Time-varying delay systems and 1-D hyperbolic equations, Harmonic transfer function and nonlinear electric circuits

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ÉCOLE DOCTORALE
SCIENCES FONDAMENTALES ET APPLIQUÉES

THÈSE DE DOCTORAT

Systèmes à retard instationnaires et EDP hyperboliques 1-D instationnaires, fonctions de transfert harmoniques et circuits électriques non-linéaires

présentée et soutenue publiquement par

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En vue d'obtenir le grade de

Docteur en Sciences

de l'Université Côte d'Azur

Discipline : Mathématique

Equipe FACTAS et MCTAO - Inria Sophia Antipolis-Méditerranée

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Soutenue le 30 octobre 2020

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Résumé /Abstract

Résumé

Les amplificateurs contiennent des composants linéaires passifs, ainsi que non-linéaires actifs, qui peuvent tous être décrits par un nombre fini de variables d'état; ils contiennent aussi des lignes de transmission, généralement modélisées par des équations aux dérivées partielles 1-D hyperboliques comme les équations du Télégraphe sans perte qui rendent l'espace d'état de dimension infinie. En utilisant une forme intégrée des équations du Télégraphe, on obtient un modèle composé d'équations aux différence retardées et d'équations différentielles. Considérant une trajectoire périodique qui s'établit dans l'amplificateur à cause d'un signal périodique forçant, la thèse vise à caractériser la stabilité locale d'une telle trajectoire périodique. En utilisant une approximation de premier ordre, cela se réduit à étudier la stabilité exponentielle du système linéaire périodique temporel obtenu par linéarisation autour de la solution périodique, et qui est un réseau d'équations aux différences retardées dont les conditions aux limites sont couplé par des équations différentielles. La stabilité de ce type d'équations est fortement corrélée avec la stabilité d'un système périodique aux différences linéaires (via un argument de perturbation compacte). La thèse établit alors des conditions pour garantir la stabilité des systèmes retardés périodiques linéaires.

En raison du nombre énorme de composants électroniques, il est connu dans les livres d'ingénierie électronique que la stabilité ne peut pas se déterminer directement à partir du système linéarisée. Ainsi pour étudier les propriétés de stabilité du système linéarisé précédent, une famille de systèmes entrées-sorties est construite, obtenue en perturbant le système linéarisé par un petit courant i à un nœud du circuit et en observant la perturbation résultante de tension v entre deux nœuds. Via un développement de Fourier, la stabilité se ramène à étudier les singularités de la fonction de transfert harmonique (FTH) qui est une matrice infinie dépendant d'une variable complexe et à valeur banachique. Sous des hypothèses de dissipation à haute fréquence qui sont toujours vérifiées pour les amplificateurs, la thèse montre alors que la FTH possède au plus des pôles dans un demi plan droit complexe contenant strictement l'axe imaginaire. Ces pôles sont en particulier les logarithmes d'une famille finie de nombre complexe, et sous une hypothèse de contrôlabilité et d'observabilité, la solution périodique est localement stable si et seulement si la FTH n'a pas de poles dans le demi-plan droit complexe.

Mots clés: Système retardé à temps variant, Système hyperbolique 1-D à temps variant, Fonction de transfert harmonique, Stabilité, Circuits électriques.

Abstract

Amplifiers contain linear, passive components as well as nonlinear, active ones, all of which can be described by finitely many state variables; but they also contain transmission lines, typically modeled by simple hyperbolic Partial Differential Equations (PDE) like lossless telegrapher equations, that make the global state space of the circuit infinite-dimensional. Using an integrated form of telegraphers equations, one obtains a model comprised of delay difference and differential equations. Using first order approximation, this reduces to exponential stability of the time-periodic linear system obtained by linearizing around the periodic solution, which is a network of delay difference equations whose boundary conditions are coupled by differential equations. The stability of this kind of equation is strongly correlated with the stability of a periodic linear difference delay system (via a compact perturbation argument). The thesis then establishes conditions to guarantee the stability of periodic difference delay system systems.

Due to the huge number of electronic components, it is known in electronic engineering textbooks that stability cannot be determined directly from the linearized system. To study the stability properties of the previously-described linearized system, one constructs a family of input-output systems, obtained by perturbing the linearized system by a small current i at some node of the circuit and observing the resulting perturbation of the voltage v between two nodes. Via a Fourier development, stability is studied through the singularities of the harmonic transfer function (HTF) which is an infinite matrix depending on a complex variable with Banach value. Under high frequency dissipativity assumption, which are always verified for amplifiers, the HTF has at most poles in a complex right half-plane containing strictly the imaginary axis. These poles are in particular the logarithms of a finite family of complex numbers, and under an assumption of controllability and observability, the periodic solution is locally stable if and only if the HTF has no poles in the complex right half-plane.

Keywords: Time-varying difference delay equations, Time-varying 1-D hyperbolic systems, Harmonic transfer function, Stability, electric circuit.

Remerciement

Je voudrais tout d'abord remercier mes directeurs de thèse. La culture scientifique, et non scientifique, de Laurent Baratchart ont été et sont une grande source d'inspiration, bien que j'ai et ai eu besoin souvent d'un delai de trois mois pour comprendre ses idées profondes. Son éthique scientifique, sa qualité rédactionnelle et ses idées originales ont su guidé mon travail de thèse. Le sens pédagogique de Jean-Baptiste Pomet et son sens de la phrase juste m'ont permis de mieux exprimer mes idées mathématiques.

Je remercie Yacine Chitour et Jonathan Partington d'avoir accepté de rapporter cette thèse. Je les remercie pour leurs suggestions qui ont aidé à améliorer la qualité de ce manuscrit.

Je remercie Marius Tucznak d'avoir accepté de présider mon jury, ainsi que Birgit Jacob, Hoài-Minh Nguyên, Almudena Suárez et Adam Cooman d'y avoir participé.

Concernant mon travail de thèse, il doit également beaucoup à Gilles Lebeau et son expertise précieuse pour la stabilité des EDP hyperboliques.

Tout au long de ces quatre années, l'équipe FACTAS (feu APICS) a fourni des conditions de travail idéales. Si j'étais le guide Gault et Millau, je recommanderais chaudement d'aller faire un tour à la table de cette équipe aux heures de déjeuner, tout en vous invitant à réviser la situation géopolitique du Guatemala sous peine d'être possiblement perdu pendant la discussion. Je remercie plus personnellement Fabien et son sens de l'humour (ainsi qu'une pensée émue pour la salade de pomme de terre qu'il ne pourra pas manger à Noël cette année), Juliette pour les innombrables pauses cafés ainsi que sa dextérité pour dompter la machine infernale, Martine pour sa gentillesse (je n'en écrirais pas trop ici car l'octet pollue), Sylvain pour ses qualités pédagogiques et sa capacité à résoudre tout problème informatique en un temps record, ainsi que Marie-Line pour m'avoir aidé dans les taches administratives parfois abstruses.

Je remercie également l'équipe McTAO, même si l'incompatibilité du fuseau horaire concernant l'heure du déjeuner m'a amené à moins les côtoyer que l'équipe FACTAS. Je remercie Jean-Baptiste, dît le jeune, qui j'espère notera la propreté du bureau que j'ai laissé derrière moi, Lamberto et sa chaleurosité (en espérant que ce mot entre un jour dans le dictionnaire!) toute italienne, ainsi que Laetitia que j'ai moins connu.

Passons maintenant aux doctorants et posdoctorants que j'ai été ammené à côtoyer! David le Murcian aux problèmes, et aux solutions également, improbables. Adam le Flamand jovial avec qui j'avais un accord tacite: il m'expliquait l'électronique et les lignes de transmission tant que je lui donnais mon dessert à manger, ce qui tombait bien car je ne comprenais pas les lignes de transmission et lui oui, et je n'avais plus faim alors que lui était affamé. Gibin qui m'aura permis de situer la région de Kerala en Inde. Yacine qui a été un collègue de bureau fort agréable et qui m'a permis de retrouver la joie de jouer aux échecs (et de souvent perdre!). Ainsi que les inombrables doctorants qui sont passés de façon plus éphémères comme Pauline, Axel, Lucie, Sofya, Vanna Lisa et Christos. Je souhaite bonne chance pour leurs fins de thèse à Paul le constructeur d'abris pour

tortue, Masimba qui va bientôt parler un français impeccable et Alesia en espérant que la situation de son pays s'améliore.

J'ai aussi une pensée pour tous les professeurs ayant marqués mon cursus scolaire. Tout d'abord je pense à M.Delgendre professeur d'histoire-géographie au collège La Guicharde et Arnaud Beaumont professeur de philosophie, et je remercie ces derniers pour la curiosité intellectuelle qu'ils m'ont transmise. Ensuite je pense aux enseignants que j'ai eu lors de ma préparation à l'agrégation à Marseille et qui ont été décisifs notamment Pierre Bousquet, Franck Boyer et Brigitte Mossé.

Je remercie également Claudine Chaouiya et Elisabeth Remy de m'avoir initié il y a maintenant fort longtemps aux systèmes dynamiques discrets, et en espérant un jour pouvoir retravailler avec vous sur ces sujets.

Enfin, et plus personnellement, je remercie ma maman de m'avoir élevée seule et d'avoir toujours tout fait pour moi. Merci à mon oncle (dit Parrain ou Tonton Grumeau) et ma tante (dit Tatie Sylvie) pour m'avoir accueilli en leur maison en 2014 et sans qui ce travail n'aurait pas existé. Je remercie mon papa, même si les relations n'ont jamais été simple, d'être à nouveau prêt de moi.

Mes dernières pensées iront à ceux qui ne sont plus de ce monde. Papy Jo et Mamie Lucie, mes grands-parents. William, mon ami qui était le seul témoin de ma profonde incompréhension du théorème de Banach-Steinhaus.

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Introduction

Motivation, Computer Assisted Design for circuits

This work applies to more general active circuits than amplifiers, like pure oscillators, but we restrict the motivations to amplifiers for simplicity. An ideal amplifier would be an active circuit that outputs exactly the input signal multiplied by some gain that does not depend on the signal. In practice, amplifier design consists in proposing a circuit, made of available elementary elements, so as to obtain a gain that does not vary too much when the input belongs to some class of signals. The available elementary elements are of three sorts:

- passive components: resistors, capacities and inductances,
- active nonlinear components: diodes and transistors (fed by an external power source),
- lines that induce some delay effect, negligible at low frequencies.

The specification of an amplifier is always given in terms of a "frequency response", *i.e.* the output occurring for sinusoidal input signals of all possible frequencies. The purpose of computer aided design (CAD) tools is to compute this frequency response for a given circuit design proposed by the user. For RLC circuits, this may be readily computed, even analytically, possibly using computer algebra if the number of components is very large. For more complex ones, where the response of some elements may be available numerically only, specialised tools are needed, (see *e.g.* [Key]). They rely on a dynamical model of the circuit, obtained from the models of all components, and compute numerically a forced periodic solution of this dynamical system under periodic excitation, through a numerical method, often referred to as "Harmonic Balance", that we briefly describe in Section 1.2.
There is substantial recent literature on the subject in Electronic Engineering, see for instance [Kun06, SQ02, Sua09], because this is important for circuit design. Knowing whether the computed periodic solution corresponds to the steady state response to periodic excitation that will be observed in real life is obviously very important. Two related points are commonly raised. First, there may be more than one possible response to a given sinusoidal signal, at least for the mathematical/numerical model used by the CAD tool: it would be more correct to state that these tools compute *one* response. We do not investigate this point further. The second point is certainly more determining and has received a lot of attention: the stability of the computed response is not a priori guaranteed, and if the response happens to be unstable it will simply not be observed because it would occur only for very special initial conditions of inner variables of the circuit, that are in practice never exactly achieved. On the contrary, with the additional information that all the computed responses are stable, it would be granted that numerical tools predict the frequency response (at least if the basin of attraction is reasonably large in terms of the inner variables of the circuit, but we do not deal with this question here and consider local stability only, obviously necessary). Stability is by no means a straightforward side-result of the estimation of the response: it may happen that a

computed response is unstable (although the numerical process converged). Determining stability of the computed response is hence crucial, it has retained a lot of attention and some methods are indeed even implemented [STA, CSO+16, Sua15, SQ02].

The initial motivation of this thesis is the mathematical framework to determine stability of a given response as computed by these CAD tools. We hope to provide some mathematical insight.

The stability question

Let us now say a work about the model of the circuit, examine what the response for a fixed harmonic/periodic/sinusoidal signal is, and state the stability question more precisely. Without transmission lines, the equations of a circuit composed of the above mentioned elementary components are given by a set of ordinary differential equations (ODEs), hence a model with an finite dimensional state; this case is reviewed in Chapter 1. Taking the lines into account, as in most of this thesis, leads, on the contrary, to a model with an infinite dimensional state, containing for instance currents and voltages distributed along the lines, the model is of a more complex nature, it is a set of hyperbolic partial differential equations (PDEs) modeling the lines, coupled by ODEs through their boundary conditions; when the PDEs are lossless Telegrapher's equations, as always assumed here, the model can also, thanks to explicit integration of the PDEs, be viewed as a "hybrid" system including delay equations and differential equations (see (1) below), the state being then more abstract.

For a given periodic input signal, the amplifier, including the excitation by that signal, can be seen as a complex periodic time-varying dynamical system, whose state is all the currents and voltages, possibly distributed in the lines. We are interested in periodic solutions of this dynamical system, with the same period as the input. As mentioned above, there might be more than one; without discussing existence or uniqueness, we assume that we are provided with a periodic solution, *i.e.* a possible response of the amplifier, obtained numerically, and we need to decide on its stability. Recall that this stability (*i.e.* the property that starting from initial conditions close to the solution leads to remaining close and even to converge to the solution) is certainly a necessary condition for this solution to be observed in reality: without stability, the observed behaviour may be an escape to high currents and burn, or convergence to another periodic solution, or actually more chaotic behavior.

A classical step to determine stability is to linearise the dynamical system around the now known periodic solution, obtaining a periodic time-varying linear dynamical system, whose (strict exponential) stability amounts to local stability of the periodic solution. This stability could be determined by computing Floquet exponents, or the spectrum of the discrete flow pushing the state one period ahead, but these computations are out of reach, because of the great number of transmission lines and because the periodic solution we linearise around is only known numerically. The way around is to analyse this stability in an indirect manner, through an auxiliary linear input-output system that relates a fictitious input to a fictitious output and whose dynamics (drift) is the considered time-varying linear system; the input is for instance a current added between two points and the output the measurement of some voltage, in the linearised circuit. The point that makes this construction relevant is that this fictitious input-output system can be made real for the CAD tool, and this tool can compute its frequency response, that in turn may be used to assess the desired stability.

To describe the way one uses this frequency response, let us make, for a while, a simplifying assumption, that occurs for instance if the periodic signal to be amplified is the zero signal and hence the periodic solution is simply an equilibrium point: we assume in this paragraph that the the linear input-output system is time-invariant. Its frequency response is then simply the value on the imaginary axis of its transfer function. Determining stability from the knowledge of the transfer

function on the imaginary axis led to a large literature from a theoretical or numerical viewpoint. It is well known that, provided that the auxiliary input-output system is controllable and observable, the spectrum of the operator defining the linear dynamics is the locus of singularities of the transfer function, so that local exponential stability of the origin for the linear dynamical system occurs if and only if the transfer function has no singularity in the right half plane (see [PW97]). It has been proven in [BCC+18] that the transfer function is meromorphic for such time-invariant systems, even in the presence of transmission lines; this legitimates the use of rational approximation to determine the possible unstable poles, all the more so as the same reference also proves that there are at most a finite number of such poles in the right half plane if the system is "dissipative at high frequency", which is the case of any realistic circuit. Rational approximation methods are not in the scope of this thesis, we only make a rough sketch in Section 1.5 (Chapter 1), but we retain the principle that a method that amounts to find poles of a function in a domain where it is meromorphic is feasible. Modulo this principle, one now sees how to determine stability in the case of an equilibrium point, leading to a time-invariant linear approximation. The two ingredients that make this method (use of an auxiliary input-ouptut system, estimation of the transfer function of the imaginary axis, rational approximation) workable are on the one hand that stability is captured by the location of the singularities of the transfer function and on the other hand that the part of the singularities that have to be identified to decide stability are poles, so that rational approximation is relevant.

Let us go back to the more general case of a periodic excitation creating a periodic solution, leading to a periodic time-varying linear approximation and an auxiliary periodic time-varying linear input-output system. There is a generalisation of the concept of transfer function to these latter systems, it is called harmonic transfer function (further denoted HTF). It was introduced rather recently [Wer90] in the context of circuits, it is not very well known in the control theory community, probably because time-varying linear systems have retained little attention. Since it is a central object in the thesis, we define it with some care, for finite-dimensional systems (*i.e.* neglecting transmission lines) in Chapter 1, for an example (with lines) that we treat extensively in Chapter 2, and in general in Chapter 6. Its definition is based on the Fourier development of the system. The HTF is a more sophisticated object than the usual transfer function: instead of a single function of the complex variable, it is an infinite matrix with square integrable lines, whose entries are analytic function of a complex variable, analytic in some domain of the complex plane, or it can also be seen as an operator valued (operators $l^2(\mathbb{Z}) \to l^2(\mathbb{Z})$) function of the complex variable. The values on the imaginary axis of the coefficients of this infinite matrix also derive from the frequency response computed by the CAD tool. This suggests a similar route as the one we just described for time-invariant systems. However, to complete the analysis, the two "ingredients" mentioned at the end of the previous paragraph have to be provided in the case of periodic system too. On the one hand, we have to link the local stability of the periodic solution with the singularities of the harmonic transfer function, or of the entries of the infinite matrix than encodes it. On the other hand, we have to determine what the structure of the HTF is and wether this operator is meromorphic or not on the domain that matters for stability, namely the right-half plane, in order to apply, for instance to a few entries of the infinite matrix defining the HTF, rational approximation algorithms as discussed in Section 1.5. The arguments given in [BCC+18], based on complex valued function analysis, are not sufficient anymore when we consider the harmonic transfer function; in fact, as mentioned above, the HTF takes its values in a complex Banach space and the theory for these objects is less tractable than for complex functions. Solving the problem will lead us through the theory of the delay systems, functional analysis via a compact perturbation argument, the Lyapunov functions and a constant back and forth between time-domain and frequency-domain.

Mathematical point of view, models, summary of contributions

Let us move to a more mathematical viewpoint to describe our work in more details. Each lossless transmission line is modeled, in this thesis, by a Telegrapher's equation. An amplifier is a directed graph where the edges are the lossless transmission lines and the nodes are composed of the active nonlinear components and the passive components. Under the assumption that at each node we can express all the currents and voltages in function of the voltages of the condensators, the currents of the inductors and the currents of the lines which arrive at this node, the resolved form of these 1-D hyperbolic equations leads to the following type of periodic nonlinear "hybrid" delay system (this is detailed in Chapter 3):

$$\begin{cases} \frac{dx(t)}{dt} = f(t, x(t), y(t), y(t-\tau_1), \cdots, y(t-\tau_N)) \\ y(t) = g(t, y(t-\tau_1), \cdots, y(t-\tau_N), x(t)) \end{cases} \tag{1}$$

where f and g are obtained through the implicit function theorem from implicit equations, we neglect this difficulty and assume these maps reasonably smooth. This assumption is systematically made in electronic engineering, the justification is not immediate for singularities in the equations may occur, see for instance [Sma72], but it seems that algorithms in the CAD tools would not manage to find periodic solutions going through these singularities.

We assume that System (1) admits a periodic solution $(\overline{x}(t), \overline{y}(t))$ and we are interested in local stability of this periodic solution. Linearising System (1) around this periodic solution yields a linear time-varying system:

$$\begin{cases} \frac{dx(t)}{dt} = A_1(t)x(t) + \sum_{i=0}^{N} B_i^1(t)y(t-\tau_i), \\ y(t) = \sum_{i=1}^{N} B_i^2(t)y(t-\tau_i) + A_2(t)x(t), \end{cases} \tag{2}$$

where the smoothness of the functions of time depend on the smoothness of f and g and on the smoothness of the periodic solution.

Under adequate smoothness assumptions, we prove in Chapter 6 that the periodic solution $(\overline{x}(t), \overline{y}(t))$ of System (1) is locally stable if the origin of the linearised system (2) is exponentially stable. Thus local stability is reduced to exponential stability of the linearised system. We also prove that we have this exponential stability if and only if the monodromy operator of System (2) has its spectrum strictly inside the unit disk, the monodromy operator being the operator which integrates the solution one period ahead. These preliminary results are classical but needed technical adaptation. The core of the thesis is devoted to describing this spectrum in relation with the singularities of the HTF (Harmonic Transfer function, introduced above) of some input-ouptput system. Let us keep in mind that the spectrum of an operator defined on an infinite dimensional Banach space is usually more complicated than a finite set of eigenvalues.

One step in that direction is to consider the "high frequency limit" of System (2), that puts to zero the x component and leads to the following periodic linear difference delay system:

$$z(t) = \sum_{i=1}^{N} B_i^2(t)z(t-\tau_i). \tag{3}$$

For circuits, the behavior of the system at high frequency means that, the condensators become wires and the inductors become open switches. This is of help for two reasons.

- On the one hand, we prove (see Chapters 2 and 6) that the monodromy operator of System (2) is a compact perturbation of the monodromy operator of System (3). So, if one proves

exponential stability of (3), i.e. if one proves that the spectrum of the monodromy operator of (3) is contained in a disk of radius strictly less than 1, this establishes that the spectrum of the monodromy operator of System (2) is composed of a finite number of eigenvalues outside a disk with radius strictly less than one and some part of that disk, that we do not have to describe because it does not impair stability.

- On the other hand, exponential stability of difference delay systems of the type (3) is studied in a detailed manner in Chapters 4 and 5. The literature on such equations is really scarce in the time-dependent case, we believe that the material in these two chapters has an interest beyond microwave circuits, but in any case they prove that exponential stability holds for these systems, *i.e.* for (3), under a passivity property that is always, implicitely or explictely, assumed in electrical engineering.

  The fact that any real life circuit, like an amplifier, is passive, or dissipative, at high frequency roughly means that, although active, the circuit does not inject energy in high frequency enough signals. When making a "theoretical" circuit out of ideal diodes, transistors, capacitors, inductors and lines, one may easily come up with a circuit that does not have this property, hence is not "real life". Assuming this property for our models amount to not forget to add some small capacities or inductances or resistances here and there, to make "ideal" elements more real. Dissipativity is also a well known idea in mathematics to study the stability of the systems [BC16, Wil72, BLME07, TGTN+07, Wil13].

Since high frequency passivity is always assumed, this closes the needed description of the unstable part of the spectrum of the monodromy operator of (2), now known to be composed of isolated eigenvalues only.

The last point needed in the process, and addressed by the thesis, is to link the unstable part of the spectrum of the monodromy operator with the unstable singularities of the HTF of the input-ouptut system obtained by disturbing the circuit represented by System (3) with a current u and observe the voltage response v_1 at some node; the resulting input-output system is of the form:

$$\begin{cases} \frac{dx(t)}{dt} = A_1(t)x(t) + \sum_{i=0}^{N} B_i^1(t)y(t-\tau_i) + C_1(t)u(t) \\ y(t) = \sum_{i=1}^{N} B_i^2(t)y(t-\tau_i) + A_2(t)x(t) + C_2(t)u(t) \\ v_1(t) = \sum_{i=0}^{N} B_i^3(t)y(t-\tau_i) + A_3(t)x(t) + C_3(t)u(t), \end{cases} \tag{4}$$

for suitable periodic time varying matrices $B_i^3(.), A_3(.), C_1(.), C_2(.), C_3(.)$.

The development of the matrices depending periodically on time in System (4) into Fourier series and the Laplace transform allow us to better describe the concept of the HTF: it can be defined as the infinite matrix $H(\cdot)$ function of the complex variable s such that the input and the output are linked (modulo transients) by

$$\begin{pmatrix} \vdots \\ \mathcal{L}\{v_1\}(s+i\omega_0) \\ \mathcal{L}\{v_1\}(s) \\ \mathcal{L}\{v_1\}(s-i\omega_0) \\ \vdots \end{pmatrix} = H(s) \begin{pmatrix} \vdots \\ \mathcal{L}\{u\}(s+i\omega_0) \\ \mathcal{L}\{u\}(s) \\ \mathcal{L}\{u\}(s-i\omega_0) \\ \vdots \end{pmatrix}, \tag{5}$$

where $\omega_0 = 2\pi/T$ denotes the proper frequency of the system and $\mathcal{L}\{\cdot\}$ denotes the Laplace transform:

$$\mathcal{L}\{v_1\}(s) := \int_0^{+\infty} e^{-st} v_1(t) dt\,, \quad \mathcal{L}\{u\}(s) := \int_0^{+\infty} e^{-st} u(t) dt\,. \tag{6}$$

The definition of the HTF will be re-stated and made more precise in Chapters 1, 2 and 6.

Let us briefly state how this thesis links the HTF of System (4) with the monodromy operator of System (2). Under the stability at high frequency of the amplifier, we are able to prove that the HTF is meromorphic in some "extended" right half-plane $\{z \in \mathbb{C},\, \Re(z) > -\alpha\}$ with α positive, and that all its poles in that half plane are logarithms of eigenvalues of the monodromy operator of System (2). This justifies the paradigm that under the stability at high frequency, all unstable singularities are poles. This is proved into details on a class of examples in Chapter 2 (on this class of example, system (3) reads $z(t) = a\, z(t-\tau)$ with z scalar real, $\tau > 0$ and $|a| < 1$ so that its stability is not the major problem) and in Chapter 6 for general systems (4). A difficult unsolved problem is to tell in advance which logarithms will indeed be singularities and which entries of the infinite matris $H(\cdot)$ do carry these singularities. Partial results are given. The notion of singularities of this infinite matrix, or of this operator valued function, is subtle.

We describe our contributions in a more detailed way, chapter by chapter, in the next "Plan and contributions of the thesis".

Plan and contribution of the thesis

Generalities

Chapter 1 is an overview of the knowledge concerning the Harmonic Balance (HB) techniques, the Harmonic Transfer Function (HTF) and the link with the local stability for the circuits modeled by nonlinear ordinary differential equations ([Wer90, Lou14, Sua09, Hal69]). Chapter 1 establishes the major ideas to study the periodic systems in finite dimension:

- the Floquet theory which sa ys that a linear periodic differential equation is equivalent to a constant differential equation modulo a periodic change of basis.
- the fact that the local stability is given by the spectrum of the monodromy operator, *i.e.* the operator solution that we integrate after one period.
- the variation of constant formula which permits to express the solution of a periodic linear differential equation disturbed by a pertubation in function of the periodic linear differential equation itself.
- the Harmonic Transfer Function is an operator valued analytic map, where the values are the continuous operator from $l^2(\mathbb{Z})$ to $l^2(\mathbb{Z})$ with $l^2(\mathbb{Z})$ denoting the square summable series.
- the singularities of the Harmonic transfer function are at most the logarithm of the eigenvalues of the monodromy operator.
- under the classical controllability and observability assumption, if the Harmonic Transfer Function is analytic in the closed right half plane then the periodic solution is locally stable.

Chapter 2 is devoted to the study of a simple circuit which possesses only one lossless transmission line. This circuit reduces to a nonlinear neutral differential equation and we follow the theory that we can find in [HVL93]. The theory stated in [HVL93] deals with the continuous solutions (space C^0) for the neutral differential system. In fact, the theory can be generalized for the square integrable functions (L^2 space) and for the absolutely continuous functions with a square integrable derivative (W_2^1 space). The necessity to use the square integrable functions comes from the fact that the harmonic transfer function is viewed as an infinite matrix defined on the space of the square summable series. The absolutely continuous function with square integrable derivative occurs because, *via* the resolution of a first kind Volterra equation, the system is controllable on this space with a square integrable function; *i.e.* each element of W_2^1 can be reached with an element of L^2. The semigroup theory, a compact perturbation argument and the variation of constant formula permit to prove that the HTF is a meromorphic operator in the closed right half plane where the poles are the complex logarithm of a finite number of complex numbers, and if the HTF is holomorphic in the closed right half plane then the periodic solution is locally stable. Moreover if we add observability assumptions, the HTF has essential singularities view as a Banach valued operator in the left half plane when the delay and the period are not commensurable.

Chapter 3 reduces the network of lossless Telegrapher's equations with nonlinear differential boundaries to difference differential equations coupled with nonlinear difference equations. The result is not new but it seems difficult to find a proper reference in the literature. Moreover, this reduction is allowed under the assumption that at each nodes of the network, we can express all the voltages and currents of this node in function of the voltage of the condensators, the current of the inductors and the current at the end of the lines connected to this node.

Linear time-variant difference delay systems

Chapters 4 and 5 focus on the stability of the linear time-varying difference delay equations. The literature is relatively large for linear time-invariant delay equations ([CH70, Hen74, HVL93]) but becomes sparse when the equations are time dependent, where the most complete study can be found in [CMS16].

Chapter 4 gives a sufficient and necessary stability condition for a periodic linear difference delay system with continuous Hölder derivative.

The proof of the sufficiency follows the ideas of [Hen74, BC63]. More precisely, we establish a variation of constant formula which permits to express all the solutions of the periodic difference delay system in function of one so called fundamental solution and the initial data. The use of the Fourier development and the Laplace transform allow us to bound exponentially the variation of the fundamental solution and we conclude the exponential stability of the system. Contrary to the time-invariant case, the difficulties come from:

- we have to invert an analytic almost periodic operator a Banach space on a vertical strip and the theory of the analytic almost periodic complex function does not apply anymore ([Bes54]),
- we have to justify the swapping between the series coming from the Fourier developpment and the integral coming from the Laplace transform.

The proof of the necessity used in [Hen74, BC63] uses the spectral semigroup properties for time-invariant difference delay systems. In our case we cannot follow this path because the periodicity. However we are able to prove the necessity of our stability condition with the use of an input-output system.

The result is applied to the 1-D hyperbolic equations with continuous Hölder derivative boundaries, equations which are a little more general than the lossless Telegrapher's equations. We also give some conjectures for different periodic delay systems, systems more general than the periodic difference delay system with a finite number of delays that we considered in this chapter.

Chapter 5 undertake the study of the time-varying (not necessarily periodic) linear difference delay system which comes from a network of lossless Telegrapher's equations. When the system is time-invariant, we can find all the classic stability results in [BC16]. The idea of this chapter is to exploit the equivalence between the L^2 exponential stability (for the square integrable solution) and the C^0 exponential stability (for the continuous solution). This result can be found in [CMS16], but we hope to prove in a slight easier way the same result in Chapter 5. Assuming that the network dissipates energy at each node, we utilise a Lyapunov function to prove the L^2 exponential stability and so the C^0 exponential stability. In particular, the reasonning permits to give a stability criteria for the subclass of difference delay system which have the column disjoint.

Hybrid delay systems, harmonic transfer function and application to the circuits containing lossless transmission lines

Chapter 6 deals with generalities concerning the nonlinear hybrid delay equations, *i.e.* difference delay equations coupled with difference differential equations. The results are non-trivial generaliza-

tions obtained in the periodic neutral differential case in Chapter 2; the results are based on the establishment of a variation of a constant formula and a compact perturbation argument.

These theoretical results about delay systems are applied to nonlinear electric circuits containing lossless transmission lines. Under the assumption of the dissipativity at high frequency of each components of the circuit, we are able to prove that the harmonic transfer function is meromorphic in the closed right half plane and the poles are the logarithm of a finite number of complex numbers. Moreover, under controllability and observability assumptions, if the harmonic transfer function is analytic in the closed right half plane then the periodic solution is locally stable.

The outcomes of this chapter legitimate the harmonic balance techniques and the search of poles in the closed right half plane *via* meromorphic or rational approximations under the rather weak assumption of the dissipativity at high frequency of the circuit.

Chapter 1

Theory of the circuits modeled by ordinary differential equations

Although the thesis is devoted to active circuits in which the effect of transmission lines is important, we start with a chapter on the case where this effect can be neglected. The resulting models are ordinary differential equations rather than a coupling of these with propagation equations, as described in Chapter 2 for an example and Chapter 3 and further in general. In this simpler setting, we review here known facts and ideas, but we also show that the mathematical treatment still contains some challenges and open questions.

The considered circuits contain resistors, diodes (introducing nonlinearities), capacitors and inductors, as well as a periodic generator that represents a (periodic) signal to be amplified.The "amplifying" energy is brought by the diodes, that make the circuit active (non conservative).

We first recall how to obtain the equations of such a circuit. We then show that it is reasonable to assume that these equations have a periodic solution and describe a practical method to compute an approximation of this solution and of the linearized equations around this solution; the goal is to determine whether this periodic solution is stable.

Stability of this periodic solution is given by (global) exponential stability of the origin for the linear time-varying ordinary differential equation obtained by linearizing around the periodic solution. The most practical way to decide numerically on this stability is to generate the frequency response of a linear time-varying input-output system obtained by adding a perturbation to the linearized circuit, produced by an artificial additionnal current in some branch and imagining the measure of (for instance) a voltage between two nodes; there is some freedom in chosing this perturbation and this measure.

We describe briefly this process and the resulting linear time-varying input-output system, and then proceed to introduce the notion of *harmonic transfer function*, a generalisation of the well-known transfer function in the time-invariant case, that is not so classical and plays an important role in the thesis; it is more sophisticated than the transfer function: it is an *operator valued* (rather than scalar or matrix-valued) function of the complex variable. We discuss the relation between the singularities of this harmonic transfer function and exponential stability of the above mentionned time-varying linear ordinary differential equation. We prove that, if the perturbation and measure have been chosen so that the input-output system enjoys controllability and observability properties, the zero solution of the linear time-varying ordinary differential equation is exponentially stable if and only if the harmonic transfer function is holomorphic in the right half plane. Since it can be proved to be meromorphic in general, this justifies existing methods that apply rational approximation to a finite number of values of the frequency response of the input-output system (computed by Harmonic Balance methods) to locate poles. Here a discussion takes place on the fact that one only locates poles of coefficients in a Fourrier expansion of the Harmonic Transfer Function and not of the Harmonic Transfer Function as an operator value function. We also discuss rational approximation in this context.

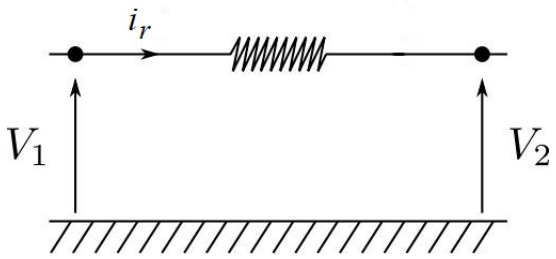
1.1 Equations in finite dimension

We consider in this chapter the simplest nonlinear circuits that we can have, ie a circuit made of with two kind of electronic components : dipoles (inductors, capacitors, resistors and diodes) and transistors. More precisely :

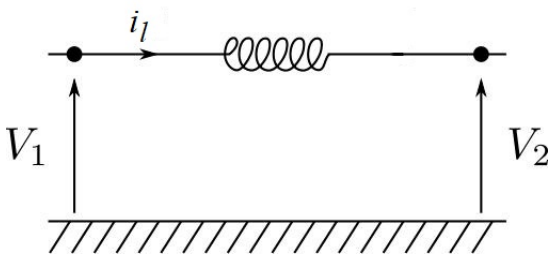
i. A dipole, also called a branch [CC98], is an electronic box with 2 terminals, labeled 1, 2, such that the current through the box, oriented from 1 to 2, is related to the potentials V_1, V_2 at the terminals by a linear, nonlinear or differential form. Elementary dipoles considered in this thesis are the following (see Figure 1.1):

- Ideal resistor, with :
$$v_r = Ri_r, \tag{1.1}$$
where $R > 0$ and $v_r = V_2 - V_1$.
- Ideal inductor, with $L\frac{di_l}{dt} = v_l$, $L > 0$, $v_l = V_2 - V_1$.
- Ideal capacitor, with $C\frac{dv_c}{dt} = i_c$, $C > 0$, $v_c = V_2 - V_1$.
- Ideal diode with :
$$v_d = f(i_d) \tag{1.2}$$
where f is a function and $v_d = V_2 - V_1$. In a circuit with a periodic solution, the ideal diode is often approximate by its linearization around the periodic trajectory which leads to the equation :
$$v_d = R(t)i_d, \tag{1.3}$$
where $R(t)$ is a periodic function (possibly negative).
- Ideal periodic voltage generator :
$$v(t) = p(t), \tag{1.4}$$
where $p(\cdot)$ is a C^∞ periodic function and $v = V_2 - V_1$.

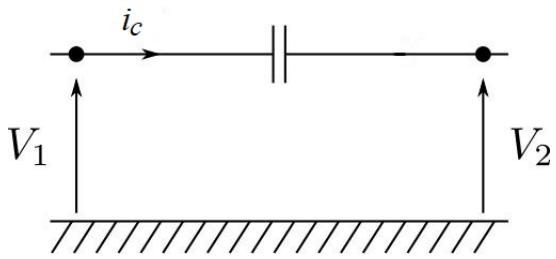
ii. A transistor is typically modeled by a controlled current source, usually combined with some resistors and (non-linear) capacitors. After linearization the latter become ordinary capacitors, so we are left to describe the current sources and their linearization. A controlled current source has 3 terminals. When the transistor if a Field Effect Transistor (FET), these terminals are called gate, source, and drain, denoted respectively by G, S and D (see Figure 1.2). Their behavior is described by a relation of the form :
$$i_D = f(v_{GS}, v_{DS}) \text{ and } i_G = h(v_{GS}, v_{DS}), \tag{1.5}$$
where f, h are a non-linear real-valued function and $v_{GS} = V_G - V_S$, $v_{DS} = V_D - V_S$. As in the case of diodes, this simple model assumes no inductive nor capacitive effect, as f and g only depends on v_{GS}, v_{DS} and not on their time derivatives, nor on the derivative of i_D. Moreover the function f are increasing in both variables. Like the diode, we will often consider linearized transistor. In a circuit with a periodic solution, the ideal diode is often approximate by its linearization around the periodic trajectory which leads to the equation :
$$i_D = g_m(t)v_{GS} + g_d(t)v_{DS} \text{ and } i_G = \tilde{g}_m(t)v_{GS} + \tilde{g}_d(t)v_{DS}, \tag{1.6}$$
where $g_m(t), g_d(t), \tilde{g}_m(t)$ and $\tilde{g}_m(t)$ are periodic functions. And $g_m(t)$ and $g_d(t)$ are strictly positive with the assumption on f.



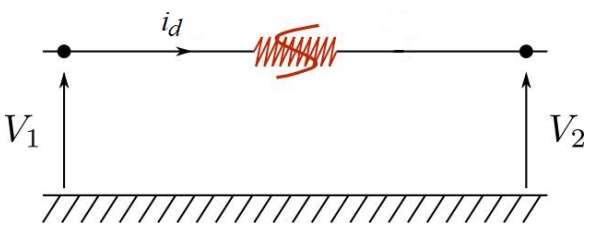
(a) Resistor



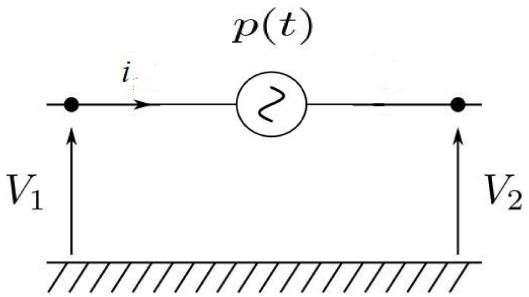
(b) Inductor



(c) Capacitor



(d) Diode



(e) Periodic Voltage Generator

Figure 1.1 : Elementary dipoles

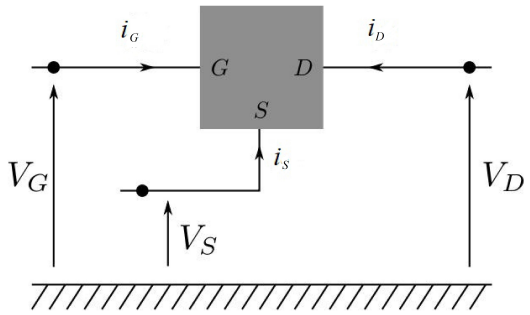


Figure 1.2 : Controlled current source.

The components defined in *i* and *ii* are put together to form a connected graph and we call this object a circuit. In a circuit, The link between the currents and the voltages in the circuit is given by the Kirchhoff law, ie the current arriving at each node is equal to the current leaving the node and the directed sum of the voltages around any closed loop is zero. We are interested to obtain the equations which governs the behavior of the circuit. Given a circuit, if we assume that all the voltages and all the currents of the circuit can be reconstructed from the knowledge of all the inductors' currents and the capacitor's voltages in the circuit thus the circuit is modeled by a nonsingular ordinary differential equation (result due to Brayon-Moser [BM64]). We have in the same way that if there exists some currents and some voltages which can give all the currents and voltages in the circuit, then the equations of the circuit are (possibly singular) ordinary differential equations (see chapter 10 of [HS74]). The more general result on the structure of the equation of such circuits has been proved by S. Smale ([Sma72]). He proved that these circuits are generically differential equation on a real submanifold.

Theorem 1.1 ([BM64])**.** *Noting by abuse of notation all the inductors current i_l and all the capacitors voltage v_c of the circuit and assuming that all voltages and currents in the circuit can be expressed by i_l and v_c, then there exists two functions h_l and h_c such that :*

$$\begin{cases} \dfrac{di_l}{dt} = h_l(t, i_l, v_c), \\ \dfrac{dv_c}{dt} = h_c(t, i_l, v_c). \end{cases} \tag{1.7}$$

Moreover h_l or h_c have the minimum regularity of the functions in equations (1.2) and (1.5).

Theorem 1.2 ([Sma72])**.** *Assuming that all voltages and currents in the circuit can be expressed by $x \in \mathbb{R}^n$ which is a vector composed of some currents and voltages of the circuit (n is equal to the number of capacitors and inductors), then there exists two functions f and g such that :*

$$\frac{df(t, x(t))}{dt} = g(t, x(t)), \tag{1.8}$$

Moreover f or g have the minimum regularity of the functions in equations (1.2) and (1.5).

Remark 1.3. *Theorems 1.1 and 1.2 stated here are a weaker version than the originals. In fact, the function h_l, h_c, f and g are a particular structure (almost like an Hamiltonian System) but we do not take advantage of this more precise information in the rest of the thesis.*

1.2 Harmonic Balance Approach to compute a periodic solution

We consider an amplifier with a periodic signal which has to be amplified and we sketch how the electronic engineers, through the Harmonic Balance Approach, approximate the amplified signal. Our exposition of the method is a bit mathematical compared with standard electronic textbooks (see for instance [SQ03] and [Sua09]), which is why we start with the formal assumptions below that make our account meaningful :

i. the circuit contains one and only one ideal periodic generator with a period T,

ii. the functions (1.2) and (1.5) associated to the nonlinear elements of the circuit (ie diodes and transistors) are at least of class C^4,

iii. the assumption of Theorem 1.2 is verified; hence,

$$\frac{df(t,x(t))}{dt} = g(t,x(t)),\ t \geq 0, \tag{1.9}$$

where f and g are at least C^4 and T-periodic in their first variable, while $x \in \mathbb{R}^n$ is a vector composed of certain currents and voltages in the circuit.

iv. the system (1.9) has a T-periodic solution $\overline{x}$, forced by the periodic generator, which does not meet a singularity of (1.9), ie $\partial_2(t,\overline{x}(t))$ is invertible for all t real.

The periodic generator in assumption *i.* represents the signal which has to be amplified. Assumption *ii.* is important because justifying the Harmonic Balance method mathematically requires some smoothness. Assumption *iii.* allows us to express the equations of the amplifier. Among electronic engineers, it is commonly accepted that for almost all active circuit of interest there exists a periodic solution; i.e., that assumption *iiii.* holds. However, from a mathematical point of view, this is not obvious and it would be interesting to prove theorems on the generic existence of periodic solutions for such circuits. It has to be noted that under a kind of dissipativity assumption at high frequency (see equation (1.10) below), one can prove the desired existence via the Brouwer fixed-point theorem :

Theorem 1.4. *Denote* $(,)$ *(resp.* $||\cdot||$*) the scalar product (resp. Euclidean norm) on* $\mathbb{R}^n$*. Assuming that* $f(t,x(t)) = x(t)$ *in the system* (1.9) *and that :*

$$(x, g(t,x)) < 0, \tag{1.10}$$

for all t *real and all* x *such that* $||x|| \geq k$ *for some fixed* $k > 0$*, then the system* (1.9) *admits at least one periodic solution of period* T*.*

Proof. The equation (1.10) permits to prove that all the solutions of the system (1.9) are bounded and then the Brouwer fixed-point theorem give the result (see [SC64] p.366). □

The assumption (1.10) is always true for realistic circuits. In fact, the realistic diodes and transistors have capacitive and inductive effects and so have just resistive effects at high frequency. Hereafter we do not discuss further the existence of a periodic solution, and we take it for granted. The Harmonic Balance method seeks an approximation of a T-periodic solution to the system (1.9). However, surprisingly perhaps, there may exist periodic solutions to system (1.9) with a period different from T (see [CGM16]). It is clear that under our regularity assumptions, a periodic solution $\overline{x}$ is of class at least C^3.

Let us quickly go through the method to compute an estimate of a truncated Fourier series of the periodic solution, following loosely the exposition in [Sua09]. The idea of this method, called harmonic balance, is simple and can be summed up as :

- using the Laplace transform, we convert the time domain into the frequency domain,
- we develop the periodic system in Fourier series and to truncate these series,
- we solve for the truncated Fourier coefficients using a numerical fixed point method.

More precisely :

i. We note

$$\omega_0 := \frac{2\pi}{T} \tag{1.11}$$

the proper frequency of the system. We take the Laplace transform of the system (1.9) evaluated in $\overline{x}$ for $s \in \mathbb{C}$ with strictly positive real part :

$$s\mathcal{L}\{f(t,\overline{x}(t))\}(s) = \mathcal{L}\{g(t,\overline{x}(t))\}(s), \tag{1.12}$$

where

$$\mathcal{L}\{f(t,\overline{x}(t))\}(s) := \int_0^{+\infty} e^{-st} f(t,\overline{x}(t))dt \text{ and } \mathcal{L}\{g(t,\overline{x}(t))\}(s) := \int_0^{+\infty} e^{-st} g(t,\overline{x}(t))dt. \tag{1.13}$$

Note that the Lapace transforms are well-defined for $\Re(s) > 0$ since $f(t,\overline{x}(t))$ and $g(t,\overline{x}(t))$ are C^3-smooth and periodic, and hence, are bounded.

ii. We develop the periodic solution $\overline{x}(t)$ in Fourier series :

$$\overline{x}(t) \quad = \quad \sum_{j\in\mathbb{Z}} \overline{x}_j e^{ij\omega_0 t}, \text{where } \overline{x}_j := \tfrac{1}{T}\int_0^T \overline{x}(t)e^{-ij\omega_0 t}dt \ \forall j \in \mathbb{Z}. \tag{1.14}$$

From now on we identify the function $\overline{x}$ with its Fourier series and putting the Fourier expression of $\overline{x}(t)$ in $f(t,\overline{x}(t))$ and $g(t,\overline{x}(t))$, we obtain :

$$f(t,\overline{x}(t)) = f(t, \sum_{j\in\mathbb{Z}} \overline{x}_j e^{ij\omega_0 t}) \text{ and } g(t,\overline{x}(t)) = g(t, \sum_{j\in\mathbb{Z}} \overline{x}_j e^{ij\omega_0 t}). \tag{1.15}$$

Setting $\overline{X} := (\cdots, \overline{x}_{-1}, \overline{x}_0, \overline{x}_1, \cdots)$ and expanding $f(t,\overline{x}(t))$ and $g(t,\overline{x}(t))$ in Fourier series, we obtain on using the equation (1.15):

$$\begin{aligned} f(t,\overline{x}(t)) \quad &= \quad \sum_{j\in\mathbb{Z}} \overline{f}_j(\overline{X}) e^{ij\omega_0 t}, \text{where } \overline{f}_j(\overline{X}) := \tfrac{1}{T}\int_0^T f(t, \sum_{k\in\mathbb{Z}} \overline{x}_j e^{ik\omega_0 t})e^{-ij\omega_0 t}dt, \ \forall j \in \mathbb{Z} \\ g(t,\overline{x}(t)) \quad &= \quad \sum_{j\in\mathbb{Z}} \overline{g}_j(\overline{X}) e^{ij\omega_0 t}, \text{where } \overline{g}_j(\overline{X}) := \tfrac{1}{T}\int_0^T g(t, \sum_{k\in\mathbb{Z}} \overline{x}_k e^{ik\omega_0 t})e^{-ij\omega_0 t}dt, \ \forall j \in \mathbb{Z} \end{aligned} \tag{1.16}$$

Replacing (1.16) in (1.12), integrating termwise the Fourier series which is permitted because of the regularity of the periodic solution, we can remark that the equation (1.12) has a meromorphic continuation for $\Re(s) < a$ where a is a strictly negative real where the only poles are in $s = ik\omega_0$ for all $k \in \mathbb{Z}$. Evaluating the system (1.12) in each frequency $s = ik\omega_0$ for all $k \in \mathbb{Z}$, we obtain the following infinite system :

$$(ik\omega_0)\overline{f}_k(\overline{X}) = \overline{g}_k(\overline{X}), \text{ for all } k \in \mathbb{Z}. \tag{1.17}$$

The equation (1.17) symbolizes the fact that the numerator of the partial fraction decomposition of each side of the equation (1.12) are equal. The equation (1.17) can be interpreted like the behavior of the system (1.9) on the multiple of the proper frequency.

We truncate the system (1.17); i.e. we fix $N \in \mathbb{N}$ and note $\overline{X}_N = (\overline{x}_{-N}, \cdots, \overline{x}_{-1}, \overline{x}_0, \overline{x}_1, \cdots, \overline{x}_N)$ and we consider the system :

$$(ik\omega_0)\check{f}_k(\overline{X}_N) = \check{g}_k(\overline{X}_N), \text{ for all } k \in [-N, \cdots, N], \tag{1.18}$$

where

$$\begin{aligned}\check{f}_k(\overline{X}_N) &:= \frac{1}{T}\int_0^T f(t, \sum_{j=-N}^{N} \overline{x}_j e^{ij\omega_0 t})e^{-ik\omega_0 t}dt,\\ \check{g}_k(\overline{X}_N) &:= \frac{1}{T}\int_0^T g(t, \sum_{j=-N}^{N} \overline{x}_j e^{-ij\omega_0 t})e^{ik\omega_0 t}dt.\end{aligned} \tag{1.19}$$

Noting $\tilde{f}_k = (ik\omega_0)\check{f}_k - \check{g}_k$ for $k \in [-N, \cdots, N]$ and $\widehat{f}_N = (\tilde{f}_{-N}, \cdots, \tilde{f}_N)^*$ where the * denote the transposition, we can rewrite the system (1.18) as :

$$\widehat{f}_N(\overline{X}_N) = 0. \tag{1.20}$$

The truncation transforms the infinite equations with an infinite number of unknowns (1.17) in $2N+1$ equations with $2N+1$ unknowns and we have just to determine a solution to a nonlinear equation in finite dimension. In practice, the solution of the system (1.20) is obtain via a numerical approximation.

iii. Determining an approximation of $\overline{x}$ can be done by performing a fixed point Newton method. In fact, the Jacobian in (1.20) is easily computed: noting $\partial_2 f$ (resp. $\partial_2 g$) the derivative with respect to the second argument of f (resp. g), we have that the Jacobian of $\widehat{f}_N$ at the point $\overline{X}_N$ is equal to

$$Diag(-iN\omega_0, \cdots, iN\omega_0)L_{\partial_2 f,N} - L_{\partial_2 g,N}, \tag{1.21}$$

where for h equal to f or g, we have :

$$\left[L_{\partial_2 h,N}(\overline{X}_N)\right]_{k,n\in[|-N,N|]} = \left(\frac{1}{T}\int_0^T \partial_2 h\left(t, \sum_{j\in -N}^{N} \overline{x}_j e^{ij\omega_0 t}\right)e^{i(n-k)\omega_0 t}dt\right), \tag{1.22}$$

and

$$Diag(-iN\omega_0, \cdots, iN\omega_0) \text{ is the diagonal matrix with diagonal } (-iN\omega_0, \cdots, iN\omega_0). \tag{1.23}$$

The knowledge of the Jacobian matrix allows one to numerically solve the system (1.18) efficiently through a Newton method, and thus to obtain an approximation of the Fourier coefficient of the periodic solution $\overline{x}$.

