

Two-loop RGEs for general gauge theories and some aspects of BSM Higgs phenomenology

Kseniia Svirina

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Université Grenoble Alpes

THÈSE

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Kseniia SVIRINA

Thèse dirigée par **Ingo SCHIENBEIN, Université Grenoble Alpes**

préparée au sein du **Laboratoire de Physique Subatomique et Cosmologie**
dans **l'École Doctorale Physique**

Les équations de groupe de renormalisation à deux boucles pour les théories de jauge générales et quelques aspects de la phénoménologie du Higgs dans des modèles au-delà du MS

Two-loop RGEs for general gauge theories and some aspects of BSM Higgs phenomenology

Thèse soutenue publiquement le **26 juin 2020**,
devant le jury composé de :

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Abstract

We revisit the renormalisation group equations (RGE) for general renormalisable gauge theories at one- and two-loop accuracy. We identify and correct various mistakes in the literature for the β-functions for both dimensionless and dimensionful Lagrangian parameters. The discrepancies arise form the assumption of a diagonal wave-function renormalisation in the literature, which is not appropriate for models with mixing in the scalar sector, and due to inaccurate use of the dummy field method employed in the literature for the derivation of the β-functions for dimensionful parameters. We perform an independent cross-check using well-tested supersymmetric RGEs which confirms our results. The numerical impact of the changes in the β-function for the fermion mass terms is illustrated using a toy model with a heavy vector-like fermion pair coupled to a scalar gauge singlet. Unsurprisingly, the correction to the running of the fermion mass becomes sizeable for large Yukawa couplings of the order of $\mathcal{O}(1)$. Furthermore, we demonstrate the importance of the correction to the β-functions of the scalar quartic couplings using a general type-III Two-Higgs-Doublet-Model. We also provide a detailed pedagogic discussion of the dummy field method and summarize all the correct expressions for the β-functions in one place.

As an independent part of the reserach, we study the BSM Higgs physics in the Randall-Sundrum (RS) model with one extra dimension, predicting a new scalar particle, the radion, the remarkable similarity of which to the Higgs boson has been noticed [73, 74]. The model is implemented in the `FeynRules` package for the derivation of the Feynman rules, which is helpful for the future study of the collider phenomenology of the RS model with `MadGraph`, in particular, the scalar pair production and Higgs coupling modifications.

Résumé

Nous revisitons les équations de groupe de renormalisation (RGE) pour les théories de jauge générales renormalisables avec une précision à une et deux boucles. Nous identifions et corrigeons les diverses fautes dans la littérature pour les fonctions β pour les paramètres du lagrangien avec et sans dimension de masse. Les contradictions résultent de l'hypothèse d'une renormalisation diagonale de la fonction d'onde, qui n'est pas appropriée pour les modèles avec mélange dans le secteur scalaire, et de l'utilisation inexacte de la méthode du 'champ fictif', employée dans la littérature pour la dérivation de β-fonctions pour les paramètres dimensionnels. Nous effectuons une contre-vérification indépendante en utilisant des RGE supersymétriques bien testées, qui confirme nos résultats. L'impact numérique des changements dans la fonction β pour les masses de fermions est illustré à l'aide d'un toy-model avec une paire de fermions massifs de type vecteur, couplée à un scalaire singlet de jauge. Sans surprise, la correction pour la fonction β pour les masses de fermions devient importante pour les gros couplages de Yukawa de l'ordre de $\mathcal{O}(1)$. De plus, nous démontrons l'importance de la correction des fonctions β pour les couplages scalaires quartiques en utilisant un modèle général à deux Higgs-Doublet de type III. Nous fournissons également une discussion pédagogique détaillée de la méthode du 'champ factice' et résumons toutes les expressions correctes pour les fonctions β en un seul endroit.

En tant que partie indépendante de la recherche, nous étudions la physique du Higgs au-delà du modèle standard, dans le modèle Randall-Sundrum (RS) avec une dimension supplémentaire, prédisant une nouvelle particule scalaire, le radion, dont la remarquable similitude avec le boson de Higgs a été remarquée [73, 74]. Le modèle RS est implémenté dans le package `FeynRules` pour la dérivation des règles de Feynman, ce qui est utile pour l'étude de la phénoménologie des collisionneurs du modèle RS avec madgraph, en particulier, la production de paires scalaires et les modifications de couplage du Higgs.

Contents

Abstract

Résumé

Preface **1**

I Revisiting RGEs in General Gauge Theories **3**

1 A basic introduction to Renormalization and the Renormalization Group Equations **7**

1.1 Regularization 7

1.2 Renormalization 10

1.3 Renormalization Group Equations 14

1.3.1 The running of the coupling 14

1.3.2 The running mass and the field strength renormalization 18

2 RGE's for a general gauge theory **23**

2.1 The Lagrangian for a general gauge theory 23

2.2 Definitions and notations 25

2.2.1 β-functions and anomalous dimensions 25

2.2.2 Useful group invariants and coupling combinations 26

2.3 β-functions of dimensionless parameters 28

2.3.1 Scalar wave function renormalization 28

2.3.2 Fermion wave function renormalization 31

2.3.3 Vector wave function renormalization 32

2.3.4 Renormalization of the proper Yukawa vertex 34

2.3.5 A full list of β-functions of dimensionless parameters 37

3 β-functions for dimensionful parameters **41**

3.1 The dummy field method 41

3.2 Derivation of β-functions for dimensionful parameters with the dummy field method . . . 43
3.2.1 Fermion mass . . . 43
3.2.2 Trilinear coupling . . . 47
3.2.3 Scalar mass . . . 49
3.3 Comparison with supersymmetric RGEs . . . 52
3.4 Numerical impact . . . 59
3.4.1 Running of fermion mass terms . . . 59
3.4.2 Off-diagonal wave-function renormalization . . . 61

4 Conclusions and Outlook 65

II Some aspects of BSM Higgs phenomenology 67

5 The Standard Model — our current understanding of nature and reasons for BSM 71
5.1 The Standard Model and Electroweak Symmetry Breaking . . . 71
5.2 Open questions of the SM . . . 77

6 Scenarios with extra dimensions 81
6.1 The Randall-Sundrum model . . . 81
6.1.1 The Higgs-radion Lagrangian before mixing . . . 83
6.1.2 Higgs-radion mixing . . . 85
6.2 Some notes on the scalar potential . . . 86

7 Collider phenomenology of the RS model 91
7.1 RS model with FeynRules package . . . 91
7.2 Constraints on the RS model parameters with Lilith . . . 93

8 Conclusions and Outlook 97

A The dummy field method at two-loop 99

B Full two-loop RGEs without SUSY relations 115

C Feynman rules, Widths and Branching ratios for the RS model 125

D `FeynRules` model-file for the RS model 133

References 137

Preface

This thesis contains two independent parts of research. Part I represents a complete study of the renormalization group equations (RGEs) for general gauge theories, correcting certain well known expressions of 1- and 2-loop beta functions that are of interest in various models for physics Beyond the Standard Model (BSM) where the running of parameters across different energy scales is relevant. This study also contains a thorough explanation of the "dummy" field method, used for the derivation of beta functions for dimensionful parameters of the Lagrangian, which had never been properly addressed in the literature before. It has been published in Nuclear Phusics B (2019) [22]. In this manuscript, this large study is extended by some examples of calculations, as well as a detailed introduction to regularization, renormalization, RGEs and examples of their derivation useful for the understanding of the rest of this first part of the thesis.

Part II of the thesis is dedicated to the BSM Higgs physics in the Randall-Sundrum (RS) model with one extra dimension. The model has been implemented in the FeynRules package (from scratch) for the derivation of the Feynman rules, and has been successfully tested. FeynRules has an interface to MadGraph, which can be applied for the RS model file (the work on it is still in progress). Certain phenomenological results have been compared with the literature [72] and confirmed. Part II of the manuscript contains all the relevant expressions and conventions that define and describe the RS model. The possibility to obtain constraints for the new physics from Higgs measurements with `Lilith` tool has also been considered, in particular, the "reduced couplings" required as an input have been introduced in an appropriate form, and working examples have been obtained. The main goal of this research remains the study of scalar pair production in models with extra dimensions, and we aim at a publication in the future.

Part I

Revisiting RGEs in General Gauge Theories

Part I of this thesis is devoted to Renormalization Group Equations (RGEs) for general gauge theories. RGEs are an important tool that allows to describe physics throughout different energy scales:

- In many cases the parameters of a model are fixed at a hight scale, and the RGEs are needed to obtain the values of these parameters at the electroweak scale.
- Conversely, in some applications the parameters are fixed at the electroweak scale using experimental results, and the RGEs are used to obtain these parameters at high energies. For example, this is the case in analyses of the stability of the effective Higgs potential, see, e.g., [34].
- The running parameters are used for improving the predictions for observables (RGE improved observables), since they effectively resum the dominant terms of the perturbation series to all orders.

The expressions for RGEs at two-loop level for all dimensionless parameters in general gauge theories have been known for more than 30 years [6, 7, 8, 9, 10, 11]. These results were later re-derived and completed in a paper by Luo *et al.* [12], where the entire set of β-functions of parameters with and without a mass dimension was presented. The results for the dimensionful parameters, such as fermion masses, scalar masses and trilinear scalar couplings, were obtained from the β-functions of dimensionless parameters (Yukawa couplings and quartic scalar couplings) by applying a so called "dummy field" method [13]. However, no independent direct calculation of the two-loop β-functions for scalar and fermion masses and scalar trilinear couplings exists so far in the literature.

In this part we perform a detailed reassessment of the 2-loop RGEs for general gauge theories, employing a diagrammatic approach. Particular attention is paid to the RGEs for the dimensionful parameters, calculated by dint of the dummy field method, which has been proposed and used before in the literature. We provide a detailed and pedagogical discussion of this procedure, showing that it is valid to all orders (due to relations at the level of the Lagrangian). As a result, the RGEs for the dimensionful parameters are critically examined and the β-functions for the fermion masses are corrected at 1- and 2-loop level. We also identify an issue for the purely scalar couplings that emerges in certain models with respect to the literature. This issue is related to not always justified assumption about the properties of the wave-function renormalization. The corrected expressions are independently cross-checked and confirmed using well tested supersymmetric RGEs and the numerical estimation of the changes is provided.

Our results are implemented in the Mathematica package `SARAH` [15, 16, 17, 18, 19] and in the Python package `PyR@TE` [20, 21], both of which have been updated with respect

to the provided corrections. The results of this study are published in Nuclear Physics B in 2019 [22].

Taking into account the importance and the wide use of the running in various models, this summary of all the corrected expressions along with the detailed pedagogic discussion should be a valuable addition to the literature on RGEs.

The outline of Part I of this thesis is as follows:

- Chapter 1 provides a basic theoretical introduction to renormalization and the renormalization group that will be helpful for understanding the further material. In particular, the method of dimensional regularization is introduced using QED as an example. This is followed by a discussion of the renormalization of the fields and coupling constants. The definitions of beta- and gamma-functions are given and we exemplify how to calculate them in QED.

- Chapter 2 is devoted to RGEs in general gauge theories. After introducing the Lagrangian for a general gauge theory, defining group theoretical quantities and fixing the notations, we summarise the derivation of the beta functions for dimensionless parameters following the seminal papers by M. E. Machacek and M. T. Vaughn [6, 7, 8]. In these papers a number of tables can be found exhibiting the residues of the single and double poles of all contributions to the renormalization of the wave functions and vertices. This information can then readily be used to obtain the beta functions. Here, we hope to make the tables for the wave function renormalization and the Yukawa vertex much more accessible by including all the relevant Feynman diagrams. In passing, we correct a typo in the expression for the Yukawa beta function which has propagated through all the literature so far. We also address the issue of off-diagonal wave function renormalization in the scalar sector which leads to corrections of the beta functions for the scalar quartic couplings with respect to the literature.

- In Chapter 3 the derivation of the beta functions for dimensionful parameters by dint of the dummy method is demonstrated using a diagrammatic approach. We start by a detailed pedagogic discussion of the dummy field method which has so far been missing in the literature. We then apply this method to derive all the beta functions for the dimensionful parameters. We correct mistakes in the literature due to an improper use of the dummy field method. Our expressions are verified in comparison with supersymmetric RGEs and the estimated numerical impact in the frameworks of two different models is discussed.

- Finally, in Conclusions and Outlook we summarize our main results and provide an

outlook on possible future work.

- Some lengthy material has been relegated to the appendices. In Appendix A we list all two-loop vertex corrections which are needed to derive the beta functions for the dimensionful parameters. In all cases we provide the mapping of the corresponding group theoretical structures due to the dummy method. In some cases this mapping is non-trivial due to symmetry factors of identical particles and we provide a detailed example of how the calculation has been done. In Appendix B we list the full two-loop RGEs for the supersymmetric toy model introduced in Sec. 3.3 which we have used to validate our general results for the RGEs derived in this thesis.

Chapter 1

A basic introduction to Renormalization and the Renormalization Group Equations

In this chapter we give a basic introduction to renormalization and renormalization group equations (RGEs) which will be helpful for the understanding of the following chapters 2 and 3, where we discuss the two-loop RGEs of general gauge theories. The material discussed in this chapter is covered in many textbooks on Quantum Field Theory (QFT), for example the one by M. Peskin and D. Schroeder [1] or the more recent one by Matthew D. Schwartz [2]. Here we follow closely L. H. Ryder [3] for the introduction to renormalization and S. Pokorski [4] for the renormalization group equations. We have also found the lecture notes by D. Soper useful [5]. We first discuss dimensional regularization in Sec. 1.1 before we turn to renormalization at the example of QED in Sec. 1.2. Finally, Sec. 1.3 is devoted to renormalization group equations, in particular, the calculation of beta- and gamma-functions.

1.1 Regularization

It is well known that the integration over internal loops in Feynman diagrams can give divergent results. It is therefore necessary to introduce a consistent procedure of regularization in order to mathematically control the divergences. Different ways to regularize a QFT are known (Cut-off, Pauli-Villars, Dimensional regularization/reduction, ...). In the following we will focus on the most widely used dimensional regularization which is particularly simple and elegant. In particular, it allows to treat all possible types of divergences (UV, IR, Collinear) at the same time and it preserves Lorentz symmetry and gauge symmetry.

The idea of this method is to consider the whole theory in d space-time dimensions,

where $d = 4 - 2\epsilon$, $d \in \mathbb{N}$. The theory will be further analytically continued to arbitrary complex d, $d \in \mathbb{C}$. The divergences will then appear as poles in $d - 4$ in the complex d-plane:

$$\frac{1}{d-4} = \frac{-1}{2\epsilon} \underset{\epsilon \to 0}{\longrightarrow} \infty \,. \tag{1.1}$$

Later, after removing the UV divergences by the renormalization procedure (see Sec. 1.2), the limit $\epsilon \to 0$ ($d \to 4$) will be recovered.

Let us demonstrate this for QED as a simple example. We consider the 4-dimensional QED Lagrangian in the following form:

$$\begin{aligned} \mathcal{L}_{\text{QED}} &= i\bar{\psi} \not{D} \psi - m\bar{\psi}\psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \\ &= i\bar{\psi}\gamma^\mu \partial_\mu \psi - m\bar{\psi}\psi - eA^\mu \bar{\psi}\gamma_\mu \psi \\ &\quad - \frac{1}{4} \left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)^2 \,. \end{aligned} \tag{1.2}$$

Here A_μ is the photon field, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ the field strength tensor, and ψ describes a free electron field with mass m. Furthermore, $\not{D} = D_\mu \gamma^\mu$ where γ^μ are the Dirac matrices and $D_\mu = \partial_\mu + ieA_\mu$ is the covariant derivative leading to an interaction between the electron and the photon with coupling e ($-eA^\mu \bar{\psi}\gamma_\mu \psi$).

The Lagrangian (1.2) will be generalised to the case of d dimensions, which will change the dimensions of all its parameters. We keep in mind that the mass dimension of $\mathcal{L}_{\text{QED}}$ is then equal to d, i.e. $[\mathcal{L}_{\text{QED}}] = d$, such that the action is dimensionless (in natural units). Thus, one can see that the mass dimensions of the electron and electromagnetic fields become, respectively,

$$[\psi] = \frac{d-1}{2}, \; [A_\mu] = \frac{d-2}{2} \,,$$

and all terms in this Lagrangian have the correct mass dimension, except for the third one, assuming that the coupling constant e is dimensionless:

$$[eA^\mu \bar{\psi}\gamma_\mu \psi] = \frac{d-2}{2} + 2\frac{d-1}{2} = \frac{3d-4}{2} \,.$$

In order to compensate this and to get the correct dimension for this term, we introduce an arbitrary parameter μ with dimension of mass (called 't Hooft mass), and multiply the constant e by $\mu^{\frac{4-d}{2}}$, so that

$$[\mu^{\frac{4-d}{2}} eA^\mu \bar{\psi}\gamma_\mu \psi] = \frac{4-d}{2} + \frac{3d-4}{2} = d \,.$$

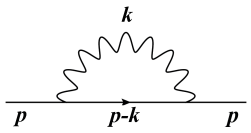


Figure 1.1: 1-loop electron self energy $-i\Sigma(p,m)$.

Thus, the d-dimensional Lagrangian takes the following form:

$$\mathcal{L}^d_{\rm QED} = i\bar\psi\gamma^\mu\partial_\mu\psi - m\bar\psi\psi - e\mu^{2-\frac{d}{2}}A^\mu\bar\psi\gamma_\mu\psi \\ - \frac{1}{4}\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)^2 + (\text{gauge fixing term})\,. \tag{1.3}$$

Now the divergent QED graphs can be calculated. In the framework of this QED example we will demonstrate only one of them - the electron self-energy (Fig. 1.1) - that is usually denoted as $\Sigma(p)$.

The electron self-energy can be generalized to the case of d dimensions in the following way:

$$\Sigma(p) = -ie^2\int\frac{d^4k}{(2\pi)^4}\gamma_\mu\frac{1}{\not p - \not k - m}\gamma_\nu\frac{g^{\mu\nu}}{k^2} \Longrightarrow -ie^2\mu^{4-d}\int\frac{d^dk}{(2\pi)^d}\gamma_\mu\frac{1}{\not p - \not k - m}\gamma_\nu\frac{g^{\mu\nu}}{k^2}\,, \tag{1.4}$$

with the commonly used notations $\not p = p_\mu\gamma^\mu$ and $\not k = k_\mu\gamma^\mu$.

We introduce the Feynman parametrization by combining the denominators according to the well known formula

$$\frac{1}{ab} = \int_0^1\frac{dz}{[az + b(1-z)]^2}\,. \tag{1.5}$$

Now we can rewrite Eq. (1.4) as follows:

$$\Sigma(p) = -i\mu^{4-d}e^2\int_0^1 dz\int\frac{d^dk}{(2\pi)^d}\frac{\gamma_\mu(\not p - \not k + m)\gamma^\mu}{\left[(p-k)^2z - m^2z + k^2(1-z)\right]^2}\,, \tag{1.6}$$

and further, performing a shift $k' = k - pz$, one gets

$$\Sigma(p) = -i\mu^{4-d}e^2\int_0^1 dz\int_0^1\frac{d^dk'}{(2\pi)^d}\frac{\gamma_\mu(\not p - \not pz - \not k' + m)\gamma^\mu}{\left[k'^2 - m^2z + p^2z(1-z)\right]^2} \\ = -i\mu^{4-d}e^2\int_0^1 dz\gamma_\mu(\not p - \not pz + m)\gamma^\mu\times\int\frac{d^dk'}{(2\pi)^d}\frac{1}{\left[k'^2 - m^2z + p^2z(1-z)\right]^2}\,. \tag{1.7}$$

The integral can be evaluated using the standard formula

$$\int\frac{d^dl}{(l^2 + 2lq - M^2)^\alpha} = i\pi^{d/2}\frac{\Gamma\left(\alpha - \frac{d}{2}\right)}{\Gamma(\alpha)}\frac{1}{[-q^2 - M^2]^{\alpha - d/2}}\,, \tag{1.8}$$

where $\Gamma(\alpha)$ is the Euler gamma function ($\Gamma(2) = 1$), and the parameters in our case are $l = k'$, $q = 0$, $M^2 = m^2 z - p^2 z(1-z)$ and $\alpha = 2$.

This yields[1]

$$\Sigma(p) = \mu^{4-d} e^2 \frac{\Gamma(2-\frac{d}{2})}{(4\pi)^{d/2}} \int_0^1 dz \gamma_\mu [\not{p}(1-z) + m] \gamma^\mu \times |m^2 z - p^2 z(1-z)|^{\frac{d}{2}-2} \,. \tag{1.9}$$

The gamma function $\Gamma(2-\frac{d}{2})$ has a pole in $d = 4$. Assuming that $\epsilon = 2 - \frac{d}{2}$ and using the expansion $\Gamma(2-\frac{d}{2}) = \Gamma(\epsilon) = \frac{1}{\epsilon} - \gamma_E + O(\epsilon)$ with the Euler constant $\gamma_E = 0.577$, as well as the relations $\gamma^\mu \gamma_\mu = d$ and $\gamma_\mu \gamma_\nu \gamma^\mu = (2-d)\gamma_\nu$, we perform the following calculation

$$\begin{aligned}
\Sigma(p) &= -\frac{e^2}{16\pi^2}\Gamma(\epsilon) \int_0^1 dz \{2\not{p}(1-z) - 4m - 2\epsilon[\not{p}(1-z)+m]\} \left(\frac{m^2 - p^2 z(1-z)}{4\pi\mu^2}\right)^{-\epsilon} \\
&= -\frac{e^2}{16\pi^2}\Gamma(\epsilon) \int_0^1 dz \{2\not{p}(1-z) - 4m - 2\epsilon[\not{p}(1-z)+m]\} \left[1 - \epsilon \ln\left(\frac{m^2 - p^2 z(1-z)}{4\pi\mu^2}\right)\right] \\
&= -\frac{e^2}{16\pi^2}\left(\frac{1}{\epsilon} - \gamma_E\right)\left\{\not{p} - 4m - \epsilon(\not{p} + 2m)\right\} \\
&\quad - \frac{e^2}{16\pi^2}\left(\frac{1}{\epsilon} - \gamma_E\right)\epsilon \int_0^1 dz \{2\not{p}(1-z) - 4m - 2\epsilon[\not{p}(1-z)+m]\} \ln\left(\frac{m^2 - p^2 z(1-z)}{4\pi\mu^2}\right) \\
&= \frac{e^2}{16\pi^2\epsilon}\left(-\not{p} + 4m\right) + \frac{e^2}{16\pi^2}\left\{\not{p}(1+\gamma_E) - 2m(1+2\gamma_E) + \cancelto{\epsilon\to 0}{\epsilon\gamma_E(\ldots)}\right. \\
&\qquad\qquad \left. + \int_0^1 dz \{2\not{p}(1-z) - 4m\} \ln\left(\frac{m^2 - p^2 z(1-z)}{4\pi\mu^2}\right)\right\} \\
&= \frac{e^2}{16\pi^2\epsilon}\left(-\not{p} + 4m\right) + (\text{finite part})\,.
\end{aligned} \tag{1.10}$$

We have obtained the result for the electron self-energy in the regularized theory. This expression will diverge for $\epsilon \to 0$. In the next section we will discuss how to treat this divergence in a consistent manner.

1.2 Renormalization

To make expression Eq. (1.10) finite, one has to complement the Lagrangian with compensating terms — the so called counter terms. The electron self energy modifies the inverse effective electron propagator $[S'_F(p)]^{-1}$ (Fig. 1.2) to

$$[S'_F(p)]^{-1} = [S_F(p)]^{-1} - \Sigma(p)\,, \tag{1.11}$$

where $S_F(p) = 1/(\not{p} - m + i\epsilon)$.

[1] To be strict, the fermion and the photon propagators in Eq. (1.4) should, respectively, have the forms $\frac{1}{\not{p}-\not{k}-m+i\epsilon}$ and $\frac{g^{\mu\nu}}{k^2+i\epsilon}$, and thus, $M^2 = |M^2|e^{i\phi}$ with the phase $\phi = 0$ or $\pm\pi$. However, here we neglect the potential imaginary part, being interested only in the pole structure of $\Sigma(p)$.

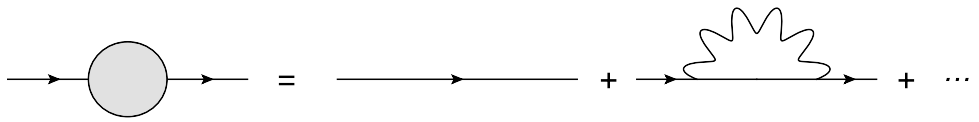


Figure 1.2: Perturbative expansion of the inverse effective electron propagator $i[S'_F(p)]^{-1} = i[S_F(p)]^{-1} - i\Sigma(p,m)$.

Inserting the expression for the electron self-energy Eq. (1.10), we get

$$\begin{aligned}
[S'_F(p)]^{-1} &= (\not{p} - m) - \Sigma(p) \\
&= \not{p} - m - \frac{e^2}{16\pi^2\epsilon}\left(-\not{p} + 4m\right) + (\text{finite part}) \\
&= \not{p}\left(1 + \frac{e^2}{16\pi^2\epsilon}\right) - m\left(1 + \frac{e^2}{4\pi^2\epsilon}\right) + (\text{finite part})\,.
\end{aligned} \tag{1.12}$$

Two counter terms are needed: one for the full propagator value (which will contribute to the electron wave function normalization) and one for the electron mass, which leads to the following Lagrangian (expressed in terms of the renormalized quantities)

$$\mathcal{L} = i\bar{\psi}\partial\!\!\!/\psi - m\bar{\psi}\psi + iB\bar{\psi}\partial\!\!\!/\psi - A\bar{\psi}\psi = i(1+B)\bar{\psi}\partial\!\!\!/\psi - (m+A)\bar{\psi}\psi\,, \tag{1.13}$$

where the choice of the constants A and B guarantees that the electron propagator is finite to the order of e^2. This is diagrammatically represented in Fig. 1.3.

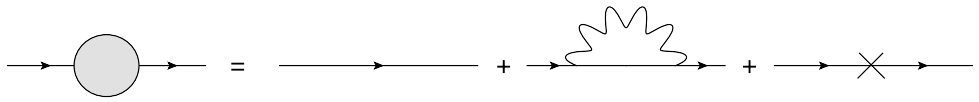


Figure 1.3: Perturbative expansion of the inverse electron propagator up to the order $O(e^2)$ including the counter term: $i[S'_F(p)]^{-1} = i[S_F(p)]^{-1} - i\Sigma(p,m) - i\Sigma^{\mathrm{c}}(p,m)$, with $\Sigma^{\mathrm{c}} = A - B\not{p}$.

Thus, the renormalized electron self-energy takes the form

$$\hat{\Sigma}(p) = \Sigma(p) + \Sigma^{\mathrm{c}}(p) = \frac{e^2}{16\pi^2\varepsilon}\left(-\not{p} + 4m\right) + A - B\not{p} + (\text{finite part})\,. \tag{1.14}$$

In the minimal subtraction scheme (MS) A and B are chosen so that only the pole terms are compensated:

$$A = -\frac{me^2}{4\pi^2\varepsilon},\ B = -\frac{e^2}{16\pi^2\varepsilon}\,. \tag{1.15}$$

Now we can define the wave function renormalization constant Z_ψ in the MS scheme:

$$Z_\psi = 1 + B = 1 - \frac{e^2}{16\pi^2\varepsilon}\,, \tag{1.16}$$

and the bare electron wave function ψ_B can be related to the renormalized electron wave

function ψ in the following way

$$\psi_B = \sqrt{Z_\psi}\psi\,. \tag{1.17}$$

Thus, the Lagrangian takes the form

$$\mathcal{L} = i\bar{\psi}_B \partial\!\!\!/ \psi_B - m_B \bar{\psi}_B \psi_B\,, \tag{1.18}$$

where the bare mass is defined by

$$m_B = Z_\psi^{-1}(m+A) = m\left(1-\frac{e^2}{4\pi^2\varepsilon}\right)\left(1+\frac{e^2}{16\pi^2\varepsilon}\right) = m\left(1-\frac{3e^2}{16\pi^2\varepsilon}\right) = Z_m m = m+\delta m\,, \tag{1.19}$$

with the mass renormalization constant Z_m,

$$Z_m = 1 - \frac{3e^2}{16\pi^2\varepsilon}\,. \tag{1.20}$$

There are two more UV divergent graphs shown in Fig. 1.4.

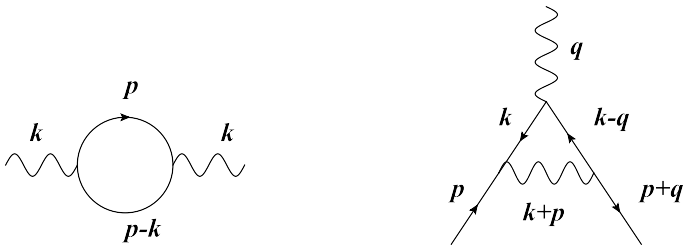


Figure 1.4: Vacuum polarization $\Pi_{\mu\nu}$ (left) and vertex correction Λ_μ (right) diagrams at one-loop.

Similarly, as for the electron self-energy, it can be shown that corresponding expressions for the vacuum polarisation $\Pi_{\mu\nu}$ and the vertex correction Λ_μ in the regularized theory have the following form, respectively:

$$\Pi_{\mu\nu}(k) = \frac{e^2}{12\pi^2\epsilon}(k_\mu k_\nu - g_{\mu\nu}k^2) + (\text{finite part})\,, \tag{1.21}$$

$$\Lambda_\mu^{(1)}(p,q,p') = \frac{e^2}{16\pi^2\epsilon}\gamma_\mu + (\text{finite part})\,, \tag{1.22}$$

and the divergent terms can be eliminated by adding the necessary counterterms. One obtains the following Lagrangian that gives a finite photon propagator to order e^2 (as $\Pi_{\mu\nu}$ in Eq. (1.21) contributes to the renormalized photon propagator)

$$\mathcal{L}_A = -\frac{Z_A}{4}F_{\mu\nu}F^{\mu\nu} + (\text{gauge terms})\,, \tag{1.23}$$

and the definition of the bare electromagnetic field

$$A_B^\mu = Z_A^{1/2} A^\mu \,, \tag{1.24}$$

with

$$Z_A = 1 - \frac{e^2}{12\pi^2\epsilon} \,. \tag{1.25}$$

The divergent part of $\Lambda_\mu^{(1)}$ (in Eq. (1.22)) can be treated by writing the Lagrangian in the following way

$$(\mathcal{L}_{\rm int}) = -Z_{\rm int} e \mu^\epsilon A^\mu \bar{\psi} \gamma_\mu \psi \,, \tag{1.26}$$

with

$$Z_{\rm int} = 1 - \frac{e^2}{16\pi^2\epsilon} \,. \tag{1.27}$$

Now the full Lagrangian of QED, as it follows from Eqs. (1.18), (1.23) and (1.26), has the form

$$\begin{aligned} \mathcal{L} = {} & i Z_\psi \bar{\psi} \gamma^\mu \partial_\mu \psi - (m + A) \bar{\psi} \psi - Z_{\rm int}\ e \mu^\epsilon A^\mu \bar{\psi} \gamma_\mu \psi \\ & - \frac{Z_A}{4} \left(\partial_\mu A_\nu - \partial_\nu A_\mu \right)^2 + (\text{gauge terms}) \,, \end{aligned} \tag{1.28}$$

where the renormalization constants are

$$\begin{aligned} Z_{\rm int} = Z_\psi &= 1 - \frac{e^2}{16\pi^2\epsilon} = 1 - \frac{1}{4} \frac{\alpha_{\rm em}}{\pi} \frac{1}{\epsilon} + \mathcal{O}(\alpha_{\rm em}^2) \,, \\ Z_A &= 1 - \frac{e^2}{12\pi^2\epsilon} = 1 - \frac{1}{3} \frac{\alpha_{\rm em}}{\pi} \frac{1}{\epsilon} + \mathcal{O}(\alpha_{\rm em}^2) \,, \\ A &= -\frac{m e^2}{4\pi^2\epsilon} \,. \end{aligned} \tag{1.29}$$

This is known as multiplicative renormalization.

We can also define the bare charge e_B (which is a dimensionful parameter in d dimensions):

$$e_B = e \mu^\epsilon \frac{Z_{\rm int}}{Z_\psi Z_A^{1/2}} = e \mu^\epsilon Z_A^{-1/2} = \mu^\epsilon Z_e e \,, \tag{1.30}$$

with

$$Z_e \equiv Z_A^{-1/2} = 1 + \frac{e^2}{24\pi^2\epsilon} = 1 + \frac{1}{6} \frac{\alpha_{\rm em}}{\pi} \frac{1}{\epsilon} + \mathcal{O}(\alpha_{\rm em}^2) \,. \tag{1.31}$$

Using relations Eqs. (1.17), (1.19), (1.24) and (1.30), we finally obtain the Lagrangian in terms of the bare parameters in the following form

$$\begin{aligned} \mathcal{L} = {} & i \bar{\psi}_B \gamma^\mu \psi_B - m_B \bar{\psi}_B \psi_B - e_B A_B^\mu \bar{\psi}_B \gamma_\mu \psi_B \\ & - \frac{1}{4} \left(\partial_\mu A_{B\nu} - \partial_\nu A_{B\mu} \right)^2 \,. \end{aligned} \tag{1.32}$$

As we can see, all the infinite quantities have been absorbed into the definitions of bare quantities. The fact that we were able to do this, keeping the Lagrangian (1.32) of the same form as the original Eq. (1.3) means, that, to this order, QED is renormalizable. (Commonly, one starts to discuss renormalization with a Lagrangian in terms of bare quantities (1.32) and then defines the renormalized quantities. However, in this thesis we have chosen the inverse way, following the book by L. H. Ryder [3]. This approach, in our opinion, more gradually leads the reader to the understanding of the idea, in particular, of the need for bare parameters and counter terms.)

1.3 Renormalization Group Equations

In this section we will only discuss the RGEs for parameters of the Lagrangian (coupling e, mass m), but not the RGEs for physical observables and Green's functions, the solution of which would depend on the running parameters and would give renormalization improved observables. For the renormalization of Green's functions see, *e.g.*, chapter 4 in the book by S. Pokorski [4].

1.3.1 The running of the coupling

Let us now consider the coupling e as a function of the renormalization scale μ and the parameter ϵ. In order to see how e scales with μ, we define its beta function as

$$\beta(e,\epsilon) \equiv \frac{\partial e(\mu,\epsilon)}{\partial \ln \mu} = \mu \frac{d}{d\mu} e(\mu,\epsilon)\,. \tag{1.33}$$

The final result for the beta function is then obtained in the limit $\epsilon \to 0$, which therefore must be finite:

$$\beta(e) = \lim_{\epsilon\to 0} \beta(e,\epsilon)\,. \tag{1.34}$$

Therefore, we can calculate $\beta(e,\epsilon)$ using Eq. (1.30):

$$e_B = Z_e e \mu^\epsilon \quad \Rightarrow \quad e(\mu,e) = e_B Z_e^{-1} \mu^{-\epsilon}\,, \tag{1.35}$$

such that

$$\begin{aligned}
\beta(e,\epsilon) &= \mu\frac{d}{d\mu}\left[e_B Z_e^{-1}\mu^{-\epsilon}\right] \\
&= e_B\left(\mu\frac{d}{d\mu}Z_e^{-1}\right)\mu^{-\epsilon} + e_B Z_e^{-1}\left(\mu\frac{d}{d\mu}\mu^{-\epsilon}\right) \\
&= Z_e\, e\,\mu^{\epsilon}\,\frac{-1}{Z_e^2}\left(\mu\frac{d}{d\mu}Z_e\right)\mu^{-\epsilon} + Z_e\, e\,\mu^{\epsilon}\; Z_e^{-1}\mu\,(-\epsilon)\,\mu^{-\epsilon-1} \\
&= -e\frac{1}{Z_e}\left(\mu\frac{d}{d\mu}Z_e\right) - e\epsilon\mu^{\epsilon}\mu^{-\epsilon} \\
&= -\epsilon\, e - e\frac{1}{Z_e}\left(\frac{d}{de}Z_e\right)\underbrace{\mu\frac{de}{d\mu}}_{\beta_e} \\
&= -\epsilon\, e - e\left(\frac{d\ln Z_e}{de}\right)\beta_e\,, \qquad (1.36)
\end{aligned}$$

from where we can write the equation

$$\beta(e,\epsilon) + \epsilon\, e + e\left(\frac{d\ln Z_e}{de}\right)\beta(e,\epsilon) = 0\,, \qquad (1.37)$$

and, consequently,

$$Z_e\beta(e,\epsilon) + \epsilon\, e\, Z_e + e\,\frac{dZ_e}{de}\,\beta(e,\epsilon) = 0\,. \qquad (1.38)$$

We are looking for a solution for $\beta(e,\epsilon)$ in the form of a series $\beta(e,\epsilon) = \sum_\nu \beta_\nu\epsilon^\nu$ which is regular at $\epsilon = 0$ since our theory is renormalizable and there exist a finite limit $\beta(e,\epsilon)|_{\epsilon\to 0}$.

Expanding Z_e in terms of poles:

$$Z_e = 1 + \sum_{\nu=1}^{\infty}\frac{a_\nu(e)}{\epsilon^\nu}\,, \quad \text{with} \quad a_\nu(e) = \sum_{\rho=1}^{\infty} a_{\nu,\rho}\, e^{2\rho}\,, \qquad (1.39)$$

and taking the solution for $\beta(e,\epsilon)$ in the following form:

$$\beta(e,\epsilon) = \sum_{\nu=0}^{\infty}\bar\beta_\nu\epsilon^\nu = \bar\beta_0 + \bar\beta_1\epsilon + \dots\,, \qquad (1.40)$$

we write (1.38) explicitly as

$$\left(\bar\beta_0 + \epsilon\bar\beta_1 + \epsilon^2\bar\beta_2 + \dots\right)\left(1 + \sum_{\nu=1}^{\infty}\frac{a_\nu}{\epsilon^\nu} + e\frac{dZ_e}{de}\right) + e\,\epsilon\left(1 + \sum_{\nu=1}^{\infty}\frac{a_\nu}{\epsilon^\nu}\right) = 0\,. \qquad (1.41)$$

Defining the following notations

$$\frac{dZ_e}{de} = \sum_{\nu=1}^{\infty} \frac{a'_\nu(e)}{\epsilon^\nu}\,, \quad a'_\nu(e) = \frac{da_\nu}{de}\,, \quad c_\nu \equiv a_\nu + e\,a'_\nu\,, \tag{1.42}$$

we obtain

$$\left(\bar\beta_0 + \epsilon\bar\beta_1 + \epsilon^2\bar\beta_2 + \dots\right)\left(1 + \sum_{\nu=1}^{\infty}\frac{1}{\epsilon^\nu}(\overbrace{a_\nu + e\,a'_\nu}^{c_\nu})\right) + e\,\epsilon + e\sum_{\nu=0}^{\infty}\frac{a_{\nu+1}}{\epsilon^\nu} = 0\,. \tag{1.43}$$

We now group the terms by the order of ϵ and find the coefficients:

$$\mathcal{O}(\epsilon^2): \quad \bar\beta_2 + \bar\beta_3\,c_1 + \bar\beta_4\,c_2 + \dots = 0 \quad \Rightarrow \quad \bar\beta_i = 0 \ \text{ for } \ i \geq 2\,, \tag{1.44}$$

$$\mathcal{O}(\epsilon): \quad \bar\beta_1 + \bar\beta_2\,c_1 + \bar\beta_3\,c_2 + \dots + e\ = 0 \quad \Rightarrow \quad \bar\beta_1 = -e\,, \tag{1.45}$$

$$\mathcal{O}(1): \quad \bar\beta_0 + \bar\beta_1\,c_1 + e\,a_1\ = 0 \quad \Rightarrow \quad \bar\beta_0 = -\bar\beta_1\,c_1 - e\,a_1 = e(c_1 - a_1) \tag{1.46}$$

$$= e^2 a'_1 = e^2\frac{da_1}{de} \equiv \beta(e)\,, \tag{1.47}$$

$$\mathcal{O}(1/\epsilon): \quad \bar\beta_0 c_1 + \bar\beta_1\,c_2 + e\,a_2\ = 0 \quad \Rightarrow \quad \bar\beta_0 = e^2 a'_1\,,\ \bar\beta_1 = -e\,. \tag{1.48}$$

Therefore, Eq. (1.48) can be written as

$$e^2 a'_1(a_1 + e\,a'_1) - e\,(a_2 + e\,a'_2) + e\,a_2 = 0\,, \tag{1.49}$$

$$\Leftrightarrow e^2 a_1\,a'_1 + e^3\,{a'_1}^2 - e^2\,a'_2 = 0\,, \tag{1.50}$$

$$\begin{aligned}\Leftrightarrow e^2 a'_2 &= e^2 a_1\,a'_1 + e^3\,{a'_1}^2 \\ &= \underbrace{e^2 a'_1}_{=\bar\beta_0 \equiv \beta(e)}\ (a_1 + e\,a'_1) \\ &= \beta(e)(\underbrace{a_1 + e\,a'_1}_{=\frac{d}{de}(e\,a_1)})\,,\end{aligned} \tag{1.51}$$

and, finally,

$$e^2\frac{da_2}{de} = \beta(e)\frac{d}{de}(e\,a_1)\,, \qquad \beta(e) = e^2\frac{da_1}{de}\,, \tag{1.52}$$

$$e^2\frac{da_{\nu+1}}{de} = \beta(e)\frac{d}{de}(e\,a_\nu)\,. \tag{1.53}$$

Thus, we arrive at two important conclusions:

- $\beta(e)$ is totally determined by the residue a_1 of the simple pole in ϵ of Z_e.
- The residua of the higher order poles of Z_e are totally determined by $a_1(e)$.

An important feature in the analysis of a beta function is its fixed points. As illustrated in Fig. 1.5, fixed points are the points in which the beta function turns to zero. They are classified as ultra-violet fixed points (UV-FP) and infra-red fixed points (IR-FP) depending on the behaviour of the parameters when approaching the fixed point. For example:

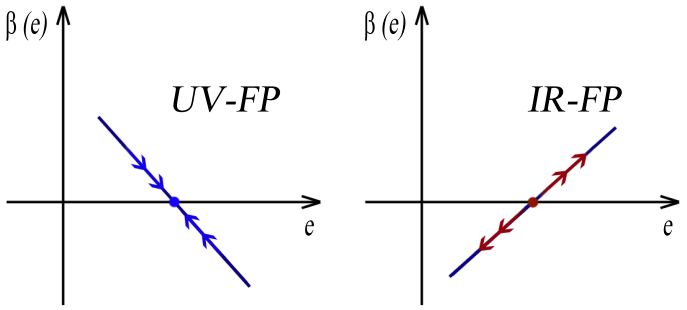


Figure 1.5: A general example of the UV stable fixed point (on the left) and the IR stable fixed point (on the right) for a beta function $\beta(e)$.

- Approaching the fixed point in Fig. 1.5 (left graph) from both the left-hand side and the right-hand side, while increasing the energy scale μ, the value of the coupling e approaches the fixed point (first, for $\beta > 0$ the coupling e is increasing, and then, for $\beta < 0$ the coupling e is decreasing). Such a fixed point is called an UV stable fixed point and acts as an 'attractor'.

- Approaching the fixed point in Fig. 1.5 (right graph) from both the left-hand side and the right-hand side, while increasing the energy scale μ, the value of the coupling e goes away from the fixed point (first, for $\beta < 0$ the coupling e is decreasing, and then, for $\beta > 0$ the coupling e is increasing). Such a fixed point is called an IR stable fixed point and acts as a 'repulser'.

Returning to the real QED case in 4 dimensions, $\epsilon = 0$, and using Eqs. (1.31), (1.42), (1.52), we get

$$\frac{dZ_{e,1}}{de} = \frac{a_1'}{\epsilon} = \frac{2e}{24\pi^2} \cdot \frac{1}{\epsilon}\,, \tag{1.54}$$

and, finally, we obtain the function $\beta(e)$ in the following form

$$\beta(e) = e^2 a_1' + ... = \frac{e^3}{12\pi^2} + \mathcal{O}(e^5)\,, \tag{1.55}$$

and the approximate (1-loop) RGE:

$$\frac{de(\mu)}{d\ln\mu} = \beta_0\, e(\mu)^3\,, \tag{1.56}$$

with $\beta_0 = 1/(12\pi^2)$.
The solution of this equation has the form[2]

$$e(\mu)^2 = \frac{e(\mu_0)^2}{1 - \beta_0\, e(\mu_0)^2 \ln(\mu^2/\mu_0^2)}\,, \tag{1.57}$$

or, in terms of $\alpha_{\rm em}$,

$$\alpha_{\rm em}(\mu) = \frac{\alpha_{\rm em}(\mu_0)}{1 - \frac{\alpha_{\rm em}(\mu_0)}{3\pi} \ln(\mu^2/\mu_0^2)}\,. \tag{1.58}$$

There is an IR stable fixed point at $e = 0$, and $e(\mu)$ slowly goes to zero with decreasing μ, like $1/\ln(\mu^2)$.

The perturbatively calculated QED beta function would predict a pole (the so-called Landau pole) at some scale $\mu \gg M_{Pl}$ (where M_{Pl} is the Planck mass). However, when the coupling gets large, the perturbative calculation of the beta function becomes unreliable.

1.3.2 The running mass and the field strength renormalization

As we have seen in Eq. (1.19), the mass parameter also depends on the scale μ:

$$m(\mu, \epsilon) = Z_m(e(\mu, \epsilon), \epsilon)^{-1}\, m_B\,, \tag{1.59}$$

where m_B is the bare mass.

We define the gamma function, also called the anomalous dimension, as

$$\gamma_m(e, \epsilon) \equiv \frac{\partial \ln m(\mu, \epsilon)}{\partial \ln \mu} = \frac{\partial \ln[m_B Z_m^{-1}]}{\partial \ln \mu} = -\mu \frac{d}{d\mu} \ln Z_m\,, \tag{1.60}$$

from where, by analogy with (1.36), we get

$$\gamma_m(e, \epsilon) = -\frac{\partial \ln Z_m(e(\mu), \epsilon)}{\partial e} \beta(e, \epsilon)\,. \tag{1.61}$$

Again, the final result for the gamma function is obtained in the limit $\epsilon \to 0$, which has to be finite.

Similarly to the previous section, we express Z_m as an expansion in terms of poles:

$$Z_m(e, \epsilon) = 1 + \sum_{n=1}^{\infty} \sum_{j=1}^{n} Z_{n,j}^{(m)} \frac{e^{2n}}{\epsilon^j}\,. \tag{1.62}$$

Then

$$\ln Z_m = \sum_{n=1}^{\infty} e^{2n} \left\{ Z_{n,1}^{(m)} \frac{1}{\epsilon} + \mathcal{O}(1/\epsilon^2) \right\}\,, \tag{1.63}$$

[2]Note that there is only one flavour of fermions in the current example.

and

$$\gamma_m(e,\epsilon) = -\sum_{n=1}^{\infty} 2n\ e^{2n-1}\left\{Z_{n,1}^{(m)}\frac{1}{\epsilon}+\mathcal{O}(1/\epsilon^2)\right\}[-\epsilon\, e+\beta(e)]$$
$$= \sum_{n=1}^{\infty} 2n\ e^{2n}\left\{Z_{n,1}^{(m)}+\mathcal{O}(1/\epsilon)\right\}. \tag{1.64}$$

Since $\gamma_m(e,\epsilon)$ has a finite $\epsilon\to 0$ limit, the $1/\epsilon$ terms cancel, and $\gamma_m(e,\epsilon)$ becomes independent of ϵ and is given at each order of perturbation theory by the $1/\epsilon$ term in Z_m:

$$\gamma_m(e,\epsilon)|_{\epsilon\to 0} = \gamma_m(e) = \sum_{n=1}^{\infty} 2n\ e^{2n} Z_{n,1}^{(m)}\,. \tag{1.65}$$

In our QED case, using (1.20), we get

$$\gamma_m(e) = -\frac{3e^2}{8\pi^2}+\mathcal{O}(e^4)\,. \tag{1.66}$$

Running mass Coming back to equation Eq. (1.60) (where, as we have just seen, the ϵ-dependence can be ommited), let us analyse the behaviour of the mass $m(\mu)$ in the case when the coupling has an UV-FP at $e=e_\star$. With increasing μ the function $e(\mu)$ approaches $e_\star$ like a power, and for e near $e_\star$ the equation for the evolution of m can be approximated by

$$\frac{\partial \ln m(\mu)}{\partial \ln \mu} = \gamma_m(e_\star)\,, \tag{1.67}$$

the solution of which has the form

$$m(\mu) = m(\mu_0)\left(\frac{\mu}{\mu_0}\right)^{\gamma_m}. \tag{1.68}$$

Depending on the sign of γ_m, the function $m(\mu)$ approaches either zero or infinity like a power.

In the real QED, the physical coupling is near to an IR stable fixed point at $e=0$. In this case γ_m approaches zero as e approaches the fixed point.

Using Eqs. (1.65) and (1.57), we get

$$\begin{aligned}\frac{\partial \ln m(\mu)}{\partial \ln \mu} &= \gamma_m(e(\mu))\\ &\simeq \gamma_m^{(0)} e(\mu)^2\\ &\simeq \frac{\gamma_m^{(0)}}{\frac{1}{e(\mu_0)^2}-2\beta_0\ln\left(\frac{\mu}{\mu_0}\right)}\,.\end{aligned} \tag{1.69}$$

If we introduce Λ as a very large mass scale for QED, much bigger than any μ of practical

interest, we can write

$$\frac{\partial \ln m(\mu)}{\partial \ln \mu} \simeq -\frac{\gamma_m^{(0)}}{2\beta_0 \ln\left(\frac{\mu}{\Lambda}\right)}\,, \tag{1.70}$$

where $\ln\left(\frac{\mu}{\Lambda}\right)$ is negative.

