produire des algèbres hypercycliques.

13.1.1 Algèbres hypercycliques communes pour le produit CPC

L'une des applications les plus basiques du Théorème 13.1 est le Critère d'Hypercyclicité Commune (dont le Critère de Costakis-Sambarino est une conséquence).

Théorème 13.3 (Critère Basique [33]) *On suppose que, pour tout compact $K \subset \Lambda$, pour toute paire $(u, v) \in \mathcal{D} \times \mathcal{D}$ et pour tout voisinage O de l'origine 0 dans X , on peut trouver $q \in \mathbb{N}$, des paramètres $\lambda_1, \dots, \lambda_q \in \Lambda$, des sous-ensembles $\Lambda_1, \dots, \Lambda_q \subset \Lambda$ avec $\lambda_i \in \Lambda_i$, $i = 1, \dots, q$, et des entiers $n_1, \dots, n_q \in \mathbb{N}$ tels que:*

- (i) $\bigcup_i \Lambda_i \supset \Lambda$;
- (ii) $\sum_{j=1}^q S_{\lambda_j}^{n_j}(v) \in O$;
- (iii) Pour tout $i \in \{1, \dots, q\}$ et tout $\lambda \in \Lambda_i$,

- $\sum_{j \neq i} T_{\lambda}^{n_i} S_{\lambda_j}^{n_j}(u) \in O$;

- $T_{\lambda_i}^{n_i}(u) \in O$;
- $T_{\lambda}^{n_i} S_{\lambda_i}^{n_i}(v) - v \in O$.

Alors $\bigcap_{\lambda \in \Lambda} HC(T_{\lambda})$ est un sous-ensemble coaigre de X .

Ces conditions ne sont pas très différentes de celles qu'on a adaptées antérieurement aux algèbres avec le produit CPC. On peut donc espérer qu'on puisse répéter cette adaptation encore une fois, le premier pas étant de reformuler le Théorème 13.1 pour produire des algèbres hypercycliques. Nous avons trouvé la formulation suivante.

Théorème 13.4 *Soit Λ un espace métrique σ -compact et soit $(T_{\lambda})_{\lambda \in \Lambda}$ une famille continue d'opérateurs sur un F -espace X . Supposons que, pour tout compact $K \subset \Lambda$ et pour tout $I \in \mathcal{P}_f(\mathbb{N}^*)$, il existe $m_0 \in I$ tel que, pour toute paire (U, V) d'ouverts non vides de X et pour tout voisinage W de l'origine, il existe $u \in U$ tel que, pour tout $\lambda \in K$, on peut trouver $N \in \mathbb{N}$ satisfaisant*

$$\begin{cases} T_{\lambda}^N(u^n) \in W \text{ pour } n \in I \setminus \{m_0\}, \\ T_{\lambda}^N u^{m_0} \in V. \end{cases}$$

Alors, l'ensemble des vecteurs engendrant une algèbre hypercyclique commune pour $(T_{\lambda})_{\lambda \in \Lambda}$ est coaigre dans X .

Ce théorème nous permet d'énoncer une version algébrique du Critère Basique dans le contexte d'algèbres avec le produit CPC (voir Article B, Lemma 3.2), ainsi que des applications pratiques, dérivées du résultat suivant.

Théorème 13.5 *Soit $\Lambda = [a, b]$ un intervalle compact et soit $(w(\lambda))_{\lambda \in \Lambda}$ une famille de poids admissibles. On suppose que*

- (e_i) est une base inconditionnelle de X ;
- pour chaque $n \in \mathbb{N}$ la fonction $\lambda \mapsto \log(w_n(\lambda))$ est L_n -Lispchitz pour une certaine constante $L_n > 0$;
- pour tout $p > 0$ et tout $m_0 \geq 1$, il existe une suite $(n_k)_k$ telle que

- les séries $\sum_{k \geq 1} \frac{1}{(w_{l+1}(a) \cdots w_{l+n_k}(a))^{1/m_0}} e_{l+n_k}$ sont inconditionnellement convergentes pour tout $l = 0, \dots, p$;
- les séries $\sum_{k \geq 1} \frac{1}{w_{l+1}(a) \cdots w_{l+n_k+i-n_i}(a)} e_{l+n_k+i-n_i}$ sont uniformément inconditionnellement convergentes pour tous $i \geq 1, l = 0, \dots, p$;
- pour tous $k \geq 1, l \geq 0, w_{l+1}(a) \cdots w_{l+n_k}(a) \geq 1$;
- $\sum_{i > 1} 1/C_{n_i} = +\infty$, où $C_n = \sum_{k=1}^n L_k$.

Alors l'ensemble des points qui engendrent une algèbre hypercyclique commune pour $(T_{\lambda})_{\lambda \in [a, b]}$ est coaigre dans X .

Ce théorème s'applique aux familles classiques $(\lambda B)_{\lambda>1}$ sur $\ell_p(\mathbb{N})$ ou $c_0(\mathbb{N})$ et $(\lambda D)_{\lambda>0}$ sur $HC(\mathbb{C})$, ainsi que $(B_{w(\lambda)})_{\lambda>0}$, où $w_n(\lambda) = 1 + \frac{\lambda}{n}$, sur $\ell_p(\mathbb{N})$ ou $c_0(\mathbb{N})$. Ce dernier exemple est particulièrement intéressant parce que l'existence d'un vecteur hypercyclique commun pour cette famille sur $\ell_p(\mathbb{N})$ n'était connu que pour $\lambda > 1/p$. Notre méthode implique en particulier le corollaire suivant.

Corollaire 13.2 *Considérons la famille de poids $(w(\lambda))_{\lambda>0}$ définie par $w_n(\lambda) = 1 + \frac{\lambda}{n}, n \geq 1$. Alors, $(B_{w(\lambda)})_{\lambda>0}$ admet un vecteur hypercyclique commun sur $\ell_p(\mathbb{N}), 1 \leq p < \infty$.*

13.1.2 Algèbres hypercycliques communes pour le produit de Cauchy

Le Théorème 13.5 nous permet d'énoncer des résultats pratiques pour les algèbres avec le produit CPC, similaires aux résultats trouvés dans la littérature pour les vecteurs hypercycliques communs. Ceci est possible parce que le produit CPC se comporte bien pour les sommes, ce qui n'est pas le cas quand on travaille avec le produit de Cauchy. Le problème est encore plus compliqué pour la recherche d'algèbres hypercycliques communes puisque le vecteur u auquel est appliqué le Théorème 13.4 s'écrit de façon naturelle

$$u = x + z_1 + z_2 + \cdots + z_q,$$

où les z_i sont associés à la partition. Plus de termes sont donc susceptibles de se mélanger quand on considère une puissance u^n de u . Néanmoins, en utilisant toujours de façon essentielle le terme de décalage et le Théorème 13.4, on obtient le résultat général suivant.

Théorème 13.6 *Soit $\Lambda \subset \mathbb{R}$ un intervalle, X une F -algèbre de suites avec le produit de Cauchy et $(w(\lambda))_{\lambda \in \Lambda}$ une famille de poids admissibles telle que chaque fonction $\lambda \mapsto \log(w_n(\lambda))$ est croissante et Lipschitz sur les ensembles compacts avec des constantes de Lipschitz uniformément bornées. Supposons que (e_n) est une base inconditionnelle de X et que*

(a) *pour tout $\lambda \in \Lambda$,*

$$\sum_{n=1}^{\infty} \frac{1}{w_1(\lambda) \cdots w_n(\lambda)} e_n \in X;$$

(b) *pour tous $m \in \mathbb{N}$ et $[a_0, b_0] \subset \Lambda$, il existe $c \in (0, 1)$ et $\kappa_0 > 1$ tels que*

$$\lim_{N \rightarrow \infty} \sum_{n=cN}^N \frac{[w_1(\kappa_0 a) \cdots w_{mN}(\kappa_0 a)]^{\frac{m-1}{m}}}{w_1(a) \cdots w_{(m-1)N+n}(a)} = 0, \quad \text{pour tout } a \in [a_0, b_0].$$

Alors, $(B_{w(\lambda)})_{\lambda \in \Lambda}$ admet une algèbre hypercyclique commune.

Le Théorème 13.6 garantit une algèbre hypercyclique commune avec le produit de Cauchy pour les familles classiques d'opérateurs de Rolewicz $(\lambda B)_{\lambda>1}$ sur $\ell_1(\mathbb{N})$ et MacLane $(\lambda D)_{\lambda>0}$ sur $H(\mathbb{C})$, mais laisse ouvert le cas $(B_{w(\lambda)})_{\lambda>0}$ où $w_n(\lambda) = 1 + \frac{\lambda}{n}$, qui ne satisfait pas la condition (b). Cette question a été résolue plus tard dans l'Article D, où nous avons montré que cette famille admet aussi une algèbre hypercyclique commune pour le produit de Cauchy. La preuve va être discuté dans la partie 5.3.

13.2 Partitions de carrés et hypercyclicité commune

Dans cette partie nous allons discuter le problème d'hypercyclicité commune quand l'ensemble de paramètres Λ est un sous-ensemble de \mathbb{R}^d avec $d > 1$. Il est possible d'aborder ce problème de différentes façons, comme dans [52] ou [29], où l'on augmente l'ensemble de paramètres sans changer l'espace. Ici nous sommes intéressés par les opérateurs comme dans l'exemple de Borichev: les sommes directes d'opérateurs de décalage agissant sur un espace produit. Son exemple montre que l'ensemble de paramètre ne peut pas être trop grand, mais quelles formes peut-il avoir ? Bayart et Matheron montrent dans [32] que $(sB \times tB)_{(s,t) \in \Lambda}$ admet un vecteur hypercyclique commun lorsque Λ est une courbe de Lipschitz croissante. Dans cet exemple, un ensemble unidimensionnel de paramètres permet l'existence d'un vecteur hypercyclique commun dans l'environnement bidimensionnel. Plus généralement, les mêmes auteurs montrent que, dans le cas d -dimensionnel $(t_1B \times \dots \times t_dB)_{(t_1, \dots, t_d) \in \Lambda}$, il existe un vecteur hypercyclique commun lorsque Λ est une surface de Lipschitz croissante, ce qui est un sous-ensemble $(d-1)$ -dimensionnel de \mathbb{R}^d . Toutefois, savoir si $\bigcap_{s \in [2,3]} HC(sB \times (4-s)B)$ est vide ou non est une question laissée ouverte. On ne sait pas non plus comment estimer la « taille maximale » de l'ensemble de paramètres Λ pour que l'existence d'un vecteur hypercyclique soit possible. On sait juste que la mesure de Lebesgue d -dimensionnelle de Λ doit être nulle, mais le traitement des produits de décalages pondérés sur un espace produit n'a pas été exploré auparavant.

Les questions que nous venons de discuter motivent les résultats de l'Article C. D'une part, nous cherchons des conditions nécessaires sur $\Lambda \subset \mathbb{R}^d$ pour que $\bigcap_{\lambda \in \Lambda} HC(\lambda B \times \mu B) \neq \emptyset$. L'analyse de la preuve de l'Exemple 13.1 révèle que, ce qui limite la taille de Λ dans l'exemple $(\lambda B \times \mu B)_{(\lambda, \mu) \in \Lambda}$ est le fait que la condition (2) du critère de Costakis-Sambarino est « saturée » pour cette famille T_λ : pour de nombreux vecteurs x ,

$$\|T_\lambda^n S_\mu^s(x) - x\| \geq cn|\lambda - \mu|.$$

Cette observation nous a conduit à obtenir le résultat suivant.

Théorème 13.7 *Soit $\Lambda \subset \mathbb{R}^d$ et considérons une famille d'opérateurs $(T_\lambda)_{\lambda \in \Lambda}$ bornées sur un espace de Banach X . Supposons qu'il existe $\alpha > 0$, $v \in X$, $\delta > 0$ and $C > 0$ tels que, pour tous $\lambda, \mu \in \Lambda$, tout $n \in \mathbb{N}$ et tout $u \in X$ satisfaisant*

$$\|T_\lambda^n(u) - v\| < \delta \quad \text{and} \quad \|T_\mu^n(u) - v\| < \delta,$$

on a

$$\|T_\lambda^n(u) - T_\mu^n(u)\| \geq Cn^\alpha \|\lambda - \mu\|.$$

Si $\bigcap_{\lambda \in \Lambda} HC(T_\lambda) \neq \emptyset$, alors $\dim_{\mathcal{H}}(\Lambda) \leq \frac{1}{\alpha}$.

D'autre part, nous cherchons des conditions suffisantes sur $\Lambda \subset \mathbb{R}^d$ pour que $\bigcap_{\lambda \in \Lambda} HC(T_\lambda) \neq \emptyset$. La différence essentielle avec la dimension 1 est l'absence d'ordre naturel sur \mathbb{R}^d pour produire un recouvrement par des parties suffisamment fines et bien placées. Nous avons défini un recouvrement dyadique par des cubes de coté $\frac{\tau}{n_i^\alpha}$ avec une suite associée $(n_i)_i$ choisis en considérant le meilleur emplacement possible des parties (voir paragraphe 4.1 de l'Article C

pour une comparaison de trois positionnements possibles des cubes dyadiques). En particulier on obtient le théorème suivant.

Théorème 13.8 *Soit $d \geq 1$ et $\Lambda \subset \mathbb{R}^d$ compact. On suppose qu'il existe $\alpha \in (0, 1/d)$, $\beta > \alpha d$ et $D > 0$ tels que, pour tout $u \in \text{span}(e_i)$,*

(a) *il existe $C, N > 0$ tels que, pour tous $\lambda, \mu \in \Lambda$ et tous $n \geq 0, k \geq N$ satisfaisant $\|\lambda - \mu\|_\infty \leq D \frac{k^\alpha}{(n+k)^\alpha}$, nous avons*

$$\left\| T_\lambda^{n+k} S_\mu^n u \right\| \leq \frac{C}{k^\beta}$$

$$\left\| T_\lambda^n S_\mu^{n+k} u \right\| \leq \frac{C}{k^\beta}.$$

(b) *pour tout $\varepsilon > 0$, il existe $\tau > 0$ tel que, pour tout $n \geq 1$, pour tous $\lambda, \mu \in \Lambda$,*

$$\|\lambda - \mu\|_\infty \leq \frac{\tau}{n^\alpha} \implies \|T_\lambda^n S_\mu^n u - u\| < \varepsilon.$$

Alors $\bigcap_{\lambda \in \Lambda} HC(T_\lambda)$ est un sous-ensemble comeagre de X .

Puisque les corollaires de ce théorème ont été généralisées au contexte algébrique dans l'Article D, on les discutera dans la section suivante. L'une des conséquences de ce résultat est l'existence un vecteur hypercyclique commun pour $(B_{w(\lambda_1)} \times \cdots \times B_{w(\lambda_d)})_{\lambda \in \Lambda}$ dans le cas $w_1(a) \cdots w_n(a) = \exp(an^\alpha)$ lorsque $\alpha < 1/d$. D'autre part, le Théorème 3.1 implique que, dans le même cas $w_1(a) \cdots w_n(a) = \exp(an^\alpha)$, l'ensemble $\bigcap_{\lambda \in \Lambda} (B_{w(\lambda_1)} \times \cdots \times B_{w(\lambda_d)})$ est vide lorsque $\alpha > 1/d$. En somme, le seul cas pour lequel on n'a pas de réponse est $\alpha = 1/d$.

Encore dans l'Article C, nous avons obtenu une caractérisation des produits de décalages pondérés admettant un vecteur hypercyclique commun. Ce résultat résout le problème ouvert de [32] en montrant que $\bigcap_{s \in [2,3]} HC(sB \times (4-s)B) \neq \emptyset$. Aussi dans le paragraphe suivant, on discutera une version algébrique de ce résultat.

13.2.1 Produits de décalages pondérés admettant une algèbre hypercyclique commune pour le produit CPC

La caractérisation du Théorème 12.1 est une généralisation du critère d'hypercyclicité [46, Theorem 4.8(a)] dans le contexte des algèbres avec le produit CPC. On a ensuite essayé d'appliquer la même démarche à une caractérisation pour l'hypercyclicité simultanée, sauf qu'une telle caractérisation n'existait pas encore dans la littérature. On a donc obtenu la première caractérisation de l'hypercyclicité commune pour les décalages à poids dans l'Article C, Théorème 2.7, et on l'a généralisé au contexte des algèbres avec le produit CPC dans l'Article D sous la forme suivante.

Théorème 13.9 *Soit X une F -algèbre de suites munie du produit CPC avec une norme continue et admettant (e_n) comme base inconditionnelle. On considère $I \subset \mathbb{R}$ un intervalle non vide, $d \geq 1$ et $\Lambda \subset I^d$ σ -compact. Soit $(B_{w(a)})_{a \in I}$ une famille de décalages pondérés sur X telle que chaque fonction $a \in I \mapsto w_n(a)$ est croissante. On suppose qu'il existe $F : \mathbb{N}^* \rightarrow \mathbb{R}_+$ ainsi que $c, C > 0$ tels que, pour tous $n \geq 1$ et $(a, b) \in I^2$, en définissant $f_n(a) = \sum_{k=1}^n \log(w_k(a))$,*

$$cF(n)|a - b| \leq |f_n(a) - f_n(b)| \leq CF(n)|a - b|,$$

$$\frac{w_n(a)}{w_n(b)} \geq c.$$

Alors, toutes les assertions suivantes sont équivalentes.

- (a) $(B_{w(\lambda(1))} \times \cdots \times B_{w(\lambda(d))})_{\lambda \in \Lambda}$ admet une algèbre hypercyclique commune dans X ;
- (b) pour tout $m \geq 1$ il existe $u \in X^d$ tel que u^m est un vecteur hypercyclique commun pour $(B_{w(\lambda(1))} \times \cdots \times B_{w(\lambda(d))})_{\lambda \in \Lambda}$;
- (c) pour tous $m, N \geq 1$, tous $\tau, \varepsilon > 0$ et tout $K \subset \Lambda$ compact, il existe $N \leq n_1 < n_1 + N \leq n_2 < \cdots < n_{q-1} + N \leq n_q$ ainsi que $(\lambda_k)_{k=1, \dots, q} \in I^d$ tels que:

(i) $K \subset \bigcup_{k=1}^q \prod_{l=1}^d [\lambda_k(l) - \frac{\tau}{F(n_k)}, \lambda_k(l)]$;

(ii) pour tous $i = 1, \dots, d$,

$$\left\| \sum_{k=1}^q \frac{1}{[w_1(\lambda_k(i)) \cdots w_{n_k}(\lambda_k(i))]^{1/m} e_{n_k}} \right\| < \varepsilon;$$

(iii) pour tous $k = 1, \dots, q-1$, $i = 1, \dots, d$ et $l = 0, \dots, N$,

$$\left\| \sum_{j=k+1}^q \frac{w_{n_j-n_k+l+1}(\lambda_k(i)) \cdots w_{n_j+l}(\lambda_k(i))}{w_{l+1}(\lambda_j(i)) \cdots w_{n_j+l}(\lambda_j(i))} e_{n_j-n_k+l} \right\| < \varepsilon.$$

Le critère pour vecteurs hypercycliques communs est obtenu à partir des conditions ci-dessus en considérant $m = 1$. Bien que techniques, les hypothèses du Théorème 13.9 deviennent beaucoup plus simples quand on se restreint au produit des multiples de l'opérateur de Rolewicz $(\lambda_1 B \times \cdots \times \lambda_d B)_{\lambda \in (1, +\infty)^d}$ sur $\ell_p(\mathbb{N})$, $1 \leq p < +\infty$, ou $c_0(\mathbb{N})$. De façon à rendre l'énoncé plus lisible, on regardera cette famille sous la forme équivalente $(e^{\lambda_1} B \times \cdots \times e^{\lambda_d} B)_{\lambda \in (0, +\infty)^d}$.

Théorème 13.10 Soit $d \geq 1$, $\Lambda \subset (0, +\infty)^d$, $X = \ell_p(\mathbb{N})$, $p \in [1, +\infty)$, ou $X = c_0(\mathbb{N})$. Les assertions suivantes sont équivalentes.

1. $(e^{\lambda_1} B \times \cdots \times e^{\lambda_d} B)_{\lambda \in (1, +\infty)^d}$ admet une algèbre hypercyclique commune.
2. Pour tous $N \geq 1$ et $\tau, \varepsilon > 0$, pour tout $K \subset \Lambda$ compact, il existe $N \leq n_1 < n_1 + N \leq n_2 < \cdots < n_{q-1} + N \leq n_q$ ainsi que $(\lambda_k)_{k=1, \dots, q} \in (0, +\infty)^d$ tels que

(i) $K \subset \bigcup_{k=1}^q \prod_{l=1}^d [\lambda_k(l) - \frac{\tau}{n_k}, \lambda_k(l)]$,

(ii) pour tous $k = 1, \dots, q$ et $l = 1, \dots, d$,

$$\lambda_{k+1}(l)n_{k+1} - \lambda_k(l)n_k \geq N.$$

On note qu'il n'y a plus la présence de $m \geq 1$ dans le dernier énoncé. En effet, l'existence d'une algèbre hypercyclique commune avec le produit CPC pour les multiples de l'opérateur de décalage est équivalente à l'existence d'un vecteur hypercyclique commun. Cela est toujours vrai dans le cas plus général $w_1(a) \cdots w_n(a) = \exp(\lambda n^\alpha)$, comme montre la Proposition 2.9 de l'Article D. L'une des conséquences de ces résultats, comme il est prouvé dans le Théorème 1.3 de

l'Article C, est que pour toute courbe de Lipschitz $\Lambda \subset (1, +\infty)^d$, la famille $(\lambda_1 B \times \cdots \times \lambda_d B)_{\lambda \in \Lambda}$ admet une algèbre hypercyclique commune pour le produit CPC.

D'autres résultats de l'Article C qui ont été généralisés au cas algébrique dans l'Article D sont des conséquences du Théorème 1.5 (voir l'Article C, paragraphe 4.3). Une version unifiée est présentée dans l'Article D sous la forme suivante.

Théorème 13.11 *Soient $\alpha \in (0, 1/d)$ et $I \subset (0, +\infty)$ σ -compact. On suppose que, pour tous $m_0 \in \mathbb{N}$ et $I_0 \subset I$ compact, il existe $C_1, C_2 > 0, \beta > \alpha d$, $F : \mathbb{N} \rightarrow (0, +\infty)$ avec $F(n) \leq C_1 n^\alpha$ et $N_0, M_0 > 0$ tels que, pour tout $n \geq N_0$,*

(i) $a \in I_0 \mapsto \sum_{i=1}^n \log(w_i(a))$ est $F(n)$ -Lipschitz,

(ii) pour tout $a \in I_0$, $w_1(a) \cdots w_n(a) \rightarrow +\infty$ quand $n \rightarrow +\infty$,

(iii) pour tous $a \in I_0$ et $k \geq N_0$,

$$\left\| \frac{\exp\left(C_2 \frac{F(n+k)}{(n+k)^\alpha} k^\alpha\right)}{w_1(a) \cdots w_k(a)} \right\| \leq \frac{M_0}{k^\beta},$$

$$\left\| \frac{1}{[w_1(a) \cdots w_k(a)]^{1/m_0}} e_k \right\| \leq \frac{M_0}{k^\beta}.$$

Alors, $(B_{w(\lambda_1)} \times \cdots \times B_{w(\lambda_d)})_{\lambda \in I^d}$ admet une algèbre hypercyclique commune dans X^d avec le produit CPC.

Ceci inclut les multiples de l'opérateur de Rolewicz ainsi que la famille induite par les poids $w_n(a) = 1 + \frac{a}{n^{1-\alpha}}$, $\lambda > 0$, pour tout $\alpha \in [0, 1/d)$.

13.3 Une algèbre hypercyclique commune pour $(B_{w(\lambda_1)} \times \cdots \times B_{w(\lambda_d)})_{\lambda \in (0, +\infty)^d}$ où $w_n(a) = 1 + \frac{a}{n}$ avec le produit de Cauchy

Dans cette dernière partie du paragraphe 13, on discute la réponse donnée dans l'Article D à une question laissée ouverte dans l'Article B sur l'existence d'une algèbre hypercyclique commune pour la famille des opérateurs de décalage à poids $(B_{w(\lambda)})_{\lambda > 0}$ induite par $w_n(\lambda) = 1 + \frac{\lambda}{n}$ et agissant sur $\ell_1(\mathbb{N})$ avec le produit de Cauchy. Afin de simplifier la discussion, quand on dit que la famille de poids admissibles $(w(a))_{a \in I}$ satisfait la propriété $F(n)$ -Lipschitz pour une certaine fonction $F : \mathbb{N}^* \rightarrow \mathbb{R}_+$, on veut dire que chaque fonction $a \in I \mapsto f_n(a) = \sum_{k=1}^n \log(w_k(a))$ est $F(n)$ -Lipschitz, $n \in \mathbb{N}$.

Après avoir obtenu plusieurs résultats en dimension plus grande établis pour le cas n^α -Lipschitz, nous avons pensé qu'il serait possible d'obtenir une algèbre hypercyclique avec le produit de Cauchy dans le cas $w_n(a) = 1 + \frac{a}{n}$ en dimension d comme application de la méthode développée dans les paragraphes 4.4 et 4.5 de l'Article C. Cependant, nous avons fait face à des complications techniques relatives au nombre de termes dans la suite n_1, \dots, n_q . Ces difficultés

apparaissent même dans le cas unidimensionnel. La suite de puissances $(n_i)_i$ considérée dans la preuve du Théorème 13.6 est linéaire par rapport à i , et cette linéarité est cachée dans la condition (b). Il semble que $n_i = iN$ fonctionne bien dans le cas n -Lipschitz car on définit une partition de taille $\frac{\tau}{iN}$, mais ne fonctionne pas aussi bien pour le cas $\log(n)$ -Lipschitz si l'on maintient la même taille de partition. Ce qu'on a fait dans l'Article D consiste à choisir la suite $(n_i)_i$ spécialement pour le poids $w_n(a) = 1 + \frac{a}{n}$ en considérant la condition $\log(n)$ -Lipschitz. De même, le recouvrement dyadique utilisé dans l'Article C ne semble pas adapté pour produire une algèbre hypercyclique commune dans le cas où $w_n(\lambda) = 1 + \frac{\lambda}{n}$ et où l'on travaille avec $\ell_1(\mathbb{N})^d$. En effet, il contient « trop » d'éléments. À nouveau, le fait que le poids est $\log(n)$ -Lipschitz permet de construire un recouvrement adapté à ce cas. Ceci nous conduit au théorème suivant, démontré dans le paragraphe 3 de l'Article D

Théorème 13.12 *Soit $(w(\lambda))_{\lambda>0}$ la famille de poids définie par $w_n(a) = 1 + \frac{a}{n}, n \geq 1$. Alors, $(B_{w(\lambda_1)} \times \cdots \times B_{w(\lambda_d)})_{\lambda \in (0, +\infty)^d}$ admet une algèbre hypercyclique commune dans $\ell_1(\mathbb{N})$ muni du produit de Cauchy.*

14 Algèbres hypercycliques supérieurement fréquentes

Un autre concept relatif à l'hypercyclicité et que nous avons aussi exploré est l'hypercyclicité supérieurement fréquente, introduit par S. Shkarin dans [50] comme une forme faible de l'hypercyclicité fréquente (concept introduit par F. Bayart et S. Grivaux dans [31]). Voyons les définitions.

Les *densités supérieure* $\bar{d}(A)$ et *inférieure* $\underline{d}(A)$ d'un sous-ensemble $A \subset \mathbb{N}$ sont définies par

$$\bar{d}(A) = \limsup_{n \rightarrow \infty} \frac{\#\{m \in A : m \leq n\}}{n} \quad \text{et} \quad \underline{d}(A) = \liminf_{n \rightarrow \infty} \frac{\#\{m \in A : m \leq n\}}{n},$$

où $\#E$ désigne le cardinal de E . Soit X un F -espace et T un opérateur linéaire borné sur X . On dit que $x \in X$ est un *vecteur hypercyclique fréquent* pour T si, pour tout $V \subset X$ ouvert non-vide, la densité inférieure de l'ensemble $\{n \in \mathbb{N} : T^n(x) \in V\}$ est strictement positive. Dans ce cas on dit que T est un *opérateur hypercyclique fréquent* et on dénote par $FHC(T)$ l'ensemble des vecteurs hypercycliques fréquents de T . D'autre part, si la densité supérieure de l'ensemble $\{n \in \mathbb{N} : T^n(x) \in V\}$ est strictement positive pour tout $V \subset X$ ouvert et non-vide, on dit que x est un *vecteur hypercyclique supérieurement fréquent* de T et que T est un *opérateur hypercyclique supérieurement fréquent*. L'ensemble des vecteurs avec cette propriété est noté $UFHC(T)$. La différence entre ces deux concepts peut sembler subtile, mais la distinction dans la pratique est énorme. Effectivement, comme Bonnilla et Grosse-Erdmann ont souligné dans [42], on peut obtenir des vecteurs hypercycliques supérieurement fréquents à l'aide du théorème de Baire alors que l'ensemble des vecteurs hypercycliques fréquents est toujours maigre. En particulier, nous avons pu adapter les techniques mises en œuvre précédemment pour construire des algèbres de vecteurs hypercycliques supérieurement fréquents.

Le premier pas, comme d'habitude, consiste à obtenir une version de l'argument de Baire pour l'hypercyclicité supérieurement fréquente. Nous avons utilisé la proposition suivante.

Proposition 14.1 *Soit T un opérateur borné sur une F -algèbre X . Supposons que, pour tout $I \in \mathcal{P}(\mathbb{N}^*) \setminus \{\emptyset\}$, il existe $m_0 \in I$ tel que, pour tout ouvert non-vidé $V \subset X$ et tout voisinage de l'origine W , on peut trouver $\delta > 0$ tel que, pour tout ouvert non-vidé $U \subset X$ et tout $N_0 \in \mathbb{N}$, on peut trouver $u \in U$ et $N \geq N_0$ satisfaisant*

$$\frac{\#\{p \leq N : T^p(u^m) \in W, m \in I \setminus \{m_0\} \text{ et } T^p(u^{m_0}) \in V\}}{N+1} > \delta.$$

Alors, T admet une algèbre hypercyclique supérieurement fréquente.

Ce même critère va être appliqué pour les deux produits CPC et Cauchy. Comme toujours, pour le premier on choisit $m_0 = \min(I)$ et pour le deuxième on choisit $m_0 = \max(I)$.

14.1 Produit coordonnée par coordonnée

Dans le cas présent, nous avons trouvé la conséquence suivante du critère précédent.

Théorème 14.1 *Soit X une algèbre de Fréchet de suites munie du produit CPC et avec une norme continue. Supposons que $(e_n)_n$ engendre un sous-espace dense de X et que B_w est un opérateur de décalage pondéré induit par un poids admissible $(w_n)_n$ tel que, pour tout $m \geq 1$, la série $\sum_{n \geq 1} (w_1 \cdots w_n)^{-1/m} e_n$ converge inconditionnellement. Alors B_w admet une algèbre hypercyclique supérieurement fréquente.*

Parmi les applications de ce résultat, on obtient assez facilement une algèbre hypercyclique supérieurement fréquente pour les opérateurs classiques $\lambda B, \lambda > 1$, sur $\ell_p(\mathbb{N}), 1 \leq p < \infty$, ou $c_0(\mathbb{N})$ avec le produit CPC, ainsi que $\lambda D, \lambda > 0$, sur $H(\mathbb{C})$ avec le produit d'Hadamard. On trouve aussi que B_w , où $w_n = 1 + \frac{\lambda}{n}$, admet une algèbre hypercyclique supérieurement fréquente dans $c_0(\mathbb{N})$ pour tout $\lambda > 0$. Par contre, sur $\ell_p(\mathbb{N})$ la condition n'est pas satisfaite par ce poids car la série diverge lorsque $p\lambda < m$. L'existence d'une algèbre hypercyclique supérieurement fréquente dans ce cas reste un problème ouvert.

14.2 Produit de Cauchy

Pour le produit de convolution, nous avons trouvé la conséquence suivante.

Théorème 14.2 *Soit X une algèbre de Fréchet de suites régulière munie du produit de Cauchy et soit B_w un opérateur de décalage pondéré borné sur X tel que son poids $(w_n)_n$ satisfait:*

- (a) $\sum_{n \geq 1} (w_1 \cdots w_n)^{-1/m} e_n$ converge inconditionnellement;
- (b) pour tout $m \geq 2$, il existe $c \in (0, 1)$ tel que

$$\lim_{\sigma \rightarrow +\infty} \sup_{z \in c_{00} \cap B_{\ell_\infty}} \left\| \sum_{n \geq c\sigma} \frac{z_n (w_1 \cdots w_n)^{\frac{m-1}{m}}}{w_1 \cdots w_{(m-1)\sigma+n}} e_n \right\| = 0.$$

Alors B_w admet une algèbre hypercyclique supérieurement fréquente dans X .

Il est remarquable la similitude entre la condition (b) ci-dessus et la condition (b) du Théorème 13.6 pour les algèbres hypercycliques communes, ce qui corrobore la proximité des méthodes. Comme pour le produit CPC, le Théorème 14.2 s'applique aux opérateurs classiques λB , $\lambda > 1$, sur $\ell_1(\mathbb{N})$ ainsi que λD , $\lambda > 0$, sur $H(\mathbb{C})$, mais pas à l'opérateur B_w où $w_n = 1 + \frac{\lambda}{n}$. À nouveau, l'existence d'une algèbre hypercyclique supérieurement fréquente dans $\ell_1(\mathbb{N})$ avec le produit de Cauchy pour l'opérateur B_w où $w_n = 1 + \frac{\lambda}{n}$, $\lambda \geq 0$, reste une question ouverte à étudier.

15 Algèbres hypercycliques disjointes

Le concept d'hypercyclité disjointe, apparu par la première fois de manière indépendante dans les articles de J. Bès et A. Peris [39] et L. Bernal-González [34], est une sorte d'hypercyclité diagonale. Soit X un espace de Fréchet de suites et soient T_1 et T_2 deux opérateurs continus sur X . On dit que T_1 et T_2 sont *hypercycliques disjointes* quand il existe $x \in X$ (appelé *vecteur hypercyclique disjoint* pour T_1 et T_2) tel que la suite $(T_1^n(x), T_2^n(x))_{n \geq 0}$ est dense dans X^2 . Plusieurs types d'opérateurs sont explorés dans ces deux articles initiaux de la théorie, comme les opérateurs de composition, opérateurs différentiels et, ce qui nous intéresse le plus, opérateurs de décalage à gauche. Par simplicité, nous avons préféré explorer seulement le cas de deux opérateurs, mais les définitions ainsi que quelques résultats fonctionnent aussi dans le contexte général de plusieurs opérateurs hypercycliques disjointes. En plus, nous nous concentrerons dans le cas où T_1 et T_2 sont puissances de deux opérateurs de décalage pondérés, $T_1 = B_w^{r_1}$ et $T_2 = B_w^{r_2}$, disons $0 < r_1 \leq r_2$. Comme il est souligné dans [39] et postérieurement confirmé dans [38], le cas $r_1 \neq r_2$ se comporte de manière assez différente de $r_1 = r_2$. Ici on parlera seulement du cas $r_1 < r_2$ (mais le cas $r_1 = r_2$ soit aussi traité dans l'Article B).

Notre but est d'amener au contexte algébrique des résultats connus. La définition d'algèbre hypercyclique disjointe est la plus naturelle possible: dans le contexte où X est une F -algèbre, on dit que deux opérateurs T_1 et T_2 admettent une *algèbre hypercyclique disjointe* quand il existe une sous-algèbre non triviale A de X telle que tout élément non-nul x est un vecteur hypercyclique disjoint pour T_1 et T_2 . Le premier pas, comme avant, consiste à obtenir un argument de Baire en fournissant une algèbre hypercyclique disjointe. Nous obtenons la formulation suivante.

Proposition 15.1 *Soit T_1, T_2 deux opérateurs continus sur la même F -algèbre X . Supposons que, pour tout $I \subset \mathcal{P}(\mathbb{N}^*) \setminus \{\emptyset\}$, il existe $m_0 \in I$ tel que, pour tous ouverts non-vides $U, V_1, V_2 \subset X$ et tout voisinage W de l'origine, on peut trouver $u \in U$ et $N \in \mathbb{N}$ tels que*

$$\begin{cases} T_1^N(u^n) \in W, T_2^N(u^n) \in W \text{ pour tout } n \in I \setminus \{m_0\}; \\ T_1^N(u^m) \in V_1, T_2^N(u^m) \in V_2. \end{cases}$$

Alors T_1 et T_2 admettent une algèbre hypercyclique disjointe.

Ce résultat a été appliqué dans les deux cas qui nous intéressent: produit CPC (où on choisit $m_0 = \min(I)$) et produit de Cauchy (où on choisit $m_0 = \max(I)$ et le terme distingué est toujours présent).

15.1 Produit coordonnée par coordonnée

Pour le produit CPC, en utilisant des idées similaires à celles appliquées au Théorème 12.1, nous avons caractérisé les puissances de décalages à poids B_w et B_w^2 admettant une algèbre hypercyclique disjointe.

Théorème 15.1 *Soit X une algèbre de Fréchet de suites munie du produit CPC, avec une norme continue et telle que $(e_n)_n$ est une base de Schauder. Soit $B_1 = B_w$ et $B_2 = B_w$ deux opérateurs de décalage pondérés bornés sur X . Les assertions suivantes sont équivalentes.*

(i) B_1 et B_2^2 supportent une algèbre hypercyclique disjointe;

(ii) Il existe une suite $(n_k)_k$ telle que, pour tout $l \geq 0$ et tout $\gamma > 0$,

$$\begin{aligned} (w'_{l+1} \cdots w'_{n_k+l})^{-\gamma} e_{n_k+l} &\xrightarrow{k \rightarrow \infty} 0, \\ (w''_{l+1} \cdots w''_{2n_k+l})^{-\gamma} e_{2n_k+l} &\xrightarrow{k \rightarrow \infty} 0, \\ \frac{w'_{n_k+l+1} \cdots w'_{2n_k+l}}{w''_{l+1} \cdots w''_{2n_k+l}} e_{n_k+l} &\xrightarrow{k \rightarrow \infty} 0. \end{aligned}$$

Sur les espaces où $(e_n)_n$ est bornée, on peut enlever les termes e_{n_k+l} et donc $-\gamma$. Les conditions qui restent suffisent pour montrer l'existence d'un vecteur hypercyclique disjoint, selon [39]. Ainsi, sur les espaces $\ell_p(\mathbb{N})$, $1 \leq p < \infty$, ou $c_0(\mathbb{N})$, par exemple, l'existence d'une algèbre hypercyclique disjointe est équivalente à l'existence d'un vecteur hypercyclique disjoint. Ce résultat s'applique également à $B_1 = B_{w(\lambda)}$ et $B_2 = B_{w(\mu)}$, où $w_n(a) := 1 + \frac{a}{n}$ et $\lambda, \mu > 0$, sur $\ell_p(\mathbb{N})$ ou $c_0(\mathbb{N})$ avec le produit CPC. On peut aussi facilement l'appliquer pour obtenir une algèbre hypercyclique disjointe dans $H(\mathbb{C})$ avec le produit d'Hadamard pour $\lambda D, \mu D^2$, pour tous $\lambda, \mu > 0$.

15.2 Produit de Cauchy

Comme l'on s'y attendait, les résultats pour le produit de Cauchy sont plus difficiles à obtenir et moins génériques. En effet, on n'a pas exploré le cas général des décalages pondérés. Même pour l'exemple $B_1 = B_{w(\lambda)}$ et $B_2 = B_{w(\mu)}$, où $w_n(a) := 1 + \frac{a}{n}$ et $\lambda, \mu > 0$, nous ne savons pas s'il existe une algèbre hypercyclique disjointe sur $\ell_1(\mathbb{N})$ avec le produit de Cauchy. Néanmoins, dans le cas particulier des multiples $\lambda B, \mu B^2$, où B est l'opérateur de décalage, il est montré dans [39] que ce couple admet un vecteur hypercyclique disjoint si et seulement si $1 < \lambda < \mu$. Nous avons montré que cette condition est aussi suffisante pour l'obtention d'une algèbre hypercyclique disjointe dans $\ell_1(\mathbb{N})$ avec le produit de Cauchy. Un résultat similaire a été trouvé aussi pour $\lambda D, \mu D^2$, mais sans restrictions sur $\lambda, \mu > 0$. Nos découvertes sont résumées dans l'énoncé suivant.

Théorème 15.2 *Soit $\lambda, \mu > 0$.*

(1) *Les opérateurs λB et μB^2 supportent une algèbre hypercyclique disjointe dans $\ell_1(\mathbb{N})$ avec le produit de Cauchy si et seulement si $1 < \lambda < \mu$.*

(2) λD et λD^2 supportent une algèbre hypercyclique disjointe dans $H(\mathbb{C})$ avec le produit de convolution.

Un fait intéressant dans la démonstration du point (1) du théorème précédent est l'emplacement du terme distingué εe_σ . Comme toujours, il est présent car nous traitons le produit de Cauchy, toutefois la preuve a été séparée en deux cas sous la condition $1 < \lambda < \mu$, soit: (1) $1 < \lambda < \mu^{1/2}$; et (2) $\mu^{1/2} \leq \lambda < \mu$. Grosso modo cette division dérive de $\mu B^2 = (\mu^{1/2} B)^2$. Dans le cas (1), le terme εe_σ est positionné à la fin de la définition de l'élément $u \in U$ juste comme nous avons fait auparavant, c'est-à-dire, u a la forme $u = x + z + \varepsilon e_\sigma$. Dans le cas (2), le terme est positionné avant le terme d'approximation z , c'est-à-dire, on définit $u = x + \varepsilon e_\sigma + z$. Cette position nous a permis de faire fonctionner comme auparavant le jeu de supports.

16 Algèbres hypercycliques fermées

Le dernier sujet traité dans nos études a été l'existence des algèbres hypercycliques fermées (voir l'Article A, Section 6.1). Cette question est le prolongement naturel de la recherche de sous-espaces hypercycliques, c'est-à-dire de sous-espaces fermés et de dimension infinie dans $HC(T) \cup \{0\}$ (pour un certain opérateur hypercyclique T). Il n'est pas clair comment adapter notre méthode pour obtenir une algèbre fermée. Néanmoins, on a trouvé quelques exemples où la réponse est négative.

Le premier résultat répond négativement à une question posée par S. Shkarin dans [51] à propos de l'existence d'une algèbre hypercyclique fermée pour l'opérateur de dérivation D sur $H(\mathbb{C})$. Non seulement la réponse est négative mais, plus généralement, aucun opérateur de convolution $P(D)$ induit par un polynôme non-constant P n'admet une algèbre hypercyclique fermée dans $H(\mathbb{C})$. Le problème de trouver une fonction entière ϕ de type exponentiel fini et telle que $\phi(D)$ admet une algèbre hypercyclique fermée dans $H(\mathbb{C})$ reste néanmoins ouvert.

À propos de nos opérateurs favoris, les décalages pondérés sur les espaces de suites, on a montré que la réponse est toujours négative lorsque l'on considère le produit CPC sur $\ell_p(\mathbb{N})$, $1 \leq p < \infty$, sur $c_0(\mathbb{N})$ ou même sur l'espace plus permissif ω . Dans ces cas, aucun opérateur de décalage pondéré n'admet une algèbre hypercyclique fermée. Il reste quand même ouvert le cas où $\ell_1(\mathbb{N})$ est muni du produit de Cauchy, c'est-à-dire, on ne sait pas s'il existe un opérateur de décalage pondéré admettant une algèbre hypercyclique fermée dans $\ell_1(\mathbb{N})$ muni du produit de Cauchy.

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Part III

Included Articles

Article A

Baire theorem and hypercyclic algebras

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Abstract

The question of whether a hypercyclic operator T acting on a Fréchet algebra X admits or not an algebra of hypercyclic vectors (but 0) has been addressed in the recent literature. In this paper we give new criteria and characterizations in the context of convolution operators acting on $H(\mathbb{C})$ and backward shifts acting on a general Fréchet sequence algebra. Analogous questions arise for stronger properties like frequent hypercyclicity. In this trend we give a sufficient condition for a weighted backward shift to admit an upper frequently hypercyclic algebra and we find a weighted backward shift acting on c_0 admitting a frequently hypercyclic algebra for the coordinatewise product. The closed hypercyclic algebra problem is also covered.

1 Introduction

Among the many problems in linear dynamics, understanding the structure of the set of hypercyclic vectors is a major one. Let us introduce the relevant definitions. Let (X, T) be a linear dynamical system, namely X is a topological vector space and T is a bounded linear operator on X . A vector $x \in X$ with dense orbit under T is called a hypercyclic vector, and we denote by $HC(T)$ the set of hypercyclic vectors for T :

$$HC(T) = \{x \in X : \{x, Tx, T^2x, \dots\} \text{ is dense in } X\}.$$

This set $HC(T)$ possesses interesting properties. When X is a Baire space, its nonemptiness implies its residuality, preventing it from being a non-trivial proper linear subspace of X . However, it is well known that, whenever $HC(T)$ is nonempty, then $HC(T) \cup \{0\}$ contains a dense linear manifold (see [15]). In many cases (not always) $HC(T) \cup \{0\}$ contains a closed, infinite dimensional linear subspace (see [8, 20–22]). These properties reflect that we are working in a linear space.

Suppose now that X has a richer structure: it is an F -algebra, namely a metrizable and complete topological algebra. It is natural to ask whether $HC(T) \cup \{0\}$ also contains a non-trivial subalgebra of X . Such an algebra will be called a *hypercyclic algebra*. The pioneering work in that direction has been done independently by Shkarin in [22] and by Bayart and Matheron in [5]: they showed that the derivation operator $D : f \mapsto f'$, acting on the Fréchet algebra $H(\mathbb{C})$ of entire functions endowed with the pointwise multiplication, supports a hypercyclic algebra. However, this is not the case for all hypercyclic operators acting on an F -algebra: for instance, as pointed out in [3], the translation operators, acting on $H(\mathbb{C})$, do not support a hypercyclic algebra. Recent papers (see e.g. [4, 9–11, 16, 17]) give other examples of operators admitting a hypercyclic algebra.

Our aim, in this paper, is to shed new light on this problem and to study how it interacts with popular problems arising in linear dynamics. We are particularly interested in two questions.

1.1 Existence of hypercyclic algebras

All examples in the literature of operators supporting a hypercyclic algebra are generalizations of D . There are several ways to extend it. You may see D as a special convolution operator acting on $H(\mathbb{C})$. By [19], such an operator may be written $\phi(D)$, where ϕ is an entire function with exponential type; if ϕ is not constant, then $\phi(D)$ is hypercyclic. When $|\phi(0)| < 1$, the existence of hypercyclic algebras is well-understood since [4]: such an algebra does exist if and only if ϕ is not a multiple of an exponential function. When $|\phi(0)| = 1$, sufficient conditions are given in [4] or in [12] but almost nothing, except a very specific example, is known when $|\phi(0)| > 1$. We partly fill this gap by proving the existence of a hypercyclic algebra when ϕ goes to zero along some half-line.

Theorem 1.1 *Let ϕ be a nonconstant entire function with exponential type, not a multiple of an exponential function. Assume that $|\phi(0)| > 1$ and that there exists some $w \in \mathbb{C}$ such that $|\phi(tw)| \rightarrow 0$ as $t \rightarrow +\infty$. Then $\phi(D)$ supports a hypercyclic algebra.*

In particular, we shall see that if $\phi(z) = P(z)e^z$ for some non-constant polynomial P , then $\phi(D)$ supports a hypercyclic algebra.

Another way to generalize D is to see it as a weighted backward shift acting on $H(\mathbb{C})$ considered as a sequence space. This was explored in [16]. The general context is that of a Fréchet sequence algebra X . Precisely we assume that X is a subspace of the space $\omega = \mathbb{C}^{\mathbb{N}_0}$ of all complex sequences, whose topology is induced by a non-decreasing sequence of seminorms $(\|\cdot\|_q)_{q \geq 1}$ and that X is endowed with a product \cdot such that, for all $x, y \in X$, all $q \geq 1$,

$$\|x \cdot y\|_q \leq \|x\|_q \times \|y\|_q.$$

There are two natural products on a Fréchet sequence space: the coordinatewise product and the convolution or Cauchy product. It is clear that ℓ_p and c_0 are Fréchet sequence algebras for the coordinatewise product, and that ℓ_1 is also a Fréchet sequence algebra for the convolution product. Endowing $H(\mathbb{C})$ with

$$\left\| \sum_{n \geq 0} a_n z^n \right\|_q = \sum_{n \geq 0} |a_n| q^n$$

and ω with

$$\|(x_n)\|_q = \sum_{n=0}^q |x_n|,$$

we also obtain that $H(\mathbb{C})$ and ω are Fréchet sequence algebras for both products (on $H(\mathbb{C})$, the Cauchy product of f and g is nothing else but the product of the two functions f and g). Another interesting source of examples for us will be the sequence spaces $X = \{(x_n) \in \omega : \gamma_n x_n \rightarrow 0\}$ endowed with $\|x\| = \sup_n \gamma_n |x_n|$, where $(\gamma_n) \in \mathbb{R}_+^{\mathbb{N}_0}$. Provided $\gamma_n \geq 1$ for all n , X is a Fréchet sequence algebra for the coordinatewise product.

Given a sequence of nonzero complex numbers $w = (w_n)_{n \in \mathbb{N}}$, the (unilateral) weighted backward shift B_w with weight w is defined by

$$B_w(x_0, x_1, \dots) = (w_1 x_1, w_2 x_2, \dots).$$

The weight w will be called *admissible* (for X) if B_w is a bounded operator on X . It is known that, provided the canonical basis (e_n) is a Schauder basis of X , B_w is hypercyclic if and only if there exists a sequence (n_k) such that for all $l \in \mathbb{N}$, $((w_{l+1} \cdots w_{n_k+l})^{-1} e_{n_k+l})$ goes to zero.

Let us first assume that X is a Fréchet algebra under the coordinatewise product. Under a supplementary technical condition on X , a sufficient condition on w is given in [16] so that B_w supports a hypercyclic algebra. It turns out that we shall give a very natural characterization of this property when X admits a continuous norm. We recall that a Fréchet space $(X, (\|\cdot\|_q))$ admits a continuous norm if there exists a norm $\|\cdot\| : X \rightarrow \mathbb{R}$ that is continuous for the topology of X , namely there exists $q \in \mathbb{N}$ and $C > 0$ with $\|x\| \leq C\|x\|_q$ for all $x \in X$. In particular, for any q large enough, $\|\cdot\|_q$ itself is a norm.

Theorem 1.2 *Let X be a Fréchet sequence algebra for the coordinatewise product and with a continuous norm. Assume that (e_n) is a Schauder basis for X . Let also B_w be a bounded weighted shift on X . The following assumptions are equivalent.*

- (i) B_w supports a dense and not finitely generated hypercyclic algebra.
- (ii) There exists a sequence of integers (n_k) such that for all $\gamma > 0$, for all $l \in \mathbb{N}$, $((w_{l+1} \cdots w_{n_k+l})^{-\gamma} e_{n_k+l})$ tends to zero.

In particular, this theorem implies that on ℓ_p or c_0 , any hypercyclic weighted shift supports a hypercyclic algebra.

When X is a Fréchet algebra for the Cauchy product, it is shown in [16] that, under additional technical assumptions on X , B_w supports a hypercyclic algebra as soon as it is mixing, namely as soon as $(w_1 \cdots w_n)^{-1} e_n$ tends to zero. We shall improve that theorem by showing that any hypercyclic backward shift on a Fréchet sequence algebra for the Cauchy product supports a hypercyclic algebra. We will only require a supplementary assumption on X (to be regular) which is less strong than the assumption required in [16].

Theorem 1.3 *Let X be a regular Fréchet sequence algebra for the Cauchy product and let B_w be a bounded weighted shift on X . The following assertions are equivalent.*

- (i) B_w is hypercyclic.
- (ii) B_w supports a dense and not finitely generated hypercyclic algebra.

In particular, if we compare this statement with Theorem 1.2, we see that, for the convolution product, we do not need extra assumptions on the weight, which was not the case for the coordinatewise product. Even if the proofs of Theorems 1.2, 1.3 share some similarities (the latter one being much more difficult), they also have strong differences, the main one being that, under the coordinatewise product, any power of $x \in \omega$ keeps the same support, which is far from being the case if we work with the Cauchy product.

We also point out that this detailed study of the existence of hypercyclic algebras for weighted shifts has interesting applications. For instance, working with a bilateral shift, it will allow us to exhibit an invertible operator on a Banach algebra supporting a hypercyclic algebra and such that its inverse does not (see Example 4.12).

1.2 Frequently and upper frequently hypercyclic algebras

Another fruitful subject in linear dynamics is frequent and upper frequent hypercyclicity. We say that T is frequently hypercyclic (resp. upper frequently hypercyclic) if there exists a vector $x \in X$ such that, for all $U \subset X$ open and non-empty, the set $\{n \in \mathbb{N} : T^n x \in U\}$ has positive lower density (resp. positive upper density). Again, linearity allows to give a nice criterion to prove that an operator is (upper) frequently hypercyclic and gives rise to nice examples. For instance, if X is a Fréchet sequence space and B_w is a bounded weighted shift acting on X , it is known that the unconditional convergence of $\sum_{n \geq 1} (w_1 \cdots w_n)^{-1} e_n$ implies that B_w is frequently hypercyclic. Moreover, in some spaces (for instance, on ℓ_p -spaces), this condition is even necessary for the upper frequent hypercyclicity of B_w .

Of course, it is natural to ask if a (upper) frequently hypercyclic operator defined on an F -algebra X admits a (upper) frequently hypercyclic algebra, namely an algebra consisting only, except 0, of (upper) frequently hypercyclic vectors. Falcó and Grosse-Erdmann have shown recently ([17]) that this is not always the case: for instance, λB , $\lambda > 1$, acting on any ℓ_p space ($1 \leq p < +\infty$) or on c_0 , endowed with the coordinatewise product, does not admit a frequently hypercyclic algebra. Nevertheless, this leaves open the possibility for λB , $\lambda > 1$, to admit an upper frequently hypercyclic algebra.

We shall give two general results implying that a weighted shift on a Fréchet sequence algebra admits an upper frequently hypercyclic algebra. The first one deals with Fréchet sequence algebras endowed with the coordinatewise product. In view of Theorem 1.2, the natural extension of the above result for the existence of an upper frequently hypercyclic vector is to ask now for the unconditional convergence of the series $\sum_{n \geq 1} (w_1 \cdots w_n)^{-1/m} e_n$ for all $m \geq 1$. This is sufficient!

Theorem 1.4 *Let X be a Fréchet sequence algebra for the coordinatewise product and with a continuous norm. Assume that (e_n) spans a dense subspace of X . Let also B_w be a bounded weighted shift on X such that, for all $m \geq 1$, $\sum_{n \geq 1} (w_1 \cdots w_n)^{-1/m} e_n$ converges unconditionally. Then B_w admits an upper frequently hypercyclic algebra.*

In particular, for $\lambda > 1$, λB admits on any ℓ_p -space ($1 \leq p < +\infty$) and on c_0 an upper frequently hypercyclic algebra. This last result was independently obtained by Falcó and Grosse-Erdmann in [17] in a different context (they concentrate themselves on λB but allow different notions of hypercyclicity) and with a completely different proof.

Regarding Fréchet sequence algebras endowed with the convolution product, we also have been able to get a general statement (see the forthcoming Theorem 5.7). Its main feature is that we will only need the unconditional convergence of $\sum_{n \geq 1} (w_1 \cdots w_n)^{-1} e_n$ and a technical

condition to ensure the existence of an upper frequently hypercyclic algebra. As a corollary, we can state the following.

Corollary 1.5 (i) *Let $X = \ell_1$ endowed with the convolution product and let $\lambda > 1$. Then λB admits an upper frequently hypercyclic algebra.*

(ii) *Let $X = H(\mathbb{C})$ endowed with the convolution product. Then D admits an upper frequently hypercyclic algebra.*

Coming back to our initial problem, we show that it is possible to exhibit a weighted shift supporting a frequently hypercyclic algebra. The place to do this will be c_0 endowed with the coordinatewise product; of course, the weight sequence will be much more complicated than that of the Rolewicz operator.

Theorem 1.6 *There exists a weight (w_n) such that B_w , acting on c_0 endowed with the coordinatewise product, supports a frequently hypercyclic algebra.*

The proof of this theorem will need the construction of disjoint subsets of \mathbb{N} with positive lower density and with some other extra properties, which seems interesting by itself.

1.3 Organization of the paper

Up to now, there were two ways to produce hypercyclic algebras: by a direct construction (this is the method devised in [22] and in [16]) or by using a Baire argument (this method was initiated in [5]). In this paper, we improve the latter. We first give in Section 2.1 a general result for the existence of a hypercyclic algebra, enhancing the main lemma proved in [5]. This general theorem will be suitable to our new examples of operators supporting a hypercyclic algebra. Next, we adapt the Baire argument to produce upper frequently hypercyclic algebras as well. Since the set of frequently hypercyclic vectors is always meagre, Theorem 1.6 cannot be proved using such an argument; it follows from a careful construction both of the weight and of the algebra.

We finish in the last section by making some remarks and asking some questions. In particular, we give a negative answer to a question raised by Shkarin about the existence of a closed hypercyclic algebra for the derivation operator.

1.4 Notations

The symbol \mathbb{N} will stand for the set of positive integers, whereas $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. We shall denote by $\mathcal{P}_f(A)$ the set of finite subsets of a given set A .

For $x = \sum_{n=0}^{+\infty} x_n e_n \in \omega$, the support of x is equal to $\text{supp}(x) = \{n \in \mathbb{N}_0 : x_n \neq 0\}$. The notation c_{00} will denote the set of sequences in ω with finite support.

For $u \in X^d$ and $\alpha \in \mathbb{N}_0^d$, u^α will mean $u_1^{\alpha_1} \cdots u_d^{\alpha_d}$. If z is any complex number and $m \in \mathbb{N}$, $z^{1/m}$ will denote any m th root of z .

When working on a Fréchet space $(X, \|\cdot\|_p)$, it is often convenient to endow X with an F -norm $\|\cdot\|$ defining the topology of X (see [18, Section 2.1]). Such an F -norm can be defined

by the formula

$$\|x\| = \sum_{p=1}^{+\infty} \frac{1}{2^p} \min(1, \|x\|_p).$$

In particular, an F -norm satisfies the triangle inequality and the inequality

$$\forall \lambda \in \mathbb{C}, \forall x \in X, \|\lambda x\| \leq (|\lambda| + 1)\|x\|, \quad (\text{A.1})$$

a property which replaces the positive homogeneity of the norm.

We finally recall some results on unconditional convergence in Fréchet spaces (see for instance [18, Appendix A]). A series $\sum_{n=0}^{+\infty} x_n$ in a Fréchet space X is called unconditionally convergent if for any bijection $\pi : \mathbb{N}_0 \rightarrow \mathbb{N}_0$, the series $\sum_{n=0}^{+\infty} x_{\pi(n)}$ is convergent. This amounts to saying that, for any $\varepsilon > 0$, there is some $N \in \mathbb{N}$ such that, whenever $\sup_n |\alpha_n| \leq 1$, the series $\sum_{n=0}^{+\infty} \alpha_n x_n$ converges and

$$\left\| \sum_{n=N}^{+\infty} \alpha_n x_n \right\| < \varepsilon.$$

2 A general criterion

2.1 A transitivity criterion to get hypercyclic algebras

We first give a general statement which may be thought of as a Birkhoff transitivity theorem for hypercyclic algebras. This criterion will be the main ingredient for the results from Section 3.1 and some from Section 2. A different version of this approach will give rise to a new criterion for the existence of upper frequently hypercyclic algebras in Section 5.1.

Theorem 2.1 *Let T be a continuous operator on a separable commutative F -algebra X and let $d \geq 1$. Assume that for any $A \subset \mathbb{N}_0^d \setminus \{(0, \dots, 0)\}$ finite and non-empty, for any non-empty open subsets U_1, \dots, U_d, V of X , for any neighbourhood W of 0, there exist $u = (u_1, \dots, u_d) \in U_1 \times \dots \times U_d$, $\beta \in A$ and $N \geq 1$ such that $T^N(u^\beta) \in V$ and $T^N(u^\alpha) \in W$ for all $\alpha \in A$, $\alpha \neq \beta$. Then the set of d -tuples that generate a hypercyclic algebra for T is residual in X^d . Moreover, if the assumptions are satisfied for all $d \geq 1$, then T admits a dense and not finitely generated hypercyclic algebra.*

Proof. Let (V_k) be a basis of open neighbourhoods of X . For $A \in \mathcal{P}_f(\mathbb{N}_0^d)$, $A \neq \emptyset$, $(0, \dots, 0) \notin A$, for $s, k \geq 1$, for $\beta \in A$, define

$$E(A, \beta, s) = \left\{ \sum_{\alpha \in A} \hat{P}(\alpha) z^\alpha \in \mathbb{C}[z_1, \dots, z_d] : \hat{P}(\beta) = 1 \text{ and } \sup_{\alpha \in A} |\hat{P}(\alpha)| \leq s \right\}$$

$$\mathcal{A}(A, \beta, s, k) = \left\{ u \in X^d : \forall P \in E(A, \beta, s), \exists N \geq 1, T^N(P(u)) \in V_k \right\}.$$

The sets $E(A, \beta, s)$ are compact subsets of $\mathbb{C}[z_1, \dots, z_d]$. By continuity of the maps $(u, P) \mapsto T^N(P(u))$, this implies that each set $\mathcal{A}(A, \beta, s, k)$ is open. Let us show that, for all A, k and s , $\bigcup_{\beta \in A} \mathcal{A}(A, \beta, s, k)$ is dense in X^d . Indeed, pick U_1, \dots, U_d non-empty open subsets of X . Let $V \subset V_k$ and $W = B(0, \varepsilon)$ be a neighbourhood of 0 such that $V + B(0, (s+1) \text{card}(A)\varepsilon) \subset V_k$.

The assumptions of the proposition give the existence of $u = (u_1, \dots, u_d) \in U_1 \times \dots \times U_d$, $\beta \in A$ and $N \geq 1$. We claim that u belongs to $\mathcal{A}(A, \beta, s, k)$. Indeed,

$$T^N P(u) = \sum_{\alpha \neq \beta} \hat{P}(\alpha) T^N(u^\alpha) + T^N(u^\beta) \in V + B(0, (s+1) \text{card}(A)\varepsilon) \subset V_k$$

(observe that we have used (B.1)). Hence, $\bigcap_{A,s,k} \bigcup_{\beta} \mathcal{A}(A, \beta, s, k)$ is a residual subset of X^d . Pick $u \in \bigcap_{A,s,k} \bigcup_{\beta} \mathcal{A}(A, \beta, s, k)$.

We show that for all non-zero polynomials $P \in \mathbb{C}[z_1, \dots, z_d]$ with $P(0) = 0$, $P(u)$ belongs to $HC(T)$. We set $A = \{\alpha : \hat{P}(\alpha) \neq 0\}$ and we first prove that $\bigcup_{\beta \in A} \text{Orb}\left(T, \frac{1}{\hat{P}(\beta)} P(u)\right)$ is dense. Let us fix some k and let us set $s = \sup_{\alpha, \beta \in A} |\hat{P}(\alpha)| / |\hat{P}(\beta)|$. Let $\beta \in A$ be such that $u \in \mathcal{A}(A, \beta, s, k)$. Define $Q = P/\hat{P}(\beta)$. Then Q belongs to $E(A, \beta, s)$ so that there exists $N \geq 1$ satisfying

$$T^N \left(\frac{1}{\hat{P}(\beta)} P(u) \right) = T^N(Q(u)) \in V_k.$$

By the Bourdon-Feldman theorem, we deduce that there is some $\beta_0 \in A$ such that $\text{Orb}\left(T, \frac{1}{\hat{P}(\beta_0)} P(u)\right)$ is dense in X . Since any non-zero multiple of a hypercyclic vector remains hypercyclic, we finally deduce that $P(u)$ is a hypercyclic vector for T .

The modification to obtain dense and infinitely generated algebras is easy. For $A \in \mathcal{P}_f(\mathbb{N}_0^d)$, $A \neq \emptyset$, $(0, \dots, 0) \notin A$, we now let

$$\mathcal{A}(A, \beta, s, k) = \left\{ u \in X^{\mathbb{N}} : \forall P \in E(A, \beta, s), \exists N \geq 1, T^N(P(u)) \in V_k \right\}$$

and we still consider the set $\bigcap_{A,s,k} \bigcup_{\beta \in A} \mathcal{A}(A, \beta, s, k)$ where now the intersection runs over all non-empty and finite sets $A \subset \mathbb{N}^d \setminus \{(0, \dots, 0)\}$ with $d \geq 1$ arbitrary. This intersection is still residual in $X^{\mathbb{N}}$. We also know from [11] that the set of u in $X^{\mathbb{N}}$ that induce a dense algebra in X is residual in $X^{\mathbb{N}}$. Hence we may pick $u \in X^{\mathbb{N}}$ belonging to $\bigcap_{A,s,k} \bigcup_{\beta} \mathcal{A}(A, \beta, s, k)$ and inducing a dense algebra in X . It is plain that for any non-zero polynomial P with $P(0) = 0$, $P(u)$ is hypercyclic for T .

It remains to show that the algebra generated by u is not finitely generated. Assume on the contrary that it is generated by a finite number of $P_1(u), \dots, P_p(u)$. In particular, it is generated by a finite number of u_1, \dots, u_q . Then there exists a polynomial $Q \in \mathbb{C}[z_1, \dots, z_q]$ such that $Q(0) = 0$ and $u_{q+1} = Q(u_1, \dots, u_q)$. Define $P(z) = z_{q+1} - Q(z)$. Then P is a non-zero polynomial with $P(0) = 0$. Nevertheless, $P(u) = 0$, which contradicts the fact that $P(u)$ is a hypercyclic vector for T . \square

Remark 2.2 Theorem 2.1 remains true if the algebra is not commutative. This is clear if $d = 1$. For the remaining cases, we have to replace in the proof polynomials in d commutative variables by polynomials in d non-commutative variables. Details are left to the reader.

We point out that, unlike [11, Lemma 3.1], in the previous theorem, the index β may depend on A, U_1, \dots, U_d, V and W . We will never use this possibility: we will only need that β may depend on A and we will denote $\beta = \beta_A$. For this particular case, we could give an easier proof avoiding the use of the Bourdon-Feldman theorem (see the proof of [11, Lemma 3.1]).

Let us give a couple of corollaries. The first one comes from [5, Remark 5.28] and was the key lemma in [5], [10] or [4] to get hypercyclic algebras.

Corollary 2.3 *Let T be a continuous operator on a separable F -algebra X . Assume that, for any pair (U, V) of non-empty open sets in X , for any open neighbourhood W of zero, and for any positive integer m , one can find $u \in U$ and an integer N such that $T^N(u^n) \in W$ for all $n < m$ and $T^N(u^m) \in V$. Then T admits a hypercyclic algebra.*

Proof. It is straightforward to show that the assumptions of Theorem 2.1 with $d = 1$ and $\beta_A = \max A$ are satisfied. \square

In this work, we will often use the inverse choice for β_A .

Corollary 2.4 *Let T be a continuous operator on a separable F -algebra X . Assume that, for any pair (U, V) of non-empty open sets in X , for any open neighbourhood W of zero, and for any positive integers $m_0 < m_1$, one can find $u \in U$ and an integer N such that $T^N(u^m) \in W$ for all $m \in \{m_0 + 1, \dots, m_1\}$ and $T^N(u^{m_0}) \in V$. Then T admits a hypercyclic algebra.*

Proof. This is now Theorem 2.1 with $d = 1$ and $\beta_A = \min A$. \square

2.2 Countably generated, free hypercyclic algebras.

The conclusion of Theorem 2.1 about infinitely generated hypercyclic algebras does not prevent the possibility for such an algebra to be contained in a finitely generated algebra. Furthermore, it is well known that every at least two generated algebra contains an infinitely generated subalgebra. For all of the examples of this paper we may avoid this scenario thanks to the following result. We notice that due to [11, Corollary 2.7], a countably generated free algebra is not contained in a finitely generated one.

Corollary 2.5 *Let X be a separable commutative F -algebra that contains a dense freely generated subalgebra. Let T be a continuous operator on X and let $d \geq 1$. Assume that for any $A \subset \mathbb{N}_0^d \setminus \{(0, \dots, 0)\}$ finite and non-empty, for any non-empty open subsets U_1, \dots, U_d, V of X , for any neighbourhood W of 0, there exist $u = (u_1, \dots, u_d) \in U_1 \times \dots \times U_d$, $\beta \in A$ and $N \geq 1$ such that $T^N(u^\beta) \in V$ and $T^N(u^\alpha) \in W$ for all $\alpha \in A$, $\alpha \neq \beta$. Then T admits a d -generated, free hypercyclic algebra. Moreover, if the assumptions are satisfied for all $d \geq 1$, then T admits a dense, countably generated, free hypercyclic algebra.*

Proof. By Theorem 2.1 the set of d -tuples generating a hypercyclic algebra for T is residual in X^d . By [11, Proposition 2.4], the set of u in X^d that induce a d -generated, free algebra is residual in X^d . For the conclusion we just need to pick an element in the intersection of those two sets.

For the second claim, we consider $X^{\mathbb{N}}$ endowed with the product topology. By the assumption, for each $N \in \mathbb{N}$, the set

$$H_N = \{(u_n) \in X^{\mathbb{N}} : (u_1, \dots, u_N) \text{ generates a hypercyclic algebra for } T\}$$

is residual in $X^{\mathbb{N}}$ and hence, by the Baire category theorem, the set $H = \bigcap_{N=1}^{\infty} H_N$ is residual as well. The algebra generated by any $u \in H$ is hypercyclic for T . Furthermore, by [11, Proposition 2.4], the set of sequences of $X^{\mathbb{N}}$ which generate a dense and free algebra is also residual. The conclusion follows by one more application of the Baire category theorem. \square

Hence, we need to provide for our examples a dense and freely generated subalgebra. This is quite easy for a Fréchet sequence algebra endowed with the Cauchy product: provided $\text{span}(e_n)$ is dense, it always contain a dense and freely generated subalgebra, namely the unital algebra generated by the sequence e_1 . This covers the case of $H(\mathbb{C})$ and of the disc algebra. This is slightly more difficult for a Fréchet sequence algebra endowed with the coordinatewise product.

Lemma 2.6 *Let X be a Fréchet sequence algebra, endowed with the pointwise product, and for which the sequence (e_n) is a Schauder basis. Then X has a dense freely generated subalgebra.*

Proof. Let $(b_n) \subset (0, 1)$ be such that the series $\sum_{n=0}^{\infty} b_n \|e_n\|_n$ converges. Consider the sequence of natural numbers (a_n) such that $a_0 = 0$ and $a_n = a_{n-1} + n, n \in \mathbb{N}$ and define $c_n := \min_{l \in [a_{m-1}, a_m]} b_l$, if $n \in [a_{m-1}, a_m)$. The series $\sum_{n=0}^{\infty} c_n e_n$ converges absolutely since $c_n \leq b_n, n \in \mathbb{N}$.

