Formality theorems for Lie algebroids and quantization of dynamical r-matrices

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THÈSE

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par

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Théorèmes de formalité pour les algébroïdes de Lie et quantification des *r*-matrices dynamiques

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Un progrès rapide et régulier des sciences n'est possible que si un individu n'est pas obligé d'être *trop méfiant*, de vérifier un à un les calculs et les assertions des autres dans des domaines qui ne lui sont pas familiers ; mais la condition en est que chacun ait dans sa propre sphère des concurrents *extrêmement méfiants*, et qui le surveillent de très près.

Friedrich Nietzsche, *Opinions et sentences mêlées.*

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Introduction

Les algébroïdes de Lie ont été introduits par Pradines [**56**] comme les objets infinitésimaux associés aux groupoïdes de Lie, qui permettent de réunir dans un même formalisme les symétries internes et externes. En particulier, ils revêtent une importance particulière dans l'étude des feuilletages [**8, 10, 68**], des orbifolds [**49**] et des quotients singuliers [**8, 10**]. Remarquons que la structure algébrique sous-jacente à la notion d'algébroïde de Lie était déjà présente dans un travail de Rinehart [**57**]. Un telle structure est aujourd'hui appelée une *algèbre de Lie-Rinehart.*

Les algébroïdes de Lie constituent le principal objet d'étude de cette thèse, dans laquelle nous nous intéressons à leurs déformations dans la direction d'une structure de Poisson donnée.

La quantification d'un système physique classique donné consiste en la donnée d'un système physique quantique tel que si l'on néglige la constante de Planck $\hbar$ (mathématiquemet parlant, on considère la limite $\hbar \to 0$) alors on retrouve le système classique de départ. Dans [**3**], F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowics et D. Sternheimer ont découvert une formulation extrêmement élégante de cette situation en termes purement algébriques. C'est ce qu'on appelle habituellement la *quantification par deformation.*

D'une part, en mécanique classique les états sont les points d'une variété de Poisson et les observables sont les fonctions sur cette variété. Ces fonctions forment une algèbre commutative munie d'un crochet de Poisson $\{\,,\}$. D'autre part, en mécanique quantique les états sont des éléments d'un espace de Hilbert (en général un espace de fonctions) et les observables sont les opérateurs auto-adjoints sur cet espace de Hilbert. Ces opérateurs forment une algèbre associative (non commutative). L'idée principale de la quantification par déformation est d'oublier les espaces de Hilbert (les états) et de ne s'occuper que des algèbres (les observables) : partant d'une algèbre commutative A munie d'un crochet de Poisson $\{\,,\}$, on se demande s'il existe un produit associatif $\hbar$-linéaire $*$ sur $A[[\hbar]]$ (les séries formelles en $\hbar$ à coefficients dans A) tel que $a*b = ab+\hbar\{a,b\}+o(\hbar)$. Dans le cas où $A = C^\infty(M)$ est l'algèbre des fonctions sur une variétés de Poisson, une réponse positive a été donnée à ce problème par M. Kontsevich [**39**]. En réalité, le résultat démontré par M. Kontsevich dans [**39**] est beaucoup plus fort : il s'agit de la *formalité* de l'algèbre de Lie différentielle graduée des opérateurs polydifférentiels (ou cochaînes de Hochschild locales) sur une variété donnée M (ce qui signifie que cette algèbre de Lie différentielle graduée est L_∞-quasi-isomorphe à sa cohomologie), qui implique non seulement l'existence d'une quantification pour n'importe quelle structure de Poisson sur M, mais permet également de classifier ces quantifications. Une version pour les chaînes de Hochschild (qui forment un module différentiel gradué sur l'algèbre de Lie différentielle graduée des cochaînes) de la conjecture de formalité a été formulée par B. Tsygan [**66**] et démontrée par B. Shoikhet [**58**] puis V. Dolgushev [**17**] respectivement pour $M = \mathbb{R}^d$ et dans le cas général ; qui permet de calculer l'homologie de Hochschild des algèbres déformées obtenues (et notament d'obtenir des traces quantiques).

Formuler puis démontrer de telles conjectures de formalité (pour les cochaînes et pour les chaînes de Hochschild) dans le cas des algebroides de Lie permettrait ainsi d'obtenir des

quantifications préservant les symétries. Cela constitue le premier axe du travail présenté ici.

La quantification des variétés de Poisson holomorphes pose plus de problèmes, du fait du très petit nombre de fonctions holomorphes globales (dans la plupart des cas, il ne reste que les fonctions constantes). Le bon problème dans cette situation est la déformation du *faisceau* des fonctions holomorphes sur la variété. Un résultat dans cette direction a été annoncé par M. Kontsevich [**40**] et démontré dans le cas symplectique par P. Polesello et P. Schapira [**55**]. Un théorème de classification a également été démontré dans le cas symplectique par P. Polesello [**54**].

La généralisation de ces résultats aux variétés de Poisson holomorphes (et même aux algébroïdes de Lie poissoniens holomorphes) constitue le deuxième axe de notre travail.

Enfin, il a été démontré par P. Etingof et A. Varchenko qu'à toute *r-matrice dynamique* classique peut être associé un groupoïde de Lie-Poisson. Par ailleurs, les r-matrices dynamiques apparaissent comme des cas particuliers de la procédure de *réduction*. Aussi bien les r-matrices dynamiques que la procédure de réduction admettent des contreparties quantiques, accompagnées des problèmes de quantification associés (voir [**70, 72**] pour les r-matrices dynamiques et [**69, 53**] pour la réduction). C'est très naturellement que nous nous penchons sur ces questions dans le troisième axe de notre travail.

Dans ce contexte, l'objectif de cette thèse est triple :

- dans un premier temps (chapitres 1 et 2) on formule puis on démontre la formalité pour les cochaînes et pour les chaînes de Hochschild associées à un algébroïde de Lie ;
- on discute dans un deuxième temps (chapitre 3) la généralisation des précédents résultats au cadre holomorphe ;
- on applique enfin (chapitre 4) ces résultats au problème de la quantification des r-matrices dynamiques.

Le chapitre 1 est logiquement consacré à la construction des éléments principaux qui permettent de formuler les théorèmes de formalité du chapitre suivant. Il contient essentiellement des rappels de constructions déjà connues, ainsi que quelques nouveaux objets (comme les cochaînes et les chaînes de Hochschild associées à un algébroïde de Lie, respectivement introduites dans [**5**] et [**7**]).

Le chapitre 2 est consacré à la démonstration du(des) théorème(s) de formalité pour les cochaînes et les chaînes de Hochschild associées à un algébroïde de Lie. Ce chapitre est une compilation des articles [**5**] et [**7**]. On utilise les techniques introduites par B. Fedosov [**26**] dans le cas symplectique, et généralisées par A. Cattaneo, G. Felder et L. Tomassini [**9**], puis par V. Dolgushev [**16, 17**]. La possibilité d'utiliser ces techniques repose essentiellement sur l'existence d'une connexion (au sens des algébroïdes) sans torsion. On s'attarde également sur les applications de ces résultats pour la quantifications par déformation.

Dans le chapitre 3 on s'attache à démontrer une version des résultats précédents dans le cadre holomorphe. Les principales difficultés proviennent du fait qu'il n'existe pas nécessairement de connexion holomorphe (ce problème est équivalent à celui, évoqué plus haut, du trop petit nombre de sections globales). Dans cette situation, on verra que des applications a la quantification par déformation sont toujours possibles, mais dans un sens plus faible que précédemment.

On démontre dans le chapitre 4 un théorème d'existence et de classification pour la quantification des r-matrices dynamiques, sous une hypothèse de réductivité. Nous utilisons pour cela le résultat principal du chapitre 2, qui nous permet de construire un quasi-isomorphisme L_∞ entre deux algèbres de Lie différentielles graduées appropriées.

Un annexe est consacré à quelques rappels utiles concernant les algèbere de Lie-Rinehart et les bialgébroïdes d'une part, et les *algèbres de Lie à homotopie près* (ou *algèbres* L_∞) d'autre part.

Les définitions et résultats relatifs aux cochaines dans les chapitres 1 et 2 sont tirés de l'article [**5**] écrit seul et paru dans *Communications in Mathematical Physics.*

Dans ces mêmes chapitres, les définitions et résultats relatifs aux chaines sont tirées de l'article [**7**] écrit en collaboration avec Vasiliy Dolgushev et Gilles Halbout et soumis aux *Annales de l'Institut Fourier.*

La sections 1, 2 et 3 du chapitre 3 sont également tirées de l'article [**7**] en collaboration avec Vasiliy Dolgushev et Gilles Halbout. La quatrième et dernière section de ce chapitre, à la quantification des variétés de Poisson holomorphes a fait l'objet d'un travail personnel.

Enfin, le chapitre 4 est tiré de l'article [**6**] écrit seul et à paraitre dans *Advances in Mathematics.*

La suite du présent texte est rédigée en anglais.

Notations

We summarize here the conventions we adopt in the text.

- We assume Einstein's convention for the summation over repeated indices. For example,
$$\alpha^l_{ijk} e^i e^k f_l$$
means
$$\sum_{i,k,l} \alpha^l_{ijk} e^i e^k f_l$$

- We sometimes omit the symbol $\wedge$ referring to a local basis of exterior forms.

- For any graded vector space V, $V[k]$ ($k \in \mathbb{Z}$) denote the graded vector space defined by
$$V[k]^n = V^{k+n}$$

- The arrow $\succ\!\!\rightarrow$ denotes a L_∞-morphism of L_∞-algebras, the arrow $\succ\!\!\succ\!\!\rightarrow$ denotes a L_∞-morphism of L_∞-modules, and the notation
$$\begin{array}{c} \mathcal{L} \\ \downarrow_{mod} \\ \mathcal{M} \end{array}$$
means that $\mathcal{M}$ is a L_∞-module over the L_∞-algebra $\mathcal{L}$.

- Throughout the dissertation (except Chapter 3) M denotes a smooth real manifold and $\mathcal{O}_M$ denotes the structure sheaf of M.

- The abbreviation "DG" stands for "differential graded". In particular, the abbreviation "DGLA" stands for "differential graded Lie algebra" and the abbreviation "DGAA" stands for "differential graded associative algebra".

- The words "deformation" and "quantization" are considered as synonym.

- We denote by the same symbol a vector bundle and its sheaf of sections.

CHAPITRE 1

Algebraic structures associated to Lie algebroids

RÉSUMÉ. Ce premier chapitre est consacré au rappel des concepts et notions usuels relatifs aux algébroïdes de Lie, ainsi qu'à la construction des structures algébriques associées à un algébroïde de Lie donné. On y définit également le complexe des cochaînes (respectivement, des chaînes) de Hochschild correspondant et on calcule sa cohomologie (respectivement, son homologie).

Lie algebroids and Lie groupoids provide a natural framework for developing analysis on differentiable foliations ([**10, 51, 68**]), orbifolds and singular quotients. This motivates our interest to the natural analogues of Hochschild and cyclic (co)homological complexes in the setting of Lie algebroids.

An appropriate analogue of the Hochschild cochain (respectively, chain) complex associated with a Lie algebroid E is the complex of E-polydifferential operators (respectively, Hochschild E-chains) (see definitions 1.11 and 1.18 in what follows). It turns out that the complex of E-polydifferential operators is naturally a DGLA and the complex of E-chains is naturally a DG module over this DGLA.

In this chapter we recall from [**5, 7**] and [**8, 44**] some basic facts about Lie algebroids, associated sheaves and define algebraic structures on these sheaves. It is organized as follows.
Section 1 is devoted to the definition of the analogues of usual sheaves from differential geometry in the Lie algebroid setting: polyvector fields, differential forms and differential operators. In Section 2 we recall some basic facts about Lie algebroid connections [**30**]. Algebraic structures on Hochschild E-(co)chains are described in Section 3, and the corresponding (co)homology is computed in Section 4.

1.1. Lie algebroids and associated sheaves

Let us recall the following

DEFINITION 1.1. *A* Lie algebroid *over a smooth manifold M is a smooth vector bundle E of finite rank whose sheaf of sections is a sheaf of Lie algebras equipped with a $\mathcal{O}_M$-linear morphism of sheaves of Lie algebras*

$$\rho : E \to TM.$$

The $\mathcal{O}_M$-module structure and the Lie algebra structure on the sheaf E are compatible in the following sense: for any open subset $U \subset M$, any function $f \in \mathcal{O}_M(U)$ and any sections $u, v \in \Gamma(U, E)$

$$[u,fv] = f[u,v] + \rho(u)(f)v. \tag{1.1}$$

The map ρ is called the anchor.

In other words, a Lie algebroid is a vector bundle E over a smooth manifold M whose sheaf of sections is a sheaf of Lie-Rinehart algebras over $\mathcal{O}_M$.

EXAMPLES 1.2. (i) The tangent bundle TM on M is the simplest example of a Lie algebroid. The bracket is the usual Lie bracket of vector fields and the anchor is the identity map $\mathrm{id} : TM \to TM$.

(ii) Suppose the anchor ρ is injective. It is equivalent to $E \simeq \rho(E) \subset TM$ being a foliation, and the bracket on E is completey determined by that on TM.

(iii) A Lie algebra is a Lie algebroid over a one-point manifold $M = \{\mathrm{pt}\}$.

1.1.1. The sheaf of E-polyvector fields.

DEFINITION 1.3. *The bundle* ${}^E T^*_{poly}$ *of* E-polyvector fields *is the exterior algebra of the bundle* E *with the shifted grading*

$$ {}^E T_{poly} = \bigoplus_{k\geq -1} {}^E T^k_{poly}\,, \qquad {}^E T^*_{poly} = \begin{cases} \wedge^{*+1} E\,, & * \geq 0\,, \\ \mathcal{O}_M\,, & * = -1. \end{cases} \tag{1.2} $$

It turns out that the Lie bracket $[,]$ on $\Gamma(M, {}^E T^0_{poly}) = \Gamma(M,E)$ can be naturally extended to a Lie bracket on the whole vector space $\Gamma(M, {}^E T^*_{poly})$ of E-polyvectors. Indeed, first, we define a Lie bracket $[,]$ on $\Gamma(M, {}^E T^{-1}_{poly} \oplus {}^E T^0_{poly})$ as follows

$$ \begin{array}{ll} [f,g] = 0, & \forall\ f,g \in \Gamma(M, {}^E T^{-1}_{poly})\,, \\ [u,f] = \rho(u)f, & \forall\ u \in \Gamma(M, {}^E T^0_{poly})\,,\ f \in \Gamma(M, {}^E T^{-1}_{poly})\,, \\ [u,v] = [u,v], & \forall\ u,v \in \Gamma(M, {}^E T^0_{poly}). \end{array} \tag{1.3} $$

Next, we extend $[,]$ to $\Gamma(M, {}^E T^*_{poly})$ by requiring the graded Leibniz rule with respect to the $\wedge$-product

$$ [u, v\wedge w] = [u,v]\wedge w + (-1)^{k(l+1)} v \wedge [u,w]\,, \tag{1.4} $$

$$ \forall\ u \in \Gamma(M, {}^E T^k_{poly}),\ v \in \Gamma(M, {}^E T^l_{poly}),\ w \in \Gamma(M, {}^E T^*_{poly}). $$

In the simplest example $E = TM$ the Lie bracket $[,]$ coincides with the well known Schouten-Nijenhuis bracket of ordinary polyvector fields.

1.1.2. The sheaf of E-differential forms. The exterior algebra $\wedge^* E^\vee$ of the dual bundle $E^\vee$ to E is a natural candidate for the bundle ${}^E\Omega^*_M$ of *E-differential forms* or just *E-forms* for short. The bundle ${}^E\Omega^*_M$ of E-forms is endowed with the following *E-de Rham differential*

$$ \begin{aligned} {}^E d\omega(\sigma_0,\dots,\sigma_k) \quad = \quad & \sum_i (-1)^i \rho(\sigma_i)\omega(\sigma_0,\dots,\hat{\sigma}_i,\dots,\sigma_k) \\ & + \sum_{i<j} (-1)^{i+j} \omega([\sigma_i,\sigma_j],\sigma_0,\dots,\hat{\sigma}_i,\dots,\hat{\sigma}_j,\dots,\sigma_k)\,, \\ & \sigma_i \in \Gamma(M,E). \end{aligned} \tag{1.5} $$

Let $(e_1,\dots,e_r)$ and $(\xi^1,\dots,\xi^r)$ be dual local basis of E and $E^\vee$ respectively. Then the local expression for ${}^E d$ is

$$ {}^E d = \xi^i \rho(e_i) - \frac{1}{2}\xi^i\xi^j c^k_{ij}(x)\frac{\vec{\partial}}{\partial \xi^k} $$

where $[e_i,e_j] = c^k_{ij}(x)e_k$, and the arrow over ∂ denotes the left derivative with respect to the anti-commuting variable ξ^i.

Another operation defined on E-forms is a contraction with E-polyvector fields. For an E-polyvector field $u \in \Gamma(M, {}^E T^k_{poly})$ we denote by ι_u the contraction with u. Using this contraction, the E-de Rham differential (1.5), and the Cartan-Weil formula

$$ {}^E L_u = {}^E d \circ \iota_u + (-1)^k \iota_u \circ {}^E d \tag{1.6} $$

we define the *E-Lie derivative* of E-forms by the E-polyvector field $u \in \Gamma(M, {}^E T^k_{poly})$.

For our purposes it is more convenient to use the reversed grading in the bundle of E-forms. Thus we denote by

$$ {}^E A_* = {}^E \Omega^{-*}_M\,, \qquad {}^E A_0 = \mathcal{O}_M \tag{1.7} $$

the corresponding bundle with reversed grading and observe that ${}^EA_*$ is equipped with a structure of a graded module over the sheaf of graded Lie algebras ${}^ET^*_{poly}$ via the E-Lie derivative (1.6). Namely,

LEMMA 1.4. *For any E-polyvector fields $u \in \Gamma(M,{}^ET^k_{poly})$ and $v \in \Gamma(M,{}^ET^l_{poly})$ one has*

$$ {}^EL_u \circ {}^EL_v - (-1)^{kl}\,{}^EL_v \circ {}^EL_u = {}^EL_{[u,v]}. \tag{1.8}$$

PROOF. First, it is immediate from the definition (1.6) that for any $u \in \Gamma(M,{}^ET^k_{poly})$

$$ {}^Ed \circ {}^EL_u = (-1)^k\,{}^EL_u \circ {}^Ed. \tag{1.9}$$

Second, we claim that for any $u \in \Gamma(M,{}^ET^k_{poly})$ and $v \in \Gamma(M,{}^ET^l_{poly})$ we have

$$ {}^EL_u \circ \iota_v - (-1)^{k(l+1)}\iota_v \circ {}^EL_u = (-1)^k\iota_{[u,v]}. \tag{1.10}$$

Using (1.9) and (1.10) it is not hard to show that for any $u \in \Gamma(M,{}^ET^k_{poly})$ and $v \in \Gamma(M,{}^ET^l_{poly})$

$${}^EL_u({}^Ed\iota_v + (-1)^l\iota_v{}^Ed) - (-1)^{kl}({}^Ed\iota_v + (-1)^l\iota_v{}^Ed){}^EL_u = ({}^Ed\iota_{[u,v]} + (-1)^{k+l}\iota_{[u,v]}{}^Ed).$$

Thus it suffices to prove that equation (1.10) holds.

The proof of (1.10) goes as follows. First, direct computations show that (1.10) holds for any sections u and v of the subsheaf ${}^ET^{-1}_{poly} \oplus {}^ET^0_{poly}$. Second, using the Leibniz rule (1.4) we prove the desired identity by induction on the degrees of E-polyvector fields u and v. In doing this, we need another simple identity

$$ {}^EL_{u_1\wedge u_2} = {}^EL_{u_1}\iota_{u_2} - (-1)^{k_1}\iota_{u_1}{}^EL_{u_2}\,, \qquad \forall\ u_i \in \Gamma(M,{}^ET^{k_i}_{poly})\,,$$

which follows easily from the fact that for any $u \in \Gamma(M,{}^ET^k_{poly})$ and $v \in \Gamma(M,{}^ET^l_{poly})$

$$ \iota_{u\wedge v} = \iota_u \circ \iota_v. $$

□

REMARK 1.5. One can also naturally consider E-tensors, that are just sections of the bundle $(\otimes^*E) \otimes (\otimes^*E^\vee)$.

1.1.3. The sheaf of E-differential operators. One can also define the $\mathcal{O}_M$-module $\mathcal{U}E$ of *E-differential operators* to be the sheaf of algebras locally generated by functions and E-vector fields. More precisely, $\mathcal{U}E$ is the sheaf associated with the following presheaf

$$ U \longmapsto \frac{T\Big(\mathcal{O}_M(U) \oplus \Gamma(U,E)\Big)}{\left\{ \begin{array}{c} f\otimes g - fg,\ f\otimes u - fu, \\ u\otimes f - f\otimes u - \rho(u)f, \\ u\otimes v - v\otimes u - [u,v], \end{array} \right\}} \tag{1.11}$$

$$ f,g \in \mathcal{O}_M(U), \qquad u,v \in \Gamma(U,E).$$

As a sheaf of $\mathcal{O}_M$-modules, $\mathcal{U}E$ is endowed with an increasing filtration

$$ \mathcal{O}_M = \mathcal{U}E^0 \subset \mathcal{U}E^1 \subset \mathcal{U}E^2 \subset \cdots \subset \mathcal{U}E, \tag{1.12}$$

which is defined by assigning the degree 1 to the E-vector fields.

In other words $\mathcal{U}E$ is the universal enveloping algebroid of the sheaf of Lie-Rinehart algebras E. Thus, besides the fact that $\mathcal{U}E$ is a sheaf of algebras, $\mathcal{U}E$ is also equipped with a coassociative $\mathcal{O}_M$-linear coproduct $\Delta : \mathcal{U}E \to \mathcal{U}E \otimes_{\mathcal{O}_M} \mathcal{U}E$ which is defined as follows

$$ \Delta(1) = 1\otimes 1\,,$$

$$ \Delta(u) = u\otimes 1 + 1\otimes u, \qquad \Delta(PQ) = \Delta(P)\Delta(Q)\,, \tag{1.13}$$

$$ \forall\ u \in \Gamma(M,E),\ P,Q \in \Gamma(M,\mathcal{U}E).$$

Notice that, in the simplest example $E = TM$, $\mathcal{U}E$ is the sheaf of usual differential operators on M.

1.2. Lie algebroids connections

By the word *connection* on a vector bundle $\mathcal{B}$ over M we always mean *E-connection*, that is a linear operator

$$\nabla : \Gamma(M,\mathcal{B}) \to {}^{E}\Omega^1(M,\mathcal{B}) \tag{1.14}$$

satisfying the following equation

$$\nabla(fu) = {}^{E}d(f)u + f\nabla(u) \tag{1.15}$$

for any $f \in \mathcal{O}_M(M)$ and $u \in \Gamma(M,\mathcal{B})$.

Locally, ∇ is completely determined by its *Christophel's symbols* Γ^k_{ij}. Namely, let $(e_1,\dots,e_r)$ and $(\xi^1,\dots,\xi^r)$ be dual local basis of E and $E^\vee$ respectively, and $(b_1,\dots,b_s)$ be a local base of $\mathcal{B}$, then

$$\nabla(b_j) = \xi^i \Gamma^k_{ij} b_k \tag{1.16}$$

For any $u \in \Gamma(M,E)$ we denote by ∇_u the associated map $\Gamma(M,\mathcal{B}) \to \Gamma(M,\mathcal{B})$.

REMARK 1.6. As with usual connections, one can extend this *covariant derivative* on E-tensor in a unique way such that ∇_u is a derivation with respect to the tensor product of E-tensors, commutes with the contraction of E-tensors, acts as $\rho(u)$ on functions, and is $\mathbb{R}$-linear.

DEFINITION 1.7. *The* curvature *R of a connection ∇ with value in $\mathcal{B}$ is the section R of the bundle $E^\vee \otimes E^\vee \otimes \mathcal{B}^\vee \otimes \mathcal{B}$ defined by*

$$R(u,v)w = \big(\nabla_u\nabla_v - \nabla_v\nabla_u - \nabla_{[u,v]}\big)w \tag{1.17}$$

for any $u,v \in \Gamma(M,E)$ and $w \in \Gamma(M,\mathcal{B})$.

Locally, the curvature is given by

$$R(e_i,e_j)b_k = (R_{ij})^l_k b_l$$

with

$$(R_{ij})^l_k = \Gamma^l_{im}\Gamma^m_{jk} - \Gamma^m_{ik}\Gamma^l_{jm} + \rho(e_i)\cdot\Gamma^l_{jk} - \rho(e_j)\cdot\Gamma^l_{ik} - c^m_{ij}\Gamma^l_{mk} \tag{1.18}$$

For a connection ∇ on E itself one has the following

DEFINITION 1.8. *The* torsion *T of ∇ is a E-tensor of type* (1,2) *defined by*

$$T(u,v) = \nabla_u v - \nabla_v u - [u,v] \tag{1.19}$$

for any $u,v \in \Gamma(M,E)$.

One can write the local coefficients of this tensor very easily:

$$T^k_{ij} = \Gamma^k_{ij} - \Gamma^k_{ji} - c^k_{ij} \tag{1.20}$$

PROPOSITION 1.9. *A torsion free connection on E exists.*

PROOF. Let $(U_\alpha)_\alpha$ be a cover of M by trivializing opens for E. On each α U_α one has a base $(e_i)_i$ of sections and then can define $\nabla^{(\alpha)}_{e_i} e_j = \frac{1}{2}[e_i,e_j]$. Let $(f_\alpha)_\alpha$ be partition of unity for $(U_\alpha)_\alpha$ and define $\nabla = f_\alpha \nabla^{(\alpha)}$. ∇ is a torsion free connection on E. □

PROPOSITION 1.10 (Bianchi's identities). *Let ∇ be connection on E. For any $u,v,w \in \Gamma(M,E)$ one has*

$$\nabla_u R(v,w) + R(T(u,v),w) + c.p.(u,v,w) = 0 \tag{1.21}$$

and

$$R(u,v)w - T(T(u,v),w) - \nabla_u T(v,w) + c.p.(u,v,w) = 0 \tag{1.22}$$

PROOF. See for example [**30**]. □

1.3. Algebraic structures on E-polydifferential operators and E-polyjets

We start with the definition of E-polydifferential operators, which will play the role of Hochschild E-cochains.

1.3.1. The DGLA of E-polydifferential operators.

DEFINITION 1.11. *The bundle* ${}^{E}D^{*}_{poly}$ *of* E-polydifferential operators *is the tensor algebra of the bundle* $\mathcal{U}E$ *with a shifted grading:*

$$ {}^{E}D_{poly} = \bigoplus_{k\geq -1} {}^{E}D^{k}_{poly}\,, \qquad {}^{E}D^{*}_{poly} = \begin{cases} \otimes^{*+1}_{\mathcal{O}_M}\mathcal{U}E\,, & *\geq 0\,, \\ \mathcal{O}_M\,, & *=-1. \end{cases} $$

It is easy to see that in the case $E = TM$ the sheaf ${}^{E}D^{*}_{poly}$ is the sheaf of polydifferential operators on M.

Using the coproduct (A.4) in $\mathcal{U}E$ we endow the graded sheaf ${}^{E}D^{*}_{poly}$ of E-polydifferential operators with a Lie bracket $[\,,]_G$. To introduce this bracket we first define the following bilinear product of degree 0

$$ \bullet \,:\, {}^{E}D_{poly}\otimes {}^{E}D_{poly} \to {}^{E}D_{poly}\,, $$

$$ \begin{aligned} P\bullet Q &= \sum_{i=0}^{|P|}(-1)^{i|Q|}\big(1^{\otimes i}\otimes\Delta^{(|Q|)}\otimes 1^{\otimes|P|-i}\big)(P)\cdot(1^{\otimes i}\otimes Q\otimes 1^{\otimes|P|-i})\,, \\ P\bullet f &= \sum_{i=0}^{|P|}(-1)^{i}(1^{\otimes i}\otimes\rho\otimes 1^{\otimes|P|-i})(P)(1^{\otimes i}\otimes f\otimes 1^{\otimes|P|-i})\,, \\ f\bullet g &= 0\,, \qquad f\bullet P = 0\,, \end{aligned} \tag{1.23} $$

for any $P,Q\in\Gamma(M,{}^{E}D^{\geq 0}_{poly})$ and $f,g\in\Gamma(M,{}^{E}D^{-1}_{poly})=\mathcal{O}_M$. Here $\Delta^{(n)}=(\Delta\otimes 1^{\otimes n-1})\circ\cdots\circ\Delta$, $\Delta^{(0)}$ is by convention the identity map, and ρ denotes the representation of $\mathcal{U}E$ on $\mathcal{O}_M$ induced via the anchor map[1]

Although the bilinear product is not associative, the graded commutator

$$ [P,Q]_G = P\bullet Q-(-1)^{|P||Q|}Q\bullet P\,, \qquad P,Q\in\Gamma(M,{}^{E}D^{*}_{poly}). \tag{1.24} $$

defines a graded Lie bracket between the E-polydifferential operators.

It is not hard to see that in the case $E=TM$ the above bracket reduces to the well known Gerstenhaber bracket [**31**] between polydifferential operators on M.

Notice that an element $1\otimes 1\in\Gamma(M,{}^{E}D^{1}_{poly})$ is distinguished by the following remarkable identity $[1\otimes 1,1\otimes 1]_G=0$. Using this observation we define the following differential

$$ \partial = [1\otimes 1,\,]_G \,:\, {}^{E}D^{*}_{poly}\to {}^{E}D^{*+1}_{poly} \tag{1.25} $$

on the sheaf of E-polydifferential operators.

We see from its definition that ∂ is compatible with the Lie bracket (1.24). Thus, $({}^{E}D^{*}_{poly},\partial,[\,,]_G)$ is a sheaf of differential graded Lie algebras (DGLA for short).

We would like to mention that the tensor product of sections (over $\mathcal{O}_M$) turns the sheaf ${}^{E}D_{poly}[-1]^{*}$ with the shifted grading into a sheaf of graded associative algebras. Moreover, it is not hard to see that the differential ∂ (1.25) is compatible with this product. Thus ${}^{E}D^{*}_{poly}$ can be also viewed as a sheaf of DG associative algebras (DGAA).

---

1. These four equations reduce to a single-one if we assume the convention $\Delta^{(-1)}=\rho$, that is done in the rest of the dissertation.

1.3.2. The DG-module of E-polyjets and the Grothendieck connection. Let us now define the vector bundle of *E-polyjets*.

DEFINITION 1.12. *The bundle ${}^E J^{poly}_*$ of* E-polyjets *is the following graded bundle placed in nonnegative degrees*

$$ {}^E J^{poly} = \bigoplus_{k\geq 0} {}^E J^{poly}_k \,, \qquad {}^E J^{poly}_* := Hom_{\mathcal{O}_M}(\mathcal{U}E^{\otimes^{*+1}_{\mathcal{O}_M}}, \mathcal{O}_M). $$

Since the sheaf ${}^E D^*_{poly}$ of E-polydifferential operators is an ind-finite dimensional graded vector bundle the sheaf ${}^E J^{poly}_*$ of E-polyjets is a profinite dimensional graded vector bundle. Furthermore, the sheaf ${}^E J^{poly}_*$ is endowed with a canonical flat connection ∇^G which is called the *Grothendieck connection* and defined by the formula

$$ \nabla^G_u(j)(P) := \rho(u)(j(P)) - j(u\bullet P)\,, \tag{1.26} $$

where $u \in \Gamma(M,E)$, $j \in \Gamma(M, {}^E J^{poly}_k)$, $P \in \Gamma(M, {}^E D^k_{poly})$, and the operation $\bullet$ is defined in (1.23).

For this connection we have the following standard

PROPOSITION 1.13. *Let χ be a map of sheaves*

$$ \chi : {}^E J^{poly}_k \to \begin{cases} {}^E J^{poly}_{k-1}\,, & \text{if} \quad k>0\,, \\ \mathcal{O}_M\,, & \text{if} \quad k=0 \end{cases} $$

defined by the formula

$$ \chi(a)(P) = a(1\otimes P)\,, \quad P\in\Gamma(M, {}^E D^{k-1}_{poly})\,, \quad a\in\Gamma(M, {}^E J^{poly}_k). \tag{1.27} $$

The restriction of the map χ to the ∇^G-flat E-polyjets gives the isomorphism of sheaves

$$ \chi : \ker\nabla^G \cap {}^E J^{poly}_k \xrightarrow{\sim} \begin{cases} {}^E J^{poly}_{k-1}\,, & \text{if} \quad k>0\,, \\ \mathcal{O}_M\,, & \text{if} \quad k=0. \end{cases} \tag{1.28} $$

PROOF. To see that the map (1.28) is surjective one has to notice that for any E-polyjet b of degree $k-1$ (respectively, a function $b\in\Gamma(M,\mathcal{O}_M)$) the equations

$$ a(1\otimes P) = b(P)\,, \quad P\in\Gamma(M, {}^E D^{k-1}_{poly}) $$

and

$$ a(u\cdot Q\otimes P) = \rho(u)a(Q\otimes P) - a(Q\otimes(\Delta^{(k-1)}(u)\cdot P))\,, \tag{1.29} $$
$$ Q\in\Gamma(M,\mathcal{U}E)\,, \qquad u\in\Gamma(M,E) $$

define a ∇^G-flat E-polyjet a of degree k (respectively, a ∇^G-flat E-jet a).

On the other hand, if a is a ∇^G-flat E-polyjet of degree k equation (1.29) is automatically satisfied. Thus a is uniquely determined by its image $\chi(a)$. □

Let t be the cyclic permutation acting on the sheaf ${}^E J^{poly}_*$ of E-polyjets

$$ t(a)(P_0\otimes\cdots\otimes P_l) := a(P_1\otimes\cdots\otimes P_l\otimes P_0)\,, \tag{1.30} $$
$$ a\in\Gamma(M, {}^E J^{poly}_l)\,, \qquad P_i\in\Gamma(M,\mathcal{U}E). $$

Using this operation and the bilinear product (1.23) we define the map

$$ {}^E S : {}^E D^k_{poly}\otimes {}^E J^{poly}_l \to {}^E J^{poly}_{l-k}\,, $$
$$ P\otimes a \mapsto {}^E S_P(a) \text{ such that} $$

$$ {}^E S_P(a)(Q) = a(Q\bullet P) + \sum_{j=1}^{k}(-1)^{lj} t^j(a)\big((\Delta^{(k)}\otimes 1^{\otimes(l-k)})(Q)\cdot(P\otimes 1^{\otimes(l-k)})\big)\,, \tag{1.31} $$

$$\text{for } P \in \Gamma(M, {}^E D^k_{poly}), \qquad a \in \Gamma(M, {}^E J^{poly}_l), \qquad Q \in \Gamma(M, {}^E D^{l-k}_{poly}).$$

Due to the following proposition the map (1.31) defines an action of the sheaf of graded Lie algebras ${}^E D^*_{poly}$ of E-polydifferential operators on the graded sheaf ${}^E J^{poly}_*$ of E-polyjets. Namely,

PROPOSITION 1.14. *For any pair P_1,$P_2 \in \Gamma(M, {}^E D^*_{poly})$ of E-polydifferential operators and any E-polyjet $a \in \Gamma(M, {}^E J^{poly}_*)$*

$$ {}^E S_{P_1} {}^E S_{P_2}(a) - (-1)^{|P_1||P_2|} {}^E S_{P_2} {}^E S_{P_1}(a) = {}^E S_{[P_1,P_2]_G}(a). \tag{1.32}$$

Moreover, the action (1.31) is compatible with the Grothendieck connection (1.26)

$$\nabla^G_u \left({}^E S_P(a) \right) = {}^E S_P(\nabla^G_u(a)), \quad u \in \Gamma(M, E), \quad P \in \Gamma(M, {}^E D^*_{poly}). \tag{1.33}$$

PROOF. It is not hard to show that for any $a \in \Gamma(M, {}^E J^{poly}_n)$

$$ {}^E S_{P_1} {}^E S_{P_2}(a) = {}^E S_{P_1 \bullet P_2}(a) + H(P_1,P_2)(a) + (-1)^{|P_1||P_2|} H(P_2,P_1)(a), \tag{1.34}$$

where[2]

$$H(P_1, P_2) : {}^E J^{poly}_* \to {}^E J^{poly}_{*-|P_1|-|P_2|}$$

is a graded $\mathcal{O}_M$-linear endomorphism of the sheaf ${}^E J^{poly}_*$ defined by the following formula

$$(H(P_1,P_2)(a))(Q) =$$

$$\sum_{i,j} (-1)^{i|P_1|+j|P_2|} a \Big[(1^{\otimes i} \otimes \Delta^{|P_1|} \otimes 1^{\otimes(j-i-|P_1|-1)} \otimes \Delta^{|P_2|} \otimes 1^{\otimes(n-j-|P_2|)}(Q)) \cdot$$

$$(1^{\otimes i} \otimes P_1 \otimes 1^{\otimes(j-i-|P_1|-1)} \otimes P_2 \otimes 1^{\otimes(n-j-|P_2|)}) \Big] +$$

$$\sum_{k,l} (-1)^{k|P_2|+l(n-|P_2|)} t^l(a) \Big[(\Delta^{|P_1|} \otimes 1^{k+l-|P_1|-1} \otimes \Delta^{|P_2|} \otimes 1^{\otimes n-k-l-|P_2|} (Q)) \cdot$$

$$P_1 \otimes 1^{\otimes(k+l-|P_1|-1)} \otimes P_2 \otimes 1^{\otimes(n-k-l-|P_2|)} \Big],$$

the sums run over all i, j, k, l satisfying the conditions

$$0 \le i \le j - |P_1| - 1, \qquad j \le n - |P_2|,$$

$$1 \le l \le |P_1|, \qquad |P_1| - l + 1 \le k \le n - |P_2| - l,$$

and

$$Q \in \Gamma(M, {}^E D^{n-|P_1|-|P_2|}_{poly}).$$

Equation (1.34) obviously implies identity (1.32).