The solution of this equation is

$$\ln\left(\frac{m(\mu)}{m(\mu_0)}\right) = -\frac{\gamma_m^{(0)}}{2\beta_0} \ln\left(\frac{\ln(\mu/\Lambda)}{\ln(\mu_0/\Lambda)}\right)\,, \tag{1.71}$$

from where we get

$$m(\mu) = m(\mu_0)\left(\frac{\ln(\mu/\Lambda)}{\ln(\mu_0/\Lambda)}\right)^{-\gamma_m^{(0)}/(2\beta_0)}\,, \tag{1.72}$$

or

$$m(\mu) = m(\mu_0)\left(\frac{e(\mu_0)}{e(\mu)}\right)^{-\gamma_m^{(0)}/\beta_0}\,. \tag{1.73}$$

We can see that $m(\mu)$ varies slowly as μ changes, like a power of $e(\mu)$ which itself does not change very quickly (the latter is due to the fact that the derivative of e with respect to $\ln\mu$ is proportional to $e \times e^2$ (Eq. (1.56)) and e^2 is small).

Field strength renormalization In order to see how the field strengths change with a change of the parameter μ, we recall the relations between the bare and renormalized fields from the previous section (Eqs. (1.17), (1.24))

$$\psi = \frac{1}{\sqrt{Z_\psi(e(\mu,\epsilon),\epsilon)}}\psi_B\,, \tag{1.74}$$

$$A^\mu = \frac{1}{Z_A(e(\mu,\epsilon),\epsilon)}A^\mu_B\,. \tag{1.75}$$

First of all, we write the γ function of ψ by definition:

$$\begin{aligned}\gamma_\psi(e,\epsilon) &\equiv -\frac{\partial \ln\left(Z_\psi(e(\mu,\epsilon),\epsilon)^{-1/2}\right)}{\partial \ln\mu} \\ &= \frac{1}{2}\frac{\partial \ln Z_\psi(e(\mu),\epsilon)}{\partial e}\beta(e,\epsilon)\,.\end{aligned} \tag{1.76}$$

We take again Z_ψ in the form of an expansion in $1/\epsilon$ and the coupling e:

$$Z_\psi(e,\epsilon) \sim 1 + \sum_{n=1}^{\infty}\sum_{j=1}^{n} Z_{n,j}^{(\psi)}\frac{e^{2n}}{\epsilon^j}\,. \tag{1.77}$$

Then

$$\gamma_\psi(e,\epsilon) = \frac{1}{2}\sum_{n=1}^{\infty} 2n\ e^{2n-1}\left\{Z_{n,1}^{(\psi)}\frac{1}{\epsilon}+\mathcal{O}(1/\epsilon^2)\right\}[-\epsilon\, e+\beta(e)]$$
$$= -\sum_{n=1}^{\infty} n\ e^{2n}\left\{Z_{n,1}^{(\psi)}+\mathcal{O}(1/\epsilon)\right\}. \tag{1.78}$$

Similarly to the previous example, the $1/\epsilon$ terms cancel since $\gamma_\psi(e,\epsilon)$ has to have a finite $\epsilon\to 0$ limit, so it is independent of ϵ and is given at each order of perturbation theory by the $1/\epsilon$ term in Z_ψ:

$$\gamma_\psi(e) = -\sum_{n=1}^{\infty} n\ e^{2n} Z_{n,1}^{(\psi)}\,. \tag{1.79}$$

In the same way, for the electromagnetic field A we find the anomalous dimension

$$\gamma_A(e) = -\sum_{n=1}^{\infty} n\ e^{2n} Z_{n,1}^{(A)}\,. \tag{1.80}$$

Therefore, for QED, using (1.29) we get

$$\gamma_\psi(e) = \frac{e^2}{16\pi^2}+\mathcal{O}(e^4)\,,\quad \gamma_A(e) = \frac{e^2}{12\pi^2}+\mathcal{O}(e^4)\,. \tag{1.81}$$

Chapter 2

RGE's for a general gauge theory

In this chapter we summarise the derivation of the beta functions for dimensionless parameters following the papers by M. E. Machacek and M. T. Vaughn [6, 7, 8]. Sections 2.1 and 2.2 contain the Lagrangian for a general gauge theory along with some necessary notations and definitions. In Sec. 2.3, we discuss the wave function renormalization and the Yukawa vertex renormalization and provide the tables with all the relevant Feynman diagrams in an attempt to illustrate the coupling structure of the contributions and to make it easier to follow the diagrammatic approach used in Chapter 3. We correct a mistake in the beta functions for the Yukawa couplings already present in the work of M. E. Machacek and M. T. Vaughn. This error has propagated through all the relevant literature and was only very recently uncovered. We also address the issue of off-diagonal wave function renormalization in the scalar sector which leads to corrections of the beta functions for the scalar quartic couplings with respect to the literature. The full list of corrected beta functions for dimensionless parameters is provided at the end of this chapter.

2.1 The Lagrangian for a general gauge theory

Let us consider a general renormalizable field theory with the following particle content:

- $V_\mu^A(x)$ $(A = 1, \ldots, d)$ are gauge fields of a compact simple[1] group G where d is the dimension of G.
- $\phi_a(x)$ $(a = 1, \ldots, N_\phi)$ denote real scalar fields transforming under a (in general) reducible representation of G. The Hermitian generators of G in this representation will be denoted Θ_{ab}^A $(A = 1, \ldots, d;\ a, b = 1, \ldots, N_\phi)$. The assumption of real scalars

[1] All results of this work can be extended to the case of semi-simple groups. The corresponding substitution rules are provided in Ref. [12] and not affected by the corrections discussed in the current study.

comes out of the fact that any complex scalars can be decomposed in terms of real scalars. The generators Θ^A are consequently purely imaginary and antisymmetric.

- $\psi_j(x)$ $(j = 1, \ldots, N_\psi)$ are left-handed complex two-component fermion fields transforming under a representation of G which is in general reducible as well. The Hermitian generators are denoted by t^A_{jk} $(A = 1, \ldots, d;\ j, k = 1, \ldots, N_\psi)$.

For convenience, the general Lagrangian of this theory can be decomposed into three parts [12],

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1 + (\text{gauge fixing} + \text{ghost terms})\,, \tag{2.1}$$

where $\mathcal{L}_0$ is free of dimensional parameters and $\mathcal{L}_1$ contains all terms with dimensional parameters. Explicitly, $\mathcal{L}_0$ reads

$$\begin{aligned}\mathcal{L}_0 = &-\frac{1}{4}F^{\mu\nu}_A F^A_{\mu\nu} + \frac{1}{2}D^\mu\phi_a D_\mu\phi_a + i\psi^\dagger_j\sigma^\mu D_\mu\psi_j \\ &- \frac{1}{2}\left(Y^a_{jk}\psi_j\zeta\psi_k\phi_a + Y^{a*}_{jk}\psi^\dagger_j\zeta\psi^\dagger_k\phi_a\right) - \frac{1}{4!}\lambda_{abcd}\phi_a\phi_b\phi_c\phi_d\,,\end{aligned} \tag{2.2}$$

where $F^A_{\mu\nu}(x)$ is the gauge field strength tensor defined in the usual way in terms of the structure constants f^{ABC} of the gauge group and the gauge coupling constant g:

$$F^A_{\mu\nu} = \partial_\mu V^A_\nu - \partial_\nu V^A_\mu + g f^{ABC} V^B_\mu V^C_\nu\,. \tag{2.3}$$

The covariant derivatives of the scalar and fermion fields are given by

$$D_\mu\phi_a = \partial_\mu\phi_a - ig\Theta^A_{ab}V^A_\mu\phi_b, \tag{2.4}$$

$$D_\mu\psi_j = \partial_\mu\psi_j - igt^A_{jk}V^A_\mu\psi_k\,. \tag{2.5}$$

The matrices Y^a_{jk} $(a = 1, \ldots, N_\phi; j, k = 1, \ldots, N_\psi)$ are complex Yukawa couplings and $\zeta = i\sigma_2$ is the two-component spinor metric (σ_2 is the second Pauli matrix). Finally, λ_{abcd} denotes quartic scalar couplings which are real and invariant under permutations of the set of indices $\{a, b, c, d\}$.

Another part of the Lagrangian that includes the dimensionful parameters has the following form

$$\mathcal{L}_1 = -\frac{1}{2}\left[(m_f)_{jk}\psi_j\zeta\psi_k + (m_f)^*_{jk}\psi^\dagger_j\zeta\psi^\dagger_k\right] - \frac{m^2_{ab}}{2!}\phi_a\phi_b - \frac{h_{abc}}{3!}\phi_a\phi_b\phi_c\,, \tag{2.6}$$

where m_f is a complex matrix of fermion masses, m^2 is a real matrix of scalar masses squared, and h_{abc} are real cubic scalar couplings.

The one- and two-loop β-functions for these dimensionful couplings were addressed in Ref. [12] where the authors introduced an indirect way of their derivation: the so-called

"dummy field" method which has been initially proposed in Ref. [13].

2.2 Definitions and notations

2.2.1 β-functions and anomalous dimensions

Let us get familiar with the general algorithm for the calculation of renomalization constants at 2-loop order in dimensional regularisation with modified minimal subtraction ($\overline{\text{MS}}$ scheme). In this scheme, the renormalized Lagrangian parameters in $d \equiv 4 - 2\varepsilon$ dimensions, Θ_i, are related to the corresponding bare parameters, Θ_{iB}, by the following expansion:

$$\Theta_{iB}\mu^{-\rho_i\varepsilon} = \Theta_i Z_i = \Theta_i + \sum_{n=1}^{\infty} a_i^{(n)}(\Theta)\frac{1}{\varepsilon^n}\,, \tag{2.7}$$

with $\rho = 1$ for gauge and Yukawa couplings and $\rho = 2$ for scalar quartic couplings. The coefficients $a_i^{(n)}(\Theta)$ (where Θ denotes the list/vector of all parameters $\Theta_1, \Theta_2, \dots$) are to be calculated in perturbation theory.

As discussed in Sec. 1.3.1, the bare couplings are independent of the renormalization scale, whereas the renormalized couplings depend on the choice of the scale parameter μ, and their logarithmic derivatives define the β-functions

$$\beta_i(\Theta) \equiv \mu\frac{d\Theta_i}{d\mu}\bigg|_{\varepsilon=0}\,. \tag{2.8}$$

Inserting Eq. (2.7) in Eq. (2.8) and collecting coefficients of like powers of ε (see the derivation of eqs. (1.52), (1.53)), one can show that

$$\beta_i(\Theta) = \sum_l \rho_l\Theta_l\frac{\partial a_i^{(1)}}{\partial\Theta_l} - \rho_i a_i^{(1)}(\Theta)\,, \tag{2.9}$$

i.e. the β-functions are completely determined by the coefficients of the single order pole in the expansion Eq. (2.7).
The β-functions can be expanded in a perturbative series:

$$\beta_i = \frac{1}{16\pi^2}\beta_i^I + \frac{1}{(16\pi^2)^2}\beta_i^{II} + \dots = \sum_n \frac{1}{(16\pi^2)^n}\beta_i^{(n)}\,, \tag{2.10}$$

where β_i^I and β_i^{II} are the one- and two-loop contributions to the running which we consider in the current study.

The wave function renormalization constant Z_i of the i-th field is associated with an

irreducible self-energy part and can be written as the following expansion

$$Z_i = 1 + \sum_{n=1}^{\infty} C_i^{(n)}(\Theta)\frac{1}{\varepsilon^n}\,. \tag{2.11}$$

The following derivative, as we showed in Sec. 1.3.2, is the the anomalous dimension for the i-th field

$$\gamma_i = \frac{1}{2}\mu\frac{d}{d\mu}\ln Z_i = -\frac{1}{2}\sum_l \rho_l \Theta_l \frac{\partial C_i^{(1)}}{\partial \Theta_l}\,, \tag{2.12}$$

and is also determined from the the single order pole term in Eq. (2.11).

Similarly to the β-functions, γ_i can be expanded perturbatively:

$$\gamma_i = \frac{1}{16\pi^2}\gamma_i^I + \frac{1}{(16\pi^2)^2}\gamma_i^{II} + \cdots = \sum_n \frac{1}{(16\pi^2)^n}\gamma_i^{(n)}\,, \tag{2.13}$$

where γ_i^I and γ_i^{II} are the one- and two-loop contributions, respectively.

2.2.2 Useful group invariants and coupling combinations

Let us list notations for group invariants and definitions for certain combinations of coupling constants commonly found throughout all this study. These definitions allow one to write the expressions for the β-functions in a more compact form.

Group invariants $C_2(F)$ is the quadratic Casimir operator for the (in general) reducible fermion representation:

$$C_2(F) := \sum_{A=1}^{d} t^A t^A \quad \text{, i.e. } [C_2(F)]_{ij} \equiv C_2^{ij}(F) = \sum_{A=1}^{d}\sum_{k=1}^{N_\psi} t^A_{ik} t^A_{kj}\,, \tag{2.14}$$

where $i, j = 1, \ldots, N_\psi$. Due to Schur's lemma, $C_2(F)$ is a diagonal $N_\psi \times N_\psi$ matrix with the same eigenvalues for each irreducible part of the representation. Similarly, $C_2(S)$ is the quadratic Casimir operator for the (in general) reducible scalar representation:

$$C_2(S) := \sum_{A=1}^{d} \theta^A \theta^A \quad \text{, i.e. } [C_2(S)]_{ab} \equiv C_2^{ab}(S) = \sum_{A=1}^{d}\sum_{c=1}^{N_\phi} \theta^A_{ac}\theta^A_{cb}\,, \tag{2.15}$$

where $a, b = 1, \ldots, N_\phi$. Again due to Schur's lemma, $C_2(S)$ is a diagonal $N_\phi \times N_\phi$ matrix. Furthermore, $S_2(S)$ and $S_2(F)$ denote the Dynkin index of the scalar and fermion representations, respectively,

$$\mathrm{Tr}[\theta^A\theta^B] =: S_2(S)\delta^{AB}\,, \quad \mathrm{Tr}[t^A t^B] =: S_2(F)\delta^{AB}\,, \tag{2.16}$$

and $C_2(G)$ is the quadratic Casimir operator of the (irreducible) adjoint representation

$$C_2(G)\delta^{AB} := \sum_{C,D=1}^{d} f^{ACD} f^{BCD} \,. \tag{2.17}$$

Coupling combinations We start with two $N_\psi \times N_\psi$ matrices formed out of the Yukawa matrices Y^a_{ij}:

$$Y_2(F) := \sum_{a=1}^{N_\phi} Y^{\dagger a} Y^a \,, \quad Y_2^\dagger(F) := \sum_{a=1}^{N_\phi} Y^a Y^{\dagger a} \,, \tag{2.18}$$

where the sum includes all 'active' (propagating) scalar indices but not the dummy index. It should be noted that $Y_2^\dagger(F) \neq [Y_2(F)]^\dagger$; instead it represents the quantity $Y_2(F)$ where the Yukawa coupling Y^a has been replaced by its conjugate $Y^{\dagger a}$. Furthermore, the following $N_\phi \times N_\phi$ matrices are needed below:

$$Y_2^{ab}(S) := \frac{1}{2} \operatorname{Tr}[Y^{\dagger a} Y^b + Y^{\dagger b} Y^a] \,, \tag{2.19}$$

$$H^2_{ab}(S) := \frac{1}{2} \sum_{c=1}^{N_\phi} \operatorname{Tr}[Y^a Y^{\dagger b} Y^c Y^{\dagger c} + Y^{\dagger a} Y^b Y^{\dagger c} Y^c] \,, \tag{2.20}$$

$$\overline{H}^2_{ab}(S) := \frac{1}{2} \sum_{c=1}^{N_\phi} \operatorname{Tr}[Y^a Y^{\dagger c} Y^b Y^{\dagger c} + Y^{\dagger a} Y^c Y^{\dagger b} Y^c] \,, \tag{2.21}$$

$$\Lambda^2_{ab}(S) := \frac{1}{6} \sum_{c,d,e=1}^{N_\phi} \lambda_{acde} \lambda_{bcde} \,, \tag{2.22}$$

$$Y^{2F}_{ab}(S) := \frac{1}{2} \operatorname{Tr}[C_2(F)(Y^a Y^{\dagger b} + Y^b Y^{\dagger a})] \,. \tag{2.23}$$

There is one crucial comment in order concerning the properties of these objects: in previous works [6, 7, 8, 12] it is assumed that $Y_2^{ab}(S) = Y_2(S)\delta_{ab}$ and $\Lambda^2_{ab}(S) = \Lambda^2(S)\delta_{ab}$ holds. These properties are derived from group theoretical arguments. We agree with them as long as the considered model does not contain several scalar particles with identical quantum numbers. However, if this is the case than these relations are no longer valid. Or, in other words, the matrices Y_2^{ab} and Λ^2_{ab} are diagonal in the space of irreducible representations but not necessarily in the space of particles in the considered model. The consequence is that contributions from off-diagonal wave-function corrections may arise which are not included in Refs. [6, 7, 8, 12]. This is one source for the discrepancies between our results and previous ones. This does not only affect the dimensionful parameters but also the quartic scalar couplings.

2.3 β-functions of dimensionless parameters

In this section we present an overview of the derivation of the beta functions for the Yukawa couplings, following the works by M.E. Machacek and M.T. Vaughn, Refs. [6, 7]. For this purpose we first discuss the wave function renormalization for scalar, fermion and vector fields, and then we turn to the renormalization of the fermion-fermion-scalar vertex. These ingredients are further used to derive the Yukawa coupling beta functions, which explicitly have the following structure:

$$\beta^a = \underbrace{{}^{\mathrm{Y}}\gamma^a}_{\substack{\text{anomalous dimensions of the operators } \bar{\psi}_j\psi_k\phi_a}} + \underbrace{\gamma^{\dagger F}}_{\substack{\text{anomalous dimension of the fermion field } \bar{\psi}}} Y^a + Y^a \underbrace{\gamma^F}_{\substack{\text{anomalous dimension of the fermion field } \psi}} + \underbrace{\gamma^S_{ab}}_{\substack{\text{anomalous dimensions of the scalar field } \phi}} Y^b \tag{2.24}$$

2.3.1 Scalar wave function renormalization

In order to calculate the anomalous dimensions of the fields to the desired order, one has to evaluate the dimensionally regularized Feynman diagrams to this order, extract the coefficients of the single pole terms and use Eq. (2.12).

Let us demonstrate it on the example of the (diagonal) scalar wave function renormalization, conducted by M.E. Machacek and M.T. Vaughn in Ref. [6]. The contribution of a diagram to the singular part of the scalar wave function renormalization matrix at 2-loop has the following form

$$(Z_S^{-1})_{ab} = \frac{1}{(4\pi)^4} S_{ab} \left(\frac{A}{\eta^2} + \frac{B}{\eta} \right) , \tag{2.25}$$

where S_{ab} is a group theoretic factor associated with the diagram, and η is the $\overline{\mathrm{MS}}$ expansion parameter defined as

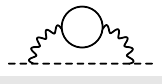
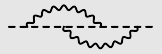
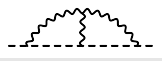
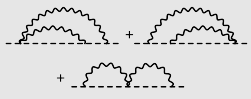
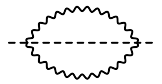
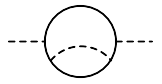
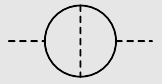
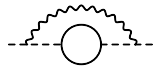
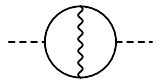
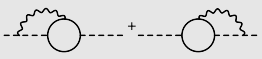
$$\frac{1}{\eta} = \frac{1}{\varepsilon} + \ln 4\pi - \gamma_E \, , \tag{2.26}$$

with the Euler constant γ_E.

The relevant singular parts along with the corresponding diagrams are collected in Table (2.1) below.

The diagonal wave-function renormalization in this example assumes that $S_{ab} = S\delta_{ab}$, which, however, would be valid only for models with no mixing in the scalar sector. This problem is discussed further, and the relevant expressions are generalized according to the non-diagonal wave-function renormalization in Sec. 2.3.5.

Table 2.1: Scalar wave function renormalization diagrams and corresponding singular parts of the renormalization matrix Z_S^{-1} defined by (2.25) (with the appropriate 1-loop counterterms subtracted).

Order	**Diagram**	$\mathbf{S_{ab} = S\delta_{ab}}$	**A**	**B**
g^4		$g^4 C_2(S) C_2(G)$	$\frac{1}{2}(5+\frac{3}{2}\alpha)$	$-\left(\frac{37}{12}-\frac{17}{8}\alpha+\frac{3}{8}\alpha^2\right)$
g^4		$\kappa g^4 C_2(S) S_2(F)$	-2	$\frac{5}{3}$
g^4		$g^4 C_2(S) S_2(S)$	$-\frac{1}{4}$	$\frac{11}{24}$
g^4		$g^4 [C_2(S)]^2$	$\frac{1}{2}(2+\alpha)^2$	$-\frac{1}{4}(2+\alpha)^2$
g^4		$g^4 C_2(S)\left[C_2(S)-\frac{1}{2}C_2(G)\right]$	$(2+\alpha)(1-\alpha)$	$\frac{1}{2}-3\alpha+\frac{7}{4}\alpha^2$
g^4		$\frac{1}{2} g^4 C_2(S) C_2(G)$	$\frac{3}{2}(2+\alpha)(1-\alpha)$	$-\frac{17}{8}-\frac{31}{4}\alpha+2\alpha^2$
g^4		$g^4 C_2(S)\left[2C_2(S)-\frac{1}{2}C_2(G)\right]$	$-\frac{3}{2}(2+\alpha)$	$-\frac{3}{8}+\frac{5}{2}\alpha-\alpha^2$
g^4		$g^4 C_2(S)\left[2C_2(S)-\frac{1}{2}C_2(G)\right]$	0	$1-\frac{1}{2}\alpha+\frac{1}{4}\alpha^2$
λ^2		$\frac{1}{6}\lambda_{acde}\lambda_{bcde}$	0	$\frac{1}{4}$
Y^4		$\frac{1}{2}\kappa\,\mathrm{Tr}[Y^b Y^{\dagger a} Y^c Y^{\dagger c} + Y^{\dagger b} Y^a Y^{\dagger c} Y^c]$	1	$-\frac{3}{2}$
Y^4		$\frac{1}{2}\kappa\,\mathrm{Tr}[Y^b Y^{\dagger c} Y^a Y^{\dagger c} + Y^{\dagger b} Y^c Y^{\dagger a} Y^c]$	2	-1
g^2Y^2		$\frac{1}{2}\kappa g^2 C_2(S)\,\mathrm{Tr}[Y^a Y^{\dagger b} + Y^b Y^{\dagger a}]$	$-(2+\alpha)$	$1+\frac{5}{2}\alpha$
g^2Y^2		$\frac{1}{2}\kappa g^2\,\mathrm{Tr}[C_2(F)(Y^a Y^{\dagger b} + Y^b Y^{\dagger a})]$	$2(1-\alpha)$	$-1+\alpha$
g^2Y^2		$\frac{1}{2}\kappa g^2\,\mathrm{Tr}[C_2(F)(Y^a Y^{\dagger b} + Y^b Y^{\dagger a})]$ $-\frac{1}{4}C_2(S)\,\mathrm{Tr}[Y^a Y^{\dagger b} + Y^b Y^{\dagger a}]$	$-2(4-\alpha)$	$2(3-\frac{1}{2}\alpha)$
g^2Y^2		$\frac{1}{2}\kappa g^2 C_2(S)\,\mathrm{Tr}[Y^a Y^{\dagger b} + Y^b Y^{\dagger a}]$	$2(1+2\alpha)$	$2(1-\frac{3}{2}\alpha)$

Here α is a gauge parameter and the factor $\kappa = \frac{1}{2}$ for two-component fermions and $\kappa = 1$ for four-component fermions. The notations $C_2(F)$, $C_2(S)$, $C_2(G)$ and $S_2(S)$, $S_2(F)$ for quadratic Casimir operators for the fermion/scalar/adjoint representations and Dynkin indices of the scalar/fermion representations, respectively, have been listed in Sec. 2.2.2 of the current chapter.

From the contributions above, one can obtain the following result for the 2-loop anoma-

lous dimension matrix for the scalar fields

$$\gamma^S_{ab} = \frac{2}{(4\pi)^4} \sum_{diagrams} BS_{ab}\,, \tag{2.27}$$

and explicitly,

$$\begin{aligned}(4\pi)^4\gamma^S_{ab}\big|_{2-loop} = \frac{1}{12}\lambda_{acde}\lambda_{bcde} - 3\kappa\cdot\frac{1}{2}\,\mathrm{Tr}[Y^bY^{\dagger a}Y^cY^{\dagger c} + Y^{\dagger b}Y^aY^{\dagger c}Y^c] \\ - 2\kappa\cdot\frac{1}{2}\,\mathrm{Tr}[Y^bY^{\dagger c}Y^aY^{\dagger c} + Y^{\dagger b}Y^cY^{\dagger a}Y^c] \\ + 10\kappa g^2\cdot\frac{1}{2}\,\mathrm{Tr}[C_2(F)(Y^aY^{\dagger b} + Y^bY^{\dagger a})] \\ - g^4C_2(S)\left\{\left(\frac{113}{12} + \frac{5}{2}\alpha - \frac{1}{4}\alpha^2\right)C_2(G) - \frac{10}{3}\kappa S_2(F)\right. \\ \left. -\frac{11}{12}S_2(S) - \frac{3}{2}C_2(S)\right\}\delta_{ab}\,.\end{aligned} \tag{2.28}$$

The anomalous dimension of the scalar field remains dependent on the gauge parameter α, however, this gauge dependence cancels out in the explicit expression of the Yukawa and scalar quartic coupling beta functions by the gauge dependence of the proper vertex parts.

Using notations (2.20)–(2.23), which have been introduced in Refs. [8, 12][2], one can rewrite the result (2.28) as follows

$$\begin{aligned}(4\pi)^4\gamma^S_{ab}\big|_{2-loop} = \frac{1}{2}\Lambda^2(S)\delta_{ab} - 3\kappa H^2(S)\delta_{ab} - 2\kappa\bar{H}^2(S)\delta_{ab} + 10\kappa g^2Y^{2F}(S)\delta_{ab} \\ - g^4C_2(S)\left\{\left(\frac{113}{12} + \frac{5}{2}\alpha - \frac{1}{4}\alpha^2\right)C_2(G) - \frac{10}{3}\kappa S_2(F)\right. \\ \left. -\frac{11}{12}S_2(S) - \frac{3}{2}C_2(S)\right\}\delta_{ab}\,.\end{aligned} \tag{2.29}$$

This diagonal anomalous dimension of the scalar field enters the 2-loop beta function for the scalar quartic couplings as shown in Refs. [8, 12]:

$$\beta_{abcd} \equiv \mu\frac{d}{d\mu}\lambda_{abcd} = \gamma_{abcd} + \sum_k \gamma^S(k)\lambda_{abcd}\,, \tag{2.30}$$

where $\gamma^S(k)$ is the anomalous dimension of the scalar field k, given by (2.29), and γ_{abcd} is the anomalous dimension of the operator $\phi_a\phi_b\phi_c\phi_d$, the derivation of which is provided in Ref. [8] and will not be demonstrated here due to the large number of diagrams.

Such an assumption of a diagonal wave-function renormalization is reasonable only

[2]Note that the notations are not exactly the same in these two papers: they have been modified in Ref. [12] in order to be hermitian for complex fermions. We follow the same notations as in Ref. [12], which are repeated in Sec. 2.2.2.

for models which do not contain scalar mixing. The same must be taken into account for the one-loop result, mentioned in Refs. [8, 12]. We provide more general expressions that include the contributions from off-diagonal wave-function corrections in Sec. 2.3.5.

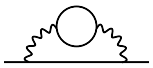
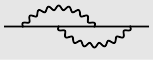
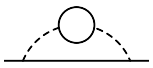
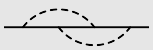
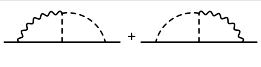
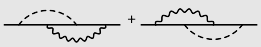
2.3.2 Fermion wave function renormalization

Let us now turn to the fermion wave function renormalization at 2-loop level, the contributing diagrams for which are shown in Table (2.2). The contribution of a diagram to the singular part of the fermion wave function renormalization matrix can be expressed in the following form

$$(Z_F^{-1}) = \frac{1}{(4\pi)^4} S \left(\frac{A}{\eta^2} + \frac{B}{\eta} \right) , \tag{2.31}$$

where, similarly to the previous subsection, S is a group theoretic factor associated with the diagram and η is defined as before in (2.26).

Table 2.2: Fermion wave function renormalization diagrams and corresponding singular parts of the renormalization matrix Z_F^{-1} (with the gauge parameter α and the factor $\kappa = \frac{1}{2}$ or 1 for two- or four-component fermions, respectively).

Order	Diagram	S	A	B
g^4		$g^4 C_2(F) C_2(G)$	0	$\frac{1}{4}\left(5+\frac{3}{2}\alpha\right)$
g^4		$\kappa g^4 C_2(F) S_2(F)$	0	-1
g^4		$g^4 C_2(F) S_2(S)$	0	$-\frac{1}{8}$
g^4		$g^4 \left[C_2(F)\right]^2$	$-\frac{1}{2}(1-\alpha)^2$	$-\frac{1}{4}(1-\alpha)^2$
g^4		$g^4 C_2(F)\left[C_2(F) - \frac{1}{2}C_2(G)\right]$	$-(1-\alpha)^2$	$-\frac{1}{2}(1+\alpha-\frac{1}{2}\alpha^2)$
g^4		$\frac{1}{2} g^4 C_2(F) C_2(G)$	$-\frac{3}{2}(2-\alpha)(1-\alpha)$	$\frac{11}{2} - \frac{15}{4}\alpha + \frac{1}{2}\alpha^2$
Y^4		$Y^a Y^{\dagger b} Y^b Y^{\dagger a}$	$\frac{1}{8}$	$-\frac{1}{16}$
Y^4		$\frac{1}{2}\kappa Y^a Y^{\dagger b} \mathrm{Tr}[Y^{\dagger a} Y^b + Y^{\dagger b} Y^a]$	$\frac{1}{2}$	$-\frac{3}{4}$
Y^4		$Y^a Y^{\dagger b} Y^a Y^{\dagger b}$	$\frac{1}{2}$	0
g^2Y^2		$g^2 C_2(F) Y^a Y^{\dagger a}$	$\frac{1}{4}(1-\alpha)$	$-\frac{3}{8}(1-\alpha)$
g^2Y^2		$g^2 Y^a C_2(F) Y^{\dagger a}$	$\frac{1}{4}(1-\alpha)$	$\frac{1}{8}(1-\alpha)$
g^2Y^2		$g^2 Y^b Y^{\dagger a} \theta^A_{bc} \theta^A_{ca}$	$-\frac{1}{4}(2+\alpha)$	$\frac{1}{4}(3-\frac{1}{2}\alpha)$
g^2Y^2		$g^2 Y^b t^A Y^{\dagger a} \theta^A_{ba}$	$\frac{1}{2}(1-\alpha)$	$-\frac{1}{4}(5+\alpha)$
g^2Y^2		$g^2(Y^b Y^{\dagger a} t^A \theta^A_{ba}$ $+ t^A Y^b Y^{\dagger a} \theta^A_{ba})$	$-\frac{1}{2}(1-\alpha)$	$\frac{1}{8}(4-\alpha)$
g^2Y^2		$g^2(t^A Y^a t^A Y^{\dagger a}$	$-\frac{1}{4}(5-2\alpha)$	$-\frac{1}{8}(3+\alpha)$

Order	Diagram	S	A	B
		$+Y^a t^A Y^{\dagger a} t^A$		

The 2-loop fermion anomalous dimension matrix is given by

$$\gamma^F = \frac{2}{(4\pi)^4} \sum_{diagrams} BS\,, \tag{2.32}$$

and explicitly,

$$\begin{aligned}(4\pi)^4 \gamma^F\big|_{2-loop} = &-\frac{1}{8} Y^a Y^{\dagger b} Y^b Y^{\dagger a} - \frac{3}{2}\kappa \cdot \frac{1}{2} \operatorname{Tr}[Y^a Y^{\dagger b} + Y^b Y^{\dagger a}] Y^{\dagger a} Y^b \\ &+ g^2 \left[\frac{9}{2} C_2(S) Y^a Y^{\dagger a} - \frac{7}{4} C_2(F) Y^a Y^{\dagger a} - \frac{1}{4} Y^a C_2(F) Y^{\dagger a}\right] \\ &+ g^4 C_2(F) \left[\frac{1}{2}\left(17 - 5\alpha + \frac{1}{2}\alpha^2\right) C_2(G) - 2\kappa S_2(F) - \frac{1}{4} S_2(S)\right] \\ &- \frac{3}{2} g^4 [C_2(F)]^2\,. \end{aligned} \tag{2.33}$$

Here the gauge dependence also remains and will be cancelled out later on in the calculation of the Yukawa and scalar quartic coupling constants.

2.3.3 Vector wave function renormalization

The anomalous dimension for the gauge field is defined as

$$\gamma_A \equiv \frac{1}{2} \mu \frac{d}{d\mu} \ln Z_A\,, \tag{2.34}$$

with the vector wave function renormalization constant Z_A.

This is not needed for the derivation of the beta functions for the Yukawa and scalar quartic couplings, however, let us include the corresponding equations and diagrams for completeness. The contribution of a diagram to the singular part of the vector wave function renormalization constant is given by

$$Z_A^{-1} = \frac{1}{(4\pi)^4} S \left(\frac{A}{\eta^2} + \frac{B}{\eta}\right)\,, \tag{2.35}$$

where S is a group theoretic factor and η is defined by (2.26).

The 2-loop vector field anomalous dimension has the following form

$$\gamma_A = \frac{2}{(4\pi)^4} \sum_{diagrams} BS\,. \tag{2.36}$$

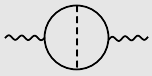
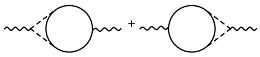
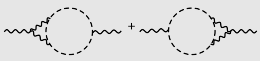
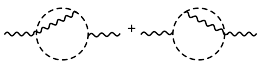
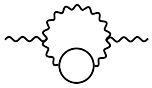
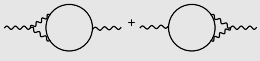
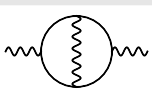
It is useful to know the contributions for diagrams with external (background) gauge

field lines both in the usual R_ξ gauge and in the corresponding background field gauge B_ξ with modified Feynman rules. This simplifies the derivation of the beta function β_g for the gauge coupling to the short relation [14]

$$\beta_g = g\gamma_A \,. \tag{2.37}$$

The relevant singular parts along with the corresponding diagrams are shown in Table (2.3).

Table 2.3: Vector wave function renormalization diagrams and corresponding singular parts of the renormalization matrix Z_A^{-1}. The additional contributions from the extra vertex terms present in the B_ξ gauge are noted explicitly.

Order	**Diagram**	$\mathbf{S_{AB} \equiv S\delta_{AB}}$	**A**	**B**
g^2Y^2		$\kappa g^2 \,\mathrm{Tr}[Y^a Y^{\dagger a} t^A t^B]$	$\frac{2}{3}$	$-\frac{10}{9}$
g^2Y^2		$\kappa g^2 \theta^A_{ab}\theta^B_{ca} \,\mathrm{Tr}[Y^b Y^{\dagger c}]$	$\frac{1}{3}$	$-\frac{7}{18}$
g^2Y^2		$\kappa g^2 \,\mathrm{Tr}[Y^a t^A Y^{\dagger a} t^B]$	$-\frac{2}{3}$	$\frac{1}{9}$
g^2Y^2		$\kappa g^2 \theta^A_{ba} \,\mathrm{Tr}[Y^b Y^{\dagger a} t^A]$	$-\frac{4}{3}$	$\frac{8}{9}$
g^4		$g^4 C_2(S) S_2(S)$	$-\frac{1}{6}(2+\alpha)$	$\frac{1}{9}(4-\alpha)$
g^4		$g^4 C_2(G) S_2(S)$ (B_ξ)	$-\frac{7}{24} - \frac{1}{24}\alpha$ $-\frac{1}{8}$	$\frac{55}{144} + \frac{31}{144}\alpha$ $\frac{5}{16}$
g^4		$g^4 C_2(G) S_2(S)$ (B_ξ)	$\frac{1}{8} + \frac{5}{24}\alpha$ $\frac{1}{3} - \frac{1}{12}\alpha$	$-\frac{5}{48} - \frac{29}{144}\alpha$ $-\frac{1}{12}$
g^4		$g^4 \left[C_2(S) - \frac{1}{2}C_2(G)\right] S_2(S)$	$-\frac{1}{6}(1-\alpha)$	$\frac{29}{36} - \frac{1}{18}\alpha$
g^4		$g^4 \left[2C_2(S) - \frac{1}{2}C_2(G)\right] S_2(S)$	0	$\frac{1}{3} - \frac{1}{12}\alpha$
g^4		$g^4 \left[2C_2(S) - \frac{1}{2}C_2(G)\right] S_2(S)$	$\frac{1}{4}$	$-\frac{11}{24} + \frac{1}{6}\alpha$
g^4		$\kappa g^4 C_2(F) S_2(F)$	$\frac{4}{3}(1-\alpha)$	$-\frac{10}{9}(1-\alpha)$
g^4		$\kappa g^4 C_2(G) S_2(F)$ (B_ξ)	$-\frac{7}{3} - \frac{1}{3}\alpha$ -1	$\frac{13}{18} + \frac{25}{18}\alpha$ $\frac{3}{2}$
g^4		$\kappa g^4 C_2(G) S_2(F)$ (B_ξ)	$\frac{5}{3}\alpha$ $\frac{2}{3}(4-\alpha)$	$\frac{10}{3} - \frac{35}{18}\alpha$ $-\frac{2}{3}$
g^4		$\kappa g^4 \left[C_2(F) - \frac{1}{2}C_2(G)\right] S_2(F)$	$-\frac{4}{3}(1-\alpha)$	$\frac{28}{9} - \frac{10}{9}\alpha$

With the pure gauge theory contributions from [14], one can obtain

$$(4\pi)^4\gamma_A\big|_{2-loop} = -\frac{34}{3}g^4[C_2(G)]^2 + \kappa g^4\left[4C_2(F)+\frac{20}{3}C_2(G)\right]S_2(F) + g^4\left[2C_2(S)+\frac{1}{3}C_2(G)\right]S_2(S) - 2\kappa g^2 Y_4(F)\,, \tag{2.38}$$

where $Y_4(F) \equiv \frac{1}{d(G)}\,\mathrm{Tr}[C_2(F)Y^aY^{\dagger a}]$ with the dimension of the group $d(G)$.

The result (2.38) for the anomalous dimension of the background gauge field is proportional to the beta function for the gauge coupling (2.37) and is gauge independent.

2.3.4 Renormalization of the proper Yukawa vertex

The last necessary step for computing the beta functions for the Yukawa couplings is the renormalization of the proper Yukawa vertex [7].

The singular part of each contributing proper vertex diagram can be expressed as

$${}^{\mathrm{Y}}Z^a = \frac{1}{(4\pi)^4}S^a\left(\frac{A}{\eta^2}+\frac{B}{\eta}\right)\,, \tag{2.39}$$

where S^a is a group theoretic factor associated with the diagram and η is the $\overline{\mathrm{MS}}$ expansion parameter defined by (2.26).

The two-loop diagrams which modify the proper Yukawa vertex along with corresponding singular parts of the vertex renormalization Z^a are listed below. They are collected in three tables: one-loop diagrams with one-loop propagator insertions (Tab. 2.4), one-loop diagrams with one-loop vertex insertions (Tab. 2.5), and diagrams with crossed ladder topology (Tab. 2.6).

Table 2.4: 1-loop vertex diagrams with 1-loop propagator insertions and corresponding singular parts of the vertex renormalization Z^a. The one-loop insertions include the appropriate $\overline{\mathrm{MS}}$ counterterms.

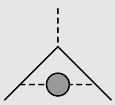
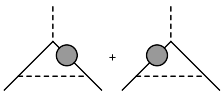
Diagram	Insertion type	$\mathbf{S^a}$	A	B
		$\kappa Y_2^{bc}(S)Y^bY^{\dagger a}Y^c$	-1	1
		$C_2^{bc}(S)Y^bY^{\dagger a}Y^c$	$1+\frac{1}{2}\alpha$	$-1+\frac{1}{2}\alpha$
		$Y^bY^{\dagger a}Y_2^\dagger(F)Y^b + Y^bY_2(F)Y^{\dagger a}Y^b$	$-\frac{1}{4}$	$\frac{1}{4}$
		$Y^b\{C_2(F),Y^{\dagger a}\}Y^b$	$-\frac{1}{2}(1-\alpha)$	0
		$t^AY^at^AS_2(S)$	$\frac{1}{4}$	$-\frac{7}{12}$
		$t^AY^at^AS_2(F)$	2κ	$-\frac{8}{3}\kappa$

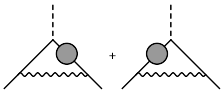
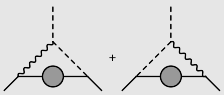
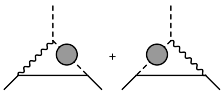
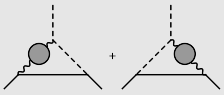
Diagram	Insertion type	$\mathbf{S^a}$	A	B
		$t^A Y^a t^A C_2(G)$	$-\frac{1}{4}(10+3\alpha)$	$\frac{13}{3}-\frac{7}{4}\alpha+\frac{3}{8}\alpha^2$
		$t^A Y^a t^A Y_2(F)+Y_2^\dagger(F)t^A Y^a t^A$	$1-\frac{1}{4}\alpha$	$-\frac{3}{2}+\frac{1}{4}\alpha$
		$\{C_2(F), t^A Y^a t^A\}$	$\frac{1}{2}(1-\alpha)(4-\alpha)$	$-(1-\alpha)$
		$\theta^A_{ab} t^A Y_2^\dagger(F) Y^b - \theta^A_{ab} Y^b Y_2(F) t^A$	$-\frac{1}{4}(1-\alpha)$	$\frac{1}{4}(1-\alpha)$
		$\theta^A_{ab}[C_2(F)t^A, Y^b]$	$-\frac{1}{2}(1-\alpha)^2$	0
		$\kappa\theta^A_{ab} Y_2^{bc}(S)[t^A, Y^c]$	$-(1-\alpha)$	$1-\alpha$
		$\theta^A_{ab} C_2^{bc}(S)[t^A, Y^c]$	$\frac{1}{2}(1-\alpha)(2+\alpha)$	$-\frac{1}{2}(1-\alpha)(2-\alpha)$
	–	–	0	0

Table 2.5: 1-loop vertex diagrams with 1-loop vertex insertions and corresponding singular parts of the vertex renormalization Z^a. The one-loop insertions include the appropriate $\overline{\text{MS}}$ counterterms.

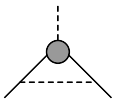
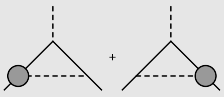
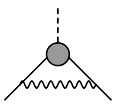
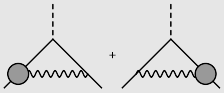
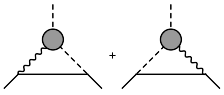
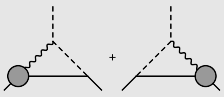
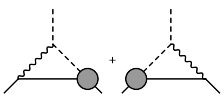
Diagram	Insertion type	$\mathbf{S^a}$	A	B
		$Y^c Y^{\dagger b} Y^a Y^{\dagger b} Y^c$	$-\frac{1}{2}$	$\frac{1}{2}$
		$Y^b\{C_2(F), Y^{\dagger a}\}Y^b$	$1-\frac{1}{4}\alpha$	$-\frac{1}{4}(2-\alpha)$
		$C_2^{ac}(S) Y^b Y^{\dagger c} Y^b$	$-\frac{1}{4}(2+\alpha)$	$\frac{1}{4}(2-\alpha)$
		$Y^b Y^{\dagger c} Y^b Y^{\dagger a} Y^c$ $+Y^c Y^{\dagger a} Y^b Y^{\dagger c} Y^b$	$-\frac{1}{2}$	0
		$\{C_2(F), Y^b Y^{\dagger a} Y^b\}$	$1-\frac{1}{4}\alpha$	$\frac{5}{4}$
		$Y^b\{C_2(F), Y^{\dagger a}\}Y^b$	$1-\frac{1}{4}\alpha$	$-\frac{1}{4}$
		$C_2^{bc}(S) Y^b Y^{\dagger a} Y^c$	$-\left(1+\frac{1}{2}\alpha\right)$	$-\left(\frac{3}{2}+\alpha\right)$
		$t^A Y^b Y^{\dagger a} Y^b t^A$	$1-\frac{1}{4}\alpha$	$-3+\frac{1}{2}\alpha$
		$t^A\{C_2(F), Y^a\}t^A$	$-\frac{1}{4}(4-\alpha)^2$	$\frac{1}{4}(4-\alpha)^2$
		$C_2^{ab}(S) t^A Y^b t^A$	$\frac{1}{4}(4-\alpha)(2+\alpha)$	$-\frac{1}{4}(12-4\alpha+\alpha^2)$

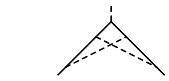
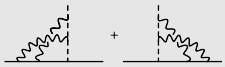
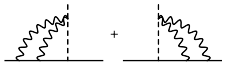
Diagram	Insertion type	S^a	A	B
		$Y_2^\dagger(F)t^AY^bt^A + t^AY^at^AY_2(F)$	$-(1-\frac{1}{4}\alpha)$	$\frac{1}{2}\left(3-\frac{1}{2}\alpha\right)$
		$t^AY^aY^{\dagger b}t^AY^b + Y^bt^AY^{\dagger b}Y^at^A$	0	$-\frac{3}{2}$
		$\{C_2(F), t^AY^at^A\}$	$-\frac{1}{2}(1-\alpha)(4-\alpha)$	$-(2+\alpha)$
		$C_2(G)t^AY^at^A$	$-\frac{1}{4}(4-\alpha)^2$	$\frac{1}{2}(2+\alpha)(7-\alpha)$
		$\kappa\theta^A_{ab}Y_2^{bc}(S)[t^A, Y^c]$	$1-\alpha$	$-(1-\alpha)$
		$\theta^A_{ab}C_2^{bc}(S)[t^A, Y^c]$	$-\frac{1}{2}(1-\alpha)(2+\alpha)$	$\frac{1}{2}(1-\alpha)(2-\alpha)$
		$C_2(G)\theta^A_{ab}[t^A, Y^b]$	$\frac{1}{8}(1-\alpha)(4-\alpha)$	$-\frac{1}{8}(1-\alpha)(2-\alpha)$
		$\theta^A_{ab}Y_2^\dagger(F)t^AY^b - \theta^A_{ab}Y^bt^AY_2(F)$	$\frac{1}{4}(1-\alpha)$	$-\frac{1}{4}(1-\alpha)$
		$\theta^A_{ab}C_2(F)t^AY^b - \theta^A_{ab}Y^bt^AC_2(F)$	$\frac{1}{2}(1-\alpha)^2$	0
		$C_2(G)\theta^A_{ab}[t^A, Y^b]$	$\frac{1}{8}(1-\alpha)(4-\alpha)$	$-\frac{1}{8}(1-\alpha)(2-\alpha)$
		$\theta^A_{ab}[t^A, Y^cY^{\dagger b}Y^c]$	$-\frac{1}{2}(1-\alpha)$	0
		$\theta^A_{ab}C_2(F)t^AY^b - \theta^A_{ab}Y^bt^AC_2(F)$	$\frac{1}{4}(1-\alpha)(4-\alpha)$	$-\frac{1}{4}(1-\alpha)$
		$\theta^A_{ab}t^AY^bC_2(F) - \theta^A_{ab}C_2(F)Y^bt^A$	$\frac{1}{4}(1-\alpha)(4-\alpha)$	$\frac{5}{4}(1-\alpha)$
		$\theta^A_{ab}C_2^{bc}(S)[t^A, Y^c]$	$-\frac{1}{4}(1-\alpha)(2+\alpha)$	$-\frac{1}{4}(1-\alpha)(3+2\alpha)$

Note that the not-1-particle-irreducible graphs and corresponding coupling structures appear here only **effectively**, due to the gauge invariance relations in Appendix A in [7] !

The diagrams with crossed ladder topology in Tab. 2.6 have at most a single pole in ϵ, since the integration over the first loop momenta is convergent for any routing of the loop momenta, and the coefficient $A = 0$ in each case.

Table 2.6: Crossed ladder topology 2-loop diagrams and corresponding singular parts of the vertex renormalization Z^a.

Diagram	S^a	B
	$\lambda_{abcd}Y^bY^{\dagger c}Y^d$	$\frac{1}{2}$

Diagram	S^a	B
	$Y^c Y^{\dagger b} Y^a Y^{\dagger c} Y^b$	$-\frac{1}{2}$
	$Y^b t^A Y^{\dagger a} Y^b t^A + t^A Y^b Y^{\dagger a} t^A Y^b$	$-\left(1+\frac{1}{2}\alpha\right)$
	$t^B t^A Y^a t^B t^A$	$-\left(2-4\alpha+\frac{1}{2}\alpha^2\right)$
	$\theta^A_{ad}\theta^A_{bc} Y^b Y^{\dagger d} Y^c$	0
	$if^{ABC}\theta^C_{ab} t^B Y^b t^A$	0
	$(\theta^A\theta^B)_{ab} Y^b t^A t^B + t^A t^B Y^b (\theta^A\theta^B)_{ba}$	$-\frac{3}{2}(1-\alpha)$
	$\{\theta^A,\theta^B\}_{ab} Y^b t^A t^B + t^A t^B Y^b \{\theta^A\theta^B\}_{ba}$	$1-\frac{1}{2}\alpha+\frac{1}{4}\alpha^2$
	$\{\theta^A,\theta^B\}_{ab} t^A Y^b t^B$	$-\left(2-\alpha+\frac{1}{2}\alpha^2\right)$

The 2-loop anomalous dimension ${}^{\mathrm{Y}}\gamma^a$ of the operators $\bar{\psi}_j\psi_k\phi_a$ is given by

$$ {}^{\mathrm{Y}}\gamma^a = \frac{4}{(4\pi)^4}\sum_{diagrams} BS^a\,. \tag{2.40}$$

After writing down the explicit sum of the single-pole contributions, collected in Tab. 2.4–2.6, one can see that the result is gauge dependent. This gauge dependence will be cancelled out in the expression of the beta function for the Yukawa couplings (2.24) by the gauge dependence of anomalous dimensions of the scalar (2.28) and fermion (2.33) fields.

The explicit 2-loop result for the beta function for the Yukawa couplings is provided in the following subsection.