Define a sequence $(\lambda_n) \subset (0, 1)$ inductively as follows: choose $\lambda_0 \in (0, 1)$ and, for $n \in \mathbb{N}$, take $\lambda_n \in (0, 1) \setminus \{\lambda_0^{p_0} \dots \lambda_{n-1}^{p_{n-1}} : p_0, \dots, p_{n-1} \in \mathbb{Q}\}$. Observe that if $\lambda = (\lambda_0, \dots, \lambda_p), p \in \mathbb{N}_0$ and $\alpha \neq \beta \in \mathbb{N}_0^p$, then $\lambda^\alpha \neq \lambda^\beta$. Let now, for each $n \in \mathbb{N}_0$,

$$g_n = e_n + \sum_{k=1}^{\infty} \lambda_n^{n+k} c_{n+k} e_{n+k},$$

where the convergence of the series is ensured by the convergence of $\sum_{n=0}^{\infty} b_n \|e_n\|_n$ and the inequality $\lambda_n^{n+k} c_{n+k} < b_{n+k}$. We claim that the algebra generated by the $g_n, n \in \mathbb{N}$ is dense and free.

First, we show that the sequence $\{g_n : n \in \mathbb{N}\}$ is algebraically independent. For that reason, let

$$\sum_{\alpha \in A} a_\alpha g^\alpha = 0, \quad (\text{A.2})$$

where $A = \{\alpha(1), \dots, \alpha(q)\} \subset \mathbb{N}_0^p, g = (g_1, \dots, g_p) \in X^p$, and $p \in \mathbb{N}$. If we consider the coordinate $N = n + k$ in equation (A.2), we get the following equation which holds for all N sufficiently large.

$$\sum_{i=1}^q a_{\alpha(i)} (\lambda_1^N c_N)^{\alpha_1(i)} \dots (\lambda_p^N c_N)^{\alpha_p(i)} = 0. \quad (\text{A.3})$$

Now we may choose N sufficiently big such that $N, \dots, N + q - 1 \in [a_{m-1}, a_m)$ for some m , which means that $c_N = \dots = c_{N+q-1} = b_m$. Equation (A.3) then becomes

$$\sum_{i=1}^q a_{\alpha(i)} (\lambda_1^M b_m)^{\alpha_1(i)} \dots (\lambda_p^M b_m)^{\alpha_p(i)} = 0,$$

where M ranges over $N, \dots, N + q - 1$. Setting A the matrix

$$\begin{bmatrix} (\lambda_1^N b_m)^{\alpha_1(1)} \dots (\lambda_p^N b_m)^{\alpha_p(1)} & \dots & (\lambda_1^N b_m)^{\alpha_1(q)} \dots (\lambda_p^N b_m)^{\alpha_p(q)} \\ \vdots & \ddots & \vdots \\ (\lambda_1^{N+q-1} b_m)^{\alpha_1(1)} \dots (\lambda_p^{N+q-1} b_m)^{\alpha_p(1)} & \dots & (\lambda_1^{N+q-1} b_m)^{\alpha_1(q)} \dots (\lambda_p^{N+q-1} b_m)^{\alpha_p(q)} \end{bmatrix},$$

we find the matrix equality

$$A \begin{bmatrix} a_{\alpha(1)} \\ \vdots \\ a_{\alpha(q)} \end{bmatrix} = 0.$$

The determinant of the square matrix A , after making use of the Vandermonde identity, is

$$\prod_{i=1}^q \prod_{j=1}^p (b_m \lambda_j^N)^{\alpha_j(i)} \prod_{i>j} [(\lambda_1^{\alpha_1(i)} \dots \lambda_p^{\alpha_p(i)}) - (\lambda_1^{\alpha_1(j)} \dots \lambda_p^{\alpha_p(j)})] \neq 0.$$

Hence, we get that $a_{\alpha(i)} = 0$ for all $i = 1, \dots, q$.

Next, we show that the algebra generated by $\{g_n : n \in \mathbb{N}\}$ is dense in X . We will show that the elements $e_n, n \in \mathbb{N}$, are in the closure of this algebra. Let us fix $n \in \mathbb{N}$ and observe that, for all $p \in \mathbb{N}$,

$$g_n^p - e_n = \sum_{k=1}^{\infty} \lambda_n^{(n+k)p} c_{n+k}^p e_{n+k}.$$

Fix $q \in \mathbb{N}$ and let $\varepsilon > 0$. There exists $N \geq q$ such that, for all $p \in \mathbb{N}$,

$$\left\| \sum_{k>N} \lambda_n^{(n+k)p} c_{n+k}^p e_{n+k} \right\|_q \leq \sum_{k>N} b_{n+k} \|e_{n+k}\|_q < \varepsilon.$$

Since furthermore

$$\sum_{k=1}^N \lambda_n^{(n+k)p} c_{n+k}^p e_{n+k} \xrightarrow{p \rightarrow \infty} 0,$$

we conclude that $g_n^p \xrightarrow{p \rightarrow \infty} e_n$. □

We conclude this subsection by comparing Corollary 2.5 with [11, Remark 3.4]. Corollary 2.5 allows the index β to depend on A, U_1, \dots, U_d, V and W providing, at least theoretically, an extra flexibility and range of application for the result. Practically, throughout the paper, β will depend only on A in which case Corollary 2.5 and [11, Remark 3.4] coincide. We were unable to find an example where Corollary 2.5 applies while [11, Remark 3.4] does not.

3 Convolution operators with $|\phi(0)| > 1$

3.1 Operators with many eigenvectors

In this section we shall deduce Theorem 1.1 from a more general assertion on operators having many eigenvalues. As Theorem 2.1 does for Corollary 2.3, this generalized statement also includes [4, Theorem 2.1] as a particular case. Before stating and proving it, let us add some notation.

Given $p, d \in \mathbb{N}$, we denote each set $\{1, \dots, p\}$ by I_p , each d -tuple $(j_1, \dots, j_d) \in I_p^d$ by the multi-index $\mathbf{j} \in I_p^d$ and each product $a_{j_1} \dots a_{j_m}$ by the symbol $a_{\mathbf{j}}$. We allow $d = 0$ with the convention that, in this case, $a_{\mathbf{j}} = 1$.

Theorem 3.1 *Let X be an F -algebra and let $T \in \mathcal{L}(X)$. Assume that there exist a function $E : \mathbb{C} \rightarrow X$ and an entire function $\phi : \mathbb{C} \rightarrow \mathbb{C}$ satisfying the following assumptions:*

- (a) for all $\lambda \in \mathbb{C}$, $TE(\lambda) = \phi(\lambda)E(\lambda)$;
- (b) for all $\lambda, \mu \in \mathbb{C}$, $E(\lambda)E(\mu) = E(\lambda + \mu)$;
- (c) for all $\Lambda \subset \mathbb{C}$ with an accumulation point, the linear span of $\{E(\lambda) : \lambda \in \Lambda\}$ is dense in X ;

(d) ϕ is not a multiple of an exponential function;

(e) for all $I \in \mathcal{P}_f(\mathbb{N}) \setminus \{\emptyset\}$, there exist $m \in I$ and $a, b \in \mathbb{C}$ such that $|\phi(mb)| > 1$ and, for all $n \in I$ and $d \in \{0, \dots, n\}$, with $(n, d) \neq (m, m)$, $|\phi(db + (n-d)a)| < |\phi(mb)|^{d/m}$.

Then T supports a hypercyclic algebra.

The proof of this result follows the lines of that of Theorem 2.1 in [4], replacing Corollary 2.3 by the more general Theorem 2.1. For the sake of completeness, we include the details.

Proof. Let (U, V, W) be a triple of non-empty open sets in X , with $0 \in W$, and let $I \in \mathcal{P}_f(\mathbb{N}) \setminus \{\emptyset\}$. By the hypothesis there are $m \in I$ and $a, b \in \mathbb{C}$ satisfying (e). Define $w_0 := mb$ and let $\delta > 0$ be small enough and $w_1, w_2 \in B(w_0, \delta)$ so that

- (i) $|\phi| > 1$ on $B(w_0, \delta)$;
- (ii) $t \mapsto \log |\phi(tw_1 + (1-t)w_2)|$ is strictly convex (the existence of $w_1, w_2 \in B(w_0, \delta)$ comes from [4, Lemma 2.2] and is a consequence of (d));
- (iii) for all $n \in I$ and $d \in \{0, \dots, n\}$, with $(n, d) \neq (m, m)$, and for all $\lambda_1, \dots, \lambda_d \in B(w_0, \delta)$ and $\gamma_1, \dots, \gamma_{n-d} \in B(a, \delta)$,

$$\left| \phi \left(\frac{\lambda_1 + \dots + \lambda_d}{m} + \gamma_1 + \dots + \gamma_{n-d} \right) \right| < (|\phi(\lambda_1)| \times \dots \times |\phi(\lambda_d)|)^{1/m}. \quad (\text{A.4})$$

The last condition can be satisfied because

$$\frac{\lambda_1 + \dots + \lambda_d}{m} + \gamma_1 + \dots + \gamma_{n-d} = db + (n-d)a + z,$$

where the size of z can be controlled through δ . Now, since $B(a, \delta)$ and $[w_1, w_2]$ have accumulation points, we can find $p, q \in \mathbb{N}$, $a_1, \dots, a_p, b_1, \dots, b_q \in \mathbb{C}$, $\gamma_1, \dots, \gamma_p \in B(a, \delta)$ and $\lambda_1, \dots, \lambda_q \in [w_1, w_2]$ with

$$\sum_{l=1}^p a_l E(\gamma_l) \in U \quad \text{and} \quad \sum_{j=1}^q b_j E(\lambda_j) \in V.$$

For some big $N \in \mathbb{N}$ (which will be determined later in the proof) and each $j \in \{1, \dots, q\}$, let $c_j := c_j(N)$ be any complex number satisfying

$$c_j^m(N) = \frac{b_j}{\phi(\lambda_j)^N}$$

and define

$$u := u(N) = \sum_{l=1}^p a_l E(\gamma_l) + \sum_{j=1}^q c_j E\left(\frac{\lambda_j}{m}\right).$$

For the powers of u we have the formula

$$u^n = \sum_{d=0}^n \sum_{\substack{\mathbf{l} \in I_p^{n-d} \\ \mathbf{j} \in I_q^d}} \alpha(\mathbf{l}, \mathbf{j}, d, n) a_1 c_j E\left(\gamma_{l_1} + \dots + \gamma_{l_{n-d}} + \frac{\lambda_{j_1} + \dots + \lambda_{j_d}}{m}\right). \quad (\text{A.5})$$

We claim that, if N is taken large enough, $u = u(N)$ satisfies the conditions of the general criterion with $d = 1$, what will complete the proof.

That $u \in U$ for large N is clear since, from (i), $c_j(N) \rightarrow 0$ as $N \rightarrow \infty$. Applying T^N to u^n we see that we need to study the behaviour (as N grows) of

$$c_j(N) \left[\phi \left(\gamma_{l_1} + \cdots + \gamma_{l_{n-d}} + \frac{\lambda_{j_1} + \cdots + \lambda_{j_d}}{m} \right) \right]^N. \quad (\text{A.6})$$

For $n \in I \setminus \{m\}$ we have that (A.6) goes to 0 as N grows by the inequality (A.4) and the definition of c_j , $j = 1, \dots, q$. This way we get $T^N(u^n) \in W$ for all $n \in I \setminus \{m\}$ if N is large enough. Now let us consider the case $n = m$. We have

$$\begin{aligned} u^m &= \sum_{d=0}^{m-1} \sum_{\substack{\mathbf{l} \in I_p^{m-d} \\ \mathbf{j} \in I_q^d}} \alpha(\mathbf{l}, \mathbf{j}, d, m) a_{\mathbf{l}} c_{\mathbf{j}} E \left(\gamma_{l_1} + \cdots + \gamma_{l_{m-d}} + \frac{\lambda_{j_1} + \cdots + \lambda_{j_d}}{m} \right) \\ &+ \sum_{\mathbf{j} \in I_q^m \setminus D_q} \alpha(\mathbf{j}, m) c_{\mathbf{j}} E \left(\frac{\lambda_{j_1} + \cdots + \lambda_{j_m}}{m} \right) \\ &+ \sum_{j=1}^q c_j^m E(\lambda_j) \\ &=: v_1 + v_2 + v_3, \end{aligned}$$

where

$$\begin{aligned} v_1 &:= \sum_{d=0}^{m-1} \sum_{\substack{\mathbf{l} \in I_p^{m-d} \\ \mathbf{j} \in I_q^d}} \alpha(\mathbf{l}, \mathbf{j}, d, m) a_{\mathbf{l}} c_{\mathbf{j}} E \left(\gamma_{l_1} + \cdots + \gamma_{l_{m-d}} + \frac{\lambda_{j_1} + \cdots + \lambda_{j_d}}{m} \right), \\ v_2 &:= \sum_{\mathbf{j} \in I_q^m \setminus D_q} \alpha(\mathbf{j}, m) c_{\mathbf{j}} E \left(\frac{\lambda_{j_1} + \cdots + \lambda_{j_m}}{m} \right), \quad v_3 := \sum_{j=1}^q c_j^m E(\lambda_j) \end{aligned}$$

and D_q is the diagonal of I_q^m , that is, the set of all m -tuples (j, \dots, j) with $1 \leq j \leq q$. Again we have $T^N(v_1) \rightarrow 0$ as $N \rightarrow \infty$ from (A.4). Furthermore, since $t \in [0, 1] \mapsto \log |\phi(tw_1 + (1-t)w_2)|$ is strictly convex, we have

$$\left| \phi \left(\frac{\lambda_{j_1} + \cdots + \lambda_{j_m}}{m} \right) \right| < |\phi(\lambda_{j_1})|^{1/m} \cdots |\phi(\lambda_{j_m})|^{1/m}.$$

From this we conclude that

$$\begin{aligned} &|c_j(N)| \cdot \left| \phi \left(\frac{\lambda_{j_1} + \cdots + \lambda_{j_m}}{m} \right) \right|^N \\ &= \left| \frac{b_{\mathbf{j}}^{1/m}}{|\phi(\lambda_{j_1})|^{N/m} \cdots |\phi(\lambda_{j_n})|^{N/m}} \right| \cdot \left| \phi \left(\frac{\lambda_{j_1} + \cdots + \lambda_{j_m}}{m} \right) \right|^N \\ &= |b_{\mathbf{j}}^{1/m}| \cdot \left| \frac{\phi \left(\frac{\lambda_{j_1} + \cdots + \lambda_{j_m}}{m} \right)}{|\phi(\lambda_{j_1})|^{1/m} \cdots |\phi(\lambda_{j_m})|^{1/m}} \right|^N \rightarrow 0 \text{ as } N \rightarrow \infty, \end{aligned}$$

what shows that $T^N(v_2)$ also tends to 0 as $N \rightarrow \infty$. Finally, by the definition of c_j , $j = 1, \dots, q$, we get

$$T^N v_3 = \sum_{j=1}^q b_j E(\lambda_j) \in V$$

for all $N \in \mathbb{N}$. This completes the proof. \square

We now deduce a more readable corollary, when the entire function ϕ is “well behaved” in some half-line of the complex plane (like in the Figure A.1 for example).

Corollary 3.2 *Let X be an F -algebra and let $T \in \mathcal{L}(X)$. Assume that there exist a function $E : \mathbb{C} \rightarrow X$ and an entire function $\phi : \mathbb{C} \rightarrow \mathbb{C}$ satisfying the following assumptions:*

- (a) *for all $\lambda \in \mathbb{C}$, $TE(\lambda) = \phi(\lambda)E(\lambda)$;*
- (b) *for all $\lambda, \mu \in \mathbb{C}$, $E(\lambda)E(\mu) = E(\lambda + \mu)$;*
- (c) *for all $\Lambda \subset \mathbb{C}$ with an accumulation point, the linear span of $\{E(\lambda) : \lambda \in \Lambda\}$ is dense in X ;*
- (d) *ϕ is not a multiple of an exponential function;*
- (e) *there exist $v \in \mathbb{C}$ and a real number $p > 0$ such that $|\phi(v)| > 1$ and $|\phi(tv)| \leq 1$ for all $t > p$.*

Then T supports a hypercyclic algebra.

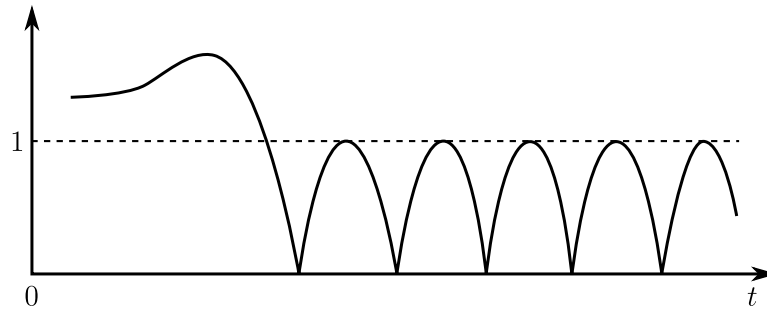


Fig. A.1 – Graph of $t \mapsto |\phi(tv)|$

For the proof of Corollary 3.2, we are going to need the following simple lemma.

Lemma 3.3 *Let ϕ be entire and $\Lambda \subset \mathbb{R}$ have an accumulation point in it. If $|\phi(t)| = 1$ for all $t \in \Lambda$ then $|\phi(t)| = 1$ for all $t \in \mathbb{R}$.*

Proof. Since $|\phi(t)| = 1$ for all $t \in \Lambda \subset \mathbb{R}$, we have $\overline{\phi(\bar{t})} \cdot \phi(t) = \overline{\phi(\bar{t})} \cdot \phi(t) = 1$, hence

$$\overline{\phi(\bar{t})} = \phi(t)^{-1}, \forall t \in \Lambda \subset \mathbb{R}.$$

Since this is a holomorphic equality ($t \mapsto \overline{\phi(\bar{t})}$ is entire), it extends to the whole complex plane. In particular it holds for all $t \in \mathbb{R}$, that is,

$$\overline{\phi(\bar{t})} = \overline{\phi(\bar{t})} = \phi(t)^{-1}, \forall t \in \mathbb{R},$$

thus

$$|\phi(t)| = 1, \forall t \in \mathbb{R},$$

as we wanted. □

Proof of Corollary 3.2. We may assume without loss of generality that $v = 1$. Let $t_0 > 0$ be the smallest positive real number such that $|\phi(t)| \leq 1$ for all $t \geq t_0$. We just need to prove that condition (e) of Theorem 3.1 is satisfied. So let $I \in \mathcal{P}_f(\mathbb{N}) \setminus \{\emptyset\}$ be arbitrary and set $m = \min I$. We can find $t_1 < t_0$ near enough to t_0 so that $|\phi(t_1)| > 1$ and $t_1 + \frac{t_1}{m} > t_0$. Letting $b := \frac{t_1}{m}$ we have $|\phi(mb)| > 1$. Now fix $a_0 > t_0$ and take $\epsilon \in (0, 1/2)$ such that $a_0 - \epsilon > t_0$. There exists $a \in [a_0 - \epsilon, a_0 + \epsilon]$ such that $|\phi(db + (n - d)a)| < 1$ for all $n \in I$ and $d \in \{0, \dots, n\}$ with $(n, d) \neq (m, m)$. In fact, if this is not the case then, for each $a \in [a_0 - \epsilon, a_0 + \epsilon]$, we can find a point $t_a = d_a b + (n_a - d_a)a$, with $n_a \in I$, $d_a \in \{0, \dots, n_a\}$ and $(n_a, d_a) \neq (m, m)$, such that $|\phi(t_a)| \geq 1$. Since $m = \min(I)$, $b = t_1/m$ and $a > t_0$, we get $t_a > t_0$ so that $|\phi(t_a)| \leq 1$. This way, varying $a \in [a_0 - \epsilon, a_0 + \epsilon]$ we find infinitely many points t_a within $[t_0, \max I(a_0 + \epsilon)]$ in which $|\phi|$ assumes the value 1. The set Λ composed by these points is infinite, closed and subset of the compact $[t_0, \max(I)(a_0 + \epsilon)]$, hence Λ has an accumulation point. By Lemma 3.3 we conclude that $|\phi(t)| = 1$ for all $t \in \mathbb{R}$, which contradicts the fact that $|\phi(t_1)| > 1$. This completes the proof. \square

3.2 Applications to convolution operators

We now observe that we may apply Corollary 3.2 to convolution operators $\phi(D)$ with $|\phi(0)| > 1$, where $E(\lambda)(z) = e^{\lambda z}$. This yields immediately Theorem 1.1. We may also apply Corollary 3.2 to handle the case $|\phi(0)| = 1$.

Corollary 3.4 *Let $P \in \mathbb{C}[z]$ be a non-constant polynomial and let $\phi(z) = P(z)e^z$. Then $\phi(D)$ supports a hypercyclic algebra.*

Proof. The case $|P(0)| < 1$ is done in [4], the case $|P(0)| > 1$ is settled by Theorem 1.1. It remains to consider the case $|P(0)| = 1$. Since $|P(it)|$ tends to $+\infty$ as t tends to $+\infty$, there exists $t_0 \in \mathbb{R}$ such that $|\phi(it_0)| > 1$. By continuity of $|\phi|$, there exists $v = |v|e^{i\theta}$ with $\theta \in (\pi/2, 3\pi/2)$ such that $|\phi(v)| > 1$. Now, because v lies in the left half-plane, $|\phi(tv)|$ tends to 0 as t tends to $+\infty$. We may conclude with Corollary 3.2. \square

We finish this section by pointing out that Theorem 3.1 can also handle functions which do not satisfy the properties described above.

Example 3.5 The convolution operator induced by $\phi(z) = \frac{1}{2}e^z + e^{iz} - \frac{1}{4}$ supports a hypercyclic algebra (let us observe that $|\phi(0)| > 1$ and that ϕ does not tend to 0 along any ray). Indeed, for any $I \in \mathcal{P}_f(\mathbb{N}) \setminus \{\emptyset\}$ we choose $m = \max(I)$ and take $a = k(2\pi i)$ and $b = k2\pi$ for some large integer k . Let $n \in I$ and $d \in \{0, \dots, n\}$ with $(n, d) \neq (m, m)$. Then

$$\phi(db + (n - d)a) = \frac{1}{2}e^{2dk\pi} + e^{-2(n-d)k\pi} - \frac{1}{4}.$$

In particular,

$$|\phi(mb)| = \frac{1}{2}e^{2mk\pi} + \frac{3}{4} > 1.$$

When $d = 0$,

$$|\phi(na)| = \left| e^{-2nk\pi} + \frac{1}{4} \right| < 1.$$

Finally,

$$|\phi(db + (n - d)a)| \leq \frac{1}{2}e^{2dk\pi} + \frac{3}{4}$$

and we have, for all $d = 1, \dots, m - 1$,

$$\left(\frac{1}{2}e^{2dk\pi} + \frac{3}{4}\right)^m \leq \left(\frac{1}{2}e^{2mk\pi} + \frac{3}{4}\right)^d$$

if k is large enough.

Remark 3.6 Combining the previous arguments with that of [4, Section 6], under the assumptions of Theorem 1.1, $\phi(D)$ admits a dense, countably generated, free hypercyclic algebra.

4 Weighted shifts on Fréchet sequence algebras

4.1 Fréchet sequence algebras with the coordinatewise product

We begin with the proof of Theorem 1.2. We first explain where the property of admitting a continuous norm comes into play.

Lemma 4.1 *Let X be a Fréchet sequence algebra for the coordinatewise product and with a continuous norm. Then the sequence (e_n) is bounded below.*

Proof. Let $q \geq 1$ be such that $\|\cdot\|_q$ is a norm on X . Then for all $n \in \mathbb{N}_0$,

$$0 < \|e_n\|_q = \|e_n \cdot e_n\|_q \leq \|e_n\|_q^2$$

which shows that $\|e_n\|_q \geq 1$. □

We shall prove the following precised version of Theorem 1.2.

Theorem 4.2 *Let X be a Fréchet sequence algebra for the coordinatewise product and with a continuous norm. Assume that (e_n) is a Schauder basis for X . Let also B_w be a bounded weighted shift on X . The following assumptions are equivalent.*

- (i) B_w supports a dense, countably generated, free hypercyclic algebra.
- (ii) B_w supports a hypercyclic algebra.
- (iii) For all $m \geq 1$, there exists $x \in X$ such that x^m is a hypercyclic vector for B_w .
- (iv) For all $m \geq 1$, for all $L \in \mathbb{N}$, there exists a sequence of integers (n_k) such that, for all $l = 0, \dots, L$, $((w_{l+1} \cdots w_{n_k+l})^{-1/m} e_{n_k+l})$ tends to zero.

(v) There exists a sequence of integers (n_k) such that, for all $\gamma > 0$ and for all $l \in \mathbb{N}$, $((w_{l+1} \cdots w_{n_k+l})^{-\gamma} e_{n_k+l})$ tends to zero.

Proof. The implications (i) \implies (ii) and (ii) \implies (iii) are trivial. The proof of (iii) \implies (iv) mimics that of the necessary condition for hypercyclicity. Let $m \geq 1$ and $x \in X$ be such that $x^m \in HC(B_w)$. Write $x = \sum_{n=0}^{+\infty} x_n e_n$. Since (e_n) is a Schauder basis, the sequence $(x_n e_n)$ goes to zero. Moreover, there exists a sequence of integers (n_k) such that $(B_w^{n_k}(x^m))_k$ goes to $e_0 + \cdots + e_L$. Since convergence in X implies coordinatewise convergence, for all $l = 0, \dots, L$, $(w_{l+1} \cdots w_{n_k+l} x_{n_k+l}^m)$ converges to 1. Hence the sequences $((w_{l+1} \cdots w_{n_k+l})^{1/m} x_{n_k+l})$ are bounded below. Writing

$$(w_{l+1} \cdots w_{n_k+l})^{-1/m} e_{n_k+l} = \frac{1}{(w_{l+1} \cdots w_{n_k+l})^{1/m} x_{n_k+l}} \cdot x_{n_k+l} e_{n_k+l}$$

we get the result.

To prove that (iv) \implies (v), observe that a diagonal argument ensure the existence of a sequence (n_k) such that, for all $m \geq 1$ and all $l \in \mathbb{N}$, the sequence $((w_{l+1} \cdots w_{n_k+l})^{-1/m} e_{n_k+l})$ tends to zero. Now we can conclude by observing that, since the sequence (e_n) is bounded below, if $((w_{l+1} \cdots w_{n_k+l})^{-1/m} e_{n_k+l})$ tends to zero for some m , then $(w_{l+1} \cdots w_{n_k+l})$ tends to $+\infty$ and, in particular, $((w_{l+1} \cdots w_{n_k+l})^{-\gamma} e_{n_k+l})$ tends to zero for all $\gamma \geq 1/m$.

It remains to prove the most difficult implication, (v) \implies (i). We start by fixing a sequence of integers (n_k) such that for all $\gamma > 0$ and for all $l \in \mathbb{N}$, $((w_{l+1} \cdots w_{n_k+l})^{-\gamma} e_{n_k+l})$ goes to zero. We intend to apply Theorem 2.1. Thus, let $d \geq 1$ and $A \subset \mathbb{N}_0^d \setminus \{(0, \dots, 0)\}$ be finite and non-empty. For $\alpha \in A$ we define the linear form L_α on \mathbb{R}^d by $L_\alpha(\kappa) = \sum_{j=1}^d \alpha_j \kappa_j$. Since L_α and $L_{\alpha'}$ coincide only on a hyperplane for $\alpha \neq \alpha'$, there exist $\kappa \in (0, +\infty)^d$ and $\beta = \beta_A \in A$ such that $0 < L_\beta(\kappa) < L_\alpha(\kappa)$ for all $\alpha \neq \beta$, $\alpha \in A$. Without loss of generality, we may assume that $L_\beta(\kappa) = 1$.

Let now U_1, \dots, U_d, V be non-empty open subsets of X and let W be a neighbourhood of zero. Let x_1, \dots, x_d belonging respectively to U_1, \dots, U_d with finite support and let $y = \sum_{l=0}^p y_l e_l$ belonging to V . We set, for $j = 1, \dots, d$,

$$u_j := u_j(n_k) = x_j + \sum_{l=0}^p \frac{y_l^{\kappa_j}}{(w_{l+1} \cdots w_{n_k+l})^{\kappa_j}} e_{n_k+l}.$$

Our assumption implies that, provided n_k is large enough, u_j belongs to U_j for all $j = 1, \dots, d$. Moreover, again if n_k is large enough (larger than the size of the support of each x_j), for all $\alpha \in A$,

$$B_w^{n_k}(u^\alpha) = \sum_{l=0}^p \frac{y_l^{L_\alpha(\kappa)}}{(w_{l+1} \cdots w_{n_k+l})^{L_\alpha(\kappa)-1}} e_l.$$

In particular, for $\alpha = \beta$, $B_w^{n_k}(u^\beta) = y \in V$ whereas, for $\alpha \neq \beta$, since $L_\alpha(\kappa) - 1 > 0$ and since the sequences $(w_{l+1} \cdots w_{n_k+l})$ tend to $+\infty$, we get $B_w^{n_k}(u^\alpha) \in W$ provided n_k is large enough. Hence, B_w admits a dense and not finitely generated hypercyclic algebra. \square

As recalled in the introduction, the hypercyclicity of B_w on X is equivalent to the existence of a sequence (n_k) such that, for all $l \in \mathbb{N}$, $((w_{l+1} \cdots w_{n_k+l})^{-1} e_{n_k+l})$ tends to zero. It is well known that this last condition is equivalent to the following one, which seems much weaker:

there exists a sequence of integers (n_k) such that $((w_1 \cdots w_{n_k})^{-1} e_{n_k})$ tends to zero. In view of this and of Theorem 4.2, it is tempting to conjecture that B_w supports a hypercyclic algebra if and only if there exists a sequence of integers (n_k) such that, for all $\gamma > 0$, $((w_1 \cdots w_{n_k})^{-\gamma} e_{n_k})$ tends to zero. Unfortunately, this is not the case, as the following example points out.

Example 4.3 Let $X = \{(x_n) \in \omega : |x_n| a_n \rightarrow 0\}$ where $a_{2n} = 1$ and $a_{2n+1} = 2^n$ endowed with $\|x\| = \sup_n |x_n| a_n$ and let w be the weight such that $w_1 \cdots w_{2n} = 2^{n-1}$ and $w_1 \cdots w_{2n+1} = 2^{2n}$. Then w is an admissible weight on X , $((w_1 \cdots w_{2n})^{-\gamma} e_{2n})$ tends to zero for all $\gamma > 0$ but B_w does not admit a hypercyclic algebra.

Proof. We first observe that, endowed with the coordinatewise product, X is a Fréchet sequence algebra (since $a_n \geq 1$ for all n). To prove that w is admissible, it suffices to observe that $w_k \|e_{k-1}\| \leq 2 \|e_k\|$ for all k . The construction of w ensures that $w_{2n} = 2^{-(n-1)}$ and $w_{2n+1} = 2^{n+1}$. Hence the previous inequality is clearly satisfied if we separate the case k even and k odd. Moreover for all $\gamma > 0$,

$$(w_1 \cdots w_{2n})^{-\gamma} \|e_{2n}\| = 2^{-\gamma(n-1)} \xrightarrow{n \rightarrow +\infty} 0.$$

To prove that B_w does not support a hypercyclic algebra, it suffices to observe that, for all $n \geq 1$, $(w_1 \cdots w_{2n+1})^{-1/2} \|e_{2n+1}\| = 1$, which implies that condition (v) of Theorem 4.2 cannot be satisfied. \square

Nevertheless, if we add an extra assumption on X , then we get the expected result.

Corollary 4.4 *Let X be a Fréchet sequence algebra for the coordinatewise product and with a continuous norm. Assume that (e_n) is a Schauder basis for X . Assume also that, for all admissible weights w , for all $\gamma > 0$, w^γ is admissible. Let B_w be a bounded weighted shift on X . The following assumptions are equivalent.*

(i) B_w supports a hypercyclic algebra.

(ii) For all $\gamma > 0$, there exists a sequence (n_k) such that $((w_1 \cdots w_{n_k})^{-\gamma} e_{n_k})$ tends to zero.

Proof. We assume that (ii) is satisfied and we show that, for all $\gamma > 0$ and for all $L \in \mathbb{N}$, there exists a sequence (m_k) such that $((w_1 \cdots w_{m_k+l})^{-\gamma} e_{m_k+l})$ tends to zero. An application of Theorem 4.2 will then allow to conclude. It is easy to get this sequence m_k . Indeed, it is sufficient to set $m_k = n_k - L$, since in that case

$$(w_1 \cdots w_{m_k+l})^{-\gamma} e_{m_k+l} = (B_w^\gamma)^{L-l} ((w_1 \cdots w_{n_k})^{-\gamma} e_{n_k})$$

which goes to zero by continuity of B_w^γ . \square

We may observe that our favorite sequence spaces (namely unweighted ℓ_p -spaces or $H(\mathbb{C})$) satisfy the assumptions of the last corollary. We may also observe that on unweighted ℓ_p -spaces as well as on any Fréchet sequence algebra with a continuous norm such that (e_n) is bounded, the convergence of $((w_1 \cdots w_{n_k})^{-\gamma} e_{n_k})$ to zero is equivalent to the convergence of $(w_1 \cdots w_{n_k})$ to $+\infty$. Hence, we may formulate the following corollary.

Corollary 4.5 *Let X be a Fréchet sequence algebra for the coordinatewise product and with a continuous norm. Assume that (e_n) is a Schauder basis for X and that (e_n) is bounded. Assume also that, for all admissible weights w , for all $\gamma > 0$, w^γ is admissible. Let B_w be a bounded weighted shift on X . The following assumptions are equivalent.*

(i) B_w supports a hypercyclic algebra.

(ii) B_w is hypercyclic.

(iii) There exists a sequence (n_k) such that $(w_1 \cdots w_{n_k})$ tends to $+\infty$.

Remark 4.6 On $H(\mathbb{C})$, the sequence (z^n) is unbounded. Nevertheless, any hypercyclic weighted shift B_w on $H(\mathbb{C})$ supports a hypercyclic algebra. Indeed, for a sequence of integers (n_k) ,

$$\begin{aligned} & (w_1 \cdots w_{n_k})^{-1} z^{n_k} \text{ tends to } 0 \text{ in } H(\mathbb{C}) \\ \iff & \forall q \geq 1, (w_1 \cdots w_{n_k})^{-1} q^{n_k} \text{ tends to } 0 \\ \iff & \forall q \geq 1, \forall \gamma > 0, (w_1 \cdots w_{n_k})^{-\gamma} q^{n_k} \text{ tends to } 0 \\ \iff & \forall \gamma > 0, (w_1 \cdots w_{n_k})^{-\gamma} z^{n_k} \text{ tends to } 0 \text{ in } H(\mathbb{C}). \end{aligned}$$

Remark 4.7 Theorem 4.2 points out one difficulty when dealing with hypercyclic algebras: to admit a hypercyclic algebra is not a property preserved by similarity. Indeed, let $X = \{(x_n) \in \omega : |x_n|2^n \rightarrow 0\}$ endowed with $\|x\| = \sup_n |x_n|2^n$ and let w be the weight such that $w_1 \cdots w_n = n \cdot 2^n$ for all $n \geq 1$. Then $(w_1 \cdots w_n)^{-1} 2^n$ goes to zero whereas $(w_1 \cdots w_n)^{-1/2} 2^n$ tends to $+\infty$, showing that B_w is hypercyclic but that no square vector x^2 belongs to $HC(B_w)$.

Let now (ρ_n) be defined by $\rho_1 = 1$ and $\rho_n = n/(n-1)$ for $n \geq 2$. Then B_w acting on X is similar to B_ρ acting on c_0 , the similarity being given by $S : X \rightarrow c_0, (x_n) \mapsto (2^n x_n)$. But B_ρ admits a hypercyclic algebra, which is not the case of B_w . Of course, the problem is that S is not a morphism of algebra.

When X does not admit a continuous norm, one cannot apply Theorem 4.2. The space ω is the prototypal example of a Fréchet space without a continuous norm (in fact, by a result of Bessaga and Pelczinski [13], a Fréchet space fails to admit a continuous norm if and only if it has a subspace isomorphic to ω) and we shall now concentrate on this space. On ω , for all weight sequences $w = (w_n)$, the weighted shift B_w is bounded, hypercyclic and satisfies (iv). If the sequences $(w_l \cdots w_{n+l})$ converge to $+\infty$ for all $\ell \geq 0$, then an easy modification of the proof of the previous theorem shows that B_w admits a hypercyclic algebra. On the other hand, if the sequences $(w_l \cdots w_{n+l})$ converge to 0 for all $\ell \geq 0$, then we may modify the previous proof using Corollary 2.3 instead of Corollary 2.4 to prove that B_w still admits a hypercyclic algebra. A completely different case is that of the unweighted shift B . It is a hypercyclic multiplicative operator on ω . By [10, Theorem 16], B supports a hypercyclic algebra if and only if for each nonconstant polynomial $P \in \mathbb{C}[X]$ with $P(0) = 0$, the map $\tilde{P} : \omega \rightarrow \omega, x \mapsto P(x)$ has dense range. This is clearly true.

We now show that every weighted shift on ω admits a hypercyclic algebra showing that, coordinate by coordinate, B_w behaves like one of the three previous models.

Theorem 4.8 *Every weighted shift B_w on ω endowed with the coordinatewise product supports a hypercyclic algebra.*

Proof. For V a non-empty open subset of ω , $I \subset \mathbb{N}$ finite and non-empty and $s > 0$, let us define

$$E(I, s) = \left\{ P \in \mathbb{C}[z] : |\hat{P}(\min I)| \geq 1/s, |\hat{P}(\max I)| \geq 1/s, \right. \\ \left. |\hat{P}(n)| \leq s \text{ for all } n \in \mathbb{N}, \right. \\ \left. \hat{P}(n) = 0 \text{ when } n \notin I \right\}$$

$$\mathcal{A}(I, s, V) = \left\{ u \in \omega : \forall P \in E(I, s), \exists N \geq 1, T^N(P(u)) \in V \right\}.$$

As in the proof of Theorem 2.1, it is enough to prove that each set $\mathcal{A}(I, s, V)$ is dense and open. The last property follows easily from the compactness of $E(I, s)$. Thus, let us fix I, s and V and let us prove that $\mathcal{A}(I, s, V)$ is dense. We set $m_0 = \min(I)$ and $m_1 = \max(I)$. Let U be a non-empty open subset of ω . Let $p \in \mathbb{N}_0$, $u_0, \dots, u_p, v_0, \dots, v_p \in \mathbb{C}$ and $\varepsilon > 0$ be such that, for all $x, y \in \omega$,

$$|x_l - u_l| < \varepsilon \text{ for all } l = 0, \dots, p \text{ implies } x \in U,$$

$$|y_l - v_l| < \varepsilon \text{ for all } l = 0, \dots, p \text{ implies } y \in V.$$

Let us first look at the sequence $(w_1 \cdots w_n)$. Three possibilities (which are not mutually exclusive) can occur:

- either $(w_1 \cdots w_n)$ is bounded and bounded below;
- or it admits a subsequence going to zero;
- or it admits a subsequence going to $+\infty$.

Thus, we get the existence of a subsequence $(w_1 \cdots w_{n_k})$ going to $a_0 \in [0, +\infty]$. We then do the same with $(w_2 \cdots w_{n_k+1})$ and so on. By successive extractions, we get the existence of a sequence of integers (n_k) (we can assume that $n_{k+1} - n_k > p$ for all k and that $n_0 > p$) and of $a_0, \dots, a_p \in [0, +\infty]$ such that, for all $l = 0, \dots, p$, $(w_{l+1} \cdots w_{n_k+l})$ tends to a_l . We set $A_1 = \{l \in \{0, \dots, p\} : a_l = +\infty\}$, $A_2 = \{l \in \{0, \dots, p\} : a_l = 0\}$ and $A_3 = \{l \in \{0, \dots, p\} : a_l \in (0, +\infty)\}$.

We fix now $(\alpha(k))$, $(\beta(k))$ two sequences of non-zero complex numbers and $(z(k))$ a sequence in \mathbb{C}^{p+1} such that $(\alpha(k), \beta(k), z(k))$ is dense in \mathbb{C}^{p+3} . We set

$$x = u + \sum_{k=0}^{+\infty} y(k)$$

where, for $l = 0, \dots, p$,

$$y_{n_k+l}(k) = \begin{cases} \frac{v_l^{1/m_0}}{\alpha(k)^{1/m_0} (w_{l+1} \cdots w_{n_k+l})^{1/m_0}} & \text{provided } l \in A_1, \\ \frac{v_l^{1/m_1}}{\beta(k)^{1/m_1} (w_{l+1} \cdots w_{n_k+l})^{1/m_1}} & \text{provided } l \in A_2, \\ z_l(k) & \text{provided } l \in A_3 \end{cases}$$

and $y_i(k) = 0$ if $i \neq n_k, \dots, n_k + p$.

We claim that $x \in U \cap \mathcal{A}(I, s, V)$. The definition of ε and p ensure that $x \in U$. Let $P \in E(I, s)$. There exists an increasing function $\phi : \mathbb{N} \rightarrow \mathbb{N}$ such that $\alpha(\phi(k)) \rightarrow \hat{P}(m_0)$, $\beta(\phi(k)) \rightarrow \hat{P}(m_1)$ and $a_l P(z_l(\phi(k))) \rightarrow v_l$ for all $l \in A_3$. We claim that $(B_w^{n_{\phi(k)}}(P(x)))$ belongs to V provided k is large enough. It suffices to prove that for $l = 0, \dots, p$, the l -th coordinate of $B_w^{n_{\phi(k)}}(P(x))$ tends to v_l . Assume first that $l \in A_1$. This l -th coordinate is equal to

$$w_{l+1} \cdots w_{n_{\phi(k)}+l} P \left(\frac{v_l^{1/m_0}}{\alpha(\phi(k))^{1/m_0} (w_{l+1} \cdots w_{n_{\phi(k)}+l})^{1/m_0}} \right).$$

Now, since $w_{l+1} \cdots w_{n_{\phi(k)}+l}$ tends to $+\infty$, and $m_0 = \min(I)$,

$$w_{l+1} \cdots w_{n_{\phi(k)}+l} P \left(\frac{v_l^{1/m_0}}{\alpha(\phi(k))^{1/m_0} (w_{l+1} \cdots w_{n_{\phi(k)}+l})^{1/m_0}} \right) = \hat{P}(m_0) \frac{v_l}{\alpha(\phi(k))} + o(1)$$

and this tends to v_l . When $l \in A_2$, the proof is similar since now, because $w_{l+1} \cdots w_{n_{\phi(k)}+l}$ tends to 0, and $m_1 = \max(I)$,

$$w_{l+1} \cdots w_{n_{\phi(k)}+l} P \left(\frac{v_l^{1/m_1}}{\beta(\phi(k))^{1/m_1} (w_{l+1} \cdots w_{n_{\phi(k)}+l})^{1/m_1}} \right) = \hat{P}(m_1) \frac{v_l}{\beta(\phi(k))} + o(1)$$

and this also goes to v_l . Finally, when $l \in A_3$, the l -th coordinate of $B_w^{n_{\phi(k)}}(P(x))$ is equal to $w_{l+1} \cdots w_{n_{\phi(k)}+l} P(z_l(\phi(k)))$ which tends again to v_l . \square

Theorem 4.8 has an analogue (with a completely different proof!) if we endow ω with the Cauchy product: see [16, Corollary 3.9]. We also point out that the existence of a continuous norm is an important assumption in several problems in linear dynamics, for instance for the existence of a closed infinite dimensional subspace of hypercyclic vectors (see [20]).

4.2 Bilateral shifts on Fréchet sequence algebras with the coordinatewise product

In this section, we investigate the case of bilateral shifts on a Fréchet sequence algebra X on \mathbb{Z} ; namely, X is a subset of $\mathbb{C}^{\mathbb{Z}}$ endowed with the coordinatewise product under which it is an F -algebra. We intend to give an analogue of Theorem 4.2 for bilateral shifts on X . The statement and the methods are close to what happens for unilateral shifts. Since we do not want to give an exhaustive list of examples in this work, there is an extra interest for looking at bilateral shifts: a small subtlety appears in this case, since the condition that appears is not symmetric for the positive part of the weight and for the negative one. This will lead us to an interesting example of a hypercyclic operator T supporting a hypercyclic algebra such that T^{-1} does not.

Theorem 4.9 *Let X be a Fréchet sequence algebra on \mathbb{Z} for the coordinatewise product, with a continuous norm. Assume that $(e_n)_{n \in \mathbb{Z}}$ is a Schauder basis for X . Let also B_w be a bounded bilateral shift on X such that, for all $\gamma \in (0, 1)$, B_{w^γ} is bounded. The following assertions are equivalent.*

(i) B_w supports a hypercyclic algebra.

(ii) For all $m \geq 1$, for all $L \in \mathbb{N}$, there exists a sequence of integers (n_k) such that, for all $l = -L, \dots, L$, $((w_{l+1} \cdots w_{n_k+l})^{-1/m} e_{n_k+l})$ and $(w_l \cdots w_{-n_k+l+1} e_{-n_k+l})$ tend to zero.

Proof. (ii) \implies (i). We intend to apply Corollary 2.4. Let $1 \leq m_0 < m_1$, let U, V be nonempty open subsets of X and let W be a neighbourhood of zero. Let x, y belonging to U and V respectively, with finite support contained in $[-p, p]$. Write $y = \sum_{l=-p}^p y_l e_l$ and let (n_k) be the sequence given in (ii) for $m = m_0$ and $L = p$. Define

$$u := u(n_k) = x + \sum_{l=-p}^p \frac{y_l^{1/m_0}}{(w_{l+1} \cdots w_{n_k+l})^{1/m_0}} e_{n_k+l}.$$

Provided k is large enough, u belongs to U . Moreover, for $m \in \{m_0, \dots, m_1\}$,

$$B_w^{n_k}(u^m) = \sum_{l=-p}^p w_l \cdots w_{-n_k+l+1} x_l^m e_{-n_k+l} + \sum_{l=-p}^p \frac{y_l^{m/m_0}}{(w_{l+1} \cdots w_{n_k+l})^{\frac{m}{m_0}-1}} e_l.$$

For all values of m , it is clear that

$$\sum_{l=-p}^p w_l \cdots w_{-n_k+l+1} x_l^m e_{-n_k+l} \xrightarrow{k \rightarrow +\infty} 0.$$

Hence, for $m = m_0$ and provided k is large enough, $B_w^{n_k}(u^{m_0})$ belongs to V . Furthermore, if $m > m_0$, since each sequence $(w_{l+1} \cdots w_{n_k+l})^{-1}$ goes to zero (recall that (e_n) is bounded below), then $B_w^{n_k}(u^m)$ belongs to W for large values of k , showing that B_w admits a hypercyclic algebra.

(i) \implies (ii). The proof is slightly more difficult than for unilateral shifts. Fix m and L and let $x \in X$ be such that $x^m \in HC(B_w)$. Let (s_k) be an increasing sequence of integers such that $B_w^{s_k}(x^m)$ tends to $e_{-L} + \cdots + e_L$. We fix some $s \in \mathbb{N}$ (which can be taken equal to some s_{k_0}) such that, for all $l = -L, \dots, L$, the l -th coordinate of $B_w^s(x)$ is not equal to zero. We then consider $y \in X$ defined by $y_l = (w_{l+1} \cdots w_{l+s})^{1/m} x_{l+s}$ (namely, $y = B_{w^{1/m}}^s(x)$) and we set $n_k = s_k - s$. It is easy to check that $B_w^{n_k}(y^m) = B_w^{s_k}(x^m)$. Hence, it goes to $e_{-L} + \cdots + e_L$. This implies that

- for all $l = -L, \dots, L$,

$$w_l \cdots w_{-n_k+l+1} y_l^m e_{-n_k+l} \text{ tends to } 0.$$

- for all $l = -L, \dots, L$,

$$w_{l+1} \cdots w_{n_k+l} y_{n_k+l}^m \text{ tends to } 1.$$

We conclude as in the unilateral case, using that y_l is never equal to zero for $l = -L, \dots, L$. \square

We can then state corollaries similar to what happens in the unilateral case.

Corollary 4.10 *Let X be a Fréchet sequence algebra on \mathbb{Z} for the coordinatewise product and with a continuous norm. Assume that $(e_n)_{n \in \mathbb{Z}}$ is a Schauder basis for X . Assume also that, for all admissible weights w , for all $\gamma \in (0, 1)$, w^γ is admissible. Let B_w be a bounded bilateral weighted shift on X . The following assumptions are equivalent.*

(i) B_w supports a hypercyclic algebra.

(ii) For all $\gamma > 0$, there exists a sequence (n_k) such that $((w_1 \cdots w_{n_k})^{-\gamma} e_{n_k})$ tends to zero and $(w_{-1} \cdots w_{-n_k} e_{-n_k})$ tends to zero.

Corollary 4.11 *Let X be a Fréchet sequence algebra on \mathbb{Z} for the coordinatewise product and with a continuous norm. Assume that (e_n) is a Schauder basis for X and that (e_n) is bounded. Assume also that, for all admissible weights w , for all $\gamma \in (0, 1)$, w^γ is admissible. Let B_w be a bounded bilateral weighted shift on X . The following assumptions are equivalent.*

- (i) B_w supports a hypercyclic algebra.
- (ii) B_w is hypercyclic.
- (iii) There exists a sequence (n_k) such that $(w_1 \cdots w_{n_k})$ and $(w_{-1} \cdots w_{-n_k})$ tend to $+\infty$.

On the contrary, the nonsymmetry of the conditions in (ii) of Theorem 4.9 proves to be useful to get the following example.

Example 4.12 There exists an invertible operator T on a Banach algebra such that T supports a hypercyclic algebra and T^{-1} does not.

Proof. Let

$$X = \left\{ x \in \mathbb{C}^{\mathbb{Z}} : x_n(|n| + 1) \xrightarrow{n \rightarrow \pm\infty} 0 \right\},$$

endowed with

$$\|x\| = \sup_n |x_n|(|n| + 1).$$

Equipped with the coordinatewise product, X is a Fréchet sequence algebra. Let w be the weight defined by $w_0 = 1$, $w_n = 2$ and $w_{-n} = n^2/(n + 1)^2$ for $n > 0$. For all $\gamma > 0$, the weighted shift B_{w^γ} is bounded on X . Moreover, it satisfies the assumptions of Theorem 4.9 with (n_k) equal to the whole sequence of integers. In particular, $w_{-1} \cdots w_{-n} \|e_{-n}\| = (n + 1)^{-1}$ tends to zero.

It is plain that B_w is invertible and that its inverse is the forward shift F_ρ , defined by $F_\rho(e_n) = \rho_{n+1} e_{n+1}$ with $\rho_n = 1/w_n$. Assume that F_ρ supports a hypercyclic algebra. Then we apply the symmetrized version of Theorem 4.9 adapted to forward shifts with $m = 2$ to get the existence of a sequence (n_k) such that $(\rho_{-1} \cdots \rho_{-n_k})^{-1/2} e_{-n_k}$ tends to zero. This is impossible since

$$\left\| (\rho_{-1} \cdots \rho_{-n_k})^{-1/2} e_{-n_k} \right\| \sim_{k \rightarrow +\infty} \frac{n_k}{n_k^{2 \times 1/2}} = 1.$$

□

4.3 Fréchet sequence algebras for the convolution product

This subsection is devoted to the proof of Theorem 1.3. We first have to give the meaning of a regular Fréchet sequence algebra. Let $(X, (\|\cdot\|_q))$ be such a Fréchet sequence algebra for the Cauchy product. We will say that X is *regular* provided that it satisfies the following three properties:

- (a) X admits a continuous norm;
- (b) (e_n) is a Schauder basis of X ;
- (c) for any $r \geq 1$, there exists $q \geq 1$ and $C > 0$ such that, for all $n, k \geq 0$,

$$\|e_n\|_r \cdot \|e_k\|_r \leq C \|e_{n+k}\|_q.$$

Let us make some comments on these assumptions. Conditions (a) and (b) are standard in this work. We shall use (a) by assuming that $\|e_n\|_q > 0$ for all $n \in \mathbb{N}_0$ and all $q > 0$. Regarding (c), it should be thought as a reverse inequality for the continuity of the product in X . Observe also that $H(\mathbb{C})$ and ℓ_1 are clearly regular. However, this is not the case of all Fréchet sequence algebras for the Cauchy product. Pick for instance any sequence (a_n) of positive real numbers such that, for all $n, p, q \in \mathbb{N}_0$, with $n = p + q$, $a_n \leq a_p \cdot a_q$ and $a_n^2/a_{2n} \rightarrow +\infty$ (the sequence $a_n = 1/n!$ does the job). Then the Banach space $X = \{x \in \omega : \|x\|_X = \sum_{n \geq 0} a_n |x_n| < +\infty\}$ is a Fréchet sequence algebra for the Cauchy product which does not satisfy (c).

A consequence of (c) is the following technical lemma which will be crucial later.

Lemma 4.13 *Let X be a regular Fréchet sequence algebra for the Cauchy product and let (w_n) be an admissible weight sequence on X . Then, for all $M \geq 1$, for all $r \geq 1$, and for all $\rho \geq 0$, there exist $C > 0$ and $q \geq r$ such that, for all $n \geq M$, for all $u < v$ in $\{n - M, \dots, n\}$, for all $k \in \{n - M + \rho, \dots, n + \rho\}^{v-u}$,*

$$\prod_{j=1}^{v-u} w_{k_j} \|e_u\|_r \leq C \|e_v\|_q. \quad (\text{A.7})$$

Before to proceed with the proof, let us comment the statement of Lemma 4.13. The inequality (A.7) is nothing else than the continuity of B_w if we assume that $k_j = u + j$ for $j = 1, \dots, v - u$. The regularity of X (and more precisely the third condition) will imply that we may slightly move the indices k_j .

Proof. Let us fix $M \geq 1$ and observe that $v - u$ may only take the values $1, \dots, M$. Then, upon doing a finite induction and taking suprema, we need only to prove that, for all $r \geq 1$, and for all $\rho \geq 0$, there exist $C > 0$ and $q \geq r$ such that, for all $n \geq M$, for all $u \in \{n - M, \dots, n - 1\}$, for all $k \in \{n - M + \rho, \dots, n + \rho\}$,

$$w_k \|e_u\|_r \leq C \|e_{u+1}\|_q, \quad (\text{A.8})$$

a property which should be thought as a strong version of the continuity of B_w . Assume first that $u \geq k - 1$. Then, writing $e_u = e_{k-1} \cdot e_{u-(k-1)}$ and using the continuity of the product and of B_w , we get the existence of $C > 0$ and $q_1 \geq r$ such that

$$\begin{aligned} w_k \|e_u\|_r &\leq w_k \|e_{k-1}\|_r \cdot \|e_{u-(k-1)}\|_r \\ &\leq C_1 \|e_k\|_{q_1} \cdot \|e_{u-(k-1)}\|_{q_1}. \end{aligned}$$

We now use property (c) for $r = q_1$ to deduce the existence of $C_2 > 0$ and $q_2 \geq q_1$ such that

$$w_k \|e_u\|_r \leq C_1 C_2 \|e_{u+1}\|_{q_2}.$$

Hence, (A.8) is proved for $q = q_2$ and $C = C_1C_2$. If we assume that $u < k - 1$, then the argument is completely similar by exchanging the place where we use the continuity of the product and property (c). Precisely,

$$\begin{aligned} w_k \|e_u\|_r &= w_k \frac{\|e_u\|_r \cdot \|e_{(k-1)-u}\|_r}{\|e_{(k-1)-u}\|_r} \\ &\leq C_1 w_k \frac{\|e_{k-1}\|_{q_1}}{\|e_{(k-1)-u}\|_r} \\ &\leq C_1 C_2 \frac{\|e_k\|_{q_2}}{\|e_{k-(u+1)}\|_r} \\ &\leq C_1 C_2 \frac{\|e_{k-(u+1)}\|_{q_2}}{\|e_{k-(u+1)}\|_r} \|e_{u+1}\|_{q_2}. \end{aligned}$$

Hence, (A.8) is proved for $q = q_2$ and

$$C = \max \left\{ C_1 C_2 \frac{\|e_l\|_{q_2}}{\|e_l\|_r} : 1 \leq l \leq M + \rho - 1 \right\}.$$

□

Lemma 4.13 will be used through the following more particular form.

Corollary 4.14 *Let X be a regular Fréchet sequence algebra for the Cauchy product and let (w_n) be an admissible weight sequence on X . Then, for all $m \geq 1$, for all $N \geq 1$, for all $r \geq 1$, and for all $\rho \geq 0$, there exist $C > 0$ and $q \geq 1$ such that, for all $n \geq mN$, for all $s \in \{1, \dots, N\}$,*

$$(w_{n-s+1+\rho})^{m-1} \cdots (w_{n-1+\rho})^{m-1} (w_{n+\rho})^{m-1} \|e_{n-ms+m\rho}\|_r \leq C \|e_{n-s+\rho}\|_q.$$

Proof. We apply the previous lemma with $M = mN$ to get $q' \geq r$ and $C' > 0$ such that

$$(w_{n-s+1+\rho})^{m-1} \cdots (w_{n-1+\rho})^{m-1} (w_{n+\rho})^{m-1} \|e_{n-ms}\|_r \leq C' \|e_{n-s}\|_{q'}.$$

Now, using property (c) and the continuity of the product on a Fréchet algebra, we get $q \geq q' \geq r$ and $C'' > 0$ with

$$\begin{aligned} &(w_{n-s+1+\rho})^{m-1} \cdots (w_{n-1+\rho})^{m-1} (w_{n+\rho})^{m-1} \|e_{n-ms+m\rho}\|_r \\ &\leq (w_{n-s+1+\rho})^{m-1} \cdots (w_{n-1+\rho})^{m-1} (w_{n+\rho})^{m-1} \|e_{n-ms}\|_r \|e_\rho\|_r \|e_{(m-1)\rho}\|_r \\ &\leq C' \|e_{n-s}\|_{q'} \|e_\rho\|_{q'} \|e_{(m-1)\rho}\|_r \\ &\leq C' C'' \|e_{n-s+\rho}\|_q \|e_{(m-1)\rho}\|_r \\ &= C \|e_{n-s+\rho}\|_q, \end{aligned}$$

where $C = C' C'' \|e_{(m-1)\rho}\|_r$. □

Before proceeding with the proof of Theorem 1.3, let us make some comments on the difference between the coordinatewise product and the convolution product. Let $P(z) = \sum_{m \in I} \hat{P}(m) z^m$ be a nonzero polynomial, x, y with finite support. The work done in Section 2.1 shows that it is important for us to find u close to x and N such that $B_w^N(P(u))$ is close to y . In both cases, u will be of the form $u = x + z$, where z has finite support and $\min(\text{supp}(z)) \gg \max(\text{supp}(x))$. For the coordinatewise product, each u^m has the same support as u . Moreover, since z has to be

small, z^m becomes smaller as m increases. Hence, in $P(u)$, the most important term was u^{m_0} , where $m_0 = \min(I)$ (we assume $\hat{P}(m) \neq 0 \iff m \in I$) and it was natural to apply Corollary 2.4.

Regarding the convolution product, the support of u^m is now moving to the right: $\max(\text{supp}(u^{m+1})) \geq \max(\text{supp}(u^m))$. By using a specific translation term (an idea coming from [22] and [17]), we will arrange the choice of N such that $B_w^N(u^m) = 0$ when $m < m_1 := \max(I)$ and $B_w^N(u^{m_1})$ is close to y . This explains why we will rather use Corollary 2.3.

Proof of Theorem 1.3. We start with a hypercyclic weighted shift B_w and prove that B_w supports a dense hypercyclic algebra which is not contained in a finitely generated algebra. Let $d \geq 1$, $A \in \mathcal{P}_f(\mathbb{N}_0^d) \setminus \{\emptyset\}$ and $U_1, \dots, U_d, V, W \subset X$ be open and non-empty, with $0 \in W$. We choose $\beta = \max A$ under the lexicographical order, say $\beta = (m, \beta_2, \dots, \beta_d)$. Upon interchanging the coordinates in \mathbb{N}^d , we may and will assume that $m > 0$. Let x_1, \dots, x_d belonging respectively to U_1, \dots, U_d and let $y = \sum_{i=1}^p y_i e_i$ belonging to V . We can find $r \geq 1$, $\delta > 0$ and a ball $B \subset W$ for the seminorm $\|\cdot\|_r$ and with radius δ such that $y + B \subset V$ and $x_i + B \subset U_i$, for all $i = 1, \dots, d$.

Since $\beta > \alpha$ for all $\alpha \in A \setminus \{\beta\}$ under the lexicographical order, we may find integers s_1, \dots, s_d with $s_i > 4p$ such that

$$(m - \alpha_1)s_1 + (\beta_2 - \alpha_2)s_2 + \dots + (\beta_d - \alpha_d)s_d > 3p, \text{ for all } \alpha \in A \setminus \{\beta\}. \quad (\text{A.9})$$

The procedure to do this is the following. First find s_d such that $(\beta_d - \alpha_d)s_d > 3p$, for all $\alpha_d < \beta_d$, $\alpha_d \in \pi_d(A)$. Next, find s_{d-1} such that $(\beta_{d-1} - \alpha_{d-1})s_{d-1} + (\beta_d - \alpha_d)s_d > 3p$, for all $\alpha_{d-1} < \beta_{d-1} \in \pi_{d-1}(A)$ and $\alpha_d \in \pi_d(A)$. Continuing inductively, after finitely many steps we define s_2 such that $(\beta_2 - \alpha_2)s_2 + \dots + (\beta_d - \alpha_d)s_d > 3p$, for all $\alpha_2 < \beta_2$, $\alpha_2 \in \pi_2(A)$, and $\alpha_i \in \pi_i(A)$, for $i = 3, \dots, d$. Finally we chose s_1 large enough so that $(m - \alpha_1)s_1 + (\beta_2 - \alpha_2)s_2 + \dots + (\beta_d - \alpha_d)s_d > 3p$, for all $\alpha_1 < m$, $\alpha_1 \in \pi_1(A)$, $\alpha_i \in \pi_i(A)$, for $i = 2, \dots, p$. This way we get (A.9).

These s_i being fixed, we now choose positive real numbers η_2, \dots, η_d such that

$$\|\eta_i e_{s_i}\|_r < \delta, \text{ for all } i = 2, \dots, d. \quad (\text{A.10})$$

We will distinguish two cases in order to apply Corollary 2.5. The most difficult one is $m > 1$, an assumption that we now make. We set $\rho = \beta_2 s_2 + \dots + \beta_d s_d$ and we consider a sequence (J_k) going to $+\infty$ such that, for all $l = 0, \dots, p$,

$$(w_1 \cdots w_{mJ_k - 3p + l + \rho})^{-1} e_{mJ_k - 3p + l + \rho} \xrightarrow{k \rightarrow +\infty} 0.$$

Indeed, let (m_k) be a sequence of integers such that $(w_1 \cdots w_{m_k})^{-1} e_{m_k}$ goes to zero and $m_k \geq m + \rho$ for all k . Define J_k as the single integer such that $m_k - m < mJ_k - 2p + \rho \leq m_k$. Then, for all $l = 0, \dots, p$,

$$(w_1 \cdots w_{mJ_k - 3p + \rho + l})^{-1} e_{mJ_k - 3p + \rho + l} = B_w^{m_k - mJ_k + 3p - \rho - l} ((w_1 \cdots w_{m_k})^{-1} e_{m_k})$$

which tends to zero by the continuity of B_w and because

$$0 \leq m_k - mJ_k + 3p - \rho - l \leq m + p.$$

We now proceed with the construction of the vectors u_1, \dots, u_d required to apply Corollary 2.5. We set, for k large enough, $N = mJ_k - 3p + \rho$ and

$$\varepsilon = \max_{0 \leq l \leq p} \left(\frac{\|e_{J_k-3p+l}\|_r}{w_1 \cdots w_{mJ_k-3p+l+\rho}} \right)^{\frac{1}{2(m-1)}} \times \min \left(\frac{1}{\|e_{J_k}\|_r}, \frac{1}{(w_1 \cdots w_{mJ_k+\rho})^{1/m}} \right)^{\frac{1}{2}},$$

$$d_j = \frac{w_1 \cdots w_j y_j}{\eta_2^{\beta_2} \cdots \eta_d^{\beta_d} m \varepsilon^{m-1} w_1 \cdots w_{mJ_k-3p+j+\rho}}.$$

We also define

$$u_1 = x_1 + \sum_{j=0}^p d_j e_{J_k-3p+j} + \varepsilon e_{J_k},$$

$$u_i = x_i + \eta_i e_{s_i}, \text{ for } i = 2, \dots, d.$$

Let us postpone the proof of the following facts.

$$\varepsilon \|e_{J_k}\|_r \rightarrow 0 \text{ as } k \rightarrow +\infty, \quad (\text{A.11})$$

$$|d_j| \cdot \|e_{J_k-3p+j+\rho}\|_r \rightarrow 0 \text{ as } k \rightarrow +\infty, \text{ for all } j = 0, \dots, p, \quad (\text{A.12})$$

$$\varepsilon^m w_1 \cdots w_{mJ_k+\rho} \rightarrow 0 \text{ as } k \rightarrow +\infty. \quad (\text{A.13})$$

From (A.11) and (A.12) we get $u_1 \in U_1$ if k is large enough and from (A.10) we get $u_i \in U$ for $i = 2, \dots, d$. We claim that $u^\alpha \in \ker B_w^N$ for all $\alpha \in A$ with $\alpha \neq \beta$. In fact, for a given $\alpha \in A \setminus \{\beta\}$, say $\alpha = (\alpha_1, \dots, \alpha_d)$, we have

$$\max(\text{supp}(u^\alpha)) \leq \alpha_1 J_k + \alpha_2 s_2 + \cdots + \alpha_d s_d,$$

so the claim follows by (A.9) since $\alpha_1 \leq m$ and for k large enough, $J_k \geq s_1$. Finally, for the main power β we write

$$u^\beta = z + \sum_{j=0}^p \eta_2^{\beta_2} \cdots \eta_d^{\beta_d} m \varepsilon^{m-1} d_j e_{mJ_k-3p+j+\rho} + \varepsilon^m \eta_2^{\beta_2} \cdots \eta_d^{\beta_d} e_{mJ_k+\rho},$$

where the maximum of the support of z is less than $N = mJ_k - 3p + \rho$. Indeed, a term in z can come

- either from a term in $u_1^{\beta_1}$ with support in $[0, mJ_k - 4p]$ so that the maximum of the support of this term is at most $mJ_k - 4p + \rho < N$;
- or from a term in some $u_i^{\beta_i}$, $i = 2, \dots, d$, with support in $[0, (\beta_i - 1)s_i + p]$. The maximum of the support of such a term is then at most $mJ_k + \rho + p - s_i < N$ since $s_i > 4p$.

Hence, we get

$$B_w^N u^\beta = y + \frac{\varepsilon^m \eta_2^{\beta_2} \cdots \eta_d^{\beta_d} w_1 \cdots w_{mJ_k+\rho}}{w_1 \cdots w_{3p}} e_{3p},$$

which belongs to V by (A.13) if k is big enough. It remains now to show that properties (A.11), (A.12) and (A.13) hold true.

Let us first prove (A.11). By property (c) and an easy induction, there exist $q \geq 1$ and $C > 0$ (depending on r and m) such that, for all $k \geq 1$ and all $l \in \{0, \dots, p\}$,

$$\begin{aligned} \frac{\|e_{J_k-3p+l}\|_r^{\frac{1}{2(m-1)}} \cdot \|e_{J_k}\|_r}{\|e_{J_k}\|_r^{\frac{1}{2}}} &= \frac{\left(\|e_{J_k-3p+l}\|_r \cdot \|e_{J_k}\|_r^{m-1} \cdot \|e_\rho\|_r\right)^{\frac{1}{2(m-1)}}}{\|e_\rho\|_r^{\frac{1}{2(m-1)}}} \\ &\leq C \|e_{mJ_k-3p+l+\rho}\|_q^{\frac{1}{2(m-1)}}. \end{aligned}$$

Hence,

$$\varepsilon \|e_{J_k}\|_r \leq C \max_{0 \leq l \leq p} \left(\frac{\|e_{mJ_k-3p+l+\rho}\|_q}{w_1 \cdots w_{mJ_k-3p+l+\rho}} \right)^{\frac{1}{2(m-1)}}$$

and this goes to zero as k tends to $+\infty$.

Regarding (A.12), we first write

$$\begin{aligned} |d_j| \cdot \|e_{J_k-3p+j+\rho}\|_r &\leq C \frac{\|e_{J_k-3p+j+\rho}\|_r}{w_1 \cdots w_{mJ_k-3p+j+\rho}} \times \min_{0 \leq l \leq p} \left(\frac{w_1 \cdots w_{mJ_k-3p+l+\rho}}{\|e_{J_k-3p+l}\|_r} \right)^{\frac{1}{2}} \\ &\quad \times \max \left(\|e_{J_k}\|_r, (w_1 \cdots w_{mJ_k+\rho})^{\frac{1}{m}} \right)^{\frac{m-1}{2}} \\ &\leq C \frac{\|e_\rho\|_r^{\frac{1}{2}} \cdot \|e_{J_k-3p+j+\rho}\|_r^{\frac{1}{2}}}{(w_1 \cdots w_{mJ_k-3p+j+\rho})^{\frac{1}{2}}} \\ &\quad \times \max \left(\|e_{J_k}\|_r, (w_1 \cdots w_{mJ_k+\rho})^{\frac{1}{m}} \right)^{\frac{m-1}{2}}, \end{aligned}$$

where the last line comes from the continuity of the product, more precisely from

$$\|e_{J_k-3p+l+\rho}\|_r \leq \|e_{J_k-3p+l}\|_r \cdot \|e_\rho\|_r.$$

Assume first that the maximum is attained for $\|e_{J_k}\|_r$. In that case, using (c) in a similar way we write

$$\frac{\|e_{J_k-3p+j+\rho}\|_r}{w_1 \cdots w_{mJ_k-3p+j+\rho}} \|e_{J_k}\|_r^{m-1} \leq C \frac{\|e_{mJ_k-3p+j+\rho}\|_q}{w_1 \cdots w_{mJ_k-3p+j+\rho}}$$

and the last parcel goes to zero. If the maximum is attained for $(w_1 \cdots w_{mJ_k+\rho})^{\frac{1}{m}}$, we now write

$$\begin{aligned} \frac{\|e_{J_k-3p+j+\rho}\|_r (w_1 \cdots w_{mJ_k+\rho})^{\frac{m-1}{m}}}{w_1 \cdots w_{mJ_k-3p+j+\rho}} &= \frac{\left(\|e_{J_k-3p+j+\rho}\|_r^m (w_1 \cdots w_{mJ_k+\rho})^{m-1}\right)^{\frac{1}{m}}}{w_1 \cdots w_{mJ_k-3p+j+\rho}} \\ &\leq C_1 \frac{\left((w_1 \cdots w_{mJ_k+\rho})^{m-1} \|e_{mJ_k-m(3p-j)+m\rho}\|_q\right)^{\frac{1}{m}}}{w_1 \cdots w_{mJ_k-3p+j+\rho}}. \end{aligned}$$

Now, using Corollary 4.14 for $n = mJ_k$, $N = 3p$ and $s = 3p - j$, we get the existence of $C_2 > 0$ and $q' \geq q$ (which does not depend on k) such that

$$\begin{aligned} (w_1 \cdots w_{mJ_k+\rho})^{m-1} \|e_{mJ_k-m(3p-j)+m\rho}\|_q &\leq C_2 (w_1 \cdots w_{mJ_k-3p+j+\rho})^{m-1} \\ &\quad \times \|e_{mJ_k-3p+j+\rho}\|_{q'} \end{aligned}$$

so that

$$\frac{\|e_{J_k-3p+j+\rho}\|_r (w_1 \cdots w_{mJ_k+\rho})^{\frac{m-1}{m}}}{w_1 \cdots w_{mJ_k-3p+j+\rho}} \leq C_1 C_2^{\frac{1}{m}} \left(\frac{\|e_{mJ_k-3p+j+\rho}\|_{q'}}{w_1 \cdots w_{mJ_k-3p+j+\rho}} \right)^{\frac{1}{m}}$$

and this goes to zero.