Equation (1.33) follows immediately from the fact that the coproduct (A.4) is compatible with the multiplication of the E-differential operators and the fact that the Grothendieck connection (1.26) commutes with the cyclic permutation (1.30). □

As in the previsous subsection, we use the distinguished element $1 \otimes 1 \in \Gamma(M, {}^E D^1_{poly})$ to define a differential

$$\mathfrak{b} := {}^E S_{1 \otimes 1} : {}^E J^{poly}_* \to {}^E J^{poly}_{*-1} \tag{1.35}$$

on the sheaf of E-polyjets.

From the definition of the differential (1.35) and equation (1.32), we see that $\mathfrak{b}$ is compatible with the action (1.31) in the sense of the following equation

$$\mathfrak{b} \left({}^E S_P(a) \right) = {}^E S_{\partial P}(a) + (-1)^{|P|} {}^E S_P(\mathfrak{b}(a)).$$

$$\forall\, a \in \Gamma(M, {}^E J^{poly}_*), \quad P \in \Gamma(M, {}^E D^*_{poly}).$$

Thus, $({}^E J^{poly}_*, \mathfrak{b}, {}^E S)$ is a sheaf of differential graded modules (DG modules for short) over the sheaf of DGLA ${}^E D^*_{poly}$.

2. Formula (1.34) is essentially borrowed from paper **[32]** of E. Getzler.

1.3.3. Hochschild E-chains. The complex of sheaves $(\Gamma(M, {}^E J_*^{poly}), \mathfrak{b})$ is not a good candidate for the Hochschild chain complex in the Lie algebroid setting. Indeed, if our Lie algebroid E is TM then the complex $(\Gamma(M, {}^E J_*^{poly}), \mathfrak{b})$ boils down to the Hochschild chain complex of $C^\infty(M)$ without the zeroth term and the action (1.31) does not coincide with the standard action of Hochschild cochains on Hochschild chains (see eq. (3.4) in [**17**]). To cure these problems simultaneously we introduce a graded sheaf ${}^E C_*^{poly}$ of $\mathcal{O}_M$-modules placed in non-positive degrees

$$ {}^E C_*^{poly} = \begin{cases} \mathcal{O}_M\,, & *=0\,, \\ {}^E J_{-*-1}^{poly} & *\leq 0. \end{cases} \tag{1.36} $$

and the following $\mathbb{R}$-linear isomorphism of sheaves

$$ \varrho : {}^E C_*^{poly} \to \ker \nabla^G \cap {}^E J_{-*}^{poly} \tag{1.37} $$

obtained by inverting the map (1.28).

Due to propositions 1.14 the action (1.31) and the differential $\mathfrak{b}$ (1.40) commute with the Grothendieck connection ∇^G. Thus, the ∇^G-flat E-polyjets form a sheaf of DG submodule of $({}^E J_*^{poly}, \mathfrak{b}, {}^E S)$ over the sheaf of DGLA $({}^E D_{poly}^*, \partial, [\,,\,]_G)$. Combining this observation with Proposition 1.13 we conclude that the isomorphism (1.37) allows us to endow the sheaf (1.36) with a structure of a sheaf of DG modules over the sheaf of DGLA ${}^E D_{poly}^*$. Namely,

PROPOSITION 1.15. *The map*

$$ {}^E R_\bullet : {}^E D_{poly}^k \otimes {}^E C_l^{poly} \to {}^E C_{k+l}^{poly} \tag{1.38} $$

given by the formula

$$ {}^E R_P(a) = \chi^E S_P(\varrho(a)), \quad P \in \Gamma(M, {}^E D_{poly}^k), \quad a \in \Gamma(M, {}^E C_l^{poly}) \tag{1.39} $$

and the differential

$$ \mathfrak{b}(a) = \chi^E S_{1\otimes 1}(\varrho(a)) : \Gamma(M, {}^E C_*^{poly}) \to \Gamma(M, {}^E C_{*+1}^{poly}) \tag{1.40} $$

turn ${}^E C_*^{poly}$ (1.36) into a sheaf of DG modules over the sheaf of DGLA ${}^E D_{poly}^*$. □

REMARK 1.16. Since the map ϱ is NOT $\mathcal{O}_M$-linear the DG module structure (1.39), (1.40) on ${}^E C_*^{poly}$ is only $\mathbb{R}$-linear unlike the DG module structure (1.31) (1.40) on the sheaf ${}^E J_*^{poly}$.

REMARK 1.17. It is not hard to see that in the case $E = TM$ the global sections of the sheaf ${}^E C_*^{poly}$ give the jet version [**66**] of the homological Hochschild complex of the algebra $\mathcal{O}_M$ of functions on M.

The second remark motivates the following definition:

DEFINITION 1.18. *We refer to the sheaf ${}^E C_*^{poly}$ of DG modules over the sheaf of DGLA ${}^E D_{poly}^*$ of E-polydifferential operators as* the sheaf of the Hochschild E-chains *or just* E-chains *for short.*

1.4. Computation of Hochschild (co)homology

The (co)homology of the complexes ${}^E D_{poly}^*$ and ${}^E C_*^{poly}$ are described by Hochschild-Kostant-Rosenberg type theorems. The original version of this theorem [**36**] says that the module of Hochschild homology of a smooth affine algebra is isomorphic to the module of exterior forms of the corresponding affine variety. In [**10**] A. Connes proved an analogous statement for the algebra of smooth functions on any compact real manifold, and in [**65**], N. Teleman was able to get rid of the assumption of compactness. The similar question about Hochschild cohomology turns out to be tractable if we replace the Hochschild cochains by polydifferential operators. We believe that the cohomology of this complex of polydifferential

operators was originally computed by J. Vey [**67**]. All these computations correspond to the case when $E = TM$. In our general case we have the following proposition:

PROPOSITION 1.19 ([**5, 7**]). *The natural maps*

$$\begin{aligned} \mathcal{V} : ({}^E T^*_{poly}, 0) &\longrightarrow ({}^E D^*_{poly}, \partial) \\ v_0 \wedge \cdots \wedge v_k &\longmapsto \frac{1}{(k+1)!} \sum_{\sigma \in S_{k+1}} \epsilon(\sigma) v_{\sigma_0} \otimes \cdots \otimes v_{\sigma_k} \end{aligned} \tag{1.41}$$

and

$$\begin{aligned} \mathfrak{C} : ({}^E C^{poly}_*, \mathfrak{b}) &\longrightarrow ({}^E A_*, 0) \\ a &\longmapsto (v \mapsto \varrho(a)(\mathcal{V}(v))) \end{aligned} \tag{1.42}$$

are quasi-isomorphisms of (sheaves of) complexes.

PROOF. We only need to prove that $\mathcal{V}$ is a quasi-isomorphism. The fact that $\mathfrak{C}$ is also a quasi-isomorphism follows immediatly.

First, one can immediately check that the image of $\mathcal{V}$ is annihilated by ∂, i.e. that it is a morphism of complexes.

Now remark that the complex ${}^E D^*_{poly}$ is filtered by the total degree of polydifferential operators. ${}^E T^*_{poly}$ carries also a natural filtration (which is in fact a gradation), namely by degree of polyvector fields. Then $\mathcal{V}$ is compatible with filtrations. Thus we have to prove that $Gr(\mathcal{V}) : Gr({}^E T^*_{poly}) \to Gr({}^E D^*_{poly})$ is a quasi-isomorphism of complexes. In $Gr({}^E D^*_{poly})$ all components are sections of some vector bundle on M and ∂ is $\mathcal{O}_M$-linear (the same is obviously true for ${}^E T^*_{poly}$), therefore we have to show that $Gr(\mathcal{V})$ is a quasi-isomorphism fiberwise.

Fix $x \in M$ and consider the vector space $V = E_x$. One has

$$Gr({}^E D^*_{poly})_x = \bigoplus_{n \geq 0} S(V)^{\otimes n}$$

but it is better to indentify $S(V)$ with the cofree cocommutative coalgebra with counit $C(V) \oplus (\mathbb{R}1)^*$. As above the differential can be expressed in terms of the cocommutative coproduct Δ; namely

$$(-1)^{n-1}\partial = 1^* \otimes \mathrm{id}^{\otimes n} - \sum_{i=1}^{n-1} (-1)^i \mathrm{id} \otimes \cdots \otimes \Delta_i \otimes \cdots \otimes \mathrm{id} + (-1)^{n-1} \mathrm{id}^{\otimes n} \otimes 1^*$$

Now let us recall a standard result in homological algebra:

LEMMA 1.20. *Let $S(V)$ be the cofree cocommutative coalebra with counit cogenerated by a vector space V. Then the natural homomorphism of complexes $(\wedge^* V, 0) \to (\otimes^* S(V), \partial)$ is a quasi-isomorphism.* □

Apply this lemma in the case when $V = E_x$ and remark that $Gr({}^E T^*_{poly})_x = ({}^E T^*_{poly})_x = \wedge^* V$. Consequently $\mathcal{V}$ is a quasi-isomorphism of complexes. □

REMARK 1.21. Since we work in the C^∞ setting all sheaves considered here have partition of unity. Consequently, for any two sheaves of complexes $\mathcal{C}_1$ and $\mathcal{C}_2$, $\mathcal{C}_1$ and $\mathcal{C}_2$ are quasi-isomorphic as sheaves of complexes if and only if $\Gamma(M, \mathcal{C}_1)$ and $\Gamma(M, \mathcal{C}_2)$ are quasi-isomorphic as complexes. For this reason, we will sometimes make no difference between sheaves and there spaces of global sections.

This remark will be of a particular importance in Chapter 3, where we also deal with holomorphic sheaves (that do not admit partition of unity).

CHAPITRE 2

Formality theorems for Lie algebroids and applications

RÉSUMÉ. Dans ce chaptitre on démontre une version de la conjecture de formalité des cochaînes et des chaînes pour les algébroïdes de Lie, en utilisant le quasi-isomorphisme de Kontsevich pour les cochaînes de Hochschild de $\mathbb{R}[[y^1,\dots,y^d]]$, le quasi-isomorphisme de Shoikhet pour les chaînes de Hochschild de $\mathbb{R}[[y^1,\dots,y^d]]$, et des résolutions à la Fedosov des analogues naturels des complexes de (co)chaînes de Hochschild associés définis au chapitre précédent. On discute les applications de ce résultat à des problèmes de quantificaton par déformation. Précisons que les résultats de ce chapitre font l'objet de [**5**] en ce qui concerne la formalité pour les cochaînes et de [**7**] en ce qui concerne la formalité pour les chaînes.

Unfortunately, the maps (1.41) and (1.42) respect neither the Lie brackets nor the actions. This defect can be cured using the notion of Lie algebras and their modules *up to homotopy* (see [**35**] for a detailed discussion of the general theory and its applications, and annexe A.2 for a quick review of the notions and results we need). In this chapter we prove that ${}^{E}D^{*}_{poly}$ (respectively, ${}^{E}C^{poly}_{*}$) is quasi-isomorphic as a L_∞-algebra (respectively, as a L_∞-module) to its cohomology (respectively, its homology). For short, we say that ${}^{E}D^{*}_{poly}$ and ${}^{E}C^{poly}_{*}$ are formal.

The formality theorem for the differential graded Lie algebra (DGLA) of Hochschild cochains in the Lie algebroid setting (first proved in [**5**]) allows us to quantize an arbitrary Poisson Lie algebroid[1]. The formality of the DGLA module of Hochschild chains in the Lie algebroid setting (first proved in [**7**]) allows us to give a description of the quantum traces for Poisson Lie algebroids. Notice that the formality of the cyclic complex in the setting of Lie algebroids would imply the algebraic index theorem [**50**], [**62**] for the deformations associated with an arbitrary Poisson Lie algebroid.

This chapter is organized as follows. In Section 1 we state our formality theorem, giving an equivariant version of it, and recall Kontsevich's [**39**] and Shoikhet's [**58**] formality theorems for $\mathbb{R}^d_{formal}$. In sections 2 and 3 we use these theorems and 'Fedosov-like' [**28**] globalization technique [**9, 16, 17, 50**] to prove our main result. Section 4 is devoted to the application of the formality to deformation quantization theory: existence of a deformation quantization for any Poisson Lie algebroid, classification of these quantizations, description of quantum traces associated to such a quantization, Hochschild (co)homology of a deformation.

2.1. The formality of ${}^{E}D^{*}_{poly}$ and ${}^{E}C^{poly}_{*}$

2.1.1. Statement of the main result. The main result of this chapter is the following theorem:

1. According to the terminology of P. Xu [**71**] we have to call this object a triangular Lie bialgebroid. However, since we do not mention the bialgebroid structure, we refer to this object as a Poisson Lie algebroid.

THEOREM 2.1 ([**5, 7**]). *For any C^∞ Lie algebroid $(E,[\,,],\rho)$ there exists a commutative diagram of sheaves of DGLA and DGLA modules over M*

$$\begin{array}{ccc} {}^E T^*_{poly} & \overset{\Phi}{\succ\!\!\rightarrow} & {}^E D^*_{poly} \\ \downarrow mod & & \downarrow mod \\ {}^E A_* & \overset{\Psi}{\leftarrow\!\!\prec} & {}^E C^{poly}_* \end{array} \tag{2.1}$$

such that $\Phi^{[1]} = \mathcal{V}$ is Vey's quasi-isomorphism (1.41) and $\Psi^{[0]} = \mathfrak{C}$ is the quasi-isomorphism of Connes (1.42).

The proof of this theorem occupies the next two sections. It is based on the construction of a bigger commutative diagram of sheaves of DGLA and DGLA modules over M

$$\begin{array}{ccccccc} {}^E T^*_{poly} & \succ\!\!\rightarrow & \mathcal{L}_1 & \succ\!\!\rightarrow & \mathcal{L}_2 & \leftarrow\!\!\prec & {}^E D^*_{poly} \\ \downarrow mod & & \downarrow mod & & \downarrow mod & & \downarrow mod \\ {}^E A_* & \succ\!\!\succ\!\!\rightarrow & \mathcal{M}_1 & \leftarrow\!\!\prec\!\!\prec & \mathcal{M}_2 & \leftarrow\!\!\prec\!\!\prec & {}^E C^{poly}_* \end{array} \tag{2.2}$$

Moreover, it will appear clearly in the proof that the terms ($\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{M}_1$, $\mathcal{M}_2$) and the quasi-isomorphisms of the diagram (2.2) are functorial for isomorphisms of pairs (E,∇), where E is a C^∞ Lie algebroid and ∇ is a torsion free E-connection on E.

We would like to mention that this functoriality of the chain of quasi-isomorphisms (2.2) between the pair of sheaves of DGLA modules implies the following interesting result

THEOREM 2.2. *Let $(E,[\,,],\rho)$ be a C^∞ Lie algebroid equipped with a smooth action of a group G. If one can construct a G-invariant connection ∇ on E then there exists a chain of G-equivariant quasi-isomorphisms between the sheaves of DGLA modules $({}^E T^*_{poly}, {}^E A_*)$ and $({}^E D^*_{poly}, {}^E C^{poly}_*)$.* □

In particular,

COROLLARY 2.3. *If $(E,[\,,],\rho)$ is a C^∞ Lie algebroid equipped with a smooth action of a finite or compact group G then the DGLA modules $(\Gamma(M,{}^E T^*_{poly})^G,\ \Gamma(M,{}^E A_*)^G)$ and $(\Gamma(M,{}^E D^*_{poly})^G,\ \Gamma(M,{}^E C^{poly}_*)^G)$ are quasi-isomorphic.* □

EXAMPLES 2.4. (i) Consider the case of a Lie algebra $\mathfrak{g}$ (i.e., a Lie algebroid over a point) with the adjoint action of its Lie group G (which is a good action). Then the Lie algebroid connection given by half the Lie bracket on $\mathfrak{g}$ is a torsion free G-invariant connection and we obtain a G-equivariant L_∞-quasi-isomorphism of DGLA from $\wedge^*\mathfrak{g}$ to $\otimes^* U\mathfrak{g}$. In particular for any subgroup $H \subset G$ one obtains a quasi-isomorphism of DGLA from $(\wedge^*\mathfrak{g})^H$ to $(\otimes^* U\mathfrak{g})^H$.

(ii) If a group G acts smoothly on a manifold M, then it induces an action on the Lie algebroid $E = TX$. In this particular case, and only considering cochains, we recover theorem 5 in [**16**]; taking chains into account we obtain an equivariant version of the main theorem of [**17**].

(iii) Now if $E \to M$ is a Lie algebroid with injective anchor (i.e., E is the Lie algebroid of a foliation), then any smooth action of a group G on M that respects the foliation (i.e., that sends a leaf to a leaf) gives rise to an action on E. In this context we obtain a leafwise version of the previous example.

2.1.2. Formality theorems for the Hochschild complexes of $\mathbb{R}[[y^1,\ldots,y^d]]$. In order to prove Theorem 2.1 we construct the Fedosov resolutions of the sheaves of DGLA ${}^E T^*_{poly}$ and ${}^E D^*_{poly}$ and of the sheaves of DG modules ${}^E A_*$ and ${}^E C^{poly}_*$. These resolutions allow us to reduce the problem to the case of the Lie algebroid $T\mathbb{R}^d \to \mathbb{R}^d$. For the latter

case the desired result follows from the combination of Kontsevich's [**39**] and Shoikhet's [**58**] formality theorems.

First, we recall the required version of Kontsevich's formality theorem. Let $M = \mathbb{R}^d_{formal}$ be the formal completion of $\mathbb{R}^d$ at the origin. In other words we set $\mathcal{O}_M = \mathbb{R}[[y^1, \dots, y^d]]$ and $E = \mathrm{Der}(\mathcal{O}_M)$. Let us denote by $T^*_{poly}(\mathbb{R}^d_{formal})$ and $D^*_{poly}(\mathbb{R}^d_{formal})$ the DGLA of polyvector fields and polydifferential operators on $\mathbb{R}^d_{formal}$, respectively, then

THEOREM 2.5 (Kontsevich, [**39**]). *There exists a L_∞-quasi-isomorphism $\mathcal{K}$ from $T^*_{poly}(\mathbb{R}^d_{formal})$ to $D^*_{poly}(\mathbb{R}^d_{formal})$ such that*

(1) *The first structure map $\mathcal{K}^{[1]}$ is Vey's quasi-isomorphism (1.41) of complexes $\mathcal{V}$.*
(2) *$\mathcal{K}$ is $GL_d(\mathbb{R})$-equivariant.*
(3) *If $n > 1$ then for any vector fields $v_1, \dots, v_n \in T^0_{poly}(\mathbb{R}^d_{formal})$*

$$\mathcal{K}^{[n]}(v_1, \dots, v_n) = 0$$

(4) *If $n > 1$ then for any vector field $v \in T^0_{poly}(\mathbb{R}^d_{formal})$ linear in the coordinates y^i and any polyvector fields $\chi_2, \dots, \chi_n \in T^*_{poly}(\mathbb{R}^d_{formal})$*

$$\mathcal{K}^{[n]}(v, \chi_2, \dots, \chi_n) = 0.$$

We denote by

$$A^*(\mathbb{R}^d_{formal}) = \mathbb{R}[[y^1, \dots, y^d]] \otimes \bigwedge(\mathbb{R}^d)$$

the complex of exterior forms on $\mathbb{R}^d_{formal}$ with the vanishing differential and by

$$J^{poly}_*(\mathbb{R}^d_{formal}) = \mathbb{R}[[y^1, \dots, y^d]]^{\hat{\otimes}\,(*+1)}$$

the complex of Hochschild chains of $\mathbb{R}[[y^1, \dots, y^d]]$, where the notation $\hat{\otimes}$ stands for the tensor product completed in the adic topology on $\mathbb{R}[[y^1, \dots, y^d]]$.

Using the Lie derivative (1.6) of exterior forms by a polyvector field, we can regard $A^*(\mathbb{R}^d_{formal})$ as a graded module over the graded Lie algebra $T^*_{poly}(\mathbb{R}^d_{formal})$. Furthermore, the action of Hochschild cochains on Hochschild chains (see formula (3.4) in [**17**]) allows us to regard $J^{poly}_*(\mathbb{R}^d_{formal})$ as a DG modules over the DGLA $D^*_{poly}(\mathbb{R}^d_{formal})$. Finally, using Kontsevich's L_∞-quasi-isomorphism $\mathcal{K}$ we get a L_∞-module structure on $J^{poly}_*(\mathbb{R}^d_{formal})$ over $T^*_{poly}(\mathbb{R}^d_{formal})$. For this L_∞-module, we have the following theorem:

THEOREM 2.6 (Shoikhet, [**58**]). *There exists a quasi-isomorphism $\mathcal{S}$ of L_∞-modules over $T^*_{poly}(\mathbb{R}^d_{formal})$ from $J^{poly}_*(\mathbb{R}^d_{formal})$ to $A^*(\mathbb{R}^d_{formal})$ such that*

(1) *The 0-th structure map $\mathcal{S}^{[0]}$ is the quasi-isomorphism of Connes (1.42).*
(2) *The structure maps of $\mathcal{S}$ are $GL_d(\mathbb{R})$-equivariant.*
(3) *If $n > 1$ then for any vector field $v \in T^0_{poly}(\mathbb{R}^d_{formal})$ linear in the coordinates, any polyvector fields $\chi_2, \dots, \chi_n \in T^*_{poly}(\mathbb{R}^d_{formal})$ and any chain $j \in J^{poly}_*(\mathbb{R}^d_{formal})$*

$$\mathcal{S}^{[n]}(v, \chi_2, \dots, \chi_n; j) = 0$$

REMARK 2.7. The third assertion of the above theorem is proved in [**17**, Theorem 3].

REMARK 2.8. Hopefully, one can prove the assertions of Theorem 2.6 along the lines of Tamarkin and Tsygan [**61, 62, 63**].

2.2. The Fedosov resolutions

Let, as above, $E \to M$ be a C^∞ Lie algebroid with bracket $[,]$ and anchor ρ.

2.2.1. Weyl-like bundles. Following [**26, 16, 17, 5, 7**] we introduce the formally completed symmetric algebra $\hat{S}(E^\vee)$ of the dual bundle $E^\vee$ and bundles $\mathcal{T}, \mathcal{D}, \mathcal{A}, \mathcal{J}$ naturally associated to $\hat{S}(E^\vee)$.

- $\hat{S}(E^\vee)$ is the formally completed symmetric algebra of the bundle $E^\vee$. Local sections are given by formal power series
$$\sum_{l=0}^{\infty} s_{i_1\dots i_l}(x) y^{i_1}\cdots y^{i_l}$$
where y^i are coordinates on the fibers of E and $s_{i_1\dots i_l}$ are components of a symmetric covariant E-tensor.
- $\mathcal{T}^* := \hat{S}(E^\vee)\otimes \wedge^{*+1} E$ is the graded bundle of formal fiberwise polyvector fields on E. Local homogeneous sections of degree k are of the form
$$\sum_{l=0}^{\infty} v_{i_1\dots i_l}^{j_0\dots j_k}(x) y^{i_1}\cdots y^{i_l}\frac{\partial}{\partial y^{j_0}}\wedge\cdots\wedge\frac{\partial}{\partial y^{j_k}}, \tag{2.3}$$
where $v_{i_1\dots i_l}^{j_0\dots j_k}$ are components of an E-tensor with symmetric covariant part (indices $i_1,\dots,i_l$) and antisymmetric contravariant part (indices $j_0,\dots,j_k$).
- $\mathcal{D}^* := \hat{S}(E^\vee)\otimes T^{*+1}(SE)$ is the graded bundle of formal fiberwise polydifferential operators on E with the shifted grading. A local homogeneous section of degree k looks as follow
$$\sum_{l=0}^{\infty} P_{i_1\dots i_l}^{\alpha_0,\dots,\alpha_k}(x) y^{i_1}\cdots y^{i_l}\frac{\partial^{|\alpha_0|}}{\partial y^{\alpha_0}}\otimes\cdots\otimes\frac{\partial^{|\alpha_k|}}{\partial y^{\alpha_k}}, \tag{2.4}$$
where α_s are multi-indices, $P_{i_1\dots i_l}^{\alpha_0\cdots\alpha_k}$ are components of an E-tensor with the obvious symmetry of the corresponding indices, and
$$\frac{\partial^{|\alpha_s|}}{\partial y^{\alpha_s}} = \frac{\partial}{\partial y^{j_1}}\cdots\frac{\partial}{\partial y^{j_{|\alpha_s|}}}$$
for $\alpha_s = (j_1\dots j_{|\alpha_s|})$.
- $\mathcal{A}_* := \hat{S}(E^\vee)\otimes \wedge^{-*}(E^\vee)$ is the graded bundle of formal fiberwise differential forms on E with the reversed grading. Any local homogeneous section of degree $-k$ can be written as
$$\sum_{l=0}^{\infty}\omega_{i_1\dots i_l, j_1\dots j_k}(x) y^{i_1}\cdots y^{i_l} dy^{j_1}\wedge\cdots\wedge dy^{j_k}, \tag{2.5}$$
where $\omega_{i_1\dots i_l, j_1\dots j_k}$ are components of a covariant E-tensor symmetric in indices $i_1,\dots,i_l$ and antisymmetric in indices $j_1,\dots,j_k$.
- $\mathcal{J}_*$ is the bundle of Hochschild chains of $\hat{S}(E^\vee)$ over $\mathcal{O}_M$.
$$\mathcal{J} = \bigoplus_{k\geq 0}\mathcal{J}_k, \qquad \mathcal{J}_k := (\hat{S}E^\vee)^{\hat{\otimes}_{\mathcal{O}_M}(k+1)}, \tag{2.6}$$
where $\hat{\otimes}$ stands for the tensor product completed in the adic topology. Local sections of homogeneous degree k are formal power series
$$a_{\alpha_0,\dots,\alpha_k}(x) y_0^{\alpha_0} y_1^{\alpha_1}\cdots y_k^{\alpha_k} \tag{2.7}$$
in $k+1$ copies $y_0,\dots,y_k$ of coordinates on the fibers of E. Here α_s are multi-indices, $a_{\alpha_0,\dots,\alpha_k}$ are components of a tensor with an obvious symmetry in the corresponding indices, and
$$y_m^{\alpha_m} = y_m^{j_1}\cdots y_m^{j_{|\alpha_m|}}$$
for $\alpha_m = (j_1\dots j_{|\alpha_m|})$.

For our purposes, we consider E-differential forms with values in the sheaves $\hat{S}(E^\vee)$, $\mathcal{T}$, $\mathcal{D}$, $\mathcal{A}$, $\mathcal{J}$. Below we list these sheaves of E-forms together with the algebraic structures they carry.

- ${}^E\Omega(\hat{S}(E^\vee))$ is a bundle of graded commutative algebras with grading given by the exterior degree of E-forms. ${}^E\Omega(\hat{S}(E^\vee))$ is also filtered by the degree of monomials in fiber coordinates y^i.
- ${}^E\Omega(\mathcal{T})$ is a sheaf of graded Lie algebras and ${}^E\Omega(\mathcal{A})$ is a sheaf of graded modules over ${}^E\Omega(\mathcal{T})$. These structures are induced by those of $T^*_{poly}(\mathbb{R}^d_{formal})$ and $A^*(\mathbb{R}^d_{formal})$, respectively and the grading is given by the sum of the exterior degree and the degree of an E-polyvector (respectively, an E-form). $[,]_{SN}$ will denote the Lie bracket between sections of the sheaf ${}^E\Omega(\mathcal{T})$ and L_u (the Lie derivative) will denote the action of a fiberwise polyvector $u \in {}^E\Omega(\mathcal{T})$ on the sections of ${}^E\Omega(\mathcal{A})$. ${}^E\Omega(\mathcal{T})$ is also a sheaf of graded commutative algebras. The multiplication of sections in ${}^E\Omega(\mathcal{T})$ is given by the exterior product in the space $T^*_{poly}(\mathbb{R}^d_{formal})$ of E-polyvector fields on $\mathbb{R}^d_{formal}$. The Lie bracket and the product in ${}^E\Omega(\mathcal{T})$ turn ${}^E\Omega(\mathcal{T})$ into a sheaf of Gerstenhaber algebras [2].
- ${}^E\Omega(\mathcal{D})$ is a sheaf of DGLA and ${}^E\Omega(\mathcal{J})$ is a sheaf of DG modules over ${}^E\Omega(\mathcal{D})$. These structures are induced by those of $D^*_{poly}(\mathbb{R}^d_{formal})$ and $J^*_{poly}(\mathbb{R}^d_{formal})$, respectively and the grading is given by the sum of the exterior degree and the degree of a (co)chain. We denote by ∂ and $[,]_G$ respectively the differential and the Lie bracket on ${}^E\Omega(\mathcal{D})$, $\mathfrak{b}$ will stand for the differential on ${}^E\Omega(\mathcal{J})$ and $\mathcal{R}_P$ will denote the action of $P \in {}^E\Omega(\mathcal{D})$ on the sections of ${}^E\Omega(\mathcal{J})$. ${}^E\Omega(\mathcal{D})$ is also a sheaf of DGAA. The multiplication of sections is induced by the cup product in the space $D^*_{poly}(\mathbb{R}^d_{formal})$ of E-polydifferential operators on $\mathbb{R}^d_{formal}$.

REMARK 2.9. Notice that $\mathcal{A}$ is a sheaf of exterior forms with values in $\hat{S}(E^\vee)$. However, we would like to distinguish $\mathcal{A}$ from ${}^E\Omega(\hat{S}(E^\vee))$. For this purpose we use two copies of a local basis of exterior forms. Those are $\{dy^i\}$ and $\{\xi^i\}$ for $\mathcal{A}$ and ${}^E\Omega(\hat{S}(E^\vee))$, respectively.

The following proposition shows that we have a distinguished sheaf of graded Lie algebras which acts on the sheaves ${}^E\Omega(\hat{S}(E^\vee))$, ${}^E\Omega(\mathcal{A})$, ${}^E\Omega(\mathcal{T})$, ${}^E\Omega(\mathcal{D})$, and ${}^E\Omega(\mathcal{J})$.

PROPOSITION 2.10. *The sheaf ${}^E\Omega(\mathcal{T}^0)$ of E-forms with values in fiberwise vector fields is a sheaf of graded Lie algebras. The sheaves ${}^E\Omega(\hat{S}(E^\vee))$, ${}^E\Omega(\mathcal{A})$, ${}^E\Omega(\mathcal{T})$, ${}^E\Omega(\mathcal{D})$, and ${}^E\Omega(\mathcal{J})$ are sheaves of modules over ${}^E\Omega(\mathcal{T}^0)$ and the action of sections in ${}^E\Omega(\mathcal{T}^0)$ is compatible with the DG algebraic structures on ${}^E\Omega(\hat{S}(E^\vee))$, ${}^E\Omega(\mathcal{A})$, ${}^E\Omega(\mathcal{T})$, ${}^E\Omega(\mathcal{D})$, and ${}^E\Omega(\mathcal{J})$.*

PROOF. Since the Schouten-Nijenhuis bracket (1.3), (1.4) has degree zero ${}^E\Omega(\mathcal{T}^0) \subset {}^E\Omega(\mathcal{T}) \subset {}^E\Omega(\mathcal{D})$ is a subsheaf of graded Lie algebras. While the action of ${}^E\Omega(\mathcal{T}^0)$ on the sections of ${}^E\Omega(\hat{S}(E^\vee))$ is obvious, the action on ${}^E\Omega(\mathcal{A})$ is given by the Lie derivative, the action on ${}^E\Omega(\mathcal{T})$ is the adjoint action corresponding to the Schouten-Nijenhuis bracket, the action on ${}^E\Omega(\mathcal{D})$ is given by the Gerstenhaber bracket and the action on ${}^E\Omega(\mathcal{J})$ is induced by the action of Hochschild cochains on Hochschild chains (see formula 3.4 in paper [**17**]). The compatibility of the action with the corresponding DGLA and DGLA-module structures follows from the construction. The compatibility of the action with the product in ${}^E\Omega(\mathcal{T})$ follows from the axioms of the Gerstenhaber algebra [**31**] and the compatibility with the product in ${}^E\Omega(\mathcal{D})$ can be verified by a straightforward computation. □

Due to the above proposition the following 2-nilpotent derivation

$$\delta := \xi^i \frac{\partial}{\partial y^i} : {}^E\Omega^*(\hat{S}(E^\vee)) \to {}^E\Omega^{*+1}(\hat{S}(E^\vee)) \tag{2.8}$$

2. The definition of the Gerstenhaber algebra can be found in section 4.1 of the second part of [**13**] or in the original paper [**31**].

of the sheaf of algebras ${}^E\Omega(\hat{S}(E^\vee))$ obviously extends to 2-nilpotent differentials on ${}^E\Omega(\mathcal{T})$, ${}^E\Omega(\mathcal{D})$, ${}^E\Omega(\mathcal{A})$ and ${}^E\Omega(\mathcal{J})$. Furthermore, it follows from Proposition 2.10 that δ is compatible with the DG algebraic structures on ${}^E\Omega(\mathcal{T})$, ${}^E\Omega(\mathcal{A})$, ${}^E\Omega(\mathcal{D})$, and ${}^E\Omega(\mathcal{J})$.

Note that

$$\ker\delta\cap\hat{S}(E^\vee)\cong\mathcal{O}_M\,,\qquad \ker\delta\cap\mathcal{A}^*\cong{}^EA_* \tag{2.9}$$

as sheaves of (graded) commutative algebras over $\mathcal{O}_M$. Similarly, $\ker\delta\cap\mathcal{T}$, (respectively, $\ker\delta\cap\mathcal{D}$) is a sheaf of fiberwise polyvector fields (2.3) (respectively, fiberwise polydifferential operators (2.4)) whose components do not depend on the fiber coordinates y^i. In other words,

$$\ker\delta\cap\mathcal{T}^*\cong\wedge^{*+1}(E) \tag{2.10}$$

as sheaves of graded commutative algebras and

$$\ker\delta\cap\mathcal{D}^*\cong\otimes^{*+1}(S(E))\,, \tag{2.11}$$

as sheaves of DGAA over $\mathcal{O}_M$.

In fact, one can prove a more stronger statement:

PROPOSITION 2.11. *For $\mathcal{B}$ being either of the sheaves $\hat{S}(E^\vee)$, $\mathcal{A}$, $\mathcal{T}$ or $\mathcal{D}$*

$$H^{\geq 1}({}^E\Omega(\mathcal{B}),\delta)=0.$$

Furthermore,

$$\begin{aligned} H^0({}^E\Omega(\hat{S}(E^\vee)),\delta)&\cong\mathcal{O}_M\,,\\ H^0({}^E\Omega(\mathcal{A}^*),\delta)&\cong{}^EA_*\,,\\ H^0({}^E\Omega(\mathcal{T}^*),\delta)&\cong\wedge^{*+1}(E) \end{aligned} \tag{2.12}$$

as sheaves of (graded) commutative algebras and

$$H^0({}^E\Omega(\mathcal{D}^*),\delta)\cong\otimes^{*+1}(S(E)) \tag{2.13}$$

as sheaves of DGAA over $\mathcal{O}_M$.