2.3.5 A full list of β-functions of dimensionless parameters

In this section, we provide a complete list of the beta functions for the dimensionless parameters at one- and two-loop in order to have all the expressions in one place. We highlight differences with respect to the literature by underlining the corresponding terms. These expressions provide the starting point for the derivation of the beta functions for the dimensionful parameters in the next Chapter 3 using the so-called dummy field method.

RGEs for dimensionless parameters

The Yukawa couplings. The one- and two-loop expressions for the running of the

Yukawa couplings are given by

$$\beta_a^I = \frac{1}{2}\left[Y_2^+(F)Y^a + Y^a Y_2(F)\right] + 2Y^b Y^{+a} Y^b + 2\kappa Y^b Y_2^{ab}(S) - 3g^2\{C_2(F), Y^a\}\,, \quad (2.41)$$

$$\begin{aligned}
\beta_a^{II} =& 2Y^c Y^{+b} Y^a (Y^{+c} Y^b - Y^{+b} Y^c) - Y^b \left[Y_2(F)Y^{+a} + Y^{+a} Y_2^+(F)\right] Y^b \\
&- \frac{1}{8}\left[Y^b Y_2(F) Y^{+b} Y^a + Y^a Y^{+b} Y_2^+(F) Y^b\right] - \underline{4\kappa Y_2^{bc}(S) Y^b Y^{+a} Y^c} - 2\kappa Y^b \bar{H}_{ab}^2(S) \\
&- \frac{3}{2}\kappa Y_2^{bc}(S)(Y^b Y^{+c} Y^a + Y^a Y^{+c} Y^b) - 3\kappa Y^b H_{ab}^2(S) - 2\lambda_{abcd} Y^b Y^{+c} Y^d \\
&+ \frac{1}{2}\Lambda_{ab}^2(S) Y^b + 3g^2\{C_2(F), Y^b Y^{+a} Y^b\} + 5g^2 Y^b \{C_2(F), Y^{+a}\} Y^b \\
&- \frac{7}{4} g^2 [C_2(F) Y_2^+(F) Y^a + Y^a Y_2(F) C_2(F)] \\
&- \frac{1}{4} g^2 [Y^b C_2(F) Y^{+b} Y^a + Y^a Y^{+b} C_2(F) Y^b] + 6g^2 H_{2t}^a + 10\kappa g^2 Y^b Y_{ab}^{2F}(S) \\
&+ 6g^2 [C_2^{bc}(S) Y^b Y^{+a} Y^c - 2C_2^{ac}(S) Y^b Y^{+c} Y^b] + \frac{9}{2} g^2 C_2^{bc}(S)(Y^b Y^{+c} Y^a + Y^a Y^{+c} Y^b) \\
&- \frac{3}{2} g^4 \{[C_2(F)]^2, Y^a\} + 6g^4 C_2^{ab}(S)\{C_2(F), Y^b\} \\
&+ g^4 \left[-\frac{97}{6} C_2(G) + \frac{10}{3}\kappa S_2(F) + \frac{11}{12} S_2(S)\right] \{C_2(F), Y^a\} - \frac{21}{2} g^4 C_2^{ab}(S) C_2^{bc}(S) Y^c \\
&+ g^4 C_2^{ab}(S) \left[\frac{49}{4} C_2(G) - 2\kappa S_2(F) - \frac{1}{4} S_2(S)\right] Y^b\,, \qquad (2.42)
\end{aligned}$$

where the definition of H_{2t}^a can be found in App. A.1 and the factor $\kappa = 1/2$ for 2-component fermions and $\kappa = 1$ for 4-component fermions.

The underlined term differs from the literature by a swapped index: due to a typo, the term $-4\kappa Y_2^{ac} Y^b Y^{\dagger c} Y^b$ was included in 2-loop beta function for the Yukawa couplings in [7], which would correspond to a not-1-particle-irreducible diagram as depicted in Fig. 2.1. It should be replaced by $-4\kappa Y_2^{bc} Y^b Y^{\dagger a} Y^c$. This typo has also propagated in [12], where it caused a mistake in the fermion mass beta function (see more in detail in Sec. 3.2.1).

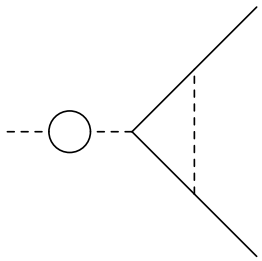


Figure 2.1: The two-loop diagram corresponding to the term $-4\kappa Y_2^{ac} Y^b Y^{\dagger c} Y^b$ which should not contribute to the β-function for the Yukawa couplings.

The scalar quartic couplings. For the quartic coupling, we are going to use the following expressions:

$$\beta^I_{abcd} = \Lambda^2_{abcd} - 8\kappa H_{abcd} + 2\kappa\underline{\Lambda^Y_{abcd}} - 3g^2\Lambda^S_{abcd} + 3g^4 A_{abcd}\,, \tag{2.43}$$

$$\begin{aligned}\beta^{II}_{abcd} =& \underline{\frac{1}{12}\sum_{per}\Lambda^2_{af}\lambda_{fbcd}} - \bar{\Lambda}^3_{abcd} - 4\kappa\bar{\Lambda}^{2Y}_{abcd} + \kappa\left[8\bar{H}^{\lambda}_{abcd} - \underline{\frac{1}{6}\sum_{per}\left[3H^2_{af} + 2\bar{H}^2_{af}\right]\lambda_{fbcd}}\right] \\ &+ 4\kappa(H^Y_{abcd} + 2\bar{H}^Y_{abcd} + 2H^3_{abcd}) \\ &+ g^2\left[2\bar{\Lambda}^{2S}_{abcd} - 6\Lambda^{2g}_{abcd} + 4\kappa(H^S_{abcd} - H^F_{abcd}) + \underline{\frac{5}{3}\kappa\sum_{per}Y^{2F}_{af}\lambda_{fbcd}}\right] \\ &- g^4\Big\{\left[\frac{35}{3}C_2(G) - \frac{10}{3}\kappa S_2(F) - \frac{11}{12}S_2(S)\right]\Lambda^S_{abcd} - \frac{3}{2}\Lambda^{SS}_{abcd} - \frac{5}{2}A^{\lambda}_{abcd} - \frac{1}{2}\bar{A}^{\lambda}_{abcd} \\ &\qquad + 4\kappa(B^Y_{abcd} - 10\bar{B}^Y_{abcd})\Big\} \\ &+ g^6\left\{\left[\frac{161}{6}C_2(G) - \frac{32}{3}\kappa S_2(F) - \frac{7}{3}S_2(S)\right]A_{abcd} - \frac{15}{2}A^S_{abcd} + 27A^g_{abcd}\right\}\,. \end{aligned} \tag{2.44}$$

The following quantities are used in (2.43):

$$\Lambda^2_{abcd} = \frac{1}{8}\sum_{per}\lambda_{abef}\lambda_{efcd}\,, \tag{2.45}$$

$$H_{abcd} = \frac{1}{4}\sum_{per}\mathrm{Tr}(Y^a Y^{\dagger b} Y^c Y^{\dagger d})\,, \tag{2.46}$$

$$\Lambda^Y_{abcd} = \frac{1}{6}\sum_{per}Y_2^{af}(S)\lambda_{fbcd}\,, \tag{2.47}$$

$$\Lambda^S_{abcd} = \sum_i C_2(i)\lambda_{abcd}\,, \tag{2.48}$$

$$A_{abcd} = \frac{1}{8}\sum_{per}\{\theta^A,\theta^B\}_{ab}\{\theta^A,\theta^B\}_{cd}\,, \tag{2.49}$$

(the mapping to the case with one dummy scalar for these quantities is discussed in Sec. 3.2.2).

The definitions for the quantities $\bar{\Lambda}^3_{abcd}$, …, A^g_{abcd} in Eq. (2.44) can be found in App. A.2. Here, $\sum_{per}$ denotes a sum over all permutations of uncontracted scalar indices. Unlike the approach employed in Refs. [8, 12], we include the off-diagonal renormalization, and our equations (2.43) and (2.44) differ from the results in those papers in the terms which are underlined.

The gauge coupling. Finally, let us list here the β-functions for the gauge coupling:

$$\beta_g^I = -\, g^3 \left[\frac{11}{3} C_2(G) - \frac{4}{3} \kappa S_2(F) - \frac{1}{6} S_2(S) \right] , \tag{2.50}$$

$$\begin{aligned} \beta_g^{II} = &-\, 2\kappa g^3 Y_4(F) - g^5 \left[\frac{34}{3} C_2(G)^2 - \kappa \left(4C_2(F) + \frac{20}{3} C_2(G) \right) S_2(F) \right. \\ &\left. - \left(2C_2(S) + \frac{1}{3} C_2(G) \right) S_2(S) \right] . \end{aligned} \tag{2.51}$$

Chapter 3

β-functions for dimensionful parameters

In this chapter we derive the beta functions for dimensionful parameters using the dummy field method. The dummy field method was developed to obtain the β-functions for the dimensionful parameters from those of the dimensionless parameters. We start by a detailed pedagogic discussion of the method in Sec. 3.1. In the further sections we apply the method, using the generic results for the dimensionless parameters given in Chapter 2. The result for the fermion mass term is derived in Sec. 3.2.1 from the β-functions of the Yukawa couplings; for the derivation of the running of the trilinear scalar couplings (Sec. 3.2.2) and for the scalar mass terms (Sec. 3.2.3), the β-functions for the scalar quartic couplings are used. We correct mistakes in the literature due to an improper use of the dummy field method and verify our expressions in comparison with supersymmetric RGEs in Sec. 3.3 and estimate the numerical impact in the frameworks of two different models in Sec. 3.4.

3.1 The dummy field method

The calculation of the renormalization constants for the dimensionful couplings (the fermion masses $(m_f)_{jk}$, the squared scalar masses m^2_{ab}, and the cubic scalar couplings h_{abc}) with the following derivation of the β-functions can be carried out directly, however, most of the authors agree that this would be rather tedious. So far it has not been attempted in the literature, whereas in the most known and cited work [12] addressing these β-functions, another approach is used. This approach is called a "dummy field" method and applies an idea, to our knowledge, first briefly mentioned in Ref. [13] and used for the derivation of RGEs of supersymmetric theories. However, there has never appeared any detailed description of this method in the literature, so we found it useful to have a close look into it and to provide a careful discussion.

The idea of this method is to introduce a scalar "dummy field", i.e. a non-propagating real scalar field with no gauge interactions. The dummy field will be denoted by an index

with a hat, $\phi_{\hat{d}}$, and satisfies the condition $D_\mu \phi_{\hat{d}} = 0$. As a consequence, expressions with two identical internal dummy indices (corresponding to a propagating dummy field) have to vanish. Furthermore, since $D_\mu \phi_{\hat{d}} = 0$, all gauge boson - dummy scalar vertices vanish as well: $\Gamma_{V\phi_a\phi_{\hat{d}}} = \Gamma_{V\phi_{\hat{d}}\phi_{\hat{d}}} = \Gamma_{VV\phi_a\phi_{\hat{d}}} = \Gamma_{VV\phi_{\hat{d}}\phi_{\hat{d}}} = 0$.

Let us now consider the Lagrangian $\mathcal{L}_0$ (2.2) in the presence of the same particle content plus one extra scalar dummy field ($\phi_{\hat{d}}$) and separate the terms with the dummy field. Using $D_\mu \phi_{\hat{d}} = 0$, $\lambda_{ab\hat{d}\hat{d}} + \lambda_{a\hat{d}b\hat{d}} + \lambda_{\hat{d}ab\hat{d}} + \lambda_{a\hat{d}\hat{d}b} + \lambda_{\hat{d}a\hat{d}b} + \lambda_{\hat{d}\hat{d}ab} = 6\lambda_{ab\hat{d}\hat{d}}$, $\lambda_{abc\hat{d}} + \lambda_{ab\hat{d}c} + \lambda_{a\hat{d}bc} + \lambda_{\hat{d}abc} = 4\lambda_{abc\hat{d}}$, and $\lambda_{a\hat{d}\hat{d}\hat{d}} + \lambda_{\hat{d}a\hat{d}\hat{d}} + \lambda_{\hat{d}\hat{d}a\hat{d}} + \lambda_{\hat{d}\hat{d}\hat{d}a} = 4\lambda_{a\hat{d}\hat{d}\hat{d}}$ one easily finds (writing the sums over the scalar indices explicitly):

$$
\begin{aligned}
\mathcal{L}_0 = & -\frac{1}{4} F_A^{\mu\nu} F_{\mu\nu}^A + \sum_{a=1}^{N_\phi} \frac{1}{2} D^\mu \phi_a D_\mu \phi_a + i\psi_j^\dagger \sigma^\mu D_\mu \psi_j \\
& - \frac{1}{2} \Big(\sum_{a=1}^{N_\phi} Y_{jk}^a \psi_j \zeta \psi_k \phi_a + \text{h.c.} \Big) - \sum_{a,b,c,d=1}^{N_\phi} \frac{1}{4!} \lambda_{abcd} \phi_a \phi_b \phi_c \phi_d \\
& - \frac{1}{2} (Y_{jk}^{\hat{d}} \psi_j \zeta \psi_k \phi_{\hat{d}} + \text{h.c.}) - 6 \sum_{a,b=1}^{N_\phi} \frac{1}{4!} \lambda_{ab\hat{d}\hat{d}} \phi_a \phi_b \phi_{\hat{d}} \phi_{\hat{d}} - 4 \sum_{a,b,c=1}^{N_\phi} \frac{1}{4!} \lambda_{abc\hat{d}} \phi_a \phi_b \phi_c \phi_{\hat{d}} \\
& - 4 \sum_{a=1}^{N_\phi} \frac{1}{4!} \lambda_{a\hat{d}\hat{d}\hat{d}} \phi_a \phi_{\hat{d}} \phi_{\hat{d}} \phi_{\hat{d}} - \frac{1}{4!} \lambda_{\hat{d}\hat{d}\hat{d}\hat{d}} \phi_{\hat{d}} \phi_{\hat{d}} \phi_{\hat{d}} \phi_{\hat{d}} \,.
\end{aligned}
\tag{3.1}
$$

A few comments are in order:

- The first two lines reproduce the Lagrangian $\mathcal{L}_0$ (2.2) with the original particle content without the dummy field.

- The terms in the third line reproduce the Lagrangian $\mathcal{L}_1$ (2.6) if one makes the following identifications:

$$
Y_{jk}^{\hat{d}} \phi_{\hat{d}} = (m_f)_{jk} \,, \quad \lambda_{ab\hat{d}\hat{d}} \phi_{\hat{d}} \phi_{\hat{d}} = 2m_{ab}^2 \,, \quad \lambda_{abc\hat{d}} \phi_{\hat{d}} = h_{abc} \,. \tag{3.2}
$$

  Note that we believe these are the correct relations while the notation below Eq. (21) in [12] is rather sloppy:

$$
Y_{jk}^{\hat{d}} = (m_f)_{jk} \,, \quad \lambda_{ab\hat{d}\hat{d}} = 2m_{ab}^2 \,, \quad \lambda_{abc\hat{d}} = h_{abc} \,. \tag{3.3}
$$

- The terms in the fourth line of Eq. (3.1) do not spoil the relations in Eq. (3.2) or (3.3). First of all, the second last term is only gauge invariant if ϕ_a is a gauge singlet. Furthermore, it is an effective tadpole term which can be removed by a shift of the field ϕ.[1] The last term is just a constant. In any case, contributions

[1] For the same reason such a term is not included in $\mathcal{L}_1$ in Eq. (2.6).

from the interactions in the fourth line to the β-functions of the other dimensionful parameters would involve at least one internal dummy line which gives a vanishing result.

The relations (3.3) have been used in Ref. [12] to derive the β-functions for the fermion masses from the known ones for the Yukawa interactions. Likewise, the β-functions for the scalar masses and the trilinear scalar couplings were obtained from the scalar quartic β-functions. This was achieved by removing contributions with a summation of $\hat{d}$-type indices and terms with $\hat{d}$ indices appearing on the generators Θ. However, a subtlety arises due to the wave-function renormalization of external dummy scalar lines which leads to effective tadpole contributions. Such contributions should be removed from the β-functions for the Yukawa interactions and quartic couplings but are not necessarily eliminated by just suppressing the summation over $\hat{d}$-indices and associated gauge couplings. For this reason, we re-examine in the current chapter all the β-functions for the dimensionful parameters by verifying the dummy method on a diagram by diagram basis.

3.2 Derivation of β-functions for dimensionful parameters with the dummy field method

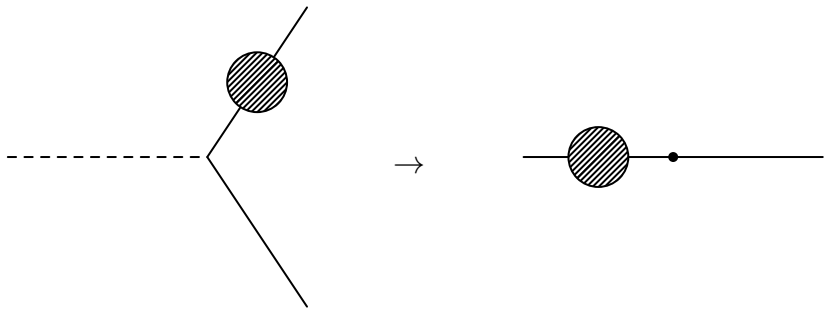
3.2.1 Fermion mass

The β-function of the fermion mass term can be obtained from the expressions of the Yukawa coupling by considering the external scalar as dummy field. We follow a diagrammatic approach; for each class of diagrams we provide the coupling structure and show the resulting diagram together with its expression after applying the dummy field method. In accord with the discussion in Sec. 3.1, the following mappings are performed:

$$a \to \hat{d}\,,\ Y^a \to Y^{\hat{d}} \to m_f\,,\ Y^{\dagger a} \to Y^{\dagger \hat{d}} \to m_f^\dagger\,,\ \lambda_{abcd} \to \lambda_{\hat{d}bcd} \to h_{bcd}\,.$$

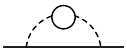
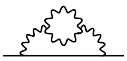
The fermion mass insertions will be represented by black dots in the Feynman diagrams. We recall that dummy scalars do neither couple to gauge bosons nor propagate. There are two generically different wave function correction diagrams contributing to the running of the Yukawa couplings: those stemming from either external fermions or scalars. For external fermions, the transition between the Yukawa coupling (with a dummy scalar) and the fermion mass term is represented in detail by Tab. 3.1 and looks as follows, where

the grey blob depicts all loop corrections to the external line:



(3.4)

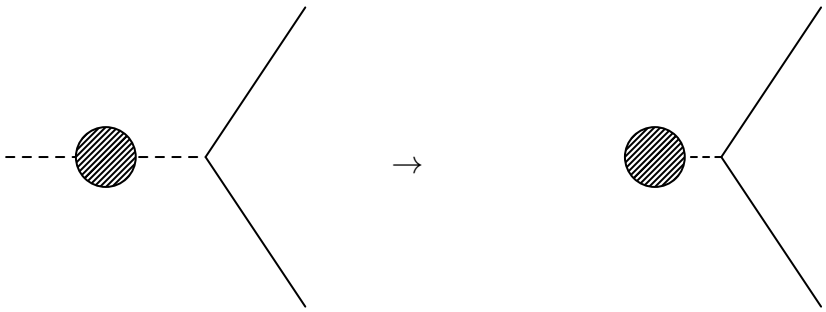
Table 3.1: The transition between the Yukawa coupling (with a dummy scalar) and the fermion mass term with all types of one- and two-loop corrections.

Blob	Original coupling structure		New coupling structure
	$Y_2^\dagger(F)Y^a + Y^aY_2(F)$	$\to$	$Y_2^\dagger(F)m_f + m_fY_2(F)$
	$\{C_2(F), Y^a\}$	$\to$	$\{C_2(F), m_f\}$
	$Y^bY_2(F)Y^{\dagger b}Y^a + Y^aY^{\dagger b}Y_2^\dagger(F)Y^b$	$\to$	$Y^bY_2(F)Y^{\dagger b}m_f + m_fY^{\dagger b}Y_2^\dagger(F)Y^b$
	$Y_2^{bc}(S)(Y^bY^{\dagger c}Y^a + Y^aY^{\dagger c}Y^b)$	$\to$	$Y_2^{bc}(S)(Y^bY^{\dagger c}m_f + m_fY^{\dagger c}Y^b)$
	$g_2^2(C_2(F)Y_2^\dagger(F)Y^a + Y^aY_2(F)C_2(F))$	$\to$	$g_2^2(C_2(F)Y_2^\dagger(F)m_f + m_fY_2(F)C_2(F))$
	$g_2^2(Y^bC_2(F)Y^{\dagger b}Y^a + Y^aY^{\dagger b}C_2(F)Y^b)$	$\to$	$g_2^2(Y^bC_2(F)Y^{\dagger b}m_f + m_fY^{\dagger b}C_2(F)Y^b)$
	$g^2C_2^{bc}(S)(Y^bY^{\dagger c}Y^a + Y^aY^{\dagger c}Y^b)$	$\to$	$g^2C_2^{bc}(S)(Y^bY^{\dagger c}m_f + m_fY^{\dagger c}Y^b)$
	$g^4\{\lvert C_2(F)\rvert^2, Y^a\}$	$\to$	$g^4\{\lvert C_2(F)\rvert^2, m_f\}$
	$g^4C_2(G)\{C_2(F), Y^a\}$	$\to$	$g^4C_2(G)\{C_2(F), m_f\}$
			
	$g^4(x_1C_2(G) + x_2S_2(F) + x_3S_2(S))\{C_2(F), Y^a\}$	$\to$	$g^4(x_1C_2(G) + x_2S_2(F) + x_3S_2(S))\{C_2(F), m_f\}$
			

Here, x_1, x_2 and x_3 are real numbers (cf. Eq. (2.42)).

The wave-function renormalization part stemming from the external scalar is completely different: after applying the replacement with dummy fields, we find only tadpole contributions. However, those are usually absorbed into a re-definition of the vacuum, i.e., they don't contribute to the β-function of the fermion mass term, and the correct

replacements are



$$Y^b Y_2^{ab}(S) \quad \to \quad 0\,, \tag{3.5}$$

$$Y^b \overline{H}_{ab}^2(S) \quad \to \quad 0\,, \tag{3.6}$$

$$Y^b H_{ab}^2(S) \quad \to \quad 0\,, \tag{3.7}$$

$$\Lambda_{ab}^2(S) Y^b \quad \to \quad 0\,, \tag{3.8}$$

$$g^2 Y^b Y_{ab}^{2F}(S) \quad \to \quad 0\,, \tag{3.9}$$

$$g^4 C_2^{ab}(S)\{C_2(F), Y^b\} \quad \to \quad 0\,, \tag{3.10}$$

$$g^4 C_2^{ab}(S) C_2^{bc}(S) Y^c \quad \to \quad 0\,, \tag{3.11}$$

$$g^4 C_2^{ab}(S)[x_1 C_2(G) + x_2 S_2(F) + x_3 S_2(S)] Y^b \quad \to \quad 0\,. \tag{3.12}$$

However, we find differences compared to the results of Ref. [12], where the following replacements have been made:

$$Y^b Y_2^{ab}(S) \to \frac{1}{2} Y^b \,\mathrm{Tr}[m_f^\dagger Y^b + Y^{\dagger b} m_f]\,, \tag{3.13}$$

$$Y^b \overline{H}_{ab}^2(S) \to \frac{1}{2} Y^b \,\mathrm{Tr}[m_f Y^{\dagger c} Y^b Y^{\dagger c} + m_f^\dagger Y^c Y^{\dagger b} Y^c]\,, \tag{3.14}$$

$$Y^b H_{ab}^2(S) \to \frac{1}{2} Y^b \,\mathrm{Tr}[m_f Y^{\dagger b} Y^b Y_2^\dagger(F) + m_f^\dagger Y^b Y_2(F)]\,, \tag{3.15}$$

$$\Lambda_{ab}^2(S) Y^b \to \frac{1}{6} h_{cde} \lambda_{bcde} Y^b\,, \tag{3.16}$$

$$g^2 Y^b Y_{ab}^{2F}(S) \to \frac{1}{2} g^2 Y^b \,\mathrm{Tr}[C_2(F)(m_f Y^{\dagger b} + Y^b m_f^\dagger)]\,, \tag{3.17}$$

$$g^4 C_2^{ab}(S)\{C_2(F), Y^b\} \to 0\,, \tag{3.18}$$

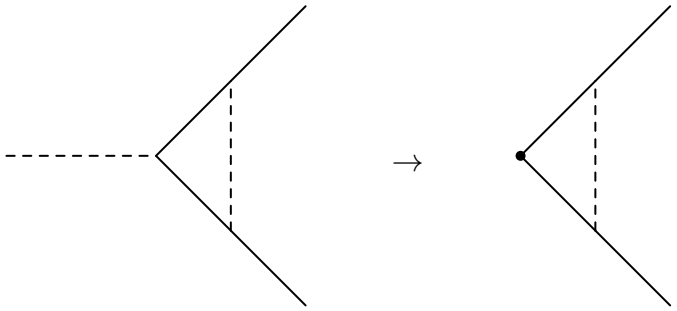
$$g^4 C_2^{ab}(S) C_2^{bc}(S) Y^c \to 0\,, \tag{3.19}$$

$$g^4 C_2^{ab}(S)[\ldots] Y^b \to 0\,. \tag{3.20}$$

Thus, there is a disagreement between contributions (3.5) and (3.13) entering the one-loop beta function for m_f. Furthermore, there are differences between contributions (3.6)–(3.9) and (3.14)–(3.17) affecting the two-loop beta function.

We now turn to the vertex corrections. At one-loop level, there is only one diagram

which needs to be considered:



$$Y^b Y^{\dagger a} Y^b \qquad\qquad Y^b m_f^\dagger Y^b \tag{3.21}$$

At the two-loop level, there are many more contributions. The explicit diagrams are given in Appendix A.1. While we completely agree with Ref. [12] for the one-loop vertex corrections, we also found differences at the two-loop level. Those stem from diagrams involving both, wave-function corrections of scalars as well as vertex corrections.

As we have mentioned in Sec. 2.3.5, there is a typo in the term $-4\kappa Y_2^{ac} Y^b Y^{\dagger c} Y^b$ of the 2-loop beta function for the Yukawa couplings in Refs. [7, 12], which then caused a mistake in the fermion mass beta function. Such a term would correspond to a diagram as depicted in Fig. 3.1, which is not one-particle-irreducible, and should be replaced by $-4\kappa Y_2^{bc} Y^b Y^{\dagger a} Y^c$, see (2.42). Applying the replacement $Y^{\dagger a} \to m_f^\dagger$ due to the dummy field method it then gives $-4\kappa Y_2^{bc}(S) Y^b m_f^\dagger Y^c$ (see (A.3)).

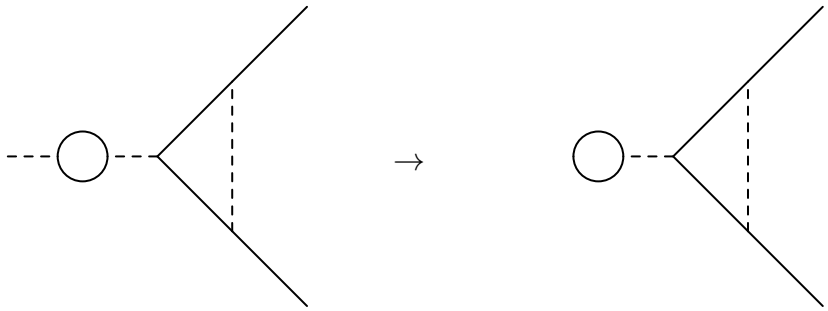


Figure 3.1: Two-loop diagram which does not contribute to the β-function for the Yukawa couplings (on the left) and, consequently, for the fermion mass when replacing the external scalar by a dummy field (on the right). A (non-zero) contribution corresponding to the tadpole diagram on the right hand side was included in Ref. [12]. It should be replaced by (A.3).

Summarising our results, we find that the one-loop β-functions of fermion masses have

one term less than the expression given in Ref. [12] and are given by the following form:

$$\beta^I_{m_f} = \frac{1}{2}\left[Y_2^\dagger(F)m_f + m_f Y_2(F)\right] + 2Y^b m_f^\dagger Y^b - 3g^2\{C_2(F), m_f\}. \tag{3.22}$$

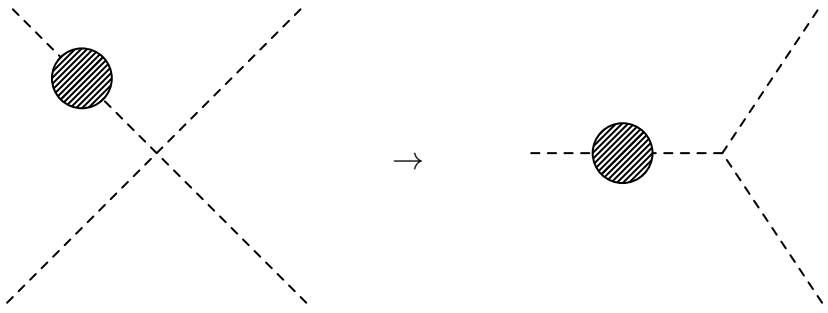
At the two-loop level, we obtain

$$\begin{aligned}
\beta^{II}_{m_f} = {} & 2Y^c Y^{\dagger b} m_f (Y^{\dagger c} Y^b - Y^{\dagger b} Y^c) - Y^b \left[Y_2(F) m_f^\dagger + m_f^\dagger Y_2^\dagger(F)\right] Y^b \\
& - \frac{1}{8}\left[Y^b Y_2(F) Y^{\dagger b} m_f + m_f Y^{\dagger b} Y_2^\dagger(F) Y^b\right] - \underline{4\kappa Y_2^{bc}(S) Y^b m_f^\dagger Y^c} \\
& - \frac{3}{2}\kappa Y_2^{bc}(S)(Y^b Y^{\dagger c} m_f + m_f Y^{\dagger c} Y^b) \\
& - 2h_{bcd} Y^b Y^{\dagger c} Y^d + 3g^2\{C_2(F), Y^b m_f^\dagger Y^b\} + 5g^2 Y^b \{C_2(F), m_f^\dagger\} Y^b \\
& - \frac{7}{4}g^2\left[C_2(F) Y_2^\dagger(F) m_f + m_f Y_2(F) C_2(F)\right] \\
& - \frac{1}{4}g^2\left[Y^b C_2(F) Y^{\dagger b} m_f + m_f Y^{\dagger b} C_2(F) Y^b\right] \\
& + 6g^2\left[t^{A*} m_f Y^{\dagger b} t^{A*} Y^b + Y^b t^A Y^{\dagger b} m_f t^A\right] + 6g^2 C_2^{bc}(S) Y^b m_f^\dagger Y^c \\
& - \frac{3}{2}g^4\{[C_2(F)]^2, m_f\} + \frac{9}{2}g^2 C_2^{bc}(S)(Y^b Y^{\dagger c} m_f + m_f Y^{\dagger c} Y^b) \\
& + g^4\left[-\frac{97}{6}C_2(G) + \frac{10}{3}\kappa S_2(F) + \frac{11}{12}S_2(S)\right]\{C_2(F), m_f\}\,.
\end{aligned} \tag{3.23}$$

Here, we disagree with the literature in several places, having removed the terms as discussed above and added one missing term, which is underlined. The numerical impact of these differences compared to earlier results is briefly discussed at the example of a specific model in Sec. 3.4.

3.2.2 Trilinear coupling

We now turn to the purely scalar interactions. The β-functions of the cubic interactions are obtained from the expressions for the quartic couplings by replacing one external scalar by a dummy field. The translation of the wave-function contributions between both cases is straightforward and can be summarized as follows:



$$\Lambda^{Y}_{abcd} = \frac{1}{6}\sum_{per} Y_2^{af}(S)\lambda_{fbcd} \quad \rightarrow \quad \Lambda^{Y}_{abc} = \frac{1}{2}\sum_{per} Y_2^{af}(S)h_{fbc}\,, \tag{3.24}$$

$$\Lambda^{S}_{abcd} = \sum_{i} C_2(i)\lambda_{abcd} \quad \rightarrow \quad \Lambda^{S}_{abc} = \sum_{i} C_2(i)h_{abc}\,, \tag{3.25}$$

$$\frac{1}{6}\sum_{per} \Lambda^2_{af}(S)\lambda_{fbcd} \quad \rightarrow \quad \frac{1}{2}\sum_{per} \Lambda^2_{af}(S)h_{fbc}\,, \tag{3.26}$$

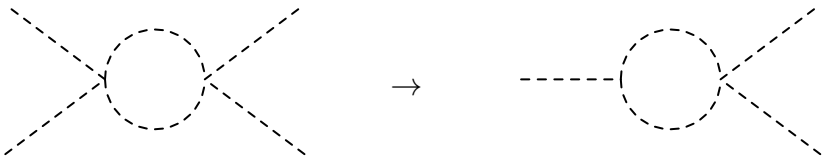
$$\frac{1}{6}\sum_{per} (3H^2_{af}(S) + 2\overline{H}^2_{af}(S))\lambda_{fbcd} \quad \rightarrow \quad \frac{1}{2}\sum_{per} (3H^2_{af}(S) + 2\overline{H}^2_{af}(S))h_{fbc}\,, \tag{3.27}$$

$$\frac{1}{6}\sum_{per} Y^{2F}_{af}(S)\lambda_{fbcd} \quad \rightarrow \quad \frac{1}{2}\sum_{per} Y^{2F}_{af}(S)h_{fbc}\,, \tag{3.28}$$

$$X\Lambda^{S}_{abcd} \quad \rightarrow \quad X\Lambda^{S}_{abc}\,, \tag{3.29}$$

$$\Lambda^{SS}_{abcd} = \sum_{i} |C_2(i)|^2\lambda_{abcd} \quad \rightarrow \quad \Lambda^{SS}_{abc} = \sum_{i} |C_2(i)|^2 h_{abc}\,. \tag{3.30}$$

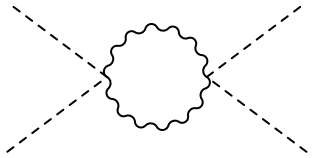
In this notation, the index i is summed over all uncontracted scalar indices. Furthermore, 'X' denotes the combination of group invariants multiplying Λ^S_{abcd} in Eq. (2.44). As discussed above, we have modified the parts which involve Yukawa or quartic couplings compared to Ref. [12]. The reason is that in these cases new contributions can be present due to off-diagonal wave-function renormalization corrections. There are three generically different vertex corrections which contribute to the RGE of the quartic interaction. However, since the dummy field does not interact with the gauge sector, those kind of contributions do not appear in the case of the cubic interaction. Therefore, the translation at the one-loop level becomes:



$$\Lambda^2_{abcd} = \frac{1}{8}\sum_{per} \lambda_{abef}\lambda_{efcd} \qquad \Lambda^2_{abc} = \frac{1}{2}\sum_{per} \lambda_{abef}h_{efc} \tag{3.31}$$



$$H_{abcd} = \frac{1}{4}\sum_{per} \mathrm{Tr}(Y^a Y^{\dagger b} Y^c Y^{\dagger d}) \qquad \begin{aligned} H_{abc} = {} & \tfrac{1}{2}\textstyle\sum_{per} \mathrm{Tr}(m_f Y^{\dagger a} Y^b Y^{\dagger c} \\ & + m_f^\dagger Y^a Y^{\dagger b} Y^c) \end{aligned} \tag{3.32}$$



$\rightarrow$



$$A_{abcd} = \frac{1}{8}\sum_{per}\{\theta^A,\theta^B\}_{ab}\{\theta^A,\theta^B\}_{cd} \qquad 0 \tag{3.33}$$

The explicit form of the two-loop diagrams as well as their expressions before and after the mapping are given in Appendix A.2. We find agreement between our results and those of Ref. [12] at the one- and two-loop level up to the differences from off-diagonal wave-function renormalizations. Thus, the β-functions at the one- and two-loop levels are

$$\beta^I_{h_{abc}} = \Lambda^2_{abc} - 8\kappa H_{abc} + 2\kappa\Lambda^Y_{abc} - 3g^2\Lambda^S_{abc}\,, \tag{3.34}$$

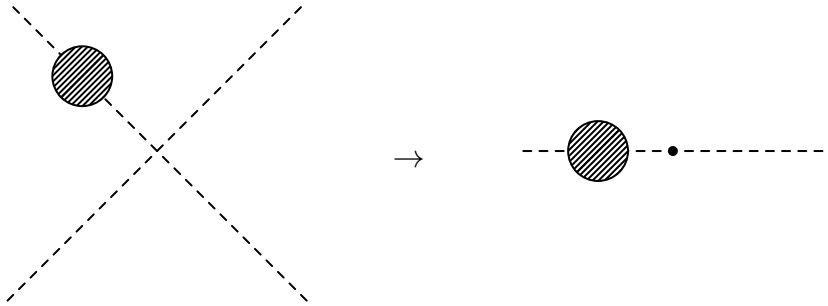
$$\begin{aligned}
\beta^{II}_{h_{abc}} =& \frac{1}{4}\sum_{per}\Lambda^2_{af}(S)h_{fbc} - \bar{\Lambda}^3_{abc} - 4\kappa\bar{\Lambda}^{2Y}_{abc} \\
&+ \kappa\left[8\bar{H}^{\lambda m}_{abc} + 8\bar{H}^h_{abc} - \frac{1}{2}\sum_{per}\left[3H^2_{af}(S) + 2\bar{H}^2_{af}(S)\right]h_{fbc}\right] \\
&+ 4\kappa(H^Y_{abc} + 2\bar{H}^Y_{abc} + 2H^3_{abc}) \\
&+ g^2\left[2\bar{\Lambda}^{2S}_{abc} - 6\Lambda^{2g}_{abc} + 4\kappa(H^S_{abc} - H^F_{abc}) + 5\kappa\sum_{per}Y^{2F}_{af}(S)h_{fbc}\right] \\
&- g^4\left\{\left[\frac{35}{3}C_2(G) - \frac{10}{3}\kappa S_2(F) - \frac{11}{12}S_2(S)\right]\Lambda^S_{abc}\right. \\
&\left. \quad -\frac{3}{2}\Lambda^{SS}_{abc} - \frac{5}{2}A^\lambda_{abc} - \frac{1}{2}\bar{A}^\lambda_{abc} + 4\kappa(B^Y_{abc} - 10\bar{B}^Y_{abc})\right\}\,,
\end{aligned} \tag{3.35}$$

where the invariants are defined in Eqs. (3.31)–(3.32) and (A.10)–(A.26).

3.2.3 Scalar mass

Finally, we turn to the terms involving two scalar couplings. The procedure is very similar to the case of the cubic scalar coupling, and we find the following relations for the

wave-function corrections to the terms appearing for the quartic scalar coupling:



$$\Lambda^Y_{abcd} = \frac{1}{6}\sum_{per} Y_2^{ae}(S)\lambda_{ebcd} \quad \rightarrow \quad \Lambda^Y_{ab} = 2\sum_{per} Y_2^{ae}(S) m^2_{eb}\,, \tag{3.36}$$

$$\Lambda^S_{abcd} = \sum_i C_2(i)\lambda_{abcd} \quad \rightarrow \quad \Lambda^S_{ab} = 2\sum_i C_2(i) m^2_{ab}\,, \tag{3.37}$$

$$\frac{1}{6}\sum_{per} \Lambda^2_{ae}(S)\lambda_{ebcd} \quad \rightarrow \quad 2\sum_{per} \Lambda^2_{ae}(S) m^2_{eb}\,, \tag{3.38}$$

$$\frac{1}{6}\sum_{per} (3H^2_{af}(S) + 2\overline{H}^2_{af}(S))\lambda_{fbcd} \quad \rightarrow \quad 2\sum_{per} (3H^2_{af}(S) + 2\overline{H}^2_{af}(S)) m^2_{fb}\,, \tag{3.39}$$

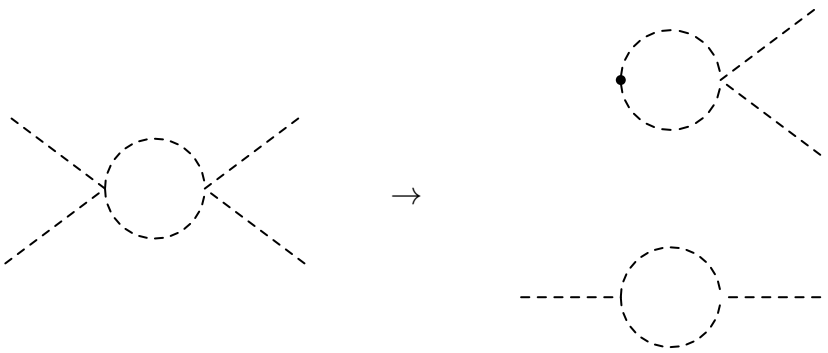
$$\frac{1}{6}\sum_{per} Y^{2F}_{af}(S)\lambda_{fbcd} \quad \rightarrow \quad 2\sum_{per} Y^{2F}_{af}(S) m^2_{fb}\,, \tag{3.40}$$

$$X\Lambda^S_{abcd} \quad \rightarrow \quad X\Lambda^S_{ab}\,, \tag{3.41}$$

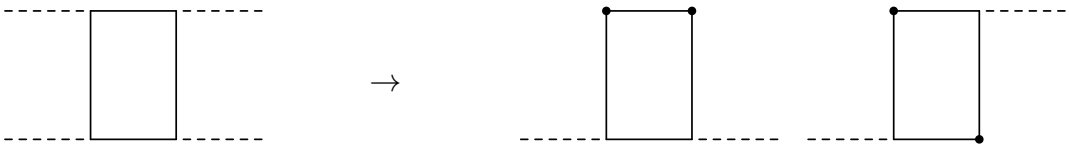
$$\Lambda^{SS}_{abcd} = \sum_i |C_2(i)|^2 \lambda_{abcd} \quad \rightarrow \quad \Lambda^{SS}_{ab} = 2\sum_i |C_2(i)|^2 m^2_{ab}\,. \tag{3.42}$$

Again, 'X' denotes the combination of group invariants multiplying Λ^S_{abcd} in Eq. (2.44).

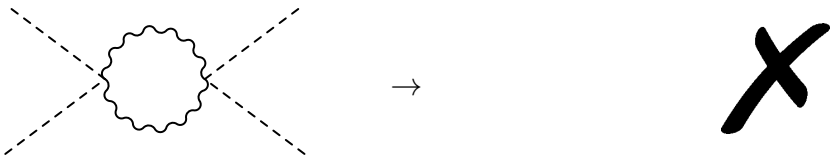
As before, we need to consider the three generically different diagrams which contribute to the running of the quartic functions. The one with vector bosons in the loop vanishes due to inserting dummy fields, while for the other two diagrams additional terms arise.



$$\Lambda^2_{abcd} = \frac{1}{8}\sum_{per} \lambda_{abef}\lambda_{efcd} \qquad 2m^2_{ef}\lambda_{abef} + 2h_{aef}h_{bef} \tag{3.43}$$



$$H_{abcd} = \frac{1}{4}\sum_{per} \mathrm{Tr}(Y^a Y^{\dagger b} Y^c Y^{\dagger d}) \qquad H_{ab} = \sum_{per} \mathrm{Tr}(Y^a Y^{\dagger b} m_f m_f^\dagger + Y^{\dagger a} Y^b m_f^\dagger m_f + \tfrac{1}{2} Y^{\dagger a} m_f Y^{b\dagger} m_f + \tfrac{1}{2} Y^a m_f^\dagger Y^b m_f^\dagger) \tag{3.44}$$



$$A_{abcd} = \frac{1}{8}\sum_{per} \{\theta^A, \theta^B\}_{ab} \{\theta^A, \theta^B\}_{cd} \qquad 0 \tag{3.45}$$

The two-loop diagrams are given in Appendix A.3. We also find agreement between our results here and the ones given in Ref. [12] up to the wave-function renormalization. One needs to be careful about some factor of $\frac{1}{2}$ due to $\beta_{m^2_{ab}} = \frac{1}{2}\beta_{\lambda_{ab\hat{d}\hat{d}}}$, which we have included here explicitly into the definition of the β-function for m^2_{ab}, while it has been partially absorbed into other definitions in Ref. [12]. Thus, with our conventions the one- and two-loop β-functions read

$$\beta^I_{m^2_{ab}} = m^2_{ef}\lambda_{abef} + h_{aef}h_{bef} - 4\kappa H_{ab} + \kappa\Lambda^Y_{ab} - \frac{3}{2}g^2\Lambda^S_{ab}\,, \tag{3.46}$$

$$\begin{aligned}
\beta^{II}_{m^2_{ab}} =& \frac{1}{2}\sum_{per}\Lambda^2_{af}(S)m^2_{fb} - \frac{1}{2}\bar{\Lambda}^3_{ab} - 2\kappa\bar{\Lambda}^{2Y}_{ab} \\
&+ \kappa\left[4\bar{H}^\lambda_{ab} - \sum_{per}\left[3H^2_{af}(S) + 2\bar{H}^2_{af}(S)\right]m^2_{fb}\right] \\
&+ 2\kappa(H^Y_{ab} + 2\bar{H}^Y_{ab} + 2H^3_{ab}) \\
&+ g^2\left[\bar{\Lambda}^{2S}_{ab} - 3\Lambda^{2g}_{ab} + 2\kappa(H^S_{ab} - H^F_{ab}) + 10\kappa\sum_{per}Y^{2F}_{af}(S)m^2_{fb}\right] \\
&- g^4\left\{\left[\frac{35}{6}C_2(G) - \frac{5}{3}\kappa S_2(F) - \frac{11}{24}S_2(S)\right]\Lambda^S_{ab}\right. \\
&\left. \quad -\frac{3}{4}\Lambda^{SS}_{ab} - \frac{5}{4}A^\lambda_{ab} - \frac{1}{4}\bar{A}^\lambda_{ab} + 2\kappa(B^Y_{ab} - 10\bar{B}^Y_{ab})\right\}\,,
\end{aligned} \tag{3.47}$$

where we used the objects defined in Eqs. (3.43)–(3.44) and (A.27)–(A.43).

3.3 Comparison with supersymmetric RGEs

We have now re-derived the full one- and two-loop RGEs for the dimensionful parameters. While we agree with Ref. [12] concerning the bilinear and cubic scalar interactions (up to wave-function renormalization), we find differences in the fermion mass terms. Therefore, we want to double-check our results by comparing to those obtained using supersymmetric (SUSY) RGEs. The general RGEs for a softly broken SUSY model have been independently calculated in Refs. [13, 26, 27] and the general agreement between all results has been discussed in Ref. [28]. Thus, there is hardly any doubt that these RGEs are absolutely correct. Therefore, we want to test our results with a model in which we enforce SUSY relations among parameters. After a translation from the $\overline{\text{MS}}$ to the $\overline{\text{DR}}$ scheme one should recover the SUSY results.