Finally, let us prove (A.13). The proof is very similar. Indeed, for all $l = 0, \dots, p$,

$$\begin{aligned}
 & \left(\frac{\|e_{J_k-3p+l+\rho}\|_r}{w_1 \cdots w_{mJ_k-3p+l+\rho}} \right)^{\frac{m}{2(m-1)}} (w_1 \cdots w_{mJ_k+\rho})^{\frac{1}{2}} \\
 & \leq \frac{(\|e_{J_k-3p+l+\rho}\|_r^m (w_1 \cdots w_{mJ_k+\rho})^{m-1})^{\frac{1}{2(m-1)}}}{(w_1 \cdots w_{mJ_k-3p+l+\rho})^{\frac{m}{2(m-1)}}} \\
 & \leq C_1 \frac{(\|e_{mJ_k-m(3p-l)+m\rho}\|_q (w_1 \cdots w_{mJ_k+\rho})^{m-1})^{\frac{1}{2(m-1)}}}{(w_1 \cdots w_{mJ_k-3p+l+\rho})^{\frac{m}{2(m-1)}}} \\
 & \leq C_1 C_2^{\frac{1}{2(m-1)}} \frac{(\|e_{mJ_k-3p+l+\rho}\|_{q'} (w_1 \cdots w_{mJ_k-3p+l+\rho})^{m-1})^{\frac{1}{2(m-1)}}}{(w_1 \cdots w_{mJ_k-3p+l+\rho})^{\frac{m}{2(m-1)}}} \\
 & \leq C_1 C_2^{\frac{1}{2(m-1)}} \left(\frac{\|e_{mJ_k-3p+l+\rho}\|_{q'}}{w_1 \cdots w_{mJ_k-3p+l+\rho}} \right)^{\frac{1}{2(m-1)}},
 \end{aligned}$$

which achieves the proof of (A.13).

We now sketch briefly the proof when $m = 1$. We still set $\rho = \beta_2 s_2 + \cdots + \beta_d s_d$ and we now consider a sequence (J_k) satisfying

$$(w_1 \cdots w_{J_k+l+\rho})^{-1} e_{J_k+l+\rho} \xrightarrow{k \rightarrow +\infty} 0$$

for all $l = 0, \dots, p$. We define

$$\begin{aligned}
 u_1 &= x_1 + \sum_{j=0}^p d_j e_{J_k+j+\rho}, \\
 u_i &= x_i + \eta_i e_{s_i}, \text{ for } i = 2, \dots, d,
 \end{aligned}$$

where

$$d_j = \frac{y_j w_1 \cdots w_j}{\eta_2^{s_2} \cdots \eta_d^{s_d} w_1 \cdots w_{J_k+j+\rho}}.$$

Setting $N = J_k + \rho$ for k sufficiently large, it is easy to check that $u_i \in U_i$, $i = 1, \dots, d$, $B_w^N(u^\alpha) = 0$ if $\alpha \in A \setminus \{\beta\}$, and $B_w^N(u^\beta) \in V$, which concludes the proof. \square

5 Frequently and upper frequently hypercyclic algebras

5.1 How to get upper frequently hypercyclic algebras

In [14], following the proof made in [6] that the set of upper frequently hypercyclic vectors is either empty or residual, the authors gave an analogue to Birkhoff's transitivity theorem for upper frequent hypercyclicity. We adapt it in order to get upper frequently hypercyclic algebras.

Proposition 5.1 *Let T be a continuous operator on an F -algebra X satisfying the following condition: for each $I \in \mathcal{P}_f(\mathbb{N}) \setminus \{\emptyset\}$, there exists $m_0 \in I$ such that, for each non-empty open*

subset V of X and each neighbourhood W of the origin, there is $\delta > 0$ such that for each non-empty open subset U and each $N_0 \in \mathbb{N}$, there is $u \in U$ and $N \geq N_0$ satisfying

$$\frac{1}{N+1} \text{card} \{p \leq N : T^p(u^m) \in W \text{ for } m \in I \setminus \{m_0\} \text{ and } T^p(u^{m_0}) \in V\} > \delta.$$

Then T admits an upper frequently hypercyclic algebra.

Proof. Let $(V_k)_k$ be a basis for the topology of X and let $(W_j)_j$ be a basis of open neighbourhoods of the origin. By the assumption for each $I \in \mathcal{P}_f(\mathbb{N}) \setminus \{\emptyset\}$, there exists $m_0 = m_0(I)$ such that, for each k, j , there is $\delta_{k,j,I} > 0$ such that for each non-empty open subset U and each $N_0 \in \mathbb{N}$, there is $u \in U$ and $N \geq N_0$ satisfying

$$\frac{1}{N+1} \text{card} \{p \leq N : T^p(u^m) \in W_j \text{ for } m \in I \setminus \{m_0\} \text{ and } T^p(u^{m_0}) \in V_k\} > \delta_{k,j,I}.$$

We set

$$A = \bigcap_{k,j,I,N_0 \geq 1} \bigcup_{N \geq N_0} \left\{ u \in X : \frac{1}{N+1} \text{card} \{p \leq N : T^p(u^m) \in W_j \text{ for } m \in I \setminus \{m_0\} \text{ and } T^p(u^{m_0}) \in V_k\} > \delta_{k,j,I} \right\}$$

and show that A is residual.

For fixed k, j, I, N_0 the set

$$\bigcup_{N \geq N_0} \left\{ u \in X : \frac{1}{N+1} \text{card} \{p \leq N : T^p(u^m) \in W_j \text{ for } m \in I \setminus \{m_0\} \text{ and } T^p(u^{m_0}) \in V_k\} > \delta_{k,j,I} \right\}$$

is clearly open, and the definition of $\delta_{k,j,I}$ implies that it is also dense. By the Baire category theorem A is residual.

Next, we check that if u belongs to A and if $P \in \mathbb{C}[z]$ is not constant, with $P(0) = 0$, then $P(u) \in UFHC(T)$. We write $P(z) = \sum_{m \in I} \hat{P}(m)z^m$ for some $I \in \mathcal{P}_f(\mathbb{N}) \setminus \{\emptyset\}$ and $\hat{P}(m) \neq 0$ for all $m \in I$. Since $UFHC(T)$ is invariant under multiplication by a scalar we may assume that $\hat{P}(m_0) = 1$. Let V be a non-empty open set and find k, j such that $V_k + (\text{card}(I) - 1)\|\hat{P}\|_\infty W_j \subset V$. For each $N_0 \in \mathbb{N}$ there is $N \geq N_0$ such that

$$\frac{1}{N+1} \text{card} \{p \leq N : T^p(u^m) \in W_j \text{ for } m \in I \setminus \{m_0\} \text{ and } T^p(u^{m_0}) \in V_k\} > \delta_{k,j,I}.$$

But if $T^p(u^m) \in W_j$ for $m \in I \setminus \{m_0\}$ and $T^p(u^{m_0}) \in V_k$, then $T^p(P(u)) \in V$. Therefore,

$$\frac{1}{N+1} \text{card} \{p \leq N : T^p(P(u)) \in V\} > \delta_{k,j,I}$$

which yields that $\overline{\text{dens}}(\{p \in \mathbb{N} : T^p(P(u)) \in V\}) > \delta_{k,j,I} > 0$. \square

We will apply this lemma either for $m_0(I) = \min(I)$ or $m_0 = \max(I)$. The proposition gets the simpler forms:

Corollary 5.2 *Let X be an F -algebra. If for each nonempty subset V of X , for each neighbourhood W of the origin, for any positive integers $m_0 < m_1$, there is $\delta > 0$ such that for each nonempty open subset U and each $N_0 \in \mathbb{N}$, there is $u \in U$ and $N \geq N_0$ satisfying*

$$\frac{1}{N+1} \text{card} \{p \leq N : T^p(u^m) \in W \text{ for } m \in \{m_0+1, \dots, m_1\} \text{ and } T^p(u^{m_0}) \in V\} > \delta$$

then T admits an upper frequently hypercyclic algebra.

Corollary 5.3 *Let X be an F -algebra. If for each nonempty subset V of X , for each neighbourhood W of the origin, for any positive integer m , there is $\delta > 0$ such that for each nonempty open subset U and each $N_0 \in \mathbb{N}$, there is $u \in U$ and $N \geq N_0$ satisfying*

$$\frac{1}{N+1} \text{card} \{p \leq N : T^p(u^n) \in W \text{ for } n \in \{1, \dots, m-1\} \text{ and } T^p(u^m) \in V\} > \delta$$

then T admits an upper frequently hypercyclic algebra.

5.2 Existence of upper frequently hypercyclic algebras for weighted backward shifts - coordinatewise products

We intend to apply the previous method to backward shift operators and prove Theorem 1.4. The unconditional convergence of the series $\sum_{n \geq 1} (w_1 \cdots w_n)^{-1} e_n$ will be used throughout the following lemma.

Lemma 5.4 *Let (w_n) be a weight sequence such that $\sum_{n \geq 1} (w_1 \cdots w_n)^{-1} e_n$ converges unconditionally. Then, for all $\varepsilon > 0$, for all $p > 0$, for all $M > 0$, there exists $N \geq p$ such that, for each sequence of complex numbers $(y(n, l))_{n \geq N, 0 \leq l \leq p}$ with $|y(n, l)| \leq M$ for all n, l , then*

$$\left\| \sum_{n \geq N} \sum_{l=0}^p \frac{y(n, l)}{w_{l+1} \cdots w_{n+l}} e_{n+l} \right\| \leq \varepsilon.$$

Proof. We first observe that the convergence of the series involved follows from the unconditional convergence of each series $\sum_{n \geq 1} (w_{l+1} \cdots w_{n+l})^{-1} e_{n+l}$. Setting $z(n, l) = y(n, l)/M$ and using the triangle inequality, the (almost) homogeneity of the F -norm implies that

$$\left\| \sum_{n \geq N} \sum_{l=0}^p \frac{y(n, l)}{w_{l+1} \cdots w_{n+l}} e_{n+l} \right\| \leq (M+1) \sum_{l=0}^p \left\| \sum_{n \geq N} \frac{z(n, l)}{w_{l+1} \cdots w_{n+l}} e_{n+l} \right\|.$$

The existence of an N such that the last term is less than ε now follows directly from the unconditional convergence of the series $\sum_n (w_{l+1} \cdots w_{n+l})^{-1} e_{n+l}$ (see the introduction). \square

Proof of Theorem 1.4. Let $m_0 < m_1$ be two positive integers. Let $V, W \subset X$ be open and non-empty with $0 \in W$. Let $p \geq 0$, $\varepsilon > 0$ and $v = \sum_{l=0}^p v_l e_l$ be such that $B(v, \varepsilon) \subset V$ and $B(0, 2\varepsilon) \subset W$. We also set $M = \max(1, \|v\|_\infty)^{m_1/m_0}$. Let $N \in \mathbb{N}$ be given by Lemma 5.4 for

these values of ε , p and M . Without loss of generality, we may assume $N > p$. We set $\delta = \frac{1}{2N}$. Let $U \subset X$ be open and non-empty and let $x \in U$ with finite support. We also define, for $k \geq 0$,

$$v(k) = \sum_{l=0}^p \frac{v_l^{1/m_0}}{(w_{l+1} \cdots w_{k+l})^{1/m_0}} e_{k+l}.$$

Let N_1 be very large (precise conditions on it will be given later; for the moment we just assume that N_1 is bigger than the maximum of the support of x). We finally set

$$u = x + \sum_{k \geq N_1} v(Nk).$$

The unconditional convergence of the series $\sum_k (w_{l+1} \cdots w_{k+l})^{-1/m_0} e_{k+l}$ ensures that u is well-defined and that $\|u - x\| < \varepsilon$ provided N_1 is large enough. Let now $m \in \{m_0, \dots, m_1\}$ and $j \geq N_1$. Then, since x and the $v(kN)$, $k \geq N_1$, have pairwise disjoint support,

$$\begin{aligned} B_w^{Nj} u^m &= \sum_{l=0}^p \frac{v_l^{m/m_0}}{(w_{l+1} \cdots w_{jN+l})^{\frac{m}{m_0}-1}} e_l \\ &\quad + \sum_{k>j} \sum_{l=0}^p \frac{v_l^{m/m_0} w_{(k-j)N+l+1} \cdots w_{kN+l}}{(w_{l+1} \cdots w_{kN+l})^{\frac{m}{m_0}}} e_{(k-j)N+l} \\ &=: z(1, j, m) + z(2, j, m). \end{aligned}$$

If $m = m_0$, then $z(1, j, m) = v$ whereas, if $m \in \{m_0 + 1, \dots, m_1\}$, then since $|v_l|^{m/m_0} \leq M$ and since the sequences $(w_{l+1} \cdots w_{l+n})_n$ go to $+\infty$ (recall that X has a continuous norm), we may adjust N_1 so that $\|z(1, j, m)\| < \varepsilon$ for all $j \geq N_1$. On the other hand, we may write

$$z(2, j, m) = \sum_{n \geq N} \sum_{l=0}^p \frac{y(j, n, l, m)}{w_{l+1} \cdots w_{n+l}} e_{n+l}$$

where, for $s = k - j \geq 1$, $l = 0, \dots, p$,

$$y(j, sN, l, m) = \frac{v_l^{m/m_0}}{(w_{l+1} \cdots w_{(s+j)N})^{\frac{m}{m_0}-1}}$$

and $y(j, n, l, m) = 0$ if n is not a multiple of N . Again taking N_1 large enough guarantees that $|y(j, n, l, m)| \leq M$ for all $j \geq N_1$, all $n \geq N$, all $l = 0, \dots, p$ and all $m = m_0, \dots, m_1$. By the choice of N , we get $\|z(2, j, m)\| < \varepsilon$ for all $j \geq N_1$. Summarizing, we have proved that, for all $j \geq N_1$, $B_w^{Nj} u^{m_0} \in V$ and $B_w^{Nj} u^m \in W$ for all $m \in \{m_0 + 1, \dots, m_1\}$. Hence we may apply Corollary 5.2 to prove that B_w supports an upper frequently hypercyclic algebra. \square

The Theorem 1.4 can be applied to the following examples: λB on ℓ_p for $\lambda > 1$, λD on $H(\mathbb{C})$ for $\lambda > 0$ or B_w on c_0 with $w_n = 1 + \lambda/n$, $\lambda > 0$. Regarding this last weight, on ℓ_p , B_w is upper frequently hypercyclic if and only if $\lambda > 1/p$. However we do not know the answer to the following question because of the divergence of $\sum_n (w_1 \cdots w_n)^{-1/m} e_n$ for $m > \lambda p$.

Question 5.5 *Let $X = \ell_p$ and $w_n = 1 + \lambda/n$ for $\lambda > 1/p$. Does B_w supports an upper frequently hypercyclic algebra?*

On ℓ_p , it is known that B_w is (upper) frequently hypercyclic if and only if $\sum_n (w_1 \cdots w_n)^{-p}$ is convergent (see [6]).

Question 5.6 *Let $X = \ell_p$ endowed with the coordinatewise product and let $w = (w_n)$ be an admissible weight sequence. Assume that B_w supports an upper frequently hypercyclic algebra. Does this imply that $\sum_n (w_1 \cdots w_n)^{-\gamma}$ is convergent for all $\gamma > 0$?*

5.3 Existence of upper frequently hypercyclic algebras for weighted backward shifts - convolution product

We now study the existence of an upper frequently hypercyclic algebra for weighted backward shifts when the underlying Fréchet algebra is endowed with the convolution product. We shall give a general statement encompassing the case of the multiples of the backward shift and of the derivation operator.

Theorem 5.7 *Let X be a regular Fréchet sequence algebra for the Cauchy product and let (w_n) be an admissible weight sequence. Assume that*

(a) $\sum_{n \geq 1} (w_1 \cdots w_n)^{-1} e_n$ converges unconditionally.

(b) for all $m \geq 2$, there exists $c \in (0, 1)$ such that

$$\lim_{\sigma \rightarrow +\infty} \sup_{z \in c_{00} \cap B_{\ell_\infty}} \left\| \sum_{n \geq c\sigma} \frac{z_n (w_1 \cdots w_{m\sigma})^{(m-1)/m}}{w_1 \cdots w_{(m-1)\sigma+n}} e_n \right\| = 0.$$

Then B_w admits an upper frequently hypercyclic algebra.

Proof. We shall prove that the assumptions of Corollary 5.3 are satisfied. For $m = 1$, this follows from condition (a) which implies that T admits a dense set of (upper) frequently hypercyclic vectors. Thus, let us assume that $m \geq 2$ and let $c \in (0, 1)$ be given by (b). We also consider $d \in (c, (1+c)/2) \subset (0, 1)$. Let V be a non-empty open subset of X , W a neighbourhood of 0, $y = \sum_{l=0}^p y_l e_l \in V$ and $\eta > 0$ such that $B(0, 2\eta) + y \subset V$. Let finally $q > p$ be such that, for all $z \in \ell_\infty$ with $\|z\|_\infty \leq \|y\|_\infty (\max(1, w_1, \dots, w_p))^{p+1}$,

$$\left\| \sum_{n \geq q} \frac{z_n}{w_1 \cdots w_n} e_n \right\| < \eta.$$

We intend to prove that, for each non-empty open subset U and each $N_0 \in \mathbb{N}$, there is $u \in U$ and $N \geq N_0$ satisfying

$$\frac{1}{N+1} \text{card} \{s \leq N : B_w^s(u^n) \in W \text{ for } n < m \text{ and } B_w^s(u^m) \in V\} \geq \frac{d-c}{2((m-1)q + qd)}.$$

More precisely, we shall prove that, for all σ large enough, setting

$$E_\sigma = \{(m-1)q\sigma + qj : c\sigma \leq j < d\sigma\},$$

there exists $u \in U$ such that, for all $s \in E_\sigma$, $B_w^s(u^n) = 0$ for $n < m$ and $B_w^s(u^m) \in V$. Since

$$\lim_{\sigma \rightarrow +\infty} \frac{\text{card}(E_\sigma)}{\max(E_\sigma)} = \frac{d-c}{(m-1)q+qd},$$

we will get the claimed result.

We thus fix $x \in U$ with finite support (we denote by p' the maximum of the support of x) and let $\sigma > 0$ be such that $p' < c\sigma$. Inspired by the proof of Theorem 1.3, we set

$$u = x + \sum_{j=c\sigma}^{d\sigma-1} \sum_{l=0}^p d_{j,l} e_{qj+l} + \varepsilon e_{q\sigma}$$

where

$$\varepsilon = \frac{1}{(w_1 \cdots w_{mq\sigma})^{1/m}}$$

$$d_{j,l} = \frac{y_l}{m\varepsilon^{m-1} w_{l+1} \cdots w_{(m-1)q\sigma+qj+l}}.$$

Let us first prove that, provided that σ is large enough, u belongs to U . Let $r \geq 1$. Since X is regular, there exists $\rho \geq r$ and $C > 0$ such that

$$\|\varepsilon e_{q\sigma}\|_r \leq \frac{C}{(w_1 \cdots w_{mq\sigma})^{1/m}} \|e_{mq\sigma}\|_\rho^{1/m} \xrightarrow{\sigma \rightarrow +\infty} 0.$$

Furthermore,

$$\left\| \sum_{j=c\sigma}^{d\sigma-1} \sum_{l=0}^p d_{j,l} e_{qj+l} \right\| = \left\| \sum_{j=c\sigma}^{d\sigma-1} \sum_{l=0}^p \frac{y_l w_1 \cdots w_l (w_1 \cdots w_{mq\sigma})^{(m-1)/m}}{m w_1 \cdots w_{(m-1)q\sigma+qj+l}} e_{qj+l} \right\|$$

$$= \left\| \sum_{n \geq cq\sigma} \frac{z_n (w_1 \cdots w_{mq\sigma})^{(m-1)/m}}{w_1 \cdots w_{(m-1)q\sigma+n}} e_n \right\|$$

for some eventually null sequence (z_n) such that

$$\|z\|_\infty \leq \|y\|_\infty (\max(1, w_1, \dots, w_p))^p / m.$$

Assumption (b) allows us to conclude that $\sum_{j=c\sigma}^{d\sigma-1} \sum_{l=0}^p d_{j,l} e_{qj+l}$ tends to zero as σ goes to $+\infty$.

Observe now that, for $n < m$, the support of u^n is contained in $[0, nq\sigma]$ so that, for $s \in E_\sigma$, $B_w^s(u^n) = 0$. On the other hand,

$$u^m = z + m\varepsilon^{m-1} \sum_{j=c\sigma}^{d\sigma-1} \sum_{l=0}^p d_{j,l} e_{(m-1)q\sigma+qj+l} + \varepsilon^m e_{mq\sigma}$$

with $\text{supp}(z) \subset [0, (m-2)q\sigma + 2qd\sigma] \cup [0, (m-1)q\sigma + p']$. It is not difficult to see that, because $d < (1+c)/2$, $\max(\text{supp}(z)) \leq (m-1)q\sigma + qc\sigma$ for σ large enough. Thus, for any $s = (m-1)q\sigma + qk \in E_\sigma$,

$$B_w^s(u^m) = y + \sum_{j=k+1}^{d\sigma-1} \sum_{l=0}^p \frac{y_l}{w_{l+1} \cdots w_{q(j-k)+l}} e_{q(j-k)+l} + \frac{1}{w_1 \cdots w_{q\sigma-qk}} e_{q\sigma-qk}.$$

We handle the second term of the right hand side of the equality by writing

$$\sum_{j=k+1}^{d\sigma-1} \sum_{l=0}^p \frac{y_l}{w_{l+1} \cdots w_{q(j-k)+l}} e_{q(j-k)+l} = \sum_{n \geq q} \frac{z_n}{w_1 \cdots w_n} e_n$$

for some sequence (z_n) such that $\|z\|_\infty \leq \|y\|_\infty (\max(1, w_1, \dots, w_p))^p$. By our choice of q , this has F -norm less than η . Finally, $\|e_{q\sigma - qk}\| / (w_1 \cdots w_{q\sigma - qk})$ becomes also less than η provided σ , and thus $q\sigma - qd\sigma$, becomes large enough. \square

Assumption (a) in Theorem 5.7 is what we need to get an (upper) frequently hypercyclic vector. Assumption (b) is also an assumption around unconditional convergence of the series $\sum_{n \geq 1} (w_1 \cdots w_n)^{-1} e_n$. This looks clearer by writing

$$\sum_{n \geq c\sigma} \frac{z_n (w_1 \cdots w_{m\sigma})^{(m-1)/m}}{w_1 \cdots w_{n+(m-1)\sigma}} e_n = \sum_{n \geq c\sigma} \frac{z_n (w_1 \cdots w_{m\sigma})^{(m-1)/m}}{w_{n+1} \cdots w_{n+(m-1)\sigma}} \times \frac{e_n}{w_1 \cdots w_n}.$$

Although it looks quite technical, it is satisfied by three natural examples (where we always endow the F -algebra with the Cauchy product).

Example 5.8 Let $X = \ell_1$ and $w_n = \lambda > 1$ for all $n \in \mathbb{N}$. Then B_w supports an upper frequently hypercyclic algebra.

Proof. The situation is very simple here because, for all $n \geq 1$,

$$\frac{(w_1 \cdots w_{m\sigma})^{(m-1)/m}}{w_{n+1} \cdots w_{n+(m-1)\sigma}} = 1,$$

so that (b) is clearly satisfied. \square

Example 5.9 Let $X = \ell_1$ and $w_1 \cdots w_n = \exp(n^\alpha)$ for all $n \in \mathbb{N}$ and some $\alpha \in (0, 1)$. Then B_w supports an upper frequently hypercyclic algebra.

Proof. That (b) is satisfied follows from the classical asymptotic behavior

$$\sum_{n \geq N} \exp(-n^\alpha) \sim_{N \rightarrow +\infty} \frac{1}{\alpha} N^{1-\alpha} \exp(-N^\alpha). \quad (\text{A.14})$$

Assuming (A.14) is true, we just write

$$\begin{aligned} & \left\| \sum_{n \geq c\sigma} \frac{z_n (w_1 \cdots w_{m\sigma})^{(m-1)/m}}{w_1 \cdots w_{(m-1)\sigma + n}} e_n \right\| \\ &= \exp\left(\frac{m-1}{m} m^\alpha \sigma^\alpha\right) \sum_{n \geq c\sigma} \exp(-((m-1)\sigma + n)^\alpha) \\ &\sim_{+\infty} C \sigma^{1-\alpha} \exp\left(\left(\frac{m-1}{m} m^\alpha - (m-1+c)^\alpha\right) \sigma^\alpha\right). \end{aligned}$$

Assumption (b) is satisfied for c close enough to 1, since in that case

$$(m-1+c)^\alpha > \frac{(m-1)m^\alpha}{m}.$$

For the sake of completeness, we just mention that (A.14) follows from the formula of integration by parts:

$$\int_N^{+\infty} \exp(-x^\alpha) dx = \frac{1}{\alpha} N^{1-\alpha} \exp(-N^\alpha) + \frac{1-\alpha}{\alpha} \int_N^{+\infty} x^{-\alpha} \exp(-x^\alpha) dx.$$

□

Example 5.10 The derivation operator D supports an upper frequently hypercyclic algebra on $H(\mathbb{C})$.

Proof. It is sufficient to verify (b) replacing $\|\cdot\|$ by any seminorm $\|\cdot\|_r$. Now,

$$\begin{aligned} \left\| \sum_{n \geq c\sigma} \frac{z_n (w_1 \cdots w_{m\sigma})^{(m-1)/m}}{w_1 \cdots w_{(m-1)\sigma+n}} e_n \right\|_r &= \sum_{n \geq c\sigma} \frac{(m\sigma)!^{(m-1)/m}}{((m-1)\sigma+n)!} r^n \\ &= \frac{(m\sigma)!^{(m-1)/m}}{r^{(m-1)\sigma}} \sum_{n \geq c\sigma} \frac{r^{(m-1)\sigma+n}}{((m-1)\sigma+n)!} \\ &\leq Cr^{c\sigma} \frac{(m\sigma)!^{(m-1)/m}}{(m-1+c)\sigma}. \end{aligned}$$

Since for all $\varepsilon > 0$ Stirling's formula implies

$$\begin{aligned} (m\sigma)!^{(m-1)/m} &\leq C_\varepsilon \sigma^{(m-1+\varepsilon)\sigma} \\ ((m-1+c)\sigma)! &\geq C_\varepsilon \sigma^{(m-1+c-\varepsilon)\sigma}, \end{aligned}$$

choosing $\varepsilon < 2c$ it follows that, for all $c \in (0, 1)$ and all $r \geq 1$, we have

$$\left\| \sum_{n \geq c\sigma} \frac{z_n (w_1 \cdots w_{m\sigma})^{(m-1)/m}}{w_1 \cdots w_{(m-1)\sigma+n}} e_n \right\|_r \leq C \frac{r^{c\sigma}}{\sigma^{(c-2\varepsilon)\sigma}} \xrightarrow{\sigma \rightarrow +\infty} 0.$$

Hence, assumption (b) is verified. □

Another natural operator that could admit an upper frequently hypercyclic algebra is the backward shift B_w with $w_n = (1 + \frac{\lambda}{n})$, $\lambda > 1$, acting over ℓ_1 with the convolution product. Unfortunately, for this weight, the assumptions in Theorem 5.7 are not verified.

Question 5.11 Let $X = \ell_1$ endowed with the convolution product and $w_n = (1 + \frac{\lambda}{n})$, $\lambda > 1$. Does B_w admit an upper frequently hypercyclic algebra?

We can ask a similar question for convolution operators $\phi(D)$ on $H(\mathbb{C})$, $|\phi(0)| < 1$, which are frequently hypercyclic and admit a hypercyclic algebra.

Question 5.12 Let $X = H(\mathbb{C})$ and let $\phi : \mathbb{C} \rightarrow \mathbb{C}$ be a nonconstant entire function with exponential type, not a multiple of an exponential function, with $|\phi(0)| < 1$. Does $\phi(D)$ supports an upper frequently hypercyclic algebra?

5.4 Weighted shifts with a frequently hypercyclic algebra on ω

Despite the result of Falcó and Grosse-Erdmann, it is not so difficult to exhibit operators supporting a frequently hypercyclic algebra if we work on the big space ω .

Theorem 5.13 *Let $w = (w_n)_{n \geq 1}$ be a weight sequence such that $(w_1 \cdots w_n)$ either tends to $+\infty$ or to 0. Then B_w , acting on ω endowed with the coordinatewise product, supports a frequently hypercyclic algebra.*

Proof. We first assume that $(w_1 \cdots w_n)$ tends to $+\infty$ and observe that this clearly implies that, for all $l \geq 0$, $(w_{l+1} \cdots w_{n+l})$ tends to $+\infty$. Let $(v(p), m(p))$ be a dense sequence in $\omega \times \mathbb{N}$, where each $v(p)$ has finite support contained in $[0, p]$. We then write $v(p) = \sum_{l=0}^p v_l(p) e_l$. For $(n, p) \in \mathbb{N}^2$, we define

$$y(n, p) = \sum_{l=0}^p \frac{v_l(p)^{1/m(p)}}{(w_{l+1} \cdots w_{n+l})^{1/m(p)}} e_{n+l}.$$

By [5, Lemma 6.19] (see the forthcoming Lemma 5.14), there exists a sequence $(A(p))$ of pairwise disjoint subsets of \mathbb{N} , with positive lower density, and such that $|n - n'| \geq p + q + 1$ whenever $n \neq n'$ and $(n, n') \in A(p) \times A(q)$. In particular, the vectors $y(n, p)$ for $p \in \mathbb{N}$ and $n \in A(p)$ have disjoint support. Hence, we may define $u = \sum_{p \in \mathbb{N}} \sum_{n \in A(p)} y(n, p)$ and we claim that u generates a frequently hypercyclic algebra.

Indeed, let $P \in \mathbb{C}[z]$ be non-constant with $P(0) = 0$, $P(z) = \sum_{m=m_0}^{m_1} \hat{P}(m) z^m$, $\hat{P}(m_0) \neq 0$, and let V be a non-empty open subset of ω . Let $p \in \mathbb{N}$, $\varepsilon > 0$ be such that $m(p) = m_0$ and any vector $x \in \omega$ satisfying $|x_l - \hat{P}(m_0) v_l(p)| < \varepsilon$ for all $l = 0, \dots, p$ belongs to V . Now, for $l = 0, \dots, p$ and $n \in A(p)$,

$$(B_w^n P(u))_l = \hat{P}(m_0) v_l(p) + \sum_{m=m_0+1}^{m_1} \frac{\hat{P}(m) v_l(p)^{\frac{m}{m_0}}}{(w_{l+1} \cdots w_{n+l})^{\frac{m}{m_0} - 1}}.$$

Since $(w_{l+1} \cdots w_{n+l})$ tends to $+\infty$ for all l , $B_w^n P(u)$ belongs to V for all n in a cofinite subset of $A(p)$. Hence, $P(u)$ is a frequently hypercyclic vector for B_w .

The proof is completely similar if we assume that $(w_1 \cdots w_n)$ tends to 0. The only difference is that the dominant term is now given by the term of highest degree of P , namely we choose p such that $m(p) = m_1$ and we write

$$(B_w^n P(u))_l = \hat{P}(m_1) v_l(p) + \sum_{m=m_0}^{m_1-1} \frac{\hat{P}(m) v_l(p)^{\frac{m}{m_1}}}{(w_{l+1} \cdots w_{n+l})^{\frac{m}{m_1} - 1}}.$$

We will conclude because, for $m < m_1$, $(w_{l+1} \cdots w_{n+l})^{\frac{m}{m_1} - 1}$ tends to $+\infty$. Details are left to the reader. \square

The unweighted backward shift on ω (still endowed with the coordinatewise product) supports a frequently hypercyclic algebra. Indeed, more generally, let T be a multiplicative operator on an F -algebra X with the property that for every non-zero polynomial P vanishing at the origin, the map

$$\tilde{P} : X \rightarrow X, x \mapsto P(x)$$

has dense range. Then if T is frequently hypercyclic, it supports a frequently hypercyclic algebra. The reason for that is the simple observation that if U is a non-empty open set of X and P a non-zero polynomial vanishing at the origin,

$$\{n \in \mathbb{N} : T^n(P(x)) \in U\} = \{n \in \mathbb{N} : T^n x \in \tilde{P}^{-1}(U)\}.$$

From the same observation we may conclude that the translation operators T_a acting on $C^\infty(\mathbb{R}, \mathbb{C})$ for $a \in \mathbb{R}, a \neq 0$, admit a frequently hypercyclic algebra. The fact that $C^\infty(\mathbb{R}, \mathbb{C})$ has the above mentioned property is proven in [10, Proposition 20].

5.5 A sequence of sets with positive lower density which are very far away from each other

The remaining part of this section is devoted to the proof of Theorem 1.6. The starting point to exhibit frequently hypercyclic vectors is the following lemma on the existence of subsets of \mathbb{N} with positive lower density which are sufficiently separated.

Lemma 5.14 (Lemma 6.19 in [5]) *Let $(a(p))$ be any sequence of positive real numbers. Then one can find a sequence $(A(p))$ of pairwise disjoint subsets of \mathbb{N} such that*

- (i) *each set $A(p)$ has positive lower density;*
- (ii) *$\min A(p) \geq a(p)$ and $|n - n'| \geq a(p) + a(q)$ whenever $n \neq n'$ and $(n, n') \in A(p) \times A(q)$.*

To produce a frequently hypercyclic algebra for a weighted shift on c_0 , we will need a refined version of this lemma where we add new conditions of separation.

Theorem 5.15 *Let $(a(p))$ be any sequence of positive real numbers. Then one can find a sequence $(A(p))$ of pairwise disjoint subsets of \mathbb{N} such that*

- (i) *each set $A(p)$ has positive lower density;*
- (ii) *$\min A(p) \geq a(p)$ and $|n - n'| \geq a(p) + a(q)$ whenever $n \neq n'$ and $(n, n') \in A(p) \times A(q)$.*
- (iii) *for all $C > 0$, there exists $\kappa > 0$ such that, for all $(n, n') \in A(p) \times A(q)$ with $p \neq q$ and $\max(n, n') \geq \kappa$, then $|n - n'| \geq C$.*

The proof of this theorem is rather long. The strategy is to construct a sequence of sets satisfying only (i) and (iii), and then to modify them to add (ii). We begin with two sets.

Lemma 5.16 *Let $E \subset \mathbb{N}$ be a set with positive lower density. There exist $A, B \subset E$ disjoint, with positive lower density, and such that, for all $C > 0$, there exists $\kappa > 0$ such that, for all $n \in A$ and all $n' \in B$ with $\max(n, n') \geq \kappa$, then $|n - n'| \geq C$.*

Proof. We write $E = \{n_j : j \in \mathbb{N}\}$ in an increasing order. We set, for $k \geq 1$, $u_k = k$, $v_k = \lfloor \sqrt{k} \rfloor$. We define sequences (M_k) , (N_k) , (P_k) and (Q_k) by setting $M_1 = 1$ and, for $k \geq 1$,

$$N_k = M_k + u_k, \quad P_k = N_k + v_k, \quad Q_k = P_k + u_k, \quad M_{k+1} = Q_k + v_k.$$

We then define

$$I = \bigcup_k [M_k, N_k), \quad J = \bigcup_k [P_k, Q_k),$$

$$A = \{n_j : j \in I\}, \quad B = \{n_j : j \in J\}.$$

The sets I and J have positive lower density. Indeed, for $N \in \mathbb{N}$, let k be such that $N \in [M_k, M_{k+1})$. Then

$$\frac{\text{card}(I \cap [1, \dots, N])}{N} \geq \frac{u_1 + \dots + u_{k-1}}{2(u_1 + \dots + u_k + v_1 + \dots + v_k)} \geq \frac{1}{4}$$

provided k is large enough. The same is true for J . Since E has positive lower density, this yields that A and B have positive lower density. Moreover, let $C > 0$. There exists $k \geq 0$ such that $v_{k-1} \geq C$. We set $\kappa = n_{M_k}$. Let $(n, n') \in A \times B$ with $\max(n, n') \geq \kappa$. Assume for instance that $n \geq n_{M_k}$ and write $n = n_j$, $n' = n_{j'}$. Then $j \geq M_k$ and the construction of the sets I and J ensure that j' does not belong to $[j - v_{k-1}, j + v_{k-1}]$. Thus, $|n - n'| \geq |j - j'| \geq C$. \square

It is not difficult to require that (ii) in Theorem 5.15 holds when we restrict ourselves to $p = q$.

Lemma 5.17 *Let $A \subset \mathbb{N}$ with positive lower density and $a > 0$. There exists $B \subset A$ with positive lower density, $\min(B) \geq a$ and $|n - n'| \geq a$ for all $n, n' \in B$, $n \neq n'$.*

Proof. Write $A = \{n_j : j \in \mathbb{N}\}$ in an increasing order and define $B = \{n_{ka} : k \in \mathbb{N}\}$. \square

We then go inductively from two sets to a sequence of sets.

Lemma 5.18 *There exists a sequence $(A(p))$ of pairwise disjoint subsets of \mathbb{N} such that*

- (i) *each set $A(p)$ has positive lower density;*
- (ii) *for all $C > 0$, there exists $\kappa > 0$ such that, for all $(n, n') \in A(p) \times A(q)$ with $p \neq q$ and $\max(n, n') \geq \kappa$, then $|n - n'| \geq C$.*

Proof. We shall construct by induction two sequences of sets $(A(p))$ and $(B(p))$ and a sequence of integers (κ_k) such that, at each step r ,

- (a) for all $1 \leq p \leq r$, $A(p)$ and $B(p)$ are disjoint and have positive lower density.
- (b) for all $1 \leq p < q \leq r$, $A(q) \subset B(p)$ and $B(q) \subset B(p)$.
- (c) for all $C > 0$, there exists $\kappa > 0$ such that, for all $1 \leq p \leq q \leq r$, for all $n \in A(p)$ and $n' \in B(q)$, $\max(n, n') \geq \kappa \implies |n - n'| \geq C$.
- (d) for all $k \in \{1, \dots, r\}$, for all $1 \leq p \leq q \leq r$, for all $n \in A(p)$ and $n' \in B(q)$, $\max(n, n') \geq \kappa_k \implies |n' - n| \geq k$.

It is straightforward to check that the resulting sequence $(A(p))$ satisfies the conclusions of Lemma 5.18. Observe nevertheless that it is condition (d) together with the inclusion $A(q) \subset B(p)$ for $q > p$ which gives (ii) in this lemma (which is uniform with respect to p and q). Condition (c) is only helpful for the induction hypothesis.

We initialize the construction by applying Lemma 5.16 to $E = \mathbb{N}$. We set $A(1) = A$ and $B(1) = B$ which satisfy (a), (b) and (c). In particular, applying (c) for $C = 1$ we find some κ that we call κ_1 .

Assume now that the construction has been done until step r and let us perform it for step $r + 1$. Let E be a subset of $B(r)$ with positive lower density and $|n - n'| \geq r + 1$ provided $n \neq n'$ are in E . We apply Lemma 5.16 to this set E and we set $A(r + 1) = A$ and $B(r + 1) = B$, so that (a) and (b) are clearly satisfied. Upon taking a maximum, (c) is also easily satisfied: indeed, the only case which is not settled by the induction hypothesis is $p = q = r + 1$ (when $p < r + 1$ and $q = r + 1$, use $B(q) \subset B(r)$); this case is solved by the construction of $A(r + 1)$ and $B(r + 1)$.

The proof of (d) is slightly more delicate. For $k = 1, \dots, r$, we have to verify that for $1 \leq p \leq r + 1$, $n \in A(p)$ and $n' \in B(r + 1)$, $\max(n, n') \geq \kappa_k \implies |n - n'| \geq k$. When $p \leq r$, again this follows from $B(r + 1) \subset B(r)$. For $p = r + 1$, this follows from $A(r + 1), B(r + 1) \subset E$ and the fact that distinct elements of E have distance greater than or equal to $r + 1$. Finally, applying (c) for $C = k + 1$, we define κ_{k+1} . \square

We need now to ensure property (ii) in Theorem 5.15. This will be done again inductively, the main step being the following lemma.

Lemma 5.19 *Let $(A(p))$ be a sequence of pairwise disjoint subsets of \mathbb{N} with positive lower density and let $(a(p))$ be a sequence of positive real numbers. There exists a sequence $(B(p))$ of subsets of \mathbb{N} with positive lower density such that each $B(p)$ is contained in $A(p)$ and, for all $n \in B(1)$, for all $n' \in B(p)$, $p \geq 2$, $|n - n'| \geq a(1) + a(p)$.*

Proof. Since $A(1)$ has positive lower density, there exists $N \in \mathbb{N}$ and $\delta > 0$ such that, for all $n \geq N$,

$$\frac{\text{card}(A(1) \cap [0, \dots, n])}{n + 1} \geq \delta.$$

For $p \geq 2$, let $B(p)$ be a subset of $A(p)$ such that, for all $n \in \mathbb{N}$,

$$\frac{\text{card}((B(p) + [-a(1) - a(p), a(1) + a(p)]) \cap [0, \dots, n])}{n + 1} < \frac{\delta}{2^p}$$

but $B(p)$ still has positive lower density. This is possible if, writing $A(p) = \{n_j; j \in \mathbb{N}\}$, we set $B(p) = \{n_{ka}; k \geq 1\}$ for some sufficiently large a . We then define $B(1) = A(1) \setminus \bigcup_{p \geq 2} (B(p) + [-a(1) - a(p), a(1) + a(p)])$. Then, for all $n \in B(1)$ and all $n' \in B(p)$, $p \geq 2$, one clearly has $|n - n'| \geq a(1) + a(p)$ whereas, for all $n \geq N$,

$$\frac{\text{card}(B(1) \cap [0, \dots, n])}{n + 1} \geq \delta - \sum_{p \geq 2} \frac{\delta}{2^p} \geq \frac{\delta}{2}$$

so that $B(1)$ still has positive lower density. \square

Proof of Theorem 5.15. Applying Lemmas 5.18 and 5.17, we start from a sequence $(A(p))$ of pairwise disjoint subsets of \mathbb{N} , satisfying properties (i) and (iii) of Theorem 5.15 and property (ii) when $p = q$. We construct by induction on r sets $B(1), \dots, B(r)$, $A_r(k)$ for $k \geq r + 1$ such that

- $B(k) \subset A(k)$ for all $k \leq r$, $A_r(k) \subset A(k)$ for all $k \geq r + 1$;
- $B(k)$ and $A_r(k)$ have positive lower density;
- for all $p \in \{1, \dots, r\}$, for all $q \geq p + 1$, for all $n \in A(p)$, for all $n' \in B(q)$ if $q \leq r$, for all $n' \in A_r(q)$ if $q \geq r + 1$, $|n - n'| \geq a(p) + a(q)$.

The sequence $(B(p))$ that we get at the end will answer the problem. Now the construction is easily done by successive applications of Lemma 5.19 first with the sequence $(A(p))_{p \geq 1}$, then with the sequence $(A_1(p))_{p \geq 2}$, and so on. \square

5.6 A weighted shift with a frequently hypercyclic algebra on c_0

Let us now define a weight (w_n) such that B_w , acting on c_0 endowed with the coordinatewise product, supports a frequently hypercyclic algebra. We start with the sequence $(A(p))_{p \geq 1}$ given by Theorem 5.15 for $a_p = p$. We then construct inductively a sequence of integers (M_k) such that, for all $(n, n') \in A(p) \times A(q)$ with $p \neq q$, $\max(n, n') \geq M_{k+1} \implies |n - n'| \geq M_k$. This follows directly from property (iii) of Theorem 5.15, applied successively with $C = M_1 = 1$ to get M_2 , $C = M_2$ to get M_3 , and so on. We may also assume that the sequence $(M_{k+1} - M_k)$ is non-decreasing.

We define the weight $(w_n)_{n \geq 1}$ by the following inductive formulas:

- $w_n = 2$ for all $n \leq M_2$;
- for all $k \geq 2$, for all $n \in \{M_k + 1, \dots, M_{k+1}\}$,

$$w_n = (w_1 \cdots w_{M_k})^{\frac{1}{k(M_{k+1} - M_k)}}$$

so that, and this is the crucial point,

$$w_{M_k+1} \cdots w_{M_{k+1}} = (w_1 \cdots w_{M_k})^{\frac{1}{k}}.$$

Let us summarize the properties of the weight which will be useful later.

Lemma 5.20 *The weight (w_n) satisfies the following properties:*

- for all $n \geq 1$, $w_n \geq 1$;
- (w_n) is non-increasing;
- $(w_1 \cdots w_n)$ tends to $+\infty$;

- for all $\alpha > 0$, for all $l \geq 0$, $\frac{w_{M_{k-1}+l+1} \cdots w_{M_{k+1}+l}}{(w_{l+1} \cdots w_{M_{k+1}+l})^\alpha} \xrightarrow{k \rightarrow +\infty} 0$.

Proof. The first property is clear. For the second one, it suffices to prove that if $n \in \{M_k + 1, \dots, M_{k+1}\}$ and $n' \in \{M_{k+1} + 1, \dots, M_{k+2}\}$ for some k , then $w_{n'} \leq w_n$. We now write

$$\begin{aligned} w_{n'} &= (w_1 \cdots w_{M_{k+1}})^{\frac{1}{(k+1)(M_{k+2}-M_{k+1})}} \\ &= (w_1 \cdots w_{M_k})^{\frac{1}{(k+1)(M_{k+2}-M_{k+1})}} (w_{M_{k+1}} \cdots w_{M_{k+1}})^{\frac{1}{(k+1)(M_{k+2}-M_{k+1})}} \\ &= (w_1 \cdots w_{M_k})^{\frac{1}{k(M_{k+2}-M_{k+1})}} \\ &\leq w_n. \end{aligned}$$

To prove that $(w_1 \cdots w_n)$ tends to $+\infty$, we just observe that, for all $k \geq 2$,

$$w_1 \cdots w_{M_k} = (w_1 \cdots w_{M_2})^{\prod_{j=2}^{k-1} (1 + \frac{1}{j})},$$

and this goes to $+\infty$ as k tends to $+\infty$. Finally, since (w_n) is bounded and bounded below, we need only to prove the last property for $l = 0$. Now we write

$$\begin{aligned} w_{M_{k-1}+1} \cdots w_{M_{k+1}} &= w_{M_{k-1}+1} \cdots w_{M_k} (w_1 \cdots w_{M_k})^{1/k} \\ &= (w_{M_{k-1}+1} \cdots w_{M_k})^{1 + \frac{1}{k}} (w_1 \cdots w_{M_{k-1}})^{\frac{1}{k}} \\ &= (w_1 \cdots w_{M_{k-1}})^{\frac{1}{k-1} (1 + \frac{1}{k}) + \frac{1}{k}} \\ &= (w_1 \cdots w_{M_{k-1}})^{\frac{2}{k-1}} \end{aligned}$$

so that

$$\frac{w_{M_{k-1}+1} \cdots w_{M_{k+1}}}{(w_1 \cdots w_{M_{k+1}})^\alpha} \leq \frac{1}{(w_1 \cdots w_{M_{k-1}})^{\alpha - \frac{2}{k-1}}}$$

which indeed tends to zero. \square

We now prove that the operator B_w acting on c_0 endowed with the coordinatewise product supports a frequently hypercyclic algebra. Let $(v(p), m(p))$ be a sequence dense in $c_0 \times \mathbb{N}$ such that each $v(p)$ has finite support contained in $[0, p]$. We shall need a last technical lemma involving all the objects we constructed until now.

Lemma 5.21 *There exists a sequence of integers $(N(r))_{r \geq 1}$ satisfying the following properties:*

- (i) for all $r \geq 1$,

$$\sup_{n \geq N(r), l=0, \dots, r} \left| \frac{v_l(r)}{(w_{l+1} \cdots w_{n+l})^{\frac{1}{m(r)+1}}} \right|^{\frac{1}{m(r)}} < \frac{1}{r}.$$

- (ii) for all $r \geq 2$, for all $s \in \{1, \dots, r-1\}$, for all $(j, j') \in A(r) \times A(s)$ with $j \geq N(r)$, for all $l \in \{0, \dots, r\}$, for all $\alpha \geq \min\left(\frac{1}{m(r)}, \frac{1}{m(s)}\right)$,

$$\begin{aligned} j > j' &\implies \left| \frac{w_{l+(j-j')+1} \cdots w_{j+l} v_l(r)^\alpha}{(w_{l+1} \cdots w_{j+l})^\alpha} \right| < \frac{1}{r} \\ j' > j &\implies \left| \frac{w_{l+(j'-j)+1} \cdots w_{j'+l} v_l(s)^\alpha}{(w_{l+1} \cdots w_{j'+l})^\alpha} \right| < \frac{1}{r}. \end{aligned}$$

Proof. Let $r \geq 1$ be fixed, We first observe that it is easy to ensure (i), just by assuming that $N(r)$ is large enough. Let us choose $N(r)$ to ensure (ii). Upon taking a supremum, we may fix s and l and to simplify the notations, we will assume $l = 0$. Let $\alpha_0 = \min\left(\frac{1}{m(r)}, \frac{1}{m(s)}\right)$ and $C = \max(1, |v_0(r)|, |v_0(s)|)$. We define three integers N_0, k_0 and k_1 satisfying the following three conditions:

$$\begin{aligned} n \geq N_0 &\implies \frac{C^2}{w_1 \cdots w_n} < \frac{1}{r} \\ n \geq M_{k_0} &\implies \frac{C^2}{w_1 \cdots w_n} < \frac{1}{r} \\ k \geq k_1 &\implies \frac{w_{M_{k-1}+1} \cdots w_{M_{k+1}} C}{(w_1 \cdots w_{M_{k+1}})^{\alpha_0}} < \frac{1}{r}. \end{aligned}$$

We set $N(r) = \max(N_0, M_{k_0+1}, M_{k_1})$. Let $(j, j') \in A(r) \times A(s)$ with $j \geq N(r)$. To fix the ideas, we assume that $j > j'$. If $\alpha \geq 2$, then

$$\left| \frac{w_{(j-j')+1} \cdots w_j v_0(r)^\alpha}{(w_1 \cdots w_j)^\alpha} \right| \leq \frac{w_{(j-j')+1} \cdots w_j}{w_1 \cdots w_j} \times \left| \frac{v_0(r)^2}{w_1 \cdots w_j} \right|^{\alpha/2} < 1 \times \left(\frac{1}{r}\right)^{\frac{\alpha}{2}} \leq \frac{1}{r}.$$

If $\alpha \in [1, 2]$, then

$$\left| \frac{w_{(j-j')+1} \cdots w_j v_0(r)^\alpha}{(w_1 \cdots w_j)^\alpha} \right| \leq \frac{C^2}{w_1 \cdots w_{j-j'}}.$$

Since $j \geq N(r)$, $j - j' \geq M_{k_0}$ so that the last term is less than $1/r$. For $\alpha < 1$, since $j \geq N(r) \geq M_{k_1}$, there exists a single integer $k \geq k_1$ such that $j \in [M_k, M_{k+1})$. Then $j - j' \geq M_{k-1}$ and

$$\begin{aligned} \left| \frac{w_{(j-j')+1} \cdots w_j v_0(r)^\alpha}{(w_1 \cdots w_j)^\alpha} \right| &\leq \frac{w_{M_{k-1}+1} \cdots w_j C}{(w_1 \cdots w_j)^{\alpha_0}} \\ &\leq \frac{w_{M_{k-1}+1} \cdots w_{M_{k+1}} C}{(w_1 \cdots w_{M_{k+1}})^{\alpha_0}} \\ &< \frac{1}{r}. \end{aligned}$$

□

Proof of Theorem 1.6. We are now ready for the proof that B_w supports a frequently hypercyclic algebra. By Lemma 5.17, for each $p \geq 1$, let $B(p)$ be a subset of $A(p)$ with positive lower density such that $\min(B(p)) \geq N(p)$ and $|n - n'| \geq N(p)$ for all $n \neq n' \in B(p)$. We set

$$u(p) = \sum_{n \in B(p)} \sum_{l=0}^p \frac{1}{(w_{l+1} \cdots w_{n+l})^{1/m(p)}} v_l(p)^{1/m(p)} e_{n+l}.$$

Since $(w_1 \cdots w_n)$ tends to $+\infty$, $u(p)$ belongs to c_0 . Moreover, the choice of $N(p)$ (here, (i) of Lemma 5.21) ensures that $\|u(p)\| < 1/p$. We also observe that the $u(p)$ have pairwise disjoint support. Hence we may define $u = \sum_{p \geq 1} u(p)$ which still belongs to c_0 . We claim that the following property is true: for all $p \geq 1$, for all $q \neq p$, for all $n \in B(p)$,

$$\left\| B_w^n u(p)^{m(p)} - v(p) \right\| < \frac{1}{p}, \tag{A.15}$$

$$\forall m > m(p), \|B_w^n u(p)^m\| < \frac{1}{p}, \tag{A.16}$$

$$\forall m \geq m(p), \|B_w^n u(q)^m\| < \frac{1}{p}. \quad (\text{A.17})$$

Assume that these properties have been proved. Let P be a non-constant polynomial with $P(0) = 0$ and write it $P(z) = \sum_{m=m_0}^{m_1} \hat{P}(m)z^m$ with $\hat{P}(m_0) \neq 0$. We aim to prove that $P(u)$ is a frequently hypercyclic vector for B_w . Without loss of generality, we can assume $\hat{P}(m_0) = 1$. Let V be a non-empty open subset of c_0 . There exists $p \geq 1$ such that $B(v(p), (2 + \sum_{m=m_0+1}^{m_1} |\hat{P}(m)|) / p) \subset V$ and $m(p) = m_0$. Then, for all $n \in B(p)$,

$$\begin{aligned} \|B_w^n P(u) - v(p)\| &\leq \left\| B_w^n u(p)^{m(p)} - v(p) \right\| + \left\| \sum_{q \neq p} B_w^n u(q)^{m(p)} \right\| \\ &\quad + \sum_{m=m_0+1}^{m_1} |\hat{P}(m)| \left\| \sum_{q \geq 1} B_w^n u(q)^m \right\| \\ &\leq \frac{2 + \sum_{m=m_0+1}^{m_1} |\hat{P}(m)|}{p}, \end{aligned}$$

where the last inequality follows from (A.15), (A.16), (A.17) and the fact that the $B_w^n u(q)$ have pairwise disjoint support. Therefore, for all n in a set of positive lower density, $B_w^n P(u)$ belongs to V , showing that $P(u)$ is a frequently hypercyclic vector for B_w . Hence, it remains to prove (A.15), (A.16) and (A.17). We first observe that

$$B_w^n u(p)^{m(p)} - v(p) = \sum_{\substack{n' \in B(p) \\ n' \geq n}} \sum_{l=0}^p \frac{v_l(p)}{w_{l+1} \cdots w_{(n'-n)+l}} e^{(n'-n)+l}.$$

Since $n' - n > N(p)$ for all $n' > n$, $n' \in B(p)$, (A.15) follows from (i) in Lemma 5.21. Next, for $m > m(p)$, we may write $B_w^n u(p)^m$ as

$$\sum_{\substack{n' \in B(p) \\ n' \geq n}} \sum_{l=0}^p \frac{v_l(p)^{\frac{m}{m(p)}}}{(w_{l+1} \cdots w_{(n'-n)+l})^{\frac{m}{m(p)}} (w_{(n'-n)+l+1} \cdots w_{n'+l})^{\frac{m}{m(p)} - 1}} e^{(n'-n)+l}.$$

There is an additional difficulty since now we may have $n' = n$. We overcome this difficulty by writing

$$\begin{aligned} &\left| \frac{v_l(p)^{\frac{m}{m(p)}}}{(w_{l+1} \cdots w_{(n'-n)+l})^{\frac{m}{m(p)}} (w_{(n'-n)+l+1} \cdots w_{n'+l})^{\frac{m}{m(p)} - 1}} \right| \\ &\leq \left| \frac{v_l(p)^{\frac{m}{m(p)}}}{(w_{l+1} \cdots w_{n'+l})^{\frac{m}{m(p)} - 1}} \right| \\ &\leq \left| \frac{v_l(p)^{\frac{1}{m(p)}}}{(w_{l+1} \cdots w_{n'+l})^{\frac{1}{m(p)} - \frac{1}{m}}} \right|^m \\ &\leq \left| \frac{v_l(p)^{\frac{1}{m(p)}}}{(w_{l+1} \cdots w_{n'+l})^{\frac{1}{m(p)(m(p)+1)}}} \right|^m \\ &< \frac{1}{p}. \end{aligned}$$

Finally, for $m \geq m(p)$ and $q \neq p$, we write

$$B_w^n u(q)^m = \sum_{\substack{n' \in B(q) \\ n' > n}} \sum_{l=0}^q \frac{w_{l+(n'-n)+1} \cdots w_{n'+l}}{(w_{l+1} \cdots w_{n'+l})^{\frac{m}{m(q)}}} v_l(q)^{\frac{m}{m(q)}} e_{(n'-n)+l}.$$

For $q > p$, we apply (ii) of Lemma 5.21 with $r = q$, $s = p$, $j = n'$, $j' = n$ and $\alpha = m/m(q)$. For $q < p$, we apply (ii) of Lemma 5.21 with $r = p$, $s = q$, $j = n$, $j' = n'$ and $\alpha = m/m(q)$. In both cases, we immediately find that all the coefficients of $B_w^n u(q)^m$ are smaller than $1/p$, yielding

$$\|B_w^n u(q)^m\| < \frac{1}{p}.$$

This closes the proof of Theorem 1.6. □

This technical construction leads to an example over the not so difficult space c_0 , but the following question remains open.

Question 5.22 *Does there exist a weighted shift on ℓ_p endowed with the pointwise product admitting a frequently hypercyclic algebra?*

Of course, it would also be nice to get simpler examples! On the other hand, for sequence spaces endowed with the convolution product, we have neither positive nor negative examples. For instance, it would be very interesting to solve the following questions.

Question 5.23 *Does B on ω endowed with the convolution product support a frequently hypercyclic algebra?*

Question 5.24 *Does $2B$ on ℓ_1 endowed with the convolution product support a frequently hypercyclic algebra?*

6 Concluding remarks and open questions

6.1 Closed hypercyclic algebras

As pointed out in the introduction, provided T is hypercyclic, $HC(T) \cup \{0\}$ always contains a dense subspace. When moreover T satisfies the hypercyclicity criterion, there is a necessary and sufficient condition to determine whether $HC(T) \cup \{0\}$ contains an infinite-dimensional closed subspace (see for instance [5]). In our context, it is natural to ask whether, for some of our examples, $HC(T) \cup \{0\}$ contains a closed non-trivial algebra (we will say that T supports a closed hypercyclic algebra).

The third author and K. Grosse-Erdmann have shown that it is the case if T is a translation operator acting on the space $\mathcal{C}^\infty(\mathbb{R}, \mathbb{C})$. The fact that T is an algebra homomorphism plays an important role here. We now give several negative results. The first one solves a question of [22].

Proposition 6.1 *No convolution operator $P(D)$ induced by a nonconstant polynomial $P \in \mathbb{C}[z]$ admits a closed hypercyclic algebra.*

Proof. We write $P(z) = \sum_{s=0}^t \hat{P}(s)z^s$, with $\hat{P}(t) \neq 0$, and let $f \in HC(P(D))$. We shall prove that the closed algebra generated by f contains a non-zero and non-hypercyclic vector. Write $f(z) = a_0 + \sum_{n \geq p} a_n z^n$, with $a_p \neq 0$. Without loss of generality, we may assume that $a_p = 1$. We shall construct by induction a sequence of complex numbers (b_k) such that

$$|b_k| \leq \left(\frac{|\hat{P}(0)| + 1}{|\hat{P}(t)|} \right)^{kp} \times \frac{1}{(ktp)!}$$

for all k and, setting $P_k(z) = \sum_{l=1}^k b_l(z - a_0)^{lt}$, then

$$|P(D)^{lp}(P_k \circ f)(0)| \geq (|\hat{P}(0)| + 1)^{lp}$$

for all $1 \leq l \leq k$. The conclusion follows easily. In fact, (P_k) converges uniformly on compact subsets of \mathbb{C} to some entire function g . From the uniformity of the convergence, we conclude that the function $g \circ f$ satisfies

$$\left| P(D)^{lp}(g \circ f)(0) \right| \geq (|\hat{P}(0)| + 1)^{lp}$$

for all $l \geq 1$. Let us set $h = g - g(0)$. The function $h \circ f$, which is in the algebra generated by f , satisfies

$$\begin{aligned} \left| P(D)^{lp}(h \circ f)(0) \right| &\geq \left| P(D)^{lp}(g \circ f)(0) \right| - \left| P(D)^{lp}(g(0)) \right| \\ &\geq (|\hat{P}(0)| + 1)^{lp} - |\hat{P}(0)|^{lp}|g(0)| \xrightarrow{l \rightarrow +\infty} +\infty. \end{aligned}$$

Hence, $h \circ f$ is nonzero and it cannot be hypercyclic for $P(D)^p$. In particular, since $HC(P(D)) \subset \bigcap_{n \geq 1} HC(P(D)^n)$ (see [1, Theorem 1]), $h \circ f$ cannot be hypercyclic for $P(D)$ as well.

For the proof we will use the formula

$$P(D)^q = \sum_{\mathbf{j} \in I_q} \binom{q}{\mathbf{j}} \hat{P}(0)^{j_0} \dots \hat{P}(t)^{j_t} D^{j_1 + 2j_2 + \dots + tj_t},$$

where $I_q = \{\mathbf{j} = (j_0, \dots, j_t) \in \mathbb{N}_0^{t+1} : j_0 + \dots + j_t = q\}$ and $\binom{q}{\mathbf{j}}$ denote the multinomial coefficient

$$\binom{q}{j_0, \dots, j_t} = \frac{q!}{j_0! \dots j_t!}.$$

Let us set $P_0(z) = 0$ and let us assume that the construction has been done until step $k - 1$. Then

$$\begin{aligned} P_{k-1} \circ f + b(f - a_0)^{kt} &= P_{k-1} \circ f + b(z^p + a_{p+1}z^{p+1} + \dots)^{kt} \\ &= P_{k-1} \circ f + bz^{ktp} + \sum_{j \geq ktp+1} c_j z^j. \end{aligned}$$

Hence, for $1 \leq l \leq k$,

$$\begin{aligned} P(D)^{lp}(P_{k-1} \circ f + b(f - a_0)^{kt})(0) &= P(D)^{lp}(P_{k-1} \circ f)(0) + P(D)^{lp}(b(f - a_0)^{kt})(0) \\ &= P(D)^{lp}(P_{k-1} \circ f)(0) + g_l(0), \end{aligned}$$

where

$$g_l(z) = P(D)^{lp} \left(bz^{ktp} + \sum_{j \geq ktp+1} c_j z^j \right).$$

If $l \leq k - 1$ then $\deg P^{lp} \leq (k - 1)tp < ktp$, hence $g_l(0) = 0$. By the induction hypothesis it follows that

$$\begin{aligned} |P(D)^{lp}(P_{k-1} \circ f + b(f - a_0)^{kt})(0)| &= |P(D)^{lp}(P_{k-1} \circ f)(0) + g_l(0)| \\ &= |P(D)^{lp}(P_{k-1} \circ f)(0)| \geq (|\hat{P}(0)| + 1)^{lp} \end{aligned}$$

whatever the value of b is. On the other hand, if $l = k$, then

$$\begin{aligned} g_k(z) &= b\hat{P}(t)^{kp}(ktp)! \\ &+ \sum_{\mathbf{j} \in I_{kp} \setminus \{(0, \dots, 0, ktp)\}} \binom{kp}{\mathbf{j}} \hat{P}(0)^{j_0} \dots \hat{P}(t)^{j_t} D^{j_1+2j_2+\dots+j_t} \left(bz^{ktp} + \sum_{j \geq ktp+1} c_j z^j \right), \end{aligned}$$

hence $g_k(0) = b\hat{P}(t)^{kp}(ktp)!$, that is,

$$\begin{aligned} |P(D)^{kp}(P_{k-1} \circ f + b(f - a_0)^{kt})(0)| &= |P(D)^{kp}(P_{k-1} \circ f)(0) + g_k(0)| \\ &= |P(D)^{kp}(P_{k-1} \circ f)(0) + b\hat{P}(t)^{kp}(ktp)!|, \end{aligned}$$

so we can find b satisfying

$$|b| \leq \left(\frac{|\hat{P}(0)| + 1}{|\hat{P}(t)|} \right)^{kp} \times \frac{1}{(ktp)!}$$

such that

$$|P(D)^{lp}(P_{k-1} \circ f + b(f - a_0)^{kt})(0)| \geq (|\hat{P}(0)| + 1)^{lp}.$$

The proof is now done by taking $b_k = b$. □

Question 6.2 *Does there exist an entire function ϕ of exponential type such that $\phi(D)$ supports a closed hypercyclic algebra?*

Proposition 6.3 *Let $X = \ell_p$, $X = c_0$ or $X = \omega$, endowed with the coordinatewise product. No backward shift on X supports a closed hypercyclic algebra.*

Proof. We first consider the case $X = c_0$. Let $x \in X$ be a non-zero sequence, and let $D \subset \mathbb{C}$ be a compact disc centered at the origin and omitting at least one of the terms of x . For each $n \in \mathbb{N}$, consider the compact set K_n as in Figure A.2 and f a holomorphic function defined on a neighbourhood of K_n and satisfying that $f(z) = 0$ if $z \in D$ and $f(z) = 1$ if $z \in K_n \setminus D$.

By Runge's approximation theorem we get a polynomial P_n such that $\|P_n - f\|_{K_n} < \frac{1}{n}$. We end up with a sequence of polynomials (P_n) , satisfying that $P_n(z) \rightarrow 0$ uniformly on D , and $P_n(z) \rightarrow 1$ pointwise on $\mathbb{C} \setminus D$. Redefining P_n by $P_n - P_n(0)$, we may also assume that $P_n(0) = 0$, for every $n \in \mathbb{N}$.

Since $x \in c_0$, it follows that eventually all the terms of x belong to D which yields that $P_n(x) \rightarrow y = (y_k)$ in c_0 , where $y_k = 0$, if $x_k \in D$ and $y_k = 1$ otherwise. We conclude, that y is a

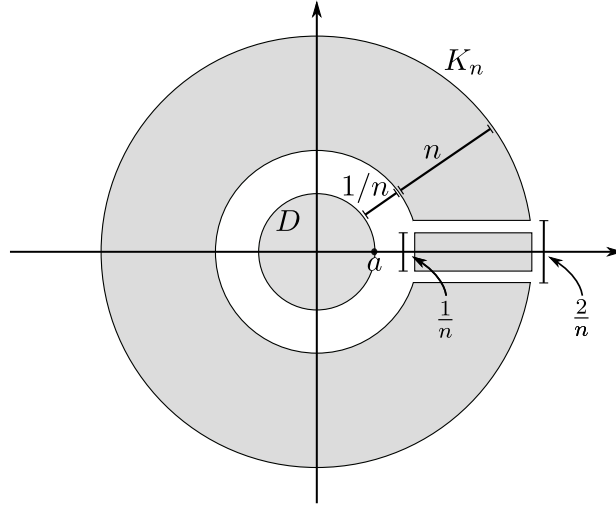


Fig. A.2

non-zero element in the closed algebra generated by x which is not hypercyclic for any weighted backward shift on c_0 .

Let now $X = \ell^p, p \geq 1$. Consider an $x \in X, x \neq 0$, and D and (P_n) defined as above. Cauchy's formula ensures that $P'_n \rightarrow 0$ uniformly on $\frac{1}{2}D$. Let

$$C = \sup \left\{ |P'_n(z)| : n \in \mathbb{N}, z \in \frac{1}{2}D \right\}.$$

Fix $\varepsilon > 0$ and let $k_0 \in \mathbb{N}$ be such that, for $k > k_0, x_k \in \frac{1}{2}D$ and

$$2^p C^p \sum_{k > k_0} |x_k|^p < \frac{\varepsilon}{2}.$$

Find $N \in \mathbb{N}$ such that for all $m, n \geq N$,

$$\sum_{k=1}^{k_0} |P_m(x_k) - P_n(x_k)|^p < \frac{\varepsilon}{2}.$$

We have

$$\begin{aligned} \|P_m(x) - P_n(x)\|_p^p &\leq \sum_{k=1}^{k_0} |P_m(x_k) - P_n(x_k)|^p + \sum_{k > k_0} (|P_m(x_k)| + |P_n(x_k)|)^p \\ &\leq \sum_{k=1}^{k_0} |P_m(x_k) - P_n(x_k)|^p + 2^p C^p \sum_{k > k_0} |x_k|^p < \varepsilon. \end{aligned}$$

That means that the sequence $(P_n(x))$ is Cauchy in ℓ_p and the conclusion follows exactly as in the previous case.

Finally, we consider the case $X = \omega$. Letting $D = \{0\}$ and K_n be as above, and by using Runge's approximation theorem, we get a sequence of polynomials (Q_n) such that $Q_n(0) = 0$, for every $n \in \mathbb{N}$ and $Q_n(z) \rightarrow 1$ for each $z \in \mathbb{C} \setminus \{0\}$. If $x \in \omega$ is a non-zero sequence, then $Q_n(x) \rightarrow y = (y_k)$, where $y_k = 0$ if $x_k = 0$, and $y_k = 1$ otherwise. It is immediate that y is a non-zero element in the closed algebra generated by x which fails to be hypercyclic for any weighted backward shift on ω . \square

Question 6.4 *Does there exist a weight (w_n) such that B_w , acting on ℓ_1 endowed with the Cauchy product, supports a closed hypercyclic algebra?*

6.2 Hypercyclic algebras in the ideal of compact operators

Beyond the examples given in the paper, there are other examples where the existence of a hypercyclic algebra would be natural. One of them is given by hypercyclic operators acting on separable ideals of operators. For instance, assume that H is a separable Hilbert space and denote by $X = \mathcal{K}(H)$ the (non-commutative) algebra of compact operators in H , endowed with the norm topology.

For $T \in \mathcal{L}(H)$, denote by L_T the operator of left multiplication by T , defined on $\mathcal{K}(H)$. It is known (see for instance [5, Chapter 8]) that if T satisfies the hypercyclicity criterion, then L_T is a hypercyclic operator on $\mathcal{K}(H)$. This latter space being an algebra, it is natural to study whether L_T supports a hypercyclic algebra. We do not know the answer to this question, but we point out that a positive answer would require different techniques. Indeed, Theorem 2.1 can never be applied to these operators.

Proposition 6.5 *Let $T \in \mathcal{L}(H)$. Then L_T , acting on $\mathcal{K}(H)$, does not satisfy the assumptions of Theorem 2.1 even for $d = 1$.*

Proof. We fix $x \in H$, $x^* \in H^*$ with $x^*(x) = 1$, $\|x\| = 1$, $\|x^*\| = 1$. Using the notations of Theorem 2.1, let $A = \{1, 2\}$,

$$U = V = \{u \in \mathcal{L}(H) : \|u - x^* \otimes x\| < 1/4\},$$

$$W = \{u \in \mathcal{L}(H) : \|u\| < 1/8\}.$$

Assume first that $\beta = 1$ and that there exist $u \in U$, $N \in \mathbb{N}$ with $T^N u \in V$ and $T^N u^2 \in W$. Then we know that

$$\|T^N u^2(x) - x^*(u(x))x\| < \frac{\|u(x)\|}{4}$$

since $T^N u \in V$. Now, $\|u(x) - x\| < 1/4$ so that $\|u(x)\| < 5/4$ and $|x^*(u(x))| > 3/4$. Hence,

$$\|T^N u^2(x)\| > \frac{3}{4} - \frac{5}{16} > \frac{1}{8}.$$

This contradicts $T^N u^2 \in W$.