To conclude, the harmonic balance method permits to approximate the periodic solution $\overline{x}$ via the approximation of a finite number of its Fourier coefficient. From now, we assume that the harmonic balance has been performed and thus we consider that the approximation of the periodic solution $\overline{x}$ via its Fourier coefficients is the knowledge of the exact solution of the system.

1.3 Stability and monodromy operator

Assuming that we have calculated the periodic solution $\overline{x}$ by Harmonic Balance, we want to know if it is "physically observable". In fact, it is possible that after the physical construction of the circuit, the periodic solution calculated by Harmonic Balance does not occur because of the variability of the electrical components of the circuit. So to ensure that the periodic solution calculated by Harmonic Balance appears in a real electrical circuit we need to ensure that the periodic solution is stable under the variability of the electrical components of the circuit. In mathematical terms, to be locally stable means that if we consider a solution of the circuit x which starts near the periodic trajectory thus this solution convergence exponentially to $\overline{x}$. The classical way to determine the local stability is to linearize the system (1.9) around the periodic solution and to prove that the zero of the linearized system is exponentially stable. For a linear periodic system the zero exponentially stability is given by the eigenvalues of the monodromy operator, ie the operator which integrates a solution during a period T. All the following theorems and definitions can be found in a lot of classical books on ordinary differential equation (see [Hal69] for example).

Definition 1.5 (Exponential Local Stability)**.** We say that the solution $\overline{x}$ of the system (1.9) is exponentially locally stable if there exists $\delta > 0$ such that there exists a K and γ strictly positive such that :

$$||x(0) - \overline{x}(0)||_2 \leq \delta \Rightarrow ||x(t) - \overline{x}(t)||_2 \leq Ke^{-\gamma t}, \tag{1.24}$$

where x is a solution of the equation (1.9).

To study local stability, we linearise system (1.9) around the periodic solution calculated by harmonic balance which does not pass through a singularity :

$$\frac{d}{dt}y(t) = A(t)y(t),\ t \geq 0\ , \tag{1.25}$$

where $A(\cdot)$ is a $n \times n$ periodic matrix at least C^2.

Definition 1.6. The system (1.25) is said exponentially stable if there exists $K > 0$ and $\gamma > 0$ such that :

$$||y(t)||_2 \leq Ke^{-\gamma t},\ t \geq 0\ . \tag{1.26}$$

Proposition 1.7. *If the system (1.25) is exponentially stable then the solution $\overline{x}$ of the system (1.9) is locally stable.*

We introduce the fundamental solution $X(t, \tau)$ which checks :

$$\frac{d}{dt}X(t,\tau) = A(t)X(t,\tau),\ t \geq \tau \tag{1.27}$$

$$X(\tau,\tau) = Id,\ \text{for all } \tau \in \mathbb{R}, \tag{1.28}$$

where Id is the $n \times n$ identity matrix. The fundamental solution is central in the following of this chapter. First of all, the fundamental solution contains all the solution of the system (1.25) because of :

$$y(t) = X(t, 0)y(0). \tag{1.29}$$

Moreover the fundamental solution contains entirely the stability information of the system through the monodromy operator and the theory of Floquet, and it permits to link the solution of the system (1.25) that we disturb by a function $h \in L^1_{loc}([0,+\infty[,\mathbb{R}^n)$:

$$\frac{d}{dt}y(t) = A(t)y(t) + h(t),\ t \geq 0 \tag{1.30}$$

with the system (1.25).

These considerations are summed up in the following theorems and definitions :

Definition 1.8. The monodromy operator of the system (1.25) is the operator $X(T,0)$.

Proposition 1.9. *The system (1.25) is exponentially stable if and only if the spectrum of the monodromy operator is strictly included in the unit disk.*

Theorem 1.10 (Floquet)**.** *The fundamental solution has the form :*

$$X(t,\tau) = P(t)e^{Q(t-\tau)}P(\tau) \tag{1.31}$$

where $P(t)$, Q are $n \times n$ matrices with P invertible, $P(t+T) = P(t)$ for all t, and Q constant. Moreover, putting $z(t) = P(t)y(t)$ we have that z check the equation :

$$\frac{dz}{dt} = Qz, \tag{1.32}$$

and the zero of the system (1.25) is exponentially stable if and only if the eigenvalues of the matrix Q are in the open left half plane.

Theorem 1.11 (Variation of constant formula)**.** *We have :*

$$y(t) = X(t,0)y(0) + \int_0^t X(t,\tau')h(\tau')d\tau' \tag{1.33}$$

1.4 Input-output system and Wereley's Harmonic Transfer Function matrix

By Proposition 1.9, the local stability of system (1.25) depends on the eigenvalues of the monodromy operator $X(T,0)$. From a numerical analysis point of view, we could perform time-domain analysis in order to approximate the monodromy operator and its eigenvalues for the system (1.25). This strategy is possible when looking at periodic differential systems but fails when looking at circuits containing transmission lines. Indeed such circuits induces delays effects, and because of the large number of components of the circuit and the periodicity of the system, the discretization time step become to small to be able to provide computation in the time domain. Since the main subject of this thesis is to deal with periodic differential systems coupled with transmission lines, we do not detail such numerical methods. The harmonic balance method is entangled with frequency analysis and we give quickly the numerical methods in the frequency domain through a fictitious input-output system which permits to approximate the eigenvalues of the monodromy operator (1.25). Obviously it requires a back and forth between the time-domain and the frequency-domain.

To introduce the frequency stability methods, we disturb the linearized circuit by a small source of current $u \in L^2_{loc}([0,+\infty[,\mathbb{R})$ at time zero, where $L^2_{loc}([0,+\infty[,\mathbb{R})$ is the space of square integrable function on each compact, and we obtain the voltage response to this current perturbation :

$$\begin{cases} \frac{d}{dt}y(t) = A(t)y(t) + B(t)u(t) \\ z(t) = C(t)y(t) + D(t)u(t),\ t \geq 0, \end{cases} \tag{1.34}$$

where $A(\cdot)$, $B(\cdot)$, $C(\cdot)$, $D(\cdot)$ are $n \times n$, $n \times 1$, $1 \times n$ and 1×1 T-periodic matrices, at least C^2. Thus we have an input-output system, called a linear time periodic system, where the output is for the instance considered to be zero for t strictly negative. When the system (1.34) is time-invariant we can take Laplace transforms and express in a simple way via the transfer function the output in the frequency domain in terms of the input in the frequency domain. For the periodic case, we can proceed in the same way after developing the system (1.34) in Fourier series. This leads to the Wereley harmonic transfer function matrix. Apparently, it initially occurred in the thesis of Norman Wereley (see [Wer90]).

Another way to recover the harmonic transfer function is to write the output as a convolution operator applied to the input. The kernel of the convolution is periodic and can be developed in Fourier series. Performing a Laplace transformation, we obtain the harmonic transfer function which is an infinite matrix linking frequency domain outputs to frequency domains inputs. One can prove that the singularities of the harmonic transfer function has just poles. Moreover the poles are among the logarithms of the Floquet exponents. Since we deal with an input output system, we can "lose" poles because of a lack of controllability or observability. However, the link between the observability or controllability and the missing poles in the harmonic transfer function, has apparently not been studied much, even for linear periodic dynamical system in finite dimension.

1.4.1 The time-invariant constant case

To introduce Wereley's Harmonic Transfer Function Matrix for periodic systems, we start by recalling some results about linear time independent systems which can be found in classical books of control theory (see for example [Son98] or [PW97]). We consider an autonomous system :

$$\frac{d}{dt}y(t) = Ay(t),\ y(t) \in \mathbb{R}^n,\ A \in \mathcal{C}_{n\times n}(\mathbb{R})\ , t \geq 0. \tag{1.35}$$

We know that the zero solution of the system (1.35) is exponentially stable if and only if the spectrum of the matrix A is in the open strict complex left half plane. Consider now an input-output system :

$$\begin{cases} \frac{d}{dt}y(t) = Ay(t) + Bu(t) \\ z(t) = Cy(t) + Du(t)\ , t \geq 0. \end{cases} \tag{1.36}$$

Taking the Laplace transform in the system (1.36), we obtain :

$$\mathcal{L}\{z\}(s) = H(s)\mathcal{L}\{u\}(s) \tag{1.37}$$

with

$$H(s) := C(sI_n - A)^{-1}B + D \tag{1.38}$$

and we call this matrix $H(s)$ the transfer function. We already now by definition that the matrix $H(s)$ is a rational function. Unfortunately, in bad cases, we can lose some eigenvalues of the matrix A in the function $H(s)$. However if the system (1.35) is observable and controllable (see Definition 1.12 below for a definition of the observability and controllability), then the singularities of the matrix H are poles and are the eigenvalues of the spectrum of A. So the singularities of the matrix H give the stability of System (1.35). Moreover we have the fact that it is a generic property to be observable and controllable in the class of linear constant system (which is not entirely relevant when the linear system comes from a linearization of a nonlinear system).

Definition 1.12. The system (1.35) is said :

- *observable*, if knowing that the output and the input are zero, then we have y must be identically zero,
- *controllable*, if there exists an input u from state y_0 at time t_0 to state y_1 at time $t_1 > t_0$.

The definition of the controllability and observability is a little too stronger for us purposes, so we give the following definition :

Definition 1.13. The system (1.35) is said :

- *detectable*, if knowing that the output z is zero and the input u is zero, then we have y which tends to zero at the infinity,
- *stabilizable*, if any initial state can be asymptotically steered to the origin by choosing the input u appropriately.

Theorem 1.14. *Under the assumption that the system (1.35) is observable and controllable, we have that the singularities of the transfer function (1.38) are exactly the eigenvalue of the matrix A. Moreover the system (1.35) is exponentially stable if and only if the poles of the transfer function are contained strictly in the left half plane.*

Theorem 1.15. *Under the assumption that the system (1.35) is detectable and stabilizable, we have that the system (1.35) is exponentially stable if and only if the poles of the transfer function are contained strictly in the left half plane.*

1.4.2 Periodic case

We want to apply the same previous reasoning in the periodic case, ie for the system (1.34). We would like to take the Laplace Transform in the system (1.34). Unfortunately, since the system is time dependant, we cannot express directly $\mathcal{L}\{z\}(s)$ in function of $\mathcal{L}\{u\}(s)$. However, developing in Fourier series the system (1.34), we obtain :

$$\begin{cases} \frac{d}{dt}y(t) = \sum\limits_{k=-\infty}^{\infty} a_k e^{i\omega_0 kt} y(t) + \sum\limits_{k=-\infty}^{\infty} b_k e^{i\omega_0 kt} u(t) \\ z(t) = \sum\limits_{k=-\infty}^{\infty} c_k e^{i\omega_0 kt} y(t) + \sum\limits_{k=-\infty}^{\infty} d_k e^{i\omega_0 kt} u(t) \end{cases} \tag{1.39}$$

Taking the Laplace transform in (1.39), and swapping Fourier series and the Laplace transform (it is allowed because of the regularity of the system), we have :

$$\begin{cases} s\mathcal{L}\{y\}(s) = \sum\limits_{k=-\infty}^{\infty} a_k \mathcal{L}\{y\}(s - ik\omega_0) + \sum\limits_{k=-\infty}^{\infty} b_k \mathcal{L}\{u\}(s - ik\omega_0) \\ \mathcal{L}\{z\}(s) = \sum\limits_{k=-\infty}^{\infty} c_k \mathcal{L}\{y\}(s - ik\omega_0) + \sum\limits_{k=-\infty}^{\infty} d_k \mathcal{L}\{u\}(s - ik\omega_0) \end{cases} \tag{1.40}$$

In the equation (1.40), it is not possible to express directly $\mathcal{L}\{y\}(s)$ because it appears together with the expressions $\mathcal{L}\{y\}(s + ik\omega_0)$, $k \in \mathbb{Z}$. These have to be considered *together* in the following way: first define the following infinite vectors:

$$Y(s) = \begin{pmatrix} \vdots \\ \mathcal{L}\{y\}(s + i\omega_0) \\ \mathcal{L}\{y\}(s) \\ \mathcal{L}\{y\}(s - i\omega_0) \\ \vdots \end{pmatrix}, U(s) = \begin{pmatrix} \vdots \\ \mathcal{L}\{u\}(s + i\omega_0) \\ \mathcal{L}\{u\}(s) \\ \mathcal{L}\{u\}(s - i\omega_0) \\ \vdots \end{pmatrix} \text{ and } Z(s) = \begin{pmatrix} \vdots \\ \mathcal{L}\{z\}(s + i\omega_0) \\ \mathcal{L}\{z\}(s) \\ \mathcal{L}\{z\}(s - i\omega_0) \\ \vdots \end{pmatrix}, \tag{1.41}$$

and then write a countable number of copies of (1.40) where s is replaced by $s + ik\omega_0$, $k \in \mathbb{Z}$, to obtain the following relation between infinite vectors:

$$\begin{cases} D_{\omega_0}(s)Y(s) = L_A Y(s) + L_B U(s)\,, \\ Z(s) = L_C Y(s) + L_D U(s)\,, \end{cases} \tag{1.42}$$

where $D_{\omega_0}(s) := Diag(\cdots, s + i\omega_0, s, s - i\omega_0, \cdots)$ and
$L_A := (a_{j-i})_{i,j\in\mathbb{Z}}$, $L_B := (b_{j-i})_{i,j\in\mathbb{Z}}$, $L_C := (c_{j-i})_{i,j\in\mathbb{Z}}$ and $L_D := (d_{j-i})_{i,j\in\mathbb{Z}}$.

Definition 1.16. The infinite matrix $H(s) := L_C[D_{\omega_0}(s) - L_A]^{-1}L_B + L_D$ is called the Wereley Harmonic Transfer Function matrix.

We have proved that that the Harmonic transfer function links the input at all the multiple of the proper frequency of the system with the output at all the multiple of the proper frequency of the system :

Proposition 1.17. *The harmonic transfer function matrix verifies the following equality :*

$$Z(s) = H(s)U(s) \tag{1.43}$$

According to Equation (1.43), the harmonic transfer function $H(s)$ relates the output at each frequencies $s + \frac{2i\pi k}{T}$ for a fixed $k \in \mathbb{Z}$ with the input at all frequencies $s + \frac{2i\pi n}{T}$ for all $n \in \mathbb{Z}$. In the autonomous case, we have seen that the transfer function permits to recover the eigenvalues of the matrix A under some assumption of controllability and observability. The questions that arise are :

- What is the sense of the matrix $H(s)$? More precisely on which space acts the function ?
- What is the link between the Wereley Harmonic Transfer function matrix in the Definition 1.16 and the monodromy operator $X(T, 0)$?

Concerning question 1, it seems relevant to consider that the matrix $H(s)$ acts on the space $l^2(\mathbb{Z})$ because of the Fourier development of the system. Concerning question 2, it has not been studied extensively. In fact the only results we have is that the matrix $H(s)$ has at most poles with the form $\frac{ln(\Psi)+i\omega_0}{T}$ with for each Ψ complex which is an eigenvalue of the monodromy operator.

1.4.3 Link between the Wereley matrix and the monodromy operator

Using the variation of constants formula (see equation (1.33)), we can express in the system (1.34) the output z as a convolution of the input u, where the kernel of the convolution is strongly linked with the fundamental solution. We sum up these remarks in the following theorem :

Theorem 1.18. *The solution $z(\cdot)$ of (1.34) with initial condition $z(0) = y(0) = 0$ is given by :*

$$z(t) = \int_0^{t^+} \tilde{X}(t, \tau)u(\tau)d\tau, \tag{1.44}$$

where,

$$\tilde{X}(t, \tau) = C(t)X(t, \tau)B(\tau) + D(t)\delta_0(t - \tau),\ t \geq \tau\ , \tag{1.45}$$

and X is the fundamental solution defined in the equation (1.27) and δ_0 is the Dirac distribution in zero.

The kernel $\tilde{X}$ is called the impulse response of the system and satisfies :

- the function $t \mapsto \tilde{X}(t, t-\tau)$ is T-periodic for all fixed $\tau \in \mathbb{R}$.
- the function $t \mapsto \tilde{X}(t, t-\tau)$ is at least C^2 because of the assumption on the regularity of the periodic function.
- $\tilde{X}(t, t-\tau)$ and its derivative in t are exponentially bounded, ie there exists $K > 0$ and $\gamma > 0$ such that :

$$\int_0^\tau \left\| \frac{\partial^j}{\partial t^j} \tilde{X}(t, t-s) \right\| ds \leq K e^{\gamma \tau}, \tag{1.46}$$

  for $j \in [0, 1, 2]$ and where $||\cdot||$ is the norm induced by the euclidean norm $||\cdot||_2$ on $\mathbb{R}^n$.

Definition 1.19 (ITF)**.** The function :

$$G(t, s) = \int_0^{+\infty} \tilde{X}(t, t-\tau) e^{-s\tau} d\tau, \tag{1.47}$$

for $s \in \mathbb{C}$ is called the instantaneous transfer function.

The properties of the ITF are similar to the classical transfer function. We list here the most important (see [Lou14]) :

- The response to an exponential complex function $e^{i\omega t}$ is :

$$z(t) = G(t, i\omega) e^{i\omega t} \tag{1.48}$$

- If we consider zero as an initial data, we have :

$$z(t) = \mathcal{L}^{-1}\{G(t, s)\mathcal{L}\{u\}(s)\}, \tag{1.49}$$

  where $\mathcal{L}^{-1}$ is the inverse of the Laplace transform.

Assuming that $\tilde{X}(\cdot, \cdot - \tau)$ is at least C^2 for all $\tau \in \mathbb{R}$ and developing in Fourier series the ITF (which is licit because of (1.46)) :

$$G(s, t) = \sum_{n \in \mathbb{Z}} G_n(s) e^{i\omega_0 n t}, \tag{1.50}$$

putting $H_{m,n}(s) := G_{m-n}(s + in\omega_0)$, we have :

$$\mathcal{L}\{z\}(s + im\omega_0) = \sum_{n \in \mathbb{Z}} H_{n,m}(s) \mathcal{L}\{u\}(s + in\omega_0). \tag{1.51}$$

And we have that if we evaluate the equation (1.51) for $s \to s + \frac{2ik\pi}{T}$ for all k integer :

$$\begin{pmatrix} \vdots \\ \mathcal{L}\{z\}(s + i\omega_0) \\ \mathcal{L}\{z\}(s) \\ \mathcal{L}\{z\}(s - i\omega_0) \\ \vdots \end{pmatrix} = \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \iddots \\ \cdots & G_0(s + i\omega_0) & G_1(s) & G_2(s - i\omega_0) & \cdots \\ \cdots & G_{-1}(s + i\omega_0) & G_0(s) & G_1(s - i\omega_0) & \cdots \\ \cdots & G_{-2}(s + i\omega_0) & G_{-1}(s) & G_0(s - i\omega_0) & \cdots \\ \iddots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} \vdots \\ \mathcal{L}\{u\}(s + i\omega_0) \\ \mathcal{L}\{u\}(s) \\ \mathcal{L}\{u\}(s - i\omega_0) \\ \vdots \end{pmatrix}$$

Comparing the equations (1.51) and (1.43), we have that the infinite matrix $(H_{m,n}(s))_{m,n \in \mathbb{Z}}$ is the harmonic transfer function. We denote $\lambda_1, \cdots, \lambda_r$ the eigenvalues of the monodromy operator and

$m_1, \cdots, m_r$ their multiplicities. Developing Q in its Jordan basis, ie $Q = \tilde{P}J\tilde{P}^{-1}$ where J denote the Jordan form of Q. Thus inserting the Floquet representation (1.31) in the equation (1.45) and using the block jordan form of Q we have :

$$\tilde{X}(t,\tau) = C(t)P(t)\tilde{P}e^{J(t-\tau)}\tilde{P}^{-1}P(\tau)^{-1}B(\tau) + D(t)\delta(t-\tau). \tag{1.52}$$

Developing $C(t)P(t)\tilde{P}$, $\tilde{P}^{-1}P(\tau)^{-1}B(\tau)$ and $D(t)$, and inserting in (2.65), we obtain :

$$G_n(s) = \left[\sum_{k\in\mathbb{Z}}\sum_{j=1}^{r}\sum_{q=1}^{m_r}\frac{\tilde{L}_{k,n,j,q}}{(s-\lambda_j+ki\omega_0)^q}\right] + D_n, \tag{1.53}$$

where $\tilde{L}_{k,m,j,q}$ and D_n are complex numbers which depend of the Fourier development of $C(\cdot)P(\cdot)\tilde{P}$, $\tilde{P}^{-1}P(\cdot)^{-1}B(\cdot)$ and $D(\cdot)$. The equation (1.53) prove :

Theorem 1.20. *Each elements of the Wereley matrix is a meromorphic function. And its poles are included (possibly strictly) in the set of the complex numbers of the form* $\frac{ln(\zeta)+2i\pi k}{T}$ *for all* $k \in \mathbb{Z}$ *and where* ζ *is an eigenvalue of the monodromy operator.*

Theorem 1.21. *Assuming that the system (1.39) is observable and controllable in sense of the definition 1.12, then for all* ζ *an eigenvalue of the monodromy operator there exists* n *and* k *two integers such that* $G_n(\cdot)$ *has a pole in* $\frac{ln(\zeta)+2i\pi k}{T}$.

Proof. Let λ_r be an eigenvalue of the monodromy operator. Thus the system (1.39) has a solution which has the form $y(t) = v(t)e^{\lambda_r t}$ with $v(t)$ T-periodic. Since we assumed that the system is controllable, there exists a control u_λ on a finite time t_f which leads to this periodic solution. Thus for $t \geq t_f$, we have that :

$$z(t) = C(t)v(t)e^{\lambda_r t}. \tag{1.54}$$

Since the system is observable, we have that $C(t)v(t)$ is non zero and so the development in Fourier series of $C(t)v(t)$ admits one non zero element. Thus there exists n' integer such that $\mathcal{L}\{z\}(\lambda_r + n'i\omega_0)$ is equal to the $+\infty$ and so we have the result. □

Remark 1.22. *Since we can see the linear autonomous differential system (1.35) as a periodic one with any arbitrary period* T*, we have that in this case the development of the instantaneous function is reduced to the unique* $G_0(s)$ *which is the usual transfer function (see equation (1.38))and the wereley harmonic transfer function matrix is the diagonal matrix* $(\cdots, G_0(s+i\omega_0), G_0(s), G_0(s-i\omega_0), \cdots)$. *Considering a pole of* A, λ, *then* $G_0(s)$ *possesses* λ *as pole and it is analytical in* $\lambda + \frac{2i\pi k}{T}$ *for all* k *non null integer. Thus the controllability and the observability fail to assure that for all* n *and* k *two integers* $G_n(\cdot)$ *has a pole in* $\frac{ln(\zeta)+2i\pi k}{T}$. *This ascertainment was obvious with the equation (1.53). In fact, ascertain that for all* n *and* k *two integers* $G_n(\cdot)$ *has a pole in* $\frac{ln(\zeta)+2i\pi k}{T}$ *is strongly related with the Fourier development of the Floquet and the matrices* $B(t)$ *and* $C(t)$. *It would be interesting to give some conditions to ensure that the previous property is verified.*

1.4.4 Approximation of the first column Harmonic transfer function on the imaginary axis

The harmonic balance method permits to approach the first column of the Wereley matrix or equivalently to approach the Fourier coefficient of the instantaneous transfer function $G(t,s)$ (see equation (1.50)) on the imaginary axis. In fact we recall that if the input is of the form $e^{i\omega t}$ then the

output is of the form $G(t, i\omega)e^{i\omega t}$. Using the Fourier development of $G(t, i\omega)$ (*i.e* put the equation (1.50) in the equation (1.48)), we have that the output z is of the form :

$$z(t) = \sum_{n\in\mathbb{Z}} G_n(i\omega)e^{i(\omega_0 n+\omega)t}. \tag{1.55}$$

Taking the Laplace transform in the previous equation (1.55) and evaluating in each frequency $i\omega_0 n + \omega$ for all integer $n \in \mathbb{Z}$, we obtain :

$$Z(i\omega) = H(i\omega)U(i\omega), \tag{1.56}$$

where $U(i\omega) = [\cdots, 0, 1, 0, \cdots]^*$.

On the other side, we have the system (1.42) evaluated in $i\omega$:

$$\begin{cases} D_{\omega_0}(i\omega)Y(i\omega) = L_A Y(i\omega) + L_B U(i\omega) \\ Z(i\omega) = L_C Y(i\omega) + L_D U(i\omega) \end{cases} \tag{1.57}$$

Fix N a positive integer and consider the truncated system :

$$\begin{cases} D_{\omega_0,N}(i\omega)Y_N(i\omega) = L_{A,N} Y_N(i\omega) + L_{B,N} U_N(i\omega) \\ Z_N(i\omega) = L_{C,N} Y_N(i\omega) + L_{D,N} U(i\omega) \end{cases} \tag{1.58}$$

Where $U_N(i\omega) = [0, \cdots, 0, 1, 0, \cdots, 0]^*$ is a vector column of size $2N + 1$. Making the assumption that the matrix $D_{\omega_0,N}(i\omega) - L_{A,N}$ is invertible, we have :

$$Z_N(i\omega) = L_{C,N}\left(D_{\omega_0,N}(i\omega) - L_{A,N}\right)^{-1} L_{D,N} U_N(i\omega), \tag{1.59}$$

and we have that $Z_N(i\omega)$ is an approximation of the first column of the wereley matrix thanks to the equation (1.56).

1.5 Rational approximation

In the previous Section 1.4.4 we saw how the harmonic balance method permits to compute the first column of the harmonic transfer function on the imaginary axis. We assume here that we know the $G_k(s)$ on all the imaginary axis for $k \in [|-N, N|]$, *i.e.* we have $G_k(i\omega)$ for all ω real and $k \in [|-N, N|]$. How to recover the unstable poles of the Wereley matrix with this data? The Theorem 1.20 states that the $G_k(\cdot)$ are meromorphic functions. There exists a large literature to recover a meromorphic (or rational) function on all the complex plane from the knowledge of the function on the imaginary axis *via* rational approximations. We cite the following non-exhaustive list of papers or thesis which deal with this kind of problem [PB04, PGZC88, MP99, Sua15, AAC+10, BS02, SQ03, Sua09, JPA+01, Bar87, BCT16, PMP99, YL97, HM98, CC98].

Even so far, the more natural way to approximate the meromorphic function $G_k(\cdot)$ on the complex plane would be to perform rational approximation techniques with a high order. Unfortunately, this strategy is not entirely satisfactory because it might create spurious poles [Sta98] (see also [GS92]) which could be misleading to determine the stability [AAC+10]. In order to overcome this problem [AAC+10] proposed to divide the complex plane in small vertical band and to perform a rational approximation on each one. This algorithm is commercially available in STAN tool [STA, DGSP+12, MISPM13].

We present too succinctly a method due to Cooman and al (see [CSO+18]). The Hardy space $H^2(\mathbb{C}^+)$ is defined as the set of all functions g defined on $\mathbb{C}^+$ such that :

- $\forall z \in \mathbb{C}^+$, g is holomorphic at z

- $sup_{x>0} \int_{-\infty}^{+\infty} |g(x+i\omega)|^2 d\omega < +\infty$.

We have the decomposition :

$$L^2(i\mathbb{R}) = H^2(\mathbb{C}^+) \bigoplus H^2(\mathbb{C}^-), \tag{1.60}$$

If we suppose that the system has not a pole on its imaginary axis, we have that each element $G_k(\cdot)$ for $k \in [|-N, N|]$ are in $L^2(i\omega)$. So to find the unstable poles, we project in a good basis the $G_k(s)$ for $k \in [|-N, N|]$ and we perform a rational approximation algorithm to find the unstable poles. Obviously, if we perform the result for N larger, thus the result is more accurate (see [CFR+16]). Unfortunately this way to determine the stability is strongly correlated to the assumption that there is a "small" k and n such that $G_n(\cdot)$ has a pole in $\frac{ln(\zeta)+2i\pi k}{T}$ for each ζ eigenvalue of the monodromy operator and it is not obvious it is always the case (see Remark 1.22).

1.6 Conclusion

We gave the heuristic method of harmonic balance for the circuits modeled by ordinary nonlinear differential equations and how the harmonic balance method coupled with the rational approximation permits to determine the local stability of a forced periodic signal by a periodic signal entry. In practice, only few (G_n) computed and a rational approximation is performed. We saw that the classical concepts of controllability and observability lead to the fact that at least one G_k possesses the pole $z_{j,k}$ and hardly more because the example of the equilibrium case. An interesting question is then :

- For fixed j, which $z_{j,k}$ is a pole of which G_n?

To answer to this question, it seems necessary to introduce a stronger concept of observability and controllability, maybe using the controllability and observability on the harmonics. In fact, with the formula (1.53) it seems strongly correlated to the fact that Fourier development are always non-zero. We do not give an answer to that question in this thesis.

Chapter 2

A simple circuit containing one lossless transmission line

We resume the path followed in Chapter 1 for a simple circuit which possesses a lossless transmission line modeled by a lossless Telegrapher's equation and a periodic solution. Since the Telegrapher's equation is a linear differential equation in the frequency domain, we show how the harmonic balance technique of Chapter 1 is applied on this simple circuit.

We have that the problem reduces to a neutral delay system and we prove some facts, on this specific example, which are less obvious than for the ODEs system. In fact, these kind of equations are infinite dimensional and we need to define functional spaces on which the system occurs. We use the square integrable functions (L^2), the continuous functions (C) and the absolute continuous functions with a square integrable derivative (W_1^2); and we prove that the convergence is equivalente for these spaces.

The monodromy operator of a neutral system is related, *via* a compact perturbation, to the monodromy operator of a scalar difference delay system. On this simple example, we are able to compute exactly the spectrum of this last system, thanks to the spectral semigroup theory. So we are able to give the spectrum of the monodromy operator of the neutral system. More precisely the spectrum is composed of a finite number of eigenvalues outside a disk of radius strictly less than one and, when the delay is commensurable with the period, there is a disk of spectrum elements.

The discretization of the system permits to link the spectrum of the monodromy operator and the harmonic transfer function (HTF). Since we are able to prove controllability and the detectability on the W_1^2 functions, the harmonic transfer function is then a meromorphic operator in the closed right half plane where the poles are the complex logarithm of eigenvalues of the monodromy operator, and if the HTF is holomorphic in the closed right half plane then the periodic solution is locally stable. Moreover, under observability assumption, the HTF has a vertical line of essential singularities view as a Banach valued operator in the left half plane when the delay and the period are not commensurable

2.1 A simple circuit

We consider the following simple circuit due to Brayton [Bra67].

It contains the following elements :

- A voltage generator, that delivers a tension $p(t)$, assumed to be smooth, and periodic with period $T > 0$, with respect to time. This represents the periodic signal to be amplified.
- An active nonlinear element, namely a diode with characteristic $i = g(v)$, where v is indicated on Figure 2.1 and i, is the current through the diode in the indicated direction. The map g is assumed to be smooth.

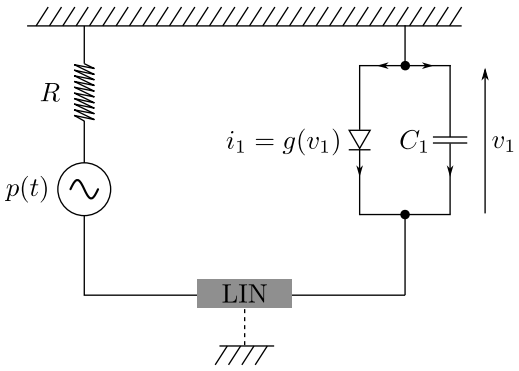


Figure 2.1 : Nonlinear circuit containing a lossless transmission line

- A resistor R and a capacitor C_1.
- A lossless transmission line ("LIN" on Figure 2.1). Its model is described below by a "Telegrapher's" PDE, whose boundary conditions are described by i, v, i_0 and v_0. It is a linear but infinite dimensional element. It has two normalized characteristic numbers : L and C.

We introduced the following data :

$$R, C_1, L, C \text{ are positive numbers} \tag{2.1}$$

$$p(\cdot),\ g(\cdot) \text{ are elements of } C^k(\mathbb{R}, \mathbb{R}),\ k \in \mathbb{N},\ k \geq 3. \tag{2.2}$$

The lossless transmission line is modeled by the telegrapher's equation :

$$\begin{cases} L\frac{\partial i}{\partial t} = -\frac{\partial v}{\partial x} \\ C\frac{\partial v}{\partial t} = -\frac{\partial i}{\partial x}, \end{cases} \tag{2.3}$$

where i and v are functions of $(x,t) \in]0,1[\times]0,+\infty[$ and the boundary conditions at $x = 0$ and $x = 1$ are given by the Kirchhoff laws :

$$\begin{cases} p(t) - v(0,t) - Ri(0,t) = 0 \\ C_1\frac{dv}{dt}(1,t) = i(1,t) - g(v(1,t)). \end{cases} \tag{2.4}$$

2.2 Harmonic balance approach

On this circuit, the harmonic balance technique of the Chapter 1 can be applied. We take the Laplace transform with respect to time in the equations (2.3) and (2.4). Noting $P(s) := \mathcal{L}\{p(\cdot)\}(s)$, $V(s,x) := \mathcal{L}\{v(x,\cdot)\}(s)$, $I(s,x) := \mathcal{L}\{i(x,\cdot)\}(s)$, $V_0(s) := \mathcal{L}\{v(0,\cdot)\}(s)$, $V_1(s) := \mathcal{L}\{v(1,\cdot)\}(s)$, $I_0(s) := \mathcal{L}\{i(0,\cdot)\}(s)$ and $I_1(s) := \mathcal{L}\{i(1,\cdot)\}(s)$, we obtain the following equations in the frequency domain where we assumed that the derivatives and the Laplace transform can be permuted, and $\mathcal{L}^{-1}$ denotes the inverse Laplace transform:

$$\begin{cases} \partial_x V(s,x) = -sLI(s,x) \\ \partial_x I(s,x) = -sCV(s,x), \end{cases} \tag{2.5}$$

and

$$\begin{cases} V_0 = -I_0 R + P \\ I_1 = \mathcal{L}\{g(\mathcal{L}^{-1}\{V_1\})\} + sV_1C_1. \end{cases} \tag{2.6}$$

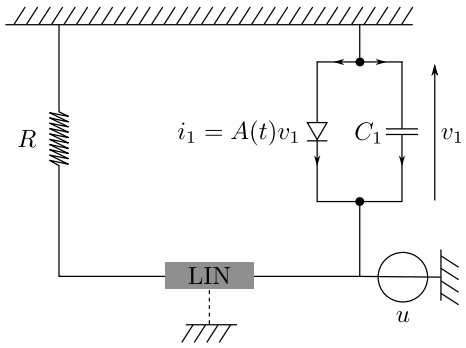


Figure 2.2 : Linearised circuit around the periodic solution disturbed by a current u

We define $K := \sqrt{C/L}$ the characteristic impedance of the line and $\tau := \sqrt{LC}$ the constant of propagation of the line. Resolving the linear system of constant differential equations (2.5), we obtain that we can model in the frequency domain a lossless transmission line as a linear matrix :

$$\begin{pmatrix} V_0 \\ V_1 \end{pmatrix} = \begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix} \begin{pmatrix} I_0 \\ I_1 \end{pmatrix}, \tag{2.7}$$

where Z_{11}, Z_{12}, Z_{21} and Z_{22} depends only on the characteristic impedance and the constant of propagation of the transmission line, and of the complex variable s. Thus putting the equation (2.7) in the equation (2.6), we have the equation :

$$V_1\Big(1 + sC_1\Big(\frac{Z_{12}Z_{21}}{R + Z_{11}} - Z_{22}\Big)\Big) = \frac{PZ_{21}}{R + Z_{11}} + \mathcal{L}\{g(\mathcal{L}^{-1}\{V_1\})\}\Big(Z_{22} - \frac{Z_{12}Z_{21}}{R + Z_{11}}\Big). \tag{2.8}$$

The equation (2.8) is almost like the equation (1.12) in Chapter 1 except it depends on Z_{11}, Z_{21},Z_{12} and Z_{22} which are known. Assuming that this circuit has a periodic solution $\overline{v}_1(t) := \overline{v}(1,t)$, we could apply the same technique to approximate a periodic solution $\overline{v}_1$ as the Subsection 1.2 harmonic balance approach of Chapter 1. In the way to check the local stability, we linearise the circuit around the periodic solution $v(1,t)$ and we disturb the linearised circuit by a current at the end of the line that we note $u \in L^2_{loc}([0,+\infty[,\mathbb{R})$ and the state response is $v(1,t)$ (see Figure 2.2 where $A(t) = g'(\overline{v}_1(t))$). We took u as notation for the input because it is the convention in the control theory field even if it is a current.

The boundary condition (2.4) when the system is linearised becomes :

$$\begin{cases} p(t) - v(0,t) - Ri(0,t) = 0 \\ C_1 \frac{dv}{dt}(1,t) = i(1,t) - g'(\overline{v}_1(t))v(1,t) - u(t). \end{cases} \tag{2.9}$$

Taking the Laplace transform in the equation (2.9), noting $U := \mathcal{L}\{u(\cdot)\}$ and repeating exactly the same reasoning as before, we have that the solution V_1 of the linearised circuit now depends on the perturbation u, we obtain:

$$V_1\Big(1 + sC_1\Big(\frac{Z_{12}Z_{21}}{R + Z_{11}} - Z_{22}\Big)\Big) = \mathcal{L}\{g'(\overline{v}_1(t))\mathcal{L}^{-1}\{V_1\}\}\Big(Z_{22} - \frac{Z_{12}Z_{21}}{R + Z_{11}}\Big) - U\Big(Z_{22} - \frac{Z_{12}Z_{21}}{R + Z_{11}}\Big).$$

Developing in Fourier series $g'(\overline{v}_1(t))$, taking $u(t) = e^{i\omega t}$ for $\omega \in \mathbb{R}$ and performing the same reasoning as in the Subsection 1.4.4 of the Chapter 1, we can approximate the first column of the harmonic transfer function H on the imaginary axis, where H is the infinite matrix which verifies :

$$\begin{pmatrix} \vdots \\ \mathcal{L}\{v_1\}(s+i\omega_0) \\ \mathcal{L}\{v_1\}(s) \\ \mathcal{L}\{v_1\}(s-i\omega_0) \\ \vdots \end{pmatrix} = H(s) \begin{pmatrix} \vdots \\ \mathcal{L}\{u\}(s+i\omega_0) \\ \mathcal{L}\{u\}(s) \\ \mathcal{L}\{u\}(s-i\omega_0) \\ \vdots \end{pmatrix},$$

It is well known for ODE's systems (see theorem 1.7 of Chapter 1) that the local exponentially stability of the periodic solution is equivalent to the exponentially stability of the origin of the circuit linearised around the periodic solution. To obtain this last stability, we disturb the linearised system by a small current signal $u \in L^2_{loc}([0,+\infty[,\mathbb{R})$ which is the input and we observe the output v_1 which leads to the concept of the harmonic transfer function. The structure of this object is well known and it is a meromorphic function (see Theorem 1.20), where the poles are the logarithm of a finite family of complex numbers. Moreover with the assumptions of controllability and observability, if the harmonic transfer function is analytic in the right half plane, then we have the local stability of the system.

When we add a lossless transmission line, the following questions arise :

- Is the harmonic transfer function always a meromorphic operator?
- Does the analyticity in the right half plane of the harmonic transfer function imply the local stability of the periodic solution?

2.3 Results

In the case of circuit that contains transmission line, the structure of the harmonic transfer is not known. In this chapter, we legitimate all the previous reasoning that we made for the ODE system in Chapter 1 and we give the structure of the HTF for the simple circuit proposed (2.1). We start by giving few notations :

- $a := \frac{1-RK}{1+RK}$
- $D_a := \left\{ s \in \mathbb{C}, Re(s) > \frac{ln|a|}{T} \right\}$, and ∂D_a the boundary of D_a.

Theorem 2.1 (Structure HTF)**.** *In D_a,*

- *H is a meromorphic operator $l^2(\mathbb{Z}) \to l^2(\mathbb{Z})$ which possibly poles at $\left\{ z_{j,k} = \frac{ln(\zeta_j)+2ik\pi}{T}, j \in \{1...n\}, k \in \mathbb{Z} \right\}$ where $(\zeta_j)_{j\in\{1...n\}}$ is a finite family of complex numbers.*
- *For all $j = 1, \cdots, n$ there exists at least one $k \in \mathbb{Z}$ such that $z_{j,k}$ is a pole of a coefficient of the matrix H.*
- *Under observability assumption, if $T/r \notin \mathbb{Q}$, all points of the vertical line ∂D_a are essential singularities of the HTF H, as an operator valued analytic map.*

Theorem 2.2. *We have :*

- *If the HTF H is holomorphic in the closed right half plane then the periodic solution is locally stable. More precisely there exists $\delta > 0$ such that there exists a K and γ strictly positive such that :*

$$|v_1(0) - \overline{v}_1(0)| \leq \delta \Rightarrow |v_1(t) - \overline{v}_1(t)| \leq Ke^{-\gamma t}, \tag{2.10}$$

where $v_1 = v(1,t)$ is a continuous solution of the equations (2.3) and (2.4), where (2.3) is understood in a distributional sense.

- *If the HTF H has a pole in the open right half plane, then the periodic solution is locally unstable i.e. there exists $\epsilon_0 > 0$ such that for all $\delta > 0$:*

$$|v_1(0) - \overline{v}_1(0)| \leq \delta \Rightarrow \sup_{0\leq\tau} |v_1(\tau) - \overline{v}_1(\tau)| \geq \epsilon_0 \tag{2.11}$$

where $v_1 = v(1,t)$ is a continuous solution of the equations (2.3) and (2.4), where (2.3) is understood in a distributional sense.

2.4 Equation in time domain : Scalar neutral differential equation

As is well known, the general solution of (2.3) in a distributional sense can be expressed from two general functions of $x - t/\tau$ and $x + t/\tau$. After some transformations, and specializing to $x = 1$ and $x = 0$, one obtains :

$$\begin{cases} i(0,t) = K[x_0(t) - x_1(t-\tau)] \\ i(1,t) = K[x_0(t-\tau) - x_1(t)] \\ v(0,t) = x_0(t) + x_1(t-\tau) \\ v(1,t) = x_0(t-\tau) + x_1(t), \end{cases} \tag{2.12}$$

where $x_0(\cdot)$ and $x_1(\cdot)$ are two functions of one variable that we retain to describe the state of the system. We can then write Kirchhoff's law as :

$$\begin{cases} p(t) - v(0,t) - Ri(0,t) = 0 \\ C_1 \frac{dv}{dt}(1,t) = i(1,t) - g(v(1,t)). \end{cases} \tag{2.13}$$

Substituting (2.12) and expressing x_0 from the first equation yields :

$$x_0(t) = Z_0 p(t) - a x_1(t-\tau) \tag{2.14}$$

and the second equation then yields :

$$\begin{aligned} C_1 \frac{d}{dt}(x_1(t) - a x_1(t-2\tau)) = K[-a x_1(t-2\tau) + Z_0 p(t-\tau) - x_1(t)] - C_1 Z_0 p'(t-\tau) \\ -g(-a x_1(t-2\tau) + Z_0 p(t-\tau) + x_1(t)). \end{aligned} \tag{2.15}$$

with $Z_0 = \frac{1}{1+RK}$ and $r = 2\tau$.
For simplicity, we rewrite the Equation (2.15) as :

$$\frac{d}{dt}(x(t) - a x(t-r)) = f(t, x(t), x(t-r)) \tag{2.16}$$

where the constant a and the map f satisfy :

$$|a| < 1,\ f \in \mathcal{C}^3(\mathbb{R} \times \mathbb{R} \times \mathbb{R}, \mathbb{R}), \tag{2.17}$$

and f is periodic in its first variable with period T. *With no loss of generality, we assume that $T > r$ (otherwise we would have just to consider the smallest integer p such that $pT > r$ and take as a period of the system pT).* Equations like equation (2.16) are called neutral differential equation and all classical results about these equations can be found in the Hale's book [HVL93]. We transformed

the PDE telegraph equation into a delay differential system. We keep here ambiguous the link between this two systems and keep focus our study on neutral differential equation. Nevertheless, we will give the link between in a more accurate way in Chapter 5 for similar equations.

This equation defines a periodic time-varying flow on $C := \mathcal{C}^0([-r, 0], \mathbb{R})$. More precisely, for all $\phi \in C$ and $s \in \mathbb{R}$, there exists a unique continuous $x(\cdot)$ of (2.16) (for all $t \in \mathbb{R}$) such that $x_s = \phi$ where $x_s(\theta) := x(s+\theta)$. All the flow $\varphi_{s,t}$ maps ϕ to x_t where $x(\cdot)$ is the only solution of (2.16) such that $x_s = \phi$. We endow the space C with the uniform norm, ie :

$$\text{for all } \phi \in C, \text{ we have } \|\phi\| := \sup_{-r \le \theta \le 0} |\phi(\theta)|. \tag{2.18}$$

Assumption (periodic solution) : there exists a continuous $T-$*periodic* solution $t \to \overline{x}(t)$ of the equation (2.16). From the fact that f is $\mathcal{C}^3$, we deduce from [HVL93, ch. 9, lem 7.1] that the continuous periodic solution $\overline{x}(\cdot)$ is in fact $\mathcal{C}^3$. Existence of a periodic solution is an *hypothesis*, and we are interested in characterizing its local stability. Conditions for existence and/or uniqueness of such periodic solution for periodically forced systems like (2.16) are considered for instance in Yuan [YG13], but this is not the purpose of the present thesis.

We are interested to study the local stability of this periodic solution $\overline{x}$. For that we linearise the equation (2.16) around the periodic trajectory. Thus we have the equation :

$$\frac{d}{dt}(y(t) - ay(t-r)) = B(t)y(t) + C(t)y(t-r), \tag{2.19}$$

where $B(t) := \partial_2 f(t, \overline{x}(t), \overline{x}(t-r))$ and $C(t) := \partial_3 f(t, \overline{x}(t), \overline{x}(t-r))$. We denote by $U(t,s)$ the linear operator :

$$U(t,s) : C \to C, \tag{2.20}$$

that maps any ϕ in C to y_t, where $y(\cdot)$ is the unique solution of (2.19) such that $y_s = \phi$. Periodicity of $B(\cdot)$ and $C(\cdot)$ in (2.19) give the following properties for the operator U :

- $U(t+T, s+T) = U(t,s)$,
- $U(t,s)U(s,\tau) = U(t,\tau)$ for $t \ge s \ge \tau$,
- $U(t+T, s) = U(t,s)U(s+T,s)$ for $t \ge s$.

Moreover, U is a continuous bounded operator. The system (2.16) can be written as the system (2.19) plus a function. More precisely :

$$\frac{d}{dt}(x(t) - ax(t-r)) = B(t)x(t) + C(t)x(t-r) + h(t, x_t), \tag{2.21}$$

with :

$$h(t, x_t) := f(t, x(t), x(t-r)) - B(t)x(t) - C(t)x(t-r). \tag{2.22}$$

Our motivation in this chapter is to deal with the stability for the continuous solution of (2.16). However to prove some results, we will have to use bigger or smaller space that is why we introduce also the following Banach spaces where I denotes an interval of $\mathbb{R}$:

- $L^p(I, \mathbb{R})$, with $p \in [1, \infty[$, the space of the p-integrable function defined on I and with values in $\mathbb{R}$ endowed with the following norm

$$\text{for all } \phi \in L^p(I, \mathbb{R}), \text{ we have } \|\phi\|_p =: \left(\int_I |\phi|^p \right)^{1/p}. \tag{2.23}$$

- $\mathcal{C}_{ABS}(I,\mathbb{R})$ the space of the absolutely continuous functions defined on I and with values in $\mathbb{R}$, *i.e.* $f \in \mathcal{C}_{ABS}(I,\mathbb{R})$ if and only if its distributional derivative is an L^1 function (see [EG92]). Moreover if $f \in \mathcal{C}_{ABS}(I,\mathbb{R})$ then f admits a derivative almost everywhere f' and :

$$f(b) - f(a) = \int_a^b f'(x)dx, \ \forall a, b \in [-r, 0]. \tag{2.24}$$

  We endow $\mathcal{C}_{ABS}(I,\mathbb{R})$ with the following norm :

$$\text{for all } \phi \in \mathcal{C}_{ABS}(I,\mathbb{R}), \text{ we have } \|\phi\|_{ABS} := \int_I (|\phi| + |\phi'|). \tag{2.25}$$

- $W_2^1(I,\mathbb{R})$ the Sobolev space of the absolute continuous functions on I with a square integrable derivative. We endow $W_2^1(I,\mathbb{R})$ with the following norm :

$$\text{for all } \phi \in W_2^1(I,\mathbb{R}), \text{ we have } \|\phi\|_{W_2^1} := \left(\int_I \left(|\phi|^2 + |\phi'|^2\right)\right)^{1/2}. \tag{2.26}$$

We note $L^2 := L^2([-r,0],\mathbb{R})$, $\mathcal{C}_{\mathcal{ABS}} := \mathcal{C}_{ABS}([-r,0],\mathbb{R})$ and $W_2^1 := W_1^2([-r,0],\mathbb{R})$. Until now the operator $U(t,s)$ was defined on C, but can also be defined on the space L^2, $\mathcal{C}_{ABS}$ or W_2^1 and we will note these operators $U(t,s)_{|L^2}$, $U(t,s)_{|\mathcal{C}_{ABS}}$ and $U(t,s)_{|W_2^1}$ respectively.

Let $U(T,0)$ the monodromy operator. The spectrum of this operator give us the stability or instability of the periodic solution of the system (2.16) (see Section 2.3). Moreover we can have exactly the spectrum of $U(T,0)$ (see Section 2.5.3).