Since a supersymmetric extension of the SM yields many couplings which are generically all of the same form, we opt a theory with a $U(1)$ gauge symmetry. Our toy model contains one vector superfield $\hat{B}$ and three chiral superfields

$$\hat{H}_d: \; Q = -\frac{1}{2}\,, \tag{3.48}$$

$$\hat{H}_u: \; Q = \frac{1}{2}\,, \tag{3.49}$$

$$\hat{S}: \; Q = 0\,, \tag{3.50}$$

where Q denotes the electric charge. The superpotential consists of two terms[2]

$$W = \lambda \hat{H}_u \hat{H}_d \hat{S} + \mu \hat{H}_u \hat{H}_d \tag{3.51}$$

and the soft-breaking terms are

$$\begin{aligned} -L_{SB} = &\left(B_\mu H_d H_u + T_\lambda H_d H_u S + \frac{1}{2} M_B \tilde{B}^2 + \text{h.c.} \right) + \\ &m_{H_d}^2 |H_d|^2 + m_{H_u}^2 |H_u|^2 + m_S^2 |S|^2\,. \end{aligned} \tag{3.52}$$

This model contains all of the relevant generic structure we need to test. Making use of the results of Ref. [13], which are also implemented in the package `SARAH`, we find the following expressions for the one- and two-loop RGEs for the different parts of the model:

1. Gauge Couplings

$$\beta_g^{(1)} = \frac{1}{2} g^3\,, \tag{3.53}$$

[2]We neglect terms $\sim \hat{S}^2$, $\hat{S}^3$ which are not essential for our argument.

$$\beta_g^{(2)} = \frac{1}{2}g^3\Big(-2|\lambda|^2 + g^2\Big)\,. \tag{3.54}$$

2. Gaugino Mass Parameters

$$\beta_{M_B}^{(1)} = g^2 M_B\,, \tag{3.55}$$
$$\beta_{M_B}^{(2)} = 2g^2\Big(g^2 M_B + \lambda^*\Big(-M_B\lambda + T_\lambda\Big)\Big)\,. \tag{3.56}$$

3. Trilinear Superpotential Parameters

$$\beta_\lambda^{(1)} = \lambda\Big(3|\lambda|^2 - g^2\Big)\,, \tag{3.57}$$
$$\beta_\lambda^{(2)} = \lambda\Big(-6|\lambda|^4 + g^2|\lambda|^2 + g^4\Big)\,. \tag{3.58}$$

4. Bilinear Superpotential Parameters

$$\beta_\mu^{(1)} = -\mu\Big(-2|\lambda|^2 + g^2\Big)\,, \tag{3.59}$$
$$\beta_\mu^{(2)} = \mu\Big(-4|\lambda|^4 + g^4\Big)\,. \tag{3.60}$$

5. Trilinear Soft-Breaking Parameters

$$\beta_{T_\lambda}^{(1)} = 2g^2 M_B\lambda - \Big(-9|\lambda|^2 + g^2\Big)T_\lambda\,, \tag{3.61}$$
$$\beta_{T_\lambda}^{(2)} = -30|\lambda|^4 T_\lambda + g^2|\lambda|^2\Big(-2M_B\lambda + 3T_\lambda\Big) + g^4\Big(-4M_B\lambda + T_\lambda\Big)\,. \tag{3.62}$$

6. Bilinear Soft-Breaking Parameters

$$\beta_{B_\mu}^{(1)} = 2g^2 M_B\mu + 4|\lambda|^2 B_\mu + 4\mu\lambda^* T_\lambda - g^2 B_\mu\,, \tag{3.63}$$
$$\beta_{B_\mu}^{(2)} = \Big(2g^2|\lambda|^2 - 8|\lambda|^4 + g^4\Big)B_\mu - 2\mu\Big(10|\lambda|^2\lambda^* T_\lambda + 2g^4 M_B + g^2 M_B|\lambda|^2\Big)\,. \tag{3.64}$$

7. Soft-Breaking Scalar Masses

$$\beta_{m_{H_d}^2}^{(1)} = -2g^2|M_B|^2 + 2\Big(m_{H_d}^2 + m_{H_u}^2 + m_S^2\Big)|\lambda|^2 + 2|T_\lambda|^2 - \frac{1}{2}g^2\Big(-m_{H_d}^2 + m_{H_u}^2\Big)\,, \tag{3.65}$$

$$\begin{aligned}\beta_{m_{H_d}^2}^{(2)} &= 6g^4|M_B|^2 - 8\Big(m_{H_d}^2 + m_{H_u}^2 + m_S^2\Big)|\lambda|^4 - 16|\lambda|^2|T_\lambda|^2 \\ &\quad + g^4 m_{H_d}^2 + g^2|\lambda|^2\Big(m_{H_u}^2 - m_{H_d}^2\Big)\,,\end{aligned} \tag{3.66}$$

$$\beta_{m_{H_u}^2}^{(1)} = -2g^2|M_B|^2 + 2\Big(m_{H_d}^2 + m_{H_u}^2 + m_S^2\Big)|\lambda|^2 + 2|T_\lambda|^2 + \frac{1}{2}g^2\Big(-m_{H_d}^2 + m_{H_u}^2\Big)\,, \tag{3.67}$$

$$\beta^{(2)}_{m^2_{H_u}} = 6g^4|M_B|^2 - 8\Big(m^2_{H_d} + m^2_{H_u} + m^2_S\Big)|\lambda|^4 - 16|\lambda|^2|T_\lambda|^2 + g^4 m^2_{H_u} + g^2|\lambda|^2\Big(- m^2_{H_u} + m^2_{H_d}\Big)\,, \tag{3.68}$$

$$\beta^{(1)}_{m^2_S} = 2\Big(\Big(m^2_{H_d} + m^2_{H_u} + m^2_S\Big)|\lambda|^2 + |T_\lambda|^2\Big)\,, \tag{3.69}$$

$$\beta^{(2)}_{m^2_S} = 2\Big(- 4\Big(m^2_{H_d} + m^2_{H_u} + m^2_S\Big)|\lambda|^4 + \lambda^*\Big(g^2 M_B^*\Big(2M_B\lambda - T_\lambda\Big) + \lambda\Big(- 8|T_\lambda|^2 + g^2\Big(m^2_{H_d} + m^2_{H_u} + m^2_S\Big)\Big)\Big) + g^2 T^*_\lambda\Big(- M_B\lambda + T_\lambda\Big)\Big)\,. \tag{3.70}$$

As before, we have suppressed the pre-factors $\frac{1}{16\pi^2}$ and $\frac{1}{(16\pi^2)^2}$ for the one- and two-loop β-functions. With these functions, the running of all parameters at the one- and two-loop level is fixed. However, for later comparison, it will be convenient to know the β-functions for some products of parameters as well. That is done by applying the chain rule:

$$\beta^{(1)}_{\frac{1}{8}g^2} = \frac{1}{4} g \beta^{(1)}_g = \frac{1}{8} g^4\,, \tag{3.71}$$

$$\beta^{(2)}_{\frac{1}{8}g^2} = \frac{1}{4} g \beta^{(2)}_g = -\frac{1}{4}|\lambda|^2 g^4 + \frac{1}{8} g^6\,, \tag{3.72}$$

$$\beta^{(1)}_{|\lambda|^2} = \lambda(\beta^{(1)}_\lambda)^* + \lambda^* \beta^{(1)}_\lambda = 2|\lambda|^2\Big(3|\lambda|^2 - g^2\Big)\,, \tag{3.73}$$

$$\beta^{(2)}_{|\lambda|^2} = \lambda(\beta^{(2)}_\lambda)^* + \lambda^* \beta^{(2)}_\lambda = 2|\lambda|^2\Big(- 6|\lambda|^4 + g^2|\lambda|^2 + g^4\Big)\,, \tag{3.74}$$

$$\beta^{(1)}_{|\lambda|^2 - \frac{1}{4}g^2} = 2\lambda\beta^{(1)}_\lambda - \frac{1}{2} g \beta^{(1)}_g = -2|\lambda|^2 g^2 + 6|\lambda|^4 - \frac{1}{4} g^4\,, \tag{3.75}$$

$$\beta^{(2)}_{|\lambda|^2 - \frac{1}{4}g^2} = 2\lambda\beta^{(2)}_\lambda - \frac{1}{2} g \beta^{(2)}_g = -12|\lambda|^6 - \frac{1}{4} g^6 + \frac{5}{2} g^4|\lambda|^2 + 2g^2|\lambda|^4\,, \tag{3.76}$$

$$\beta^{(1)}_{\lambda\mu^*} = \lambda(\beta^{(1)}_\mu)^* + \mu^* \beta^{(1)}_\lambda = \mu^*\lambda(-2g^2 + 5|\lambda|^2)\,, \tag{3.77}$$

$$\beta^{(2)}_{\lambda\mu^*} = \lambda(\beta^{(2)}_\mu)^* + \mu^* \beta^{(2)}_\lambda = \mu^*\lambda(-10|\lambda|^4 + 2g^4 + g^2|\lambda|^2)\,, \tag{3.78}$$

$$\beta^{(1)}_{|\mu|^2} = \mu(\beta^{(1)}_\mu)^* + \mu^* \beta^{(1)}_\mu = -2|\mu|^2\Big(- 2|\lambda|^2 + g^2\Big)\,, \tag{3.79}$$

$$\beta^{(2)}_{|\mu|^2} = \mu(\beta^{(2)}_\mu)^* + \mu^* \beta^{(2)}_\mu = 2|\mu|^2\Big(- 4|\lambda|^4 + g^4\Big)\,. \tag{3.80}$$

We now consider the same model written as non-supersymmetric version. In this case, we have one gauge boson B, four fermions

$$\tilde{H}_d:\ Q = -\frac{1}{2}\,, \tag{3.81}$$

$$\tilde{H}_u:\ Q = \frac{1}{2}\,, \tag{3.82}$$

$$\tilde{S}:\ Q = 0\,, \tag{3.83}$$

$$\tilde{B}:\ Q = 0\,, \tag{3.84}$$

and three scalars

$$H_d: \; Q = -\frac{1}{2}\,, \tag{3.85}$$

$$H_u: \; Q = \frac{1}{2}\,, \tag{3.86}$$

$$S: \; Q = 0\,. \tag{3.87}$$

The full potential for this models involves a substantial amount of different couplings

$$\begin{aligned} V = & \left(T_1 S|H_d|^2 + T_2 S|H_u|^2 + T_3 H_d H_u S + \text{h.c.}\right) \\ & + m_1^2|H_d|^2 + m_2^2|H_u|^2 + m_3^2|S|^2 \\ & \quad + \lambda_1|S|^2|H_d|^2 + \lambda_2|S|^2|H_u|^2 + \lambda_3|H_d|^2|H_u|^2 + \lambda_4|H_d|^4 + \lambda_5|H_u|^4 \\ & + \left(M_1\tilde{B}\tilde{B} + M_2\tilde{H}_d\tilde{H}_u + BH_dH_u + \text{h.c.}\right) \\ & + \left(Y_1 S\tilde{H}_d\tilde{H}_u + Y_2\tilde{S}H_d\tilde{H}_u + Y_3\tilde{S}\tilde{H}_d H_u - \frac{1}{\sqrt{2}}g_d\tilde{B}\tilde{H}_d H_d^* + \frac{1}{\sqrt{2}}g_u\tilde{B}\tilde{H}_u H_u^* + \text{h.c.}\right). \end{aligned} \tag{3.88}$$

We think that this rather lengthy form justifies our approach to consider only a toy model, but not a realistic SUSY theory. We have neglected couplings that would be allowed by the symmetry of this theory, but vanish as we match to the SUSY model. In particular, the CP even and odd parts of the complex field S will run differently unless specific (SUSY) relations among the parameters exist. Therefore, one would need to decompose S into its real components and write down all possible potential terms involving these fields. However, we are only interested in the β functions in the SUSY limit where no splitting between these fields is introduced. Therefore, we retain the more compact notation in (3.88). We can now make use of our revised expressions to calculate the RGEs up to two-loop. For this purpose, we modified the packages `SARAH` and `PyR@TE` accordingly. The lengthy expressions in the general case are given in Appendix B. In order to make connection to the SUSY case, we can make the following associations between parameters of these models:

$$g_d = g_u = \; g\,, \tag{3.89}$$

$$Y_1 = Y_2 = Y_3 = \; \lambda\,, \tag{3.90}$$

$$\lambda_1 = \lambda_2 = \; |\lambda|^2\,, \tag{3.91}$$

$$\lambda_3 = \; |\lambda|^2 - \frac{1}{4}g^2\,, \tag{3.92}$$

$$\lambda_4 = \lambda_5 = \; \frac{1}{8}g^2\,, \tag{3.93}$$

$$T_1 = T_2 = \; \mu^*\lambda\,, \tag{3.94}$$

$$T_3 = T_\lambda\,, \tag{3.95}$$
$$M_1 = \frac{1}{2}M_B\,, \tag{3.96}$$
$$M_2 = \mu\,, \tag{3.97}$$
$$m_1^2 = m_{H_d}^2 + |\mu|^2\,, \tag{3.98}$$
$$m_2^2 = m_{H_u}^2 + |\mu|^2\,, \tag{3.99}$$
$$m_3^2 = m_S^2\,, \tag{3.100}$$
$$B = B_\mu\,. \tag{3.101}$$

By doing that, we obtain the following RGEs:

1. Gauge Couplings

$$\beta_g^{(1)} = \frac{1}{2}g^3\,, \tag{3.102}$$
$$\beta_g^{(2)} = \frac{1}{2}g^3\Big(-2|\lambda|^2 + g^2\Big)\,. \tag{3.103}$$

2. Quartic scalar couplings

$$\beta_{\lambda_1}^{(1)} = \beta_{\lambda_2}^{(1)} = 2|\lambda|^2\Big(3|\lambda|^2 - g^2\Big)\,, \tag{3.104}$$
$$\beta_{\lambda_1}^{(2)} = \beta_{\lambda_2}^{(2)} = 2|\lambda|^2\Big(-6|\lambda|^4 + \frac{5}{4}g^2|\lambda|^2 + \frac{17}{8}g^4\Big)\,, \tag{3.105}$$

$$\beta_{\lambda_3}^{(1)} = -2g^2|\lambda|^2 + 6|\lambda|^4 - \frac{1}{4}g^4\,, \tag{3.106}$$
$$\beta_{\lambda_3}^{(2)} = -12|\lambda|^6 - \frac{17}{8}g^6 + \frac{31}{4}g^4|\lambda|^2 + g^2|\lambda|^4\,, \tag{3.107}$$

$$\beta_{\lambda_4}^{(1)} = \beta_{\lambda_5}^{(1)} = \frac{1}{8}g^4\,, \tag{3.108}$$
$$\beta_{\lambda_4}^{(2)} = \beta_{\lambda_5}^{(2)} = \frac{7}{8}g^4|\lambda|^2 - g^2|\lambda|^4 + \frac{1}{16}g^6\,. \tag{3.109}$$
$$\tag{3.110}$$

3. Yukawa Couplings

$$\beta_{g_d}^{(1)} = \beta_{g_u}^{(1)} = \frac{1}{2}g^3\,, \tag{3.111}$$
$$\beta_{g_d}^{(2)} = \beta_{g_u}^{(2)} = \frac{1}{2}g^3\Big(-\frac{22}{8}|\lambda|^2 + \frac{11}{8}g^2\Big)\,, \tag{3.112}$$

$$\beta_{Y_1}^{(1)} = \lambda\Big(3|\lambda|^2 - g^2\Big)\,, \tag{3.113}$$
$$\beta_{Y_1}^{(2)} = \lambda\Big(-6|\lambda|^4 + \frac{1}{4}g^2|\lambda|^2 + \frac{11}{8}g^4\Big)\,, \tag{3.114}$$

$$\beta_{Y_2}^{(1)} = \beta_{Y_3}^{(1)} = \lambda\Big(3|\lambda|^2 - g^2\Big)\,, \tag{3.115}$$

$$\beta_{Y_2}^{(2)} = \beta_{Y_3}^{(2)} = \lambda\Big(-6|\lambda|^4 + \frac{11}{8}g^2|\lambda|^2 + \frac{13}{16}g^4\Big)\,. \tag{3.116}$$

4. Fermion Mass Terms

$$\beta_{M_1}^{(1)} = \frac{1}{2}g^2 M_B\,, \tag{3.117}$$

$$\beta_{M_1}^{(2)} = g^2\Big(\frac{9}{8}g^2 M_B + \lambda^*\Big(-M_B\lambda + T_\lambda\Big)\Big)\,, \tag{3.118}$$

$$\beta_{M_2}^{(1)} = -\mu\Big(-2|\lambda|^2 + g^2\Big)\,, \tag{3.119}$$

$$\beta_{M_2}^{(2)} = \mu\Big(-4|\lambda|^4 + \frac{11}{8}g^4 - \frac{1}{4}g^2|\lambda|^2\Big)\,. \tag{3.120}$$

5. Trilinear Scalar couplings

$$\beta_{T_1}^{(1)} = \beta_{T_2}^{(1)} = \lambda\mu^*\Big(-2g^2 + 5|\lambda|^2\Big)\,, \tag{3.121}$$

$$\beta_{T_1}^{(2)} = \beta_{T_2}^{(2)} = \lambda\mu^*\Big(-10|\lambda|^4 + \frac{17}{4}g^4 + 2g^2|\lambda|^2\Big)\,, \tag{3.122}$$

$$\beta_{T_3}^{(1)} = 2g^2 M_B\lambda - \Big(-9|\lambda|^2 + g^2\Big)T_\lambda\,, \tag{3.123}$$

$$\beta_{T_3}^{(2)} = -30|\lambda|^4 T_\lambda + g^2|\lambda|^2\Big(-2M_B\lambda + 3T_\lambda\Big) + g^4\Big(-4M_B\lambda + \frac{7}{4}T_\lambda\Big)\,. \tag{3.124}$$

6. Scalar Mass Terms

$$\beta_B^{(1)} = 2g^2 M_B\mu + 4B_\mu|\lambda|^2 + 4\mu\lambda^* T_\lambda - B_\mu g^2\,, \tag{3.125}$$

$$\beta_B^{(2)} = \Big(-8|\lambda|^4 + \frac{5}{2}g^2|\lambda|^2 + \frac{7}{4}g^4\Big)B_\mu - 2\mu\Big(10|\lambda|^2\lambda^* T_\lambda + 2g^4 M_B + g^2|\lambda|^2 M_B\Big)\,, \tag{3.126}$$

$$\begin{aligned}\beta_{m_1^2}^{(1)} =& -2g^2|M_B|^2 + 2|\lambda|^2\Big(m_{H_d}^2 + m_{H_u}^2 + m_S^2\Big) + 2|T_\lambda|^2 + \frac{1}{2}g^2\Big(m_{H_d}^2 - m_{H_u}^2\Big)\\ &+ \Big(4|\lambda|^2 - 2g^2\Big)|\mu|^2\,,\end{aligned} \tag{3.127}$$

$$\begin{aligned}\beta_{m_1^2}^{(2)} =& \frac{11}{2}g^4|M_B|^2 - 8\Big(m_{H_d}^2 + m_{H_u}^2 + m_S^2\Big)|\lambda|^4 - 16|T_\lambda|^2|\lambda|^2\\ &+ \frac{1}{2}|\lambda|^2 g^2\Big(2m_{H_u}^2 - m_{H_d}^2\Big) + \frac{1}{4}g^4\Big(+2m_{H_u}^2 + 9m_{H_d}^2\Big)\\ &+ |\mu|^2\Big(\frac{3}{2}|\lambda|^2 g^2 - 8|\lambda|^4 + \frac{17}{4}g^4\Big)\,,\end{aligned} \tag{3.128}$$

$$\begin{aligned}\beta_{m_2^2}^{(1)} =& -2g^2|M_B|^2 + 2|\lambda|^2\Big(m_{H_d}^2 + m_{H_u}^2 + m_S^2\Big) + 2|T_\lambda|^2 + \frac{1}{2}g^2\Big(m_{H_u}^2 - m_{H_d}^2\Big)\\ &+ \Big(4|\lambda|^2 - 2g^2\Big)|\mu|^2\,,\end{aligned} \tag{3.129}$$

$$\begin{aligned}\beta^{(2)}_{m_2^2} =&\frac{11}{2}g^4|M_B|^2 - 8\Big(m^2_{H_d}+m^2_{H_u}+m^2_S\Big)|\lambda|^4 - 16|T_\lambda|^2|\lambda|^2\\ &+\frac{1}{2}|\lambda|^2 g^2\Big(2m^2_{H_d}-m^2_{H_u}\Big)+\frac{1}{4}g^4\Big(+2m^2_{H_d}+9m^2_{H_u}\Big)\\ &+|\mu|^2\Big(\frac{3}{2}|\lambda|^2g^2-8|\lambda|^4+\frac{17}{4}g^4\Big),\end{aligned} \tag{3.130}$$

$$\beta^{(1)}_{m_3^2} =2\Big(\Big(m^2_{H_d}+m^2_{H_u}+m^2_S\Big)|\lambda|^2+|T_\lambda|^2\Big), \tag{3.131}$$

$$\begin{aligned}\beta^{(2)}_{m_3^2} =&-2\Big(4\Big(m^2_{H_d}+m^2_{H_u}+m^2_S\Big)\lambda^2+g^2\mu^2\Big)\lambda^{*\,2}-2g^2\Big(\lambda^2\mu^{*\,2}+T^*_\lambda\Big(M_B\lambda-T_\lambda\Big)\Big)\\ &+\lambda^*\Big(g^2\lambda\Big(2m^2_{H_d}+2m^2_{H_u}+4|M_B|^2+8|\mu|^2+m^2_S\Big)-2\Big(8\lambda T^*_\lambda+g^2M^*_B\Big)T_\lambda\Big).\end{aligned} \tag{3.132}$$

We see that all one-loop expressions as well as the two-loop β-function of the gauge coupling agree with the SUSY expressions. The remaining discrepancies at two-loop are due to the differences between the $\overline{\text{MS}}$ and $\overline{\text{DR}}$ scheme. In order to translate the non-SUSY expressions to the $\overline{\text{DR}}$-scheme, we need to apply the following shifts [29]

$$g_{d,u} \;\to\; g_{d,u}\Big(1-\frac{1}{16\pi^2}\cdot\frac{1}{8}g^2\Big), \tag{3.133}$$

$$Y_1 \;\to\; Y_1\Big(1+\frac{1}{16\pi^2}\cdot\frac{1}{4}g^2\Big), \tag{3.134}$$

$$Y_{2,3} \;\to\; Y_{2,3}\Big(1-\frac{1}{16\pi^2}\cdot\frac{1}{8}g^2\Big), \tag{3.135}$$

$$\lambda_3 \;\to\; \lambda_3-\frac{1}{16\pi^2}\cdot\frac{1}{4}g^4, \tag{3.136}$$

$$\lambda_{4,5} \;\to\; \lambda_{4,5}-\frac{1}{16\pi^2}\cdot\frac{1}{8}g^4, \tag{3.137}$$

$$M_2 \;\to\; M_2\Big(1+\frac{1}{16\pi^2}\cdot\frac{1}{4}g^2\Big), \tag{3.138}$$

which have to be applied to the expressions of the one-loop β functions to obtain the corresponding two-loop shifts. In addition, one must take into account that for the quartic couplings and the Yukawa couplings an additional shift appears 'on the left hand side' of the expression, e.g.

$$\beta_Y^{\overline{\text{DR}}}=\frac{d}{dt}Y^{\overline{\text{DR}}}=\frac{d}{dt}\left(Y^{\overline{\text{MS}}}\Big(1+\frac{c}{16\pi^2}g^2\Big)\right)=\beta_Y^{\overline{\text{MS}}}\left(1+\frac{c}{16\pi^2}g^2\right)+2gY^{\overline{\text{MS}}}\frac{c}{16\pi^2}\beta_g \tag{3.139}$$

with some coefficient c depending on the charges of the involved fields.
We find the following shifts for the different couplings:

$$\Delta\lambda_1=-\frac{1}{2}g^2|\lambda|^4-\frac{9}{4}g^4|\lambda|^2, \tag{3.140}$$

$$\Delta\lambda_3=\frac{15}{8}g^6-\frac{21}{4}g^4|\lambda|^2+g^2|\lambda|^4, \tag{3.141}$$

$$\Delta\lambda_4 = \frac{1}{16}g^6 - \frac{9}{8}g^4|\lambda|^2 + g^2|\lambda|^4\,, \tag{3.142}$$

$$\Delta g_d = -\frac{3}{16}g^5 + \frac{3}{8}g^3|\lambda|^2\,, \tag{3.143}$$

$$\Delta Y_1 = \frac{3}{4}g^2\lambda|\lambda|^2 - \frac{3}{8}g^4\lambda\,, \tag{3.144}$$

$$\Delta Y_2 = \frac{3}{16}g^4\lambda - \frac{3}{8}g^2\lambda|\lambda|^2\,, \tag{3.145}$$

$$\Delta M_1 = -\frac{1}{8}g^4 M_B\,, \tag{3.146}$$

$$\Delta M_2 = \frac{1}{4}g^2\mu|\lambda|^2 - \frac{3}{8}g^4\mu\,, \tag{3.147}$$

$$\Delta T_1 = -\frac{1}{4}g^2\lambda\Big(4|\lambda|^2 + 9g^2\Big)\mu^*\,, \tag{3.148}$$

$$\Delta T_3 = -\frac{3}{4}g^4 T_\lambda\,, \tag{3.149}$$

$$\Delta B = -\frac{1}{4}Bg^2\Big(2|\lambda|^2 + 3g^2\Big)\,, \tag{3.150}$$

$$\Delta m_1^2 = -\frac{1}{4}g^2\Big(-2g^2|M_B|^2 + 2|\lambda|^2\Big(3|\mu|^2 + m_{H_d}^2\Big) + g^2\Big(2m_{H_u}^2 + 5m_{H_d}^2 + 9|\mu|^2\Big)\Big)\,, \tag{3.151}$$

$$\Delta m_2^2 = -\frac{1}{4}g^2\Big(-2g^2|M_B|^2 + 2|\lambda|^2\Big(3|\mu|^2 + m_{H_u}^2\Big) + g^2\Big(2m_{H_d}^2 + 5m_{H_u}^2 + 9|\mu|^2\Big)\Big)\,, \tag{3.152}$$

$$\Delta m_3^2 = g^2\Big(2\lambda^2\mu^{*2} + 2\mu^2(\lambda^*)^2 + |\lambda|^2\Big(-8|\mu|^2 + m_S^2\Big)\Big)\,. \tag{3.153}$$

This gives a complete agreement between the two-loop β-functions of both calculations. Thus, our revised results for the RGEs of a general quantum field theory are confirmed.

3.4 Numerical impact

3.4.1 Running of fermion mass terms

We briefly want to discuss the numerical impact on the changes in the β-function for the fermion mass term. Differences in the running will only appear in models in which the Lagrangian contains fermionic terms

$$\mathcal{L} \supset YSf_1f_2 + \mu f_1 f_2 + \text{h.c.} \tag{3.154}$$

with a Yukawa-like coupling Y between two Weyl fermions f_1, f_2 and a scalar S as well as a fermion mass term μ. Both terms can only be present if S is a gauge singlet and if f_1, f_2 form a vector-like fermion pair. As concrete example, we consider the case of heavy

top-like states and a real singlet, i.e.

$$T': \quad (\mathbf{3},\mathbf{1})_{-\frac{1}{3}}\,, \tag{3.155}$$

$$\bar{T}': \quad (\overline{\mathbf{3}},\mathbf{1})_{\frac{1}{3}}\,, \tag{3.156}$$

$$S: \quad (\mathbf{1},\mathbf{1})_{0}\,, \tag{3.157}$$

and the potential reads

$$\begin{aligned} V =& V_{SM} + \frac{1}{4}\lambda_S S^4 + \frac{1}{2}\lambda_{SH}|H|^2 S^2 + \kappa_{SH}|H|^2 S + \frac{1}{3}\kappa_S S^3 + \frac{1}{2}m_S^2 S^2 \\ &+ \left(Y_T S \bar{T}' T' + \mu_T \bar{T}' T' + \text{h.c.}\right)\,. \end{aligned} \tag{3.158}$$

The one- and two-loop β-functions are computed using our corrected expression and read

$$\beta^{(1)}_{\mu_T} = 2Y_T^2\mu_T^* - \frac{2}{5}\Big(20g_3^2 + g_1^2\Big)\mu_T + \mu_T|Y_T|^2\,, \tag{3.159}$$

$$\begin{aligned} \beta^{(2)}_{\mu_T} =& \frac{1}{450}\Big(667g_1^4 - 240g_1^2g_3^2 - 46600g_3^4\Big)\mu_T + \frac{4}{15}\Big(2g_1^2\mu_T + 40g_3^2\mu_T - 15\kappa_S Y_T\Big)|Y_T|^2 \\ &- \frac{37}{4}\mu_T|Y_T|^4 + \frac{2}{15}Y_T^2\Big(-105|Y_T|^2 + 8\Big(20g_3^2 + g_1^2\Big)\Big)\mu_T^*\,, \end{aligned} \tag{3.160}$$

while the differences compared to the old results are

$$\Delta\beta^{(1)}_{\mu_T} = -6\mu_T Y_T^2\,, \tag{3.161}$$

$$\Delta\beta^{(2)}_{\mu_T} = Y_T(-2\kappa_{HS}\lambda_{HS} - \kappa_S\lambda_S + \mu_T Y_T(27Y_T^2 - 2g_1^2 - 40g_3^2) - 12\mu_T^* Y_T|Y_T|^2)\,. \tag{3.162}$$

The numerical impact of this difference is depicted in Fig. 3.2 where we assumed a value of 1 TeV for μ_T at the scale $Q = 1$ TeV and used different values Y_T. As expected from Eq. (3.161), the discrepancy between the old and new results rapidly grows with increasing Y_T. Thus, the correction in the RGEs is crucial for instance to study grand unified theories which also predict additional vector-like fermions with large Yukawa couplings to a gauge singlet.

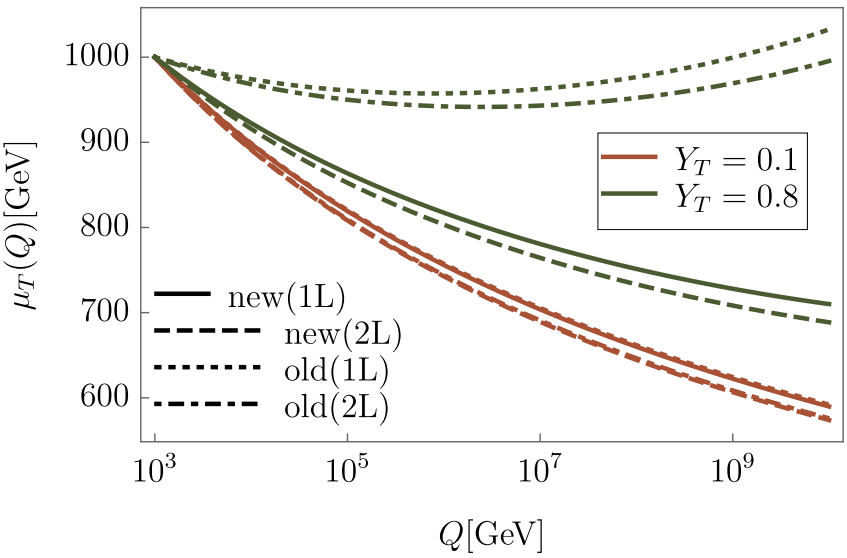


Figure 3.2: The running mass μ_T of the vector-like top partners at one- and two-loop level for two different choices of the Yukawa coupling Y_T. Here, we show the results using the incorrect ('old') expressions in literature as well as our derived expressions ('new'). The other parameters are set to $\lambda_{HS} = 0$, $\lambda_S = 1$, $\kappa_{HS} = \kappa = 1$ TeV.

3.4.2 Off-diagonal wave-function renormalization

We now turn to the numerical impact of the off-diagonal wave-function renormalization which is not included in the previous works. For this purpose, we consider the general Two-Higgs-Doublet-Model type-III with the following scalar potential:

$$\begin{aligned} V =& m_1^2|H_1|^2 + m_2^2|H_2|^2 + \lambda_1|H_1|^4 + \lambda_2|H_2|^4 + \lambda_3|H_1|^2|H_2|^2 + \lambda_4|H_2^\dagger H_1|^2 \\ &+ \left(\frac{1}{2}\lambda_5(H_2^\dagger H_1)^2 + \lambda_6|H_1|^2(H_1^\dagger H_2) + \lambda_7|H_2|^2(H_1^\dagger H_2) - M_{12}^2 H_1^\dagger H_2 + \text{h.c.}\right) . \end{aligned} \tag{3.163}$$

Here the parameters M_{12}^2, $\lambda_{5,6,7}$ are complex whereas all the other parameters have to be real in order to yield a real potential.
Furthermore, the Yukawa interactions can be written as

$$\begin{aligned} \mathcal{L}_Y = -\Big(& \bar{Q}_L Y_d H_1^\dagger d_R + \bar{L}_L Y_e H_1^\dagger e_R - \bar{Q}_L Y_u H_2 u_R \\ &+ \bar{Q}_L \epsilon_d H_2^\dagger d_R + \bar{L}_L \epsilon_e H_2^\dagger e_R - \bar{Q}_L \epsilon_u H_1 u_R + \text{h.c.}\Big) . \end{aligned} \tag{3.164}$$

Due to the presence of all Yukawa interactions allowed by gauge invariance, the anomalous dimensions of the Higgs doublets H_1 and H_2 are no longer diagonal, but a mixing is induced proportional to $\text{Tr}(Y_i \epsilon_i)$ with $i = e, d, u$. If we neglect for the moment all terms involving either the electroweak gauge couplings (g_1, g_2), a lepton or down-quark Yukawa

coupling (Y_d, Y_e, ϵ_d, ϵ_e), the one-loop β-functions for the quartic coupling read

$$\begin{aligned}
\beta^{(1)}_{\lambda_1} &= 24\lambda_1^2 + 2\lambda_3^2 + 2\lambda_3\lambda_4 + \lambda_4^2 + |\lambda_5|^2 + 12|\lambda_6|^2 \\
&\quad + 12\lambda_1 \mathrm{Tr}\Big(\epsilon_u\epsilon_u^\dagger\Big) + \underline{6\mathrm{Re}(\lambda_6)\mathrm{Tr}\Big(\epsilon_u Y_u^\dagger\Big)} - 6\mathrm{Tr}\Big(\epsilon_u\epsilon_u^\dagger\epsilon_u\epsilon_u^\dagger\Big)\,, \qquad (3.165)\\
\beta^{(1)}_{\lambda_2} &= 24\lambda_2^2 + 2\lambda_3^2 + 2\lambda_3\lambda_4 + \lambda_4^2 + |\lambda_5|^2 + 12|\lambda_7|^2 \\
&\quad + 12\lambda_2 \mathrm{Tr}\Big(Y_u Y_u^\dagger\Big) + \underline{6\mathrm{Re}(\lambda_7)\mathrm{Tr}\Big(\epsilon_u Y_u^\dagger\Big)} - 6\mathrm{Tr}\Big(Y_u Y_u^\dagger Y_u Y_u^\dagger\Big)\,, \qquad (3.166)\\
\beta^{(1)}_{\lambda_3} &= 2|\lambda_5|^2 + 2\lambda_4^2 + 4\lambda_3^2 + \underline{6\mathrm{Re}\Big(\lambda_6+\lambda_7\Big)\mathrm{Tr}\Big(\epsilon_u Y_u^\dagger\Big)} + 4|\lambda_7|^2 + 4|\lambda_6|^2 + 16\mathrm{Re}(\lambda_6\lambda_7^*) \\
&\quad + 6\lambda_3 \mathrm{Tr}\Big(\epsilon_u\epsilon_u^\dagger + Y_u Y_u^\dagger\Big) + 4\Big(\lambda_1+\lambda_2\Big)\Big(3\lambda_3+\lambda_4\Big) - 12\mathrm{Tr}\Big(\epsilon_u\epsilon_u^\dagger Y_u Y_u^\dagger\Big)\,, \qquad (3.167)\\
\beta^{(1)}_{\lambda_4} &= 4\lambda_4\Big(2\lambda_3+\lambda_1+\lambda_2+\lambda_4\Big) + 8|\lambda_5|^2 + \underline{6\mathrm{Re}\Big(\lambda_6+\lambda_7\Big)\mathrm{Tr}\Big(\epsilon_u Y_u^\dagger\Big)} + 2\lambda_6^*\Big(5\lambda_6+\lambda_7\Big) \\
&\quad + 2\lambda_7^*\Big(5\lambda_7+\lambda_6\Big) + 6\lambda_4 \mathrm{Tr}\Big(\epsilon_u\epsilon_u^\dagger + Y_u Y_u^\dagger\Big) - 12\mathrm{Tr}\Big(\epsilon_u Y_u^\dagger Y_u\epsilon_u^\dagger\Big)\,, \qquad (3.168)\\
\beta^{(1)}_{\lambda_5} &= 2\Big(2\Big(2\lambda_3+3\lambda_4+\lambda_1+\lambda_2\Big)\lambda_5 + 5\lambda_6^{*\,2} + 2\lambda_6^*\lambda_7^* + 5\lambda_7^{*\,2} + \underline{3\Big(\lambda_6^*+\lambda_7^*\Big)\mathrm{Tr}\Big(\epsilon_u Y_u^\dagger\Big)} \\
&\quad + 3\lambda_5\Big(\mathrm{Tr}\Big(\epsilon_u\epsilon_u^\dagger\Big) + \mathrm{Tr}\Big(Y_u Y_u^\dagger\Big)\Big) - 6\mathrm{Tr}\Big(\epsilon_u Y_u^\dagger\epsilon_u Y_u^\dagger\Big)\Big)\,, \qquad (3.169)\\
\beta^{(1)}_{\lambda_6} &= 24\lambda_1\lambda_6 + 6\lambda_3\Big(\lambda_6+\lambda_7\Big) + 4\lambda_4\Big(2\lambda_6+\lambda_7\Big) + \lambda_5^*\Big(10\lambda_6^*+2\lambda_7^*\Big) + \underline{3\lambda_5^*\mathrm{Tr}\Big(\epsilon_u Y_u^\dagger\Big)} \\
&\quad + \underline{3\Big(2\lambda_1+\lambda_3+\lambda_4\Big)\mathrm{Tr}\Big(Y_u\epsilon_u^\dagger\Big)} + 3\lambda_6 \mathrm{Tr}\Big(3\epsilon_u\epsilon_u^\dagger + Y_u Y_u^\dagger\Big) - 12\mathrm{Tr}\Big(\epsilon_u\epsilon_u^\dagger Y_u\epsilon_u^\dagger\Big)\,, \qquad (3.170)\\
\beta^{(1)}_{\lambda_7} &= 4\lambda_4\lambda_6 + 8\Big(3\lambda_2+\lambda_4\Big)\lambda_7 + 6\lambda_3\Big(\lambda_6+\lambda_7\Big) + \lambda_5^*\Big(10\lambda_7^*+2\lambda_6^*\Big) + \underline{3\lambda_5^*\mathrm{Tr}\Big(\epsilon_u Y_u^\dagger\Big)} \\
&\quad + \underline{3\Big(2\lambda_2+\lambda_3+\lambda_4\Big)\mathrm{Tr}\Big(Y_u\epsilon_u^\dagger\Big)} + 3\lambda_7 \mathrm{Tr}\Big(3Y_u Y_u^\dagger + \epsilon_u\epsilon_u^\dagger\Big) - 12\mathrm{Tr}\Big(Y_u\epsilon_u^\dagger Y_u Y_u^\dagger\Big)\,. \qquad (3.171)
\end{aligned}$$

The underlined terms stem from the off-diagonal wave-function renormalization and are missing in the results of Refs. [6, 7, 8, 12]. In Fig. 3.3 we show the numerical impact of the additional one-loop contributions on the running of the quartic couplings for two different points. The chosen sets of the quartic couplings, $\tan\beta$ and M_{12} result in a tree-level Higgs mass of 125 GeV[3]. We see that the additional terms can lead to sizeable differences already for $\epsilon_{u,33} = 0.5$ and small $\tan\beta = 2$. This is due to $\mathrm{Tr}(\epsilon_u Y_u^\dagger)$. When increasing $\epsilon_{u,33}$ to 1 and $\tan\beta = 50$, one obtains $\mathrm{Tr}(\epsilon_u Y_u^\dagger) \simeq 1$ and the impact on the running couplings is tremendous.

Of course, there are also differences at the two-loop level. Those read within the same approximation:

$$\begin{aligned}
\Delta\beta^{(2)}_{\lambda_1} &= \frac{1}{4}(6\lambda_5^*\lambda_6^{*\,2} + 6\lambda_6((2\lambda_2+\lambda_3+\lambda_4)\lambda_7^* + \lambda_5\ (\lambda_6+\lambda_7)) \\
&\quad + \lambda_6\epsilon_t Y_t(-27\epsilon_u^2 - 27Y_u^2 + 80g_3^2) + \lambda_6^*(12\lambda_2\lambda_7 + 24\lambda_1\lambda_6 - 27\epsilon_u^3 Y_t - 27\epsilon_t Y_u^3
\end{aligned}$$

[3]While it is in principle possible to renormalise the Higgs sector of the THDM-III on-shell, large radiative corrections can occur when extracting the $\overline{\mathrm{MS}}$ parameters which enter the RGEs [30]. Therefore, the given example is meant as an illustration on the difference in the running, but the input parameters in the running will change when including those corrections.

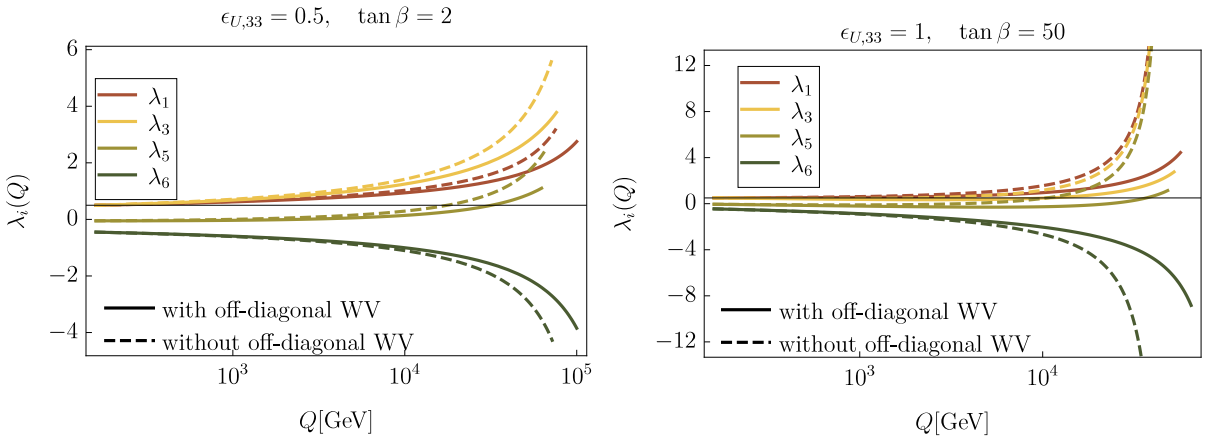


Figure 3.3: The running of different quartic couplings in the THDM-III with and without the contributions of off-diagonal wave-function renormalization to the β-functions of the quartic couplings. Here, we have used the input parameters $\lambda_1 = \lambda_3 = \lambda_4 = 0.5$, $\lambda_5 = -0.05$, $\lambda_6 = \lambda_7 = -0.45$, $\tan\beta = 2$ and $M_{12} = 500^2$ GeV^2 at $Q = m_t$. On the left, we have used $\epsilon_{U,33} = 0.5$, $\lambda_2 = 0.5$, $\tan\beta = 2$, and on the right $\epsilon_{U,33} = 1$, $\lambda_2 = 0.15$, $\tan\beta = 50$. All other ϵ_i are zero.

$$+ 6(\lambda_3 + \lambda_4)(2\lambda_6 + \lambda_7) + 6\lambda_5^*\lambda_7^* + 80\epsilon_t g_3^2 Y_t)) \,, \tag{3.172}$$

$$\begin{aligned}\Delta\beta_{\lambda_2}^{(2)} = \frac{1}{4}(&\lambda_7(6\lambda_5(\lambda_6 + \lambda_7) + \epsilon_t Y_t(-27\epsilon_u^2 - 27Y_u^2 + 80g_3^2\ )) \\ &+ 6\lambda_6^*((2\lambda_1 + \lambda_3 + \lambda_4)\lambda_7 + \lambda_5^*\lambda_7^*) + \lambda_7^*(12\lambda_1\lambda_6 + 24\lambda_2\lambda_7 - 27\epsilon_u^3 Y_t - 27\epsilon_t Y_u^3 \\ &+ 6(\lambda_3 + \lambda_4)(2\lambda_7 + \lambda_6) + 6\lambda_5^*\lambda_7^* + 80\epsilon_t g_3^2 Y_t)) \,,\end{aligned} \tag{3.173}$$

$$\begin{aligned}\Delta\beta_{\lambda_3}^{(2)} = \frac{1}{4}(&(\lambda_6 + \lambda_7)(6\lambda_5(\lambda_6 + \lambda_7) + \epsilon_t Y_t(-27\epsilon_u^2 - 27Y_u^2 + 80g_3^2)) + 6\lambda_5^*\lambda_6^{*\,2} + 6\lambda_5^*\lambda_7^{*\,2} \\ &+ 12(\lambda_5^*\lambda_6^*\lambda_7^* + (2\lambda_2 + \lambda_3 + \lambda_4)(|\lambda_7|^2 + |\lambda_6|^2) + 2(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4)\mathrm{Re}(\lambda_7^*\lambda_6)) \\ &+ (\lambda_7^* + \lambda_6^*)(-27\epsilon_u^3 Y_t - 27\epsilon_t Y_u^3 + 80\epsilon_t g_3^2 Y_t) \,,\end{aligned} \tag{3.174}$$

$$\begin{aligned}\Delta\beta_{\lambda_4}^{(2)} = \frac{1}{4}(&(\lambda_6 + \lambda_7)(6\lambda_5(\lambda_6 + \lambda_7) + \epsilon_t Y_t(-27\epsilon_u^2 - 27Y_u^2 + 80g_3^2)) + 6\lambda_5^*\lambda_6^{*\,2} + 6\lambda_5^*\lambda_7^{*2} \\ &+ 12(\lambda_5^*\lambda_6^*\lambda_7^* + (2\lambda_2 + \lambda_3 + \lambda_4)(|\lambda_7|^2 + |\lambda_6|^2) + 2(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4)\mathrm{Re}(\lambda_7^*\lambda_6)) \\ &+ (\lambda_7^* + \lambda_6^*)(-27\epsilon_u^3 Y_t - 27\epsilon_t Y_u^3 + +80\epsilon_t g_3^2 Y_t) \,,\end{aligned} \tag{3.175}$$

$$\begin{aligned}\Delta\beta_{\lambda_5}^{(2)} = \frac{1}{2}(&\lambda_6^* + \lambda_7^*)(6(2\lambda_1 + \lambda_3 + \lambda_4)\lambda_6^* + 6(2\lambda_2 + \lambda_3 + \lambda_4)\lambda_7^* + 6\lambda_5(\lambda_6 + \lambda_7) \\ &+ \epsilon_t Y_t(-27\epsilon_u^2 - 27Y_u^2 + 80g_3^2)) \,,\end{aligned} \tag{3.176}$$

$$\begin{aligned}\Delta\beta_{\lambda_6}^{(2)} = \frac{1}{4}(&(2\lambda_1 + \lambda_3 + \lambda_4)(12(\lambda_1\lambda_6 + \lambda_2\lambda_7) - 27(\epsilon_u^3 Y_t + \epsilon_t Y_u^3) + 6(\lambda_3 + \lambda_4)(\lambda_6 + \lambda_7) \\ &+ 80\epsilon_t g_3^2 Y_t) + \lambda_5^*(12(2\lambda_1 + \lambda_3 + \lambda_4)\lambda_6^* + 12(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4)\lambda_7^* \\ &+ 6\lambda_5(\lambda_6 + \lambda_7) + \epsilon_t Y_t(-27(\epsilon_u^2 + Y_u^2) + 80g_3^2))) \,,\end{aligned} \tag{3.177}$$

$$\begin{aligned}\Delta\beta_{\lambda_7}^{(2)} = \frac{1}{4}(&6\lambda_5^*\lambda_6^{*,2} + \lambda_6(6(2\lambda_2 + \lambda_3 + \lambda_4)\lambda_7^* + 6\lambda_5(\lambda_6 + \lambda_7) + \epsilon_t Y_t(-27(\epsilon_u^2 + Y_u^2) \\ &+ 80g_3^2)) + \lambda_6^*(12\lambda_2\lambda_7 + 24\lambda_1\lambda_6 - 27(\epsilon_u^3 Y_t + \epsilon_t Y_u^3) + 6(\lambda_3 + \lambda_4)(2\lambda_6 + \lambda_7) \\ &+ 6\lambda_5^*\lambda_7^* + 80\epsilon_t g_3^2 Y_t)) \,.\end{aligned} \tag{3.178}$$

Chapter 4

Conclusions and Outlook

In Part I of this thesis, we have revisited the RGEs for general renormalizable gauge theories with the goal to present the current state-of-the-art and to correct some mistakes in the literature. In particular, the known expressions for the scalar quartic couplings [8, 12] assume a diagonal wave-function renormalization which is not appropriate for models with mixing in the scalar sector. We therefore have corrected/generalized the expressions for the β-functions of the quartic couplings in (2.43) and (2.44). Very recently, a related paper appeared on the arxiv [31] which confirms our findings concerning the couplings in the scalar sector. Furthermore, we have carefully re-examined the dummy field method and have provided a detailed description of it, which has so far been missing in the literature. We then have used this method to re-derive the β-functions for the dimensionful parameters (fermion masses, scalar masses, and the cubic scalar couplings). For cubic scalar couplings and scalar masses, the only differences to Ref. [12] are due to the aforementioned off-diagonal wave-function renormalization. However, discrepancies for the fermion mass β-functions in [12] have been found and reconciled in (3.22) and (3.23). We have also performed an independent cross-check of our results using well-tested supersymmetric RGEs and we have found complete agreement.

We have illustrated the numerical impact on the changes in the β-function for the fermion mass terms using a toy model with a heavy vector-like fermion pair coupled to a scalar gauge singlet. Unsurprisingly, the correction to the running of the fermion mass rapidly grows with increasing Yukawa coupling. Thus it is crucial to use the corrected RGEs if one wants to study for instance grand unified theories which predict additional vector-like fermions with large Yukawa couplings to a gauge singlet. In addition, we have demonstrated the importance of the correction to the β-functions of the scalar quartic couplings using a general type-III Two-Higgs-Doublet-Model. As can be seen in Fig. 3.3, the corrections to the running couplings are non-negligible and can become very large in certain regions of the parameter space.

All the corrected expressions have been implemented in updated versions of the Math-

ematica package SARAH and the Python package PyR@TE. We hope that this study will be a useful resource in which all the relevant information on the two-loop β-functions is at hand in one place.

Recently, a paper by C. Poole and A. E. Thomsen [32] has appeared which also presents the general beta functions for the dimensionless couplings. The derivation and organization of these results is different from the approach in the papers by Machacek and Vaughn. Moreover, the results presented in the paper by Poole and Thomsen automatically satisfy certain consistency conditions (so-called Weyl consistency conditions) which provide relations between the beta functions for the gauge couplings, Yukawa couplings and quartic scalar couplings. Therefore, we plan to perform a detailed comparison between those results and our results. In addition, we intend to use the dummy field method in order to obtain the full set of beta functions available in this formalism.

Part II

Some aspects of BSM Higgs phenomenology

Part II of this thesis is devoted to the Higgs boson phenomenology beyond the Standard Model (SM). It is crucial to understand the mechanism of electroweak symmetry breaking (EWSB) which is an essential ingredient of the SM and its extensions. Thus, an important task is to investigate the Higgs potential, in particular, the triple Higgs vertex where some large deviations from the SM may arise in Beyond Standard Model (BSM) scenarios. The key process for this study is the Higgs boson pair production that is sensitive to deviations in the trilinear Higgs vertex and to the presence of new scalar particles with Higgs-like properties. We consider the class of models with extra dimensions (such as the Randall-Sundrum scenario) which predicts the existence of an additional scalar (the radion) that can take part in the Higgs boson pair production and enhance the cross section as well as mimic deviations in the trilinear Higgs coupling. In order to study the processes that occur in the framework of the RS model, it has been implemented in the FeynRules package for the derivation of the Feynman rules, and has been successfully tested. The possibility to obtain constraints for the new physics from Higgs measurements with the `Lilith` tool has also been considered, in particular, the "reduced couplings" have been introduced, and working examples have been obtained. The main goal of this research in progress remains the study of scalar pair production.

The outline of Part II of this thesis is as follows:

- In Chapter 5 we discuss the EWSB mechanism in the Standard Model (SM) and the reasons of the search for extensions of the SM.

- In Chapter 6 we provide the description of the Randall-Sundrum (RS) model as one of the attractive extensions of the SM, being spared from the hierarchy problem and predicting a new scalar particle, very similar to the Higgs boson.

- Chapter 7 summarizes the work with such tools as `FeynRules` and `Lilith`. `FeynRules` is a package for the derivation of the Feynman rules, in which the RS model has been successfully implemented and the automated calculation of the underlying Feynman rules has been achieved, opening the opportunity to study various processes in the RS model, in particular, the scalar pair production, using the interface to MadGraph (the work on it is in progress). `Lilith` is engaged as a tool for constraining the parameters of the RS model from Higgs measurements at the LHC. The reduced couplings have been introduced in an appropriate form and working examples have been obtained.