PROOF. Due to equations (2.9), (2.10), and (2.11) the proposition will follow immediately if we construct an operator

$$\kappa:{}^E\Omega^*(\mathcal{B})\to{}^E\Omega^{*-1}(\mathcal{B}) \tag{2.14}$$

such that for any section u of ${}^E\Omega(\mathcal{B})$

$$u=\delta\kappa(u)+\kappa\delta(u)+\mathcal{H}(u)\,, \tag{2.15}$$

where

$$\mathcal{H}(u)=u\Big|_{y^i=\xi^i=0}. \tag{2.16}$$

First, we define this operator on the sheaf ${}^E\Omega(\hat{S}(E^\vee))$

$$\kappa(a)=y^k\frac{\vec{\partial}}{\partial\xi^k}\int_0^1 a(x,ty,t\xi)\frac{dt}{t},\qquad a\in{}^E\Omega^{>0}(\hat{S}(E^\vee))\,,\qquad \kappa\Big|_{\hat{S}(E^\vee)}=0\,, \tag{2.17}$$

where the arrow over ∂ denotes the left derivative with respect to the anti-commuting variable ξ^i. Remark that one can also define κ on a homogeneous element $a\in{}^E\Omega^k(\hat{S}^l(E^\vee))$ by

$$\kappa(a)=\begin{cases}\frac{1}{k+l}\iota_{\theta_E}(a)\text{ if }k+l\neq 0\\ 0\text{ if }k+l=0\end{cases}$$

where $\theta_E=y^ie_i$ is the so called Euler vector field of E. Next, we extend κ to sections of the sheaves ${}^E\Omega(\mathcal{A})$, ${}^E\Omega(\mathcal{T})$, ${}^E\Omega(\mathcal{D})$ in the componentwise manner. A direct (and very standard) computation shows that equation (2.15) holds and the proposition follows. □

2.2.2. The Fedosov differential. Since we work in the C^∞ setting, we know from Proposition 1.9 that our Lie algebroid E admits a torsion free connection ∇. Using this connection we define the following derivation (which we denote be the same symbol) of the DG sheaves ${}^E\Omega(\hat{S}(E^\vee))$, ${}^E\Omega(\mathcal{A})$, ${}^E\Omega(\mathcal{T})$, ${}^E\Omega(\mathcal{D})$, and ${}^E\Omega(\mathcal{J})$:

$$\nabla = {}^E d + \Gamma\cdot : {}^E\Omega^*(\mathcal{B}) \to {}^E\Omega^{*+1}(\mathcal{B}), \qquad \Gamma = -\xi^i \Gamma^k_{ij} y^j \frac{\partial}{\partial y^k}, \tag{2.18}$$

where $\mathcal{B}$ is either of the sheaves $\hat{S}(E^\vee)$, $\mathcal{A}$, $\mathcal{T}$, $\mathcal{D}$, or $\mathcal{J}$, $\Gamma^k_{ij}(x)$ are Christoffel's symbols of the connection and $\Gamma\cdot$ denotes the action of Γ on the sections of the sheaves ${}^E\Omega(\mathcal{B})$ (see Proposition 2.10). It is not hard to see that ∇ (2.18) is compatible with the DG algebraic structures on ${}^E\Omega(\hat{S}(E^\vee))$, ${}^E\Omega(\mathcal{T})$, ${}^E\Omega(\mathcal{A})$, ${}^E\Omega(\mathcal{D})$, and ${}^E\Omega(\mathcal{J})$. Furthermore, the torsion freeness of the connection implies that

$$\nabla\delta + \delta\nabla = 0. \tag{2.19}$$

The standard curvature E-tensor $(R_{ij})^l_k(x)$ (2.20) of the connection provides us with the following fiberwise vector field:

$$R = -\frac{1}{2}\xi^i\xi^j (R_{ij})^l_k(x) y^k \frac{\partial}{\partial y^l} \in {}^E\Omega^2(\mathcal{T}^0). \tag{2.20}$$

A direct computation shows that for $\mathcal{B}$ being any of the sheaves ${}^E S$, $\mathcal{A}$, $\mathcal{T}$, $\mathcal{D}$, or $\mathcal{J}$, we have

$$\nabla^2 = R\cdot : {}^E\Omega^*(\mathcal{B}) \to {}^E\Omega^{*+2}(\mathcal{B}), \tag{2.21}$$

where $R\cdot$ denotes the action of the vector field R in the sense of Proposition 2.10.

Although ∇ is not flat the following theorem shows that the combination $\nabla - \delta$ can be extended to a flat connection on the sheaves ${}^E\Omega(\hat{S}(E^\vee))$, ${}^E\Omega(\mathcal{T})$, ${}^E\Omega(\mathcal{A})$, ${}^E\Omega(\mathcal{D})$, and ${}^E\Omega(\mathcal{J})$.

THEOREM 2.12. *Let $\mathcal{B}$ be either of the sheaves ${}^E S$, $\mathcal{A}$, $\mathcal{T}$, $\mathcal{D}$, or $\mathcal{J}$. There exists a section*

$$A = \sum_{s=2}^{\infty} \xi^k A^j_{k,i_1\dots i_s} y^{i_1}\cdots y^{i_s} \frac{\partial}{\partial y^j} \tag{2.22}$$

of the sheaf ${}^E\Omega^1(\mathcal{T}^0)$ such that the derivation

$$D := \nabla - \delta + A\cdot : {}^E\Omega^*(\mathcal{B}) \to {}^E\Omega^{*+1}(\mathcal{B}) \tag{2.23}$$

is 2-nilpotent

$$D^2 = 0,$$

and (2.23) is compatible with the DG algebraic structure on ${}^E\Omega(\mathcal{B})$.

PROOF. The proof goes essentially along the lines of [**16**, Theorem 2].

Thanks to equation (2.21) the condition $D^2 = 0$ is equivalent to the equation

$$R + \nabla A - \delta A + \frac{1}{2}[A,A]_{SN} = 0. \tag{2.24}$$

We claim that a solution of (2.24) can be obtained by iterations of the following equation

$$A = \kappa R + \kappa(\nabla A + \frac{1}{2}[A,A]_{SN}) \tag{2.25}$$

in degrees in the fiber coordinates y^i. Indeed, equation (2.15) implies that iterating (2.25) we get a solution of the equation

$$\kappa(R + \nabla A - \delta A + \frac{1}{2}[A,A]_{SN}) = 0.$$

We denote by C the left hand side of (2.24)

$$C = R + \nabla A - \delta A + \frac{1}{2}[A,A]_{SN},$$

and mention that due to Bianchi's identities $\nabla R = \delta R = 0$

$$\nabla C - \delta C + [A,C] = 0. \tag{2.26}$$

Applying κ (2.17) to (2.26) and using the homotopy property (2.15) we get

$$C = \kappa(\nabla C + [A,C]).$$

The latter equation has the unique vanishing solution since the operator κ (2.17) raises the degree in the fiber coordinates y^i.

Proposition 2.10 implies that the differential (2.23) is compatible with the DG algebraic structures on ${}^E\Omega(\mathcal{B})$. Thus, the theorem is proved. □

In what follows we refer to the differential D (2.23) as *the Fedosov differential*. Since D satisfies the condition (1.15) for connexions we also sometimes refer to it as *the Fedosov connexion* on $\mathcal{B}$.

The following theorem describes the cohomology of the Fedosov differential D for the sheaves ${}^E\Omega(\hat{S}(E^\vee))$, ${}^E\Omega(\mathcal{A})$, ${}^E\Omega(\mathcal{T})$, and ${}^E\Omega(\mathcal{D})$

THEOREM 2.13. *For* $\mathbf{B}$ *being either of the sheaves* ${}^E\Omega(\hat{S}(E^\vee))$, ${}^E\Omega(\mathcal{A})$, ${}^E\Omega(\mathcal{T})$, *or* ${}^E\Omega(\mathcal{D})$

$$H^{\geq 1}(\mathbf{B},D) = 0. \tag{2.27}$$

Furthermore,

$$H^0({}^E\Omega(\hat{S}(E^\vee)),D) \cong \mathcal{O}_M\,,$$

$$H^0({}^E\Omega(\mathcal{A}_*),D) \cong {}^EA_*\,, \tag{2.28}$$

$$H^0({}^E\Omega(\mathcal{T}^*),D) \cong \wedge^{*+1}E\,,$$

as sheaves of graded commutative algebras

$$H^0({}^E\Omega(\mathcal{D}^*),D) \cong \otimes^{*+1}(S(E)) \tag{2.29}$$

as sheaves of DGAA (over $\mathbb{R}$*).*

PROOF. The first statement follows easily from the spectral sequence argument. Indeed, using the fiber coordinates y^i we introduce the decreasing filtration

$$\cdots \subset F^{p+1}\mathbf{B} \subset F^p\mathbf{B} \subset F^{p-1}\mathbf{B} \subset \cdots \subset F^0\mathbf{B} = \mathbf{B}\,,$$

where the components of the sections of the sheaf $F^p\mathbf{B}$ have degree in $y^i \geq p$.

Since $D(F^p\mathbf{B}) \subset F^{p-1}\mathbf{B}$ the corresponding spectral sequence starts with

$$E_{-1}^{p,q} = F^p\mathbf{B}^{p+q}.$$

It is easy to see that

$$d_{-1} = \delta.$$

Thus using Proposition 2.11 we conclude that for any p,q satisfying the condition $p+q > 0$

$$E_0^{p,q} = E_1^{p,q} = \cdots = E_\infty^{p,q} = 0$$

and the first statement (2.27) follows.

Let $\mathcal{B}$ denote either of the bundles $\hat{S}(E^\vee)$, $\mathcal{A}$, $\mathcal{T}$, or $\mathcal{D}$. We claim that iterating the equation

$$\lambda(u) = u + \kappa(\nabla\lambda(u) + A\cdot\lambda(u))\,, \qquad u \in \Gamma(M,\mathcal{B}) \cap \ker\delta \tag{2.30}$$

we get a map of sheaves of graded vector spaces

$$\lambda : \mathcal{B} \cap \ker\delta \to \mathcal{B} \cap \ker D. \tag{2.31}$$

Here $A\cdot$ denotes the action of the fiberwise vector field A, defined in Proposition 2.10. Indeed, let u be a section of $\mathcal{B}$. Then, due to formula (2.15) $\lambda(u)$ satisfies the following equation

$$\kappa(D(\lambda(u))) = 0. \tag{2.32}$$

Let us denote $D\lambda(u)$ by Y

$$Y = D\lambda(u).$$

The equation $D^2 = 0$ implies that

$$DY = 0$$

which is equivalent to

$$\delta Y = \nabla Y + A \cdot Y \tag{2.33}$$

Applying (2.15) to Y and using equations (2.32), (2.33) we get

$$Y = \kappa(\nabla Y + A \cdot Y).$$

The latter equation has the unique vanishing solution since the operator κ (2.17) raises the degree in the fiber coordinates y^i.

The map (2.31) is obviously injective. To prove that the map is surjective we notice that $\mathcal{H}$

$$\mathcal{H} : \mathcal{B} \to \mathcal{B} \cap \ker \delta$$

is a left inverse of the map (2.31). Thus it suffices to prove that if $a \in \Gamma(M, \mathcal{B}) \cap \ker D$ and

$$\mathcal{H}a = 0 \tag{2.34}$$

then a vanishes.

The condition $a \in \ker D$ is equivalent to the equation

$$\delta a = \nabla a + A \cdot a.$$

Hence, applying (2.15) to a and using (2.34) we get

$$a = \kappa(\nabla a + A \cdot a).$$

The latter equation has the unique vanishing solution since the operator κ (2.17) raises the degree in the fiber coordinates y^i. Thus, the map (2.31) is bijective and the map $\mathcal{H}$

$$\mathcal{H} : \mathcal{B} \cap \ker D \to \mathcal{B} \cap \ker \delta \tag{2.35}$$

is the inverse of (2.31).

It remains to prove that the map (2.31) is compatible with the multiplication of the sections of the sheaf $\mathcal{B}$, where $\mathcal{B}$ is either $\hat{S}(E^\vee)$, $\mathcal{A}$, $\mathcal{T}$, or $\mathcal{D}$. The latter follows immediately from the fact that the inverse map $\mathcal{H}$

$$\mathcal{H} : \mathcal{B} \to \mathcal{B} \cap \ker \delta \tag{2.36}$$

respects the corresponding algebra structures on $\hat{S}(E^\vee)$, $\mathcal{A}$, $\mathcal{T}$, and the DGAA structure on $\mathcal{D}$. □

2.2.3. Compatibilities of the resolutions with the algebraic structures. Let us now mention that since the Fedosov differential (2.23) is compatible with the graded algebraic structures on the sheaves ${}^E\Omega(\mathcal{T})$ and ${}^E\Omega(\mathcal{A})$ we conclude that $H^*({}^E\Omega(\mathcal{T}), D)$ is a sheaf of graded Lie algebras and $H^*({}^E\Omega(\mathcal{A}), D)$ is a sheaf of graded modules over $H^*({}^E\Omega(\mathcal{T}), D)$. On the other hand the above theorem tells us that

$$H^*({}^E\Omega(\mathcal{A}), D) = {}^EA_* \,,$$

and

$$H^*({}^E\Omega(\mathcal{T}), D) = \wedge^{*+1} E,$$

Thus, it is natural to ask whether the graded algebraic structures on the sheaves $\wedge^{*+1} E = {}^ET^*_{poly}$ and ${}^EA_*$ coincide with the ones given by Lie bracket (1.3) (1.4) and the Lie derivative (1.6). A positive answer to this question is given by the following proposition:

PROPOSITION 2.14 ([**5, 7**]). *The map*

$$\mathcal{H} : \mathcal{T}^* \cap \ker D \to \mathcal{T}^* \cap \ker \delta \cong {}^ET^*_{poly} \tag{2.37}$$

induces an isomorphism of the sheaves of graded Lie algebras $H^*({}^E\Omega(\mathcal{T}),D) \cong {}^E T^*_{poly}$.

And the map

$$\mathcal{H} : \mathcal{A}_* \cap \ker D \to \mathcal{A}_* \cap \ker \delta \cong {}^E A_* \tag{2.38}$$

induces an isomorphism of the sheaves of graded modules $H^*({}^E\Omega(\mathcal{A}), D) \cong {}^E A_*$ *over the sheaf of graded Lie algebras* $H^*({}^E\Omega(\mathcal{T}), D) \cong {}^E T^*_{poly}$.

PROOF. Let us recall the proof of the first part of the proposition from [**5**, proposition 2.4]. We have to show that for any D-closed fiberwise polyvector field $u, v \in \Gamma(M, {}^E T^*_{poly})$ one has

$$\mathcal{H}\big([u\,,v]_{SN}\big) = [\mathcal{H}(u)\,,\mathcal{H}(v)]$$

Since $\mathcal{H}$ is a morphism of graded commutative algebra, it is sufficient to prove it for functions and vector fields:

- *First case.* Let f be a function on M and $u_0 = u^i(x)e_i$ a vector field and

  $$\omega = \lambda(f)\,,\ u = \lambda(u_0)\,.$$

  A direct computation shows that

  $$\lambda(f) = f + y^i\rho(e_i)f \mod |y|^2\,,$$

  and

  $$\lambda(u_0) = u^i\frac{\partial}{\partial y^i} \bmod |y|\,,$$

  Therefore $[u\,,\omega]_{SN} = u^i\rho(e_i)f \bmod |y|$ and hence $\mathcal{H}\big([u\,,\omega]_{SN}\big) = [u_0,f]$.
- *Second case.* Let $u_0 = u^i(x)e_i$ and $v_0 = v^i(x)e_i$ two vector fields and

  $$u = \lambda(u_0)\ \ v = \lambda(v_0)\,.$$

  It is not hard to show that

  $$\lambda(u_0) = u^i\frac{\partial}{\partial y^i} + y^i\big(\rho(e_i)u^k + \Gamma^k_{ij}u^j\big)\frac{\partial}{\partial y^k} \mod |y|^2\,.$$

  Therefore

$$\begin{aligned}[u,v]_{SN} &= u^i\big(\rho(e_i)v^k + \Gamma^k_{ij}v^j\big)\frac{\partial}{\partial y^k} - v^i\big(\rho(e_i)u^k + \Gamma^k_{ij}u^j\big)\frac{\partial}{\partial y^k} \mod |y| \\ &= \big(u^i\rho(e_i)v^k + c^j_{ij}u_iv_j - v^i\rho(e_i)u^k\big)\frac{\partial}{\partial y^k} \mod |y| \ = \ [u_0,v_0] \mod |y|\end{aligned}$$

  And hence $\mathcal{H}\big([u,v]_{SN}\big) = [u_0,v_0]$.

Next recall the proof of the second part the proposition from [**7**, proposition 2.5] . We have to show that for any D-closed fiberwise differential form $\omega \in \Gamma(M,\mathcal{A})$ one has

$$\mathcal{H}(d^f\omega) = {}^E d\mathcal{H}(\omega)\,,$$

where $d^f = dy^i\frac{\partial}{\partial y^i}$ is the fiberwise De Rham differential on $\mathcal{A}$. Since $\mathcal{H}$ is a morphism of graded commutative algebras, it is sufficient to prove it for functions and 1-forms:

- *First case.* Let f be a function on M and

  $$\omega = \lambda(f).$$

  A direct computation shows that

  $$\lambda(f) = f + y^i\rho(e_i)f \mod |y|^2.$$

  Therefore $d^f\omega = \rho(e_i)f dy^i \mod |y|$, and hence, $\mathcal{H}(d^f\omega) = {}^E df$.

- *Second case.* Let $\alpha = \alpha_i(x)\xi^i$ be a E-1-form and
$$\omega = \lambda(\alpha).$$
It is not hard to show that
$$\lambda(\alpha) = \alpha + y^i\big(\rho(e_i)\alpha_j - \Gamma_{ij}^k\alpha_k\big)dy^j \mod |y|^2.$$
Therefore,
$$d^f\omega = \big(\rho(e_i)\alpha_j - \Gamma_{ij}^k\alpha_k\big)dy^i\wedge dy^j \mod |y| = \big(\rho(e_i)\alpha_j - \frac{1}{2}c_{ij}^k\alpha_k\big)dy^i\wedge dy^j \mod |y|,$$
and hence,
$$\mathcal{H}(d^f\omega) = {}^Ed\alpha.$$

To finish the proof we notice that for any fiberwise polyvector field $u\in\Gamma(M,\mathcal{T}^*)$ and any fiberwise differential form $\omega\in\Gamma(M,\mathcal{A})$, the equation
$$\mathcal{H}(\iota_u\omega) = \iota_{\mathcal{H}(u)}\circ\mathcal{H}(\omega)$$
is obviously satisfied. The latter implies that for any pair of D-closed sections $u\in\Gamma(M,\mathcal{T}^*)$, $\omega\in\Gamma(M,\mathcal{A}_*)$
$$\mathcal{H}(L_u\omega) = {}^EL_{\mathcal{H}(u)}\circ\mathcal{H}(\omega),$$
and the proposition follows. □

REMARK 2.15. Actually, we have proved a slightly stronger statement. Namely, we shown that the maps (2.38) and (2.37) induce an isomorphism of sheaves of *calculi.*
$$(H^*({}^E\Omega(\mathcal{T}),D),H^*({}^E\Omega(\mathcal{A}),D))\cong({}^ET^*_{poly},{}^EA_*).$$
The precise definition of the calculi can be found in section 4.3 of the second part of [**13**].

Let us now recall that $\mathcal{T}^{-1} = \hat{S}(E^\vee)$, $\mathcal{T}^0$ is a sheaf of Lie-Rinehart algebras over the sheaf of algebras $\mathcal{T}^{-1} = \hat{S}(E^\vee)$, and $\mathcal{D}^0$ is the universal enveloping algebroid of $\mathcal{T}^0$. Therefore, the inverse $\lambda = (\mathcal{H})^{-1}$ of the map (2.37) induces the morphism
$$\mu : \mathcal{U}E\to\mathcal{D}^0. \tag{2.39}$$
of sheaves of bialgebroids and for any $P\in\Gamma(M,\mathcal{U}E)$
$$D(\mu(P)) = 0. \tag{2.40}$$
We claim that

PROPOSITION 2.16. *The map (2.39) gives the isomorphism*
$$\mu : \mathcal{U}E\to\mathcal{D}^0\cap\ker D. \tag{2.41}$$
of sheaves of bialgebroids.

PROOF. Notice that $\mathcal{U}E$ and $\mathcal{D}^0$ are both filtered sheaves of algebras. The filtration on $\mathcal{U}E$ is defined in (1.12) and the filtration on $\mathcal{D}^0$ is given by the degree of differential operators.

Thanks to the results of [**51**] and [**57**] we have the PBW theorem for Lie algebroids. This theorem says that the associated graded module of the filtration (1.12) on $\mathcal{U}E$ is
$$Gr(\mathcal{U}E) = S(E)$$
the symmetric algebra of the bundle E.

Furthermore, it is not hard to see that the map μ is compatible with the filtrations on $\mathcal{U}E$ and $\mathcal{D}^0$ and due to Theorem 2.13 and Proposition 2.11 μ induces the isomorphism
$$S(E)\cong\mathcal{D}^0\cap\ker D$$
of the associated graded sheaves of vector spaces. Therefore, the snake lemma argument implies that the map (2.41) is also an isomorphism onto the sheaf $\mathcal{D}^0\cap\ker D$ of D-flat sections of $\mathcal{D}^0$. □

Let us recall that ${}^{E}D^{*}_{poly}$ (respectively, $\mathcal{D}^{*}$) is the tensor algebras of $\mathcal{U}E$ over $\mathcal{O}_M$ (respectively, the tensor algebra of $\mathcal{D}^{0}$ over $\hat{S}(E^{\vee})$). Using this fact we extend (2.39) to the morphism

$$\mu' : {}^{E}D^{*}_{poly} \to \mathcal{D}^{*}. \tag{2.42}$$

of sheaves of DGAA algebras (over $\mathbb{R}$) by setting

$$\mu'\Big|_{{}^{E}D^{0}_{poly}} = \mu\,, \qquad \mu'\Big|_{\mathcal{O}_M} = \lambda\,,$$

where λ is defined in (2.31).

Let us also observe that since the map (2.39) is a morphism of sheaves of bialgebroids the map (2.42) a morphism of sheaves of DGLA (over $\mathbb{R}$). Furthermore, Theorem 2.13 implies that the sheaf of DGAA $\mathcal{D}^{*} \cap \ker D$ is generated by the sheaf $\mathcal{D}^{0} \cap \ker D$ over the sheaf of commutative algebras $\hat{S}(E^{\vee}) \cap \ker D \cong \mathcal{O}_M$. Therefore using Proposition 2.16 we get the following result:

PROPOSITION 2.17 (proposition 2.5, [**5**]). *The map (2.42) gives an isomorphism of sheaves of DGLA*

$$\mu' : {}^{E}D^{*}_{poly} \xrightarrow{\sim} \mathcal{D}^{*} \cap \ker D. \tag{2.43}$$

This map is also compatible with the DGAA structures on the sheaves ${}^{E}D^{*}_{poly}$ *and* $\mathcal{D}^{*} \cap \ker D$ *by construction.* □

2.2.4. Fedosov resolution of Hochschild E-chains. Let us consider the map of sheaves of graded vector spaces

$$\gamma : \mathcal{J}^{*} \to {}^{E}J^{poly}_{*}\,, \qquad \gamma(j)(P) = (\mu'(P))(j)\Big|_{y^i=0}\,, \tag{2.44}$$

$$j \in \Gamma(M, \mathcal{J}^{k})\,, \quad P \in \Gamma(M, {}^{E}D^{k}_{poly}).$$

We claim that

THEOREM 2.18 ([**7**]). *For any $q \geq 1$*

$$H^{q}({}^{E}\Omega(\mathcal{J}), D) = 0\,, \tag{2.45}$$

and the map (2.44) gives an isomorphism of sheaves of DG modules over the sheaf of DGLA ${}^{E}D^{*}_{poly} \cong \mathcal{D}^{*} \cap \ker D$

$$\gamma : \mathcal{J}_{*} \xrightarrow{\sim} {}^{E}J^{poly}_{*}. \tag{2.46}$$

This isomorphism sends the Fedosov connection (2.23) on $\mathcal{J}^{*}$ to the the Grothendieck connection (1.26) on ${}^{E}J^{poly}_{*}$.

PROOF. The first statement (2.45) follows easily from the spectral sequence argument. Indeed, using the zeroth collection of the fiber coordinates y^{i}_{0} (2.7) we introduce the decreasing filtration on the sheaf ${}^{E}\Omega(\mathcal{J})$

$$\cdots \subset F^{p+1}({}^{E}\Omega(\mathcal{J})) \subset F^{p}({}^{E}\Omega(\mathcal{J})) \subset F^{p-1}({}^{E}\Omega(\mathcal{J})) \subset \cdots \subset F^{0}({}^{E}\Omega(\mathcal{J})) = {}^{E}\Omega(\mathcal{J})\,,$$

where the components of the sections (2.7) of the sheaf $F^{p}({}^{E}\Omega(\mathcal{J}))$ have degree in $y^{i}_{0} \geq p$.

Since $D(F^{p}({}^{E}\Omega(\mathcal{J}))) \subset F^{p-1}({}^{E}\Omega(\mathcal{J}))$ the corresponding spectral sequence starts with

$$E^{p,q}_{-1} = F^{p}({}^{E}\Omega(\mathcal{J})^{p+q}).$$

Next, we observe that

$$d_{-1} = \xi^{i}\frac{\partial}{\partial y^{i}_{0}}\,,$$

and hence, due to the Poincaré lemma for the formal disk we have

$$E^{p,q}_{0} = E^{p,q}_{1} = \cdots = E^{p,q}_{\infty} = 0$$

whenever $p+q>0$. Thus, the first statement (2.45) of the theorem follows.

Since (2.39) is a morphism of sheaves of bialgebroids

$$\mu'(P\bullet Q)=\mu'(P)\bullet\mu'(Q)\,,\qquad P,Q\in\Gamma(M,{}^E\!D^*_{poly}).$$

Furthermore, μ' is obviously compatible with cyclic permutations

$$t\,\mu'(P_0\otimes P_1\otimes\cdots\otimes P_l)=\mu'(P_1\otimes P_2\otimes\cdots\otimes P_l\otimes P_0)\,,\qquad P_i\in\Gamma(M,\mathcal{U}E).$$

Hence, for any $P\in\Gamma(M,{}^E\!D^*_{poly})$ and any $a\in\Gamma(M,\mathcal{J}_*)$

$$ {}^E\!S_P(\gamma(a))=\gamma(\mathcal{R}_{\mu'(P)}(a)). \tag{2.47}$$

Since $\mathcal{J}_*$ is dual to $\mathcal{D}^*\cap\ker\delta$ and $\mathcal{D}^*\cap\ker\delta\cong\mathcal{D}^*\cap\ker D\cong{}^E\!D^*_{poly}$ the map (2.46) is an isomorphism. It remains to prove that the map (2.46) sends the Fedosov connection (2.23) to the Grothendieck connection (1.26). This statement is proved by the following line of equations:

$$\begin{aligned}\gamma(D_uj)(P)&=(\mu'(P))(D_uj)\Big|_{y^i=0}=(D_u[\mu'(P)(j)])\Big|_{y^i=0}\\&=\rho(u)[\mu'(P)(j)]\Big|_{y^i=0}-(\iota_u\delta\bullet[\mu'(P)(j)])\Big|_{y^i=0}\\&=\rho(u)[\mu'(P)(j)]\Big|_{y^i=0}-(\mu'(u)\bullet\mu'(P))(j)\Big|_{y^i=0}\\&=\rho(u)[\mu'(P)(j)]\Big|_{y^i=0}-\mu'(u\bullet P)(j)\Big|_{y^i=0}\\&=\rho(u)(\gamma(j))(P)-(\gamma(j))(u\bullet P)=(\nabla^G_u\gamma(j))(P)\,,\end{aligned}$$

where $u\in\Gamma(M,E)$, $j\in\Gamma(M,\mathcal{J}_k)$, $P\in\Gamma(M,{}^E\!D^k_{poly})$, ι denotes the contraction of an E-vector field with E-differential forms, ρ is the anchor map, and u is viewed both as a section of E and an E-differential operator. □

2.3. Proof of the formality theorem

Let us denote

- $\lambda_A\colon {}^E\!A_*\to{}^E\Omega(\mathcal{A}_*)$, the inverse of the map $\mathcal{H}$ (2.38),
- $\lambda_T\colon {}^E\!T^*_{poly}\to{}^E\Omega(\mathcal{T})$, the inverse of the map $\mathcal{H}$ (2.37),
- $\lambda_D\colon {}^E\!D^*_{poly}\to{}^E\Omega(\mathcal{D})$, the map μ' (2.42) and
- $\lambda_C\colon {}^E\!C^{poly}_*\to{}^E\Omega(\mathcal{J})$, the composition $\gamma^{-1}\circ\varrho$ of the inverse of the map γ (2.44) with the map ϱ (1.37).

The results of the previous section can be represented in the form of the following commutative diagrams of sheaves of DG Lie algebras, their modules, and morphisms

$$\begin{array}{ccc}({}^E\!T^*_{poly},0,[\,,])&\overset{\lambda_T}{\rightarrowtail}&({}^E\Omega(\mathcal{T}),D,[\,,]_{SN})\\\downarrow{\scriptstyle {}^E\!L_{mod}}&&\downarrow{\scriptstyle L_{mod}}\\({}^E\!A_*,0)&\overset{\lambda_A}{\twoheadrightarrow}&({}^E\Omega(\mathcal{A}),D),\end{array}\tag{2.48}$$

$$\begin{array}{ccc}({}^E\Omega(\mathcal{D}),D+\partial,[\,,]_G)&\overset{\lambda_D}{\leftarrowtail}&({}^E\!D^*_{poly},\partial,[\,,]_G)\\\downarrow{\scriptstyle \mathcal{R}_{mod}}&&\downarrow{\scriptstyle {}^E\!R_{mod}}\\({}^E\Omega(\mathcal{J}),D+\mathfrak{b})&\overset{\lambda_C}{\twoheadleftarrow}&({}^E\!C^{poly}_*,\mathfrak{b}),\end{array}$$

where the horizontal arrows correspond to embeddings of the sheaves of DG Lie algebras (respectively, of DGLA modules) constructed in the previous section. These embeddings are quasi-isomorphisms by theorems 2.13, 2.18 and propositions 2.14, 2.17.

Next, due to claims *1* and *2* in Theorem 2.5 we have a fiberwise quasi-isomorphism

$$\mathcal{K} : ({}^E\Omega(\mathcal{T}), 0, [\,,\,]_{SN}) \succ\!\!\rightarrow ({}^E\Omega(\mathcal{D}), \partial, [\,,\,]_G) \tag{2.49}$$

from the sheaf of DGLA $({}^E\Omega(\mathcal{T}), 0, [\,,\,]_{SN})$ to the sheaf of DGLA $({}^E\Omega(\mathcal{D}), \partial, [\,,\,]_G)$. Composing quasi-isomorphism (2.49) with the action of ${}^E\Omega(\mathcal{D})$ on ${}^E\Omega(\mathcal{J})$ we get a L_∞-module structure on ${}^E\Omega(\mathcal{J})$ over ${}^E\Omega(\mathcal{T})$.

Due to claims *1* and *2* in Theorem 2.6 we have a fiberwise quasi-isomorphism

$$\mathcal{S} : ({}^E\Omega(\mathcal{J}), \mathfrak{b}) \succ\!\succ\!\rightarrow ({}^E\Omega(\mathcal{A}), 0) \tag{2.50}$$

from the sheaf of L_∞-modules ${}^E\Omega(\mathcal{J})$ to the sheaf of DGLA modules ${}^E\Omega(\mathcal{A})$ over ${}^E\Omega(\mathcal{T})$.

Thus we get the following commutative diagram

$$\begin{array}{ccc} ({}^E\Omega(\mathcal{T}), 0, [\,,\,]_{SN}) & \overset{\mathcal{K}}{\succ\!\!\rightarrow} & ({}^E\Omega(\mathcal{D}), \partial, [\,,\,]_G) \\ \downarrow^{L}_{mod} & & \downarrow^{\mathcal{R}}_{mod} \\ ({}^E\Omega(\mathcal{A}), 0) & \overset{\mathcal{S}}{\leftarrow\!\!\prec} & ({}^E\Omega(\mathcal{J}), \mathfrak{b}), \end{array} \tag{2.51}$$

where by commutativity we mean that $\mathcal{S}$ is a morphism of the sheaves of L_∞-modules $({}^E\Omega(\mathcal{J}), \mathfrak{b})$ and $({}^E\Omega, 0)$ over the sheaf of DGLA $({}^E\Omega(\mathcal{T}), 0, [\,,\,]_{SN})$ and the L_∞-module structure on $({}^E\Omega(\mathcal{J}), \mathfrak{b})$ over $({}^E\Omega(\mathcal{T}), 0, [\,,\,]_{SN})$ is obtained by composing the L_∞-quasi-isomorphism $\mathcal{K}$ with the action $\mathcal{R}$ of $({}^E\Omega(\mathcal{D}), \partial, [\,,\,]_G)$ on $({}^E\Omega(\mathcal{J}), \mathfrak{b})$.

Let us now restrict ourselves to an open subset $V \subset M$ such that $E\Big|_V$ is trivial. Over any such subset the E-de Rham differential (1.5) is well defined for either of the sheaves ${}^E\Omega(\mathcal{A})$, ${}^E\Omega(\mathcal{T})$, ${}^E\Omega(\mathcal{J})$, and ${}^E\Omega(\mathcal{D})$. Furthermore, since the quasi-isomorphisms (2.49) and (2.50) are fiberwise we can add to all the differentials in diagram (2.51) the E-de Rham differential (1.5). Thus we get a new commutative diagram

$$\begin{array}{ccc} ({}^E\Omega(\mathcal{T})\Big|_V, {}^Ed, [\,,\,]_{SN}) & \overset{\mathcal{K}}{\succ\!\!\rightarrow} & ({}^E\Omega(\mathcal{D})\Big|_V, {}^Ed + \partial, [\,,\,]_G) \\ \downarrow^{L}_{mod} & & \downarrow^{\mathcal{R}}_{mod} \\ ({}^E\Omega(\mathcal{A})\Big|_V, {}^Ed) & \overset{\mathcal{S}}{\leftarrow\!\!\prec} & ({}^E\Omega(\mathcal{J})\Big|_V, {}^Ed + \mathfrak{b}) \end{array} \tag{2.52}$$

of the L_∞-morphism $\mathcal{K}$ and the morphism of L_∞-modules $\mathcal{S}$.

We claim that

PROPOSITION 2.19. *The L_∞-morphism $\mathcal{K}$ and the morphism of L_∞-modules $\mathcal{S}$ in (2.52) are quasi-isomorphisms.*

PROOF. This statement follows easily from the standard argument of the spectral sequence. Indeed, we can naturally regard ${}^E\Omega(\mathcal{T})$ and ${}^E\Omega(\mathcal{D})$ (respectively, ${}^E\Omega(\mathcal{J})$ and ${}^E\Omega(\mathcal{A})$) as sheaves of double complexes and the exterior degree provides us with the following descending filtration

$$F^p({}^E\Omega^*(\mathcal{B})) = \bigoplus_{k \geq p} {}^E\Omega^k(\mathcal{B}),$$

where $\mathcal{B}$ is either $\mathcal{T}$ or $\mathcal{D}$ (resp. $\mathcal{J}$ or $\mathcal{A}$).

The corresponding versions of Vey's [**67**] and Hochschild-Kostant-Rosenberg-Connes-Teleman [**10**], [**36**], [**65**] theorems for $\mathbb{R}^d_{formal}$ imply that $\mathcal{K}$ (respectively, $\mathcal{S}$) induces a quasi-isomorphism on the level of E_0. Hence, $\mathcal{K}$ (respectively, $\mathcal{S}$) induces a quasi-isomorphism on the level of E_∞. The standard snake lemma argument of homological algebra implies that $\mathcal{K}$ (respectively, $\mathcal{S}$) in (2.52) is a quasi-isomorphism. □

On the open subset V we can represent the Fedosov differential (2.23) in the following (non-covariant) form

$$D = {}^E d + B\,, \tag{2.53}$$

$$B = \sum_{p=0}^{\infty} \xi^i B^k_{i;j_1\ldots j_p}(x) y^{j_1}\ldots y^{j_p}\frac{\partial}{\partial y^k}.$$

If we regard B as a section of ${}^E\Omega^1(\mathcal{T}^0)\Big|_V$ then the nilpotency condition $D^2 = 0$ says that B is a Maurer-Cartan section of the sheaf of DGLA $({}^E\Omega(\mathcal{T})\Big|_V, {}^E d, [\,,\,]_{SN})$. Then following appendix A.2.3 this means that the sheaf of DGLA $({}^E\Omega(\mathcal{T})\Big|_V, D, [\,,\,]_{SN})$ is obtained from $({}^E\Omega(\mathcal{T})\Big|_V, {}^E d, [\,,\,]_{SN})$ via the twisting procedure by the Maurer-Cartan element B (see claim *2* in Proposition A.23).