2.5 Stability and instability

It is well known that the spectrum of the operator $U(T,0)$ gives the stability or instability for the origin to the system (2.19) (see Lemma 2.8 and Lemma 2.11 below). It is almost as well known that this spectrum determines the local (in)stability of the periodic solution for the linear time-varying system (2.16). In fact, according to the following two theorems (where the Section 2.5.1 and 2.5.2 are devoted to the proofs), this flow also determines the local stability or the instability of the periodic solution $\overline{x}(\cdot)$ for the nonlinear system (2.16) :

Theorem 2.3. *Assuming that the spectral radius of $U(T,0)$ is strictly less than 1, there exists $\delta > 0$ such as for all $\phi \in C$ verifying $||\phi - \overline{\phi}|| \leq \delta$, then there are $K > 0$ and $\alpha > 0$ such that for all $s \geq 0$ and $t \geq s$:*

$$|x(t) - \overline{x}(t)| \leq Ke^{-\alpha(t-s)},$$

for all $t \geq s$, where x is the solution of (2.16) starting from ϕ at the initial time s.

Theorem 2.4. *Assuming that the spectral radius of $U(T,0)$ is strictly superior than 1, then the solution $x(t)$ from time 0 is unstable. More precisely, there exists $\epsilon_0 > 0$ such that for all $\delta > 0$, there exists $\phi \in C$ such that $||\phi - \overline{\phi}|| < \delta$ and $\sup_{t\in\mathbb{R}^+} ||x_t - \overline{x}_t|| \geq \epsilon_0$ where x is the solution of (2.16) starting from ϕ.*

Before to give the proofs of the Theorem 2.3 and Theorem 2.4, we give the variation of constant formula in the case of the neutral differential system which is just a generalization of the Theorem 1.38 in the Chapter 1. To give this formula, we need to introduce the fundamental solution of the system (2.19) which is just a particular solution of this system with discontinuous initial data.

Proposition 2.5 (Fundamental solution)**.** *Be two real numbers s and $t \geq s$. There exists an unique solution measurable to the equation (2.19) with initial data $X_0(\theta) = 0$ for $\theta < s$ and $X_0(s) = 1$. We note this solution $X(t,s)$. This solution is continuous on all the domains $\{(t,s) \in \mathbb{R}^2,\ nr \leq t-s < (n+1)r\}$, $n \in \mathbb{N}$, and the restriction to this domain extends into a continuous function on $\{(t,s) \in \mathbb{R}^2,\ nr \leq t-s \leq (n+1)r\}$. Moreover it is also continuous on $\{(t,s) \in \mathbb{R}^2,\ t-s < 0\}$ since identically zero.*

Proof. We can remark that for $t \geq s$:

$$X(t,s) = aX(t-r,s) + e^{\int_s^t B(u)du} + \int_s^t e^{\int_u^t B(v)dv}[C(u) + B(u)A(u)]X(u-r,s)du,$$

is solution to the equation (2.19) with initial data X_0. The uniqueness and the continuous properties come from directly from this formula. □

We have that the fundamental solution is exponentially bounded [HVL93, ch. 9, formula 1.14] :

Lemma 2.6. *There exists $K, \gamma \in \mathbb{R}$ such that:*

$$|X(t,s)| \leq Ke^{\gamma(t-s)}.$$

Proof. After integration of the equation (2.19) we have just to apply Gronwall lemma. □

We can find in [HVL93, ch. 9, formula 1.16] the following variation of constant formula :

Theorem 2.7 (Variation of constant formula)**.** *For all $t \geq 0$, the solution of the equation (2.16) with the initial time 0 satisfies:*

$$x(t) = U(t,0)\phi(0) + \int_0^t X(t,s)h(s,x_s)ds.$$

2.5.1 Stability

We start by proving that the spectrum of the monodromy operator U gives the exponential stability of the origin of the system (2.19) :

Lemma 2.8 ([Bu])**.** *Let $\mu = e^{\lambda}$ an element of the spectrum of $U(T,0)$ such that $|\mu|$ is equal to the spectral radius of this operator. Thus, for all $\epsilon > 0$, there exists $C_\epsilon > 1$ such that for all $s \geq 0$ and $t \geq s$, we have :*

$$||U(t,s)|| \leq C_\epsilon e^{\frac{(\Re(\lambda)+\epsilon)}{T}(t-s)}.$$

Proof. By the Gelfand's formula, we have :

$$\lim_{m \to +\infty} ||U^m(T,0)||^{1/m} = e^{\Re(\lambda)}$$

Thus for $\epsilon > 0$ there exists an integer m such that for all integer $M \geq m$ we have :

$$||U^M(T,0)|| \leq e^{(Re(\lambda)+\epsilon)M}$$

Let $t \geq mT$. There exists q integer greater than m and $\beta \in [0,T[$ such that $t = qT + \beta$, thus :

$$\begin{aligned} ||U(t,0)|| &= ||U(\beta,0)U^q(T,0)|| \\ &\leq M_1 e^{\frac{(\Re(\lambda)+\epsilon)}{T}t}, \end{aligned}$$

where $M_1 > 1$ is a bound of $||U(r,0)||$ for all $r \in [0,T]$ (it exists because the Banach-Steinhaus theorem and the continuity of the operator). In the same way there exists a constant $M_2 > 1$ which is an upper bound of $||U(t,0)||$ for $t \in [0, mT]$. Then there exists a constant $M_3 > 1$ such that for all $t \geq 0$ we have:

$$||U(t,0)|| \leq M_3 e^{\frac{(\Re(\lambda)+\epsilon)}{T}t}.$$

Let $\zeta \geq 0$ and $t \geq \zeta + T$. There exists $\beta \in [0, T[$ and q integer such that $\zeta = qT + \beta$. We have :

$$\begin{aligned} ||U(t,\zeta)|| &= ||U(t,(q+1)T)U((q+1)T,\zeta)|| \\ &= ||U(t-(q+1)T,0)U(T,\beta)|| \\ &\leq M_3 e^{\frac{(\Re(\lambda)+\epsilon)}{T}(t-\zeta)} M_4, \end{aligned}$$

where $M_4 > 1$ a upper bound of $||U(T,\beta)||$ for $\beta \in [0,T]$. For $\zeta \geq 0$ and $t < \zeta + T$, we have the existence of β, β' two real such that $0 \leq \beta, \beta' < T$ and :

$$||U(t,\zeta)|| = ||U(\beta' + \beta, \beta)||.$$

Then if we take an uniform upper bound of $||U(z,e)||$ for $0 \leq z \leq 2T$ and $0 \leq e \leq T$, we proved that there exists $M_\epsilon > 1$ such that for all $s \geq 0$ and $t \geq s$, we have :

$$||U(t,s)|| \leq C_\epsilon e^{\frac{(\Re(\lambda)+\epsilon)}{T})(t-s)}.$$

□

Lemma 2.9. *Let $\eta > 0$. There exists $\delta_1 > 0$ such that if for $S > 0$ we have $||x_t - \overline{x}_t|| \leq \delta_1$ for all $t \in [0,S]$ and for all function x continuous defined on $[-r,S]$, then :*

$$|h(t,x_t) - h(t,\overline{x}_t)| \leq 2\eta ||x_t - \overline{x}_t||.$$

Proof. For $t \in \mathbb{R}^+$, we let $F_t(\beta) = f(t, \overline{x}(t) + \beta(x(t) - \overline{x}(t)), \overline{x}(t-r)\beta(x(t-r) - \overline{x}(t-r)))$ for $\beta \in [0,1]$. By the mean value theorem, there exists $\beta \in]0,1[$ (which depends of t) such that :

$$F_t(1) - F_t(0) = \frac{d}{d\beta}(F(\beta)). \tag{2.27}$$

Then we deduce from the equation (2.27) :

$$\begin{aligned} |h(t,x_t) - h(t,\overline{x}_t)| &= |F_t(1) - F_t(0) - B(t)(x(t) - \overline{x}(t)) - C(t)(x(t-r) - \overline{x}(t-r))| \\ &\leq \tilde{h}_1(t,x_t,\overline{x}_t) + \tilde{h}_2(t,x_t,\overline{x}_t), \end{aligned} \tag{2.28}$$

with :

$$\tilde{h}_1(t,x_t,\overline{x}_t) := \sup_{0<\beta<1} |\partial_2 f(t,\overline{x}(t)+\beta(x(t)-\overline{x}(t)),\overline{x}(t-r)+\beta(x(t-r)-\overline{x}(t-r))) \\ -\partial_2 f(t,\overline{x}(t),\overline{x}(t-r)))(x(t)-\overline{x}(t))|,$$

$$\tilde{h}_2(t,x_t,\overline{x}_t) := \sup_{0<\beta<1} |\partial_3 f(t,\overline{x}(t)+\beta(x(t)-\overline{x}(t)),\overline{x}(t-r)+\beta(x(t-r)-\overline{x}(t-r))) \\ -\partial_3 f(t,\overline{x}(t),\overline{x}(t-r)))(x(t)-\overline{x}(t))|.$$

By the continuity of the partial derivatives of f and the fact it is periodic in its first variable, we deduce from the equation (2.28) that there exists, for $\eta > 0$, one $\delta_1 > 0$ such that if for $S > 0$, we have $||x_t - \overline{x}_t|| \leq \delta_1$ for all $t \in [0, S]$, then :

$$|h(t,x_t) - h(t,\overline{x}_t)| \leq 2\eta ||x_t - \overline{x}_t||.$$

□

If the operator $U(t,s)$ is exponentially bounded then we can prove that the fundamental solution $X(t,s)$ is exponentially bounded too with the same exponential bound. This result is classic when the neutral differential system is time-invariant because the theory of the semigroup can be applied. In the periodic case the result is still true and it is the subject of the following lemma :

Lemma 2.10. *If there exists $C' > 1$ and $\alpha \in \mathbb{R}$ such that for all $s \geq 0$ and $t \geq s$, we have :*

$$||U(t,s)|| \leq C' e^{\alpha(t-s)}.$$

then there exists $K_0 > 1$ such that :

$$||U(t,s)|| \leq K_0 e^{\alpha(t-s)}.$$

and

$$|X(t,s)| \leq K_0 e^{\alpha(t-s)}.$$

Proof. By the equation (1.10) of [HVL93, ch. 9, thm 1.2], we have for $s \in \mathbb{R}^+$ and $t \geq s$:

$$U(t,s)\phi(0) = X(t,s)f(s) + \int_s^t X(t,\alpha)df(\alpha),$$

Where f is defined as :

$$f(t) = \begin{cases} \phi(0) - a\phi(-r) + a\phi(t-s-r) + \int_s^t C(u)\phi(u-s-r)du \text{ for } s \leq t \leq s+r \\ \phi(0) - a\phi(-r) + \int_s^{s+r} C(u)\phi(u-s-r)du \text{ for } t > s+r \ . \end{cases}$$

Let ϕ a real continuously differentiable function defined on $[-r, 0]$ such that there exists a constant $S > 0$ verifying : $\left|\frac{1}{f(s)}\right| \leq S$ for all s.

We have :

$$|X(t,s)| \leq SC' e^{\alpha(t-s)} + S\int_t^s |X(t,\alpha)||df(\alpha)|,$$

and applying the Gronwall lemma ([EK09, p. 498, lem 7.1]), we have the result. □

Proof Theorem 2.3. With no loss of generality we can take 0 as initial data. For a fixed $t \in \mathbb{R}^+$, following the Lemma 2.8 and the Lemma 2.10, there exists $K_0 \geq 1$ and $\alpha_0 > 0$ such that :

$$||U(t,s)|| \leq K_0 e^{-\alpha_0(t-s)},$$
$$|X(t,s)| \leq K_0 e^{-\alpha_0(t-s)},$$

for $0 \leq s \leq t$. Let $\epsilon > 0$ and $0 < \alpha < \alpha_0$. We define :

$$\eta = \frac{\epsilon(\alpha_0 - \alpha)}{2(1+\epsilon)K_0} e^{-\alpha r} \text{ and } K_1 = (1+\epsilon)K_0.$$

Let $\delta_1 > 0$ the δ_1 from the Lemma 2.9. We define $\delta = \frac{\delta_1}{K_1}$. Let $\overline{\phi}(\theta) := \overline{x}(\theta)$ where $\overline{x}$ is the periodic solution of (2.16). Let $\phi \in C$ and $||\phi - \overline{\phi}|| < \delta$ and x the solution of (2.16) with initial data ϕ. We assume that there exists a $S > r$ such that for $t \in [0, S[$ on ait $|x(t) - \overline{x}(t)| < \delta_1$ and $|x(S) - \overline{x}(S)| = \delta_1$. By the variation constant formula (Theorem 2.7), we have :

$$|x(t) - \overline{x}(t)| \leq |U(t,0)||\phi(0) - \overline{\phi}(0)| + \int_0^t |X(t,s)||h(s,x_s) - h(s,\overline{x}_s)|ds.$$

And for all $t \leq S$, we have :

$$|x(t) - \overline{x}(t)| \leq K_0 e^{-\alpha_0 t}||\phi - \overline{\phi}|| + 2\eta K_0 \int_0^t e^{-\alpha_0(t-s)}||x_s - \overline{x}_s||ds.$$

Multiplying the last equation by $e^{\alpha t}$, we obtain :

$$e^{\alpha t}|x(t) - \overline{x}(t)| \leq K_0||\phi - \overline{\phi}|| + 2\eta K_0 e^{(\alpha-\alpha_0)t} \int_0^t e^{\alpha_0 s}||x_s - \overline{x}_s||ds.$$

We let $v(t) = max\{e^{\alpha t}|x(s) - \overline{x}(s)| : -r \leq s \leq t\}$ and we have :

$$e^{\alpha t}|x(t) - \overline{x}(t)| \leq K_0||\phi - \overline{\phi}|| + 2\eta K_0 v(t) e^{(\alpha-\alpha_0)t+\alpha r} \int_0^t e^{(\alpha_0-\alpha)s}ds.$$

Then :

$$e^{\alpha t}|x(t) - \overline{x}(t)| \leq K_0||\phi - \overline{\phi}|| + \frac{2\eta K_0 v(t) e^{\alpha r}}{\alpha_0 - \alpha}.$$

Since the right-hand member of the equation is increasing in t, we have :

$$v(t) \leq K_0||\phi - \overline{\phi}|| + \frac{2\eta K_0 v(t) e^{\alpha r}}{\alpha_0 - \alpha}.$$

And thus :

$$v(t) \leq K_0||\phi - \overline{\phi}|| + \frac{\epsilon v(t)}{1+\epsilon}.$$

We deduce :

$$|x(t) - \overline{x}(t)| \leq K_1 e^{-\alpha t}||\phi - \overline{\phi}||.$$

Then $\delta_1 = |x(S) - \overline{x}(S)| < K_1\delta = \delta_1$ which is impossible. We have $S = +\infty$.

□

2.5.2 Instability

The instability theorem 2.4 says that even if we can approach infinitely close the periodic solution $\overline{v}_1$ we always end up by deviating from a fixed constant $\epsilon > 0$. The instability theorem 2.4 is mainly due to the ideas of [SS00] and [Str12] where they give a proof in the case of autonomous differential equation on Banach spaces. The proof that we give here is exactly the same except that it is perform in the case of scalar periodic neutral differential equation which generates very slight changes.

Lemma 2.11. *Let $\mu = e^{\lambda}$ in the spectrum of $U(T,0)$ and such that $|\mu|$ is equal to the spectral radius of $U(T,0)$. Then for each $\gamma > 0$ and each integer m, there exists $v \in C$ such that :*

$$||(U(mT,0) - e^{\lambda m})v|| < \gamma||v|| \tag{2.29}$$

and

$$||(U(t,0))v|| \leq 2Ke^{t\Re(\lambda)/T}||v|| \text{ for all } 0 \leq t \leq mT, \tag{2.30}$$

where $K = \sup_{r\in[0,T]} ||U(r,0)||$.

Proof. Since the boundary of the spectrum of $U(T,0)$ is included in the approximate spectrum, there exists $v_n \in C$ such that $(U(T,0) - e^{\lambda})v_n \to 0$ when $n \to +\infty$. Let $\gamma > 0$ and m integer, we have :

$$\begin{aligned}(U(mT,0) - e^{\lambda m})v_n &= (U^m(T,0) - e^{\lambda m})v_n \\ &= \sum_{j=0}^{m-1} U^j(T,0)e^{\lambda(m-1-j)}[U(T,0) - e^{\lambda}]v_n\end{aligned}$$

If we take n enough large, we obtain the equation (2.29). For $\gamma > 0$, we take n enough large to have moreover :

$$||(U(jT,0) - e^{\lambda j})v_n|| < 1 \text{ for } j = 1, ..., m$$

We have for $0 \leq t \leq mT$, $t = jT + r$, $0 \leq r < T$ and $j \in [0, 1, ..., n]$.

$$\begin{aligned}||U(t,0)v_n|| &\leq ||U(r,0)||||U^j(T,0)||||v_n|| \\ &\leq K(1 + e^{j\Re(\lambda)})||v_n|| \\ &\leq 2Ke^{\frac{t}{T}\Re(\lambda)}||v_n||,\end{aligned}$$

for $0 \leq t \leq mT$ and where $K \geq 1$ is a bound of $||U(r,0)||$ for $r \in [0,T]$.

□

Lemma 2.12. *There exists a $\rho_0 > 0$ and $c > 0$ such that if $S > 0$ with $||x_t - \overline{x}_t|| < \rho_0$ then for all $t \in [0,S]$ we have:*

$$|h(t,x_t) - h(t,\overline{x}_t)| \leq c||x_t - \overline{x}_t||^2.$$

Proof. The proof is similar to the proof of Lemma 2.9.

□

Proof Theorem 2.4. Let $\mu = e^{\lambda}$ an element of the spectrum $U(T,0)$ such that $|\mu|$ equal to the spectral of this operator. For k integer and ρ_0 to be defined later, that is $0 < \delta < min\{1/k, \rho_0/2, 1\}$. The δ is free for power parameter is arbitrarily small. Let us choose t_1 positive integer from δ defined by:

$$\frac{1}{k} < \delta e^{t_1 \Re(\lambda)} \leq \frac{|\mu|}{k}.$$

We can choose t_1 in the interval $]b, b+1]$ where $b = \frac{ln(\frac{1}{\delta k})}{ln|\mu|} > 0$.

Moreover, let v given by the lemma 2.11 with $m = t_1$ and $\gamma = (4k)^{-1}$. We normalize $||v|| = \delta$. We have :

$$||(U(t_1 T, 0) - e^{\lambda t_1})v|| < \frac{\delta}{4k}.$$

Then by the lemma 2.11, we have :

$$||U(t_1 T, 0)v|| > ||e^{\lambda t_1} v|| - \frac{\delta}{4k} > \frac{1}{k} - \frac{\delta}{4k}$$

and

$$||U(t,0)v|| \leq 2K\delta e^{\frac{t}{T}\Re(\lambda)} \text{for all } 0 \leq t \leq t_1 T.$$

Fix $\delta_0 > \delta$ and let $\phi \in C$ such that $\phi = \overline{\phi} + v$. We note $x(t)$ the solution to the system (2.16) associated to ϕ. We have $||v|| = \delta < \delta_0$. By the variation of constant formula (Theorem 2.7) :

$$\begin{aligned} |x(t) - \overline{x}(t) - U(t,0)v(0)| &\leq \int_0^t |X(t,s)||h(s,x_s) - h(s,\overline{x}_s)|ds \\ &\leq c\int_0^t |X(t,s)||x_s - \overline{x}_s||^2 ds. \end{aligned}$$

for $||x_s - \overline{x_s}|| < \rho_0$ on the interval $[0,S]$ where $S > 0$ is a real in the lemma 2.12. We have two possibilities : there exists a $S > 0$ such that $||x_S - \overline{x_S}|| = \rho_0$ or $S = +\infty$. We assume that $S = +\infty$. Thanks to the lemma 2.8 and 2.10, we have :

$$|x(t) - \overline{x}(t) - U(t,0)v(0)| \leq K_0 c \int_0^t e^{\frac{3}{2}\Re(\lambda)\frac{t-s}{T}} ||x_s - \overline{x}_s||^2 ds.$$

We define:

$$t_2 = sup\{t : ||x_s - \overline{x_s} - U(s,0)v|| < \frac{1}{2|\mu|}\delta e^{\Re(\lambda)\frac{s}{T}} \text{ and } ||U(s,0)v|| < \frac{\rho_0}{2} \text{ pour } 0 \leq s \leq Tt\}.$$

We have $t_2 > 0$. For $t \leq min\{Tt_2, Tt_1\}$, we have :

$$\begin{aligned}
|x(t)-\overline{x}(t)-U(t,0)v(0)| &\leq K_0c\int_0^t e^{\frac{3}{2}\Re(\lambda)\frac{t-s}{T}}(||U(s,0)v||+||x_s-\overline{x}_s-U(s,0)v||)^2ds\\
&\leq K_0c\int_0^t e^{\frac{3}{2}\Re(\lambda)\frac{t-s}{T}}(2\delta Ke^{\frac{s}{T}\Re(\lambda)}+\frac{1}{2|\mu|}\delta e^{\Re(\lambda)\frac{s}{T}})^2ds\\
&< K_0c(2K+\frac{1}{2|\mu|})^2\delta^2e^{\frac{3}{2}\Re(\lambda)\frac{t}{T}}\frac{2T}{\Re(\lambda)}e^{\Re(\lambda)\frac{t}{2T}}\\
&= \frac{k}{2|\mu|^2}(\delta e^{\Re(\lambda)\frac{t}{T}})^2,
\end{aligned}$$

where $k=4|\mu|^2K_0c(2K+\frac{1}{2|\mu|})^2\frac{T}{Re(\lambda)}$. we deduce :

$$||x_t-\overline{x_t}-U(t,0)v||<\frac{k}{2|\mu|^2}(\delta e^{Re(\lambda)\frac{t}{T}})^2.$$

We have $Tt_1<Tt_2$ or $||x_{Tt_2}-\overline{x_{Tt_2}}||=\frac{\rho_0}{2}$. If we assume the contrary : $Tt_1\geq Tt_2$ and $||x_{Tt_2}-\overline{x_{Tt_2}}||<\frac{\rho_0}{2}$. By definition of t_2, we have that :

$$\frac{1}{2|\mu|}\delta e^{Re(\lambda)t_2}=||x_{Tt_2}-\overline{x_{Tt_2}}-U(Tt_2,0)v||<\frac{k}{2|\mu|^2}(\delta e^{Re(\lambda)t_2})^2.$$

Then $(\delta e^{Re(\lambda)t_2})>\frac{|\mu|}{k}\geq(\delta e^{Re(\lambda)t_1})$.We deduce $t_2>t_1$, which is a contradiction. We assume that$||x_{Tt_2}-\overline{x_{Tt_2}}||\neq\frac{\rho_0}{2}$, then $Tt_1<Tt_2$. Then :

$$\begin{aligned}
||x_{t_1T}-\overline{x_{t_1T}}-U(t_1T,0)v|| &< \frac{k}{2|\mu|^2}(\delta e^{Re(\lambda)t_1})^2\\
&< \frac{k}{2|\mu|^2}\frac{|\mu|^2}{k^2}\\
&= \frac{1}{2k}
\end{aligned}$$

Then :

$$\begin{aligned}
||x_{Tt_1}-\overline{x_{Tt_1}}|| &> ||U(Tt_1,0)v||-\frac{1}{2k}\\
&> \frac{1}{k}-\frac{\delta}{4k}-\frac{1}{2k}\\
&= \frac{1}{2k}-\frac{\delta}{4k}\\
&> \frac{1}{4k}\text{ car }\delta<1.
\end{aligned}$$

We deduce that there exists a time $t>0$(Tt_2 or Tt_1 or S) such that $||x_t-\overline{x_t}||\geq min\{\frac{1}{4k},\frac{\rho_0}{2}\}$, which is the result.

□

If one access to the spectral radius of $U(T,0)$, Theorem 2.3 and Theorem 2.4 would determine the local stability of the periodic solution $\overline{x}(\cdot)$ and this was our goal. To estimate the spectral radius it can be useful to know the nature of the spectrum. In fact, it is easier to approximate eigenvalue than the other type of the element of the spectrum. These considerations lead to know more about the spectrum of $U(T,0)$.

2.5.3 Spectrum of monodromy operator

We are interested in knowing more precisely how is the spectrum of the monodromy operator $U(T,0)$.

In finite dimension, the rank theorem says that it is equivalent to be injective, surjective and bijective. Moreover the spectrum is composed of eigenvalues where the dimension of the kernel and the codimension are equal. In infinite dimension, this result does not hold and there is three way to fail to be bijective. Let $A : E \to E$ a continuous linear operator acting on E a Banach space and Id the identity operator on E, we have the following decomposition of the spectrum :

i. if, for $\lambda \in \mathbb{C}$, $A-\lambda Id$ is not injective, then λ is an eigenvalue of A and the kernel, $ker(A-\lambda Id) := \{\phi \in E | A\phi - \lambda\phi = 0\}$, is not empty. The dimension of the kernel is called the multiplicity of the eigenvalue and the set $P\sigma(A) := \{\lambda \in \mathbb{C} | ker(A-\lambda) \neq \varnothing\}$ is called the point spectrum of A.

ii. if, for $\lambda \in \mathbb{C}$, $A - \lambda Id$ is injective and its range is a dense subset different of E then λ is an approximate eigenvalue. We note the set of all the approximate eigenvalues $App\sigma(A)$.

iii. if, for $\lambda \in \mathbb{C}$, $A - \lambda Id$ is injective and its range is not a dense subset of E, then λ is an element of the residual spectrum A that we call $R\sigma(A)$.

We note $\sigma(A)$ the spectrum of A, *i.e.* the union of the sets $P\sigma(A)$, $App\sigma(A)$ and $R\sigma(A)$. We call the nullity of A, $nul(A)$, the dimension of the kernel of A and the deficiency of A, $def(A)$, the dimension of the codimension of A, *i.e.* the dimension of $E/R(A)$ where $R(A)$ denotes the range of A. We define the index of A, noted $Ind(A)$, as the difference between $nul(A)$ and $def(A)$. A is said to be semi-Fredholm if $R(A)$ is closed and at least one of the quantity $nul(A)$ or $def(A)$ is finite. The semi-Fredholmness property and the index are preserved under compact perturbation, more precisely:

Theorem 2.13. *[Kat95, ch. 4, thm 5.26] Let K a compact operator from E into E. If A is semi-Fredholm then $A+K$ is semi-Fredholm and $Ind(A) = Ind(A+K)$.*

Nullity and deficiency of a semi-Fredholm operator are constant in a neighborhood of this operator, more precisely :

Theorem 2.14. *[Kat95, ch. 4, thm 5.31] If A is semi-Fredholm and B is a bounded operator from E into E then $A+xB$ is semi-Fredholm and $nul(A+xB)$, $def(A+xB)$ are constant for sufficiently small $|x| > 0$ real.*

After these rapid recalls of spectral theory, we are able to characterize the spectrum of $U(T,0)$. In a first time, we introduce the operator solution T_a associated to the solution of the system :

$$z(t) = az(t-r). \tag{2.31}$$

We can see that for $\phi \in C_a := \{\phi \in C | \phi(0) = a\phi(-r)\}$ this problem has a unique solution z, defined on $]-\infty, +\infty[$, with $z(\theta) = \phi(\theta)$ for $\theta \in [-r, 0]$. We have the following result which says the operator $U(T,0)$ is a compact perturbation of the operator T_a.

Lemma 2.15. *Let $\varphi \in C^1$ such that $\varphi(0) = a\varphi(-r) + 1$ and we define the operator $\Psi : C \to C_a$ with $\Psi(f) = f - \varphi[f(0) - af(-r)]$ for $f \in C$. Thus we have :*

$$U(T,0) = T_a(T)\Psi + K(T,0)$$

with $K(T,0)$ compact operator. Moreover, the restriction of the operator $K(T,0)$ to $\mathcal{C}_{ABS}$ or W_2^1 is also compact. We also have :

$$U(T,0)_{|L^2} = T_a(T)_{|L^2} + K(T,0)_{|L^2},$$

where $K(T,0)_{|L^2}$ is a compact operator.

Proof. When the space is C the proof is in [HVL93, ch. 3, thm 7.3]. The proof for the spaces L^2, $\mathcal{C}_{ABS}$ and W_2^1 are not really different. Since the lemma [HVL93, ch. 3, lem 7.2] can be generalized for the L^2 functions, we have the existence of a $b, \gamma > 0$ such that for t, τ real and $\phi \in L^2$:

$$\begin{aligned}\|K(T+\tau,0)\phi - K(T,0)\phi\|_2 \leq be^{\gamma T}\Bigg(&\|K(\tau,0)\phi\|_2 \\ &+ \sup_{0\leq u\leq T}\int_u^{u+\tau} |f(s,U(s,0)\phi(0),U(s,0)\phi(-r))|ds\Bigg).\end{aligned} \tag{2.32}$$

Taking ϕ in a bounded set B of L^2, we can have the right hand side as small as we want by taking τ enough small. Thus, we have that $K(T,0)_{|L^2}B$ is an equicontinuous family and uniformly bounded family, and so relatively compact by the Kolmogorov-Riesz-Fréchet theorem ([Bre10, ch. 4, thm 4.26]).

Taking ϕ in a bounded set B of $\mathcal{C}_{ABS}$. By definition, we have that the operator $\frac{d}{d\theta}K(t,s)\phi(\theta)$ exists almost everywhere and $\frac{d}{d\theta}K(t,s)\phi(\cdot)$ is integrable. Then the generalization of the lemma [HVL93, ch. 3, lem 7.2] leads to :

$$\left\|\frac{d}{d\theta}K(T+\tau,0)\phi - \frac{d}{d\theta}K(T,0)\phi\right\|_1 \leq be^{\gamma T}\left(\left\|\frac{d}{d\theta}K(\tau,0)\phi - \frac{d}{d\theta}K(0,0)\phi\right\|_1 + h_1(T,\tau)\right), \tag{2.33}$$

where :

$$h_1(T,\tau) := \sup_{0\leq u\leq T} |f(u,U(u+\tau,0)\phi(0),U(u+\tau,0)\phi(-r)) - f(u,U(u,0)\phi(0),U(u,0)\phi(-r))|,$$

for some γ and b positive real. We can have the right hand side of (2.33) as small as we want by taking τ enough small. Thus, we have that $\frac{d}{d\theta}K(T,0)_{|\mathcal{C}_{ABS}}B$ is an equicontinuous and uniformly bounded family on L^1 and so relatively compact by the Kolmogorov-Riesz-Fréchet theorem ([Bre10, ch. 4, thm 4.26]). We deduce that $K(T,0)_{|\mathcal{C}_{ABS}}B$ is also relatively compact for the norm $\|\cdot\|_{ABS}$.

The proof of the compacity of $K(T,0)_{|W_2^1}$ is similar and is left to the reader.

□

Remark 2.16. *We can see that the spectrum of $T_a(T)\Psi$ is composed of the spectrum of $T_a(T)$ plus 0 where $dim(ker(T_a(T)\Psi)) = 1$ and $codim(Im(T_a(T)\Psi)) = 1$. Thus, with the nature of the spectrum of $T_a(T)$ we will have the nature of the spectrum of $U(T,0)$ by theory of compact perturbation (see [Kat95]). As $|a| < 1$, we have that the spectral radius of $T_a(T)$ is bounded by a. And we can conclude immediately that $U(T,0)$ has a most a finite number of eigenvalue outside the disk of the radius a. But we can be more precise to determine entirely the spectrum of $U(T,0)$ by studying more precisely the spectrum of $T_a(T)$.*

In the view to compute the spectrum $T_a(T)$ we use the classic theory of the semigroups where all the useful properties can be found in the book [EN00]. In fact the system (2.31) is time-invariant and then the family $\{T_a(t)\}_{t\in\mathbb{R}}$ defines a semigroup. The interest of the semigroup is to transform the system (2.31) in an abstract differential equation on a dense subset of the continuous functions

where the spectrum can be computed. In particular, the link between the spectrum of the generator of the semigroup and the semigroup is stated in [EN00, ch. 4, thm 3.6 and 3.7].

The reasoning that we can find in [HVL93] leads to the following proposition :

Proposition 2.17. *The family $\{T_a(t)\}_{t\in\mathbb{R}}$ is a strongly continuous group. Its generator A verifies :*

$$A\phi = \dot{\phi}$$

and

$$D(A) = \{\phi \in C^1 | \dot{\phi}(0) = a\dot{\phi}(-r) \text{ and } \phi(0) = a\phi(-r)\}.$$

Moreover the spectrum of A is composed of point of spectrum with finite multiplicity :

$$\sigma(A) = \{\lambda \in \mathbb{C} | a = e^{\lambda r}\}.$$

Theorem 2.18. *We have two cases :*

i. If $\frac{T}{r} := \frac{p}{q}$ is rational, then $\sigma(T_a(T)) = \{a^{p/q}e^{i2k\pi/q} | k \in [0, ..., q-1]\}$ and furthermore all the elements of the spectrum have infinite multiplicity.

ii. If $\frac{T}{r}$ is irrational, then we have $\sigma(T_a(T)) = \{\lambda \in \mathbb{C},\ |\lambda| = |a|^{T/r}\}$. Furthermore, all the elements of the spectrum are not semi-Fredholm.

Proof. *i.* We can remark that $\sigma(T_a(r)) = \{a\}$, thus by semigroup property and spectral mapping theorem, we have :

$$\sigma(T_a(T))^q = a^p,$$

and thus :

$$\sigma(T_a(T)) \subseteq \{a^{p/q}e^{i2k\pi/q} | k \in [0, ..., q-1]\}.$$

Moreover as $P\sigma(T_a(T))\backslash\{0\} = e^{TP\sigma(A)}$, where $P\sigma$ is the point spectrum, and that 0 is not in the spectrum of $T_a(t)$, we have by the spectrum of A in Proposition 2.17 :

$$\sigma(T_a(T)) \supseteq \{a^{p/q}e^{i2k\pi/q} | k \in [0, ..., q-1]\}.$$

By corollary 3.8 p278 of [EN00], we have for all λ complex :

$$ker(e^{\lambda T} - T_a(T)) = \overline{\underset{n\in\mathbb{Z}}{lin}} ker(\lambda + i\frac{2\pi n}{T} - A),$$

where $\overline{lin}$ is the closure of the subspace generated by the kernels. As $\frac{T}{r}$ is rational, for $\mu \in \sigma(T_a(T))$ there exists a subset countable (infinite) J such that :

$$ker(\mu - T_a(T)) = \overline{\underset{j\in J}{lin}} ker(\mu_j - A),$$

with μ_j distinct eigenvalue of A. Thus we have that all the elements of $\sigma(T_a(T))$ have infinite multiplicity.

ii. By [Hen74], we have :

$$\overline{e^{T\sigma(A)}} \subseteq \sigma(T_a(T)) \subseteq |a|^{\frac{T}{r}} e^{i\mathbb{R}} \cup \{0\}, \tag{2.34}$$

and by the Proposition 2.17, the density of irrational rotation and the fact that $T_a(T)$ is invertible, we have :

$$\sigma(T_a(T)) = \{\lambda \in \mathbb{C},\ |\lambda| = |a|^{T/r}\}. \tag{2.35}$$

Moreover we have by the spectral mapping theorem for residual and point spectrum, and Proposition 2.17, that $R\sigma(T_a(T)) = \{\varnothing\}$ and $P\sigma(T_a(T)) = \{a^{T/r}e^{2i\pi kT/r} | k \in \mathbb{Z}\}$. As the complementary of $\{a^{T/r}e^{2i\pi kT/r} | k \in \mathbb{Z}\}$ is dense in the circle of radius $|a|^{T/r}$, we have that all element of this circle must not be semi-fredholm because $T_a(T)$ is bounded, and the nullity and the defficiency of a semi-Fredholm operator are constant in a neighborhood (theorem 2.14). □

Corollary 2.19. *The spectrums of $U(T,0)$, $U(T,0)_{|L^2}$, $U(T,0)_{|\mathcal{C}_{ABS}}$ and $U(T,0)_{|W_2^1}$ are equal to the spectrum of $T_a(T)$ plus possibly the same finite number of eigenvalue. We note $\zeta_1, \cdots, \zeta_N$ the eigenvalues which are outside the disk with the spectral radius of $T_a(T)$.*

Proof. We can remark that the family $\{T_a(t)\}_{t\in\mathbb{R}}$ acting on L^2, or $\{\phi \in \mathcal{C}_{ABS} | \phi(0) = a\phi(-r)\}$ or $\{\phi \in W_2^1 | \phi(0) = a\phi(-r)\}$ defines also a strongly continuous group with the same domain than in Proposition 2.17. Then Proposition 2.17 and Theorem 2.18 are also true for the family $\{T_a(t)\}_{t\in\mathbb{R}}$ acting on L^2 , $\{\phi \in \mathcal{C}_{ABS} | \phi(0) = a\phi(-r)\}$ and $\{\phi \in W_2^1 | \phi(0) = a\phi(-r)\}$.

The theorem 2.13, the lemma 2.15 and the theorem 2.18 imply that the spectrums of $U(T,0)$, $U(T,0)_{|L^2}$, $U(T,0)_{|\mathcal{C}_{ABS}}$ and $U(T,0)_{|W_2^1}$ are equal to the spectrum of $T_a(T)$ plus possibly a finite number of eigenvalues. Moreover we have $W_2^1 \subseteq \mathcal{C}_{ABS} \subseteq C \subseteq L^2$, with W_2^1, $\mathcal{C}_{ABS}$ and C dense in L^2 and stable through the operator $U(T,0)$. We can deduce that the spectrum of $U(T,0)_{|L^2}$ is included in the spectrum of $U(T,0)$ which is included in the spectrum of $U(T,0)_{|\mathcal{C}_{ABS}}$ which is included in the spectrum of $U(T,0)_{|W_2^1}$, and thus the operators $U(T,0)$, $U(T,0)_{|L^2}$, $U(T,0)_{|\mathcal{C}_{ABS}}$ and $U(T,0)_{|W_2^1}$ have the same eigenvalues. □

The corollary 2.19 is important because, through the theorem 2.3, it says that the stability of the zero solution for the linear system (2.19) is equivalent for the spaces L^2, $\mathcal{C}_{ABS}$, W_2^1 and C.

2.5.4 HTF and link with the monodromy operator

We introduce the following (fictitious) input-output system. Linearising this system (2.15) around the periodic solution calculated by harmonic balance and disturbing the zero signal at time zero of the linearised circuit by small periodic current entries $u \in L^2_{loc}([0,+\infty[,\mathbb{R})$, $L^2_{loc}([0,+\infty[,\mathbb{R})$ denotes the space of the square integrable function on each compact of $[0,\infty[$ with values in $\mathbb{R}$, like what we did for the frequential case on the right of the line. The input-output system that harmonic balance compute have the equations for $t \geq 0$:

$$\begin{cases} \frac{d}{dt}(x_1(t) - ax_1(t-r)) = \frac{-1}{C_1}[K + g'(-a\overline{x}(t-r) + Z_0 p(t-r/2) + \overline{x}(t))]x_1(t) \\ \qquad\qquad + \frac{a}{C_1}[-K + g'(-Z\overline{x}(t-r) + Z_0 p(t-r/2) + \overline{x}(t))]x_1(t-r) - \frac{u(t)}{C_1}, \\ v_1(t) = x_1 - Zx_1(t-r). \end{cases} \tag{2.36}$$

We can rewrite the input-output system (2.36) as :

$$\begin{cases} \frac{d}{dt}(x(t) - ax(t-r)) = B(t)x(t) + C(t)x(t-r) + u(t), \\ y(t) = x(t) - ax(t-r),\ t \geq 0, \end{cases} \tag{2.37}$$

where $y(t)$ and $x(t)$ are equal to zero for $t \geq 0$ and are two continuous function on $\mathbb{R}$, and for $t \geq 0$ u is a square integrable on each compact function on $[0, +\infty[$ and with values in $\mathbb{R}$. Since f is at least C^3 then B, C are at least C^2. We note $\omega_0 := \frac{2\pi}{T}$. In the context of finite dimensional systems, the definition of the transfer function and its relation to the monodromy operator is well known thanks to Floquet's theory (cf. [Möl00]). One can also connect the monodromy operator and the transfer function without going through the theory of Floquet but considering a lifting of the system. The following results generalize the results of [Möl00] in our framework of neutral equations, and give moreover a mathematical justification of the computation that we can find in [Möl00]. Firstly we give expression of the impulse response of the system :

Lemma 2.20. *We have :*

$$y(t) = \int_0^t [X(t,\tau) - aX(t-r,\tau)]u(\tau)d\tau \tag{2.38}$$

$$:= \int_0^t \tilde{X}(t,\tau)u(\tau)d\tau \tag{2.39}$$

where X is the fundamental solution of system (2.19), ie the solution satisfying the initial condition $X(t,s) = 0$ for $t < s$ and $X(s,s) = Id$. Moreover there exists $K, \gamma > 0$ such that for all $s \in \mathbb{R}$ and $t \geq s$, we have :

$$|X(t,s)| \leq Ke^{\gamma(t-s)}. \tag{2.40}$$

Proof. See Lemma 2.6 and Theorem 2.7. □

To justify all the computations, we need estimation for the impulse response like for the ODE system. More precisely, we have :

Theorem 2.21 (Regularity impulse response)**.** *Let $\mathcal{X}(\tau, s) := X(s+\tau, s)$, we have :*

i. The function $\mathcal{X}$ is continuous outside $\{(\tau, s) \in \mathbb{R}^2,\ \tau/r \in \mathbb{N}\}$, everywhere continuous compared to his second argument (s) and admits everywhere a limit to the left and right compared to his first argument(τ).

ii. If B and C are $\mathcal{C}^k$ then $\mathcal{X}$ is k continuously differentiable from its second argument s (for τ fixed) and for all j, $1 \leq j \leq k$,

$$(\tau, s) \mapsto \frac{\partial^j \mathcal{X}}{\partial s^j}(\tau, s)$$

is continuous.

iii. There exists $\gamma > 0$ and $K > 0$ such that :

$$\left|\frac{\partial \mathcal{X}^k}{\partial^k s}(\tau, s)\right| \leq Ke^{\gamma\tau}.$$

Proof. *i.* The first item is a classical property of the fundamental solution (see proposition 2.5).

ii. Assume B and C are $\mathcal{C}^k$. We put :

$$\mathcal{Y}(\tau, s) := \mathcal{X}(\tau, s) - a\,\mathcal{X}(\tau - r, s), \tag{2.41}$$

$$\widehat{C}(t) := C(t) + a\,B(t). \tag{2.42}$$

By definition of fundamental solution, we have :

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\tau}\mathcal{Y}(\tau, s) &= B(s+\tau)\,\mathcal{Y}(\tau, s) + a\,\widehat{C}(s+\tau)\,\mathcal{X}(\tau - r, s) \\ \mathcal{Y}(0, s) &= 1,\ \mathcal{Y}(\tau, s) = 0 \text{ for } \tau < 0 \\ \mathcal{X}(\tau, s) &= \mathcal{Y}(\tau, s) + a\mathcal{X}(\tau - r, s) \end{aligned}$$

- We prove that if B and C are $\mathcal{C}^k$ then $\mathcal{Y}$ is k continuously differentiable from its second argument s (for τ fixed) and for all j, $1 \leq j \leq k$,

$$(\tau, s) \mapsto \frac{\partial^j \mathcal{Y}}{\partial s^j}(\tau, s)$$

is continuous. For simplicity let us show the result for $k = 1$. By variation constant formula, we have :

$$\begin{aligned}
&\mathcal{Y}(t,s) = e^{\int_0^t B(s+u)du} + \int_0^t e^{\int_u^t B(s+v)dv} a\widehat{C}(s+u)\mathcal{X}(u-r,s)du \text{ for } t \geq 0,\\
&\mathcal{Y}(0,s) = 1\\
&\mathcal{X}(\tau,s) = \mathcal{Y}(\tau,s) + a\mathcal{X}(\tau - r, s)
\end{aligned}$$

Thus we have for $t \geq 0$:

$$\mathcal{Y}(t,s) = e^{\int_0^t B(s+u)du} + \int_0^t e^{\int_u^t B(s+v)dv} \widehat{C}(s+u) \sum_{k=1}^{+\infty} a^k \mathcal{Y}(u-(k+1)r, s)du.$$

Let's show by recurrence on n where $t \in [nr, (n+1)r[$ that $\mathcal{Y}(t,s)$ is differentiable in s and that the differential is continuous. For n negative integer, it is true because $\mathcal{Y}(t,s)$ is equal to zero eveyrwhere. If, that's true for n, so that's true for $n+1$ by hypothesis of recurrence and the previous equation.

- We have :

$$\mathcal{X}(t,s) \quad = \quad \sum_{k=1}^{+\infty} a^k \mathcal{Y}(t-kr, s) \text{ for all } t \text{ and } s. \tag{2.43}$$

According to the preceding point, we deduces the result of differentiation and continuity of $\mathcal{X}$.

iii. Put $Z(\tau, s) := \frac{\partial \mathcal{X}}{\partial s}(\tau, s)$. We have :

$$\frac{d}{dt}(\mathcal{X}(\tau,s) - a\mathcal{X}(\tau - r, s)) = B(\tau+s)\mathcal{X}(\tau,s) + C(\tau+s)\mathcal{X}(\tau - r, s). \tag{2.44}$$

Integrating this equality and differentiating with respect to s (which is legal by the point *ii*) we have :

$$\begin{aligned}
Z(\tau,s) = aZ(\tau - r, s)) + \int_s^\tau [&\dot{B}(u+s)\mathcal{X}(u,s) + \dot{C}(u+s)\mathcal{X}(u-r,s)\\
&+B(u+s)Z(u,s) + C(u+s)Z(u-r,s)]du.
\end{aligned} \tag{2.45}$$

Thus, by differentiating with respect to τ, we have :

$$\begin{aligned}
\frac{d}{dt}(Z(\tau,s) - aZ(\tau - r, s)) = B(\tau+s)Z(\tau,s) + C(\tau+s)Z(u-r,s) + \dot{B}(\tau+s)\mathcal{X}(\tau,s)\\
+\dot{C}(\tau+s)\mathcal{X}(\tau - r, s).
\end{aligned} \tag{2.46}$$

Remarking that $Z(t,s) = 0$ for $t \leq 0$, by Corollary 1.1 p259 of Hale [HVL93] there exists K' and γ' positive integer such that :

$$|Z(\tau,s)| \leq K' e^{\gamma'\tau} \int_0^\tau [\dot{B}(u+s)\mathcal{X}(u,s) + \dot{C}(u+s)\mathcal{X}(u-r,s)]du. \tag{2.47}$$

As B and C are periodic regular and the fact that the fundamental solution is exponentially bounded (see Corollary 1.1 p259 of Hale [HVL93]), thus there exists K, γ positive such that :

$$\left|\frac{\partial \mathcal{X}}{\partial s}(\tau, s)\right| \leq Ke^{\gamma\tau}.$$

We can prove in a similar way by a recursive argument :

$$\left|\frac{\partial \mathcal{X}^k}{\partial^k s}(\tau, s)\right| \leq Ke^{\gamma\tau}.$$

□

In the view to bring out the monodromy operator, we discretize the continuous dynamical system in a discrete dynamical one. For that we consider the input and the output on interval of length T. Put :

$$\widetilde{u}_k := u(kT + t) \text{ for } t \in [0, T] \text{ and } k \in \mathbb{Z},$$
$$\widetilde{y}_k := y(kT + t) \text{ for } t \in [0, T] \text{ and } k \in \mathbb{Z}.$$

Thus we have the following discrete system :

Theorem 2.22. *We have for all integer positive k :*

$$\begin{cases} x_{(k+1)T} &= \widetilde{A}x_{kT} + \widetilde{B}\widetilde{u}_k, \\ \widetilde{y}_k &= \widetilde{C}x_{kT} + \widetilde{D}\widetilde{u}_k, \end{cases} \tag{2.48}$$

where :

- $\begin{aligned} \widetilde{A} : C([-r, 0], \mathbb{R}) &\longrightarrow C([-r, 0], \mathbb{R}) \\ v &\longmapsto U(T, 0)v, \end{aligned}$
- $\begin{aligned} \widetilde{B} : L^2([0, T], \mathbb{R}) &\longrightarrow C([-r, 0], \mathbb{R}) \\ v &\longmapsto \int_0^{T+\cdot} d[K(T, \alpha)]v(\alpha), \end{aligned}$ *with* $K(t, \alpha)(\theta) = \int_0^\alpha X(t + \theta, v)dv$ *for* $\theta \in [-r, 0]$,
- $\begin{aligned} \widetilde{C} : C([-r, 0], \mathbb{R}) &\longrightarrow C([0, T], \mathbb{R}) \\ v &\longmapsto U(\cdot, 0)v(0) - aU(\cdot, 0)v(-r), \end{aligned}$
- $\begin{aligned} \widetilde{D} : L^2([0, T], \mathbb{R}) &\longrightarrow C([0, T], \mathbb{R}) \\ v &\longmapsto \int_0^\cdot X(\cdot, \tau)v(\tau)d\tau - a\int_0^\cdot X(\cdot - r, \tau)v(\tau)d\tau, \end{aligned}$
- $x_0 = 0$,
- *For all $k \in \mathbb{N}$, $x_{(k+1)T}$ is a function of $[-r, 0]$ in $\mathbb{R}$ continuous.*

Proof. Recall the system (2.37) :

$$\begin{cases} \frac{d}{dt}(x(t) - ax(t-r)) = B(t)x(t) + C(t)x(t-r) + u(t), \\ y(t) = x(t) - ax(t-r). \end{cases} \tag{2.49}$$

Let $k \in \mathbb{N}$. Recall the notation : $x_{(k+1)T}(\theta) = x((k+1)T + \theta)$, with $\theta \in [-r, 0]$, and where x is the only solution of the system with the initial data $x_0 = 0$. Applying the variation of constant formula (Theorem 2.7), we integrate during a period T for each k the system (2.49) with initial time kT. We obtain for $t \in [0, T]$:

$$\begin{cases} x_{(k+1)T}(\theta) & = U((k+1)T, kT)x_{kT}(\theta) + \int_{kT}^{(k+1)T+\theta} d[K((k+1)T, \alpha)]u(\alpha), \\ y(t+kT) & = U(t+kT, kT)x_{kT}(0) - aU(t+kT, kT)x_{kT}(-r) \\ & + \int_{kT}^{t+kT} X(t+kT, \tau)u(\tau)d\tau - a\int_{kT}^{t+kT-r} X(t+kT-r, \tau)u(\tau)d\tau, \end{cases}$$

where $K(t, \alpha)(\theta) = \int_0^\alpha X(t+\theta, \alpha)d\alpha$ for $\theta \in [-r, 0]$. By periodicity we have the result. □

For a sequence of operator $a := (a_n)_{n\in\mathbb{N}}$ we define the $z - transform$ as:

$$\hat{a} := \sum_{n\in\mathbb{N}} a_n z^{-n}, \tag{2.50}$$

when for $z \in \mathbb{C}$ this series exists. Taking the $z - transform$ in the equation (2.48) :

$$\widehat{Y} = [\widetilde{C}(zId - \widetilde{A})^{-1}\widetilde{B} + \widetilde{D}]\widehat{U}, \tag{2.51}$$

where $Y := (y_n)_{n\in\mathbb{N}}$ and $U := (u_n)_{n\in\mathbb{N}}$. On in the other side, we can compute the impulse response of the discrete system and we have for n integer :

$$\widetilde{y}_n(t) = \sum_{k=0}^{n} H_{[k]}\widetilde{u}_{n-k}(t), \tag{2.52}$$

where :

$$H_{[k]}v(t) = \int_0^T \tilde{X}(kT + t, \tau)v(\tau)d\tau \tag{2.53}$$

is the discrete impulse response of the discretized dynamical system. Taking the $z - transform$ in the equation (2.52), we have for $z \in \mathbb{C}$ with real part enough large that :

$$\widehat{Y}(z) = \widehat{H}(z)\widehat{U}(z), \tag{2.54}$$

where $\widehat{H}(z)$ is the $z - transform$ of $\left(H_{[n]}\right)_{n\in\mathbb{N}}$. We deduce from the equation (2.51) and the equation (2.54) :

$$\widehat{H}(z) = \widetilde{C}(zId - \widetilde{A})^{-1}\widetilde{B} + \widetilde{D}. \tag{2.55}$$

Even so far the singularities of the discrete transfer function (2.55) would give the exponential stability of the origin of the neutral differential system (2.19) if there is not an eigenvalue cancelled by the input-output system (2.48). We saw in Chapter 1, *i.e* for the ordinary differential equations, that it is not the case when the system is controllable and observable. In the case of neutral differential equations, even when the system is time-invariant, it is not an easy task to give a necessary and sufficient condition to ensure the controllability. We can cite the two following references which give

such conditions [OT83] and [Yam89]. When the system is a linear scalar periodic neutral differential equation, we did not find a literature available for the contrallability of such system that is why we give the following proof of the controllability of the system (2.37). We prove, with the use of the resolution of a Volterra first kind equation, that the discrete system (2.48), and so the neutral system (2.37), is controllable in the sense that all the functions absolutely continuous wwith square integrable derivative are reachable from a locally square integrable input.