If we assume that $\beta = 2$ and that there exist $u \in U$, $N \in \mathbb{N}$ with $T^N u \in W$ and $T^N u^2 \in V$, then we get successively

$$\|T^N u^2(x) - x\| < \frac{1}{4} \text{ (since } T^N u^2 \in V)$$

$$\|T^N u^2(x)\| < \frac{1}{8} \|u(x)\| \text{ (since } T^N u \in W)$$

$$\|u(x) - x\| < \frac{1}{4}.$$

These three inequalities yield easily a contradiction. □

6.3 Further question and remark

As independently shown by Ansari in [2] and later-on by Bernal-González in [7], every separable Banach space supports a hypercyclic operator. In the context of algebras a natural question arises.

Question 6.6 *Is it true that every separable Banach algebra supports a hypercyclic operator admitting a hypercyclic algebra?*

In all known results, the set of generators for hypercyclic algebras is either empty or residual. We observe below that this set can be non-empty and meager.

Remark 6.7 For every pair (X, T) where X is a Banach space and T a hypercyclic operator with a non-hypercyclic vector, we may define a product on X turning it into a commutative Banach algebra and such that the set of generators for a hypercyclic algebra for T is non-empty and nowhere dense.

Proof. Let $x \in HC(T)$ and y a non-hypercyclic vector for T with $\|y\| = 1$. Consider $f \in X^*$ with $\|f\| = 1$ and such that $f(x) = 0$ and $f(y) = 1$. We define the product

$$z \cdot w = f(z)f(w)y, \text{ with } z, w \in X$$

and observe that it turns X into a commutative Banach algebra. Now, $x^2 = 0$ so $A(x) = \text{span}(x)$, and thus x is a generator for a hypercyclic algebra for T . Moreover, it is easy to check that the following holds,

$$\{x \in X : A(x) \setminus \{0\} \subset HC(T)\} = HC(T) \cap Ker(f).$$

Since $Ker(f)$ is a proper, closed hyperplane of X , we conclude that the set of generators for a hypercyclic algebra for T is non-empty and nowhere dense. \square

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Article B

Disjoint and common hypercyclic algebras

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Abstract

Over time the concept of hypercyclicity has been explored in different manners and contexts, gaining new forms and applications. In particular when the space has an adjacent structure, we can always look for sets of hypercyclic vectors compatible with that framework. In this paper we deal with hypercyclic operators acting on Fréchet sequence algebras and give criteria for the existence of common and disjoint hypercyclic algebras.

1 Introduction

This paper goes over some structures of hypercyclic vectors in a linear dynamical system (X, T) , that is, a couple of a topological vector space X and a continuous linear operator T on X . We say that a vector $x \in X$ is hypercyclic for T when its orbit $Orb(x, T) := \{x, Tx, T^2x, \dots\}$ is dense in X . Since its first appearance in the '80s with the thesis of Kitai (see [10]), this concept has been the object of multiple interesting generalizations and modifications which yielded to many new related concepts (frequent hypercyclicity, common hypercyclicity, disjoint hypercyclicity, etc.).

When the underlying space has a richer structure, it is natural to investigate the existence of sets of hypercyclic vectors maintaining that framework. For instance, when X is an F -algebra, that is, a metrizable and complete topological algebra, and denoting by $HC(T)$ the set of all hypercyclic vectors for T , it is natural to ask whether $HC(T) \cup \{0\}$ contains a non-trivial algebra. This is what we call a hypercyclic algebra. The first result in this vein was done independently by Shkarin in [11] and by Bayart and Matheron in [3]. They have shown that the derivation operator $D : f \mapsto f'$, acting on the Fréchet algebra $H(\mathbb{C})$ of entire functions endowed with the pointwise multiplication, supports a hypercyclic algebra.

In this paper we are interested in the existence of algebraic structures within two concepts related to hypercyclicity, namely disjoint and common hypercyclicity. We will study them in the context of weighted backward shifts on Fréchet sequence algebras. Precisely, we assume that X is a subspace of the space $\omega = \mathbb{C}^{\mathbb{N}_0}$ of all complex sequences, whose topology is induced by a non-decreasing sequence of seminorms $(\|\cdot\|_q)_{q \geq 1}$ and that X is endowed with a product \cdot such that, for all $x, y \in X$, all $q \geq 1$,

$$\|x \cdot y\|_q \leq \|x\|_q \times \|y\|_q.$$

There are two natural products on a Fréchet sequence space: the coordinatewise product and the convolution or Cauchy product. They are defined as follows. Let $(a_n)_{n \geq 0}$ and $(b_n)_{n \geq 0}$ be sequences in ω . Their coordinatewise product is defined as

$$(a_n)_n \cdot (b_n)_n = (a_n b_n)_n.$$

Their convolution product is defined as

$$(a_n)_n \cdot (b_n)_n = (c_n)_n,$$

where $c_n = \sum_{k=0}^n a_k b_{n-k}$ for all $n \geq 0$. It is clear that ℓ_p and c_0 are Fréchet sequence algebras for the coordinatewise product, and that ℓ_1 is also a Fréchet sequence algebra for the convolution product. Endowing $H(\mathbb{C})$ with

$$\left\| \sum_{n \geq 0} a_n z^n \right\|_q = \sum_{n \geq 0} |a_n| q^n$$

and ω with

$$\|(x_n)\|_q = \sum_{n=0}^q |x_n|,$$

we also obtain that $H(\mathbb{C})$ and ω are Fréchet sequence algebras for both products (on $H(\mathbb{C})$, the Cauchy product of f and g is nothing else but the product of the two functions f and g).

Our results will cover both the coordinatewise and the Cauchy product. Before going further, we mention that they behave very differently. The main point is that the coordinatewise product preserves the support, whereas the Cauchy product mixes it. In particular, if x and y are two vectors with disjoint supports, then $(x+y)^m = x^m + y^m$ for the coordinatewise product whereas this is not at all the case for the Cauchy product. This explains why the latter case is considerably more difficult to handle.

As announced above, we will work with weighted backward shifts, one of the favourite class of examples in linear dynamics. Given a sequence of nonzero complex numbers $w = (w_n)_{n \in \mathbb{N}}$, the (unilateral) weighted backward shift B_w with weight w is defined by

$$B_w(x_0, x_1, \dots) = (w_1 x_1, w_2 x_2, \dots).$$

The weight w will be called *admissible* (for X) if B_w is a bounded operator on X . It is known that, provided the canonical basis (e_n) is a Schauder basis of X , B_w is hypercyclic if and only if there exists a sequence (n_k) such that for all $l \in \mathbb{N}$, $((w_{l+1} \cdots w_{n_k+l})^{-1} e_{n_k+l})$ goes to zero.

1.1 Disjoint hypercyclic algebras

The notion of disjointness was introduced in linear dynamics in [5] and [6]; we shall limit ourselves to the case of two operators. We say that two operators T_1 and T_2 acting on the same F -space X are disjointly hypercyclic if there exists a vector $x \in X$, called a disjointly hypercyclic vector for T_1 and T_2 , such that $(T_1^n x, T_2^n x)_{n \geq 0}$ is dense in X^2 . Many examples or counterexamples of disjointly hypercyclic operators can be found in the literature. In particular, in [6] and in [7], our favourite examples, weighted backward shifts, are studied in depth: in [6], it is characterized when two shifts $B_1^{r_1}$ and $B_2^{r_2}$ are disjointly hypercyclic on ℓ_p when they are not raised to the same power (namely $r_1 < r_2$), whereas in [7] such a characterization is given when $r_1 = r_2 = 1$. From the proofs given in [6] and in [7] we see that the two cases are very different, as it will become clearer later on in this paper.

Assume now that X is an algebra. We say that T_1 and T_2 support a disjoint hypercyclic algebra if there exists a nontrivial subalgebra A of X such that any non zero element of A is a disjointly hypercyclic vector for T_1 and T_2 . We will study whether two backward shifts acting on a Fréchet sequence algebra X support a disjoint hypercyclic algebra or not. When X is endowed with the coordinatewise product, mixing the arguments of [4] with that of [6, 7], we will get a complete characterization of the existence of a disjoint hypercyclic algebra for $B_1^{r_1}$ and $B_2^{r_2}$ (to avoid cumbersome statements, we will focus on the cases $r_1 = r_2 = 1$ and $r_1 = 1 < r_2 = 2$). Despite the problem of the support, we will also succeed to provide such a characterization when X is endowed with the convolution product and $r_1 = r_2 = 1$ (see the forthcoming Theorems 2.5, 2.10, 2.19). The most difficult case is that of the Cauchy product and two shifts raised to different powers. We shall concentrate ourselves on the multiples of B and D . In [6], it is shown that λB and μB^2 are disjointly hypercyclic on ℓ_1 if and only if $1 < \lambda < \mu$ and that, for all $\lambda, \mu > 0$, λD and μD^2 are disjointly hypercyclic on $H(\mathbb{C})$. Under these assumptions we will even get a disjointly hypercyclic algebra.

Theorem 1.1 *Let $\lambda, \mu > 0$.*

- a) λB and μB^2 support a disjoint hypercyclic algebra on ℓ_1 (endowed with the convolution product) if and only if $1 < \lambda < \mu$.
- b) λD and μD^2 support a disjoint hypercyclic algebra on $H(\mathbb{C})$.

1.2 Common hypercyclic algebras

If $(T_\lambda)_{\lambda \in \Lambda}$ is a family of hypercyclic operators acting on the same F -space X , it is natural to ask whether it admits a common hypercyclic vector, namely if $\bigcap_{\lambda \in \Lambda} HC(T_\lambda)$ is non-empty. This is trivial if Λ is countable since each $HC(T_\lambda)$ is a residual set. However, this becomes already an issue for the first natural example which comes in mind, that is the multiples of the unweighted backward shift B . It was proved by Abakumov and Gordon in [1] that indeed $\bigcap_{\lambda > 1} HC(\lambda B) \neq \emptyset$ where B is seen acting on any ℓ_p -space or on c_0 . Further examples were given in subsequent papers (for instance in [8], [2])

Suppose now that we have a family $(T_\lambda)_{\lambda \in \Lambda}$ of operators acting on the same F -algebra, each one supporting a hypercyclic algebra. It is natural to ask whether $\bigcap_{\lambda \in \Lambda} HC(T_\lambda) \cup \{0\}$ contains a nontrivial algebra, which will be called a *common hypercyclic algebra*. Even when Λ has two elements, this is unclear. In this paper, we concentrate ourselves on the case where (T_λ) is a family of weighted shifts acting on X .

Firstly, regarding Fréchet sequence algebras endowed with the coordinatewise product, we have been able to get a general statement covering all the cases (and even more!) known for the existence of a common hypercyclic vector. As a corollary of our work, we have the following statement.

Theorem 1.2 *(i) Let $X = \ell_p$, $1 \leq p < +\infty$, or $X = c_0$ endowed with the coordinatewise product. Then the family $(\lambda B)_{\lambda > 1}$ admits a common hypercyclic algebra.*

(ii) Let $X = \ell_p$, $1 \leq p < +\infty$, or $X = c_0$ endowed with the coordinatewise product. For $\lambda > 0$, define $w(\lambda) = (1 + \lambda/n)_{n \geq 1}$. Then the family $(B_{w(\lambda)})_{\lambda > 0}$ admits a common hypercyclic algebra.

(iii) Let $X = H(\mathbb{C})$ endowed with the coordinatewise product. Then the family $(\lambda D)_{\lambda > 0}$ admits a common hypercyclic algebra.

The statement of (ii) is particularly interesting. Indeed, for this family of weights, the existence of a single common hypercyclic vector was only known for $\lambda > 1/p$.

Again, it is much more difficult to handle Fréchet sequence algebras endowed with the Cauchy product. Nevertheless, we will be able to give a general sufficient condition which implies the following result.

Theorem 1.3 (i) Let $X = \ell_1$ endowed with the Cauchy product. Then $(\lambda B)_{\lambda > 1}$ admits a common hypercyclic algebra.

(ii) Let $X = H(\mathbb{C})$ endowed with the Cauchy product. Then $(\lambda D)_{\lambda > 0}$ admits a common hypercyclic algebra.

1.3 Organization of the paper

All the criteria of this paper are based on a Baire argument. This idea first appeared in [3, Remark 8.28] and is a sort of Birkhoff transitivity theorem for hypercyclic algebras. Here we adapt this criterion to produce either common or disjoint hypercyclic algebras. Each section begins with its own particular criterion followed by more practical criteria for the particular case of Fréchet sequence algebras and then we apply these results on some classical examples.

1.4 Notations

The symbol \mathbb{N} will stand for the set of positive integers, whereas $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. We shall denote by $\mathcal{P}_f(A)$ the set of finite subsets of a given set A .

For $x = \sum_{n=0}^{+\infty} x_n e_n \in \omega$, the support of x is equal to $\text{supp}(x) = \{n \in \mathbb{N}_0 : x_n \neq 0\}$. The notation c_{00} will denote the set of sequences in ω with finite support.

For $u \in X^d$ and $\alpha \in \mathbb{N}_0^d$, u^α will mean $u_1^{\alpha_1} \cdots u_d^{\alpha_d}$. If z is any complex number and $m \in \mathbb{N}$, $z^{1/m}$ will denote any m th root of z .

When working on a Fréchet space $(X, \|\cdot\|_p)$, it is often convenient to endow X with an F -norm $\|\cdot\|$ defining the topology of X (see [9, Section 2.1]). Such an F -norm can be defined by the formula

$$\|x\| = \sum_{p=1}^{+\infty} \frac{1}{2^p} \min(1, \|x\|_p).$$

In particular, an F -norm satisfies the triangle inequality and the inequality

$$\forall \lambda \in \mathbb{C}, \forall x \in X, \|\lambda x\| \leq (|\lambda| + 1)\|x\|, \tag{B.1}$$

a property which replaces the positive homogeneity of the norm.

We finally recall some results on unconditional convergence in Fréchet spaces (see for instance [9, Appendix A]). A series $\sum_{n=0}^{+\infty} x_n$ in a Fréchet space X is called unconditionally convergent if for any bijection $\pi : \mathbb{N}_0 \rightarrow \mathbb{N}_0$, the series $\sum_{n=0}^{+\infty} x_{\pi(n)}$ is convergent. This amounts to saying that, for any $\varepsilon > 0$, there is some $N \in \mathbb{N}$ such that, whenever $\sup_n |\alpha_n| \leq 1$, the series $\sum_{n=0}^{+\infty} \alpha_n x_n$ converges and

$$\left\| \sum_{n=N}^{+\infty} \alpha_n x_n \right\| < \varepsilon.$$

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2 Disjoint hypercyclic algebras

2.1 How to get a disjoint hypercyclic algebra

As described in the introduction, we adapt Birkhoff's transitivity theorem to get a criterion ensuring disjoint hypercyclicity. We may observe the similarity with [6, Proposition 2.3].

Proposition 2.1 *Let T_1 and T_2 be two operators acting on the same F -algebra X . Assume that for each $I \in \mathcal{P}_f(\mathbb{N}) \setminus \{\emptyset\}$, there exists $m_0 \in I$ such that, for all U, V_1 and V_2 nonempty open subsets of X , for all W neighbourhood of zero, there exist $u \in U$ and $N \in \mathbb{N}$ such that*

$$\begin{cases} T_1^N(u^n) \in W, T_2^N(u^n) \in W \text{ for all } n \in I, n \neq m_0 \\ T_1^N(u^{m_0}) \in V_1, T_2^N(u^{m_0}) \in V_2. \end{cases}$$

Then T_1 and T_2 admit a disjoint hypercyclic algebra.

Proof. For $V, V' \subset X$ open and nonempty, for W a neighbourhood of zero, for $I \subset \mathcal{P}_f(\mathbb{N}) \setminus \{\emptyset\}$, let us define

$$\begin{aligned} \mathcal{A}(I, V, V', W) = \left\{ u \in X : \exists N \in \mathbb{N}, T_1^N(u^n) \in W \text{ for all } n \in I \setminus \{m_0(I)\} \right. \\ \left. T_2^N(u^n) \in W \text{ for all } n \in I \setminus \{m_0(I)\} \right. \\ \left. T_1^N(u^{m_0(I)}) \in V, T_2^N(u^{m_0(I)}) \in V' \right\} \end{aligned}$$

Fixing (V_k) a basis of open subsets of X and (W_l) a basis of neighbourhoods of 0, one can verify that each set $\mathcal{A}(I, V_k, V_{k'}, W_l)$ is open and dense, hence $\bigcap_{I,k,k',l} \mathcal{A}(I, V_k, V_{k'}, W_l)$ is non-empty and we can argue exactly like in [4, Theorem 2.1] to prove that any vector in this intersection generates a disjoint hypercyclic algebra for T_1 and T_2 . \square

Similarly to [4], for the Cauchy product we will apply this proposition for $m_0 = \max I$ and for the coordinatewise product we will do the opposite choice $m_0 = \min I$. The corresponding corollaries are the following.

Corollary 2.2 *Let T_1 and T_2 be two operators acting on the same F -algebra X . Assume that for all $m \geq 1$, for all U , V_1 and V_2 nonempty open subsets of X , for all W neighbourhood of zero, there exist $u \in U$ and $N \in \mathbb{N}$ such that*

$$\begin{cases} T_1^N(u^n) \in W, T_2^N(u^n) \in W \text{ for all } n < m \\ T_1^N(u^m) \in V_1, T_2^N(u^m) \in V_2. \end{cases}$$

Then T_1 and T_2 admit a disjoint hypercyclic algebra.

Corollary 2.3 *Let T_1 and T_2 be two operators acting on the same F -algebra X . Assume that for all positive integers $m_0 < m_1$, for all U , V_1 and V_2 nonempty open subsets of X , for all W neighbourhood of zero, there exist $u \in U$ and $N \in \mathbb{N}$ such that*

$$\begin{cases} T_1^N(u^n) \in W, T_2^N(u^n) \in W \text{ for all } n \in \{m_0 + 1, \dots, m_1\} \\ T_1^N(u^{m_0}) \in V_1, T_2^N(u^{m_0}) \in V_2. \end{cases}$$

Then T_1 and T_2 admit a disjoint hypercyclic algebra.

2.2 Disjoint hypercyclic algebras for backward shifts - coordinatewise product

We now show how these general results can be applied to get disjoint hypercyclic algebras for backward shifts operators. We first begin with Fréchet sequence algebras endowed with the coordinatewise product. We get complete characterizations either for the case where the shifts are raised to different powers as for the case where the powers are the same.

We recall that a Fréchet space $(X, (\|\cdot\|_q))$ admits a continuous norm if there exists a norm $\|\cdot\| : X \rightarrow \mathbb{R}$ that is continuous for the topology of X , namely there exists $q \in \mathbb{N}$ and $C > 0$ with $\|x\| \leq C\|x\|_q$ for all $x \in X$. In particular, for any q large enough, $\|\cdot\|_q$ itself is a norm. The following lemma is taken from [4, Lemma 4.1].

Lemma 2.4 *Let X be a Fréchet sequence algebra for the coordinatewise product and with a continuous norm. Then the sequence (e_n) is bounded below.*

This follows from the simple observation that, for all $q \geq 0$, $0 < \|e_n\|_q = \|e_n \cdot e_n\|_q \leq \|e_n\|_q^2$, which implies $\|e_n\|_q \geq 1$.

2.2.1 Backward shifts raised to different powers

We first state a characterization of the existence of a disjoint hypercyclic algebra for B_1 and B_2^2 when B_1 and B_2 are two weighted shifts on the Fréchet sequence algebra X .

Theorem 2.5 *Let X be a Fréchet sequence algebra for the coordinatewise product with a continuous norm and such that (e_n) is a Schauder basis. Let $B_1 = B_{w(1)}$ and $B_2 = B_{w(2)}$ be two bounded weighted shifts on X . The following assertions are equivalent.*

- (i) B_1 and B_2^2 support a disjoint hypercyclic algebra.

(ii) For all $p \in \mathbb{N}$, for all $m \geq 1$, there exists a sequence of integers (n_k) such that, for all $l = 0, \dots, p$,

$$\begin{aligned} & (w_{l+1}^{(1)} \cdots w_{n_k+l}^{(1)})^{-1/m} e_{n_k+l} \xrightarrow{k \rightarrow +\infty} 0, \\ & (w_{l+1}^{(2)} \cdots w_{2n_k+l}^{(2)})^{-1/m} e_{2n_k+l} \xrightarrow{k \rightarrow +\infty} 0, \\ & \frac{w_{n_k+l+1}^{(1)} \cdots w_{2n_k+l}^{(1)}}{w_{l+1}^{(2)} \cdots w_{2n_k+l}^{(2)}} e_{n_k+l} \xrightarrow{k \rightarrow +\infty} 0. \end{aligned}$$

(iii) There exists a sequence (n_k) such that, for all $l \geq 0$, for all $\gamma > 0$,

$$\begin{aligned} & (w_{l+1}^{(1)} \cdots w_{n_k+l}^{(1)})^{-\gamma} e_{n_k+l} \xrightarrow{k \rightarrow +\infty} 0, \\ & (w_{l+1}^{(2)} \cdots w_{2n_k+l}^{(2)})^{-\gamma} e_{2n_k+l} \xrightarrow{k \rightarrow +\infty} 0, \\ & \frac{w_{n_k+l+1}^{(1)} \cdots w_{2n_k+l}^{(1)}}{w_{l+1}^{(2)} \cdots w_{2n_k+l}^{(2)}} e_{n_k+l} \xrightarrow{k \rightarrow +\infty} 0. \end{aligned}$$

Proof. (i) \implies (ii). Let $m \geq 1$ and let $x \in X$, $x = \sum_{n=0}^{+\infty} x_n e_n$, be such that x^m is a disjoint hypercyclic vector for B_1 and B_2^2 . Let $p \geq 1$ and let (n_k) be a sequence of integers such that $(B_j^{jn_k}(x^m))_k$ goes to $e_0 + \cdots + e_p$, $j = 1, 2$. Let us see what we obtain for $j = 1$. Since convergence in X implies coordinatewise convergence, for all $l = 0, \dots, p$, $(w_{l+1}^{(1)} \cdots w_{n_k+l}^{(1)} x_{n_k+l}^m)$ converges to 1. Hence the sequences $((w_{l+1}^{(1)} \cdots w_{n_k+l}^{(1)})^{1/m} x_{n_k+l})$ are bounded below. Writing

$$(w_{l+1}^{(1)} \cdots w_{n_k+l}^{(1)})^{-1/m} e_{n_k+l} = \frac{1}{(w_{l+1}^{(1)} \cdots w_{n_k+l}^{(1)})^{1/m} x_{n_k+l}} x_{n_k+l} e_{n_k+l}$$

we get that $(w_{l+1}^{(1)} \cdots w_{n_k+l}^{(1)})^{-1/m} e_{n_k+l} \xrightarrow{k \rightarrow +\infty} 0$. Similarly we obtain for all $l = 0, \dots, p$,

$$\begin{aligned} & (w_{l+1}^{(2)} \cdots w_{2n_k+l}^{(2)})^{-1/m} e_{2n_k+l} \xrightarrow{k \rightarrow +\infty} 0, \\ & w_{l+1}^{(2)} \cdots w_{2n_k+l}^{(2)} x_{2n_k+l}^m \xrightarrow{k \rightarrow +\infty} 1. \end{aligned} \tag{B.2}$$

Moreover, write

$$B_1^{n_k}(x^m) = z + \sum_{l=0}^p w_{n_k+l+1}^{(1)} \cdots w_{2n_k+l}^{(1)} x_{2n_k+l}^m e_{n_k+l} + z',$$

with $\text{supp}(z) \subset [0, n_k)$ and $\text{supp}(z') \subset (n_k + p, +\infty)$. Again, since convergence in X implies pointwise convergence, we get

$$w_{n_k+l+1}^{(1)} \cdots w_{2n_k+l}^{(1)} x_{2n_k+l}^m e_{n_k+l} \xrightarrow{k \rightarrow +\infty} 0 \text{ for all } l = 0, \dots, p$$

which in turn, in view of (B.2), gives

$$\frac{w_{n_k+l+1}^{(1)} \cdots w_{2n_k+l}^{(1)}}{w_{l+1}^{(2)} \cdots w_{2n_k+l}^{(2)}} e_{n_k+l} \xrightarrow{k \rightarrow +\infty} 0 \text{ for all } l = 0, \dots, p.$$

(ii) \implies (iii). This follows from a diagonal argument.

(iii) \implies (i). We intend to apply Corollary 2.3. Let $m_0 < m_1$ be two positive integers, let U, V_1, V_2 be nonempty open subsets of X and let W be a neighbourhood of 0. Let $x \in U$ with finite support, let $y_i = \sum_{l=0}^p y_{i,l} e_l$ belonging to V_i , $i = 1, 2$. We set

$$u = x + \sum_{l=0}^p \frac{y_{1,l}^{1/m_0}}{(w_{l+1}^{(1)} \cdots w_{n_k+l}^{(1)})^{1/m_0}} e_{n_k+l} + \sum_{l=0}^p \frac{y_{2,l}^{1/m_0}}{(w_{l+1}^{(2)} \cdots w_{2n_k+l}^{(2)})^{1/m_0}} e_{2n_k+l},$$

which belongs to U for all large values of k . We recall that, since the sequence (e_n) is bounded below, for all $l \geq 0$, the sequences $(w_{l+1}^{(1)} \cdots w_{n_k+l}^{(1)})$ and $(w_{l+1}^{(2)} \cdots w_{2n_k+l}^{(2)})$ tend to $+\infty$. Let $n \in \{m_0, \dots, m_1\}$ and let us compute $B_2^{2n_k}(u^n)$:

$$B_2^{2n_k}(u^n) = \sum_{l=0}^p \frac{y_{2,l}^{n/m_0}}{(w_{l+1}^{(2)} \cdots w_{2n_k+l}^{(2)})^{\frac{n}{m_0}-1}} e_l.$$

The assumptions tell us that $B_2^{2n_k}(u^{m_0}) = y_2$ while $B_2^{2n_k}(u^n)$ tends to 0 if $n > m_0$. On the other hand,

$$B_1^{n_k}(u^n) = \sum_{l=0}^p \frac{y_{1,l}^{n/m_0}}{(w_{l+1}^{(1)} \cdots w_{n_k+l}^{(1)})^{\frac{n}{m_0}-1}} e_l + \sum_{l=0}^p y_{2,l}^{n/m_0} \frac{w_{n_k+l+1}^{(1)} \cdots w_{2n_k+l}^{(1)}}{(w_{l+1}^{(2)} \cdots w_{2n_k+l}^{(2)})^{\frac{n}{m_0}}} e_{n_k+l}.$$

Writing

$$\frac{w_{n_k+l+1}^{(1)} \cdots w_{2n_k+l}^{(1)}}{(w_{l+1}^{(2)} \cdots w_{2n_k+l}^{(2)})^{\frac{n}{m_0}}} e_{n_k+l} = \frac{1}{(w_{l+1}^{(2)} \cdots w_{2n_k+l}^{(2)})^{\frac{n}{m_0}-1}} \times \frac{w_{n_k+l+1}^{(1)} \cdots w_{2n_k+l}^{(1)}}{w_{l+1}^{(2)} \cdots w_{2n_k+l}^{(2)}} e_{n_k+l},$$

we get that the second (finite) sum tends to zero for all $n \geq m_0$, whereas the first sum can be handled exactly as for B_2 . \square

Remark 2.6 If we are just interested in disjointly hypercyclic operators, our proof can be easily modified to give the following extension of [6, Theorem 4.1] : let X be a Fréchet sequence space such that (e_n) is a Schauder basis for X . Let $B_1 = B_{w^{(1)}}$ and $B_2 = B_{w^{(2)}}$ be two bounded weighted shifts on X . Then B_1 and B_2^2 are disjointly hypercyclic if and only if there exists a sequence (n_k) such that, for all $l \geq 0$,

$$\begin{aligned} (w_{l+1}^{(1)} \cdots w_{n_k+l}^{(1)})^{-1} e_{n_k+l} &\xrightarrow{k \rightarrow +\infty} 0, \\ (w_{l+1}^{(2)} \cdots w_{2n_k+l}^{(2)})^{-1} e_{2n_k+l} &\xrightarrow{k \rightarrow +\infty} 0, \\ \frac{w_{n_k+l+1}^{(1)} \cdots w_{2n_k+l}^{(1)}}{w_{l+1}^{(2)} \cdots w_{2n_k+l}^{(2)}} e_{n_k+l} &\xrightarrow{k \rightarrow +\infty} 0. \end{aligned}$$

We point out that we do not need that (e_n) is bounded below. This last assumption was only useful when $n > m_0$, where we used it to deduce that $(w_{l+1}^{(1)} \cdots w_{n_k+l}^{(1)})$ and $(w_{l+1}^{(2)} \cdots w_{2n_k+l}^{(2)})$ tend to $+\infty$. For the existence of a disjointly hypercyclic vector, we always have $n = m_0 = 1$.

This remark also leads us to an interesting example of a couple of disjointly hypercyclic operators not having a disjoint hypercyclic algebra.

Example 2.7 Let B_1 and B_2^2 be weighted shifts induced respectively by the weight sequences $w_n^{(1)} = \frac{n^2}{(n-1)^2}$ with $w_1^{(1)} = 1$ and $w_n^{(2)} = 2$, $n \geq 1$, both acting on the weighted $\ell_1(v)$ with $v_n = n$ for all $n \geq 1$. By weighted $\ell_1(v)$ we mean the Banach space

$$\ell_1(v) = \{(x_n)_n : \sum_{n=1}^{\infty} |x_n|v_n < +\infty\}$$

equipped with the weighted ℓ_1 -norm

$$\|(x_n)_n\| = \sum_{n=1}^{\infty} |x_n|v_n.$$

For this example, all conditions of the previous theorem are satisfied for $m = 1$ (which grants a disjointly hypercyclic vector) but, for $m = 2$ and for any sequence $(n_k)_k$, we have that

$$\|(w_1^{(1)} \cdots w_{n_k}^{(1)})^{-1/2} e_{n_k+l}\| = \frac{n_k}{(n_k^2)^{1/2}} = 1$$

does not go to 0 as $k \rightarrow +\infty$.

Remark 2.8 In particular, if the sequence (e_n) is bounded (it is already bounded below), the conditions of Theorem 2.5 are equivalent to say that there exists a sequence (n_k) such that, for all $l = 0, \dots, p$,

$$\begin{aligned} w_{l+1}^{(1)} \cdots w_{n_k+l}^{(1)} &\xrightarrow{k \rightarrow +\infty} +\infty, \\ w_{l+1}^{(2)} \cdots w_{2n_k+l}^{(2)} &\xrightarrow{k \rightarrow +\infty} +\infty, \\ \frac{w_{n_k+l+1}^{(1)} \cdots w_{2n_k+l}^{(1)}}{w_{l+1}^{(2)} \cdots w_{2n_k+l}^{(2)}} &\xrightarrow{k \rightarrow +\infty} 0. \end{aligned}$$

In turn, this amounts to saying that B_1 and B_2^2 are disjointly hypercyclic. Hence, all the examples given in [6] support a disjoint hypercyclic algebra.

It is plain that this result can be extended for a finite list $B_{w^{(1)}}^{n_1}, \dots, B_{w^{(r)}}^{n_r}$ raised to different powers n_1, \dots, n_r . The statement of precise conditions are left to the reader.

We close this subsection by addressing the case of ω which is the prototypical example of a Fréchet space without a continuous norm. The following result resembles [4, Theorem 4.8].

Theorem 2.9 *If $B_1 = B_{w^{(1)}}$ and $B_2 = B_{w^{(2)}}$ are weighted backward shifts on ω endowed with the coordinatewise product, then B_1 and B_2^2 support a disjoint hypercyclic algebra.*

Proof. For V_1 and V_2 non-empty open subsets of ω , $I \subset \mathbb{N}$ finite and non-empty and $s > 0$, let us define

$$\begin{aligned} E(I, s) = \{ &P \in \mathbb{C}[z] : |\hat{P}(\min I)| \geq 1/s, |\hat{P}(\max I)| \geq 1/s, \\ &|\hat{P}(n)| \leq s \text{ for all } n \in \mathbb{N}, \\ &\hat{P}(n) = 0 \text{ when } n \notin I \} \end{aligned}$$

$$\mathcal{A}(I, s, V_1, V_2) = \left\{ u \in \omega : \forall P \in E(I, s), \exists N \geq 1, B_i^{iN}(P(u)) \in V_i, i = 1, 2 \right\}.$$

By a Baire category argument, it is enough to prove that each set $\mathcal{A}(I, s, V_1, V_2)$ is dense and open. The last property follows easily from the compactness of $E(I, s)$. Thus, let us fix I, s, V_1 and V_2 and let us prove that $\mathcal{A}(I, s, V_1, V_2)$ is dense. We set $m_0 = \min(I)$ and $m_1 = \max(I)$. Let U be a non-empty open subset of ω . Let $p \in \mathbb{N}_0$, $u_0, \dots, u_p, v_{i,0}, \dots, v_{i,p} \in \mathbb{C}$, for $i = 1, 2$, and $\varepsilon > 0$ be such that, for all x, y_1 and y_2 in ω ,

$$|x_l - u_l| < \varepsilon \text{ for all } l = 0, \dots, p \text{ implies } x \in U,$$

$$|y_{i,l} - v_{i,l}| < \varepsilon \text{ for all } l = 0, \dots, p \text{ implies } y_i \in V_i, i = 1, 2.$$

Let us first look at the sequence $(w_1^{(1)} \cdots w_n^{(1)})$. Three possibilities (which are not mutually exclusive) can occur:

- either $(w_1^{(1)} \cdots w_n^{(1)})$ is bounded and bounded below;
- or it admits a subsequence going to zero;
- or it admits a subsequence going to $+\infty$.

Thus, we get the existence of a subsequence $(w_1^{(1)} \cdots w_{n_k}^{(1)})$ going to $a_{1,0} \in [0, +\infty]$. We then do the same with $(w_2^{(1)} \cdots w_{n_k+1}^{(1)})$ and so on. By successive extractions, we get the existence of a sequence of integers (n_k) (we can assume that $n_{k+1} - 2n_k > p$ and $n_k > p$ for all k) and of $a_{i,0}, \dots, a_{i,p} \in [0, +\infty]$, $i = 1, 2$, such that, for all $l = 0, \dots, p$, and $i = 1, 2$, $(w_{l+1}^{(i)} \cdots w_{n_k+l}^{(i)})$ tends to $a_{i,l}$. We set $A_{i,1} = \{l \in \{0, \dots, p\} : a_{i,l} = +\infty\}$, $A_{i,2} = \{l \in \{0, \dots, p\} : a_{i,l} = 0\}$ and $A_{i,3} = \{l \in \{0, \dots, p\} : a_{i,l} \in (0, +\infty)\}$.

We fix now $(\alpha(k))$, $(\beta(k))$ two sequences of non-zero complex numbers and $(z_i(k))$, $i = 1, 2$, sequences in \mathbb{C}^{p+1} such that $(\alpha(k), \beta(k), z_1(k), z_2(k))$ is dense in $\mathbb{C}^{2(p+2)}$. We set

$$x = u + \sum_{k=0}^{+\infty} y_1(k) + \sum_{k=0}^{+\infty} y_2(k)$$

where, for $l = 0, \dots, p$ and $i = 1, 2$,

$$y_{i, in_k+l}(k) = \begin{cases} \frac{v_{i,l}^{1/m_0}}{\alpha(k)^{1/m_0} (w_{l+1}^{(i)} \cdots w_{n_k+l}^{(i)})^{1/m_0}} & \text{provided } l \in A_{i,1}, \\ \frac{v_{i,l}^{1/m_1}}{\beta(k)^{1/m_1} (w_{l+1}^{(i)} \cdots w_{n_k+l}^{(i)})^{1/m_1}} & \text{provided } l \in A_{i,2}, \\ z_{i,l}(k) & \text{provided } l \in A_{i,3} \end{cases}$$

and $y_{i,j}(k) = 0$ if $j \neq in_k, \dots, in_k + p$.

We claim that $x \in U \cap \mathcal{A}(I, s, V_1, V_2)$. The definition of ε and p ensure that $x \in U$. Let $P \in E(I, s)$. There exists an increasing function $\phi : \mathbb{N} \rightarrow \mathbb{N}$ such that $\alpha(\phi(k)) \rightarrow \hat{P}(m_0)$, $\beta(\phi(k)) \rightarrow \hat{P}(m_1)$ and $a_{i,l}P(z_{i,l}(\phi(k))) \rightarrow v_{i,l}$ for all $l \in A_{i,3}$ and $i = 1, 2$. We claim that $(B_i^{in_{\phi(k)}}(P(x)))$ belongs to V_i , $i = 1, 2$, provided k is large enough. It suffices to prove that for $l = 0, \dots, p$ and $i = 1, 2$, the l -th coordinate of $B_i^{in_{\phi(k)}}(P(x))$ tends to $v_{i,l}$. Assume first that $l \in A_{i,1}$. This l -th coordinate is equal to

$$w_{l+1}^{(i)} \cdots w_{n_{\phi(k)}+l}^{(i)} P \left(\frac{v_{i,l}^{1/m_0}}{\alpha(\phi(k))^{1/m_0} (w_{l+1}^{(i)} \cdots w_{n_{\phi(k)}+l}^{(i)})^{1/m_0}} \right).$$

Now, since $w_{l+1}^{(i)} \cdots w_{n_{\phi(k)}+l}^{(i)}$ tends to $+\infty$, and $m_0 = \min(I)$,

$$w_{l+1}^{(i)} \cdots w_{n_{\phi(k)}+l}^{(i)} P \left(\frac{v_{i,l}^{1/m_0}}{\alpha(\phi(k))^{1/m_0} (w_{l+1}^{(i)} \cdots w_{n_{\phi(k)}+l}^{(i)})^{1/m_0}} \right) = \hat{P}(m_0) \frac{v_{i,l}}{\alpha(\phi(k))} + o(1)$$

and this tends to $v_{i,l}$. When $l \in A_{i,2}$, the proof is similar since now, because $w_{l+1}^{(i)} \cdots w_{n_{\phi(k)}+l}^{(i)}$ tends to 0, and $m_1 = \max(I)$,

$$w_{l+1}^{(i)} \cdots w_{n_{\phi(k)}+l}^{(i)} P \left(\frac{v_{i,l}^{1/m_1}}{\beta(\phi(k))^{1/m_1} (w_{l+1}^{(i)} \cdots w_{n_{\phi(k)}+l}^{(i)})^{1/m_1}} \right) = \hat{P}(m_1) \frac{v_{i,l}}{\beta(\phi(k))} + o(1)$$

and this also goes to $v_{i,l}$. Finally, when $l \in A_{i,3}$, the l -th coordinate of $B_i^{in_{\phi(k)}}(P(x))$ is equal to $w_{l+1}^{(i)} \cdots w_{n_{\phi(k)}+l}^{(i)} P(z_{i,l}(\phi(k)))$ which tends again to $v_{i,l}$. \square

2.2.2 Backward shifts raised to the same power

We now do the same for two shifts raised to the same power.

Theorem 2.10 *Let X be a Fréchet sequence algebra for the coordinatewise product with a continuous norm and such that (e_n) is a Schauder basis. Let $B_1 = B_{w^{(1)}}$ and $B_2 = B_{w^{(2)}}$ be two bounded weighted shifts on X . The following assertions are equivalent.*

(i) B_1 and B_2 support a disjoint hypercyclic algebra.

(ii) For all $p \in \mathbb{N}$, for all $m \geq 1$, there exists a sequence (n_k) such that

$$\left(w_{l+1}^{(1)} \cdots w_{n_k+l}^{(1)} \right)^{-1/m} e_{n_k+l} \xrightarrow{k \rightarrow +\infty} 0, \text{ for all } l = 0, \dots, p,$$

$$\left\{ \left(\frac{w_{l+1}^{(2)} \cdots w_{n_k+l}^{(2)}}{w_{l+1}^{(1)} \cdots w_{n_k+l}^{(1)}} \right)_{l \geq 0} : k \geq 0 \right\} \text{ is dense in } \omega.$$

(iii) There exists a sequence (n_k) such that, for all $\gamma > 0$, for all $l \geq 0$,

$$\left(w_{l+1}^{(1)} \cdots w_{n_k+l}^{(1)} \right)^{-\gamma} e_{n_k+l} \xrightarrow{k \rightarrow +\infty} 0$$

$$\left\{ \left(\frac{w_{l+1}^{(2)} \cdots w_{n_k+l}^{(2)}}{w_{l+1}^{(1)} \cdots w_{n_k+l}^{(1)}} \right)_{l \geq 0} : k \geq 0 \right\} \text{ is dense in } \omega.$$

Proof. (i) \implies (ii). Let $m \geq 1$ and let $x \in X$, $x = \sum_{n=0}^{+\infty} x_n e_n$, be such that x^m is a disjointly hypercyclic vector for B_1 and B_2 . Let also $(\lambda^{(k)})$ be a dense sequence in ω of vectors with finite support. We may assume that $\text{supp}(\lambda^{(k)}) \subset [0, k]$ and that, for any fixed k_0 , there are infinitely many k such that $\lambda^{(k)} = \lambda^{(k_0)}$. We then choose an increasing sequence of integers (n_k) such that, for all $k \geq 1$,

$$\left\| B_1^{n_k}(x^m) - \sum_{l=0}^k e_l \right\| < \frac{1}{k}, \quad (\text{B.3})$$

$$\left\| B_2^{n_k}(x^m) - \sum_{l=0}^k \lambda_l^{(k)} e_l \right\| < \frac{1}{k}. \quad (\text{B.4})$$

As we have already observed, (B.3) implies that, for all $l \geq 0$,

$$\begin{aligned} (w_{l+1}^{(1)} \cdots w_{n_k+l}^{(1)})^{-1/m} e_{n_k+l} &\xrightarrow{k \rightarrow +\infty} 0, \\ w_{l+1}^{(1)} \cdots w_{n_k+l}^{(1)} x_{n_k+l}^m &\xrightarrow{k \rightarrow +\infty} 1. \end{aligned}$$

Let us now fix U a nonempty open subset of ω and $k_0 \in \mathbb{N}$ be such that $\lambda^{(k_0)} \in U$. Let $(n_{\phi(k)})$ be a subsequence of (n_k) such that $\lambda^{(\phi(k))} = \lambda^{k_0}$ for all k . By (B.4), we know that

$$w_{l+1}^{(2)} \cdots w_{n_{\phi(k)}+l}^{(2)} x_{n_{\phi(k)}+l}^m \xrightarrow{k \rightarrow +\infty} \lambda_l^{(k_0)} \text{ for all } l \geq 0.$$

Combining the two last properties, we find that

$$\frac{w_{l+1}^{(2)} \cdots w_{n_{\phi(k)}+l}^{(2)}}{w_{l+1}^{(1)} \cdots w_{n_{\phi(k)}+l}^{(1)}} \xrightarrow{k \rightarrow +\infty} \lambda_l^{(k_0)} \text{ for all } l \geq 0.$$

Hence, for k large enough,

$$\left(\frac{w_{l+1}^{(2)} \cdots w_{n_{\phi(k)}+l}^{(2)}}{w_{l+1}^{(1)} \cdots w_{n_{\phi(k)}+l}^{(1)}} \right)_{l \geq 0} \in U.$$

(ii) \implies (iii). This follows from a diagonal argument.

(iii) \implies (i). Let $m_0 < m_1$ be two positive integers, let U, V_1, V_2 be nonempty open subsets of X and let W be a neighbourhood of 0. Let $x \in U$ with finite support, let $y_i = \sum_{l=0}^p y_{i,l} e_l$ belonging to V_i , $i = 1, 2$. Without loss of generality, we may assume that $y_{1,l} \neq 0$ for all $l = 0, \dots, p$. We then consider a sequence (n_k) of integers such that, for all $l = 0, \dots, p$,

$$\begin{aligned} (w_{l+1}^{(1)} \cdots w_{n_k+l}^{(1)})^{-1/m_0} e_{n_k+l} &\xrightarrow{k \rightarrow +\infty} 0, \\ \frac{w_{l+1}^{(2)} \cdots w_{n_k+l}^{(2)}}{w_{l+1}^{(1)} \cdots w_{n_k+l}^{(1)}} &\xrightarrow{k \rightarrow +\infty} \frac{y_{2,l}}{y_{1,l}}. \end{aligned} \quad (\text{B.5})$$

Observe that $w_{l+1}^{(1)} \cdots w_{n_k+l}^{(1)}$ goes to $+\infty$ for all $l \geq 0$. We finally set

$$u = x + \sum_{l=0}^p \frac{y_{1,l}^{1/m_0}}{(w_{l+1}^{(1)} \cdots w_{n_k+l}^{(1)})^{1/m_0}} e_{n_k+l}.$$

Clearly, u belongs to U (provided k is large enough), $B_1^{n_k}(u^{m_0}) = y_1$ and $B_1^{n_k}(u^n)$ goes to 0 for all $n > m_0$. Now, let us write

$$B_2^{n_k}(u^n) = \sum_{l=0}^p y_{1,l}^{n/m_0} \frac{w_{l+1}^{(2)} \cdots w_{n_k+l}^{(2)}}{(w_{l+1}^{(1)} \cdots w_{n_k+l}^{(1)})^{n/m_0}} e_l.$$

If $n = m_0$, then (B.5) tells us immediately that $B_2^{n_k}(u^m)$ tends to y_2 . For $n > m_0$,

$$y_{1,l}^{n/m_0} \frac{w_{l+1}^{(2)} \cdots w_{n_k+l}^{(2)}}{(w_{l+1}^{(1)} \cdots w_{n_k+l}^{(1)})^{n/m_0}} \sim_{k \rightarrow +\infty} \frac{y_{1,l}^{n/m_0 - 1} y_{2,l}}{(w_{l+1}^{(1)} \cdots w_{n_k+l}^{(1)})^{n/m_0 - 1}}.$$

Since $(w_{l+1}^{(1)} \cdots w_{n_k+l}^{(1)})$ tends to $+\infty$, this implies that $B_2^{n_k}(u^n)$ belongs to W for k large enough, so that we may apply Corollary 2.3 to prove that B_1 and B_2 support a disjoint hypercyclic algebra. \square

Remark 2.11 If we are just interested in disjointly hypercyclic operators, our proof can be easily modified to give the following result : let X be a Fréchet sequence space such that (e_n) is a Schauder basis for X . Let $B_1 = B_{w^{(1)}}$ and $B_2 = B_{w^{(2)}}$ be two bounded weighted shifts on X . Then B_1 and B_2 are disjointly hypercyclic if and only if there exists a sequence (n_k) such that $((w_{l+1}^{(1)} \cdots w_{n_k+l}^{(1)})^{-1} e_{n_k+l})$ tends to zero for all $l \geq 0$ and the set

$$\left\{ \left(\frac{w_{l+1}^{(2)} \cdots w_{n_k+l}^{(2)}}{w_{l+1}^{(1)} \cdots w_{n_k+l}^{(1)}} \right)_{l \geq 0} : k \geq 0 \right\}$$

is dense in ω .

Remark 2.12 In particular, if the sequence (e_n) is bounded away from zero, the conditions of the above theorem are all equivalent to say that B_1 and B_2 are disjointly hypercyclic. Hence, all the examples given in [7] support a disjoint hypercyclic algebra.

Finally, we address the case of ω .

Theorem 2.13 *If $B_1 = B_{w^{(1)}}$ and $B_2 = B_{w^{(2)}}$ are weighted backward shifts on ω endowed with the coordinatewise product, then the following are equivalent.*

- (i) B_1 and B_2 are disjointly hypercyclic
- (ii) B_1 and B_2 support a disjoint hypercyclic algebra.
- (iii) The set

$$\left\{ \left(\frac{w_{l+1}^{(2)} \cdots w_{n+l}^{(2)}}{w_{l+1}^{(1)} \cdots w_{n+l}^{(1)}} \right)_{l \geq 0} : n \geq 0 \right\}$$

is dense in ω .

Proof. The implication (ii) \Rightarrow (i) is immediate. Let us first show that (i) \Rightarrow (iii). Let U be a non-empty open subset of ω , $p \in \mathbb{N}_0$, $u_0, \dots, u_p \in \mathbb{C}$ and $\varepsilon > 0$ be such that, for all $y \in \omega$

$$|y_l - u_l| < \varepsilon \text{ for all } l = 0, \dots, p \text{ implies } y \in U.$$

If x is a disjoint hypercyclic vector for B_1 and B_2 , there is a subsequence (n_k) such that

$$B_1^{n_k} x \rightarrow \sum_{l=0}^p e_l$$

and

$$B_2^{n_k} x \rightarrow \sum_{l=0}^p u_l e_l.$$

That means that for all $l = 0, \dots, p$,

$$w_{l+1}^{(1)} \cdots w_{n_k+l}^{(1)} x_{n_k+l} \rightarrow 1$$

and

$$w_{l+1}^{(2)} \cdots w_{n_k+l}^{(2)} x_{n_k+l} \rightarrow u_l$$

from which we immediately get that

$$\frac{w_{l+1}^{(2)} \cdots w_{n_k+l}^{(2)}}{w_{l+1}^{(1)} \cdots w_{n_k+l}^{(1)}} \rightarrow u_l.$$

Next, we show that (iii) \Rightarrow (ii). For V_1 and V_2 non-empty open subsets of ω , $I \subset \mathbb{N}$ finite and non-empty and $s > 0$, let us define

$$E(I, s) = \left\{ P \in \mathbb{C}[z] : |\hat{P}(\min I)| \geq 1/s, |\hat{P}(\max I)| \geq 1/s, \right. \\ \left. |\hat{P}(n)| \leq s \text{ for all } n \in \mathbb{N}, \right. \\ \left. \hat{P}(n) = 0 \text{ when } n \notin I \right\}$$

$$\mathcal{A}(I, s, V_1, V_2) = \left\{ u \in \omega : \forall P \in E(I, s), \exists N \geq 1, B_i^N(P(u)) \in V_i, i = 1, 2 \right\}.$$

By a Baire category argument, it is enough to prove that each set $\mathcal{A}(I, s, V_1, V_2)$ is dense and open. The last property follows easily from the compactness of $E(I, s)$. Thus, let us fix I, s, V_1 and V_2 and let us prove that $\mathcal{A}(I, s, V_1, V_2)$ is dense. We set $m_0 = \min(I)$ and $m_1 = \max(I)$. Let U be a non-empty open subset of ω . Let $p \in \mathbb{N}_0$, $u_0, \dots, u_p, v_{i,0}, \dots, v_{i,p} \in \mathbb{C}$, for $i = 1, 2$ (we may assume that $v_{i,l} \neq 0$ for all $l = 0, \dots, p$), and $\varepsilon > 0$ be such that, for all x, y_1 and y_2 in ω ,

$$|x_l - u_l| < \varepsilon \text{ for all } l = 0, \dots, p \text{ implies } x \in U,$$

$$|y_{i,l} - v_{i,l}| < \varepsilon \text{ for all } l = 0, \dots, p \text{ implies } y_i \in V_i, i = 1, 2.$$

Let by assumption (n_k) be a subsequence such that

$$\frac{w_{l+1}^{(2)} \cdots w_{n_k+l}^{(2)}}{w_{l+1}^{(1)} \cdots w_{n_k+l}^{(1)}} \rightarrow \frac{v_{2,l}}{v_{1,l}}, \quad l = 0, \dots, p.$$

By passing to a subsequence we may assume that $n_0 > p$, that $n_{k+1} - n_k > p$ for all k , and that there are $a_0, \dots, a_p \in [0, +\infty]$ such that, for all $l = 0, \dots, p$, $(w_{l+1}^{(1)} \cdots w_{n_k+l}^{(1)})$ tends to a_l . We set $A_1 = \{l \in \{0, \dots, p\} : a_l = +\infty\}$, $A_2 = \{l \in \{0, \dots, p\} : a_l = 0\}$ and $A_3 = \{l \in \{0, \dots, p\} : a_l \in (0, +\infty)\}$.

We fix now $(\alpha(k)), (\beta(k))$ two sequences of non-zero complex numbers and $(z(k))$ a sequence in \mathbb{C}^{p+1} such that $(\alpha(k), \beta(k), z(k))$ is dense in \mathbb{C}^{p+3} . We set

$$x = u + \sum_{k=0}^{+\infty} y(k)$$

where, for $l = 0, \dots, p$,

$$y_{n_k+l}(k) = \begin{cases} \frac{v_{1,l}^{1/m_0}}{\alpha(k)^{1/m_0} (w_{l+1}^{(1)} \cdots w_{n_k+l}^{(1)})^{1/m_0}} & \text{provided } l \in A_1, \\ \frac{v_{1,l}^{1/m_1}}{\beta(k)^{1/m_1} (w_{l+1}^{(1)} \cdots w_{n_k+l}^{(1)})^{1/m_1}} & \text{provided } l \in A_2, \\ z_l(k) & \text{provided } l \in A_3 \end{cases}$$

and $y_j(k) = 0$ if $j \neq n_k, \dots, n_k + p$.

We claim that $x \in U \cap \mathcal{A}(I, s, V_1, V_2)$. The definition of ε and p ensure that $x \in U$. Let $P \in E(I, s)$. There exists an increasing function $\phi : \mathbb{N} \rightarrow \mathbb{N}$ such that $\alpha(\phi(k)) \rightarrow \hat{P}(m_0)$, $\beta(\phi(k)) \rightarrow \hat{P}(m_1)$ and $a_l P(z_l(\phi(k))) \rightarrow v_{1,l}$ for all $l \in A_3$. We claim that $(B_i^{n_{\phi(k)}}(P(x)))$ belongs to V_i , $i = 1, 2$, provided k is large enough. It suffices to prove that for $l = 0, \dots, p$ and $i = 1, 2$, the l -th coordinate of $B_i^{n_{\phi(k)}}(P(x))$ tends to $v_{i,l}$. The details for the case $i = 1$ are identical as in [4, Theorem 4.8] hence we will present the case $i = 2$. Assume first that $l \in A_1$. The l -th coordinate of $B_2^{n_{\phi(k)}}(P(x))$ is equal to

$$w_{l+1}^{(2)} \cdots w_{n_{\phi(k)}+l}^{(2)} P \left(\frac{v_{1,l}^{1/m_0}}{\alpha(\phi(k))^{1/m_0} (w_{l+1}^{(1)} \cdots w_{n_{\phi(k)}+l}^{(1)})^{1/m_0}} \right).$$

Now, since $w_{l+1}^{(1)} \cdots w_{n_{\phi(k)}+l}^{(1)}$ tends to $+\infty$, and $m_0 = \min(I)$, we have for $m > m_0$,

$$\frac{w_{l+1}^{(2)} \cdots w_{n_{\phi(k)}+l}^{(2)}}{(w_{l+1}^{(1)} \cdots w_{n_{\phi(k)}+l}^{(1)})^{m/m_0}} = \frac{w_{l+1}^{(2)} \cdots w_{n_{\phi(k)}+l}^{(2)}}{w_{l+1}^{(1)} \cdots w_{n_{\phi(k)}+l}^{(1)}} (w_{l+1}^{(1)} \cdots w_{n_{\phi(k)}+l}^{(1)})^{1-\frac{m}{m_0}} \rightarrow 0.$$

Hence,

$$\begin{aligned} & w_{l+1}^{(2)} \cdots w_{n_{\phi(k)}+l}^{(2)} P \left(\frac{v_{1,l}^{1/m_0}}{\alpha(\phi(k))^{1/m_0} (w_{l+1}^{(1)} \cdots w_{n_{\phi(k)}+l}^{(1)})^{1/m_0}} \right) = \\ & \frac{w_{l+1}^{(2)} \cdots w_{n_{\phi(k)}+l}^{(2)}}{w_{l+1}^{(1)} \cdots w_{n_{\phi(k)}+l}^{(1)}} \hat{P}(m_0) \frac{v_{1,l}}{\alpha(\phi(k))} + o(1) \end{aligned}$$

and this tends to $v_{2,l}$. When $l \in A_2$, the proof is similar. Finally, when $l \in A_3$, the l -th coordinate of $B_2^{n_{\phi(k)}}(P(x))$ is equal to

$$w_{l+1}^{(2)} \cdots w_{n_{\phi(k)}+l}^{(2)} P(z_l(\phi(k))) = \frac{w_{l+1}^{(2)} \cdots w_{n_{\phi(k)}+l}^{(2)}}{w_{l+1}^{(1)} \cdots w_{n_{\phi(k)}+l}^{(1)}} w_{l+1}^{(1)} \cdots w_{n_{\phi(k)}+l}^{(1)} P(z_l(\phi(k)))$$

which tends again to $v_{2,l}$. □

2.3 Disjoint hypercyclic algebras for backward shifts - convolution product

2.3.1 Backward shifts raised to different powers

We now turn to the case of Fréchet sequence algebras endowed with the Cauchy product, and we first study the existence of a disjoint hypercyclic algebra for two backward shifts raised to different powers. This situation seems more complicated and we choose to present only two significant examples. The first one is the multiples of B and B^2 on ℓ_1 . It is known by [6, Cor 4.2] that λB and μB^2 are disjointly hypercyclic if and only if $1 < \lambda < \mu$. We show that under this condition, they even support a disjoint hypercyclic algebra. The proof will be done in two steps. We first handle the case $\lambda < \mu^{1/2}$. The proof uses a shifting factor which is placed after the term to approximate.

Theorem 2.14 *Let $1 < \lambda < \mu^{1/2}$. Then λB and μB^2 , acting on ℓ_1 endowed with the Cauchy product, support a disjoint hypercyclic algebra.*

Proof. We shall prove that the assumptions of Corollary 2.2 are satisfied. Let $m \geq 1$, let U, V_1, V_2 be nonempty open sets and let W be a neighbourhood of 0. Let $p \geq 1$, $x \in U$, $y_1 \in V_1$, $y_2 \in V_2$ with $\text{supp}(x), \text{supp}(y_1), \text{supp}(y_2) \subset [0, p]$ and let us write $y_i = \sum_{l=0}^p y_{i,l} e_l$, $i = 1, 2$. Let finally $\delta > 0$ be such that $B(y_i, 2\delta) \subset V_i$. For σ a large integer, we set

$$\begin{aligned} N &= m\sigma - 3p \\ d_{1,l} &= \frac{y_{1,l}}{m\lambda^{N/m}\delta^{(m-1)/m}}, & \varepsilon_1 &= \frac{\delta^{1/m}}{\lambda^{N/m}} \\ d_{2,l} &= \frac{y_{2,l}}{m\mu^{N/m}\delta^{(m-1)/m}}, & \varepsilon_2 &= \frac{\delta^{1/m}}{\mu^{N/m}} \\ z_1 &= \sum_{l=0}^p d_{1,l} e_{\sigma-3p+l} + \varepsilon_1 e_\sigma \\ z_2 &= \sum_{l=0}^p d_{2,l} e_{2\sigma-6p+l} + \varepsilon_2 e_{2\sigma} \\ u &= x + z_1 + z_2. \end{aligned}$$

The terms $\varepsilon_1 e_\sigma$ and $\varepsilon_2 e_{2\sigma}$ are shifting terms which will be used to build an appropriate approximating term when we will take the m -th power, despite the mixing of the supports.

It is easy to check that, provided σ , hence N , is large enough, then $u \in U$. We shall prove that we also have

$$\begin{cases} (\lambda B)^N(u^m) \in V_1, (\lambda B)^N(u^n) \in W \text{ for } n < m, \\ (\mu B^2)^N(u^m) \in V_2, (\mu B^2)^N(u^n) \in W \text{ for } n < m. \end{cases}$$

The two last properties are easier to prove: there is no more difficulty to prove them than to prove that μB^2 admits a hypercyclic algebra. Indeed, if $n < m$, then $\text{supp}(u^n) \subset [0, 2n\sigma]$ with $2n\sigma < 2N$ whence $(\mu B^2)^N(u^n) = 0$. On the other hand,

$$u^m = z + \sum_{l=0}^p \frac{y_{2,l}}{\mu^N} e_{2m\sigma-6p+l} + \frac{\delta}{\mu^N} e_{2m\sigma}$$

with $\max(\text{supp}(z)) < 2N$, hence

$$(\mu B^2)^N(u^m) = y_2 + \delta e_{6p} \in V_2.$$

The proofs of the first two properties are more difficult: the powers of z_2 are mixed with that of $x + z_1$. Such a mixing term does not always disappear when you apply B^N . It is also not always smaller than λ^{-N} . But taking into account the strong condition $\mu > \lambda^2$, we will show that it keeps smaller than λ^{-N} if its support exceeds $[0, N]$. We start by writing

$$\begin{aligned} u^m &= (x + z_1)^m + \sum_{\substack{\alpha+\beta+\gamma=m \\ \gamma \geq 1}} \binom{m}{\alpha, \beta, \gamma} x^\alpha z_1^\beta z_2^\gamma \\ &= z + \sum_{l=0}^p \frac{y_{1,l}}{\lambda^N} e_{m\sigma-3p+l} + \frac{\delta}{\lambda^N} e_{m\sigma} + \sum_{\substack{\alpha+\beta+\gamma=m \\ \gamma \geq 1}} \binom{m}{\alpha, \beta, \gamma} x^\alpha z_1^\beta z_2^\gamma \end{aligned}$$

where

$$\binom{m}{\alpha, \beta, \gamma} = \frac{m!}{\alpha! \beta! \gamma!}$$

is the multinomial coefficient and $\max(\text{supp}(z)) < N$. For the first three terms, we proceed as before to observe that, provided σ is large enough,

$$(\lambda B)^N \left(z + \sum_{l=0}^p \frac{y_{1,l}}{\lambda^N} e_{m\sigma-3p+l} + \frac{\delta}{\lambda^N} e^{m\sigma} \right) = y_1 + \delta e_{3p}.$$

Let us now look at $(\lambda B)^N(x^\alpha z_1^\beta z_2^\gamma)$ with $\gamma \geq 1$ and $\alpha + \beta + \gamma = m$. Expanding the product and using the behaviour of the terms $d_{i,j}$ and ε_i , $x^\alpha z_1^\beta z_2^\gamma$ is a sum of a finite number of terms $c_j e_j$ (this finite number does not depend on σ), with

$$j \leq p\alpha + \beta\sigma + 2\gamma\sigma$$

$$|c_j| \leq C_{\alpha, \beta, \gamma} \frac{1}{\lambda^{\beta N/m} \mu^{\gamma N/m}}$$

where the involved constant $C_{\alpha, \beta, \gamma}$ does not depend on σ (of course, it also depends on x, y_1, y_2, δ and m). We then distinguish two cases. Assume first that $\beta + 2\gamma < m$. Then, provided σ is large enough, $p\alpha + \beta\sigma + 2\gamma\sigma < N$, hence $(\lambda B)^N(x^\alpha z_1^\beta z_2^\gamma) = 0$. On the contrary, if $\beta + 2\gamma \geq m$, then

$$\left\| (\lambda B)^N(x^\alpha z_1^\beta z_2^\gamma) \right\| \leq C_{\alpha, \beta, \gamma} \frac{\lambda^N}{\lambda^{\beta N/m} \mu^{\gamma N/m}}$$

and the right handside goes to 0 since $\mu > \lambda^2$, hence $\mu^\gamma \lambda^\beta > \lambda^m$. This shows that, provided σ is large enough, $(\lambda B)^N(u^m) \in V_1$. Finally, for $n < m$, it is easy to check that

$$(\lambda B)^N(u^n) = \sum_{\substack{\alpha + \beta + \gamma = n \\ \gamma \geq 1}} \binom{n}{\alpha, \beta, \gamma} (\lambda B)^N(x^\alpha z_1^\beta z_2^\gamma).$$

We use exactly the same arguments to prove that

- $(\lambda B)^N(x^\alpha z_1^\beta z_2^\gamma) = 0$ provided $\beta + 2\gamma \leq m$;
- $(\lambda B)^N(x^\alpha z_1^\beta z_2^\gamma)$ tends to zero as σ tends to $+\infty$ provided $\beta + 2\gamma > m$.

Hence, $(\lambda B)^N(u^n)$ belongs to W , which finishes the proof of Theorem 2.14. \square

We turn to the case $\lambda \geq \mu^{1/2}$. The proof follows a similar scheme, but we now put the shifting term before the approximating term.

Theorem 2.15 *Let $1 < \lambda < \mu \leq \lambda^2$. Then λB and μB^2 , acting on ℓ^1 endowed with the Cauchy product, support a disjoint hypercyclic algebra.*

Proof. Again we shall prove that the assumptions of Corollary 2.2 are satisfied. Thus, let $m \geq 1$, let U, V_1, V_2 be nonempty open subsets of ℓ^1 and let W be a neighbourhood of 0. Let $p \geq 1, x \in U, y_1 \in V_1, y_2 \in V_2$ with $\text{supp}(x), \text{supp}(y_1)$ and $\text{supp}(y_2) \subset [0, p]$ and let us write $y_i = \sum_{l=0}^p y_{i,l} e_l$, $i = 1, 2$. Let $\sigma > 2p$ and let $\varepsilon > 0$ be such that the ball $B(x, 2\varepsilon)$ is contained in U . For N a large integer, we define

$$z_1 = \frac{1}{m\varepsilon^{m-1}\lambda^N} \sum_{l=0}^p y_{1,l} e_{N-(m-1)\sigma+l}$$

$$z_2 = \frac{1}{m\varepsilon^{m-1}\mu^N} \sum_{l=0}^p y_{2,l} e_{2N-(m-1)\sigma+l}$$

$$u = x + \varepsilon e_\sigma + z_1 + z_2.$$

The definition of ε ensures that, for N large enough, u belongs to U . Let us show that the other assumptions of Corollary 2.2 are satisfied. First, consider $n < m$ and write

$$u^n = w_1 + w_2$$

where

$$w_1 = \sum_{\substack{\alpha+\beta+\gamma=n \\ \gamma \leq 1}} \binom{n}{\alpha, \beta, \gamma} x^\alpha (\varepsilon e_\sigma)^\beta z_1^\gamma$$

$$w_2 = \sum_{\substack{\alpha+\beta+\gamma=n \\ \gamma > 1}} \binom{n}{\alpha, \beta, \gamma} x^\alpha (\varepsilon e_\sigma)^\beta z_1^\gamma + \sum_{\substack{\alpha+\beta+\gamma+\delta=n \\ \delta \geq 1}} \binom{n}{\alpha, \beta, \gamma, \delta} x^\alpha (\varepsilon e_\sigma)^\beta z_1^\gamma z_2^\delta.$$

The support of w_1 is contained in $[0, N - (m-1)\sigma + p + (n-1)\sigma] \subset [0, N)$ since $\sigma > p$ and $n < m$. Thus $(\lambda B)^N w_1^n = 0$. Moreover, it is easy to check that

$$\|w_2\| \leq C \max\left(\frac{1}{\lambda^{2N}}, \frac{1}{\mu^N}\right)$$

where the constant C depends on $x, p, \varepsilon, y_1, y_2$ but not on N . Since $1 < \lambda < \mu$, the norm of $(\lambda B)^N w_2$ can be adjusted to be arbitrarily small, so that $(\lambda B)^N u^n$ belongs to W . In the same vein, write

$$u^n = w'_1 + w'_2$$

where

$$w'_1 = \sum_{\substack{\alpha+\beta+\gamma=n \\ \gamma \leq 2}} \binom{n}{\alpha, \beta, \gamma} x^\alpha (\varepsilon e_\sigma)^\beta z_1^\gamma + \sum_{\substack{\alpha+\beta+\delta=n \\ \delta=1}} \binom{n}{\alpha, \beta, \delta} x^\alpha (\varepsilon e_\sigma)^\beta z_2^\delta$$

$$w'_2 = \sum_{(\alpha, \beta, \gamma, \delta) \in E_n} \binom{n}{\alpha, \beta, \gamma, \delta} x^\alpha (\varepsilon e_\sigma)^\beta z_1^\gamma z_2^\delta$$

with

$$E_n = \left\{ (\alpha, \beta, \gamma, \delta) \in \mathbb{N}_0^4 : \alpha + \beta + \gamma + \delta = n, (\gamma \geq 3 \text{ and } \delta = 0) \right. \\ \left. \text{or } (\delta \geq 2 \text{ and } \gamma = 0) \text{ or } (\gamma \geq 1 \text{ and } \delta \geq 1) \right\}.$$

As before, the support of w'_1 is contained in $[0, 2N)$, so that $(\mu B^2)^N (w'_1) = 0$. Since

$$\|w'_2\| \leq C \max\left(\frac{1}{\lambda^{3N}}, \frac{1}{\mu^{2N}}, \frac{1}{\lambda^N \mu^N}\right)$$

and $\mu \leq \lambda^2 < \lambda^3$, we get that $(\mu B^2)^N (u^n)$ belongs to W provided that N is large enough. Let us now inspect u^m . We first write it

$$u^m = \tilde{w}_1 + \frac{1}{\lambda^N} \sum_{l=0}^p y_{1,l} e_{N+l} + \tilde{w}_2$$

where

$$\begin{aligned}\tilde{w}_1 &= \sum_{\alpha+\beta+\gamma \in F_m} \binom{m}{\alpha, \beta, \gamma} x^\alpha (\varepsilon e_\sigma)^\beta z_1^\gamma \\ \tilde{w}_2 &= \sum_{\substack{\alpha+\beta+\gamma=m \\ \gamma>1}} \binom{m}{\alpha, \beta, \gamma} x^\alpha (\varepsilon e_\sigma)^\beta z_1^\gamma + \sum_{\substack{\alpha+\beta+\gamma+\delta=m \\ \delta \geq 1}} \binom{m}{\alpha, \beta, \gamma, \delta} x^\alpha (\varepsilon e_\sigma)^\beta z_1^\gamma z_2^\delta\end{aligned}$$

with $F_m = \{(\alpha, \beta, \gamma) \in \mathbb{N}_0^3 : \alpha + \beta + \gamma = m, \gamma \leq 1\} \setminus \{(0, m-1, 1)\}$. The same proof shows that $\text{supp}(\tilde{w}_1) \subset [0, N - (m-1)\sigma + p + (m-2)\sigma + p) \subset [0, N)$ and that $(\lambda B)^N(\tilde{w}_2)$ tends to 0 as N tends to $+\infty$. Hence, $(\lambda B)^N(u^m) \in V_1$ for N large enough. We also write

$$\begin{aligned}u^m &= \tilde{w}'_1 + \frac{1}{\mu^N} \sum_{l=0}^p y_{2,l} e_{2N+l} + \tilde{w}'_2 \\ \tilde{w}'_1 &= \sum_{\substack{\alpha+\beta+\gamma=m \\ \gamma \leq 2}} \binom{m}{\alpha, \beta, \gamma} x^\alpha (\varepsilon e_\sigma)^\beta z_1^\gamma + \sum_{(\alpha, \beta, \delta) \in F'_m} \binom{m}{\alpha, \beta, \delta} x^\alpha (\varepsilon e_\sigma)^\beta z_2^\delta \\ \tilde{w}'_2 &= \sum_{(\alpha, \beta, \gamma, \delta) \in E_m} \binom{m}{\alpha, \beta, \gamma, \delta} x^\alpha (\varepsilon e_\sigma)^\beta z_1^\gamma z_2^\delta \\ F'_m &= \{(\alpha, \beta, \delta) \in F_m : \delta = 1\}.\end{aligned}$$

Looking at the support of \tilde{w}'_1 and at the norm of \tilde{w}'_2 , we show again that $(\mu B^2)^N(u^m)$ belongs to V_2 for N large enough, showing that the assumptions of Corollary 2.2 are satisfied. \square

We end up this section by proving the existence of a disjoint hypercyclic algebra for the multiples of D and D^2 . The proof follows the scheme of that of Theorem 2.14, but we need a more careful analysis.