According to the first statement of Proposition A.23 the element

$$B_D = \sum_{k=1}^{\infty} \frac{1}{k!}\mathcal{K}_k(B,\ldots,B)$$

is a Maurer-Cartan section of $({}^E\Omega(\mathcal{D})\Big|_V, {}^E d + \partial, [\,,\,]_G)$. Moreover, due to claim *3* in Theorem 2.5

$$B_D = B\,,$$

where B is viewed as a section of the sheaf ${}^E\Omega^1(\mathcal{D}^0)\Big|_V$.

Thus twisting the L_∞-quasi-isomorphism $\mathcal{K}$ in (2.52) by the Maurer-Cartan element B (claims *3, 4* in Proposition A.23) we get the L_∞-quasi-isomorphism

$$\mathcal{K}^{tw} : ({}^E\Omega(\mathcal{T})\Big|_V, D, [\,,\,]_{SN}) \succ\!\!\to ({}^E\Omega(\mathcal{D})\Big|_V, D + \partial, [\,,\,]_G).$$

Following Proposition A.24 (claim *1*) one can twist the DG module structure of ${}^E\Omega(\mathcal{A})$ over ${}^E\Omega(\mathcal{T})$ by B (respectively, of ${}^E\Omega(\mathcal{J})$ over ${}^E\Omega(\mathcal{D})$ by $B_D = B$). Hence, by virtue of claim *2, 3* of the same proposition the twisting procedure turns diagram (2.52) into the commutative diagram

$$\begin{array}{ccc} ({}^E\Omega(\mathcal{T})\Big|_V, D, [\,,\,]_{SN}) & \overset{\mathcal{K}^{tw}}{\succ\!\!\to} & ({}^E\Omega(\mathcal{D})\Big|_V, D + \partial, [\,,\,]_G) \\ \downarrow^L_{mod} & & \downarrow^R_{mod} \\ ({}^E\Omega(\mathcal{A})\Big|_V, D) & \overset{\mathcal{S}^{tw}}{\leftarrow\!\!\prec} & ({}^E\Omega(\mathcal{J})\Big|_V, D + \mathfrak{b}), \end{array} \tag{2.54}$$

where $\mathcal{S}^{tw}$ is a L_∞-quasi-isomorphism obtained from $\mathcal{S}$ by twisting via the Maurer-Cartan section B of the sheaf of DGLA $({}^E\Omega(\mathcal{T})\Big|_V, {}^E d, [\,,\,]_{SN})$.

We claim that the morphism $\mathcal{K}^{tw}$ (respectively, $\mathcal{S}^{tw}$) does not depend on the choice of the trivialization of E over V and hence is a well-defined L_∞-morphism of sheaves of DGLA (respectively, sheaves of DGLA modules). Indeed, the term in B that depends on the choice of the trivialization of E is linear in the fiber coordinates y^i. But due to claim *4* in Theorem 2.5 and claim *3* in Theorem 2.6 this term contribute neither to $\mathcal{K}^{tw}$ nor to $\mathcal{S}^{tw}$.

Thus the quasi-isomorphisms $\mathcal{K}^{tw}$ and $\mathcal{S}^{tw}$ are well defined and we arrive at the following commutative diagram

$$\begin{array}{ccc} ({}^E\Omega(\mathcal{T}), D, [\,,]_{SN}) & \overset{\mathcal{K}^{tw}}{\succ\!\!\rightarrow} & ({}^E\Omega(\mathcal{D}), D+\partial, [\,,]_G) \\ \downarrow^{L}_{mod} & & \downarrow^{\mathcal{R}}_{mod} \\ ({}^E\Omega(\mathcal{A}), D) & \overset{\mathcal{S}^{tw}}{\leftarrow\!\!\prec} & ({}^E\Omega(\mathcal{J}), D+\mathfrak{b}). \end{array} \tag{2.55}$$

Assembling diagrams (2.48) and (2.55) we get the desired chain (2.2) of L_∞-quasi-isomorphisms between the sheaves of DGLA modules $({}^ET^*_{poly}, {}^EA_*)$ and $({}^ED^*_{poly}, {}^EC_*^{poly})$. It is obvious from the construction that the terms and the quasi-isomorphisms of the resulting diagram (2.2) are functorial in the pair (E,∇), where ∇ is a torsion-free connection on E.

Now we can end the proof of Theorem 2.1. Using Theorem A.13 we can inverse the L_∞-quasi-isomorphism λ_D: there exists a L_∞-quasi-isomorphism

$$\alpha_D : ({}^E\Omega(\mathcal{D}), D+\partial, [\,,]_G) \succ\!\!\rightarrow ({}^ED^*_{poly}, \partial, [\,,]_G)$$

with first structure map $\alpha_D^{[1]} = \mathcal{H}$. Then define $\Phi = \alpha_D \circ \mathcal{K}^{tw} \circ \lambda_T$; its first structure map is $\Phi^{[1]} = \mathcal{H} \circ \mathcal{K}^{tw[1]} \circ \lambda_T = \mathcal{H} \circ \mathcal{K}^{[1]} \circ \lambda_T = \mathcal{V}$. In the same way using Theorem A.21 one can find a quasi-inverse

$$\alpha_A : ({}^E\Omega(\mathcal{A}), D+\mathfrak{b}) \succ\!\succ\!\rightarrow ({}^EA_*, \mathfrak{b})$$

of λ_A such that $\alpha_A^{[0]} = \mathcal{H}$. Then define $\Psi = \alpha_A \circ \mathcal{S}^{tw} \circ \lambda_A$; its first structure map is $\Psi^{[0]} = \mathcal{H} \circ \mathcal{S}^{tw[0]} \circ \lambda_A = \mathcal{H} \circ \mathcal{S}^{[0]} \circ \lambda_A = \mathfrak{C}$. The theorem is proved. □

2.4. Applications in deformation quantization theory

The obvious applications of the formality theorem for Lie algebroids are related to the deformations associated with triangular Lie bialgebroids. Moreover, Theorem 2.1 allows us to get an elegant description of the Hochschild homology and the traces of these deformations.

2.4.1. Quantization of Poisson Lie algebroids. Let $E \to M$ be a Lie algebroid with bracket $[\,,]$ and anchor ρ.

First we recall that

DEFINITION 2.20. *A Lie algebroid $(E,[\,,],\rho)$ equipped with an E-bivector $\pi \in \Gamma(M, {}^ET^1_{poly})$ satisfying the Jacobi identity*

$$[\pi,\pi] = 0 \tag{2.56}$$

is called a Poisson Lie algebroid.

In the simplest example $E = TM$ this corresponds to the definition of a Poisson manifold.

Following [**50**] a *quantization* (or *deformation*) of a Poisson Lie algebroid is a construction of an element

$$\Pi \in \Gamma(M, {}^ED^1_{poly})[[\hbar]] \tag{2.57}$$

satisfying the condition of the classical limit

$$\Pi = 1 \otimes 1 \bmod \hbar, \qquad \Pi - t(\Pi) = \hbar\pi \bmod \hbar^2, \tag{2.58}$$

and the "associativity" condition

$$[\Pi,\Pi]_G = 0. \tag{2.59}$$

Here $\hbar$ is an auxiliary variable and t denotes the (cyclic) permutation of components of $\Pi \in \Gamma(M, {}^ED^1_{poly})[[\hbar]] = \Gamma(M, \mathcal{U}E \otimes \mathcal{U}E)[[\hbar]]$.

Furthermore, two quantizations Π and Π' of $(E,[\,,],\rho,\pi)$ are called *gauge equivalent* if there exists a formal power series

$$\Psi = 1 + \hbar\Psi_1 + \hbar^2\Psi_2 + \ldots \in \Gamma(M,\mathcal{U}E)[[\hbar]]$$

such that

$$(\Delta\Psi)\,\Pi' = \Pi\,(\Psi\otimes\Psi)\,, \tag{2.60}$$

where Δ is the coproduct (A.4) in $\mathcal{U}E$.

REMARK 2.21. Following [**45**], a *Lie bialgebroid* is a Lie algebroid $(E,[\,,],\rho)$ together with a differential

$$d' : \wedge^* E \to \wedge^{*+1} E$$

which is a derivation of the bracket $[\,,]$

$$d'\big([u,v]\big) = [d'(u),v] + [u,d'(v)]\,, \qquad u,v\in\Gamma(M,E)$$

Starting from a Poisson Lie algebroid $(E,[\,,],\pi)$ one can define a canonical Lie bialgebroid structure $d'_\pi = [\pi,\cdot]$ on E. In this context, a Poisson Lie algebroid is called a *triangular Lie bialgebroid* in [**5, 45**].

Following [**19**], Ping Xu formulates in [**70**] a quantization problem for Lie bialgebroids. Namely, a quantization of a Lie bialgebroid $(E,[\,,],d',\rho)$ is a bialgebroid structure on $\mathcal{U}E[[\hbar]]$ with product $m' = m \mod \hbar^2$ and coproduct $\Delta' = \Delta + \hbar d' \mod \hbar^2$. Xu shows ([**70**]) that to quantize a triangular Lie bialgebroid it is sufficient to find an element Π satisfying (2.57), (2.58) and (2.59) (a *twist*, according to annexe A.1.2) .

Thanks to the formality of the sheaf of DGLA ${}^E D^*_{poly}$ (from Theorem 2.1) we have a bijective correspondence between the moduli spaces of Maurer-Cartan elements of the DGLA $\Gamma(M,{}^E T^*_{poly})[[\hbar]]$ of E-polyvector fields and the DGLA $\Gamma(M,{}^E D^*_{poly})[[\hbar]]$ of E-polydifferential operators. In other words, if we consider the cone

$$\begin{gathered}\pi_\hbar = \hbar\pi + \hbar^2\pi_1 + \hbar^3\pi_2 + \ldots\,,\\ [\pi_\hbar,\pi_\hbar] = 0\,,\\ \pi_i \in \Gamma(M,{}^E T^1_{poly})\end{gathered} \tag{2.61}$$

of formal power series in $\hbar$ acted upon by the Lie algebra $\hbar\Gamma(M,E)[[\hbar]]$

$$\pi_\hbar \to [u,\pi_\hbar]\,, \qquad u\in\hbar\Gamma(M,E)[[\hbar]]\,, \tag{2.62}$$

then (following annexe A.2.4)

PROPOSITION 2.22. *The map*

$$\pi_\hbar \longmapsto 1\otimes 1 + \sum_{n\geq 1}\frac{1}{n!}\Phi^{[n]}(\underbrace{\pi_\hbar,\ldots,\pi_\hbar}_{n\ times}) \tag{2.63}$$

defines a bijective correspondence between the deformations (2.57) associated with a Poisson Lie algebroid $(E,[\,,],\rho,\pi)$ modulo the relation (2.60) and the points of the cone (2.61) modulo the action (2.62) of the prounipotent group corresponding to the Lie algebra $\hbar\Gamma(M,E)[[\hbar]]$. □

An orbit $[\pi_\hbar]$ on the cone (2.61) corresponding to a deformation Π (2.57) is called *the class of the deformation* and any point $\pi_\hbar$ of this orbit is called *a representative* of the class.

2.4.2. Hochschild (co)homology of a deformation. Given a deformation Π (2.57) associated with a Poisson Lie algebroid $(E, [\,,\,], \rho, \pi)$ one can define the Hochschild (or tangent) chain complex of this deformation as the graded vector space

$$\Gamma(M, {}^E C^{poly}_*)[[\hbar]] \tag{2.64}$$

equipped with the differential

$${}^E R_\Pi : {}^E C^{poly}_* \to {}^E C^{poly}_{*+1}.$$

Furthermore, one defines the Hochschild (or tangent) cochain complex of the deformation Π as the graded vector space

$$\Gamma(M, {}^E D^*_{poly})[[\hbar]] \tag{2.65}$$

equipped with the differential

$$[\Pi, \cdot]_G : {}^E D^*_{poly} \to {}^E D^{*+1}_{poly}.$$

Due to claim *4* of Proposition A.23 and Theorem 2.1 we get the following

PROPOSITION 2.23. *Let Π be a deformation associated with a Poisson Lie algebroid $(E, [\,,\,], \rho, \pi)$ and let $\pi_\hbar$ be a representative of the class of this deformation. Then the Hochschild cochain complex (2.65) of the deformation Π is quasi-isomorphic to the cochain complex $(\Gamma(M, {}^E T^*_{poly})[[\hbar]], [\pi_\hbar, \cdot])$ of Poisson cohomology for $\pi_\hbar$.* □

And due to claim *3* of Proposition A.24 and Theorem 2.1 we get the following

PROPOSITION 2.24. *Let Π be a deformation associated with a Poisson Lie algebroid $(E, [\,,\,], \rho, \pi)$ and let $\pi_\hbar$ be a representative of the class of this deformation. Then the Hochschild chain complex (2.64) of the deformation Π is quasi-isomorphic to the chain complex $({}^E\Omega(M)[[\hbar]], {}^E L_{\pi_\hbar})$.* □

REMARK 2.25. Actually, in the case $\Pi = 1 \otimes 1 + \sum_n \frac{1}{n!} \Phi^{[n]}(\pi_\hbar, \dots, \pi_\hbar)$ the quasi-isomorphisms are "explicitly" given by

$$v \longmapsto \mathcal{V}(v) + \sum_{n>1} \Phi^{[n]}(\underbrace{\pi_\hbar, \dots, \pi_\hbar}_{n \text{ times}}, v)\,, \qquad v \in {}^E D^*_{poly}\,, \tag{2.66}$$

and

$$a \longmapsto \mathfrak{C}(a) + \sum_{n>0} \Psi^{[n]}(\underbrace{\pi_\hbar, \dots, \pi_\hbar}_{n \text{ times}}, a)\,, \qquad a \in {}^E C^{poly}_*\,, \tag{2.67}$$

2.4.3. Deformation quantization with traces. Given a deformation Π (2.57) associated with a Poisson Lie algebroid $(E, [\,,\,], \rho, \pi)$ one can define a *trace* of the deformation Π as an $\mathbb{R}[[\hbar]]$-linear functional

$$tr : \mathcal{O}(M)[[\hbar]] \to \mathbb{R}[[\hbar]] \tag{2.68}$$

satisfying the following condition

$$tr(j(\Pi) - j(t(\Pi))) = 0\,, \qquad \forall\ j \in \Gamma(M, {}^E J^{poly}_1) \cap \ker \nabla^G. \tag{2.69}$$

It is not hard to see that Proposition 2.24 implies the following statement:

COROLLARY 2.26. *Let Π be a deformation associated with a Poisson Lie algebroid $(E, [\,,\,], \rho, \pi)$ and let $\pi_\hbar$ be a representative of the class of this deformation. Then the vector space of traces of the deformation Π is isomorphic to the vector space of continuous $\mathbb{R}[[\hbar]]$-linear $\mathbb{R}[[\hbar]]$-valued functionals on $\mathcal{O}(M)[[\hbar]]$ vanishing on all functions $f \in \mathcal{O}(M)[[\hbar]]$ of the following form*

$$f = j(\pi_\hbar)\,, \qquad j \in \Gamma(M, {}^E J^{poly}_1) \cap \ker \nabla^G\,,$$

where $\pi_\hbar$ is viewed as a series E-bidifferential operators. □

REMARK 2.27. One can verify without difficulty that all our constructions still make sens (and all our results remain true) if we replace $\mathcal{O}(M)$ by the algebra of compactly supported functions $\mathcal{O}_c(M)$ (this can be usefull because in the case M is not compact, it can happen that there is no traces on $\mathcal{O}(M)$).

2.5. Further developments

2.5.1. Compatibility with cup-products on tangent cohomology. Let Π be a deformation associated with a Poisson Lie algebroid $(E, [,], \rho, \pi)$.

On one hand, recall from [**39**] that the tangent cochain complex associated to Π (called Hochschild cochain complex associated to Π in the previous section) can be equipped with an associative product $\cup_\Pi$ compatible with the differential and defined as follows

$$P \cup_\Pi Q = \Pi^{1\dots k,k+1\dots k+l}(P \otimes Q) \qquad P \in {}^E D^{k-1}_{poly}[[\hbar]], Q \in {}^E D^{l-1}_{poly}[[\hbar]] \tag{2.70}$$

On the other hand, the $\wedge$-product endows the Poisson cochain complex $({}^E T^*_{poly}[[\hbar]], [\pi_\hbar,\cdot])$ with the structure of a DG commutative algebra. It is conjectured that at the level of cohomology, the products $\wedge$ and $\cup_\Pi$ coincide:

CONJECTURE 2.28. *The (graded commutative) algebras* $\big(H^*({}^E T^*_{poly}, [\pi_\hbar,\cdot]),\ \wedge\ \big)$ *and* $\big(H^*({}^E D^*_{poly}, [\Pi,\cdot]_G),\ \cup_\Pi\ \big)$ *are isomorphic.*

This result was announced in [**39**, Section 8.2], where it is proved in the case $E = T\mathbb{R}^d$ (for a proof with details we refer to [**46**]). The global situation $E = TM$ has been treated (in [**9**]) only for the 0-th cohomology. Let us prove it for any Poisson Lie algebroid. Namely, assume that Π is the image of some $\pi_\hbar$ by the map (2.63), then

THEOREM 2.29 ([**9**]). *The algebras* $\big(H^0({}^E T^*_{poly}, [\pi_\hbar,\cdot]), \wedge\big)$ *and* $\big(H^0({}^E D^*_{poly}, [\Pi,\cdot]_G), \cup_\Pi\big)$ *are isomorphic.*

We give here a new proof of this theorem, which we hope to make work in the future for the whole cohomology.

PROOF. First observe that it follows directly from Theorem 2.13 and Proposition 2.14 that the inverse

$$\lambda_T : \big({}^E T^*_{poly}[[\hbar]], [\pi_\hbar,\cdot],\ \wedge\ \big) \longrightarrow \big({}^E\Omega(\mathcal{T})[[\hbar]], D + [\lambda_T(\pi_\hbar),\cdot],\ \wedge\ \big)$$

of the map $\mathcal{H}$ (2.37) defines a morphism of DG commutative algebras and induces an isomorphism in cohomology. In particular it restricts to an isomorphism of algebras

$${}^E T^{-1}_{poly}[[\hbar]] \cap \ker([\pi_\hbar,\cdot]) \longrightarrow \mathcal{T}^{-1}[[\hbar]] \cap \ker(D + [\lambda_T(\pi_\hbar),\cdot]) \tag{2.71}$$

In the same way λ_D restricts to an isomorphism of algebras

$$\big({}^E D^{-1}_{poly}[[\hbar]] \cap \ker([\Pi,\cdot]_G),\ \cup_\Pi\ \big) \longrightarrow \big(\mathcal{D}^{-1}[[\hbar]] \cap \ker(D + [\lambda_D(\Pi),\cdot]_G),\ \cup_{\lambda_D(\Pi)}\ \big) \tag{2.72}$$

Consider the element

$$\widetilde{\Pi} = 1 \otimes 1 + \sum_{n\geq 1} \frac{1}{n!} (\mathcal{K}^{tw})^{[n]}\big(\lambda_T(\pi_\hbar), \dots, \lambda_T(\pi_\hbar)\big)$$

of which the part of zero exterior degree is

$$\overline{\Pi} = 1 \otimes 1 + \sum_{n\geq 1} \frac{1}{n!} \mathcal{K}^{[n]}\big(\lambda_T(\pi_\hbar), \dots, \lambda_T(\pi_\hbar)\big)\,.$$

Then one has the quasi-isomorphism of complexes

$$\big((\mathcal{K}^{tw})^{\lambda(\pi_\hbar)}\big)^{[1]} : \big({}^E\Omega(\mathcal{T})[[\hbar]], D + [\lambda_T(\pi_\hbar),\cdot]_{SN}\big) \longrightarrow \big({}^E\Omega(\mathcal{D})[[\hbar]], D + [\widetilde{\Pi},\cdot]_G\big)$$

In particular it restricts to an isomorphism (of vector spaces)

$$\big((\mathcal{K}^{tw})^{\lambda(\pi_\hbar)}\big)^{[1]} : \mathcal{T}^{-1}[[\hbar]] \cap \ker(D + [\lambda_T(\pi_\hbar),\cdot]_{SN}) \longrightarrow \mathcal{D}^{-1}[[\hbar]] \cap \ker(D + [\widetilde{\Pi},\cdot]_G) \tag{2.73}$$

Due to degree considerations[3] $\big((\mathcal{K}^{tw})^{\lambda(\pi_\hbar)}\big)^{[1]} = (\mathcal{K}^{\lambda(\pi_\hbar)})^{[1]}$ on $\mathcal{T}^{-1}$. Then using the fact that $\mathcal{K}$ is compatible with cup-products [**39, 46**] we obtain that the map (2.73) sends the usual product of functions to the product $\cup_{\overline{\Pi}}$.

Finally observe that since $\lambda_D(\Pi)$ and $\widetilde{\Pi}$ are gauge equivalent one has an isomorphism

$$(\mathcal{D}^{-1}[[\hbar]] \cap \ker(D + [\widetilde{\Pi},\cdot]_G),\, \cup_{\overline{\Pi}}) \longrightarrow (\mathcal{D}^{-1}[[\hbar]] \cap \ker(D + [\lambda_D(\Pi),\cdot]_G),\, \cup_{\lambda_D(\Pi)}), \tag{2.74}$$

and composing the inverse of (2.72) with (2.74), (2.73) and (2.71) we prove the theorem. □

2.5.2. Compatibility with cup-products on tangent homology. Let Π be a deformation associated with a Poisson Lie algebroid $(E, [,], \rho, \pi)$.

On one hand, recall from [**58**] that the tangent chain complex associated to Π (called Hochschild chain complex associated to Π in the previous section) has a module structure $\bullet_\Pi$ over the tangent cochain complex (equipped with the cup-product $\cup_\Pi$) compatible with the differential and defined as follows

$$\begin{aligned} (P \bullet_\Pi j)(Q) = j\big((1 \cup_\Pi P) \otimes Q)\big) \qquad & P \in {}^E D^{k-1}_{poly}[[\hbar]], Q \in {}^E D^{l-1}_{poly}[[\hbar]], \\ & j \in {}^E J^{poly}_{k+l}[[\hbar]] \cap \ker \nabla^G \cong {}^E C^{poly}_{k+l-1}[[\hbar]]\,. \end{aligned} \tag{2.75}$$

On the other hand, the contraction of E-forms with E-polyvector fields endows the chain complex $({}^E A_*[[\hbar]], {}^E L_{\pi_\hbar})$ with a structure of a module over $({}^E T^*_{poly}[[\hbar]], [\pi_\hbar,\cdot], \wedge)$. It is conjectured that at the level of cohomology these module structures coincide. It has already been proved in degree 0 for the case $E = T\mathbb{R}^d$ (see [**60**]).

Since our construction is compatible with many algebraic structures (see Remark 2.15) one can expect to prove this compatibility at least in degree 0 for a general Poisson Lie algebroid.

2.5.3. Characteristic classes and cyclic formality. Let us mention that it would be interesting to prove the corresponding version of the algebraic index theorem [**50**], [**62**], which should relate a cyclic chain in the complex associated with a deformation Π to its principal part and characteristic classes of the Lie algebroid (E, M, ρ). It would be also interesting to investigate how other characteristic classes [**12**], [**30**], [**43**] of Lie algebroids could enter this picture.

Let us finally observe that the compatibilities with cup-products on tangent cohomology and homology, and the Lie algebroid version of the algebraic index theorem is a part of more general conjecture. Namely, the pair $({}^E T^*_{poly}, {}^E A_*)$ is a calculus [**13**], and it appears that at the level of complexes $({}^E D^*_{poly}, {}^E C^{poly}_*)$ is a calculus *up to homotopy* (see [**64**]).

The conjecture states that these two pairs are quasi-isomorphic as calculi up to homotopy.

3. Namely, $\mathcal{K}(B, \lambda(\pi_\hbar), \dots, \lambda(\pi_\hbar), f) \in {}^E\Omega^1(\mathcal{D}^{-2}) = \{0\}$.

CHAPITRE 3

Formality theorems for holomorphic Lie algebroids

RÉSUMÉ. Ce chapitre est consacré à la généralisation des résultats du précédent chapitre au cadre holomorphe. Ici la formulation des résultats en termes de faisceaux prends toute sont importance. On discute en particulier de la quantification des variétés de Poisson holomorphes. Les trois premières sections de ce chapitre sont tirées de [**7**, Section 4].

It is known ([**40, 50**]) that there are serious obstructions to deformation quantization of holomorphic (or algebraic) Poisson manifolds, and thus *a posteriori* of holomorphic Poisson Lie algebroids. Nevertheless, we prove in this chapter that the sheaf of holomorphic E-(co)chains, where E is a holomorphic Lie algebroid, is formal as a sheaf of DGLA. But unlike C^∞ sheaves, holomorphic sheaves possess nontrivial higher cohomology and thus this formality theorem does NOT imply that the DGLA of its global sections is formal.

Despite this fact we can still apply this theorem to deformation quantization and prove that any holomorphic Poisson Lie algebroid is weakly quantizable. Here weakly means broadly that we enlarge (in a reasonable way) the category in which we allow the deformation to hold [**40, 14, 55**].

This chapter is organized as follows. In Section 1 we summarize and adapt the constructions of Chapter 1 for holomorphic Lie algebroids. Sections 2 and 3 are respectively devoted to the statement and the proof of the analogue of Theorem 2.1 for holomorphic Lie algebroids; the proof goes essentially along the same lines as in Chapter 2 except that the Dolbeault differential $\bar{d}$ enter the game. In Section 4 we apply this formality theorem to deformation quantization theory in the holomorphic context: here we need a notion of weak deformation. In the case of a holomorphic Poisson manifold we allow to deform not only the product of functions but also the gluing conditions ... then the object that we obtain is no longer a sheaf but something called an *algebroid stack* in [**40**].

3.1. Holomorphic Lie algebroids

Let now M be a complex manifold. Let us write $TM = T^{1,0}M \oplus T^{0,1}M$ for the decomposition of the tangent bundle as the sum of the holomorphic tangent bundle and anti-holomorphic tangent bundle. We denote by $\mathcal{O}_M$ the structure sheaf of holomorphic functions on M and by z^α local coordinates on M. We have to adapt the definition of holomorphic Lie algebroids:

DEFINITION 3.1. *A* holomorphic Lie algebroid *over a complex manifold M is a holomorphic vector bundle E of finite rank whose sheaf of sections is a sheaf of Lie algebras equipped with a holomorphic map of sheaves of Lie algebras*

$$\rho : E \to T^{1,0} .$$

satisfying the same conditions described (for the smooth case) in formula (1.1).

In other words, a holomorphic Lie algebroid is a holomorphic vector bundle whose sheaf of sections is a sheaf of Lie-Rinehart algebras over $\mathcal{O}_M$.

Let E be a holomorphic Lie algebroid. As in chapter 1, one can define the following sheaves (which are also holomorphic vector bundles):

- ${}^E T^*_{poly}$ is the sheaf of E-polyvector fields. We regard ${}^E T^*_{poly}$ as a sheaf of DGLA with the vanishing differential and with the Lie bracket $[,]_{SN}$ defined as in (1.3), (1.4).
- ${}^E A_*$ is the sheaf of E-differential forms with converted grading:

$$ {}^E A_* = \wedge^{-*} E^\vee , \qquad {}^E A_0 = \mathcal{O}_M . \tag{3.1}$$

We regard ${}^E A_*$ as a sheaf of DGLA modules over ${}^E T^*_{poly}$ with the vanishing differential and with the action ${}^E L$ defined as in (1.6). For sections

$$ a = \sum_{m \geq 0} a_{i_1 \dots i_m}(z) dy^{i_1} \dots dy^{i_m} \tag{3.2}$$

of the sheaf ${}^E A_*$ we reserve the basis of local E-1-forms $\{dy^i\}$, where y^i are fiber coordinates on E.

- ${}^E D^*_{poly}$ is a sheaf of E-polydifferential operators. We regard ${}^E D^*_{poly}$ as a sheaf of DGLA with the bracket $[,]_G$ and the differential ∂ defined as in (1.24) and (1.25). Notice that the tensor product of sections (over $\mathcal{O}_M$) of ${}^E D^*_{poly}$ turns ${}^E D^*_{poly}$ into a sheaf of DGAA.
- ${}^E J^{poly}_*$ is the sheaf of E-polyjets

$$ {}^E J^{poly} = \bigoplus_{k \geq 0} {}^E J^{poly}_k , \qquad {}^E J^{poly}_* := Hom_{\mathcal{O}_M}({}^E D^*_{poly}, \mathcal{O}_M) , $$

which we regard as a sheaf of DGLA modules over ${}^E D^*_{poly}$ with the action ${}^E S$ and the differential $\mathfrak{b}$ defined as in (1.31) and (1.35). The sheaf ${}^E J^{poly}_*$ is also equipped with the Grothendieck connection

$$ \nabla^G : T^{1,0} \otimes {}^E J^{poly}_* \mapsto {}^E J^{poly}_* , \qquad \nabla^G_u(j)(P) := \rho(u)(j(P)) - j(u \bullet P) , \tag{3.3}$$

where $u \in \Gamma(T^{1,0})$ is a holomorphic vector field, $P \in \Gamma({}^E D^k_{poly})$, $j \in \Gamma({}^E J^{poly}_k)$ and the operation $\bullet$ is defined in (1.23). The connection (3.3) is compatible the DGLA module structure on ${}^E J^{poly}_*$.

- ${}^E C^{poly}_*$ is the graded sheaf of ∇^G-flat E-polyjets with converted grading

$$ {}^E C^{poly}_* := \ker \nabla^G \cap {}^E J^{poly}_{-*} . \tag{3.4}$$

Due to the compatibility of the Grothendieck connection (3.3) with the DGLA module structure on E-polyjets ${}^E C^{poly}_*$ can be viewed as a sheaf of DG modules over sheaf of DGLA ${}^E D^*_{poly}$. We refer to ${}^E C^{poly}_*$ as a sheaf of Hochschild E-chains or E-chains for short.

3.2. Statement of the formality theorem for holomorphic Lie algebroids

The main result of this chapter can be formulated as follows:

THEOREM 3.2. *For any holomorphic Lie algebroid E over a complex manifold M the sheaves of DGLA modules $({}^E T^*_{poly}, {}^E A_*)$ and $({}^E D^*_{poly}, {}^E C^{poly}_*)$ are quasi-isomorphic.*

Omitting the sheaves of DGLA modules ${}^E A_*$ and ${}^E C^{poly}_*$ in the above theorem we get the following corollary:

COROLLARY 3.3. *For any holomorphic Lie algebroid E over a complex manifold M the sheaves of DGLA ${}^E T^*_{poly}$ and ${}^E D^*_{poly}$ are L_∞-quasi-isomorphic.* □

We would like to mention that this corollary is parallel to the result of A. Yekutieli [**73**], who proved this statement for the tangent Lie algebroid $TM \to M$ of any smooth algebraic variety over a field $\mathbf{k}$ for which $\mathbb{R} \subset \mathbf{k}$.

Notice that applying Theorem 3.2 to the tangent algebroid $T^{1,0}M \to M$ we prove the following version of Tsygan's formality conjecture for complex manifolds:

THEOREM 3.4. *For any complex manifold M the sheaf of DGLA modules $C^{poly}(M)$ of Hochschild chains over the sheaf $D_{poly}(M)$ of (holomorphic) polydifferential operators is formal.* □

The proof of Theorem 3.2 occupies the following section.

3.3. Proof of the formality theorem

First, we observe that any holomorphic Lie algebroid E can be viewed as a smooth Lie algebroid in the sense of Definition 1.1, where the anchor map is naturally extended to the smooth sections of E. It is clear that the sheaf of Lie algebras $T^{0,1}$ acts on E and that this action commutes with ρ as ρ is holomorphic. Thus we get

PROPOSITION 3.5. *Let F be the smooth vector bundle $F = E \oplus T^{0,1}$. Then F is a smooth Lie algebroid over M with the anchor map $\rho_F\colon F \to T^{1,0} \oplus T^{0,1}$ given by $\rho_F\Big|_E = \rho$ and $\rho_F\Big|_{T^{0,1}} = \mathrm{id} : T^{0,1} \to T^{0,1}$.* □

For a holomorphic vector bundle $\mathcal{B}$ over M we consider the sheaf of smooth F-differential forms with values in $\mathcal{B}$:

$$ {}^F\Omega(\mathcal{B}) = \bigoplus_{p,q} {}^F\Omega^{p,q}(\mathcal{B}), \tag{3.5} $$

$$ {}^F\Omega^{p,q}(\mathcal{B}) = \wedge^p E^\vee \otimes \wedge^q T^{*0,1}M \otimes \mathcal{B} $$

For sections

$$ a = \sum_{p,q} a_{i_1 \dots i_p\,;\,\alpha_1,\dots,\,\alpha_q}(z,\bar{z})\xi^{i_1}\dots\xi^{i_p} d\bar{z}^{\alpha_1}\dots d\bar{z}^{\alpha_q}, \tag{3.6} $$

$$ a_{i_1 \dots i_p\,;\,\alpha_1,\dots,\,\alpha_q}(z,\bar{z}) \in \Gamma^{\mathrm{smooth}}(\mathcal{B}) $$

of ${}^F\Omega(\mathcal{B})$ we reserve the local basis $\{\xi^i\}$ of anti-commuting fiber coordinates on E and the local basis $\{d\bar{z}^\alpha\}$ of antiholomorphic exterior forms on M. We denote by $\bar{d}$ the Dolbeault differential

$$ \bar{d} = d\bar{z}^\alpha \partial_{\bar{z}^\alpha} : {}^F\Omega^{p,*}(\mathcal{B}) \mapsto {}^F\Omega^{p,*+1}(\mathcal{B}). \tag{3.7} $$

It is obvious that the (DG) algebraic structures on the sheaves ${}^E T^*_{poly}$, ${}^E A_*$, ${}^E D^*_{poly}$, and ${}^E J^{poly}_*$, can be naturally extended to the sheaves ${}^F\Omega^{0,*}({}^E T^*_{poly})$, ${}^F\Omega^{0,*}({}^E A_*)$, ${}^F\Omega^{0,*}({}^E D^*_{poly})$, and ${}^F\Omega^{0,*}({}^E J^{poly}_*)$. Similarly, the Grothendieck connection (3.3) on ${}^E J^{poly}_*$ extends to the operator

$$ \nabla^G : T^{1,0} \otimes {}^F\Omega^{0,*}({}^E J^{poly}_*) \mapsto {}^F\Omega^{0,*}({}^E J^{poly}_*), \tag{3.8} $$

which is compatible with the action ${}^E S$ of ${}^F\Omega^{0,*}({}^E D^*_{poly})$ on ${}^F\Omega^{0,*}({}^E J^{poly}_*)$ and with the differential $\mathfrak{b}$ on ${}^F\Omega^{0,*}({}^E J^{poly}_*)$.

Since ${}^E T^*_{poly}$, ${}^E A_*$, ${}^E D^*_{poly}$, and ${}^E J^{poly}_*$ are holomorphic vector bundles it makes sense to speak about the Dolbeault differential (3.7)

$$ \bar{d}\colon {}^F\Omega^{0,*}(\mathcal{B}) \mapsto {}^F\Omega^{0,*+1}(\mathcal{B}), \tag{3.9} $$

for $\mathcal{B}$ being either ${}^E T^*_{poly}$, ${}^E A_*$, ${}^E D^*_{poly}$, or ${}^E J^{poly}_*$. It is obvious that $\bar{d}$ is compatible with the (DG) algebraic structures on ${}^F\Omega^{0,*}(\mathcal{B})$ and with the Grothendieck connection (3.8) on ${}^F\Omega^{0,*}({}^E J^{poly}_*)$.