Theorem 2.23. *The discrete system (2.48) is :*

- *detectable in the sense that if there exists a K integer such that for all $k \geq K$ we have $y_k = 0$ and $u_k = 0$, then x_{kT} tends to zero when k tends to the infinity.*
- *controllable for the absolutely continuous function with square integrable derivative, i.e. for all $x_f \in W_2^1$, there exists $\phi \in L^2([0,T],\mathbb{R})$ such that $x_f = \tilde{B}\phi$.*

Proof.

- Let K integer such that for all $k \geq K$ we have $y_k = 0$ and $u_k = 0$. We have by the system (2.37) that x_{kT} tends to zero when k tends to the infinity because $|a| < 1$.
- The controllability for the discrete system is equivalent to a Volterra first kind equation. The general theory of these kind of equations can be found in the textbooks [Bru17, GLS90]. We have that $t \longmapsto d_t X(t,s) = h(t,s) + \sum_{k=1}^{+\infty} a^k \delta_0(t-\alpha-kr)$ where h is a continuous piecewise function on $\mathbb{R}^2$. Let :

$$\kappa(t,\alpha) = -\int_0^\alpha h(t,u)du - \sum_{k=1}^{+\infty} a^k \mathfrak{h}(t-\alpha+kr), \tag{2.56}$$

where $\mathfrak{h}(\tau)$ is the Heaviside function which is zero for $\tau < 0$ and 1 for $\tau \geq 0$ (so that its derivative is a Dirac delta at 0), we have that $d_\alpha \kappa(t,\alpha)$ is equal to $-d_t X(t,\alpha)$. Let $x \in W_2^1([0,T],\mathbb{R})$, we search $\phi \in L^2([0,T],\mathbb{R})$ such that for all $t \in [0,T]$:

$$x(t) = \int_0^{t^+} X(t,s)\phi(s)ds. \tag{2.57}$$

The equation (2.57) is a first kind Volterra equation which can be transformed into a second kind Volterra equation using the derivative in the equation (2.57). We obtain for almost all $t \in [0,T]$:

$$\frac{d}{dt}x(t) = \phi(t) + \int_0^{t^-} d_t X(t,s)\phi(s)ds, \tag{2.58}$$

and with the equation (2.56)

$$\phi(t) = \int_0^{t^-} d_s \kappa(t,s)\phi(s) + \frac{d}{dt}x(t). \tag{2.59}$$

The Appendix, and more precisely Lemma A.2 and Lemma A.1 (see also [GLS90, ch. 10, thm 2.5]), says that the kernel κ is a Stieltjes-Volterra kernel on $[0,T]\times[0,T]$ and admits a resolvent ρ, which is a Stieltjes-Volterra kernel on $[0,T]\times[0,T]$. Let n an integer, a solution of the equation

$$\phi_n(t) = \int_0^{t^-} d_s \kappa(t,s)\phi_n(s) + f_n(t), \tag{2.60}$$

where f_n is a function of bounded variation, is given by :

$$\phi_n(t) = f_n(t) - \int_0^{t^-} d\rho(t,\alpha) f_n(\alpha), \tag{2.61}$$

for $t \in [0,T]$ and ϕ_n is a bounded function. By density we can find a sequence $(f_n)_{n\in\mathbb{N}}$ of function of bounded variation which converges in $L^2([0,T],\mathbb{R})$ toward $\frac{d}{dt}x(\cdot)$ and we define the sequence of $(\phi_n)_{n\in\mathbb{N}}$ which verifies the equations (2.61) for all integer n. Since $d\rho$ has just a finite number of Dirac, we have that $(\phi_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^2([0,T],\mathbb{R})$ and then is convergent toward a ϕ in $L^2([0,T],\mathbb{R})$. There exists a subsequence of $(\phi_n)_{n\in\mathbb{N}}$ which converge almost everywhere toward ϕ. Then we have for almost all $t \in [0,T]$ that ϕ satisfies the equation (2.59) and

$$\phi(t) = \frac{d}{dt}x(t) - \int_0^{t^-} d\rho(t,\alpha)\frac{d}{dt}x(\alpha). \tag{2.62}$$

Moreover, if we assume that $\frac{d}{dt}x(0) = 0$, then the solution ϕ in the equation (2.62) is a solution of the Volterra equation of first kind (2.57).

Let $x_f \in W_2^1$ and we search $\phi \in L^2([0,T],\mathbb{R})$ such that :

$$x_f = \tilde{B}\phi, \tag{2.63}$$

and equivalently :

$$x_f(\theta) = \int_0^{T^+} X(T+\theta,\alpha)\phi(\alpha)d\alpha. \tag{2.64}$$

We define $x \in W_1^2([0,T],\mathbb{R})$ such that $x(t) = 0$ for $t \in [0, T-r]$ and $x(t) = x_f(t-T)$ for $t \in [T-r, T]$ and then the equation (2.64) become the equation (2.57) which admits a solution in $L^2([0,T],\mathbb{R})$ by the previous reasonning. It has to be noted that this solution is not necessarily unique.

□

Remark 2.24. *We proved that the system (2.48) is detectable. The delay can induce a lack of observability, however we can see that if $aB(t) + C(t)$ is not equal to zero for all t real then the system (2.37) is observable and so the system (2.48) is observable too.*

Corollary 2.25. *For all z complex with modulus greater than the spectral radius of T_a, we have that $\widehat{H}(z)$ is an analytic Banach valued function except in the ζ_j for $j = 1, \cdots, N$ where it has poles.*

Proof. The two equations (2.51) and (2.54) leads to the equation (2.55) for z with a modulus enough large. We have that $\tilde{A}$ and the restriction of $\tilde{A}$ to W_2^1 has exactly the same spectrum composed of the eigenvalues $\zeta_1, \cdots, \zeta_N$ (see Corollary 2.19). Then, the structure of the spectrum of $\tilde{A}$, the controllability and detectability of the discrete system (Theorem 2.23) and the analytic continuation principle permit to extend the equation (2.55) for z with modulus greater that the spectral radius of T_a except when z is an eigenvalue of $\tilde{A}$ in this domain where it is a pole. □

We recall from the Definition 1.19 that the Instantaneous Transfer function (ITF) is defined by :

$$G(t,s) = \int_0^{+\infty} \tilde{X}(t, t-\tau)e^{-s\tau}d\tau. \tag{2.65}$$

Let us now give the link between the monodromy operator and the ITF.

Lemma 2.26. *We have :*

$$G(s,t) = e^{-st}[\widehat{H}(e^{sT})e^{s\cdot}](t). \tag{2.66}$$

Proof. We have by the definition of the $z-transform$:

$$\begin{aligned} \widehat{H}(z)v(t) &= \sum_{k=0}^{+\infty} H_{[k]}v(t)z^{-k} \\ &= \sum_{k=0}^{+\infty} z^{-k}\int_0^T \tilde{X}(kT+t,\tau)v(\tau)d\tau. \end{aligned} \tag{2.67}$$

We have by the equation (2.65) and integration by substitution :

$$\begin{aligned} G(s,t) &= \int_{-\infty}^{t} \tilde{X}(t,\tau)e^{s(\tau-t)}d\tau \\ &= \int_{-\infty}^{+\infty} \tilde{X}(t,\tau)e^{s(\tau-t)}d\tau \\ &= \sum_{k=-\infty}^{+\infty}\int_0^T \tilde{X}(t,\tau-kT)e^{s(\tau-t-kT)}d\tau \\ &= \sum_{k=0}^{+\infty} e^{-s(t+kT)}\int_0^T \tilde{X}(t+kT,\tau)e^{s\tau}d\tau. \end{aligned} \tag{2.68}$$

Since the equation (2.67) is equal to the equation (2.68) with $z = e^{sT}$, we have the result. □

The link between the ITF and the monodromy operator permits to reach the link between the monodromy operator and the Harmonic Transfer Function HTF. In fact, we recall from Chapter 1 that the HTF H is the infinite matrix with coefficient $H_{m,n}(s) := G_{m-n}(s - \frac{2i\pi n}{T})$ where the $(G_n(s))_{n\in\mathbb{Z}}$ are the Fourier coefficients of the ITF $G(\cdot,s)$). Let us prove our two major theorems :

Proof of Theorem 2.1 and Theorem 2.2. Fix $s \in \mathbb{C}$ with real part enough large. Let

$$\Lambda(s) : L^2([0,T],\mathbb{R}) \to L^2([0,T],\mathbb{R}), \tag{2.69}$$

where $[\Lambda(s)\phi](t) = e^{-st}[\hat{H}(e^{sT})e^{s\cdot}\phi(\cdot)](t)$ for all $\phi \in L^2([0,T],\mathbb{R})$. Theorem 2.21 gives the existence of $K(\Re(s))$ real positive which depend only of the real part enough large of s complex :

$$|G_n(s)| \leq \frac{K(\Re(s))}{1+n^2}, \tag{2.70}$$

where G_n is the n-th Fourrier coefficient of $G(\cdot,s)$. We have that the Fourier development of $\Lambda(s)\phi(t)$ is :

$$\Lambda(s)\phi(t) = \sum_{n\in\mathbb{Z}}\left(\sum_{k\in\mathbb{Z}} a_k G_{n-k}(s+2i\pi k/T)\right)e^{2i\pi n/T}, \tag{2.71}$$

with $(a_k)_{k\in\mathbb{Z}}$ the Fourier coefficients of ϕ. In fact, when $\phi \in C^2$, we deduce from Lemma 2.26, the equation (2.70), the regularity of the Fourier coefficient of ϕ that the equation (6.68) holds. By a density argument, the equation (6.68) holds as well for a function $\phi \in L^2([0,T],\mathbb{R})$.

By Parseval's identity :

$$\begin{aligned} &\text{The HTF } H(s) \text{ is a bounded operator on } l^2(\mathbb{Z}) \Leftrightarrow \Lambda(s) \\ &\qquad\qquad \text{is a bounded operator on } L^2([0,T],\mathbb{R}) \ . \end{aligned}$$

Corollary 2.19 implies that the spectrum of $U(t,s)$ restricted to L^2, $\mathcal{C}_{ABS}$, W_2^1 and C are equal and are composed of a finite number of eigenvalues outside a disk of radius strictly than 1. Then $\Lambda(s)$ is a bounded operator on L^2 if and only if e^{sT} is not in the spectrum of the monodromy operator $U(T,0)$, we have that $H(s)$ admits an analytic continuation on the domain D_a except on the $\frac{ln\zeta+2i\pi k}{T}$ where ζ is an element of the spectrum of the monodromy operator of $U(T,0)$. In particular, under observability assumption (see Remark (6.24)), if $T/r \notin \mathbb{Q}$, Corollary 2.19 implies that all points of the vertical line ∂D_a are essential singularities of the HTF H, as an operator valued analytic map which proves Theorem 2.1.

Put ζ_i for some i in $\{1,\cdots,n\}$ an isolated eigenvalue of $U(T,0)$ and $\tilde{E}_i$ its kernel. Since the discrete system is controllable for the W_2^1 functions (Theorem 2.23), $\tilde{E}_i$ is not empty. For all $\phi \in \tilde{E}_i$ we have by the representation of the resolvent of $\tilde{A}$:

$$\hat{H}(z)\phi = \sum_{k=0}^{p} \frac{\tilde{C}N^k\tilde{B}\phi}{(z-\zeta_i)^{k+1}}, \tag{2.72}$$

where N a nilpotent operator, for all z complex in a neighborhood of ζ_i and p is an integer. Put s_0 such that $e^{s_0T} = \zeta_i$, s is a complex in the neighborhood of s_0 and $\Psi_s(\cdot) = e^{-s\cdot}\phi(\cdot)$. Thus :

$$\lim_{s\to s_0} \left(e^{sT} - e^{s_0T}\right)^{p+1} [\Lambda(s)\phi](t) = e^{-s_0t}\left[\tilde{C}\tilde{B}\Psi_{s_0}(\cdot)e^{s_0\cdot}\right](t). \tag{2.73}$$

Since the discrete system is detectable (Theorem 2.23) the right hand side of the equation (2.73) is not identically equal to zero. We deduce that there exists n and k such that $\lim_{s\to s_0}\left(e^{sT}-e^{s_0T}\right)^{p+1} G_{n-k}(s+2i\pi k/T)$ is non-zero. We deduce the Theorem 2.2 from the Theorem 2.3 and Theorem 2.4. □

2.6 Conclusion

The structure of the harmonic transfer function is more complicated than in the finite case. In fact, there are other ways for the HTF $H(s)$ to not be a bounded operator from $l^2(\mathbb{Z})$ to $l^2(\mathbb{Z})$ than to be a pole. Theorem 2.1 remarks that all the information of the stability is not contains in the coefficient of the matrix $H(s)$ but is contained entirely in the matrix $H(s)$ view as a Banach valued operator. However we can see that the HTF is meromorphic in the closed right half plane where the poles are the logarithm of a finite family of complex numbers. Since on this simple example, we proved that the system is controllable and detectable, the analyticity of the HTF in the closed right half plane implies the local stability of the periodic solution (Theorem 2.2). It would be interesting to investigate further the structure of the harmonic transfer function in all the complex plane. We rise some open interesting questions which are linked with this possibly investigation.

Open questions

- The actual knowledge of the spectral theory for autonomous constant difference delay equation and neutral differential equation permits to know that the spectrum of the monodromy operator for an autonomous difference delay system is composed of not semi-Fredholm elements. Since the monodromy operator of the neutral differential system that we considered is a compact perturbation of the monodromy operator of the autonomous delay system, we have that the spectrum of the monodromy operator of the neutral differential equation is composed of elements which are not semi-Fredholm plus a finite number of eigenvalues. Because of the elements of the spectrum which are not semi-Fredholm are highly unstable after a compact perturbation (see the article of Bouldin [Bou82] for the Hilbert case and the article of Gonzalez et al [GO86] for the Banach case) it would be interesting to characterize entirely the nature of the elements of the spectrum of the monodromy operator for the neutral differential delay

system because the nature of this spectrum is strongly linked with the singularities of the harmonic transfer function.

- Bound on the number of unstable poles?
- We saw under observability assumption and the commensurability of the delay and the period that the harmonic transfer function has a vertical line of singularities in the left half plane which is composed of essential singularities where the HTF is viewed as a Banach valued operator from $l^2(\mathbb{Z})$ to $l^2(\mathbb{Z})$. This essential singularities come from the fact that the monodromy operator of the neutral differential equation possesses an entire disk of elements of the spectrum. For the infinite HTF to be an essential singularities could be caused by different things. In fact let s_0 an essential singularity of the harmonic transfer function, then the following things could happen :
  - $H(s_0)$ is not a continuous operator from $l^2(\mathbb{Z})$ to $l^2(\mathbb{Z})$
    - all the coefficient of $H(s_0)$ are analytic complex valued function
    - some coefficients of $H(s_0)$ have poles
    - some coefficients of $H(s_0)$ have essential singularities
  - $H(s_0)$ define a continuous operator from $l^2(\mathbb{Z})$ to $l^2(\mathbb{Z})$ but fail to have an analytic continuation in an open neighborhood of s_0.

  In the way to perform rational approximation on the coefficient G_n of the harmonic transfer function *via* the harmonic balance method it must be important to answer to the following question : May the (stable) singularities of the G_n's be other than poles ?

Chapter 3

Equations of a circuit containing lossless transmission lines

In this chapter we are interested to give the equations for general circuits containing several lossless transmission lines coupled with multiport which are composed of capacitors, inductors, resistors, diodes and transistors. More precisely :

i. Putting together the components of the Chapter 1, ie capacitors, diodes, inductors, resistors and transistors we can form a multiport. Formally speaking, the multiport is a directed graph with labeled vertices (called junction nodes), and edges (called branches). Branches correspond to the components and nodes to terminals thereof.

To each junction node j is associated a potential v_j, and to each edge k an electric current i_k. One of the junction nodes, say v_n, is the ground (its potential is 0 by convention). *We always assume that the graph associated with a multiport is connected.* The currents $i_m := (i_{k_1}, \cdots, i_{k_m})$ are plugged in between the ground and the junction nodes $(k_1, \cdots, k_m)$ of the circuit containing the components (see figure 6.1 below), where m is an integer which denotes the number of output of the multiport.

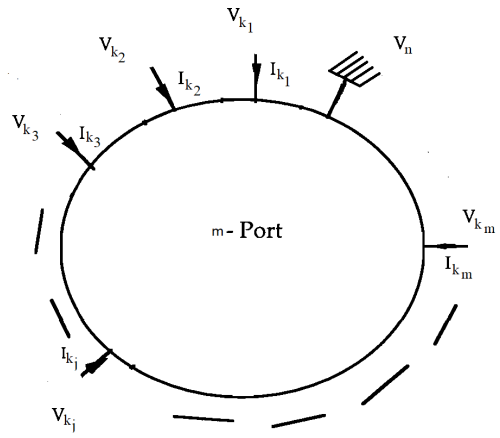


Figure 3.1 : A m multiport

To give the equations which govern the behavior of a such multiport, we assume that the voltage of such multiport $v_m := (v_{k_1}, \cdots, v_{k_m})$ can be expressed with the voltage of the capacitors, the current of the inductors and i_m the current of the multiport thus by the Brayton-Moser theorem (see [BM64]), we have that the equation of the multiport is :

$$\begin{cases} C_m \frac{dv_{c_m}}{dt} = h_c(v_{c_m}, i_{l_m}, i_m) \\ L_m \frac{di_{l_m}}{dt} = h_l(v_{c_m}, i_{l_m}, i_m), \\ v_m = h_m(v_{c_m}, i_{l_m}, i_m), \end{cases} \tag{3.1}$$

where v_{c_m} (resp. i_{l_m}) denotes the voltage (resp. current) of the capacitors (resp. inductors) contained in the multiport and the function h_c, h_l and h_m have the maximum regularity of the functions in the equation (1.2) and (1.5) that we assume at least C^4. Like in the Chapter 1, we add a smooth periodic generator and then the equations (6.71) become :

$$\begin{cases} C_m \frac{dv_{c_m}}{dt} = h_c(t, v_{c_m}, i_{l_m}, i_m) \\ L_m \frac{di_{l_m}}{dt} = h_l(t, v_{c_m}, i_{l_m}, i_m), \\ v_m = h_m(t, v_{c_m}, i_{l_m}, i_m), \end{cases} \tag{3.2}$$

ii. We already introduced the modelization of a transmission line in the Chapter 2 without explaining where comes from the lossless telegrapher's equation. Since a transmission line (see Figure 3.2) is commonly modeled as a concatenation of infinitesimal capacitors, resistors and inductors with the same impedance [Poz11] (see Figure 3.2, in which G denotes the conductance of the resistor and the hatched region is the ground).

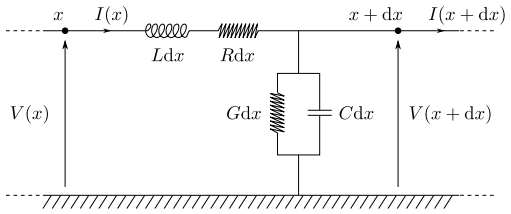


Figure 3.2 : Model of a transmission line

This model leads to the so-called telegrapher's equation [ML86, sec. 9.7.3]:

$$\begin{cases} \frac{\partial v}{\partial x} = -L\frac{\partial i}{\partial t} - Ri \\ \frac{\partial i}{\partial x} = -C\frac{\partial v}{\partial t} - Gv \end{cases} \tag{3.3}$$

When R and G are non zero and $L/R \neq G/C$, the previous equation doesn't have an analytic solution on a bounded domain in space. To be able to manipulate an analytic solution to telegrapher's equation, we assume in the following that $R = G = 0$, ie :

$$\begin{cases} \frac{\partial v}{\partial x} = -L\frac{\partial i}{\partial t} \\ \frac{\partial i}{\partial x} = -C\frac{\partial v}{\partial t} \end{cases} \tag{3.4}$$

To define the amplifier containing the lossless transmissions lines and the multiport, we have to connect each others. Formally speaking, we consider a directed graph with N edges and N' nodes, where N and N' are two positive integers. Nodes are numbered by integers from 1 to N' (that we denote as p) and edges by integers from 1 to N (that we denote as k).

Each edge figures a lossless transmission line. More precisely, we see edge number k as an oriented unitary real segment $[0,1]$ (*i.e.* a transmission line of unit length) with two real function $v_k(t,.)$ and $i_k(t,.)$ (the voltage and the current) defined on this segment at each time t, satisfying the following lossless telegrapher's equation ie R and G in the equation (3.4) are zero :

$$\begin{cases} C_k \dfrac{\partial v_k(t,x)}{\partial t} = -\dfrac{\partial i_k(t,x)}{\partial x}, \\ L_k \dfrac{\partial i_k(t,x)}{\partial t} = -\dfrac{\partial v_k(t,x)}{\partial x}, \end{cases} \qquad (t,x) \in \Omega\,, \tag{3.5}$$

where

$$\Omega \;=\; \{(t,x) \in I\!R^2,\ 0 < x < 1 \text{ and } 0 < t < +\infty\}, \tag{3.6}$$

and, for each k, L_k and C_k are two strictly positive numbers (the inductance and the capacity of the line k). In (5.3) we define numbers τ_k and K_k, often called the *delay* and *characteristic impedance* of the line k, and we assume without loss of generality after possibly re-ordering the edges, that the delays τ_k are increasing:

$$\tau_k = \sqrt{L_k C_k}\,, \quad K_k = \sqrt{C_k/L_k}\,, \quad 0 < \tau_1 \leq \tau_2 \cdots \leq \tau_N\,. \tag{3.7}$$

Each node consist in a multiport connected to the transmission line which touches this node. More precisely, if

- $j(p)$ is the out-degree of the graph at node p (i.e. the number of edges outgoing from node p) and $\tilde{j}(p)$ the in-degree of the graph at node p (i.e. the number of edges incoming at node p), and
- $k_1 < \cdots < k_{j(p)}$ are the labels of the edges *outgoing* from node p and $k_{j(p)+1} < \cdots < k_{j(p)+\tilde{j}(p)}$ the labels of the edges *incoming at* node p,
- c_p (resp. l_p) is the number of capacitors (resp. inductors) in the multiport at the node p
- v_r (resp. i_r) denotes the voltage (resp. current) of the capacitors (resp. inductors) at the node p for r belonging to $c_1, \cdots, c_p$ (resp. $l_1, \cdots, l_p$)

then node number p couples equations (3.5) together by imposing $j(p) + \tilde{j}(p) + c_p + l_p$ relations between the $2(j(p) + \tilde{j}(p)) + c_p + l_p$ entries of the following three vectors:

$$V_p(t) = \begin{pmatrix} v_{k_1}(t,0) \\ \vdots \\ v_{k_{j(p)}}(t,0) \\ v_{k_{j(p)+1}}(t,1) \\ \vdots \\ v_{k_{j(p)+\tilde{j}(p)}}(t,1) \end{pmatrix}, \qquad I_p(t) = \begin{pmatrix} -i_{k_1}(t,0) \\ \vdots \\ -i_{k_{j(p)}}(t,0) \\ i_{k_{j(p)+1}}(t,1) \\ \vdots \\ i_{k_{j(p)+\tilde{j}(p)}}(t,1) \end{pmatrix}, \tag{3.8}$$

and

$$z_p(t) = \begin{pmatrix} v_{c_1}(t) \\ \vdots \\ v_{c_p}(t) \\ i_{l_1}(t) \\ \vdots \\ i_{l_p}(t) \end{pmatrix}. \tag{3.9}$$

Thus applying the equation (3.2), we have that :

$$\begin{cases} V_p(t) & = h_{p_1}(t, I_p(t), z_p(t)), \\ \frac{dz_p(t)}{dt} & = h_{p_2}(t, I_p(t), z_p(t)) \quad p \in [|1, N'|]\,, \end{cases} \tag{3.10}$$

where h_{p_1} and h_{p_2} are at least C^4.

So far, we wrote a system of hyperbolic equations, namely (3.5), indexed by the edges of our graph, with boundary conditions given by a collection of N' linear time dependent relations, namely (3.10), indexed by the nodes of the graph. As a result, the boundary conditions at $x = 0$ and at $x = 1$ for a given telegrapher's equation of the form (3.5) are generally obtained from two different relations of the form(3.10). To offset this, we shall write the collection of all boundary conditions in lumped form, as a single linear relation between concatenated vectors $\mathbf{V}(t)$ and $\mathbf{I}(t)$ defined as

$$v(x,t) = \begin{pmatrix} v_1(x,t) \\ \vdots \\ v_N(x,t) \end{pmatrix}, \quad i(x,t) = \begin{pmatrix} i_1(x,t) \\ \vdots \\ i_N(x,t) \end{pmatrix}, \quad \mathbf{V}(t) = \begin{pmatrix} v(t,0) \\ \\ v(t,1) \end{pmatrix}, \quad \mathbf{I}(t) = \begin{pmatrix} -i(t,0) \\ \\ i(t,1) \end{pmatrix}, \tag{3.11}$$

that aggregate *all* boundary values of voltages and currents in the lines. We aggregate too all the current and the voltage of the inductors and the capacitors respectively :

$$z(t) = \begin{pmatrix} z_1(t) \\ \vdots \\ z_{N'}(t) \end{pmatrix}. \tag{3.12}$$

It follows that we can rewrite the sytem (3.10) :

$$\begin{cases} \mathbf{V}(t) & = \tilde{h}_1(t, \mathbf{I}(t), z(t)), \\ \frac{dz(t)}{dt} & = \tilde{h}_2(t, \mathbf{I}(t), z(t)), \end{cases} \tag{3.13}$$

where $\tilde{h}_1$ (resp. $\tilde{h}_2$) is a function at least C^4 from $\mathbb{R} \times \mathbb{R}^{2N} \times \mathbb{R}^{(c_1+l_1)\cdots(c_{N'}+l_{N'})}$ (resp. $\mathbb{R} \times \mathbb{R}^{2N} \times \mathbb{R}^{(c_1+l_1)\cdots(c_{N'}+l_{N'})}$) into $\mathbb{R}^{2N}$ (resp. $\mathbb{R}^{(c_1+l_1)\cdots(c_{N'}+l_{N'})}$).

3.1 Nonlinear hybrid differential delay equations

Let us now construe the system of coupled telegrapher's equations from (3.5) and (3.13) as a delay equation. Let f_k, g_k, thus a resolved form of the telegrapher's equation is :

$$\begin{cases} v_k(t,x) = f_k(x - \frac{t}{\tau_k}) + y_k(x + \frac{t}{\tau_k})\,, \\ i_k(t,x) = K_k[f_k(x - \frac{t}{\tau_k}) - y_k(x + \frac{t}{\tau_k})]\,, \end{cases} \tag{3.14}$$

and define:

$$x_k(t) = f_k(-\frac{t}{\tau_k}) \quad \text{and} \quad y_k(t) = g_k(1 + \frac{t}{\tau_k}). \tag{3.15}$$

Putting (3.15) in (3.14) we obtain :

$$\begin{cases} v_k(t,0) = x_k(t) + y_k(t - \tau_k)\,, \\ i_k(t,0) = K_k[x_k(t) - y_k(t - \tau_k)]\,, \\ v_k(t,1) = x_k(t - \tau_k) + y_k(t)\,, \\ i_k(t,1) = K_k[x_k(t - \tau_k) - y_k(t)]\,. \end{cases} \tag{3.16}$$

Putting the equation (3.16) in the equation (3.13), we obtain :

$$\left\{\begin{array}{lll} 0 & = \check{h}_1 \left(t, \begin{pmatrix} x_1(t) \\ \vdots \\ x_N(t) \\ y_1(t) \\ \vdots \\ y_N(t) \end{pmatrix}, \begin{pmatrix} x_1(t-\tau_1) \\ \vdots \\ x_N(t-\tau_N) \\ y_1(t-\tau_1) \\ \vdots \\ y_N(t-\tau_N) \end{pmatrix}, z(t) \right). \\ \frac{dz(t)}{dt} & = \check{h}_2 \left(t, \begin{pmatrix} x_1(t) \\ \vdots \\ x_N(t) \\ y_1(t) \\ \vdots \\ y_N(t) \end{pmatrix}, \begin{pmatrix} x_1(t-\tau_1) \\ \vdots \\ x_N(t-\tau_N) \\ y_1(t-\tau_1) \\ \vdots \\ y_N(t-\tau_N) \end{pmatrix}, z(t) \right), \end{array}\right. \tag{3.17}$$

where $\check{h}_1$ and $\check{h}_2$ are at least C^4 and in the space necessary to give a meaning to the equation (3.17). The equation (3.17) is called a nonlinear hybrid differential equation.

3.2 Linear hybrid differential delay equations

Assuming that the circuit contains a continuous periodic solution which is generated by the periodic generator. We can linearize the system (3.10) around this periodic solution, we obtain the following periodic system :

$$\left\{\begin{array}{ll} V_p(t) & = A_p(t)I_p(t) + A_x(t)z_p(t), \\ \frac{dz_p(t)}{dt} & = B_p(t)I_p(t) + B_x(t)z_p(t) \quad p \in [|1, N'|], \end{array}\right. \tag{3.18}$$

where $t \mapsto A_p(t), A_x(t), B_p(t)$ and $B_x(t)$ are maps from $\mathbb{R}^+$ to the set of $(j(p)+\tilde{j}(p))\times(j(p)+\tilde{j}(p))$, $(j(p)+\tilde{j}(p)) \times (c_p+l_p)$, $(c_p+l_p) \times (j(p)+\tilde{j}(p))$ and $(c_p+l_p) \times (c_p+l_p)$ matrices which are assumed to be *two time continuously differentiable* and *T-periodic*.

Since the vectors $V_p(t)$ (resp. $I_p(t)$), defined in (3.8), altogether contain each component of $\mathbf{V}(t)$ (resp. $\mathbf{I}(t)$) exactly once as p ranges from 1 to N', there is a $2N\times 2N$ permutation matrix P_1 such that

$$\begin{pmatrix} V_1(t) \\ \vdots \\ V_{N'}(t) \end{pmatrix} = P_1\,\mathbf{V}(t)\,, \quad \begin{pmatrix} I_1(t) \\ \vdots \\ I_{N'}(t) \end{pmatrix} = P_1\,\mathbf{I}(t)\,. \tag{3.19}$$

The set of equations (5.5), $1 \le p \le N'$, can now be written as

$$\left\{\begin{array}{ll} \mathbf{V}(t) & = \mathbf{A}(t)\,\mathbf{I}(t) + \mathbf{A_z}(t)z(t), \\ \frac{dz(t)}{dt} & = \mathbf{B}(t)\mathbf{I}(t) + \mathbf{B_z}(t)z(t) \end{array}\right. \tag{3.20}$$

with

$$\begin{aligned}
\mathbf{A}(t) &= P_1^{-1}\, diag(A_1(t),\ldots,A_{N'}(t))\, P_1 && (3.21)\\
\mathbf{A_z}(t) &= diag(A_{z_1}(t),\ldots,A_{z_{N'}}(t)) && (3.22)\\
\mathbf{B}(t) &= diag(B_1(t),\ldots,B_{N'}(t))\, P_1 && (3.23)\\
\mathbf{B_z}(t) &= diag(B_{z_1}(t),\ldots,B_{z_{N'}}(t)), && (3.24)
\end{aligned}$$

where $diag(A_1(t),\ldots,A_{N'}(t))$, $diag(A_{x_1}(t),\ldots,A_{x_{N'}})$, $diag(B_1(t),\ldots,B_{N'})$ and $diag(B_{x_1}(t),\ldots,B_{x_{N'}})$ are $2N\times 2N$, $2N\times(c_1+l_1)\cdots(c_{N'}+l_{N'})$, $(c_1+l_1)\cdots(c_{N'}+l_{N'})\times 2N$ and $(c_1+l_1)\cdots(c_{N'}+l_{N'})\times(c_1+l_1)\cdots(c_{N'}+l_{N'})$ block-diagonal matrices respectively. Plugging (3.16) in the first equation of (3.20) gives us

$$\begin{cases}
\begin{pmatrix} x_1(t)\\ \vdots\\ x_N(t)\\ y_1(t)\\ \vdots\\ y_N(t)\end{pmatrix} + \begin{pmatrix} y_1(t-\tau_1)\\ \vdots\\ y_N(t-\tau_N)\\ x_1(t-\tau_1)\\ \vdots\\ x_N(t-\tau_N)\end{pmatrix} = \mathbf{A}(t)\left[\begin{pmatrix} -K_1\,x_1(t)\\ \vdots\\ -K_N\,x_N(t)\\ -K_1\,y_1(t)\\ \vdots\\ -K_N\,y_N(t)\end{pmatrix} + \begin{pmatrix} K_1\,y_1(t-\tau_1)\\ \vdots\\ K_N\,y_N(t-\tau_N)\\ K_1\,x_1(t-\tau_1)\\ \vdots\\ K_N\,x_N(t-\tau_N)\end{pmatrix}\right] + \mathbf{A_z}(t)z(t)\\
\frac{dz(t)}{dt} = \mathbf{B}(t)\mathbf{K}\left[\begin{pmatrix} x_1(t)\\ \vdots\\ x_N(t)\\ x_1(t-\tau_1)\\ \vdots\\ x_N(t-\tau_N)\end{pmatrix} + \begin{pmatrix} y_1(t)\\ \vdots\\ y_N(t)\\ y_1(t-\tau_1)\\ \vdots\\ y_N(t-\tau_N)\end{pmatrix}\right] + \mathbf{B_z}(t)z(t)
\end{cases} \qquad (3.25)$$

and finally, defining

$$\mathbf{K} = diag(K_1,\ldots,K_N,K_1,\ldots,K_N) \quad \text{and} \quad P_2 = \begin{pmatrix} 0 & Id\\ Id & 0\end{pmatrix} \qquad (3.26)$$

where Id has size $N\times N$, we get as $P_2\mathbf{K} = \mathbf{K}P_2$ and

$$\begin{cases}
\begin{pmatrix} x_1(t)\\ \vdots\\ x_N(t)\\ y_1(t)\\ \vdots\\ y_N(t)\end{pmatrix} = -\left(I+\mathbf{A}(t)\,\mathbf{K}\right)^{-1}\left(I-\mathbf{A}(t)\,\mathbf{K}\right)P_2\begin{pmatrix} x_1(t-\tau_1)\\ \vdots\\ x_N(t-\tau_N)\\ y_1(t-\tau_1)\\ \vdots\\ y_N(t-\tau_N)\end{pmatrix} + \mathbf{A_z}(t)z(t).\\
\frac{dz(t)}{dt} = \mathbf{B}(t)\mathbf{K}\left[\begin{pmatrix} x_1(t)\\ \vdots\\ x_N(t)\\ x_1(t-\tau_1)\\ \vdots\\ x_N(t-\tau_N)\end{pmatrix} + \begin{pmatrix} y_1(t)\\ \vdots\\ y_N(t)\\ y_1(t-\tau_1)\\ \vdots\\ y_N(t-\tau_N)\end{pmatrix}\right] + \mathbf{B_z}(t)z(t)
\end{cases} \qquad (3.27)$$

Remark 3.1. *We needed to intervert the matrix $I+\mathbf{A}(t)\mathbf{K}$ in the equation (3.27). We just assumed that it was possible but we will see in the Chapter 6 that under some reasonable physical assumption (dissipativity of the circuit at high frequency), this matrix is always invertible.*

Remark 3.2. *The system (3.27) is slightly more complicated than the neutral one in the Chapter 2. Assuming that $\mathbf{A_z}(t)$ is invertible, then we can recover a linear differential neutral system but in general there is no reason allowing to do this assumption.*

3.3 Harmonic transfer function for a circuit containing lossless transmission lines

To check the zero exponential stability stability , we disturb the linearized circuit by adding a source of current i_ϵ at one node of the multiport which compose one node of the circuit containing the lossless transmission line and we observe the voltage u_ϵ. Thus the equation (3.18) become an input output system with the following equations :

$$\left\{\begin{array}{ll} V_p(t) & = A_p(t)I_p(t) + A_z(t)z(t) + A_\epsilon(t)i_\epsilon(t), \\ \frac{dx_p(t)}{dt} & = B_p(t)I_p(t) + B_z(t)z(t) + B_\epsilon(t)i_\epsilon(t) \quad p \in [|1, N'|], \\ u_\epsilon(t) & = C_p(t)I_p(t) + C_z(t)z(t) + C_\epsilon(t)i_\epsilon(t). \end{array}\right. \tag{3.28}$$

In the aggregated way the equations (3.20) become :

$$\left\{\begin{array}{ll} \mathbf{V}(t) & = \mathbf{A}(t)\,\mathbf{I}(t) + \mathbf{A_z}(t)z(t) + \mathbf{A}_\epsilon(t)i_\epsilon(t) \\ \frac{dz(t)}{dt} & = \mathbf{B}(t)\mathbf{I}(t) + \mathbf{B_z}(t)z(t) + \mathbf{B}_\epsilon(t)i_\epsilon(t \\ u_\epsilon & = \mathbf{C}(t)\,\mathbf{I}(t) + \mathbf{C_z}(t)z(t) + \mathbf{C}_\epsilon(t)i_\epsilon(t) \end{array}\right. \tag{3.29}$$

with

$$\begin{aligned} \mathbf{A}_\epsilon(t) &= diag(A_{\epsilon_1}(t), \ldots, A_{\epsilon_{N'}}(t)), && (3.30)\\ \mathbf{B}_\epsilon(t) &= diag(B_{\epsilon_1}(t), \ldots, B_{\epsilon_{N'}}(t)) && (3.31)\\ \mathbf{C}_\epsilon(t) &= diag(C_{\epsilon_1}(t), \ldots, C_{\epsilon_{N'}}(t)) && (3.32)\\ \mathbf{C}(t) &= P_1 diag(C_1(t), \ldots, C_{N'}(t)), && (3.33)\\ \mathbf{C_z}(t) &= diag(C_{z_1}(t), \ldots, C_{z_{N'}}(t)), && (3.34)\\ \mathbf{C}_\epsilon(t) &= diag(C_{\epsilon_1}(t), \ldots, C_{\epsilon_{N'}}(t)). && (3.35) \end{aligned}$$

Putting the equations (3.16) in (3.29), we obtain the following equations which are the input-output version of the equations (3.27) :

$$
\begin{cases}
\frac{dz(t)}{dt} = \mathbf{B}(t)\mathbf{K}\left[\begin{pmatrix} x_1(t) \\ \vdots \\ x_N(t) \\ x_1(t-\tau_1) \\ \vdots \\ x_N(t-\tau_N) \end{pmatrix} + \begin{pmatrix} y_1(t) \\ \vdots \\ y_N(t) \\ y_1(t-\tau_1) \\ \vdots \\ y_N(t-\tau_N) \end{pmatrix}\right] + \mathbf{B}_{\mathbf{z}}(t)z(t) + \mathbf{B}_{\epsilon}(t)i_{\epsilon}(t) \\
\begin{pmatrix} x_1(t) \\ \vdots \\ x_N(t) \\ y_1(t) \\ \vdots \\ y_N(t) \end{pmatrix} = -\left(I + \mathbf{A}(t)\,\mathbf{K}\right)^{-1}\left(I - \mathbf{A}(t)\,\mathbf{K}\right) P_2 \begin{pmatrix} x_1(t-\tau_1) \\ \vdots \\ x_N(t-\tau_N) \\ y_1(t-\tau_1) \\ \vdots \\ y_N(t-\tau_N) \end{pmatrix} + \mathbf{A}_{\mathbf{z}}(t)z(t) + \mathbf{A}_{\epsilon}(t)i_{\epsilon}(t) \\
u_{\epsilon}(t) = \mathbf{C}(t)\mathbf{K}\left[\begin{pmatrix} x_1(t) \\ \vdots \\ x_N(t) \\ x_1(t-\tau_1) \\ \vdots \\ x_N(t-\tau_N) \end{pmatrix} + \begin{pmatrix} y_1(t) \\ \vdots \\ y_N(t) \\ y_1(t-\tau_1) \\ \vdots \\ y_N(t-\tau_N) \end{pmatrix}\right] + \mathbf{C}_{\mathbf{z}}(t)z(t) + \mathbf{C}_{\epsilon}(t)i_{\epsilon}(t)
\end{cases} \tag{3.36}
$$

Since the system is periodic and linear we can define the harmonic transfer function $H(s)$ which is the infinite matrix satisfying the following equation :

$$U_{\epsilon}(s) = H(s)I_{\epsilon}(s), \tag{3.37}$$

where

$$U_{\epsilon}(s) = \begin{pmatrix} \vdots \\ \mathcal{L}\{u_{\epsilon}\}(s+i\omega_0) \\ \mathcal{L}\{u_{\epsilon}\}(s) \\ \mathcal{L}\{u_{\epsilon}\}(s-i\omega_0) \\ \vdots \end{pmatrix} \text{ and } I_{\epsilon}(s) = \begin{pmatrix} \vdots \\ \mathcal{L}\{i_{\epsilon}\}(s+i\omega_0) \\ \mathcal{L}\{i_{\epsilon}\}(s) \\ \mathcal{L}\{i_{\epsilon}\}(s-i\omega_0) \\ \vdots \end{pmatrix}. \tag{3.38}$$

Once again, using the frequency domain, we can express each transmission line (3.5) as a linear matrix (see equation (2.7) in the Chapter 2). Putting this linear matrices in the boundary equations (3.10) leads to a system like the system (1.12) and we can perform the Harmonic Balance analysis to approximate a periodic solution and approximate the Harmonic Transfer Function H on the imaginary axis.

The questions which arrive are :

i. Under suitable observability and controllability assumptions, does the holomorphy of the harmonic transfer function in the right half-plane give the local stability of the periodic solution?

ii. Does the harmonic transfer function have possibly only poles in the right half-plane?

Because of the regularity of the system it seems natural to think that the answer to the question *i* is yes but it is not a trivial thing because system like (3.27) hasn't been really considered until nowadays in the time-varying (here periodic) case. Concerning the question *ii* it seems to be more complicated. However, we can follow the intuition of the electronic engineers who say that, since the system at high frequency is exponentially stable, then the harmonic transfer function has possibly only poles in the right half plane. From an electronic engineers's point of view, the system at high frequency is the system where the capacitors are replaced by a wire and the inductors are replaced by an open switch. In mathematical terms, it means that x become zero in the equations (3.27) and thus we obtain the following equations, which models the behavior of the system at high frequency :

$$\begin{pmatrix} x_1(t) \\ \vdots \\ x_N(t) \\ y_1(t) \\ \vdots \\ y_N(t) \end{pmatrix} = -\left(I + \mathbf{A}(t)\,\mathbf{K}\right)^{-1}\left(I - \mathbf{A}(t)\,\mathbf{K}\right) P_2 \begin{pmatrix} x_1(t-\tau_1) \\ \vdots \\ x_N(t-\tau_N) \\ y_1(t-\tau_1) \\ \vdots \\ y_N(t-\tau_N) \end{pmatrix}. \tag{3.39}$$

The equation (3.39) is a periodic difference delay equations and are not really present in the literature. That's why before to address the issue of the questions *i* and *ii* in the Chapter 6, we study stability conditions for system like (3.39) in the Chapter 4 and Chapter 5. The Chapter 4 studies stability condition for general periodic difference delay system generalizing methods of Bellman, Cooke, Hale and Henry whereas the Chapter 5 give stability conditions using the particular structure of difference delay equations which come from a network of lossless telegrapher's equations.

The Chapter 6 relies the equations (3.27) with the equation (3.39), in fact the behavior of the system is just a compact perturbation of the system at high frequency. And since all the multiport are assumed to be passive at high frequency, we can conclude through the Chapter 4 or the Chapter 5 to the fact that the harmonic transfer function has just possible poles in the right half plane. Moreover we give some local stability results for the equations of the forms (3.27).

Chapter 4

Stability criteria of linear periodic difference delay systems

4.1 Introduction

Motivation

Physical models involving delays naturally arise in circuit theory, which has been an initial motivation of the authors for the present work. Indeed, in electric networks operating at high frequency, one cannot neglect delays induced by the transmission lines between the linear static (resistors), linear dynamic (capacitors/inductors), and nonlinear (diodes/transistors) elements of the circuit. Lossless transmission lines are typically modelled by telegrapher equations, which are the simplest transport partial differential equations (PDE). Using the integrated form of the latter, one obtains at a generic functioning rate a model for the circuit consisting of a system on nonlinear ordinary differential equations (ODE), coupled with a nonlinear delay difference system:

$$\begin{cases} \frac{dx(t)}{dt} = f(x(t), y(t-\tau_1), \cdots, y(t-\tau_N)) \\ y(t) = g(x(t), y(t-\tau_1), \cdots, y(t-\tau_N)), \end{cases} \tag{4.1}$$

where the x variable collects voltages or currents at terminals of the devices while the y variable consists of (combinations of) voltages and currents at the ends of the transmission lines [Bra67].

Periodic solutions of such dynamical systems arise spontaneously in the case of oscillators, or by periodic forcing in the case of amplifiers (the forcing is the signal to be amplified, represented for instance by a periodic voltage source). Assuming such a periodic solution $(\overline{x}(t), \overline{y}(t))$, one may linearize the equations around the latter to investigate local stability of the system, based on the exponential stability of the first order approximation [Sua09]. This linearized system consists of periodic linear ODE coupled with a periodic linear difference delay system of the form:

$$\begin{cases} \frac{dx(t)}{dt} = A(t)x(t) + \sum_{j=1}^{N} B_j(t)y(t-\tau_j) \\ y(t) = C(t)x(t) + \sum_{j=1}^{N} D_j(t)y(t-\tau_j). \end{cases} \tag{4.2}$$

As explained in [HVL93, Ch. 3, thm. 7.3] for the so-called neutral case (*i.e.* when $C(t)$ is smoothly invertible so that the x-variable can be eliminated) and more generally in [Fue20], the solution operator of (4.2) is a compact perturbation (modulo a projection) of the solution operator of the high frequency limit system which has no dynamics:

$$y(t) = \sum_{j=1}^{N} D_j(t)y(t-\tau_j). \tag{4.3}$$

Hence, the high frequency limit system must be exponentially stable for the original system to be exponentially stable, and when this is the case exponential stability of system (4.2) depends only on the sign of the real part of finitely many eigenvalues, due to the compactness of the perturbation. This motivates our study below of time-varying periodic difference delay systems of the form (4.3).

Recap of known results for time-invariant delay systems

Historically, the main efforts to study the stability of the system (4.2) have been made without considering the linear autonomous difference in itself. This efforts can be classified in two principal areas. The first consists to give a stability condition using the right complex half plane where the proofs are based on the Laplace transform techniques, the theory of the complex almost periodic function and the spectral semi-group theory (for this classical results, see the Bellman-Cooke book [BC59] or the Hale book [HVL93]). The second way to analyse the stability of this systems is to use the theory of the Lyapunov-Krasolneskii functional (see the chapter 5 of the book of [HVL93]). Nowaday, the efforts focus on the building of such functional (see [GEM19], [RN02]).
Surprisingly, the first stability of autonomous linear difference system occur late in [CH70] for a finite number of commensurable delay. This result has been improved by Henry who proved the same theorem for an infinite number of discrete delay and non necessarily commensurable (see [Hen74]), and later by Hale for distributed delays (see chapter 9 of [HVL93]). We consider the following constant difference delays system :

$$y(t) = \sum_{j=1}^{N} D_j\, y(t-\tau_j), \text{ for all } t \geq s, \tag{4.4}$$

where s is a real time initial data, d and N are positive integer, and each D_j is a constant $d \times d$ matrix and solutions $t \longmapsto y(t)$ are $\mathbb{C}^d$ valued functions of time t and the delays can be ordered as $\tau_1 < \cdots < \tau_N$. For constant difference-delay equations like (4.4), the Henry-Hale theorem below gives a necessary and sufficient condition for zero to be exponentially stable (see [Hen74, CH70]). Hereafter, $\|.\|$ indicates the Euclidean norm.

Theorem 4.1 (Henry-Hale)**.** *The following two properties are equivalent:*

(a) There exists $\gamma > 0$ and $K > 0$ such that for all continuous function y, satisfying (4.4) and verifying :

$$\sup_{s-\tau_N \leq \theta \leq s} ||y(t+\theta)|| \leq K e^{-\gamma(t-s)} \sup_{s-\tau_N \leq \theta \leq s} ||y(\theta)||, \text{ for all } t \geq s \text{ and all } s \text{ real.} \tag{4.5}$$

where $||\cdot||$ is the euclidean norm on $\mathbb{C}^d$.

(b) There exists $\alpha < 0$ such that :

$$I - \sum_{j=1}^{N} D_j\, e^{-p\tau_j} \text{ is invertible for all complex } p \text{ with real part greater than } \alpha. \tag{4.6}$$

The assumption of the Henry-Hale theorem is not always simple to check because it involves the computation of an infinite number of determinants. However the condition (4.6) is often simplified in the case of transport equation, because each delays τ_i acts only on the non zero columns of D_i for all $i \in \{1, \cdots, N\}$ and we have just to check that the spectral radius of $\sum_{j=1}^{N} D_j$ is in the unit disk. Theorem 5.17 is proved with a Laplace transformation technique for one implication and by the spectral semigroup theory for the converse.

Review of the existing results of the linear time-varying difference delay system and outline of the chapter

When the linear difference delay equation becomes non-autonomous (ie time-varying), the case turns out to be more difficult and has not been studied extensively. First, a straightforward generalization of the Theorem 5.17 in the time-varying case become obviously false. In fact, considering the following delay system :

$$y(t) = D_1(t)y(t-1), \tag{4.7}$$

where $D_1(t)$ is a 2×2 continuous $2-$*periodic* matrix. We can construct a matrix $D_1(\cdot)$ which has the spectral radius strictly less than one and however has the spectral radius of $D_1(t)D_1(t-1)$ greater than one which is an example which checks the condition 2. of the Theorem 5.17 but which is not exponentially stable.
For the time-varying linear delay system, the first article referring to a sufficient stability condition which is obtained through the Perron-Frobenius theorem can be found in [NH15]. Involving a kind of a joint spectral radius, Chitour et al ([CMS16]) give a necessary and sufficient condition in the general case of linear non-autonomous difference equation. But the condition leads to a complicated combinatories and the criterion become difficult to check. When the linear time-varying system come from a network lossless telegrapher's equations we can find a stability result for a subclass of difference delay system in [BFLP20]. To sum up, the stability of the constant linear difference equation is entirely understood via the Henry-Hale theorem, and the general time varying seem to lead to a too complicated combinatorics to have a general result like the Henry-Hale theorem. Between this two cases, it seems interesting to investigate the periodic difference delay system who provide, through the using of the Fourier developpment, an intermediate case in terms of difficulties. In this article we give a generalization of the Henry-Hale theorem in the case of linear regular periodic difference equation and show an application of this theorem to the 1-D hyperbolic equation with linear periodic boundaries.