Theorem 2.16 *Let $\lambda, \mu > 0$. Then λD and μD^2 support a disjoint hypercyclic algebra on $H(\mathbb{C})$ endowed with the convolution product.*

Proof. The scheme of the proof is similar to that of Theorem 2.14; nevertheless, we will need to be more careful when we analyze $(\lambda D)^N(x^\alpha z_1^\beta z_2^\gamma)$ for $\beta + 2\gamma = m$. We fix $m \geq 1$, U, V_1, V_2 and W as above. Let $q \geq 0$ and $x \in U$ with support in $[0, q]$. Let now $p \geq 1$ be such that $2p > qm$ and such that there exists $y_i \in V_i$ with $\text{supp}(y_i) \subset [0, p]$, $i = 1, 2$. We again write $y_i = \sum_{l=0}^p y_{i,l} e_l$ and we consider $r \geq 1$ and $\delta > 0$ such that $B_{\|\cdot\|, r}(y_i, 2\delta) \subset V_i$. We then consider $\omega > 1/m$ satisfying the following property: for all integers $\alpha, \beta, \gamma \geq 0$ with $\alpha + \beta + \gamma \leq m$ and $\beta + 2\gamma > m$, we have

$$\beta + 2\gamma(m - m(m-1)\omega) > m \tag{B.6}$$

(observe that when $\omega = 1/m$, then the left-hand side is equal to $\beta + 2\gamma$, which is greater than m , and that there is only a finite number of constraints to satisfy). For σ a large integer, we set

$$\begin{aligned}N &= m\sigma - 3p, \\ d_{1,l} &= \frac{l! y_{1,l}}{m \varepsilon_1^{m-1} \lambda^N (N+l)!}, \quad \varepsilon_1 = \frac{((3p)!)^{1/m} \delta^{1/m}}{(3p)^{r/m} \lambda^{N/m} [(N+3p)!]^{1/m}},\end{aligned}$$

$$\begin{aligned}
 d_{2,l} &= \frac{l!y_{2,l}}{m\varepsilon_2^{m-1}\mu^N(2N+l)!}, & \varepsilon_2 &= \frac{\delta^{1/m}}{\mu^{N/m}[(2N+6p)!]^\omega}, \\
 z_1 &= \sum_{l=0}^p d_{1,l}e_{\sigma-3p+l} + \varepsilon_1 e_\sigma, \\
 z_2 &= \sum_{l=0}^p d_{2,l}e_{2\sigma-6p+l} + \varepsilon_2 e_{2\sigma}, \\
 u &= x + z_1 + z_2.
 \end{aligned}$$

The first part of the proof of Theorem 2.14 carries on to our new context. First, we observe that, using Stirling's formula, we now have the crude estimates, for some $C > 0$ and some $\kappa > 0$,

$$\begin{aligned}
 |d_{1,l}| &\leq \frac{C\kappa^\sigma}{\sigma^\sigma} \\
 |d_{2,l}| &\leq \frac{C\kappa^\sigma}{\sigma^{2(m-m(m-1)\omega)\sigma}} \\
 \varepsilon_1 &\leq \frac{C\kappa^\sigma}{\sigma^\sigma} \\
 \varepsilon_2 &\leq \frac{C\kappa^\sigma}{\sigma^{2m\omega\sigma}} \leq \frac{C\kappa^\sigma}{\sigma^{2(m-m(m-1)\omega)\sigma}}
 \end{aligned}$$

(the very last inequality is a consequence of $\omega > 1/m$). Hence, we still have $u \in U$ provided σ is large enough. Looking at the support of u^n shows that $(\mu D^2)^N(u^n) = 0$ for all $n < m$, whereas the values of ε_2^m and $m\varepsilon_2^{m-1}d_{2,l}$ ensure that

$$(\mu D^2)^N(u^m) = y_2 + \frac{\delta}{((2N+6p)!)^{m\omega-1}(6p)!} e_{6p}.$$

Since $\omega > 1/m$, we clearly have $\|(\mu D^2)^N(u^m) - y_2\|_r < 2\delta$ provided N is large enough.

Regarding $(\lambda D)^N$, the support of each element and the definitions of $d_{1,l}$, ε_1 ensure that

$$\begin{aligned}
 (\lambda D)^N(u^m) &= y_1 + \frac{\delta}{(3p)^r} e_{3p} + \sum_{\substack{\alpha+\beta+\gamma=m \\ \gamma \geq 1, \beta+2\gamma \geq m}} \binom{m}{\alpha, \beta, \gamma} (\lambda D)^N(x^\alpha z_1^\beta z_2^\gamma) \\
 (\lambda D)^N(u^n) &= \sum_{\substack{\alpha+\beta+\gamma=n \\ \gamma \geq 1, \beta+2\gamma \geq m}} \binom{n}{\alpha, \beta, \gamma} (\lambda D)^N(x^\alpha z_1^\beta z_2^\gamma) \text{ for } n < m.
 \end{aligned}$$

Hence, it remains to show that $(\lambda D)^N(x^\alpha z_1^\beta z_2^\gamma)$ tends to zero as σ tends to $+\infty$ when $\alpha + \beta + \gamma \leq m$, $\gamma \geq 1$ and $\beta + 2\gamma \geq m$. We first assume that $\beta + 2\gamma > m$. The product $x^\alpha z_1^\beta z_2^\gamma$ may be written as a finite sum (whose number of terms do not depend on σ) $\sum_j c_j e_j$ where

$$\begin{aligned}
 j &\leq q\alpha + \beta\sigma + 2\gamma\sigma \\
 |c_j| &\leq \frac{C_{\alpha,\beta,\gamma} \kappa_{\alpha,\beta,\gamma}^\sigma}{\sigma^{\beta\sigma+2\gamma(m-(m-1)\omega)\sigma}}.
 \end{aligned}$$

Thus, using $\|(\lambda D)^N e_j\|_r \leq \lambda^N j^r j^N$, we get that for some different constants $C_{\alpha,\beta,\gamma}$ and $\kappa_{\alpha,\beta,\gamma}$,

$$\left\| (\lambda D)^N(x^\alpha z_1^\beta z_2^\gamma) \right\|_r \leq \frac{C_{\alpha,\beta,\gamma} \kappa_{\alpha,\beta,\gamma}^\sigma \sigma^{m\sigma}}{\sigma^{\beta\sigma+2\gamma(m-(m-1)\omega)\sigma}}.$$

The choice of ω (sufficiently close to $1/m$, see (B.6)) guarantees that this goes to 0 as σ tends to $+\infty$. Finally, assume that $\beta + 2\gamma = m$ and define

$$w_2 = \sum_{l=0}^p d_{2,l} e_{2\sigma-6p+l}$$

so that

$$\begin{aligned} x^\alpha z_1^\beta z_2^\gamma &= x^\alpha z_1^\beta (w_2 + \varepsilon_2 e_{2\sigma})^\gamma \\ &= \sum_{k=0}^{\gamma} \binom{\gamma}{k} \varepsilon_2^{\gamma-k} x^\alpha z_1^\beta w_2^k e_{2(\gamma-k)\sigma}. \end{aligned}$$

Assume first that $k \neq 0$. Then

$$\begin{aligned} \max(\text{supp}(x^\alpha z_1^\beta w_2^k e_{2(\gamma-k)\sigma})) &\leq q\alpha + \beta\sigma + 2\gamma\sigma - 5p \\ &\leq qm + m\sigma - 5p \\ &< N \end{aligned}$$

since $qm < 2p$. This implies that

$$(\lambda D)^N (x^\alpha z_1^\beta w_2^k e_{2(\gamma-k)\sigma}) = 0$$

and it only remains to handle the term $\varepsilon_2^\gamma x^\alpha z_1^\beta e_{2\gamma\sigma}$. As before, it is equal to a finite sum $\sum_j c_j e_j$ with $j \leq q\alpha + (\beta + 2\gamma)\sigma$ but now we can be slightly more precise on $|c_j|$:

$$|c_j| \leq \frac{C_{\alpha,\beta,\gamma} \kappa_{\alpha,\beta,\gamma}^\sigma}{\sigma^{\beta\sigma+2\gamma m\omega\sigma}}.$$

Therefore,

$$\left\| (\lambda D)^N (x \varepsilon_2^\gamma x^\alpha z_1^\beta e_{2\gamma\sigma}) \right\|_r \leq \frac{C_{\alpha,\beta,\gamma} \kappa_{\alpha,\beta,\gamma}^\sigma \sigma^{m\sigma}}{\sigma^{\beta\sigma+2\gamma m\omega\sigma}}.$$

Now, since $\omega > 1/m$ and $\beta + 2\gamma = m$, this tends to 0 as σ goes to $+\infty$. This finishes the proof of Theorem 2.16. \square

Theorem 2.9 has the following analogue when we are dealing with the convolution product.

Theorem 2.17 *If $B_1 = B_{w(1)}$ and $B_2 = B_{w(2)}$ are weighted backward shifts on ω endowed with the Cauchy product, then B_1 and B_2^2 support a disjoint hypercyclic algebra.*

Proof. We will apply Corollary 2.2. Let $m \geq 1$, U, V_1, V_2 be nonempty open subsets of ω , and W be a zero neighbourhood. Let also $p \in \mathbb{N}_0$, $u_0, \dots, u_p, v_{i,0}, \dots, v_{i,p} \in \mathbb{C}$, for $i = 1, 2$, and $\varepsilon > 0$ be such that, for all $x \in \omega$,

$$|x_l - u_l| < \varepsilon \text{ for all } l = 0, \dots, p \text{ implies } x \in U,$$

$$|x_l - v_{i,l}| < \varepsilon \text{ for all } l = 0, \dots, p \text{ implies } x \in V_i, i = 1, 2 \text{ and}$$

$$|x_l| < \varepsilon \text{ for all } l = 0, \dots, p \text{ implies } x \in W.$$

We set

$$x = \sum_{l=0}^p u_l e_l + e_{2p} + z_1 + z_2$$

where

$$z_1 = \sum_{l=0}^p \frac{v_{1,l}}{w_{l+1}^{(1)} \cdots w_{6pm+l}^{(1)}} e_{2p(2m+1)+l}$$

and

$$z_2 = \sum_{l=0}^p \frac{v_{2,l}}{w_{l+1}^{(2)} \cdots w_{12pm+l}^{(2)}} e_{2p(5m+1)+l}.$$

It is immediate that $x \in U$. If $N = 6pm$, it is tedious but straightforward to check that the support of $B_1^N(x^n)$, with $n \leq m$, intersects $[0, p]$ only when $n = m$. Specifically, for $l = 0, \dots, p$, the l -th coordinate of $B_1^N(x^m)$ is the l -th coordinate of $B_1^N(e_{2p}^{m-1} z_1)$ which equals to $v_{1,l}$. That means that $B_1^N(x^n) \in W$, for $n < m$ and $B_1^N(x^m) \in V_1$. Similarly, it follows that $B_2^{2N}(x^n) \in W$, for $n < m$ and $B_2^{2N}(x^m) \in V_2$ which concludes the proof. \square

2.3.2 Backward shifts raised to the same power

We conclude this section by studying the existence of disjoint hypercyclic algebras for two backward shifts (raised to the same power). We will need something similar to the converse inequality of the continuity of the product under the F -norm. This idea is captured by the notion of a regular Fréchet sequence algebra which first appeared in [4].

Definition 2.18 A Fréchet sequence algebra $(X, (\|\cdot\|_q))$ is said to be *regular* when it satisfies the following properties:

1. X admits a continuous norm;
2. (e_i) is a Schauder basis for X ;
3. for any $r \geq 1$, there exist $q \geq 1$ and $C > 0$ such that, for all $n, k \geq 0$,

$$\|e_n\|_r \cdot \|e_k\|_r \leq C \|e_{n+k}\|_q.$$

We were able to get a general result saying that two disjoint hypercyclic backward shifts support a disjoint hypercyclic algebra, a statement simpler than that regarding the coordinatewise product.

Theorem 2.19 *Let X be a regular Fréchet sequence algebra for the Cauchy product and let $B_1 = B_{w^{(1)}}$, $B_2 = B_{w^{(2)}}$ be two bounded weighted shifts on X . The following assertions are equivalent.*

- (i) B_1 and B_2 are disjointly hypercyclic.
- (ii) B_1 and B_2 support a disjoint hypercyclic algebra.
- (iii) There exists a sequence (n_k) such that $((w_{l+1}^{(1)} \cdots w_{n_k+l}^{(1)})^{-1} e_{n_k+l})$ tends to zero for all $l \geq 0$ and the set

$$\left\{ \left(\frac{w_{l+1}^{(2)} \cdots w_{n_k+l}^{(2)}}{w_{l+1}^{(1)} \cdots w_{n_k+l}^{(1)}} \right)_{l \geq 0} : k \geq 0 \right\}$$

is dense in ω .

For the proof of Theorem 2.19 we will use property (3) from the definition of regularity through the following lemma from [4, Corollary 4.14] which works as some kind of index managing tool.

Lemma 2.20 *Let X be a regular Fréchet sequence algebra for the Cauchy product and let (w_n) be an admissible weight sequence on X . Then, for all $m \geq 1$, for all $N \geq 1$, for all $r \geq 1$, and for all $\rho \geq 0$, there exist $C > 0$ and $q \geq 1$ such that, for all $n \geq mN$, for all $s \in \{1, \dots, N\}$,*

$$(w_{n-s+1+\rho})^{m-1} \cdots (w_{n-1+\rho})^{m-1} (w_{n+\rho})^{m-1} \|e_{n-ms+m\rho}\|_r \leq C \|e_{n-s+\rho}\|_q.$$

Proof of Theorem 2.19. The implication (ii) \implies (i) is trivial. Let us prove the other ones.

(iii) \implies (ii). Let $m \geq 1$, let U, V_1, V_2 be nonempty open subsets of X and let W be a neighbourhood of 0. Let $p \geq 1$, $x \in U$, $y_1 \in V_1$, $y_2 \in V_2$ with $\text{supp}(x), \text{supp}(y_1), \text{supp}(y_2) \subset [0, p]$ and let us write $y_i = \sum_{l=0}^p y_{i,l} e_l$. Without loss of generality, we may assume $y_{1,l} \neq 0$ for all $l = 0, \dots, p$. Let finally B be a ball for the seminorm $\|\cdot\|_r$ with radius δ such that $B \subset W$, $x + B \subset U$ and $y_i + B \subset V_i$, $i = 1, 2$. We consider (n_k) a sequence of integers such that

$$\begin{aligned} (w_{l+1}^{(1)} \cdots w_{n_k+l}^{(1)})^{-1} e_{n_k+l} &\text{ tends to 0 for all } l = 0, \dots, p \\ \frac{w_{l+1}^{(2)} \cdots w_{n_k+l}^{(2)}}{w_{l+1}^{(1)} \cdots w_{n_k+l}^{(1)}} &\text{ tends to } \frac{y_{2,l}}{y_{1,l}} \text{ for all } l = 0, \dots, p \\ \frac{w_{l+1}^{(2)} \cdots w_{n_k+l}^{(2)}}{w_{l+1}^{(1)} \cdots w_{n_k+l}^{(1)}} &\text{ tends to 1 for all } l = 3p, \dots, 3p + (m-1). \end{aligned}$$

We can assume without loss of generality $n_1 > (m-1)3p$. We then consider, for each $k \geq 1$, the integer J_k defined by

$$n_k + 3p \leq mJ_k < n_k + 3p + m$$

and we set $N_k = J_k - (mJ_k - n_k) \geq 1$ so that $3p \leq J_k - N_k < 3p + m$. We finally define

$$\begin{aligned} \varepsilon &= \max_{0 \leq l \leq p} \left(\frac{\|e_{N_k+l}\|_r}{w_1^{(1)} \cdots w_{n_k+l}^{(1)}} \right)^{\frac{1}{2(m-1)}} \times \min \left(\frac{1}{\|e_{J_k}\|_r}, \frac{1}{(w_1^{(1)} \cdots w_{mJ_k}^{(1)})^{1/m}} \right)^{\frac{1}{2}}, \\ d_l &= \frac{y_{1,l}}{m\varepsilon^{m-1} w_{l+1}^{(1)} \cdots w_{n_k+l}^{(1)}}, \\ u &= x + \sum_{l=0}^p d_l e_{N_k+l} + \varepsilon e_{J_k}. \end{aligned}$$

Let us assume for a while that the following facts are true:

$$\varepsilon \|e_{J_k}\|_r \rightarrow 0 \text{ as } k \rightarrow +\infty, \tag{B.7}$$

$$|d_l| \cdot \|e_{N_k+l}\|_r \rightarrow 0 \text{ as } k \rightarrow +\infty, \text{ for all } l = 0, \dots, p, \tag{B.8}$$

$$\varepsilon^m w_1^{(1)} \cdots w_{mJ_k}^{(1)} \rightarrow 0 \text{ as } k \rightarrow +\infty. \tag{B.9}$$

Then it is clear that u belongs to U for a large integer k and that $B_i^{n_k}(u^n) = 0$ for $i = 1, 2$ and $n < m$, since $\max(\text{supp}(u^n)) \leq (m-1)J_k = n_k - N_k < n_k$. We can also write

$$u^m = z + \sum_{l=0}^p \frac{y_{1,l}}{w_{l+1}^{(1)} \cdots w_{n_k+l}^{(1)}} e_{N_k+(m-1)J_k+l} + \varepsilon^m e_{mJ_k},$$

with

$$\max(\text{supp}(z)) \leq (m-2)J_k + 2N_k + 2p < N_k + (m-1)J_k = n_k.$$

Hence,

$$B_i^{n_k}(u^m) = \sum_{l=0}^p \frac{w_{l+1}^{(i)} \cdots w_{n_k+l}^{(i)}}{w_{l+1}^{(1)} \cdots w_{n_k+l}^{(1)}} y_{1,l} e_l + \varepsilon^m w_{mJ_k-n_k+1}^{(i)} \cdots w_{mJ_k}^{(i)} e_{mJ_k-n_k}.$$

When $i = 1$, the above sum is equal to y_1 ; when $i = 2$, it goes to y_2 . Hence, we have to prove that, for $i = 1, 2$,

$$\varepsilon^m w_{mJ_k-n_k+1}^{(i)} \cdots w_{mJ_k}^{(i)} e_{mJ_k-n_k} \xrightarrow{k \rightarrow +\infty} 0.$$

Since $(mJ_k - n_k)$ is a bounded sequence, this amounts to saying that

$$\varepsilon^m w_{mJ_k-n_k+1}^{(i)} \cdots w_{mJ_k}^{(i)} \xrightarrow{k \rightarrow +\infty} 0.$$

For $i = 1$, this is (B.9). For $i = 2$, we just observe that $mJ_k - n_k \in \{3p, \dots, 3p + m - 1\}$ and we write

$$\frac{w_{mJ_k-n_k+1}^{(2)} \cdots w_{mJ_k}^{(2)}}{w_{mJ_k-n_k+1}^{(1)} \cdots w_{mJ_k}^{(1)}} = \frac{w_{mJ_k-n_k+1}^{(2)} \cdots w_{n_k+(mJ_k-n_k)}^{(2)}}{w_{mJ_k-n_k+1}^{(1)} \cdots w_{n_k+(mJ_k-n_k)}^{(1)}} \xrightarrow{k \rightarrow +\infty} 1.$$

It remains to prove (B.7), (B.8), (B.9).

Let us first prove (B.7). By property (3) from Definition 2.18 and an easy induction on m , there exist $q \geq 1$ and $C > 0$ (depending on r and m) such that, for all $k \geq 1$ and all $l \in \{0, \dots, p\}$,

$$\begin{aligned} \frac{\|e_{N_k+l}\|_r^{\frac{1}{2(m-1)}} \cdot \|e_{J_k}\|_r}{\|e_{J_k}\|_r^{\frac{1}{2}}} &= (\|e_{N_k+l}\|_r \cdot \|e_{J_k}\|_r^{m-1})^{\frac{1}{2(m-1)}} \\ &\leq C \|e_{n_k+l}\|_q^{\frac{1}{2(m-1)}}. \end{aligned}$$

Hence,

$$\varepsilon \|e_{J_k}\|_r \leq C \max_{0 \leq l \leq p} \left(\frac{\|e_{n_k+l}\|_q}{w_1^{(1)} \cdots w_{n_k+l}^{(1)}} \right)^{\frac{1}{2(m-1)}}$$

and this goes to zero as k tends to $+\infty$.

Regarding (B.8), we first write

$$|d_l| \cdot \|e_{N_k+l}\|_r \leq C \left(\frac{\|e_{N_k+l}\|_r}{w_1^{(1)} \cdots w_{n_k+l}^{(1)}} \right)^{1/2} \times \max \left(\|e_{J_k}\|_r, (w_1^{(1)} \cdots w_{mJ_k}^{(1)})^{1/m} \right)^{(m-1)/2}.$$

If the maximum is attained for $\|e_{J_k}\|_r$, then using

$$\begin{aligned} \|e_{N_k+l}\|_r \cdot \|e_{J_k}\|_r^{m-1} &\leq C \|e_{(m-1)J_k+N_k+l}\|_q \\ &\leq C \|e_{n_k+l}\|_q, \end{aligned}$$

we get

$$|d_l| \cdot \|e_{N_k+l}\|_r \leq C \left(\frac{\|e_{n_k+l}\|_q}{w_1^{(1)} \cdots w_{n_k+l}^{(1)}} \right)^{1/2}$$

and this goes to 0. Otherwise, we apply Lemma 2.20 with $n = mJ_k$ and $s = mJ_k - n_k - l$ to get

$$\begin{aligned} (w_1^{(1)} \cdots w_{mJ_k}^{(1)})^{m-1} \|e_{N_k+l}\|_r^m &\leq C (w_1^{(1)} \cdots w_{mJ_k}^{(1)})^{m-1} \|e_{mN_k+ml}\|_q \\ &= C (w_1^{(1)} \cdots w_{mJ_k}^{(1)})^{m-1} \|e_{mJ_k-ms}\|_q \\ &\leq C' (w_1^{(1)} \cdots w_{mJ_k-s}^{(1)})^{m-1} \|e_{mJ_k-s}\|_{q'} \\ &= C' (w_1^{(1)} \cdots w_{n_k+l}^{(1)})^{m-1} \|e_{n_k+l}\|_{q'} \end{aligned}$$

whence

$$|d_j| \cdot \|e_{N_k+l}\|_r \leq C'' \left(\frac{\|e_{n_k+l}\|_{q'}}{w_1^{(1)} \cdots w_{n_k+l}^{(1)}} \right)^{1/2m},$$

which again goes to 0.

Finally, let us prove (B.9). Once more we apply Lemma 2.20 just as above and get, for all $l = 0, \dots, p$,

$$\begin{aligned} &\left(\frac{\|e_{N_k+l}\|_r}{w_1^{(1)} \cdots w_{n_k+l}^{(1)}} \right)^{\frac{m}{2(m-1)}} (w_1^{(1)} \cdots w_{mJ_k}^{(1)})^{\frac{1}{2}} \\ &= \frac{\left(\|e_{N_k+l}\|_r^m (w_1^{(1)} \cdots w_{mJ_k}^{(1)})^{m-1} \right)^{\frac{1}{2(m-1)}}}{(w_1^{(1)} \cdots w_{n_k+l}^{(1)})^{\frac{m}{2(m-1)}}} \\ &\leq C_1 \frac{\left(\|e_{n_k+l}\|_{q'} (w_1^{(1)} \cdots w_{n_k+l}^{(1)})^{m-1} \right)^{\frac{1}{2(m-1)}}}{(w_1^{(1)} \cdots w_{n_k+l}^{(1)})^{\frac{m}{2(m-1)}}} \\ &= C_1 \left(\frac{\|e_{n_k+l}\|_{q'}}{w_1^{(1)} \cdots w_{n_k+l}^{(1)}} \right)^{\frac{1}{2(m-1)}}, \end{aligned}$$

which achieves the proof of (B.9) since the last term goes to zero.

(i) \implies (iii). The proof is exactly the same as the proof of (i) \implies (ii) in Theorem 2.10, restricted to the case $m = 1$. \square

The situation in ω with the convolution product is much easier as the next analogue of Theorem 2.13 shows.

Theorem 2.21 *If $B_1 = B_{w^{(1)}}$ and $B_2 = B_{w^{(2)}}$ are weighted backward shifts on ω endowed with the Cauchy product, then the following are equivalent.*

- (i) B_1 and B_2 are disjointly hypercyclic
- (ii) B_1 and B_2 support a disjoint hypercyclic algebra.
- (iii) The set

$$\left\{ \left(\frac{w_{l+1}^{(2)} \cdots w_{n+l}^{(2)}}{w_{l+1}^{(1)} \cdots w_{n+l}^{(1)}} \right)_{l \geq 0} : n \geq 0 \right\}$$

is dense in ω .

Proof. The implication (ii) \Rightarrow (i) is immediate and the equivalence (i) \Leftrightarrow (iii) follows by Theorem 2.13. Hence, we establish that (iii) \Rightarrow (ii). We will apply Corollary 2.2. Let $m \geq 1$, U, V_1, V_2 be nonempty open subsets of ω , and W be a zero neighbourhood. Let also $p \in \mathbb{N}_0$, $u_0, \dots, u_p, v_{i,0}, \dots, v_{i,p} \in \mathbb{C}$, for $i = 1, 2$, and $\varepsilon > 0$ be such that, for all $x \in \omega$,

$$|x_l - u_l| < \varepsilon \text{ for all } l = 0, \dots, p \text{ implies } x \in U,$$

$$|x_l - v_{i,l}| < \varepsilon \text{ for all } l = 0, \dots, p \text{ implies } x \in V_i, i = 1, 2 \text{ and}$$

$$|x_l| < \varepsilon \text{ for all } l = 0, \dots, p \text{ implies } x \in W.$$

We may also assume that $v_{1,l} \neq 0$ for $l = 0, \dots, p$. Let (n_k) be a subsequence such that $n_0 > 4p$ and

$$\frac{w_{l+1}^{(2)} \cdots w_{n_k+l}^{(2)}}{w_{l+1}^{(1)} \cdots w_{n_k+l}^{(1)}} \rightarrow \frac{v_{2,l}}{v_{1,l}}, \quad \text{for } l = 0, \dots, p.$$

For $k \geq 0$, we set

$$x = \sum_{l=0}^p u_l e_l + z + e_{n_k}$$

where

$$z = z(k) = \sum_{l=0}^p \frac{v_{1,l}}{w_{l+1}^{(1)} \cdots w_{n_k+l}^{(1)}} e_{n_k-3p+l}.$$

We clearly have that $x \in U$. Now, if $N = mn_k - 3p$, then for $i = 1, 2$ and $n \leq m$, the support of $B_i^N(x^n)$ intersects $[0, p]$ only when $n = m$. Specifically, the l -th coordinate of $B_i^N(x^m)$ is the l -th coordinate of $B_i^N(e_{n_k}^{m-1}z)$ which equals to

$$\frac{w_{l+1}^{(i)} \cdots w_{n_k+l}^{(i)}}{w_{l+1}^{(1)} \cdots w_{n_k+l}^{(1)}} v_{1,l}$$

and tends to $v_{i,l}$ as $k \rightarrow \infty$. Hence, for k large, we have that $B_i^N(x^n) \in W$, if $n < m$, and $B_i^N(x^m) \in V_i$, which concludes the proof. \square

3 Common hypercyclic algebras

3.1 How to get a common hypercyclic algebra

We begin this section by a parametrized version of [4, Proposition 2.3]. Let us fix a topological space Λ which is a countable union of compact sets.

Proposition 3.1 *Let Λ be a countable union of compact sets and let $(T_\lambda)_{\lambda \in \Lambda}$ be a family of operators in $\mathcal{L}(X)$ such that the map $(\lambda, x) \mapsto T_\lambda(x)$ is continuous from $\Lambda \times X$ into X . Assume that, for all compact sets $K \subset \Lambda$, for all $I \in \mathcal{P}_f(\mathbb{N}) \setminus \{\emptyset\}$, there exists $m_0 \in I$ such that, for all U, V non-empty open subsets of X , for all neighborhood W of 0, one can find $u \in U$ such that, for all $\lambda \in K$, there exists $N \in \mathbb{N}$ satisfying*

$$\begin{cases} T_\lambda^N(u^m) \in W \text{ for } m \in I \setminus \{m_0\} \\ T_\lambda^N(u^{m_0}) \in V. \end{cases}$$

Then the set of points generating a common hypercyclic algebra for $(T_\lambda)_{\lambda \in \Lambda}$ is residual in X .

Proof. By the Baire Category theorem, it is enough to show the result assuming that $\Lambda = K$ is a compact set. For $V \subset X$ open and non-empty, W a neighborhood of 0, $I \subset \mathcal{P}_f(\mathbb{N}) \setminus \{\emptyset\}$, define

$$\mathcal{A}(I, V, W) = \left\{ u \in X : \forall \lambda \in \Lambda, \exists N \in \mathbb{N}, T_\lambda^N(u^m) \in W \text{ for } m \in I \setminus \{m_0(I)\} \text{ and } T_\lambda^N(u^{m_0(I)}) \in V \right\}$$

(here, $m_0(I)$ is uniquely defined by I using the assumptions of the lemma). The assumption tells us that each set $\mathcal{A}(I, V, W)$ is dense. The compactness of Λ together with the continuity of $(\lambda, x) \mapsto T_\lambda(x)$ ensure that these sets are also open. Fix now (V_k) a basis of open sets of X and (W_l) a basis of open neighborhoods of 0 and let us consider $u \in \bigcap_{I, k, l} \mathcal{A}(I, V_k, W_l)$. Let $P \in \mathbb{C}[z]$ be non-constant, $P(z) = \sum_{m \in I} \hat{P}(m)z^m$ with $\hat{P}(m) \neq 0$ for $m \in I$ and $P(0) = 0$. Let V be any non-empty open subset of X and let k, l be such that $\hat{P}(m_0(I))V_k + (\sum_{m \neq m_0(I)} |\hat{P}(m)|)W_l \subset V$. Since $u \in \mathcal{A}(I, V_k, W_l)$, for all $\lambda \in \Lambda$, there exists $N \in \mathbb{N}$ such that

$$T_\lambda^N(P(u)) = \hat{P}(m_0(I))T_\lambda^N(u^{m_0(I)}) + \sum_{m \neq m_0(I)} \hat{P}(m)T_\lambda^N(u^m) \in V.$$

Hence, $P(u)$ belongs to $\bigcap_{\lambda \in \Lambda} HC(T_\lambda)$ and the proof is complete. \square

3.2 Common hypercyclic algebras for a family of backward shifts - coordinatewise product

We now specialize our study to backward shifts. Thus we fix X a Fréchet sequence algebra under the coordinatewise product with a continuous norm in which $\text{span}(e_i)$ is dense. Let $(w(\lambda))_{\lambda \in \Lambda}$ be a family of admissible weights. For simplicity, we will denote by T_λ the operator $B_{w(\lambda)}$ and by S_{λ^r} the forward shift defined by $S_{\lambda^r}e_i = \frac{1}{w_{i+1}^r(\lambda)}e_{i+1}$. For $x = (x_n) \in \omega$, we will also denote $x^{1/m} = (x_n^{1/m})$.

Proposition 3.1 will be interpreted under the following form.

Lemma 3.2 *Assume that the map $(\lambda, x) \mapsto T_\lambda(x)$ is continuous. Assume that, for all $m_0 \leq m_1 \in \mathbb{N}$, for all $v \in X$ with finite support, for all $M > 0$, for all \mathcal{O} neighborhood of 0, one can find*

- parameters $\lambda_1, \dots, \lambda_q$,
- sets of parameters $\Lambda_1, \dots, \Lambda_q \subset \Lambda$ with $\Lambda \subset \bigcup_{i=1}^q \Lambda_i$,
- integers $N_1, \dots, N_q \in \mathbb{N}$ with $N_1 > M$ and $N_{i+1} - N_i > M$ for all i ,

such that

- (i) $\sum_{i=1}^q S_{\lambda_i^{1/m_0}}^{N_i}(v^{1/m_0}) \in \mathcal{O}$;
- (ii) $\sum_{j=i+1}^q T_\lambda^{N_i} \left(\left(S_{\lambda_j^{1/m_0}}^{N_j} v^{1/m_0} \right)^m \right) \in \mathcal{O}$ for all $m \in [m_0, m_1]$, all $i \in \{1, \dots, q\}$, all $\lambda \in \Lambda_i$;
- (iii) $T_\lambda^{N_i} \left(\left(S_{\lambda_i^{1/m_0}}^{N_i} v^{1/m_0} \right)^m \right) \in \mathcal{O}$ for all $m \in (m_0, m_1]$, all $i \in \{1, \dots, q\}$, all $\lambda \in \Lambda_i$;

(iv) $T_\lambda^{N_i} \left(\left(S_{\lambda_i^{1/m_0}}^{N_i} v^{1/m_0} \right)^{m_0} \right) - v \in \mathcal{O}$ for all $i \in \{1, \dots, q\}$, all $\lambda \in \Lambda_i$.

Then the set of points generating a common hypercyclic algebra for $(T_\lambda)_{\lambda \in \Lambda}$ is residual in X .

Proof. We apply Proposition 3.1. Let $I \subset \mathcal{P}_f(\mathbb{N}) \setminus \{\emptyset\}$. We set $m_0 = \min(I)$, $m_1 = \max(I)$. Let U, V, W be three non-empty open subsets of X with $0 \in W$. Let $x \in U$, $v \in V$ with finite support and let \mathcal{O} be a neighborhood of zero such that $x + \mathcal{O} \subset U$, $v + \mathcal{O} + \mathcal{O} \subset V$ and $\mathcal{O} + \mathcal{O} \subset W$.

Let M be bigger than any integer in the support of x and v . The assumptions of the lemma give us parameters $\lambda_1, \dots, \lambda_q$, sets of parameters $\Lambda_1, \dots, \Lambda_q$ and integers N_1, \dots, N_q . We set

$$u = x + \sum_{i=1}^q S_{\lambda_i^{1/m_0}}^{N_i} (v^{1/m_0})$$

so that assumption (i) says that u belongs to U . Fix $\lambda \in \Lambda$ and let $i \in \{1, \dots, q\}$ be such that $\lambda \in \Lambda_i$. We intend to prove that $T_\lambda^{N_i}(u^m) \in W$ for $m \in I \setminus \{m_0\}$ and $T_\lambda^{N_i}(u^{m_0}) \in V$. Since we are working with the coordinatewise product and by the choice of M , we have

$$u^m = x^m + \sum_{i=1}^q \left(S_{\lambda_i^{1/m_0}}^{N_i} (v^{1/m_0}) \right)^m$$

and

$$T_\lambda^{N_i}(u^m) = T_\lambda^{N_i} \left(\left(S_{\lambda_i^{1/m_0}}^{N_i} (v^{1/m_0}) \right)^m \right) + \sum_{j=i+1}^q T_\lambda^{N_j} \left(\left(S_{\lambda_j^{1/m_0}}^{N_j} (v^{1/m_0}) \right)^m \right).$$

Assumptions (ii), (iii) and (iv) and the choice of v and \mathcal{O} allow us to conclude that $T_\lambda^{N_i}(u^m) \in W$ for $m \in I \setminus \{m_0\}$ and $T_\lambda^{N_i}(u^{m_0}) \in V$. \square

We now give a more concrete application. We first need a definition.

Definition 3.3 Let $(x_{k,\alpha})_{k \geq 1, \alpha \in I}$ be a family of vectors of X . The series $\sum_{k \geq 1} x_{k,\alpha}$, $\alpha \in I$, are said to be *uniformly unconditionally convergent* if, for any neighborhood \mathcal{O} of 0, there exists $K \geq 1$ such that, for all $\alpha \in I$, for all sequences $(\omega_k) \subset [-1, 1]^{\mathbb{N}}$,

$$\sum_{k \geq K} \omega_k x_{k,\alpha} \in \mathcal{O}.$$

Implicitly, in the previous definition, we assume that each series $\sum_k \omega_k x_{k,\alpha}$ is convergent. Of course, if I is finite and each series $\sum_k x_{k,\alpha}$ is unconditionally convergent, then they are uniformly unconditionally convergent.

Theorem 3.4 Let $\Lambda = [a, b]$ be a compact interval and let $(w(\lambda))_{\lambda \in \Lambda}$ be a family of admissible weight sequences. Assume that

- (a) (e_i) is an unconditional basis of X ;
- (b) for each n , the function $\log(w_n)$ is non-decreasing and Lipschitz on Λ , with Lipschitz constant less than or equal to $L_n > 0$.
- (c) for all $p > 0$, for all $m_0 \geq 1$, there exists an increasing sequence of integers (n_k) such that

- the series $\sum_{k \geq 1} \frac{1}{(w_{l+1}(a) \cdots w_{l+n_k}(a))^{1/m_0}} e_{l+n_k}$ are unconditionally convergent for all $l = 0, \dots, p$;
- the series $\sum_{k \geq 1} \frac{1}{w_{l+1}(a) \cdots w_{l+n_{k+i}-n_i}(a)} e_{l+n_{k+i}-n_i}$, $i \geq 1$, $l = 0, \dots, p$ are uniformly unconditionally convergent;
- for all $k \geq 1$ and all $l = 0, \dots, p$, $w_{l+1}(a) \cdots w_{l+n_k}(a) \geq 1$;
- $\sum_{i > 1} 1/C_{n_i} = +\infty$, where $C_n = \sum_{k=1}^n L_k$.

Then the set of points generating a common hypercyclic algebra for $(T_\lambda)_{\lambda \in \Lambda}$ is residual in X .

Proof. Let us first prove that $(\lambda, x) \mapsto T_\lambda(x)$ is continuous. We first observe that the family $(T_\lambda)_{\lambda \in \Lambda}$ is pointwise bounded. Indeed, for all $x \in X$, since all functions (w_n) are non-decreasing, it follows from the unconditional convergence of $\sum_{n \geq 1} w_n(b)x_n e_{n-1}$ that the family $(T_\lambda(x))_{\lambda \in \Lambda}$ is bounded. Hence, by the Banach-Steinhaus theorem, the family $(T_\lambda)_{\lambda \in \Lambda}$ is equicontinuous. Now let us fix some $(\mu, y) \in \Lambda \times X$. For all $(\lambda, x) \in \Lambda \times X$, we write

$$T_\lambda(x) - T_\mu(y) = T_\lambda(x - y) + T_\lambda(y) - T_\mu(y).$$

The equicontinuity of $(T_\lambda)_{\lambda \in \Lambda}$ ensures that, fixing $\varepsilon > 0$, there exists $\delta > 0$ such that, provided $\|x - y\| < \delta$, then $\|T_\lambda(x - y)\| < \varepsilon$ for all $\lambda \in \Lambda$. On the other hand, we write

$$T_\lambda(y) - T_\mu(y) = \sum_{n \geq 1} (w_n(\lambda) - w_n(\mu)) y_n e_{n-1}.$$

Again, the unconditional convergence of $\sum_{n \geq 1} w_n(b) y_n e_{n-1}$ implies that there exists $N \geq 1$ such that, for all $\lambda \in \Lambda$,

$$\left\| \sum_{n > N} (w_n(\lambda) - w_n(\mu)) y_n e_{n-1} \right\| < \varepsilon.$$

Finally, we observe that the continuity of each function w_n implies, provided $|\lambda - \mu|$ is small enough,

$$\left\| \sum_{n=1}^N (w_n(\lambda) - w_n(\mu)) y_n e_{n-1} \right\| < \varepsilon.$$

We now show that the assumptions of Lemma 3.2 are satisfied. Let $v = \sum_{l=0}^p v_l e_l \in X$ with finite support, $m_0 \leq m_1$, $M > 0$ and \mathcal{O} be a neighborhood of 0. By linearity (which is not destroyed by taking powers since we are using the coordinatewise product), it is enough to verify that the assumptions are satisfied for $v = e_l$. Let (n_k) be the sequence given by the assumptions of the theorem, and let $K \geq 1$ be such that, for all $i \geq 0$, for all sequences $(\omega_k) \subset [-1, 1]^{\mathbb{N}}$,

$$\sum_{k \geq K} \frac{\omega_k}{w_{l+1}(a) \cdots w_{l+n_{i+k}-n_i}(a)} e_{l+n_{i+k}-n_i} \in \mathcal{O} \quad (\text{B.10})$$

$$\sum_{k \geq K} \frac{\omega_k}{(w_{l+1}(a) \cdots w_{l+n_k}(a))^{1/m_0}} e_{l+n_k} \in \mathcal{O}. \quad (\text{B.11})$$

Without loss of generality, we may assume that $K \geq M$. We then set $N_i = n_{K_i}$. Since the sequence (C_n) is nondecreasing, we still have

$$\sum_{i \geq 1} \frac{1}{C_{N_i+p}} = +\infty.$$

Let $\tau > 0$ be a small real number (a precised condition on τ will be fixed later) and define $\lambda_1 = a$, $\lambda_{i+1} = \lambda_i + \frac{\tau}{C_{N_i+p}}$. Let q be the first integer such that $\lambda_{q+1} \geq b$ and define $\Lambda_i = [\lambda_i, \lambda_{i+1})$ for $i = 1, \dots, q-1$, $\Lambda_q = [\lambda_q, b]$. Regarding (i) of Lemma 3.2, we write

$$\sum_{i=1}^q S_{\lambda_i^{1/m_0}}^{N_i}(e_l) = \sum_{i=1}^q \frac{1}{(w_{l+1}(\lambda_i) \cdots w_{l+N_i}(\lambda_i))^{1/m_0}} e_{l+N_i}. \quad (\text{B.12})$$

Since each (w_n) is nondecreasing, using (B.11), we get that (i) is true.

Let now $i \in \{1, \dots, q\}$, $\lambda \in \Lambda_i$ and $m \in [m_0, m_1]$. Then

$$\sum_{j=i+1}^q T_{\lambda}^{N_i} \left(\left(S_{\lambda_j^{1/m_0}}^{N_j}(e_l) \right)^m \right) = \sum_{j=i+1}^q \frac{w_{l+N_j-N_i+1}(\lambda) \cdots w_{l+N_j}(\lambda)}{(w_{l+1}(\lambda_j) \cdots w_{l+N_j}(\lambda_j))^{m/m_0}} e_{l+N_j-N_i}$$

Now, since

$$\begin{aligned} (w_{l+1}(\lambda_j) \cdots w_{l+N_j}(\lambda_j))^{m/m_0} &\geq w_{l+1}(\lambda_j) \cdots w_{l+N_j}(\lambda_j) \\ &\geq w_{l+1}(a) \cdots w_{l+N_j-N_i}(a) w_{l+N_j-N_i+1}(\lambda_j) \cdots w_{l+N_j}(\lambda_j) \end{aligned}$$

and

$$w_{l+N_j-N_i+1}(\lambda) \cdots w_{l+N_j}(\lambda) \leq w_{l+N_j-N_i+1}(\lambda_j) \cdots w_{l+N_j}(\lambda_j)$$

we get

$$\sum_{j=i+1}^q T_{\lambda}^{N_i} \left(\left(S_{\lambda_j^{1/m_0}}^{N_j}(e_l) \right)^m \right) = \sum_{j=i+1}^q \frac{\alpha_{i,j}(\lambda)}{w_{l+1}(a) \cdots w_{l+N_j-N_i}(a)} e_{l+N_j-N_i}$$

where $\alpha_{i,j}(\lambda) \in [0, 1]$. Hence, by (B.10), (ii) of Lemma 3.2 is satisfied. Finally, we have for $i \in \{1, \dots, q\}$ and $\lambda \in \Lambda_i$,

$$T_{\lambda}^{N_i} \left(\left(S_{\lambda_i^{1/m_0}}^{N_i}(e_l) \right)^m \right) = \frac{w_{l+1}(\lambda) \cdots w_{l+N_i}(\lambda)}{(w_{l+1}(\lambda_i) \cdots w_{l+N_i}(\lambda_i))^{m/m_0}} e_l.$$

If $m = m_0$, we just write

$$\begin{aligned} \left| \frac{w_{l+1}(\lambda) \cdots w_{l+N_i}(\lambda)}{w_{l+1}(\lambda_i) \cdots w_{l+N_i}(\lambda_i)} - 1 \right| &\leq \left| \exp \left(\sum_{k=1}^{N_i} (\log w_{l+k}(\lambda) - \log w_{l+k}(\lambda_i)) \right) - 1 \right| \\ &\leq \left| \exp \left(\sum_{k=1}^{N_i} L_{l+k} |\lambda - \lambda_i| \right) - 1 \right| \\ &\leq |\exp(C_{N_i+p} |\lambda - \lambda_i|) - 1| \\ &\leq |\exp(\tau) - 1|. \end{aligned}$$

Hence, provided τ is small enough, (iv) is satisfied and we may also ensure that

$$\left| \frac{w_{l+1}(\lambda) \cdots w_{l+N_i}(\lambda)}{w_{l+1}(\lambda_i) \cdots w_{l+N_i}(\lambda_i)} \right| \leq 2.$$

Hence, if $m > m_0$, using again the monotonicity of each w_n , for all $i \in \{1, \dots, q\}$ and all $\lambda \in \Lambda_i$,

$$\frac{w_{l+1}(\lambda) \cdots w_{l+N_i}(\lambda)}{(w_{l+1}(\lambda_i) \cdots w_{l+N_i}(\lambda_i))^{m/m_0}} \leq \frac{2}{(w_{l+1}(a) \cdots w_{l+N_i}(a))^{1/m_0}}.$$

Applying a last time (B.11) (with only one nonzero ω_k now), we get that (iii) is satisfied, which closes the proof of Theorem 3.4. \square

Remark 3.5 If we are only interested in the existence of a common hypercyclic vector, we have a similar statement by verifying the assumptions of Theorem 3.4 only for $m_0 = 1$.

Before giving specific examples, let us explain the dependance between the behaviour of (C_n) and the choice of (n_k) . We work on $X = \ell^1$ and we assume first that $w_n(\lambda) = \lambda$ for all $n \geq 1$, $\lambda > 1$. In that case, $C_n \sim n$ and the sequence n_i cannot grow too fast in order to ensure the divergence of $\sum_i 1/C_{n_i}$. It is then natural to set $n_i = Ni$ for some N , and this will be enough to ensure the uniform unconditional convergence of the series involved in Theorem 3.4, because the product $w_1(\lambda) \cdots w_n(\lambda) = \lambda^n$ grows very fast.

Assume now that $w_n(\lambda) = 1 + \lambda/n$. In that case, $L_n = 1/n$ and C_n behaves like $\log n$. This means that we may choose a sequence (n_i) going very fast to $+\infty$, like $n_i = 2^i$. This will be necessary to ensure uniform unconditional convergence, since now $w_1(\lambda) \cdots w_n(\lambda) \sim n^\lambda$, which grows much slowly, especially if we allow λ to be close to 0.

Let us proceed with the details. We first give a result which should be thought as a version for hypercyclic algebras of [2, Corollary 4.10].

Theorem 3.6 *Let $\Lambda \subset \mathbb{R}$ be an interval and let $(w(\lambda))_{\lambda \in \Lambda}$ be a family of admissible weight sequences. Assume that*

- (a) (e_i) is an unconditional basis of X ;
- (b) all functions $\log(w_n)$ are non-decreasing and are Lipschitz on compact sets with uniformly bounded Lipschitz constants;
- (c) for all $m \geq 1$ and all $\lambda \in \Lambda$, the series $\sum_n (w_1(\lambda) \cdots w_n(\lambda))^{-1/m} e_n$ converges.

Then $\bigcap_{\lambda \in \Lambda} HC(T_\lambda) \cup \{0\}$ contains a non-trivial algebra.

Proof. We may assume that $\Lambda = [a, b]$ is a compact interval. Let $C > 0$ be such that all functions $\log(w_n)$ are C -Lipschitz. We then set $n_k = k$, $k \geq 0$, and observe that, for all $i \geq 0$,

$$\sum_{k \geq 1} \frac{1}{(w_{l+1}(a) \cdots w_{l+n_{k+i}-n_i}(a))^{1/m}} e_{l+n_{k+i}-n_i} = \sum_{k \geq 1} \frac{1}{(w_{l+1}(a) \cdots w_{l+k}(a))^{1/m}} e_{l+k},$$

which shows that the assumptions on unconditional uniform convergence are satisfied. □

Corollary 3.7 *Let X be a Fréchet sequence algebra for the coordinatewise product with a continuous norm and such that (e_i) is an unconditional basis of X . Let $w = (w_n)$ be an admissible weight sequence and define*

$$\lambda_w = \inf \left\{ \lambda > 0 : \sum_n \lambda^{-n/m} (w_1 \cdots w_n)^{-1/m} e_n \text{ converges for all } m > 0 \right\}.$$

Then $\bigcap_{\lambda > \lambda_w} HC(\lambda B_w) \cup \{0\}$ contains a non-trivial algebra.

This corollary includes the case of the families $(\lambda B)_{\lambda > 1}$ on ℓ_p or c_0 and $(\lambda D)_{\lambda > 0}$ on $H(\mathbb{C})$. For families $(\lambda B)_{\lambda > 0}$ on ω the same result holds with an even easier proof, although this space does not admit a continuous norm.

Theorem 3.8 *The family $(\lambda B)_{\lambda>0}$ admits a common hypercyclic algebra on ω for the coordinatewise product.*

Proof. It is enough to prove that the sets of vectors generating a common hypercyclic algebra for $(\lambda B)_{0<\lambda<1}$ and $(\lambda B)_{\lambda>1}$ on ω with the coordinatewise product are G_δ dense. Since we know that B itself also has a G_δ dense set of vectors generating a hypercyclic algebra, we can gather the three cases $0 < \lambda < 1$, $\lambda = 1$ and $\lambda > 1$ by a Baire argument (a countable intersection of comeager sets is comeager).

The proof for the family $(\lambda B)_{\lambda>1}$ imitates that of Theorem 3.4 and is left to the reader. Let us consider the family $(\lambda B)_{0<\lambda<1}$. Here we need another version of Lemma 3.2 obtained by an application of Proposition 3.1 with $m = \max(I)$. The statement is the same but with m_1 in the place of m_0 . Let T_λ be λB and S_λ be $\frac{1}{\lambda}F$, where F is the unweighted forward shift. Let $\Lambda = [a, b] \subset (0, 1)$, let $m_0 \leq m_1$ in \mathbb{N} and let $v \in \omega$. Let $M > 0$ and finally let \mathcal{O} be an open neighborhood of 0 in ω . There exist $\varepsilon > 0$ and $p \in \mathbb{N}$ such that

$$|x_l| < \varepsilon \text{ for } l = 0, \dots, p \implies (x_n)_n \in \mathcal{O}.$$

We define $K_v = \max_{l=0, \dots, p} |v_l|$ and fix $\tau > 0$ satisfying, for all $m \in [m_0, m_1]$,

$$|\exp(\tau) - 1| K_v^{m/m_1} < \varepsilon.$$

We also fix a big integer $N > \max(p, M)$ (more conditions will be given later) and we define $N_i = iN$ for all $i \geq 1$, $\lambda_1 = a$ and $\lambda_i = \lambda_{i-1} + \frac{\tau}{N_i}$ for all $i > 1$. Since $\sum \frac{\tau}{N_i} = +\infty$, there exists $q \in \mathbb{N}$ such that $\lambda_{q+1} > b$. Let $\Lambda_i = [\lambda_i, \lambda_{i+1})$ for $i = 1, \dots, q$.

By construction we have $\Lambda \subset \bigcup_{i=1}^q \Lambda_i$, $N_1 > M$ and $N_{i+1} - N_i > M$. Also, since $N > p$, we get immediately that

$$\sum_{i=1}^q S_{\lambda_i^{1/m_1}}^{N_i} v^{1/m_1} \in \mathcal{O}$$

and, for all $m \in [m_0, m_1]$ and all $\lambda \in \Lambda_i, i = 1, \dots, q$,

$$\sum_{j=i+1}^q T_\lambda^{N_j} \left(S_{\lambda_j^{1/m_1}}^{N_j} v^{1/m_1} \right)^m \in \mathcal{O}.$$

Now, let $m \in [m_0, m_1]$ and $\lambda \in \Lambda_i$ for some $i = 1, \dots, q$. We write

$$T_\lambda^{N_i} \left(S_{\lambda_i^{1/m_1}}^{N_i} v^{1/m_1} \right)^m = \left(\frac{\lambda}{\lambda_i^{m/m_1}} \right)^{N_i} v^{m/m_1}.$$

If $m = m_1$, then for all $l = 0, \dots, p$,

$$\begin{aligned} \left| \left(\frac{\lambda}{\lambda_i^{m/m_1}} \right)^{N_i} v_l^{m/m_1} - v_l \right| &= \left| \left(\frac{\lambda}{\lambda_i} \right)^{N_i} - 1 \right| |v_l| \\ &\leq \left(\exp(N_i |\lambda - \lambda_i|) - 1 \right) K_v \\ &\leq \left(\exp(N_i \frac{\tau}{N_i}) - 1 \right) K_v \\ &\leq (\exp(\tau) - 1) K_v \end{aligned}$$

$$< \varepsilon,$$

that is, $T_\lambda^{N_i} \left(S_{\lambda_i^{1/m_1}}^{N_i} v^{1/m_1} \right)^m - v \in \mathcal{O}$. Moreover, if $\tau > 0$ is small enough, we have

$$\left(\frac{\lambda}{\lambda_i} \right)^{N_i} \leq 2.$$

If $m \in [m_0, m_1)$, then for all $l = 0, \dots, p$,

$$\begin{aligned} \left| \left(\frac{\lambda}{\lambda_i^{m/m_1}} \right)^{N_i} v_l^{m/m_1} \right| &= \left(\frac{\lambda}{\lambda_i^{m/m_1}} \right)^{N_i} |v_l^{m/m_1}| \\ &\leq \left(\frac{\lambda}{\lambda_i} \right)^{N_i} (\lambda_i^{1-\frac{m}{m_1}})^{N_i} |K_v^{m/m_1}| \\ &\leq 2b^{N_i(1-\frac{m}{m_1})} |K_v^{m/m_1}|. \end{aligned}$$

Hence, provided N is big enough, this is less than ε since $0 < b < 1$ and $1 - \frac{m}{m_1} \geq \frac{1}{m_1}$. \square

We can also get examples of families which are not multiples of a single operator.

Example 3.9 Let $X = c_0$ and consider $w_n(\lambda) = 1 + \frac{\lambda}{n}$, $\lambda > 0$. Then $\bigcap_{\lambda > 0} HC(B_{w(\lambda)}) \cup \{0\}$ contains a non-trivial algebra.

To include the ℓ_p -case, we need to change the choice of (n_k) .

Example 3.10 Let $X = \ell_p$, $1 \leq p < \infty$ and consider $w_n(\lambda) = 1 + \frac{\lambda}{n}$, $\lambda > 0$. Then $\bigcap_{\lambda > 0} HC(B_{w(\lambda)}) \cup \{0\}$ contains a non-trivial algebra.

Proof. Again, we need only to consider the case $\Lambda = [a, b]$ with $a > 0$. We apply Theorem 3.4 with $L_k = 1/k$ and $n_k = 2^k$. The uniform unconditional convergence of the involved series is ensured by the inequalities

$$w_1(a) \cdots w_n(a) \geq \exp(c \log n) = n^c \text{ for some } c > 0$$

and

$$\begin{aligned} &\left\| \sum_{k \geq K} \frac{1}{(w_{l+1}(a) \cdots w_{l+n_{k+i}-n_i}(a))^{1/m} e^{l+n_{k+i}-n_i}} \right\| \\ &\leq \sum_{k \geq K} \frac{1}{(w_{l+1}(a) \cdots w_{l+2^{k+i}-2^i}(a))^{1/m}} \\ &\leq C_l \sum_{k \geq K} \frac{1}{2^{ic/m} (2^k - 1)^{c/m}} \\ &\leq C_l \sum_{k \geq K} \frac{1}{(2^k - 1)^{c/m}}. \end{aligned}$$

\square

As mentioned in the introduction, this last result is interesting even for the existence of a single hypercyclic vector, which was only known for $\lambda > 1/p$. The novelty here is that, since the weight varies very slowly, we may take a sequence (n_k) increasing much more quickly to $+\infty$ without reducing the size of the intervals I_k such that, for all $\lambda, \mu \in I_k$, $T_\lambda^{n_k} S_\mu^{n_k} e_0$ is close to e_0 . Allowing (n_k) to be large helps to ensure the smallness of the sum appearing in (B.12), even if the product $w_1(\lambda) \cdots w_n(\lambda)$ is not very large.

3.3 Common hypercyclic algebras for a family of backward shifts - Cauchy product

In this subsection we are dealing with the question of the existence of common hypercyclic algebras, for families of weighted backward shifts when the underlying Fréchet sequence algebra is endowed with the Cauchy product. We will establish a general criterion for the existence of such algebras which encompasses the case of the multiples of the backward shift on ℓ_1 and of the multiples of D on $H(\mathbb{C})$ as particular cases. Again, our main tool will be Proposition 3.1. Nevertheless, we will not be able to devise a partition of each $[a, b] \subset \Lambda$ as in the proof of Theorem 3.4. We will only succeed to do this for intervals $[a, \kappa a]$, where κ will be independent of a . To come back to the assumptions of Proposition 3.1, we will need an auxiliary lemma. For a family of admissible weighted sequences $(w(\lambda))$ we will denote as usual by $T_\lambda = B_{w(\lambda)}$ the corresponding weighted backward shift.

Lemma 3.11 *Let X be a Fréchet sequence algebra under the Cauchy product in which $\text{span}(e_i)$ is dense and let $\Lambda \subset \mathbb{R}$ be an interval. Let $(w(\lambda))_{\lambda \in \Lambda}$ be a family of admissible weights and assume that the map $(\lambda, x) \mapsto T_\lambda x$ is continuous. Suppose that for all $[a_0, b_0] \subset \Lambda$, all $m \geq 1$ and all non-empty open set $V \subset X$, there exists $\kappa := \kappa(m, V, a_0, b_0) > 1$ such that, for all $a \in [a_0, b_0]$, for all \mathcal{O} open neighborhood of zero and for all $M_0 \in \mathbb{N}$, one can find $u \in X$ and $M_1 \in \mathbb{N}$ satisfying*

(i) u has finite support and $u \in \mathcal{O}$;

(ii) for each $\lambda \in [a, \kappa a] \cap [a_0, b_0]$ there is $N \leq M_1$ such that, for all $x \in X$ with $\text{supp}(x) \subset [0, M_0]$, $(T_\lambda)^N(u+x)^n = 0$ for $n \leq m-1$ and $(T_\lambda)^N(u+x)^m \in V$.

Then $(T_\lambda)_{\lambda \in \Lambda}$ admits a common hypercyclic algebra.

Proof. We are going to apply Proposition 3.1. So let $K = [a_0, b_0] \subset \Lambda$, let $I \in \mathcal{P}_f(\mathbb{N}) \setminus \{\emptyset\}$, set $m = \max(I)$ and let $U, V, W \subset X$ be non-empty open subsets, with $0 \in W$. We begin by considering κ given by the hypothesis. Let s be the first positive integer satisfying $a_0 \kappa^s \geq b$ and define $a_i = \kappa^i a_0$, for $i = 1, \dots, s-1$, and $a_s = b_0$. We will use the assumptions of the lemma to construct a sequence of pairs $(u(1), N_1), \dots, (u(s), N_s)$ satisfying, for all $i = 1, \dots, s$,

- $u(i) \in U$;
- for all $\lambda \in [a_0, a_i]$, there is $N \leq N_i$ such that $(T_\lambda)^N(u(i)^n) \in W$, for $n < m$, and $(T_\lambda)^N(u(i)^m) \in V$.

This construction being done, it is clear that $u := u(s)$ is the desired point we are looking for in order to apply Proposition 3.1.

We fix $u(0) \in U$ with finite support and we let $M(0) = \max(\text{supp}(u(0)))$ and \mathcal{O}_0 such that $u(0) + \mathcal{O}_0 \subset U$. For $a = a_0$, $\mathcal{O} = \mathcal{O}_0$ and $M_0 = M(0)$ in the lemma we find $u \in X$ and $M_1 =: N_1$ such that u has finite support, $u \in \mathcal{O}_0$ and, for each $\lambda \in [a_0, a_1]$ there is $N \leq N_1$ such that, for all $x \in X$ with $\text{supp}(x) \subset [0, M_0]$, $(T_\lambda)^N(u+x)^n = 0$ for $n \leq m-1$ and $(T_\lambda)^N(u+x)^m \in V$. In particular, for $x = u(0)$ we have $u(1) := u(0) + u \in U$ and, for all $\lambda \in [a_0, a_1]$, there is $N \leq N_1$ such that $(T_\lambda)^N(u(1)^n) = 0 \in W$, for $n < m$, and $(T_\lambda)^N(u(1)^m) \in V$.

Assume that $u(i)$ has been defined and let us define $u(i+1)$. Instead of applying directly the lemma we need to adjust things so that the corresponding interval is not just $[a_i, a_{i+1}]$ but $[a_0, a_{i+1}]$. By the inductive properties of $u(i)$, by the continuity of $(\lambda, x) \mapsto T_\lambda x$ and of the product of X and by the compactness of $[a_0, a_i]$, we can find \mathcal{O}_i an open neighborhood of zero that

- $u(i) + \mathcal{O}_i \subset U$;
- for all $y \in u(i) + \mathcal{O}_i$ and all $\lambda \in [a_0, a_i]$, there is $N \leq N_i$ such that $(T_\lambda)^N(y^n) \in W$, for $n < m$, and $(T_\lambda)^N(y^m) \in V$.

Now we apply the lemma with $a = a_i$, $M_0 = \max(\text{supp}(u(i)))$ and $\mathcal{O} = \mathcal{O}_i$ and we find $u \in X$ and M_1 such that u has finite support, $u \in \mathcal{O}_i$ and, for each $\lambda \in [a_i, a_{i+1}]$ there is $N \leq M_1$ such that, for all $x \in X$ with $\text{supp}(x) \subset [0, M_0]$, $(T_\lambda)^N(u+x)^n = 0$ for $n \leq m-1$ and $(T_\lambda)^N(u+x)^m \in V$. We set $N_{i+1} = \max(M_1, N_i)$ and $u(i+1) = u(i) + u$. Applying the previous result with $x = u(i)$ when $\lambda \in [a_i, a_{i+1}]$ or using that $u(i+1) \in u(i) + \mathcal{O}_i$ when $\lambda \in [a_0, a_{i+1}]$, we finally get that for all $\lambda \in [a_0, a_{i+1}]$, there is $N \leq N_{i+1}$ such that $(T_\lambda)^N(u(i+1)^n) = 0 \in W$, for $n < m$, and $(T_\lambda)^N(u(i+1)^m) \in V$. \square

Now we are ready to prove the main theorem of this subsection.

Theorem 3.12 *Let $\Lambda \subset \mathbb{R}$ be an interval, let X be a regular Fréchet sequence algebra under the Cauchy product and let $(w(\lambda))_{\lambda \in \Lambda}$ be a family of admissible weighted sequences such that all functions $\log(w_n)$ are non-decreasing and Lipschitz on compact sets with uniformly bounded Lipschitz constants. Suppose that (e_n) is an unconditional basis of X and that*

(a) *for all $\lambda \in \Lambda$,*

$$\sum_{n=1}^{\infty} \frac{1}{w_1(\lambda) \cdots w_n(\lambda)} e_n \in X;$$

(b) *for all $m \in \mathbb{N}$ and all $[a_0, b_0] \subset \Lambda$ there exist $c \in (0, 1)$ and $\kappa_0 > 1$ such that*

$$\lim_{N \rightarrow \infty} \sum_{n=cN}^N \frac{[w_1(\kappa_0 a) \cdots w_{mN}(\kappa_0 a)]^{\frac{m-1}{m}}}{w_1(a) \cdots w_{(m-1)N+n}(a)} e_n = 0, \text{ for all } a \in [a_0, b_0]. \quad (\text{B.13})$$

Then $(T_\lambda)_{\lambda \in \Lambda}$ admits a common hypercyclic algebra.

Proof. First we observe that an application of the Banach-Steinhaus theorem as in the proof of Theorem 3.4 yields that the map $(\lambda, x) \mapsto T_\lambda x$ is continuous. Given $[a_0, b_0] \subset \Lambda$, $m \geq 1$ and

$V \subset X$ open and non-empty, from condition (b) there exist $c \in (0, 1) \cap \mathbb{Q}$ and $\kappa_0 > 1$ such that (B.13) holds. Since the functions $\lambda \mapsto w_n(\lambda)$ are nondecreasing and by unconditionality, (B.13) holds for all $\kappa \in (1, \kappa_0)$. Fix $d \in (c, \frac{c+1}{2}) \cap \mathbb{Q}$, take $y = \sum_{j=0}^p y_j e_j \in V$ and find $\rho \geq 1$ and $\eta > 0$ so that $y + B \subset V$, where B is the ball for the seminorm $\|\cdot\|_\rho$ centered at the origin and with radius 3η . By unconditionality of the basis (e_n) , there exist $\rho_0 \geq 1$ and $C_{\rho, \rho_0} > 0$ such that, for all $x = \sum_n x_n e_n$ and all $\theta \in \ell_\infty$,

$$\left\| \sum_{n=0}^{+\infty} \theta_n x_n e_n \right\|_\rho \leq C_{\rho, \rho_0} \|\theta\|_\infty \left\| \sum_{n=0}^{+\infty} x_n e_n \right\|_{\rho_0}. \quad (\text{B.14})$$

Take $q > p$ big enough so that

$$\left\| \sum_{n \geq q} \frac{z_n}{w_1(a_0) \cdots w_n(a_0)} e_n \right\|_\rho < \eta, \quad (\text{B.15})$$

for all $z \in c_{00}$ with $\|z\|_\infty \leq \|y\|_\infty (\max(1, w_1(b_0), \dots, w_p(b_0)))^p$ (this is possible by condition (a)). We also fix $M > 0$ so that all the functions $\log(w_n)$ are M -Lipschitz on $[a_0, b_0]$. By continuity of exp we can find $\gamma > 0$ so that

$$|\exp(x) - 1| \leq \frac{\eta}{C_{\rho, \rho_0} \|y\|_{\rho_0}}, \quad \text{for all } x \in \mathbb{R} \text{ with } |x| \leq \gamma. \quad (\text{B.16})$$

We then choose $\tau > 0$ and $\kappa \in (1, \kappa_0)$ such that

$$\tau < \frac{\gamma}{M}, \quad (\kappa - 1)b_0 \leq \tau \frac{d - c}{(m - 1 + d)q}. \quad (\text{B.17})$$

Now let $a \in [a_0, b_0]$, \mathcal{O} an open neighborhood of zero and $M_0 \in \mathbb{N}$ be arbitrary. There exist $\sigma \geq 1$ and $\delta > 0$ such that $B_{\|\cdot\|_\sigma}(0, \delta) \subset \mathcal{O}$. Take $r > 0$ large enough such that $cr > M_0$ and $cr, dr \in \mathbb{N}$. Later on we will request more conditions on the size of r . For $j = cr, \dots, dr$, let λ_j be defined inductively by $\lambda_{cr} = a$ and $\lambda_{j+1} = \lambda_j + \frac{\tau}{(m-1)qr + qj}$. We notice that

$$\begin{aligned} \lambda_{dr} &= a + \sum_{j=cr}^{dr-1} \frac{\tau}{(m-1)qr + qj} \\ &\geq a + \tau \frac{d - c}{(m-1 + d)q} \\ &\geq \kappa a. \end{aligned}$$

Define

$$u := u(r) = \sum_{j=cr}^{dr-1} \sum_{l=0}^p d_{j,l} e_{qj+l} + \varepsilon e_{qr},$$

where

$$\begin{aligned} d_{j,l} &:= \frac{y_l}{m \varepsilon^{m-1} w_{l+1}(\lambda_{j+1}) \cdots w_{(m-1)qr + qj + l}(\lambda_{j+1})}, \\ \varepsilon &= \left(\frac{1}{w_1(\kappa a) \cdots w_{mqr}(\kappa a)} \right)^{\frac{1}{m}}. \end{aligned}$$

Let also $M_1 := mqr$. We shall show that $\|u\|_\sigma < \delta$ if r is big enough. For the double sum we have

$$\left\| \sum_{j=cr}^{dr-1} \sum_{l=0}^p d_{j,l} e_{qj+l} \right\|_\sigma = \left\| \sum_{j=cr}^{dr-1} \sum_{l=0}^p \frac{y_l(w_1(\kappa a) \cdots w_{mqr}(\kappa a))^{\frac{m-1}{m}}}{m w_{l+1}(\lambda_{j+1}) \cdots w_{(m-1)qr+qj+l}(\lambda_{j+1})} e_{qj+l} \right\|_\sigma.$$

Now, we write the quotient of weights as

$$\frac{w_1(a) \cdots w_{(m-1)qr+qj+l}(a)}{w_{l+1}(\lambda_{j+1}) \cdots w_{(m-1)qr+qj+l}(\lambda_{j+1})} \times \frac{(w_1(\kappa a) \cdots w_{mqr}(\kappa a))^{\frac{m-1}{m}}}{w_1(a) \cdots w_{(m-1)qr+qj+l}(a)}.$$

Using unconditionality of the basis (e_n) as in (B.14), there exist $\sigma_0 \geq \sigma$ and $C_{\sigma, \sigma_0} > 0$ depending only on σ such that

$$\begin{aligned} & \left\| \sum_{j=cr}^{dr-1} \sum_{l=0}^p d_{j,l} e_{qj+l} \right\|_{\sigma_0} \\ & \leq C_{\sigma, \sigma_0} \left\| \sum_{j=cr}^{dr-1} \sum_{l=0}^p \frac{y_l w_1(a) \cdots w_l(a)}{m} \frac{(w_1(\kappa a) \cdots w_{mqr}(\kappa a))^{\frac{m-1}{m}}}{w_1(a) \cdots w_{(m-1)qr+qj+l}(a)} e_{qj+l} \right\|_{\sigma_0} \\ & \leq C_{\sigma, \sigma_0} \left\| \sum_{n=cqr}^{qr} z_n \frac{(w_1(\kappa a) \cdots w_{mqr}(\kappa a))^{\frac{m-1}{m}}}{w_1(a) \cdots w_{(m-1)qr+n}(a)} e_n \right\|_{\sigma_0}, \end{aligned}$$

for some eventually null sequence $z = (z_n)$ satisfying

$$\|z\|_\infty \leq \|y\|_\infty (\max(1, w_1(b_0), \dots, w_p(b_0))^p) / m.$$

Therefore, by assumption (b) and the unconditionality of (e_n) , we conclude that the sum $\sum_{j=cr}^{dr-1} \sum_{l=0}^p d_{j,l} e_{qj+l}$ converges to 0 when $r \rightarrow +\infty$. On the other hand, for the term εe_{qr} , since X is regular, we find $\sigma_1 \geq \sigma$ and $C_{\sigma, \sigma_1} > 0$ depending only on σ and m such that

$$\begin{aligned} \|\varepsilon e_{qr}\|_\sigma &= \left\| \frac{1}{[w_1(\kappa a) \cdots w_{mqr}(\kappa a)]^{\frac{1}{m}}} e_{qr} \right\|_\sigma \\ &= \left[\frac{1}{w_1(\kappa a) \cdots w_{mqr}(\kappa a)} \|e_{qr}\|_\sigma^m \right]^{\frac{1}{m}} \\ &\leq C_{\sigma, \sigma_1} \left[\frac{1}{w_1(\kappa a) \cdots w_{mqr}(\kappa a)} \|e_{mqr}\|_{\sigma_1}^m \right]^{\frac{1}{m}} \\ &\leq C_{\sigma, \sigma_1} \left\| \frac{1}{w_1(\kappa a) \cdots w_{mqr}(\kappa a)} e_{mqr} \right\|_{\sigma_1}^{\frac{1}{m}}, \end{aligned}$$

and this converges to 0 as $r \rightarrow +\infty$. This shows that condition (i) of Lemma 3.11 is satisfied if r is big enough.