Furthermore, due to the $\bar{d}$-Poincaré lemma we have

PROPOSITION 3.6. *If $\mathcal{B}$ is either ${}^{E}T^{*}_{poly}$, ${}^{E}A_{*}$, ${}^{E}D^{*}_{poly}$, or ${}^{E}J^{poly}_{*}$ then the canonical inclusion of sheaves*

$$\text{inc} : \mathcal{B} \hookrightarrow {}^{F}\Omega^{0,*}(\mathcal{B}) \tag{3.10}$$

is a quasi-isomorphism of complexes of sheaves $(\mathcal{B},0)$ and $({}^{F}\Omega^{0,*}(\mathcal{B}),\bar{d})$. The inclusion inc *is compatible with the (DG) algebraic structures on $\mathcal{B}$, and ${}^{F}\Omega^{0,*}(\mathcal{B})$, and with the Grothendieck connection (3.3), (3.8).* □

Due to this proposition it now suffices to prove that the sheaves of DGLA modules $({}^{F}\Omega^{0,*}({}^{E}T^{*}_{poly}), {}^{F}\Omega^{0,*}({}^{E}A_{*}))$, and $({}^{F}\Omega^{0,*}({}^{E}D^{*}_{poly}), {}^{F}\Omega^{0,*}({}^{E}J^{poly}_{*}))$ are quasi-isomorphic. To show this we follow the lines of Section 2 and introduce the formally completed symmetric algebra $\hat{S}(E^{\vee})$ of the dual bundle $E^{\vee}$ and (holomorphic) bundles $\mathcal{T}$, $\mathcal{D}$, $\mathcal{A}$, $\mathcal{J}$ associated with $\hat{S}(E^{\vee})$ (see page 24). As in Section 2, $\mathcal{T}$ and $\mathcal{D}$ are sheaves of DGLA while $\mathcal{A}$ and $\mathcal{J}$ are sheaves of DGLA modules over $\mathcal{T}$ and $\mathcal{D}$, respectively. $\mathcal{D}$ is also a sheaf of DGAA.

Next, we consider sheaves of smooth F-differential forms with values in the bundles $\hat{S}(E^{\vee})$ $\mathcal{T}$, $\mathcal{D}$, $\mathcal{A}$, and $\mathcal{J}$. It is clear that the sheaves ${}^{F}\Omega(\hat{S}(E^{\vee}))$, ${}^{F}\Omega(\mathcal{A})$, ${}^{F}\Omega(\mathcal{T})$, ${}^{F}\Omega(\mathcal{D})$, and ${}^{F}\Omega(\mathcal{J})$ acquire the corresponding (DG) algebraic structures and the Dolbeault differential (3.7) is obviously compatible with these structures.

Furthermore, we have the following obvious analogue of Proposition 2.10

PROPOSITION 3.7. *The sheaf ${}^{F}\Omega(\mathcal{T}^{0})$ of F-forms with values in fiberwise vector fields is a sheaf of graded Lie algebras. The sheaves ${}^{F}\Omega(\hat{S}(E^{\vee}))$, ${}^{F}\Omega(\mathcal{A})$, ${}^{F}\Omega(\mathcal{T})$, ${}^{F}\Omega(\mathcal{D})$, and ${}^{F}\Omega(\mathcal{J})$ are sheaves of modules over ${}^{F}\Omega(\mathcal{T}^{0})$ and the action of ${}^{F}\Omega(\mathcal{T}^{0})$ is compatible with the DG algebraic structures on ${}^{F}\Omega(\hat{S}(E^{\vee}))$, ${}^{F}\Omega(\mathcal{A})$, ${}^{F}\Omega(\mathcal{T})$, ${}^{F}\Omega(\mathcal{D})$, ${}^{F}\Omega(\mathcal{J})$ and with the Dolbeault differential (3.7).* □

Due to this proposition one can extend the following differential

$$\delta := \xi^{i}\frac{\partial}{\partial y^{i}} : {}^{F}\Omega^{*,q}(\hat{S}(E^{\vee})) \to {}^{F}\Omega^{*+1,q}(\hat{S}(E^{\vee}))$$

of the sheaf of algebras ${}^{F}\Omega(\hat{S}(E^{\vee}))$ to the sheaves ${}^{F}\Omega(\mathcal{T})$, ${}^{F}\Omega(\mathcal{D})$, ${}^{F}\Omega(\mathcal{A})$ and ${}^{F}\Omega(\mathcal{J})$ so that δ is compatible with the (DG) algebraic structures on ${}^{F}\Omega(\mathcal{T})$, ${}^{F}\Omega(\mathcal{A})$, ${}^{F}\Omega(\mathcal{D})$, and ${}^{F}\Omega(\mathcal{J})$, and with the differential $\bar{d}$ (3.7). Here $\{y^{i}\}$ (resp. $\{\xi^{i}\}$) denote commuting (resp. anticommuting) fiber coordinates of the bundle E.

We now have an analogue of Proposition 2.11

PROPOSITION 3.8. *For $\mathcal{B}$ being either of the sheaves $\hat{S}(E^{\vee})$, $\mathcal{A}$, $\mathcal{T}$ or $\mathcal{D}$ and $q \geq 0$,*

$$H^{\geq 1}({}^{F}\Omega^{*,q}(\mathcal{B}),\delta) = 0\,.$$

Furthermore,

$$\begin{aligned} H^{0}({}^{F}\Omega^{*,q}(\hat{S}(E^{\vee})),\delta) &\cong {}^{F}\Omega^{0,q}(\mathcal{O}_{M})\,,\\ H^{0}({}^{F}\Omega^{*,q}(\mathcal{A}_{*}),\delta) &\cong {}^{F}\Omega^{0,q}({}^{E}A_{*})\,,\\ H^{0}({}^{F}\Omega^{*,q}(\mathcal{T}^{*}),\delta) &\cong {}^{F}\Omega^{0,q}(\wedge^{*+1}(E)) \end{aligned} \tag{3.11}$$

as sheaves of (graded) commutative algebras and

$$H^{0}({}^{F}\Omega^{*,q}(\mathcal{D}^{*}),\delta) \cong {}^{F}\Omega^{0,q}(\otimes^{*+1}(S(E))) \tag{3.12}$$

as sheaves of DGAA over $\mathcal{O}_{M}$.

PROOF. As in Proposition 2.11 is suffices to construct an operator ($q \geq 0$)

$$\kappa : {}^{F}\Omega^{*,q}(\mathcal{B}) \to {}^{F}\Omega^{*-1,q}(\mathcal{B}) \tag{3.13}$$

such that for any section u of ${}^F\Omega(\mathcal{B})$ equation

$$u = \delta\kappa(u) + \kappa\delta(u) + \mathcal{H}(u)\,, \tag{3.14}$$

is still true, where now

$$\mathcal{H}(u) = u\Big|_{y^i=\xi^i=0}\,. \tag{3.15}$$

and y^i are as above fiber coordinates on E. As in the proof of Proposition 2.11 we define κ on ${}^F\Omega(\hat{S}(E^\vee))$ by equation (2.14) and then extend it to ${}^F\Omega(\mathcal{T})$, ${}^F\Omega(\mathcal{A})$, and ${}^F\Omega(\mathcal{D})$ in the componentwise manner. □

3.3.1. Fedosov resolutions. Let us choose a connection ∇ on E which is compatible with the complex structure on E, locally

$$\nabla = {}^Ed + \bar{d} + \xi^i\Gamma_i : {}^F\Omega^*(E) \to {}^F\Omega^{*+1}(E)\,, \tag{3.16}$$

where $\xi^i\Gamma_i$ is locally a section of the sheaf ${}^E\Omega^1(End(E))$ and ${}^Ed : {}^F\Omega^{*,q}_M \to {}^F\Omega^{*+1,q}_M$ is defined in (1.5).

It is not hard to show that such a connection always exists, and moreover, one can always choose ∇ to be torsion free (one can use partition of unity, like in the proof of Proposition 1.9, since ${}^F\Omega(E)$ is a sheaf of C^∞ sections).

As in (2.18) we extend (3.16) to a derivation of the DG sheaves ${}^F\Omega(\hat{S}(E^\vee))$, ${}^F\Omega(\mathcal{A})$, ${}^F\Omega(\mathcal{T})$, ${}^F\Omega(\mathcal{D})$, and ${}^F\Omega(\mathcal{J})$:

$$\nabla = {}^Ed + \Gamma\cdot + \bar{d} : {}^F\Omega^*(\mathcal{B}) \to {}^F\Omega^{*+1}(\mathcal{B})\,, \tag{3.17}$$

where $\mathcal{B}$ is either of the sheaves $\hat{S}(E^\vee)$, $\mathcal{A}$, $\mathcal{T}$, $\mathcal{D}$, or $\mathcal{J}$, $\Gamma = -\xi^i\Gamma^k_{ij}y^j\frac{\partial}{\partial y^k}$, $\Gamma^k_{ij}(x)$ are Christoffel's symbols of the connection (3.16) and $\Gamma\cdot$ denotes the action of Γ on the sections of the sheaves ${}^F\Omega(\mathcal{B})$. Due to Proposition 3.7 ∇ (3.17) is compatible with the DG algebraic structures on ${}^F\Omega(\hat{S}(E^\vee))$, ${}^F\Omega(\mathcal{T})$, ${}^F\Omega(\mathcal{A})$, ${}^F\Omega(\mathcal{D})$, and ${}^F\Omega(\mathcal{J})$, and since ∇ is torsion free

$$\nabla\delta + \delta\nabla = 0\,. \tag{3.18}$$

Regarding (3.17) as a connection on $\mathcal{B}$ one can see that the curvature of (3.17) has the components of type (2,0) and (1,1)

$$\nabla^2 = R^{2,0} + R^{1,1}\,, \qquad R^{2,0} = ({}^Ed + \Gamma)^2, \qquad R^{1,1} = \bar{d}\Gamma\,. \tag{3.19}$$

We now prove the existence of a complex Fedosov differential D:

THEOREM 3.9 ([**7**]). *Let $\mathcal{B}$ be either of the sheaves $\hat{S}(E^\vee)$, $\mathcal{A}$, $\mathcal{T}$, $\mathcal{D}$, or $\mathcal{J}$. There exists a section*

$$A = \sum_{s=2}^{\infty}\xi^k A^j_{k,i_1\dots i_s}(z,\bar{z})y^{i_1}\cdots y^{i_s}\frac{\partial}{\partial y^j} \tag{3.20}$$

of the sheaf ${}^F\Omega^{1,0}(\mathcal{T}^0)$ and a section

$$\bar{A} = \sum_{s=2}^{\infty}d\bar{z}^\alpha \bar{A}^j_{\alpha,i_1\dots i_s}(z,\bar{z})y^{i_1}\cdots y^{i_s}\frac{\partial}{\partial y^j} \tag{3.21}$$

of the sheaf ${}^F\Omega^{0,1}(\mathcal{T}^0)$ such that the derivation

$$D := \nabla - \delta + A\cdot + \bar{A}\cdot : {}^F\Omega^*(\mathcal{B}) \to {}^F\Omega^{*+1}(\mathcal{B}) \tag{3.22}$$

is 2-nilpotent ($D^2 = 0$) and compatible with the DG algebraic structure on ${}^F\Omega(\mathcal{B})$.

PROOF. Let us rewrite $D = D^{1,0} + D^{0,1}$ with

$$D^{1,0} = {}^E d + \Gamma \cdot - \delta + A \cdot , \qquad D^{0,1} = \bar{d} + \bar{A} \cdot$$

and try to mimic the proof of Theorem 2.12.

Due to (3.18) and (3.19) the condition $(D^{1,0})^2 = 0$ is equivalent to the equation

$$R^{2,0} + ({}^E d + \Gamma \cdot)A - \delta A + \frac{1}{2}[A, A]_{SN} = 0 .$$

This equation has a solution obtained by iterations of the following equation (with respect to the degrees in fiber coordinates y_i's)

$$A = \kappa R^{2,0} + \kappa(({}^E d + \Gamma \cdot)A + \frac{1}{2}[A, A]_{SN})$$

(the proof is the same as for Theorem 2.12).

Using (3.19) once again we observe that the condition $D^{1,0}D^{0,1} + D^{0,1}D^{1,0} = 0$ is equivalent to

$$R^{1,1} + \bar{d}A + ({}^E d + \Gamma \cdot)\bar{A} - \delta \bar{A} + [A, \bar{A}]_{SN} = 0 ,$$

which, using the same arguments, has a solution obtained by iterations of the equation

$$\bar{A} = \kappa(R^{1,1} + \bar{d}A + ({}^E d + \Gamma \cdot)\bar{A} + [A, \bar{A}]_{SN}).$$

Indeed, denoting

$$C^{1,1} = R^{1,1} + \bar{d}A + ({}^E d + \Gamma \cdot)\bar{A} - \delta \bar{A} + [A, \bar{A}]_{SN} ,$$

and using that $\delta A = R^{2,0} + {}^E d + \Gamma \cdot A + \frac{1}{2}[A, A]_{SN}$ ($(D^{1,0})^2 = 0$), $\bar{d}R^{2,0} = 0$ and $\delta R^{1,1} = 0$ (Bianchi's identities for ∇) we get

$$({}^E d + \Gamma \cdot)C^{1,1} - \delta C^{1,1} + [A, C^{1,1}] = 0.$$

We have $\kappa C^{1,1} = 0$ by construction of $\bar{A}$ and so, by the "Hodge-de Rham" decomposition (3.14), we have

$$C^{1,1} = \kappa(({}^E d + \Gamma \cdot)C^{1,1} + [A, C^{1,1}]).$$

The latter equation has the unique vanishing solution, which gives the result.

Let us now check the condition $(D^{0,1})^2 = 0$. This will be true if the section

$$C^{0,2} = \bar{d}\bar{A} + \frac{1}{2}[\bar{A}, \bar{A}] \in {}^F\Omega^{0,2}(\mathcal{T}^0)$$

is zero. One has again $D^{1,0}C^{0,2} = 0$ and $\kappa C^{0,2} = 0$ because it does not have ξ's. As before, one can conclude that $C^{0,2} = 0$.

The compatibility of (3.22) with the corresponding DG algebraic structures follows from Proposition 3.7. □

We now describe the cohomology of the Fedosov differential D for the sheaves ${}^F\Omega(\hat{S}(E^\vee))$, ${}^F\Omega(\mathcal{A})$, ${}^F\Omega(\mathcal{T})$, and ${}^F\Omega(\mathcal{D})$

THEOREM 3.10 ([7]). *Let $\mathcal{B}$ be either of the sheaves $\hat{S}(E^\vee)$, $\mathcal{A}$, $\mathcal{T}$, or $\mathcal{D}$ and $q \geq 0$. We have*

$$H({}^F\Omega^*(\mathcal{B}), D) \cong H({}^F\Omega^{0,*}(\mathcal{B}) \cap \ker \delta, \bar{d}) .$$

as sheaves of (differential) graded (commutative) algebras.

PROOF. Let us consider the double complex $({}^F\Omega^{*,*}(\mathcal{B}), D^{1,0} + D^{0,1})$. Using the degree in the fiber coordinates y^i we introduce on this complex a decreasing filtration. Applying the spectral sequence argument (as in the proof of Theorem 2.13) and using Proposition 3.8 we conclude that for any $i \geq 0$, the cohomology of the complex $({}^F\Omega^{*,i}(\mathcal{B}), D^{1,0})$ is concentrated in degree $* = 0$. Therefore,

$$H({}^F\Omega^*(\mathcal{B}), D) = H({}^F\Omega^{0,*}(\mathcal{B}) \cap \ker D^{1,0}, D^{0,1}) . \tag{3.23}$$

Following the lines of the proof of Theorem 2.13 it is not hard to show that iterating the equation

$$\lambda(u) = u + \kappa(\nabla\lambda(u) + A\cdot\lambda(u) + \bar{A}\cdot\lambda(u)), \qquad u \in {}^{F}\Omega^{0,q}(\mathcal{B}) \cap \ker\delta \tag{3.24}$$

we get an isomorphism of sheaves (of graded vector spaces)

$$\lambda : {}^{F}\Omega^{0,q}(\mathcal{B}) \cap \ker\delta \to {}^{F}\Omega^{0,q}(\mathcal{B}) \cap \ker D^{1,0}\,, \tag{3.25}$$

and moreover, the map λ (3.25) has a natural inverse given by the map $\mathcal{H}$ (3.15).

We claim that λ gives a quasi-isomorphism of complexes

$$\lambda : ({}^{F}\Omega^{0,*}(\mathcal{B}) \cap \ker\delta, \bar{d}) \to ({}^{F}\Omega^{*}(\mathcal{B}), D).$$

Indeed, due to (3.23) it suffices to show that for any $u \in {}^{F}\Omega^{0,q}(\mathcal{B}) \cap \ker\delta$, one has

$$\lambda(\bar{d}(u)) = D^{0,1}\lambda(u).$$

The term $\lambda(\bar{d}(u))$ is the only element in ${}^{F}\Omega^{0,q}(\mathcal{B})$ such that $\mathcal{H}(\lambda(\bar{d}(u))) = \bar{d}(u)$ and $D^{1,0}\lambda(\bar{d}(u)) = 0$. It is clear that $\mathcal{H}(D^{0,1}\lambda(u)) = \bar{d}(u)$ and one has

$$D^{1,0}D^{0,1}\lambda(u) = -D^{0,1}D^{1,0}\lambda(u) = 0,$$

since map λ (3.24) lands in $\ker D^{1,0}$.

The map λ (3.25) is compatible with the corresponding multiplications in $\hat{S}(E^{\vee})$, $\mathcal{A}$, $\mathcal{T}$, or $\mathcal{D}$ since so is the map $\mathcal{H}$ (3.15). The theorem is proved. □

It is not hard to prove the following analogue of Proposition 2.14:

PROPOSITION 3.11 ([**7**]). *The map*

$$\mathcal{H} : {}^{F}\Omega^{0,*}(\mathcal{T}) \cap \ker D^{1,0} \to {}^{F}\Omega^{0,*}(\mathcal{T}) \cap \ker\delta \cong {}^{F}\Omega^{0,*}({}^{E}T^{*}_{poly}) \tag{3.26}$$

is an isomorphism of the sheaves of DGLA

$$({}^{F}\Omega^{0,*}(\mathcal{T}) \cap \ker D^{1,0}, D^{0,1}, [\,,\,]_{SN}) \cong ({}^{F}\Omega^{0,*}({}^{E}T^{*}_{poly}), \bar{d}, [\,,\,]_{SN}) \tag{3.27}$$

And the map

$$\mathcal{H} : {}^{F}\Omega^{0,*}(\mathcal{A}_*) \cap \ker D^{1,0} \to {}^{F}\Omega^{0,*}(\mathcal{A}_*) \cap \ker\delta \cong {}^{F}\Omega^{0,*}({}^{E}A_*) \tag{3.28}$$

is an isomorphism of the sheaves of DGLA modules

$$({}^{F}\Omega^{0,*}(\mathcal{A}_*) \cap \ker D^{1,0}, D^{0,1}) \cong ({}^{F}\Omega^{0,*}({}^{E}A_*), \bar{d})$$

over the sheaf of DGLA (3.27). □

Thanks to equation (3.23) this proposition implies that the inverse λ_T of the map $\mathcal{H}$ (3.26) defines gives a quasi-isomorphism of the sheaves of DGLA $({}^{F}\Omega^{0,*}({}^{E}T^{*}_{poly}), \bar{d}, [\,,\,]_{SN})$ and $({}^{F}\Omega^{*}(\mathcal{T}), D, [\,,\,]_{SN})$. In the same way the inverse λ_A of the map (3.28) defines a quasi-isomorphism of the sheaves of DGLA modules $({}^{F}\Omega^{0,*}({}^{E}A_*), \bar{d})$ and $({}^{F}\Omega^{*}(\mathcal{A}_*), D)$.

Playing with the PBW theorem for the Lie algebroids (as we did in the proof of Proposition 2.16) and with the cup product in the sheaves $\mathcal{D}$ and ${}^{E}D^{*}_{poly}$ (see equation (2.42)) one can prove the following analogue of Proposition 2.17

PROPOSITION 3.12 ([**7**]). *The exists an isomorphism of the sheaves of DGLA*

$$\mu' : ({}^{F}\Omega^{0,*}({}^{E}D^{*}_{poly}), \bar{d}, [\,,\,]_{G}) \xrightarrow{\sim} ({}^{F}\Omega^{0,*}(\mathcal{D}) \cap \ker D^{1,0}, D^{0,1}, [\,,\,]_{G})\,, \tag{3.29}$$

which is compatible with the DGAA structures on the sheaves ${}^{F}\Omega^{0,*}({}^{E}D^{*}_{poly})$ *and* ${}^{F}\Omega^{0,*}(\mathcal{D})$. □

Thanks to equation (3.23) this proposition implies that the map μ' (3.29) gives a quasi-isomorphism of the sheaves of DGLA $({}^F\Omega^{0,*}({}^ED^*_{poly}), \bar{d}, [\,,\,]_G)$ and $({}^F\Omega^*(\mathcal{D}), D, [\,,\,]_G)$.

Let us consider the map

$$\gamma : {}^F\Omega^{0,q}(\mathcal{J}_*) \to {}^F\Omega^{0,q}({}^EJ^{poly}_*), \qquad \gamma(j)(P) = (\mu'(P))(j)\Big|_{y^i=0}, \tag{3.30}$$

where $j \in {}^F\Omega^{0,q}(\mathcal{J}_k)$ and P is a holomorphic section of ${}^ED^k_{poly}$.

For this map we have the following obvious analogue of Theorem 2.18

THEOREM 3.13 ([7]). *For any $q \geq 0$*

$$H^q({}^F\Omega^*(\mathcal{J}), D) = H^q({}^F\Omega^{0,*}(\mathcal{J}) \cap \ker D^{1,0}, D^{0,1}). \tag{3.31}$$

and the map γ (3.30) provides us with an isomorphism of the sheaves of DGLA modules

$$\gamma : {}^F\Omega^{0,*}(\mathcal{J}_*) \xrightarrow{\sim} {}^F\Omega^{0,*}({}^EJ^{poly}_*) \tag{3.32}$$

over the sheaf of DGLA

$$({}^F\Omega^{0,*}(\mathcal{D}) \cap \ker D^{1,0}, D^{0,1}, [\,,\,]_G) \cong ({}^F\Omega^{0,*}({}^ED^*_{poly}), \bar{d}, [\,,\,]_G).$$

The map γ sends the component $D^{1,0}$ to the Grothendieck connection (3.8) and the component $D^{0,1}$ to the Dolbeault differential $\bar{d}$ (3.7). □

3.3.2. End of the proof. Thus we have constructed the following maps

- $\lambda_T : ({}^F\Omega^{0,*}(M, {}^ET^*_{poly}), \bar{d}, [\,,\,]_{SN}) \to ({}^F\Omega(\mathcal{T}), D, [\,,\,]_{SN})$,
- $\lambda_A : ({}^F\Omega^{0,*}(M, {}^EA_*), \bar{d}) \to ({}^F\Omega(\mathcal{A}_*), D)$,
- $\lambda_D : ({}^F\Omega^{0,*}(M, {}^ED^*_{poly}), \bar{d}, [\,,\,]_G) \to ({}^F\Omega(\mathcal{D}), D, [\,,\,]_G)$, and
- $\lambda_C : ({}^F\Omega^{0,*}(M, {}^EC^{poly}_*), \bar{d}) \to ({}^F\Omega(\mathcal{J}), D)$.

Namely, the map λ_T is the inverse of (3.26), the map λ_A is the inverse of (3.28), $\lambda_D = \mu'$ (3.29), and λ_C is composition of the identification (3.4) and the inverse of γ (3.30).

Our results can be summarized in the following commutative diagrams

$$\begin{array}{ccc} ({}^F\Omega^{0,*}({}^ET^*_{poly}), \bar{d}, [\,,\,]_{SN}) & \overset{\lambda_T}{\succ\!\!\to} & ({}^F\Omega(\mathcal{T}), D, [\,,\,]_{SN}) \\ \downarrow^{{}^EL}_{mod} & & \downarrow^{L}_{mod} \\ ({}^F\Omega^{0,*}({}^EA_*), \bar{d}) & \overset{\lambda_A}{\succ\!\!\succ\!\!\to} & ({}^F\Omega(\mathcal{A}), D), \end{array} \tag{3.33}$$

$$\begin{array}{ccc} ({}^F\Omega(\mathcal{D}), D + \partial, [\,,\,]_G) & \overset{\lambda_D}{\leftarrow\!\!\prec} & ({}^F\Omega^{0,*}({}^ED^*_{poly}), \bar{d} + \partial, [\,,\,]_G) \\ \downarrow^{\mathcal{R}}_{mod} & & \downarrow^{{}^ER}_{mod} \\ ({}^F\Omega(\mathcal{J}), D + \mathfrak{b}) & \overset{\lambda_C}{\leftarrow\!\!\prec\!\!\prec} & ({}^F\Omega^{0,*}({}^EC^{poly}_*), \bar{d} + \mathfrak{b}), \end{array}$$

where the action ${}^ER$ is obtained from the action ${}^ES$ of ${}^F\Omega^{0,*}(M, {}^ED^*_{poly})$ on ${}^F\Omega^{0,q}(M, {}^EJ^{poly}_*)$ via the identification (3.4).

Due to claims *1* and *2* in Theorem 2.5 and claims *1* and *2* in Theorem 2.6 we get the following commutative diagram

$$\begin{array}{ccc} ({}^F\Omega(\mathcal{T}), 0, [\,,\,]_{SN}) & \overset{\mathcal{K}}{\succ\!\!\to} & ({}^F\Omega(\mathcal{D}), \partial, [\,,\,]_G) \\ \downarrow^{L}_{mod} & & \downarrow^{\mathcal{R}}_{mod} \\ ({}^F\Omega(\mathcal{A}), 0) & \overset{\mathcal{S}}{\leftarrow\!\!\prec} & ({}^F\Omega(\mathcal{J}), \mathfrak{b}), \end{array} \tag{3.34}$$

where by commutativity we mean that $\mathcal{S}$ is a morphism of the sheaves of L_∞-modules $({}^F\Omega(\mathcal{J}),\mathfrak{b})$ and $({}^F\Omega,0)$ over the sheaf of DGLA $({}^F\Omega(\mathcal{T}),0,[\,,\,]_{SN})$ and the L_∞-module structure on $({}^F\Omega(\mathcal{J}),\mathfrak{b})$ over $({}^F\Omega(\mathcal{T}),0,[\,,\,]_{SN})$ is obtained by composing the L_∞-quasi-isomorphism $\mathcal{K}$ with the action $\mathcal{R}$ of $({}^F\Omega(\mathcal{D}),\partial,[\,,\,]_G)$ on $({}^F\Omega(\mathcal{J}),\mathfrak{b})$.

Let us now restrict ourselves to an open subset $V \subset M$ such that $E\Big|_V$ is trivial. Over any such subset the E-de Rham differential (1.5) is well defined for either of the sheaves ${}^F\Omega(\mathcal{A})$, ${}^F\Omega(\mathcal{T})$, ${}^F\Omega(\mathcal{J})$, and ${}^F\Omega(\mathcal{D})$. So again, we get a new commutative diagram

$$\begin{array}{ccc} ({}^F\Omega(\mathcal{T})\Big|_V, {}^Ed+\bar{d}, [\,,\,]_{SN}) & \overset{\mathcal{K}}{\succ\!\!\to} & ({}^F\Omega(\mathcal{D})\Big|_V, {}^Ed+\bar{d}+\partial, [\,,\,]_G) \\ \downarrow^L_{mod} & & \downarrow^{\mathcal{R}}_{mod} \\ ({}^F\Omega(\mathcal{A})\Big|_V, {}^Ed+\bar{d}) & \overset{\mathcal{S}}{\leftarrow\!\!\prec} & ({}^F\Omega(\mathcal{J})\Big|_V, {}^Ed+\bar{d}+\mathfrak{b}) \end{array} \tag{3.35}$$

in which the L_∞-morphism $\mathcal{K}$ and the morphism of L_∞-modules $\mathcal{S}$ are quasi-isomorphisms.

On the open subset V we can represent the Fedosov differential (2.23) in the following (non-covariant) form

$$D = {}^Ed + \bar{d} + B\cdot\, + \bar{B}\cdot\,, \tag{3.36}$$

$$B = \sum_{p=0}^{\infty} \xi^i B^k_{i;j_1\dots j_p}(z,\bar{z}) y^{j_1}\dots y^{j_p}\frac{\partial}{\partial y^k}\,,$$

and

$$\bar{B} = \sum_{p=0}^{\infty} d\bar{z}^\alpha \bar{B}^k_{\alpha;j_1\dots j_p}(z,\bar{z}) y^{j_1}\dots y^{j_p}\frac{\partial}{\partial y^k}\,,$$

where the z^α are local coordinates on M. If we regard $B+\bar{B}$ as a section of ${}^F\Omega^1(\mathcal{T}^0)\Big|_V$ then the nilpotency condition $D^2=0$ says that $B+\bar{B}$ is a Maurer-Cartan section of the sheaf of DGLA $({}^F\Omega(\mathcal{T})\Big|_V, {}^Ed+\bar{d}, [\,,\,]_{SN})$.

Thus applying the twisting procedures like in Section 2.3 (see also annexe A.2.3 and using claim *3* of Theorem 2.5 we get the following commutative diagram

$$\begin{array}{ccc} ({}^E\Omega(\mathcal{T})\Big|_V, D, [\,,\,]_{SN}) & \overset{\mathcal{K}^{tw}}{\succ\!\!\to} & ({}^E\Omega(\mathcal{D})\Big|_V, D+\partial, [\,,\,]_G) \\ \downarrow^L_{mod} & & \downarrow^{\mathcal{R}}_{mod} \\ ({}^E\Omega(\mathcal{A})\Big|_V, D) & \overset{\mathcal{S}^{tw}}{\leftarrow\!\!\prec} & ({}^E\Omega(\mathcal{J})\Big|_V, D+\mathfrak{b}), \end{array} \tag{3.37}$$

in which $\mathcal{K}^{tw}$ is a quasi-isomorphism of the sheaves of DGLA and $\mathcal{S}^{tw}$ is a quasi-isomorphism of the sheaves of DGLA modules.

Due to claim *4* in Theorem 2.5 and claim *3* in theorem 2.6 the quasi-isomorphisms do not depend on the trivialization of E over V.

Thus we constructed the following commutative diagram of sheaves of DGLA, DGLA modules and their (L_∞-)quasi-isomorphisms:

$$\begin{array}{ccc} ({}^E\Omega(\mathcal{T}), D, [\,,\,]_{SN}) & \overset{\mathcal{K}^{tw}}{\succ\!\!\to} & ({}^E\Omega(\mathcal{D}), D+\partial, [\,,\,]_G) \\ \downarrow^L_{mod} & & \downarrow^{\mathcal{R}}_{mod} \\ ({}^E\Omega(\mathcal{A}), D) & \overset{\mathcal{S}^{tw}}{\leftarrow\!\!\prec} & ({}^E\Omega(\mathcal{J}), D+\mathfrak{b}), \end{array} \tag{3.38}$$

Combining the diagrams in (3.33), (3.38) together with the Proposition 3.6 we see that the sheaves of DGLA modules $({}^{E}T^{*}_{poly}, {}^{E}A_{*})$ and $({}^{E}D^{*}_{poly}, {}^{E}C^{poly}_{*})$ are connected by chain of L_∞-quasi-isomorphisms. Thus, Theorem 3.2 is proved. □

3.4. Application to deformation quantization in the holomorphic context

Corollary 3.2 does not in general give a chain of quasi-isomorphisms between the DGLA $\Gamma(M, {}^{E}T^{*}_{poly})$ and $\Gamma(M, {}^{E}D^{*}_{poly})$ of global holomorphic sections. However, since the sheaves of smooth forms ${}^{F}\Omega^{0,*}({}^{E}T^{*}_{poly})$, ${}^{F}\Omega^{0,*}({}^{E}D^{*}_{poly})$, ${}^{F}\Omega^{0,*}({}^{E}A_{*})$ and ${}^{F}\Omega^{0,*}({}^{E}C^{poly}_{*})$ admits partition of unity one obtains a quasi-isomorphism for global sections. Using the correspondence between the Dolbeault and Cech pictures we relate these considerations to Kontsevich's algebroid picture of deformation quantization of algebraic varieties [**40**]. Namely, we construct a quantization of any holomorphic Poisson Lie algebroid in a weaker sens (that we define following [**14, 40, 55**]).

3.4.1. Algebroid stacks and weak deformations. Let us recall the usefull notion of *algebroid stack* introduced by M. Kontsevich in [**40**]. On the one hand it is a sheafified version of the categorical realization of an algebra [**47**], and on the other hand it is the linear analogue of the notion of *gerb* (groupoid stack locally connected by isomorphisms) from algebraic geometry [**33**].

Let us begin by the categorical realization of an algebra.

DEFINITION 3.14. *An* R-algebroid *is a small* R*-linear category* $\mathcal{A}$ *such that* $\mathrm{Obj}(\mathcal{A}) \neq \emptyset$ *and all objects of* $\mathcal{A}$ *are isomorphic.*

Remember that a category is *R-linear* if morphisms sets are R-module and compositions of morphisms are R-bilinear.

REMARK 3.15. A unital associative algebra is the same as an algebroid over R with only one object. Conversely, to any algebroid $\mathcal{A}$ we can associate an isomorphism class of algebras: all algebras $\mathrm{Hom}_{\mathcal{A}}(x,x)$ $(x \in \mathrm{Obj}(\mathcal{A}))$ are isomorphic since all objects of $\mathcal{A}$ are isomorphic.

Now we want to sheafify this notion of an algebroid. Since we deal with categories, the good concept to use is the one of a *stack*. Let X be a topological space[1] and R a sheaf of commutative rings. An *R-algebroid stack* $\mathcal{A}$ is a sheaf of R-categories (a stack in the terminology of Giraud [**33**]) which is

- locally non-empty: for a small enough open U, $\mathrm{Obj}(\mathcal{A}|_U) \neq \emptyset$, and
- locally connected by isomorphisms: for any two objects $x, y \in \mathrm{Obj}(\mathcal{A}|_U)$ over an open U, there exists an open covering $U = \bigcup_\alpha U_\alpha$ such that $x|_{U_\alpha} \cong y|_{U_\alpha}$ in $\mathcal{A}|_{U_\alpha}$.

Under a technical assumption (that more or less says that Yoneda lemma is satisfied) one obtains [**14**] an equivalent description of an R-algebroid stack by the following local data

- an open covering $X = \bigcup_\alpha U_\alpha$,
- an R-algebra A_α on each U_α,
- isomorphisms of R-algebras $g_{\alpha\beta} : \mathcal{A}_\beta \to \mathcal{A}_\alpha$ on $U_{\alpha\beta}$,
- invertible sections $a_{\alpha\beta\gamma} \in \mathcal{A}^{\times}_{\alpha}(U_{\alpha\beta\gamma})$ such that

$$g_{\alpha\beta} \circ g_{\beta\gamma} \circ g_{\gamma\alpha} = \mathrm{Ad}_{\mathcal{A}_\alpha}(a_{\alpha\beta\gamma}) \quad \text{as morphisms } \mathcal{A}_\alpha \to \mathcal{A}_\alpha \text{ on } U_{\alpha\beta\gamma}$$

and

$$a_{\alpha\beta\gamma} a_{\alpha\gamma\delta} = g_{\alpha\beta}(a_{\beta\gamma\delta}) a_{\alpha\beta\delta} \quad \text{in } \mathcal{A}_\alpha(U_{\alpha\beta\gamma\delta})$$

1. Actually, one can define stacks over any site. Here we consider the case of the small site of a topological space.

This discussion motivates the following definition of a *weak deformation* of a Poisson Lie algebroid:

DEFINITION 3.16. *A* weak quantization *of a Poisson Lie algebroid* $(E,[\,,],\rho,\pi)$ *over a manifold* M *is the data of*

- *an open covering* $M=\bigcup_\alpha U_\alpha$,
- *elements* $\Pi_\alpha\in\Gamma(U_\alpha,{}^ED^1_{poly})[[\hbar]]$ *satisfying the condition of the classical limit (2.58) and the associativity condition (2.59),*
- *gauge equivalences* $G_{\alpha\beta}\in\Gamma(U_{\alpha\beta},{}^ED^0_{poly})[[\hbar]]$ *from* Π_β *to* Π_α,
- *invertible sections* $f_{\alpha\beta\gamma}\in\Gamma(U_{\alpha\beta\gamma},{}^ED^{-1}_{poly})[[\hbar]]$ *such that*[2]

$$G_{\alpha\beta}G_{\beta\gamma}G_{\gamma\alpha}=\Pi^{1,23}_\alpha\Pi^{2,3}_\alpha(f^1_{\alpha\beta\gamma},(f^3_{\alpha\beta\gamma})^{-1}) \tag{3.39}$$

and

$$f_{\alpha\beta\gamma}f_{\alpha\gamma\delta}=G_{\alpha\beta}(f_{\beta\gamma\delta})f_{\alpha\beta\delta} \tag{3.40}$$

Given a weak quantization (or deformation) of a Poisson Lie algebroid $(E,[\,,],\rho,\pi)$ over a manifold M, one can obviously construct a $\mathbb{C}[[\hbar]]$-algebroid stack that is a deformation of the sheaf of functions on M, with first order term given by the image of π by the anchor. Namely, the algebra structure on $\mathcal{O}(U_\alpha)[[\hbar]]$ is given by $\rho^{\otimes 2}(\Pi_\alpha)$, isomorphisms of algebras are $g_{\alpha\beta}=\rho(G_{\alpha\beta})$, and $a_{\alpha\beta\gamma}=f_{\alpha\beta\gamma}$.