4.2 Notation

We let $\mathbb{R}$ and $\mathbb{C}$ be the real and complex fields, and $\mathbb{R}^d$ (resp. $\mathbb{C}^d$) the Euclidean (resp. Hermitian) space of real (resp. complex) dimension d. When $x, y \in \mathbb{R}^d$ (resp. $\mathbb{C}^d$), we write $\langle x, y\rangle$ for their scalar (resp. Hermitian) product and $\|x\|$ for the norm of x. The notation is the same for the real and complex case, but the context will keep the meaning clear.

For $E \subset \mathbb{R}^d$ or $\mathbb{C}^d$, we let $C^0(E)$ be the space of continuous real or complex valued functions. Likewise, for $\mathcal{B}$ a Banach space, we write $C^0(E, \mathcal{B})$ for continuous $\mathcal{B}$-valued maps on E. When E is compact, we endow $C^0(E)$ and $C^0(E, \mathcal{B})$ with sup norm. For $\alpha \in (0, 1)$, we designate by $C^\alpha(E)$ the space of real or complex valued Hölder continuous functions with exponent α; *i.e.*, $f \in C^\alpha(E)$ if and only if $|f(x) - f(y)| \leq C\|x - y\|^\alpha$ for some constant C and all $x, y \in E$, the smallest C being the Hölder constant of f. For $f \in C^\alpha(E)$, we note the partial derivative $\frac{\partial^\alpha}{\partial t^\alpha}$ the smallest C such that $|f(t) - f(y)| \leq C\|t - y\|^\alpha$ for all $y \in E$. The space $C^\alpha(E, \mathcal{B})$ is defined analogously. When E is open, we let $C^1(E)$ (resp. $C^1(E, \mathcal{B})$) designate real or complex (resp. $\mathcal{B}$-valued) functions on E whose first derivatives lie in $C^0(E)$. The space $C^{1,\alpha}(E)$ comprises functions whose first derivatives belong to $C^\alpha(E)$, and $C^1(\overline{E})$ those functions which are restriction to E of an element of $C^1(E')$, where E' is open and contains $\overline{E}$. The spaces $C^{1,\alpha}(E, \mathcal{B})$ and $C^1(\overline{E}, \mathcal{B})$ are defined analogously. The space of continuous periodic functions on $\mathbb{R}$ with period $T > 0$ identifies, for any $a \in \mathbb{R}$, with the subspace $\widetilde{C}^0([a, a + T])$ of $C^0([a, a + T])$ consisting of f for which $f(a) = f(a + T)$. Alternatively, this space may be viewed as $C^0(\mathbb{T})$ where $\mathbb{T}$ is the unit circle, up to a rescaling of the variable by $2\pi/T$. Similar definitions hold for $\mathcal{B}$-valued periodic functions.

For I a real interval, recall that a function $f : I \to \mathbb{R}$ is said to have bounded variation if

$$W_I(f) := \sup_{\substack{x_0<x_1<\cdots<x_N \\ x_i\in I, N\in\mathbb{N}}} \sum_{i=1}^{N} |f(x_i) - f(x_{i-1})| < \infty. \tag{4.8}$$

In the degenerate cases where I is empty or reduces to a point, we set $W_I(f) = 0$. The number $W_I(f)$ is called the total variation of f on I, and we let $BV(I)$ indicate the space of functions of bounded variation on I endowed with the norm $\|f\|_{BV(I)} = W_I(f) + |f(o)|$, where o is arbitrary but fixed in I. Different o give rise to equivalent norms, for which $BV(I)$ is a Banach space. When I is bounded on the right and contains its endpoint b, we choose $o = b$ for definiteness. Note that functions in $BV(I)$ are bounded and $\|.\|_{BV(I)}$ is stronger than the uniform norm, because evidently $|f(x)| \leq |f(o)| + W_I(f)$. A function of bounded variation is the difference of two nondecreasing functions, therefore it has a limit $f(x^-)$ (resp. $f(x^+)$) from the left (resp. right) at each $x \in I$ where the limit applies [Łoj88, sec. 1.4]. If $f, g \in BV(I)$, then clearly $fg \in BV(I)$ with

$$W_I(fg) \leq W_I(f) \sup_{x\in I} |g(x)| + W_I(g) \sup_{x\in I} |f(x)|. \tag{4.9}$$

To $f \in BV(I)$ one can associate a unique finite signed Borel measure ν_f on I such that, whenever $(a, b) \subset I$, we have $\nu_f((a, b)) = f(b^-) - f(a^+)$, while if I is bounded on the right (resp. left) and contains its endpoint a, then $\nu_f(\{a\}) = f(a) - f(a^-)$ (resp. $f(a^+) - f(a)$) [Łoj88, ch. 7, pp. 185–189]. For $g : I \to \mathbb{R}$ a measurable function which is summable against ν_f, the Lebesgue-Stieltjes integral $\int g df$ is defined to be $\int g d\nu_f$, whence the differential element df identifies with $d\nu_f$ [Łoj88, ch. 7, pp. 190–191]. From (4.8) and the regularity of finite positive Borel measures on $\mathbb{R}$ [Rud82, thm. 2.18], it follows that $W_I(f) = \|\nu_f\|$ when f is monotonic on I, where $\|\nu_f\|$ indicates the total variation of the measure ν_f, see [Rud82, sec. 6.1]. In general, it holds that $\|\nu_f\| \leq 2W_I(f)$: this follows from the definition of ν_f [Łoj88, eqn. 7.5.13] and the decomposition of f into its continuous and saltus part [Łoj88, thm. 1.4.4]. Note that different f may generate the same ν_f: for example if f and f_1 coincide except at finitely many interior points of I, then $\nu_f = \nu_{f_1}$. In any case $|\int g df| \leq \int |g| d|\nu_f| \leq 2W_I(f) \sup_I |g|$, where $|\nu_f|$ is the total variation measure of ν_f; that is: $|\nu_f|$ is the only positive measure on I with mass $\|\nu_f\|$ such that ν_f is absolutely continuous with respect to $|\nu_f|$ and the Radon-Nykodim derivative $d\nu_f/d|\nu_f|$ has modulus 1 at $|\nu_f|$-almost every point, see [Rud82, sec. 6.1]. We let $BV_r(I) \subset BV(I)$ denote the subpace of functions that are right-continuous. Likewise, $BV_l(I) \subset BV(I)$ is the subpace of left-continuous functions. Because $\|.\|_{BV(I)}$ is stronger than the uniform norm, $BV_r(I)$ and $BV_l(I)$ are Banach spaces in their own right. We shall lean especially on the space $BV_l[a, b])$. By what precedes, it consists of functions of the form $u(t) = \nu[a, t) + c$ with ν a finite signed measure on $[a, b]$ and c a real constant; clearly then, $\nu_u = \nu$.

Given $f \in BV(J)$, we often need to integrate a function g with respect to df over a subinterval $I \subset J$. The subinterval I is typically encoded by assigning bounds to the integral sign: a lower bound a^- (resp. a^+) means that I contains (resp. does not contain) its lower bound a, while an upper bound b^+ (resp. b^-) means that I contains (resp. does not contain) its upper bound b. Care must be used with this notion, for $\nu_{f_{|I}}$ needs not coincide with the restriction of $\nu_{f_{|J}}$ to I. More precisely, if there is an endpoint a of I that belongs to I and lies interior to J, then the two measures may differ by the weight they put on $\{a\}$: when a is the right (resp. left) endpoint of I, the two measures agree if f is right (resp. left) continuous at a. For example, when splitting an integral of the form $\int_{a^\pm}^{b^\pm} g df$ as $\int_{a^\pm}^{c^\pm} g df + \int_{c^\pm}^{b^\pm} g df$ where $c \in (a, b)$, we must use c^+ (resp. c^-) if f is right (resp. left) continuous at c.

We denote by $BV_{loc}(\mathbb{R})$ the space of functions whose restriction to any bounded interval $I \subset \mathbb{R}$ lies in $BV(I)$.

We apply the above notation and definitions to complex, vector and matrix-valued functions, replacing absolute values in (4.8) by moduli, Euclidean and operator norms, respectively. This of course means that the real and imaginary parts, or components lie in the corresponding spaces of real-valued functions.

4.3 Results

We consider the following periodic difference delay system:

$$y(t) = \sum_{j=1}^{N} D_j(t) y(t - \tau_j), \qquad t \geq s, \tag{4.10}$$

where $s \in \mathbb{R}$ is the initial time, d and N are positive integers, $D_j : \mathbb{R} \to \mathbb{C}^{d\times d}$ is a continuous T-periodic matrix-valued function for $1 \leq j \leq N$, and the solutions to (6.6) are $\mathbb{C}^d$ valued functions $t \longmapsto y(t)$ of the time t. Without loss of generality, we order the delays so that $0 < \tau_1 < \cdots < \tau_N$. Note that all delays are assumed to be strictly positive. The case where $\tau_1 = 0$ and $I - D_1(t)$ is invertible for all t quickly reduces to the present one, but when $I - D_1(t)$ is not invertible new difficulties arise that we shall not take up here.

Without loss of generality, we can assume that $T > \tau_N$ (otherwise just consider the smallest integer p such that $pT > \tau_N$ and let $T' := pT$). In order to have a unique continuous solution to (6.6), we need continuous initial data on the interval $[s - \tau_N, s]$ that satisfy a compatibility condition at the endpoints. So, we define the space :

$$C_s := \{\phi \in C_0([-\tau_N, 0], \mathbb{C}^d) | \phi(0) = \sum_{j=1}^{N} D_j(s)\phi(-\tau_j)\}. \tag{4.11}$$

Since the system is T-periodic, we have that $C_{s+T} = C_s$. We endow C_s with the supremum norm :

$$||\phi||_\infty := \sup_{-\tau_N \leq \theta \leq 0} ||\phi(\theta)||, \tag{4.12}$$

where $||\cdot||$ is the euclidean norm on $\mathbb{C}^d$. For $\phi \in C_s$, a recursive argument shows that the system (6.6) admits a unique continuous solution y with $y(s + \theta) = \phi(\theta)$ for $\theta \in [-\tau_N, 0]$. Thus, we may define the solution operator:

$$\begin{aligned} U(t, s) : C_s &\to C_t & (4.13)\\ \phi &\mapsto U(t, s)\phi, & (4.14) \end{aligned}$$

where $(U(t, s)\phi)(\theta) = \{y(t + \theta),\ \theta \in [-\tau_N, 0]\}$.

Definition 4.2. The zero solution of the system (6.6) is called C^0 asymptotically exponentially stable if there exist $\gamma > 0$ and $K > 0$ such that :

$$||U(t, s)\phi||_\infty \leq Ke^{-\gamma(t-s)}||\phi||_\infty, \text{ for all } s \in \mathbb{R}, \text{ all } t \geq s \text{ and all } \phi \in C_s. \tag{4.15}$$

Before stating our main theorem, we introduce more notation. We consider the Fourier coefficient $(\check{D}_j(k))_{k\in\mathbb{Z}}$ of D_j for $j \in \{1, \cdots, n\}$, that is:

$$\check{D}_j(k) := \frac{1}{T}\int_0^T D_j(t)e^{-\frac{i2\pi kt}{T}}dt. \tag{4.16}$$

We let L_{D_j} be the (doubly infinite) block Laurent matrix associated with $D_j(t)$, whose entries are

$$\left(L_{D_j}\right)_{k,n} := \check{D}_j(k-n), \qquad n,k \in \mathbb{Z}. \tag{4.17}$$

Since each D_j is bounded, these Laurent matrices act on the Hilbert space :

$$l_d^2(\mathbb{Z}) := \{z = (z_j)_{j\in\mathbb{Z}} | z_j \in \mathbb{C}^d, \sum_{j=-\infty}^{+\infty} ||z_j||^2 < +\infty\}, \tag{4.18}$$

equiped with the norm$\|z\|_2$ given by

$$||z||_2^2 := \sum_{j=-\infty}^{+\infty} ||z_j||^2. \tag{4.19}$$

Note that $\|L_{D_j} z\|_2 = \|D_j Z\|_{L^2(\mathbb{T})}$ where $\mathbb{T}$ is the unit circle and $L^2(\mathbb{T})$ the familiar space of square summable functions on $\mathbb{T}$, while $Z \in L^2(\mathbb{T})$ is the function whose Fourier coefficients are the z_k.

For $A : l_d^2(\mathbb{Z}) \to l_d^2(\mathbb{Z})$ a linear operator, we denote by $|||A|||_2$ its operator norm:

$$|||A|||_2 := \sup_{z\neq 0} \frac{||Az||_2}{||z||_2}. \tag{4.20}$$

Irrespectively of $k \in \mathbb{N}$, we put I_∞ for the identity operator on $l_k^2(\mathbb{Z})$ and, for $1 \leq j \leq N$, we define the (doubly infinite) block diagonal matrix $\tilde{D}_{\tau_j}$ by

$$\tilde{D}_{\tau_j} := \text{ diag}\{\cdots, e^{-4i\pi\tau_j/T} I_d, e^{-2i\pi\tau_j/T} I_d, I_d, e^{+2i\pi\tau_j/T} I_d, e^{+4i\pi\tau_j/T} I_d, \cdots\}, \tag{4.21}$$

where I_d is the identity matrix of size $d \times d$.

Theorem 4.3. *Assume that the $D_j : \mathbb{R} \to \mathbb{C}^{d\times d}$ are periodic and differentiable with Hölder continuous derivative for $1 \leq j \leq N$. We have the following equivalence :*

- *there exists a real number $\alpha < 0$ such that :*

  i. $I_\infty - \sum_{j=1}^{N} e^{-p\tau_j} L_{D_j} \tilde{D}_{\tau_j}$ *is invertible in $l_d^2(\mathbb{Z})$ to $l_d^2(\mathbb{Z})$ for all $p \in \{z \in \mathbb{C} | \Re(z) \geq \alpha\}$,*

  ii. $\left\| \left[I_\infty - \sum_{j=1}^{N} e^{-p\tau_j} L_{D_j} \tilde{D}_{\tau_j} \right]^{-1} \right\|_2$ *is uniformly bounded for all $p \in \{z \in \mathbb{C} | \Re(z) \geq \alpha\}$,*

- *the System (6.6) is C^0 exponentially stable.*

The chapter is organised as follows. The section 4.4 deals with the proof of the sufficiency of the theorem 4.3 while the section 4.5 is devoted to the proof of the necessity. The section 4.6 represents a discussion about the assumptions and the immediate corollaries of the theorem 4.3, and gives an application to the 1-D Hyperbolic Systems with linear periodic boundaries.

4.4 Sufficiency

We assume that the assumption *i* and *ii* of the theorem 4.3 are verified and we prove that the System (6.6) is C^0 exponentially stable. We proceed in four steps.

Step 1: For $s \in \mathbb{R}$ and $t \geq s$, we introduce below a function $X : \mathbb{R}^2 \to \mathbb{R}^{d\times d}$, called the fundamental solution of System (6.6), that satisfies the equation:

$$X(t,s) = \begin{cases} 0 \text{ for } t < s, \\ I_d + \sum_{j=1}^{N} D_j(t) X(t-\tau_j, s) \text{ for } t \geq s. \end{cases} \tag{4.22}$$

Proceeding inductively, it is easy to check that X uniquely exists. Introducing the set :

$$\mathcal{F} := \{\sum_{j=1}^{N} n_j \tau_j |\ n_j \in \mathbb{N} \text{ for } j = 1, \cdots, N\}, \tag{4.23}$$

we remark that $X(t,s)$ is continuous when $t - s \notin \mathcal{F}$ and has a bounded jump across each line $t - s = f$ as f ranges over $\mathcal{F}$. In particular for fixed $t \in \mathbb{R}$, $X(t,\cdot)$ is locally a function of bounded variation on $\mathbb{R}$ with respect to its second argument. The importance of the fundamental solution stems from the following integral representation formula expressing the solution y of the system (6.6) in terms of X and the initial data $\phi \in C_s$:

$$y(t) = -\sum_{j=1}^{N} \int_{s^-}^{(s+\tau_j)^-} d_\alpha X(t,\alpha) D_j(\alpha) \phi(\alpha - \tau_j - s), \qquad t \geq s, \tag{4.24}$$

where the right hand side of (4.24) is a sum of Lebesgue-Stieltjes integrals and the minus or and plus superscripts in the bounds indicates they are taken in the lower and upper limit, respectively; *i.e.*, the j-th integral above is taken on $[s, s+\tau_j($, and $d_\alpha X(t,\alpha)$ indicates differentiation of $[s, s+\tau_j(\ni \alpha \mapsto X(t,\alpha)$. We observe that the corresponding formulas for autonomous linear difference-delay systems which appear in [Hen74] and [Hal77] seem to have issues, which is why we give a proof of (4.24) in A.1. We have that the fundamental solution is exponentially bounded. More precisely, we prove by induction on k integer the existence of a $K > 2$ and a $\lambda > 0$ such that :

$$|||X(t,s)||| \leq K e^{\lambda(t-s)}, \text{ for all } t \geq s, \tag{4.25}$$

where $|||X(t,s)|||$ denote the operator norm of $X(t,s)$ induced by $||\cdot||$ on $\mathbb{C}^d$. Let K' a positive uniform bound of the periodic continuous family $(D_i(\cdot))_{i=1,\cdots,N}$ and we take λ enough large such that :

$$K' N e^{-\lambda\tau_1} < \frac{1}{2}. \tag{4.26}$$

For $k = 0$, it is obvious that the equation (4.25) holds for $0 \leq t - s < \tau_1$ for some λ and K positive real. We assume that (4.25) hold for k integer and all t, s such that $0 \leq t - s < k\tau_1$. For t, s real verifying $k\tau_1 \leq t - s < (k+1)\tau_1$, we have from the formula (4.22) :

$$\begin{aligned} |||X(t,s)||| &\leq 1 + K' N K e^{-\lambda\tau_1} e^{\lambda(t-s)} && (4.27) \\ &\leq K e^{\lambda(t-s)}, && (4.28) \end{aligned}$$

which concludes the induction. To prove that $y(t)$ is exponentially stable, we will compute the variation of $X(t,s)$ in its second variable by using the Laplace transform.

Step 2: Since we assumed that the D_j has a Hölder continuous derivative, we have the following regularity of the fourier coefficients ([Zyg02, Ch. 2, thm. 4.7]) :

$$\left|\left|\left| \check{D}_j(k) \right|\right|\right| \leq \frac{C}{1 + |k|^{1+\delta}}, \ j \in \{1, \cdots, n\}, \tag{4.29}$$

where C is a positive constant and $\delta \in]0,1[$ the Hölder exponent of the derivative. Putting the Fourier coefficient of D_j (see equation (4.16)) in the equation (4.22) we obtain :

$$X(t,s) = I + \sum_{k\in\mathbb{Z}}\sum_{j=1}^{N} e^{\frac{2i\pi kt}{T}}\check{D}_j(k)X(t-\tau_j,s) \text{ if } t \geq s \text{ and } 0 \text{ if } t < s. \tag{4.30}$$

The equation (4.30) has a sense because of the regularity of the fourier coefficient (4.29). Performing the Laplace transform in the equation (4.30), and permuting the series and the integral by the dominated convergence theorem (thanks to the equation (4.29) again), we obtain for p complex with the real part of p equal to $c \in \mathbb{R}$ largest than the λ defined in the equation (4.25) :

$$\int_{-\infty}^{+\infty} e^{-pt}X(t,s)dt = \int_{-\infty}^{+\infty} \mathbb{1}_{[s,+\infty[}(t)e^{-pt}Idt + \sum_{k\in\mathbb{Z}}\sum_{j=1}^{N}\int_{-\infty}^{+\infty} e^{(-p+\frac{2i\pi k}{T})t}\check{D}_j(k)X(t-\tau_j,s)dt, \tag{4.31}$$

where $\mathbb{1}_{[s,+\infty[}$ is the charateristic function equal to 1 on $[s,+\infty[$ and 0 elsewhere. We note the Laplace transform of $X(t,s)$ in its firt variable :

$$\hat{X}(p,s) := \int_{-\infty}^{+\infty} e^{-pt}X(t,s)dt. \tag{4.32}$$

By N substitutions ($t \to t-\tau_j$ for all $j = 1,\cdots,N$), we deduce from the equation (5.38) :

$$\hat{X}(p,s) = \frac{e^{-ps}}{p}I + \sum_{k\in\mathbb{Z}}\sum_{j=1}^{N} e^{(-p+\frac{2i\pi k}{T})\tau_j}\check{D}_j(k)\hat{X}(p-\frac{2i\pi k}{T},s). \tag{4.33}$$

We define $\hat{\mathbb{X}}(p,s) := \begin{pmatrix} \vdots \\ \hat{X}(p+\frac{2i\pi}{T},s) \\ \hat{X}(p,s) \\ \hat{X}(p-\frac{2i\pi}{T},s) \\ \vdots \end{pmatrix}$, $\hat{e}(p,s) := \begin{pmatrix} \vdots \\ \frac{e^{-(p+\frac{2i\pi}{T})s}}{p+\frac{2i\pi}{T}}I \\ \frac{e^{-ps}}{p}I \\ \frac{e^{-(p-\frac{2i\pi}{T})s}}{p-\frac{2i\pi}{T}}I \\ \vdots \end{pmatrix}$

and

$$\hat{H}(p) := I_\infty - \sum_{j=1}^{N} e^{-p\tau_j}L_{D_j}\tilde{D}_{\tau_j}. \tag{4.34}$$

Replacing p by $p+\frac{2i\pi n}{T}$ for all $n \in \mathbb{Z}$ we can rewrite equation (5.25) as

$$\hat{H}(p)\hat{\mathbb{X}}(p,s) = \hat{e}(p,s). \tag{4.35}$$

Step 3: Thanks to Fourier's series and Laplace transform, we transform the periodic delay system in finite dimension (4.22) into the linear constant infinite dimensional system (4.35). In order to express $\mathbb{X}(p,s)$ -and in particular $X(t,s)$- we have to inverse the operator $\hat{H}(p)$. Let an

infinite matrix $A := (a_{i,j})_{i,j\in\mathbb{Z}}$ where $a_{i,j}$ is a $d \times d$ matrix with its coefficients in $\mathbb{C}$ for all i and j integer. We define $B(l^2_d(\mathbb{Z}), l^2_d(\mathbb{Z}))$ the space of the infinite matrix A which define a continuous operator from $l^2_d(\mathbb{Z})$ to $l^2_d(\mathbb{Z})$ with a finite Wiener's norm $|||\cdot|||_W$ where :

$$|||A|||_W := \sum_{k\in\mathbb{Z}} \sup_{|i-j|=k} |||a_{i,j}|||. \tag{4.36}$$

The Wiener's norm is the sum of the supremum of the diagonals. The space $B(l^2_d(\mathbb{Z}), l^2_d(\mathbb{Z}))$ is a Banach space. Since the periodic functions D_j are differentiable with a Hölder continuous derivative for $j = 1, \cdots, N$ we have the equation (4.29) and thus $\hat{H}(p)$ has a finite Wiener norm. Since an operator with an infinite Wiener norm define an operator from $l^2_d(\mathbb{Z})$ in $l^2_d(\mathbb{Z})$, we have that $\hat{H}(p) \in B(l^2_d(\mathbb{Z}), l^2_d(\mathbb{Z}))$. Moreover we have that $\hat{H} : \{z \in \mathbb{C} | \Re(z) \geq \alpha\} \longrightarrow B(l^2_d(\mathbb{Z}), l^2_d(\mathbb{Z}))$ is a Banach valued holomorphic function. By the assumption *i.*, we have $\hat{H}(p)$ is invertible on $l^2_d(\mathbb{Z})$ to $l^2_d(\mathbb{Z})$ for all fixed $p \in \{z \in \mathbb{C} | \Re(z) \geq \alpha\}$. By the assumption *ii*, we have that the inverse operator of $\hat{H}(p)$ is uniformly bounded in the complex plane $\{z \in \mathbb{C} | \Re(z) \geq \alpha\}$ with the norm $|||\cdot|||_2$. By the theorem 1.2 in [GK14], we deduce that the inverse $\hat{H}^{-1}(p)$ is uniformly bounded in the Wiener norm for all $p \in \{z \in \mathbb{C} | \Re(z) \geq \alpha\}$, thanks to the equation (4.29) :

$$\left|\left|\left| \left[I_\infty - \sum_{i=j}^{N} e^{-p\tau_j} L_{D_j} \tilde{D}_{\tau_j} \right]^{-1} \right|\right|\right|_W \text{ is uniformly bounded for all } p \in \{z \in \mathbb{C} | \Re(z) \geq \alpha\}.$$

Applying the theorem 2.6 in [Kri11] (or [BK10]), which is a generalization of the Wiener's lemma, we have that $\hat{H}^{-1}(p)$ admits a Fourier developpment on the vertical line $p = \alpha + i\mathbb{R}$ which is absolutaly summable and of the form :

$$\hat{H}^{-1}(\alpha + i\omega) = \sum_{k\in\mathbb{Z}} \tilde{H}^k e^{\beta_k \omega i}, \text{ for all } \omega \in \mathbb{R} , \tag{4.37}$$

where β_k real for $k \in \mathbb{Z}$, $\tilde{H}^k \in B(l^2_d(\mathbb{Z}), l^2_d(\mathbb{Z}))$, and :

$$\sum_{k\in\mathbb{Z}} \left|\left|\left| \tilde{H}^k \right|\right|\right|_W < +\infty. \tag{4.38}$$

Following the proof of the theorem in the paragraph 3 of Dirichlet's series p147 of [Bes54] which is allow because the Cauchy's theorem is true for holomorphic banach valued function, we deduce that $\tilde{H}^k$ is equal to $H^k e^{\alpha\beta_k}$ where $H^k \in B(l^2_d(\mathbb{Z}), l^2_d(\mathbb{Z}))$ for all $k \in \mathbb{Z}$ and we can rewrite the equation (4.37) and (4.38) as :

$$\hat{H}^{-1}(\alpha + i\omega) = \sum_{k\in\mathbb{Z}} H^k e^{\alpha\beta_k} e^{\beta_k \omega i}, \text{ for all } \omega \in \mathbb{R} , \tag{4.39}$$

and

$$\sum_{k\in\mathbb{Z}} \left|\left|\left| H^k \right|\right|\right|_W e^{\alpha\beta_k} < +\infty. \tag{4.40}$$

Using the fact that for p with real part enough large we have by the Von Neumann series :

$$\hat{H}^{-1}(p) = \sum_{n=1}^{+\infty} \left(\sum_{j=1}^{N} L_{D_j} \tilde{D}_{\tau_j} e^{-p\tau_j} \right)^n , \tag{4.41}$$

and always using the proof of the theorem in the paragraph 3 of Dirichlet's series p147 of [Bes54], we deduce :

$$\{\beta_k | k \in \mathbb{Z}\} = \{\sum_{j=1}^{N} n_j \tau_j |\ n_j \text{ non positive integer for } j = 1, \cdots, N\}. \tag{4.42}$$

[1]

Step 4: We search now to compute $\mathbb{X}(t,s)$. For that, we apply the Laplace inversion formula (see lemma 5.2 in chapter 1 of [HVL93] for example). We have :

$$\mathbb{X}(t,s) = \lim_{\omega \to +\infty} \frac{1}{2\pi i} \int_{c-i\omega}^{c+i\omega} \hat{H}^{-1}(p)\hat{e}(p,s)e^{pt}dp. \tag{4.43}$$

In the view to keep the proof straightforward and meaningful we cite the two following lemmas that we delay the proofs after this proof because it is technical with the using of several time of residue theorem and dominated convergence theorem :

Lemma 4.4. *Let $\omega_m := \frac{2\pi m}{T} + \frac{\pi}{T}$ for all positive integer m, we have :*

$$\mathbb{X}(t,s) = \lim_{m \to +\infty} \frac{1}{2\pi i} \int_{\alpha-i\omega_m}^{\alpha+i\omega_m} \hat{H}^{-1}(p)\hat{e}(p,s)e^{pt}dp + Q(t), \tag{4.44}$$

where :

$$Q(t) := \cdots + \hat{H}_{-1}^{-1}(\frac{-2i\pi}{T})e^{-\frac{2i\pi t}{T}} + \hat{H}_0^{-1}(0) + \hat{H}_1^{-1}(\frac{2i\pi}{T})e^{\frac{2i\pi t}{T}} + \cdots$$

with $\hat{H}_k^{-1}$ the k th column of $\hat{H}^{-1}$.

Since $X(t,s)$ is the 0th element of the vector column $\mathbb{X}(t,s)$ and if for each $k \in \mathbb{Z}$, we call $(H_{0,n}^k)_{n\in\mathbb{Z}}$ the first line of the the matrix H^k defined in the equation (4.39), we obtain from the equation (4.44):

$$X(t,s) = \frac{1}{2\pi i} \lim_{m \to +\infty} \int_{\alpha-i\omega_m}^{\alpha+i\omega_m} \sum_{k,n\in\mathbb{Z}} H_{0,n}^k \frac{e^{-(p-\frac{2i\pi n}{T})s}}{p-\frac{2i\pi n}{T}} e^{p(t+\beta_k)}dp + Q(t)_0, \tag{4.45}$$

with $Q(t)_0$ the 0ieme element of the column vector $Q(t)$.

Lemma 4.5. *We have for all t and s such that $t + \beta_k - s \neq 0$ for all $k \in \mathbb{Z}$:*

$$X(t,s) = \sum_{\alpha(t+\beta_k-s)>0, n\in\mathbb{Z}} H_{0,n}^k e^{\frac{2i\pi n}{T}(t+\beta_k)} + Q(t)_0. \tag{4.46}$$

We want to evaluate the variation of X in its second argument when it is in $[s, s+\tau_N]$. We deduce from the equation (4.46) :

$$\underset{s\le\tau\le s+\tau_N}{Var_\tau} X(t,\tau) \le \sum_{0\le t+\beta_k-s\le\tau_N, n\in\mathbb{Z}} \left|\left|\left| H_{0,n}^k \right|\right|\right| \tag{4.47}$$

$$\le \left(\sum_{k,n\in\mathbb{Z}} \left|\left|\left| H_{0,n}^k \right|\right|\right| e^{\alpha\beta_k} e^{|\alpha|\tau_N} \right) e^{\alpha(t-s)}. \tag{4.48}$$

[1] The equations (4.39), (4.40) and (4.42) are a reminiscence of the equations (12.15.12) and (12.15.13) p 429 of [BC59] in the case of complex functions.

From the equation (4.24) and (4.47), we deduce :

$$||y(t)|| \leq N\left(\underset{s\leq\tau\leq s+\tau_N}{Var_\tau} X(t,\tau)\right)||\phi||_\infty \tag{4.49}$$

$$\leq N\left(\sum_{k,n\in\mathbb{Z}}\left|\left|\left|H_{0,n}^k\right|\right|\right|e^{\alpha\beta_k}e^{|\alpha|\tau_N}\right)e^{\alpha(t-s)}||\phi||_\infty, \tag{4.50}$$

which is the result of the sufficiency of the theorem 4.3.

□

Proof of Lemma 6.4. Considering the subsequence $\omega_m = \frac{2\pi m}{T} + \frac{\pi}{T}$, we deduce from the equation (4.43) that :

$$\mathbb{X}(t,s) = \lim_{m\to+\infty}\frac{1}{2\pi i}\int_{c-i\omega_m}^{c+i\omega_m}\hat{H}^{-1}(p)\hat{e}(p,s)e^{pt}dp. \tag{4.51}$$

We can remark that each component of the vector $\hat{H}^{-1}(\cdot)\hat{e}(\cdot,s)$ is a meromorphic function in the complex plane $\{p\in\mathbb{C}|\Re(p)\geq\alpha\}$ and has poles in the set $\{p\in\mathbb{C}|p=\frac{ik\pi}{T},k\in\mathbb{Z}\}$. Then, from the equation (4.51), we have by the analytic continuation and the residue theorem :

$$\mathbb{X}(t,s) = \lim_{m\to+\infty}\frac{1}{2\pi i}\left(\int_{\alpha-i\omega_m}^{\alpha+i\omega_m}+\int_{\alpha+i\omega_m}^{c+i\omega_m}+\int_{\alpha-i\omega_m}^{c-i\omega_m}\right)\hat{H}^{-1}(p)\hat{e}(p,s)e^{pt}dp + Q(t) \tag{4.52}$$

where we recall that :

$$Q(t) := \cdots + \hat{H}_{-1}^{-1}(\frac{-2i\pi}{T})e^{-\frac{2i\pi t}{T}} + \hat{H}_0^{-1}(0) + \hat{H}_1^{-1}(\frac{2i\pi}{T})e^{\frac{2i\pi t}{T}} + \cdots$$

with $\hat{H}_k^{-1}$ the k th column of $\hat{H}^{-1}$. We have that :

$$||Q(t)||_2 \leq \left|\left|\left|\hat{H}^{-1}(0)\right|\right|\right|_W < +\infty, \tag{4.53}$$

and then the equality (4.52) has a sense. We want to prove that :

$$\lim_{m\to+\infty}\int_{\alpha+i\omega_m}^{c+i\omega_m}\hat{H}^{-1}(p)\hat{e}(p,s)e^{pt}dp = \lim_{m\to+\infty}\int_{\alpha-i\omega_m}^{c-i\omega_m}\hat{H}^{-1}(p)\hat{e}(p,s)e^{pt}dp = 0. \tag{4.54}$$

We have that the $j-th$ line of the vector $\lim_{m\to+\infty}\int_{\alpha+i\omega_m}^{c+i\omega_m}\hat{H}^{-1}(p)\hat{e}(p,s)e^{pt}dp$ for $j\in\mathbb{Z}$ is equal to :

$$\lim_{m\to+\infty}\int_{\alpha+i\omega_m}^{c+i\omega_m}\sum_{n\in\mathbb{Z}}H_{j,n}(p)\frac{e^{-(p-2in\pi/T)s}}{p-2in\pi/T}e^{pt}dp. \tag{4.55}$$

We have :

$$\left|\left|\left|\int_{\alpha+i\omega_m}^{c+i\omega_m}\sum_{n\in\mathbb{Z}}H_{j,n}(p)\frac{e^{-(p-2in\pi/T)s}}{p-2in\pi/T}e^{pt}dp\right|\right|\right|$$

$$\leq \sum_{n\in\mathbb{Z}}\sup_{\alpha\leq p_1\leq c}|||H_{j,n}(p_1+i\omega_m)|||\sup\left(e^{-\alpha(t-s)},e^{c(t-s)}\right)\int_{\alpha+i\omega_m}^{c+i\omega_m}\frac{1}{|p-2in\pi/T|}dp. \tag{4.56}$$

Since we have :

$$|\omega_m - \tfrac{2\pi n}{T}| \geq \tfrac{\pi}{T} \text{ for all } n \text{ and } m, \tag{4.57}$$

the right member of the equation (4.56) tends uniformly in j to zero by the dominated convergence theorem and the uniform bound for the Wiener's norm when m tends to $+\infty$. Then we proved that :

$$\lim_{m\to+\infty} \int_{\alpha+i\omega_m}^{c+i\omega_m} \hat{H}^{-1}(p)\hat{e}(p,s)e^{pt}dp = 0. \tag{4.58}$$

In the same way we would prove too :

$$\lim_{m\to+\infty} \int_{\alpha-i\omega_m}^{c-i\omega_m} \hat{H}^{-1}(p)\hat{e}(p,s)e^{pt}dp = 0. \tag{4.59}$$

So we deduce from the equations (4.52) and (4.54) the result of the Lemma 6.4.

□

□

Proof of Lemma 6.5. Let t and s such that $t+\beta_k - s \neq 0$. We have by a substitution ($p \to p - \frac{2i\pi n}{T}$, for all $n \in \mathbb{Z}$) :

$$\begin{aligned} &\lim_{m\to+\infty} \int_{\alpha-i\omega_m}^{\alpha+i\omega_m} \sum_{k,n\in\mathbb{Z}} H_{0,n}^k \frac{e^{-(p-\frac{2i\pi n}{T})s}}{p-\frac{2i\pi n}{T}} e^{p(t+\beta_k)}dp \\ &= \lim_{m\to+\infty} \sum_{n\in\mathbb{Z}} \int_{\alpha-\frac{2i\pi n}{T}-i\omega_m}^{\alpha-\frac{2i\pi n}{T}+i\omega_m} \sum_{k\in\mathbb{Z}} H_{0,n}^k e^{\frac{2i\pi n}{T}(t+\beta_k)} \frac{e^{p(t+\beta_k-s)}}{p}dp. \end{aligned} \tag{4.60}$$

Since we have :

$$\left|\!\left|\!\left| \sum_{k,n\in\mathbb{Z}} H_{0,n}^k e^{\frac{2i\pi n}{T}(t+\beta_k)} \int_{\alpha-\frac{2i\pi n}{T}-i\omega_m}^{\alpha-\frac{2i\pi n}{T}+i\omega_m} \frac{e^{p(t+\beta_k-s)}}{p}dp \right|\!\right|\!\right| \leq \frac{2\omega_m}{|\alpha|} \sum_{k,n\in\mathbb{Z}} \left|\!\left|\!\left| H_{0,n}^k \right|\!\right|\!\right| e^{\alpha(t+\beta_k-s)}, \tag{4.61}$$

and the right hand side of the equation (4.61) is finite because the Wiener's norm is finite, we can swap series and integral in the equation (4.60) and we obtain :

$$\begin{aligned} &\lim_{m\to+\infty} \int_{\alpha-i\omega_m}^{\alpha+i\omega_m} \sum_{k,n\in\mathbb{Z}} H_{0,n}^k \frac{e^{-(p-\frac{2i\pi n}{T})s}}{p-\frac{2i\pi n}{T}} e^{p(t+\beta_k)}dp \\ &= \lim_{m\to+\infty} \sum_{k,n\in\mathbb{Z}} H_{0,n}^k e^{\frac{2i\pi n}{T}(t+\beta_k)} \int_{\alpha-\frac{2i\pi n}{T}-i\omega_m}^{\alpha-\frac{2i\pi n}{T}+i\omega_m} \frac{e^{p(t+\beta_k-s)}}{p}dp. \end{aligned} \tag{4.62}$$

We have by an integration by parts that :

$$\int_{\alpha-\frac{2i\pi n}{T}-i\omega_m}^{\alpha-\frac{2i\pi n}{T}+i\omega_m} \frac{e^{p(t+\beta_k-s)}}{p}dp = \left[\frac{e^{p(t+\beta_k-s)}}{p(t+\beta_k-s)}\right]_{\alpha-\frac{2i\pi n}{T}-i\omega_m}^{\alpha-\frac{2i\pi n}{T}+i\omega_m} + \int_{\alpha-\frac{2i\pi n}{T}-i\omega_m}^{\alpha-\frac{2i\pi n}{T}+i\omega_m} \frac{e^{p(t+\beta_k-s)}}{(t+\beta_k-s)p^2}dp, \tag{4.63}$$

We have from the equation (4.42) that there exists a $\delta > 0$ such that $|t+\beta_k - s| \geq \delta$ for all $k \in \mathbb{Z}$. Using the inequalities (4.57), we deduce from the equation (4.63) :

$$\left|\left[\frac{e^{p(t+\beta_k-s)}}{p(t+\beta_k-s)}\right]_{\alpha-\frac{2i\pi n}{T}-i\omega_m}^{\alpha-\frac{2i\pi n}{T}+i\omega_m}\right| \leq \frac{1}{|t+\beta_k-s|}\left(\frac{e^{\alpha(t+\beta_k-s)}}{\sqrt{\alpha^2+\left(\omega_m-\frac{2\pi n}{T}\right)^2}}+\frac{e^{\alpha(t+\beta_k-s)}}{\sqrt{\alpha^2+\left(\omega_m+\frac{2\pi n}{T}\right)^2}}\right)$$
$$\leq \frac{2e^{\alpha(t+\beta_k-s)}}{\delta\sqrt{\alpha^2+\left(\frac{\pi}{T}\right)^2}} \tag{4.64}$$

and

$$\left|\int_{\alpha-\frac{2i\pi n}{T}-i\omega_m}^{\alpha-\frac{2i\pi n}{T}+i\omega_m}\frac{e^{p(t+\beta_k-s)}}{(t+\beta_k-s)p^2}dp\right| \leq \frac{e^{\alpha(t+\beta_k-s)}}{|t+\beta_k-s|}\int_{-\omega_m-\frac{2\pi n}{T}}^{\omega_m-\frac{2\pi n}{T}}\frac{1}{\alpha^2+p_2^2}dp_2$$
$$\leq \frac{e^{\alpha(t+\beta_k-s)}}{|t+\beta_k-s|}\left[\frac{1}{\alpha}arctan\left(\frac{p_2}{\alpha}\right)\right]_{-\omega_m-\frac{2i\pi n}{T}}^{\omega_m-\frac{2i\pi n}{T}}$$
$$\leq \frac{\pi}{|\alpha|\delta}e^{\alpha(t+\beta_k-s)}. \tag{4.65}$$

We deduce from the equations (4.64) and (4.65) that there exists a constant $K>0$ independent of k, n and ω_m such that :

$$\left|\int_{\alpha-\frac{2i\pi n}{T}-i\omega_m}^{\alpha-\frac{2i\pi n}{T}+i\omega_m}\frac{e^{p(t+\beta_k-s)}}{p}dp\right| \leq Ke^{\alpha(t+\beta_k-s)}. \tag{4.66}$$

And we have from the equation (4.66):

$$\left\|\sum_{k,n\in\mathbb{Z}}H_{0,n}^k e^{\frac{2i\pi n}{T}(t+\beta_k)}\int_{\alpha-\frac{2i\pi n}{T}-i\omega_m}^{\alpha-\frac{2i\pi n}{T}+i\omega_m}\frac{e^{p(t+\beta_k-s)}}{p}dp\right\| \leq Ke^{\alpha(t-s)}\sum_{k,n\in\mathbb{Z}}\left\|H_{0,n}^k\right\|e^{\alpha\beta_k}dp. \tag{4.67}$$

Since the right hand side of the equation (4.67) exists because the Wiener's norm is finite (equation(4.40)), the dominated convergence theorem permits to swap limit and series in the right hand side of the equation (4.62) :

$$\lim_{m\to+\infty}\frac{1}{2i\pi}\int_{\alpha-i\omega_m}^{\alpha+i\omega_m}\sum_{k,n\in\mathbb{Z}}H_{0,n}^k\frac{e^{-(p-\frac{2i\pi n}{T})s}}{p-\frac{2i\pi n}{T}}e^{p(t+\beta_k)}dp$$
$$=\sum_{k,n\in\mathbb{Z}}\frac{1}{2i\pi}H_{0,n}^k e^{\frac{2i\pi n}{T}(t+\beta_k)}\lim_{m\to+\infty}\int_{\alpha-\frac{2i\pi n}{T}-i\omega_m}^{\alpha-\frac{2i\pi n}{T}-i\omega_m}\frac{e^{p(t+\beta_k-s)}}{p}dp. \tag{4.68}$$

A classical computation give :

$$\sum_{k,n\in\mathbb{Z}}\frac{1}{2i\pi}H_{0,n}^k e^{\frac{2i\pi n}{T}(t+\beta_k)}\lim_{m\to+\infty}\int_{\alpha-i\omega_m}^{\alpha+i\omega_m}\frac{e^{p(t+\beta_k-s)}}{p}dp=\sum_{\alpha(t+\beta_k-s)>0,n\in\mathbb{Z}}H_{0,n}^k e^{\frac{2i\pi n}{T}(t+\beta_k)}. \tag{4.69}$$

Putting together the equations (4.45), (4.62) and (4.69), we obtain the result of the Lemma 6.5. □

□

4.5 Necessity

The Henry-Hale Theorem 5.17 deals with a necessary and sufficient condition to ensure the stability of the linear delay system (4.4). The sufficiency is proved in the same manner as proceded in the section 4.4. The necessity is proved through the spectral theory of semigroup. We cannot follow this path to show the necessity when the system becomes periodic because we have not a semigroup. The goal of this section is to prove the necessity of the assumptions of the theorem 4.3. To tackle the problem, we will consider an input-output system. More precisely, we want to prove under the C^0 exponential stability of the System (6.6) that the assumptions *i* and *ii* are verified. Before the proof of this fact in the section 4.5.2, we start by the statement of some stability results for the periodic system (6.6) and we establish a variation of constant formula for the system (6.6) which is disturbed by a square integrable function on each compact in the section 4.5.1.

4.5.1 Stability, monodromy operator and variation of constant formula

Until now, we dealt we the C^0 stability. Here we need square integrable initial data on the interval $[s-\tau_N, s]$ to have a square integrable solution to (6.6). We define the space :

$$L^2 := \{\phi \in L^2([-\tau_N, 0], \mathbb{C}^d)\}. \tag{4.70}$$

We endow L^2 with the following norm :

$$||\phi||_2 := \left(\int_{-\tau_N}^{0} ||\phi(\theta)||^2\right)^{1/2}. \tag{4.71}$$

For $\phi \in L^2$, a recursive argument shows that the system (6.6) admits a unique square integrable solution on each compact y with $y(s+\theta) = \phi(\theta)$ for almost all $\theta \in [-\tau_N, 0]$. Thus, we may define the solution operator:

$$U_2(t,s) : L^2 \rightarrow L^2 \tag{4.72}$$

$$\phi \mapsto U_2(t,s)\phi, \tag{4.73}$$

where $(U_2(t,s)\phi)(\theta) = \{y(t+\theta), \text{ for almost all } \theta \in [-\tau_N, 0]\}$.

Definition 4.6. The zero solution of the system (6.6) is called L^2 asymptotically exponentially stable if there exist $\gamma > 0$ and $K > 0$ such that :

$$||U(t,s)\phi||_2 \leq Ke^{-\gamma(t-s)}||\phi||_2, \text{ for all } s \in \mathbb{R}, \text{ all } t \geq s \text{ and all } \phi \in L^2. \tag{4.74}$$

It is a classical fact that the spectrum of the monodromy operators give the exponential stability of the system (see for example [BP01, lem. 4.2]) :

Proposition 4.7. *The system (6.6) is L^2 (resp. C^0) exponentially stable if and only if the spectral radius of the monodromy operator $U_2(T,0)$ (resp. $U(T,0)$) is strictly less than one.*

A nice result that we can find in [CMS16, Corollary 3.29] or [BFLP20, Theorem 3.4] is that the L^2 stability is equivalent to the C^0 stability :

Proposition 4.8. *The system (6.6) is L^2 exponentially stable if and only it is C^0 exponentially stable.*

We consider the system (6.6) disturbed by a function u :

$$y(t) = \sum_{j=1}^{N} D_j(t)y(t-\tau_j) + u(t), \qquad t \geq s. \tag{4.75}$$

We note for $t \geq s$, $y_t(\theta) = y(t+\theta)$ for $\theta \in [-\tau_N, 0]$.

Proposition 4.9. *For all $s \in \mathbb{R}$, if $y(\cdot)$ and $u(\cdot)$ belong to $L^2_{loc}([s-\tau_N, +\infty[, \mathbb{C}^d)$ and $L^2_{loc}([s, +\infty[, \mathbb{C}^d)$ respectively and which verify the equation (6.19) for almost all $t \geq s$, we have for all $t \geq s$:*

$$y_t = U_2(t,s)y_s + \int_{s^-}^{t^+} d_\alpha K(t,\alpha)u(\alpha), \tag{4.76}$$

where the previous integral is understood in Bochner sense and

$$K(t,\alpha)(\theta) = \int_s^{\alpha^+} d_v X(t+\theta, v), \tag{4.77}$$

with $X(\cdot,\cdot)$ the fundamental solution defined in (4.22).