Now, taking $\lambda \in [a, \kappa a]$, there exists $k \in \{cr, \dots, dr-1\}$ such that $\lambda \in [\lambda_k, \lambda_{k+1}]$. We choose $N = (m-1)qr + qk \leq M_1$ and take $x \in X$ with $\text{supp}(x) \subset [0, M_0]$. Clearly we have $\max(\text{supp}((u+x)^n)) \leq (m-1)qr < N$ for all $n = 1, \dots, m-1$, which implies $(T_\lambda)^N (u+x)^n = 0$. On the other hand,

$$(u+x)^m = x^m + \sum_{j=cr}^{dr-1} \sum_{l=0}^p m \varepsilon^{m-1} d_{j,l} e_{(m-1)qr+qj+l} + \varepsilon^m e_{mqr},$$

with $\text{supp}(x') \subset [0, (m-2)qr+2(dqr+p)] \cup [0, (m-1)qr+M_0]$. It follows that $\max(\text{supp}(x')) < N$ if r is big enough. We also have that $(m-1)qr+qj+p < N$ if $j < k$. Hence,

$$\begin{aligned}
 & (T_\lambda)^N(u+x)^m \\
 &= \sum_{l=0}^p m\varepsilon^{m-1} d_{k,l} w_{(m-1)qr+qk-N+l+1}(\lambda) \cdots w_{(m-1)qr+qk+l}(\lambda) e_{(m-1)qr+qk-N+l} \\
 & \quad + \sum_{j=k+1}^{dr-1} \sum_{l=0}^p m\varepsilon^{m-1} d_{j,l} w_{(m-1)qr+qj-N+l+1}(\lambda) \cdots w_{(m-1)qr+qj+l}(\lambda) e_{(m-1)qr+qj-N+l} \\
 & \quad + \varepsilon^m w_{mqr-N+1}(\lambda) \cdots w_{mqr}(\lambda) e_{mqr-N} \\
 &=: P_1 + P_2 + P_3,
 \end{aligned}$$

where, after substituting $d_{j,l}$, N and ε by their values,

$$\begin{aligned}
 P_1 &= \sum_{l=0}^p \frac{w_{l+1}(\lambda) \cdots w_{(m-1)qr+qk+l}(\lambda)}{w_{l+1}(\lambda_{k+1}) \cdots w_{(m-1)qr+qk+l}(\lambda_{k+1})} y_l e_l \\
 P_2 &= \sum_{j=k+1}^{dr-1} \sum_{l=0}^p \frac{w_{(j-k)q+l+1}(\lambda) \cdots w_{(m-1)qr+qj+l}(\lambda)}{w_{l+1}(\lambda_{j+1}) \cdots w_{(m-1)qr+qj+l}(\lambda_{j+1})} y_l e_{(j-k)q+l} \\
 P_3 &= \frac{w_{(r-k)q+1}(\lambda) \cdots w_{mqr}(\lambda)}{w_1(\kappa a) \cdots w_{mqr}(\kappa a)} e_{(r-k)q}.
 \end{aligned}$$

From the definition of $\lambda_{cr}, \dots, \lambda_{dr}$, by the Lipschitz condition on the functions $\log w_n$ and by (B.17), we get

$$\begin{aligned}
 & \left| \sum_{j=1}^{(m-1)qr+qk} \log(w_{l+j}(\lambda)) - \log(w_{l+j}(\lambda_{k+1})) \right| \\
 & \leq \sum_{j=1}^{(m-1)qr+qk} |\log(w_{l+j}(\lambda)) - \log(w_{l+j}(\lambda_{k+1}))| \\
 & \leq \sum_{j=1}^{(m-1)qr+qk} M(\lambda_{k+1} - \lambda_k) \\
 & = ((m-1)qr + qk) M \frac{\tau}{(m-1)qr + qk} \\
 & < \gamma.
 \end{aligned}$$

Hence, by (B.16) we have

$$\begin{aligned}
 & \left| \frac{w_{l+1}(\lambda) \cdots w_{(m-1)qr+qk+l}(\lambda)}{w_{l+1}(\lambda_{k+1}) \cdots w_{(m-1)qr+qk+l}(\lambda_{k+1})} - 1 \right| \\
 &= \left| \exp \left(\sum_{j=1}^{(m-1)qr+qk} \log(w_{l+j}(\lambda)) - \log(w_{l+j}(\lambda_{k+1})) \right) - 1 \right| \\
 &\leq \frac{\eta}{C_{\rho, \rho_0} \|y\|_{\rho_0}}.
 \end{aligned}$$

Writing

$$P_1 - y = \sum_{l=0}^p \theta_l y_l e_l \quad \text{with } |\theta_l| \leq \frac{\eta}{C_{\rho, \rho_0} \|y\|_{\rho_0}},$$

we get by (B.14) that $\|P_1 - y\|_\rho \leq \eta$. Furthermore, let us write

$$\begin{aligned} & \frac{w_{(j-k)q+l+1}(\lambda) \cdots w_{(m-1)qr+qj+l}(\lambda)}{w_{l+1}(\lambda_{j+1}) \cdots w_{(m-1)qr+qj+l}(\lambda_{j+1})} \\ &= \frac{w_1(\lambda_{j+1}) \cdots w_l(\lambda_{j+1})}{w_1(a_0) \cdots w_{(j-k)q+l}(a_0)} \prod_{s=1}^{(j-k)q+l} \frac{w_s(a_0)}{w_s(\lambda_{j+1})} \prod_{s=(j-k)q+l+1}^{(m-1)qr+qj+l} \frac{w_s(\lambda)}{w_s(\lambda_{j+1})}. \end{aligned}$$

Then we get since the functions (w_n) are nondecreasing,

$$P_2 = \sum_{n \geq q} \frac{z_n}{w_1(a_0) \cdots w_n(a_0)} e_n$$

for some eventually null sequence (z_n) with

$$\|z\|_\infty \leq \|y\|_\infty (\max(1, w_1(b_0), \dots, w_p(b_0)))^p.$$

By (B.15) we conclude that $\|P_2\|_\rho < \eta$. Finally,

$$\begin{aligned} \frac{w_{(r-k)q+1}(\lambda) \cdots w_{mqr}(\lambda)}{w_1(\kappa a) \cdots w_{mqr}(\kappa a)} &\leq \frac{w_{(r-k)q+1}(\kappa a) \cdots w_{mqr}(\kappa a)}{w_1(\kappa a) \cdots w_{mqr}(\kappa a)} \\ &= \frac{1}{w_1(\kappa a) \cdots w_{(r-k)q}(\kappa a)} \\ &\leq \frac{1}{w_1(a_0) \cdots w_{(r-k)q}(a_0)}, \end{aligned}$$

hence $\|P_3\|_\rho < \eta$ if r is big enough. With this we conclude that

$$\|(T_\lambda)^N(u+x)^m - y\|_\rho < 3\eta,$$

which shows that $(T_\lambda)^N(u+x)^m \in V$ and completes the proof. \square

We now apply Theorem 3.12 to multiples of the backward shift and of the derivative operator.

Example 3.13 On ℓ_1 endowed with the Cauchy product, $\bigcap_{\lambda > 1} HC(\lambda B) \cup \{0\}$ contains a nontrivial algebra.

Proof. We apply Theorem 3.12 with $X = \ell_1$, $\Lambda = (1, \infty)$ and $w_n(\lambda) = \lambda$. We have that the functions $\log(w_n)$ are continuously first differentiable hence uniformly Lipschitz on compact subsets of Λ . Condition (a) trivially holds and, for (b), we have that

$$\left\| \sum_{n=cN}^{\infty} \frac{[w_1(\kappa a) \cdots w_{mN}(\kappa a)]^{\frac{m-1}{m}}}{w_1(a) \cdots w_{(m-1)N+n}(a)} e_n \right\| = \frac{a}{a-1} \times \frac{\kappa^{(m-1)N}}{a^{\lceil cN \rceil}}$$

which, for $m = 1$, tends to 0 as N goes to $+\infty$ for any $c \in (0, 1)$, $\kappa > 1$ and $a > 1$, and for $m > 1$ it goes to 0 for any $c \in (0, 1)$ and $1 < \kappa < a^{\frac{c}{m-1}}$. \square

Example 3.14 On $H(\mathbb{C})$ endowed with the Cauchy product, $\bigcap_{\lambda > 0} HC(\lambda D) \cup \{0\}$ contains a non-trivial algebra.

Proof. We apply Theorem 3.12 with $X = H(\mathbb{C})$, $\Lambda = (0, \infty)$ and $w_n(\lambda) = \lambda n$. Again the functions $\log(w_n)$ are uniformly Lipschitz on compact subsets of Λ and condition (a) is clearly satisfied. For (b) we fix $m \geq 1$, $a > 0$, $\kappa > 1$ and $r \geq 1$ and we have

$$\begin{aligned} & \left\| \sum_{n \geq cN} \frac{(w_1(\kappa a) \cdots w_{mN}(\kappa a))^{(m-1)/m}}{w_1(a) \cdots w_{(m-1)N+n}(a)} e_n \right\|_r \\ &= \kappa^{(m-1)N} \sum_{n \geq cN} \frac{(mN)!^{(m-1)/m}}{((m-1)N+n)!} \left(\frac{r}{a}\right)^n \\ &= \left(\frac{\kappa a}{r}\right)^{(m-1)N} (mN)!^{(m-1)/m} \sum_{n \geq cN} \frac{(r/a)^{n+(m-1)N}}{(n+(m-1)N)!} \\ &\leq C \left(\frac{\kappa a}{r}\right)^{(m-1)N} (mN)!^{(m-1)/m} \times \frac{(r/a)^{(m-1+c)N}}{((m-1+c)N)!}. \end{aligned}$$

Since for all $\varepsilon > 0$, Stirling's formula implies

$$\begin{aligned} (mN)!^{(m-1)/m} &\leq C_\varepsilon N^{(m-1+\varepsilon)N} \\ ((m-1+c)N)! &\geq C_\varepsilon N^{(m-1+c-\varepsilon)N}, \end{aligned}$$

choosing $\varepsilon < 2c$, it follows that for all $c \in (0, 1)$, all $\kappa > 1$, all $a > 0$, all $m \geq 1$ and all $r \geq 1$, we have

$$\left(\frac{\kappa a}{r}\right)^{(m-1)N} (mN)!^{(m-1)/m} \times \frac{(r/a)^{(m-1+c)N}}{((m-1+c)N)!} \xrightarrow{N \rightarrow +\infty} 0.$$

Hence, assumption (b) is satisfied. \square

Remark 3.15 We can make use of the same ideas and prove that $(\lambda B_w)_{\lambda > 0}$ admits a common hypercyclic algebra on ω with the Cauchy product. In fact, the separating term allows us to push all undesirable parcels to the right and that is all we need when the topology of ω comes into play. One can even simplify the definitions of εe_σ and $d_{j,l}$ as their sizes are not important since everything will vanish by the support.

Question 3.16 Let $X = \ell_1$, $\Lambda = (0, +\infty)$ and $w_n(\lambda) = 1 + \lambda/n$. In the previous section, we have shown that $\bigcap_{\lambda \in \Lambda} HC(B_w(\lambda))$ is not empty. Does it contain (except 0) a nontrivial algebra for the Cauchy product?

We cannot apply Theorem 3.12 for this family of shifts since (b) is not satisfied. Indeed, using standard tools of calculus, it can be shown that, for all $n \in \mathbb{N}$, all $a \in \Lambda$, all $c \in (0, 1)$ and all $\kappa > 1$, there exists $C > 0$ such that, for all $N \geq 1$,

$$\left\| \sum_{n=cN}^N \frac{[w_1(\kappa a) \cdots w_{mN}(\kappa a)]^{\frac{m-1}{m}}}{w_1(a) \cdots w_{(m-1)N+n}(a)} e_n \right\|_1 \geq CN^{a(\frac{\kappa(m-1)}{m}-1)+1}.$$

For $a = m = 2$, it is impossible to find $\kappa > 1$ such that the right hand-side goes to 0.

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Article C

Common hypercyclic vectors and dimension of
the parameter set

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Abstract

We investigate the existence of a common hypercyclic vector for a family $(T_\lambda)_{\lambda \in \Lambda}$ of hypercyclic operators acting on the same Banach space X . We give positive and negative results involving the dimension of Λ and the regularity of each map $\lambda \in \Lambda \mapsto T_\lambda^n x$, $x \in X$, $n \in \mathbb{N}$.

1 Introduction

Among the many problems arising in linear dynamics, that of finding a common hypercyclic vector for an uncountable family of hypercyclic operators is one of the most prominent. Let us introduce the relevant definitions. Let X be an infinite-dimensional and separable F -space and let $T \in \mathcal{L}(X)$. A vector $x \in X$ is said to be *hypercyclic* for T if its orbit under T , $\{T^n x : n \geq 0\}$ is dense in X . The set of hypercyclic vectors for T will be denoted by $HC(T)$. We refer to the two books [3] and [7] for the standard theory of hypercyclic operators.

Given $(T_\lambda)_{\lambda \in \Lambda}$ a family of hypercyclic operators acting on the same F -space, it is natural to ask whether $\bigcap_{\lambda \in \Lambda} HC(T_\lambda)$ is nonempty. The first result in that direction is due to Abakumov and Gordon who showed in [1] that $\bigcap_{a>0} HC(e^a B)$ is nonempty, where B is the unweighted backward shift acting on ℓ_p , $p \in [1, +\infty)$ or on c_0 . Soon after, Costakis and Sambarino in [5] came with a criterion for proving the common hypercyclicity of some families, which allow them to extend the results of [1] to other families of shifts or to translation operators.

The paper [1] also contains an important negative result, granted to Borichev: the two-dimensional family $(e^a B \times e^b B)_{(a,b) \in (0, +\infty)^2}$ acting on $\ell_2 \times \ell_2$, does not admit a common hypercyclic vector. It turns out that most of the examples of families admitting a common hypercyclic vector are one-dimensional families, with two notable exceptions: the Leon-Müller theorem [9] which allows to introduce an extra parameter of rotations, and translation operators which have some redundant properties (see for instance [4]). Even for one-dimensional families, several intriguing problems remain. For instance, if $\Lambda \subset (0, +\infty)^2$ is a monotonic Lipschitz curve, then $(e^a B \times e^b B)_{(a,b) \in \Lambda}$ possesses a common hypercyclic vector, whereas this is unknown for $(e^t B \times e^{2-t} B)_{t \in [1,2]}$ (see [2]).

Our ambition, in this paper, is to revisit this problem and to shed new light on common hypercyclic vectors. We begin with a review of the results we intend to prove. In what follows, the parameter set Λ will always be a subset of \mathbb{R}^d for some $d \geq 1$ and \mathbb{R}^d will be endowed with the sup-norm.

1.1 Products of multiples of the backward shift

Our first result is an answer to the problem of [2] we just recalled. More precisely, we will prove the following.

Theorem 1.1 *Let $X = \ell_p(\mathbb{N})$, $p \in [1, +\infty)$, or $X = c_0(\mathbb{N})$, and let $\Lambda \subset (0, +\infty)^d$ be a Lipschitz curve. Then $(e^{\lambda(1)}B \times \cdots \times e^{\lambda(d)}B)_{\lambda \in \Lambda}$ possesses a dense G_δ set of common hypercyclic vectors.*

The way to delete the assumption " Λ is monotonic" in Theorem 1.1 will be to obtain a characterization for the common hypercyclicity of a family of products of weighted shifts acting on a Fréchet sequence space, when these shifts satisfy some natural conditions. This condition, which is rather technical, takes a much more pleasant form when we apply it to multiples of the backward shifts. We will apply it in order to get Theorem 1.1.

1.2 Borichev result revisited

Borichev's result can be rephrased in the following more precise way:

Let $\Lambda \subset (0, +\infty)^2$ be such that $\bigcap_{(a,b) \in \Lambda} HC(e^a B \times e^b B)$ is not empty. Then Λ has measure zero.

If we analyze the proof of this result, it turns out that a key point is given by the fact that if $(e^a B)^n u$ and $(e^{a'} B)^n u$ are both close to the same nonzero vector (e.g. e_0), then $|a - a'|$ has to be small, precisely $|a - a'| \leq C/n$ for some constant C .

We will show that this can be put in a more general framework, replacing sets of zero Lebesgue measure by sets of small Hausdorff dimension.

Theorem 1.2 *Let $\Lambda \subset \mathbb{R}^d$, let $(T_\lambda)_{\lambda \in \Lambda}$ be a family of operators acting on the Banach space X . Assume that there exist $\alpha > 0$, $v \in X$, $\delta > 0$ and $C > 0$ such that, for all $\lambda, \mu \in \Lambda$, for all $n \in \mathbb{N}$ and all $u \in X$ satisfying*

$$\|T_\lambda^n u - v\| < \delta \text{ and } \|T_\mu^n u - v\| < \delta,$$

then

$$\|T_\lambda^n u - T_\mu^n u\| \geq Cn^\alpha \|\lambda - \mu\|.$$

If $\bigcap_{\lambda \in \Lambda} HC(T_\lambda) \neq \emptyset$, then $\dim_{\mathcal{H}}(\Lambda) \leq \frac{1}{\alpha}$.

In particular, this can be applied to the family $(B_{w(\lambda(1))} \times \cdots \times B_{w(\lambda(d))})_{\lambda \in (0, +\infty)^d}$ where $(w(a))_{a>0}$ is defined by $w_1(a) \cdots w_n(a) = \exp(an^\alpha)$ for some $\alpha > 0$ and for all $n \in \mathbb{N}$.

Corollary 1.3 *Let $X = \ell_p(\mathbb{N})$, $p \in [1, +\infty)$ or $X = c_0(\mathbb{N})$. Let $\alpha \in (0, 1]$, let $(w(a))_{a>0}$ be defined by $w_1(a) \cdots w_n(a) = \exp(an^\alpha)$ for all $n \in \mathbb{N}$ and let $\Lambda \subset (0, +\infty)^d$. If $(B_{w(\lambda(1))} \times \cdots \times B_{w(\lambda(d))})_{\lambda \in \Lambda}$ admits a common hypercyclic vector, then $\dim_{\mathcal{H}}(\Lambda) \leq 1/\alpha$.*

In view of the previous corollary, one may ask if the converse holds true, namely if the condition $\dim_{\mathcal{H}}(\Lambda) \leq 1/\alpha$, or $\dim_{\mathcal{H}}(\Lambda) < 1/\alpha$, implies that $\bigcap_{\lambda \in \Lambda} HC(B_{w(\lambda(1))} \times \cdots \times B_{w(\lambda(d))}) \neq \emptyset$. More specifically, we may ask if $\alpha \leq 1/d$ implies $\bigcap_{\lambda \in (0, +\infty)^d} HC(B_{w(\lambda(1))} \times \cdots \times B_{w(\lambda(d))}) \neq \emptyset$. The study of these questions motivates the remaining part of the paper.

1.3 A common hypercyclicity criterion in dimension greater than 1

In the remaining of this introduction, we will always assume that X is a separable Banach space. We discuss now common hypercyclicity criteria for a family $(T_\lambda)_{\lambda \in \Lambda}$ of operators acting on the same Banach space X . We will always assume that the following assumptions are true:

- the map $(\lambda, u) \mapsto T_\lambda u$ is continuous from $\Lambda \times X$ into X ;
- there exists a dense set $\mathcal{D} \subset X$ such that each operator T_λ has a partial right-inverse $S_\lambda : \mathcal{D} \rightarrow \mathcal{D}$, that is $T_\lambda S_\lambda(u) = u$ for all $u \in \mathcal{D}$ and all $\lambda \in \Lambda$.

These assumptions are for instance satisfied if T_λ is defined as the product of weighted shifts $B_{w^{(1)}(\lambda(1))} \times \cdots \times B_{w^{(d)}(\lambda(d))}$, $X = c_0^d(\mathbb{N})$ or $X = \ell_p^d(\mathbb{N})$, $p \in [1, +\infty)$, and, for each $i = 1, \dots, d$ and each $n \in \mathbb{N}$, $a \mapsto w_n^{(i)}(a)$ is continuous (we will call this a *continuous family of weights*).

When Λ is an interval of the real line, one of the most useful result to get common hypercyclic vectors is the Costakis-Sambarino criterion:

Costakis-Sambarino Theorem. *Let Λ be an interval of the real line. Assume that for every compact interval $K \subset \Lambda$, every $u \in \mathcal{D}$,*

(CS1) *there exist $m \geq 1$ and $(c_k)_{k \geq m}$ a sequence of positive real numbers with $\sum_{k \geq m} c_k < \infty$ such that*

$$(a) \quad \|T_\lambda^{n+k} S_\mu^n u\| \leq c_k \text{ for every } n \in \mathbb{N}, k \geq m, \mu \leq \lambda, \mu, \lambda \in K$$

$$(b) \quad \|T_\lambda^n S_\mu^{n+k} u\| \leq c_k \text{ for every } n \in \mathbb{N}, k \geq m, \mu \geq \lambda, \mu, \lambda \in K;$$

(CS2) *for all $\varepsilon > 0$, there exists $\tau > 0$ such that, for all $n \geq 1$,*

$$0 \leq \mu - \lambda \leq \frac{\tau}{n} \Rightarrow \|T_\lambda^n S_\mu^n(u) - u\| < \varepsilon.$$

Then $\bigcap_{\lambda \in \Lambda} HC(T_\lambda)$ is a dense G_δ subset of X .

We look for a substitute for this theorem when Λ is not an interval of the real line and in particular if the "dimension" of Λ is greater than 1. The continuity condition (CS2) is naturally implied by the following Lipschitz estimate: for all $u \in \mathcal{D}$, there exists $C > 0$ such that, for all $n \geq 1$, for all $\lambda, \mu \in \Lambda$,

$$\|T_\lambda^n S_\mu^n u - T_\mu^n S_\mu^n u\| \leq Cn \|\lambda - \mu\|.$$

Nevertheless, if we have the opposite inequality

$$\|T_\lambda^n S_\mu^n u - T_\mu^n S_\mu^n u\| \geq Cn \|\lambda - \mu\|,$$

then Theorem 1.2 essentially says that the Hausdorff dimension of the set of common hypercyclic vectors cannot exceed 1. Hence, to get common hypercyclic vectors for subsets of \mathbb{R}^d of bigger dimension, we will need a stronger condition, at least something like

$$\|\lambda - \mu\| \leq \frac{\tau}{n^\alpha} \Rightarrow \|T_\lambda^n S_\mu^n(u) - u\| < \varepsilon$$

for some $\alpha \in (0, 1)$. Under this last condition and an appropriate substitute for (CS1), we will be able to prove a common hypercyclic criterion when $\Lambda \subset \mathbb{R}^d$ possesses some regularity and has "dimension" less than $1/\alpha$. The notion of dimension we need is a kind of homogeneous box dimension. For $r \geq 1$, we define $I_r = \{1, \dots, r\}$.

Definition 1.4 Let $\Lambda \subset \mathbb{R}^d$ be compact. We say that Λ has *homogeneous box dimension at most* $\gamma \in (0, d]$ if there exist $r \geq 2$, $C(\Lambda) > 0$ and, for all $m \geq 1$, a family $(\Lambda_{\mathbf{k}})_{\mathbf{k} \in I_r^m}$ of compact subsets of Λ such that for all $m \geq 1$,

- for all $\mathbf{k} \in I_r^m$, $\text{diam}(\Lambda_{\mathbf{k}}) \leq C(\Lambda) \left(\frac{1}{r^{1/\gamma}}\right)^m$;
- $\Lambda \subset \bigcup_{\mathbf{k} \in I_r^m} \Lambda_{\mathbf{k}}$;
- for all $\mathbf{k} \in I_r^m$, $\Lambda_{k_1, \dots, k_m} \subset \Lambda_{k_1, \dots, k_{m-1}}$.

The *homogeneous box dimension* of Λ is defined as the infimum of the $\gamma \in (0, d]$ such that Λ has homogeneous box dimension at most γ and will be denoted $\text{dim}_{\text{HB}}(\Lambda)$.

We will discuss later the link between this notion of dimension and more classical ones; we just observe for the moment that any compact subset of \mathbb{R}^d has homogeneous box dimension at most d . Having this notion of dimension at hand, we can prove the following result.

Theorem 1.5 *Let $\gamma > 0$ and let Λ be a compact subset of \mathbb{R}^d having homogeneous box dimension at most γ . Assume moreover that there exist $\alpha \in (0, 1/\gamma)$, $\beta > \alpha\gamma$ and $D > 0$ such that, for all $u \in \mathcal{D}$,*

(a) *there exist $C > 0, N > 0$ such that, for all $\lambda, \mu \in \Lambda$, for all $n \geq 0$ and $k \geq N$ such that $\|\lambda - \mu\| \leq D \frac{k^\alpha}{(n+k)^\alpha}$, then*

$$\begin{aligned} \left\| T_\lambda^{n+k} S_\mu^n u \right\| &\leq \frac{C}{k^\beta}, \\ \left\| T_\lambda^n S_\mu^{n+k} u \right\| &\leq \frac{C}{k^\beta}. \end{aligned}$$

(b) *for all $\varepsilon > 0$, there exists $\tau > 0$ such that, for all $n \geq 1$, for all $\lambda, \mu \in \Lambda$,*

$$\|\lambda - \mu\| \leq \frac{\tau}{n^\alpha} \implies \|T_\lambda^n S_\mu^n u - u\| < \varepsilon.$$

Then $\bigcap_{\lambda \in \Lambda} HC(T_\lambda)$ is a dense G_δ subset of X .

In particular we get the following corollary, which can be seen as the desired converse of Corollary 1.3.

Corollary 1.6 *Let $X = \ell_p(\mathbb{N})$, $p \in [1, +\infty)$ or $X = c_0(\mathbb{N})$, $\alpha \in (0, 1]$, let $(w(a))_{a>0}$ be defined by $w_1(a) \cdots w_n(a) = \exp(an^\alpha)$ for all $n \in \mathbb{N}$ and let $\Lambda \subset (0, +\infty)^d$. Assume that $\text{dim}_{\text{HB}}(\Lambda) < 1/\alpha$. Then $(B_{w(\lambda(1))} \times \cdots \times B_{w(\lambda(d))})_{\lambda \in \Lambda}$ admits a common hypercyclic vector.*

Example 1.7 Let $X = \ell_p(\mathbb{N})$, $p \in [1, +\infty)$ or $X = c_0(\mathbb{N})$, $\alpha \in (0, 1/d)$ and let $(w(a))_{a>0}$ be the family of weights defined by $w_1(a) \cdots w_n(a) = \exp(an^\alpha)$ (resp. $w_n(a) = 1 + \frac{a}{n^{1-\alpha}}$) for all $n \geq 1$. Then $(B_{w(\lambda(1))} \times \cdots \times B_{w(\lambda(d))})_{\lambda \in (0, +\infty)^d}$ admits a common hypercyclic vector.

Theorem 1.5 can also be applied to Hölder curves leading to a nice complement to Theorem 1.1. Indeed, let $\alpha \in (0, 1]$. A compact set $\Lambda \subset \mathbb{R}^d$ is called an α -Hölder curve if $\Lambda = f(I)$ for some function $f : [0, 1] \rightarrow \mathbb{R}^d$ satisfying

$$\exists C > 0, \forall (s, t) \in [0, 1]^2, \|f(s) - f(t)\| \leq C|s - t|^\alpha.$$

It turns out that any α -Hölder curve has homogeneous box dimension at most $1/\alpha$. Therefore, we will also obtain the following example.

Example 1.8 Let $X = \ell_p(\mathbb{N})$, $p \in [1, +\infty)$ or $X = c_0(\mathbb{N})$. Let $\Lambda \subset (0, +\infty)^d$ be a β -Hölder curve for some $\beta \in (0, 1]$. Let $\alpha \in (0, \beta)$ and let $(w(a))_{a>0}$ be the family of weights defined by $w_1(a) \cdots w_n(a) = \exp(an^\alpha)$ (resp. $w_n(a) = 1 + \frac{a}{n^{1-\alpha}}$) for all $n \geq 1$. Then $(B_{w(\lambda(1))} \times \cdots \times B_{w(\lambda(d))})_{\lambda \in \Lambda}$ admits a common hypercyclic vector.

1.4 The Basic Criterion

Almost all results of common hypercyclicity rely on the construction of a suitable covering of the parameter space Λ and on an associated sequence (n_k) . What we need is contained in the following basic criterion (see [3, Lemma 7.12]).

Theorem 1.9 (Basic Criterion) *Let Λ' be a topological space. Let $(T_\lambda)_{\lambda \in \Lambda'}$ be a family of operators on X such that*

- *the map $(\lambda, x) \mapsto T_\lambda(x)$ is continuous from $\Lambda' \times X \rightarrow X$;*
- *there exist a dense set $\mathcal{D} \subset X$ and maps $S_\lambda : \mathcal{D} \rightarrow \mathcal{D}$ satisfying $T_\lambda S_\lambda x = x$ for every $x \in \mathcal{D}$ and every $\lambda \in \Lambda'$.*

If $\Lambda \subset \Lambda'$ is σ -compact and if for every compact set $K \subset \Lambda$, every pair $(u, v) \in \mathcal{D} \times \mathcal{D}$, every $\varepsilon > 0$, there exist $\lambda_1, \dots, \lambda_q \in \Lambda'$, sets $\Lambda_1, \dots, \Lambda_q \subset \Lambda$ and positive integers n_1, \dots, n_q such that

$$(BC1) \quad \bigcup_k \Lambda_k \supset K,$$

$$(BC2) \quad \left\| \sum_k S_{\lambda_k}^{n_k} v \right\| < \varepsilon,$$

$$(BC3) \quad \text{for all } k = 1, \dots, q \text{ and all } \lambda \in \Lambda_k, \left\| \sum_{j \neq k} T_\lambda^{n_k} S_{\lambda_j}^{n_j} v \right\| < \varepsilon,$$

$$(BC4) \quad \text{for all } k = 1, \dots, q \text{ and all } \lambda \in \Lambda_k, \|T_\lambda^{n_k} u\| < \varepsilon,$$

$$(BC5) \quad \text{for all } k = 1, \dots, q \text{ and all } \lambda \in \Lambda_k, \|T_\lambda^{n_k} S_{\lambda_k}^{n_k} v - v\| < \varepsilon,$$

then $\bigcap_{\lambda \in \Lambda} HC(T_\lambda)$ is a dense G_δ subset of X .

2 Characterization of families of products of weighted shifts admitting a common hypercyclic vector

In this section, we work in the context of a Fréchet sequence space X , namely X is a Fréchet space endowed with a family of seminorms $(\|\cdot\|_p)$, contained in the space $\mathbb{C}^{\mathbb{N}}$ of all complex

sequences and such that each coordinate functional $(x_n)_n \mapsto x_m$ is continuous. Such a space can be endowed with an F -norm $\|\cdot\|$ defining the topology of X (see [7, Section 2.1]). Such an F -norm can be defined by the formula

$$\|x\| = \sum_{p=1}^{+\infty} \frac{1}{2^p} \min(1, \|x\|_p).$$

In particular, an F -norm satisfies the triangle inequality and the inequality

$$\forall \lambda \in \mathbb{C}, \forall x \in X, \|\lambda x\| \leq (|\lambda| + 1)\|x\|, \quad (\text{C.1})$$

a property which replaces the positive homogeneity of the norm. We will also need the following property of a Fréchet sequence space in which (e_n) is an unconditional basis (see [8, Theorem 3.9]).

(UNC) If $(x_n) \in X$ and $(\alpha_n) \in \ell_\infty$, then $(\alpha_n x_n) \in X$. Moreover, for all $\varepsilon > 0$, for all $M > 0$, there exists $\delta > 0$ such that, for all $(x_n) \in X$ with $\|(x_n)\| \leq \delta$, for all sequence $(\alpha_n) \in \ell_\infty$ with $\|(\alpha_n)\|_\infty \leq M$, then $(\alpha_n x_n) \in X$ and $\|(\alpha_n x_n)\| < \varepsilon$.

Theorem 2.1 *Let X be a Fréchet sequence space in which (e_n) is an unconditional basis. Let $I \subset \mathbb{R}$ be a nonempty interval and let $\Lambda \subset I^d$ be σ -compact. Let $(B_{w(a)})_{a \in I}$ be a continuous family of weighted shifts on X and assume that $a \in I \mapsto w_n(a)$ is nondecreasing. Assume also that there exist a nondecreasing map $F : \mathbb{N} \rightarrow \mathbb{N}$ and $c, C > 0$ such that, for all $n \geq 1$, denoting by $f_n(a) = \sum_{k=1}^n \log(w_k(a))$,*

$$\forall (a, b) \in I^2, cF(n)|a - b| \leq |f_n(a) - f_n(b)| \leq CF(n)|a - b|$$

$$\forall (a, b) \in I^2, \frac{w_n(a)}{w_n(b)} \geq c.$$

Then the following assertions are equivalent:

(a) $(B_{w(\lambda(1))} \times \cdots \times B_{w(\lambda(d))})_{\lambda \in \Lambda}$ possesses a dense G_δ set of common hypercyclic vectors in X^d .

(b) $(B_{w(\lambda(1))} \times \cdots \times B_{w(\lambda(d))})_{\lambda \in \Lambda}$ admits a common hypercyclic vector in X^d .

(c) For all $\tau > 0$, for all $N \geq 1$, for all $\varepsilon > 0$, for all $K \subset \Lambda$ compact, there exist $N \leq n_1 < n_1 + N \leq n_2 < \cdots < n_{q-1} + N \leq n_q$ and $(\lambda_k)_{k=1, \dots, q} \in I^d$ such that

(i) $K \subset \bigcup_{k=1}^q \prod_{i=1}^d \left[\lambda_k(i) - \frac{\tau}{F(n_k)}, \lambda_k(i) \right]$

(ii) For all $i = 1, \dots, d$,

$$\left\| \sum_{k=1}^q \frac{1}{w_1(\lambda_k(i)) \cdots w_{n_k}(\lambda_k(i))} e_{n_k} \right\| < \varepsilon.$$

(iii) For all $k = 1, \dots, q - 1$, for all $i = 1, \dots, d$, for all $l = 0, \dots, N$,

$$\left\| \sum_{j=k+1}^q \frac{w_{n_j - n_k + l + 1}(\lambda_k(i)) \cdots w_{n_j + l}(\lambda_k(i))}{w_{l+1}(\lambda_j(i)) \cdots w_{n_j + l}(\lambda_j(i))} e_{n_j - n_k + l} \right\| < \varepsilon.$$

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To simplify the notations, we will do the proof only for $d = 2$ and we shall denote by $\lambda = (a, b)$ any element of \mathbb{R}^2 . For $\lambda = (a, b) \in I^2$, we shall denote by T_λ the operator $B_{w(a)} \times B_{w(b)}$ acting on $X \times X$ and by S_λ the operator $F_{w^{-1}(a)} \times F_{w^{-1}(b)}$, where $F_{w^{-1}(a)}$ denotes the forward shift associated to the sequence $(w_n^{-1}(a))_n$.

Proof. We first assume that $(T_\lambda)_{\lambda \in \Lambda}$ admits a common hypercyclic vector. Let $\tau > 0$, $N \geq 1$, $\varepsilon > 0$ and $K \subset \Lambda$ compact. We set K_1 the projection of K onto the first coordinate; K_1 is a compact subset of I . We consider $0 < \eta < \min(\frac{1}{2}, \frac{\varepsilon\tau}{4})$ satisfying also the following two technical conditions:

$$\eta < \frac{1}{4} \times \frac{1}{\prod_{k=1}^N \max(1, \sup_{a \in K_1} w_k(a))} \quad (\text{C.2})$$

$$c \inf_{\substack{a, a' \in K_1 \\ l, l' \in \{1, \dots, N+1\}}} \frac{w_l(a)}{w_{l'}(a')} \times \frac{1-\eta}{1+\eta} > \frac{\eta}{1-\eta}. \quad (\text{C.3})$$

By continuity of the first $N + 2$ coordinate functionals and by (UNC), we may find $\delta > 0$ such that, for all $z = (z_n) \in X$ and all $\alpha = (\alpha_n) \in \ell_\infty$,

$$\|(z_n)\| \leq \delta \implies \forall l \in \{0, \dots, N+1\}, |z_l| < \eta$$

$$\|(z_n)\| \leq \delta \text{ and } \|(\alpha_n)\| \leq 2 \implies \|(\alpha_n z_n)\| < \varepsilon.$$

Let $u = (x, y)$ be a common hypercyclic vector for $(T_\lambda)_{\lambda \in \Lambda}$. We may always assume that $\max\{\|x\|, \|y\|\} \leq \delta$ and we set

$$v = \left(\sum_{j=0}^N e_j, \sum_{j=0}^N e_j \right).$$

Let (n_k) be an increasing enumeration of

$$\{n \geq 1 : \|T_\lambda^n u - v\| < \delta \text{ for some } \lambda \in K\}.$$

Let $\Lambda_k = \{\lambda \in K : \|T_\lambda^{n_k} u - v\| < \delta\}$. Since u is a common hypercyclic vector for the family $(T_\lambda)_{\lambda \in K}$, since K is compact and each Λ_k is open, there exists $q \geq 1$ such that

$$K \subset \bigcup_{k=1}^q \Lambda_k.$$

For each $k = 1, \dots, q$, we define a_k and b_k by

$$a_k := \sup \{a : \exists b, (a, b) \in \Lambda_k\}$$

$$b_k := \sup \{b : \exists a, (a, b) \in \Lambda_k\}$$

and we set $\lambda_k = (a_k, b_k)$. We first observe that $n_1 \geq N$. Indeed, since $\|B_{w(a_1)}^x - \sum_{j=0}^N e_j\| \leq \delta$, we know that

$$w_1(a_1) \cdots w_{n_1}(a_1) |x_{n_1}| \geq 1 - \eta > 1/2.$$

Assume that $n_1 < N$. Then

$$w_1(a_1) \cdots w_{n_1}(a_1) \leq \prod_{k=1}^N \max\left(1, \sup_{a \in K_1} w_k(a)\right)$$

whereas

$$\begin{aligned} w_1(a_1) \cdots w_{n_1}(a_1) &\geq w_1(a_1) \cdots w_{n_1}(a_1) \frac{|x_{n_1}|}{\eta} \\ &> \frac{1}{2\eta}, \end{aligned}$$

which contradicts (C.2). We now show that we also have $n_{k+1} - n_k \geq N$ for all $k = 1, \dots, q-1$. On the contrary, assume that there exists some k such that $n_{k+1} - n_k < N$. We set $p = n_k + N - n_{k+1} \in \{1, \dots, N-1\}$. Then using that $\|B_{w(a_{k+1})}^{n_{k+1}} x - \sum_{j=0}^N e_j\| \leq \delta$, we get

$$\begin{cases} |w_{p+1}(a_{k+1}) \cdots w_{n_k+N}(a_{k+1}) x_{n_k+N} - 1| < \eta \\ |w_{p+2}(a_{k+1}) \cdots w_{n_k+N+1}(a_{k+1}) x_{n_k+N+1} - 1| < \eta. \end{cases}$$

Since we also know that $\|B_{w(a_k)}^{n_k} x - \sum_{j=0}^N e_j\| \leq \delta$, we also get

$$\begin{cases} |w_{N+1}(a_k) \cdots w_{n_k+N}(a_k) x_{n_k+N} - 1| < \eta \\ |w_{N+2}(a_k) \cdots w_{n_k+N+1}(a_k) x_{n_k+N+1} - 1| < \eta. \end{cases}$$

Taking respective quotients, these inequalities lead to

$$\begin{aligned} \frac{w_{N+2}(a_k) \cdots w_{n_k+N+1}(a_k)}{w_{p+2}(a_{k+1}) \cdots w_{n_k+N+1}(a_{k+1})} &\leq \frac{\eta}{1-\eta} \\ \frac{w_{N+1}(a_k) \cdots w_{n_k+N}(a_k)}{w_{p+1}(a_{k+1}) \cdots w_{n_k+N}(a_{k+1})} &\geq \frac{1-\eta}{1+\eta}. \end{aligned}$$

Consequently we obtain

$$\frac{1-\eta}{1+\eta} \times \frac{w_{n_k+N+1}(a_k)}{w_{n_k+N+1}(a_{k+1})} \times \frac{w_{p+1}(a_{k+1})}{w_{N+1}(a_k)} \leq \frac{\eta}{1-\eta}.$$

This again leads to a contradiction, with (C.3).

Let us now prove (i). We consider $\lambda = (a, b) \in \Lambda_k$ for some $k = 1, \dots, q$. The choice of δ ensures that for any $0 \leq l \leq N$,

$$\begin{cases} |w_{l+1}(a_k) \cdots w_{n_k+l}(a_k) x_{n_k+l} - 1| < \eta \\ |w_{l+1}(a) \cdots w_{n_k+l}(a) x_{n_k+l} - 1| < \eta. \end{cases}$$

Hence,

$$|w_1(a_k) \cdots w_{n_k}(a_k) - w_1(a) \cdots w_{n_k}(a)| \cdot |x_{n_k}| < 2\eta.$$

On the other hand,

$$\begin{aligned} &w_1(a_k) \cdots w_{n_k}(a_k) - w_1(a) \cdots w_{n_k}(a) \\ &= w_1(a) \cdots w_{n_k}(a) \left(\exp \left(\sum_{j=1}^{n_k} \log(w_j(a_k)) - \sum_{j=1}^{n_k} \log(w_j(a)) \right) - 1 \right) \\ &\geq w_1(a) \cdots w_{n_k}(a) (\exp(cF(n_k)(a_k - a)) - 1) \\ &\geq cw_1(a) \cdots w_{n_k}(a)(a_k - a)F(n_k). \end{aligned}$$

Since we also know that $w_1(a) \cdots w_{n_k}(a)|x_{n_k}| \geq 1/2$, we finally get

$$0 \leq a_k - a < \frac{4\eta}{cF(n_k)} < \frac{\tau}{F(n_k)}.$$

The same is true for the second coordinate and we get (i). In order to prove (ii), we define (α_n) by

$$\alpha_n = \begin{cases} \frac{1}{w_1(a_k) \cdots w_{n_k}(a_k) x_{n_k}} & \text{if } n = n_k \text{ for some } k = 1, \dots, q \\ 0 & \text{otherwise.} \end{cases}$$

Observe that $\|\alpha\|_\infty \leq 2$, hence

$$\left\| \sum_{k=1}^q \frac{1}{w_1(a_k) \cdots w_{n_k}(a_k)} e_{n_k} \right\| = \left\| \sum_{n=0}^{+\infty} \alpha_n x_n e_n \right\| < \varepsilon.$$

It remains to prove (iii). We fix $k = 1, \dots, q-1$ and $l = 0, \dots, N$, and we now set

$$\beta_n = \begin{cases} \frac{1}{w_{l+1}(a_j) \cdots w_{n_j+l}(a_j) x_{n_j+l}} & \text{if } n = n_j - n_k + l \text{ for some } j \geq k+1, \\ 0 & \text{otherwise.} \end{cases}$$

Again $\|\beta\|_\infty \leq 2$ and writing $B_{w(a_k)}^{n_k}(x) - \sum_{j=0}^N e_j$ as $(z_n)_n$, one gets

$$\sum_n \beta_n z_n e_n = \sum_{j=k+1}^q \frac{w_{n_j-n_k+l+1}(a_k) \cdots w_{n_j+l}(a_k)}{w_{l+1}(a_j) \cdots w_{n_j+l}(a_j)} e_{n_j-n_k+l}$$

since $n_j - n_k > N$ for all $j \geq k+1$. The result follows again from the choice of δ .

It remains to show that (c) implies (a) since (a) \Rightarrow (b) is obvious. To this end, we shall apply the Basic Criterion. Let $K \subset \Lambda$ be compact, let $\mathcal{D} \subset X^2$ be the set of couples of vectors with finite support and let $(u, v) \in \mathcal{D} \times \mathcal{D}$. We shall write

$$u(1) = \sum_{l=0}^N u_l e_l \text{ and } v(1) = \sum_{l=0}^N v_l e_l$$

for some $N \geq 0$. We fix $\varepsilon > 0$ and $\tau > 0$ (conditions on ε and τ will be imposed later) and we consider the two sequences $(n_k)_{k=1, \dots, q}$ and $(\lambda_k)_{k=1, \dots, q}$ given by (c) with $\lambda_k := (a_k, b_k)$. We set $\Lambda_k = K \cap ([a_k - \tau/F(n_k), a_k] \times [b_k - \tau/F(n_k), b_k])$ so that $\bigcup_k \Lambda_k \supset K$ and we show that the assumptions of the Basic Criterion are satisfied for the sequence $m_k = n_k - N$. First, we observe that

$$\left\| \sum_{k=1}^q F_{w^{-1}(a_k)}^{m_k}(v(1)) \right\| \leq \sum_{l=0}^N (|v_l| + 1) \left\| \sum_{k=1}^q \frac{1}{w_{l+1}(a_k) \cdots w_{n_k-(N-l)}(a_k)} e_{n_k-(N-l)} \right\|.$$

We fix $\tilde{a} \in I$ and we claim that for all $l = 0, \dots, N$,

$$\sum_{k=1}^q \frac{1}{w_{l+1}(a_k) \cdots w_{n_k-(N-l)}(a_k)} e_{n_k-(N-l)} = B_{w(\tilde{a})}^{N-l} \left(\sum_{k=1}^q \frac{x_{k,l}}{w_1(a_k) \cdots w_{n_k}(a_k)} e_{n_k} \right)$$

for some sequence $(x_{k,l})_k \in \ell_\infty$ with

$$\|(x_{k,l})_k\|_\infty \leq \left(\frac{M}{c} \right)^N,$$

where $M = \max\{1, |w_j(a)| : 0 \leq j \leq N, (a, b) \in K \text{ for some } b\}$. Provided this has been shown, it is easy to adjust ε so that (BC2) is satisfied, using the continuity of $B_{w(\tilde{a})}$ and the unconditionality of (e_n) . The proof of the claim follows from a rather straightforward computation:

$$\sum_{k=1}^q \frac{1}{w_{l+1}(a_k) \cdots w_{n_k-(N-l)}(a_k)} e_{n_k-(N-l)}$$

$$= \sum_{k=1}^q w_1 \cdots w_l(a_k) \frac{w_{n_k-(N-l)+1}(a_k) \cdots w_{n_k}(a_k)}{w_{n_k-(N-l)+1}(\tilde{a}) \cdots w_{n_k}(\tilde{a})} B_{w(\tilde{a})}^{N-l} \left(\frac{1}{w_1(a_k) \cdots w_{n_k}(a_k)} e_{n_k} \right).$$

The proof that the other conditions of the Basic Criterion are satisfied is rather similar. Indeed, for $k = 1, \dots, q$ and $(a, b) \in \Lambda_k$,

$$\begin{aligned} \sum_{j \neq k} B_{w(a)}^{m_k} F_{w^{-1}(a_j)}^{m_j}(v(1)) &= \sum_{j=k+1}^q B_{w(a)}^{m_k} F_{w^{-1}(a_j)}^{m_j}(v(1)) \\ &= \sum_{j=k+1}^q \sum_{l=0}^N v_l \frac{w_{n_j-n_k+l+1}(a) \cdots w_{n_j+l-N}(a)}{w_{l+1}(a_j) \cdots w_{n_j+l-N}(a_j)} e_{n_j-n_k+l}. \end{aligned}$$

For each $0 \leq l \leq N$, we now set

$$\alpha_n^{(l)} = \begin{cases} \frac{w_{n_j-n_k+l+1}(a) \cdots w_{n_j+l}(a)}{w_{n_j-n_k+l+1}(a_k) \cdots w_{n_j+l}(a_k)} \frac{w_{n_j+l-N+1}(a_j) \cdots w_{n_j+l}(a_j)}{w_{n_j+l-N+1}(a) \cdots w_{n_j+l}(a)} & \text{if } n = n_j - n_k + l \\ & \text{for some } j \geq k+1, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, since $\|\alpha^{(l)}\|_\infty \leq \frac{1}{c^N}$ (because $a \leq a_k$) and since

$$\left\| \sum_{j \neq k} B_{w(a)}^{m_k} F_{w^{-1}(a_j)}^{m_j}(v(1)) \right\| \leq \sum_{l=0}^N (|v_l| + 1) \left\| \sum_{n=0}^\infty \alpha_n^{(l)} \frac{w_{n_j-n_k+l+1}(a_k) \cdots w_{n_j+l}(a_k)}{w_{l+1}(a_j) \cdots w_{n_j+l}(a_j)} e_n \right\|,$$

(BC3) follows from the unconditionality of (e_n) if ε is sufficiently small.

We observe that (BC4) is empty and finish the proof by showing (BC5). Let $k = 1, \dots, q$ and $(a, b) \in \Lambda_k$. Then

$$\left\| B_{w(a)}^{m_k} F_{w^{-1}(a_k)}^{m_k}(v(1)) - v(1) \right\| \leq \sum_{l=0}^N (|v_l| + 1) \left\| \left(\frac{w_{l+1}(a) \cdots w_{n_k-(N-l)}(a)}{w_{l+1}(a_k) \cdots w_{n_k-(N-l)}(a_k)} - 1 \right) e_l \right\|$$

and it is easy to show that this becomes small provided $\tau > 0$ becomes small enough, using that $|a - a_k| \leq \frac{\tau}{F(n_k)}$, that F is nondecreasing, that

$$\frac{w_{l+1}(a) \cdots w_{n_k-(N-l)}(a)}{w_{l+1}(a_k) \cdots w_{n_k-(N-l)}(a_k)} = \frac{\exp(f_{n_k-(N-l)}(a) - f_{n_k-(N-l)}(a_k))}{\exp(f_l(a) - f_l(a_k))}$$

and that for every $a, b \in I$

$$|f_n(a) - f_n(b)| \leq CF(n)|a - b|.$$

□

The previous statement shows clearly that if $(F(n))_n$ grows slowly, there is more hope to get a large set $\Lambda \subset \mathbb{R}^d$ such that $(T_\lambda)_{\lambda \in \Lambda}$ possesses a common hypercyclic vector. Of course, the simplest examples of weights satisfying the assumptions of Theorem 2.1 (when I is a bounded interval) are given by $w_1(a) \cdots w_n(a) = \exp(aF(n))$ where $F(n)$ is nondecreasing and $F(n+1) - F(n)$ is bounded, which includes the case of the multiples of the backward shift. For this last example, Theorem 2.1 takes the following easier form.

Theorem 2.2 *Let $\Lambda \subset (0, +\infty)^d$ be σ -compact, $X = \ell_p(\mathbb{N})$, $p \in [1, +\infty)$ or $X = c_0(\mathbb{N})$. The following assertions are equivalent:*

(a) $(e^{\lambda^{(1)}}B \times \cdots \times e^{\lambda^{(d)}}B)_{\lambda \in \Lambda}$ admits a common hypercyclic vector.

(b) For all $\tau > 0$, for all $N \geq 1$, for all $\varepsilon > 0$, for all $K \subset \Lambda$ compact, there exist $N \leq n_1 < n_1 + N \leq n_2 < \cdots < n_{q-1} + N \leq n_q$ and $(\lambda_k)_{k=1, \dots, q} \in (0, +\infty)^d$ such that

(i) $K \subset \cup_{k=1}^q \prod_{i=1}^d [\lambda_k(i) - \frac{\tau}{n_k}, \lambda_k(i)]$

(ii) for all $k = 1, \dots, q-1$, for all $i = 1, \dots, d$,

$$\lambda_{k+1}(i)n_{k+1} - \lambda_k(i)n_k \geq N.$$

Proof. Without loss of generality, we can assume that $\Lambda \subset I^d$ for some bounded interval $I \subset (0, \infty)$. That (a) \implies (b) then follows directly from Theorem 2.1. Indeed, let us apply it for $K \subset \Lambda$ compact, $N \geq 1$ and $\varepsilon = e^{-N}$ to get $(n_k)_{k=1, \dots, q}$ and $(\lambda_k)_{k=1, \dots, q}$. We need only to verify (ii). From (c) (iii) of Theorem 2.1 with $l = 0$, we know that, for $k = 1, \dots, q-1$, for $i = 1, \dots, d$,

$$\frac{\exp(n_k \lambda_k(i))}{\exp(n_{k+1} \lambda_{k+1}(i))} = \frac{w_{n_{k+1}-n_k+1}(\lambda_k(i)) \cdots w_{n_{k+1}}(\lambda_k(i))}{w_1(\lambda_{k+1}(i)) \cdots w_{n_{k+1}}(\lambda_{k+1}(i))} \leq \varepsilon = e^{-N}$$

and we conclude by taking the logarithm. The converse direction is slightly more difficult. We fix $K \subset \Lambda$ compact, $\tau > 0$, $N \geq 1$ and $\varepsilon > 0$ and we apply (b) for K , τ and $N_0 \geq N$ whose value will be precised later. Let $a > 0$ be such that $\lambda(i) \geq a$ for all $\lambda \in K$ and all $i = 1, \dots, d$. Then, for all $i = 1, \dots, d$,

$$\begin{aligned} \left\| \sum_{k=1}^q \frac{1}{w_1(\lambda_k(i)) \cdots w_{n_k}(\lambda_k(i))} e^{n_k} \right\| &\leq \sum_{k=1}^q \frac{1}{\exp(an_k)} \\ &\leq \sum_{j=N_0}^{+\infty} \frac{1}{\exp(aj)} < \varepsilon \end{aligned}$$

provided N_0 is large enough. Regarding (iii), for all $k = 1, \dots, q-1$, all $i = 1, \dots, d$ and all $l = 0, \dots, N$,

$$\begin{aligned} &\left\| \sum_{j=k+1}^q \frac{w_{n_j-n_k+l+1}(\lambda_k(i)) \cdots w_{n_j+l}(\lambda_k(i))}{w_{l+1}(\lambda_j(i)) \cdots w_{n_j+l}(\lambda_j(i))} e^{n_j-n_k+l} \right\| \\ &\leq \sum_{j=k+1}^{+\infty} \exp(-((\lambda_j(i)n_j - \lambda_k(i)n_k))). \end{aligned}$$

Now,

$$\begin{aligned} \lambda_j(i)n_j - \lambda_k(i)n_k &= \sum_{s=k}^{j-1} (\lambda_{s+1}(i)n_{s+1} - \lambda_s(i)n_s) \\ &\geq (j-k)N_0. \end{aligned}$$

Again, provided N_0 is large enough, we get that condition (c) (iii) of Theorem 2.1 is satisfied. \square

Remark 2.3 If we work with the family of weight $(w_n(a))_{a>0}$, with $w_1(a) \cdots w_n(a) = \exp(an^\alpha)$, $\alpha \in (0, 1)$, we still have a necessary condition for common hypercyclicity if we replace (b) (ii) by: for all $k = 1, \dots, q-1$, for all $j = k+1, \dots, q$, for all $i = 1, \dots, d$,

$$(\lambda_j(i) - \lambda_k(i))n_j^\alpha + \lambda_k(i)(n_j - n_k)^\alpha > N$$

and this condition is even sufficient on c_0 . The lack of linearity when $\alpha \neq 1$ prevents us to go further.

We are now ready for the proof of Theorem 1.1.

Proof of Theorem 1.1. We may assume that $\Lambda = f([0, 1])$ with f satisfying

$$\exists C > 0, \forall (s, t) \in [0, 1]^2, \|f(s) - f(t)\| \leq C|s - t|.$$

Let $\tau > 0$, $N \geq 1$ and let $M > 0$ be very large (at least $M \geq N$). We set $n_k = kM$ for $k \geq 1$. We also define the sequence $(t_k)_{k \geq 1}$ by $t_1 = 0$ and $t_{k+1} = t_k + \frac{\tau}{Cn_k}$. Let $q \geq 1$ be the greatest integer such that $t_q \leq 1$ and define, for $k = 1, \dots, q-1$, $I_k = [t_k, t_{k+1}]$ and $I_q = [t_q, 1]$. For all $k = 1, \dots, q$, we set $\Lambda_k = f(I_k)$ and for $i = 1, \dots, d$, we define $\lambda_k(i)$ as the maximum of the i -th coordinate of the elements of Λ_k . The Lipschitz condition on f implies that

$$\forall k = 1, \dots, q, \Lambda_k \subset \prod_{i=1}^d \left[\lambda_k(i) - \frac{\tau}{n_k}, \lambda_k(i) \right]$$

$$\forall k = 1, \dots, q-1, \forall i = 1, \dots, d, |\lambda_{k+1}(i) - \lambda_k(i)| \leq C|t_{k+2} - t_k| \leq \frac{2\tau}{kM}.$$

Therefore,

$$\begin{aligned} \lambda_{k+1}(i)n_{k+1} - \lambda_k(i)n_k &\geq \lambda_k(i)n_{k+1} - \frac{2\tau}{kM}n_{k+1} - \lambda_k(i)n_k \\ &\geq \lambda_k(i)M - \frac{2\tau}{kM} \times (k+1)M \\ &\geq \lambda_k(i)M - 4\tau. \end{aligned}$$

Provided M is large enough, we can ensure that (c)(ii) of Theorem 2.2 is satisfied. \square

3 On the size of the parameter sets for common hypercyclicity

We will prove a slightly more precise result than Theorem 1.2. Let us recall the relevant definitions (we refer to [6] and [10] for more on this subject). If $\phi : (0, +\infty) \rightarrow (0, +\infty)$ is a nondecreasing continuous function satisfying $\lim_{0+} \phi = 0$ (ϕ is called a *dimension function* or a *gauge function*), the ϕ -Hausdorff outer measure of a set $E \subset \mathbb{R}^d$ is

$$\mathcal{H}^\phi(E) = \lim_{\varepsilon \rightarrow 0} \inf_{r \in R_\varepsilon(E)} \sum_{B \in r} \phi(|B|),$$

where $R_\varepsilon(E)$ is the set of (countable) coverings of E with balls B of diameter $|B| \leq \varepsilon$. When $\phi(x) = \phi_s(x) = x^s$, we write for short \mathcal{H}^s instead of \mathcal{H}^{ϕ_s} . The *Hausdorff dimension* of a set E is defined by

$$\dim_{\mathcal{H}}(E) := \sup\{s > 0 : \mathcal{H}^s(E) > 0\} = \inf\{s > 0 : \mathcal{H}^s(E) = 0\}.$$

Theorem 3.1 *Let $\Lambda \subset \mathbb{R}^d$, let $(T_\lambda)_{\lambda \in \Lambda}$ be a family of operators acting on the Banach space X . Assume that there exist a function $\psi : \mathbb{N} \rightarrow (0, +\infty)$, $v \in X$, $\delta > 0$ such that, for all $\lambda, \mu \in \Lambda$, for all $n \in \mathbb{N}$ and all $u \in X$ satisfying*

$$\|T_\lambda^n u - v\| < \delta \quad \text{and} \quad \|T_\mu^n u - v\| < \delta,$$

one has

$$\|T_\lambda^n u - T_\mu^n u\| \geq \psi(n) \|\lambda - \mu\|.$$

If $\bigcap_{\lambda \in \Lambda} HC(T_\lambda) \neq \emptyset$, then $\mathcal{H}^\phi(\Lambda) = 0$ for any gauge function ϕ such that $\sum_n \phi\left(\frac{2\delta}{\psi(n)}\right) < +\infty$.

Proof. Let ψ, v, δ be given by the assumptions and let $u \in \bigcap_{\lambda \in \Lambda} HC(T_\lambda)$. Define $\Lambda_n = \{\lambda \in \Lambda : \|T_\lambda^n u - v\| < \delta\}$. Then $\text{diam}(\Lambda_n) \leq 2\delta/\psi(n)$. Indeed, pick $\lambda, \mu \in \Lambda_n$ and observe that

$$\begin{aligned} \psi(n) \|\lambda - \mu\| &\leq \|T_\lambda^n u - T_\mu^n u\| \\ &\leq \|T_\lambda^n u - v\| + \|T_\mu^n u - v\| \\ &\leq 2\delta. \end{aligned}$$

In particular, since the condition $\sum_n \phi\left(\frac{2\delta}{\psi(n)}\right) < +\infty$ implies that $\psi(n) \rightarrow +\infty$, the theorem follows from the fact that, for any $N \geq 1$, $\Lambda \subset \bigcup_{n \geq N} \Lambda_n$. \square

Theorem 1.2 follows directly from the above theorem by considering $\psi(n) = Cn^\alpha$ and $\phi(x) = x^s$ for any $s > 1/\alpha$. Moreover, we can easily apply this to families of weighted shifts.

Corollary 3.2 *Let $X = \ell_p(\mathbb{N})$, $p \in [1, +\infty[$ or $X = c_0(\mathbb{N})$. Let $I \subset \mathbb{R}$, let $(w(a))_{a \in I}$ be a family of weights. Assume that there exist $C, \alpha > 0$ such that, for any $n \geq 1$, for any $a, b \in I$,*

$$\left| \sum_{j=1}^n \log w_j(a) - \sum_{j=1}^n \log w_j(b) \right| \geq Cn^\alpha |a - b|.$$

Then, for any $\Lambda \subset I^d$,

$$\bigcap_{\lambda \in \Lambda} HC(B_{w(\lambda(1))} \times \cdots \times B_{w(\lambda(d))}) \neq \emptyset \implies \dim_{\mathcal{H}}(\Lambda) \leq \frac{1}{\alpha}.$$

Proof. Let $v = (e_0, \dots, e_0)$. Let $\lambda, \mu \in \Lambda$, $n \in \mathbb{N}$, $u \in X \times \cdots \times X$ be such that

$$\|(B_{w(\lambda(1))} \times \cdots \times B_{w(\lambda(d))})^n u - v\| < \frac{1}{2} \quad \text{and} \quad \|(B_{w(\mu(1))} \times \cdots \times B_{w(\mu(d))})^n u - v\| < \frac{1}{2}.$$

Let $1 \leq k \leq d$. Looking at the k -th coordinate, we get

$$|w_1(\lambda(k)) \cdots w_n(\lambda(k)) u_n(k) - 1| < 1/2 \quad \text{and} \quad |w_1(\mu(k)) \cdots w_n(\mu(k)) u_n(k) - 1| < 1/2.$$

Now, setting $\varepsilon_n = \|(B_{w(\lambda(1))} \times \cdots \times B_{w(\lambda(d))})^n u - (B_{w(\mu(1))} \times \cdots \times B_{w(\mu(d))})^n u\|$, we get

$$\varepsilon_n \geq |w_1(\lambda(k)) \cdots w_n(\lambda(k)) - w_1(\mu(k)) \cdots w_n(\mu(k))| \cdot |u_n(k)|.$$

Assume for instance that $w_1(\mu(k)) \cdots w_n(\mu(k)) \geq w_1(\lambda(k)) \cdots w_n(\lambda(k))$. Then

$$\varepsilon_n \geq \left| \frac{w_1(\mu(k)) \cdots w_n(\mu(k))}{w_1(\lambda(k)) \cdots w_n(\lambda(k))} - 1 \right| \cdot w_1(\lambda(k)) \cdots w_n(\lambda(k)) \cdot |u_n(k)|$$

$$\geq \frac{1}{2} \left| \frac{w_1(\mu(k)) \cdots w_n(\mu(k))}{w_1(\lambda(k)) \cdots w_n(\lambda(k))} - 1 \right|.$$

Now,

$$\begin{aligned} \left| \frac{w_1(\mu(k)) \cdots w_n(\mu(k))}{w_1(\lambda(k)) \cdots w_n(\lambda(k))} - 1 \right| &= \exp \left(\sum_{j=1}^n \log w_j(\mu(k)) - \sum_{j=1}^n \log w_j(\lambda(k)) \right) - 1 \\ &\geq \exp(Cn^\alpha |\lambda(k) - \mu(k)|) - 1 \\ &\geq Cn^\alpha |\lambda(k) - \mu(k)|. \end{aligned}$$

We conclude by applying Theorem 1.2 □

In the case of the multiples of the backward shift, we get the following result:

Corollary 3.3 *Let $d \geq 1$, $\Lambda \subset (0, +\infty)^d$ and $X = \ell_p(\mathbb{N})$, $p \in [1, +\infty)$ or $X = c_0(\mathbb{N})$. If $\bigcap_{\lambda \in \Lambda} HC(e^{\lambda(1)}B \times \cdots \times e^{\lambda(d)}B) \neq \emptyset$ then $\dim_{\mathcal{H}}(\Lambda) \leq 1$.*

Fixing $\alpha \in (0, 1]$, we can also apply Corollary 3.2 to the more general case of weights defined by $w_1(a) \cdots w_n(a) = \exp(an^\alpha)$ or by $w_n(a) = 1 + \frac{a}{n^{1-\alpha}}$ for all $n \geq 1$ in order to get Corollary 1.3. We point out the following example which will be useful later.

Example 3.4 Let $\alpha \in (0, 1]$, and $X = \ell_p(\mathbb{N})$, $p \in [1, +\infty)$ or $X = c_0(\mathbb{N})$. Let $(w(a))_{a>0}$ be the family of weights defined by $w_1(a) \cdots w_n(a) = \exp(an^\alpha)$ (resp. by $w_n(a) = 1 + \frac{a}{n^{1-\alpha}}$) for all $n \geq 1$. If $\bigcap_{\lambda \in (0, +\infty)^d} HC(B_{w(\lambda(1))} \times \cdots \times B_{w(\lambda(d))}) \neq \emptyset$, then $\alpha \leq 1/d$.

4 A common hypercyclicity criterion in several dimensions

4.1 Why this statement? Why this proof?

This section is purely expository. We intend to explain the assumptions of Theorem 1.5 and to talk a few words to introduce its proof. Let $\alpha \in (0, 1/2)$ and let w be the weight defined by $w_1(a) \cdots w_n(a) = \exp(an^\alpha)$. In view of Corollary 3.4, a plausible statement is that $(B_{w(a)} \times B_{w(b)})_{(a,b) \in [1,2]^2}$ admits a common hypercyclic vector. To prove it and apply either the Basic Criterion or Theorem 2.1, we need a covering of $\Lambda = [1, 2]^2$. A natural covering is given by the set Γ_m of the closed dyadic cubes of width 2^{-m} . We have to order these cubes, $\Gamma_m = (\Lambda_k)_{k=1, \dots, 4^m}$, to fix $\lambda_k = (a_k, b_k) \in \Lambda_k$ and to associate an increasing sequence $(n_k)_{k=1, \dots, 4^m}$ to this covering. Because we are working on \mathbb{R}^2 , it is not clear how we have to order the dyadic cubes. Figure C.1 shows three natural candidates.

This order and the associated sequence (n_k) are very important and we know that they at least have to satisfy the following conditions:

- $n_{4^m}^\alpha$ cannot be greater than 2^m , so that $\Lambda_k \subset B(\lambda_k, C/n_k^\alpha)$ for some $C > 0$ and for all $k = 1, \dots, 4^m$ (see Theorem 2.1 (c) (i));

Fig. C.1 – How to order the dyadic covering ($m = 2$)

Λ_{13}	Λ_{14}	Λ_{15}	Λ_{16}	Λ_{16}	Λ_{15}	Λ_{14}	Λ_{13}	Λ_{16}	Λ_{15}	Λ_{12}	Λ_{11}
Λ_9	Λ_{10}	Λ_{11}	Λ_{12}	Λ_9	Λ_{10}	Λ_{11}	Λ_{12}	Λ_{13}	Λ_{14}	Λ_9	Λ_{10}
Λ_5	Λ_6	Λ_7	Λ_8	Λ_8	Λ_7	Λ_6	Λ_5	Λ_4	Λ_3	Λ_8	Λ_7
Λ_1	Λ_2	Λ_3	Λ_4	Λ_1	Λ_2	Λ_3	Λ_4	Λ_1	Λ_2	Λ_5	Λ_6
(a) First ordering				(b) Second ordering				(c) Third ordering			

- for all $k = 1, \dots, 4^m - 1$ and all $j = k + 1, \dots, 4^m$,

$$\begin{cases} (a_j - a_k)n_j^\alpha + a_k(n_j - n_k)^\alpha > 0 \\ (b_j - b_k)n_j^\alpha + b_k(n_j - n_k)^\alpha > 0 \end{cases}$$

(see Remark 2.3).

The last conditions are always satisfied if $a_j > a_k$ and $b_j > b_k$ but are relevant if there is a backward jump between Λ_k and Λ_j , namely if either $a_j < a_k$ or $b_j < b_k$. Suppose for instance that $a_j < a_k$. In that case a small computation shows that we must have

$$n_j \geq \left(\frac{1}{1 - \frac{(a_k - a_j)^{1/\alpha}}{a_k^{1/\alpha}}} \right) n_k \geq \left(\frac{1}{1 - \frac{(a_k - a_j)^{1/\alpha}}{2^{1/\alpha}}} \right) n_k. \quad (\text{C.4})$$

Let us now discuss what this implies on each of the three orderings.

1. for the first ordering, there are $2^m - 1$ backward jumps of size around 1, say at least $1/2$, between two consecutive dyadic cubes, namely we must have

$$n_{l \cdot 2^{m+1}} \geq \left(\frac{1}{1 - \frac{1}{2^{2/\alpha}}} \right) n_{l \cdot 2^m}$$

for $l = 1, \dots, 2^m - 1$. Hence we will have at least

$$n_{4^m} \geq \left(\frac{1}{1 - \frac{1}{2^{2/\alpha}}} \right)^{2^m - 1} n_1$$

which is much bigger than $2^{m/\alpha}$.

2. for the second ordering, we have 2^{m-1} backward jumps of size around 1, say again at least $1/2$, now between the cubes $\Lambda_{(2l+1) \cdot 2^{m+1}}$ and $\Lambda_{(2l+2) \cdot 2^m}$, for $l = 0, \dots, 2^{m-1} - 1$. Therefore we must have

$$n_{(2l+2) \cdot 2^m} \geq \left(\frac{1}{1 - \frac{1}{2^{2/\alpha}}} \right) n_{(2l+1) \cdot 2^{m+1}},$$

$l = 0, \dots, 2^{m-1} - 1$, which again implies that n_{4^m} will be much bigger than expected since

$$n_{4^m} \geq \left(\frac{1}{1 - \frac{1}{2^{2/\alpha}}} \right)^{2^{m-1}} n_{2^{m+1}}.$$

3. An important part of the proof of Theorem 1.5 will be to show that the third way to order the covering is much more economical from this point of view (heuristically speaking, because the big backward jumps are not consecutive, see Λ_{10} , Λ_{13} , Λ_{11} and Λ_{16}). More specifically, we will be able to exhibit an increasing sequence (n_k) such that there exists some $D > 0$ satisfying

$$\forall 1 \leq k < j \leq 4^m, \forall \lambda \in \Lambda_k, \forall \mu \in \Lambda_j, \|\lambda - \mu\| \leq D \left(\frac{n_j - n_k}{n_k} \right)^\alpha \quad (\text{C.5})$$

and such that $n_{4^m}^\alpha$ is smaller than 2^m .