REMARK 3.17. A weak quantization with $f_{\alpha\beta\gamma}=1$ exactly corresponds to a usual quantization in the sens of the previous chapter.

THEOREM 3.18. *Any holomorphic Poisson Lie algebroid is weakly quantizable.*

Before proving this theorem we have to introduce a variant of the standard Čech resolution of a sheaf, which will be compatible with algebraic structures.

3.4.2. A variant of the Čech cochain complex. Let us begin with some abstract notions about (co)simplicial objects, taking our inspiration from [**37**, Section 1].

Let Δ denote the category with objects the ordered sets $[k]:=\{0,\dots,k\}$ $(k\in\mathbb{N})$ and with sets of morphisms $\Delta^l_k:=\mathrm{Hom}_\Delta([k],[l])$ $(k,l\in\mathbb{N})$ consisting of the order preserving functions $[k]\to[l]$.

A *(co)simplicial object* (respectively, a *simplicial object*) in a given category $\mathcal{C}$ is a functor $C:\Delta\to\mathcal{C}$ (respectively, $C:\Delta^{\mathrm{op}}\to\mathcal{C}$). Such objects form a category (namely, morphisms are given by natural transformations): let $\Delta\mathcal{C}$ (respectively, $\Delta^{\mathrm{op}}\mathcal{C}$) denotes the category of cosimplicial (respectively, simplicial) objects in $\mathcal{C}$. For a cosimplical object $C=\{C_k\}_{k\in\mathbb{N}}$ (respectively, a simplicial object $C=\{C^k\}_{k\in\mathbb{N}}$) , we denote by $\alpha_*:C_k\to C_l$ (respectively, $\alpha^*:C^l\to C^k$) the morphism corresponding to $\alpha\in\Delta^l_k$.

EXAMPLE 3.19. For any $k\in\mathbb{N}$ let $\Delta^k_{\mathbb{R}}$ be the *geometric k-dimensional simplex*

$$\Delta^k_{\mathbb{R}}:=\mathrm{Spec}\,\mathbb{R}[t_0,\dots,t_k]/<t_0+\cdots+t_k-1>$$

The i-th vertex of $\Delta^k_{\mathbb{R}}$ is the $\mathbb{R}$-point $t_j\mapsto\epsilon_{i,j}$. We identify the set of vertices with the ordered set $[k]$. For any $\alpha\in\Delta^l_k$ there exists a unique linear morphism $\alpha_*:\Delta^k_{\mathbb{R}}\to\Delta^l_{\mathbb{R}}$ extending the corresponding map between the sets of vertices; namely, it is given by the morphism of algebras

$$\mathcal{O}_{\Delta^l_{\mathbb{R}}}\to\mathcal{O}_{\Delta^k_{\mathbb{R}}}:t_i\mapsto\sum_{j\text{ s.t. }\alpha_j=i}t_j$$

2. Let us explain the notation we will follow by now with an example: if $P\in{}^ED^1_{poly}$ and $f\in{}^ED^{-1}_{poly}$ then $P(f^1):=(\rho\otimes\mathrm{id})(P)\cdot(f\otimes 1)$.

which is well-defined since $\sum_i \sum_{j|\alpha_j=i} t_j = \sum_j t_j = 1$. In this way $\{\Delta^k_{\mathbb{R}}\}_{k\in\mathbb{N}}$ is a cosimplicial affine variety. As an obvious consequence we also obtain that $\{\Omega^*(\Delta^k_{\mathbb{R}})\}_{k\in\mathbb{N}}$ defines a simplicial DG commutative algebra.

EXAMPLE 3.20. Let $\mathcal{G}$ be a sheaf of (DG) vector spaces over a topological space X. Then one can associate a cosimplicial (DG) vector space to any open covering $X = \bigcup_{i=0}^m U_i$. Namely, define (here $U_{i_0\dots i_k} := U_{i_0} \cap \dots \cap U_{i_k}$)

$$C_k(\underline{U}, \mathcal{G}) := \prod_{0\le i_0\le\dots\le i_k\le m} \Gamma(U_{i_0\dots i_k}, \mathcal{G})$$

and for any $\alpha = (0 \le \alpha_0 \le \dots \le \alpha_k \le l) \in \Delta^l_k$ one has

$$\alpha_*(f)_{i_0\dots i_l} = (f_{i_{\alpha_0}\dots i_{\alpha_k}})|_{U_{i_0\dots i_l}} \qquad \big(f \in C_k(\underline{U}, \mathcal{G}) \text{ and } 0 \le i_0 \le \dots \le i_l \le m\big)$$

Moreover, if the starting sheaf is a sheaf of DGLA (respectively, a sheaf taking its values in any category $\mathcal{C}$) then $\{C_k(\underline{U}, \mathcal{G}^*)\}_{k\in\mathbb{N}}$ becomes a cosimplicial DGLA (respectively, a cosimplical object in $\mathcal{C}$).

Let $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$ be abelian categories, with an additive (bi)functor $F : \mathcal{C} \times \mathcal{D} \to \mathcal{E}$. If we assume that projective limits exist in $\mathcal{E}$ then one has a functor $\widetilde{F} : \Delta^{\mathrm{op}}\mathcal{C} \times \Delta\mathcal{D} \to \mathcal{E}$, namely

$$\widetilde{F}(U^\bullet, V_\bullet) := \ker\Big(\prod_{m\in\mathbb{N}} F(U^m, V_m) \overset{1\otimes\alpha_* - \alpha^*\otimes 1}{\longrightarrow} \prod_{k,l\in\mathbb{N}} \prod_{\alpha\in\Delta^l_k} F(U^k, V_l)\Big)$$

We now need to apply this in the following situation: $\mathcal{C} = \underline{\mathrm{DGCA}}$ is the category of DG commutative algebras, $\mathcal{D} = \mathcal{E} = \underline{\mathrm{DGmod}}$ is the category of DG vector spaces, and $F = \otimes$ is the tensor product functor. This way we have the functor

$$\widetilde{\otimes} : \Delta^{\mathrm{op}}\underline{\mathrm{DGCA}} \times \Delta\underline{\mathrm{DGmod}} \to \underline{\mathrm{DGmod}}$$

Then to any sheaf of complexes $\mathcal{G}$ one can canonically associate a complex (which is actually a bicomplex)

$$\check{\mathrm{C}}^*(\underline{U}, \mathcal{G}) = \Omega^*(\Delta^\bullet_{\mathbb{R}}) \widetilde{\otimes} C_\bullet(\underline{U}, \mathcal{G}^*)$$

that we call the *modified Čech resolution* of $\mathcal{G}$. Moreover, if $\mathcal{G}$ is a sheaf of DGLA, then its modified Čech resolution becomes a DGLA (since the tensor product of a DG commutative algebra with a DGLA is a DGLA).

LEMMA 3.21. *Let $\mathcal{G}$ be a sheaf of complexes. The 0-th cohomology of the cochain complex $\big(\check{\mathrm{C}}^*(\underline{U}, \mathcal{G}), d^\Delta_{dR}\big)$ is isomorphic to the DG vector space of global sections $\Gamma(X, \mathcal{G})$.*

PROOF. Let $f = (f_0, f_1, \dots)$ be a 0-d^Δ_{dR}-cocycle. Remember that $f_0 = (f_\alpha)_\alpha$, $f_1 = (f_{ij})_{i\le j}$, etc ..., where $f_{i_0\dots i_l}$ are functions on $\Delta^l_{\mathbb{R}}$ with values in $\Gamma(U_{\alpha_0\dots\alpha_l}, \mathcal{G})$. The cocycle property says that these functions are constants. In particular using the fact that for any $\alpha \in \Delta^k_0$, $(1\otimes\alpha_*)(f_0) = (\alpha^*\otimes 1)(f_k)$ in $\mathcal{O}_{\Delta^0_{\mathbb{R}}} \otimes C_k(\underline{\mathrm{U}}, \mathcal{G})$, it implies that $f_{i_0\dots i_k} = (f_{i_\alpha})_{|U_{i_0\dots i_k}}$. Consequently f is uniquely determined by f_0, and $(f_i)_{|U_{ij}} = f_{ij} = (f_j)_{|U_{ij}}$. □

This lemma is actually a corollary of a more general result. Namely, let $C = \{C^*_k\}_{k\in\mathbb{N}}$ be a cosimplicial complex and consider vector space $\mathrm{N}^{p,q}C = \bigcap_{i=0}^q \ker(s^i_*) \subset C^{p-q}_q$, where $s_i = (\dots, i, i, \dots) \in \Delta^q_{q+1}$ is the i-th codegeneracy map. Then the differential of C provides us a coboundary map $d := (-1)^q d_C : \mathrm{N}^{p,q}C \to \mathrm{N}^{p+1,q}$; there is also another natural coboundary map $\check{d} = \sum_{i=0}^{q+1}(-1)^i t^i_* : \mathrm{N}^{p,q}C \to \mathrm{N}^{p,q+1}C$, where $t^i = (\dots, i-1, i+1, \dots) \in \Delta^{q+1}_q$ is the i-th coface map. Thus we get a bicomplex $(\mathrm{N}^{*,*}C, \check{d} + d)$ (which is the standard Čech resolution of C) and then we have the following

THEOREM 3.22 (Simplicial de Rham theorem, [**37, 4**]). *Let $C = \{C^*_k\}_{k\in\mathbb{N}}$ be a cosimplicial DG vector space. Then the complexes $\Omega^*(\Delta^\bullet_{\mathbb{R}}) \widetilde{\otimes} C^*_\bullet$ and $\mathrm{N}^{*,*}C$ are quasi-isomorphic.* □

REMARK 3.23. The quasi-isomorphism is explicitly given by the integration of forms over the simplex (see [**4, 37**]). In the case when C is a cosimplicial DGLA, even if the complex NC is not endowed with a DGLA structure, we conjecture that it is naturally a L_∞-algebra and that the quasi-isomorphism of complexes $\Omega^*(\Delta^\bullet_{\mathbb{R}})\widetilde{\otimes}C^*_\bullet \to N^{*,*}C$ extends to a L_∞-quasi-isomorphism.

From now we assume that we are given a complex manifold M together with a covering $M = \bigcup U_\alpha$ by nice enough opens (which means trivializing contractible opens). For any holomorphic DG vector bundle $\mathcal{B}$ over M, if we assume that $\mathcal{B}\Big|_{U_\alpha}$ is trivial then

PROPOSITION 3.24. *The natural inclusions*

$$\mathrm{inc}_1 : \big(\check{\mathrm{C}}^*(\underline{U},\mathcal{B}), d^\Delta_{dR} + d_{\mathcal{B}}\big) \hookrightarrow \big(\check{\mathrm{C}}^*(\underline{U},\Omega^{0,*}(\mathcal{B})), d^\Delta_{dR} + d_{\mathcal{B}} + \bar{d}\big)$$

and

$$\mathrm{inc}_2 : \big(\Omega^{0,*}(M,\mathcal{B})), d_{\mathcal{B}} + \bar{d}\big) \hookrightarrow \big(\check{\mathrm{C}}^*(\underline{U},\Omega^{0,*}(\mathcal{B})), d^\Delta_{dR} + d_{\mathcal{B}} + \bar{d}\big)$$

are quasi-isomorphisms of complexes.

Moreover, if $\mathcal{B}$ is in fact a DGLA then inc_1 *and* inc_2 *are morphisms of graded Lie algebras.*

PROOF. First of all, due to the $\bar{d}$-Poincaré lemma

$$H^*\big(\check{\mathrm{C}}^*(\underline{U},\Omega^{0,*}(\mathcal{B})), \bar{d}\big) = H^0\big(\check{\mathrm{C}}^*(\underline{U},\Omega^{0,*}(\mathcal{B})), \bar{d}\big) = \check{\mathrm{C}}^*(\underline{U},\mathcal{B}),$$

and thus it follows from the standard argument of the spectral sequence that inc_1 is a quasi-isomorphism.

Second, since $\Omega^{0,*}(\mathcal{B})$ is a sheaf of smooth sections of a C^∞-bundle then its standard Čech cohomology is concentrated in degree 0. Then due to the simplicial de Rham theorem and Lemma 3.21

$$H^*\big(\check{\mathrm{C}}^*(\underline{U},\Omega^{0,*}(\mathcal{B})), d^\Delta_{dR}\big) = H^0\big(\check{\mathrm{C}}^*(\underline{U},\Omega^{0,*}(\mathcal{B})), d^\Delta_{dR}\big) = \Omega^{0,*}(M,\mathcal{B}),$$

and thus it follows from the standard argument of the spectral sequence that inc_2 is a quasi-isomorphism.

Finally, the last statement of the proposition is obvious. □

3.4.3. Existence of weak quantizations (proof of Theorem 3.18). Recall that we proved the existence of a L_∞-quasi-isomorphism

$$\Omega^{0,*}(M, {}^E T^*_{poly}) = {}^F\Omega^{0,*}(M, {}^E T^*_{poly}) \succ\!\!\to {}^F\Omega^{0,*}(M, {}^E D^*_{poly}) = \Omega^{0,*}(M, {}^E D^*_{poly})$$

Now using Proposition 3.24 we obtain the existence of a L_∞-quasi-isomorphism

$$\Psi : \big(\Omega^{0,*}(M, {}^E T^*_{poly}), \bar{d}, [\,,\,]\big) \succ\!\!\to \big(\check{\mathrm{C}}^*(\underline{U}, {}^E D^*_{poly}), d^\Delta_d + \partial, [\,,\,]_G\big)$$

If π is a holomorphic Poisson E-bivector then $\bar{d}(\pi) = 0$ and $[\pi,\pi] = 0$ and thus $\hbar\pi \in \hbar\Omega^{0,0}(M, {}^E T^1_{poly})[[\hbar]]$ is a Maurer-Cartan element. As usual we define

$$\overline{\Pi} := 1 \otimes 1 + \sum_{k\geq 1} \frac{1}{k!}\Psi^{[k]}(\hbar\pi, \dots, \hbar\pi),$$

and it satisfies

$$d^\Delta_{dR}\overline{\Pi} + \frac{1}{2}[\overline{\Pi},\overline{\Pi}]_G = 0. \tag{3.41}$$

Let $\overline{\Pi} = \Pi + \Pi' + \Pi''$ with $\Pi = 1 \otimes 1 + O(\hbar) \in \check{\mathrm{C}}^0(\underline{U}, {}^E D^1_{poly})$, $\Pi' \in \hbar\check{\mathrm{C}}^1(\underline{U}, {}^E D^0_{poly})$ and $\Pi'' \in \hbar\check{\mathrm{C}}^2(\underline{U}, {}^E D^{-1}_{poly})$. Then equation (3.41) rewrite

- $[\Pi,\Pi]_G = 0$,
- $d^\Delta_{dR}\Pi + [\Pi,\Pi'] = 0$,

- $(d^{\Delta}_{dR}\Pi' + \frac{1}{2}[\Pi',\Pi']_G) + [\Pi,\Pi'']_G = 0$, and
- $d^{\Delta}_{dR}\Pi'' + [\Pi',\Pi'']_G = 0$.

Remember that $\Pi = (\Pi_0, \dots, \Pi_k, \dots)$, with

$$\Pi_k = (\Pi_{\alpha_0\dots\alpha_k})_{\alpha_0\le\dots\le\alpha_k} \in \prod_{0\le\alpha_0\le\dots\le\alpha_k\le m} \mathcal{O}_{\Delta^k_{\mathbb{R}}} \otimes \Gamma(U_{\alpha_0\dots\alpha_k}, {}^E D^1_{poly})[[\hbar]]$$

such that $\Pi_{\alpha_0\dots\alpha_k}(i\text{-th vertex}) = \Pi_{\alpha_i}$ and satisfying

$$[\Pi_{\alpha_0\dots\alpha_k}, \Pi_{\alpha_0\dots\alpha_k}]_G = 0\,.$$

In particular for any $\alpha \in \{0,\dots,m\}$ we have an element $\Pi_\alpha = 1\otimes 1 + O(\hbar) \in \Gamma(U_\alpha, {}^E D^1_{poly})[[\hbar]]$ such that $[\Pi_\alpha, \Pi_\alpha]_G = 0$. The second condition of Definition 3.16 is satisfied.

In the same way $\Pi' = (0, \Pi'_1, \dots, \Pi'_k, \dots)$ with

$$\Pi'_k = (\Pi'_{\alpha_0\dots\alpha_k})_{\alpha_0\le\dots\le\alpha_k} \in \prod_{0\le\alpha_0\le\dots\le\alpha_k\le m} \hbar\,\Omega^1(\Delta^k_{\mathbb{R}}) \otimes \Gamma(U_{\alpha_0\dots\alpha_k}, {}^E D^0_{poly})[[\hbar]]$$

that satisfies

$$d^{\Delta}_{dR}\Pi_{\alpha_0\dots\alpha_k} = [-\Pi'_{\alpha_0\dots\alpha_k}, \Pi_{\alpha_0\dots\alpha_k}]_G\,.$$

Here the action of the pronilpotent Lie algebra $\hbar\Gamma(U_{\alpha_0\dots\alpha_k}, {}^E D^0_{poly})[[\hbar]]$ on $\Gamma(U_{\alpha_0\dots\alpha_k}, {}^E D^1_{poly})[[\hbar]]$ given by

$$q \cdot P := [q,P]_G = \Delta(q)P - P(1\otimes q + q\otimes 1)$$

integrates to an action of the prounipotent group

$$\mathbf{G}_{\alpha_0\dots\alpha_k} := \exp\left(\hbar\Gamma(U_{\alpha_0\dots\alpha_k}, {}^E D^0_{poly})[[\hbar]]\right) = 1 + \hbar\Gamma(U_{\alpha_0\dots\alpha_k}, {}^E D^0_{poly})[[\hbar]]$$

given by

$$G \cdot P = \Delta(G)P(G\otimes G)^{-1}$$

Then defining $G_{\alpha\beta} := P\exp\int_0^1 \Pi'_{\alpha\beta} \in \mathbf{G}_{\alpha\beta}$, it is a gauge equivalence from Π_β to Π_α:

$$\Pi_\beta = \Delta(G_{\alpha\beta})^{-1}\Pi_\alpha(G_{\alpha\beta}\otimes G_{\alpha\beta})\,.$$

For $\beta > \alpha$ we set $G_{\alpha\beta} = G^{-1}_{\beta\alpha}$, and thus the third condition of Definition 3.16 is satisfied.

Let now $\Pi'' = (0,0,\Pi''_2,\dots,\Pi''_k,\dots)$ enter the game. Here

$$\Pi''_k = (\Pi''_{\alpha_0\dots\alpha_k})_{\alpha_0\le\dots\le\alpha_k} \in \prod_{0\le\alpha_0\le\dots\le\alpha_k\le m} \hbar\,\Omega^2(\Delta^k_{\mathbb{R}}) \otimes \Gamma(U_{\alpha_0\dots\alpha_k}, {}^E D^{-1}_{poly})[[\hbar]]$$

satisfies

$$\left(d^{\Delta}_{dR}\Pi'_{\alpha_0\dots\alpha_k} + \frac{1}{2}[\Pi'_{\alpha_0\dots\alpha_k}, \Pi'_{\alpha_0\dots\alpha_k}]_G\right) = [-\Pi''_{\alpha_0\dots\alpha_k}, \Pi_{\alpha_0\dots\alpha_k}]_G \tag{3.42}$$

Let us explain the geometric meaning of this equation. First of all $\nabla := d^{\Delta}_{dR} + \Pi'_{\alpha_0\dots\alpha_k}$ defines a (usual) connection on $\Delta^k_{\mathbb{R}}$ on the trivial $\mathbf{G}_{\alpha_0\dots\alpha_k}$-bundle. It follows from what we wrote before that $\Pi_{\alpha_0\dots\alpha_k}$ is a horizontal section of ∇ with values in the trivial vector bundle with fiber $\Gamma(U_{\alpha_0\dots\alpha_k}, {}^E D^1_{poly})[[\hbar]]$. Then the action of the pronilpotent abelian Lie algebra $\hbar\Gamma(U_{\alpha_0\dots\alpha_k}, {}^E D^{-1}_{poly})[[\hbar]]$ given by

$$q \cdot P := [q,P]_G = P(q^1) - P(\cdot q^2)$$

integrates to the action of the prounipotent abelian group

$$\mathbf{A}_{\alpha_0\dots\alpha_k} := \exp\left(\hbar\Gamma(U_{\alpha_0\dots\alpha_k}, {}^E D^{-1}_{poly})[[\hbar]]\right) = 1 + \hbar\Gamma(U_{\alpha_0\dots\alpha_k}, {}^E D^{-1}_{poly})[[\hbar]]$$

given by

$$f \cdot P = (\Pi^{1,23}_{\alpha_0\dots\alpha_k}\Pi^{2,3}_{\alpha_0\dots\alpha_k})(f^1_{\alpha_0\dots\alpha_k}, (f^3_{\alpha_0\dots\alpha_k})^{-1})$$

Then equation 3.42 means that for any piecewise-smooth map $\phi : D^2 \to \Delta^k_{\mathbb{R}}$ with a given point $p \in \partial D^2 = S^1$, the holonomy of ∇ along the based oriented loop $\ell = (x = \phi(p), \phi(S^1))$

is given by $f_{\phi,p} \cdot \Pi_{\alpha_0 \dots \alpha_k}(x) \in \mathbf{G}_{\alpha_0 \dots \alpha_k}$ for a certain $f_{\phi,p} \in \mathbf{A}_{\alpha_0 \dots \alpha_k}$ [3]. Applying this in the case $k = 2$ and $\phi : D^2 \tilde{\to} \Delta^2_{\mathbb{R}}$ is the natural isomorphism sending p to the 0-th vertex in $\Delta^2_{\mathbb{R}}$, we obtain elements $f_{\alpha\beta\gamma}$ such that $G_{\alpha\beta}G_{\beta\gamma}G_{\gamma\alpha} = f_{\alpha\beta\gamma} \cdot \Pi_\alpha$. The fourth condition of definition 3.16 is satisfied for $0 \leq \alpha \leq \beta \leq \gamma \leq m$; in order to make it true for any α, β, γ we have to define $f_{\alpha\gamma\beta} := f^{-1}_{\alpha\beta\gamma}$ and $f_{\gamma\alpha\beta} := G_{\gamma\alpha}(f_{\alpha\beta\gamma})$.

Finally, the last equation

$$d^{\Delta}_{dR}\Pi''_{\alpha_0 \dots \alpha_k} = [-\Pi'_{\alpha_0 \dots \alpha_k}, \Pi''_{\alpha_0 \dots \alpha_k}]_G$$

means that the quantities $f_{\phi,p}$ depend only on the based oriented loop ℓ. Namely, if $\phi_i : D^2 \to \Delta^k_{\mathbb{R}}$ $(i = 1, 2)$ are such that $(\phi_1(p), \phi_1(S^1)) = (\phi_2(p), \phi_2(S^1))$ then $f_{\phi_1,p} = f_{\phi_2,p}$. Applying this in the case $k = 3$, $\phi_1 : D^2 \to \Delta^3_{\mathbb{R}}$ corresponds to the $(0, 1, 2)$-face, $\phi_2 : D^2 \to \Delta^3_{\mathbb{R}}$ corresponds to the union of faces $(0, 1, 3)$, $(0, 3, 2)$, $(1, 2, 3)$, and $\phi_1(p) = \phi_2(p)$ the 0-th vertex, then we obtain the last condition of Definition 3.16.
Theorem 3.18 is proved. □

REMARK 3.25. The L_∞-quasi-isomorphism Ψ should imply a stronger result than just the existence of a weak quantization, namely a classification result generalizing the one proved in [**54**] for the symplectic case.

REMARK 3.26. The way we proved the existence of weak quantizations is pedestrian and I believe there should exist a more conceptual proof of this fact. Namely, to any sheaf of DGLA one can associate the *groupoid stack* (roughly speaking, it is a sheaf of groupoids) of Maurer-Cartan elements. Then two L_∞-quasi-isomorphic sheaves of DGLA have isomorphic stacks of Maurer-Cartan elements (this was pointed out to me by Mathieu Anel). Both existence of weak quantizations and classification result should come from this fact using Theorem 3.2.

3. the explicit formula for $f_{\phi,p}$ is given by $\exp\big(-\int_{D^2} \phi^* \Pi''_{\alpha_0 \dots \alpha_k}\big)$.

CHAPITRE 4

Quantization of formal classical dynamical r-matrices

RÉSUMÉ. Dans ce chapitre, dont les résultats font l'objet de [**6**], on démontre l'existence d'une quantification par twist dynamique pour toute r-matrice dynamique formelle dans le cas réductif. On démontre également un théorème de classification de telles quantifications. Ces deux résultats découlent de l'existence d'un quasi-isomorphisme L_∞ entre deux algèbres de Lie différentielles graduées appropriées.

In [**28**], Felder introduced dynamical versions of both classical and quantum Yang-Baxter equations which has been generalized to the case of a nonabelian base in [**25**] for the classical part and in [**72**] for the quantum part. Naturally this leads to quantization problems which have been formulated in terms of twist quantization *à la* Drinfeld ([**19**]) in [**70, 72, 21, 22**].

Let us formulate this problem in the general context. Consider an inclusion $\mathfrak{h} \subset \mathfrak{g}$ of Lie algebras equipped with an element $Z \in (\wedge^3\mathfrak{g})^{\mathfrak{g}}$. A *(modified) classical dynamical r-matrix* for $(\mathfrak{g}, \mathfrak{h}, Z)$ is a regular (meaning C^∞, meromorphic, formal, ... depending on the context) $\mathfrak{h}$-equivariant map $\rho : \mathfrak{h}^* \to \wedge^2\mathfrak{g}$ which satisfies the *(modified) classical dynamical Yang-Baxter equation* (CDYBE)

$$\mathrm{CYB}(\rho) - \mathrm{Alt}(d\rho) = Z \tag{4.1}$$

where $\mathrm{CYB}(\rho) := [\rho^{1,2}, \rho^{1,3}] + [\rho^{1,2}, \rho^{2,3}] + [\rho^{1,3}, \rho^{2,3}] = \frac{1}{2}[\rho, \rho]$ and

$$\mathrm{Alt}(d\rho) := \sum_i \left(h_i^1 \frac{\partial \rho^{2,3}}{\partial \lambda^i} - h_i^2 \frac{\partial \rho^{1,3}}{\partial \lambda^i} + h_i^3 \frac{\partial \rho^{1,2}}{\partial \lambda^i}\right)$$

Here (h_i) and (λ^i) are dual basis of $\mathfrak{h}$ and $\mathfrak{h}^*$.

Let $\Phi = 1 + O(\hbar^2) \in (U\mathfrak{g}^{\otimes 3})^{\mathfrak{g}}[[\hbar]]$ be an associator quantizing Z (of which the existence was proved in [**20**, proposition 3.10]). A *dynamical twist quantization* of a (modified) classical dynamical r-matrix ρ associated to Φ is a regular $\mathfrak{h}$-equivariant map $J = 1 + O(\hbar) \in \mathrm{Reg}(\mathfrak{h}^*, U\mathfrak{g}^{\otimes 2})[[\hbar]]$ such that $\mathrm{Alt}\frac{J-1}{\hbar} = \rho \bmod \hbar$ and which satisfies the *(modified) dynamical twist equation* (DTE)

$$J^{12,3}(\lambda) * J^{1,2}(\lambda + \hbar h^3) = \Phi^{-1} J^{1,23}(\lambda) * J^{2,3}(\lambda) \tag{4.2}$$

where $*$ denotes the PBW star-product of functions on $\mathfrak{h}^*$ and

$$J^{1,2}(\lambda + \hbar h^3) := \sum_{k \geq 0} \frac{\hbar^k}{k!} \sum_{i_1, \ldots, i_k} (\partial_{\lambda^{i_1}} \cdots \partial_{\lambda^{i_k}} J)(\lambda) \otimes (h_{i_1} \cdots h_{i_k})$$

Now observe that many (modified) classical dynamical r-matrices can be viewed as formal ones by taking their Taylor expansion at 0. In this chapter we are interested in the following conjecture:

CONJECTURE 4.1 ([**21**]). *Any (modified) formal classical dynamical r-matrix admits a dynamical twist quantization.*

Let us reformulate DTE in the formal framework. A formal (modified) dynamical twist is an element $J(\lambda) = 1 + O(\hbar) \in (U\mathfrak{g}^{\otimes 2}\hat{\otimes}\hat{S}\mathfrak{h})^{\mathfrak{h}}[[\hbar]]$ which satisfies DTE, and $J^{1,2}(\lambda + \hbar h^3) \in (U\mathfrak{g}^{\otimes 3}\hat{\otimes}\hat{S}\mathfrak{h})[[\hbar]]$ is equal to $(\mathrm{id}^{\otimes 2} \otimes \tilde{\Delta})(J)$ where $\tilde{\Delta} : \hat{S}\mathfrak{h} \to (U\mathfrak{g}\hat{\otimes}\hat{S}\mathfrak{h})[[\hbar]]$ is induced by

$\mathfrak{h} \ni x \mapsto \hbar x \otimes 1 + 1 \otimes x$. Then define $K := J(\hbar\lambda) \in (U\mathfrak{g}^{\otimes 2} \otimes S\mathfrak{h})^{\mathfrak{h}}[[\hbar]]$ which we view as an element of $(U\mathfrak{g}^{\otimes 2} \otimes U\mathfrak{h})^{\mathfrak{h}}[[\hbar]]$ using the symmetrization map $S\mathfrak{h} \to U\mathfrak{h}$. Since J is a solution of DTE K satisfies the *(modified) algebraic dynamical twist equation* (ADTE)

$$K^{12,3,4}K^{1,2,34} = (\Phi^{-1})^{1,2,3}K^{1,23,4}K^{2,3,4} \tag{4.3}$$

Moreover and by construction, $K = 1 + \sum_{n\geq 1} \hbar^n K_n$ has the *$\hbar$-adic valuation property*. Namely, $U\mathfrak{h}$ is filtered by $(U\mathfrak{h})_{\leq n} = \ker(\mathrm{id} - \eta \circ \varepsilon)^{\otimes n+1} \circ \Delta^{(n)}$ where $\varepsilon : U\mathfrak{h} \to \mathbf{k}$ and $\eta : \mathbf{k} \to U\mathfrak{h}$ are the counit and unit maps, and $K_n \in (U\mathfrak{h})_{\leq n-1}$. Conversely, any algebraic dynamical twist having the $\hbar$-adic valuation property can be obtained from a unique formal dynamical twist by this procedure.

This chapter, in which we always assume $Z = 0$ and $\Phi = 1$ (non-modified case), is organized as follow.

In Section 1 we define two differential graded Lie algebras (DGLA) respectively associated to classical dynamical r-matrices and algebraic dynamical twists. In Section 2 we formulate and prove the main theorem of this chapter which states that if $\mathfrak{h}$ admits an ad$\mathfrak{h}$-invariant complement (the *reductive* case) then these two DGLA are L_∞-quasi-isomorphic. We prove that it implies Conjecture 4.1 in this case, which generalizes Theorem 5.3 of [**70**]:

THEOREM 4.2. *In the reductive case, any formal classical dynamical r-matrix for $(\mathfrak{g}, \mathfrak{h}, 0)$ admits a dynamical twist quantization (associated to the trivial associator).*

The construction of the L_∞-quasi-isomorphism makes use of an equivariant formality theorem for homogeneous spaces which is obtain from results of Chapter 2. In Section 3 we use our quasi-isomorphism to deduce some classification results, still in the reductive case. Section 4 is dedicated to a short study of what happens in the case of an abelian base: in particular we also prove that if $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ for $\mathfrak{h}$ abelian and $\mathfrak{m}$ a Lie subalgebra then the results of the previous sections are still true in this situation. We conclude the chapter with some open questions.

4.1. Algebraic structures associated to dynamical equations

Let $\mathfrak{h} \subset \mathfrak{g}$ be an inclusion of Lie algebras.

4.1.1. Algebraic structures associated to CDYBE. Let us consider the following graded vector space

$$\underline{\mathrm{CDYB}} := \wedge^*\mathfrak{g} \otimes S\mathfrak{h} = \bigoplus_{k\geq 0} \wedge^k\mathfrak{g} \otimes S\mathfrak{h}$$

equipped with the differential d defined by

$$\mathrm{d}(x_1 \wedge \cdots \wedge x_k \otimes h_1 \cdots h_l) := -\sum_{i=1}^{l} h_i \wedge x_1 \wedge \cdots \wedge x_k \otimes h_1 \cdots h_l \hat{h}_i \tag{4.4}$$

With the exterior product $\wedge$ it becomes a differential graded commutative associative algebra. Moreover, one can define a graded Lie bracket of degree -1 on $\underline{\mathrm{CDYB}}$ which is the Lie bracket of $\mathfrak{g}$ extended to $\underline{\mathrm{CDYB}}$ in the following way:

$$[a, b \wedge c] = [a, b] \wedge c + (-1)^{(|a|-1)|b|} b \wedge [a, c] \tag{4.5}$$

Thus one can observe that polynomial solutions to CDYBE are exactly elements $\rho \in \underline{\mathrm{CDYB}}$ of degree 2 such that $\mathrm{d}\rho + \frac{1}{2}[\rho, \rho] = 0$. We would like to say that such a ρ is a Maurer-Cartan element but $(\underline{\mathrm{CDYB}}[1], \mathrm{d}, [\,,\,])$ is not a differential graded Lie algebra (DGLA).

Instead, remember that we are interested in $\mathfrak{h}$-equivariant solutions of CDYBE (i.e., dynamical r-matrices) and thus consider the subspace $\mathfrak{g}_1 = (\underline{\mathrm{CDYB}})^{\mathfrak{h}}$ of $\mathfrak{h}$-invariants with the same differential and bracket.

PROPOSITION 4.3. *$(\mathfrak{g}_1[1], \mathrm{d}, [\,,\,])$ is a DGLA. Moreover $(\mathfrak{g}_1, \mathrm{d}, \wedge, [\,,\,])$ is a Gerstenhaber algebra.*

PROOF. Let $a = x_1 \wedge \cdots \wedge x_k \otimes h_1 \cdots h_s$ and $b = y_1 \wedge \cdots \wedge y_l \otimes m_1 \cdots m_t$ be $\mathfrak{h}$-invariant elements in $\mathfrak{g}_1$. We want to show that

$$\mathrm{d}[a,b] = [\mathrm{d}a, b] + (-1)^{k-1}[a, \mathrm{d}b] \tag{4.6}$$

The l.h.s. of (4.6) is equal to

$$-\Big(\sum_{i=1}^{s} h_i \wedge [x_1 \wedge \cdots \wedge x_k, y_1 \wedge \cdots \wedge y_l] \otimes h_1 \cdots h_s m_1 \cdots m_t \hat{h}_i + \sum_{j=1}^{t} m_j \wedge [x_1 \wedge \cdots \wedge x_k, y_1 \wedge \cdots \wedge y_l] \otimes h_1 \cdots h_s m_1 \cdots m_t \hat{m}_j\Big)$$

The first term in the r.h.s. of (4.6) gives

$$\sum_{i=1}^{s} \big((-1)^{k-1} x_1 \wedge \cdots \wedge x_k \wedge [h_i, y_1 \wedge \cdots \wedge y_l] - h_i \wedge [x_1 \wedge \cdots \wedge x_k, y_1 \wedge \cdots \wedge y_l]\big) \otimes h_1 \cdots h_s m_1 \cdots m_t \hat{h}_i$$

and for the second term we obtain

$$\sum_{j=1}^{t} \big((-1)^{k-1} [m_j, x_1 \wedge \cdots \wedge x_k] \wedge y_1 \wedge \cdots \wedge y_l - m_j \wedge [x_1 \wedge \cdots \wedge x_k, y_1 \wedge \cdots \wedge y_l]\big) \otimes h_1 \cdots h_s m_1 \cdots m_t \hat{m}_j$$

Thus the difference between the l.h.s. and the r.h.s. of (4.6) is equal to

$$(-1)^k\Big(\sum_{i=1}^{k} x_1 \wedge \cdots \wedge x_k \wedge [h_i, y_1 \wedge \cdots \wedge y_l] \otimes h_1 \cdots h_s m_1 \cdots m_t \hat{h}_i + \sum_{j=1}^{l} [m_j, x_1 \wedge \cdots \wedge x_k] \wedge y_1 \wedge \cdots \wedge y_l \otimes h_1 \cdots h_s m_1 \cdots m_t \hat{m}_j\Big)$$

Then using $\mathfrak{h}$-invariance of a and b one obtains

$$(-1)^{k-1} \sum_{i,j} x_1 \wedge \cdots \wedge x_k \wedge y_1 \wedge \cdots \wedge y_l \otimes \big(h_1 \cdots h_s m_1 \cdots m_t([h_i, m_j] - [m_j, h_i])\hat{h}_i \hat{m}_j\big) = 0$$

The second statement of the proposition is obvious from the definition (4.5) of the bracket. □

Let $\rho(\lambda) \in (\wedge^2 \mathfrak{g} \hat{\otimes} \hat{S}\mathfrak{h})^{\mathfrak{h}}$ be a formal classical dynamical r-matrix. Since ρ satisfies CDYBE, $\alpha := \hbar\rho(\hbar\lambda) \in \hbar\mathfrak{g}_1[[\hbar]]$ is a Maurer-Cartan element (i.e. $\mathrm{d}\alpha + \frac{1}{2}[\alpha, \alpha] = 0$).