Proof. With no loss of generality, we can assume $s = 0$. For $t = 0$, we have by the definition of the fundamental solution that :

$$U_2(0,0)y_0 + \int_{0^-}^{0^+} d_\alpha K(t,\alpha)u(\alpha) = y_0. \tag{4.78}$$

Let $\theta \in [-\tau_N, 0]$ and $t > 0$ such that $t+\theta > 0$, We prove that the formula (6.25) verifies the equation (6.19) in $t+\theta$ for almost θ. We note $\tilde{y}_t = U_2(t,s)y_s$. We have :

$$\begin{aligned} y_t(\theta) = y(t+\theta) &= \tilde{y}(t+\theta) + \int_{0^-}^{t^+} d_\alpha X(t+\theta,\alpha)u(\alpha) \\ &= \sum_{i=1}^{N} D_i(t+\theta)y(t+\theta-\tau_i) + u(t+\theta). \end{aligned}$$

□

□

Proposition 4.10. *Assume that the $D_j : \mathbb{R} \to \mathbb{C}^{d\times d}$ are periodic and differentiable with Hölder continuous derivative for $1 \leq j \leq N$, .i.e $D_j(\cdot) \in C^{1,\delta}$ with $\delta \in]0,1[$. For all s real, the distribution $t \longmapsto d_s X(t, t-s)$ is T-periodic and belongs to the space $C^{1,\delta}$. Moreover for all t, τ real there exists $K, \gamma > 0$ such that :*

$$\int_0^\tau \left\| \frac{\partial^j}{\partial t^j} d_s X(t, t-s) \right\| \leq K e^{\gamma\tau}, \text{ for } j = \{0, 1, 1+\delta\}. \tag{4.79}$$

Proof. With the definition of the fundamental solution $X(\cdot,\cdot)$ in the equation (4.22), we can see that the distribution $t \longmapsto d_s X(t, t-\tau)$ is composed of a sum of Dirac on the set $\mathcal{F}$ defined in the equation (4.23), more precisely we have :

$$d_s X(t, t-s) = \sum_{n\in\mathcal{F}} E_n(t)\delta_n(s), \tag{4.80}$$

where $E_n(t)$ represents a finite product of some elements (possibly delayed) of the family $(D_j(\cdot))_{1\leq j\leq N}$ and δ_n denotes the Dirac in $n \in \mathcal{F}$. By the assumption we deduce that the distribution $t \longmapsto d_s X(t, t-s)$ is T-periodic and belongs to the space $C^{1,\delta}$. Let τ, t real and :

$$Y(t,\tau) := \int_0^\tau d_s X(t, t-s), \tag{4.81}$$

$$Z(t,\tau) := \int_0^\tau \|d_s X(t, t-s)\|. \tag{4.82}$$

We have by the formula (4.22) that :

$$Y(t,\tau) = \sum_{j=1}^{N} D_j(t) Y(t-\tau_j, \tau-\tau_j). \tag{4.83}$$

Since $Y(\cdot,\cdot)$ is periodic in its first variable we deduce from the equation (4.83) that there exists a $K > 0$ such that for all real t, τ :

$$Z(t,\tau) \leq K \left[\sum_{j=1}^{N} Z(t, \tau-\tau_j) + 1 \right]. \tag{4.84}$$

Performe an induction like we did to prove the exponential growth of the fundamental solution $X(\cdot,\cdot)$ (see equation (4.25)), we conclude that

$$Z(t,\tau) \leq K' e^{\gamma\tau}, \tag{4.85}$$

for some γ and K' positive real, and we proved the equation (4.79) for $j = 0$. The use of the same reasonning would prove the majoration (4.79) for $j = 1$ or $1 + \delta$ as well. □

□

4.5.2 Proof necessity theorem 4.3

From now, we assume that the hypothesis of the theorem 4.3, ie the system (6.6) is C^0 exponentially stable. We give in this subsection the link between the spectrum of the monodromy operator associated to the periodic linear difference delay system (6.6) and the operator $\hat{H}^{-1}$. The ideas follow the ideas in [Möl00]. We consider the input-output system (6.19) with initial data 0 and at the time 0, ie let $u \in L^2_{loc}([0,+\infty), \mathbb{C}^d)$ the input and let the $y(t) \in \mathbb{C}^d$ the output at the time $t \geq 0$, more precisely :

$$y(t) = \sum_{j=1}^{N} D_j(t) y(t-\tau_j) + u(t), \qquad \text{for allmost all } t \geq 0, \tag{4.86}$$

with $y(t) = 0$ for $t \leq 0$. We note the Laplace transform of $y(t)$ and $u(t)$:

$$\hat{Y}(p) := \int_0^{+\infty} e^{-pt} y(t) dt, \tag{4.87}$$

$$\hat{U}(p) := \int_0^{+\infty} e^{-pt} u(t) dt. \tag{4.88}$$

It has to be noted that developping the system (6.43) in Fourier series, taking the Laplace transform and swapping the series and the Laplace transform like we did for the fundamental solution $X(\cdot,\cdot)$ in the proof of the theorem 4.3, we obtain for p complex with a real part enough large :

$$\hat{\mathbb{Y}}(p) = \hat{H}^{-1}(p) \hat{U}(p), \tag{4.89}$$

$$\hat{\mathbb{Y}}(p) := \begin{pmatrix} \vdots \\ \hat{Y}(p + \frac{2i\pi}{T}) \\ \hat{Y}(p) \\ \hat{Y}(p - \frac{2i\pi}{T}) \\ \vdots \end{pmatrix}, \hat{U}(p) := \begin{pmatrix} \vdots \\ \hat{U}(p + \frac{2i\pi}{T}) \\ \hat{U}(p) \\ \hat{U}(p - \frac{2i\pi}{T}) \\ \vdots \end{pmatrix} \text{ and } \hat{H}(p) := I_\infty - \sum_{j=1}^{N} e^{-p\tau_j} L_{D_j} \tilde{D}_{\tau_j}.$$

For the electronic engineers the operator $\hat{H}^{-1}(\cdot)$ is called the harmonic transfer function and it links the input of the system with the output [Wer90, Lou14]. Put the variation constant formula (6.25) in the equation (6.43) leads to :

$$y(t) = \int_0^{t^+} d_\alpha X(t,\alpha)u(\alpha),\ t \geq 0, \tag{4.90}$$

where $X(t,\alpha)$ is the fundamental solution defined in the equation (4.22). We define the instantaneous transfer function :

$$G(s,t) = \int_0^{+\infty} d_\tau X(t,t-\tau)e^{-s\tau}, \tag{4.91}$$

for $s \in \mathbb{C}$. We can deduce from the proposition 6.13 that the function $t \mapsto G(t,s)$ is $C^{1+\delta}$ and T periodic. We developp in Fourier series the instantaneous transfer function :

$$G(s,t) = \sum_{n\in\mathbb{Z}} G_n(s)e^{i\omega_0 nt}, \tag{4.92}$$

with $\omega_0 := 2\pi/T$. Where there exists $K(\Re(s))$ real positive which depend only of the real part enough large of s complex such that ([Zyg02, Ch. 2, thm. 4.7]) :

$$|G_n(s)| \leq \frac{K(\Re(s))}{1+|n|^{1+\delta}}. \tag{4.93}$$

We deduce from the equations (4.90), (4.92) and (6.22) :

$$Y(s+in\omega_0) = \sum_m G_{n-m}(s+\frac{2i\pi m}{T})U(s+im\omega_0). \tag{4.94}$$

Combining the equations (4.89) and (4.94), we see that the $\hat{H}^{-1}_{m,n}(p) = G_{n-m}(p+\frac{2i\pi m}{T})$. In the view to bring out the monodromy operator and the link with the operator $\hat{H}^{-1}(p)$, we discretise the continuous dynamical system in a discrete dynamical one. For that we consider the input and the output on interval of length T. Put :

$$\begin{array}{rcl} \widetilde{u}_k & := & u(kT+t) \text{ for } t \in [0,T] \text{ and } k \in \mathbb{N}, \\ \widetilde{y}_k & := & y(kT+t) \text{ for } t \in [0,T] \text{ and } k \in \mathbb{N}, \\ \widetilde{z}_{kT} & := & y(kT+\theta) \text{ for } \theta \in [-\tau_N,0] \text{ and } k \in \mathbb{N}, \end{array}$$

Thus we have the following discrete system :

Theorem 4.11. *We have for all integer positive k :*

$$\left\{ \begin{array}{rl} \tilde{z}_{(k+1)T} & = \widetilde{A}\tilde{z}_{kT} + \widetilde{B}\widetilde{u}_k \\ \widetilde{y}_k & = \widetilde{C}\tilde{z}_{kT} + \widetilde{D}\widetilde{u}_k \end{array} \right. \tag{4.95}$$

where :

- $\begin{array}{rcl} \widetilde{A} : L^2([-\tau_N,0],\mathbb{C}^d) & \longrightarrow & L^2([-\tau_N,0],\mathbb{C}^d) \\ v & \longmapsto & U_2(T,0)v. \end{array}$

- $\begin{aligned} \widetilde{B} : L^2([0,T],\mathbb{R}^d) &\longrightarrow L^2([-\tau_N,0],\mathbb{C}^d) \\ v &\longmapsto \int_0^{T+\cdot} d[K(T,\alpha)]v(\alpha), \end{aligned}$
- $\begin{aligned} \widetilde{C} : L^2([-\tau_N,0],\mathbb{C}^d) &\longrightarrow L^2([0,T],\mathbb{C}^d) \\ v &\longmapsto U_2(\cdot,0)v(0) \end{aligned}$
- $\begin{aligned} \widetilde{D} : L^2([0,T],\mathbb{C}^d) &\longrightarrow L^2([0,T],\mathbb{C}^d) \\ v &\longmapsto \int_0^{\cdot} d_\alpha X(\cdot,\alpha)v(\alpha), \end{aligned}$
- $\tilde{z}_0 = 0$

Proof. Using the variation constant formula (6.25) in the system (6.43), we obtain for $t \in [0,T]$:

$$\begin{cases} \tilde{z}_{(k+1)T} = U_2((k+1)T,kT)\tilde{z}_{kT} + \int_{kT}^{(k+1)T} d_\alpha K((k+1)T,\alpha)u(\alpha), \\ y(t+kT) = U_2(t,0)\tilde{z}_{kT}(0) + \int_0^{t+} d_\alpha X(t,\alpha)u_k(\alpha). \end{cases}$$

By the periodicity, we have the result. □

□

For a sequence of elements $a := (a_n)_{n\in\mathbb{N}}$ we define the p-transform as:

$$\mathcal{L}\{a\}(p) := \sum_{n\in\mathbb{N}} a_n p^{-n}, \tag{4.96}$$

when for $p \in \mathbb{C}$ this series exists. Take the p-transform with a great modulus in the equations (6.53), we have :

$$\mathcal{L}\{\overline{y}\}(p) = [\widetilde{C}(pId - \widetilde{A})^{-1}\widetilde{B} + \widetilde{D}]\mathcal{L}\{\overline{u}\}(p), \tag{4.97}$$

where $\overline{y} := (y_n)_{n\in\mathbb{N}}$ and $\overline{u} := (u_n)_{n\in\mathbb{N}}$. On in the other side, we can compute the impulsional response of the discrete system and we have :

$$\widetilde{y}_n(t) = \sum_{k=0}^{n} H_{[k]}\widetilde{u}_{n-k}(t), \tag{4.98}$$

where :

$$H_{[k]}v(t) = \int_0^T d_\tau X(kT+t,\tau)v(\tau) \tag{4.99}$$

is the discrete impulsional response of the discretised dynamical system. Take the p-transform in the equation (6.56) for p with great modulus (which is licit because of proposition 6.13), we have:

$$\mathcal{L}\{\overline{y}\}(p) = \mathcal{L}\{\overline{H}\}(p)\mathcal{L}\{\overline{u}\}(p), \tag{4.100}$$

where $\overline{H} := (H_{[n]})_{n\in\mathbb{N}}$. Since we assumed that the system (6.6) is C^0 exponentially stable, we have through the propositions 6.3 and 4.8 that $\tilde{A} = U_2(T,0)$ has its spectrum strictly inside the unit disk. We note a, a positive real strictly less than one, the spectral radius of the operator $U_2(T,0)$. We introduce the set $P_a := \{z \in \mathbb{C} || z| > a\}$ and for p complex with a real part enough large, we deduce from the equations (6.55) and (6.58) that :

$$\mathcal{L}\{\overline{H}\}(p) = \widetilde{C}(pId - \widetilde{A})^{-1}\widetilde{B} + \widetilde{D}, \tag{4.101}$$

and by analytic continuation, the equation (6.59) holds on P_a.

Lemma 4.12. *For $t \in [0,T]$, we have :*

$$G(s,t) = e^{-st}\left[\mathcal{L}\{\overline{H}\}(e^{sT})e^{s\cdot}\right](t) \tag{4.102}$$

Proof. We have, by the definition of the p-transform and the equation (6.57), for $v \in L^2([0,T],\mathbb{R})$ and $t \in [0,T]$ that:

$$\begin{aligned}\mathcal{L}\{\overline{H}\}(p)v(t) &= \sum_{k=0}^{+\infty} H_{[k]}v(t)p^{-k} &(4.103)\\ &= \sum_{k=-\infty}^{+\infty} p^{-k}\int_0^T d_\tau X(kT+t,\tau)v(\tau). &(4.104)\end{aligned}$$

and on the other side the equation (6.48) where we performed an integration by substition ($t \to t-\tau$) and using the periodicity of the system leads to :

$$\begin{aligned}G(s,t) &= \int_{-\infty}^{t} d_\tau X(t,\tau)e^{s(\tau-t)} &(4.105)\\ &= \int_{-\infty}^{+\infty} d_\tau X(t,\tau)e^{s(\tau-t)} &(4.106)\\ &= \sum_{k=-\infty}^{+\infty}\int_0^T d_\tau X(t,\tau-kT)e^{s(\tau-t-kT)} &(4.107)\\ &= \sum_{k=-\infty}^{+\infty} e^{-s(t+kT)}\int_0^T d_\tau X(t+kT,\tau)e^{s\tau} &(4.108)\end{aligned}$$

and thus we have the result with $p = e^{sT}$. □

□

Fix $s \in \mathbb{C}$ with real part enough large. Let

$$\Lambda(s) : L^2([0,T],\mathbb{C}^d) \to L^2([0,T],\mathbb{C}^d), \tag{4.109}$$

where $[\Lambda(s)\phi](t) = e^{-st}[\mathcal{L}\{\overline{H}\}(e^{sT})e^{s\cdot}\phi(\cdot)](t)$ for all $\phi \in L^2([0,T],\mathbb{C}^d)$. Because the equation (6.22) justifies the swapping between the series, we deduce from the lemma 6.25 and the equation (4.92) that the Fourier development of $\Lambda(s)\phi(t)$ is :

$$\Lambda(s)\phi(t) = \sum_{n\in\mathbb{Z}}\left(\sum_{k\in\mathbb{Z}} G_{n-k}(s+2i\pi k/T)a_k\right)e^{2i\pi n/T}, \tag{4.110}$$

with $(a_k)_{k\in\mathbb{Z}}$ the Fourier coefficient of ϕ. We have that the operator $[pId - U_2(T,0)]^{-1}$ is uniformly bounded for $p \in P_a$, then we deduce that there exists a constant $C > 0$ independ of $s \in \{z \in \mathbb{C} | \Re(z) > \frac{ln(a)}{T}\}$ such that :

$$||\Lambda(s)\phi||_2 \leq C||\phi||_2 \tag{4.111}$$

By Parseval's identity, we deduce from the equations (4.110) and (4.111) that there exists $C' > 0$ such that for all $s \in \{z \in \mathbb{C} | \Re(z) > \frac{ln(a)}{T}\}$:

$$\left|\left|\left|\hat{H}^{-1}(s)\right|\right|\right|_2 \leq C',$$

which conclude the proof of necessity of the theorem 4.3.

□

4.6 Discussion, Corollaries and Stability of 1-D Hyperbolic Systems with linear periodic boundaries

In the Henry-Hale Theorem 5.17 for the constant difference delay system the assumption (b) consists to invert the matrix operator $I - \sum_{j=1}^{N} D_j\, e^{-p\tau_j}$ for all complex p with real part greater than a some strictly negative α. Thanks to the determinant and the theory of complex analytic almost periodic function (see [Bes54]), we have that the assumption (b) implies :

$$\left\| \left[I - \sum_{j=1}^{N} D_j\, e^{-p\tau_j} \right]^{-1} \right\| \text{ is uniformly bounded in euclidean norm for all } p \in \{z \in \mathbb{C} | \Re(z) \geq \alpha\}. \quad (4.112)$$

In our Theorem 4.3 for the periodic delay system, the condition (b) is replaced by the inversion of the infinite matrix operator $I_\infty - \sum_{j=1}^{N} e^{-p\tau_j} L_{D_j} \tilde{D}_{\tau_j}$ on the bigger space $l^2_d(\mathbb{Z})$. More precisily the assumption (b) become the assumption *i* in the periodic case. We previously saw that in the constant case the condition (b) implies the condition (4.112). In the case of periodic difference delay system the question of knowing if the condition *i.* imply the condition *ii.* is still open. The difficulties are that there exixts few results about inverse of analytic almost periodic Banach valued function probably because the Rouché theorem fails to apply for general operator. However if we assume that the delays are commensurable, ie for each $i \in \{1, \cdots, N\}$ and $j \in \{1, \cdots, N\}$ there exists p and q two integers such that $\tau_i/\tau_j = p/q$, thus the operator $I_\infty - \sum_{i=1}^{N} e^{-p\tau_i} L_{D_i} \tilde{D}_i$ is periodic in the variable p and we can deduce that the assumption *i* implies the assumption *ii* which leads to the following corollary :

Corollary 4.13. *If the delays are commensurable then the Theorem 4.3 is true without the condition ii.*

Remark 4.14. *We have assumed that the periodic matrices $D_i(\cdot)$ were in the space of the differentiable operator with Hölder derivative. Actually, the Theorem 4.3 would be true if we omit to impose regularity assumption on D_i and we replace the assumption ii by the following which is more difficult to check :*

$$\left\| \left[I_\infty - \sum_{i=j}^{N} e^{-p\tau_j} L_{D_j} \tilde{D}_j \right]^{-1} \right\|_W \text{ is uniformly bounded for all } p \text{ in } \{z \in \mathbb{C} | \Re(z) \geq \alpha\}. \quad (4.113)$$

Hitherto we give a condition stability for the periodic delay system (6.6) in terms of the Laurent matrices associated to the fourier development of $D_j(\cdot)$ for $j = 1, \cdots, N$. In some special cases, we can deduce a stability condition directly from the $D_j(\cdot)$ for $j = 1, \cdots, N$. It is the subject of the following corollary :

Corollary 4.15. *If the column of the periodic delay system (6.6) are disjoint, ie if for some $k, n \in \mathbb{Z}$ we have $(D_i)_{k,n} \neq 0$ thus for all $j \neq i$ and for all $\tilde{k} \in \mathbb{Z}$ we have $(D_j)_{\tilde{k},n} = 0$. If there exists an invertible diagonal matrix D such that*

$$\left\| D \sum_{i=1}^{N} D_i(t) D^{-1} \right\| < 1, \quad (4.114)$$

for all t real, then the system is exponentially asymptotically stable.

Proof. If we note L_D the Laurent matrix of D, we have, because of the column are disjoint and by properties of Laurent's matrices (see [BS99]), that :

$$\left\| L_D \sum_{i=1}^{N} e^{-p\tau_i} L_{D_i} \tilde{D}_i L_D^{-1} \right\|_2 \leq \sup_{t\in[0,T]} \left\| D \sum_{i=1}^{N} e^{-p\tau_i} D_i(t) D^{-1} \right\|, \tag{4.115}$$

$$< 1. \tag{4.116}$$

for some $\gamma < 0$ and $p \in \{\tilde{p} \in \mathbb{C} | Re(\tilde{p}) \geq \gamma\}$. Thus $I_\infty - \sum_{j=1}^{N} e^{-p\tau_j} L_{D_j} \tilde{D}_{\tau_j}$ is invertible in $l_k^2(\mathbb{Z})$ to $l_k^2(\mathbb{Z})$ and uniformly bounded in the space of$\{\tilde{p} \in \mathbb{C} | Re(\tilde{p}) \geq \gamma\}$. Thus we have the result by the theorem 4.3. □

□

We give now an application to the 1-D hyperbolic PDE and more precisely to the systems of linear periodic conservation laws. We will prove that these kinds of equations are equivalent to a periodic difference delay system with the disjoint column property which permits to use the stability results of the corollary 4.15. More precisely, we consider the hyperbolic system following the notation of the chapter 3 of [BC16] :

$$\partial_t R(t,x) + \Lambda \partial_x R(t,x) = 0, (t,x) \in \Omega \tag{4.117}$$

where the diagonal matrix Λ is defined as

$$\Lambda := \begin{pmatrix} \Lambda^+ & 0 \\ 0 & -\Lambda^- \end{pmatrix}, with \begin{cases} \Lambda^+ = diag\{\lambda_1, \cdots, \lambda_m\} \\ \Lambda^- = diag\{\lambda_{m+1}, \cdots, \lambda_n\} \end{cases} \lambda_i > 0,\ i = 1, \cdots, n, \tag{4.118}$$

with

$$\Omega = \{(t,x) \in I\!R^2,\ 0 < x < 1 \text{ and } 0 < t < +\infty\}. \tag{4.119}$$

We note $R := \begin{pmatrix} R^+ \\ R^- \end{pmatrix}$, where :

$$R^+ = \begin{pmatrix} R_1 \\ \vdots \\ R_m \end{pmatrix} \text{ and } R^- = \begin{pmatrix} R_{m+1} \\ \vdots \\ R_n \end{pmatrix}. \tag{4.120}$$

And the boundary conditions :

$$\begin{pmatrix} R^+(t,0) \\ R^-(t,1) \end{pmatrix} = K(t) \begin{pmatrix} R^+(t,1) \\ R^-(t,0) \end{pmatrix}. \tag{4.121}$$

First of all, let us give a precise meaning to (4.117) when $R(t,x)$ is assumed continuous. Indeed, (4.117) a priori requires differentiability and (4.121), as well as definition of the initial conditions $R^+(0,.)$ and $R^-(0,.)$, require that R^+ and R^- prolong to some functions on the boundary of Ω (this boundary is $\{0\}\times[0,1] \cup [0,+\infty)\times\{0\} \cup [0,+\infty)\times\{1\}$). We note $\overline{\Omega}$ the set of Ω plus its boundary.

Concerning (4.117), it must simply be taken in the sense of distribution, *i.e.* $R(t,x)$ belonging to $C^0(\Omega, I\!R)$, are said to be a solution of (4.117) if one has

$$\iint_\Omega \left(\partial_t \varphi^* + \partial_x \varphi^* \Lambda\right) R \mathrm{d}t\mathrm{d}x = 0 \tag{4.122}$$

for any smooth functions $\varphi : \Omega \to \mathbb{R}^n$ with compact support contained in Ω.

By the characteristic method, it is well known that if R is continuous function checking the equation (5.14) then there exist n functions such that

$$R^+(t,x) = \begin{pmatrix} x_1(x-\lambda_1 t) \\ \vdots \\ x_m(x-\lambda_m t) \end{pmatrix} \text{ and } R^-(t,x) = \begin{pmatrix} x_1(x+\lambda_{m+1} t) \\ \vdots \\ x_m(x+\lambda_n t) \end{pmatrix}, \tag{4.123}$$

where x_k is continuous in $]-\infty, 1[$ for $k = 1, \cdots, m$ and x_k is continuous in $]0, +\infty[$ for $k = m+1, \cdots, n$. Considering the continuously continuation in 1 and 0 respectively, noting $y_k(t) = x_k(-\lambda_k t)$ for $k = 1, \cdots, m$, $y_k(t) = x_k(1+\lambda_k t)$ for $k = m+1, \cdots, n$ and $\tau_i = \frac{1}{\lambda_i}$ for all $i = 1, \cdots, n$, the equation (4.121) become :

$$\begin{pmatrix} y_1(t) \\ \vdots \\ y_n(t) \end{pmatrix} = K(t) \begin{pmatrix} y_1(t-\tau_1) \\ \vdots \\ y_n(t-\tau_n) \end{pmatrix}, \tag{4.124}$$

which is a periodic delay system and has the nice property to have the column disjoint.

Theorem 4.16. *There is a unique map $(t,x) \mapsto R(t,x)$, continuous $\overline{\Omega} \to \mathbb{R}^n$ which is a solution of (4.121)-(5.14).*

Proof. The system (4.121)-(5.14) leads to the delay difference system (4.124). Since a delay difference system admit a unique continuous solution, we have the result via the equation (4.123). □

□

Theorem 4.17. *If there exists an invertible diagonal matrix D such that $||DK(t)D^{-1}||_2 < 1$ and $K(t)$ is a differentiable function with continuous Hölder derivative then there exists $\gamma, K > 0$ such that, for all solutions given by Theorem 5.7 one has:*

$$\|R(t,\cdot))\|_{C^0([0,1],\mathbb{R}^n)} \leq Ke^{-\gamma t} \|R(0,\cdot)\|_{C^0([0,1],\mathbb{R}^n)} \quad \textit{for all } t \geq 0). \tag{4.125}$$

Proof. Since the system (4.124) has its columns disjoints we can apply the corollary 4.15 which gives the exponential stability of the system (4.124). Then we obtain the equation (4.125) via the equation (4.123). □

□

The theorem 4.17 is the generalization of the theorem 3.7 in the book of Bastin and Coron [BC16] in the case of smooth linear periodic boundaries for hyperbolic systems.

4.7 Conjectures for more general linear periodic delay systems

It might be possible to generalize the results obtained for the difference delay system for more general delay equations. In this section we give conjectures about periodic linear difference delay with a countable number of delay and periodic linear neutral differential equation. We indicate where we need more ideas than we had for the theorem 4.3 to conclude the proofs of the conjectures.

4.7.1 Infinite countable delays

$$y(t) = \sum_{j=1}^{\infty} D_j(t) y(t-\tau_j), \qquad t \geq s, \tag{4.126}$$

where $s \in \mathbb{R}$ is the initial time, d and N are positive integers, $D_j : \mathbb{R} \to \mathbb{C}^{d\times d}$ is a continuous T-periodic matrix-valued function for $1 \leq j < \infty$, and the solutions to (4.126) are $\mathbb{C}^d$ valued functions $t \longmapsto y(t)$ of the time t. We assume that $0 < \tau_j \leq r$ for some real $r > 0$, $\sum ||D_j|| < \infty$ and $\sum_{\tau_j \leq \epsilon} ||D_j(t)||$ tends uniformly in t to 0 when ϵ tends to 0^+. We define the space :

$$\overline{C}_s := \{\phi \in C_0([-\tau_N, 0], \mathbb{C}^d) | \phi(0) = \sum_{j=1}^{\infty} D_j(s)\phi(-\tau_j)\}. \tag{4.127}$$

Since the reasonning did in the appendix can be applied for this system, we have for $\phi \in \overline{C}_s$ than the system (4.126) admits a unique continuous solution y with $y(s+\theta) = \phi(\theta)$ for $\theta \in [-\tau_N, 0]$. Thus, we may define the solution operator:

$$\begin{aligned} \overline{U}(t,s) : \overline{C}_s &\to \overline{C}_t & (4.128)\\ \phi &\mapsto \overline{U}(t,s)\phi, & (4.129) \end{aligned}$$

where $\left(\overline{U}(t,s)\phi\right)(\theta) = \{y(t+\theta),\ \theta \in [-\tau_N, 0]\}$.

Definition 4.18. The zero solution of the system (4.126) is called C^0 asymptotically exponentially stable if there exist $\gamma > 0$ and $K > 0$ such that :

$$||\overline{U}(t,s)\phi||_\infty \leq K e^{-\gamma(t-s)} ||\phi||_\infty, \text{ for all } s \in \mathbb{R}, \text{ all } t \geq s \text{ and all } \phi \in \overline{C}_s. \tag{4.130}$$

Conjecture 1. *Assume that the $D_j : \mathbb{R} \to \mathbb{R}^{d\times d}$ are periodic and differentiable with Hölder continuous derivative for $1 \leq j < \infty$. We have the following equivalence :*

- *there exists a real number $\alpha < 0$ such that :*

  i. $I_\infty - \sum_{j=1}^{\infty} e^{-p\tau_j} L_{D_j} \tilde{D}_{\tau_j}$ *is invertible in $l^2_d(\mathbb{Z})$ to $l^2_d(\mathbb{Z})$ for all $p \in \{z \in \mathbb{C} | \Re(z) \geq \alpha\}$,*

  ii. $\left\| \left[I_\infty - \sum_{j=1}^{\infty} e^{-p\tau_j} L_{D_j} \tilde{D}_{\tau_j} \right]^{-1} \right\|_2$ *is uniformly bounded for all $p \in \{z \in \mathbb{C} | \Re(z) \geq \alpha\}$,*

- *the System (4.126) is C^0 exponentially stable.*

For the sufficiency the proofs of the lemmas 6.4 and 6.5 do not hold anymore because it is possible to have an accumalation point of delay. For the necessity, it must be possible to prove the conjecture 1 as we did for theorem 4.3 with no other difficulties than technical details.

4.7.2 Periodic linear neutral differential equation

We consider the periodic linear neutral differential system :

$$\frac{d}{dt}\left[y(t) - \sum_{j=1}^{N} D_j(t) y(t-\tau_j) \right] = \sum_{k=0}^{N} B_k(t) y(t-\tau_k), \qquad t \geq s, \tag{4.131}$$

where the notations follow the notations given in the section 4.3, and $\tau_0 = 0$ and $B_k : \mathbb{R} \to \mathbb{C}^{d\times d}$ is a continuous T-periodic matrix-valued function for $0 \leq k \leq N$. We may define the solution operator :

$$\begin{aligned} \tilde{U}(t,s) : C_0([-\tau_N, 0], \mathbb{C}^d) &\to C_0([-\tau_N, 0], \mathbb{C}^d) && (4.132)\\ \phi &\mapsto \tilde{U}(t,s)\phi, && (4.133) \end{aligned}$$

where $\left(\tilde{U}(t,s)\phi\right)(\theta) = \{y(t+\theta),\ \theta \in [-\tau_N, 0]\}$.

Definition 4.19. The zero solution of the system (4.131) is called C^0 asymptotically exponentially stable if there exist $\gamma > 0$ and $K > 0$ such that :

$$||\tilde{U}(t,s)\phi||_\infty \leq Ke^{-\gamma(t-s)}||\phi||_\infty, \text{ for all } s \in \mathbb{R}, \text{ all } t \geq s \text{ and all } \phi \in C_0([-\tau_N, 0], \mathbb{C}^d). \quad (4.134)$$

We define the following diagonal matrix for p complex :

$$D_{\omega_0}(p) := Diag(\cdots, (p + i\omega_0)I_d, pI_d, (p - i\omega_0)I_d, \cdots), \quad (4.135)$$

with $\omega_0 := 2\pi/T$. We define too :

$$H(p) := D_{\omega_0}(p)\left[I_\infty - \sum_{j=1}^{N} e^{-p\tau_j} L_{D_j}\tilde{D}_{\tau_j}\right] - \sum_{k=0}^{N} e^{-p\tau_k} L_{B_k}\tilde{D}_{\tau_k} \quad (4.136)$$

Conjecture 2. *Assume that the $D_j(\cdot)$ and $B_k(\cdot)$ are periodic and differentiable with Hölder continuous derivative for $1 \leq j \leq N$ and $0 \leq k \leq N$. We have the following equivalence :*

- *there exists a real number $\alpha < 0$ such that :*
  - *i. $H(p)$ is invertible in $l^2_d(\mathbb{Z})$ to $l^2_d(\mathbb{Z})$ for all $p \in \{z \in \mathbb{C} | \Re(z) \geq \alpha\}$,*
  - *ii. $\left\|\left|H(p)^{-1}\right|\right\|_2$ is uniformly bounded for all $p \in \{z \in \mathbb{C} | \Re(z) \geq \alpha\}$,*
- *the System (4.131) is C^0 exponentially stable.*

The difficulty of the sufficiency comes from the matrix $D_{\omega_0}(p)$ which do not permit to prove again the lemmas 6.4 and 6.5 as we did for the theorem 4.3. The necessity can be done in a same way than for the theorem 4.3.

Chapter 5

Sufficient Stability Conditions for Time-varying Networks of Telegrapher's Equations or Difference Delay Equations

The present Chapter 5 is based entirely on the preprint [BFLP20], conditionally accepted for publication.
The chapter deals with the stability assumptions ensuring that a time-varying network of Telegrapher's equation is exponentially stable with the norm. There is also a stability result for a subclass of linear time-varying delay equations. Only the conclusion of this chapter differs from [BFLP20], it consists in the comparison of the stability results obtained in Chapter 4 for a general periodic difference delay system and the stability results established the present chapter.

5.1 Introduction

The stability of electrical circuits operating at high frequency, that is, when delays induced by wires cannot be neglected, has received a lot of attention in the last decades, see for example references [Bra68, RN02]. At such an operating regime, wires should be considered as transmission lines, and it is customary to model each of them by a lossless telegrapher's equation (a 1-D hyperbolic partial differential equation, in short: PDE) where voltage and current are functions of abscissa and time. The other elements in the circuit, some of which may be active and nonlinear (transistors, diodes), induce couplings between the boundary conditions of these PDE consisting of a system of both differential and non-differential equations with finite-dimensional state, obtained by applying the classical laws of electricity, at each node, to the boundaries that "touch" this node.

Periodic solutions for such infinite dimensional dynamical systems occur naturally in several contexts; for instance, they arise spontaneously in the case of oscillators, or through periodic forcing in the case of amplifiers (the forcing is the signal to be amplified, represented for instance by a periodic voltage source). Assuming a periodic solution, one may linearize the equations around the latter to investigate its local exponential stability, based on the exponential stability of the first order approximation. The linearized system consists of the original collection of telegrapher's equations (which are linear already), coupled at their nodes (*i.e.* the endpoints of a line) by a set of linear differential and non-differential equations with periodic coefficients, obtained by linearizing the initial couplings, see [Sua09]. To this linear system, one associates a *high frequency limit system* (in short: HFLS), where the linear differential equation at each node degenerates into a linear, time-varying

but non-differential relation (*i.e.* there is no dynamics in the couplings at infinite frequencies), so that the state of the HFLS reduces to currents and voltages in the lines. The behavior of the HFLS is crucial to the stability of the linearized system, because the solution operator of the latter is, in natural functional spaces, a compact perturbation of the solution operator to the HFLS, see [HVL93, ch. 3, thm. 7.3] and [Fue20]. In particular, the stability of the HFLS is essentially necessary to the stability of the linearized system.

The HFLS is a system of lossless 1-D telegrapher's equations, with linear couplings that depend on time in a periodic manner. With this application in mind, the present paper is devoted, more generally, to the stability of lossless 1-D telegrapher's equations with linear time varying couplings whose coefficients are measurable and uniformly essentially bounded with respect to time, but not necessarily periodic. As is well known, integrating the telegrapher's equation yields an expression of the general solution in terms of two (essentially arbitrary) functions of one variable, and this allows one to recast the original system as a time-varying linear *difference delay system*; the two frameworks are equivalent to study issues of stability.

Stability of networks of hyperbolic PDEs has been addressed extensively, including more general systems of conservation laws than telegrapher's equations (possibly nonlinear), but almost[1] only when the boundary conditions (*i.e.* the couplings) consist of *time-independent* relations, see [BC16, CN15] and the bibliography therein. Another possible, different application of these criteria is to stabilization of such equations with control, like in [CVKB13, HVL02] for instance. As far as methods are concerned, Lyapunov functions are a classical tool to obtain sufficient stability conditions, see [BC16] where they are applied to certain systems of hyperbolic PDEs with conservation laws, or for instance [FT09], where Lyapunov functions are constructed through linear matrix inequalities, to retarded delay systems. We are not aware of attempts in this direction for difference delay systems.

In another connection, a typical way of obtaining necessary and sufficient stability conditions for a time-invariant network of telegrapher's equations is to apply the Henry-Hale theorem [Hen74, HVL93] or variants thereof (*cf.* Section 5.3) to the equivalent difference delay system with constant coefficients. However, no analog for the time-dependent case seems to be known.

The main contribution of this paper is to establish sufficient conditions for exponential stability of networks of telegrapher's equations, in the form of a dissipativity assumption on the couplings at each node of the network, which is fairly natural in a circuit-theoretic context. We also derive sufficient conditions for exponential stability of time-varying difference delay systems, that are a consequence of the former and of independent interest. To our knowledge, this is the first result of this kind in the time-varying case. The proof, which involves going back and forth between the PDE formulation and the difference delay system formulation, has interesting features that should be useful in other contexts as well. Roughly speaking, we rely on classical energy estimates to first obtain a Lyapunov function in the L^2 sense for each telegrapher's equation, using the dissipativity condition at each node; this allows us to show L^2 exponential stability of the system of PDE, therefore also of the associated delay system. In a second step, we deduce from the L^2 exponential stability of the difference delay system its exponential stability in the L^∞ sense (and in fact in the L^p sense for all $p \in [1, \infty]$). This second step is actually subsumed under the work in [CMS16], but we feel our derivation is simpler and worthy in its own. Note that applications to the local stability of a periodic trajectory in an electrical network indeed require L^∞ stability (or C^0 stability) and not just L^2 stability, for the state along a perturbed trajectory of the linearized system must remain uniformly close to the state along the periodic trajectory of the original system, in order that linearization remains meaningful. This paper makes no attempt at handling more general PDEs or

[1] As an anonymous reviewer pointed out to us, although the paper [CN15] deals with local stability of an equilibrium point for nonlinear *time-invariant* hyperbolic systems, it contains a statement (Lemma 3.2) about stability of smoothly *time-varying* linear systems of hyperbolic PDEs for some Sobolev norm. We discuss this further in Section 5.3.2 and sketch in Section 5.4.4 how the proof of that lemma may be adapted here.

coupling conditions. We rather tried to remain as elementary as possible in treating the problem at hand. In particular, our arguments for well posedness may fail for general hyperbolic 1-D equations, for which notions like broad solutions were introduced in [Bre00] and used *e.g.* in [CN19], see also [BC16] for other approaches.

The paper is organised as follows. Section 5.2 introduces networks of telegrapher's equations coupled by time-varying boundary conditions, gives well-posedness results that we could not find in the literature, discusses the construction of equivalent difference delay systems and defines the notions of stability under examination here. Section 5.3 contains our main result, both in terms of networks of telegrapher's equations and in terms of difference delay equations, while Section 5.4 is devoted to the proofs.

5.2 Problem statement

5.2.1 A time-varying network of hyperbolic equations

Consider a directed graph with N edges and N' nodes, where N and N' are two positive integers. Nodes are numbered by integers $p \in \{1, \cdots, N'\}$, and edges by integers $k \in \{1, \cdots, N\}$.

Figure 5.2.1 represents a graph with 3 nodes and 4 edges whose only purpose is to illustrate the definitions.

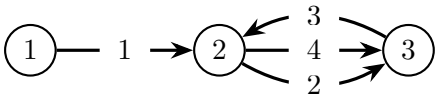


Figure 5.1 : A graph that induces coupling boundary conditions for (5.1) with $N = 4$.

Each edge figures a telegrapher equation. More precisely, we see edge number k as a copy of the real segment $[0, 1]$ (*i.e.* a transmission line of unit length) on which two real function $v_k(t, .)$ and $i_k(t, .)$ are defined for each time $t>0$ (the voltage and the current) in such a way that the lossless telegrapher's equation is satisfied:

$$\begin{cases} C_k \dfrac{\partial v_k(t,x)}{\partial t} = -\dfrac{\partial i_k(t,x)}{\partial x}, \\ L_k \dfrac{\partial i_k(t,x)}{\partial t} = -\dfrac{\partial v_k(t,x)}{\partial x}, \end{cases} \qquad (t,x) \in \Omega\,, \tag{5.1}$$

where

$$\Omega \;=\; \{(t,x) \in \mathbb{R}^2,\ 0 < x < 1 \text{ and } 0 < t < +\infty\}, \tag{5.2}$$

and, for each k, L_k and C_k are two strictly positive numbers (the inductance and the capacity of the line k). In (5.3) we define the numbers τ_k and K_k, often called the *delay* and *characteristic impedance* of the line k, and we assume without loss of generality, possibly after a re-ordering of the edges, that the delays τ_k are increasing:

$$\tau_k = \sqrt{L_k C_k}\,, \quad K_k = \sqrt{C_k/L_k}\,, \quad 0 < \tau_1 \leq \tau_2 \cdots \leq \tau_N\,. \tag{5.3}$$

Each node couples the edges adjacent to it through boundary conditions involving, for each such edge, the endpoint of $[0, 1]$ that touches the node. More precisely, if

- $j(p)$ is the out-degree of the graph at node p (i.e. the number of edges outgoing from node p) and $\tilde{j}(p)$ the in-degree of the graph at node p (i.e. the number of edges incoming at node p), and

- $k_1 < \cdots < k_{j(p)}$ are the labels of the edges *outgoing* from node p and $k_{j(p)+1} < \cdots < k_{j(p)+\tilde{\jmath}(p)}$ the labels of the edges *incoming at* node p,

then node p couples the equations (5.1) together by imposing $j(p) + \tilde{\jmath}(p)$ relations between the $2(j(p) + \tilde{\jmath}(p))$ entries of the following two vectors:

$$V_p(t) = \begin{pmatrix} v_{k_1}(t,0) \\ \vdots \\ v_{k_{j(p)}}(t,0) \\ v_{k_{j(p)+1}}(t,1) \\ \vdots \\ v_{k_{j(p)+\tilde{\jmath}(p)}}(t,1) \end{pmatrix}, \qquad I_p(t) = \begin{pmatrix} -i_{k_1}(t,0) \\ \vdots \\ -i_{k_{j(p)}}(t,0) \\ i_{k_{j(p)+1}}(t,1) \\ \vdots \\ i_{k_{j(p)+\tilde{\jmath}(p)}}(t,1) \end{pmatrix}. \tag{5.4}$$

These relations read

$$V_p(t) = A_p(t) I_p(t), \tag{5.5}$$

where $t \mapsto A_p(t)$ is a map from $\mathbb{R}^+$ (the nonnegative real numbers) to the set of square $(j(p) + \tilde{\jmath}(p)) \times (j(p) + \tilde{\jmath}(p))$ matrices which is assumed to be measurable and *bounded*. In many cases of interest (*e.g.* when modeling an electrical circuit), it is moreover *continuous*. Our results rest on the following condition characterizing *dissipativity* at (each) node p:

$$A_p(t) + A_p^*(t) \geq \alpha_p \, Id, \quad \alpha_p > 0\,, \quad p \in \{1, \cdots, N'\}, \tag{5.6}$$

where superscript $*$ denotes the transpose of a matrix and α_p is independent of t. Inequality (5.6) is meant to hold between symmetric matrices, for a.e. t. We shall have an occasion to deal also with complex matrices, in which case superscript $*$ denotes the transpose conjugate. Here and below, the symbol Id stands for the identity operator or the identity matrix of appropriate size, while the context will keep the meaning clear.

Example 1. *For the graph in Figure 5.2.1 it holds that $N = 4$, i.e. we have four telegrapher's equations of the form (5.1), numbered with $k \in \{1, 2, 3, 4\}$, and we have that $N' = 3$, hence we get three sets of boundary conditions. Let us detail the latter.*
- For $p = 1$, we have $j(1) = 1$, $\tilde{\jmath}(1) = 0$, and we see from the graph that $k_1 = 1$,
- for $p = 2$, we have $j(2) = 2$, $\tilde{\jmath}(2) = 1$, and we see from the graph that $k_1 = 2$, $k_2 = 4$, $k_3 = 3$,
- for $p = 3$, we have $j(3) = 1$, $\tilde{\jmath}(3) = 2$, and we see from the graph that $k_1 = 3$, $k_2 = 2$ and $k_3 = 4$.
This yields three equations of the form (5.5) as follows, with $A_1(t)$ a scalar, $A_2(t)$ is a 4×4 matrix and $A_3(t)$ a 3×3 matrix:

$$v_1(t,0) = -A_1(t)\, i_1(t,0), \quad \begin{pmatrix} v_2(t,0) \\ v_4(t,0) \\ v_1(t,1) \\ v_3(t,1) \end{pmatrix} = A_2(t) \begin{pmatrix} -i_2(t,0) \\ -i_4(t,0) \\ i_1(t,1) \\ i_3(t,1) \end{pmatrix}, \quad \begin{pmatrix} v_3(t,0) \\ v_2(t,1) \\ v_4(t,1) \end{pmatrix} = A_3(t) \begin{pmatrix} -i_3(t,0) \\ i_2(t,1) \\ i_4(t,1) \end{pmatrix}. \tag{5.7}$$

Remark 5.1 (On the minus signs in the vector I_p in (5.4))**.** *We shall see later why (5.6) amounts to energy dissipation in some sense. This is one justification for the minus signs in the first entries of the vector $I_p(t)$ defined by (5.4): removing the minus signs, one may still express the (same) boundary conditions by an equation like (5.5) with a different $A_p(t)$, but then condition (5.6) will assume an intricate form. Alternatively, from a circuit-theoretic viewpoint, the minus signs are justified by Kirchhoff's law of currents.*

Remark 5.2 (On the normalization of line lengths)**.** *We have assumed that the space variable x belongs to the interval $[0,1]$ for every k in equation (5.1) rather than $[0,\ell_k]$ for some positive ℓ_k. This is no loss of generality, for such a normalization can always be achieved by a linear change of variable on x. With this normalization, τ_k given by equation (5.3) has the meaning of a time delay.*

Remark 5.3 (On the possibility of loops)**.** *In the above framework, nothing prevents an edge from being both outgoing from, and incoming to a given node p. In this case, the index k of this edge appears twice in equation (5.4), once as a k_j with $j \leq j(p)$ and once as a $k_{j(p)+l}$ with $l \leq \tilde{\jmath}$.*

So far, we endowed a system consisting of N PDE, indexed by the edges of our graph (namely: (5.1)), with boundary conditions given by a collection of N' linear time-dependent relations, indexed by the nodes of the graph (namely: (5.5)). As a result, the boundary conditions at $x=0$ and at $x=1$ for a given telegrapher's equation of the form (5.1) are generally obtained from two different relations of the form(5.5). To compactify the notation, we shall rewrite the boundary conditions in lumped form, as a single linear relation between concatenated vectors $\mathbf{V}(t)$ and $\mathbf{I}(t)$ defined by:

$$v(x,t)=\begin{pmatrix} v_1(x,t)\\ \vdots \\ v_N(x,t)\end{pmatrix}, \quad i(x,t)=\begin{pmatrix} i_1(x,t)\\ \vdots \\ i_N(x,t)\end{pmatrix}, \quad \mathbf{V}(t)=\begin{pmatrix} v(t,0)\\ \\ v(t,1)\end{pmatrix}, \quad \mathbf{I}(t)=\begin{pmatrix} -i(t,0)\\ \\ i(t,1)\end{pmatrix}, \tag{5.8}$$

that aggregate *all* boundary values of voltages and currents in the lines. Since the concatenation of all vectors $V_p(t)$ (resp. $I_p(t)$) defined in (5.4) contains each component of $\mathbf{V}(t)$ (resp. $\mathbf{I}(t)$) exactly once, as p ranges from 1 to N', there is a $2N\times 2N$ permutation matrix P_1 such that

$$\begin{pmatrix} V_1(t)\\ \vdots \\ V_{N'}(t)\end{pmatrix} = P_1\,\mathbf{V}(t)\,, \quad \begin{pmatrix} I_1(t)\\ \vdots \\ I_{N'}(t)\end{pmatrix} = P_1\,\mathbf{I}(t)\,. \tag{5.9}$$

The set of equations (5.5), $1\leq p\leq N'$, can now be written as

$$\mathbf{V}(t)=\mathbf{A}(t)\,\mathbf{I}(t) \tag{5.10}$$

with

$$\mathbf{A}(t)=P_1^{-1}\,\mathrm{diag}(A_1(t),\dots,A_{N'})\,P_1 \tag{5.11}$$

where $\mathrm{diag}(A_1(t),\dots,A_{N'}(t))$ is a block-diagonal $2N\times 2N$ matrix. Note that this "aggregated" notation may be understood as collapsing all the nodes into a single one; all edges are then "loops" as described in Remark 5.3. Clearly, the hypotheses on A_p made in (5.6) translate into the following assumption on the matrix $\mathbf{A}(t)$ that will be used throughout the paper :

Assumption 5.4. *The map $t\mapsto \mathbf{A}(t)$ is measurable and essentially bounded $[0,+\infty)\to\mathbb{R}^{2N\times 2N}$, moreover there exists a positive number α, independent of t, such that*

$$\mathbf{A}(t)+\mathbf{A}^*(t)\geq \alpha\, Id, \qquad \alpha>0 \qquad t\in\mathbb{R}. \tag{5.12}$$

5.2.2 Well posedness of evolution problem in the L^p and C^0 cases

Equations (5.1) ($1\leq k\leq N$) and (5.10)-(5.8) define a linear time-varying dynamical system, whose state at time t consists of a collection of $2N$ real functions on $[0,1]$, namely $x\mapsto v_k(t,x)$ and $x\mapsto i_k(t,x)$ for $1\leq k\leq N$. Before we can study the stability of this dynamical system, we need

to address the issue of well posedness, *i.e.* of existence and uniqueness of solutions given initial conditions $v_k(0,.)$ and $i_k(0,.)$ (the Cauchy problem). When the matrices $A_p(t)$ (or equivalently the matrix $\mathbf{A}(t)$) do not actually depend on t, well-posedness results are classical, see for instance the textbooks [BC16, Daf10].

In the time-varying case, which is our concern here, a very definition of well-posedness seems hard to find in the literature, perhaps because the introduction of time dependent boundary conditions leads to a failure of classical semigroup theory. We shall consider two cases according to whether the state at time t consists of continuous functions or merely L^p-summable functions on $[0,1]$, $1 \leq p \leq \infty$.

To fix notation, we denote respectively by $\mathbb{N}$ and $\mathbb{R}$ the sets of nonnegative integers and real numbers. We also recall the notation $\mathbb{R}^+$ for nonnegative real numbers. We write the Euclidean norm of $x \in \mathbb{R}^l$ as $\|x\|$, and the Euclidean scalar product of $x, y \in \mathbb{R}^l$ as $\langle x, y\rangle$, irrespectively of l. We put $C^0(E)$ for the space of real continuous functions on any (topological) space E. When E is compact we endow $C^0(E)$ with the *sup* norm. Also, whenever $E \subset \mathbb{R}^l$ is measurable and $1 \leq p < \infty$, we put $L^p(E)$ for the familiar Lebesgue space of (equivalent classes of a.e. coinciding) real-valued measurable functions on E whose absolute value to the p^{th} power is integrable, endowed with the norm $\|f\|_{L^p(E)} = (\int_E |f(x)|^p dx)^{1/p}$ where dx indicates the differential of Lebesgue measure (restricted to E). The space $L^\infty(E)$ corresponds to real, essentially bounded Lebesgue measurable functions, normed with the essential supremum of their absolute value on E. More generally, for F a Banach space with norm $\|.\|_F$, we let $C^0(E,F)$ be the space of F-valued continuous functions on E, and if E is compact we set $\|f\|_{C^0(E,F)} = \sup_E \|f\|_F$. In a similar way, $L^p(E,F)$ is the space of F-valued measurable functions f on E such that $\|f\|_F \in L^p(E)$. We also define locally integrable functions: $L^p_{loc}(E)$ designates the space of functions whose restriction $f_{|K}$ to any compact set $K \subset E$ belongs to $L^p(K)$. Likewise, we let $L^p_{loc}(E,F)$ be the space of F-valued measurable functions f on E such that $\|f\|_F \in L^p_{loc}(E)$. Since $\mathbb{R}^l$ is σ-compact, the topology of L^p-convergence on every compact set is metrizable on $L^p_{loc}(\mathbb{R}^l, F)$. The spectral norm of a linear operator $B : F_1 \to F_2$ between two Banach spaces is $|||B||| = \sup_{x \in F_1} \|Bx\|_{F_2}/\|x\|_{F_1}$, keeping the notation independent of F_1, F_2 for simplicity.

Next, let us make precise the meaning of (5.1) and (5.10) when v_k and i_k lie in $L^1_{loc}(\overline{\Omega})$, where Ω is defined by (5.2) and $\overline{\Omega}$ indicates the closure of Ω in $\mathbb{R}^2$. Later, we shall see this space is big enough to accomodate cases we have in mind. Note that $\overline{\Omega} = [0,\infty) \times [0,1]$, and that $L^1_{loc}(\overline{\Omega})$ identifies with a subspace of $L^1_{loc}(\Omega)$, since $[0,\infty) \times [0,1] \setminus \Omega$ has 2-D Lebesgue measure zero. Indeed, the latter set is just the boundary $\partial\Omega$ of Ω in $\mathbb{R}^2$:

$$\partial\Omega = (\{0\} \times (0,1)) \cup ([0,+\infty) \times \{0\}) \cup ([0,+\infty) \times \{1\})\,. \tag{5.13}$$

Equation (5.1) is understood in the distributional sense as soon as $(v_k, i_k) \in L^1_{loc}(\Omega) \times L^1_{loc}(\Omega)$. That is, (v_k, i_k) is a solution to (5.1) if, for all C^∞-smooth functions $\varphi : \Omega \to \mathbb{R}$ with compact support, it holds that

$$\begin{aligned} \iint_\Omega \Big(L_k\, i_k(t,x) \frac{\partial \varphi}{\partial t}(t,x) + v_k(t,x) \frac{\partial \varphi}{\partial x}(t,x) \Big) \mathrm{d}t\mathrm{d}x = 0\,, \\ \iint_\Omega \Big(C_k\, v_k(t,x) \frac{\partial \varphi}{\partial t}(t,x) + i_k(t,x) \frac{\partial \varphi}{\partial x}(t,x) \Big) \mathrm{d}t\mathrm{d}x = 0\,. \end{aligned} \tag{5.14}$$

As to (5.10), the definition (5.8) of $\mathbf{V}$ and $\mathbf{I}$, as well as the choice of initial conditions $v_k(0,.)$ and $i_k(0,.)$, require that v_k and i_k extend in some way to $\partial\Omega$ described in (5.13), and this is where their membership to $L^1_{loc}(\overline{\Omega})$ (not just to $L^1_{loc}(\Omega)$) is useful. In fact, when $h \in L^1_{loc}(\overline{\Omega}) = L^1_{loc}([0,\infty) \times [0,1])$, we get from Fubini's theorem that $\tau \mapsto h(\tau, x)$ belongs to $L^1_{loc}([0,\infty))$ for a.e. $x \in [0,1]$ and that

$s \mapsto h(t,s)$ lies in $L^1([0,1])$ for a.e. $t \in [0,\infty)$. For such x and t, we set

$$\widehat{h}(0,x) = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\int_0^\varepsilon h(s,x)\mathrm{d}s\,, \qquad \widehat{h}(t,0) = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\int_0^\varepsilon h(t,s)\mathrm{d}s\,, \qquad \widehat{h}(t,1) = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\int_{1-\varepsilon}^1 h(t,s)\mathrm{d}s\,, \tag{5.15}$$

whenever the limits exist.