- In Conclusions and Outlook we provide a discussion and give an outlook on the

future work.

- The interaction Lagrangian, Feynman rules, widths and branching ratios in the RS model are provided in Appendix C.
- The created `FeynRules` model-file for the RS model is included in Appendix D.

Chapter 5

The Standard Model — our current understanding of nature and reasons for BSM

In this chapter we give some background on the Standard Model and Electroweak Symmetry Breaking (EWSB) mechanism, following mostly lectures by E. Boos [33] (Sec. 5.1). In Sec. 5.2 we discuss the flaws of the SM, in particular, the hierarchy problem.

5.1 The Standard Model and Electroweak Symmetry Breaking

The Standard Model (SM) of particle physics is the modern theory that classifies all known elementary particles and describes three of the four known fundamental forces. It has been tested in a wide class of experiments, including a very large number of precision measurements, and has shown a full compatibility with the experimental data. The SM is rightfully considered to be the best-tested theory in the history of science.

The SM is a renormalizable gauge quantum field theory with spontaneously broken electroweak symmetry. It is based on a symmetry group $SU(3)_c \times SU(2)_L \times U(1)_Y$, where the color group $SU(3)_c$ represents the QCD, $SU(2)_L$ describes the weak interaction, and $U(1)_Y$ — the so-called hypercharge interaction. The symmetry $SU(2)_L \times U(1)_Y$ is broken to $U(1)_{\text{em}}$ by the EWSB mechanism, which we describe below.

Let us first consider the following Lagrangian of a scalar model as a simple example,

$$\mathcal{L} = \partial_\mu \phi^\dagger \partial^\mu \phi - \mu^2 \phi^\dagger \phi - \lambda (\phi^\dagger \phi)^2 \,, \tag{5.1}$$

where ϕ is a complex scalar field, the second term is a mass term, and the third term

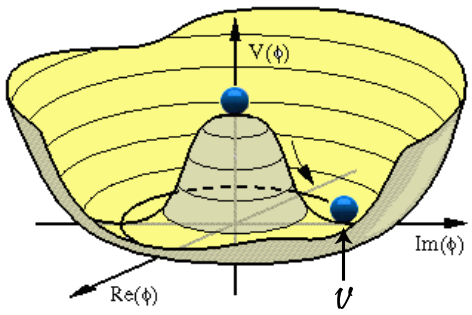


Figure 5.1: The Higgs ("Mexican hat") potential. A small perturbation transfers the system from the higher energy symmetric local maximum ($v = 0$, $\phi_0 = 0$) to the lower energy asymmetric local minimum $\left(v = \frac{|\mu^2|}{\lambda},\ \phi_0 = \pm\frac{v}{\sqrt{2}}\right)$.

describes the self-coupling of the scalar fields with a constant λ.

We recall that Lagrangians of a gauge theory must be invariant under the global phase shift

$$\phi \to e^{i\alpha}\phi\,, \tag{5.2}$$

where α is a constant. Postulating an x-dependence of α, we introduce a local gauge transformation:

$$\phi(x) \to e^{i\alpha(x)}\phi(x)\,. \tag{5.3}$$

Let us focus on the case of negative μ^2 in (5.1): $\mu^2 \to -|\mu^2| < 0$. The potential

$$V(\phi) = \mu^2\phi^\dagger\phi + \lambda(\phi^\dagger\phi)^2 \tag{5.4}$$

has a non trivial minimum

$$\left.\frac{dV}{d\phi^\dagger}\right|_{\phi_0} = -|\mu^2|\phi_0 + 2\lambda(\phi_0^\dagger\phi_0)\phi_0 = 0 \quad \Rightarrow \quad |\phi_0| = \sqrt{\frac{|\mu^2|}{2\lambda}} = \frac{v}{\sqrt{2}} > 0\,, \tag{5.5}$$

where v is the ground state of the potential, which is called the vacuum expectation value (VEV).

A concrete value for the vacuum solution, for example $\phi_0 = \frac{v}{\sqrt{2}}$ or $\phi_0 = -\frac{v}{\sqrt{2}}$, violates the phase transformation symmetry (Fig. 5.1).

A complex scalar field can be parametrized by two real fields $h(x)$ and $\zeta(x)$ as follows:

$$\phi = \frac{1}{\sqrt{2}}\left(v + h(x)\right)e^{-i\frac{\zeta(x)}{v}}\,. \tag{5.6}$$

Then the Lagrangian (5.1) has the form

$$\begin{aligned}\mathcal{L} = {} & \frac{1}{2}\partial_\mu h\partial^\mu h - \lambda v^2h^2 - \lambda vh^3 - \frac{1}{4}\lambda h^4 \\ & + \frac{1}{2}\partial_\mu\zeta\partial^\mu\zeta + \frac{2}{v}(\partial_\mu\zeta)(\partial^\mu\zeta)h + \frac{1}{v^2}(\partial_\mu\zeta)(\partial^\mu\zeta)h^2 + \frac{1}{4}\lambda v^4\,,\end{aligned} \tag{5.7}$$

and now describes a system of a massive scalar h with mass $m_h^2 = 2\lambda v^2$ interacting with a massless scalar field $\zeta(x)$, which is called the Nambu-Goldstone boson field.

According to the Goldstone's theorem, if the theory is invariant under a global group with m generators, but the vacuum is invariant under transformations generated only by l generators ($l < m$), then there exist $m - l$ massless Nambu-Goldstone bosons.

Let us consider a system described by the Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi)\,, \tag{5.8}$$

which is invariant under $i = 1, \ldots, m$ transformations:

$$\phi \to \phi' = \phi + \delta\phi, \quad \delta\phi_i = i\delta\Theta_A t^A_{ij}\phi_j\,. \tag{5.9}$$

The invariance of the potential V means

$$\delta V = \frac{\partial V}{\partial\phi_i}\delta\phi_i = i\delta\Theta_A\frac{\partial V}{\partial\phi_i}t^A_{ij}\phi_j = 0\,. \tag{5.10}$$

Let us assume that the potential V has a minimum (vacuum) at some field value $\phi_i = \phi_i^0$:

$$\left.\frac{\partial V}{\partial\phi_i}\right|_{\phi_i=\phi_i^0} = 0\,, \tag{5.11}$$

and consider the case where the vacuum is invariant under transformations generated by only l generators from all m generators corresponding to the symmetry, i.e.

$$t^A_{ij}\phi^0_j = 0 \quad \text{only for } i = 1, \ldots, l,\ (l < m)\,. \tag{5.12}$$

The second derivative of the invariance condition (5.10) at the minimun leads to

$$\left.\frac{\partial^2 V}{\partial\phi_k\partial\phi_i}\right|_{\phi_i=\phi_i^0} t^A_{ij}\phi^0_j + \underbrace{\left.\frac{\partial V}{\partial\phi_i}\right|_{\phi_i=\phi_i^0}}_{0 \text{ due to (5.11)}} t^A_{ik} = 0\,, \tag{5.13}$$

and therefore

$$\left.\frac{\partial^2 V}{\partial\phi_k\partial\phi_i}\right|_{\phi_i=\phi_i^0} t^A_{ij}\phi^0_j = 0\,. \tag{5.14}$$

This equality holds due to the condition $t^A_{ij}\phi^0_j = 0$ for fields with $i = 1, \ldots, l$, (5.12), however, for the other $m - l$ fields (with $i = (l+1), \ldots, m$) the following equality has to be valid:

$$\left.\frac{\partial^2 V}{\partial\phi_k\partial\phi_i}\right|_{\phi_i=\phi_i^0} = 0\,. \tag{5.15}$$

The second derivative of the potential in (5.15) is nothing but the mass term for these

$m - l$ fields. Therefore, the masses of these fields are equal to zero.

We have shown that in such a situation, when the vacuum of a system is not invariant under all the symmetry transformations of the Lagrangian, there exist $m - l$ massless fields (Nambu-Goldstone bosons), corresponding to the number of generators violating the symmetry.

Let us finally consider the SM gauge group $SU(2)_L \times U(1)_Y$, where we introduce one complex scalar field $\Phi(x)$, which is an $SU(2)_L$ doublet and carries a hypercharge Y_H, with the following gauge invariant Lagrangian

$$\mathcal{L}_\Phi = D_\mu \Phi^\dagger D^\mu \Phi - \mu^2 \Phi^\dagger \Phi - \lambda (\Phi^\dagger \Phi)^4 \,, \tag{5.16}$$

where the covariant derivative D_μ has the form

$$D_\mu = \partial_\mu - i g_2 W_\mu^i \tau^i - i g_1 \frac{Y_H}{2} B_\mu \,. \tag{5.17}$$

As before, we consider the case of a negative μ^2, i.e. $\mu^2 \to -|\mu^2| < 0$, so that the field potential has a non-trivial minimum at $|\Phi| = \frac{v}{\sqrt{2}}$.

The complex field doublet Φ can be parametrized by four real scalar fields (ζ^i, $i = 1, 2, 3$ and h) in the following generic way

$$\Phi = \exp\left[-i \frac{\zeta^i(x) t^i}{v}\right] \begin{pmatrix} 0 \\ (v + h(x))/\sqrt{2} \end{pmatrix} \,, \tag{5.18}$$

where $t^i = \sigma^i/2$ are the generators of the $SU(2)_L$ gauge group (σ^i are the Pauli spin matrices).

The Lagrangian (5.16) is invariant under the $SU(2)_L$ transformation

$$\Phi(x) \to \Phi'(x) = e^{i g_2 \alpha^i t^i} \Phi(x) \,. \tag{5.19}$$

Comparing (5.18) and (5.19), we notice that the unitary factor $\exp\left[-i \frac{\zeta^i(x) t^i}{v}\right]$ can be removed by choosing a special gauge — $i g_2 \alpha^i(x) = \zeta^i(x)/v$ — the unitary gauge. In the unitary gauge the field Φ takes the form

$$\Phi = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v + h(x) \end{pmatrix} \,. \tag{5.20}$$

The field Φ is called the Higgs field and has a non-zero VEV, which leads to the violation of the symmetry of the system.

Let us now rewrite Lagrangian (5.16), using (5.20) and the explicit form of the covariant derivative (5.17) (i.e. after EWSB):

$$\mathcal{L} = \frac{1}{2}(\partial_\mu h)^2 - \frac{1}{2}(2\lambda v^2)h^2 - \lambda v h^3 - \frac{\lambda}{4}h^4 + m_W^2 W_\mu^+ W^{\mu-}\left(1+\frac{h}{v}\right)^2 + \frac{1}{2}m_Z^2 Z_\mu Z^\mu \left(1+\frac{h}{v}\right)^2 , \tag{5.21}$$

with

$$m_h = \sqrt{2\lambda v^2}\,, \quad m_W = \frac{1}{2}g_2 v\,, \quad m_Z = \frac{1}{2}(g_2\cos\theta_W + g_1 Y_H \sin\theta_W)v\,, \tag{5.22}$$

which are, respectively, the masses of the scalar field h (the Higgs boson) and vector fields $W_\mu^\pm$, Z_μ.

After adding the kinetic terms and self-interactions of the gauge fields $W_\mu^\pm$, A_μ, and Z_μ (which appear from the terms $(-\frac{1}{4}W^i_{\mu\nu}W^{i,\mu\nu} - \frac{1}{4}B_{\mu\nu}B^{\mu\nu})$ of the SM Lagrangian), the Lagrangian (5.21) describes the massive Higgs boson field h, massive vector fields W_μ^i, Z_μ and the massless vector field A_μ. The three Nambu-Goldstone bosons (ζ_i, $i = 1,2,3$) are "eaten" by the three longitudinal components of the fields W_μ^-, W_μ^+ and Z_μ. This is known as the Brout-Englert-Higgs mechanism of spontaneous symmetry breaking. The Higgs-like boson of a mass $125.09 \pm 0.21(stat) \pm 0.11(syst)$ GeV was discovered by the ATLAS and CMS experiments at the LHC in 2012 [35]. The value of the VEV of the Higgs field has been found experimentally [36, 37] and is equal approximately to 246 GeV in the given normalization conventions.

It can be shown [33], that the photon field A_μ remains massless and has correct electromagnetic interactions only under the following condition: the hypercharge Y_H of the Higgs field has to be equal to the hypercharge Y_L^l of the charged lepton with the opposite sign

$$Y_H = -Y_L^l\,. \tag{5.23}$$

If the vacuum $\Phi_{\text{vac}} = \frac{1}{\sqrt{2}}\binom{0}{v}$ is invariant under a group transformation with the generator of the group T_i

$$e^{iT_i\Theta_i}\Phi_{\text{vac}} = \Phi_{\text{vac}}\,, \tag{5.24}$$

then

$$T_i\Phi_{\text{vac}} = 0\,, \tag{5.25}$$

which leads to the following form of the (hermitian) generator of the unbroken symmetry:

$$T\Phi_{\text{vac}} = \frac{1}{\sqrt{2}}\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{pmatrix}\begin{pmatrix}0\\ v\end{pmatrix} = 0, \quad \Rightarrow \quad T = \begin{pmatrix} a_{11} & 0 \\ 0 & 0\end{pmatrix}. \tag{5.26}$$

For $SU(2)_L \times U(1)_Y$ such a generator is

$$T_3 + \frac{1}{2}Y_H = \frac{1}{2}\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) + \frac{1}{2}Y_H\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) \quad \text{only if} \quad Y_H = +1\,. \tag{5.27}$$

This means that the vacuum is neutral, and the physical vacuum is broken from an $SU(2)_L \times U(1)_Y$-invariant vacuum in the absence of a Higgs field to a $U(1)_{\rm em}$ symmetry in the presence of a non-zero Higgs VEV:

$$\begin{aligned} SU(2)_L \times U(1)_Y &\to U(1)_{\rm em}\,, \\ Q_H = T_3 + \frac{1}{2}Y_H &= \frac{1}{2} + \frac{1}{2} = 0\,, \end{aligned} \tag{5.28}$$

and, besides,

$$g_2 \sin\theta_W = g_1 \cos\theta_W \quad \Rightarrow \quad m_W = m_Z \cos\theta_W\,. \tag{5.29}$$

The value of the Higgs hypercharge ($Y_H = 1$) fixes the lepton hypercharge: $Y_L^l = -1$, and we find the values for the hypercharges of the left and right leptons and quarks:

$$Y_L^l = -3Y_L^q \quad \Rightarrow \quad Y_R^l = -2,\ Y_L^q = Y_L^u = Y_L^d = \frac{1}{3},\ Y_R^u = \frac{4}{3},\ Y_R^d = -\frac{2}{3}\,, \tag{5.30}$$

which confirms the Gelmann-Nishijima relation for all leptons and quarks of both chiralities:

$$I_3 + \frac{Y}{2} = Q\,. \tag{5.31}$$

Let us now demonstrate how the SM leptons and quarks obtain their masses after the EWSB, without violating the gauge invariance.
The SM left-handed quarks and leptons are $SU(2)$ doublets $\left(Q_L = \binom{u_L}{d_L}\right.$ and $\left. L_L = \binom{\nu_L}{e_L}\right)$, and the right-handed fermions are $SU(2)$ singlets (u_R and e_R). There are only two options how to write a gauge invariant dimension 4 operator of the Yukawa type:

$$\bar{Q}_L \Phi d_R \quad \text{and} \quad \bar{Q}_L \Phi^C u_R\,, \tag{5.32}$$

where

$$\Phi = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v+h \end{pmatrix} \quad \text{and} \quad \Phi^C = i\sigma^2\Phi^\dagger = \frac{1}{\sqrt{2}}\begin{pmatrix} v+h \\ 0 \end{pmatrix} \tag{5.33}$$

are, respectively, the Higgs and the conjugated Higgs $SU(2)_L$ doublet fields in the unitary gauge. After spontaneous symmetry breaking,

$$\bar{Q}_L \Phi d_R + \text{h.c.} = (\bar{u}_L\, \bar{d}_L)\begin{pmatrix} 0 \\ v \end{pmatrix} d_R + \bar{d}_R (0\ v)\begin{pmatrix} u_L \\ d_L \end{pmatrix} = v(\bar{d}_L d_R + \bar{d}_R d_L) = v\bar{d}d\,, \tag{5.34}$$

which is the Dirac mass term for the down-type fermion.

Similarly, for the up-type fermions:

$$\bar{Q}_L \Phi^C u_R + \text{h.c.} = v\bar{u}u\,. \tag{5.35}$$

The SM Yukawa Lagrangian in the general form, including the mixing of the fermions from different generations with generic mixing coefficients $\Gamma_{u,d,e}$, can be expressed as

$$\mathcal{L}_{\text{Yukawa}} = -\Gamma_d^{ij} \bar{Q'}^i_L \Phi d'^j_R + \text{h.c.} - \Gamma_u^{ij} \bar{Q'}^i_L \Phi^C u'^j_R + \text{h.c.} - \Gamma_e^{ij} \bar{L'}^i_L \Phi e'^j_R + \text{h.c.}\,, \tag{5.36}$$

where the states with ($'$) are the interaction eigenstates.

After the EWSB, Eq. (5.36) takes the form

$$\mathcal{L}_{\text{Yukawa}} = -\left[M_d^{ij} \bar{d'}^i_L d'^j_R + M_u^{ij} \bar{u'}^i_L u'^j_R + M_e^{ij} \bar{e'}^i_L e'^j_R + \text{h.c.}\right] \cdot \left(1 + \frac{h}{v}\right)\,, \tag{5.37}$$

with $M^{ij} = \Gamma^{ij} v/\sqrt{2}$.

After diagonalization, which is usually done in order to get the physical states for quarks and leptons with definite masses, the Yukawa Lagrangian contains masses of the fermions and describes their interaction with the Higgs boson:

$$\mathcal{L}_{\text{Yukawa}} = -\left[m_d^i \bar{d}^i d^i + m_u^i \bar{u}^i u^i + m_l^i \bar{l}^i l^i\right] \cdot \left(1 + \frac{h}{v}\right)\,. \tag{5.38}$$

The experimental detection of a Higgs particle completes the SM in the sense that all predicted SM particles have been observed. However, this discovery, as a messenger of the underlying scalar field, is only the first step in our quest of understanding the mechanism of EWSB. The ultimate goal is to reconstruct the entire Higgs potential. For this purpose it is necessary to get information on the trilinear Higgs vertex, for example via a measurement of Higgs pair production at the LHC. In the SM, the scalar sector takes a minimal form by introducing a single Higgs doublet field which implies, after expanding this field around the EWSB vacuum, that the trilinear Higgs coupling λ_{3h}^{SM} is related to the Higgs mass and vacuum expectation value in a specific way ($\lambda_{3h}^{SM} h^3 \equiv \lambda v h^3$ in (5.21), so $m_h^2 = 2\lambda v^2 = 2v\lambda_{3h}^{SM}$, and thus, $\lambda_{3h}^{SM} = \frac{m_h^2}{2v}$). Clearly an experimental verification of this relation will be crucial. At the same time, any deviation would signal new physics beyond the SM.

5.2 Open questions of the SM

Despite the remarkable success of the SM, there are still facts that can not be explained in its minimal setup. Among these open questions are:

- Why gravity is so weak compared to other interactions (hierarchy problem)?
- The SM Higgs mechanism is not stable with respect to quantum corrections (naturalness problem).
- Why do fermion masses vary in such a wide range ($M_e = 0.5\text{MeV}$, $M_t = 173\text{GeV}$), though they appear from the same mechanism?
- There are no Dark Matter candidates in the SM.
- There is no explanation for neutrino masses and oscillations in the SM.

Let us focus on the hierarchy problem, namely, the large gap between the energy scales of various fundamental interactions (Fig. 5.2).



Figure 5.2: The hierarchy of scales of gauge interactions.

This problem is deeply connected to the problem of the naturalness of the Higgs mass [38] and becomes visible when assuming the SM to be valid up to very hight scales and considering the radiative corrections to the Higgs mass

$$m^2_{h,\text{obs}} = m^2_h + \delta m^2_h \,, \tag{5.39}$$

where $m_{h,\text{obs}}$ is the observable Higgs mass, m_h is the tree-level Higgs mass, and δm^2_h stands for the some of contributions from various loop corrections (e.g. Fig. 5.3).

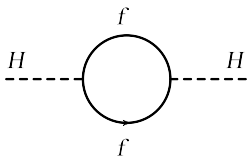


Figure 5.3: One-loop correction to the Higgs squared mass parameter m^2_h with a massive fermion loop.

Such corrections appear to be quadratically dependent on the scale and extremely large, in particular, if we consider the SM valid up to the reduced Planck scale ($M_{Pl} \sim 2.4 \times 10^{18}\text{GeV}$), where gravity becomes comparable with other interactions, the Higgs squared mass gets the correction of order M^2_{Pl} and has to be fine-tuned in order to match the observable value $m^2_{h,\text{obs}} \approx (125\text{ GeV})^2$.

An elegant solution of this problem was proposed by L. Randall and R. Sundrum in 1999 [39] as a multidimensional extension of the SM — the so-called Randall-Sundrum model, which is discussed in Chapter 6.

Chapter 6

Scenarios with extra dimensions

Multidimensional "brane-world" models based on the Randall-Sundrum (RS) model with two branes stabilized by a bulk scalar field [39, 40, 41] are quite attractive extensions of the SM since they suggest a way of unification of the gravity and electromagnetic interaction, considering the comparably strong gravity that propagates in the entire 5-dimensional space-time and becomes weak only on the 4-dimensional brane, where our world is localized. The additional predicted scalar boson, the radion, interacting with the SM particles in the experimentally accessible range of energies, makes the hypothesis of extra dimensions testable at the LHC. It is reasonable to probe modifications of the Higgs boson properties that arise due to the presence of this new scalar that has the same quantum numbers.

This chapter is devoted to the description of the Randall-Sundrum model including the Higgs-radion mixing (Sec. 6.1) and a discussion of the possible forms of the scalar potential in this model (Sec. 6.2).

6.1 The Randall-Sundrum model

Let us consider a five-dimensional (5D) space-time $E = M_4 \times S^1/Z_2$ with two four-dimensional (4D) surfaces containing matter, which are called "branes".

The coordinates system is defined as follows:
$\{x^M\} \equiv \{x^\mu, y\}$, $M = 0, 1, 2, 3, 4$, where $\{x^\mu\}$, $\mu = 0, 1, 2, 3$ are the coordinates in the 4D space-time and $x^4 \equiv y$ — the coordinate corresponding to the 5th dimension, which is limited by $-L \leq y \leq L$.
The 5th dimension forms an orbifold S^1/Z_2, which represents a circle of circumference $2L$ with the points y and $-y$ identified. The 4D branes are placed at $y = 0$ and $y = L$.

All the SM particles and forces are assumed to be localized on one of the branes (the "infra-red" (IR), "TeV" or "visible" brane), whereas gravity and a stabilizing scalar field live on both the IR brane and the other (the "ultra-violet" (UV), "Planck" or "hidden")

brane, and can also propagate in the space between the branes, defined by the 5th coordinate and called the "bulk".
All mass scales in the 5D theory are of the order of the Planck mass and are rescaled on the IR brane by an exponential suppression factor (the "warp factor") $e^{-kb_0/2}$, which reduces them down to the weak scale $\mathcal{O}(1\text{TeV})$. The 5D non-flat metric is usually written in the following form

$$ds^2 = e^{-2kb_0|y|}\eta_{\mu\nu}dx^\mu dx^\nu - b_0^2 dy^2\,, \tag{6.1}$$

where k is the curvature of the 5D geometry, b_0 is a length parameter for the 5th dimension, and $-1/2 \leq y \leq 1/2$ ($L = 1/2$).

In such a model, a ratio 1 TeV$/M_{Pl}$ between the weak energy scale and the reduced Planck mass ($M_{Pl} \sim 2.4 \times 10^{18}$GeV) is significantly smaller than in the original scenario and corresponds to the parameter $kb_0/2 \sim 35$.

One of the characteristic features of the RS model with a stabilization of the extra space dimension is the existence of the massive radion [40, 59, 42] – the lowest Kaluza-Klein (KK) mode of the five-dimensional scalar field appearing from the fluctuations of the metric component corresponding to the extra dimension. The radion might be significantly lighter than the other KK modes [66]. Moreover, the radion has the same quantum numbers as the neutral Higgs field, which opens up the possibility of the radion-Higgs mixing. Therefore, such a model is of a special interest for collider phenomenology (see, e.g., [43] – [57]).
Due to its origin the radion interacts with the trace of the energy-momentum tensor of the SM

$$\mathcal{L} = -\frac{\phi(x)}{\Lambda_\phi}T^\mu_\mu\,, \tag{6.2}$$

where ϕ is the radion field (sometimes can be denoted as r), Λ_ϕ is a dimensional scale parameter that can be interpreted as the VEV of the radion, and T^μ_μ is the trace of the SM energy-momentum tensor:

$$\begin{aligned} T^\mu_\mu = &\frac{\beta(g_s)}{2g_s}G^a_{\rho\sigma}G^{\rho\sigma}_a + \frac{\beta(e)}{2e}F_{\rho\sigma}F^{\rho\sigma} \\ &+ \sum_f \left[\frac{3i}{2}\left(\left(D_\mu\bar{f}\right)\gamma^\mu f - \bar{f}\gamma^\mu\left(D_\mu f\right)\right) + 4m_f\bar{f}f\left(1+\frac{h}{v}\right)\right] \\ &- \left(\partial_\mu h\right)\left(\partial^\mu h\right) + 2m_h^2h^2\left(1+\frac{h}{2v}\right)^2 \\ &- \left(2m_W^2W^+_\mu W^{\mu\,-} + m_Z^2Z_\mu Z^\mu\right)\left(1+\frac{h}{v}\right)^2\,, \end{aligned} \tag{6.3}$$

where the first two terms correspond to the conformal anomaly of massless gluon and

photon fields, $\beta(g_s)$, $\beta(e)$ are the QCD and QED β-functions respectively, h, $W^{\pm}$ and Z are the SM Higgs, W- and Z-boson fields, D_μ is the SM covariant derivative and the summation here is taken over all the SM fermions.

Remarkably, despite the differences in the Higgs boson and radion origin and the structure of the interactions with the SM particles, it was shown earlier [73, 74] that there exists an incredible Higgs-radion similarity for certain classes of processes at the level of the amplitudes (even for the rather complicated case of off-shell fermions involving additional non-SM diagrams):

- Single radion and single Higgs boson production processes in association with an arbitrary number of SM gauge bosons (V_i, $i = 1, \ldots, N$) are similar

$$\bar{f}f \to \phi\, V_1 \ldots V_N \quad \Longleftrightarrow \quad \bar{f}f \to h\, V_1 \ldots V_N$$

  up to a replacement of masses and coupling constants $m_\phi \to m_h$ and $\Lambda_\phi \to v$;

- Associated Higgs boson-radion and Higgs boson pair production including processes with an arbitrary number of SM Higgs (h_i, $i = 1, \ldots, N$) and gauge bosons (V_i, $i = 1, \ldots, M$) are similar

$$\begin{aligned} gg \to \phi h \quad &\Longleftrightarrow \quad gg \to hh\,, \\ \bar{f}f \to \phi\, h_1 \ldots h_N\, V_1 \ldots V_M \quad &\Longleftrightarrow \quad \bar{f}f \to h\, h_1 \ldots h_N\, V_1 \ldots V_M \end{aligned}$$

  up to a replacement of masses and coupling constants $m_\phi \to m_h$, $\Lambda_\phi \to v$ and a rescaling of the trilinear Higgs coupling $\lambda_{3h}^{SM} \to \left(1 + \frac{m_\phi^2 - m_h^2}{3m_h^2}\right) \lambda_{3h}^{SM}$, where m_ϕ is the radion mass and $m_h \simeq 125\,\text{GeV}$ is the Higgs boson mass.

Whereas in those earlier works [73, 74] the contributions from the radion anomalous interaction as well as the possibility of the Higgs-radion mixing were deliberately not taken into account, in the current study we are interested in the more general case.

6.1.1 The Higgs-radion Lagrangian before mixing

In this section, we discuss the mixing between gravity and the Higgs sector following Ref. [71]. The simplest example of an action for the mixing between gravity and the electroweak sector is given by (see e.g. [71], [68, 67]):

$$S_\xi = \xi \int d^4x \sqrt{g_{vis}}\, \mathcal{R}(\, g_{vis})\, \hat{H}^\dagger\, \hat{H}, \tag{6.4}$$

where $\hat{H}$ is the Higgs field in the 5D context before rescaling to canonincal normalization on the brane, and $\mathcal{R}(\, g_{vis})$ is the Ricci scalar for the metric induced on the IR brane:

$$g^{\mu\nu}_{vis} = \Omega_b^2(x)(\eta^{\mu\nu} + \varepsilon h^{\mu\nu})\,. \tag{6.5}$$

Using $H_0 = \Omega_0\,\hat{H}$ and $\Omega_b(x) = \Omega_0\Omega(x)$ one can obtain [66]:

$$\xi\sqrt{g_{vis}}\,\mathcal{R}(\, g_{vis})\,\hat{H}^\dagger\,\hat{H} = 6\xi\Omega(x)(-\Box\Omega(x) + \varepsilon h_{\mu\nu}\partial^\mu\partial^\nu\Omega(x) + \dots)\,H_0^\dagger\,H_0\,, \tag{6.6}$$

with[1]

$$\Omega(x) = 1 + \frac{\phi_0}{\Lambda_\phi} = 1 + \gamma\frac{\phi_0}{v}\;, \quad H_0 \to \frac{1}{\sqrt{2}}(+\,h_0)\,. \tag{6.7}$$

Above we have introduced the ratio of vacuum expectation values

$$\gamma = \frac{v}{\Lambda_\phi} \lesssim 0.1\,. \tag{6.8}$$

The $h_{\mu\nu}$-term in Eq. (6.6) does not contribute to the kinetic energy since a partial integration would lead to $h_{\mu\nu}\partial^\mu\partial^\nu\Omega(x) = -\partial^\mu h_{\mu\nu}\partial^\nu\Omega = 0$ due to the gauge choice $\partial^\mu h_{\mu\nu} = 0$.

This leads to the following 4D effective Lagrangian for the scalar sector (see Eq. (2.1) in [72], and [69]):

$$\mathcal{L}_{eff} = \frac{1}{2}(\partial_\mu\,\phi_0)^2 - \frac{1}{2}\,m_{\phi_0}^2\,\phi_0^2 - 6\xi\Omega(x)\Box\Omega(x)\,H_0^\dagger\,H_0 + |D_\mu\,H_0|^2 - \Omega^4 V_H(\,H_0) \tag{6.9}$$

where

$$V_H(\,H_0) = V_0 - \mu^2\,H_0^\dagger\,H_0 + \lambda(\,H_0^\dagger\,H_0)^2\,, \tag{6.10}$$

with the constant $V_0 = \lambda v^4/4$.

The term $\Omega^4 V_H(\,H_0)$ is model-dependent and could be generalized (see Sec. 6.2). Note that if the constant term V_0 is omitted, $V_0 = 0$, the $\Omega^4 V_H(\,H_0)$ term would give a contribution to the radion mass term $\frac{3}{4}\,\gamma^2\,m_{h_0}^2\,\phi_0^2$:
$-\frac{1}{2}\,m_{\phi_0}^2\,\phi_0^2 \to -\frac{1}{2}\,m_{\phi_0}^2\,\phi_0^2 + \frac{3}{4}\,\gamma^2\,m_{h_0}^2\,\phi_0^2$ so that $m_{\phi_0}^2 \to m_{\phi_0}^2 - \frac{3}{2}\,\gamma^2\,m_{h_0}^2$.

Using $\mu^2 = v^2\lambda = m_{h_0}^2/2$ and $\lambda = m_{h_0}^2/(2v^2)$, equation (6.9) leads to the following Higgs-Radion Lagrangian at the quadratic level:

$$\mathcal{L}^{(2)}_{eff} = -\frac{\beta}{2}\,\phi_0\Box\,\phi_0 - \frac{1}{2}\,m_{\phi_0}^2\,\phi_0^2 - 6\,\gamma\xi\,\phi_0\Box\,h_0 - \frac{1}{2}\,h_0\Box\,h_0 - \frac{1}{2}\,m_{h_0}^2\,{h_0}^2 \tag{6.11}$$

[1]Note that in [72], $\Omega(x)$ is defined as $\Omega(x) = 1 - \frac{\phi_0}{\Lambda_\phi}$.

$$= \frac{\beta}{2}\partial^\mu \phi_0 \partial_\mu \phi_0 - \frac{1}{2} m_{\phi_0}^2 \phi_0^2 + 6\,\gamma\xi \partial^\mu \phi_0 \partial_\mu h_0 + \frac{1}{2}\partial^\mu h_0 \partial_\mu h_0 - \frac{1}{2} m_{h_0}^2 {h_0}^2 + \partial_\mu[\ldots]\,, \tag{6.12}$$

where $\phi_0(x)$, $h_0(x)$ are the radion and the Higgs fields before the Higgs-radion mixing, and $\beta = 1 + 6\xi\,\gamma^2$. Eq. (6.11) is in agreement with Eq. (29) in [71]. Eq. (2.3) in Ref. [72] can be recovered from Eq. (6.11) by replacing $\phi_0 \to -\phi_0$. Similarly, the sign convention for ϕ_0 is opposite to that in Ref. [66]. Eq. (6.12) agrees with Eq. (12) in [70].

6.1.2 Higgs-radion mixing

The term $-6\,\gamma\xi\,\phi_0 \Box\, h_0$ that mixes the Higgs and the radion in Lagrangian (6.11) can be removed by rotating the scalar fields into the mass eigenstate basis:

$$\begin{pmatrix} \phi_0 \\ h_0 \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \phi \\ h \end{pmatrix} \tag{6.13}$$

where the coefficients are given by

$$A = -\frac{1}{Z}\cos\theta\,, \quad B = \frac{1}{Z}\sin\theta\,, \quad C = \sin\theta + \frac{6\,\gamma\xi}{Z}\cos\theta\,, \quad D = \cos\theta - \frac{6\,\gamma\xi}{Z}\sin\theta \tag{6.14}$$

with

$$Z^2 = \beta - (6\,\gamma\xi)^2 = 1 + 6\xi\,\gamma^2(1 - 6\xi) \tag{6.15}$$

and the mixing angle θ defined by

$$\tan 2\theta = \frac{12\xi\,\gamma Z\, m_{h_0}^2}{m_{\phi_0}^2 - m_{h_0}^2(Z^2 - 36\xi^2\,\gamma^2)}\,. \tag{6.16}$$

Note that in [72], the coefficient a is equal to our A (and to that in [71]) but enters the mixing matrix with the opposite sign.

The mixing parameter ξ is constrained by the requirement $Z^2 > 0$ in order to get a real mixing angle.

The mass eigenvalues of the radion dominated (ϕ) and Higgs dominated (h) physical eigenstates are given by

$$m_\phi^2 = \frac{1}{2Z^2}\left[m_{\phi_0}^2 + \beta\, m_{h_0}^2 + \sqrt{(m_{\phi_0}^2 + \beta\, m_{h_0}^2)^2 - 4Z^2\, m_{\phi_0}^2\, m_{h_0}^2} \right]\,, \tag{6.17}$$

$$m_h^2 = \frac{1}{2Z^2}\left[m_{\phi_0}^2 + \beta\, m_{h_0}^2 - \sqrt{(m_{\phi_0}^2 + \beta\, m_{h_0}^2)^2 - 4Z^2\, m_{\phi_0}^2\, m_{h_0}^2} \right]\,. \tag{6.18}$$

Alternatively, the physical masses can be used as external input parameters (see, e.g.,

Sec. 7.1) and the unphysical mass parameters m_{ϕ_0} and m_{h_0} can be calculated as

$$m^2_{\phi_0} = \frac{Z^2}{2}\left[m^2_\phi + m^2_h + \sqrt{(m^2_\phi + m^2_h)^2 - \frac{4\beta\, m^2_\phi\, m^2_h}{Z^2}}\right], \tag{6.19}$$

$$m^2_{h_0} = \frac{Z^2}{2\beta}\left[m^2_\phi + m^2_h - \sqrt{(m^2_\phi + m^2_h)^2 - \frac{4\beta\, m^2_\phi\, m^2_h}{Z^2}}\right]. \tag{6.20}$$

The parts of the Lagrangian responsible for the different radion and Higgs interactions in the RS model, as well as the Feynman rules and the re-derived amplitudes, widths and branching ratios for certain processes involving the radion dominated and Higgs dominated states are collected in Appendix C.

6.2 Some notes on the scalar potential

The scalar potential in Ref. [72] is of the form

$$V(\phi_0, h_0) = \frac{1}{2}\, m^2_{\phi_0}\, \phi_0^2 + \Omega^4 V_H(H_0)\,, \tag{6.21}$$

where

$$\Omega = 1 - \frac{\phi_0}{\Lambda_\phi}\,, \tag{6.22}$$

and $V_H(H_0)$ is given in Eq. (6.10):

$$V_H(H_0) = V_0 - \mu^2\, H_0^\dagger\, H_0 + \lambda (H_0^\dagger\, H_0)^2\,.$$

Note the different sign convention for the radion field ϕ_0 in [72] compared to [71]. Using the replacement $H_0 \to (0 \quad (v + h_0)/\sqrt{2})^T$ (unitary gauge) and a) $\mu^2 = \lambda v^2$ or b) $\mu^2 = \lambda v^2, \lambda = m^2_{h_0}/(2v^2)$ one finds:

$$V_H \to V_0 + \lambda\left[-\frac{v^4}{4} + v^2\, h_0^2 + v\, h_0^3 + \frac{1}{4}\, h_0^4\right] \tag{6.23}$$

$$= V_0 - \frac{1}{8}\, m^2_{h_0} v^2 + \frac{1}{2}\, m^2_{h_0}\, h_0^2 + \frac{m^2_{h_0}}{2v}\, h_0^3 + \frac{m^2_{h_0}}{8v^2}\, h_0^4 \tag{6.24}$$

The scalar potential is then given in the broken phase by

$$V(\phi_0, h_0) \to \frac{1}{2}\, m^2_{\phi_0}\, \phi_0^2 + \left[1 - 4/\,\Lambda_\phi\, \phi_0 + 6/\,\Lambda^2_\phi\, \phi_0^2 - 4/\,\Lambda^3_\phi\, \phi_0^3 + 1/\,\Lambda^4_\phi\, \phi_0^4\right] \times \lambda\left[\frac{V_0}{\lambda} - \frac{v^4}{4} + v^2\, h_0^2 + v\, h_0^3 + \frac{1}{4}\, h_0^4\right] \tag{6.25}$$

$$= \sum_{i,j=0}^{4} c_{i,j}\, \phi_0^i\, h_0^j \tag{6.26}$$

with

$$\begin{aligned}
&c_{0,0} = V_0 - \frac{\lambda v^4}{4}\,,\; c_{1,0} = c_{0,0}\frac{-4}{\Lambda_\phi}\,,\; c_{2,0} = \frac{m_{\phi_0}^2}{2} + c_{0,0}\frac{6}{\Lambda_\phi^2}\,,\; c_{3,0} = c_{0,0}\frac{-4}{\Lambda_\phi^3}\,,\; c_{4,0} = c_{0,0}\frac{1}{\Lambda_\phi^4}\,,\\
&c_{i,1} = 0 \quad (i = 0,1,2,3,4)\,,\\
&c_{0,2} = \lambda v^2\,,\; c_{1,2} = c_{0,2}\frac{-4}{\Lambda_\phi}\,,\; c_{2,2} = c_{0,2}\frac{6}{\Lambda_\phi^2}\,,\; c_{3,2} = c_{0,2}\frac{-4}{\Lambda_\phi^3}\,,\; c_{4,2} = c_{0,2}\frac{1}{\Lambda_\phi^4}\,,\\
&c_{0,3} = \lambda v\,,\; c_{1,3} = c_{0,3}\frac{-4}{\Lambda_\phi}\,,\; c_{2,3} = c_{0,3}\frac{6}{\Lambda_\phi^2}\,,\; c_{3,3} = c_{0,3}\frac{-4}{\Lambda_\phi^3}\,,\; c_{4,3} = c_{0,3}\frac{1}{\Lambda_\phi^4}\,,\\
&c_{0,4} = \frac{\lambda}{4}\,,\; c_{1,4} = c_{0,4}\frac{-4}{\Lambda_\phi}\,,\; c_{2,4} = c_{0,4}\frac{6}{\Lambda_\phi^2}\,,\; c_{3,4} = c_{0,4}\frac{-4}{\Lambda_\phi^3}\,,\; c_{4,4} = c_{0,4}\frac{1}{\Lambda_\phi^4}\,.
\end{aligned} \tag{6.27}$$

The choice $V_0 = \lambda v^4/4$ then leads to a vanishing $c_{0,0}$ so that $c_{2,0} = \frac{m_{\phi_0}^2}{2}$.[2]

Considering only operators up to a mass dimension 4, the potential takes the form:

$$V(\,\phi_0,\, h_0) = \frac{1}{2}\, m_{\phi_0}^2\, \phi_0^2 + \lambda(v^2\, h_0^2 + v\, h_0^3 + \frac{1}{4}\, h_0^4) + c_{1,2}\, \phi_0\, h_0^2 + c_{2,2}\, \phi_0^2\, h_0^2 + c_{1,3}\, \phi_0\, h_0^3\,. \tag{6.28}$$

In particular, there is no term proportional to ϕ_0^3 (unless the constant term $c_{3,0}$ is kept in the Higgs potential).

The scalar potential considered in Ref. [71] (see Eqs. (54) and (59)) is of the form:

$$V(\,\phi_0,\, h_0) = \frac{1}{2}\, m_{\phi_0}^2\, \phi_0^2 + X_3 \frac{m_{\phi_0}^2}{2\,\Lambda_\phi}\, \phi_0^3 + \ldots + \lambda(\, H_0^\dagger\, H_0 - \frac{1}{2}v^2)^2 \tag{6.29}$$

$$= \frac{1}{2}\, m_{\phi_0}^2\, \phi_0^2 + X_3 \frac{m_{\phi_0}^2}{2\,\Lambda_\phi}\, \phi_0^3 + \ldots + \lambda(v^2\, h_0^2 + v\, h_0^3 + \frac{1}{4}\, h_0^4)\,, \tag{6.30}$$

where X_3 is a constant.
Note that $\lambda(v^2\, h_0^2 + v\, h_0^3 + \frac{1}{4}\, h_0^4)$ agrees with $V_H(\, H_0)$ in Eq. (6.23) with the constant $V_0 = \lambda v^4/4$. This potential doesn't contain any mixed terms proportional to $\phi_0\, h_0^2$, $\phi_0^2\, h_0^2$.

There are a few possible generalizations of the scalar potential. For example, one could replace V_H by the potential for a 2HDM, or consider the most general renormalizable potential for an SU(2) Higgs doublet and a real scalar SM singlet, an SU(2) Higgs doublet and a complex scalar SM singlet, two SU(2) Higgs doublets plus a real/complex scalar SM singlet. One could also add more scalar singlets or consider higher-dimensional operators

[2] Of course, a shift of the full scalar potential by a constant term has no observable consequences. However, shifting the "Higgs potential" V_H by modifying the value of V_0 would lead to observable differences. In particular, a contribution to the radion mass and a triple radion vertex would be generated.

(impacting vacuum stability considerations).

Let us explicitly provide the two following examples: the SM scalar potential complemented by a real or a complex scalar singlet.

SM+real scalar singlet: The most general renormalizable scalar potential containing the SM SU(2) doublet Φ and a real scalar SM singlet S with a vanishing vev is given by [60]

$$V(\Phi, S) = \frac{m^2}{2}\,\Phi^\dagger\,\Phi + \frac{\lambda}{4}(\Phi^\dagger\,\Phi)^2 + \frac{1}{2}\delta_1\,\Phi^\dagger\,\Phi S + \frac{1}{2}\delta_2\,\Phi^\dagger\,\Phi S^2 + \kappa_1 S + \frac{1}{2}\kappa_2 S^2 + \frac{1}{3}\kappa_3 S^3 + \frac{1}{4}\kappa_4 S^4\,, \tag{6.31}$$

where $\kappa_1 = \delta_1 m^2/(2\lambda)$ so that the field S doesn't acquire a vev. (Note: $m^2 = -\mu^2$.) After spontaneous symmetry breaking, in unitary gauge, we have

$$\Phi = \begin{pmatrix} 0 \\ \frac{v+h_0}{\sqrt{2}} \end{pmatrix}\,, \quad S = v_s + s_0\,. \tag{6.32}$$

This leads to the following scalar potential in the broken phase

$$V(s_0,\, h_0) \to \sum_{i,j=0}^{4} c_{i,j}\, s_0^i\, h_0^j \tag{6.33}$$

with

$$\begin{aligned}
&c_{0,0} = V_0 - \frac{\lambda v^4}{4}\,,\; c_{1,0} = c_{0,0}\frac{-4}{\Lambda_\phi}\,,\; c_{2,0} = \frac{m_{\phi_0}^2}{2} + c_{0,0}\frac{6}{\Lambda_\phi^2}\,,\; c_{3,0} = c_{0,0}\frac{-4}{\Lambda_\phi^3}\,,\; c_{4,0} = c_{0,0}\frac{1}{\Lambda_\phi^4}\,,\\
&c_{i,1} = 0 \quad (i = 0, 1, 2, 3, 4)\,,\\
&c_{0,2} = \lambda v^2\,,\; c_{1,2} = c_{0,2}\frac{-4}{\Lambda_\phi}\,,\; c_{2,2} = c_{0,2}\frac{6}{\Lambda_\phi^2}\,,\; c_{3,2} = c_{0,2}\frac{-4}{\Lambda_\phi^3}\,,\; c_{4,2} = c_{0,2}\frac{1}{\Lambda_\phi^4}\,,\\
&c_{0,3} = \lambda v\,,\; c_{1,3} = c_{0,3}\frac{-4}{\Lambda_\phi}\,,\; c_{2,3} = c_{0,3}\frac{6}{\Lambda_\phi^2}\,,\; c_{3,3} = c_{0,3}\frac{-4}{\Lambda_\phi^3}\,,\; c_{4,3} = c_{0,3}\frac{1}{\Lambda_\phi^4}\,,\\
&c_{0,4} = \frac{\lambda}{4}\,,\; c_{1,4} = c_{0,4}\frac{-4}{\Lambda_\phi}\,,\; c_{2,4} = c_{0,4}\frac{6}{\Lambda_\phi^2}\,,\; c_{3,4} = c_{0,4}\frac{-4}{\Lambda_\phi^3}\,,\; c_{4,4} = c_{0,4}\frac{1}{\Lambda_\phi^4}\,.
\end{aligned} \tag{6.34}$$

SM+complex scalar singlet: The most general renormalizable scalar potential containing the SM SU(2) doublet Φ and a complex scalar SM singlet S_c reads [61, 62]

$$V(\Phi,\, S_c) = \frac{m^2}{2}\,\Phi^\dagger\,\Phi + \frac{\lambda}{4}(\Phi^\dagger\,\Phi)^2 + \left(\frac{1}{4}\delta_1\,\Phi^\dagger\,\Phi\,S_c + \frac{1}{4}\delta_3\,\Phi^\dagger\,\Phi\,S_c^2 + a_1\,S_c + \frac{1}{4}b_1\,S_c^2 + \frac{1}{6}c_1\,S_c^4 + \frac{1}{6}c_2\,S_c|\,S_c|^2 + \frac{1}{8}d_1\,S_c^4 + \frac{1}{8}d_3\,S_c^2|\,S_c|^2 + \text{h.c.}\right)$$

$$+\frac{1}{4}d_2(|\,S_c^2|)^2+\frac{1}{2}\delta_2\,\Phi^\dagger\,\Phi|\,S_c|^2+\frac{1}{2}b_2|\,S_c|^2\,, \tag{6.35}$$

where $a_1, b_1, c_1, c_2, d_1, d_3, \delta_1$ and δ_3 are complex. After spontaneous symmetry breaking, in unitary gauge, we have

$$\Phi=\begin{pmatrix}0\\ \frac{v+h_0}{\sqrt{2}}\end{pmatrix}\,,\quad S_c=\frac{1}{\sqrt{2}}\left(v_s+S+i(v_A+A)\right) \tag{6.36}$$

A few comments are in order:

- Since all allowed terms have been included in Eq. (6.35) the coefficients can always be redefined such that $v_S = v_A = 0$. This makes the potential of Eq. (6.35) identical to that obtained by adding two real scalar singlets to the SM and there is no CP violation.

- If a global U(1) symmetry or a Z_2 symmetry is imposed to eliminate some of the terms in the potential, making the shift to $v_S = v_A = 0$ in general is not possible.

Chapter 7

Collider phenomenology of the RS model

In this chapter we summarize the efforts that have been made to implement the RS model in such HEP tools as `FeynRules` (Sec. 7.1) and `Lilith` (Sec. 7.1) and to perform a first phenomenological analysis of the RS model.

7.1 RS model with FeynRules package

`FeynRules` [78] is a `Mathematica`-based package designed for the calculation of the underlying Feynman rules of any implemented QFT model with a possibility of making an output in a form appropriate for various HEP tools such as `MadGraph`, `CalcHep`, `FeynArts`, `Sherpa` and `Whizard` in order to get the observables to be further compared with experimental data.
The implementation of a new model implies the creation of a new model-file from scratch with description of all fields, parameters, symmetries and Lagrangians of the model.

The RS model has been implemented in `FeynRules` and the the automated calculation of the relevant Feynman rules has been achieved, the obtained vertices were verified on examples from literature. The structure of the created model-file for the RS model is discussed below, and the full code is provided in Appendix D.

Since we are interested in a scenario that includes the SM and one additional scalar particle with possible mixing with the Higgs boson, it is convenient to use the SM model-file (SM.fr) already included into the distribution of the `FeynRules` package (or available in the FeynRules model database on the `FeynRules` homepage), i.e. the full SM implementation. Thus, it is sufficient to describe only the scalar fields and relevant RS parameters in the new file (say, Radion_Higgs_Model.fr).
Both files should be included together when running `FeynRules` in the `Mathematica` notebook, using the command **LoadModel** ["SM.fr","Radion_Higgs_Model.fr"] [78].

The model-file contains a few essential blocks (that must be written in a valid Mathematica syntax):

- **Information.** This block is used for defining the name of the model as well as the names and addresses of the authors, date of creation, references, *etc.*

- **Particle Classes.** In this block all fields are implemented.
  In the RS model, there are five classes corresponding to unphysical and physical scalars.