4.1.2. Algebraic structures associated to ADTE. Let us now consider the graded vector space

$$\underline{\mathrm{ADT}} := T^* U\mathfrak{g} \otimes U\mathfrak{h} = \bigoplus_{k \geq 0} \otimes^k U\mathfrak{g} \otimes U\mathfrak{h}$$

equipped with the differential b given by

$$b(P) := P^{2,\ldots,k+2} + \sum_{i=1}^{k+1} (-1)^i P^{1,\ldots,ii+1,\ldots,k+2} \quad \text{for} \quad P \in \otimes^k U\mathfrak{g} \otimes U\mathfrak{h} \tag{4.7}$$

REMARK 4.4. This is just the coboundary operator of Hochschild's cohomology with value in a comodule; and $b^2 = 0$ follows directly from an easy calculation.

One can define on $\underline{\mathrm{ADT}}$ an associative product $\cup$ (the *cup* product) which is given on homogeneous elements $P \in \otimes^k U\mathfrak{g} \otimes U\mathfrak{h}$ and $Q \in \otimes^l U\mathfrak{g} \otimes U\mathfrak{h}$ by

$$P \cup Q := P^{1,\dots,k,k+1\dots k+l+1} Q^{k+1,\dots,k+l+1} \tag{4.8}$$

PROPOSITION 4.5. *$(\underline{\mathrm{ADT}}, b, \cup)$ is a differential graded associative algebra.*

PROOF. The cup product is obviously associative. Thus the only thing we have to check is that

$$b(P \cup Q) = bP \cup Q + (-1)^{|P|} P \cup bQ \tag{4.9}$$

Let $k = |P|$ and $l = |Q|$. The l.h.s. of (4.9) is equal to

$$P^{2,\dots,k+1,k+2\dots k+l+2} Q^{k+2,\dots,k+l+2} + \sum_{i=1}^{k} (-1)^i P^{1,\dots,ii+1,\dots,k+1,k+2\dots k+l+2} Q^{k+2,\dots,k+l+2} + \sum_{i=k+1}^{k+l+1} (-1)^i P^{1,\dots,k,k+1\dots k+l+2} Q^{k+1,\dots,ii+1,\dots,k+l+2}$$

The first line of this expression is equal to

$$bP \cup Q - (-1)^{k+1} P^{1,\dots,k,k+1\dots k+l+2} Q^{k+2,\dots,k+l+2}$$

and the last term of the same expression gives

$$(-1)^k \big(P \cup bQ - P^{1,\dots,k,k+1\dots k+l+2} Q^{k+2,\dots,k+l+2} \big)$$

The proposition is proved. □

Recall that in the case $\mathfrak{h} = \{0\}$ one can define a brace algebra structure on $(T^*U\mathfrak{g})[1]$ (see [**32**]). Unfortunately we are not able to extend this structure to $\underline{\mathrm{ADT}}$ in general. Since we deal with $\mathfrak{h}$-equivariant solutions of ADTE we can consider the subspace $\mathfrak{g}_2 = (\underline{\mathrm{ADT}})^{\mathfrak{h}}$ of $\mathfrak{h}$-invariants. Let us now define a collection of linear homogeneous maps of degree zero $\{-|-,\dots,-\} : \mathfrak{g}_2[1] \otimes \mathfrak{g}_2[1]^{\otimes m} \to \mathfrak{g}_2[1]$ indexed by $m \geq 0$, and $\{P|Q_1,\dots,Q_m\}$ is given by

$$\sum_{\substack{0 \leq i_1, i_m + k_m \leq n \\ i_l + k_l \leq i_{l+1}}} (-1)^{\epsilon} P^{1,\dots,i_1+1\dots i_1+k_1,\dots,i_m+1\dots i_m+k_m,\dots,n+1} \prod_{s=i}^{m} Q_s^{i_s+1,\dots,i_s+k_s,i_s+k_s+1\dots n+1}$$

where $k_s = |Q_s|$, $n = |P| + \sum_s k_s - m$ and $\epsilon = \sum_s (k_s - 1) i_s$.

PROPOSITION 4.6. *$(\mathfrak{g}_2[1], \{-|-,\dots,-\})$ is a brace algebra.*

PROOF. Since we work with $\mathfrak{h}$-invariant elements one can remark that if $i_s + k_s \leq i_t$ then $Q_s^{i_s+1,\dots,i_s+k_s,i_s+k_s+1\dots n+1}$ and $Q_t^{i_t+1,\dots,i_t+k_t,i_t+k_t+1\dots n+1}$ commute. Using this the proof becomes identical to the case when $\mathfrak{h} = 0$ (see [**32**] for example). □

Now observe that since $m = 1^{\otimes 3} \in (\otimes^2 U\mathfrak{g} \otimes U\mathfrak{h})^{\mathfrak{h}}$ is such that $\{m|m\} = 0$ one obtains a B_∞-algebra structure ([**2**]) on $\mathfrak{g}_2$ (see [**38**]). More precisely, we have a differential graded bialgebra structure on the cofree tensorial coalgebra $T^c(\mathfrak{g}_2[1])$ of which structure maps $a^n, a^{p,q}$ are given by

- $a^1(P) = bP = (-1)^{|P|-1}[m, P]_G$, where
$$[P, Q]_G := \{P|Q\} - (-1)^{(|P|-1)(|Q|-1)} \{Q|P\}$$
- $a^2(P, Q) = \{m|P, Q\} = P \cup Q$
- $a^{0,1} = a^{1,0} = \mathrm{id}$
- $a^{1,n}(P; Q_1, \dots, Q_n) = \{P|Q_1, \dots, Q_n\}$ for $n \geq 1$
- all other maps are zero

In particular, we have

COROLLARY 4.7. *$(\mathfrak{g}_2[1], b, [\,,]_G)$ is a DGLA.* □

REMARK 4.8. Since that for any graded vector space V, DG bialgebra structures on the cofree coassociative coalgebra T^cV are in one-to-one correspondence with DG Lie bialgebra structures on the cofree Lie coalgebra L^cV (see [**61**], section 5), then $L^c(\mathfrak{g}_2[1])$ becomes a DG Lie bialgebra with differential and Lie bracket given by maps l^n, $l^{p,q}$ such that $l^1 = b$ and $l^{1,1} = [\,,]_G$. Therefore $d_2 := \sum_{i\geq 0} l^i + \sum_{p,q\geq 0} l^{p,q} : C^c(L^c(\mathfrak{g}_2[1])) \to C^c(L^c(\mathfrak{g}_2[1]))$ defines a G_∞-algebra structure on $\mathfrak{g}_2$ ($d_2 \circ d_2 = 0$ since d_2 is just the Chevalley-Eilenberg differential on the DG Lie algebra $L^c(\mathfrak{g}_2[1])$).

4.2. Existence of a twist quantization in the reductive case

In this section we assume that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ with $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$. Let us denote by $\mathrm{p} : \mathfrak{g} \to \mathfrak{m}$ the projection on $\mathfrak{m}$ along $\mathfrak{h}$; it is $\mathfrak{h}$-equivariant.

4.2.1. Main result and proof of Theorem 4.2. First of all, observe that CDYB, $\mathfrak{g}_1$ and $\mathcal{G}_1 := C^c(\mathfrak{g}_1[2])$ have a natural grading induced by the one of $S\mathfrak{h}$. In the same way ADT, $\mathfrak{g}_2$ and $\mathcal{G}_2 := C^c(\mathfrak{g}_2[2])$ have a natural filtration induced by the one of $U\mathfrak{h}$. Our main goal is to prove the following theorem, which is sufficient to obtain algebraic dynamical twists from formal dynamical r-matrices.

THEOREM 4.9. *In the reductive case, there exists a L_∞-quasi-isomorphism*

$$\Psi : (\mathfrak{g}_1[1], \mathrm{d}, [\,,]) \succ\!\!\to (\mathfrak{g}_2[1], b, [\,,]_G)$$

with the following two filtration properties:

(F1) $\forall X \in (\mathfrak{g}_1)_k$, $\Psi^{[1]}(X) = (\mathrm{alt} \otimes \mathrm{sym})(X) \bmod (\mathfrak{g}_2)_{\leq k-1}$

(F2) $\forall X \in (\Lambda^n \mathfrak{g}_1)_k$, $\Psi^{[n]}(X) \in (\mathfrak{g}_2)_{\leq n+k-1}$

PROOF OF THEOREM 4.2. Now consider a formal solution $\rho(\lambda) \in (\wedge^2\mathfrak{g}\hat{\otimes}\hat{S}\mathfrak{h})^{\mathfrak{h}}$ to CDYBE. Let us define $\alpha := \hbar\rho(\hbar\lambda) \in \hbar\mathfrak{g}_1[[\hbar]]$ which is a Maurer-Cartan element in $\hbar\mathfrak{g}_1[[\hbar]]$. The L_∞-morphism property implies that $\widetilde{\alpha} := \sum_{n=1}^{\infty} \frac{1}{n!}\Psi^{[n]}(\alpha, \ldots, \alpha)$ is a Maurer-Cartan element in $\hbar\mathfrak{g}_2[[\hbar]]$; this exactly means that $K := 1^{\otimes 3} + \widetilde{\alpha} \in (\otimes^2 U\mathfrak{g} \otimes U\mathfrak{h})^{\mathfrak{h}}[[\hbar]]$ satisfies ADTE. Moreover, due to (F2) the coefficient K_n of $\hbar^n$ in K lies in $(\mathfrak{g}_2)_{\leq n-1}$. It means that there exists $J \in (U\mathfrak{g}^{\otimes 2}\hat{\otimes}\hat{S}\mathfrak{h})^{\mathfrak{h}}[[\hbar]]$ satisfying DTE and such that $K = (\mathrm{id}^{\otimes 2} \otimes \mathrm{sym})(J(\hbar\lambda))$. Finally, property (F1) obviously implies that the semi-classical limit condition $\frac{J - J^{\mathrm{op}}}{\hbar} = \rho \bmod \hbar$ is satisfied. □

EXAMPLE 4.10. Let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ a reductive Lie algebra (i.e. $\mathfrak{h}$ is a Lie subalgebra and $[\mathfrak{h},\mathfrak{m}] \subset \mathfrak{m}$). Following [**22**] we have a map $\mathfrak{h}^\vee \to (\wedge^2\mathfrak{m})^\vee$, taking $\lambda \in \mathfrak{h}^\vee$ to $\omega(\lambda) : x \wedge y \mapsto \lambda([x,y])$. The reductive decomposition is called *nondegenrate* if for generic λ, $\omega(\lambda)$ is nondegenerate (by this we mean that if we identify $(\wedge^2\mathfrak{m})^\vee$ with a subspace of $\mathrm{End}(\mathfrak{m})$ using any isomorphism $\mathfrak{m} \cong \mathfrak{m}^\vee$, then the map $\lambda \mapsto \det\omega(\lambda)$ does not vanish identically). In this situation one can define a rational map $\rho_{\mathfrak{h}}^{\mathfrak{g}} : \mathfrak{h}^\vee \to \wedge^2\mathfrak{m} \subset \wedge^2\mathfrak{g}$; $\lambda \mapsto -\omega(\lambda)^{-1}$. This map is a dynamical r-matrix [**22**, proposition 1.1]. Unfortunately, $\rho_{\mathfrak{h}}^{\mathfrak{g}}$ is singular at the origin. But in the case $\mathfrak{h}$ is equipped with a nontrivial character $\chi \in (\mathfrak{h}^\vee)^{\mathfrak{h}}$ then the map $\lambda \mapsto \rho_{\mathfrak{h}}^{\mathfrak{g}}(\lambda + \chi)$ is a dynamical r-matrix that is regular at the origin.

4.2.2. Resolutions. Let us first observe that the bilinear map $[\,,]_{\mathfrak{m}} := (\wedge\cdot\mathrm{p}) \circ [\,,]$ defines a graded Lie bracket of degree -1 on $(\wedge^*\mathfrak{m})^{\mathfrak{h}}$. Then we prove

PROPOSITION 4.11. *The natural map $p_1 : (\mathfrak{g}_1[1], \mathrm{d}, [\,,]) \to ((\wedge^*\mathfrak{m})^{\mathfrak{h}}[1], 0, [\,,]_{\mathfrak{m}})$ is a morphism of DGLA. Moreover, there exists an operator $\delta : \mathfrak{g}_1^* \to \mathfrak{g}_1^{*-1}$ such that $\delta \circ \mathrm{d} + \mathrm{d} \circ \delta = \mathrm{id} - p_1$, $\delta \circ \delta = 0$ and $\delta\big((\mathfrak{g}_1)_k\big) \subset (\mathfrak{g}_1)_{k+1}$. In particular, p_1 induces an isomorphism at the level of cohomology.*

PROOF. The projection $p_1 := (\wedge^{\cdot}\mathrm{p}) \otimes \varepsilon : (\underline{\mathrm{CDYB}}, \mathrm{d}) \to (\wedge^*\mathfrak{m}, 0)$ is a $\mathfrak{h}$-equivariant morphism of complexes, and it obviously restricts to a morphism of (differential) graded Lie algebras at the level of $\mathfrak{h}$-invariants.

Moreover, $\wedge^n\mathfrak{g} \otimes S\mathfrak{h} \cong \bigoplus_{p+q=n} \wedge^p\mathfrak{m} \otimes \wedge^q\mathfrak{h} \otimes S\mathfrak{h}$ as a $\mathfrak{h}$-module; and under this identification d becomes $-\mathrm{id} \otimes d_K$, where $d_K : \wedge^*\mathfrak{h} \otimes S\mathfrak{h} \to \wedge^{*+1}\mathfrak{h} \otimes S\mathfrak{h}$ is Koszul's coboundary operator, and p_1 corresponds to the projection on the part of zero antisymmetric and symmetric degrees in $\mathfrak{h}$. Let us define $\delta = \mathrm{id} \otimes \delta_K$ with $\delta_K : \wedge^*\mathfrak{h} \otimes S^*\mathfrak{h} \to \wedge^{*-1}\mathfrak{h} \otimes S^{*+1}\mathfrak{h}$ defined by

$$\delta_K(x_1 \wedge \cdots \wedge x_n \otimes h_1 \cdots h_m) = \begin{cases} \frac{1}{m+n} \sum_i (-1)^i x_1 \wedge \cdots \hat{x_i} \cdots \wedge x_n \otimes h_1 \cdots h_m x_i & \text{if } m+n \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

Finally remark that δ is a $\mathfrak{h}$-equivariant homotopy operator: $\delta \circ \mathrm{d} + \mathrm{d} \circ \delta = \mathrm{id} - p_1$ and $\delta \circ \delta = 0$. The proposition is proved. □

Now we prove a similar result for $\mathfrak{g}_2$. Let us first define $U\mathfrak{m} := \mathrm{sym}(S\mathfrak{m}) \subset U\mathfrak{g}$; this is a sub-coalgebra of $U\mathfrak{g}$ and thus $T^*U\mathfrak{m}$ equipped with its Hochschild coboundary operator $b_\mathfrak{m}$ becomes a cochain subcomplex of the Hochschild complex $(T^*U\mathfrak{g}, b_\mathfrak{g})$ of $U\mathfrak{g}$. We also have the following

LEMMA 4.12. *$U\mathfrak{g} = U\mathfrak{g} \cdot \mathfrak{h} \oplus U\mathfrak{m}$ as a filtered $\mathfrak{h}$-module. Moreover $[,]_{G,\mathfrak{m}} := (\otimes^{\cdot}\mathrm{p}) \circ [,]$ defines a graded Lie bracket of degree -1 on $(T^*U\mathfrak{m})^\mathfrak{h}$*

PROOF. See [**34**, Ch.II §4.2] for the first statement. The second statement follows from a direct computation. □

Then we prove the

PROPOSITION 4.13. *The natural map $p_2 : (\mathfrak{g}_2[1], b, [,]_G) \to ((T^*U\mathfrak{m})^\mathfrak{h}[1], b_\mathfrak{m}, [,]_{G,\mathfrak{m}})$ is a morphism of DGLA. Moreover, there exists an operator $\kappa : \mathfrak{g}_2^* \to \mathfrak{g}_2^{*-1}$ such that $\kappa \circ b + b \circ \kappa = \mathrm{id} - p_2$, $\kappa \circ \kappa = 0$ and $\kappa\big((\mathfrak{g}_2)_{\leq k}\big) \subset (\mathfrak{g}_2)_{\leq k+1}$. In particular, p_2 induces an isomorphism at the level of cohomology.*

PROOF. The projection $p_2 := (\otimes^{\cdot}\mathrm{p}) \otimes \varepsilon : (\underline{\mathrm{ADT}}, b) \to (T^*U\mathfrak{m}, b_\mathfrak{m})$ is a $\mathfrak{h}$-equivariant morphism of complexes, and it obviously restricts to a morphism of DGLA at the level of $\mathfrak{h}$-invariants (by Lemma 4.12).

Remember that $\mathfrak{g}_2$ has a natural filtration induced by the one of $U\mathfrak{h}$. Then one obtains a spectral sequence of which we compute the first terms:

$$\begin{array}{ll} E_0^{*,*} = (T^*U\mathfrak{g} \otimes S^*\mathfrak{h})^\mathfrak{h} & d_0 = b_\mathfrak{g} \otimes \mathrm{id} \\ E_1^{*,*} = (\wedge^*\mathfrak{g} \otimes S^*\mathfrak{h})^\mathfrak{h} & d_1 = \mathrm{d} \\ E_2^{*,*} = E_2^{*,0} = (\wedge^*\mathfrak{m})^\mathfrak{h} & d_2 = 0 \end{array}$$

Then the proposition follows from Proposition 4.11. □

4.2.3. Inverting p_2. In this subsection, taking our inspiration from [**48**, appendix], we prove the following

PROPOSITION 4.14. *There exists a L_∞-quasi-isomorphism*

$$\mathcal{Q} : ((T^*U\mathfrak{m})^\mathfrak{h}[1], b_\mathfrak{m}, [,]_{G,\mathfrak{m}}) \rightarrowtail (\mathfrak{g}_2[1], b, [,]_G)$$

such that $\mathcal{Q}^{[1]}$ is the natural inclusion and $\mathcal{Q}^{[n]}$ takes values in $(\mathfrak{g}_2)_{\leq n-1}$.

PROOF. Let $(N, b_N) \subset (\mathfrak{g}_2, b)$ be the kernel of the surjective morphism of complexes $p_2 : (\mathfrak{g}_2, b) \to ((T^*U\mathfrak{m})^\mathfrak{h}, b_\mathfrak{m})$. It follows from Proposition 4.13 that there exists an operator $H : N^* \to N^{*-1}$ such that $H(N_{\leq n}) \subset N_{\leq n+1}$, $H \circ H = 0$ and $b_N \circ H + H \circ b_N = \mathrm{id}$.

Now let us construct a L_∞-isomorphism

$$\mathcal{F} : \big(C^c(\mathfrak{g}_2[2]), b + [,]_G\big) \xrightarrow{\sim} \big(C^c((T^*U\mathfrak{m})^\mathfrak{h}[2] \oplus N[2]), b_\mathfrak{m} + b_N + [,]_{G,\mathfrak{m}}\big)$$

with structure maps $\mathcal{F}^{[n]} : \Lambda^n \mathfrak{g}_2 \to ((T^*U\mathfrak{m})^\mathfrak{h} \oplus N)[1-n]$ such that

- $\mathcal{F}^{[1]}$ is the sum of p_2 with the projection on N along $(T^*U\mathfrak{m})^\mathfrak{h}$ (in some sense $\mathcal{F}^{[1]}$ is the identity),
- for any $n > 1$ and $X \in (\Lambda^n \mathfrak{g}_2)_{\leq k}$, $\mathcal{F}^{[n]}(X) \in N_{\leq n+k-1}$.

Let us prove it by induction on n. First $\mathcal{F}^{[1]}$ is a morphism of complexes by definition. Then let us define $\mathcal{K}_2 : \Lambda^2 \mathfrak{g}_2 \to ((T^*U\mathfrak{m})^\mathfrak{h} \oplus N)[1]$ by

$$\mathcal{K}_2(x,y) = [\mathcal{F}^1(x), \mathcal{F}^1(y)]_{G,\mathfrak{m}} - \mathcal{F}^1([x,y]_G)$$

It takes values in $N[1]$ and is such that $b_N \mathcal{K}_2(x,y) + \mathcal{K}_2(bx,y) + \mathcal{K}_2(x,by) = 0$. Consequently $\mathcal{F}^{[2]} := H \circ \mathcal{K}_2 : \Lambda^2 \mathfrak{g}_2 \to N$ is such that

$$b_N \mathcal{F}^{[2]}(x,y) - \mathcal{F}^{[2]}(bx,y) - \mathcal{F}^{[2]}(x,by) = \mathcal{K}_2(x,y) \quad (L_\infty\text{-condition for } \mathcal{F}^{[2]})$$

and for any $X \in (\Lambda^2 \mathfrak{g}_2)_{\leq k}$, $\mathcal{F}^{[2]}(X) \in N_{\leq k+1}$. After this, suppose we have constructed $\mathcal{F}^{[1]}, \dots, \mathcal{F}^{[n]}$ and let us define

$$\mathcal{K}_{n+1} := [\,,]_{G,\mathfrak{m}} \circ \mathcal{F}^{[\leq n]} - \mathcal{F}^{[\leq n]} \circ [\,,]_G : \Lambda^{n+1} \mathfrak{g}_2 \to ((T^*U\mathfrak{m})^\mathfrak{h} \oplus N)[1]$$

It obviously takes values in $N[1]$ and is such that $b_N \circ \mathcal{K}_{n+1} + \mathcal{K}_{n+1} \circ b = 0$. Consequently $\mathcal{F}^{[n+1]} := H \circ \mathcal{K}_{n+1}$ satisfies the L_∞-condition (where we omit the composition symbol $\circ$)

$$b_N \mathcal{F}^{[n+1]} - \mathcal{F}^{[n+1]} b = b_N H \mathcal{K}_{n+1} - H \mathcal{K}_{n+1} b = (b_N H + H b_N) \mathcal{K}_{n+1} = \mathcal{K}_{n+1}$$

and for any $X \in (\Lambda^n \mathfrak{g}_2)_{\leq n+1}$, $\mathcal{F}^{[n+1]}(X) \in N_{\leq n+k}$ (since $\mathcal{K}_{n+1}(X) \in N_{\leq n+k-1}$).

Now let $\mathcal{H}$ be the inverse of the isomorphism $\mathcal{F}$, it is such that for any $n \geq 1$ and $X \in (\Lambda^n \mathfrak{g}_2)_{\leq k}$, $\mathcal{H}^{[n]}(X) \in N_{\leq n+k-1}$. Finally we obtain $\mathcal{Q}$ by composing $\mathcal{H}$ with the inclusion of DGLA $(T^*U\mathfrak{m})^\mathfrak{h}[1] \hookrightarrow ((T^*U\mathfrak{m})^\mathfrak{h} \oplus N)[1]$. □

4.2.4. Proof of Theorem 4.9. Recall from [**34**, Ch.II §4.2] that

$$(T^{*+1}U\mathfrak{m})^\mathfrak{h} = \mathrm{Diff}^{*+1}(G/H)^G \quad \text{and} \quad (\wedge^{*+1}\mathfrak{m})^\mathfrak{h} = \Gamma(G/H, \wedge^{*+1} T(G/H))^G$$

as DGLA. Remember also from [**52**, Ch.II §8] that G-invariant connections on G/H are in one-to-one correspondence with $\mathfrak{h}$-equivariant linear maps $\alpha : \mathfrak{m} \otimes \mathfrak{m} \to \mathfrak{m}$, and that the torsion tensor is given by $\alpha - \alpha^{21} - \mathrm{p} \circ [\,,]$. Thus G/H is equipped with a G-invariant torsion free connection ∇, corresponding to the map $\alpha := \frac{1}{2}\mathrm{p} \circ [\,,]$. Then using the equivariant version of Theorem 2.1 (see Corollary 2.2) we obtain a G-equivariant L_∞-quasi-isomorphism

$$\phi : \Gamma(G/H, \wedge^{*+1} T(G/H)) \succ\!\!\to \mathrm{Diff}^{*+1}(G/H)$$

with first structure map $\phi^{[1]} = \mathcal{V}$, which restricts to a L_∞-quasi-isomorphism at the level of G-invariants. Let us define

$$\psi := \mathcal{Q} \circ \phi \circ p_1 : (\mathfrak{g}_1[1], \mathrm{d}, [\,,]) \succ\!\!\to (\mathfrak{g}_2[1], b, [\,,]_G)\,;$$

it is a L_∞-quasi-isomorphism with first structure map $\psi^{[1]} = (\mathrm{alt} \otimes 1) \circ (\wedge \cdot \mathrm{p} \otimes \varepsilon)$.

Finally define $V := (\mathrm{alt} \otimes \mathrm{sym}) \circ \delta : \mathfrak{g}_1 \to \mathfrak{g}_2[-1]$ and use Lemma 4.12 to construct a L_∞-quasi-morphism $\Psi : (\mathfrak{g}_1[1], \mathrm{d}, [\,,]) \succ\!\!\to (\mathfrak{g}_2[1], b, [\,,]_G)$ with first structure map $\Psi^{[1]} = \psi^{[1]} + b \circ V + V \circ \mathrm{d}$. Since for any $X \in (\mathfrak{g}_1)_k$ we have

$$b \circ (\mathrm{alt} \otimes \mathrm{sym})(X) = (\mathrm{alt} \otimes \mathrm{sym}) \circ \mathrm{d}(X) \bmod (\mathfrak{g}_2)_{\leq k-1}\,,$$

$$\begin{aligned} \text{then} \quad \Psi^1(X) &= \psi^1(X) + bV(X) + V(\mathrm{d}X) \\ &= (\mathrm{alt} \otimes \mathrm{sym}) \circ (p_1 + \mathrm{d}\delta + \delta\mathrm{d})(X) \bmod (\mathfrak{g}_2)_{\leq k-1} \\ &= (\mathrm{alt} \otimes \mathrm{sym})(X) \bmod (\mathfrak{g}_2)_{\leq k-1} \end{aligned}$$

Consequently Ψ satisfies (F1). Moreover, it follows from the proof of Lemma 4.12 that Ψ also satisfies (F2). □

4.3. Classification

Theorem 4.9 implies a stronger result than just the existence of the twist quantization. Namely, since Ψ is a L_∞-quasi-isomorphism there is a bijection between the moduli spaces of Maurer-Cartan elements of the DGLA $(\mathfrak{g}_1[1])[[\hbar]]$ and $(\mathfrak{g}_2[1])[[\hbar]]$.

4.3.1. Classification of algebraic and formal dynamical twists. Following [**21**], two dynamical twists $J(\lambda)$ and $J'(\lambda)$ are said to be *gauge equivalent* if there exists a regular $\mathfrak{h}$-equivariant map $T(\lambda) = \exp(q) + O(\hbar) \in \mathrm{Reg}(\mathfrak{h}^*, U\mathfrak{g})^{\mathfrak{h}}[[\hbar]]$, with $q \in \mathrm{Reg}(\mathfrak{h}^*, \mathfrak{g})^{\mathfrak{h}}$ such that $q(0) = 0$, and satisfying

$$J'(\lambda) = T^{12}(\lambda) * J(\lambda) * T^2(\lambda)^{-1} * T^1(\lambda + \hbar h^2)^{-1} \tag{4.10}$$

Dealing with formal functions one can easily derive an equivalence relation for the corresponding algebraic dynamical twists $K = J(\hbar\lambda)$ and $K' = J'(\hbar\lambda)$:

$$K' = Q^{12,3} K (Q^{2,3})^{-1} (Q^{1,23})^{-1} \tag{4.11}$$

in $(U\mathfrak{g}^{\otimes 2} \otimes U\mathfrak{h})^{\mathfrak{h}}[[\hbar]]$, with $Q = 1 + O(\hbar) \in (U\mathfrak{g} \otimes U\mathfrak{h})^{\mathfrak{h}}[[\hbar]]$ given by $T(\hbar\lambda)$.

Assume now we are in the reductive case.

Since the composition $\mathcal{Q}_2 \circ \phi : ((\wedge\mathfrak{m})^{\mathfrak{h}}[1], 0, [\,,\,]_{\mathfrak{m}}) \succ\!\!\rightarrow (\mathfrak{g}_2[1]), b\,[\,,\,]_G)$ in the previous section is a L_∞-quasi-isomorphism then we have a bijective correspondance

$$\frac{\{\pi \in \hbar(\wedge^2\mathfrak{m})^{\mathfrak{h}}[[\hbar]] \text{ s.t. } [\pi,\pi]_{\mathfrak{m}} = 0\}}{G_0} \longleftrightarrow \frac{\{\text{algebraic dynamical twists}\}}{\text{gauge equivalence (4.11)}} \tag{4.12}$$

where G_0 is the prounipotent group corresponding to the Lie algebra $\hbar\,\mathfrak{m}^{\mathfrak{h}}[[\hbar]]$. Moreover, since the structure maps $\mathcal{Q}_2^{[n]}$ take values in $(\mathfrak{g}_2)_{\leq n-1}$ then it appears that any algebraic dynamical twist is gauge equivalent to a one with the $\hbar$-adic valuation property and thus we have a bijection

$$\frac{\{\text{algebraic dynamical twists}\}}{\text{gauge equivalence (4.11)}} \longleftrightarrow \frac{\{\text{formal dynamical twists}\}}{\text{gauge equivalence (4.10)}} \tag{4.13}$$

4.3.2. Classical counterpart. Assume that we are in the reductive case. Since p_1 is a L_∞-quasi-isomorphism by propostion 4.11 then we have a bijection

$$\frac{\{\alpha \in \hbar(\wedge^2\mathfrak{g} \otimes S\mathfrak{h})^{\mathfrak{h}}[[\hbar]] \text{ s.t. } \mathrm{d}\alpha + \frac{1}{2}[\alpha,\alpha] = 0\}}{G_1} \longleftrightarrow \frac{\{\pi \in \hbar(\wedge^2\mathfrak{m})^{\mathfrak{h}}[[\hbar]] \text{ s.t. } [\pi,\pi]_{\mathfrak{m}} = 0\}}{G_0}$$

where G_1 is a prounipotent group and its action (by affine transformations) is given by the exponentiation of the infinitesimal action of its Lie algebra $\hbar(\mathfrak{g} \otimes S\mathfrak{h})^{\mathfrak{h}}[[\hbar]]$:

$$q \cdot \alpha = \mathrm{d}q + [q, \alpha] \qquad \big(q \in \hbar(\mathfrak{g} \otimes S\mathfrak{h})^{\mathfrak{h}}[[\hbar]]\big) \tag{4.14}$$

Then going along the lines of Subsection 4.2.3 one can prove the following

PROPOSITION 4.15. *There exists a L_∞-quasi-isomorphism*

$$\mathcal{Q}_1 : ((\wedge^*\mathfrak{m})^{\mathfrak{h}}[1], 0, [\,,\,]_{\mathfrak{m}}) \succ\!\!\rightarrow (\mathfrak{g}_1[1], \mathrm{d}, [\,,\,])$$

such that $\mathcal{Q}_1^{[1]}$ is the natural inclusion and $\mathcal{Q}_1^{[n]}$ takes values in $(\mathfrak{g}_1)_{\leq n-1}$. □

Consequently any Maurer-Cartan element in $(\mathfrak{g}_1[1])[[\hbar]]$ is equivalent to a one of the form $\hbar\rho_\hbar(\hbar\lambda)$, where $\rho_\hbar \in (\wedge^2\mathfrak{g}\hat{\otimes}\hat{S}\mathfrak{h})^{\mathfrak{h}}[[\hbar]]$ satisfies CDYBE. In other words $\rho_\hbar$ is $\hbar$-dependant formal dynamical r-matrix. On such a $\rho_\hbar$ the infinitesimal action (4.14) becomes

$$q \cdot \rho_\hbar = -\sum_i h_i \wedge \frac{\partial q}{\partial \lambda^i} + [q, \rho_\hbar] \qquad (q \in \mathfrak{g}\hat{\otimes}\hat{S}\mathfrak{m})^{\mathfrak{h}}[[\hbar]] \tag{4.15}$$

This action integrates in an affine action of some group $\widetilde{G_1}$ of $\mathfrak{h}$-equivariant formal maps with values in the Lie group G of $\mathfrak{g}$. And then we have a bijection

$$\frac{\{\pi \in \hbar(\wedge^2\mathfrak{m})^{\mathfrak{h}}[[\hbar]] \text{ s.t. } [\pi,\pi]_{\mathfrak{m}} = 0\}}{G_0} \longleftrightarrow \frac{\{\text{form. dynam. } r\text{-matrices}/\mathbb{R}[[\hbar]]\}}{\widetilde{G_1}} \tag{4.16}$$

REMARK 4.16. This bijection has to be compared with Proposition 2.13 in [**70**] and [**24**, Section 3]

Finally, combining (4.16), (4.12) and (4.13) we obtain the following generalization of Theorem 6.11 in [**70**] to the case of a nonabelian base:

THEOREM 4.17. *Let $\pi \in (\wedge^2\mathfrak{m})^{\mathfrak{h}}$ such that $[\pi,\pi]_{\mathfrak{m}} = 0$. Then there are bijective correspondances between*

1. *the set of $\hbar$-dependant and G-invariant Poisson structures $\pi_\hbar = \hbar\pi \bmod \hbar^2$ on G/H, modulo the action of G_0,*
2. *the set of $\hbar$-dependant formal dynamical r-matrices $\rho_\hbar(\lambda)$ such that $\rho_\hbar(0) = \pi \bmod \hbar$ in $\wedge^2(\mathfrak{g}/\mathfrak{h})[[\hbar]]$, modulo the action (4.15) of $\widetilde{G_1}$,*
3. *the set of formal dynamical twists $J(\lambda)$ satisfying $(\otimes^2 \mathrm{p})\big(Alt^{\frac{J(0)-1}{\hbar}}\big) = \pi \bmod \hbar$, modulo gauge equivalence (4.10).* □

4.4. The case of an abelian base

In this section we assume that $\mathfrak{h}$ is abelian.

4.4.1. A classification result for dynamical r-matrices. Let $\mathfrak{m}$ be any complement of $\mathfrak{h}$, denote by p the projection on $\mathfrak{m}$ along $\mathfrak{h}$ (that is NOT $\mathfrak{h}$-equivariant), and observe that $(\wedge\mathfrak{g})^{\mathfrak{h}} \cap \wedge\mathfrak{m}$ is naturally equipped with a graded Lie bracket $[\,,]_{\mathfrak{m}} = \wedge\cdot\mathrm{p} \circ [\,,]$ of degree -1. Then we prove

PROPOSITION 4.18. *The natural map $p_1 : (\mathfrak{g}_1[1], \mathrm{d}, [\,,]) \to ((\wedge\mathfrak{g})^{\mathfrak{h}} \cap \wedge^*\mathfrak{m}[1], 0, [\,,]_{\mathfrak{m}})$ is a morphism of DGLA. Moreover, there exists an operator $\delta : \mathfrak{g}_1^* \to \mathfrak{g}_1^{*-1}$ such that $\delta\circ\mathrm{d}+\mathrm{d}\circ\delta = \mathrm{id} - p_1$, $\delta\circ\delta = 0$ and $\delta\big((\mathfrak{g}_1)_k\big) \subset (\mathfrak{g}_1)_{k+1}$. In particular, p_1 induces an isomorphism at the level of cohomology.*

PROOF. As in the proof of Proposition 4.11, the projection $p_1 := (\wedge\cdot\mathrm{p})\otimes\varepsilon : (\underline{\mathrm{CDYB}}, \mathrm{d}) \to (\wedge^*\mathfrak{m}, 0)$ is a quasi-isomorphism of complexes, and it restricts to a morphism of DGLA $(\mathfrak{g}_1[1], \mathrm{d}, [\,,]) \to ((\wedge\mathfrak{g})^{\mathfrak{h}} \cap \wedge^*\mathfrak{m}[1], 0, [\,,]_{\mathfrak{m}})$. But in this case it is *a priori* NOT $\mathfrak{h}$-equivariant.