Definition 5.5. *We say that $h \in L^1_{loc}(\overline{\Omega})$* has a strict extension to $\partial\Omega$ *if and only if the limits in (5.15) exist for almost all $x \in (0,1)$ and almost all $t \in (0,\infty)$, and then the functions $x \mapsto \widehat{h}(0,x)$, $t \mapsto \widehat{h}(t,0)$ and $t \mapsto \widehat{h}(t,1)$ define the strict extension of h to $\partial\Omega$, almost everywhere with respect to $\mathcal{H}^1$-Hausdorff measure[2], see e.g. [EG92, ch. 2] for the definition of Hausdorff measures.*

Remark 5.6. *Definition 5.5 may look strange at first glance, since when $h \in L^1_{loc}(\overline{\Omega})$ it seems to be defined already on $\partial\Omega \subset \overline{\Omega}$; but of course it is not so, because $\partial\Omega$ has 2-D Lebesgue measure zero, hence the values assumed by h there are immaterial. When the limits in (5.15) exist for a.e. x and t, they produce a specific definition of h on $\partial\Omega$, a.e. with respect to $\mathcal{H}^1$, that we call the strict extension. If $h : \Omega \to \mathbb{R}$ is continuous and extends continuously $\overline{\Omega} \to \mathbb{R}$, clearly the strict extension exists and it is the natural one. Even then, we sometimes use the notation $\widehat{h}(0,x)$, $\widehat{h}(t,0)$ and $\widehat{h}(t,1)$ for reasons of consistency, although writing $h(0,x)$, $h(t,0)$ and $h(t,1)$ is more appropriate in this case.*

If all v_k and i_k have a strict extension to $\partial\Omega$, then we interpret the boundary conditions (5.10) to mean the following set of equalities between (a.e. defined) measurable functions of a single variable t:

$$\begin{pmatrix} \widehat{v}_1(t,0) \\ \vdots \\ \widehat{v}_N(t,0) \\ \widehat{v}_1(t,1) \\ \vdots \\ \widehat{v}_N(t,1) \end{pmatrix} = \mathbf{A}(t) \begin{pmatrix} -\widehat{i}_1(t,0) \\ \vdots \\ -\widehat{i}_N(t,0) \\ \widehat{i}_1(t,1) \\ \vdots \\ \widehat{i}_N(t,1) \end{pmatrix}, \qquad \text{a.e. } t \in (0,\infty). \tag{5.16}$$

We can now state a well-posedness result for the system (5.1)-(5.10). Part I deals with solutions belonging to $L^1_{loc}([0,\infty), L^p([0,1]))$, and part II is about continuous solutions. They do not run completely parallel to each other, because continuity requires a compatibility relation on the initial conditions, see (5.18). The theorem is standard in nature but, as mentioned already, we could not find a reference in the literature for the case of *time-varying* boundary conditions (5.16). To connect the statement with the previous discussion, we observe that $L^1_{loc}([0,\infty), L^p([0,1])) \subset L^1_{loc}([0,\infty)\times[0,1]) = L^1_{loc}(\overline{\Omega})$ for $1 \le p \le \infty$, by Hölder's inequality and the Fubini theorem.

Theorem 5.7 (Well-posedness)**.** *Let $\mathbf{A} : [0,\infty) \to \mathbb{R}^{2N\times 2N}$ meet Assumption 5.4 and $1 \le p \le \infty$.*
I) *If $i_k^0, v_k^0 \in L^p([0,1])$, $1 \le k \le N$, there is a unique map $(t,x) \mapsto (v_1(t,x),\dots,v_N(t,x),i_1(t,x),\dots,i_N(t,x))$ from Ω into $\mathbb{R}^{2N\times 2N}$ such that:*

- *$t \mapsto (v_1(t,.),\dots,v_N(t,.),i_1(t,.),\dots,i_N(t,.))$ belongs to $L^1_{loc}([0,\infty),(L^p([0,1]))^{2N})$ and v_k, i_k have a strict extension to $\partial\Omega$ satisfying the initial conditions*

$$\widehat{v}_k(0,x) = v_k^0(x), \quad \widehat{i}_k(0,x) = i_k^0(x) \quad k = 0,\dots,N\,, \tag{5.17}$$

- *$(t,x) \mapsto (v_1(t,x),\dots,v_N(t,x),i_1(t,x),\dots,i_N(t,x))$ is a solution of (5.1)-(5.10), $1 \le k \le N$, in the sense of (5.14) and (5.16).*

[2] $\mathcal{H}^1$ restricted to $\partial\Omega \subset \mathbb{R}^2$ is simply the measure whose restriction to each curve $\{0\}\times(0,1)$, $[0,+\infty)\times\{0\}$ and $[0,+\infty)\times\{1\}$ coincides with arc length.

II) *If, in addition, $t \mapsto \mathbf{A}(t)$ is continuous and $v_1^0, \dots, v_N^0, i_1^0, \dots, i_N^0$ are elements of $C^0([0,1])$ satisfying*

$$\begin{pmatrix} v_1^0(0) \\ \vdots \\ v_N^0(0) \\ v_1^0(1) \\ \vdots \\ v_N^0(1) \end{pmatrix} = \mathbf{A}(0) \begin{pmatrix} -i_1^0(0) \\ \vdots \\ -i_N^0(0) \\ i_1^0(1) \\ \vdots \\ i_N^0(1) \end{pmatrix}, \tag{5.18}$$

then the map $(t,x) \mapsto (v_1(t,x), \dots, v_N(t,x), i_1(t,x), \dots, i_N(t,x))$ from part **I** *is continuous $\overline{\Omega} \to \mathbb{R}^{2N}$ (equivalently: $t \mapsto (v_1(t,.), \dots, v_N(t,.), i_1(t,.), \dots, i_N(t,.))$ is continuous $[0,\infty) \to C^0([0,1])^{2N\times 2N}$) and satisfies the initial conditions in the strong sense:*

$$v_k(0,x) = v_k^0(x), \;\; i_k(0,x) = i_k^0(x)\,, \qquad x \in [0,1], \quad k = 0, \dots, N\,. \tag{5.19}$$

Remark 5.8. *Assumption 5.4 is stronger than needed for the previous result to hold. In fact, it is enough for part I that the maps $t \mapsto \mathbf{A}(t)$ and $t \mapsto (I + \mathbf{A}(t)\,\mathbf{K})^{-1}$ be well defined, measurable and bounded (**K** is defined in (5.28)), and for part II that they be continuous and bounded. We do not dwell on such generalizations.*

The proof of Theorem 5.7 is given at the end of Section 5.2.3, after establishing the equivalence of (5.1)-(5.10) with a suitable difference delay system. As a first step in this direction, we stress below the special form of solutions to (5.1) in $L^p_{loc}(\Omega)$, and show they have a strict extension to $\partial\Omega$ if, moreover, they lie in $L^p_{loc}(\overline{\Omega})$.

Proposition 5.9. *Let i_k and v_k belong to $L^p_{loc}(\Omega)$ (resp. $C^0(\Omega)$) for some $p \in [1,\infty]$, and satisfy (5.1) on Ω in the sense of (5.14). Then, the following properties hold.*

i. There exists two functions f_k and g_k in $L^p_{loc}((-\infty,1))$ and $L^p_{loc}((0,\infty))$ (resp. in $C^0((-\infty,1))$ and $C^0((0,\infty))$) such that

$$v_k(t,x) = f_k(x - \frac{t}{\tau_k}) + g_k(x + \frac{t}{\tau_k})\,, \quad i_k(t,x) = K_k\left(f_k(x - \frac{t}{\tau_k}) - g_k(x + \frac{t}{\tau_k})\right), \tag{5.20}$$

for almost every (resp. every) (x,t) in Ω, where τ_k, K_k are defined by (5.3).

ii. If, in addition, v_k and i_k lie in $L^p_{loc}(\overline{\Omega})$ (resp. extend continuously $\overline{\Omega} \to \mathbb{R}$), then f_k and g_k lie in $L^p_{loc}((-\infty,1])$ and $L^p_{loc}([0,\infty))$ (resp. in $C^0((-\infty,1])$ and $C^0([0,\infty))$), moreover v_k, i_k have a strict extension to $\partial\,\Omega$ according to Definition 5.5. More precisely, we have that

$$\begin{aligned}
&\widehat{v}_k(t,0) = f_k(-\frac{t}{\tau_k}) + g_k(\frac{t}{\tau_k})\,, && \widehat{i}_k(t,0) = K_k\left(f_k(-\frac{t}{\tau_k}) - g_k(\frac{t}{\tau_k})\right), \\
&\widehat{v}_k(t,1) = f_k(1 - \frac{t}{\tau_k}) + g_k(1 + \frac{t}{\tau_k})\,, && \widehat{i}_k(t,1) = K_k\left(f_k(1 - \frac{t}{\tau_k}) - g_k(1 + \frac{t}{\tau_k})\right), \\
&\widehat{i}_k(0,x) = K_k\left(f_k(x) - g_k(x)\right), && \widehat{v}_k(0,x) = f_k(x) + g_k(x)\,,
\end{aligned} \tag{5.21}$$

where the first four relations hold for almost all (resp. all) t in $(0,+\infty)$ and the last two for almost all (resp. all) x in $(0,1)$.

iii. Conversely, if f_k and g_k lie in $L^p_{loc}((-\infty,1))$ and $L^p_{loc}((0,\infty))$ (resp. in $C^0((-\infty,1))$ and $C^0((0,\infty))$), then v_k and i_k given by (5.20) belong to $L^p_{loc}(\Omega)$ (resp. $C^0(\Omega)$) and satisfy (5.1). If, moreover, f_k and g_k lie in $L^p_{loc}((-\infty,1])$ and $L^p_{loc}([0,\infty))$ (resp. in $C^0((-\infty,1])$ and $C^0([0,\infty))$), then v_k and i_k belong to $L^p_{loc}(\overline{\Omega})$ (resp. $C^0(\overline{\Omega})$) and (5.21) holds.

Proof. Point (i) is classical: if we introduce new variables $r = x - t/\tau_k$, $s = x + t/\tau_k$ and define two functions f_k, g_k on $\Omega_1 := \{(r,s) \in \mathbb{R}^2,\ 0 < r+s < 2 \text{ and } -\infty < r-s < 0\}$, in one-to-one correspondence with v_k, i_k *via*

$$\begin{pmatrix} f_k(r,s) \\ g_k(r,s) \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2K_k} \\ \frac{1}{2} & -\frac{1}{2K_k} \end{pmatrix} \begin{pmatrix} v_k\Big(\frac{-r+s}{2}\tau_k\,,\frac{r+s}{2}\Big) \\ i_k\Big(\frac{-r+s}{2}\tau_k\,,\frac{r+s}{2}\Big) \end{pmatrix},\ \begin{pmatrix} v_k(t,x) \\ i_k(t,x) \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ K_k & -K_k \end{pmatrix} \begin{pmatrix} f_k(x-t/\tau_k, x+t/\tau_k) \\ g_k(x-t/\tau_k, x+t/\tau_k) \end{pmatrix},$$

then $f_k, g_k \in L^p_{loc}(\Omega_1)$ by the change of variables formula, moreover they are continuous $\Omega_1 \to \mathbb{R}$ when i_k and v_k are continuous on Ω, and System (5.1) gets transformed into the distributional identity

$$\frac{\partial f_k}{\partial s} = \frac{\partial g_k}{\partial r} = 0\,. \tag{5.22}$$

Equation (5.22) means that f_k does not depend on the second argument nor g_k on the first one, hence the form (5.20) for v_k and i_k. We turn to Point (ii). First, we observe that if v_k and i_k lie in $L^p_{loc}(\overline{\Omega})$ (resp. extend continuously $\overline{\Omega} \to \mathbb{R}$), then f_k and g_k lie in $L^p_{loc}((-\infty,1])$ and $L^p_{loc}([0,\infty))$ (resp. in $C^0((-\infty,1])$ and $C^0([0,\infty))$), by the change of variable formula (resp. by inspection). The case where v_k and i_k extend continuously $\overline{\Omega} \to \mathbb{R}$ is now obvious. To handle the case where $v_k, i_k \in L^p_{loc}(\overline{\Omega})$, recall that a (non-centered) Lebesgue point of a function $\ell \in L^1_{loc}(\mathbb{R})$ is a point $x \in \mathbb{R}$ such that $\lim_{|I|\to 0, I\ni x} \frac{1}{|I|}\int_I |\ell(y)-\ell(x)|\mathrm{d}y = 0$, where the limit is taken over all closed intervals I containing x and $|I|$ indicates the length of I. Let $\widetilde{f}_k$ and $\widetilde{g}_k$ be the extensions by 0 of f_k and g_k to the whole real line. Using (5.20) in (5.15), we see that the first two relations in (5.21) certainly hold for $t \in (0,\infty)$ such that $-t/\tau_k$ is a Lebesgue point of $\widetilde{f}_k$ and t/τ_k is a Lebesgue point of $\widetilde{g}_k$, the third and fourth relations if $1 - t/\tau_k$ is a Lebesgue point of $\widetilde{f}_k$ and $1 + t/\tau_k$ is a Lebesgue point of $\widetilde{g}_k$, and the last two if $x \in (0,1)$ is a Lebesgue point of both $\widetilde{f}_k$ and $\widetilde{g}_k$. Since almost all points are Lebesgue points of a given function in $L^1_{loc}(\mathbb{R})$ [EG92, thm. 1.34], while $L^p_{loc}(\mathbb{R}) \subset L^1_{loc}(\mathbb{R})$ by Hölder's inequality, this proves Point (ii). Point (iii) is obvious, reverting computations. □

Remark 5.10. *The weak formulation (5.14) defines solutions v_k, i_k to (5.1) as locally integrable functions $\Omega \to \mathbb{R}$, while Theorem 5.7 stresses their representation as functions $\mathbb{R}^+ \to L^p([0,1])$. The two points of view are essentially equivalent by Fubini's theorem, but suggestive of different moods. In this connection, it is worth mentioning that if $p < \infty$, then the solution set forth in Part I of Theorem 5.7 not only belongs to $L^1_{loc}([0,\infty), (L^p([0,1]))^{2N})$, but in fact is continuous $[0,\infty) \to (L^p([0,1]))^{2N\times 2N}$. Indeed, granted that f_k and g_k lie in $L^p_{loc}((-\infty,1])$ and $L^p_{loc}([0,\infty))$ by Proposition 5.9, this follows from the very proof of the theorem (cf. (5.31) and (5.26) below) and the fact that $\tau \mapsto f(.-\tau)$ is continuous $\mathbb{R} \to L^p(\mathbb{R})$, whenever $f \in L^p(\mathbb{R})$, $p < \infty$.*

5.2.3 Difference delay equations and their relation with networks of telegrapher's equations

A general linear time-varying difference delay equation in the variable z is of the form

$$z(t) = \sum_{i=1}^{M} D_i(t)\, z(t-\eta_i) \quad \text{for all (or almost all) } t \geq 0\,, \tag{5.23}$$

where the delays $0 < \eta_1 \leq \cdots \leq \eta_M$ are arranged in nondecreasing order, each $t \mapsto D_i(t)$ is a $d \times d$ matrix-valued function, and solutions $t \mapsto z(t)$ are $\mathbb{R}^d$-valued functions. Hereafter, we make the following assumption.

Assumption 5.11. *The maps $t \mapsto D_i(t)$ belong to $L^\infty(\mathbb{R}^+, \mathbb{R}^{d\times d})$ for $1 \leq i \leq M$.*

Given initial conditions on $[-\eta_M, 0]$, we recap existence and uniqueness of solutions to (5.23) in the following theorem. The existence of continuous solutions requires an additional continuity assumption on the D_i, as well as compatibility relations on the initial conditions; this is why we introduce the following space:

$$\mathcal{C} := \{\phi \in C^0([-\eta_M, 0], \mathbb{R}^d) \,|\, \phi(0) = \sum_{i=1}^{M} D_i(0)\phi(-\eta_i)\}\,. \tag{5.24}$$

Theorem 5.12. *Let Assumption 5.11 hold and ϕ be an element of $L^p([-\eta_M, 0], \mathbb{R}^d)$ with $1 \le p \le \infty$.*

(i) There is a unique solution z to (5.23) in $L^p_{loc}([-\eta_M, +\infty), \mathbb{R}^d)$ meeting the initial condition $z_{|[-\eta_M, 0]} = \phi$.

(ii) If, moreover, the maps $D_i : \mathbb{R}^+ \to \mathbb{R}^{d\times d}$ are continuous and $\phi \in \mathcal{C}$, then $z \in C^0([-\eta_M, +\infty), \mathbb{R}^d))$.

Proof. This is a classical, elementary inductive argument, see *e.g.* [HVL93]: for any $T \ge 0$, if a solution has been found on $[-\eta_M, T]$, it clearly can be extended to $[-\eta_M, T + \eta_1]$ in a unique manner using (5.23). When the $D_i(.)$ are continuous, $\phi \in \mathcal{C}$ is clearly necessary and sufficient for the unique solution to be continuous. □

Remark 5.13 (merging repeated delays)**.** *In (5.23), we allow for repeated delays, i.e. it may be that $\eta_i = \eta_{i+1}$ for some i. This to comply with (5.1)-(5.3), where it would be too restrictive to require that the numbers τ_k are distinct, and because we are about to convert (5.1)-(5.3) into (5.23) in such a way that $\eta_i = \tau_i$. However, when dealing with (5.23), it is better to avoid repetition by merging terms with the same delay. Since it will be needed in the statement of Theorem 5.22, let us formalize this: first, define an enumeration without repetition of the original list of delays, say, $0 < \widehat{\eta}_1 < \widehat{\eta}_2 < \cdots < \widehat{\eta}_{\widehat{M}}$ with $\widehat{M} \le M$, then define for each j*

$$\widehat{D}_j(t) = \sum_{\{i,\ \eta_i = \widehat{\eta}_j\}} D_i(t)\,. \tag{5.25}$$

It is clear that (5.23) can be re-written as $z(t) = \sum_{j=1}^{\widehat{M}} \widehat{D}_j(t)\, z(t - \widehat{\eta}_j)$, and if the η_i were distinct already, then the system is left unchanged.

We now construe the system of coupled telegrapher's equations from Sections 5.2.1 and 5.2.2 as a difference delay system of the form (5.23). For this, let $(v_k, i_k) \in L^1_{loc}(\overline{\Omega}) \times L^1_{loc}(\overline{\Omega})$ (resp. $C^0(\overline{\Omega}) \times C^0(\overline{\Omega})$) be, for $1 \le k \le N$, solutions of (5.1)-(5.10), observing from Proposition 5.9 (*ii*) that the boundary conditions (5.10) indeed make sense. Let f_k, g_k be as in Proposition 5.9, and define:

$$x_k(t) = f_k(-\frac{t}{\tau_k}) \quad \text{and} \quad y_k(t) = g_k(1 + \frac{t}{\tau_k}). \tag{5.26}$$

The functions f_k and g_k lie in $L^1_{loc}((-\infty, 1])$ and $L^1_{loc}([0, \infty))$ (resp. in $C^0((-\infty, 1])$ and $C^0([0, \infty))$) by Proposition 5.9, therefore x_k and y_k lie in $L^1_{loc}([-\tau_k, +\infty))$ (resp. $C^0([-\tau_k, +\infty))$). Moreover, the boundary values of v_k and i_k are related to x_k and y_k as follows (substitute (5.26) in (5.21)):

$$\begin{cases} \widehat{v}_k(t, 0) = x_k(t) + y_k(t - \tau_k)\,, \\ \widehat{\imath}_k(t, 0) = K_k[x_k(t) - y_k(t - \tau_k)]\,, \\ \widehat{v}_k(t, 1) = x_k(t - \tau_k) + y_k(t)\,, \\ \widehat{\imath}_k(t, 1) = K_k[x_k(t - \tau_k) - y_k(t)]\,. \end{cases} \tag{5.27}$$

Plugging (5.27) in (5.16) gives us

$$\begin{pmatrix} x_1(t) \\ \vdots \\ x_N(t) \\ y_1(t) \\ \vdots \\ y_N(t) \end{pmatrix} + \begin{pmatrix} y_1(t-\tau_1) \\ \vdots \\ y_N(t-\tau_N) \\ x_1(t-\tau_1) \\ \vdots \\ x_N(t-\tau_N) \end{pmatrix} = \mathbf{A}(t) \left[\begin{pmatrix} -K_1\, x_1(t) \\ \vdots \\ -K_N\, x_N(t) \\ -K_1\, y_1(t) \\ \vdots \\ -K_N\, y_N(t) \end{pmatrix} + \begin{pmatrix} K_1\, y_1(t-\tau_1) \\ \vdots \\ K_N\, y_N(t-\tau_N) \\ K_1\, x_1(t-\tau_1) \\ \vdots \\ K_N\, x_N(t-\tau_N) \end{pmatrix} \right].$$

Thus, if we define

$$\mathbf{K} = \mathrm{diag}(K_1, \ldots, K_N, K_1, \ldots, K_N), \qquad P_2 = \begin{pmatrix} 0 & Id \\ Id & 0 \end{pmatrix} \tag{5.28}$$

where Id has size $N \times N$, and observe that $P_2\mathbf{K} = \mathbf{K}P_2$ while noting that relation $\mathbf{K} = \mathbf{K}^* > 0$ together with the dissipativity condition (5.12) entail that $I + \mathbf{A}(t)\,\mathbf{K}$ is invertible, we obtain:

$$\begin{pmatrix} x_1(t) \\ \vdots \\ x_N(t) \\ y_1(t) \\ \vdots \\ y_N(t) \end{pmatrix} = -\left(I + \mathbf{A}(t)\,\mathbf{K}\right)^{-1} \left(I - \mathbf{A}(t)\,\mathbf{K}\right) P_2 \begin{pmatrix} x_1(t-\tau_1) \\ \vdots \\ x_N(t-\tau_N) \\ y_1(t-\tau_1) \\ \vdots \\ y_N(t-\tau_N) \end{pmatrix}. \tag{5.29}$$

Setting $d = 2N$ and letting $z(t)$ be the vector $[x_1(t), \cdots, x_N(t),\, y_1(t), \cdots, y_N(t)]^*$ and, for each $i \in \{1, \cdots, , N\}$, the $d \times d$ matrix $D_i(t)$ have the same i^{th} and $(i+N)^{\text{th}}$ columns as the matrix $-\left(I + \mathbf{A}(t)\,\mathbf{K}\right)^{-1}\left(I - \mathbf{A}(t)\,\mathbf{K}\right) P_2$, the other columns being zero, it is obvious that system (5.29) can be rewritten in the form (5.23) with $M = N$ and $\eta_i = \tau_i$, $1 \le i \le N$. As for initial conditions, we obtain from (5.26) and the last line of (5.21) that

$$x_k(t) = \frac{1}{2K_k}\left(K_k\, v_k^0(-\frac{t}{\tau_k}) + i_k^0(-\frac{t}{\tau_k})\right),\ y_k(t) = \frac{1}{2K_k}\left(K_k\, v_k^0(1+\frac{t}{\tau_k}) - i_k^0(1+\frac{t}{\tau_k})\right),\ t \in [-\tau_k, 0]\,, \tag{5.30}$$

where we note that both $-t/\tau_k$ and $1 + t/\tau_k$ range over $[0,1]$ when t ranges over $[-\tau_k, 0]$. The only difference with the situation in Theorem 5.12 is that initial values for x_k, y_k are only provided over $[-\tau_k, 0]$ through (5.26) and (5.21), not over $[-\tau_N, 0]$. However, with the previous definitions of $z(t)$ and $D_i(t)$, $1 \le i \le N$, the values of x_k and y_k on $[-\tau_N, -\tau_k)$ when $\tau_k < \tau_N$ are unimportant to the dynamics of (5.23) for $t \ge 0$, because the columns of $D_i(t)$ other than i^{th} and $(i+N)^{\text{th}}$ are identically zero. Thus, we may pick initial conditions for x_k and y_k on $[-\tau_N, -\tau_k)$ arbitrarily, provided that we comply with summability or continuity requirements. For instance, we can extend x_k and y_k to $[-\tau_N, 0]$ using the operators $J_{[-\tau_k,0]}^{[-\tau_N,0]}$ defined as follows. For $a < b < c$ three real numbers, $J_{[a,b]}^{[a,c]}$ be an extension operator mapping functions on $[a,b]$ to functions on $[a,c]$ so that $L^p([a,b])$ gets mapped into $L^p([a,c])$ and $C^0([a,b])$ into $C^0([a,c])$, in a continuous manner. Such an operator is easily constructed by choosing a smooth function $\varphi : \mathbb{R} \to \mathbb{R}$ which is 1 on $(-\infty, b]$ and 0 on $[\min\{2b-a, c\}, +\infty)$; then, for $f : [a,b] \to \mathbb{R}$, define $J_{[a,b]}^{[a,c]} f$ to be f on $[a,b]$ and $\varphi(t) f(2b-t)$ for $t \in (b,c]$, where the product is interpreted as zero if $2b - t < a$. Similarly we define $J_{[b,c]}^{[a,c]}$ mapping functions on $[b,c]$ to functions on $[a,c]$.

We have now reduced the Cauchy problem for (5.1)-(5.10), $1 \leq k \leq N$, to the Cauchy problem for a particular equation of the form (5.23). Moreover, it is obvious from what precedes that initial conditions in $L^p([0,1],\mathbb{R})$ (resp. $C^0([0,1])$ meeting (5.18)) for v_k, i_k correspond to initial conditions in $L^p([-\tau_N,0],\mathbb{R}^{2N})$ (resp. $\mathcal{C}$) for z, and that solutions v_k, i_k in $L^p_{loc}([0,\infty),L^p([0,1]))$ (resp. $C^0([0,\infty),C^0([0,1]))$) correspond to solutions $z \in L^p_{loc}([0,\infty),\mathbb{R}^{2N})$ (resp. $C^0([0,\infty),\mathbb{R}^{2N})$).

Proof of Theorem 5.7. The above discussion (starting after Theorem 5.12) shows that the function $(t,x) \mapsto (v_1(t,x),\ldots,v_N(t,x),i_1(t,x),\ldots,i_N(t,x))$ is a solution of (5.1)-(5.10)-(5.17) for Part I or (5.1)-(5.10)-(5.19) for part II if and only if

$$v_k(t,x) = x_k(t-x\tau_k) + y_k((x-1)\tau_k + t)\,, \quad i_k(t,x) = K_k\left(x_k(t-x\tau_k) - y_k((x-1)\tau_k+t)\right), \tag{5.31}$$

where $t \mapsto (x_1(t),\ldots,x_N(t),y_i(t),\ldots,y_N(t))$ is a solution of the difference delay system (5.29) in $L^p_{loc}([0,\infty),\mathbb{R}^{2N})$ or in $C^0([0,\infty),\mathbb{R}^{2N})$, with initial conditions given by (5.30) and extended if necessary to $[-\tau_N,0]$ using the operator $J^{[-\tau_N,0]}_{[-\tau_k,0]}$ constructed just before this proof. The result now follows from Theorem 5.12. □

5.2.4 Exponential stability: definitions

Definition 5.14. *Let $\mathbf{A} : [0,\infty) \to \mathbb{R}^{2N\times 2N}$ meet Assumption 5.4 (resp. meet Assumption 5.4 and be continuous). For $1 \leq p \leq \infty$, System (5.1)-(5.10) is said to be L^p (resp. C^0) exponentially stable if and only if there exist $\gamma, K > 0$ such that, for all solutions given by Theorem 5.7 part I (resp. part II), one has:*

$$\begin{aligned} &\left\|(\widehat{i}(t,\cdot),\widehat{v}(t,\cdot))\right\|_{L^p([0,1],\mathbb{R}^2)} \leq Ke^{-\gamma t}\left\|(\widehat{i}(0,\cdot),\widehat{v}(0,\cdot))\right\|_{L^p([0,1],\mathbb{R}^2)}, \text{ for all } t \geq 0 \\ &(\text{ resp. } \left\|(i(t,\cdot),v(t,\cdot))\right\|_{C^0([0,1],\mathbb{R}^2)} \leq Ke^{-\gamma t}\left\|(i(0,\cdot),v(0,\cdot))\right\|_{C^0([0,1],\mathbb{R}^2)} \quad \text{for all } t \geq 0). \end{aligned} \tag{5.32}$$

Definition 5.15. *Let the maps $t \mapsto D_i(t)$ meet assumption (5.11) (resp. meet assumption (5.11) and be continuous). System (5.23) is said to be L^p (resp. C^0) exponentially stable, $p \in [1,\infty]$, if and only if there exist $\gamma, K > 0$ such that, for all solutions given by Theorem 5.12 part (i) (resp.part (ii)), one has:*

$$\left(\int_{-\tau_N}^0 \|z(t+\theta)\|^p d\theta\right)^{1/p} \leq Ke^{-\gamma t}\left(\int_{-\tau_N}^0 \|z(\theta)\|^p d\theta\right)^{1/p}, \ t \geq 0\,, \qquad \text{if } 1 \leq p < \infty, \tag{5.33}$$

$$\operatorname*{ess\,sup}_{\theta\in[-\tau_N,0]} \|z(t+\theta)\| \leq Ke^{-\gamma t} \operatorname*{ess\,sup}_{\theta\in[-\tau_N,0]} \|z(\theta)\|, \ t \geq 0\,, \qquad \text{if } p = \infty, \tag{5.34}$$

$$(\text{resp. } \sup_{\theta\in[-\tau_N,0]} \|z(t+\theta)\| \leq Ke^{-\gamma t} \sup_{\theta\in[-\tau_N,0]} \|z(\theta)\|, \ t \geq 0\ ). \tag{5.35}$$

Our main concern in this paper is the exponential stability of system (5.1)-(5.10), but we shall need the equivalent formulation as a difference delay system of the form (5.29), which is a particular case of (5.23). In fact, exponential stability of the two systems are equivalent properties, as asserted by the following proposition.

Proposition 5.16. *System (5.1)-(5.10) is L^p exponentially stable (resp. C^0 exponentially stable) if and only if System (5.29) is L^p exponentially stable (resp. C^0 exponentially stable), $1 \leq p \leq \infty$.*

Proof. This follows at once from (5.31) expressing solutions of (5.1)-(5.10) from solutions of (5.29) and vice-versa. □

5.3 Results

5.3.1 Known results in the time-invariant case

The exponential stability of difference delay systems like (5.23) when the D_i are constant matrices is well understood. Indeed, the following necessary and sufficient condition is classical.

Theorem 5.17 (Henry-Hale Theorem, [Hen74, HVL93])**.** *If the matrices D_i in system (5.23) do not depend on t, the following properties are equivalent.*

i. System (5.23) is L^p exponentially stable for all $p \in [1, +\infty]$.

ii. System (5.23) is C^0 exponentially stable.

iii. There exists $\beta < 0$ for which

$$Id - \sum_{i=1}^{N} D_i \, e^{-\lambda \tau_i} \text{ is invertible for all } \lambda \in \mathbb{C} \text{ such that } \Re(\lambda) > \beta. \tag{5.36}$$

Theorem 5.17 is usually stated for C^0 exponential stability only. However, the proof yields L^p exponential stability as well for $1 \leq p \leq \infty$, see the discussion after [CN15, eq. (1.11)]. To study the stability of time-invariant networks of 1-dimensional hyperbolic systems, it is standard to convert them into a delay system, much like we did in the previous section, and to apply Theorem 5.17. There is a sizeable literature on this topic, dealing with more general equations with conservation laws than telegrapher's ones, see for instance the textbook [BC16] and references therein.

For systems of the form (5.29), if we assume on top of the dissipativity condition (5.12) that the coupling matrix $\mathbf{A}(t)$ in fact does not depend on t, then Theorem 5.17 applies to yield exponential stability. This is the content of the following proposition, whose (elementary) proof is given in section 5.4.2 for completeness:

Proposition 5.18. *If the matrix $\mathbf{A}(t)$ is constant and condition (5.12) holds, then the constant matrices D_i obtained when putting (5.29) into the form (5.23) satisfy (5.36) for some $\beta < 0$.*

5.3.2 Sufficient stability condition in the time-varying case

Unfortunately, there is no generalization of the Henry-Hale theorem to time-varying difference delay systems of the form (5.23), even if we assume that the $D_i(t)$ are periodic with the same period, as is the case in the application to electrical networks outlined in the introduction. To the best of our knowledge, there are very few results on the stability of such systems; let us mention two. One is [CN15, Lemma 3.2]. It gives exponential stability results in Sobolev norms for the class of time-varying delay systems (5.23) which come from 1-D hyperbolic equations, where the matrices $D_i(\cdot)$ are continuously differentiable and the delays may be time-dependent. Another, extensive reference is [CMS16], which gives a necessary and sufficient condition for L^p exponential stability when $1 \leq p \leq +\infty$ that obviously remains valid for C^0 exponential stability as well. It is stated in terms of the boundedness of sums of products of the $D_i(t_j)$ at delayed time intants t_j, where the number of terms in the sums and products can be arbitrary large. This is akin to an expression of the solution to (5.23) in terms of the matrices $D_i(.)$ and the initial conditions (see (5.61) and (5.62) further below), which looks difficult to bound efficiently in practice because of the tremendous combinatorics and the many cancellations that can occur. In contrast, we only deal here with telegrapher's equations, or with difference delay systems that can be recast as such, but Assumption 5.4 is a much more manageable sufficient condition for exponential stability.

The main result of the paper —see Theorems 5.21 and 5.22 below— asserts L^p exponential stability for all $p \in [1, \infty]$, as well as C^0 exponential stability, for networks of telegrapher's equations with time-varying coupling conditions of the form or (5.1)-(5.10) (or (5.1)-(5.5)) under Assumption 5.4 (dissipativity at the nodes), and for difference delay systems (5.23) under conditions that imply that they can be put in the form (5.29) with the same dissipativity conditions.

It may be interesting to note that the sufficient condition for stability that we give here is independent of the delays when speaking of a difference delay system (Theorem 5.22) or independent of the caracteristics of the lines (constants C_k and L_k) when speaking of networks of telegrapher's equations (Theorem 5.21). Also, these sufficient conditions are not claimed to be necessary.

Let us state these results, preceded by some auxiliary results of independent interest. The proofs not given right after the theorems can be found in Sections 5.4.3 through 5.4.6.

The first step is to establish L^2 exponential stability of System (5.1)-(5.10) asserted in the following theorem. We give in Section 5.4.3 a proof using a natural energy functional as Lyapunov function for the telegrapher equations (5.1). Condition (5.12) in Assumption 5.4, which has been termed dissipativity without much explanation so far, expresses dissipativity in the sense of this energy functional. We also sketch, in Section 5.4.4, a second proof, ellaborating on [CN15, Lemma 3.2], which is exclusively based on the time-varying delay system (5.29); see the remark at the end of Section 5.4.4 for a comparison of the two proofs.

Theorem 5.19. *Under Assumption 5.4, the time-varying network of telegrapher's equations (5.1)-(5.10)-(5.8) is L^2 exponentially stable.*

In view of Proposition 5.16, we get as a corollary that L^2 exponential stability holds for difference delay systems of the form (5.29).

To deduce from Theorem 5.19 L^p exponential stability for all p, we rely on the following result.

Theorem 5.20. *Under Assumption 5.11, System (5.23) is L^p exponentially stable for some $p \in [1, \infty]$ if and only if it is L^p exponentially stable for all such p. Moreover, if the maps $t \mapsto D_i(t)$ are continuous, then this is also if and only if System (5.23) is C^0 exponentially stable.*

The only original bit here is that C^0 exponential stability implies L^p exponential stability for all p, because the first assertion of Theorem 5.20 is essentially contained in [CMS16, Corollary 3.29]. We do consider C^0 stability, because it is the natural one in the application to electronic circuits mentioned in the introduction. Although, again, the first assertion is a consequence of [CMS16, Cor. 3.29], we nevertheless give an independent proof in Section 5.4.5. Indeed, we feel our argument is simpler than in [CMS16] (the latter paper contains of course other results), and of independent interest. Moreover, our proof shows (for better readability it is not stated in the theorem) that if System (5.23) is L^p (resp. C^0) polynomially stable of degree $m > N$ for some $p \in [1, \infty]$ (*i.e.* if (5.33) or (5.34) (resp. (5.35)) holds with $e^{-\gamma t}$ replaced by $(1+t)^{-m}$), then it is L^p polynomially stable of degree 1 for all such p (and also C^0 polynomially stable of degree 1).

An obvious corollary of Theorem 5.20, based on Proposition 5.16, is that System (5.1)-(5.10) (network of telegrapher's equations) is L^p exponentially stable for some $p \in [1, \infty]$ if and only if it is C^0 exponentially stable and also L^q exponentially stable for all $q \in [1, \infty]$. This leads to our main result regarding network of telegrapher's equations:

Theorem 5.21. *Under Assumption 5.4, the time-varying network of telegrapher's equations (5.1)-(5.10)-(5.8) is L^p exponentially stable for $1 \leq p \leq \infty$, and also C^0 exponentially stable if the maps $t \mapsto \mathbf{A}(t)$ are continuous.*

Proof. This is a straightforward consequence of Theorem 5.19 and the "obvious corollary" to Theorem 5.20 mentioned just before Theorem 5.21. □

A direct consequence of Theorem 5.21 and Proposition 5.16 is that the same stability properties hold for difference delay systems of the special form (5.29). It is interesting to restate this in terms of general delay systems of the form (5.23), making additional assumptions to fall under the scope of the previous result. This is the purpose of Theorem 5.22 below, whose proof is given in Section 5.4.6. Recall that the matrices $\widehat{D}_j(t)$ were defined from the matrices $D_i(t)$ in Remark 5.13 (*cf.* (5.25)), and that they differ from the D_i only when some of the delays η_i appear with repetition in (5.23)).

Theorem 5.22. *Under Assumption 5.11, if Conditions (i) and (ii) below are satisfied, then the time-varying difference delay system (5.23) is L^p exponentially stable for all $p \in [1,\infty]$. Moreover, if the maps $t \mapsto D_i(t)$ are continuous, then it is also C^0 exponentially stable.*

i. The columns of the matrices $\widehat{D}_j(t)$ are disjoint, i.e. there is a partition $\{1,\dots,d\} = \mathcal{I}_1 \cup \cdots \cup \mathcal{I}_{\widehat{M}}$ (with $i \neq j \Rightarrow \mathcal{I}_i \cap \mathcal{I}_j = \varnothing$) such that the k^{th} column of $\widehat{D}_j(t)$ is identically zero if $k \notin \mathcal{I}_j$.

ii. The sum of the matrices $D_i(t)$ is uniformly contractive:

$$\text{there is a number } \nu < 1 \text{ such that } \left|\!\left|\!\left| \sum_{i=1}^{M} D_i(t) \right|\!\right|\!\right| \leq \nu \text{ for almost all positive } t\,.$$

Here, $|\!|\!|\cdot|\!|\!|$ is the spectral norm for matrices associated to the Euclidean norm on $\mathbb{R}^d$, like in section 5.2.2.

To recap, Theorem 5.21 offers a sufficient condition for exponential stability of networks of coupled telegrapher's equations, relevant to the study of oscillations in circuits with transmission lines as explained in Section 5.1, while Theorem 5.22 deals with difference delay systems and applies to an admittedly narrow class thereof (the disjoint columns assumption is clearly restrictive), but is still worth stating for it points at a class of time-varying systems for which relatively simple sufficient conditions for exponential stability can be given. These results are apparently first to give fairly manageable sufficient conditions for exponential stability in the time-varying case. Another contribution is the somewhat simpler approach, provided by Theorem 5.20 and its proof, to the fact that all types of L^p exponential stability, $1 \leq p \leq \infty$, are equivalent for general time-varying difference delay systems.

5.4 Proofs

5.4.1 A technical lemma

Here, the superscript $*$ denotes the transpose of a real matrix, and the spectral norm $|\!|\!|\cdot|\!|\!|$ defined at the beginning of section 5.2.2 is with respect to the canonical Euclidean norm $\|x\| = (x^*x)^{1/2}$.

Lemma 5.23. *If Q is a square matrix satisfying $Q + Q^* > \kappa > 0$, there is a unique square matrix R solution of*

$$(Id + Q)R = Id - Q\,, \tag{5.37}$$

and it satisfies $|\!|\!|R|\!|\!| < (1-\kappa)/(1+\kappa) < 1$. Conversely, if R is a square matrix satisfying $|\!|\!|R|\!|\!| < 1$, there is a unique square matrix Q solution of (5.37) and it satisfies $Q + Q^* \geq \frac{1-|\!|\!|R|\!|\!|}{1+|\!|\!|R|\!|\!|}\, Id$.

Proof. It is clear that -1 cannot be an eigenvalue of Q if $Q + Q^* > 0$ or an eigenvalue of R if $|\!|\!|R|\!|\!| < 1$. This allows to solve for R or Q using the inverse of $Id + Q$ or $Id + R$.

Now suppose that R and Q satisfy (5.37). Then $(Id + Q)(Id + R) = 2\,Id$, hence both $Id + Q$ and $Id + R$ are invertible and R commutes with Q, (5.37) can be re-written $R(Id + Q) = Id - Q$ that readily implies

$$\frac{\|R\,(Id + Q)y\|^2}{\|(Id + Q)y\|^2} = 1 - 2\,\frac{y^*(Q + Q^*)y}{\|(Id + Q)y\|^2}$$

for any nonzero y. Since on the one hand, using invertibility of $Id+Q$, $|||R||| < 1$ if and only if the left-hand side is less than 1 for any nonzero y and on the other hand the right-hand side is less than one if and only if $y^*(Q+Q^*)y$ is positive, one deduces that $Q+Q^* > 0$ and $|||R||| < 1$ are equivalent. □

5.4.2 Proof of Proposition 5.18

From the very definition of D_i in terms of $\mathbf{A}$, $\mathbf{K}$ and P_2 —see discussion after (5.29)— we get that

$$\sum_{i=1}^{N} D_i\, e^{-\lambda\tau_i} = [Id+\mathbf{AK}]^{-1}[Id-\mathbf{AK}]\, P_2\, \mathrm{diag}(e^{-\lambda\tau_1},\dots,e^{-\lambda\tau_N},e^{-\lambda\tau_1},\dots,e^{-\lambda\tau_N})\,. \tag{5.38}$$

In view of (5.12), (5.28) and the strict positivity of the K_j, it holds if we set $Q=\mathbf{K}^{1/2}\mathbf{AK}^{1/2}$ that $Q+Q^*\geq\tilde{\alpha}Id$ with $\tilde{\alpha}=\alpha\min_{1\leq j\leq N}K_j>0$, hence Lemma 6.4 gives us:

$$\left|\left|\left|(Id+\mathbf{K}^{1/2}\mathbf{AK}^{1/2})^{-1}(Id-\mathbf{K}^{1/2}\mathbf{AK}^{1/2})\right|\right|\right| < 1. \tag{5.39}$$

Consider now the $\mathbf{K}$-norm on $\mathbb{R}^{2N}$, defined for $x\in\mathbb{R}^{2N}$ by $\|x\|_{\mathbf{K}}=\|\mathbf{K}^{\frac{1}{2}}x\|$. Clearly, for any $2N\times 2N$ complex matrix B, the corresponding operator norm is $|||B|||_{\mathbf{K}}=\left|\left|\left|\mathbf{K}^{\frac{1}{2}}B\mathbf{K}^{-\frac{1}{2}}\right|\right|\right|$; it is obviously multiplicative.

Since $[Id+\mathbf{AK}]^{-1}[Id-\mathbf{AK}]=\mathbf{K}^{-\frac{1}{2}}(Id+\mathbf{K}^{\frac{1}{2}}\mathbf{AK}^{\frac{1}{2}})^{-1}(Id-\mathbf{K}^{\frac{1}{2}}\mathbf{AK}^{\frac{1}{2}})\mathbf{K}^{\frac{1}{2}}$, equation (5.39) implies that

$$\left|\left|\left|[Id+\mathbf{AK}]^{-1}[Id-\mathbf{AK}]\right|\right|\right|_{\mathbf{K}} < 1, \tag{5.40}$$

consequently there is $\beta<0$ such that

$$\left|\left|\left|[Id+\mathbf{AK}]^{-1}[Id-\mathbf{AK}]\right|\right|\right|_{\mathbf{K}}\, e^{-\beta\,\tau_N} < 1\,. \tag{5.41}$$

To see that (5.36) holds for this β, pick $\lambda\in\mathbb{C}$ with $\Re(\lambda)>\beta$ and observe that

$$\left|\left|\left|P_2\,\mathrm{diag}(e^{-\lambda\tau_1},\dots,e^{-\lambda\tau_N},e^{-\lambda\tau_1},\dots,e^{-\lambda\tau_N})\right|\right|\right|_{\mathbf{K}} \leq e^{-\beta\tau_N}$$

by (5.3), the multiplicativity of the $\mathbf{K}$-norm and the fact that P_2 commute with $\mathbf{K}^{1/2}$. Hence, using (5.38) and (5.41) together with the multiplicativity of the $\mathbf{K}$-norm, we see that $\left|\left|\left|\sum_{i=1}^{N} D_i\, e^{-\lambda\tau_i}\right|\right|\right|_{\mathbf{K}} < 1$ which implies (5.36). □

5.4.3 Proof of Theorem 5.19 *via* a Lyapunov functional approach

Let $(v_1(t,x),\dots,v_N(t,x),i_1(t,x),\dots,i_N(t,x))\in L^1_{loc}([0,\infty),(L^2([0,1]))^{2N})$ be the solution to (5.1)-(5.10)-(5.8) set forth in Part I of Theorem 5.7, with initial condition $i_k^0,v_k^0\in L^2([0,1])$ for $1\leq k\leq N$. We define the energy functional E_k in the line k and the global energy E by

$$E_k(t)=\frac{1}{2}\int_0^1\left[C_k v_k^2(t,x)+L_k i_k^2(t,x)\right]dx\,,\qquad E(t)=\sum_{k=1}^{N}E_k(t)\,. \tag{5.42}$$

Fact. Each function E_k is locally absolutely continuous and its derivative is given by:

$$\frac{d}{dt}E_k(t)=-\widehat{i}_k(t,1)\widehat{v}_k(t,1)+\widehat{i}_k(t,0)\widehat{v}_k(t,0),\qquad \text{a.e. } t. \tag{5.43}$$

Proof of the Fact. This would be easy if the solution were smooth (differentiating under the integral sign and using (5.1) would readily yield (5.43)), but we have only proved so far, according to Remark 5.10, that E_k is continuous $[0,+\infty) \to \mathbb{R}$ for each k. In particular it defines a distribution on $(0,+\infty)$; let us compute the derivative of this distribution by approximation. By Proposition 5.9, points (i)-(ii), the functions v_k, i_k are of the form (5.20) with $f_k \in L^2_{loc}((-\infty,1])$ and $g_k \in L^2_{loc}([0,\infty))$. Let $\check{f}_k$ and $\check{g}_k$ extend f_k and g_k by zero to the whole of $\mathbb{R}$, and pick $\phi : \mathbb{R} \to \mathbb{R}$ a positive, C^∞-smooth function, supported on $[-1,1]$ and such that $\int_{\mathbb{R}} \phi = 1$. For each $\varepsilon > 0$, we set $\phi_\varepsilon(x) := \phi(x/\varepsilon)/\varepsilon$ (hence, $\int_{\mathbb{R}} \phi_\varepsilon = 1$) and define

$$\widetilde{f}_{k,\varepsilon}(s) := \int_{\mathbb{R}} \check{f}(y)\phi_\varepsilon(s-y)\,dy, \qquad \widetilde{g}_{k,\varepsilon}(s) := \int_{\mathbb{R}} \check{g}(y)\phi_\varepsilon(s-y)\,dy, \qquad y \in \mathbb{R}. \tag{5.44}$$

Thus, $\widetilde{f}_{k,\varepsilon}$ is a C^∞ smooth functions in $L^2_{loc}(\mathbb{R})$ satisfying $\|\widetilde{f}_{k,\varepsilon}\|_{L^2(K)} \le \|f_k\|_{L^2(K+[-\varepsilon,\varepsilon])}$ for any compact $K \subset \mathbb{R}$, and similarly for $\widetilde{g}_{k,\varepsilon}$. Moreover, $\widetilde{f}_{k,\varepsilon}$ and $\widetilde{g}_{k,\varepsilon}$ converge, both pointwise a.e. and in $L^2_{loc}(\mathbb{R})$, respectively to $\check{f}_k$ and $\check{g}_k$, when $\varepsilon \to 0$. Indeed, it is enough to check this on an arbitrary compact set $K \subset \mathbb{R}$, and since ϕ_ε is supported on $[-\varepsilon,\varepsilon]$ we may redefine $\check{f}_k$ and $\check{g}_k$ as being zero outside the compact set $K + [-\varepsilon,\varepsilon]$ without changing the values of $\widetilde{f}_{k,\varepsilon}$ nor $\widetilde{g}_{k,\varepsilon}$ on K. Thus, it is enough to prove the desired pointwise and L^2_{loc} convergence when $\check{f}_k$ and $\check{g}_k$ lie in $L^2(\mathbb{R})$, in which case the result is standard [Ste70, ch. III, thm. 2]. Next, let us put

$$\widetilde{v}_{k,\varepsilon}(t,x) := \widetilde{f}_{k,\varepsilon}(x - \frac{t}{\tau_k}) + \widetilde{g}_{k,\varepsilon}(x + \frac{t}{\tau_k}), \quad \widetilde{i}_{k,\varepsilon}(t,x) = K_k \left(\widetilde{f}_{k,\varepsilon}(x - \frac{t}{\tau_k}) - \widetilde{g}_{k,\varepsilon}(x + \frac{t}{\tau_k}) \right), \tag{5.45}$$

so that $\widetilde{v}_{k,\varepsilon}$ and $\widetilde{i}_{k,\varepsilon}$ lie in $L^2_{loc}(\mathbb{R}^2)$ and are C^∞ smooth solutions to (5.1) on $\mathbb{R}^2$, by Proposition 5.9 point (iii). Because $(t,x) \mapsto (x - t/\tau_k, x + t/\tau_k)$ is a bi-Lipschitz homeomorphism of $\mathbb{R}^2$, it preserves compact sets and sets of measure zero. Thus, since $\check{f}_k$ and $\check{g}_k$ coincide respectively with f_k and g_k on $[0,+\infty) \times [0,1]$, the properties of $\widetilde{f}_{k,\varepsilon}$ and $\widetilde{g}_{k,\varepsilon}$ indicated after (5.44) imply that $\widetilde{v}_{k,\varepsilon}$, $\widetilde{i}_{k,\varepsilon}$ respectively converge pointwise a.e. to v_k, i_k on $[0,+\infty) \times [0,1]$, in such a way that $\|\widetilde{v}_{k,\varepsilon}(t,.)\|_{L^2([0,1])}$ and $\|\widetilde{i}_{k,\varepsilon}(t,.)\|_{L^2([0,1])}$ remain essentially bounded with t. Therefore, by dominated convergence, we get for every C^∞ smooth compactly supported function $\psi : (0,+\infty) \to \mathbb{R}$ that

$$\lim_{\varepsilon \to 0} \int_0^{+\infty} \int_0^1 [C_k \widetilde{v}^2_{k,\varepsilon}(t,x) + L_k \widetilde{i}^2_{k,\varepsilon}(t,x)]\psi(t)\,\mathrm{d}t\,\mathrm{d}x = \int_0^{+\infty} \int_0^1 [C_k v_k^2(t,x) + L_k i_k^2(t,x)]\psi(t)\,\mathrm{d}t\,\mathrm{d}x.$$

In other words: when $\varepsilon \to 0$, then $\widetilde{E}_{k,\varepsilon}(t) := \int_0^1 [C_k \widetilde{v}^2_{k,\varepsilon}(t,x) + L_k \widetilde{i}^2_{k,\varepsilon}(t,x)]dx$ converges to $E_k(t)$, as a distribution on $(0,+\infty)$. Now, since $\widetilde{i}_{k,\varepsilon}$ and $\widetilde{v}_{k,\varepsilon}$ are smooth, the derivative of $t \mapsto \widetilde{E}_{k,\varepsilon}(t)$ can be computed in the strong sense by differentiating under the integral sign; since $\widetilde{i}_{k,\varepsilon}$ and $\widetilde{v}_{k,\varepsilon}$ are solutions of the telegrapher's equation (5.1), an elementary integration yields:

$$\frac{d}{dt}\widetilde{E}_{k,\varepsilon}(t) = -\widetilde{i}_{k,\varepsilon}(t,1)\widetilde{v}_{k,\varepsilon}(t,1) + \widetilde{i}_{k,\varepsilon}(t,0)\widetilde{v}_{k,\varepsilon}(t,0). \tag{5.46}$$

By (5.45) and the Schwarz inequality, the properties of $\widetilde{f}_{k,\varepsilon}$ and $\widetilde{g}_{k,\varepsilon}$ indicated after (5.44) imply that the right hand side of (5.46) converges pointwise a.e. and in $L^1_{loc}(\mathbb{R})$ to the function

$$F(t) := K_k \left(f_k^2(-\frac{t}{\tau_k}) - g_k^2(\frac{t}{\tau_k}) - f_k^2(1 - \frac{t}{\tau_k}) + g_k^2(1 + \frac{t}{\tau_k}) \right),$$

and since we know that $\frac{d}{dt}\widetilde{E}_{k,\varepsilon}$ converges to $\frac{d}{dt}E_k$ as a distribution we conclude that $\frac{d}{dt}E_k = F$. In particular, since E_k is a distribution in dimension 1 whose derivative is a locally integrable function, [DD07, thm. 6.74] implies local absolute continuity and we get from what precedes that $\frac{d}{dt}E_k(t) = F(t)$ for a.e. t, which can be rewritten as (5.43) in view of (5.21). This ends the proof of the above fact.