  The unphysical scalars are the Higgs doublet (Φ in the code) and the radion ϕ_0 ($R0$ in the code) fields before mixing, as well as the parameter Ω that is considered as a field being proportional to ϕ_0 (6.22).

  The physical scalars are the Higgs-dominated state (H) with the mass 125 GeV and the radion-dominated state (R).

  All necessary information on the fields (such as — whether the particle has an antiparticle or not, is physical or unphysical; symmetry groups under which the field transforms, quantum numbers, physical masses, decay widths, *etc.*) is introduced in the options of the classes, including, in particular, the EWSB and the Higgs-radion mixing — by using the corresponding 'Definitions' options, which consist of sets of replacement rules that are applied to the Lagrangian before the computation of the Feynman rules.

  According to (6.13), for the unphysical Higgs:

$$\Phi = \begin{pmatrix} \Phi[1] \\ \Phi[2] \end{pmatrix},$$

$$\Phi[1] \to 0\,,$$

$$\Phi[2] \to \frac{(v+h_0)}{\sqrt{2}} = \frac{1}{\sqrt{2}}\left(v + \left(\sin\theta + 6\gamma\frac{\xi}{Z}\cos\theta\right) R + \left(\cos\theta - 6\gamma\frac{\xi}{Z}\sin\theta\right) H\right);$$

  for the unphysical radion:

$$R0 \to \frac{1}{Z}\cos\theta\, R - \frac{1}{Z}\sin\theta\, H\,.$$

- **Model Parameters.** In this block, all the model parameters (such as coupling constants, mixing angles, VEVs, *etc.*) are implemented. The parameters are classified as external (independent) and internal (depending on one or several of the other internal and/or external parameters of the model). For example, the masses of the unphysical radion and Higgs (6.19), (6.19) and the mixing angle (6.16) are defined as internal parameters.

- **Lagrangian.** The relevant Lagrangian should be defined in terms of the names of the particle classes, specifying the indices and eventually using 'Special symbols for Lagrangians' (see the rules in [78]). A Lagrangian can be defined as a full expression or as additive pieces.

  The parts of the Lagrangian responsible for different radion and Higgs interactions in the RS model are collected in Appendix C.

The Feynman rules that had been obtained after running the `FeynRules` package with the RS model-file were verified on examples from the literature [71]. Thereby, the automated calculation of all the underlying Feynman rules of the RS model has been achieved with a possibility of easy variation of the parameters and the Lagrangian structure, which significantly simplifies the study of the Higgs and radion phenomenology.

7.2 Constraints on the RS model parameters with Lilith

The possible presence of BSM physics around the electroweak scale can be tested in the study of the properties of the Higgs boson with mass around 125 GeV observed at the LHC. Indeed, the parameter spaces of the BSM scenarios affecting the properties of the Higgs boson can be constrained from the measurements presented in terms of signal strengths.

Following this idea, we have employed `Lilith` [79, 80] — a public tool developed for constraining a wide class of new physics scenarios from signal strength measurements performed at the LHC and the Tevatron. It is a Python library that can also be used in C and C++/ROOT programs. The Higgs likelihood is based on numerous experimental results stored in XML database and is evaluated from the user input given in XML format in terms of reduced couplings or signal strengths.

Deviations from the SM can be parametrized by introducing the reduced couplings, which can be demonstrated by rewriting the usual expression for the signal strength μ in the following way [79]

$$\begin{aligned}\mu &= \sum_{X,Y} \mathrm{eff}_{X,Y} \frac{\sigma(X)\mathcal{B}(H\to Y)}{\sigma^{SM}(X)\mathcal{B}^{SM}(H\to Y)} \\ &= \sum_{X,Y} \mathrm{eff}_{X,Y} \times \frac{C_X^2\sigma^{SM}(X)}{\sigma^{SM}(X)} \times \frac{C_Y^2\Gamma_Y^{SM}}{\Gamma_Y^{SM}} \times \frac{\Gamma_H^{SM}}{\sum_Y C_Y^2\Gamma_Y^{SM}} \\ &= \frac{1}{\sum_Y C_Y^2\mathcal{B}^{SM}(H\to Y)} \sum_{X,Y} \mathrm{eff}_{X,Y} C_X^2 C_Y^2 \,, \end{aligned} \tag{7.1}$$

where X and Y stand for various production and decay modes of the SM Higgs boson, respectively, $X \in (ggH,\ \mathrm{VBF},\ VH,\ ttH)$ and $Y \in (\gamma\gamma,\ ZZ^*,\ WW^*,\ b\bar{b},\ \tau\tau, \dots)$; the $\mathrm{eff}_{X,Y}$

are "reduced efficiencies" corresponding to relative contribution of each combination for the production and decay of a Higgs boson to the signal; Γ_H^{SM} is the total decay width of the SM Higgs boson, and the cross section (partial width) for each process $X(Y)$ is scaled with a factor C_X^2 (C_Y^2) compared to the SM expectation. The term $\sum_Y C_Y^2 \mathcal{B}^{SM}(H \to Y)$ accounts for the scaling of the total width of the Higgs boson (in the assumption of the narrow-width approximation for the BSM scenarios).

Thus, a BSM Higgs interaction Lagrangian can be expressed in terms of bosonic ($C_{W,Z}$) and fermionic ($C_{t,b,c,\tau}$) reduced couplings as

$$\begin{aligned}\mathcal{L} =& g\left[C_W m_W W^\mu W_\mu + C_Z \frac{m_Z}{\cos\theta_W} Z^\mu Z_\mu\right] H \\ &+ g\left[-C_t \frac{m_t}{2m_W} t\bar{t} - C_b \frac{m_b}{2m_W} b\bar{b} - C_c \frac{m_c}{2m_W} c\bar{c} - C_\tau \frac{m_\tau}{2m_W} \tau\bar{\tau}\right] H\,,\end{aligned} \tag{7.2}$$

which recovers the SM case in the limit $C_{W,Z,t,b,c,\tau} \to 1$.

At leading order in perturbation theory, the scaling factors $C_{X,Y}$ from (7.1) can be directly identified with the reduced couplings $C_{W,Z,t,b,c,\tau}$ from (7.2) for processes involving just one coupling to the Higgs boson,

$$C_{WH}^2 = C_W^2,\ C_{ZH}^2 = C_Z^2,\ C_{ttH}^2 = C_t^2,\ C_{ff}^2 = C_f^2,\ C_{VV}^2 = C_V^2\,, \tag{7.3}$$

with $f = b, c, \tau$ and $V = W, Z$.

There is no direct identification for ggH and VBF production and loop-induced decays $H \to gg$, $\gamma\gamma$, $Z\gamma$ in the general case, and the reduced couplings for these processes are given by a combination of reduced couplings C_i, weighted according to the contribution of the particle i to the process:

$$C_{ggH}^2 = \frac{\sum_{i,j=t,b,c} C_i C_j \sigma_{ij}^{SM}(ggH)}{\sum_{i,j=t,b,c} \sigma_{ij}^{SM}(ggH)}\,, \quad C_{VBF}^2 = \frac{\sum_{i,j=W,Z} C_i C_j \sigma_{ij}^{SM}(\text{VBF})}{\sum_{i,j=W,Z} \sigma_{ij}^{SM}(\text{VBF})}\,, \tag{7.4}$$

$$C_{gg}^2 = \frac{\sum_{i,j=t,b,c} C_i C_j \Gamma_{ij}^{SM}(H \to gg)}{\sum_{i,j=t,b,c} \Gamma_{ij}^{SM}(H \to gg)}\,, \quad C_{\gamma\gamma,Z\gamma}^2 = \frac{\sum_{i,j=W,t,b,c,\tau} C_i C_j \Gamma_{ij}^{SM}(H \to \gamma\gamma, Z\gamma)}{\sum_{i,j=W,t,b,c,\tau} \Gamma_{ij}^{SM}(H \to \gamma\gamma, Z\gamma)}\,, \tag{7.5}$$

where the σ_{ij}^{SM} are the different contributions to the cross section in the SM (for $i = j$ it corresponds to the cross section from the particle i alone, and for $i \neq j$ — to that from interference between the particles i and j); Γ_{ij}^{SM} are the SM partial widths of the considered processes.

Note that such a parametrization is possible only in case if the BSM couplings of our interest have the same tensor structure as the SM ones.
This is not the case for the HWW and HZZ couplings in the RS model (the full list is

provided in Appendix C):

$$V_{HWW} = igm_W \left(g_h - g_h^r \kappa_W\right) \left[\eta^{\mu\nu} - 2g_h^W \left(\eta^{\mu\nu} p_1 \cdot p_2 - p_1^\nu p_2^\mu\right)\right] , \tag{7.6}$$

$$V_{HZZ} = ig\frac{m_Z}{\cos\theta_W} \left(g_h - g_h^r \kappa_Z\right) \left[\eta^{\mu\nu} - 2g_h^Z \left(\eta^{\mu\nu} p_1 \cdot p_2 - p_1^\nu p_2^\mu\right)\right] , \tag{7.7}$$

where $g_h = (D + \gamma B)$, $g_h^r = \gamma B$ with coefficients B, D and γ from the Higgs-radion mixing matrix (6.13) in Sec. 6.1.2 (for all notations see Sec. C.2 in Appendix C).
As we can see, these couplings differ from the SM-like ones by the tensor contributions $-2g_h^{W,Z} \left(\eta^{\mu\nu} p_1 \cdot p_2 - p_1^\nu p_2^\mu\right)$ (or just $2g_h^{W,Z} p_1^\nu p_2^\mu$ if we extract the SM-like part $\eta^{\mu\nu}(1 - 2g_h^{W,Z} p_1 \cdot p_2)$). However, the numeric estimation of either of these terms gives a negligible result (see Fig. 7.1) and, therefore, these contributions can be omitted in the further analysis.

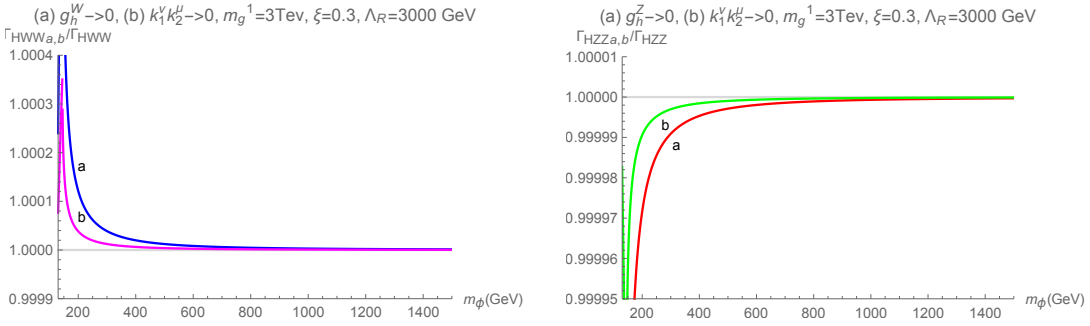


Figure 7.1: Ratio of the width without the contribution $-2g_h^{W,Z} \left(\eta^{\mu\nu} p_1 \cdot p_2 - p_1^\nu p_2^\mu\right)$ (a) and without the contribution $2g_h^{W,Z} p_1^\nu p_2^\mu$ (b) to the full widths expression, for the precesses $H \to WW$ (on the left) and $H \to ZZ$ (on the right). As we can see, for all cases this ratio is very close to 1.

Thereby, the reduced couplings for the RS model are defined as

$$C_W = g_h - g_h^r \kappa_W , \quad C_Z = g_h - g_h^r \kappa_Z , \tag{7.8}$$

$$C_g = g_h - 2g_h^r \frac{\left(b_3 + \frac{4\pi}{\alpha_S k b_0}\right)}{F_{1/2}(\tau_t)} , \quad C_\gamma = g_h - g_h^r \frac{\left(b_2 + b_Y + \frac{4\pi}{\alpha_{EW} k b_0}\right)}{F_1(\tau_W) + F_{1/2}(\tau_t)} , \tag{7.9}$$

$$C_F = C_t = C_b = C_c = C_\tau = g_h , \tag{7.10}$$

where the loop functions $F_1, F_{1/2}$ are defined in Sec.C.2 in Appendix C and include only the contribution from the top quark in the fermion loop.

The reduced couplings (7.8)-(7.10) are used as an input for `Lilith` via the XML user input file and depend on the four parameters of the RS model: the mass of the radion dominated state m_ϕ, the radion VEV Λ_ϕ, the ξ parameter related to the mixing angle (6.16), and the first KK-gluon mass m_1^g related to Λ_ϕ and k/M_{Pl} (see [72, 76, 77]). As a

result, the likelihood L ($-2\log(L) = \chi^2$) is obtained. The scan of the parameter space (m_ϕ, Λ_ϕ, ξ, m_1^g) can give the best-fit values of the reduced couplings. For this purpose, minimization of $-2\log L$ can be performed using iminuit (a Python interface to the Minuit 2 package).

An example of such a minimization of the log-likelihood for the radion mass and the mixing parameter ξ in ranges 130 GeV $\leq m_\phi \leq$ 300 GeV, $-0.6 \leq \xi \leq 0.6$, and with fixed $\Lambda_\phi = 2.5$ TeV, $m_1^g = 3$ TeV (see Fig. 7.2) gave the following best fit:
$m_{\phi,min} = 130.0$ GeV, $\xi = 0.00808080808081$, $-2\log L_{min} = 51.0467887402$
$C_W = 1.00336932206$, $C_Z = 1.00323429515$, $C_F = 1.003812078$, $C_\gamma = 0.989982773203$, $C_g = 1.07344612964$.

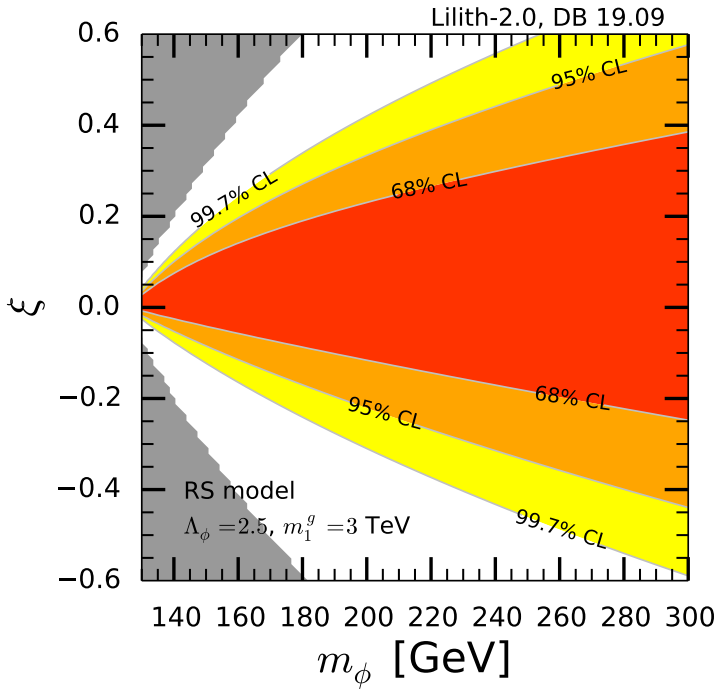


Figure 7.2: Constraints on the radion mass and parameter ξ in the (ξ, m_ϕ) plane for $\Lambda_\phi = 2.5$ TeV and $m_1^g = 3$ TeV. The red, orange and yellow filled surfaces correspond to the allowed 68%, 95% and 99.7% CL regions, respectively.

As can be seen, the constraints from the Higgs signal strength measurements impose only mild constraints on the RS model. In a next step, it would therefore be necessary to include limits from direct BSM searches before studying the scalar pair production (hh, ϕh, $\phi\phi$) in the RS model at the LHC. While working on the LHC Higgs constraints on the RS model a paper appeared [81] which, unfortunately, covers to a good degree what we wanted to do.

Chapter 8

Conclusions and Outlook

We have considered the Randall-Sundrum model with one extra dimension which is one of the possible extensions of the SM. The model solves the hierarchy problem and predicts the existence of an additional scalar particle, the radion, and allows mixing between the radion and the SM Higgs boson. The processes of our main interest are the pair radion-radion, Higgs-Higgs and radion-Higgs production, which involve the trilinear Higgs coupling, being sensitive to deviations from the SM. In order to study the scalar pair production, we have implemented the RS model in the FeynRules package and achieved the automatic calculation of the Feynman rules with a possibility to vary the model parameters and the Lagrangian and to obtain the results rather quickly. In order to identify the relevant parameter space we have used `Lilith` 2.0 to derive limits from the LHC Higgs measurements, which are however weak. Therefore, it would be necessary to obtain more limits from direct BSM searches. This has been realized in the recent study by A. Ahmed, A. Mariotti and S. Najjari [81] which turned out to be quite similar to what we wanted to do. Once the allowed parameter space has been identified, our plan was to use the `FeynRules` interface to `MadGraph` in order to get the relevant observables and to study the collider phenomenology of the RS model, in particular, the radion-radion, Higgs-Higgs and radion-Higgs production at loop level and the Higgs coupling modifications.

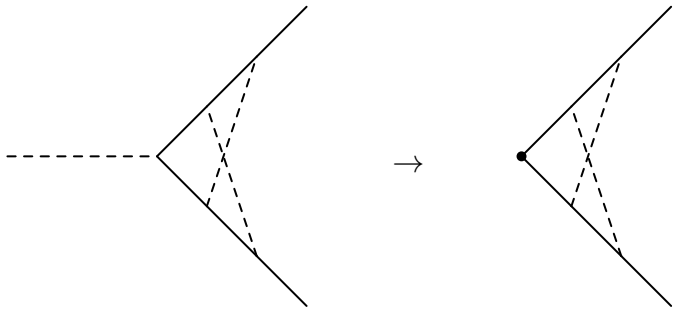
Apart from the original RS model discussed here, it would be interesting to consider more general scalar sectors discussed in Sec. 6.2, which is possible due to the created `FeynRules` model-file that can be easily adapted to such cases.

Appendix A

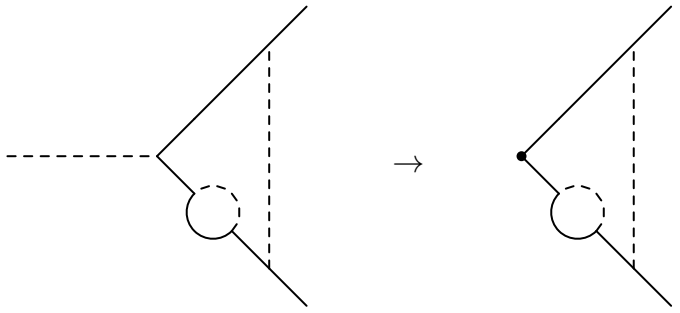
The dummy field method at two-loop

In this appendix, we list all two-loop vertex corrections which are needed to obtain the β functions for dimensionful parameters.

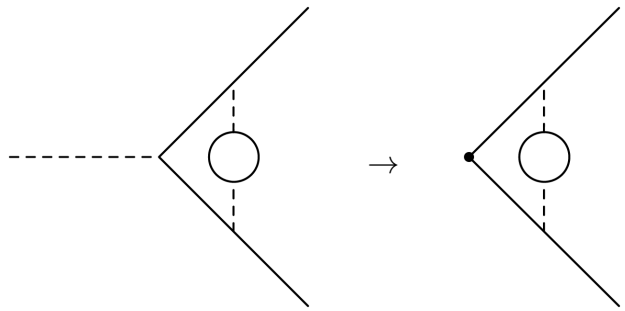
A.1 Fermion mass



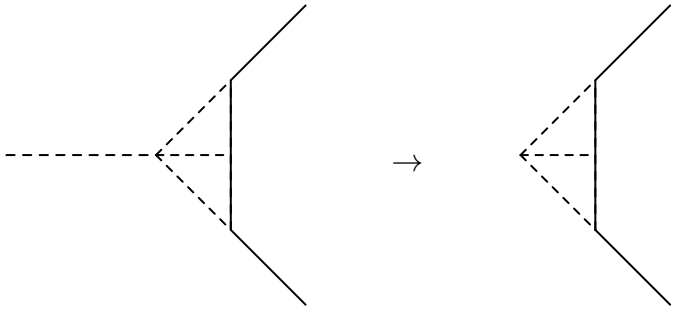
$$Y^c Y^{\dagger b} Y^a (Y^{\dagger c} Y^b - Y^{\dagger b} Y^c) \qquad Y^c Y^{\dagger b} m_f (Y^{\dagger c} Y^b - Y^{\dagger b} Y^c) \tag{A.1}$$



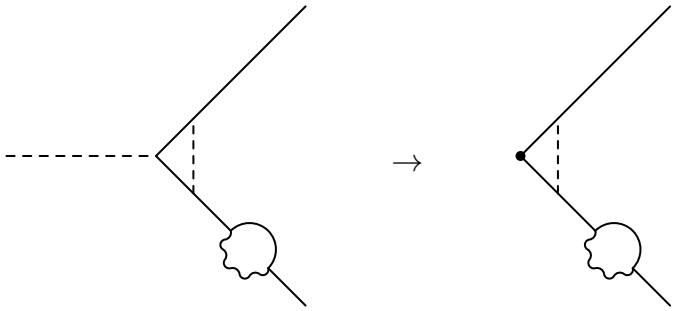
$$Y^b (Y_2(F) Y^{\dagger a} + Y^{\dagger a} Y_2^\dagger(F)) Y^b \qquad Y^b (Y_2(F) m_f^\dagger + m_f^\dagger Y_2^\dagger(F)) Y^b \tag{A.2}$$



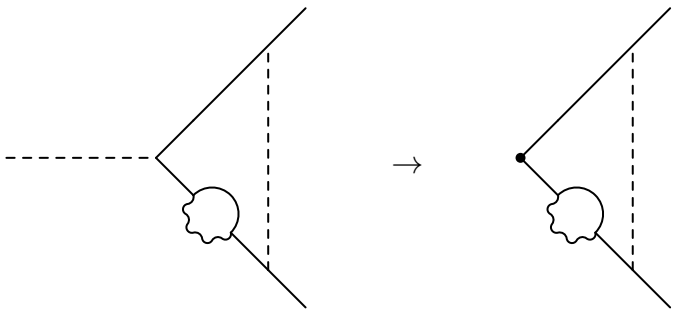
$$Y_2^{bc}(S)Y^bY^{\dagger a}Y^c \qquad Y_2^{bc}(S)Y^bm_f^\dagger Y^c \tag{A.3}$$



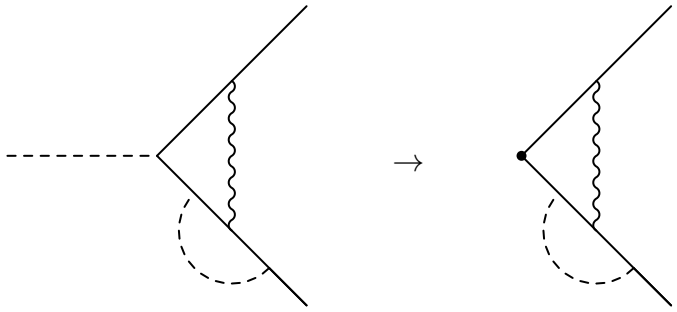
$$\lambda_{abcd}Y^bY^{\dagger c}Y^d \qquad h_{abc}Y^aY^{\dagger b}Y^c \tag{A.4}$$



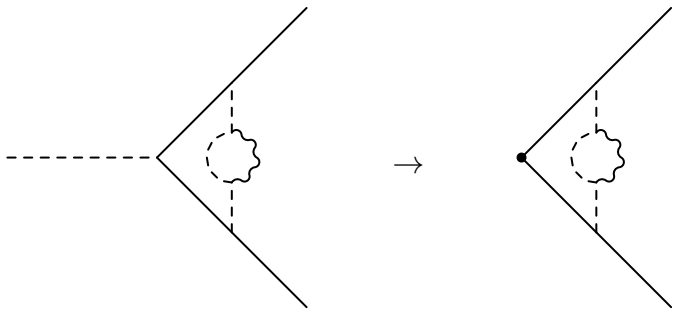
$$g^2\{C_2(F), Y^bY^{\dagger a}Y^b\} \qquad g^2\{C_2(F), Y^bm_fY^b\} \tag{A.5}$$



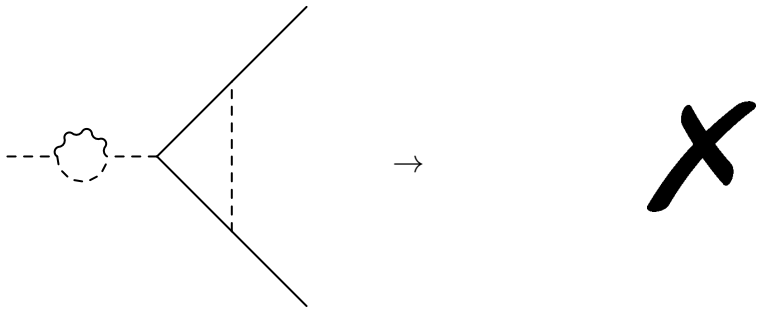
$$g^2Y^b\{C_2(F), Y^{\dagger a}\}Y^b \qquad g^2Y^b\{C_2(F), m_f^\dagger\}Y^b \tag{A.6}$$



$$g^2 H^a_{2t} = g^2(t^{A*} Y^a Y^{\dagger b} t^{A*} Y^b + Y^b t^A Y^{\dagger b} Y^a t^A) \qquad g^2(t^{A*} m_f Y^{\dagger b} t^{A*} Y^b + Y^b t^A Y^{\dagger b} m_f t^A) \tag{A.7}$$



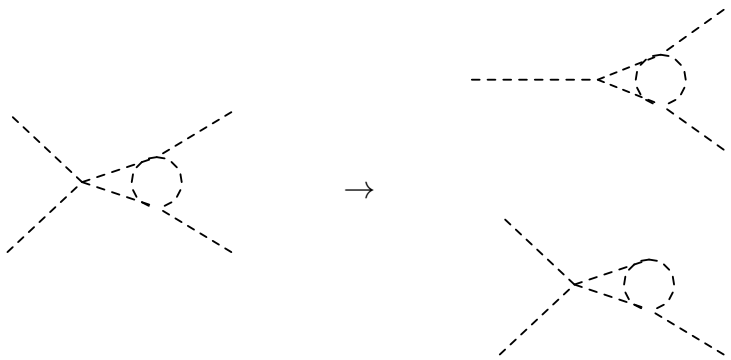
$$g^2 C_2^{bc}(S) Y^b Y^{\dagger a} Y^c \qquad g^2 C_2^{bc}(S) Y^b m_f^\dagger Y^c \tag{A.8}$$



$$g^2 C_2^{ac} Y^b Y^{\dagger c} Y^b \qquad 0 \tag{A.9}$$

A.2 Cubic scalar coupling

1. Scalar-only contributions:



$$\overline{\Lambda}^3_{abcd} = \frac{1}{4}\sum_{per} \lambda_{abef}\lambda_{cegh}\lambda_{dfgh} \qquad \overline{\Lambda}^3_{abc} = \frac{1}{2}\sum_{per}[h_{aef}\lambda_{begh}\lambda_{cfgh} + \lambda_{abef}\lambda_{cegh}h_{fgh}]$$

Let us show explicitly how this calculation, involving the permutations over indices a, b, c, $\hat{d}$, has been conducted:

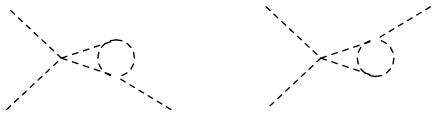
$$\begin{aligned}
\overline{\Lambda}^3_{abc\hat{d}} &= \frac{1}{4}\sum_{per} \lambda_{abef}\lambda_{cegh}\lambda_{\hat{d}fgh} \\
&= \frac{1}{4}\Big[\lambda_{abef}\lambda_{cegh}\cancel{\lambda_{\hat{d}fgh}}^{h_{fgh}} + \lambda_{baef}\lambda_{cegh}\cancel{\lambda_{\hat{d}fgh}}^{h_{fgh}} \\
&\quad + \lambda_{caef}\lambda_{begh}h_{fgh} + \lambda_{acef}\lambda_{begh}h_{fgh} \\
&\quad + \lambda_{bcef}\lambda_{aegh}h_{fgh} + \lambda_{cbef}\lambda_{aegh}h_{fgh} \\
&\quad + \\
&\quad + \lambda_{abef}\cancel{\lambda_{\hat{d}egh}}^{h_{egh}}\lambda_{cfgh} + \lambda_{baef}\cancel{\lambda_{\hat{d}egh}}^{h_{egh}}\lambda_{cfgh} \\
&\quad + \lambda_{caef}h_{egh}\lambda_{bfgh} + \lambda_{acef}h_{egh}\lambda_{bfgh} \\
&\quad + \lambda_{bcef}h_{egh}\lambda_{afgh} + \lambda_{cbef}h_{egh}\lambda_{afgh} \\
&\quad + \\
&\quad + \cancel{\lambda_{\hat{a}def}}^{h_{aef}}\lambda_{begh}\lambda_{cfgh} + \cancel{\lambda_{\hat{a}aef}}^{h_{aef}}\lambda_{begh}\lambda_{cfgh} \\
&\quad + h_{aef}\lambda_{cegh}\lambda_{bfgh} + h_{aef}\lambda_{cegh}\lambda_{bfgh} \\
&\quad + h_{bef}\lambda_{cegh}\lambda_{afgh} + h_{bef}\lambda_{cegh}\lambda_{afgh} \\
&\quad + h_{bef}\lambda_{aegh}\lambda_{cfgh} + h_{bef}\lambda_{aegh}\lambda_{cfgh}
\end{aligned}$$

$$
\begin{aligned}
&+ h_{cef}\lambda_{begh}\lambda_{afgh} + h_{cef}\lambda_{begh}\lambda_{afgh} \\
&+ h_{cef}\lambda_{aegh}\lambda_{bfgh} + h_{cef}\lambda_{aegh}\lambda_{bfgh}] \\
&= \frac{1}{4}\left[\sum_{per}\lambda_{abef}\lambda_{cegh}h_{fgh} + \sum_{per}\lambda_{abef}h_{egh}\lambda_{cfgh}\right. \\
&\left. + 2\sum_{per}h_{aef}\lambda_{begh}\lambda_{cfgh}\right] \\
&= \frac{1}{4}\left[2\sum_{per}\lambda_{abef}\lambda_{cegh}h_{fgh} + 2\sum_{per}h_{aef}\lambda_{begh}\lambda_{cfgh}\right] \\
&= \frac{1}{2}\sum_{per}[h_{aef}\lambda_{begh}\lambda_{cfgh} + \lambda_{abef}\lambda_{cegh}h_{fgh}]\,. \qquad \text{(A.10)}
\end{aligned}
$$

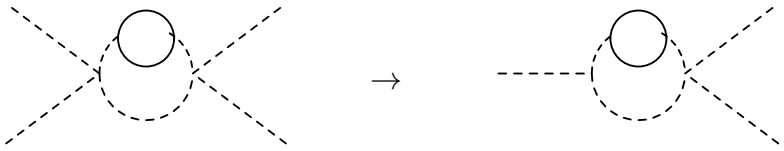
Here we took into account that

$$\sum_{per}\lambda_{abef}\lambda_{cegh}h_{fgh} = \sum_{per}\lambda_{abef}h_{egh}\lambda_{cfgh}$$

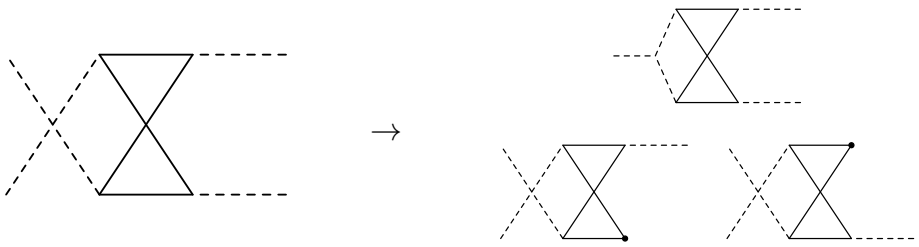
due to the symmetry of the corresponding diagrams:



2. Scalar-Fermion contributions:



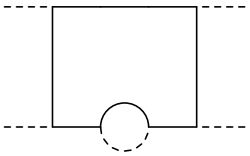
$$\overline{\Lambda}^{2Y}_{abcd} = \frac{1}{8}\sum_{per}Y_2^{fg}(S)\lambda_{abef}\lambda_{cdeg} \qquad \overline{\Lambda}^{2Y}_{abc} = \frac{1}{2}\sum_{per}Y_2^{fg}(S)\lambda_{abef}h_{ceg} \qquad \text{(A.11)}$$



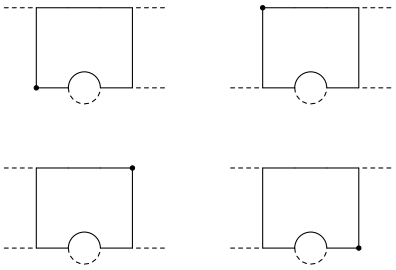
$$\overline{H}^{\lambda}_{abcd} = \frac{1}{8}\sum_{per}\lambda_{abef}\mathrm{Tr}(Y^c Y^{\dagger e} Y^d Y^{\dagger f} + Y^{\dagger c} Y^e Y^{\dagger d} Y^f)$$

$$\overline{H}^{h}_{abc} + \overline{H}^{\lambda m}_{abc} = \frac{1}{4}\sum_{per} h_{aef}\mathrm{Tr}(Y^b Y^{\dagger e} Y^c Y^{\dagger f} + Y^{\dagger b} Y^e Y^{\dagger c} Y^f) + \frac{1}{4}\sum_{per}\lambda_{abef}\mathrm{Tr}(m_f Y^{\dagger e} Y^c Y^{\dagger f} + Y^{\dagger c} Y^e m_f^{\dagger} Y^f)$$

(A.12)



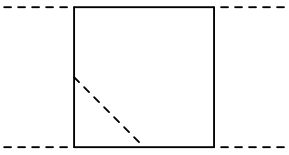
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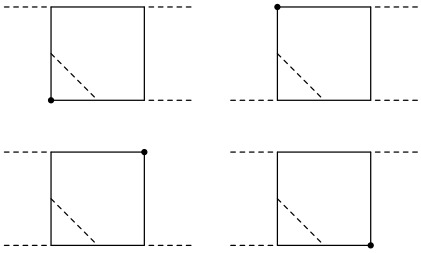
$$H^{Y}_{abcd} = \sum_{per}\mathrm{Tr}(Y_2(F) Y^{\dagger a} Y^b Y^{\dagger c} Y^d)$$

$$H^{Y}_{abc} = \sum_{per}\mathrm{Tr}\Big(Y_2(F)[m_f^{\dagger} Y^a Y^{\dagger b} Y^c + Y^{\dagger a} m_f Y^{\dagger b} Y^c + Y^{\dagger a} Y^b m_f^{\dagger} Y^c + Y^{\dagger a} Y^b Y^{\dagger c} m_f]\Big)$$

(A.13)



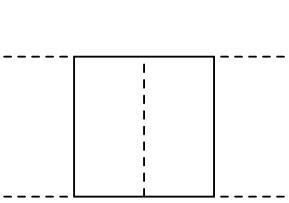
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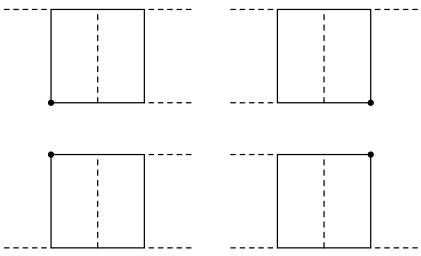
$$\overline{H}^{Y}_{abcd} = \sum_{per}\frac{1}{2}\mathrm{Tr}(Y^e Y^{\dagger a} Y^e Y^{\dagger b} Y^c Y^{\dagger d} + Y^{\dagger e} Y^a Y^{\dagger e} Y^b Y^{\dagger c} Y^d)$$

$$\overline{H}^{Y}_{abc} = \frac{1}{2}\sum_{per}\mathrm{Tr}(Y^e m_f^{\dagger} Y^e Y^{\dagger a} Y^b Y^{\dagger c} + Y^e Y^{\dagger a} Y^e m_f^{\dagger} Y^b Y^{\dagger c} + Y^e Y^{\dagger a} Y^e Y^{\dagger b} m_f Y^{\dagger c} + Y^e Y^{\dagger a} Y^e Y^{\dagger b} Y^c m_f^{\dagger} + \mathrm{h.c.})$$

(A.14)



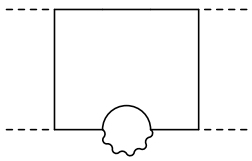
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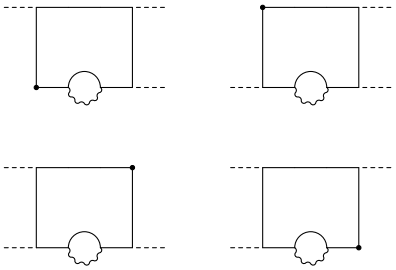
$$H^3_{abcd} = \frac{1}{2}\sum_{per} \mathrm{Tr}(Y^a Y^{\dagger b} Y^e Y^{\dagger c} Y^d Y^{\dagger e})$$

$$\begin{aligned} H^a_{abc} = \frac{1}{2}\sum_{per} \mathrm{Tr}(&m_f Y^{a\dagger} Y^e Y^{\dagger b} Y^c Y^{\dagger e} \\ &+ Y^a m_f^\dagger Y^e Y^{\dagger b} Y^c Y^{\dagger e} \\ &+ Y^a Y^{\dagger b} Y^e m_f^\dagger Y^c Y^{\dagger e} \\ &+ Y^a Y^{\dagger b} Y^e Y^{\dagger c} m_f Y^{\dagger e}) \end{aligned}$$

(A.15)



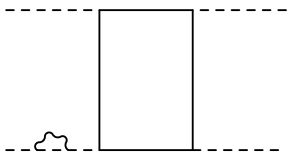
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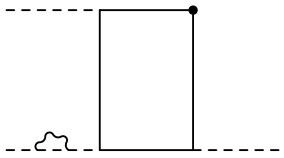
$$H^F_{abcd} = \sum_{per} \mathrm{Tr}(\{C_2(F), Y^a\} Y^{\dagger b} Y^c Y^{\dagger d})$$

$$\begin{aligned} H^F_{abc} = \sum_{per} \mathrm{Tr}(&\{C_2(F), m_f\} Y^{a\dagger} Y^b Y^{\dagger c} \\ &+ \{C_2(F), Y^a\} m_f^\dagger Y^b Y^{\dagger c} \\ &+ \{C_2(F), Y^a\} Y^{\dagger b} m_f Y^{\dagger c} \\ &+ \{C_2(F), Y^a\} Y^{\dagger b} Y^c m_f^\dagger) \end{aligned}$$

(A.16)

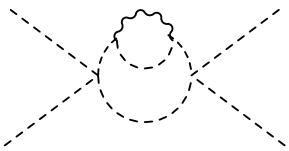


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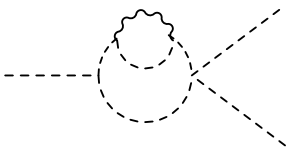


$$H^S_{abcd} = \sum_i C_2(i) H_{abcd} \qquad H^S_{abc} = \sum_i C_2(i) H_{abc} \tag{A.17}$$

3. Scalar-Vector contributions

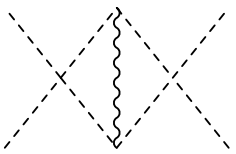


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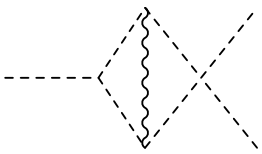


$$\overline{\Lambda}^{2S}_{abcd} = \frac{1}{8}\sum_{per} C_2^{fg}(S) \lambda_{abef} \lambda_{cdeg} \qquad \overline{\Lambda}^{2S}_{abc} = \frac{1}{2}\sum_{per} C_2^{fg}(S) h_{aef} \lambda_{bceg}$$

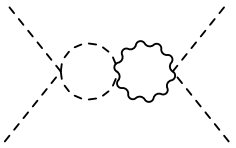
(A.18)



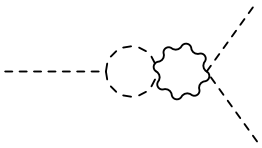
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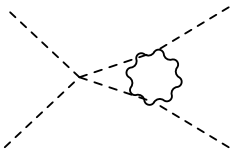
$$\overline{\Lambda}^{2g}_{abcd} = \frac{1}{8}\sum_{per} \lambda_{abef}\lambda_{cdgh}\theta^A_{eg}\theta^A_{fh} \qquad \overline{\Lambda}^{2g}_{abc} = \frac{1}{2}\sum_{per} h_{aef}\lambda_{bcgh}\theta^A_{eg}\theta^A_{fh} \tag{A.19}$$



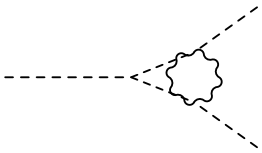
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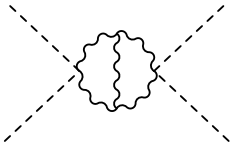
$$A^{\lambda}_{abcd} = \frac{1}{4}\sum_{per} \lambda_{abef}\{\theta^A,\theta^B\}_{ef}\{\theta^A,\theta^B\}_{cd} \qquad A^{\lambda}_{abc} = \frac{1}{2}\sum_{per} h_{aef}\{\theta^A,\theta^B\}_{ef}\{\theta^A,\theta^B\}_{bc} \tag{A.20}$$



$\rightarrow$



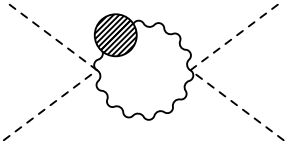
$$\overline{A}^{\lambda}_{abcd} = \frac{1}{4}\sum_{per} \lambda_{abef}\{\theta^A,\theta^B\}_{ce}\{\theta^A,\theta^B\}_{df} \qquad \overline{A}^{\lambda}_{abc} = \frac{1}{2}\sum_{per} h_{aef}\{\theta^A,\theta^B\}_{be}\{\theta^A,\theta^B\}_{cf} \tag{A.21}$$



$\rightarrow$



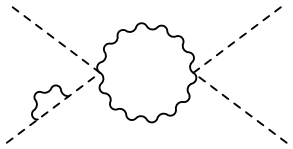
$$A^{g}_{abcd} = \frac{1}{8}f^{ACE}f^{BDE}\sum_{per}\{\theta^A,\theta^B\}_{ab}\{\theta^C,\theta^D\}_{cd} \qquad 0 \tag{A.22}$$



$\rightarrow$



$$XA_{abcd} = X\{\theta^A,\theta^B\}_{ab}\{\theta^A,\theta^B\}_{cd} \qquad 0 \tag{A.23}$$

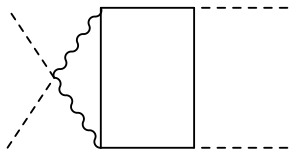


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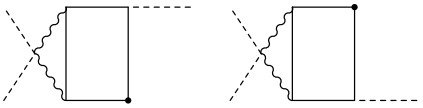


$$A^S_{abcd} = \sum_i C_2(i)\{\theta^A,\theta^B\}_{ab}\{\theta^A,\theta^B\}_{cd} \qquad 0 \qquad \text{(A.24)}$$

4. Scalar-Fermion-Vector contributions

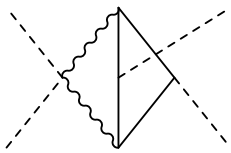


$\rightarrow$



$$B^Y_{abcd} = \frac{1}{4}\sum_{per}\{\theta^A,\theta^B\}_{ab}\mathrm{Tr}(t^{A*}t^{B*}Y^cY^{\dagger d} + Y^ct^At^BY^{\dagger d})$$

$$B^Y_{abc} = \frac{1}{4}\sum_{per}\{\theta^A,\theta^B\}_{ab}\mathrm{Tr}(t^{A*}t^{B*}m_fY^{\dagger c} + m_ft^At^BY^{\dagger c} + t^{A*}t^{B*}Y^cm_f^\dagger + Y^ct^At^Bm_f^\dagger) \qquad \text{(A.25)}$$



$\rightarrow$

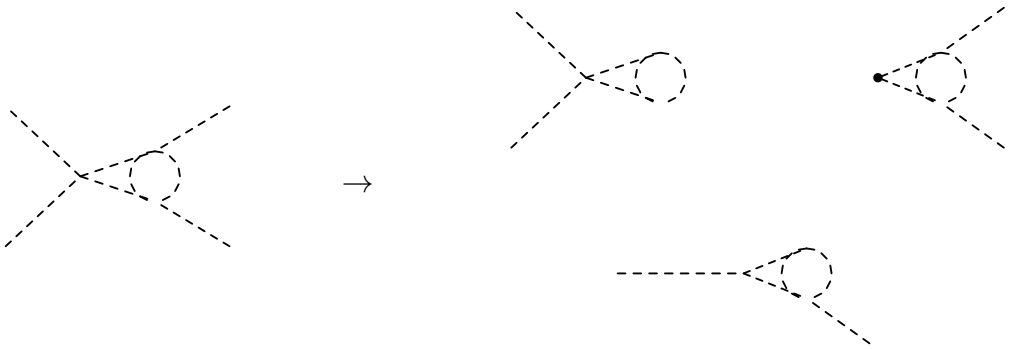


$$\overline{B}^Y_{abcd} = \frac{1}{4}\sum_{per}\{\theta^A,\theta^B\}_{ab}\mathrm{Tr}(t^{A*}Y^ct^BY^{\dagger d})$$

$$\overline{B}^Y_{abc} = \frac{1}{4}\sum_{per}\{\theta^A,\theta^B\}_{ab}\mathrm{Tr}(t^{A*}m_ft^BY^{\dagger c} + t^{A*}Y^ct^Bm_f^\dagger) \qquad \text{(A.26)}$$

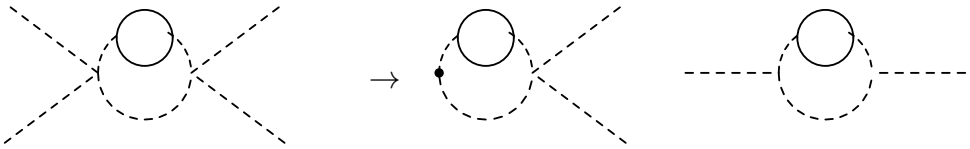
A.3 Bilinear scalar

1. Scalar-only contributions:

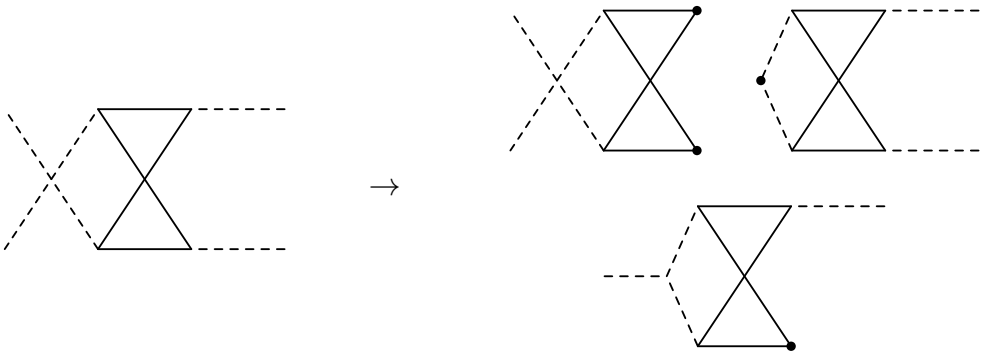


$$\overline{\Lambda}^3_{abcd} = \frac{1}{4}\sum_{per}\lambda_{abef}\lambda_{cegh}\lambda_{dfgh} \qquad \overline{\Lambda}^3_{ab} = \lambda_{abef}h_{egl}h_{fgl} + 2m^2_{ef}\lambda_{aegl}\lambda_{bfgl} + 2\sum_{per}h_{aef}h_{fgl}\lambda_{begl} \tag{A.27}$$

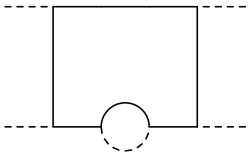
2. Scalar-Fermion contributions:



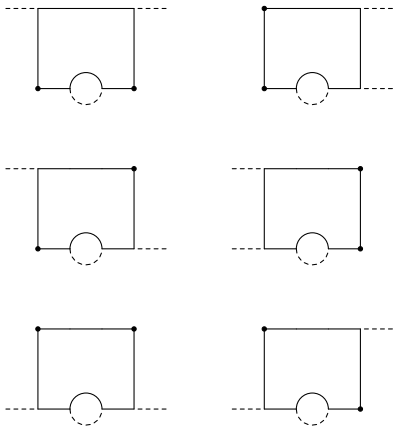
$$\overline{\Lambda}^{2Y}_{abcd} = \frac{1}{8}\sum_{per}Y_2^{fg}(S)\lambda_{abef}\lambda_{cdeg} \qquad \overline{\Lambda}^{2Y}_{ab} = 2Y_2^{fg}(S)(m^2_{eg}\lambda_{abef} + h_{aef}h_{beg}) \tag{A.28}$$



$$\overline{H}^{\lambda}_{abcd} = \frac{1}{8}\sum_{per}\lambda_{abef}\mathrm{Tr}(Y^cY^{\dagger e}Y^dY^{\dagger f} + Y^{\dagger c}Y^eY^{\dagger d}Y^f) \qquad \overline{H}^{\lambda}_{ab} = \frac{1}{2}\lambda_{abef}\mathrm{Tr}(m_fY^{\dagger e}m_fY^{\dagger f} + \mathrm{h.c.}) + m^2_{ef}\mathrm{Tr}(Y^aY^{\dagger e}Y^bY^{\dagger f} + \mathrm{h.c.}) + \sum_{per}h_{aef}\mathrm{Tr}(Y^bY^{\dagger e}m_fY^{\dagger f} + \mathrm{h.c.}) \tag{A.29}$$