Thus the third way to order the covering is very well adapted to the problem of finding a common hypercyclic vector for the family $(B_{w(a)} \times B_{w(b)})_{(a,b) \in [1,2]^2}$. That is why we will use this ordering in the general case, so the assumption (a) in Theorem 1.5 becomes very natural.

4.2 About the homogeneous box dimension

In this subsection, we discuss the property of having homogeneous box dimension at most γ . We first recall the classical notion of the upper box dimension. Let $\Lambda \subset \mathbb{R}^d$ be compact. Its upper box dimension is defined by

$$\overline{\dim}_B(\Lambda) = \limsup_{\varepsilon \rightarrow 0} \frac{\log N(\varepsilon)}{\log(1/\varepsilon)},$$

where $N(\varepsilon)$ denotes the smallest number of cubes of size $\varepsilon > 0$ which are needed to cover Λ . We do not change the definition if we only allow ε to be equal to $c\rho^m$ for some $c > 0$, some $\rho \in (0, 1)$ and all $m \in \mathbb{N}$. Namely, for all $\rho \in (0, 1)$,

$$\overline{\dim}_B(\Lambda) = \limsup_{m \rightarrow +\infty} \frac{\log N(c\rho^m)}{-m \log \rho}.$$

Let us now have a look on the property of having homogeneous box dimension at most γ . The first two conditions could be easily rephrased by saying that Λ has upper box dimension at most γ : setting $\rho = 1/r^{1/\gamma}$, for each $m \geq 1$, you can cover Λ by $1/\rho^{\gamma m}$ balls of radius $C(\Lambda)\rho^m$. The last condition adds the homogeneity requirement: the covering of depth m should be, in a precise sense, a refinement of the covering of depth $m - 1$.

It is therefore clear that

$$\dim_{\mathcal{H}}(\Lambda) \leq \overline{\dim}_B(\Lambda) \leq \dim_{\text{HB}}(\Lambda).$$

It turns out that, in many cases, one has equality or at least we can prove that Λ has homogeneous box dimension at most γ . Also, any compact subset of \mathbb{R}^d has homogeneous box dimension at most d : we may assume that $\Lambda \subset [0, 1]^d$ and we define $r = 2^d$ and $\Lambda_{\mathbf{k}}$ as the intersections of Λ with the dyadic subcubes of $[0, 1]^d$ with width 2^{-m} .

We can also provide positive results for compact selfsimilar sets. A compact set $\Lambda \subset \mathbb{R}^d$ is called *selfsimilar* provided there exists r similarities s_1, \dots, s_r with respective ratio $\rho_1, \dots, \rho_r \in (0, 1)$ such that $\Lambda = \bigcup_{i=1}^r s_i(\Lambda)$. For $\mathbf{k} \in I_r^m$, define $s_{\mathbf{k}} = s_{k_1} \circ \dots \circ s_{k_r}$. Let γ be defined by

$$\gamma = \max \left\{ \frac{-\log r}{\log(\rho_i)} : i = 1, \dots, r \right\}.$$

Then setting $\Lambda_{\mathbf{k}} = s_{\mathbf{k}}(\Lambda)$, one can show that Λ has homogeneous box dimension at most γ . If all the ratios are equal to the same ρ , then Λ has homogeneous box dimension at most $-\log r / \log \rho$, and when Λ satisfies the open set condition (namely there exists $V \subset \mathbb{R}^d$ open such that $\bigcup_{i=1}^r s_i(V) \subset V$ and $s_i(V) \cap s_j(V) = \emptyset$ for $i \neq j$), it is well-known that the Hausdorff dimension of Λ equals this value. Hence, in that case

$$\dim_{\mathcal{H}}(\Lambda) = \overline{\dim}_{\mathbb{B}}(\Lambda) = \dim_{\text{HB}}(\Lambda) = \frac{-\log r}{\log \rho}.$$

Another interesting example is that of Hölder curves. Assume that $\Lambda = f([0, 1])$ where $f : [0, 1] \rightarrow \mathbb{R}^d$ satisfies

$$\|f(s) - f(t)\| \leq C|s - t|^\alpha,$$

$C > 0$, $\alpha \in (0, 1)$. We set $r = 2$ and for $\mathbf{k} \in I_r^m$, we define $I_{\mathbf{k}}$ as the dyadic interval

$$I_{\mathbf{k}} = \left[\sum_{i=1}^m \frac{k_i - 1}{2^i}, \sum_{i=1}^m \frac{k_i - 1}{2^i} + \frac{1}{2^m} \right].$$

Then define $\Lambda_{\mathbf{k}} = f(I_{\mathbf{k}})$. We get immediately that

$$\text{diam}(\Lambda_{\mathbf{k}}) \leq C \text{diam}(I_{\mathbf{k}})^\alpha = C \left(\frac{1}{2^\alpha} \right)^m.$$

Therefore, Λ has homogeneous box dimension at most $1/\alpha$ and there are well-known examples where the box dimension of such a curve (hence, its homogeneous box dimension) is exactly equal to $1/\alpha$.

4.3 Examples

Before proceeding with the proof of Theorem 1.5, let us show how this theorem can be applied to a direct sum of weighted shifts. Let $X = \ell_p(\mathbb{N})$ or $c_0(\mathbb{N})$, $p \in [1, +\infty)$, $I \subset \mathbb{R}$ compact and $(w(a))_{a \in I}$ be a continuous family of positive weights. We keep the notations of Section 2, namely for $\lambda \in I^d$, we denote by $T_\lambda = B_{w(\lambda(1))} \times \cdots \times B_{w(\lambda(d))}$ and by $S_\lambda = F_{w^{-1}(\lambda(1))} \times \cdots \times F_{w^{-1}(\lambda(d))}$. We also set $\mathcal{D} = c_{00}^d$ and we endow X^d with

$$\|u\| = \max(\|u(1)\|, \dots, \|u(d)\|).$$

We first point out that (b) of Theorem 1.5 is implied by a Lipschitz inequality on f_n where $f_n(a) = \sum_{k=1}^n \log(w_k(a))$ as in Theorem 2.1. Indeed, suppose that there exist $\alpha > 0$ and $C > 0$ such that, for all $a, b \in I$,

$$|f_n(a) - f_n(b)| = \left| \sum_{j=1}^n (\log w_j(a) - \log w_j(b)) \right| \leq C n^\alpha |a - b|. \quad (\text{C.6})$$

Then observe first that (C.6) implies that, for all $L > 0$, there exists $C' > 0$ such that, for all $a, b \in I$, for all $l \in [0, L]$,

$$\left| \sum_{j=l+1}^{n+l} (\log w_j(a) - \log w_j(b)) \right| \leq C' n^\alpha |a - b|.$$

Let $u = (u(1), \dots, u(d)) \in \mathcal{D}$ and consider $L > 0$ such that the support of each $u(i)$ is contained in $[0, L]$. For all $\lambda, \mu \in I^d$,

$$\begin{aligned} \|T_\lambda^n S_\mu^n u - u\| &\leq \|u\| \max_{i=1, \dots, d} \max_{l=0, \dots, L} \left| \frac{w_{l+1}(\lambda(i)) \cdots w_{l+n}(\lambda(i))}{w_{l+1}(\mu(i)) \cdots w_{l+n}(\mu(i))} - 1 \right| \\ &\leq \|u\| \max_{i=1, \dots, d} \max_{l=0, \dots, L} \left| \exp \left(\sum_{j=l+1}^{n+l} (\log w_j(\lambda(i)) - \log w_j(\mu(i))) \right) - 1 \right| \\ &\leq \|u\| (\exp(C'n^\alpha \|\lambda - \mu\|) - 1). \end{aligned}$$

Clearly, for all $\varepsilon > 0$, we can choose a sufficiently small $\tau > 0$ (depending on $\varepsilon, \|u\|, C'$) for all $n \geq 1$, if $\|\lambda - \mu\| \leq \tau/n^\alpha$, then

$$\|T_\lambda^n S_\mu^n u - u\| < \varepsilon.$$

Hence, it is enough to assume (C.6) to get (b) of Theorem 1.5. Let us now turn to (a), under the assumption (C.6). What we need is the product $w_1(a) \cdots w_n(a)$ not to be too small.

Lemma 4.1 *Let $\alpha \in (0, 1]$ and assume that there exist C_1, C_2 and $C_3 > 0$ such that*

- $a \in I \mapsto \sum_{j=1}^n \log(w_j(a))$ is $C_1 n^\alpha$ -Lipschitz;
- $\inf_{a \in I} w_1(a) \cdots w_n(a) \geq C_2 \exp(C_3 n^\alpha)$.

There exists $D > 0$ such that, for all $u \in \mathcal{D}$, there exist $M > 0$ and $N > 0$ such that, for all $\lambda, \mu \in I^d$, for all $n \geq 0$ and $k \geq N$ such that $\|\lambda - \mu\| \leq D \frac{k^\alpha}{(n+k)^\alpha}$, then

$$\|T_\lambda^{n+k} S_\mu^n u\| \leq \frac{M}{k} \quad \text{and} \quad \|T_\lambda^n S_\mu^{n+k} u\| \leq \frac{M}{k}.$$

Proof. Again we fix $L > 0$ such that the support of each $u(i)$ is contained in $[0, L]$. Choosing $N > L$, we will have $T_\lambda^{n+k} S_\mu^n u = 0$ provided $k \geq N$. On the other hand,

$$\|T_\lambda^n S_\mu^{n+k} u\| \leq \|u\| \max_{i=1, \dots, d} \max_{l=0, \dots, L} \underbrace{\frac{w_{l+1}(\lambda(i)) \cdots w_{l+n+k}(\lambda(i))}{w_{l+1}(\mu(i)) \cdots w_{l+n+k}(\mu(i))}}_{F_1} \times \underbrace{\frac{1}{w_{l+1}(\lambda(i)) \cdots w_{l+k}(\lambda(i))}}_{F_2}$$

We write

$$\begin{aligned} F_1 &\leq \frac{w_1(\lambda(i)) \cdots w_{l+n+k}(\lambda(i))}{w_1(\mu(i)) \cdots w_{l+n+k}(\mu(i))} \times \frac{w_1(\mu(i)) \cdots w_l(\mu(i))}{w_1(\lambda(i)) \cdots w_l(\lambda(i))} \\ &\leq \exp(C_1((l+n+k)^\alpha + l^\alpha) \|\lambda - \mu\|) \\ &\leq \exp(2C_1(n+k)^\alpha \|\lambda - \mu\|) \end{aligned}$$

provided N , hence k , is large enough. If we add the assumption $\|\lambda - \mu\| \leq \frac{Dk^\alpha}{(n+k)^\alpha}$, we get

$$F_1 \leq \exp(2DC_1 k^\alpha).$$

On the other hand

$$F_2 \leq \left(\sup_{a \in I} \sup_{l \in [1, L]} \max(1, w_l(a)) \right)^L \frac{1}{C_2} \exp(-C_3 k^\alpha).$$

Hence

$$\|T_\lambda^n S_\mu^{n+k} u\| \leq M \exp((2DC_1 - C_3)k^\alpha)$$

for some constant M depending only on u and on the weight, but not on k and n . Thus, we get the result by picking D sufficiently small. \square

Summarizing we have obtained the following readable corollary.

Corollary 4.2 *Let $\gamma \in (0, d]$ and let $\Lambda \subset I^d$ be a compact set with homogeneous box dimension at most γ . Let $\alpha \in (0, 1/\gamma)$ and let $(w(a))_{a \in I}$ be a continuous family of positive weights. Assume that there exist $C_1, C_2, C_3 > 0$ and $N \geq 0$ such that, for all $n \geq N$,*

- $a \in I \mapsto \sum_{j=1}^n \log(w_j(a))$ is $C_1 n^\alpha$ -Lipschitz;
- $\inf_{a \in I} w_1(a) \cdots w_n(a) \geq C_2 \exp(C_3 n^\alpha)$.

Then $\bigcap_{\lambda \in \Lambda} HC(B_{w(\lambda(1))} \times \cdots \times B_{w(\lambda(d))})$ is a dense G_δ subset of X^d .

Corollary 4.2 yields immediately Corollary 1.6 stated in the introduction. Combining Example 1.7 and Example 3.4, we get the following interesting corollary.

Corollary 4.3 *Let $d \geq 1$. There exists a family of operators $(T_a)_{a \in (0, +\infty)}$ on $\ell_p(\mathbb{N})$, $p \in [1, +\infty)$, or on $X = c_0(\mathbb{N})$ such that $(T_{\lambda(1)} \oplus \cdots \oplus T_{\lambda(d)})_{\lambda \in (0, +\infty)^d}$ admits a common hypercyclic vector but $(T_{\lambda(1)} \oplus \cdots \oplus T_{\lambda(d+1)})_{\lambda \in (0, +\infty)^{d+1}}$ does not.*

Proof. Choose $T_a = B_{w(a)}$ with $w_1(a) \cdots w_n(a) = \exp(an^\alpha)$ and

$$\frac{1}{d+1} < \alpha < \frac{1}{d}.$$

\square

Observe that Example 1.7 and Example 3.4 do not settle the case $\alpha = 1/d$.

Question 4.4 *Let $d \geq 2$, $X = \ell_p(\mathbb{N})$, $p \in [1, +\infty)$, or $X = c_0(\mathbb{N})$ and let $(w(a))_{a > 0}$ be the family of weights defined by $w_1(a) \cdots w_n(a) := \exp(an^{1/d})$. Does $(B_{w(\lambda(1))} \times \cdots \times B_{w(\lambda(d))})_{\lambda \in (0, +\infty)^d}$ admit a common hypercyclic vector?*

Note that it is also possible to have a family of operators $(T_a)_{a \in (0, +\infty)}$ such that for every $d \geq 1$, $(T_{\lambda(1)} \oplus \cdots \oplus T_{\lambda(d)})_{\lambda \in (0, +\infty)^d}$ admits a common hypercyclic vector.

Example 4.5 Let $d \geq 1$, $X = \ell_p(\mathbb{N})$, $p \in [1, +\infty)$, or on $X = c_0(\mathbb{N})$ and $(w(a))_{a > 0}$ be the family of weights defined by $w_1(a) \cdots w_n(a) = 2^n n^a$. Then $\bigcap_{\lambda \in (0, +\infty)^d} HC(B_{w(\lambda(1))} \times \cdots \times B_{w(\lambda(d))}) \neq \emptyset$.

Proof. Let $\alpha \in (0, 1)$. By definition, $a \in (0, +\infty) \mapsto \sum_{i=1}^n \log(w_i(a))$ is $\log(n)$ -Lipschitz for all $n \geq 1$. In particular it is $C_1 n^\alpha$ -Lipschitz if C_1 is big enough. We then observe that $w_1(a) \cdots w_n(a) = 2^n n^a \geq \exp(C_2 n^\alpha)$ for some sufficiently small $C_2 > 0$. Hence the result follows from Corollary 4.2. \square

Following the same lines we can generalize the previous example by taking $\rho > 1$, $\alpha > 0$ and defining $w_1(a) \cdots w_n(a) := \rho^{n^\alpha} n^a$.

Corollary 4.2 can also be applied to products of weighted shifts, in exactly the same way, when $\Lambda \subset \mathbb{R}^d$ is a β -Hölder curve, leading to Example 1.8 since a β -Hölder curve has homogeneous box dimension at most $1/\beta$. When $\beta = 1$, this last result is slightly weaker than Theorem 1.1, leading to the following question.

Question 4.6 *Let $\Lambda \subset (0, +\infty)^d$ be a β -Hölder curve for some $\beta \in (0, 1)$. Let $(w(a))_{a>0}$ be the family of weights defined by $w_1(a) \cdots w_n(a) = \exp(an^\beta)$ for all $n \geq 1$. Does $(B_{w(\lambda(1))} \times \cdots \times B_{w(\lambda(d))})_{\lambda \in \Lambda}$ admit a common hypercyclic vector?*

Now let us focus on the case $w_n(a) = 1 + \frac{a}{n}$. The product $w_1(a) \cdots w_n(a)$ behaves like n^a , therefore $a \mapsto \sum_{i=1}^n \log(w_i(a))$ is $\log(n)$ -Lipschitz. In particular, it is Cn^α -Lipschitz for all $\alpha > 0$, which means that we may verify (a) of Theorem 1.5 with arbitrarily small values of β . Nevertheless the product $w_1(a) \cdots w_n(a)$ does not grow sufficiently fast in order to apply Corollary 4.2? This leads us to the forthcoming result, suitable for weights with slow varying weights.

Corollary 4.7 *Let $X = \ell_p(\mathbb{N})$, $p \in [1, +\infty)$, or $X = c_0(\mathbb{N})$ and let $(w(a))_{a \in I}$ be a continuous family of positive weights. Assume that there exist $C_1, C_2, \kappa > 0$ and $N \geq 0$ such that, for all $n \geq N$,*

- $a \in I \mapsto \sum_{j=1}^n \log(w_j(a))$ is $C_1 \log(n)$ -Lipschitz;
- $\inf_{a \in I} w_1(a) \cdots w_n(a) \geq C_2 n^\kappa$.

Then $\bigcap_{\lambda \in I^d} HC(B_{w(\lambda(1))} \times \cdots \times B_{w(\lambda(d))}) \neq \emptyset$.

Proof. We follow the proof of Corollary 4.2. Let C_1, C_2, κ, N be given by the assumptions. Let $\alpha \in (0, 1/d)$ be such that $\kappa/\alpha > d$ and let $\beta \in (0, \kappa)$ be such that $\beta/\alpha > d$. Condition (b) of Theorem 1.5 is clearly satisfied. To prove (a), we keep the same notations. Provided $\|\lambda - \mu\| \leq \frac{Dk^\alpha}{(n+k)^\alpha}$, we may write

$$\begin{aligned} F_1 &\leq \exp(2C_1 \log(n+k) \|\lambda - \mu\|) \\ &\leq \exp\left(2DC_1 \log(n+k) \frac{k^\alpha}{(n+k)^\alpha}\right). \end{aligned}$$

Now, provided k is large enough (we require $\alpha \log k > 1$), the function $n \mapsto \frac{\log(n+k)}{(n+k)^\alpha}$ is decreasing on $[0, +\infty)$, so that

$$F_1 \leq \exp(2DC_1 \log k) = k^{2DC_1}.$$

This implies that

$$F_1 F_2 \leq C' k^{2DC_1 - \kappa} \leq C' k^{-\beta}$$

provided D has been chosen so small that $\beta + 2DC_1 < \kappa$. Hence, condition (a) of Theorem 1.5 is also satisfied. \square

Example 4.8 Let $X = \ell_p(\mathbb{N})$, $p \in [1, +\infty)$, or $X = c_0(\mathbb{N})$ and let $(w(a))_{a>0}$ be the family of weights defined by $w_n(a) = 1 + \frac{a}{n}$ (resp. $w_n(a) := (1 + \frac{1}{n})^a$). Then $\bigcap_{\lambda \in (0, +\infty)^d} HC(B_{w(\lambda(1))} \times \cdots \times B_{w(\lambda(d))}) \neq \emptyset$.

Let us also show how we may apply Theorem 1.5 to get a common hypercyclic vector for $(e^a B \times e^b B)_{(a,b) \in \Lambda}$ with Λ a classical fractal set.

Example 4.9 Let $X = \ell_p(\mathbb{N})$, $p \in [1, +\infty)$, or $X = c_0(\mathbb{N})$. Let Λ be a homogeneous Cantor subset of $(0, +\infty)^2$ with dissection ratio $\rho \in (0, 1/4)$. Then $\bigcap_{(a,b) \in \Lambda} HC(e^a B \times e^b B) \neq \emptyset$.

Proof. We may apply Corollary 4.2 since Λ has homogeneous box dimension at most $-\ln 4 / \ln \rho < 1$ (we apply the definition with $r = 4$). \square

Corollary 3.3 and Example 4.9 leave open the case of the Cantor set with dissection ratio $\rho = 1/4$. More generally, for Λ a compact subset of $(0, +\infty)^2$, we know that

$$\dim_{\mathcal{H}}(\Lambda) > 1 \implies \bigcap_{(a,b) \in \Lambda} HC(e^a B \times e^b B) = \emptyset$$

$$\text{and } \dim_{\text{HB}}(\Lambda) < 1 \implies \bigcap_{(a,b) \in \Lambda} HC(e^a B \times e^b B) \neq \emptyset.$$

It is natural to ask whether we can go further. In the first implication, we cannot replace the Hausdorff dimension by the homogeneous box dimension.

Proposition 4.10 *There exists a compact subset $\Lambda \subset (0, +\infty)^2$ with $\dim_{\text{HB}}(\Lambda) = 2$ such that $\bigcap_{(a,b) \in \Lambda} HC(e^a B \times e^b B) \neq \emptyset$.*

Proof. Let $I = [1, 2] \times \{1\}$ and for any $n \geq 1$, any $0 < k < 2^n$, $I_{n,k} = \{1 + \frac{k}{2^n}\} \times [1, 1 + \frac{1}{n}]$. We consider

$$\Lambda = I \cup \bigcup_{n \geq 1} \bigcup_{0 < k < 2^n} I_{n,k}.$$

We first remark that Λ is closed and thus compact. Let $(\lambda_m, \mu_m)_{m \geq 1} \subset \Lambda$ be a sequence converging to $(\lambda, \mu) \in [1, 2]^2$. If $\mu_m = 1$ infinitely often then $\mu = 1$ and thus $(\lambda, \mu) \in \Lambda$. If we now assume that $\mu_m \neq 1$ for any m , then $(\lambda_m, \mu_m) \in I_{n_m, 2k_m+1}$ for a unique $n_m \geq 1$ and a unique $0 \leq k_m \leq 2^{n_m-1} - 1$. In particular, $\mu_m \in [1, 1 + \frac{1}{n_m}]$. Therefore, if $\sup_m n_m = \infty$, we get $(\lambda, \mu) \in \Lambda$ since $\mu = 1$ and if $\sup_m n_m < \infty$, up to an extraction, the sequences $(n_m)_{m \geq 1}$ and $(k_m)_{m \geq 1}$ are ultimately constant and $(\lambda, \mu) \in \Lambda$ since each $I_{n,k}$ is closed.

Since Λ is a countable union of Lipschitz curves, it is a consequence of Theorem 1.1 that $\bigcap_{(a,b) \in \Lambda} HC(e^a B \times e^b B) \neq \emptyset$. It remains to show that $\dim_{\text{HB}}(\Lambda) = 2$. Note that it suffices to show that $\overline{\dim}_B(\Lambda) \geq 2$. Let $m \geq 2$. How many cubes of size $\frac{1}{2^m}$ are needed to cover Λ ? To cover each fiber $I_{m-1,k}$, $0 < k < 2^{m-1}$, we need at least $2^m / (m-1)$ cubes of size 2^{-m} . Note that such a cube cannot intersect another fiber of the same generation $I_{m-1,l}$ with $l \neq k$. Therefore, in order to cover $\bigcup_{0 < k < 2^{m-1}} I_{m-1,k}$, and thus Λ , we need at least $(2^{m-1} - 1) \cdot 2^m / (m-1)$ cubes of size 2^{-m} . We conclude that

$$N(2^{-m}) \geq c \frac{4^m}{m}$$

for some $c > 0$. Therefore,

$$\overline{\dim}_B(\Lambda) \geq \lim_{m \rightarrow +\infty} \frac{m \log 4 - \log m}{m \log 2} = 2.$$

□

However the following question is open.

Question 4.11 *Let Λ be a compact subset of $(0, +\infty)^2$ such that $\dim_{\mathcal{H}}(\Lambda) < 1$. Does the family $(e^a B \times e^b B)_{(a,b) \in \Lambda}$ admit a common hypercyclic vector?*

At least, we can show that the condition $\dim_{\mathcal{H}}(\Lambda) \leq 1$ is not sufficient to obtain a common hypercyclic vector.

Proposition 4.12 *There exists a set $\Lambda \subset (0, +\infty)^2$ such that $\dim_{\mathcal{H}}(\Lambda) = 1$ and nevertheless $\bigcap_{(a,b) \in \Lambda} HC(e^a B \times e^b B) = \emptyset$.*

Proof. Let $\phi(x) = x/\log^2(x)$. Applying Theorem 3.1 as in the proof of Corollary 3.2, we know that $\mathcal{H}^\phi(\Lambda) = 0$ for any $\Lambda \subset (0, +\infty)^2$ such that $\bigcap_{(a,b) \in \Lambda} HC(e^a B \times e^b B) \neq \emptyset$. Consider now for Λ the Cantor set starting from $[1, 2]^2$ and with non-constant dissection ratio $\frac{1}{4} \times \frac{(j+1)^2}{j^2}$. Namely, $\Lambda = \bigcap_{m \geq 1} \Lambda_m$ where Λ_m consists in 4^m squares of width $(\frac{1}{4})^m (m+1)^2$. Then $\dim_{\mathcal{H}}(\Lambda) = 1$ and using the mass transference principle as in [6, Example 4.3], $\mathcal{H}^\phi(\Lambda) > 0$. Hence, $\bigcap_{(a,b) \in \Lambda} HC(e^a B \times e^b B) = \emptyset$. □

For this last example, it is easy to show that one also has $\dim_{\text{HB}}(\Lambda) = 1$.

4.4 A lemma on sequences of integers

We now proceed with the proof of Theorem 1.5. Let us start with Λ a compact subset of \mathbb{R}^d with homogeneous box dimension at most $\gamma \in (0, d]$. In order to apply the Basic Criterion, we will need a covering of Λ . Natural coverings are given by the definition of the homogeneous box dimension, namely by the sets $(\Lambda_{\mathbf{k}})_{\mathbf{k} \in I_r^m}$ for a given value of m . As pointed out above, the way we order these sets is very important. We will choose the ordering obtained by endowing I_r^m with its natural lexicographic order:

$$(i_1, \dots, i_m) < (j_1, \dots, j_m) \iff \exists p \in \{1, \dots, m\}, i_1 = j_1, \dots, i_{p-1} = j_{p-1} \text{ and } i_p > j_p.$$

We first define the sequence $(n_{\mathbf{k}})_{\mathbf{k} \in I_r^m}$.

Lemma 4.13 *Let $\alpha > 0$, $\rho \in (0, 1)$ and $r \geq 2$ be such that $\rho^{1/\alpha} r < 1$. Then there exist $c_1 > 0$ and $c_2 > 0$ such that, for all $m \geq 1$, for all $n_1 \geq 1$, for all $A > 0$, the sequence $(n_{\mathbf{k}})_{\mathbf{k} \in I_r^m}$ defined by*

$$\begin{aligned} n_{1, \dots, 1} &= n_1 \\ n_{k_1, \dots, k_m} &= \left\lfloor \frac{1}{1 - \rho^{p/\alpha}} n_{k_1, \dots, k_{p-1}, r, \dots, r} \right\rfloor + A \end{aligned}$$

for $p = 1, \dots, m$, $k_{p+1} = \dots = k_m = 1$, $k_p \neq 1$ satisfies

$$n_{r, \dots, r} \leq c_1 n_1 + c_2 r^m A.$$

A key point of this lemma is that c_1 and c_2 depend neither on m nor on n_1 nor on A . We will do the proof by induction on m . Nevertheless, we need to introduce auxiliary sequences to keep track of the involved constants at each step.

Lemma 4.14 *Let $\alpha > 0$, $\rho \in (0, 1)$ and $r \geq 2$. Let also $B \in (0, 1]$, $m \geq 1$, $n_1 \geq 1$ and $A > 0$. Then the sequence $(n_{\mathbf{k}})_{\mathbf{k} \in I_r^m}$ defined by*

$$\begin{aligned} n_{1, \dots, 1} &= n_1 \\ n_{k_1, \dots, k_m} &= \left\lfloor \frac{1}{1 - B^{1/\alpha} \rho^{p/\alpha}} n_{k_1, \dots, k_{p-1}, r, \dots, r} \right\rfloor + A \end{aligned}$$

for $p = 1, \dots, m$, $k_{p+1} = \dots = k_m = 1$, $k_p \neq 1$ satisfies

$$n_{r, \dots, r} \leq C(m, B)n_1 + D(m, B)A,$$

where

$$\begin{aligned} C(1, B) &= \left(\frac{1}{1 - B^{1/\alpha} \rho^{1/\alpha}} \right)^{r-1}, \\ D(1, B) &= r \left(\frac{1}{1 - B^{1/\alpha} \rho^{1/\alpha}} \right)^{r-1}, \\ C(m, B) &= \left(\frac{1}{1 - B^{1/\alpha} \rho^{1/\alpha}} \right)^{r-1} C(m-1, B\rho)^r, \\ D(m, B) &= r \left(\frac{1}{1 - B^{1/\alpha} \rho^{1/\alpha}} \right)^{r-1} C(m-1, B\rho)^{r-1} (1 + D(m-1, B\rho)). \end{aligned}$$

Proof. We proceed by induction on m . To simplify the notation, let $q_B := \frac{1}{1 - B^{1/\alpha} \rho^{1/\alpha}}$. The first step $m = 1$ is easy. Indeed, for $k = 1, \dots, r-1$, we have

$$n_{k+1} \leq q_B n_k + A$$

what leads to

$$n_r \leq q_B^{r-1} n_1 + \sum_{j=0}^{r-2} q_B^j A,$$

which itself gives the (nonoptimal) values for $C(1, B)$ and $D(1, B)$ as in the statement.

Let us now assume that the property is true at rank $m-1$ and let us verify it at rank m . For $\mathbf{k} \in I_r^{m-1}$ and $i \in \{1, \dots, r\}$, define

$$m_{\mathbf{k}}(i) := n_{i, k_1, \dots, k_{m-1}}$$

and observe that, for $p = 1, \dots, m-1$ and $k_p \neq 1$,

$$\begin{aligned} m_{k_1, \dots, k_p, 1, \dots, 1}(i) &= n_{i, k_1, \dots, k_p, 1, \dots, 1} \\ &= \left\lfloor \frac{1}{1 - B^{1/\alpha} \rho^{(p+1)/\alpha}} n_{i, k_1, \dots, k_{p-1}, r, \dots, r} \right\rfloor + A \\ &= \left\lfloor \frac{1}{1 - (B\rho)^{1/\alpha} \rho^{p/\alpha}} m_{k_1, \dots, k_{p-1}, r, \dots, r}(i) \right\rfloor + A. \end{aligned}$$

Therefore, the induction hypothesis yields, for each $i = 2, \dots, r$,

$$\begin{aligned} n_{i,r,\dots,r} &\leq C(m-1, B\rho)n_{i,1,\dots,1} + D(m-1, B\rho)A \\ &\leq q_B C(m-1, B\rho)n_{i-1,r,\dots,r} + (C(m-1, B\rho) + D(m-1, B\rho))A. \end{aligned}$$

Hence, proceeding as in the initial step and using a last time the induction hypothesis for $i = 1$, we find

$$\begin{aligned} n_{r,\dots,r} &\leq (q_B C(m-1, B\rho))^{r-1} n_{1,r,\dots,r} \\ &\quad + (r-1)q_B^{r-1} (C(m-1, B\rho) + D(m-1, B\rho))A \\ &\leq q_B^{r-1} C(m-1, B\rho)^r n_1 + r q_B^{r-1} C(m-1, B\rho)^{r-1} (1 + D(m-1, B\rho))A. \end{aligned}$$

□

Proof that Lemma 4.14 \Rightarrow Lemma 4.13. A simple induction yields

$$C(m, 1) \leq \prod_{j=1}^m \left(\frac{1}{1 - \rho^{j/\alpha}} \right)^{(r-1) \cdot r^{j-1}} \leq \prod_{j=1}^{\infty} \left(\frac{1}{1 - \rho^{j/\alpha}} \right)^{(r-1) \cdot r^{j-1}} =: c_1,$$

the last infinite product being convergent by the assumption $\rho^{1/\alpha} r < 1$. More precisely, we have

$$\begin{aligned} \log C(m, B) &\leq -(r-1) \sum_{j=1}^m \log(1 - B^{1/\alpha} \rho^{j/\alpha}) r^{j-1} \\ &\leq (r-1) \cdot C \sum_{j=1}^m B^{1/\alpha} \rho^{j/\alpha} r^{j-1} \\ &\leq C' B^{1/\alpha} \end{aligned}$$

for some constants $C, C' > 0$ which only depend on ρ, α and r (recall that $B \in (0, 1]$). We use this bound to estimate $D(m, B)$:

$$D(m, B) \leq r C(m, B) (1 + D(m-1, B\rho)) \leq r \exp(C' B^{1/\alpha}) D(m-1, B\rho) + r \exp(C' B^{1/\alpha}).$$

By another induction, we get

$$\begin{aligned} D(m, B) &\leq r^{m-1} \exp\left(C' B^{1/\alpha} \sum_{j=0}^{m-2} \rho^{j/\alpha}\right) D(1, B\rho^{m-1}) \\ &\quad + r \sum_{j=0}^{m-2} r^j \exp\left(C' B^{1/\alpha} \sum_{i=0}^j \rho^{i/\alpha}\right). \end{aligned}$$

The convergence of $\sum_j \rho^{j/\alpha}$ yields the existence of c_2 , depending only on α, ρ and r , such that $D(m, 1) \leq c_2 r^m$. □

4.5 A covering lemma

We now produce the desired covering together with the sequence of integers. We thus fix Λ a compact subset of \mathbb{R}^d with homogeneous box dimension at most $\gamma \in (0, d]$. Let $r \geq 2$ and

$C(\Lambda) > 0$ be such that, for all $m \geq 1$, one can construct the compact sets $(\Lambda_{\mathbf{k}})_{\mathbf{k} \in I_r^m}$ as in Definition 1.4. We also fix $\rho = (\frac{1}{r})^{1/\gamma}$, $D > 0$, $\alpha \in (0, 1/\gamma)$ and $\beta > \alpha\gamma$. Let c_1, c_2 be the constants given by Lemma 4.13. We will assume in this subsection that

$$C(\Lambda) \leq \frac{D}{(2c_1)^{\alpha} r^{1/\gamma}}. \quad (\text{C.7})$$

Lemma 4.15 *For all $\tau > 0$, for all $\delta > 0$, for all $N \geq 1$, there exist $q \geq 1$, an increasing sequence of integers $(n_k)_{k=1, \dots, q}$, a sequence of parameters $(\lambda_k)_{k=1, \dots, q} \subset \Lambda$, a sequence $(\Lambda_k)_{k=1, \dots, q}$ of compact subsets of Λ such that*

(a) $n_1 \geq N, n_{j+1} - n_j \geq N$;

(b) $\Lambda = \bigcup_{k=1}^q \Lambda_k$ and, for all $k = 1, \dots, q$, $\Lambda_k \subset \bar{B}(\lambda_k, \tau/n_k^\alpha)$;

(c) for all $1 \leq k < j \leq q$, for all $\lambda \in \Lambda_k$, for all $\mu \in \Lambda_j$,

$$\|\lambda - \mu\| \leq \frac{D(n_j - n_k)^\alpha}{n_j^\alpha};$$

(d) for all $k \in \{1, \dots, q\}$, $\sum_{j \neq k} \frac{1}{|n_j - n_k|^\beta} \leq \delta$;

(e) $\sum_{j=1}^q \frac{1}{n_j^\beta} \leq \delta$.

Proof. The inequality $\alpha\gamma < \beta$ implies that $\rho^{\beta/\alpha} r < 1$. We consider $\kappa > 0$, $s > 0$, $A \geq N$ and $m \geq 1$ satisfying the following constraints:

$$\kappa := \frac{\tau^{1/\alpha}}{4c_1(C(\Lambda))^{1/\alpha}},$$

$$r\kappa^{-\beta} \sum_{p=s}^{+\infty} (r\rho^{\beta/\alpha})^p < \frac{\delta}{3},$$

$$\sum_{l=1}^{r^{s+1}} \frac{1}{l^\beta A^\beta} < \frac{\delta}{3},$$

$$\left\lfloor \frac{1}{2c_1} \left(\frac{\tau}{\rho^m C(\Lambda)} \right)^{1/\alpha} \right\rfloor \geq \max \left(\left(\frac{3}{\delta} \right)^{1/\beta}, 2 + \frac{c_2}{c_1} A r^m, N \right).$$

Observe that the conditions imposed on ρ , α , β and r allow us to define successively s , A and m . We set

$$n_1 := \left\lfloor \frac{1}{2c_1} \left(\frac{\tau}{\rho^m C(\Lambda)} \right)^{1/\alpha} \right\rfloor$$

and we consider the sequence $(n_{\mathbf{k}})_{\mathbf{k} \in I_r^m}$ defined by Lemma 4.13. We first remark that (a) is satisfied since $n_1 \geq N$ and $n_{j+1} - n_j \geq A$ for every $j \geq 1$. We then set $q = r^m$ and we observe that

$$\begin{aligned} \max(n_{\mathbf{k}} : \mathbf{k} \in I_r^m) &= n_{r, \dots, r} \\ &\leq \frac{1}{2} \left(\frac{\tau}{\rho^m C(\Lambda)} \right)^{1/\alpha} + c_2 A r^m \end{aligned}$$

$$\begin{aligned} &\leq \left(\frac{\tau}{\rho^m C(\Lambda)} \right)^{1/\alpha} - 2c_1 \\ &\leq 2c_1 n_1 \leq \left(\frac{\tau}{\rho^m C(\Lambda)} \right)^{1/\alpha}. \end{aligned}$$

We then consider the covering $(\Lambda_{\mathbf{k}})_{\mathbf{k} \in I_r^m}$ of Λ given by Definition 1.4 and we fix $\lambda_{\mathbf{k}} \in \Lambda_{\mathbf{k}}$. Since

$$\text{diam}(\Lambda_{\mathbf{k}}) \leq \rho^m C(\Lambda) \leq \frac{\tau}{n_{\mathbf{k}}^\alpha},$$

we get (b). Let now $\mathbf{k}, \mathbf{j} \in I_r^m$ with $\mathbf{k} < \mathbf{j}$ and let $\lambda \in \Lambda_{\mathbf{k}}, \mu \in \Lambda_{\mathbf{j}}$. Let p be the biggest integer such that $k_1 = j_1, \dots, k_{p-1} = j_{p-1}$ so that $k_p < j_p$. Then λ and μ both belong to $\Lambda_{k_1, \dots, k_{p-1}}$ which has diameter less than $\rho^{p-1} C(\Lambda)$. On the other hand the definition of the sequence (n_i) ensures that

$$\begin{aligned} n_{\mathbf{j}} - n_{\mathbf{k}} &\geq \left(\frac{1}{1 - \rho^{p/\alpha}} - 1 \right) n_{\mathbf{k}} \\ &\geq \rho^{p/\alpha} n_{\mathbf{k}} \\ &\geq \frac{\rho^{p/\alpha}}{2c_1} n_{\mathbf{j}} \end{aligned}$$

so that (c) is satisfied, since

$$\begin{aligned} \|\lambda - \mu\| &\leq \frac{C(\Lambda)}{\rho} \rho^p \\ &\leq (2c_1)^\alpha \frac{C(\Lambda)}{\rho} \left(\frac{n_{\mathbf{j}} - n_{\mathbf{k}}}{n_{\mathbf{j}}} \right)^\alpha \\ &\leq D \left(\frac{n_{\mathbf{j}} - n_{\mathbf{k}}}{n_{\mathbf{j}}} \right)^\alpha. \end{aligned}$$

Let us now prove (d) and (e). For $\mathbf{j} \neq \mathbf{k} \in I_r^m$, we denote by $\gamma(\mathbf{j}, \mathbf{k})$ the biggest integer p such that $k_1 = j_1, \dots, k_{p-1} = j_{p-1}$, with $\gamma(\mathbf{j}, \mathbf{k}) = 1$ if $k_1 \neq j_1$. In particular, if we fix $\mathbf{k} \in I_r^m$ and $p \in \{1, \dots, m\}$, we can observe that

$$\text{card}(\{\mathbf{j} \in I_r^m : \gamma(\mathbf{j}, \mathbf{k}) = p\}) \leq r^{m+1-p}.$$

Moreover, if $\gamma(\mathbf{j}, \mathbf{k}) = p$, then the computation done above shows that

$$\begin{aligned} |n_{\mathbf{j}} - n_{\mathbf{k}}| &\geq \rho^{p/\alpha} n_1 \\ &\geq \kappa \rho^{-(m-p)/\alpha}. \end{aligned}$$

We then split the sum appearing in (d) into two parts. On the one hand, using this last estimation,

$$\begin{aligned} \sum_{\substack{\mathbf{j} \neq \mathbf{k} \\ \gamma(\mathbf{j}, \mathbf{k}) \leq m-s}} \frac{1}{|n_{\mathbf{j}} - n_{\mathbf{k}}|^\beta} &\leq \sum_{p=1}^{m-s} \sum_{\substack{\mathbf{j} \neq \mathbf{k} \\ \gamma(\mathbf{j}, \mathbf{k}) = p}} \kappa^{-\beta} \rho^{(m-p)\beta/\alpha} \\ &\leq \sum_{p=1}^{m-s} r \kappa^{-\beta} (r \rho^{\beta/\alpha})^{m-p} \\ &\leq r \kappa^{-\beta} \sum_{p=s}^{+\infty} (r \rho^{\beta/\alpha})^p < \frac{\delta}{3} \end{aligned}$$

by the choice of s . On the other hand, we observe that there are at most $r + \dots + r^s \leq r^{s+1}$ elements $\mathbf{j} \in I_r^m$ such that $\mathbf{j} \neq \mathbf{k}$ and $\gamma(\mathbf{j}, \mathbf{k}) \geq m - s + 1$. Moreover, the difference between two consecutive terms of the sequence $(n_{\mathbf{j}})$ is at least A . Thus,

$$\sum_{\substack{\mathbf{j} \neq \mathbf{k} \\ \gamma(\mathbf{j}, \mathbf{k}) \geq m - s + 1}} \frac{1}{|n_{\mathbf{j}} - n_{\mathbf{k}}|^\beta} \leq \sum_{l=1}^{r^{s+1}} \frac{1}{l^\beta A^\beta} < \frac{\delta}{3}.$$

This achieves the proof of (d) with the stronger bound $2\delta/3$. Moreover, we can use this improved estimate to get easily (e):

$$\sum_{\mathbf{k} \in I_r^m} \frac{1}{n_{\mathbf{k}}^\beta} \leq \frac{1}{n_{1, \dots, 1}^\beta} + \sum_{\substack{\mathbf{k} \in I_r^m \\ \mathbf{k} > (1, \dots, 1)}} \frac{1}{|n_{\mathbf{k}} - n_{1, \dots, 1}|^\beta} < \delta.$$

□

4.6 Proof of Theorem 1.5

Proof. We shall prove that the assumptions of the Basic Criterion are satisfied. Let $r \geq 2$ be such that, for all $m \geq 0$, there exists a sequence of compact sets $(\Lambda_{\mathbf{k}})_{\mathbf{k} \in I_r^m}$ satisfying the assumptions of Definition 1.4. Since for each $m \geq 1$ and each $\mathbf{k} \in I_r^m$, the set $\Lambda' = \Lambda_{\mathbf{k}}$ satisfies the same assumptions as Λ with $C(\Lambda') = C(\Lambda) \left(\frac{1}{r^{1/\gamma}}\right)^m$ (just define, for $\mathbf{j} \in I_r^{m'}$, $\Lambda'_{\mathbf{j}} = \Lambda_{\mathbf{k}, \mathbf{j}}$) and since the assumptions of Theorem 1.5 are satisfied by Λ' for the same constants α , β and D , we may assume that

$$C(\Lambda) \leq \frac{D}{(2c_1)^\alpha r^{1/\gamma}}.$$

Let $\varepsilon > 0$, $u, v \in \mathcal{D}$. Let $C, \tau > 0$ and $N \in \mathbb{N}$ be such that the assumptions of Theorem 1.5 are satisfied for both u and v . We then consider the sequences (n_k) , (λ_k) and (Λ_k) given by Lemma 4.15 applied with τ, N and $\delta = \varepsilon/C$ (we may always assume that $C \geq 1$). It is now an easy exercise to prove that the assumptions of the Basic Criterion are satisfied. The most difficult point is to prove that, for all $k \in \{1, \dots, q\}$, for all $\lambda \in \Lambda_k$, one has

$$\left\| \sum_{j \neq k} T_\lambda^{n_k} S_{\lambda_j}^{n_j}(v) \right\| < \varepsilon.$$

When $j > k$,

$$\|\lambda - \lambda_j\| \leq D \frac{(n_j - n_k)^\alpha}{n_j^\alpha}$$

so that

$$\left\| \sum_{j > k} T_\lambda^{n_j} S_{\lambda_k}^{n_k} v \right\| \leq \sum_{j > k} \frac{C}{(n_j - n_k)^\beta},$$

whereas, when $j < k$,

$$\|\lambda - \lambda_j\| \leq D \frac{(n_k - n_j)^\alpha}{n_k^\alpha}$$

so that

$$\left\| \sum_{j < k} T_\lambda^{n_k} S_{\lambda_j}^{n_j} v \right\| \leq \sum_{j < k} \frac{C}{(n_k - n_j)^\beta}.$$

Property (d) of Lemma 4.15 now finishes the job. \square

Question 4.16 *Condition (b) of Theorem 1.5 does not perfectly match (CS2) of the Costakis-Sambarino theorem because we cannot take $\alpha = 1/\gamma$. Is it possible to cover this last case (changing if necessary condition (a))?*

Observe that in the previous proof the condition $\alpha < 1/\gamma$ was needed to get the convergence of the infinite product defining $C(m, 1)$.

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Article D

Common hypercyclic algebras for families of products of backward shifts

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Abstract

In this paper, we transfer to the context of algebras some recent results by F. Bayart, Q. Menet and the author on the existence of common hypercyclic vectors for families of products of backward shift operators. We also give, in a multi-dimensional setting, a positive answer to a question raised by F. Bayart, D. Papathanasiou and the author about the existence of a common hypercyclic algebra on $\ell_1(\mathbb{N})$ with the convolution product for the family of backward shifts $(B_{w(\lambda)})_{\lambda>0}$ induced by the weights $w_n(\lambda) = 1 + \lambda/n$.

1 Introduction

The study of hypercyclicity consists of analysing the behaviour of the iterates of an operator T on a vector x in a topological vector space X . The set of iterates is called the orbit of x under T and denoted by $Orb(T; x) := \{T^n(x) : n \geq 1\}$. When the latter is dense in X , we say that T is a hypercyclic operator and that x is one of its hypercyclic vectors. The set of hypercyclic vectors of T is denoted by $HC(T)$. When we are dealing with a family $(T_\lambda)_{\lambda \in \Lambda}$ of hypercyclic operators acting on the same space X , it is natural to ask whether there is a single vector $x \in X$ which is hypercyclic for each member T_λ of the family. We usually assume that the parameter set Λ is σ -compact and that $(\lambda, x) \mapsto T_\lambda(x)$ is continuous from $\Lambda \times X$ into X . Such a vector is called common hypercyclic vector for the continuous family $(T_\lambda)_{\lambda \in \Lambda}$, which in turn is called common hypercyclic family of operators.

This question is interesting and non trivial. On the one hand, many classical family of operators are common hypercyclic, for example $(\lambda B)_{\lambda>1}$ on $\ell_p(\mathbb{N})$, where B is the backward shift, and $(T_a)_{a \neq 0}$ on $H(\mathbb{C})$, where T_a is the operator of translation by $a \neq 0$. On the other hand, an example by A. Borichev (quoted in [1]) show that the family $(sB \times tB)_{(s,t) \in \Lambda}$ does not have common hypercyclic vectors on $\ell_2(\mathbb{N}) \times \ell_2(\mathbb{N})$ whenever Λ has positive Lebesgue measure. Of course the smaller the parameter set is, the better are the chances of finding a common hypercyclic vector.

When X has a structure of algebra, it is natural to ask whether or not $HC(T) \cup \{0\}$ contains a non-trivial subalgebra of X , what we shall name as hypercyclic algebra for T . Although the subject is not new, appearing with a negative result in [4, 5] and with a positive result in [2, 3], it is continuously catching the attention of many researchers on the field in the last few years. In this paper, we are exclusively interested in the case where X is a Fréchet sequence algebra, that is, we will assume that X is a Fréchet subspace of the space of all sequences $\omega = \mathbb{C}^{\mathbb{N}}$ and on which a well defined product \cdot satisfies, for all $x, y \in X$ and $q \geq 1$,

$$\|x \cdot y\|_q \leq \|x\|_q \times \|y\|_q,$$

where the non-decreasing separating sequence of seminorms $(\|\cdot\|_q)_{q \geq 1}$ induces the complete topology of X . We are mainly interested on families of *weighted backward shift* operators, that

is, maps $B_w : X \rightarrow X$ defined by

$$B_w(x_0, x_1, x_2, \dots) = (w_1x_1, w_2x_2, \dots), \quad x \in X,$$

where $w = (w_n)_n$ is a so called *weight sequence*. We say that w is *admissible* when B_w is continuous.

Two classical products are commonly considered on sequence algebras: the coordinatewise product and the convolution (or Cauchy) product. The first one transforms $\ell_p(\mathbb{N}), p \geq 1$, and $c_0(\mathbb{N})$ into Fréchet sequence algebras and the second one does the same with $\ell_1(\mathbb{N})$. These products also make $H(\mathbb{C})$ a Fréchet sequence algebra when we endow it with the sequence of seminorms $(\|\cdot\|_q)_{q \geq 1}$ given by

$$\left\| \sum_{n=0}^{\infty} a_n z^n \right\|_q = \sum_{n=0}^{\infty} |a_n| q^n.$$

The coordinatewise product on $H(\mathbb{C})$ is also known as Hadamard product.

A powerful tool to find common hypercyclic vectors is the well known Costakis-Sambarino criterion (see [10]). Its hypothesis give a clear way of constructing the partition of the parameter set required to apply the Basic Criterion for common hypercyclicity [3, Lemma 7.12] (or Theorem 2.1 for a version adapted to algebras). On higher dimensions, on the other hand, the construction of the partition is more delicate as there is no trivial way of ordering the sets in the partition. While in one dimension, one can arrange the partition in a way that inequalities such as $\lambda < \lambda'$ can be used to get rid of otherwise problematic factors (see the proof of [9, Theorem 2.11]), in two dimensions one cannot expect to do anything similar with two parameters (λ, μ) and (λ', μ') . Hence, proximity conditions on the elements of the partitions become essential. In this paper, we make use of the ideas in the recent work [7] to obtain similar results for common hypercyclic algebras for families of product of backward shift operators. Let us summarize our main findings.

1.1 Coordinatewise product

In the same vein as the results from [7], we have obtained a more general statement which includes both the following corollaries.

Corollary 1.1 *Let $d \geq 1$, $X = \ell_p(\mathbb{N})$ or $c_0(\mathbb{N})$, $\alpha \in (0, 1/d)$, $I \subset (0, +\infty)$ be σ -compact and $(w(\lambda))_{\lambda \in I}$ be a continuous family of weights. Assume that, for all $I_0 \subset I$ compact, there exist $D_1, D_2, D_3 > 0$ and $N \geq 0$ such that, for all $n \geq N$,*

- $a \in I_0 \mapsto \sum_{i=1}^n \log(w_i(a))$ is $D_1 n^\alpha$ -Lipschitz;
- $\inf_{a \in I_0} w_1(a) \cdots w_n(a) \geq D_2 \exp(D_3 n^\alpha)$.

Then $(B_{w(\lambda_1)} \times \cdots \times B_{w(\lambda_d)})_{\lambda \in I^d}$ admits a common hypercyclic algebra for the coordinatewise product.

Corollary 1.2 *Let $d \geq 1$, $X = \ell_p(\mathbb{N})$ or $c_0(\mathbb{N})$, $I \subset (0, +\infty)$ be σ -compact and $(w(\lambda))_{\lambda \in I}$ be a continuous family of weights. Assume that, for all $I_0 \subset I$ compact, there exist $D_1, D_2, \gamma > 0$ and $N \geq 0$ such that, for all $n \geq N$,*

- $a \in I_0 \mapsto \sum_{i=1}^n \log(w_i(a))$ is $D_1 \log(n)$ -Lipschitz;
- $\inf_{a \in I_0} w_1(a) \cdots w_n(a) \geq D_2 n^\gamma$.

Then $(B_{w(\lambda_1)} \times \cdots \times B_{w(\lambda_d)})_{\lambda \in I^d}$ admits a common hypercyclic algebra for the coordinatewise product.

Thus we have found a large class of families of operators parametrized by subsets of \mathbb{R}^d admitting common hypercyclic algebras. Particularly, for both $w_n(a) = 1 + \frac{a}{n^{1-\alpha}}$, with $\alpha \in (0, 1/d)$, and $w_n(a) = 1 + \frac{a}{n}$, the family $(B_{w(\lambda_1)} \times \cdots \times B_{w(\lambda_d)})_{\lambda \in (0, +\infty)^d}$ admits a common hypercyclic algebra for the coordinatewise product.

As for the characterization [7, Theorem 2.1], the same holds true for the hypercyclic algebras for the coordinatewise product under natural modifications. For a particular case of weights $(w_n(\lambda))_\lambda$ with the property that the function $a \mapsto \sum_{k=1}^n \log(w_k(a))$ is n^α -Lipschitz, we obtained the following equivalence.

Proposition 1.3 *Let $\alpha \in (0, 1]$ and $w_1(a) \cdots w_n(a) = \exp(an^\alpha)$ and $X = \ell_p(\mathbb{N})$ or $c_0(\mathbb{N})$ endowed with the coordinatewise product. The following assertions are equivalent:*

- (a) $(B_{w(a)} \times B_{w(b)})_{(a,b) \in \Lambda}$ has a common hypercyclic vector in X^2 ;
- (b) $(B_{w(a)} \times B_{w(b)})_{(a,b) \in \Lambda}$ has a common hypercyclic algebra in X^2 .

1.2 Convolution product

As we might expect, the convolution product is more complicated to be dealt with. With the coordinatewise product, since it satisfies $(\sum a_n e_n)^m = \sum a_n^m e_n$, one has more liberty to play with the coefficients without worrying about changes on the supports, we can easily define m -roots for vectors and, as a consequence, the adapted conditions in the criteria for algebras change only for a few new powers in comparison with the original criteria for vectors. On the flip side, the convolution product mixes the supports and is hence less predictable. This behavior usually lead to more complicated criteria for the existence of algebras under this product, which are in general not analogous to classical results for the existence of single vectors.

As remarked by its authors, criterion [9, Theorem 5.11] for the existence of common hypercyclic algebras fails to apply to the family of weights $w_n(\lambda) = 1 + \frac{\lambda}{n}$, what can be thought as a surprise as this weight should not be hard to work with. In fact, for both examples given by the authors, which are $(\lambda B)_{\lambda > 1}$ and $(\lambda D)_{\lambda > 0}$, the functions $\lambda \mapsto \sum_{k=1}^n \log(w_k(\lambda))$ are n -Lipschitz and the divergence of $\sum 1/n$ is used in the construction of the partition. For the weights $w_n(\lambda) = 1 + \frac{\lambda}{n}$, however, the function $\lambda \mapsto \sum_{k=1}^n \log(w_k(\lambda))$ is $\log(n)$ -Lipschitz, and the series $\sum_n \frac{1}{\log(n)}$ diverges faster than $\sum 1/n$. Hence, what one can expect to do is to construct a partition specifically for this last example and to try to use the faster divergence to avoid using [9, Lemma 2.10]. This method should also work with the family of weights $w_n(\lambda) = (1 + \frac{1}{n})^\lambda$ or any other family for which $w_1(\lambda) \cdots w_n(\lambda)$ behaves like n^λ .

One could think that the fast divergence of $\sum \frac{1}{\log(n)^d}$ could lead to a positive result in higher dimensions. This is indeed true, as we are going to prove in this section, but not by using the dyadic covering discussed in Section 2.1.1 below. In fact, this covering doesn't seem to be

adapted to $w_n(a) = 1 + \frac{a}{n}$ on $\ell_1(\mathbb{N})^d$, as it contains too many elements. Fortunately, the fact that the family induced by $w_n(a) = 1 + \frac{a}{n}$ satisfies a $\log(n)$ -Lipschitz condition (rather than n^α -Lipschitz) will allow us to construct a (trivial) covering adapted to this case and obtain the following result.

Theorem 1.4 *If $w_1(a) \cdots w_n(a) = n^a$ (or if $w_n(a) = 1 + \frac{a}{n}$ with a few modifications in the proof), then, for all $d \geq 1$, the family $(B_{w(\lambda_1)} \times \cdots \times B_{w(\lambda_d)})_{\lambda \in (0, +\infty)^d}$ admits a common hypercyclic algebra on $\ell_1(\mathbb{N})^d$ with the convolution product.*

2 Coordinatewise product

In what follows we assume that X is a Fréchet sequence algebra under the coordinatewise product in which $\text{span}(e_i)$ is dense, $\Lambda \subset \mathbb{R}^d$ is a σ -compact set of parameters and, for each $\lambda = (\lambda_1, \dots, \lambda_d) \in \Lambda$, the operator $T_\lambda : X^d \rightarrow X^d$ is defined by $T_\lambda := B_{w(\lambda_1)} \times \cdots \times B_{w(\lambda_d)}$, where $(w(x))_x$ is a continuous family of admissible weights and the map $(\lambda, v) \mapsto T_\lambda(v)$ is continuous. Also, for each $\lambda \in \Lambda$ and $m \in \mathbb{N}$, we let $\mathcal{D} := \prod_d \text{span}(e_i)$ and define $S_{\lambda, m} : \mathcal{D} \rightarrow \mathcal{D}$ by

$$S_{\lambda, m} := F_{w(\lambda_1)^{-1/m}} \times \cdots \times F_{w(\lambda_d)^{-1/m}},$$

where each $F_{w(x)^{-1/m}}$, $x \in I$, is the weighted forward shift on X with weight sequence $(w_n^{-1/m}(x))_n$. The symbol $\|\cdot\|$ represents the F -norm of X and thus satisfies, for all $u \in X$ and $\lambda \in \mathbb{C}$,

- $|\lambda| \leq 1 \Rightarrow \|\lambda u\| \leq \|u\|$;
- $\lim_{\lambda \rightarrow 0} \|\lambda u\| = 0$;
- $\|\lambda u\| \leq (|\lambda| + 1)\|u\|$.

We will make use of the following criterion from [9].

Theorem 2.1 (Basic Criterion for algebras) *Assume that, for every compact $K \subset \Lambda$, every $m_0 \leq m_1$ in \mathbb{N} , every $v \in \mathcal{D}$, every $M > 0$ and every $\varepsilon > 0$, there exist $q \in \mathbb{N}$, $\lambda_0, \dots, \lambda_q \in \Lambda'$, sets $\Lambda_0, \dots, \Lambda_q \subset \Lambda$ and positive integers n_0, \dots, n_q with $n_0 > M$ and $n_{i+1} - n_i > M$ for $i = 0, \dots, q$ such that*

(I) $\bigcup_i \Lambda_i \supset K$;

(II) for all $\lambda \in \Lambda_i$, $i = 0, \dots, q$, and all $m \in [m_0, m_1]$,

$$\left\| \sum_j S_{\lambda_j, m_0}^{n_j} (v^{1/m_0}) \right\| \leq \varepsilon \quad \text{and} \quad \left\| \sum_{j \neq i} T_\lambda^{n_i} \left(S_{\lambda_j, m_0}^{n_j} (v^{1/m_0}) \right)^m \right\| \leq \varepsilon;$$

(III) for all $\lambda \in \Lambda_i$, $i = 0, \dots, q$, and all $m \in (m_0, m_1]$,

$$\left\| T_\lambda^{n_i} \left(\left(S_{\lambda_i, m_0}^{n_i} (v^{1/m_0}) \right)^m \right) \right\| \leq \varepsilon;$$

(IV) for all $\lambda \in \Lambda_i$, $i = 0, \dots, q$,

$$\left\| T_\lambda^{n_i} \left(S_{\lambda_i, m_0}^{n_i} (v^{1/m_0}) \right)^{m_0} - v \right\| \leq \varepsilon.$$

Then $\bigcap_{\lambda \in \Lambda} HC(T_\lambda)$ contains an algebra (but 0).

2.1 A practical criterion for algebras

The following result is a practical criterion which leads to the same applications as found in [7], but now for hypercyclic algebras.

Theorem 2.2 *Let $\alpha \in (0, 1/d)$ and let $I \subset (0, +\infty)$ be σ -compact. Suppose that, for all $m_0 \in \mathbb{N}$ and all $I_0 \subset I$ compact, there exist $C_1, C_2 > 0$, $\beta > \alpha d$, $F : \mathbb{N} \rightarrow (0, +\infty)$ with $F(n) \leq C_1 n^\alpha$ and $N_0, M_0 > 0$ such that, for all $n \geq N_0$,*

(i) $a \in I_0 \mapsto \sum_{i=1}^n \log(w_i(a))$ is $F(n)$ -Lipschitz;

(ii) for all $a \in I_0$, $\inf_{a \in I_0} w_1(a) \cdots w_k(a) \rightarrow +\infty$ as $k \rightarrow +\infty$;

(iii) for all $a \in I_0$ and $k \geq N_0$,

$$\left\| \frac{\exp \left(C_2 \frac{F(n+k)}{(n+k)^\alpha} k^\alpha \right)}{w_1(a) \cdots w_k(a)} e_k \right\| \leq \frac{M_0}{k^\beta}, \quad (\text{D.1})$$

$$\left\| \frac{1}{[w_1(a) \cdots w_k(a)]^{1/m_0}} e_k \right\| \leq \frac{M_0}{k^\beta}. \quad (\text{D.2})$$

Then $(B_{w(\lambda_1)} \times \cdots \times B_{w(\lambda_d)})_{\lambda \in I^d}$ admits a common hypercyclic algebra on X^d for the coordinatewise product.

If we assume that X admits a continuous norm, then condition (ii) follows from (iii) so it can be removed. Indeed, if X has a continuous norm then the sequence (e_n) is bounded below.

Before proving this result, we will show that the two practical Corollaries 3.2 and 3.6 from [7] (here stated as Corollaries 1.1 and 1.2) give not only a common hypercyclic vector but also a common hypercyclic algebra for the coordinatewise product. The first corollary is proven as follows.

Proof of Corollary 1.1. Let $m_0 \in \mathbb{N}$ and $I_0 \subset I$ compact. By hypothesis there are $D_1, D_2, D_3 > 0$ and $N \geq 0$ as in the statement. We define $F(n) := D_1 n^\alpha$, $C_1 := D_1$ and fix $C_2 < \frac{D_3}{D_1}$ and $\beta > \alpha d$. Also we let $N_0 > \max(N, p)$ big enough so that, for all $k \geq N_0$,

$$\min \left\{ D_2 \exp \left((D_3 - C_2 D_1) k^\alpha \right), [D_2 \exp(D_3 k^\alpha)]^{1/m_0} \right\} \geq k^\beta.$$

Conditions (i) and (ii) are satisfied by hypothesis and, for all $n, k \geq N_0$ and $a \in I_0$, we have

$$\left\| \frac{\exp \left(C_2 \frac{F(n+k)}{(n+k)^\alpha} k^\alpha \right)}{w_1(a) \cdots w_k(a)} e_k \right\| = \frac{\exp \left(C_2 \frac{D_1 (n+k)^\alpha}{(n+k)^\alpha} k^\alpha \right)}{w_1(a) \cdots w_k(a)}$$

$$\begin{aligned} &\leq \frac{\exp\left(C_2 D_1 k^\alpha\right)}{D_2 \exp\left(D_3 k^\alpha\right)} \\ &= \frac{1}{D_2 \exp\left((D_3 - C_2 D_1) k^\alpha\right)} \\ &\leq \frac{1}{k^\beta} \end{aligned}$$

and

$$\left\| \frac{1}{[w_1(a) \cdots w_k(a)]^{1/m_0}} e_k \right\| \leq \frac{1}{[D_2 \exp(D_3 k^\alpha)]^{1/m_0}} \leq \frac{1}{k^\beta},$$

what verifies (iii) and completes the proof. \square

Example 2.3 Let $d \geq 1$, $X = \ell_p(\mathbb{N})$ or $c_0(\mathbb{N})$, $\alpha \in (0, 1/d)$ and $(w(\lambda))_{\lambda>0}$ be the family of weights defined by $w_1(\lambda) \cdots w_n(\lambda) = \exp(\lambda n^\alpha)$ (or $w_n(\lambda) = 1 + \frac{\lambda}{n^{1-\alpha}}$) for all $n \geq 1$. Then the family $(B_{w(\lambda_1)} \times \cdots \times B_{w(\lambda_d)})_{\lambda \in (0, +\infty)^d}$ admits a common hypercyclic algebra for the coordinatewise product.

The second corollary is proven as follows.

Proof of Corollary 1.2. Let $m_0 \in \mathbb{N}$, $I_0 \subset I$ compact and $p \in \mathbb{N}$. From the hypothesis there are $D_1, D_2, \gamma > 0$ and $N \geq 0$ as in the statement. Fixing $\alpha < \frac{\gamma}{dm_0}$, one can find C_1 big enough so that $D_1 \log(n) \leq C_1 n^\alpha$ for all $n \in \mathbb{N}$. Let $\beta \in (\alpha d, \frac{\gamma}{m_0})$ and $C_2 < \frac{\gamma - \beta}{D_1}$. Finally let $N_0 > N$ big enough so that $n \mapsto \frac{\log(n)}{n^\alpha}$ is decreasing on $[N_0, +\infty)$. It is now easy to apply Theorem 2.2. Conditions (i) and (ii) are satisfied by hypothesis and, for all $a \in I_0$ and $k \geq N_0$,

$$\left\| \frac{\exp\left(C_2 \frac{F(n+k)}{(n+k)^\alpha} k^\alpha\right)}{w_1(a) \cdots w_k(a)} e_k \right\| = \frac{\exp\left(C_2 D_1 \frac{\log(n+k)}{(n+k)^\alpha} k^\alpha\right)}{w_1(a) \cdots w_k(a)} \leq \frac{k^{C_2 D_1}}{D_2 k^\gamma} = \frac{D_2^{-1}}{k^{\gamma - C_2 D_1}} \leq \frac{D_2^{-1}}{k^\beta}$$

and

$$\left\| \frac{1}{[w_1(a) \cdots w_k(a)]^{1/m_0}} e_k \right\| = \frac{1}{[w_1(a) \cdots w_k(a)]^{1/m_0}} \leq \frac{1}{[D_2 k^\gamma]^{1/m_0}} \leq \frac{D_2^{-1/m_0}}{k^\beta},$$

what verifies (iii) and completes the proof. \square

Example 2.4 Let $d \geq 1$, $X = \ell_p(\mathbb{N})$ or $c_0(\mathbb{N})$ and $(w(\lambda))_{\lambda>0}$ be the family of weights defined by $w_n(\lambda) = 1 + \frac{\lambda}{n}$ (or $w_n(\lambda) := (1 + \frac{\lambda}{n})^\lambda$). Then the family $(B_{w(\lambda_1)} \times \cdots \times B_{w(\lambda_d)})_{\lambda \in (0, +\infty)^d}$ admits a common hypercyclic algebra for the coordinatewise product.

The problem of finding a common hypercyclic *vector* in the limit case $\alpha = 1/d$ remains open. Although we don't intend to explore this difficulty, it remains as an interesting open problem. We state it here in the form of a bi-dimensional example.

Question 2.5 Does there exist a set of parameters $\Lambda \subset \mathbb{R}^2$ with positive Lebesgue measure such that the family $(B_{w(\lambda)} \times B_{w(\mu)})_{(\lambda, \mu) \in \Lambda}$, with $w_n(\lambda) = 1 + \frac{\lambda}{\sqrt{n}}$, admits a common hypercyclic vector on $c_0(\mathbb{N}) \times c_0(\mathbb{N})$?

2.1.1 A covering lemma

In order to prove that the conditions of Theorem 2.2 are enough to apply the Basic Criterion, we need a covering result taken from [7]. The d -dimensional version of this lemma holds true but, for simplicity, we will state it for $d = 2$. Let us fix $\alpha \in (0, 1/2)$, $\beta > 2\alpha$ and $D > 0$. An element λ of \mathbb{R}^2 will be written $\lambda = (x, y)$.

Lemma 2.6 *There exists a constant $c > 0$ such that, for all $D > 0$ and all compact squares $K \subset \mathbb{R}^2$ with $\text{diam}(K) \leq cD$, then, for all $\tau > 0$, for all $\eta > 0$, for all $N \geq 1$, there exist $q \geq 1$, a sequence of integers $(n_j)_{j=0, \dots, q-1}$, a sequence of parameters $(\lambda_j)_{j=0, \dots, q-1}$, a sequence of compact squares $(\Lambda_j)_{j=0, \dots, q-1}$ such that*

(a) $n_0 \geq N$, $n_{j+1} - n_j \geq N$;

(b) $K = \bigcup_{j=0}^{q-1} \Lambda_j$ and, for all $j = 0, \dots, q-1$, writing $\lambda_j = (x_j, y_j)$, we have $\Lambda_j \subset \left[x_j, x_j + \frac{\tau}{n_j^\alpha} \right] \times \left[y_j, y_j + \frac{\tau}{n_j^\alpha} \right]$;

(c) for all $0 \leq j < l \leq q-1$, for all $\lambda \in \Lambda_j$, for all $\mu \in \Lambda_l$,

$$\|\lambda - \mu\|_\infty \leq \frac{D(n_l - n_j)^\alpha}{n_l^\alpha};$$

(d) $\sum_{j=0}^{q-1} \frac{1}{n_j^\beta} \leq \eta$;

(e) for all $j \in \{0, \dots, q-1\}$, $\sum_{l \neq j} \frac{1}{|n_l - n_j|^\beta} \leq \eta$.

This covering works well for families of weights $(w_n(a))_a$ such that $a \mapsto \sum_{k=1}^n \log(w_n(a))$ is $F(n)$ -Lipschitz for some function $F : \mathbb{N} \rightarrow \mathbb{N}$ satisfying $F(n) \leq Cn^\alpha$, $C > 0$, $0 < \alpha < 1/d$, giving common hypercyclic vectors for the family of products of d backward shifts as it was done in [7]. As usual, one can transfer these results to the context of hypercyclic algebras for the coordinatewise product with little effort, what lead us to the formulation of Theorem 2.2. As we have discussed, the convolution product is more delicate to work with.