Nevertheless, we want to prove that it still is a quasi-ismorphism at the level of $\mathfrak{h}$-invariants. Namely, since $\mathfrak{h}$ is abelian one has

$$(\wedge^n\mathfrak{g}\otimes S\mathfrak{h})^{\mathfrak{h}} = (\wedge^n\mathfrak{g})^{\mathfrak{h}}\otimes S\mathfrak{h} \cong \bigoplus_{p+q=n} \big[(\wedge\mathfrak{g})^{\mathfrak{h}} \cap \wedge^p\mathfrak{m}\big] \otimes \wedge^q\mathfrak{h} \otimes S\mathfrak{h}\,.$$

Under this identification d becomes $-\mathrm{id}\otimes d_K$, where $d_K : \wedge^*\mathfrak{h}\otimes S\mathfrak{h} \to \wedge^{*+1}\mathfrak{h}\otimes S\mathfrak{h}$ is Koszul's coboundary operator, and p_1 corresponds to the projection on the part of zero antisymmetric and symmetric degrees in $\mathfrak{h}$. Let us define $\delta = \mathrm{id}\otimes\delta_K$ with $\delta_K : \wedge^*\mathfrak{h}\otimes S^*\mathfrak{h} \to \wedge^{*-1}\mathfrak{h}\otimes S^{*+1}\mathfrak{h}$ defined by

$$\delta_K(x_1\wedge\cdots\wedge x_n\otimes h_1\cdots h_m) = \begin{cases} \frac{1}{m+n}\sum_i (-1)^i x_1\wedge\cdots\hat{x_i}\cdots\wedge x_n\otimes h_1\cdots h_m x_i & \text{if } m+n\neq 0 \\ 0 & \text{otherwise} \end{cases}$$

and remark that δ is a homotopy operator: $\delta\circ\mathrm{d}+\mathrm{d}\circ\delta = \mathrm{id}-p_1$ and $\delta\circ\delta = 0$. The proposition is proved. □

In particular p_1 is a L_∞-quasi-isomorphism, and thus we have a bijection

$$\frac{\{\alpha \in \hbar(\wedge^2\mathfrak{g}\otimes S\mathfrak{h})^{\mathfrak{h}}[[\hbar]] \text{ s.t. } \mathrm{d}\alpha + \frac{1}{2}[\alpha,\alpha] = 0\}}{G_1} \longleftrightarrow \frac{\{\pi \in \hbar(\wedge^2\mathfrak{g})^{\mathfrak{h}} \cap \wedge^2\mathfrak{m}[[\hbar]] \text{ s.t. } [\pi,\pi]_{\mathfrak{m}} = 0\}}{G_0}$$

where G_1 is as in Subsection 4.3.2 and G_0 is the prounipotent group corresponding to the Lie algebra $\hbar\,(\mathfrak{g}^{\mathfrak{h}} \cap \mathfrak{m})[[\hbar]]$.

Again one can inverse p_1, namely

PROPOSITION 4.19. *There exists a L_∞-quasi-isomorphism*

$$\mathcal{Q}_1 : ((\wedge^{\mathfrak{g}})^{\mathfrak{h}} \cap \wedge^* \mathfrak{m}[1], 0, [\,,\,]_{\mathfrak{m}}) \succ\!\!\rightarrow (\mathfrak{g}_1[1], \mathrm{d}, [\,,\,])$$

such that $\mathcal{Q}_1^{[1]}$ is the natural inclusion and $\mathcal{Q}_1^{[n]}$ takes values in $(\mathfrak{g}_1)_{\leq n-1}$. □

Consequently any Maurer-Cartan element in $(\mathfrak{g}_1[1])[[\hbar]]$ is equivalent to a one of the form $\hbar\rho_\hbar(\hbar\lambda)$ and then we have a bijection

$$\frac{\{\pi \in \hbar(\wedge^2\mathfrak{g})^{\mathfrak{h}} \cap \wedge^2\mathfrak{m}[[\hbar]] \text{ s.t. } [\pi,\pi]_{\mathfrak{m}} = 0\}}{G_0} \longleftrightarrow \frac{\{\text{form. dynam. } r\text{-matrices}/\mathbb{R}[[\hbar]]\}}{\widetilde{G_1}} \tag{4.17}$$

where $\widetilde{G_1}$ is as in Subsection 4.3.2.

4.4.2. Another case when the twist quantization exists. In this subsection we assume that the abelian subalgebra $\mathfrak{h}$ admits a Lie subalgebra $\mathfrak{m}$ as a complement. Then the natural inclusions $i : (\wedge\mathfrak{g})^{\mathfrak{h}} \cap \wedge\mathfrak{m} \hookrightarrow \wedge\mathfrak{g}$ and $\mathrm{id} \otimes 1 : (T^*U\mathfrak{g})^{\mathfrak{h}} \hookrightarrow \mathfrak{g}_2$ are compatible with all algebraic structures. Now recall from example 2.4 (i) that there exists a L_∞-quasi-isomorphism $\mathcal{F} : (\wedge^*\mathfrak{g})^{\mathfrak{h}}[1] \succ\!\!\rightarrow (T^*U\mathfrak{g})^{\mathfrak{h}}[1]$ with $\mathcal{F}^{[1]} = \mathrm{alt}$. By composing these maps together with p_1 one obtains a L_∞-morphism

$$\widetilde{\mathcal{F}} = (\mathrm{id} \otimes 1) \circ \mathcal{F} \circ i \circ p_1 : (\mathfrak{g}_1[1], \mathrm{d}, [\,,\,]) \succ\!\!\rightarrow (\mathfrak{g}_2[1], b, [\,,\,]_G)$$

with $\widetilde{\mathcal{F}}^{[1]} = (\mathrm{alt} \otimes 1) \circ p_1$ and $\widetilde{F}^{[n]}$ taking values in $(\mathfrak{g}_2)_{\leq 0}$.

THEOREM 4.20. *There exists a L_∞-quasi-isomorphism*

$$\Psi : (\mathfrak{g}_1[1], \mathrm{d}, [\,,\,]) \succ\!\!\rightarrow (\mathfrak{g}_2[1], b, [\,,\,]_G)$$

with properties (F1) *and* (F2) *of Theorem 4.9.*

PROOF. First recall that since $\mathfrak{h}$ is abelian then $\mathfrak{g}_1 \cong ((\wedge\mathfrak{g})^{\mathfrak{h}} \cap \wedge\mathfrak{m}) \otimes \wedge\mathfrak{h} \otimes S\mathfrak{h}$ as a vector space. Thus if δ_K is as in the proof of Proposition 4.11 then $\delta := \mathrm{id} \otimes \delta_K$ is a homotopy operator such that $\delta\mathrm{d} + \mathrm{d}\delta = \mathrm{id} - \wedge^{\cdot} p \otimes \varepsilon$.

Now we proceed like in Subsection 4.2.4: use Lemma 4.12 to construct Ψ with first structure map $\Psi^1 = \widetilde{\mathcal{F}}^1 + b \circ V + V \circ \mathrm{d}$, where $V := (\mathrm{alt} \otimes \mathrm{sym}) \circ \delta : \mathfrak{g}_1 \to \mathfrak{g}_2[-1]$.

Finally, Ψ obviously induces an isomorphism in cohomology. □

Then using the same argumentation as in the proof of Theorem 4.2 (Subsection 4.2.1) one obtains the

THEOREM 4.21. *If $\mathfrak{h}$ is an abelian subalgebra of $\mathfrak{g}$ with a Lie subalgebra as a complement, then any formal classical dynamical r-matrix for $(\mathfrak{g}, \mathfrak{h}, 0)$ admits a dynamical twist quantization (associated to the trivial associator).* □

EXAMPLE 4.22. In particular, this allows us to quantize dynamical r-matrices arizing from semi-direct products $\mathfrak{g} = \mathfrak{m} \ltimes \mathbb{C}^n$ like in [**23**, example 3.7].

Moreover we also obtain a classification of quantizations like in Section 4.3:

THEOREM 4.23. *Let $\pi \in (\wedge^2\mathfrak{g})^{\mathfrak{h}} \cap \wedge^2\mathfrak{m}$ such that $[\pi,\pi]_{\mathfrak{m}} = 0$. Then there are bijective correspondances between*

1. *the set $\pi_\hbar = \hbar\pi \mod \hbar^2 \in \hbar(\wedge^2\mathfrak{g})^{\mathfrak{h}} \cap \wedge^2\mathfrak{m}[[\hbar]]$ such that $[\pi_\hbar, \pi_\hbar]_{\mathfrak{m}} = 0$, modulo the action of G_0,*
2. *the set of $\hbar$-dependant formal dynamical r-matrices $\rho_\hbar(\lambda)$ such that $(\wedge^2\mathrm{p})(\rho_\hbar(0)) = \pi \mod \hbar$, modulo the action (4.15) of $\widetilde{G_1}$,*
3. *the set of formal dynamical twists $J(\lambda)$ satisfying $(\otimes^2\mathrm{p})\big(Alt\frac{J(0)-1}{\hbar}\big) = \pi \mod \hbar$, modulo gauge equivalence (4.10).* □

4.5. Concluding remarks

Open questions. Let us then mention that one can consider a non-triangular (i.e., non-antisymmetric) version of non-modified classical dynamical r-matrices. Namely, $\mathfrak{h}$-equivariant maps $r \in \mathrm{Reg}(\mathfrak{h}^*, \mathfrak{g} \otimes \mathfrak{g})$ such that $\mathrm{CYB}(r) - \mathrm{Alt}(dr) = 0$. According to [**72**], a quantization of such a r is a $\mathfrak{h}$-equivariant map $R = 1 + \hbar r + O(\hbar^2) \in \mathrm{Reg}(\mathfrak{h}^*, U\mathfrak{g}^{\otimes 2})[[\hbar]]$ that satisfies the *quantum dynamical Yang-Baxter equation* (QDYBE)

$$R^{1,2}(\lambda) * R^{1,3}(\lambda + \hbar h^2) * R^{2,3}(\lambda) = R^{2,3}(\lambda + \hbar h^1) * R^{1,3}(\lambda) * R^{1,2}(\lambda + \hbar h^3) \tag{4.18}$$

QUESTION 4.24. *Does such a quantization always exist?*

The most famous example of non-triangular dynamical r-matrices was found in [**1**] by Alekseev and Meinrenken, then extended successively to a more general context in [**25, 24, 21**], and quantized in [**21**].

Following [**21**], remark that for any non-triangular dynamical r-matrix r such that $r + r^{\mathrm{op}} = t \in (S^2\mathfrak{g})^{\mathfrak{g}}$ (quasi-triangular case) one can define $\rho := r - t/2$ and $Z := \frac{1}{4}[t^{1,2}, t^{2,3}]$. Then ρ is a modified dynamical r-matrix for $(\mathfrak{g}, \mathfrak{h}, Z)$; morever the assignment $r \longmapsto \rho$ is a bijective correspondence between quasi-triangular dynamical r-matrices for $(\mathfrak{g}, \mathfrak{h}, t)$ and modified dynamical r-matrices for $(\mathfrak{g}, \mathfrak{h}, Z)$. Now observe that if $J(\lambda)$ is a dynamical twist quantizing ρ, then $R(\lambda) = J^{\mathrm{op}}(\lambda)^{-1} * e^{\hbar t/2} * J(\lambda)$ is a quantum dynamical R-matrix quantizing the classical one r.

In this chapter we have constructed such a dynamical twist in the triangular case $t = 0$. One can ask

QUESTION 4.25. *Does such a dynamical twist exist for any quasi-triangular dynamical r-matrix? At least in the reductive and abelian cases?*

This question seems to be more reasonable than the previous one.
More generally one can ask if Conjecture 4.1 (and its smooth and meromorphic versions) is true in general. A positive answer was given in [**21**] when $\mathfrak{h} = \mathfrak{g}$; but unfortunately it is not known in general, even for the non-dynamical case $\mathfrak{h} = \{0\}$ (which is the last problem of Drinfeld [**19**]: quantization of coboundary Lie bialgebras).

Momentum maps. Finally let us mention that if $r(\lambda)$ is a triangular dynamical r-matrix for $(\mathfrak{g},\mathfrak{h})$, then the bivector field

$$\pi := \overrightarrow{r(\lambda)} + \sum_i \frac{\partial}{\partial \lambda^i} \wedge \overrightarrow{h_i} + \pi_{\mathfrak{h}^*}$$

is a $G \times H$-biinvariant Poisson structure on $G \times \mathfrak{h}^*$ and the projection $p : G \times \mathfrak{h}^* \to \mathfrak{h}^*$ is a momentum map. Moreover, according to [**72**] any dynamical twist quantization $J(\lambda)$ of $r(\lambda)$ allows us to define a $G \times H$-biinvariant star-product $*$ quantizing π on $G \times \mathfrak{h}^*$ as follows:

$$\begin{aligned}
f * g &= f *_{PBW} g \quad \text{if} \quad f, g \in C^\infty(\mathfrak{h}^\vee) \\
f * g &= fg \quad \text{if} \quad f \in C^\infty(G), g \in C^\infty(\mathfrak{h}^\vee) \\
f * g &= \exp\big(\hbar \sum_i \frac{\partial}{\partial \lambda^i} \otimes \overrightarrow{h_i}\big) \cdot (f \otimes g) \quad \text{if} \quad f \in C^\infty(\mathfrak{h}^\vee), g \in C^\infty(G) \\
f * g &= \overrightarrow{J(\lambda)}(f \otimes g) \quad \text{if} \quad f, g \in C^\infty(G)
\end{aligned}$$

Therefore the map $p^* : (\mathrm{Fct}(\mathfrak{h}^*)[[\hbar]], *_{PBW}) \to (\mathrm{Fct}(G \times \mathfrak{h}^*)[[\hbar]], *)$ becomes a quantum momentum map in the sens of [**69**]. Consequently, there might be a way to look at momentum maps and their quantum analogues as Maurer-Cartan elements in DGLA.

ANNEXE A

Basic materials

A.1. Lie-Rinehart algebras and bialgebroids

A.1.1. Lie-Rinehart algebras. In this subsection we recall some ideas from [**57**].

DEFINITION A.1. *A* Lie-Rinehart algebra *is the data of a commutative algebra B, a Lie algebra L which is also a B-module, and a B-linear morphism of Lie algebras*

$$\rho : L \to DerB$$

such that the B-module and Lie algebra structures on L are compatible in the following sense: for any $f \in B$ and any $u,v \in L$

$$[u,fv] = f[u,v] + \rho(u)(f)v \tag{A.1}$$

The notion of a morphism of Lie-Rinehart algebras is obvious.

We then define the *enveloping algebra* $U_R L$ of a Lie-Rinehart algebra (R, L, ρ) as the quotient algebra

$$T(R \oplus L)/\left\{ \begin{array}{c} f \otimes g - fg,\ f \otimes u - fu, \\ u \otimes f - f \otimes u - \rho(u)f, \\ u \otimes v - v \otimes u - [u,v], \\ f,g \in R, \qquad u,v \in L. \end{array} \right\} \tag{A.2}$$

REMARK A.2. $U_R L$ can be obtained from the universal enveloping algebra of the Lie algebra $R \oplus L$, with bracket given by

$$[f+u, g+v] = \rho(u)(g) - \rho(v)(f) + [u,v]$$

for any $f,g \in R$ and any $u,v \in L$. Namely,

$$U_R L = U(R \oplus L)/\{f \otimes u - fu, f \in R, u \in R \oplus L\}$$

$U_R L$ is endowed with an increasing filtration

$$R = U_R^0 L \subset U_R^1 L \subset U_R^2 L \subset \cdots \subset U_R L, \tag{A.3}$$

which is defined by assigning the degree 1 to the elements of L.

PROPOSITION A.3. *Any morphism $(\varphi, \Phi) : (R, L) \to (R', L')$ of Lie-Rinehart algebras lifts to a morphism of algebras*

$$\widetilde{\Phi} : U_R L \to U_{R'} L'$$

compatible with the filtration and such that $\widetilde{\Phi}\Big|_R = \varphi$ and $\widetilde{\Phi}\Big|_L = \Phi$.

Remark that for any $P \in U_R^k L$ and any $u \in R \oplus L$, $Pu - uP \in U_R^{k-1} L$. It implies in particular that the right and left R-module structures on $Gr(U_R L)$ are the same, and we may regard $Gr(U_R L)$ as a R-algebra. Thus we can formulate the Poincaré-Birkhoff-Witt theorem for Lie-Rinehart algebras:

THEOREM A.4 (Rinehart, [**57**]). *If L is a projective R-module, then the canonical map*

$$S_R(L) \to Gr(U_R L)$$

is an isomorphism of R-algebras.

A.1.2. Bialgebroids.

DEFINITION A.5 ([**70**], see also [**42**]). *A* bialgebroid *is an associative algebra with unit H together with a base algebra R, an algebra homomorphism $s : R \to H$ and an algebra antihomomorphism $t : R \to H$ whose respective images commute together (the source and target maps, which give H an R-bimodule structure), and R-bimodule maps $\Delta : H \to H \otimes_R H$ (the coproduct) and $\varepsilon : H \to R$ (the counit) such that*

(1) $\Delta(1) = 1 \otimes_R 1$ *and* $(\Delta \otimes_R id) \circ \Delta = (id \otimes_R \Delta) \circ \Delta$
(2) $\forall a \in R, \forall h \in H, \Delta(h)(t(a) \otimes_R 1 - 1 \otimes_R s(a)) = 0$
(3) $\forall h_1, h_2 \in H, \Delta(h_1 h_2) = \Delta(h_1)\Delta(h_2)$
(4) $\varepsilon(1_H) = 1_R$ *and* $(\varepsilon \otimes_R id_H) \circ \Delta = (id_H \otimes_R \varepsilon) \circ \Delta = id_H$

Given a bialgebroid H over a base R, an *anchor* is a representation $\rho : H \to \mathrm{End}(R)$ which is also a R-bimodule map and satisfies

$$\begin{array}{ll} s(\rho(x_1) \cdot a)x_2 = xs(a) & x \in H, a \in R \\ x_1 t(\rho(x_2) \cdot a) = xt(a) & x \in H, a \in R \\ \rho(x) \cdot 1_R = \varepsilon(x) & x \in H \end{array}$$

EXAMPLE A.6. Besides the fact that $U_A L$ is an algebra, $U_A L$ is also equipped with a coassociative A-linear coproduct $\Delta : U_A L \to U_A L \otimes_A U_A L$ which is defined as follows

$$\Delta(1) = 1 \otimes 1\,,$$

$$\Delta(u) = u \otimes 1 + 1 \otimes u, \qquad \Delta(PQ) = \Delta(P)\Delta(Q)\,, \tag{A.4}$$

$$\forall\ u \in L,\ P, Q \in U_A L.$$

A *twist* ([**70**]) in a Hopf algebroid H over a base R is an invertible element $J \in H \otimes_R H$ that satisfies

$$\begin{array}{l} J^{12,3} J^{1,2} = J^{1,23} J^{2,3} \\ (\varepsilon \otimes_R \mathrm{id})(J) = (\mathrm{id} \otimes_R \varepsilon)(J) = 1_H \end{array} \tag{A.5}$$

Let H be a bialgebroid over a base R (respectively, with anchor ρ), and let $J = \sum_i x_i \otimes_R y_i$ be a twist. Then one can define a new product on R given by $a *_J b = \sum_i (\rho(x_i)a)(\rho(y_i)b)$, a new coproduct $\Delta_J = J^{-1}\Delta J$, and new source and target maps given by $s_J(a) = \sum_i s(\rho(x_i)a)y_i$ and $t_J(a) = t(\rho(y_i)a)x_i$. Denote $R_J = (R, *_J)$.

THEOREM A.7 ([**70**], theorem 4.14). *Let $(H, R, \Delta, s, t, \varepsilon)$ be a bialgebroid (respectively with anchor ρ). If J is a twist, then $(H, R_J, \Delta_J, s_J, t_J, \varepsilon)$ is again a bialgebroid (respectively with the same anchor ρ).*

A.2. L_∞-algebras, morphisms and modules

Here, by a *graded vector space* we mean a $\mathbb{Z}$-graded vector

$$V = \bigoplus_{k \in \mathbb{Z}} V^k$$

such that $V^k = 0$ for $k \ll 0$.

For a detailed study of L_∞-structures we refer to [**35, 41**].

A.2.1. L_∞-algebras and their morphisms.

DEFINITION A.8. *A* L_∞-structure *on a graded vector space $\mathcal{L}$ is a degree 1 and 2-nilpotent coderivation Q on the cofree cocommutative coalgebra $C^c(\mathcal{L}[1])$ cofreely cogenerated by $\mathcal{L}$ with the shifted parity.*

We call such a couple $(\mathcal{L}, Q)$ a L_∞-algebra.

By cofreeness, such a coderivation Q is uniquely determined by structure maps

$$Q^{[n]} : \Lambda^n \mathcal{L} \to \mathcal{L}[2-n] \qquad (n \geq 1) \tag{A.6}$$

which satisfy a (semi-)infinite collection of quadratic equations. In particular, $(\mathcal{L}, Q^1)$ is a cochain complex.

EXAMPLE A.9. Any DGLA $(\mathfrak{g}, d, [\,,\,])$ is obviously a L_∞-algebra. Namely, the coderivation Q is given by structure maps $Q^{[1]} = d$, $Q^{[2]} = [\,,\,]$ and $Q^{[n]} = 0$ for $n > 2$.

DEFINITION A.10. *Let $(\mathcal{L}_1, Q_1)$ and $(\mathcal{L}_2, Q_2)$ be two L_∞-algebras. A* morphism of L_∞-algebras $F : (\mathcal{L}_1, Q_1) \rightarrowtail (\mathcal{L}_2, Q_2)$, *or* L_∞-morphism, *is a degree* 0 *morphism of coalgebras*

$$F : C^c(\mathcal{L}_1[1]) \to C^c(\mathcal{L}_2[1])$$

such that

$$F \circ Q_1 = Q_2 \circ F \tag{A.7}$$

Again by cofreeness, such a morphism is uniquely determined by structure maps

$$F^{[n]} : \Lambda^n \mathcal{L}_1 \to \mathcal{L}_2[1-n] \qquad (n \geq 1) \tag{A.8}$$

which satisfy a (semi-)infinite collection of quadratic equations also involving $Q_i^{[n]}$'s. In particular $F^{[1]} : \mathcal{L}_1 \to \mathcal{L}_2$ is a morphism of complexes.

EXAMPLE A.11. Any morphism of DGLA is a L_∞-morphism with all structure maps equal to zero except the first one.

DEFINITION A.12. *A* (L_∞-)quasi-isomorphism *is a L_∞-morphism of which the first structure map is a quasi-isomorphism of cochain complexes (i.e. induces an isomorphism at the level of cohomology).*

Two L_∞-algebras are said to be *quasi-isomorphic* if they are connected by a chain of quasi-isomorphisms (in a pedant formulation we should have said "if they are isomorphic in the localized category with respect to quasi-isomorphisms"). A L_∞-algebra is *formal* if it is quasi-isomorphic to the graded Lie algebra of its cohomology.

Actually, L_∞-quasi-isomorphisms are really invertible, namely

THEOREM A.13. *Let $F : (\mathcal{L}_1, Q_1) \rightarrowtail (\mathcal{L}_2, Q_2)$ be a L_∞-quasi-isomorphism. Then there exists a L_∞-quasi-isomorphism $H : (\mathcal{L}_2, Q_2) \rightarrowtail (\mathcal{L}_1, Q_1)$ which induces the inverse isomorphisms between the cohomology of complexes $(\mathcal{L}_i, Q_i^{[1]})$.* □

In this dissertation we need the following lemma, that allows us to modify the first structure map of a L_∞-morphism:

LEMMA A.14 ([**16**]). *Let $F : C^c(\mathcal{L}_1[1]) \to C^c(\mathcal{L}_2[1])$ be a L_∞-morphism. For any linear map $L : \mathcal{L}_1 \to \mathcal{L}_2[-1]$ there exists a L_∞-morphism $\Psi : C^c(\mathcal{L}_1[1]) \to C^c(\mathcal{L}_2[1])$ with first structure map $\Psi^{[1]} = F^{[1]} + Q_2^{[1]} \circ L + L \circ Q_1^{[1]}$. Moreover, if F is a L_∞-quasi-isomorphism then Ψ is also.*

PROOF. First remark that L extends uniquely to a linear map $C^c(\mathcal{L}_1[1]) \to C^c(\mathcal{L}_2[1])$ of degree -1 such that

$$\Delta_2 \circ L = \Big(F \otimes L + L \otimes F + \frac{1}{2} L \otimes (Q_2 \circ L + L \circ Q_1) + \frac{1}{2}(Q_2 \circ L + L \circ Q_1) \otimes L\Big) \circ \Delta_1$$

where Δ_1 and Δ_2 denote comultiplications in $C^c(\mathfrak{g}_1[1])$ and $C^c(\mathfrak{g}_2[1])$, respectively.
Then define $\Psi := F + Q_2 \circ L + L \circ Q_1$. □

REMARK A.15. Assume that in the previous lemma $\mathcal{L}_1$ and $\mathcal{L}_2$ are filtrated, F is such that $F^{[n]}$ takes values in $(\mathcal{L}_2)_{\leq n-1}$, and $L\big((\mathcal{L}_1)_{\leq k}\big) \subset (\mathcal{L}_2)_{\leq k+1}$. Then one can obviously check that for any $X \in (\Lambda^n \mathcal{L}_1)_{\leq k}$, $F^{[n]}(X) \in (\mathfrak{g}_2)_{\leq n+k-1}$.

A.2.2. L_∞-modules and their morphisms. Let $(\mathcal{L}, Q)$ be a L_∞-algebra.

DEFINITION A.16. *A L_∞-*module *over $(\mathcal{L}, Q)$ is a graded vector space $\mathcal{M}$ together with a degree 1 and 2-nilpotent coderivation P on the cofree $C^c(\mathcal{L}[1])$-comodule $C^c(\mathcal{L}[1]) \otimes \mathcal{M}$ cogenerated by $\mathcal{M}$.*

Again by cofreeness, such a P is uniquely determined by structure maps

$$P^{[n]} : \Lambda^n \mathcal{L} \otimes \mathcal{M} \to \mathcal{M}[1-n] \qquad (n \geq 0) \tag{A.9}$$

which satisfy a (semi-)infinite collection of quadratic equations also involving $Q^{[n]}$'s. In particular $(\mathcal{M}, P^{[0]})$ is a chain complex.

EXAMPLE A.17. Any DG module $(\mathfrak{m}, b, \bullet)$ over a DGLA $(\mathfrak{g}, d, [,])$ is obviously a L_∞-module over $(\mathfrak{g}, Q = d + [,])$. Namely, the coderivation P is given by structure maps $P^{[0]} = b$, $P^{[1]} = \bullet$ and $P^{[n]} = 0$ for $n > 1$.

DEFINITION A.18. *Let $(\mathcal{M}_1, P_1)$ and $(\mathcal{M}_2, P_2)$ be two L_∞-modules over a L_∞-algebra $(\mathcal{L}, Q)$. A* morphim of L_∞-modules $G : (\mathcal{M}_1, P_1) \succ\!\succ\!\to (\mathcal{M}_2, P_2)$, *or* L_∞-morphism, *is a degree 0 morphism of comodules*

$$G : C^c(\mathcal{L}[1]) \otimes \mathcal{M}_1 \to C^c(\mathcal{L}[1]) \otimes \mathcal{M}_1$$

such that

$$G \circ P_1 = P_2 \circ G \tag{A.10}$$

By cofreeness, such a morphism is uniquely determined by structure maps

$$P^{[n]} : \Lambda^n \mathcal{L} \otimes \mathcal{M} \to \mathcal{M}[-n] \qquad (n \geq 0) \tag{A.11}$$

which satisfy a (semi-)infinit collection of quadratic equations also involving $Q^{[n]}$'s and $P_i^{[n]}$'s. In particular $P^{[0]} : \mathcal{M}_1 \to \mathcal{M}_2$ is a morphism of complexes.

EXAMPLE A.19. Any mophism of DG modules is a L_∞-morphism with all structure maps equal to zero except the 0-th one.

DEFINITION A.20. *A* (L_∞-)quasi-isomorphism *between two L_∞-modules is a L_∞-morphism of which the 0-th structure map is a quasi-isomorphism of chain complexes (i.e. induces an isomorphism at the level of homology).*

As for L_∞-algebras, L_∞-quasi-isomorpisms of L_∞-modules are invertible, namely

THEOREM A.21. *Let $G : (\mathcal{M}_1, P_1) \succ\!\succ\!\to (\mathcal{M}_2, P_2)$ be a L_∞-quasi-isomorphism of L_∞-modules. Then there exists a L_∞-quasi-isomorphism $H : (\mathcal{M}_2, P_2) \succ\!\succ\!\to (\mathcal{M}_1, P_1)$ which induces the inverse isomorphism between the homology of complexes $(\mathcal{M}_i, P_i^{[0]})$.* □

A.2.3. Maurer-Cartan elements and twisting procedures. This subsection is inspired by [**17**, Section 2.3].

Let $\mathfrak{l}$ be a local (pro-)Artinian algebra with maximal ideal $\mathfrak{m}$. In the typical situation $\mathfrak{m}$ is a (pro-)nilpotent algebra) and $\mathfrak{l} = \mathbf{k} \oplus \mathfrak{m}$.

DEFINITION A.22. *Let $(\mathcal{L}, Q)$ be a L_∞-algebra. Then $\pi \in \mathcal{L}^1 \otimes \mathfrak{m}$ is called a Maurer-Cartan element if*

$$\sum_{k \geq 1} \frac{1}{k!} Q^{[k]}(\underbrace{\pi, \ldots, \pi}_{k \text{ times}}) = 0 \tag{A.12}$$

Now assume we have a L_∞-morphism of L_∞-algebras $F : (\mathcal{L}_1, Q_1) \succ\!\to (\mathcal{L}_2, Q_2)$. Then using a Maurer-Cartan element $\pi \in \mathcal{L}_1^1 \otimes \mathfrak{m}$ one can *twist* everything, namely:

PROPOSITION A.23.

(1) *The element*

$$\Pi = \sum_{k\geq 1} \frac{1}{k!} F^{[k]}(\underbrace{\pi, \ldots, \pi}_{k \text{ times}}) \in \mathcal{L}_2^1 \otimes \mathfrak{m}$$

is a Maurer-Cartan element.

(2) *The degree* 1 *coderivation* Q_1^π *defined by structure maps*

$$(Q_1^\pi)^{[n]}(x_1, \ldots, x_n) = \sum_{k\geq 0} \frac{1}{k!} Q^{[k+n]}(\underbrace{\pi, \ldots, \pi}_{k \text{ times}}, x_1, \ldots, x_n) \qquad (n \geq 1)$$

is 2*-nilpotent. In other words,* $(\mathcal{L}_1 \otimes \mathfrak{l}, Q_1^\pi)$ *is a* L_∞*-algebra.*

(3) *The degree* 0 *morphism of coalgebras* F^π *defined by structure maps*

$$(F^\pi)^{[n]}(x_1, \ldots, x_n) = \sum_{k\geq 0} \frac{1}{k!} F^{[k+n]}(\underbrace{\pi, \ldots, \pi}_{k \text{ times}}, x_1, \ldots, x_n) \qquad (n \geq 1)$$

is such that $F^\pi \circ Q_1^\pi = Q_2^\Pi \circ F^\pi$*. In other words, one has*

$$F^\pi : (\mathcal{L}_1 \otimes \mathfrak{l}, Q_1^\pi) \succ\!\!\rightarrow (\mathcal{L}_2 \otimes \mathfrak{l}, Q_2^\Pi)$$

(4) *If* F *is a* L_∞*-quasi-isomorphism, then so is* F^π*.* □

Now let $(\mathcal{M}_1, P_1)$ and $(\mathcal{M}_2, P_2)$ be two L_∞-modules over a L_∞-algebra $(\mathcal{L}, Q)$, with a L_∞-morphism $G : (\mathcal{M}_1, P_1) \succ\!\succ\!\rightarrow (\mathcal{M}_2, P_2)$. Again, using a Maurer-Cartan element $\pi \in \mathcal{L}^1 \otimes \mathfrak{m}$ one can also twist these objects:

PROPOSITION A.24.

(1) *Let* $i = 1, 2$*. The degree* 1 *coderivation* P_i^π *defined by structure maps*

$$(P_i^\pi)^{[n]}(x_1, \ldots, x_n, v) = \sum_{k\geq 0} \frac{1}{k!} G^{[k+n]}(\underbrace{\pi, \ldots, \pi}_{k \text{ times}}, x_1, \ldots, x_n, v) \qquad (n \geq 0)$$

is 2*-nipotent. In other words,* $(\mathcal{M}_i \otimes \mathfrak{l}, P_i^\pi)$ *is a* L_∞*-module over* $(\mathcal{L} \otimes \mathfrak{l}, Q^\pi)$*.*

(2) *The degree* 0 *morphism of comodules* G^π *defined by structres maps*

$$(G^\pi)^{[n]}(x_1, \ldots, x_n, v) = \sum_{k\geq 0} \frac{1}{k!} G^{[k+n]}(\underbrace{\pi, \ldots, \pi}_{k \text{ times}}, x_1, \ldots, x_n, v) \qquad (n \geq 0)$$

is such that $G^\pi \circ P_1^\pi = P_2^\pi \circ G^\pi$*. In other words, one has*

$$G^\pi : (\mathcal{M}_1 \otimes \mathfrak{l}, P_1^\pi) \succ\!\!\rightarrow (\mathcal{M}_2 \otimes \mathfrak{l}, P_2^\pi)$$

(3) *If* G *is a* L_∞*-quasi-isomorphism, then so is* G^π*.* □

A.2.4. Gauge equivalences of Maurer-Cartan elements. In this subsection we restrict ourself to the case of DGLA.

Let $(\mathfrak{g}, d, [,])$ be a DGLA. The infinitesimal action of the Lie algebra $\hbar\mathfrak{g}^0[[\hbar]]$ on $\hbar\mathfrak{g}^1[[\hbar]]$ defined by

$$q \cdot \alpha = dq + [q, \alpha]$$

integrates into an affine action of the corresponding prounipotent group $\mathbf{G}$. An element of this group is called a *gauge equivalence*. Since the subspace of Maurer-Cartan elements in $\hbar\mathfrak{g}^1[[\hbar]]$ is stable under the action of $\mathbf{G}$ then one can define the *moduli space* $\mathbf{MC}$ of Maurer-Cartan elements up to gauge equivalence by

$$\mathbf{MC} := \frac{\{\pi_\hbar \in \hbar\mathfrak{g}^1[[\hbar]] \text{ s.t. } d\pi_\hbar + \frac{1}{2}[\pi_\hbar, \pi_\hbar]\}}{\mathbf{G}}$$

THEOREM A.25. *Let* $\Psi : (\mathfrak{g}_1, d_1 + [\,,]_1) \succ\!\!\rightarrow (\mathfrak{g}_2, d_2 + [\,,]_2)$ *be a* L_∞*-quasi-isomorphism of DGLA. Then the map*

$$\pi_\hbar \longmapsto \sum_{k\geq 1} \frac{1}{k!} \Psi^{[k]}(\pi_\hbar, \ldots, \pi_\hbar)$$

defines an isomorphism $\mathbf{MC}_1 \longrightarrow \mathbf{MC}_2$.

A.2.5. What about sheaves? All the definitions are easy to sheafify. But when dealing with sheaves one has to be carefull with the following:

- a quasi-isomorphism of sheaves of complexes does NOT induce a quasi-isomorphism between the complexes of global sections (but it does if the sheaves have their standard Čech cohomology concentrated in degree 0, like sheaves of smooth sections of C^∞-bundles for example),
- in particular theorems A.13, A.21 and A.25 are not true for sheaves.

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