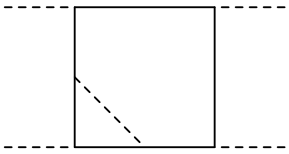
$\rightarrow$



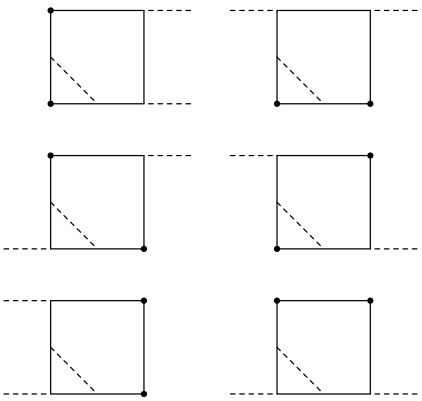
$$H^Y_{abcd} = \sum_{per} \mathrm{Tr}(Y_2(F) Y^{\dagger a} Y^b Y^{\dagger c} Y^d)$$

$$H^Y_{ab} = 2 \sum_{per} \Big[\mathrm{Tr}(\{Y_2(F), m_f^\dagger m_f\} Y^{\dagger a} Y^b) + \mathrm{Tr}(Y_2(F) Y^{\dagger a} m_f (Y^{\dagger b} m_f + m_f^\dagger Y^b) + Y_2(F) m_f^\dagger Y^a (Y^{\dagger b} m_f + m_f^\dagger Y^b)) \Big]$$

(A.30)



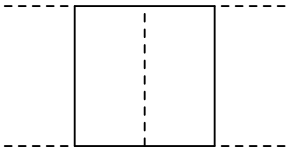
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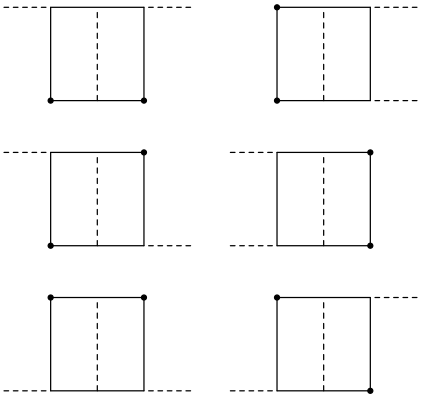
$$\overline{H}^Y_{abcd} = \sum_{per} \tfrac{1}{2} \mathrm{Tr}(Y^e Y^{\dagger a} Y^e Y^{\dagger b} Y^c Y^{\dagger d} + Y^{\dagger e} Y^a Y^{\dagger e} Y^b Y^{\dagger c} Y^d)$$

$$\overline{H}^Y_{ab} = \sum_{per} \Big[\mathrm{Tr}\Big(Y^e Y^{\dagger a} Y^e Y^{\dagger b} m_f m_f^\dagger + Y^e m_f^\dagger Y^e m_f^\dagger Y^a Y^{\dagger b} + (Y^e Y^{\dagger a} Y^e m_f^\dagger + Y^e m_f^\dagger Y^e Y^{\dagger a}) \times (Y^b m_f^\dagger + m_f Y^{\dagger b}) + \mathrm{h.c.} \Big) \Big]$$

(A.31)



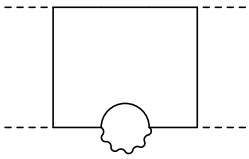
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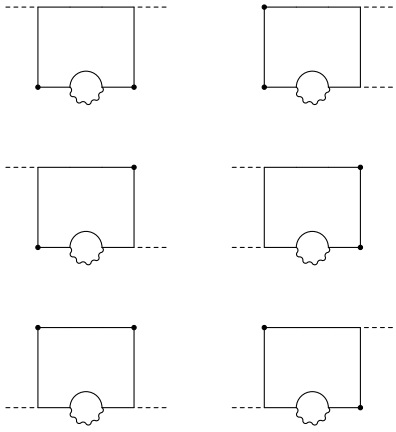
$$H^3_{abcd} = \tfrac{1}{2}\sum_{per} \mathrm{Tr}(Y^a Y^{\dagger b} Y^e Y^{\dagger c} Y^d Y^{\dagger e})$$

$$\begin{aligned} H^3_{ab} = \sum_{per} \Big[\mathrm{Tr}\Big(& Y^a Y^{\dagger b} Y^e m_f^\dagger m_f Y^{\dagger e} \\ & + m_f m_f^\dagger Y^e Y^{\dagger a} Y^b Y^{\dagger e} \\ & Y^a m_f^\dagger Y^e (Y^{\dagger b} m_f + m_f^\dagger Y^b) Y^{\dagger e} \\ & + m_f Y^{\dagger a} Y^e (Y^{\dagger b} m_f + m_f^\dagger Y^b) Y^{\dagger e} \Big) \Big] \end{aligned}$$

(A.32)



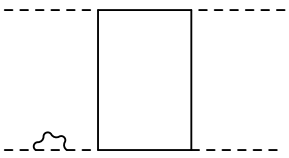
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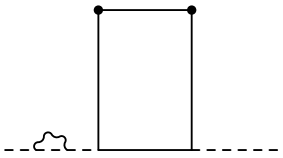
$$H^F_{abcd} = \sum_{per} \mathrm{Tr}(\{C_2(F), Y^a\} Y^{\dagger b} Y^c Y^{\dagger d})$$

$$\begin{aligned} H^F_{ab} = 2\sum_{per} \mathrm{Tr}\big[& \{C_2(F), m_f\} Y^{a\dagger} (Y^b m_f^\dagger + \text{h.c.}) \\ & + \{C_2(F), Y^a\} m_f^\dagger (Y^b m_f^\dagger + \text{h.c.}) \\ & + \{C_2(F), Y^a\} Y^{\dagger b} m_f m_f^\dagger \\ & + \{C_2(F), m_f\} m_f^\dagger Y^a Y^{\dagger b} \big] \end{aligned}$$

(A.33)



$\rightarrow$

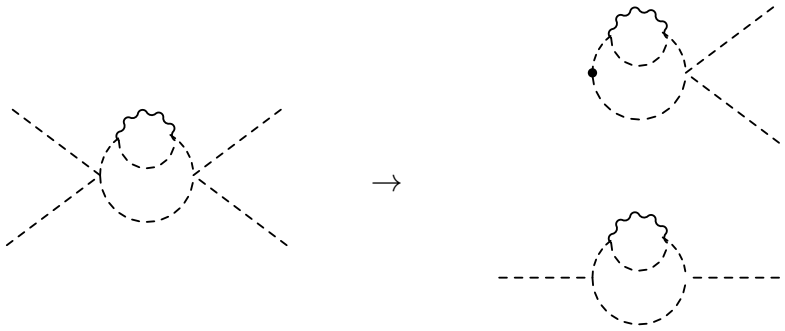


$$H^S_{abcd} = \sum_i C_2(i) H_{abcd} \qquad H^S_{ab} = \sum_i C_2(i) H_{ab}$$

(A.34)

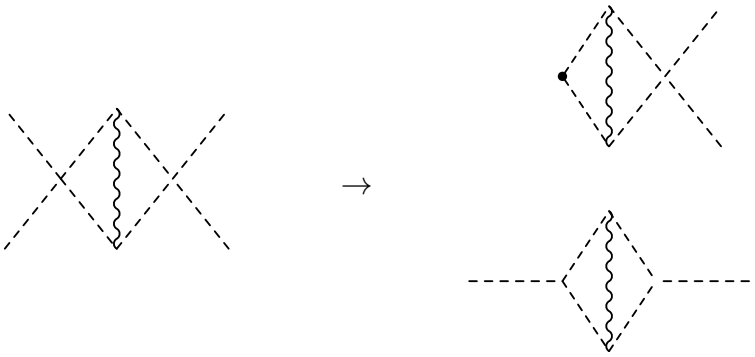
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3. Scalar-Vector contributions



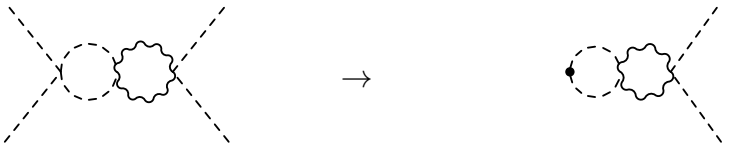
$$\overline{\Lambda}^{2S}_{abcd} = \frac{1}{8}\sum_{per} C_2^{fg}(S)\lambda_{abef}\lambda_{cdeg} \qquad \overline{\Lambda}^{2S}_{ab} = 2C_2^{fg}(S)(\lambda_{abef}m^2_{eg} + h_{aef}h_{beg})$$

(A.35)



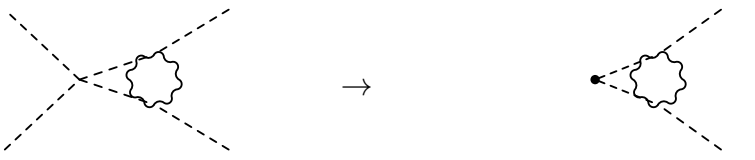
$$\overline{\Lambda}^{2g}_{abcd} = \frac{1}{8}\sum_{per} \lambda_{abef}\lambda_{cdgh}\theta^A_{eg}\theta^A_{fh} \qquad \overline{\Lambda}^{2g}_{ab} = 2(\lambda_{abef}m^2_{gh} + h_{aef}h_{bgh})\theta^A_{eg}\theta^A_{fh}$$

(A.36)



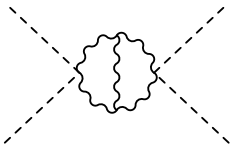
$$A^{\lambda}_{abcd} = \frac{1}{4}\sum_{per} \lambda_{abef}\{\theta^A,\theta^B\}_{ef}\{\theta^A,\theta^B\}_{cd} \qquad A^{\lambda}_{ab} = 2m^2_{ef}\{\theta^A,\theta^B\}_{ef}\{\theta^A,\theta^B\}_{ab}$$

(A.37)



$\overline{A}^{\lambda}_{abcd} =$ $\frac{1}{4}\sum_{per}\lambda_{abef}\{\theta^A,\theta^B\}_{ce}\{\theta^A,\theta^B\}_{df}$	$\overline{A}^{\lambda}_{ab} = 2m^2_{ef}\{\theta^A,\theta^B\}_{ae}\{\theta^A,\theta^B\}_{bf}$

(A.38)

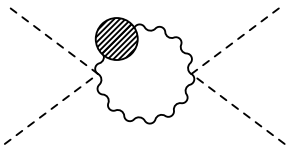


$\rightarrow$



$A^g_{abcd} =$ $\frac{1}{8}f^{ACE}f^{BDE}\sum_{per}\{\theta^A,\theta^B\}_{ab}\{\theta^C,\theta^D\}_{cd}$	0

(A.39)

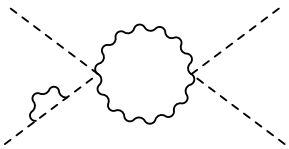


$\rightarrow$



$XA_{abcd} = X\{\theta^A,\theta^B\}_{ab}\{\theta^A,\theta^B\}_{cd}$	0

(A.40)



$\rightarrow$

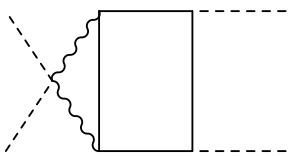


$A^S_{abcd} =$ $\sum_i C_2(i)\{\theta^A,\theta^B\}_{ab}\{\theta^A,\theta^B\}_{cd}$	0

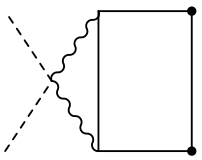
(A.41)

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4. Scalar-Fermion-Vector contributions

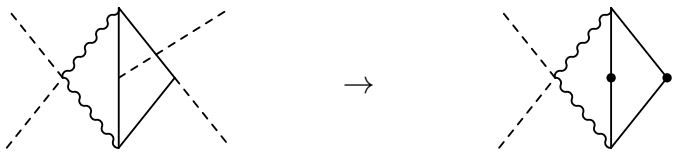


$\rightarrow$



$B^Y_{abcd} = \frac{1}{4}\sum_{per}\{\theta^A,\theta^B\}_{ab}\mathrm{Tr}(t^{A*}t^{B*}Y^cY^{\dagger d}$ $+Y^ct^At^BY^{\dagger d})$	$B^Y_{ab} = \{\theta^A,\theta^B\}_{ab}\mathrm{Tr}(t^{A*}t^{B*}m_fm_f^{\dagger}$ $+m_ft^At^Bm_f^{\dagger})$

(A.42)



$$\overline{B}^{Y}_{abcd} = \frac{1}{4}\sum_{per}\{\theta^A,\theta^B\}_{ab}\mathrm{Tr}(t^{A*}Y^c t^B Y^{\dagger d}) \qquad \overline{B}^{Y}_{abc} = \{\theta^A,\theta^B\}_{ab}\mathrm{Tr}(t^{A*}m_f t^B m_f^\dagger)$$

(A.43)

Appendix B

Full two-loop RGEs without SUSY relations

In this appendix, the full β-functions for all parameters of the non-supersymmetric toy model in Sec. 3.3 of Part I are listed up to two-loop order.

B.1 Gauge couplings

$$\beta_g^{(1)} = \frac{1}{2}g^3 \tag{B.1}$$

$$\beta_g^{(2)} = \frac{1}{8}g^3\Big(-2|Y_2|^2 - 2|Y_3|^2 - 4|Y_1|^2 + 6g^2 - |g_d|^2 - |g_u|^2\Big) \tag{B.2}$$

B.2 Quartic scalar couplings

$$\begin{aligned}
\beta_{\lambda_4}^{(1)} &= 20\lambda_4^2 + 2\lambda_4|g_d|^2 - 2|Y_2|^4 - 3g^2\lambda_4 + 4\lambda_4|Y_2|^2 - \frac{1}{2}|g_d|^4 + \frac{3}{8}g^4 + \lambda_1^2 + \lambda_3^2 \qquad \text{(B.3)}\\
\beta_{\lambda_4}^{(2)} &= -\frac{25}{16}g^6 - 4\lambda_1^3 + \frac{5}{4}g^4\lambda_3 + 2g^2\lambda_3^2 - 4\lambda_3^3 + \frac{63}{8}g^4\lambda_4 - 10\lambda_1^2\lambda_4 - 10\lambda_3^2\lambda_4 + 28g^2\lambda_4^2\\
&\quad - 240\lambda_4^3 - 2\lambda_1^2|Y_1|^2 - \frac{1}{4}g^4|Y_2|^2 + \frac{5}{2}g^2\lambda_4|Y_2|^2 - 40\lambda_4^2|Y_2|^2 - 2\lambda_3^2|Y_3|^2 + 2\lambda_4|Y_2|^4\\
&\quad + 2|Y_1|^2|Y_2|^4 + 2|Y_3|^2|Y_2|^4 + g_d^3 g_d^{*\,3} - 3\lambda_4 Y_2|Y_1|^2 Y_2^* + 8Y_2^3 Y_2^{*\,3}\\
&\quad + \frac{1}{4}g_d g_d^{*\,2}\Big(2\Big(-2g_u Y_3 Y_2^* + g_d\lambda_4 + g_d|Y_1|^2 + g_d|Y_3|^2\Big) + g_d|g_u|^2\Big) - 3\lambda_4 Y_3|Y_2|^2 Y_3^*\\
&\quad - \frac{1}{2}g_u^*\Big(2g_u\lambda_3^2 - 2g_u|Y_2|^4 + 4g_d\Big(-\lambda_4 + \lambda_3\Big)Y_2 Y_3^* + |Y_2|^2\Big(3g_u\lambda_4 + 4g_d Y_2 Y_3^*\Big)\Big)\\
&\quad - \frac{1}{8}g_d^*\Big(g^4 g_d - 10g^2 g_d\lambda_4 + 160 g_d\lambda_4^2 + 12 g_d\lambda_4|Y_3|^2 + 16 g_u\lambda_3 Y_3 Y_2^* - 16 g_u\lambda_4 Y_3 Y_2^*\\
&\quad + 16 g_u Y_2 Y_3 Y_2^{*\,2} + 4 g_d|Y_1|^2\Big(3\lambda_4 - 4Y_2 Y_2^*\Big) - 16 g_d Y_3|Y_2|^2 Y_3^*
\end{aligned}$$

$$+ 2g_d g_u^*\Big(3g_u\lambda_4 + 4g_d Y_2 Y_3^* - 4g_u|Y_2|^2\Big)\Big) \tag{B.4}$$

$$\begin{aligned}\beta_{\lambda_3}^{(1)} &= +\frac{3}{4}g^4 + 2\lambda_1\lambda_2 - 3g^2\lambda_3 + 4\lambda_3^2 + 8\lambda_3\lambda_4 + 8\lambda_3\lambda_5 + 2\lambda_3|Y_2|^2 + 2\lambda_3|Y_3|^2 \\ &+ g_d^*\Big(-2g_d|Y_3|^2 + 2g_u Y_3 Y_2^* - g_d|g_u|^2 + g_d\lambda_3\Big) - 4Y_3|Y_2|^2 Y_3^* \\ &+ g_u^*\Big(2g_d Y_2 Y_3^* - 2g_u|Y_2|^2 + g_u\lambda_3\Big)\end{aligned} \tag{B.5}$$

$$\begin{aligned}\beta_{\lambda_3}^{(2)} &= -\frac{25}{8}g^6 - 4\lambda_1^2\lambda_2 - 4\lambda_1\lambda_2^2 + \frac{43}{8}g^4\lambda_3 - \lambda_1^2\lambda_3 - 8\lambda_1\lambda_2\lambda_3 - \lambda_2^2\lambda_3 + 2g^2\lambda_3^2 - 10\lambda_3^3 \\ &+ 5g^4\lambda_4 + 16g^2\lambda_3\lambda_4 - 48\lambda_3^2\lambda_4 - 40\lambda_3\lambda_4^2 + 5g^4\lambda_5 + 16g^2\lambda_3\lambda_5 - 48\lambda_3^2\lambda_5 - 40\lambda_3\lambda_5^2 \\ &- 4\lambda_1\lambda_2|Y_1|^2 - \frac{1}{4}g^4|Y_2|^2 + \frac{5}{4}g^2\lambda_3|Y_2|^2 - 4\lambda_3^2|Y_2|^2 - 16\lambda_3\lambda_4|Y_2|^2 - \frac{1}{4}g^4|Y_3|^2 \\ &+ \frac{5}{4}g^2\lambda_3|Y_3|^2 - 4\lambda_3^2|Y_3|^2 - 16\lambda_3\lambda_5|Y_3|^2 - 3\lambda_3|Y_2|^4 + 10|Y_3|^2|Y_2|^4 - 3\lambda_3|Y_3|^4 \\ &+ 10|Y_2|^2|Y_3|^4 - \frac{3}{2}\lambda_3 Y_2|Y_1|^2 Y_2^* - \frac{1}{4}g_d^2 g_d^{*2}\Big(3\lambda_3 - 10|Y_3|^2 - 5|g_u|^2\Big) \\ &- 3g_u g_d g_d^{*2} Y_3 Y_2^* - \frac{3}{2}\lambda_3 Y_3|Y_1|^2 Y_3^* + 5\lambda_3 Y_3|Y_2|^2 Y_3^* + 12Y_2 Y_3|Y_1|^2 Y_2^* Y_3^* \\ &- \frac{1}{4}g_u g_u^{*2}\Big(-10g_u|Y_2|^2 + 12g_d Y_2 Y_3^* + 3g_u\lambda_3\Big) \\ &+ g_d^*\Big(-\frac{1}{8}g^4 g_d + \frac{5}{8}g^2 g_d\lambda_3 - 2g_d\lambda_3^2 - 8g_d\lambda_3\lambda_4 + \frac{5}{2}g_d\lambda_3|Y_3|^2 + \frac{5}{4}g_d|g_u|^4 + 5g_d|Y_3|^4 \\ &+ 2g_u\lambda_3 Y_3 Y_2^* - 8g_u\lambda_4 Y_3 Y_2^* - 8g_u\lambda_5 Y_3 Y_2^* - 6g_u Y_2 Y_3 Y_2^{*2} \\ &+ Y_1^*\Big(2g_d Y_1|Y_3|^2 - 4g_u Y_1 Y_3 Y_2^* - \frac{3}{4}g_d\lambda_3 Y_1\Big) + 6g_d Y_3|Y_2|^2 Y_3^* - 6g_u Y_3^2 Y_2^* Y_3^* \\ &+ g_u^*\Big(3g_d g_u|Y_1|^2 - 3g_d^2 Y_2 Y_3^* + 3g_d g_u|Y_3|^2 + 3g_u\Big(g_d Y_2 - g_u Y_3\Big)Y_2^* + \frac{5}{4}g_d g_u\lambda_3\Big)\Big) \\ &- \frac{1}{8}g_u^*\Big(g^4 g_u - 5g^2 g_u\lambda_3 + 16g_u\lambda_3^2 + 64g_u\lambda_3\lambda_5 - 40g_u|Y_2|^4 - 16g_d\lambda_3 Y_2 Y_3^* \\ &+ 64g_d(\lambda_4 Y_2 Y_3^* + \lambda_5 Y_2 Y_3^*) + 48g_d Y_2 Y_3 Y_3^{*2} + 16|Y_1|^2\Big(2g_d Y_2 Y_3^* + \tfrac{3}{8}g_u\lambda_3 \\ &- g_u Y_2 Y_2^*\Big) + 4|Y_2|^2\Big(12\Big(g_d Y_2 - g_u Y_3\Big)Y_3^* - 5g_u\lambda_3\Big)\Big)\end{aligned} \tag{B.6}$$

$$\begin{aligned}\beta_{\lambda_1}^{(1)} &= 2\lambda_1|Y_2|^2 + 2\lambda_2\lambda_3 + 2|Y_1|^2\Big(\lambda_1 - 2Y_2 Y_2^*\Big) + 4\lambda_1^2 + 8\lambda_1\lambda_4 - \frac{3}{2}g^2\lambda_1 \\ &+ |g_d|^2\Big(\lambda_1 - 2Y_1 Y_1^*\Big)\end{aligned} \tag{B.7}$$

$$\begin{aligned}\beta_{\lambda_1}^{(2)} &= \frac{39}{16}g^4\lambda_1 + g^2\lambda_1^2 - 10\lambda_1^3 + \frac{5}{4}g^4\lambda_2 - \lambda_1\lambda_2^2 + 4g^2\lambda_2\lambda_3 - 8\lambda_1\lambda_2\lambda_3 - 4\lambda_2^2\lambda_3 - \lambda_1\lambda_3^2 \\ &- 4\lambda_2\lambda_3^2 + 16g^2\lambda_1\lambda_4 - 48\lambda_1^2\lambda_4 - 40\lambda_1\lambda_4^2 - 3g^4|Y_1|^2 + \frac{5}{2}g^2\lambda_1|Y_1|^2 - 4\lambda_1^2|Y_1|^2 \\ &+ \frac{5}{4}g^2\lambda_1|Y_2|^2 - 4\lambda_1^2|Y_2|^2 - 16\lambda_1\lambda_4|Y_2|^2 - 4\lambda_2\lambda_3|Y_3|^2 - \frac{1}{4}\Big(-10|Y_1|^2 + 3\lambda_1\Big)|g_d|^4 \\ &- 3\lambda_1|Y_1|^4 + 10|Y_2|^2|Y_1|^4 - 3\lambda_1|Y_2|^4 + 10|Y_1|^2|Y_2|^4 - 2g^2 Y_2|Y_1|^2 Y_2^* \\ &+ 5\lambda_1 Y_2|Y_1|^2 Y_2^* - \frac{3}{2}\lambda_1 Y_3|Y_1|^2 Y_3^* - \frac{3}{2}\lambda_1 Y_3|Y_2|^2 Y_3^* + 12Y_2 Y_3|Y_1|^2 Y_2^* Y_3^* \\ &- \frac{1}{4}g_u^*\Big(8g_u\lambda_2\lambda_3 + 3g_u\lambda_1|Y_2|^2 - 4g_d\lambda_1 Y_2 Y_3^* + 8g_d\lambda_2 Y_2 Y_3^* \\ &+ |Y_1|^2\Big(24g_d Y_2 Y_3^* + 3g_u\lambda_1 - 8g_u Y_2 Y_2^*\Big)\Big)\end{aligned}$$

$$+\frac{1}{8}g_d^*\Big(5g^2g_d\lambda_1-16g_d\lambda_1^2-64g_d\lambda_1\lambda_4-6g_d\lambda_1|Y_3|^2+40g_d|Y_1|^4$$
$$-3g_d|g_u|^2\Big(\lambda_1-8Y_1Y_1^*\Big)+8g_u\lambda_1Y_3Y_2^*-16g_u\lambda_2Y_3Y_2^*$$
$$+4|Y_1|^2\Big(12\Big(g_dY_2-g_uY_3\Big)Y_2^*+g_d\Big(-2g^2+4Y_3Y_3^*+5\lambda_1\Big)\Big)\Big) \tag{B.8}$$

$$\beta_{\lambda_5}^{(1)}=20\lambda_5^2+2\lambda_5|g_u|^2-2|Y_3|^4-3g^2\lambda_5+4\lambda_5|Y_3|^2-\frac{1}{2}|g_u|^4+\frac{3}{8}g^4+\lambda_2^2+\lambda_3^2 \tag{B.9}$$

$$\beta_{\lambda_5}^{(2)}=-\frac{25}{16}g^6-4\lambda_2^3+\frac{5}{4}g^4\lambda_3+2g^2\lambda_3^2-4\lambda_3^3+\frac{63}{8}g^4\lambda_5-10\lambda_2^2\lambda_5-10\lambda_3^2\lambda_5+28g^2\lambda_5^2$$
$$-240\lambda_5^3-2\lambda_2^2|Y_1|^2-2\lambda_3^2|Y_2|^2-\frac{1}{4}g^4|Y_3|^2+\frac{5}{2}g^2\lambda_5|Y_3|^2-40\lambda_5^2|Y_3|^2+2\lambda_5|Y_3|^4$$
$$+2|Y_1|^2|Y_3|^4+2|Y_2|^2|Y_3|^4+g_u^3g_u^{*3}-3\lambda_5Y_3|Y_1|^2Y_3^*-3\lambda_5Y_3|Y_2|^2Y_3^*+8Y_3^3Y_3^{*3}$$
$$+\frac{1}{2}g_ug_u^{*2}\Big(-2g_dY_2Y_3^*+g_u\lambda_5+g_u|Y_1|^2+g_u|Y_2|^2\Big)$$
$$-\frac{1}{8}g_u^*\Big(g^4g_u-10g^2g_u\lambda_5+160g_u\lambda_5^2+16g_d\lambda_3Y_2Y_3^*-16g_d\lambda_5Y_2Y_3^*+16g_dY_2Y_3Y_3^{*2}$$
$$+4g_u|Y_1|^2\Big(3\lambda_5-4Y_3Y_3^*\Big)+4g_u|Y_2|^2\Big(3\lambda_5-4Y_3Y_3^*\Big)\Big)$$
$$-\frac{1}{4}g_d^*\Big(2g_d\Big(2\lambda_3^2-2|Y_3|^4+3\lambda_5|Y_3|^2\Big)-g_d|g_u|^4+8g_uY_3\Big(-\lambda_5+\lambda_3+|Y_3|^2\Big)Y_2^*$$
$$+|g_u|^2\Big(3g_d\lambda_5-4g_dY_3Y_3^*+4g_uY_3Y_2^*\Big)\Big) \tag{B.10}$$

$$\beta_{\lambda_2}^{(1)}=2\lambda_1\lambda_3+2\lambda_2|Y_3|^2+2|Y_1|^2\Big(-2Y_3Y_3^*+\lambda_2\Big)+4\lambda_2^2+8\lambda_2\lambda_5-\frac{3}{2}g^2\lambda_2$$
$$+|g_u|^2\Big(-2Y_1Y_1^*+\lambda_2\Big) \tag{B.11}$$

$$\beta_{\lambda_2}^{(2)}=g^4\Big(\frac{5}{4}\lambda_1+\frac{39}{16}\lambda_2\Big)-\lambda_1^2\lambda_2+g^2\lambda_2^2-10\lambda_2^3-4\Big(\lambda_1\lambda_3^2+\lambda_1^2\lambda_3-g^2\lambda_1\lambda_3\Big)-8\lambda_1\lambda_2\lambda_3$$
$$-\lambda_2\lambda_3^2+16g^2\lambda_2\lambda_5-48\lambda_2^2\lambda_5-40\lambda_2\lambda_5^2-3g^4|Y_1|^2+\frac{5}{2}g^2\lambda_2|Y_1|^2-4\lambda_2^2|Y_1|^2$$
$$-4\lambda_1\lambda_3|Y_2|^2+\frac{5}{4}g^2\lambda_2|Y_3|^2-4\lambda_2^2|Y_3|^2-16\lambda_2\lambda_5|Y_3|^2-\frac{1}{4}\Big(3\lambda_2-10|Y_1|^2\Big)|g_u|^4$$
$$-3\lambda_2|Y_1|^4+10|Y_3|^2|Y_1|^4-3\lambda_2|Y_3|^4+10|Y_1|^2|Y_3|^4-\frac{3}{2}\lambda_2Y_2|Y_1|^2Y_2^*$$
$$-2g^2Y_3|Y_1|^2Y_3^*+5\lambda_2Y_3|Y_1|^2Y_3^*-\frac{3}{2}\lambda_2Y_3|Y_2|^2Y_3^*+12Y_2Y_3|Y_1|^2Y_2^*Y_3^*$$
$$+\frac{1}{8}g_u^*\Big(5g^2g_u\lambda_2-16g_u\lambda_2^2-64g_u\lambda_2\lambda_5-6g_u\lambda_2|Y_2|^2+40g_u|Y_1|^4-16g_d\lambda_1Y_2Y_3^*$$
$$+8g_d\lambda_2Y_2Y_3^*+4|Y_1|^2\Big(-12g_dY_2Y_3^*+12g_uY_3Y_3^*-2g^2g_u+4g_uY_2Y_2^*+5g_u\lambda_2\Big)\Big)$$
$$-\frac{1}{8}g_d^*\Big(3g_d|g_u|^2\Big(-8Y_1Y_1^*+\lambda_2\Big)$$
$$+2\Big(8g_d\lambda_1\lambda_3+3g_d\lambda_2|Y_3|^2+8g_u\lambda_1Y_3Y_2^*-4g_u\lambda_2Y_3Y_2^*$$
$$+|Y_1|^2\Big(24g_uY_3Y_2^*+3g_d\lambda_2-8g_dY_3Y_3^*\Big)\Big)\Big) \tag{B.12}$$

B.3 Yukawa couplings

$$\beta_{g_d}^{(1)} = \frac{1}{4}\Big(2g_d|Y_1|^2 + 2g_d|Y_3|^2 - 3g^2 g_d + 4g_d^2 g_d^* + 4g_d|Y_2|^2 - 8g_u Y_3 Y_2^* + g_d|g_u|^2\Big) \tag{B.13}$$

$$\begin{aligned}\beta_{g_d}^{(2)} = \frac{1}{32}\Big(&- 7g_d|g_u|^4 - 10g_d^3 {g_d^*}^2 + g_u^*\Big(28g_d g_u|Y_1|^2 + 2g_u\Big(- 7g_d Y_2 + 24g_u Y_3\Big)Y_2^* \\ &+ g_d\Big(16g_d Y_2 Y_3^* - 32g_u\lambda_3 - 56g_u|Y_3|^2 - 7g^2 g_u\Big)\Big) - |g_d|^2\Big(2\Big(12g_d Y_2 Y_2^* \\ &- 8g_u Y_3 Y_2^* - 31g^2 g_d + 64g_d\lambda_4 + 7g_d Y_1 Y_1^* + 7g_d Y_3 Y_3^*\Big) + 7g_d g_u g_u^*\Big) \\ &- 2\Big(14g_d|Y_1|^4 - 2\Big(16g_u\lambda_3 Y_3 + \Big(24g_u Y_3 - 7g_d Y_2\Big)|Y_3|^2 + g^2\Big(5g_d Y_2 - 4g_u Y_3\Big)\Big)Y_2^* \\ &+ 8Y_2\Big(3g_d Y_2 - 4g_u Y_3\Big){Y_2^*}^2 + 2|Y_1|^2\Big(4g_d\Big(4\lambda_1 + g^2\Big) + \Big(7g_d Y_2 - 8g_u Y_3\Big)Y_2^*\Big) \\ &+ g_d\Big(\Big(- 11g^2 Y_3 + 32\lambda_3 Y_3\Big)Y_3^* + 14|Y_3|^4 + 2\Big(- 4\Big(8\lambda_4^2 + \lambda_1^2 + \lambda_3^2\Big) + g^4\Big)\Big)\Big)\Big)\end{aligned} \tag{B.14}$$

$$\beta_{g_u}^{(1)} = \frac{1}{4}\Big(2g_u|Y_1|^2 + 2g_u|Y_2|^2 - 3g^2 g_u + 4g_u^2 g_u^* + 4g_u|Y_3|^2 - 8g_d Y_2 Y_3^* + g_u|g_d|^2\Big) \tag{B.15}$$

$$\begin{aligned}\beta_{g_u}^{(2)} = \frac{1}{32}\Big(&- 7g_u|g_d|^4 \\ &- g_d^*\Big(7g^2 g_d g_u + 32g_d g_u\lambda_3 - 28g_d g_u|Y_1|^2 + 56g_d g_u|Y_2|^2 + 14g_d g_u|Y_3|^2 + 7g_d g_u^2 g_u^* \\ &- 16g_u^2 Y_3 Y_2^* - 48g_d^2 Y_2 Y_3^*\Big) \\ &- 2\Big(2g^4 g_u - 8g_u\lambda_2^2 - 8g_u\lambda_3^2 - 64g_u\lambda_5^2 - 11g^2 g_u|Y_2|^2 + 32g_u\lambda_3|Y_2|^2 - 10g^2 g_u|Y_3|^2 \\ &+ 14g_u|Y_1|^4 + 14g_u|Y_2|^4 + 24g_u|Y_3|^4 + 5g_u^3 {g_u^*}^2 + 8g^2 g_d Y_2 Y_3^* - 32g_d\lambda_3 Y_2 Y_3^* \\ &+ 14g_u Y_3|Y_2|^2 Y_3^* - 48g_d Y_2^2 Y_2^* Y_3^* - 32g_d Y_2 Y_3 {Y_3^*}^2 \\ &+ |g_u|^2\Big(12g_u Y_3 Y_3^* - 8g_d Y_2 Y_3^* - 31g^2 g_u + 64g_u\lambda_5 + 7g_u Y_1 Y_1^* + 7g_u Y_2 Y_2^*\Big) \\ &+ 2|Y_1|^2\Big(4g_u\Big(4\lambda_2 + g^2\Big) + \Big(7g_u Y_3 - 8g_d Y_2\Big)Y_3^*\Big)\Big)\Big)\end{aligned} \tag{B.16}$$

$$\beta_{Y_3}^{(1)} = \frac{1}{4}\Big(\Big(2g_u Y_3 - 4g_d Y_2\Big)g_u^* + Y_3\Big(2|Y_1|^2 + 2|Y_2|^2 - 3g^2 + 8|Y_3|^2\Big) + Y_3|g_d|^2\Big) \tag{B.17}$$

$$\begin{aligned}\beta_{Y_3}^{(2)} = \frac{1}{32}\Big(&- 7Y_3|g_d|^4 + g_d^*\Big(g_d\Big(- 7g_u Y_3 + 24g_d Y_2\Big)g_u^* + Y_3\Big(- 8\Big(- 2g_u Y_3 \\ &+ 7g_d Y_2\Big)Y_2^* + g_d\Big(11g^2 - 14|Y_3|^2 - 32\lambda_3\Big)\Big)\Big) - 2\Big(2g_u\Big(3g_u Y_3 - 4g_d Y_2\Big){g_u^*}^2 \\ &+ g_u^*\Big(4g^2 g_d Y_2 - 16g_d\lambda_3 Y_2 - 5g^2 g_u Y_3 + \Big(7g_u Y_3 - 24g_d Y_2\Big)|Y_2|^2 - 8g_d Y_2|Y_3|^2 \\ &+ \Big(7g_u Y_1 Y_3 - 8g_d Y_1 Y_2\Big)Y_1^* + 12g_u Y_3^2 Y_3^*\Big) + Y_3\Big(14|Y_1|^4 + 14|Y_2|^4 \\ &+ 2|Y_1|^2\Big(- 14Y_2 Y_2^* + 4\Big(4\lambda_2 + g^2\Big) + 7Y_3 Y_3^*\Big) + |Y_2|^2\Big(14Y_3 Y_3^* + 32\lambda_3 + 7g^2\Big) \\ &+ 2\Big(10|Y_3|^4 + \Big(- 31g^2 Y_3 + 64\lambda_5 Y_3\Big)Y_3^* - 4\Big(8\lambda_5^2 + \lambda_2^2 + \lambda_3^2\Big) + g^4\Big)\Big)\Big)\Big)\end{aligned} \tag{B.18}$$

$$\beta_{Y_2}^{(1)} = \frac{1}{4}\Big(2\Big(- 2g_u Y_3 + g_d Y_2\Big)g_d^* + Y_2\Big(2|Y_1|^2 + 2|Y_3|^2 - 3g^2 + 8|Y_2|^2 + |g_u|^2\Big)\Big) \tag{B.19}$$

$$\beta_{Y_2}^{(2)} = \frac{1}{32}\Big(- 4g_d\Big(3g_d Y_2 - 4g_u Y_3\Big){g_d^*}^2 + g_d^*\Big(\Big(- 7g_d Y_2 + 24g_u Y_3\Big)|g_u|^2$$

$$+ 2\Big(5g^2g_dY_2 - 4g^2g_uY_3 + 16g_u\lambda_3Y_3 - 12\Big(-\frac{2}{3}g_uY_3 + g_dY_2\Big)|Y_2|^2 - 7g_dY_2|Y_3|^2$$
$$+ \Big(8g_uY_1Y_3 - 7g_dY_1Y_2\Big)Y_1^* + 24g_uY_3^2Y_3^*\Big)\Big)$$
$$- Y_2\Big(7|g_u|^4 + g_u^*\Big(-11g^2g_u + 14g_u|Y_2|^2 + 32g_u\lambda_3 - 8\Big(2g_dY_2 - 7g_uY_3\Big)Y_3^*\Big)$$
$$+ 2\Big(2g^4 - 8\lambda_1^2 - 8\lambda_3^2 - 64\lambda_4^2 + 7g^2|Y_3|^2 + 32\lambda_3|Y_3|^2 + 14|Y_1|^4 + 20|Y_2|^4 + 14|Y_3|^4$$
$$+ 2|Y_1|^2\Big(16\lambda_1 - 14Y_3Y_3^* + 4g^2 + 7Y_2Y_2^*\Big) + 2|Y_2|^2\Big(64\lambda_4 - 31g^2 + 7Y_3Y_3^*\Big)\Big)\Big)\Big) \tag{B.20}$$

$$\beta_{Y_1}^{(1)} = \frac{1}{4}Y_1\Big(2|Y_2|^2 + 2|Y_3|^2 - 6g^2 + 8|Y_1|^2 + |g_d|^2 + |g_u|^2\Big) \tag{B.21}$$

$$\beta_{Y_1}^{(2)} = -\frac{1}{32}Y_1\Big(7|g_d|^4 + 7|g_u|^4 + g_d^*\Big(32g_d\lambda_1 - 11g^2g_d - 14g_d|g_u|^2 + 14g_d|Y_1|^2$$
$$- 32g_uY_3Y_2^* + 56g_d|Y_2|^2\Big) + g_u^*\Big(14g_u|Y_1|^2 + (32\lambda_2 - 11g^2)g_u - 8\Big(4g_dY_2$$
$$- 7g_uY_3\Big)Y_3^*\Big) + 2\Big(20|Y_1|^4 + 14(|Y_2|^4 + |Y_3|^4) - 8(\lambda_1^2 + \lambda_2^2) - 11g^2(g^2 + |Y_3|^2)$$
$$+ 32\lambda_2|Y_3|^2 + |Y_2|^2\Big(32\lambda_1 - 11g^2 - 28Y_3Y_3^*\Big) + 14|Y_1|^2\Big(Y_2Y_2^* + Y_3Y_3^* - \frac{26}{7}g^2\Big)\Big)\Big) \tag{B.22}$$

B.4 Fermion mass terms

$$\beta_{M_1}^{(1)} = \frac{1}{2}M_1\Big(|g_d|^2 + |g_u|^2\Big) \tag{B.23}$$

$$\beta_{M_1}^{(2)} = \frac{1}{16}\Big(M_1|g_d|^4 + g_u\Big(16g_dT_3Y_1^* + g_uM_1g_u^{*\,2} + M_1\Big(-12|Y_3|^2 + 17g^2$$
$$- 2|Y_1|^2 - 2|Y_2|^2\Big)g_u^*\Big) + M_1|g_d|^2\Big(-12Y_2Y_2^* + 17g^2 - 2Y_1Y_1^* - 2Y_3Y_3^*\Big)\Big) \tag{B.24}$$

$$\beta_{M_2}^{(1)} = \frac{1}{4}M_2\Big(2|Y_2|^2 + 2|Y_3|^2 + 4|Y_1|^2 - 6g^2 + |g_d|^2 + |g_u|^2\Big) \tag{B.25}$$

$$\beta_{M_2}^{(2)} = \frac{1}{32}\Big(22g^4M_2 + 64g^2M_2|Y_1|^2 + 22g^2M_2|Y_2|^2 + 22g^2M_2|Y_3|^2 - 7M_2|g_d|^4$$
$$- 7M_2|g_u|^4 + 8M_2|Y_1|^4 - 28M_2|Y_2|^4 - 28M_2|Y_3|^4 - 64Y_1|Y_2|^2T_1^* - 64Y_1|Y_3|^2T_2^*$$
$$+ g_d^*\Big(14g_dM_2|g_u|^2 - 32g_dY_1T_1^* + M_2\Big(11g^2g_d - 2g_d|Y_1|^2 + 32g_uY_3Y_2^*$$
$$- 56g_d|Y_2|^2\Big)\Big) - 4M_2Y_2|Y_1|^2Y_2^* - 4M_2Y_3|Y_1|^2Y_3^* + 56M_2Y_3|Y_2|^2Y_3^*$$
$$+ g_u^*\Big(-32g_uY_1T_2^* + M_2\Big(11g^2g_u - 2g_u|Y_1|^2 + 8\Big(4g_dY_2 - 7g_uY_3\Big)Y_3^*\Big)\Big)\Big) \tag{B.26}$$

B.5 Trilinear scalar couplings

$$\beta_{T_3}^{(1)} = \frac{1}{2}\Big(g_d^*\Big(8M_1Y_1g_u^* + g_dT_3\Big) + T_3\Big(2|Y_1|^2 + 2|Y_2|^2 + 2|Y_3|^2 - 3g^2 + 4\lambda_1 + 4\lambda_2 + 4\lambda_3 + |g_u|^2\Big)\Big) \tag{B.27}$$

$$\begin{aligned}\beta_{T_3}^{(2)} = \frac{1}{16}\Big(&- 2g_d^{*2}\Big(3g_d^2T_3 + 40g_dM_1Y_1g_u^* - 64M_1Y_1Y_3Y_2^*\Big) \\ &+ g_d^*\Big(- 80g_uM_1Y_1g_u^{*2} + 2g_u^*\Big(- 32\lambda_3M_1Y_1 \\ &- 32M_1Y_1^2Y_1^* - 48M_1Y_1|Y_2|^2 - 48M_1Y_1|Y_3|^2 + 5g_dg_uT_3 + 8g^2M_1Y_1\Big) \\ &+ T_3\Big(- 12g_d|Y_1|^2 + 16g_uY_3Y_2^* + g_d\Big(- 12|Y_3|^2 - 16\Big(\lambda_1 + \lambda_3\Big) + 5g^2\Big)\Big)\Big) \\ &+ T_3g_u^*\Big(- 12g_u|Y_1|^2 - 12g_u|Y_2|^2 + 16g_dY_2Y_3^* - 16g_u\lambda_2 - 16g_u\lambda_3 + 5g^2g_u\Big) \\ &+ g_u^{*2}\Big(128M_1Y_1Y_2Y_3^* - 6g_u^2T_3\Big) + T_3\Big(19g^4 + 8g^2\lambda_1 \\ &- 16\lambda_1^2 + 8g^2\lambda_2 - 96\lambda_1\lambda_2 - 16\lambda_2^2 + 64g^2\lambda_3 - 96\lambda_1\lambda_3 - 96\lambda_2\lambda_3 - 16\lambda_3^2 \\ &- 64(2\lambda_1\lambda_4 + 2\lambda_3\lambda_4 - \lambda_4^2 + 2\lambda_2\lambda_5 + 2\lambda_3\lambda_5 - \lambda_5^2) + 10g^2|Y_3|^2 - 32\lambda_2|Y_3|^2 \\ &- 32\lambda_3|Y_3|^2 - 24(|Y_1|^4 + |Y_2|^4 + |Y_3|^4) + 4|Y_1|^2\Big(10(Y_2Y_2^* + Y_3Y_3^*) + 5g^2 \\ &- 8(\lambda_1 + \lambda_2)\Big) + 2|Y_2|^2\Big(- 16\Big(\lambda_1 + \lambda_3\Big) + 20Y_3Y_3^* + 5g^2\Big)\Big)\Big)\end{aligned} \tag{B.28}$$

$$\begin{aligned}\beta_{T_1}^{(1)} = &\ 2\lambda_3T_2 + 2T_1|Y_2|^2 + 4\lambda_1T_1 - 4Y_1|Y_2|^2M_2^* + 8\lambda_4T_1 - \frac{3}{2}g^2T_1 \\ &+ |g_d|^2\Big(- 2Y_1M_2^* + T_1\Big) + T_1|Y_1|^2\end{aligned} \tag{B.29}$$

$$\begin{aligned}\beta_{T_1}^{(2)} = &\ \frac{39}{16}g^4T_1 + g^2\lambda_1T_1 - \frac{21}{2}\lambda_1^2T_1 + \frac{1}{2}\lambda_2^2T_1 - 4\lambda_2\lambda_3T_1 - \lambda_3^2T_1 + 16g^2\lambda_4T_1 \\ &- 48\lambda_1\lambda_4T_1 - 40\lambda_4^2T_1 + \frac{5}{4}g^4T_2 - 2\lambda_1\lambda_2T_2 + 4g^2\lambda_3T_2 - 4\lambda_1\lambda_3T_2 - 4\lambda_2\lambda_3T_2 \\ &- 4\lambda_3^2T_2 + \frac{5}{4}g^2T_1|Y_1|^2 - 4\lambda_1T_1|Y_1|^2 + \frac{5}{4}g^2T_1|Y_2|^2 - 4\lambda_1T_1|Y_2|^2 - 16\lambda_4T_1|Y_2|^2 \\ &- 4\lambda_3T_2|Y_3|^2 - \frac{3}{2}T_1|Y_1|^4 - 3T_1|Y_2|^4 - 3g^4Y_1M_2^* - 2g^2Y_1|Y_2|^2M_2^* + 4\lambda_1Y_1|Y_2|^2M_2^* \\ &+ 10Y_1|Y_2|^4M_2^* - \frac{1}{4}|g_d|^4\Big(- 10Y_1M_2^* + 3T_1\Big) + 10Y_1^2|Y_2|^2M_2^*Y_1^* + \frac{7}{4}T_1Y_2|Y_1|^2Y_2^* \\ &+ \frac{1}{8}g_d^*\Big(5g^2g_dT_1 - 16g_d\lambda_1T_1 - 64g_d\lambda_4T_1 + 7g_dT_1|Y_1|^2 - 6g_dT_1|Y_3|^2 \\ &- 3g_d|g_u|^2\Big(- 8Y_1M_2^* + T_1\Big) + 8g_uT_1Y_3Y_2^* - 16g_uT_2Y_3Y_2^* \\ &+ 8Y_1M_2^*\Big(2g_d\lambda_1 + 2g_d|Y_3|^2 + 5g_d|Y_1|^2 + 6g_d|Y_2|^2 - 6g_uY_3Y_2^* - g^2g_d\Big)\Big) \\ &- \frac{3}{4}T_1Y_3|Y_1|^2Y_3^* - \frac{3}{2}T_1Y_3|Y_2|^2Y_3^* + 12Y_1Y_3|Y_2|^2M_2^*Y_3^* - \frac{3}{8}g_u^*g_uT_1|Y_1|^2 \\ &- \frac{1}{8}g_u^*\Big(2\Big(4g_dY_2\Big(2T_2 + 6Y_1M_2^* - T_1\Big)Y_3^* + 8g_u\lambda_3T_2 + g_u|Y_2|^2\Big(3T_1 - 8Y_1M_2^*\Big)\Big)\Big)\end{aligned} \tag{B.30}$$

$$\beta_{T_2}^{(1)} = 2\lambda_3T_1 + 2T_2|Y_3|^2 + 4\lambda_2T_2 - 4Y_1|Y_3|^2M_2^* + 8\lambda_5T_2 - \frac{3}{2}g^2T_2$$

$$+ |g_u|^2\Big(- 2Y_1 M_2^* + T_2\Big) + T_2|Y_1|^2 \tag{B.31}$$

$$\begin{aligned}
\beta_{T_2}^{(2)} = {} & \frac{5}{4}g^4 T_1 - 2\lambda_1\lambda_2 T_1 + 4g^2\lambda_3 T_1 - 4\lambda_1\lambda_3 T_1 - 4\lambda_2\lambda_3 T_1 - 4\lambda_3^2 T_1 + \frac{39}{16}g^4 T_2 + \frac{1}{2}\lambda_1^2 T_2 \\
& + g^2\lambda_2 T_2 - \frac{21}{2}\lambda_2^2 T_2 - 4\lambda_1\lambda_3 T_2 - \lambda_3^2 T_2 + 16g^2\lambda_5 T_2 - 48\lambda_2\lambda_5 T_2 - 40\lambda_5^2 T_2 \\
& + \frac{5}{4}g^2 T_2|Y_1|^2 - 4\lambda_2 T_2|Y_1|^2 - 4\lambda_3 T_1|Y_2|^2 + \frac{5}{4}g^2 T_2|Y_3|^2 - 4\lambda_2 T_2|Y_3|^2 \\
& - 16\lambda_5 T_2|Y_3|^2 - \frac{3}{2}T_2|Y_1|^4 - 3T_2|Y_3|^4 - 3g^4 Y_1 M_2^* - 2g^2 Y_1|Y_3|^2 M_2^* \\
& + 4\lambda_2 Y_1|Y_3|^2 M_2^* + 10Y_1|Y_3|^4 M_2^* - \frac{1}{4}|g_u|^4\Big(- 10Y_1 M_2^* + 3T_2\Big) + 10Y_1^2|Y_3|^2 M_2^* Y_1^* \\
& - \frac{3}{4}T_2 Y_2|Y_1|^2 Y_2^* - \frac{1}{8}g_d^*\Big(3g_d T_2|Y_1|^2 + 3g_d|g_u|^2\Big(- 8Y_1 M_2^* + T_2\Big) \\
& + 2\Big(4g_u Y_3\Big(2T_1 + 6Y_1 M_2^* - T_2\Big)Y_2^* + g_d\Big(8\lambda_3 T_1 + |Y_3|^2\Big(3T_2 - 8Y_1 M_2^*\Big)\Big)\Big)\Big) \\
& + \frac{7}{4}T_2 Y_3|Y_1|^2 Y_3^* - \frac{3}{2}T_2 Y_3|Y_2|^2 Y_3^* + 12Y_1 Y_3|Y_2|^2 M_2^* Y_3^* + \frac{1}{8}g_u^*\Big(5g^2 g_u T_2 \\
& - 16g_u\lambda_2 T_2 - 64g_u\lambda_5 T_2 + 7g_u T_2|Y_1|^2 - 6g_u T_2|Y_2|^2 - 16g_d T_1 Y_2 Y_3^* + 8g_d T_2 Y_2 Y_3^* \\
& + 8Y_1 M_2^*\Big(2g_u\lambda_2 + 2g_u|Y_2|^2 + 5g_u|Y_1|^2 - 6g_d Y_2 Y_3^* + 6g_u|Y_3|^2 - g^2 g_u\Big)\Big)
\end{aligned} \tag{B.32}$$