Proof of Theorem 2.2. For simplicity we prove the result for $d = 2$, but it is clear that same can be done for any $d \in \mathbb{N}$. Also, in order to shrink the notations, we will write $\widehat{w}_n(x) = w_1(x) \cdots w_n(x)$ for any x and n . We aim to apply the Basic Criterion for algebras with $\Lambda = (0, +\infty)^d$ and some compact square $K \subset I_0^d$ where $I_0 \subset (0, +\infty)$ is a compact interval. Let $m_0 \leq m_1$ in \mathbb{N} and $(u, v) \in \mathcal{D}$ with support in $[0, p]$ for some $p \in \mathbb{N}$, say $u = \sum_{l=0}^p u_l e_l$ and $v = \sum_{l=0}^p v_l e_l$. By hypothesis we find $C_1, C_2 > 0$, $\beta > \alpha d$, $F : \mathbb{N} \rightarrow (0, +\infty)$ and $N_0, M_0 > 0$ as in the statement. Let us fix $D < C_2/2^\alpha$ and define

$$A_p := \sup \left\{ \left[\widehat{w}_l(x) \|(u, v)\|_\infty \right]^{m/m_0} : m \in [m_0, m_1] \cup \{1\}, l = 0, \dots, p \text{ and } x \in I_0 \right\},$$

$$A'_p := \inf \{ \widehat{w}_l(x) : l = 0, \dots, p \text{ and } x \in I_0 \}.$$

We can suppose that $\text{diam}(K) \leq cD$, otherwise we subdivide it into smaller squares. In order to verify the basic criterion, let $M > 0$ and $\varepsilon > 0$ arbitrary. Let $\tau_0 > 0$ small enough so that

$$\max_{l=0, \dots, p} (\|\tau_0 u_l e_l\|, \|\tau_0 v_l e_l\|) \leq \frac{\varepsilon}{2(p+1)}$$

and let $\tau > 0$ small enough so that

$$\left| \exp(C_1 2^\alpha \tau) - 1 \right| \leq \frac{\tau_0}{A_p/A'_p + \|(u)\|_\infty}.$$

Finally let us fix $N > \max\{N_0, M, p\}$ big enough so that, for all $n \geq N, l = 0, \dots, p, m \in [m_0, m_1]$ and $a \in I_0$, we have $\widehat{w}_{l+n}(a) \geq 1$ and

$$\left\| \frac{1}{(\widehat{w}_{l+n}(a))^{1/m_0}} e_l \right\| \leq \frac{\varepsilon}{2(p+1)((A_p/A'_p) \exp(\tau C_1 2^\alpha) + 1)}.$$

We then apply Lemma 2.6 for $\tau > 0$, for

$$\eta = \frac{\varepsilon}{2M_0(p+1)(A_p+1)}$$

and for N and find $q \geq 1$, integers n_0, \dots, n_{q-1} , parameters $\lambda_0, \dots, \lambda_{q-1}$ and squares $\Lambda_0, \dots, \Lambda_{q-1}$ such that properties (a)-(e) hold true. We claim that conditions (I)-(IV) of Basic Criterion are satisfied.

Condition (I) is automatically true. For condition (II), let $\lambda \in \Lambda_i$ for some $i \in \{0, \dots, q\}$, and let $m \in [m_0, m_1]$. We have

$$\left\| \sum_{j=0}^{q-1} S_{\lambda_j, m_0}^{n_j}(u, v) \right\| = \left\| \sum_{j=0}^{q-1} F_{w(x_j)^{-1/m_0}}^{n_j}(u^{1/m_0}) \right\| + \left\| \sum_{j=0}^{q-1} F_{w(y_j)^{-1/m_0}}^{n_j}(v^{1/m_0}) \right\|.$$

Let us treat each parcel separately. Firstly we apply (D.2) and get

$$\begin{aligned} \left\| \sum_{j=0}^{q-1} F_{w(x_j)^{-1/m_0}}^{n_j}(u^{1/m_0}) \right\| &= \left\| \sum_{j=0}^{q-1} \sum_{l=0}^p \frac{u_l^{1/m_0}}{[w_{l+1}(x_j) \cdots w_{l+n_j}(x_j)]^{1/m_0}} e_{l+n_j} \right\| \\ &\leq \sum_{j=0}^{q-1} \sum_{l=0}^p (|\widehat{w}_l(x_j) u_l|^{1/m_0} + 1) \left\| \frac{1}{[\widehat{w}_{l+n_j}(x_j)]^{1/m_0}} e_{l+n_j} \right\| \\ &\leq \sum_{j=0}^{q-1} \sum_{l=0}^p (A_p + 1) \frac{M_0}{(l + n_j)^\beta} \\ &\leq \sum_{j=0}^{q-1} \frac{(p+1)(A_p+1)M_0}{n_j^\beta} \\ &\leq (p+1)(A_p+1)M_0\eta \\ &\leq \frac{\varepsilon}{2}. \end{aligned}$$

Analogously we get $\left\| \sum_{j=0}^{q-1} F_{w(y_j)^{-1/m_0}}^{n_j}(v^{1/m_0}) \right\| \leq \frac{\varepsilon}{2}$, what proves the first part of condition (II). For the second, writing $\lambda = (x, y)$, we have

$$\begin{aligned} \left\| \sum_{j \neq i} T_\lambda^{n_i} \left(S_{\lambda_j, m_0}^{n_j}((u, v)^{1/m_0}) \right)^m \right\| &\leq \left\| \sum_{j \neq i} B_{w(x)}^{n_i} \left(F_{w(x_j)^{-1/m_0}}^{n_j}(u^{1/m_0}) \right)^m \right\| \\ &\quad + \left\| \sum_{j \neq i} B_{w(y)}^{n_i} \left(F_{w(y_j)^{-1/m_0}}^{n_j}(v^{1/m_0}) \right)^m \right\|. \end{aligned}$$

As before, we treat these parcels separately and get

$$\begin{aligned}
 \left\| \sum_{j \neq i} T_x^{n_i} \left(S_{x_j, m_0}^{n_j} (u^{1/m_0}) \right)^m \right\| &= \left\| \sum_{j > i} T_x^{n_i} \left(S_{x_j, m_0}^{n_j} (u^{1/m_0}) \right)^m \right\| \\
 &= \left\| \sum_{j > i} \sum_{l=0}^p u_l^{m/m_0} \frac{w_{l+n_j-n_i+1}(x) \cdots w_{l+n_j}(x)}{[w_{l+1}(x_j) \cdots w_{l+n_j}(x_j)]^{m/m_0}} e_{l+n_j-n_i} \right\| \\
 &\leq \sum_{j > i} \sum_{l=0}^p \left\| u_l^{m/m_0} \frac{w_{l+n_j-n_i+1}(x) \cdots w_{l+n_j}(x)}{[w_{l+1}(x_j) \cdots w_{l+n_j}(x_j)]^{m/m_0}} e_{l+n_j-n_i} \right\|.
 \end{aligned}$$

Notice that, using $m/m_0 \geq 1$ and $D < C_2/2^\alpha$,

$$\begin{aligned}
 &\left| u_l^{m/m_0} \frac{w_{l+n_j-n_i+1}(x) \cdots w_{l+n_j}(x)}{[w_{l+1}(x_j) \cdots w_{l+n_j}(x_j)]^{m/m_0}} \right| \\
 &\leq |\widehat{w}_l(x_j) u_l|^{m/m_0} \frac{\widehat{w}_{l+n_j}(x)}{\widehat{w}_{l+n_j-n_i}(x)} \times \frac{1}{\widehat{w}_{l+n_j-n_i}(x)} \\
 &\leq A_p \exp \left(\left| \sum_{i=1}^{l+n_j} \log(w_i(x)) - \sum_{i=1}^{l+n_j} \log(w_i(x_j)) \right| \right) \times \frac{1}{\widehat{w}_{l+n_j-n_i}(x)} \\
 &\leq A_p \exp(F(l+n_j)|x-x_j|) \times \frac{1}{\widehat{w}_{l+n_j-n_i}(x)} \\
 &\leq A_p \exp \left(D \frac{F(l+n_j)}{n_j^\alpha} (n_j-n_i)^\alpha \right) \times \frac{1}{\widehat{w}_{l+n_j-n_i}(x)} \\
 &\leq A_p \frac{\exp \left(D \frac{F(l+n_j)}{(l+n_j)^\alpha} (l+n_j-n_i)^\alpha \times \frac{(l+n_j)^\alpha}{n_j^\alpha} \times \frac{(n_j-n_i)^\alpha}{(l+n_j-n_i)^\alpha} \right)}{\widehat{w}_{l+n_j-n_i}(x)} \\
 &\leq A_p \frac{\exp \left(D 2^\alpha \frac{F(l+n_j)}{(l+n_j)^\alpha} (l+n_j-n_i)^\alpha \right)}{\widehat{w}_{l+n_j-n_i}(x)} \\
 &\leq A_p \frac{\exp \left(C_2 \frac{F(l+n_j)}{(l+n_j)^\alpha} (l+n_j-n_i)^\alpha \right)}{\widehat{w}_{l+n_j-n_i}(x)}.
 \end{aligned}$$

Hence,

$$\begin{aligned}
 \left\| \sum_{j \neq i} B_{w(x)}^{n_i} \left(F_{w(x_j)^{-1/m_0}}^{n_j} (u^{1/m_0}) \right)^m \right\| &\leq \sum_{j > i} \sum_{l=0}^p \left\| u_l^{m/m_0} \frac{w_{l+n_j-n_i+1}(x) \cdots w_{l+n_j}(x)}{[w_{l+1}(x_j) \cdots w_{l+n_j}(x_j)]^{m/m_0}} e_{l+n_j-n_i} \right\| \\
 &\leq \sum_{j > i} \sum_{l=0}^p \left\| A_p \frac{\exp \left(C_2 \frac{F(l+n_j)}{(l+n_j)^\alpha} (l+n_j-n_i)^\alpha \right)}{\widehat{w}_{l+n_j-n_i}(x)} e_{l+n_j-n_i} \right\| \\
 &\leq \sum_{j > i} \sum_{l=0}^p (A_p + 1) \left\| \frac{\exp \left(C_2 \frac{F(l+n_j)}{(l+n_j)^\alpha} (l+n_j-n_i)^\alpha \right)}{\widehat{w}_{l+n_j-n_i}(x)} e_{l+n_j-n_i} \right\| \\
 &\leq \sum_{j > i} \sum_{l=0}^p (A_p + 1) \frac{M_0}{(l+n_j-n_i)^\beta} \\
 &\leq M_0(p+1)(A_p+1) \sum_{j > i} \frac{1}{(n_j-n_i)^\beta} \\
 &\leq M_0(p+1)(A_p+1)\eta
 \end{aligned}$$

$$\leq \frac{\varepsilon}{2}.$$

Analogously, $\left\| \sum_{j \neq i} B_{w(y)}^{n_i} \left(F_{w(y_j)^{-1/m_0}}^{n_j} (v^{1/m_0}) \right)^m \right\| \leq \frac{\varepsilon}{2}$, what shows that condition (II) is satisfied. For condition (III) and (IV), let $\lambda \in \Lambda_i$ for some $i \in \{0, \dots, q-1\}$. We write $\lambda = (x, y)$. Notice that, for all $l = 0, \dots, p$ and $(z, z_i) \in \{(x, x_i), (y, y_i)\}$,

$$\begin{aligned} \frac{\widehat{w}_{l+n_i}(z)}{\widehat{w}_{l+n_i}(z_i)} &\leq \exp \left(\left| \sum_{j=1}^{l+n_i} \log(w_j(z)) - \sum_{j=1}^{l+n_i} \log(w_j(z_i)) \right| \right) \\ &\leq \exp(F(l+n_i)|z-z_i|) \\ &\leq \exp\left(\tau \frac{F(l+n_i)}{n_i^\alpha}\right) \\ &\leq \exp\left(\tau C_1 \frac{(l+n_i)^\alpha}{n_i^\alpha}\right) \\ &\leq \exp(\tau C_1 2^\alpha). \end{aligned}$$

Given $m \in (m_0, m_1]$, we have

$$\begin{aligned} \left\| B_{w(x)}^{n_i} \left(F_{w(x_i)^{-1/m_0}}^{n_i} (u^{1/m_0}) \right)^m \right\| &= \left\| \sum_{l=0}^p u_l^{m/m_0} \frac{w_{l+1}(x) \cdots w_{l+n_i}(x)}{[w_{l+1}(x_i) \cdots w_{l+n_i}(x_i)]^{\frac{m}{m_0}}} e_l \right\| \\ &\leq \sum_{l=0}^p \left\| \frac{(\widehat{w}_l(x_i) u_l)^{m/m_0}}{\widehat{w}_l(x)} \times \frac{\widehat{w}_{l+n_i}(x)}{\widehat{w}_{l+n_i}(x_i)} \times \frac{1}{[\widehat{w}_{l+n_i}(x_i)]^{\frac{m}{m_0}-1}} e_l \right\| \\ &\leq \sum_{l=0}^p \left\| (A_p/A'_p) \exp(\tau C_1 2^\alpha) \frac{1}{[\widehat{w}_{l+n_i}(x_i)]^{\frac{m}{m_0}-1}} e_l \right\| \\ &\leq ((A_p/A'_p) \exp(\tau C_1 2^\alpha) + 1) \sum_{l=0}^p \left\| \frac{1}{[\widehat{w}_{l+n_i}(x_i)]^{1/m_0}} e_l \right\| \\ &\leq \frac{\varepsilon}{2}. \end{aligned}$$

Analogously, $\left\| B_{w(y)}^{n_i} \left(F_{w(y_i)^{-1/m_0}}^{n_i} (v^{1/m_0}) \right)^m \right\| \leq \frac{\varepsilon}{2}$, what verifies condition (III). Finally, for condition (IV), we first notice that, for any x, y with $|x-y| < \frac{\tau}{n^\alpha}$, we have

$$\begin{aligned} \left| \left(\frac{w_{l+1}(x) \cdots w_{l+n}(x)}{w_{l+1}(y) \cdots w_{l+n}(y)} - 1 \right) u_l \right| &= \left| \left(\frac{\widehat{w}_l(y)}{\widehat{w}_l(x)} \times \frac{\widehat{w}_{l+n}(x)}{\widehat{w}_{l+n}(y)} - \frac{\widehat{w}_l(y)}{\widehat{w}_l(x)} + \frac{\widehat{w}_l(y)}{\widehat{w}_l(x)} - 1 \right) u_l \right| \\ &\leq \left| \frac{\widehat{w}_l(y)}{\widehat{w}_l(x)} \right| \|u\|_\infty \times \left| \frac{\widehat{w}_{l+n}(x)}{\widehat{w}_{l+n}(y)} - 1 \right| + \left| \frac{\widehat{w}_l(y)}{\widehat{w}_l(x)} - 1 \right| \|u\|_\infty \\ &\leq \frac{A_p}{A'_p} \left| \exp(F(n+l)|x-y|) - 1 \right| + \left| \exp(F(l)|x-y|) - 1 \right| \|u\|_\infty \\ &\leq \frac{A_p}{A'_p} \left| \exp(F(n+l) \frac{\tau}{n^\alpha}) - 1 \right| + \left| \exp(F(l) \frac{\tau}{n^\alpha}) - 1 \right| \|u\|_\infty \\ &\leq \frac{A_p}{A'_p} \left| \exp(C_1 \tau \frac{(n+p)^\alpha}{n^\alpha}) - 1 \right| + \left| \exp(C_1 \tau \frac{p^\alpha}{n^\alpha}) - 1 \right| \|u\|_\infty \\ &\leq \frac{A_p}{A'_p} \left| \exp(C_1 2^\alpha \tau) - 1 \right| + \left| \exp(C_1 \tau) - 1 \right| \|u\|_\infty \end{aligned}$$

$$\begin{aligned} &\leq (A_p/A'_p + \|u\|_\infty) \left| \exp(C_1 2^\alpha \tau) - 1 \right| \\ &\leq \tau_0 \end{aligned}$$

Hence,

$$\begin{aligned} &\left\| T_\lambda^{n_i} \left(S_{\lambda_i, m_0}^{n_i} (v^{1/m_0}) \right)^{m_0} - v \right\| \\ &\leq \left\| B_{w(x)}^{n_i} \left(F_{w(x_i)-1/m_0}^{n_i} (u^{1/m_0}) \right)^{m_0} - u \right\| + \left\| B_{w(y)}^{n_i} \left(F_{w(y_i)-1/m_0}^{n_i} (v^{1/m_0}) \right)^{m_0} - v \right\| \\ &= \left\| \sum_{l=0}^p \left(\frac{\widehat{w}_{l+n_i}(x)}{\widehat{w}_{l+n_i}(x_i)} - 1 \right) u_l e_l \right\| + \left\| \sum_{l=0}^p \left(\frac{\widehat{w}_{l+n_i}(y)}{\widehat{w}_{l+n_i}(y_i)} - 1 \right) v_l e_l \right\| \\ &\leq \sum_{l=0}^p \left(\|\tau_0 u_l e_l\| + \|\tau_0 v_l e_l\| \right) \\ &\leq \sum_{l=0}^p 2 \frac{\varepsilon}{2^{(p+1)}} \\ &= \varepsilon. \end{aligned}$$

This proves the claim and completes the proof. \square

2.2 Characterization of families of products of weighted shifts admitting a common hypercyclic algebra for the coordinatewise product

As done in [8], many results for vectors can be brought to the context of algebras with the coordinatewise product under almost the same conditions, the difference being the presence of some natural powers. Here we repeat this formula with [7, Theorem 2.1] and obtain Theorem 2.7. For the proof we need the following property, held true whenever $(e_n)_n$ is an unconditional basis of a Fréchet space X .

(UB) If $(x_n) \in X$ and $(\alpha_n) \in \ell_\infty$, then $(\alpha_n x_n) \in X$. Moreover, for all $\varepsilon > 0$, for all $M > 0$, there exists $\delta > 0$ such that, for all $x \in X$ with $\|x\| \leq \delta$, for all sequence $(\alpha_n) \in \ell_\infty$ with $\|(\alpha_n)\|_\infty \leq M$, then $(\alpha_n x_n) \in X$ and $\|(\alpha_n x_n)\| < \varepsilon$.

Theorem 2.7 *Let X be a Fréchet sequence algebra for the coordinatewise product with a conditnous norm and admitting (e_n) as an unconditional basis. Let $I \subset \mathbb{R}$ be a nonempty interval, let $d \geq 1$ and let $\Lambda \subset I^d$ be σ -compact. Let $(B_{w(a)})_{a \in I}$ be a family of weighted shifts on X and assume that $a \in I \mapsto w_n(a)$ is non-decreasing with $\inf_{n \in \mathbb{N}, a \in I} w_n(a) > 0$. Assume also that, there exist $F : \mathbb{N} \rightarrow \mathbb{R}_+$ and $c, C > 0$ such that, for all $n \geq 1$, denoting by $f_n(a) = \sum_{k=1}^n \log(w_k(a))$,*

$$\begin{aligned} \forall (a, b) \in I^2, \quad cF(n)|a - b| &\leq |f_n(a) - f_n(b)| \leq CF(n)|b - a| \\ \forall (a, b) \in I^2, \quad \frac{w_n(a)}{w_n(b)} &\geq c. \end{aligned}$$

Then the following assertions are equivalent:

- (a) $(B_{w(\lambda(1))} \times \cdots \times B_{w(\lambda(d))})_{\lambda \in \Lambda}$ admits a common hypercyclic algebra in X ;
- (b) For all $m \geq 1$, there exist $u \in X^d$ such that u^m is a common hypercyclic vector for $(B_{w(\lambda(1))} \times \cdots \times B_{w(\lambda(d))})_{\lambda \in \Lambda}$;
- (c) For all $m \in \mathbb{N}$, $\tau > 0$, for all $N \geq 1$, for all $\varepsilon > 0$, for all $K \subset \Lambda$ compact, there exists $N \leq n_1 < n_1 + N \leq n_2 < \cdots < n_{q-1} + N \leq n_q$ and $(\lambda_k)_{k=1, \dots, q} \in I^d$ such that

(i) $K \subset \cup_{k=1}^q \prod_{l=1}^d \left[\lambda_k(l) - \frac{\tau}{F(n_k)}, \lambda_k(l) \right]$

(ii) For all $i = 1, \dots, d$,

$$\left\| \sum_{k=1}^q \frac{1}{[w_1(\lambda_k(i)) \cdots w_{n_k}(\lambda_k(i))]^{1/m}} e_{n_k} \right\| < \varepsilon.$$

(iii) For all $k = 1, \dots, q$, for all $i = 1, \dots, d$, for all $l = 0, \dots, N$,

$$\left\| \sum_{j=k+1}^q \frac{w_{n_j-n_k+l+1}(\lambda_k(i)) \cdots w_{n_j+l}(\lambda_k(i))}{w_{l+1}(\lambda_j(i)) \cdots w_{n_j+l}(\lambda_j(i))} e_{n_j-n_k} \right\| < \varepsilon.$$

To simplify the notations, we will prove the result for $d = 2$, the general case is a straightforward adaptation with more terms and coordinates. We write $\lambda = (a, b)$ any element of \mathbb{R}^2 . For $\lambda = (a, b) \in I^2$, we will denote by T_λ the operator $B_{w(a)} \times B_{w(b)}$ acting on $X \times X$ and S_λ the operator $F_{w^{-1}(a)} \times F_{w^{-1}(b)}$, where $F_{w^{-1}(a)}$ is the weighted forward shift associated to the weight sequence $(w_n^{-1}(a))_n$. Moreover, when estimating the norms appearing in the Basic Criterion, like for example $\|T_\lambda^n(u)\|$ where $u = (x, y) \in X \times X$, it is worth noticing that it can be made coordinate by coordinate by estimating $\|B_{w(a)}(x)\|$ and $\|B_{w(b)}(y)\|$. They are often analogous to each other so we only need to estimate one of these coordinates.

Proof. The proof of (a) \Rightarrow (b) is trivial. Let us assume (b) and prove (c). Given $m \geq 1$, there exists $u = (x, y) \in X \times X$ such that u^m is a common hypercyclic vector for $(B_{w(a)} \times B_{w(b)})_{(a,b) \in \Lambda}$. We may always assume that $\|u\| \leq \delta$ where $\delta > 0$ is a small real number whose value will be precised later. Let $\tau > 0$, $N \geq 1$, $\varepsilon > 0$ and $K \subset \Lambda$ compact. We set

$$v = \left(\sum_{k=0}^N e_k, \sum_{k=0}^N e_k \right).$$

Let (n_k) be an increasing enumeration of

$$\{n \geq 1 : \|T_\lambda^n u^m - v\| < \delta \text{ for some } \lambda \in K\}.$$

Let $\Lambda_k = \{\lambda \in K : \|T_\lambda^{n_k} u^m - v\| < \delta\}$. Since u^m is a common hypercyclic vector for the family $(T_\lambda)_{\lambda \in K}$, since K is compact and each Λ_k is open, there exists $q \geq 1$ such that

$$K \subset \bigcup_{k=1}^q \Lambda_k.$$

For each $k = 1, \dots, q$, we define a_k and b_k by

$$a_k := \sup \{a : \exists b, (a, b) \in \Lambda_k\}$$

$$b_k := \sup \{b : \exists a, (a, b) \in \Lambda_k\}$$

and we set $\lambda_k = (a_k, b_k)$. Let us show that $n_1 \geq N$ and $n_{k+1} - n_k \geq N$ for all $k = 1, \dots, q-1$. First, since $\|B_{w(a_1)}^{n_1} x^m - \sum_{j=0}^N e_j\| \leq \delta$, since δ is small and since the $N+1$ first coordinate functionals are continuous, we get

$$w_1(a_1) \cdots w_{n_1}(a_1) |x_{n_1}^m| \geq 1 - \eta.$$

Hence, by making η small (depending on N), the quotient $\frac{1-\eta}{\eta}$ is as big as we want so that n_1 cannot be bounded by N because of

$$w_1(a_1) \cdots w_{n_1}(a_1) \geq w_1(a_1) \cdots w_{n_1} \frac{|x_{n_1}^m|}{\eta} \geq \frac{1-\eta}{\eta}.$$

This proves $n_1 \geq N$. Now, assume by contradiction that there exists some k such that $n_{k+1} - n_k < N$. We set $p = n_k + N - n_{k+1} \in \{0, \dots, N-1\}$ and we use that $\|B_{w(a_{k+1})}^{n_{k+1}}(x^m) - \sum_{j=0}^N e_j\| \leq \delta$ to obtain

$$\begin{cases} |w_{p+1}(a_{k+1}) \cdots w_{n_k+N}(a_{k+1}) x_{n_k+N}^m - 1| < \eta \\ |w_{p+2}(a_{k+1}) \cdots w_{n_k+N+1}(a_{k+1}) x_{n_k+N+1}^m - 1| < \eta. \end{cases}$$

Since we also know that $\|B_{w(a_k)}^{n_k}(x^m) - \sum_{j=0}^N e_j\| \leq \delta$, we get

$$\begin{cases} |w_{N+1}(a_k) \cdots w_{n_k+N}(a_k) x_{n_k+N}^m - 1| < \eta \\ |w_{N+2}(a_k) \cdots w_{n_k+N+1}(a_k) x_{n_k+N+1}^m - 1| < \eta \end{cases}$$

Taking respective quotients, these inequalities lead to

$$\begin{aligned} \frac{w_{N+2}(a_k) \cdots w_{n_k+N+1}(a_k)}{w_{p+2}(a_{k+1}) \cdots w_{n_k+N+1}(a_{k+1})} &\leq \frac{\eta}{1-\eta} \\ \frac{w_{N+1}(a_k) \cdots w_{n_k+N}(a_k)}{w_{p+1}(a_{k+1}) \cdots w_{n_k+N}(a_{k+1})} &\geq \frac{1-\eta}{1+\eta}. \end{aligned}$$

Consequently we get

$$\frac{1-\eta}{1+\eta} \times \frac{w_{n_k+N+1}(a_k)}{w_{n_k+N+1}(a_{k+1})} \times \frac{w_{p+1}(a_{k+1})}{w_{N+1}(a_k)} \leq \frac{\eta}{1-\eta}.$$

Hence, provided η is small enough, using that $w_n(a)/w_n(b)$ is bounded and bounded away from zero independently of a, b and n , using that $p \in \{0, \dots, N-1\}$ and also that $a \in I \mapsto w_n(a)$ is non-decreasing, we get a contradiction. Therefore $n_{k+1} - n_k \geq N$ for all $k = 1, \dots, q-1$.

Let us now prove (i). Let $\lambda = (a, b) \in \Lambda_k$ for some $k = 1, \dots, q$. Provided δ is small enough, we know that for any $0 \leq l \leq N$,

$$\begin{cases} |w_{l+1}(a_k) \cdots w_{n_k+l}(a_k) x_{n_k+l}^m - 1| < \eta \\ |w_{l+1}(a) \cdots w_{n_k+l}(a) x_{n_k+l}^m - 1| < \eta, \end{cases}$$

with $\eta < \min(\frac{1}{2}, \frac{c\tau}{4})$. Hence,

$$|w_1(a_k) \cdots w_{n_k}(a_k) - w_1(a) \cdots w_{n_k}(a)| \cdot |x_{n_k}^m| < 2\eta.$$

On the other hand,

$$w_1(a_k) \cdots w_{n_k}(a_k) - w_1(a) \cdots w_{n_k}(a)$$

$$\begin{aligned}
&= w_1(a) \cdots w_{n_k}(a) \left(\exp \left(\sum_{j=1}^{n_k} \log(w_j(a_k)) - \sum_{j=1}^{n_k} \log(w_j(a)) \right) - 1 \right) \\
&\geq w_1(a) \cdots w_{n_k}(a) (\exp(cF(n_k)(a_k - a)) - 1) \\
&\geq cw_1(a) \cdots w_{n_k}(a)(a_k - a)F(n_k).
\end{aligned}$$

Since we also know that $w_1(a) \cdots w_{n_k}(a)|x_{n_k}^m| \geq 1/2$, we finally get

$$0 \leq a_k - a \leq \frac{4\eta}{cF(n_k)} < \frac{\tau}{F(n_k)}.$$

The same is true for the second coordinate and we get (i). In order to prove (ii), we define (α_n) by

$$\alpha_n = \begin{cases} \frac{1}{w_1(a_k) \cdots w_{n_k}(a_k)x_{n_k}^m}, & \text{if } n = n_k \text{ for some } k = 1, \dots, q, \\ 0 & \text{otherwise.} \end{cases}$$

Observe that $\|\alpha\|_\infty \leq 2$. Hence, by choosing δ small enough and using (UB), we get

$$\begin{aligned}
\left\| \sum_{k=1}^q \frac{1}{[w_1(a_k) \cdots w_{n_k}(a_k)]^{1/m}} e_{n_k} \right\| &= \left\| \sum_{k=1}^q \frac{1}{[w_1(a_k) \cdots w_{n_k}(a_k)]^{1/m} x_{n_k}} x_{n_k} e_{n_k} \right\| \\
&= \left\| \sum_{k=1}^q \left(\frac{1}{w_1(a_k) \cdots w_{n_k}(a_k)x_{n_k}^m} \right)^{1/m} x_{n_k} e_{n_k} \right\| \\
&= \left\| \sum_{n \geq N}^{+\infty} \alpha_n^{1/m} x_n e_n \right\| < \varepsilon.
\end{aligned}$$

It remains to prove (iii). We fix $k = 1, \dots, q$ and we now set

$$\beta_n = \begin{cases} \frac{1}{w_{l+1}(a_j) \cdots w_{n_j+l}(a_j)x_{n_j+l}^m} & \text{if } n \neq n_j - n_k + l \text{ for some } j \geq k + 1 \\ 0 & \text{otherwise.} \end{cases}$$

Again $\|\beta\|_\infty \leq 2$ and, writing $B_{w(a_k)}^{n_k}(x^m) - \sum_{j=0}^N e_j$ as (z_n) , we get

$$\begin{aligned}
\sum_n \beta_n z_n e_n &= \sum_{j=k+1}^q \frac{w_{n_j-n_k+l+1}(a_k) \cdots w_{n_j+l}(a_k)x_{n_j+l}^m}{w_{l+1}(a_j) \cdots w_{n_j+l}(a_j)x_{n_j+l}^m} e_{n_j-n_k+l} \\
&= \sum_{j=k+1}^q \frac{w_{n_j-n_k+l+1}(a_k) \cdots w_{n_j+l}(a_k)}{w_{l+1}(a_j) \cdots w_{n_j+l}(a_j)} e_{n_j-n_k+l}.
\end{aligned}$$

Once more the result follows from (UB) by using that $\lambda_k \in \Lambda_k$ and that $n_j - n_k > N$ for $j \geq k + 1$.

We finish the proof of the theorem by showing that (c) \Rightarrow (a). We shall apply the Basic Criterion for algebras. Let $K \subset \Lambda$ be compact, let $\mathcal{D} \subset X^2$ be the set of couple of vectors with finite support and let $(u, v) \in \mathcal{D} \times \mathcal{D}$. Since we are going to estimate the norms coordinate by coordinate, let us write

$$u(1) = \sum_{l=0}^{N_0} u_l e_l \quad \text{and} \quad v(1) = \sum_{l=0}^{N_0} v_l e_l$$

for some $N \geq 0$. We fix $m' \leq m''$, $\varepsilon > 0$ and $\tau > 0$ (conditions on ε and τ will be imposed later) and we apply (c) with $m = m'$ and obtain two sequences $(n_k)_{k=1, \dots, q}$ and $(\lambda_k)_{k=1, \dots, q}$ satisfying

(i), (ii) and (iii). We write $\lambda_k = (a_k, b_k)$ and set $\Lambda_k = K \cap ([a_k - \tau/F(n_k); a_k] \times [b_k - \tau/F(n_k); b_k])$ so that $\bigcup_k \Lambda_k \supset K$. Let us show that the assumptions of the Basic Criterion for algebras are satisfied for the sequence $m_k = n_k - N$. First, we observe that the condition $\inf_n w_n(\tilde{a})$ and the unconditionality of $(e_n)_n$ ensures the continuity of $B_{w(\tilde{a})}^{1/m'}$. We also notice that

$$\begin{aligned} & \left\| \sum_{k=1}^q F_{w^{-1/m'}(a_k)}^{m_j} (v(1)^{1/m'}) \right\| \\ & \leq \sum_{l=0}^{N_0} (|v_l|^{1/m'} + 1) \left\| \sum_{k=1}^q \frac{1}{[w_{l+1}(a_k) \cdots w_{n_k-(N-l)}(a_k)]^{1/m'} e_{n_k-(N-l)}} \right\|. \end{aligned}$$

We fix $\tilde{a} \in I$ and we claim that

$$\begin{aligned} & \sum_{k=1}^q \frac{1}{[w_{l+1}(a_k) \cdots w_{n_k-(N-l)}(a_k)]^{1/m'} e_{n_k-(N-l)}} \\ & = B_{w(\tilde{a})}^{N-l} \left(\sum_{k=1}^q \frac{x_{k,l}}{[w_1(a_k) \cdots w_{n_k}(a_k)]^{1/m'} e_{n_k}} \right) \end{aligned} \quad (\text{D.3})$$

for some sequence $(x_{k,l})_k \in \ell_\infty$ with

$$\|(x_{k,l})_k\|_\infty \leq \left(\frac{M}{c}\right)^{N/m'}$$

where $M = \max(1, |w_j(a)| : 0 \leq j \leq N, (a, b) \in K \text{ for some } b)$. Effectively,

$$\begin{aligned} & \sum_{k=1}^q \frac{1}{[w_{l+1}(a_k) \cdots w_{n_k-(N-l)}(a_k)]^{1/m'} e_{n_k-(N-l)}} \\ & = \sum_{k=1}^q [w_1 \cdots w_l(a_k)]^{1/m'} \times \left[\prod_{j=n_k-(N-l)+1}^{n_k} \frac{w_j(a_k)}{w_j(\tilde{a})} \right]^{1/m'} \\ & \quad \times B_{w(\tilde{a})}^{N-l} \left(\frac{1}{[w_1(a_k) \cdots w_{n_k}(a_k)]^{1/m'} e_{n_k}} \right) \\ & \leq \sum_{k=1}^q M^{N/m'} \left(\frac{1}{c}\right)^{(N-l)/m'} B_{w(\tilde{a})}^{N-l} \left(\frac{1}{[w_1(a_k) \cdots w_{n_k}(a_k)]^{1/m'} e_{n_k}} \right). \end{aligned}$$

Hence, the first condition in (I) follows from the continuity of $B_{w(\tilde{a})}^{1/m'}$ and the unconditionality of (e_n) .

The other conditions follow from a similar calculation. Indeed, for $k = 1, \dots, q$, $(a, b) \in \Lambda_k$ and $m \in [m', m'']$,

$$\begin{aligned} & \sum_{j \neq k} B_{w(a)}^{m_k} \left(F_{w^{-1/m_0}(a_j)}^{m_j} (v(1)^{1/m'}) \right)^m \\ & = \sum_{j=k+1}^q B_{w(a)}^{m_k} \left(F_{w^{-1/m'}(a_j)}^{m_j} (v(1)^{1/m'}) \right)^m \\ & = \sum_{j=k+1}^q \sum_{l=0}^N [v_l(1)]^{m/m'} \frac{w_{n_j-n_k-(N_0-l)+1}(a) \cdots w_{n_j-(N-l)}(a)}{[w_{l+1}(a_j) \cdots w_{n_j-(N-l)}(a_j)]^{m/m'}} e_{n_j-n_k-(N-l)} \end{aligned}$$

$$= \sum_{j=k+1}^q \sum_{l=0}^N [v_l(1)]^{m/m'} y_{j,l} \frac{w_{n_j-n_k-(N-l)+1}(a) \cdots w_{n_j-(N-l)}(a)}{w_{l+1}(a_j) \cdots w_{n_j-(N-l)}(a_j)} e_{n_j-n_k-(N-l)},$$

where

$$y_{j,l} = \frac{1}{[w_{l+1}(a_j) \cdots w_{n_j-(N-l)}(a_j)]^{m/m'-1}}.$$

Since X has a continuous norm, from (ii) and as $m/m' - 1 \geq 0$ we get that $(y_{j,l})$ is bounded uniformly on j and l . In order to apply (iii) there is an extra difficulty to handle because we have

$$w_{n_j-n_k-(N-l)+1}(a) \cdots w_{n_j-(N-l)}(a)$$

and not

$$w_{n_j-n_k-(N-l)+1}(a_k) \cdots w_{n_j-(N-l)}(a_k).$$

But this is solved easily using that

$$w_{n_j-n_k-(N-l)+1}(a) \cdots w_{n_j-(N-l)}(a) \leq w_{n_j-n_k-(N-l)+1}(a_k) \cdots w_{n_j-(N-l)}(a_k)$$

and the unconditionality of (e_n) . This proves (II). To prove (III) and (IV), we take $k \in \{1, \dots, q\}$, $(a, b) \in \Lambda_k$, $m \in [m', m'']$ and write

$$B_{w(a)}^{m_k} \left(F_{w^{-1/m'}(a_k)}^{m_k} (v(1))^{1/m'} \right)^m = \sum_{l=0}^N v_l \frac{w_{l+1}(a) \cdots w_{n_k-(N-l)}(a_k)}{[w_{l+1}(a_k) \cdots w_{n_k-(N-l)}(a_k)]^{m/m'}} e_l.$$

If $m > m'$, then $m/m' - 1 \geq 1/m'$ and we get

$$\begin{aligned} \frac{w_{l+1}(a) \cdots w_{n_k-(N-l)}(a)}{[w_{l+1}(a_k) \cdots w_{n_k-(N-l)}(a_k)]^{m/m'}} &\leq \frac{1}{[w_{l+1}(a_k) \cdots w_{n_k-(N-l)}(a_k)]^{m/m'-1}} \\ &\leq \frac{1}{[w_{l+1}(a_k) \cdots w_{n_k-(N-l)}(a_k)]^{1/m'}} e_l. \end{aligned}$$

Here we can use the fact that (e_n) is bounded bellow and repeat what we did with (D.3), now with $(x_{k,l})$ having only one non-zero term. Hence (III) follows if we choose $\varepsilon > 0$ small enough.

If $m = m'$, then we write

$$\begin{aligned} &\left\| B_{w(a)}^{m_k} \left(F_{w^{-1/m_0}(a_k)}^{m_k} (v(1))^{1/m'} \right)^{m'} - v(1) \right\| \\ &\leq \sum_{l=0}^N (|v_l(1) + 1|) \left\| \left(\frac{w_{l+1}(a) \cdots w_{n_k-(N-l)}(a)}{w_{l+1}(a_k) \cdots w_{n_k-(N-l)}(a_k)} - 1 \right) e_l \right\| \end{aligned}$$

and it is easy to show that this becomes smaller than ε provided $\tau > 0$ is taken small enough and using

$$|f_n(a) - f_n(b)| \leq CF(n)|a - b|.$$

□

If we restrict the m -th power in the statement to $m = 1$, we obtain a characterization of families admitting a common hypercyclic *vector*. Hence, what distinguish vectors from algebras as far as this characterization goes is the presence of the power $1/m$ in (ii). As soon as these conditions are equivalent for a specific family of weights, one has that the family admits a common hypercyclic vector if and only if it admits a common hypercyclic algebra.

Proposition 2.8 *Let $\alpha \in (0, 1]$ and $w_1(a) \cdots w_n(a) = \exp(an^\alpha)$ and $X = \ell_p(\mathbb{N})$ or $c_0(\mathbb{N})$. The following assertions are equivalent:*

(a) $(B_{w(a)} \times B_{w(b)})_{(a,b) \in \Lambda}$ has a common hypercyclic vector in X ;

(b) $(B_{w(a)} \times B_{w(b)})_{(a,b) \in \Lambda}$ has a common hypercyclic algebra in X .

Proof. The proof of (b) \Rightarrow (a) is trivial. Let us assume (a) and prove (b). This family then satisfy (c) of Theorem 2.7 for $m = 1$. Let m be arbitrary, let $\tau > 0$, let $N \geq 1$, $\varepsilon > 0$ and let $K \subset \Lambda$ compact. There is N_0 such that

$$\sum_{n=N_0}^{+\infty} \frac{1}{\exp(an^\alpha)^{1/m}} < \varepsilon. \tag{D.4}$$

From Theorem 2.7(c, $m = 1$) there exist $(n_k, \lambda_k)_{k=1, \dots, q}$ satisfying (i) and (iii) of the same theorem. Condition (ii) for this arbitrary m follows automatically from (D.4), what completes the proof. \square

This particular case includes, for example, multiples of the backward shift.

Example 2.9 As shown in [7], $(sB \times (5-s)B)_{s \in [2,3]}$ just as any family $(\lambda B \times \mu B)_{(\lambda, \mu) \in \Gamma}$ indexed by a Lipschitz curve $\Gamma \subset (1, +\infty)^2$ admit a common hypercyclic vector. Hence, they admit as well common hypercyclic algebras for the coordinatewise product.

3 Convolution product

We will obtain Theorem 1.4 by applying the following more general key result from [9], which is an adapted version of Birkoff’s hypercyclic theorem for common hypercyclic algebras.

Proposition 3.1 *Let Λ be a countable union of compact sets and let $(T_\lambda)_{\lambda \in \Lambda}$ be a family of operators in $\mathcal{L}(X)$ such that the map $(\lambda, x) \mapsto T_\lambda(x)$ is continuous from $\Lambda \times X$ into X . Assume that, for all compact sets $K \subset \Lambda$, for all $m \geq 1$, for all U, V non-empty open subsets of X and for all neighborhood W of 0, one can find $u \in U$ such that, for all $\lambda \in K$, there exists $N \in \mathbb{N}$ satisfying*

$$\begin{cases} T_\lambda^N(u^n) \in W \text{ when } n \leq m - 1, \\ T_\lambda^N(u^m) \in V. \end{cases}$$

Then the set of points generating a common hypercyclic algebra for the family $(T_\lambda)_{\lambda \in \Lambda}$ is residual in X .

As for the covering, we didn’t manage to apply Lemma 2.6 to $w_n(\lambda) = 1 + \frac{\lambda}{n}$. As this result was made having in mind the case $w_1(a) \cdots w_n(a) = \exp(an^\alpha)$, it well fits this setting and naturally grants a common hypercyclic vector. The case $w_n(\lambda) = 1 + \frac{\lambda}{n}$ is rather different. Now, the function $a \mapsto \sum_{k=1}^n \log(w_n(a))$ is $\log(n)$ -Lipschitz and a partition made of cubes of side $\frac{\tau}{n_k^\alpha}$ doesn’t seem to work well. To overcome this apparent incompatibility we will insist on defining cubes of side $\frac{\tau}{\log(n_k)}$ instead.

In one dimension, we could define something like $\lambda_{j+1} = \lambda_j + \frac{\tau}{\log(n_k)}$ and proceed with similar calculations as in [9, Theorem 2.11]. The tricky part being to define a suitable sequence $(n_k)_k$, but once it is done, this gives a common hypercyclic algebra for $(B_{w(\lambda)})_{\lambda>0}$ and answer [9, Question 2.14] in the affirmative. What interest us, however, is the possibility of finding such an algebra for $(B_{w(\lambda_1)} \times \cdots \times B_{w(\lambda_d)})_{\lambda \in (0, +\infty)^d}$ for any $d \geq 1$.

As we have done before, we will prove the result for $d = 2$, but it is clear that the same can be done for any $d \geq 1$.

Proof of Theorem 1.4. Let $(T_{\lambda, \mu})_{(\lambda, \mu) \in (0, +\infty)^2}$ defined by $T_{\lambda, \mu} := B_{w(\lambda)} \oplus B_{w(\mu)}$ on $X \times X$, where $X = \ell_1(\mathbb{N})$ and $w_n(\lambda) = 1 + \frac{\lambda}{n}$. We aim to apply Proposition 3.1 to this family.

We fix $[a', b'] \times [a'', b''] \subset (0, +\infty)^2$ with $b' < 2a'$ and $b'' < 2a''$ and let $m \in \mathbb{N}$. We also fix an integer $r > \max\{\frac{1}{a'}, \frac{1}{a''}, 1\}$ (as we will see, one could just take $r = 1$ in the case $\min\{a', a''\} \geq 1$). As the products $w_1(x) \cdots w_n(x)$ behave like n^x , is it enough to prove the claim for $w_1(x) \cdots w_n(x) = n^x$. This will simplify the proof as there will be less constant factors to be dealt with.

For $m = 1$ the proposition is equivalent to the existence of a common hypercyclic vector (which is know to be true from Article C). Let us assume $m \geq 2$. Let $U, V, W \subset X \times X$ open and non-empty, with $0 \in W$. Let $(u_1, u_2) \in U$, $(v_1, v_2) \in V$, $p \in \mathbb{N}$ and $\eta > 0$ such that $B((u_1, u_2); 2\eta) \subset U$, $B((v_1, v_2); 3\eta) \subset V$ and $\max \text{supp}(u_1, u_2, v_1, v_2) \leq p$. We write $v_i = \sum_{l=0}^p v_{i,l} e_l$ for $i = 1, 2$ and we choose σ a very big integer of the form n^m (conditions on its size later). We can choose σ such that $\lfloor (\log \sigma)^3 + 1 \rfloor$ is the square of some integer q . We then have $\sqrt{q} \in \mathbb{N}$ and

$$(\log \sigma)^3 \leq q \leq (\log \sigma)^3 + 1.$$

We now define $a = \min\{a', a''\}$, $b = \max\{b', b''\}$, and we consider a covering of $[a, b]^2$ by q cubes of side $\frac{b-a}{\sqrt{q}}$. We choose the sequence of powers

$$N_j = (m-1)\sigma + \sigma^{\frac{m-1}{m}}(j+1)^r, \quad j = 1, \dots, q.$$

Let

$$(u', u'') = (u_1, u_2) + \sum_{j=1}^q \sum_{l=0}^p (d'_{l,j} e_{N_j - (m-1)\sigma + l}, d''_{l,j} e_{N_j - (m-1)\sigma + l}) + (\varepsilon_1 e_\sigma, \varepsilon_2 e_\sigma),$$

where

$$d'_{l,j} := \frac{v_{1,l}}{m \varepsilon^{m-1} w_{l+1}(\lambda_j) \cdots w_{l+N_j}(\lambda_j)}, \quad d''_{l,j} := \frac{v_{2,l}}{m \varepsilon^{m-1} w_{l+1}(\mu_j) \cdots w_{l+N_j}(\mu_j)}$$

$$\varepsilon_1 := \left(\frac{1}{w_1(a') \cdots w_{m\sigma}(a')} \right)^{\frac{1}{m}}, \quad \varepsilon_2 := \left(\frac{1}{w_1(a'') \cdots w_{m\sigma}(a'')} \right)^{\frac{1}{m}}.$$

Let us first show that $(u', u'') \in U$ if σ is big enough. We can calculate it coordinate by coordinate and get, for some constant $C'_v > 0$ depending on v ,

$$\|u' - u_1\| \leq \left\| \sum_{j=1}^q \sum_{l=0}^p d'_{l,j} e_{N_j - (m-1)\sigma + l} \right\| + \|\varepsilon_1 e_\sigma\|$$

$$\leq \sum_{j=1}^q \sum_{l=0}^p \frac{|v_{1,l}|}{m} \times \frac{[w_1(a') \cdots w_{m\sigma}(a')]^{\frac{m-1}{m}}}{w_{l+1}(\lambda_j) \cdots w_{l+N_j}(\lambda_j)} + \varepsilon_1$$

$$\begin{aligned}
 &= C'_v \sum_{j=1}^q \frac{(w_1(a') \cdots w_{m\sigma}(a'))^{\frac{m-1}{m}}}{w_1(a') \cdots w_{N_j}(a')} + \left(\frac{1}{w_1(a') \cdots w_{m\sigma}(a')} \right)^{\frac{1}{m}} \\
 &= C'_v \sum_{j=1}^q \left(\frac{(m\sigma)^{\frac{m-1}{m}}}{(m-1)\sigma + \sigma^{\frac{m-1}{m}}(j+1)^r} \right)^{a'} + \left(\frac{1}{w_1(a') \cdots w_{m\sigma}(a')} \right)^{\frac{1}{m}} \\
 &\leq C'_v \sum_{j=1}^{+\infty} \left(\frac{m^{\frac{m-1}{m}}}{(m-1)\sigma^{1/m} + (j+1)^r} \right)^a + \left(\frac{1}{w_1(a') \cdots w_{m\sigma}(a')} \right)^{\frac{1}{m}},
 \end{aligned}$$

and this goes to zero as $\sigma \rightarrow +\infty$, hence $\|u' - u_1\| \leq \eta$ if σ is big enough. Analogously $\|u'' - u_2\| \leq \eta$, what proves that $(u', u'') \in U$.

We now have to show that, for all $(\lambda, \mu) \in [a', b'] \times [a'', b'']$, there is N satisfying

$$\begin{cases} T_{\lambda, \mu}^N((u', u'')^n) \in W \text{ for } n < m, \\ T_{\lambda, \mu}^N((u', u'')^m) \in V. \end{cases} \quad (\text{D.5})$$

Given $(\lambda, \mu) \in [a', b'] \times [a'', b'']$, let $i \in \{1, \dots, q\}$ such that $(\lambda, \mu) \in B((\lambda_i, \mu_i), d_i)$. We then choose $N = N_i$. Let us verify conditions (D.5).

We first notice that, if $n < m$, since $\max\text{supp}(u', u'')^n \leq (m-1)\sigma < N_i$, we have that $\mathcal{T}_{\lambda, \mu}^N((u', u'')^n) = 0 \in W$. For the m -th power we get

$$(u', u'')^m = (P'_0, P''_0) + \sum_{j=1}^q \sum_{l=0}^p (m\varepsilon_1^{m-1} d'_{l,j} e_{N_j+l}, m\varepsilon_2^{m-1} d'_{l,j} e_{N_j+l}) + (\varepsilon_1^m e_{m\sigma}, \varepsilon_2^m e_{m\sigma}),$$

where

$$\text{supp}(P'_0, P''_0) \subset [0, (m-1)\sigma + p] \cup [0, (m-2)\sigma + 2(N_q - (m-1)\sigma + p)].$$

We aim to apply $\mathcal{T}_{\lambda, \mu}^{N_i}$ on $(u', u'')^m$, many parcels will be eliminated by the support. If σ is big enough, we'll have $(m-1)\sigma + p < N_i$. Moreover,

$$\begin{aligned}
 N_i - [(m-2)\sigma + 2(N_q - (m-1)\sigma + p)] &\geq N_0 - [(m-2)\sigma + 2(N_q - (m-1)\sigma + p)] = \\
 &= (m-1)\sigma + \sigma^{\frac{m-1}{m}} - (m-2)\sigma - 2((m-1)\sigma + \sigma^{\frac{m-1}{m}}(q+1)^r - (m-1)\sigma + p) = \\
 &\geq \sigma + \sigma^{\frac{m-1}{m}} - 2\left(\sigma^{\frac{m-1}{m}} \left([\log(\sigma)]^3 + 2\right)^r + p\right) \geq \sigma - 2C' \sigma^{\frac{m-1}{m}} [\log(\sigma)]^{3r},
 \end{aligned}$$

for some constant $C' > 0$. Then we will have $N_i > \max\text{supp}(P_0)$ if σ is taken big enough. Furthermore, in the sum $\sum_{j=1}^q \sum_{l=0}^p (m\varepsilon_1^{m-1} d'_{l,j} e_{N_j+l}, m\varepsilon_2^{m-1} d'_{l,j} e_{N_j+l})$, we can use that $N_i > N_j + p$ when $j < i$ to conclude that all parcels indexed by $j = 1, \dots, i-1$ have maximum support less than N_i . The parcel $j = i$ is the one we will use to approach (v_1, v_2) and the final parcels $j = i+1, \dots, q$, such as the separated term $(\varepsilon_1^m e_{m\sigma}, \varepsilon_2^m e_{m\sigma})$, will be handled in our next calculations. All in all, we will write

$$\mathcal{T}_{\lambda, \mu}^{N_i}((u', u'')^m) = (P'_1, P''_1) + (P'_2, P''_2) + (P'_3, P''_3),$$

where

$$(P'_1, P''_1) = \sum_{l=0}^p \left(\frac{w_{l+1}(\lambda) \cdots w_{l+N_i}(\lambda)}{w_{l+1}(\lambda_i) \cdots w_{l+N_i}(\lambda_i)} v_{1,l} e_l, \frac{w_{l+1}(\mu) \cdots w_{l+N_i}(\mu)}{w_{l+1}(\mu_i) \cdots w_{l+N_i}(\mu_i)} v_{2,l} e_l \right),$$

$$\begin{aligned}
 P'_2 &= \sum_{j=i+1}^q \sum_{l=0}^p \frac{w_{N_j-N_i+l+1}(\lambda) \cdots w_{l+N_j}(\lambda)}{w_{l+1}(\lambda_j) \cdots w_{l+N_j}(\lambda_j)} v_{1,l} e_{N_j-N_i+l}, \\
 P''_2 &= \sum_{j=i+1}^q \sum_{l=0}^p \frac{w_{N_j-N_i+l+1}(\mu) \cdots w_{l+N_j}(\mu)}{w_{l+1}(\mu_j) \cdots w_{l+N_j}(\mu_j)} v_{2,l} e_{N_j-N_i+l}, \\
 P'_3 &= \varepsilon_1^m w_{m\sigma-N_i+1}(\lambda) \cdots w_{m\sigma}(\lambda) e_{m\sigma-N_i}, \\
 P''_3 &= \varepsilon_2^m w_{m\sigma-N_i+1}(\lambda) \cdots w_{m\sigma}(\lambda) e_{m\sigma-N_i}.
 \end{aligned}$$

The proof will be finished if we manage to prove the following:

- (I) $\|(P'_1, P''_1) - (v_1, v_2)\| < \eta$;
- (II) $\|(P'_2, P''_2)\| < \eta$;
- (III) $\|(P_3, P_3)\| < \eta$.

Let us begin with (I) and show that

$$\|(P'_1, P''_1) - (v_1, v_2)\| \leq \|P'_1 - v_1\| + \|P''_1 - v_2\| < \eta.$$

We have, for some constant C_v depending only on v (the one appearing in the definition of d'_i, d''_i),

$$\begin{aligned}
 \|P'_1 - v_1\| &= \left\| \sum_{l=0}^p \left(\frac{w_{l+1}(\lambda) \cdots w_{l+N_i}(\lambda)}{w_{l+1}(\lambda_i) \cdots w_{l+N_i}(\lambda_i)} - 1 \right) v_{1,l} e_l \right\| \\
 &\leq C_v \left| \frac{w_1(\lambda) \cdots w_{N_i}(\lambda)}{w_1(\lambda_i) \cdots w_{N_i}(\lambda_i)} - 1 \right| \\
 &\leq C_v \left| \exp \left(\sum_{j=1}^{N_i} \log(w_j(\lambda)) - \sum_{j=1}^{N_i} \log(w_j(\lambda_i)) \right) - 1 \right| \\
 &\leq 2C_v \left| \sum_{j=1}^{N_i} \log(w_j(\lambda)) - \sum_{j=1}^{N_i} \log(w_j(\lambda_i)) \right| \\
 &\leq 2C_v \log(N_i) |\lambda - \lambda_i| \\
 &< 2C_v \log(N_i) d'_i \\
 &= 2C_v \log(N_i) \frac{\eta}{4C_v \log(N_i)} = \frac{\eta}{2}.
 \end{aligned}$$

Notice that we used the estimate $|\exp(x) - 1| \leq 2|x|$ which holds for all $x \in [-1, 1]$. Analogously, $\|P''_1 - v_2\| < \frac{\eta}{2}$, what shows (I).

For (II), we first notice that, for some constant $C''_v > 0$ depending on v ,

$$\begin{aligned}
 &\left\| \frac{w_{N_j-N_i+l+1}(\lambda) \cdots w_{l+N_j}(\lambda)}{w_{l+1}(\lambda_j) \cdots w_{l+N_j}(\lambda_j)} v_{1,l} e_{N_j-N_i+l} \right\| = \\
 &= |v_{1,l}| \frac{w_{N_j-N_i+l+1}(\lambda) \cdots w_{N_j+l}(\lambda)}{w_{l+1}(\lambda_j) \cdots w_{N_j+l}(\lambda_j)} \\
 &\leq C''_v \frac{(N_j+l)^{\lambda-\lambda_j}}{(N_j-N_i+l)^\lambda} \\
 &= C''_v \frac{((m-1)\sigma + \sigma^{\frac{m-1}{m}}(j+1)^r + l)^{\lambda-\lambda_j}}{\left(((j+1)^r - (i+1)^r) \sigma^{\frac{m-1}{m}} + l \right)^\lambda}
 \end{aligned}$$

$$\begin{aligned}
 & \stackrel{l \geq 0}{\leq} C_v'' \frac{\sigma^{\lambda - \lambda_j} (j+1)^{r(\lambda - \lambda_j)} \left(\frac{(m-1)}{(j+1)^r} + \sigma^{\frac{-1}{m}} + \frac{l}{\sigma(j+1)^r} \right)^{\lambda - \lambda_j}}{\sigma^{\lambda \frac{m-1}{m}} (j+1)^{r\lambda} \left(1 - \frac{(i+1)^r}{(j+1)^r} \right)^\lambda} \\
 & \stackrel{r \geq 1}{\leq} C_v'' \frac{\left(\frac{(m-1)}{(j+1)^r} + \sigma^{\frac{-1}{m}} + \frac{l}{\sigma(j+1)^r} \right)^{\lambda - \lambda_j}}{\sigma^{\lambda_j - \frac{\lambda}{m}} (j+1)^{r\lambda_j} \left(1 - \frac{i+1}{j+1} \right)^\lambda} \\
 & \stackrel{j \geq i+1}{\leq} \frac{C_v'' C_0}{\sigma^{\lambda_j - \frac{\lambda}{m}} (j+1)^{r\lambda_j} \left(1 - \frac{i+1}{i+2} \right)^\lambda} \\
 & \stackrel{i \leq q}{\leq} \frac{C_v'' C_0}{(j+1)^{r\lambda_j} \left(\frac{\sigma^{\frac{\lambda_j - \frac{\lambda}{m}}{\lambda}}}{q+2} \right)^\lambda} \\
 & \stackrel{q \leq \lceil \log(\sigma) \rceil^3 + 1}{\leq} \frac{C_v'' C_0}{(j+1)^{r\lambda_j} \left(\frac{\sigma^{c'}}{\lceil \log(\sigma) \rceil^3 + 3} \right)^\lambda},
 \end{aligned}$$

for some constant C_0 depending on a, b, r, m and $c' = \frac{a'}{b'} - \frac{1}{m} \leq \frac{\lambda_i}{\lambda} - \frac{1}{m}$ is positive since $m \geq 2$ and $2a' \geq b'$. These estimates show that the above norm is the general term of a convergent series on j and which value can be made small by controlling σ . More precisely,

$$\begin{aligned}
 \|P_2'\| &= \left\| \sum_{j=i+1}^q \sum_{l=0}^p \frac{w_{N_j - N_i + l + 1}(\lambda) \cdots w_{l + N_j}(\lambda)}{w_{l+1}(\lambda_j) \cdots w_{l + N_j}(\lambda_j)} v_{1,l} e_{N_j - N_i + l} \right\| \\
 &\leq \frac{(p+1)C_v C_i}{\left(\frac{\sigma^{c'} B}{4C_{a,b} \lceil \log(\sigma) \rceil^3 + 3} \right)^\lambda} \sum_{j=i+1}^q \frac{1}{(j+1)^{ra'}} < \frac{\eta}{2}
 \end{aligned}$$

if σ is big enough. Analogously we get $\|P_2''\| \leq \frac{\eta}{2}$, what shows (II).

Let us proceed to the proof of (III). We have

$$\begin{aligned}
 \|P_3'\| &= \|\varepsilon^m w_{m\sigma - N_i + 1}(\lambda) \cdots w_{m\sigma}(\lambda) e_{m\sigma - N_i}\| \\
 &\leq \frac{w_{m\sigma - N_q + 1}(b') \cdots w_{m\sigma}(b')}{w_1(a') \cdots w_{m\sigma}(a')} \\
 &= \frac{w_1(b') \cdots w_{m\sigma}(b')}{w_1(a') \cdots w_{m\sigma}(a')} \times \frac{1}{w_1(b') \cdots w_{m\sigma - N_q}(b')} \\
 &\leq \frac{(m\sigma)^{b' - a'}}{(m\sigma - N_q)^{b'}} \\
 &\leq \frac{(m\sigma)^{b' - a'}}{(\sigma - \sigma^{\frac{m-1}{m}} (4C_{a,b} \lceil \log(\sigma) \rceil^3 + 2)^r)^{b'}},
 \end{aligned}$$

which goes to 0 as $\sigma \rightarrow +\infty$, then $\|P_3'\| \leq \frac{\eta}{2}$ if σ is big enough. Analogously, $\|P_3''\| \leq \frac{\eta}{2}$ and the proof is done. \square

We finish this paper by returning to the one dimensional framework. Let I be an interval and let $\lambda \in I \mapsto (w_n(\lambda))_n$ be a continuously parametrized family of weights such that $\lambda \mapsto \sum_{k=1}^n \log(w_k(\lambda))$ is $F(n)$ -Lipschitz for some function $F: \mathbb{N} \rightarrow \mathbb{N}$. As exemplified by its authors, [9, Theorem 2.11] can be applied for some cases where $F(n) = n$. It turns out that it can be more generally applied for the case where $w_1(\lambda) \cdots w_n(\lambda) = \exp(\lambda n^\alpha)$ with $\alpha \in (0, 1]$, which

is a particular case where $F(n) = n^\alpha$. However, the same result doesn't work for the case $F(n) = \log(n)$, although $w_n(\lambda) = 1 + \frac{\lambda}{n}$ does admit a common hypercyclic algebra as we proved here (through similar, yet different, calculations).

Question 3.2 *Could one obtain a unified one dimensional criterion on $F(n)$ granting a common hypercyclic subalgebra of $\ell_1(\mathbb{N})$ for the convolution product and including $F(n) = n^\alpha, \alpha \leq 1$, and $F(n) = \log(n)$ as particular cases?*

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Part IV
Conclusion

Since the beginning, our studies were motivated by the algebras of hypercyclic vectors, as it was planned in the doctorate project. Along our work, some of the problems formulated were unknown even for vectors, what lead to the inclusion of the word “vectors” in the manuscript title. The main subject that made us put aside the algebraic problems for a while was the search for common hypercyclic vectors on a higher dimensional setting. Many problems remain open on this subject, not only for algebras. We will discuss only some of them here. Of course all the other topics covered in our work also have many related open question. In what follows we will focus on the ones coming naturally from what we have done.

1 Basic hypercyclicity

Although many authors have found multiple criteria for obtaining a hypercyclic algebra for convolution operators $\phi(D)$ acting on $H(\mathbb{C})$, a general method for the case $|\phi(0)| > 1$ is not known. For instance, it is not clear if $2I + D$ satisfy Theorem 3.1 of Article A or not, and other criteria don't apply to this case.

Question 1.1 *Does the convolution operator $2I + D$ admit a hypercyclic algebra on $H(\mathbb{C})$?*

More generally, we could ask if not being a multiple of an exponential function is enough to get a hypercyclic algebra in general, as it is in the case $|\phi(0)| < 1$. We don't know, however, examples of entire functions ϕ satisfying $\phi(0) \geq 1$ and such that $\phi(D)$ does not have a hypercyclic algebra.

Question 1.2 *Does $\phi(D)$ admit a hypercyclic algebra on $\mathcal{H}(\mathbb{C})$ as soon as $\phi \in \mathcal{H}(\mathbb{C})$ is not a multiple of an exponential function?*

Maybe a different method is required just like in the case of left multiplication operators. As discussed in the section 6.2 of Article A, other techniques are required.

Question 1.3 *Let $\mathcal{K}(H)$ be the ideal of compact operators acting a Hilbert space H and equipped with the SOT-topology. Does there exist a bounded linear operator T such that the operator of left multiplication by T*

$$\begin{aligned} L_T: \mathcal{K}(H) &\longrightarrow \mathcal{K}(H) \\ L &\longmapsto TL \end{aligned}$$

has a hypercyclic algebra?

We can also ask if our methods could be applied to more classes of operators or maybe modified so that it fits in more contexts. For instance, it is not difficult to see that we can proceed as it is done with regular shifts and find hypercyclic algebras for the Taylor shift. Of course, as its name suggests, it behaves just like a normal shift so such an application is expected.

Question 1.4 *Can we find other hypercyclic operators acting on an F -algebra such that the Baire arguments developed in this work lead to the existence of an algebra of hypercyclic vectors?*

2 Common hypercyclicity

Many problems in literature remain open when it comes to common hypercyclicity in higher dimensions. Our studies have also lead to some new question, but of a more specific kind.

Looking at the covering developed in Article C, one could try to find a way of creating a partition based on some general function $F : \mathbb{N} \rightarrow \mathbb{N}$ rather than in the specific case $F(n) = n^\alpha$. However, as it was shown in article D, for $F(n) = \log(n)$ such a complex covering is not needed as a trivial one does the work. One could anyways try to improve the covering and the sequences and try to cover the case $\alpha = 1/d$ for which we don't have an answer.

Question 2.1 *Is there a way of improving our dyadic covering and its associated sequence in such a way that $w_1(a) \cdots w_n(a) = \exp(an^\alpha)$ induces a d -dimensional family $(B_{w(\lambda_1)} \times \cdots \times B_{w(\lambda_d)})_{\lambda \in [2,3]^d}$ admitting a common hypercyclic vector in the limit case $\alpha = 1/d$?*

More generally, by using the idea of constructing a covering for a general function $F : \mathbb{N} \rightarrow \mathbb{N}$, one could ask the following question.

Question 2.2 *Let I be an interval and consider a family $(w(a))_{a \in I}$ of admissible weights on $c_0(\mathbb{N})$ such that there exists a function $F : \mathbb{N} \rightarrow \mathbb{N}$ and $d \geq 1$ satisfying the following:*

- (1) *for every $n \in \mathbb{N}$, the map $a \in I \mapsto \sum_{k=1}^n \log(w_k(a))$ is $F(n)$ -Lipschitz;*
- (2) *the series $\sum_{k=1}^{+\infty} \frac{1}{F(n)^d}$ is divergent.*

Does that imply that $(B_{w(\lambda_1)} \times \cdots \times B_{w(\lambda_d)})_{\lambda \in I^d}$ admits a common hypercyclic vector?

Let us now turn to the problem of finding curves $\Lambda \subset (0, +\infty)^d$, $d \geq 1$, such that the product $(e^{\lambda(1)}B \times \cdots \times e^{\lambda(d)}B)_{\lambda \in \Lambda}$ has a common hypercyclic vector. As proven in Article C, Example 1.8, whenever $d \geq 1$, $\beta \in (0, 1]$, $w_1(a) \cdots w_n(a) = \exp(an^\alpha)$, with $\alpha \in (0, \beta)$, and Λ is a β -Hölder curve, the family $(B_{w(\lambda(1))} \times \cdots \times B_{w(\lambda(d))})_{\lambda \in \Lambda}$ admits a common hypercyclic vector on $X = \ell_p(\mathbb{N})^d$, $1 \leq p < +\infty$, or $X = c_0(\mathbb{N})^d$. The equality $\alpha = \beta$ works as a not covered limit case in this statement. Hence, the particular case $\alpha = \beta = 1$ is not covered, although it is known to be true by Theorem 1.1 also from Article C. The following question is a natural next step.

Question 2.3 *Can we show that, for all $d \geq 1$, the product family $(B_{w(\lambda(1))} \times \cdots \times B_{w(\lambda(d))})_{\lambda \in \Lambda}$ induced by the weights $w_1(a) \cdots w_n(a) = \exp(an^\beta)$ admits a common hypercyclic vector on $c_0(\mathbb{N})$ whenever $\Lambda \subset (0, +\infty)^d$ is a β -Hölder curve with $\beta \in (0, 1]$?*

When it comes to algebras, one could try to improve the dyadic covering and reduce the number q of terms in the partition. This is specially important when we work with the Cauchy product as the method imposes conditions on the size of the last power n_q . For the weight sequence $w_n(a) = 1 + \frac{a}{n}$, $n \geq 1, a > 0$, we were able to use the $\log(n)$ -Lipschitz property to construct a partition with much less terms, but the question remains open for $w_n(a) = 1 + \frac{a}{n^{1-\alpha}}$. This lead us to the following question.

Question 2.4 *Consider $d \geq 2$ and $w_1(a) \cdots w_n(a) = \exp(an^\alpha)$, $\alpha < 1/d$. Does $(B_{w(\lambda_1)} \times \cdots \times B_{w(\lambda_d)})_{\lambda \in [2,3]^d}$ admit a common hypercyclic algebra on $\ell_1(\mathbb{N})^d$ with the Cauchy product?*

We can also try to use these ideas to work on different frameworks, for instance one could look for common hypercyclic algebras for convolution operators in one and more dimensions. Also, for other related topics like common upper frequent hypercyclicity, the dyadic covering seems to be promising and could potentially give new results in higher dimensions.

3 Upper frequent hypercyclicity

We have managed to find algebras of upper frequently hypercyclic vectors for some of the classical families of operators, but the interesting case of the weighed shifts $B_{w(\lambda)}$ given by $w_n(\lambda) = 1 + \frac{\lambda}{n}$ is still open (the only positive answer is on $c_0(\mathbb{N})$ with the coordinatewise product). This problem seems to be quite doable, as an adaptation of the sequence of powers $(n_k)_k$ to this specific case has shown to be effective in the context of common hypercyclicity (see Section 3, Article D).

Question 3.1 *For each $\lambda > 0$, let the family of weights $(w_n(\lambda))$ be given by $w_n(\lambda) = 1 + \frac{\lambda}{n}$. If $X = \ell_p(\mathbb{N}), 1 \leq p < +\infty$, is equipped with the coordinatewise product, does $B_{w(\lambda)}$ admit an upper frequently hypercyclic algebra (one could assume $\lambda > 1/p$ if needed)? And if $X = \ell_1(\mathbb{N})$ is equipped with the Cauchy product, does $B_{w(\lambda)}$ admit an upper frequently hypercyclic algebra?*

As for more classes of operators, it is natural to look for sufficient conditions for a convolution operator $\phi(D)$ to admit an upper hypercyclic algebra.

Question 3.2 *How can we get upper frequently hypercyclic algebras in $H(\mathbb{C})$ for convolution operators $\phi(D)$?*

A last further possibility is to mix the properties of upper frequent and common hypercyclicity and look for criteria and examples of operators admitting an algebra of common upper frequent hypercyclic vectors.

Question 3.3 *Can one obtain a Baire argument giving common upper frequent hypercyclic algebras with applications to the classical families of operators?*

4 Disjoint hypercyclicity

Of course there are many problems for basic disjoint hypercyclic which could be stated here, but as our framework is algebraic, we will only mention related problems.

As with other concepts of hypercyclicity, when we work over sequence spaces with the coordinatewise product, the results have an analogous for algebras. For the Cauchy product, on the other hand, this is almost never the case. We have obtained a satisfying characterization for multiples of the form $\lambda B, \mu B^2$ to have a disjoint hypercyclic algebra over $\ell_1(\mathbb{N})$ with the Cauchy product, but the general case of N powers of backward shifts $B_{w(1)}^{r_1}, \dots, B_{w(N)}^{r_N}$ is far from being settled, even when the weights are the same and the powers are different.

Question 4.1 *Let B_w be a bounded backward shift on $\ell_1(\mathbb{N})$ with the Cauchy product. and let $0 < r_1 < r_2$. Under which conditions on w do $B_w^{r_1}, B_w^{r_2}$ admit a disjoint hypercyclic algebra in the case $0 < r_1 < r_2$? What happens in the case $r_1 = r_2$?*

We also can look for more contexts where the existence of a disjoint hypercyclic algebra could be possible.

Question 4.2 *Can we get a hypercyclic algebra in more contexts rather than backward shift operators on sequence spaces? Maybe composition or convolution operators?*

5 Closed hypercyclic algebras

We finish this work by pointing out one last time the limitations of the techniques used along the text. As for left multiplication maps on the ideal of compact operators, we might need a different approach to get closed hypercyclic algebras, or maybe they do not exist in general and we can find a fundamental property denying their existence.

Question 5.1 *Can we get a general result giving a closed hypercyclic algebra? Otherwise, can we get a general statement excluding many cases?*

We have managed to prove that the convolution operator $P(D)$ induced by a polynomial P never has a closed hypercyclic algebra, but it is not clear if $\phi(D)$ could have a closed hypercyclic algebra for some entire function ϕ .

Question 5.2 *Does there exist $\phi \in H(\mathbb{C})$ which induces a convolution operator $\phi(D)$ admitting a closed hypercyclic algebra?*

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