B.6 Scalar mass terms

$$\begin{aligned}
\beta_B^{(1)} = {} & \frac{1}{2}\Big(2B|Y_2|^2 + 2B|Y_3|^2 - 3Bg^2 + 4B\lambda_3 + 4T_3 T_1^* + 4T_3 T_2^* + B|g_u|^2 \\
& + g_d^*\Big(8M_1 M_2 g_u^* + B g_d\Big)\Big)
\end{aligned} \tag{B.33}$$

$$\begin{aligned}
\beta_B^{(2)} = {} & + \frac{19}{16}Bg^4 + \frac{1}{2}B\lambda_1^2 - 2B\lambda_1\lambda_2 + \frac{1}{2}B\lambda_2^2 + 4Bg^2\lambda_3 - B\lambda_3^2 - 8B\lambda_3\lambda_4 + 4B\lambda_4^2 \\
& - 8B\lambda_3\lambda_5 + 4B\lambda_5^2 + \frac{5}{8}Bg^2|Y_2|^2 - 2B\lambda_3|Y_2|^2 + \frac{5}{8}Bg^2|Y_3|^2 - 2B\lambda_3|Y_3|^2 - \frac{3}{2}B|Y_2|^4 \\
& - \frac{3}{2}B|Y_3|^4 + \frac{1}{2}g^2 T_3 T_1^* - 2\lambda_1 T_3 T_1^* - 2\lambda_2 T_3 T_1^* - 6\lambda_3 T_3 T_1^* - 8\lambda_4 T_3 T_1^* - 2T_3|Y_1|^2 T_1^* \\
& - 2T_3|Y_2|^2 T_1^* + \frac{1}{2}g^2 T_3 T_2^* - 2\lambda_1 T_3 T_2^* - 2\lambda_2 T_3 T_2^* - 6\lambda_3 T_3 T_2^* - 8\lambda_5 T_3 T_2^* \\
& - 2T_3|Y_1|^2 T_2^* - 2T_3|Y_3|^2 T_2^* + 4M_2 T_3|Y_2|^2 Y_1^* + 4M_2 T_3|Y_3|^2 Y_1^* - \frac{3}{4}BY_2|Y_1|^2 Y_2^* \\
& + g_d^{*2}\Big(- 5g_d M_1 M_2 g_u^* + 8M_1 M_2 Y_3 Y_2^* - \frac{3}{8}Bg_d^2\Big) \\
& - \frac{1}{16}g_d^*\Big(\Big(64\lambda_3 M_1 M_2 + 64M_1 M_2|Y_1|^2 + 96M_1 M_2(|Y_2|^2 + |Y_3|^2) - 10Bg_d g_u \\
& - 16g^2 M_1 M_2\Big)g_u^* + 80g_u M_1 M_2 g_u^{*2} + 16g_d T_3 T_1^* + B\Big(12g_d|Y_3|^2 + 16(g_d\lambda_3 - g_u Y_3 Y_2^*) \\
& - 5g^2 g_d + 6g_d|Y_1|^2\Big)\Big) - BY_3(\frac{3}{4}|Y_1|^2 - \frac{5}{2}|Y_2|^2)Y_3^* + g_u^{*2}\Big(8M_1 M_2 Y_2 Y_3^* - \frac{3}{8}Bg_u^2\Big)
\end{aligned}$$

$$+ g_u^*\Big(\frac{1}{16}B\Big(16g_dY_2Y_3^* - 12g_u|Y_2|^2 - 16g_u\lambda_3 + 5g^2g_u - 6g_u|Y_1|^2\Big) - g_uT_3T_2^*\Big) \quad \text{(B.34)}$$

$$\beta^{(1)}_{m_1^2} = -\frac{3}{2}g^2m_1^2 + 8\lambda_4m_1^2 + 2\lambda_3m_2^2 + 2\lambda_1m_3^2 + 4|T_1|^2 + 2|T_3|^2 + 2m_1^2|Y_2|^2$$
$$+ |g_d|^2\Big(- 2M_2M_2^* - 8M_1M_1^* + m_1^2\Big) - 4Y_2|M_2|^2Y_2^* \quad \text{(B.35)}$$

$$\beta^{(2)}_{m_1^2} = \Big(\frac{39}{16}g^4 - \lambda_1^2 - \lambda_3^2 + 16g^2\lambda_4 - 40\lambda_4^2\Big)m_1^2 + \Big(\frac{5}{4}g^4 + 4g^2\lambda_3 - 4\lambda_3^2\Big)m_2^2 - 4\lambda_1^2m_3^2$$
$$- 3g^4|M_2|^2 + g^2|T_1|^2 - 10\lambda_1|T_1|^2 - 48\lambda_4|T_1|^2 - 2\lambda_1|T_2|^2 - 4\lambda_3|T_2|^2$$
$$+ \frac{1}{2}g^2|T_3|^2 - 6\lambda_1|T_3|^2 - 6\lambda_3|T_3|^2 - 8\lambda_4|T_3|^2 - 4\lambda_1m_3^2|Y_1|^2 + \frac{5}{4}g^2m_1^2|Y_2|^2$$
$$- 16\lambda_4m_1^2|Y_2|^2 - 4\lambda_3m_2^2|Y_3|^2 - \frac{1}{4}\Big(3m_1^2 - 10|M_2|^2 - 64|M_1|^2\Big)|g_d|^4 - 3m_1^2|Y_2|^4$$
$$+ 10|M_2|^2|Y_2|^4 - 4\lambda_3T_2T_1^* + 4Y_1|Y_2|^2M_2^*T_1^* - 4\lambda_3T_1T_2^* - 4Y_1|T_1|^2Y_1^*$$
$$- 2Y_1|T_3|^2Y_1^* + 4M_2T_1|Y_2|^2Y_1^* - 2g^2Y_2|M_2|^2Y_2^* - 4Y_2|T_1|^2Y_2^* - \frac{3}{2}m_1^2Y_2|Y_1|^2Y_2^*$$
$$+ 10Y_1Y_2|M_2|^2Y_1^*Y_2^* - 2Y_3|T_3|^2Y_3^* - \frac{3}{2}m_1^2Y_3|Y_2|^2Y_3^* + 12Y_2Y_3|M_2|^2Y_2^*Y_3^*$$
$$+ \frac{1}{8}g_d^*\Big(5g^2g_dm_1^2 - 64g_d\lambda_4m_1^2 - 16g_d|T_1|^2 - 6g_dm_1^2|Y_1|^2 - 6g_dm_1^2|Y_3|^2$$
$$- 3g_d|g_u|^2\Big(- 16M_1M_1^* - 8M_2M_2^* + m_1^2\Big) + 16g_dM_2T_1Y_1^* + 32g_dY_1|M_1|^2Y_1^*$$
$$+ 8g_um_1^2Y_3Y_2^* - 16g_um_2^2Y_3Y_2^* - 64g_uY_3|M_1|^2Y_2^*$$
$$+ 8M_2^*\Big(2g_dY_1T_1^* + M_2\Big(2g_d|Y_3|^2 + 5g_d|Y_1|^2 + 6g_d|Y_2|^2 - 6g_uY_3Y_2^* - g^2g_d\Big)\Big)$$
$$+ 32g_dY_3|M_1|^2Y_3^*\Big) - \frac{1}{4}g_u^*\Big(4g_u|T_3|^2 + g_u|Y_2|^2\Big(3m_1^2 - 32M_1M_1^* - 8M_2M_2^*\Big)$$
$$+ 4\Big(2g_u\lambda_3m_2^2 + g_dY_2\Big(2m_2^2 + 6|M_2|^2 + 8|M_1|^2 - m_1^2\Big)Y_3^*\Big)\Big) \quad \text{(B.36)}$$

$$\beta^{(1)}_{m_2^2} = 2\lambda_3m_1^2 - \frac{3}{2}g^2m_2^2 + 8\lambda_5m_2^2 + 2\lambda_2m_3^2 + 4|T_2|^2 + 2|T_3|^2 + 2m_2^2|Y_3|^2$$
$$+ |g_u|^2\Big(- 2M_2M_2^* - 8M_1M_1^* + m_2^2\Big) - 4Y_3|M_2|^2Y_3^* \quad \text{(B.37)}$$

$$\beta^{(2)}_{m_2^2} = \Big(\frac{5}{4}g^4 + 4g^2\lambda_3 - 4\lambda_3^2\Big)m_1^2 + \Big(\frac{39}{16}g^4 - \lambda_2^2 - \lambda_3^2 + 16g^2\lambda_5 - 40\lambda_5^2\Big)m_2^2$$
$$- 4\lambda_2^2m_3^2 - 3g^4|M_2|^2 - 2\lambda_2|T_1|^2 - 4\lambda_3|T_1|^2 + g^2|T_2|^2 - 10\lambda_2|T_2|^2 - 48\lambda_5|T_2|^2$$
$$+ \frac{1}{2}g^2|T_3|^2 - 6\lambda_2|T_3|^2 - 6\lambda_3|T_3|^2 - 8\lambda_5|T_3|^2 - 4\lambda_2m_3^2|Y_1|^2 - 4\lambda_3m_1^2|Y_2|^2$$
$$+ \frac{5}{4}g^2m_2^2|Y_3|^2 - 16\lambda_5m_2^2|Y_3|^2 - \frac{1}{4}\Big(- 10|M_2|^2 + 3m_2^2 - 64|M_1|^2\Big)|g_u|^4$$
$$- 3m_2^2|Y_3|^4 + 10|M_2|^2|Y_3|^4 - 4\lambda_3T_2T_1^* - 4\lambda_3T_1T_2^* + 4Y_1|Y_3|^2M_2^*T_2^*$$
$$- 4Y_1|T_2|^2Y_1^* - 2Y_1|T_3|^2Y_1^* + 4M_2T_2|Y_3|^2Y_1^* - 2Y_2|T_3|^2Y_2^*$$
$$- \frac{1}{8}g_d^*\Big(3g_d|g_u|^2\Big(m_2^2 - 16M_1M_1^* - 8M_2M_2^*\Big) + 2\Big(4g_d|T_3|^2$$
$$+ g_d\Big(8\lambda_3m_1^2 + |Y_3|^2\Big(3m_2^2 - 32M_1M_1^* - 8M_2M_2^*\Big)\Big)$$
$$+ 4g_uY_3\Big(2m_1^2 + 6|M_2|^2 + 8|M_1|^2 - m_2^2\Big)Y_2^*\Big)\Big) - 2g^2Y_3|M_2|^2Y_3^* - 4Y_3|T_2|^2Y_3^*$$

$$
\begin{aligned}
&- \frac{3}{2} m_2^2 Y_3 |Y_1|^2 Y_3^* - \frac{3}{2} m_2^2 Y_3 |Y_2|^2 Y_3^* + 10 Y_1 Y_3 |M_2|^2 Y_1^* Y_3^* + 12 Y_2 Y_3 |M_2|^2 Y_2^* Y_3^* \\
&+ \frac{1}{8} g_u^* \Big(5 g^2 g_u m_2^2 - g_u \Big(64 \lambda_5 m_2^2 + 16 |T_2|^2 + 6 m_2^2 |Y_1|^2 + 6 m_2^2 |Y_2|^2 - 16 M_2 T_2 Y_1^* \Big) \\
&+ 32 g_u \Big(Y_1 |M_1|^2 Y_1^* + Y_2 |M_1|^2 Y_2^* \Big) - g_d \Big(16 m_1^2 Y_2 Y_3^* - 8 m_2^2 Y_2 Y_3^* + 64 Y_2 |M_1|^2 Y_3^* \Big) \\
&+ 8 M_2^* \Big(2 g_u Y_1 T_2^* + M_2 \Big(2 g_u |Y_2|^2 + 5 g_u |Y_1|^2 - 6 g_d Y_2 Y_3^* + 6 g_u |Y_3|^2 - g^2 g_u \Big) \Big) \Big) \qquad \text{(B.38)}
\end{aligned}
$$

$$
\begin{aligned}
\beta^{(1)}_{m_3^2} &= 2 \lambda_1 m_1^2 + 2 \lambda_2 m_2^2 - T_1^2 - T_2^2 + 2 |T_1|^2 + 2 |T_2|^2 + 2 |T_3|^2 + 2 m_3^2 |Y_1|^2 + 2 Y_1^2 M_2^{*\,2} \\
&- T_1^{*\,2} - T_2^{*\,2} - 8 Y_1 |M_2|^2 Y_1^* + 2 M_2^2 Y_1^{*\,2} \qquad \text{(B.39)}
\end{aligned}
$$

$$
\begin{aligned}
\beta^{(2)}_{m_3^2} &= 4 g^2 \lambda_1 m_1^2 - 4 \lambda_1^2 m_1^2 + 4 g^2 \lambda_2 m_2^2 - 4 \lambda_2^2 m_2^2 - \lambda_1^2 m_3^2 - \lambda_2^2 m_3^2 - 2 g^2 T_1^2 + 4 \lambda_1 T_1^2 \\
&- 2 g^2 T_2^2 + 4 \lambda_2 T_2^2 + 4 g^2 |T_1|^2 - 12 \lambda_1 |T_1|^2 + 4 g^2 |T_2|^2 - 12 \lambda_2 |T_2|^2 + 4 g^2 |T_3|^2 \\
&- 6 \lambda_1 |T_3|^2 - 6 \lambda_2 |T_3|^2 + \frac{5}{2} g^2 m_3^2 |Y_1|^2 - 4 \lambda_1 m_1^2 |Y_2|^2 + 2 T_1^2 |Y_2|^2 - 4 \lambda_2 m_2^2 |Y_3|^2 \\
&+ 2 T_2^2 |Y_3|^2 - 3 m_3^2 |Y_1|^4 + 32 |M_2|^2 |Y_1|^4 + 2 g^2 Y_1^2 M_2^{*\,2} - 2 Y_1^2 \Big(|Y_2|^2 + |Y_3|^2 \Big) M_2^{*\,2} \\
&- 2 g^2 T_1^{*\,2} + 4 \lambda_1 T_1^{*\,2} + 2 |Y_2|^2 T_1^{*\,2} - 2 g^2 T_2^{*\,2} + 4 \lambda_2 T_2^{*\,2} + 2 |Y_3|^2 T_2^{*\,2} \\
&- 8 g^2 Y_1 |M_2|^2 Y_1^* - 4 g_d g_u T_3 M_1^* Y_1^* - 8 Y_1^3 M_2^{*\,2} Y_1^* + 2 g^2 M_2^2 Y_1^{*\,2} - 2 M_2^2 |Y_2|^2 Y_1^{*\,2} \\
&- 2 M_2^2 |Y_3|^2 Y_1^{*\,2} - 8 M_2^2 Y_1 Y_1^{*\,3} \\
&- \frac{1}{4} |g_u|^2 \Big(8 \lambda_2 m_2^2 - 4 T_2^2 + 4 Y_1^2 M_2^{*\,2} + 8 T_2 T_2^* - 4 T_2^{*\,2} + 4 T_3 T_3^* + 3 m_3^2 Y_1 Y_1^* \\
&- 32 M_1 Y_1 M_1^* Y_1^* - 16 M_2 Y_1 M_2^* Y_1^* + 4 M_2^2 Y_1^{*\,2} \Big) \\
&+ g_d^* \Big(- 2 g_d \lambda_1 m_1^2 + g_d T_1^2 - 2 g_d |T_1|^2 - g_d |T_3|^2 - \frac{3}{4} g_d m_3^2 |Y_1|^2 - g_d Y_1^2 M_2^{*\,2} + g_d T_1^{*\,2} \\
&- 4 M_1 Y_1 g_u^* T_3^* + 8 g_d Y_1 |M_1|^2 Y_1^* + 4 g_d Y_1 |M_2|^2 Y_1^* - g_d M_2^2 Y_1^{*\,2} \Big) \\
&- 4 Y_2 |T_1|^2 Y_2^* - 2 Y_2 |T_3|^2 Y_2^* - \frac{3}{2} m_3^2 Y_2 |Y_1|^2 Y_2^* + 8 Y_1 Y_2 |M_2|^2 Y_1^* Y_2^* - 4 Y_3 |T_2|^2 Y_3^* \\
&- 2 Y_3 |T_3|^2 Y_3^* - \frac{3}{2} m_3^2 Y_3 |Y_1|^2 Y_3^* + 8 Y_1 Y_3 |M_2|^2 Y_1^* Y_3^* \qquad \text{(B.40)}
\end{aligned}
$$

Appendix C

Feynman rules, Widths and Branching ratios for the RS model

In this appendix we discuss interactions of the radion-dominant (ϕ) and the higgs-dominant (h) states in the RS model and provide the relevant Feynman rules: for the couplings of ϕ and h to the SM fermions and gauge bosons, as well as the trilinear self-couplings ϕhh, $h\phi\phi$, $\phi\phi\phi$, hhh .

C.1 Radion Interaction Lagrangian

Interaction with fermions

The interaction of the radion with two SM fermions is described by the following part of the trace of the SM stress tensor [73]

$$T^{\mu}_{\mu} = \sum_{f} \left[-\frac{3i}{2} \left(\left(D_{\mu} \bar{f} \right) \gamma^{\mu} f - \bar{f} \gamma^{\mu} \left(D_{\mu} f \right) \right) + 4 m_{f} \bar{f} f \right] \tag{C.1}$$

Most of the authors consider the case of on-shell fermions, thus the effective Lagrangian (including the SM Yukawa contribution due to the Higgs-radion mixing) takes the following form

$$L_{\phi ff} = \frac{\phi_0}{\Lambda_r} \sum_{f} m_{f} \bar{f} f + L_{Yukawa} \tag{C.2}$$

Interaction with massive vector bosons

Similarly, the interaction of the radion with two SM massive gauge bosons includes a contribution from the the trace of the SM stress tensor and the SM Higgs kinetic term, respectively:

$$L_{\phi VV} = \frac{\phi_0}{\Lambda_r} \left[-2 m_W^2 W^{+}_{\mu} W^{\mu -} - m_Z^2 Z_{\mu} Z^{\mu} \right] + \left| D_{\mu} \, \Phi \right|^2 \tag{C.3}$$

Anomalous couplings

The effective vertex for the radion-gluon-gluon interaction (at momentum transfer q along the scalar line) appears from the following Lagrangian [43]

$$L_{\phi gg} = \left[-\frac{\phi_0}{\Lambda_r} b_3 - \frac{1}{2} \left(-\frac{\phi_0}{\Lambda_r} + \frac{h_0}{v} \right) F_{1/2}(\tau_f) \right] \frac{\alpha_s}{8\pi} G_{\mu\nu} G^{\mu\nu} \tag{C.4}$$

with $\tau_f = 4m_f^2/q^2$ and the form factor $F_{1/2}(\tau_f) = -2\tau_f \left[1 + (1-\tau_f) f(\tau_f)\right]$, where

$$f(\tau) = -\frac{1}{4} \ln \left[-\frac{1+\sqrt{1-\tau}}{1-\sqrt{1-\tau}} \right]^2 = \begin{cases} \arcsin^2(1/\sqrt{\tau}), & \tau \geq 1 \\ -\frac{1}{4} \ln \left[-\frac{1+\sqrt{1-\tau}}{1-\sqrt{1-\tau}} - i\pi \right]^2, & \tau \leq 1 \,. \end{cases} \tag{C.5}$$

The first term here describes the QCD trace anomaly (with the SM QCD β-function coefficient $b_3 = 7$), the second term represents the effective contribution of 1-loop diagrams with virtual fermions circulating in the loop.

Similarly, the radion-photon-photon interaction is given by

$$L_{\phi\gamma\gamma} = \left[-\frac{\phi_0}{\Lambda_r} (b_2 + b_Y) - \left(-\frac{\phi_0}{\Lambda_r} + \frac{h_0}{v} \right) \left(F_1(\tau_W) + \sum_i e_i^2 N_c^i F_{1/2}(\tau_i) \right) \right] \frac{\alpha_{EM}}{8\pi} F_{\mu\nu} F^{\mu\nu} \tag{C.6}$$

where the first term describes the QED trace anomaly [75] with the SM $SU(2) \times U(1)_Y$ β-function coefficients $b_2 = 19/6$ and $b_Y = -41/6$, the second term comes from 1-loop diagrams with virtual fermions and virtual W-bosons in the loop with the form factors $F_{1/2}(\tau_i)$ and $F_1(\tau_W) = 2 + 3\tau_W + 3\tau_W(2-\tau_W) f(\tau_W)$ respectively. The summation $\sum_i$ is taken over all SM fermions with an electric charge e_i and a color N_c^i. (For the observations on conformal anomalies see [58].)

C.2 Vertices, Widths and branching ratios

The following notations are used [71, 72]:

$$g_h = (D + \gamma B)\,, \quad g_\phi = (C + \gamma A)\,, \quad g_h^r = \gamma B\,, \quad g_\phi^r = \gamma A\,,$$

$$\kappa_{W,Z} = \frac{3 m_{W,Z}^2 k b_0}{2\Lambda_\phi^2 (k/M_{Pl})^2}\,, \quad g_{h,\phi}^{W,Z} = \frac{g_{h,\phi}^r}{(g_{h,\phi} - \kappa_{W,Z} g_{h,\phi}^r) m_{W,Z}^2} \left(\frac{1}{2kb_0} + g_{W,Z}^{\rm anom} \right),$$

$$g_W^{\rm anom} = \frac{\alpha}{8\pi} \frac{b_2}{\sin^2\theta_W}\,, \quad g_Z^{\rm anom} = \frac{\alpha}{8\pi} \left(\frac{b_2}{\tan^2\theta_W} + b_Y \tan^2\theta_W \right),$$

where $b_2 = 19/6$, $b_Y = -41/6$, $b_3 = 7$ are the $SU(3) \times SU(2) \times U(1)$ beta function coefficients, θ_W is the Weinberg angle, and A, B, C, D are the parameters of the Higgs-

radion mixing matrix (6.13) in Sec. 6.1.2 of Part II.

Vertices $V_{\phi ZZ}$ and V_{hZZ}

$$V_{\phi ZZ} = ig\frac{m_Z}{\cos\theta_W}\left(g_\phi - g_\phi^r\kappa_Z\right)\left[\eta^{\mu\nu} - 2g_\phi^Z\left(\eta^{\mu\nu}p_1\cdot p_2 - p_1^\nu p_2^\mu\right)\right] \tag{C.7}$$

Vertex V_{hZZ}

$$V_{hZZ} = ig\frac{m_Z}{\cos\theta_W}\left(g_h - g_h^r\kappa_Z\right)\left[\eta^{\mu\nu} - 2g_h^Z\left(\eta^{\mu\nu}p_1\cdot p_2 - p_1^\nu p_2^\mu\right)\right] \tag{C.8}$$

Squared amplitudes $M_{\phi ZZ}$ and M_{hZZ}

$$\begin{aligned}\mid M_{\phi ZZ}\mid^2 = {}& \frac{1}{2}\frac{g^2}{c_w^2}m_Z^2(g_\phi - g_\phi^r\kappa_Z)^2 \\ & \left[\eta_{\mu\nu} - 2g_\phi^Z\left(\eta_{\mu\nu}p_1\cdot p_2 - p_1^\nu p_2^\mu\right)\right]\left[\eta_{\rho\sigma} - 2g_\phi^Z\left(\eta_{\rho\sigma}p_1\cdot p_2 - p_1^\sigma p_2^\rho\right)\right] \\ & \left(\frac{p_1^\mu p_1^\rho}{m_Z^2} - \eta_{\mu\rho}\right)\left(\frac{p_2^\nu p_2^\sigma}{m_Z^2} - \eta_{\nu\sigma}\right)\end{aligned} \tag{C.9}$$

$$\begin{aligned}\mid M_{hZZ}\mid^2 = {}& \frac{1}{2}\frac{g^2}{c_w^2}m_Z^2(g_h - g_h^r\kappa_Z)^2 \\ & \left[\eta_{\mu\nu} - 2g_h^Z\left(\eta_{\mu\nu}p_1\cdot p_2 - p_1^\nu p_2^\mu\right)\right]\left[\eta_{\rho\sigma} - 2g_h^Z\left(\eta_{\rho\sigma}p_1\cdot p_2 - p_1^\sigma p_2^\rho\right)\right] \\ & \left(\frac{p_1^\mu p_1^\rho}{m_Z^2} - \eta_{\mu\rho}\right)\left(\frac{p_2^\nu p_2^\sigma}{m_Z^2} - \eta_{\nu\sigma}\right)\end{aligned} \tag{C.10}$$

Squared amplitudes $M_{\phi ZZ}$ and M_{hZZ} (simplified)

$$\begin{aligned}\mid M_{\phi ZZ}\mid^2 = {}& \frac{g^2}{8c_w^2m_Z^2}\left(g_\phi - g_\phi^r\kappa_Z\right)^2\Big[12m_Z^4\left(1+2g_\phi^Z m_Z^2\right)^2 + m_\phi^4\left(1+8\mid g_\phi^Z\mid^2 m_Z^4\right) \\ & -4\,m_\phi^2\left(m_Z^2 + 6g_\phi^Z m_Z^4 + 8\mid g_\phi^Z\mid^2 m_Z^6\right)\Big]\end{aligned} \tag{C.11}$$

$$\begin{aligned}\mid M_{hZZ}\mid^2 = {}& \frac{g^2}{8c_w^2m_Z^2}\left(g_h - g_h^r\kappa_Z\right)^2\Big[12m_Z^4\left(1+2g_h^Z m_Z^2\right)^2 + m_h^4\left(1+8\mid g_h^Z\mid^2 m_Z^4\right) \\ & -4\,m_h^2\left(m_Z^2 + 6g_h^Z m_Z^4 + 8\mid g_h^Z\mid^2 m_Z^6\right)\Big]\end{aligned} \tag{C.12}$$

Partial widths $\Gamma_{\phi ZZ}$ and Γ_{hZZ}

$$\Gamma_{\phi ZZ} = \frac{1}{16\pi\, m_\phi}\mid M_{\phi ZZ}\mid^2\sqrt{1-\frac{4m_Z^2}{m_\phi^2}} \tag{C.13}$$

$$\Gamma_{hZZ} = \frac{1}{16\pi\, m_h} \mid M_{hZZ} \mid^2 \sqrt{1 - \frac{4m_Z^2}{m_h^2}} \tag{C.14}$$

Vertices $V_{\phi WW}$ and V_{hWW}

$$V_{\phi WW} = igm_W \left(g_\phi - g_\phi^r \kappa_W\right) \left[\eta^{\mu\nu} - 2g_\phi^W \left(\eta^{\mu\nu} p_1 \cdot p_2 - p_1^\nu p_2^\mu\right)\right] \tag{C.15}$$

$$V_{hWW} = igm_W \left(g_h - g_h^r \kappa_W\right) \left[\eta^{\mu\nu} - 2g_h^W \left(\eta^{\mu\nu} p_1 \cdot p_2 - p_1^\nu p_2^\mu\right)\right] \tag{C.16}$$

Squared amplitudes $M_{\phi WW}$ and M_{hWW}

$$\begin{aligned} \mid M_{\phi WW} \mid^2 = {} & g^2 m_W^2 (g_\phi - g_\phi^r \kappa_W)^2 \\ & \left[\eta_{\mu\nu} - 2g_\phi^W \left(\eta_{\mu\nu} p_1 \cdot p_2 - p_1^\nu p_2^\mu\right)\right] \left[\eta_{\rho\sigma} - 2g_\phi^W \left(\eta_{\rho\sigma} p_1 \cdot p_2 - p_1^\sigma p_2^\rho\right)\right] \\ & \left(\frac{p_1^\mu p_1^\rho}{m_W^2} - \eta_{\mu\rho}\right) \left(\frac{p_2^\nu p_2^\sigma}{m_W^2} - \eta_{\nu\sigma}\right) \end{aligned} \tag{C.17}$$

$$\begin{aligned} \mid M_{hWW} \mid^2 = {} & g^2 m_W^2 (g_h - g_h^r \kappa_W)^2 \\ & \left[\eta_{\mu\nu} - 2g_h^W \left(\eta_{\mu\nu} p_1 \cdot p_2 - p_1^\nu p_2^\mu\right)\right] \left[\eta_{\rho\sigma} - 2g_h^W \left(\eta_{\rho\sigma} p_1 \cdot p_2 - p_1^\sigma p_2^\rho\right)\right] \\ & \left(\frac{p_1^\mu p_1^\rho}{m_W^2} - \eta_{\mu\rho}\right) \left(\frac{p_2^\nu p_2^\sigma}{m_W^2} - \eta_{\nu\sigma}\right) \end{aligned} \tag{C.18}$$

Squared amplitudes $M_{\phi WW}$ and M_{hWW} (simplified)

$$\begin{aligned} \mid M_{\phi WW} \mid^2 = {} & \frac{g^2}{4m_W^2} \left(g_\phi - g_\phi^r \kappa_W\right)^2 \Big[12m_W^4 \left(1 + 2g_\phi^W m_W^2\right)^2 + m_\phi^4 \left(1 + 8 \mid g_\phi^W \mid^2 m_W^4\right) \\ & -4\, m_\phi^2 \left(m_W^2 + 6g_\phi^W m_W^4 + 8 \mid g_\phi^W \mid^2 m_W^6\right)\Big] \end{aligned} \tag{C.19}$$

$$\begin{aligned} \mid M_{hWW} \mid^2 = {} & \frac{g^2}{4m_W^2} \left(g_h - g_h^r \kappa_W\right)^2 \Big[12m_W^4 \left(1 + 2g_h^W m_W^2\right)^2 + m_h^4 \left(1 + 8 \mid g_h^W \mid^2 m_W^4\right) \\ & -4\, m_h^2 \left(m_W^2 + 6g_h^W m_W^4 + 8 \mid g_h^W \mid^2 m_W^6\right)\Big] \end{aligned} \tag{C.20}$$

Partial widths $\Gamma_{\phi WW}$ and Γ_{hWW}

$$\Gamma_{\phi WW} = \frac{1}{16\pi\, m_\phi} \mid M_{\phi WW} \mid^2 \sqrt{1 - \frac{4m_W^2}{m_\phi^2}} \tag{C.21}$$

$$\Gamma_{hWW} = \frac{1}{16\pi\, m_h} \mid M_{hWW} \mid^2 \sqrt{1 - \frac{4m_W^2}{m_h^2}} \tag{C.22}$$

Vertices $V_{\phi\gamma\gamma}$ and $V_{h\gamma\gamma}$

$$V_{\phi\gamma\gamma} = i\frac{\alpha_{EW}}{2\pi v_0}\left[g_\phi^r\left(b_2 + b_Y + \frac{4\pi}{\alpha_{EW} k b_0}\right) - g_\phi \cdot S_{loop,r}\right](\eta^{\mu\nu} p_1 \cdot p_2 - p_1^\nu p_2^\mu) \tag{C.23}$$

$$V_{h\gamma\gamma} = i\frac{\alpha_{EW}}{2\pi v_0}\left[g_h^r\left(b_2 + b_Y + \frac{4\pi}{\alpha_{EW} k b_0}\right) - g_h \cdot S_{loop,h}\right](\eta^{\mu\nu} p_1 \cdot p_2 - p_1^\nu p_2^\mu) \tag{C.24}$$

with

$$S_{loop,r,h} = \left(\sum_i e_i^2 N_c^i F_{1/2} + F_1\right) \tag{C.25}$$

or (if only the top-quark is circulating in the loop)

$$S_{t-loop,r,h} = \frac{2}{3}^2 \cdot 3 \cdot F_{1/2}(M_t, m_{r,h}) + F_1(m_W, m_{r,h}), \tag{C.26}$$

$$F_{1/2}(\tau_{r,h}) = -2\tau_{r,h}\left[1 + (1 - \tau_{r,h}) f(\tau_{r,h})\right], \tag{C.27}$$

$$F_1(\tau_{r,h}) = 2 + 3\tau_{r,h} + 3\tau_{r,h}(2 - \tau_{r,h}) f(\tau_{r,h}), \tag{C.28}$$

$\tau_{r,h} = 4m_i^2/m_{r,h}^2$, m_i - fermion mass.

Squared amplitudes $M_{\phi\gamma\gamma}$ and $M_{h\gamma\gamma}$

$$\mid M_{\phi\gamma\gamma} \mid^2 = \frac{\alpha_{EW}^2\, m_\phi^4}{16\pi^2 v_0^2}\left|g_\phi^r\left(b_2 + b_Y + \frac{4\pi}{\alpha_{EW} k b_0}\right) - g_\phi \cdot S_{loop,r}\right|^2 \tag{C.29}$$

$$\mid M_{h\gamma\gamma} \mid^2 = \frac{\alpha_{EW}^2\, m_h^4}{16\pi^2 v_0^2}\left|g_h^r\left(b_2 + b_Y + \frac{4\pi}{\alpha_{EW} k b_0}\right) - g_h \cdot S_{loop,h}\right|^2 \tag{C.30}$$

Partial widths $\Gamma_{\phi\gamma\gamma}$ and $\Gamma_{h\gamma\gamma}$

$$\Gamma_{\phi\gamma\gamma} = \frac{1}{16\pi\, m_\phi} \mid M_{\phi\gamma\gamma} \mid^2 \tag{C.31}$$

$$\Gamma_{h\gamma\gamma} = \frac{1}{16\pi\, m_h} \mid M_{h\gamma\gamma} \mid^2 \tag{C.32}$$

Vertices $V_{\phi\gamma Z}$ and $V_{h\gamma Z}$

$$V_{\phi\gamma Z} = i\frac{\alpha_{EW}}{2\pi v_0}\left[2g_\phi^r\left(\frac{b_2}{\tan\theta_W} - b_Y\tan\theta_W\right) - g_\phi(A_F + A_W)\right](\eta^{\mu\nu}p_1\cdot p_2 - p_1^\nu p_2^\mu) \quad \text{(C.33)}$$

$$V_{h\gamma Z} = i\frac{\alpha_{EW}}{2\pi v_0}\left[2g_h^r\left(\frac{b_2}{\tan\theta_W} - b_Y\tan\theta_W\right) - g_h(A_F + A_W)\right](\eta^{\mu\nu}p_1\cdot p_2 - p_1^\nu p_2^\mu) \quad \text{(C.34)}$$

For A_F and A_W see [82] or "The Higgs Hunter's Guide" [83].

Squared amplitudes $M_{\phi\gamma Z}$ and $M_{h\gamma Z}$

$$\mid M_{\phi\gamma Z}\mid^2 = \frac{\alpha_{EW}^2\, m_\phi^4}{8\pi^2 v_0^2}\left(1 - \frac{m_Z^2}{m_\phi^2}\right)^2\left|2g_\phi^r\left(\frac{b_2}{\tan\theta_W} - b_Y\tan\theta_W\right) - g_\phi(A_F + A_W)\right|^2 \quad \text{(C.35)}$$

$$\mid M_{h\gamma Z}\mid^2 = \frac{\alpha_{EW}^2\, m_h^4}{8\pi^2 v_0^2}\left(1 - \frac{m_Z^2}{m_h^2}\right)^2\left|2g_h^r\left(\frac{b_2}{\tan\theta_W} - b_Y\tan\theta_W\right) - g_h(A_F + A_W)\right|^2 \quad \text{(C.36)}$$

Partial widths $\Gamma_{\phi\gamma Z}$ and $\Gamma_{h\gamma Z}$

$$\Gamma_{\phi\gamma Z} = \frac{1}{16\pi\, m_\phi}\mid M_{\phi\gamma Z}\mid^2\sqrt{1 - \frac{m_Z^2}{m_\phi^2}} \quad \text{(C.37)}$$

$$\Gamma_{h\gamma Z} = \frac{1}{16\pi\, m_h}\mid M_{h\gamma Z}\mid^2\sqrt{1 - \frac{m_Z^2}{m_h^2}} \quad \text{(C.38)}$$

Vertices $V_{\phi gg}$ and V_{hgg}

$$V_{\phi gg} = i\delta^{ab}\frac{\alpha_S}{4\pi v_0}\left[2g_\phi^r\left(b_3 + \frac{4\pi}{\alpha_S k b_0}\right) - g_\phi\sum_i F_{1/2}\right](\eta^{\mu\nu}p_1\cdot p_2 - p_1^\nu p_2^\mu) \quad \text{(C.39)}$$

$$V_{hgg} = i\delta^{ab}\frac{\alpha_S}{4\pi v_0}\left[2g_h^r\left(b_3 + \frac{4\pi}{\alpha_S k b_0}\right) - g_h\sum_i F_{1/2}\right](\eta^{\mu\nu}p_1\cdot p_2 - p_1^\nu p_2^\mu) \quad \text{(C.40)}$$

Squared amplitudes $M_{\phi gg}$ and M_{hgg}

$$\mid M_{\phi gg}\mid^2 = \frac{\alpha_S^2\, m_\phi^4}{8\pi^2 v_0^2}\left|2g_\phi^r\left(b_3 + \frac{4\pi}{\alpha_S k b_0}\right) - g_\phi\sum_i F_{1/2}\right|^2 \quad \text{(C.41)}$$

$$\mid M_{hgg} \mid^2 = \frac{\alpha_S^2 \, m_h^4}{8\pi^2 v_0^2} \left| 2 g_h^r \left(b_3 + \frac{4\pi}{\alpha_S k b_0} \right) - g_h \sum_i F_{1/2} \right|^2 \tag{C.42}$$

Partial widths $\Gamma_{\phi gg}$ and Γ_{hgg}

$$\Gamma_{\phi gg} = \frac{1}{16\pi \, m_\phi} \mid M_{\phi gg} \mid^2 \tag{C.43}$$

$$\Gamma_{hgg} = \frac{1}{16\pi \, m_h} \mid M_{hgg} \mid^2 \tag{C.44}$$

Vertices $V_{\phi f\bar{f}}$ and $V_{hf\bar{f}}$ (for $f = t, b, c, \tau$)

$$V_{\phi f\bar{f}} = -i \frac{g}{2} \frac{M_f}{m_W} g_\phi \tag{C.45}$$

$$V_{hf\bar{f}} = -i \frac{g}{2} \frac{M_f}{m_W} g_h \tag{C.46}$$

Squared (on-shell) amplitudes $M_{\phi f\bar{f}}$ and $M_{hf\bar{f}}$

$$\mid M_{\phi f\bar{f}} \mid^2 = -\frac{n_c}{2} g^2 g_\phi^2 \frac{M_f^2}{m_W^2} \left(4M_f^2 - m_\phi^2 \right) \tag{C.47}$$

where n_c is the number of colors, $n_c = 3$ for quarks and $n_c = 1$ for leptons.

$$\mid M_{hb\bar{b},hc\bar{c}} \mid^2 = -\frac{3}{2} g^2 g_h^2 \frac{M_{b,c}^2}{m_W^2} \left(4M_{b,c}^2 - m_h^2 \right) \tag{C.48}$$

$$\mid M_{h\tau\bar{\tau}} \mid^2 = -\frac{1}{2} g^2 g_h^2 \frac{M_\tau^2}{m_W^2} \left(4M_\tau^2 - m_h^2 \right) \tag{C.49}$$

Partial widths $\Gamma_{\phi f\bar{f}}$ and $\Gamma_{hf\bar{f}}$

$$\Gamma_{\phi f\bar{f}} = \frac{1}{2 \, m_\phi} \frac{1}{8\pi} \mid M_{\phi f\bar{f}} \mid^2 \sqrt{1 - \frac{4M_f^2}{m_\phi^2}} \tag{C.50}$$

$$\Gamma_{hb\bar{b},hc\bar{c}} = \frac{1}{2 \, m_h} \frac{1}{8\pi} \mid M_{hb\bar{b},hc\bar{c}} \mid^2 \sqrt{1 - \frac{4M_{b,c}^2}{m_h^2}} \tag{C.51}$$

$$\Gamma_{h\tau\bar{\tau}} = \frac{1}{2 \, m_h} \frac{1}{8\pi} \mid M_{h\tau\bar{\tau}} \mid^2 \sqrt{1 - \frac{4M_\tau^2}{m_h^2}} \tag{C.52}$$

Vertices $V_{\phi hh}$ and V_{hhh} [71]

$$V_{\phi hh} = \frac{i}{\Lambda_\phi}\Big[\{6b\xi\,(\gamma(ad+bc)+cd)+ad^2\}\left(p_1^2+p_2^2\right) \\ +d\{12ab\gamma\xi+2bc+ad(6\xi-1)\}p_3^2-4d(ad+2bc)\,m_{h_0}^2 \\ -3\gamma^{-1}cd^2\,m_{h_0}^2-3X_3ab^2\,m_{\phi_0}^2\Big] \tag{C.53}$$

$$V_{hhh} = \frac{i}{\Lambda_\phi}\Big[bd\{[12b\gamma\xi+d(6\xi+1)]\left(p_1^2+p_2^2+p_3^2\right)-12d\,m_{h_0}^2\} \\ -3\gamma^{-1}d^3\,m_{h_0}^2-3X_3b^3\,m_{\phi_0}^2\Big] \tag{C.54}$$

Squared amplitudes $M_{\phi hh}$ and M_{hhh}

$$\mid M_{\phi hh}\mid^2 = \frac{1}{2}V_{\phi hh}^2\Big|_{p_1^2\to m_h^2, p_2^2\to m_h^2, p_3^2\to m_\phi^2} \tag{C.55}$$

$$\mid M_{hhh}\mid^2 = \frac{1}{2}V_{hhh}^2 \tag{C.56}$$

Partial width $\Gamma_{\phi hh}$

$$\Gamma_{\phi hh} = \frac{1}{2\,m_\phi}\frac{1}{8\pi}\mid M_{\phi hh}\mid^2\sqrt{1-\frac{4\,m_h^2}{m_\phi^2}} \tag{C.57}$$

Appendix D

FeynRules model-file for the RS model

```
(* ********************************************************************** *)
(* *****                                                          ***** *)
(* *****  FeynRules model file: SM + radion                       ***** *)
(* *****                                                          ***** *)
(* ********************************************************************** *)

(* ************************** *)
(* *****  Change log    ***** *)
(* ************************** *)

(* 2017.10.10 v1.0   - Release of version 1
                                                                    *)

(* ************************** *)
(* *****  Information   ***** *)
(* ************************** *)

M$ModelName = "Radion_Higgs_Model_v1.3.5";

M$Information = {
  Authors      -> {"K. Svirina", "I. Schienbein", "B. Fuks"},
  Institutions -> {"LPSC Grenoble", "LPSC Grenoble", "LPTHE / Sorbonne Universite"},
  Emails       -> {"svirinal@lpsc.in2p3.fr", "schien@lpsc.in2p3.fr",
      "fuks@lpthe.jussieu.fr"},
  Date         -> "2018.04.03",
  Version      -> "1.3.5",
  References   -> {"..."}
 };


(* ************************** *)
(* *****     Fields     ***** *)
(* ************************** *)
M$ClassesDescription = {

(* Higgs and radion: unphysical scalars  *)

  S[11] == {
    ClassName      -> Phi,
    Unphysical     -> True,
    Indices        -> {Index[SU2D]},
```

```
    FlavorIndex    -> SU2D,
    SelfConjugate  -> False,
    QuantumNumbers -> {Y -> 1/2},
    Definitions    -> { Phi[1]->0, Phi[2]->(vev + (Sin[th] + 6 gam xi/ZZ Cos[th]) RR +
        (Cos[th] - 6 gam xi/ZZ Sin[th]) H)/Sqrt[2] }
  },

  S[12] == {
    ClassName      -> R0,
    Unphysical     -> True,
    SelfConjugate  -> False,
    Definitions    -> { R0 ->  1/ZZ Cos[th] RR - 1/ZZ Sin[th] H  }
  },

(* Omega *)
S[13] == {
    ClassName      -> Omega,
    Unphysical     -> True,
    SelfConjugate  -> False,
    Definitions    -> { Omega -> 1 - gam R0/vev  }
  },

(* Higgs and radion: physical scalars *)
  S[1] == {
    ClassName      -> H,
    SelfConjugate  -> True,
    Mass           -> {MH,125.0},
    Width          -> {WH,0.00407},
    PDG            -> 25
  },
  S[2] == {
    ClassName      -> RR,
    SelfConjugate  -> True,
    Mass           -> {MR,5000.},
    Width          -> {WR,1.1},
    PDG            -> 35
  }
}

(* ************************* *)
(* *****   Parameters   ***** *)
(* ************************* *)
M$Parameters = {
  (* External parameters *)
  gam == {
    ParameterType -> External,
    Value         -> 0.1,
    Description   -> "Higgs vev/Radion vev",
    TeX           -> \[Gamma]
  },
  xi == {
    ParameterType -> External,
    Value         -> 0.15,
    Description   -> "dimless parameter in the Higgs-gravity coupling",
    TeX           -> \[Xi]

  },
X3 == {
```

```
    ParameterType -> External,
    Value         -> 3,
    Description   -> "radion self-coupling constant",
    TeX           -> Subscript[X,3]

  },
LambdaR == {
    ParameterType -> External,
    Value         -> 2500,
    Description   -> "radion vev",
    TeX           -> Subscript[\[CapitalLambda],R]

  },


  (* Internal parameters *)
  lam == {
    ParameterType    -> Internal,
    Value            -> mh0^2/(2*vev^2),
    InteractionOrder -> {QED, 2},
    Description      -> "Higgs quartic coupling",
    TeX              -> \[Lambda]
  },
  muH == {
    ParameterType -> Internal,
    Value         -> Sqrt[vev^2 lam],
    TeX           -> \[Mu],
    Description   -> "Coefficient of the quadratic piece of the Higgs potential"
  },
  beta == {
    ParameterType -> Internal,
    Value         -> 1 + 6 gam^2 xi,
    TeX           -> \[Beta]
  },
  ZZ == {
    ParameterType -> Internal,
    Value         -> Sqrt[beta - 36 gam^2 xi^2],
    TeX           -> Z
  },
  mh0 == {
    ParameterType -> Internal,
    TeX           -> Subscript[M,h0],
    Value -> Sqrt[ZZ^2/(2 beta) (MR^2 + MH^2 -Sqrt[(MR^2+ MH^2)^2 -4 beta MR^2
        MH^2/ZZ^2])],
    Description   -> "mass of the unphysical higgs"
  },
  mr0 == {
    ParameterType -> Internal,
    TeX           -> Subscript[M,r0],
    Value -> Sqrt[ZZ^2/2 (MR^2 + MH^2 + Sqrt[(MR^2+ MH^2)^2 -4 beta MR^2 MH^2/ZZ^2])],
    Description   -> "mass of the unphysical radion"
  },
  tan2th == {
    ParameterType -> Internal,
    Value         -> 12 gam xi ZZ mh0^2/(mr0^2 - mh0^2 (ZZ^2- 36 gam^2 xi^2)),
    Description   -> "tan of 2 * the mixing angle"
  },
  th == {
```

```
    ParameterType -> Internal,
    Value         -> ArcTan[tan2th]/2,
    TeX           -> \[Theta]
  }

};

(* ************************ *)
(* *****   Lagrangian   ***** *)
(* ************************ *)

LRadion := Block[{ii,jj,mu},
  1/2 del[R0, mu] del[R0, mu] - 1/2 mr0^2 R0^2 -
  6 xi Omega del[del[Omega,mu],mu] Phibar[ii] Phi[ii] +
  DC[Phibar[ii],mu] DC[Phi[ii],mu] -
  Omega^4 (lam vev^4/4 - muH^2 Phibar[ii] Phi[ii] + lam Phibar[ii] Phi[ii] Phibar[jj]
      Phi[jj])
];


TmumuH:= R0 / LambdaR ( Block[{mu}, -del[H0, mu] del[H0, mu]] + 2 mh0^2 H0^2
    (1+H0/2/vev)^2 );

TmumuF:= R0 / LambdaR (Block[{mu,s,f,i}, Mu[f]uqbar[s,f,i].uq[s,f,i]
+ Md[f] dqbar[s,f,i].dq[s,f,i]
+ Ml[f] lbar[s,f].l[s,f]
]);
TmumuV:= R0 / LambdaR (Block[{mu}, -2 MW^2
 W[mu] Wbar[mu]- MZ^2 Z[mu] Z[mu]
]);


LHR := LRadion + TmumuH +TmumuF;

LanomG := (-R0 / LambdaR b3 - 1/2(-R0 / LambdaR + H0/vev) F12) aS/8/Pi (Block[{mu,nu,
    aa}, FS[G,mu,nu,aa] FS[G,mu,nu,aa]]);

LanomA:=(-R0 / LambdaR (b2+bY) - (-R0 / LambdaR + H0/vev) (F1W + 4/3 F12up + 1/3 F12down
    + F12lept)) aEW/8/Pi (Block[{mu,nu}, FS[A,mu,nu] FS[A,mu,nu]]);
```

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