Proof of Theorem 5.19. Adding equalities (5.43) for $1 \leq k \leq N$ and considering (5.42) together with the boundary conditions (5.16) yields the following equation, where one may indifferently use $\frac{1}{2}\left(\mathbf{A}(t)+\mathbf{A}(t)^*\right)$ or $\mathbf{A}(t)$:

$$\frac{d}{dt}E(t) = -\begin{pmatrix} -\widehat{\imath}_1(t,0) \\ \vdots \\ -\widehat{\imath}_N(t,0) \\ \widehat{\imath}_1(t,1) \\ \vdots \\ \widehat{\imath}_N(t,1) \end{pmatrix}^* \frac{\mathbf{A}(t)+\mathbf{A}(t)^*}{2} \begin{pmatrix} -\widehat{\imath}_1(t,0) \\ \vdots \\ -\widehat{\imath}_N(t,0) \\ \widehat{\imath}_1(t,1) \\ \vdots \\ \widehat{\imath}_N(t,1) \end{pmatrix}, \qquad \text{a.e. } t, \tag{5.47}$$

Using the dissipativity condition (5.12) in (5.47) readily implies:

$$\frac{d}{dt}E(t) \leq -\frac{\alpha}{2}\sum_{k=1}^{N}\left[\widehat{\imath}_k^2(t,0)+\widehat{\imath}_k^2(t,1)\right], \qquad \text{a.e. } t. \tag{5.48}$$

This entails that the global energy E is decreasing. In order to show that it tends to zero exponentially, let us express E in terms of the functions f_k, g_k as follows. Substituting (5.20) in (5.42), we get since $L_kK_k^2 = C_k$ that

$$E_k(t) = C_k\left(\int_0^1 g_k^2(x+\frac{t}{\tau_k})dx + \int_0^1 f_k^2(x-\frac{t}{\tau_k})dx\right). \tag{5.49}$$

Changing variables to $\tau = x\tau_k + t$ in the first integral and to $\tau = (1-x)\tau_k + t$ in the second, we obtain:

$$E_k(t) = C_k\int_t^{t+\tau_k}\left(g_k^2(\frac{\tau}{\tau_k}) + f_k^2(1-\frac{\tau}{\tau_k})\right)d\tau. \tag{5.50}$$

Thus, if we define $G:(0,+\infty)\to\mathbb{R}$ by $G(\tau) := \sum_{k=1}^{N} C_k(g_k^2(\tau/\tau_k) + f_k^2(1-\tau/\tau_k))$, we deduce from (5.50) that

$$E(t) \leq \int_t^{t+\tau_N} G(\tau)\,d\tau. \tag{5.51}$$

In another connection, we get from (5.20) that $G(\tau)$ can be expressed as a non-negative quadratic form in the $4N$ variables $\widehat{v}_k(\tau,0)$, $\widehat{v}_k(\tau,1)$, $\widehat{\imath}_k(\tau,0)$, $\widehat{\imath}_k(\tau,1)$, for $1\leq k\leq N$, with constant coefficients. Hence, using (5.16) to substitute the $\widehat{v}_k$'s for the $\widehat{\imath}_k$'s, the same $G(\tau)$ can be expressed as a non-negative quadratic form in the $2N$ variables $\widehat{\imath}_k(\tau,0)$, $\widehat{\imath}_k(\tau,1)$, for $1\leq k\leq N$, with time-varying essentially bounded coefficients (Assumption 5.4) depending on the matrix $\mathbf{A}(\tau)$ and the constants K_k. This implies:

$$G(\tau) \;\leq\; \widetilde{a}\sum_{k=1}^{N}\left(\widehat{\imath}_k^2(\tau,0)+\widehat{\imath}_k^2(\tau,1)\right), \qquad \text{a.e. } \tau>0. \tag{5.52}$$

with a positive constant $\widetilde{a}$ that depends only on the coefficients K_k and the bounds on the coefficients of $\mathbf{A}(.)$. Using this inequality in (5.48) yields

$$\frac{d}{d\tau}E(\tau) \leq -\frac{\alpha}{2\,\widetilde{a}}\,G(\tau), \qquad \text{a.e. } \tau>0. \tag{5.53}$$

Integrating (5.53) between t and $t+\tau_N$ we gather, in view of (5.51), that

$$0 \leq E(t+\tau_N) \leq (1-\frac{\alpha}{2\,\widetilde{a}})\,E(t), \qquad t>0. \tag{5.54}$$

Comparing the expression of $E(t)$ in (5.42), this last inequality readily implies that system (5.1)-(5.10) is L^2 exponentially stable. □

5.4.4 Sketch of an alternative proof of Theorem 5.19 *via* time-delay systems exclusively

First note that, from a straightforward generalization of Equation (5.40) using the fact that P_2 and $\mathbf{K}$ commute, there is a $\gamma \in (0,1)$ independent of $t \in \mathbb{R}$ such that :

$$\left|\!\left|\!\left| [Id + \mathbf{A}(t)\mathbf{K}]^{-1}[Id - \mathbf{A}(t)\mathbf{K}]\, P_2 \right|\!\right|\!\right|_{\mathbf{K}} \leq \gamma < 1. \tag{5.55}$$

By applying $\|\cdot\|^2_{\mathbf{K}}$ to each side of Equation (5.29), using (5.55) above and integrating the resulting inequality between t and t_2 ($-\tau_N < t < t_2$), one gets after simple algebraic manipulation the following inequality, valid for any $t > 0$ and $t_2 > t$:

$$\int_t^{t_2} \|z(s)\|^2_{\mathbf{K}} \mathrm{d}s \leq \frac{1}{1-\gamma^2} \int_t^{t+\tau_N} \|z(s)\|^2_{\mathbf{K}} \mathrm{d}s\,, \tag{5.56}$$

in which one may then take $t_2 = +\infty$. This implies L^2 exponential stability of system (5.29) (one first proves that, for T large enough, $\int_t^{t+T} \|z(s)\|^2_{\mathbf{K}} \mathrm{d}s$ converges exponentially to zero) and thus L^2 exponential stability of system (5.1)-(5.10) *via* the equivalence between stability of the difference delay system and of the PDE network, see Proposition 5.16.

Remark 5.24. *The above proof expounds that of [CN15, Lemma 3.2], but in essence is not so different from the previous one. Indeed, the quantity* $\int_t^{t+T} \|z(s)\|^2_{\mathbf{K}} \mathrm{d}s$ *acts as a Lyapunov function for* (5.29)*, although it is not proved be non-increasing with respect to continuous time, while* E*, that has a similar expression in terms of the delay system (see* (5.49)*), is a Lyapunov function in the usual sense for the network of telegrapher's equations, see* (5.48)*.*

5.4.5 Proof of Theorem 5.20

Before proceeding with the proof, we take a closer look at the structure of solutions to System (5.23).

Given the ordered collection of delays $0 < \tau_1 \leq \tau_2 \leq \ldots \leq \tau_N$, we define the following subsets of $\mathbb{R}$:

$$\Sigma = \{\sum_{i=1}^{N} q_i \tau_i\,,\ (q_1, \ldots, q_N) \in \mathbb{N}^N\} \quad \text{and} \quad \Sigma_t = [0,t] \cap \Sigma \quad \text{for } t \text{ in } [0, +\infty). \tag{5.57}$$

Call $Q(t) \in \mathbb{N}$ the cardinality of Σ_t. Clearly, $Q(t)$ is no larger than the number of N-tuples $(q_1, \ldots, q_N) \in \mathbb{N}^N$ satisfying $\sum_{i=1}^N q_i \leq t/\tau_1$, and the latter is bounded from above by $(1 + [[t/\tau_1]])^N$, where $[[r]]$ indicates the integer part of the real number r. Hence, we have that

$$Q(t) \leq \left(1 + \frac{t}{\tau_1}\right)^N, \qquad t \in [0, +\infty). \tag{5.58}$$

We enumerate the elements of Σ as a sequence $0 = \sigma_1 < \sigma_2 < \sigma_3 \cdots$, so that Σ_t is described as:

$$\Sigma_t = \{\sigma_1, \sigma_2, \ldots, \sigma_{Q(t)}\}\,, \qquad t \in [0, +\infty)\,. \tag{5.59}$$

Our proof of Theorem 5.20 will dwell on the following observation.

Fact. There is a collection of maps $(M_q)_{q\in\mathbb{N}}$ from $\mathbb{R}$ into $\mathbb{R}^{d\times d}$ enjoying properties (*i*), (*ii*), (*iii*) below.

i. The map M_q lies in $L^\infty_{loc}(\mathbb{R}, \mathbb{R}^{d\times d})$,

ii. M_q satisfies

$$t \notin (\sigma_q - \tau_N, \sigma_q] \ \Rightarrow\ M_q(t) = 0\,, \tag{5.60}$$

iii. the solution $t \mapsto z(t)$ of (5.23) with initial condition $z(t') = \phi(t')$, $t' \in [-\tau_N, 0]$, is given by

$$z(t) = \sum_{q=1}^{Q(t+\tau_N)} M_q(t)\, \phi(t-\sigma_q) = \sum_{q=1}^{+\infty} M_q(t)\, \phi(t-\sigma_q), \qquad t \geq 0. \tag{5.61}$$

Formula (5.61) applies equally well to continuous solutions and to solutions in $L^p_{loc}([0,\infty), \mathbb{R}^d)$, but in the latter case the equality is understood for almost every t. Note that (5.60) ensures that the two sums in (5.61) are equal, and also that they do not depend on the values $\phi(y)$ for $y \notin [-\tau_N, 0]$ (which are not defined).

Proof of the Fact. For $0 \leq t < \tau_1$, Equation (5.23) is of the form (5.61), with $M_q(t) = D_i(t)$ if $\sigma_q = \tau_i$ and $M_q(t) = 0$ otherwise. If we assume inductively such a formula for $0 \leq t < \sigma_{q_0}$ and substitute it in the right hand side of (5.23) when $\sigma_{q_0} \leq t < \sigma_{q_0+1}$ to express $z(t)$ as a linear combination of the $\phi(t-\sigma_q)$ for $-\tau_N \leq t - \sigma_q < 0$, a moment's thinking will convince the reader that we get a formula of the same type over the interval $\sigma_{q_0} \leq t < \sigma_{q_0+1}$ by defining $M_q(t)$ as the sum of the coefficients corresponding, after the above substitution, to one and the same $\phi(t-\sigma_q)$ (the latter may arise as many times as there are decompositions $\sigma_q = \tau_i + \sigma_{q'}$ with $i \in \{1, \cdots, N\}$ and $\sigma_{q'} \in \Sigma_{\sigma_{q_0}+\tau_N}$. Such coefficients are of the form $D_i(t)M_{q'}(t)$, and therefore properties (i), (ii) and (iii) are obviously met.

Although we will not need this, it is instructive to derive an explicit expression for M_q that should be compared with [CMS16, thm. 3.14] or, in the continuous case, with [HVL93, ch. 9, eqns. (1.4)-(1.5)]. Namely, we can take $M_q(t)$ to be the sum of all terms

$$\mathbb{1}_{(\rho_{s-1},\rho_s]}(t)\, D_{k_1}(t) D_{k_2}(t-\rho_1) D_{k_3}(t-\rho_2) \cdots D_{k_s}(t-\rho_{s-1}) \tag{5.62}$$

for all s in $\mathbb{N} \setminus \{0\}$ and all s-tuples $(k_1, \ldots, k_s) \in \{1, \cdots, N\}^s$ such that $\sum_{j=1}^s \tau_{k_j} = \sigma_q$, where the numbers ρ_j are defined by $\rho_0 = 0$ and $\rho_j = \sum_{i=1}^j \tau_{k_i}$ for $j \geq 1$ (in particular $\rho_s = \sigma_q$), and $\mathbb{1}_{(\rho_{s-1},\rho_s]}$ is the characteristic function of the interval $(\rho_{s-1}, \rho_s]$. These maps M_q satisfies (5.60) because $(\rho_{s-1}, \rho_s] = (\sigma_q - \tau_{k_s}, \sigma_q]$ is a subset of $(\sigma_q - \tau_N, \sigma_q]$, and formula (5.61) is easily checked from (5.23), by induction on j such that $t \in (\sigma_{j-1}, \sigma_j]$.

Proof of Theorem 5.20. Assume first that $1 \leq p < \infty$. If System (5.23) is L^p exponentially stable, there is by definition $\gamma > 0$ and $C_0 > 0$ such that, for all $\phi \in L^p([-\tau_N, 0], \mathbb{R}^d)$ and all $t > 0$, one has

$$\Big(\int_{t-\tau_N}^t \|z(u)\|^p du \Big)^{1/p} \leq C_0\, e^{-\gamma t} \|\phi\|_{L^p([-\tau_N,0],\mathbb{R}^d)} \tag{5.63}$$

for $z(.)$ the unique solution of (5.23) with initial condition ϕ given by Theorem 5.12. Pick $t_\star \in (-\tau_N, 0)$, $v \in \mathbb{R}^d$, $\varepsilon > 0$, and define a function $\phi_{t_\star,v,\varepsilon} \in L^p([-\tau_N, 0], \mathbb{R}^d)$ by

$$\phi_{t_\star,v,\varepsilon}(\theta) = \frac{1}{\varepsilon^{1/p}} \mathbb{1}_{(t_\star-\varepsilon, t_\star)}(\theta)\, v\,, \qquad \theta \in [-\tau_N, 0]. \tag{5.64}$$

Let $z_{t_\star,v,\varepsilon}(.)$ be the solution to (5.23) with initial condition $\phi_{t_\star,v,\varepsilon}$ on $[-\tau_N, 0]$. By (5.63), it holds that

$$\Big(\int_{t-\tau_N}^t \|z_{t_\star,v,\varepsilon}(u)\|^p du \Big)^{1/p} \leq C_0\, e^{-\gamma t} \|v\|\,, \qquad t > 0, \tag{5.65}$$

and from (5.61) we get for all $u > 0$ that

$$z_{t_\star,v,\varepsilon}(u) \;=\; \frac{1}{\varepsilon^{1/p}} \left(\sum_{q=1}^{+\infty} \mathbb{1}_{(t_\star-\varepsilon,t_\star)}(u-\sigma_q)\; M_q(u) \right) v. \tag{5.66}$$

Let us fix $t>0$ for a while. By (5.60), the only terms in the sum on the right of (5.66) which may not be zero for a.e. $u\in(t-\tau_N,t)$ are such that $E(q,t_\star,\varepsilon):=(\sigma_q+t_\star-\varepsilon,\sigma_q+t_\star)\cap(\sigma_q-\tau_N,\sigma_q]\cap(t-\tau_N,t)$ has strictly positive measure. The set of integers q for which this holds for some $t_\star\in(-\tau_N,0)$ and some $\varepsilon>0$ consists exactly of those q such that $t-\tau_N<\sigma_q<t+\tau_N$. If we pick one of them and, say, $\sigma_q\geq t$, it is easy to check that for ε small enough $E(q,t_\star,\varepsilon)=(t_\star+\sigma_q-\varepsilon,t_\star+\sigma_q)$ when $t_\star\leq t-\sigma_q$ and $E(q,t_\star,\varepsilon)=\varnothing$ when $t_\star>t-\sigma_q$. If on the contrary $\sigma_q<t$, then $E(q,t_\star,\varepsilon)=(t_\star+\sigma_q-\varepsilon,t_\star+\sigma_q)$ when $t_\star>t-\sigma_q-\tau_N$ and $E(q,t_\star,\varepsilon)=\varnothing$ when $t_\star\leq t-\sigma_q-\tau_N$. Altogether, since there are finitely many q under examination (*i.e.* at most $Q(t+\tau_N)$), we can take $\varepsilon>0$ so small that all intervals $E(q,t_\star,\varepsilon)$ are disjoint, and then we deduce from (5.66) and the previous discussion that

$$\int_{t-\tau_N}^{t}\|z_{t_\star,v,\varepsilon}(u)\|^p du=\sum_{\{q:\ t-\tau_N-t_\star<\sigma_q\leq t-t_\star\}}\frac{1}{\varepsilon}\int_{-\varepsilon}^{0}\|M_q(t_\star+\sigma_q+\theta)\,v\|^p\mathrm{d}\theta. \tag{5.67}$$

Observe next that a.e. $t_\star\in(-\tau_N,0)$ is a Lebesgue point of (each entry of) $s\mapsto M_q(s+\sigma_q)$ for all $q\in\mathbb{N}$, and let E denote the set of such points. By the triangle inequality, E *a fortiori* consists of Lebesgue points of $s\mapsto\|M_q(s+\sigma_q)v\|$, and since $M_q\in L^\infty_{loc}(\mathbb{R},\mathbb{R}^{d\times d})$ it also consists of Lebesgue points of $s\mapsto\|M_q(s+\sigma_q)v\|^p$, by the smoothness of $x\to x^p$ for $x>0$. Thus, from (5.65) and (5.67), we deduce on letting $\varepsilon\to 0$ that

$$t-\tau_N-t_\star<\sigma_q\leq t-t_\star\ \Rightarrow\ \|M_q(t_\star+\sigma_q)v\|\leq C_0e^{-\gamma t}\|v\|,\qquad t_\star\in E. \tag{5.68}$$

Now, choose $\sigma_q\in\Sigma$ and $t_\star\in E$. We can find $t>0$ such that $t-\tau_N<\sigma_q+t_\star<t$ and then, applying what precedes with this t and this $t_\star$, we obtain in view of (5.68) that

$$\|M_q(t_\star+\sigma_q)v\|\leq C_0e^{-\gamma t}\|v\|\leq C_0e^{-\gamma(t_\star+\sigma_q)}\|v\|. \tag{5.69}$$

As E has full measure in $(-\tau_N,0]$ and $v\in\mathbb{R}^d$ is arbitrary, we conclude from (5.69) and (5.60) that

$$|||M_q(s)|||\leq C_0\,e^{-\gamma s},\qquad\text{a.e. } s>0, \tag{5.70}$$

where $|||\cdot|||$ is the spectral norm for matrices on Euclidean space. Because the number of summands in the middle term of (5.61) is $Q(t+\tau_N)$ which is bounded above by Ct^N for some constant C, as asserted in (5.58), the inequality (5.70) implies that to any $\gamma'\in(0,\gamma)$ there is a constant $C_1>0$ for which

$$\|z\|_{L^\lambda((t-\tau_N,t),\mathbb{R}^d)}\leq C_1\,e^{-\gamma' t}\|\phi\|_{L^\lambda((-\tau_N,0),\mathbb{R}^d)},\qquad t\geq 0,\qquad 1\leq\lambda<\infty, \tag{5.71}$$

$$\text{and also}\quad \|z\|_{L^\infty((t-\tau_N,t),\mathbb{R}^d)}\leq C_1e^{-\gamma' t}\|\phi\|_{L^\infty((-\tau_N,0),\mathbb{R}^d)},\qquad t\geq 0. \tag{5.72}$$

Since (5.72) readily implies C^0 exponential stability when the maps $D_i(.)$ are continuous, this achieves the proof when $1\leq p<\infty$.

Assume now that $p=\infty$, so that (5.63) gets replaced by $\|z\|_{L^\infty((t-\tau_N,t),\mathbb{R}^d)}\leq C_0e^{-\gamma t}\|\phi\|_{L^\infty((-\tau_N,0),\mathbb{R}^d)}$ for all $t>0$. The goal is again to prove (5.70) from which the result follows, as we just saw. For this, we argue much like we did before, defining $\phi_{t_\star,v,\varepsilon}$ as in (5.64) except that we do not divide by $\varepsilon^{1/p}$. Then, (5.65) becomes

$$\operatorname*{ess.\,sup}_{\alpha\in(t-\tau_N,t)}\|z_{t_\star,v,\varepsilon}(\alpha)\|\leq C_0e^{-\gamma t}\|v\|, \tag{5.73}$$

and the discussion that led us to (5.67) now yields for $\varepsilon>0$ small enough:

$$\operatorname*{ess.\,sup}_{\alpha\in(t-\tau_N,t)}\|z_{t_\star,v,\varepsilon}(\alpha)\|=\max_{\{q:\ t-\tau_N-t_\star<\sigma_q\leq t-t_\star\}}\ \operatorname*{ess.\,sup}_{\theta\in(-\varepsilon,0)}\|M_q(t_\star+\sigma_q+\theta)\,v\|. \tag{5.74}$$

We need now to replace Lebesgue points by points of approximate continuity. Recall that a function $f:\mathbb{R}\to\mathbb{R}^m$ is approximately continuous at x if, for every $\epsilon>0$,

$$\lim_{r\to 0^+}\frac{1}{2\,r}\,\mathcal{H}^1\Big((x-r,x+r)\cap\{y:\|f(y)-f(x)\|>\epsilon\}\Big)=0$$

where $\mathcal{H}^1$ is the Lebesgue measure on $\mathbb{R}$, and that a measurable f is approximately continuous at almost every point [EG92, thm. 1.37]. Thus, if we define $E\subset(-\tau_N,0)$ to be the set of approximate continuity points of all maps $s\mapsto M_q(s+\sigma_q)$ (say, extended by zero off $(-\tau_N,0)$) as q ranges over $\mathbb{N}$, then E has full measure in $(-\tau_N,0)$ and letting $\varepsilon\to 0$ in (5.74) we find that (5.68) holds. From the latter we obtain (5.70) by the same reasoning as before, thereby completing the proof when $p=\infty$.

It remains to handle the case where the $D_i(t)$ are continuous and System (5.23) is C^0 exponentially stable. Then, the previous argument needs adjustment because $\phi_{t_\star,v,\varepsilon}\notin\mathcal{C}$. However, it is easy to construct a sequence of continuous functions $\varphi_k:[-\tau_N,0]\to[0,1]$, with $\varphi_k(0)=\varphi_k(-\tau_j)=0$ for $1\le j\le N$, such that φ_k converges pointwise a.e. to $\mathbb{1}_{(t_\star-\varepsilon,t_\star)}$ when $k\to+\infty$ (for instance, we may take piecewise linear φ_k). Then, the $\phi_k(\theta):=\varphi_k(\theta)v$ lie in $\mathcal{C}$, and if z_k denotes the solution to System (5.23) with initial condition ϕ_k, we get by assumption that $\sup_{\alpha\in(t-\tau_N,t)}\|z_k(\alpha)\|\le C_0e^{-\gamma t}\|v\|$. As ϕ_k converges pointwise a.e. to $\phi_{t_\star,v,\varepsilon}$ on $[-\tau_N,0]$, we see from (5.61) that z_k converges to $z_{t_\star,v,\varepsilon}$ pointwise a.e. on $\mathbb{R}$. Thus, letting $k\to+\infty$, we deduce that (5.73) holds and we conclude as before. □

5.4.6 Proof of Theorem 5.22

First assume that each set $\mathcal{I}_j$ has even cardinality $2\,n_j$, and put $N=\sum_{j=1}^{\widehat{M}}n_j$ so that $d=2N$. Let P_3 be the permutation matrix sending $\mathcal{I}_1$ to $\{1,\dots,n_1\}\cup\{N+1,\dots,N+n_1\}$ and, more generally, $\mathcal{I}_j$ to $\{1+\sum_{\ell=1}^{j-1}n_\ell,\dots,\sum_{\ell=1}^{j}n_\ell\}\cup\{N+1+\sum_{\ell=1}^{j-1}n_\ell,\dots,N+\sum_{\ell=1}^{j}n_\ell\}$ for each j. Set $\tau_k=\eta_j$ for each k in $\{1+\sum_{\ell=1}^{j-1}n_\ell,\dots,\sum_{\ell=1}^{j}n_\ell\}$. Using P_3 as change of basis and denoting by $(x_1,\dots,x_N,y_1,\dots,y_N)$ the new coordinates, one can (by gathering the matrices with disjoint nonzero columns into a single one) re-write (5.75) as (5.29) where $-(I+\mathbf{A}(t)\,\mathbf{K})^{-1}(I-\mathbf{A}(t)\,\mathbf{K})\,P_2$ has been replaced with $\sum_{j=1}^{\widehat{M}}{P_3}^{-1}\,\widehat{D}_j(t)\,P_3$. We want now to find $\mathbf{A}(t)$ and $\mathbf{K}$ so that these two matrices coincide. For this, we fix $\mathbf{K}=Id$ and solve $(I+\mathbf{A}(t))^{-1}(I-\mathbf{A}(t))=R(t)$ with respect to $\mathbf{A}(t)$, where $R(t)=-{P_3}^{-1}\left(\sum_{j=1}^{\widehat{M}}\widehat{D}_j(t)\right)P_3{P_2}^{-1}=-{P_3}^{-1}\left(\sum_{i=1}^{M}D_i(t)\right)P_3{P_2}^{-1}$ (the last equality is clear from the definition of $\widehat{D}_i(t)$ in (5.25)). Assumption (ii) implies $|||R(t)|||\le\nu<1$ because P_2 and P_3 are orthogonal matrices, hence, according to Lemma 6.4, setting $\mathbf{A}(t)=(Id-R(t))(Id+R(t))^{-1}$ solves the above and satisfies Assumption 5.4 with $\alpha=(1-\nu)/(1+\nu)$; (5.28) is satisfied too with $\mathbf{K}=Id$, setting all the numbers K_k to 1. By virtue of Proposition 5.16 and Theorem 5.21, the difference delay equation (5.29) with these $\mathbf{A}(.)$ and τ_k is L^p exponentially stable for all $p\in[1,\infty]$, as well as C^0 exponentially stable if the maps $D_i(.)$ (hence $\mathbf{A}(.)$) are continuous. This proves the result if all the sets $\mathcal{I}_j$ has even cardinality.

If some of the sets $\mathcal{I}_j$ have odd cardinality, define $d'>d$ so that $d'-d$ is the number of such sets $\mathcal{I}_j$. By adjoining to each such $\mathcal{I}_j$ one element of $\{d+1,\dots,d'\}$, one constructs a partition $\widetilde{\mathcal{I}}_1,\dots,\widetilde{\mathcal{I}}_{\widehat{M}}$ of $\{1,\dots,d'\}$ such that, for each j, $\widetilde{\mathcal{I}}_j$ has even cardinality and contains $\mathcal{I}_j$. Constructing some $d'\times d'$ matrices $\widetilde{D}_j(t)$ by adding $d'-d$ zero last lines and $d'-d$ zero last columns to $\widehat{D}_j(t)$, the following difference delay system (with state $\widetilde{z}$ in $\mathbb{R}^{d'}$):

$$\widetilde{z}(t)=\sum_{j=1}^{\widehat{M}}\widetilde{D}_j(t)\widetilde{z}(t-\eta_j)\tag{5.75}$$

satisfies the assumptions of the theorem: (i) with the sets $\widetilde{\mathcal{I}}_j$ instead of the original sets $\mathcal{I}_j$ and (ii) because adding zero lines and columns to a matrix does not increase its norm, hence the first part of the proof gives exponential stability, that yields exponential stability of the original system because, since the last $d' - d$ columns are zero, the evolution of the d first entries of z does not depend on the last ones. □

5.5 Conclusion

In this chapter, we have studied time-varying network of Telegrapher's equations satisfying a dissipativity hypothesis. More precisely, we have shown that this type of system is equivalent to a system of a linear time-varying difference delay equations. By going back and forth between the results on PDE's systems and delay systems, we proved non-trivial results for PDE's systems or difference delay equations.

The existence and uniqueness of integrable or continuous solutions, a well-known result of delay systems, allowed us to prove the existence and uniqueness of integrable or continue solutions (modulo Lebesgue points) for PDE's systems. Moreover we have been able to give stability results for such PDE's systems (Theorem 5.21) or a subclass of difference delay systems (Theorem 5.22) under a dissipativity assumption.

5.5.1 Comparison with the results of the Chapter 4

The result of Chapter 5 has to be compared with the results of Chapter 4. The results of Chapter 4 give a sufficient and necessary stability condition about the periodic difference delay systems, and give a sufficient stability condition when the periodic difference delay system is enough regular (Hölder continuous derivative) with the column disjoint property (see Corollary 4.15). Chapter 5 permits to give a stronger result for these kind of systems. In fact, a trivial change of variable leads to the generalization of Theorem 5.22 :

Theorem 5.25. *Assuming that the $D_i(\cdot)$ are continuous and bounded, if Conditions (i) and (ii) below are satisfied, then the time-varying difference delay system (5.23) is L^p exponentially stable for all $p \in [1, \infty]$. Moreover, if the maps $t \mapsto D_i(t)$ are continuous, then it is also C^0 exponentially stable.*

i. The columns of the matrices $\widehat{D}_j(t)$ are disjoint, i.e. there is a partition $\{1, \dots, d\} = \mathcal{I}_1 \cup \cdots \cup \mathcal{I}_{\widehat{M}}$ (with $i \neq j \Rightarrow \mathcal{I}_i \cap \mathcal{I}_j = \varnothing$) such that the k^{th} column of $\widehat{D}_j(t)$ is identically zero if $k \notin \mathcal{I}_j$.

ii. The exists a diagonal invertible matrix D and a number $\nu < 1$ such that :
$$\left\| D \sum_{i=1}^{M} D_i(t) D^{-1} \right\| \leq \nu \text{ for all positive } t\,.$$

In particular, Theorem 5.25 can be reinterpreted in the case of the network of the Telegrapher's equations (equations (5.1) $(1 \leq k \leq N)$ and (5.10)-(5.8)). Defining the following Assumption 5.26 (which is weaker than the assumption 5.4) :

Assumption 5.26. *The map $t \mapsto \mathbf{A}(t)$ is continuous and bounded $[0, +\infty) \to \mathbb{R}^{2N \times 2N}$, moreover :*

i. $Id + \mathbf{A}(t)\mathbf{K}$ is invertible for all t real,

ii. there exists D a diagonal invertible matrix and ν positive strictly less than one such that :
$$\left\| D[Id + \mathbf{A}(t)\mathbf{K}]^{-1}[Id - \mathbf{A}(t)\mathbf{K}]\, P_2]D^{-1} \right\| \leq \nu \tag{5.76}$$

An application of Theorem 5.25 proves the following theorem :

Theorem 5.27. *Under Assumption 5.26, the time-varying network of telegrapher's equations (5.1)-(5.10)-(5.8) is C^0 exponentially stable.*

Chapter 6

General circuit containing lossless transmission lines

Let us sum up quickly the path that we have been following until now in this thesis. Chapter 1 introduced a heuristic of the Harmonic Balance (HB) and the harmonic transfer function (HTF) methods for the electrical circuit modeled by ordinary differential equations. Chapter 2 was devoted to a simple circuit containing a transmission line. Chapter 3 established the equations for a general electrical circuit containing a several number of lossless Telegrapher's equations, while Chapter 4 and Chapter 5 dealt with stability of time-varying linear difference delay equations. At this point, the link between Chapters 4/5 and Chapter 3 might seem vague. The present chapter makes it more evident by proving that the operator solution associated to the equations the stability of which we want to prove is a compact perturbation of the operator associated to some time-varying linear difference delay equations, and so ensuring the stability of the difference delay equations through Chapters 4 and 5 is crucial. The goal of this Chapter 6 is to give a mathematical justification of the Harmonic Balance method which searches poles in the right half plane. This chapter is divided into two differents part. The first part states results for a specific class of delay equations and the second part is oriented towards electrical circuits.

The first part of this chapter (Section 6.1) deals with general facts concerning nonlinear hybrid delay equations, *i.e.* difference delay equations coupled with difference differential equations, see (6.1) (or (1) page 18). There is few literature about this kind of systems even when they are time-invariant; that is why the results are worthy in themselves. The difficulties come from the fact that the functional spaces become harder to handle and the fact that the spectral semigroup theory does not apply anymore. The results of this section are non-trivial generalization of results for neutral differential equations [HVL93].

A first result is that the linearised hybrid system around a periodic trajectory is a compact perturbation of the difference delay system, result which permits to have just eigenvalues outside the unit circle for the monodromy operator when the difference delay system is exponentially stable; it also permits to prove that the exponential stability for the continuous function is equivalent to the stability for the square integrable function.

A second result is that the exponential stability of the linearised system around a periodic trajectory implies the local stability of the periodic solution. Even if it not surprising, this result is important for the electronic applications because we want to ensure that if we start near the periodic solution then we converge towards it. The proof relies on two facts. The first fact is the existence of a variation of constant formula, involving a fundamental solution, which is not a straightforward generalization of the variation of constant formula in the neutral case because the state space needs boundary conditions to ensure the continuity of the solution. The second fact is that the fundamental solution is exponentially stable when the linearised system is exponentially stable. In the time-invariant case it was obviously true from the theory of semigroups, but here we need to find an other argument to ensure that this result is always true.

A third result says, considering a linear periodic hybrid delay system whose associated difference delay system is exponentially stable, that the harmonic transfer function (HTF) of the input-output system obtained when disturbing the hybrid delay system by an input and observing an output has no other singularities than poles in the closed right half plane, moreover these poles are the complex logarithm of a finite family of complex numbers, complex numbers which are eigenvalues of the monodromy operator.

The second part of this chapter (Section 6.2) applies these results on hybrid delay equation to "general" nonlinear circuits with lossless transmission lines under periodic forcing. We consider such a circuit with its periodic forcing and assume that this dynamical system admits a periodic solution with the same period and are interested in determining its local stability. Introducing the concept of dissipativity at high frequency for the electronic components of the circuit and assuming this property, we can prove with the help of the first part and of Chapter 5, that the harmonic transfer function has just poles in the closed right half plane and, under controllability and observability assumptions, that the periodic solution is locally stable if and only if the harmonic transfer function is analytic in the closed right half plane.

These results give a justification of the Harmonic Balance (HB) method. In fact, we introduced quickly the HB techniques in Chapter 1 for ordinary differential equations. Theses techniques operate in the frequency domain and permit to approximate a periodic solution for a nonlinear circuit (the periodic solution we assumed). Disturbing by a source of current the linearised circuit around the periodic trajectory and observing the output voltage response at some node, HB gives an approximation of the harmonic transfer function linking the frequencies of the input with the frequencies of the output on the imaginary axis. Searching poles through rational approximation, we can rely on the HB techniques to determine the local stability of the periodic solution. As seen in Chapter 2, we can perform the same HB techniques for the nonlinear containing lossless transmission line. As also explained in Chapter 2, the structure of the harmonic transfer function becomes more complex than in the absence of transmission lines: it is no longer meromorphic in all the complex plane. However, on this simple example we can see that the HTF remains meromorphic in the closed right half plane, where the poles are just the logarithms of a finite family of eigenvalues of the mononodromy operator. This permits to use the rational approximation techniques showed in Chapter 1. The present Chapter 6 justifies, like Chapter 2 did on a simple example, the determination of stability through searching for poles of the HTF in the closed right half plane for general circuits, under the rather natural assumption of the dissipativity at high frequency of the circuit. In fact, all well modeled circuit satisfy this property because the capacitive and inductive effects "kill" the nonlinearies at high frequency and thus the circuit is resistive at high frequency.

6.1 Nonlinear hybrid delay systems

6.1.1 Equations and stability results

We saw that Chapter 3 leads to the hybrid delay systems, *i.e.* we consider the following nonlinear system :

$$\begin{cases} \frac{dx(t)}{dt} = f(t, x(t), y(t), y(t-\tau_1), \cdots, y(t-\tau_N)) \\ 0 = g(t, y(t), y(t-\tau_1), \cdots, y(t-\tau_N), x(t)), \ t \geq s, \end{cases} \tag{6.1}$$

where the delays are ordered as $\tau_1 < \cdots < \tau_N$, $s, t \in \mathbb{R}$ and $s \in \mathbb{R}$ denotes the initial time of the system, k, N and n are three integers, $f \in \mathcal{C}^3(\mathbb{R} \times \mathbb{R}^k \times (\mathbb{R}^n)^{N+1}, \mathbb{R}^k)$ and $g \in \mathcal{C}^4(\mathbb{R} \times (\mathbb{R}^n)^{N+1} \times \mathbb{R}^k, \mathbb{R}^n)$, moreover f and g are T periodic in their first variable. *With no loss of generality, we*

assume that $T > \tau_N$ (otherwise we would have just to consider the smallest integer p such that $pT > \tau_N$ and take as a period of the system pT). The assumption on the regularity of the functions f and g might seem strong but it permits to simplify some proofs and it will be noted when we can weaken these hypothesis. We admit that the system (6.1) has a C^3 periodic solution $(\overline{x}(\cdot), \overline{y}(\cdot))$ with a period T (in fact through some assumptions, it is just sufficient to have a continuous periodic solution to ensure that it is C^3, see Corollary 6.16 below). It should be interesting to investigate an equivalent existence theorem like Theorem 1.4 in Chapter 1 in the case of the hybrid delay systems. We note :

- $A_1(t) = \partial_2 f(t, x(t), y(t), y(t-\tau_1), \cdots, y(t-\tau_N))$,
- $B_i^1(t) = \partial_{i+3} f(t, x(t), y(t), y(t-\tau_1), \cdots, y(t-\tau_N))$ for $i = 0, \cdots, N$,
- $\tilde{B}_i^2(t) = \partial_{i+2} g(t, y(t), y(t-\tau_1), \cdots, y(t-\tau_N), x(t))$ for $i = 0, \cdots, N$,
- $\tilde{A}_2 = \partial_{N+3} g(t, y(t), y(t-\tau_1), \cdots, y(t-\tau_N), x(t))$,
- $\tau_0 = 0$ by convention.

Then the linearisation of the system (6.1) around the periodic trajectory leads to the system :

$$\begin{cases} \frac{dx(t)}{dt} = A_1(t)x(t) + \sum_{i=0}^{N} B_i^1(t)y(t-\tau_i) \\ 0 = \sum_{i=0}^{N} \tilde{B}_i^2(t)y(t-\tau_i) + \tilde{A}_2(t)x(t), \ t \geq s. \end{cases} \tag{6.2}$$

We assume that $\tilde{B}_0^2(t)$ is invertible and noting :

- $B_i^2(t) = -\tilde{B}_0^2(t)^{-1}\tilde{B}_i^2(t)$ for $i = 1, \cdots, N$,
- $A_2(t) = -\tilde{B}_0^2(t)^{-1}\tilde{A}_2(t)$,

we can rewrite the system (6.2) as follows :

$$\begin{cases} \frac{dx(t)}{dt} = A_1(t)x(t) + \sum_{i=0}^{N} B_i^1(t)y(t-\tau_i) \\ y(t) = \sum_{i=1}^{N} B_i^2(t)y(t-\tau_i) + A_2(t)x(t), \ t \geq s. \end{cases} \tag{6.3}$$

The system (6.3) is a periodic linear hybrid delay system with a real positive period T. The system is composed of two states noted $(x(t), y(t))$ where the notation denote a column vector. The state $x(t)$ represent the derivative state and it has its values in $\mathbb{R}^k$ with k natural integer. The state $y(t)$ represents the delayed state and it has its values in $\mathbb{R}^n$ with n natural integer. The delayed state occurs in the system with a finite number of delays. Due to the dimension of $x(t)$ and $y(t)$, and the regularity of the functions f, g and the periodic solution, we have that $A_1(\cdot)$ (resp. $A_2(\cdot)$) is a $k \times k$ (resp. $n \times k$) T-periodic C^3 matrix. The $(B_i^1(t))_{i\in[|0,N|]}$ (resp. $(B_i^2(t))_{i\in[|1,N|]}$) represent a family of size $k \times n$ (resp. $n \times n$) T-periodic C^3 matrices.

Since the delayed state $y(t)$ occurs in the system with delayed values, it is simpler to rewrite the system (6.3) in more compact way. Putting $y_t(\theta) = y(t+\theta)$ for $\theta \in [-\tau_N, 0]$ and introducing the two operators :

$$\mathbf{B_1}(s)\phi := \sum_{i=0}^{N} B_i^1(s)\phi(-\tau_i), \tag{6.4}$$

$$\mathbf{B_2}(s)\phi := \phi(0) - \sum_{i=1}^{N} B_i^2(s)\phi(-\tau_i), \tag{6.5}$$

for ϕ a measurable function defined on $[-\tau_N, 0]$ and admitting a value in τ_i for all $i = 0, \cdots, N$. Moreover the values of ϕ are in $\mathbb{R}^n$. We can rewrite the system (6.3) with the state space $(x(t), y_t)$ as :

$$\begin{cases} \frac{dx(t)}{dt} = A_1(t)x(t) + \mathbf{B_1}(t)y_t \\ \mathbf{B_2}(t)y_t = A_2(t)x(t), \ t \geq s. \end{cases} \tag{6.6}$$

We consider too the linear T periodic difference delay system associated to the system (6.3) :

$$z(t) = \sum_{i=1}^{N} B_i^2(t)z(t-\tau_i), \ t \geq s, \tag{6.7}$$

where $z(\cdot)$ is a function defined on $[-\tau_N, 0]$ and with values in $\mathbb{R}^n$. Noting $z_t(\theta) = z(t+\theta)$ for $\theta \in [-\tau_N, 0]$, the system (6.7) can be written in the following functional form :

$$\mathbf{B_2}(t)z_t = 0, \ t \geq s. \tag{6.8}$$

The systems (6.6) and (6.8) need initial data defined on an interval of length τ_N to define a solution for $t \geq s$. To ensure that there exists an unique p-integrable solution on each compact to the system (6.6) and (6.8) the initial data must be p-integrable for $p \in [1, +\infty]$. Moreover, if we want a continuous solution, the initial data must be continuous and satisfy a boundary condition. More formally we introduce the following spaces for $s \in \mathbb{R}$ and $p \in [1, \infty]$:

- $C_s := \{(x, \phi) \in \mathbb{R}^k \times C^0([-\tau_N, 0], \mathbb{R}^n) | \mathbf{B_2}(s)\phi = A_2(s)x\}$,
- $L_s^p := \{(x, \phi) \in \mathbb{R}^k \times L^p([-\tau_N, 0], \mathbb{R}^n)\}$,
- $\tilde{C}_s := \{(0, \phi) \in \{0_k\} \times C^0([-\tau_N, 0], \mathbb{R}^n) | \mathbf{B_2}(s)\phi = 0\}$,
- $\tilde{L}_s^p := \{(0, \phi) \in \{0_k\} \times L^p([-\tau_N, 0], \mathbb{R}^n)\}$,

where $C^0([-\tau_N, 0], \mathbb{R}^n)$ (resp. $L^p([-\tau_N, 0], \mathbb{R}^n)$) denotes the space of the continuous (resp. p-integrable) function defined on the interval $[-\tau_N, 0]$ and with values in $\mathbb{R}^n$. For i and j integers, we note $0_{i,j}$ (resp. $I_{i,j}$) the zeros (resp. identity) matrix with i lines and j columns. By abuse of notation, we will write 0_i or 0 (resp. I_i or I) for $0_{i,1}$ (resp. $I_{i,i}$). We have $C_{s+T} = C_s$, $L_{s+T}^p = L_s^p$, $\tilde{C}_{s+T} = \tilde{C}_s$ and $\tilde{L}_{s+T}^p = \tilde{L}_s^p$ because the periodicity of the system. We define the solution operator $U(t, s)$ (resp. $U_p(t, s)$) which acts from C_s (resp. L_s^p) to C_t (resp. L_t^p) such that for $(x_0, \phi) \in C_s$ (resp. L_s^p) we have $U(t, s)(x_0, \phi) = (x(t), y_t)$ (resp. $U_p(t, s)(x_0, \phi) = (x(t), y_t)$) with $x(t)$ and y_t the unique solution of the system (6.6) which starts from $(x_0, \phi) \in C_s$ (resp. L_s^p) at the time s. Moreover, because of the periodicity of the system we have $U(t + T, s + T) = U(t, s)$ (resp. $U_p(t + T, s + T) = U_p(t, s)$). We define too the solution operator $T_D(t, s)$ (resp. $T_{D,p}(t, s)$) which acts from $\tilde{C}_s$ (resp. $\tilde{L}_s^p$) to $\tilde{C}_t$ (resp. $\tilde{L}_t^p$) the solution operator of the system (6.8) which starts from a initial data $\phi \in \tilde{C}_s$ (resp. $\tilde{L}_s^p$) at the time s. By a recursive argument it is immediate to prove that the operator $U(t, s)$, $U_p(t, s)$, $T_D(s, t)$ and $T_{D,p}(s, t)$ are well defined.

Definition 6.1. We define for $s \in \mathbb{R}$, the monodromy operators :

$$\begin{aligned} V_p(s) : &= U_p(s+T,s), \text{ for } p \in [1,\infty], && (6.9)\\ \tilde{V}_p(s) : &= T_{D,p}(s+T,s), \text{ for } p \in [1,\infty], && (6.10)\\ V(s) : &= U(s+T,s), && (6.11)\\ \tilde{V}(s) : &= T_D(s+T,s). && (6.12) \end{aligned}$$

We endow the space $\mathbb{R}^k \times \mathbb{R}^n$ with the euclidean norm noted $||\cdot||$ and we give the definition of the zero exponential stability for all the spaces that we introduced.

Definition 6.2. The system (6.6) and (6.8) are respectively said L^p for $p \in [1,+\infty[$, L^∞ and C^0 exponentially stable if there exists a $\gamma > 0$ and a $K > 0$ such that :

$$\begin{aligned} \left(\int_{-\tau_N}^0 \|U(t,s)(x_0,\phi)(\theta)\|^p d\theta\right)^{1/p} &\leq Ke^{-\gamma t}\left(\int_{-\tau_N}^0 ||(x_0,\phi(\theta))||^p d\theta\right)^{1/p}, \ t \geq 0 \text{ and } (x_0,\phi) \in L^p_s,\\ \left(\int_{-\tau_N}^0 ||T_D(t,s)(0,\phi)(\theta)||^p d\theta\right)^{1/p} &\leq Ke^{-\gamma t}\left(\int_{-\tau_N}^0 ||(0,\phi(\theta))||^p d\theta\right)^{1/p}, t \geq 0 \text{ and } (0,\phi) \in \tilde{L}^2_s,\\ ess.\sup_{\theta\in[-\tau_N,0]} ||U_\infty(t,s)(x_0,\phi)(\theta)|| &\leq Ke^{-\gamma t} ess.\sup_{\theta\in[-\tau_N,0]} ||(x_0,\phi(\theta))||, t \geq 0 \text{ and } (x_0,\phi) \in L^\infty_s,\\ ess.\sup_{\theta\in[-\tau_N,0]} ||T_{D,\infty}(t,s)(0,\phi)(\theta)|| &\leq Ke^{-\gamma t} ess.\sup_{\theta\in[-\tau_N,0]} ||(0,\phi(\theta))||, t \geq 0 \text{ and } (x_0,\phi) \in \tilde{L}^\infty_s,\\ \sup_{\theta\in[-\tau_N,0]} ||U(t,s)(x_0,\phi)(\theta)|| &\leq Ke^{-\gamma t} \sup_{\theta\in[-\tau_N,0]} ||(x_0,\phi(\theta))||, \ t \geq 0 \text{ and } (x_0,\phi) \in C^0_s,\\ \sup_{\theta\in[-\tau_N,0]} ||T_D(t,s)(0,\phi)(\theta)|| &\leq Ke^{-\gamma t} \sup_{\theta\in[-\tau_N,0]} ||(0,\phi(\theta))||, \ t \geq 0 \text{ and } (x_0,\phi) \in \tilde{C}^0_s. \end{aligned}$$

The exponential stability of the system (6.6) and (6.8) is known to be entirely determined by the spectrum of the monodromy operators. We cite the following classical result :

Proposition 6.3. *We have :*

i. The system (6.8) is L^p for $p \in [1,\infty]$ (resp. C^0) exponentially stable if and only if the spectral radius of the monodromy operators $\tilde{V}_p(s)$ (resp. $\tilde{V}(s)$) is strictly less than one.

ii. The system (6.6) is L^p for $p \in [1,\infty]$ (resp. C^0) exponentially stable if and only if the spectral radius of the monodromy operators $V_p(s)$ (resp. $V(s)$) is strictly less than one.

Proof. The proof is identical to the proofs of the Lemma 2.8 and Lemma 2.11 in Chapter 2. □

The link between the monodromy operators of a neutral differential system and the monodromy operators of the difference delay system associated is well known (see [Hen74] and [HVL93]). In the same spirit we have that the monodromy operator of the sytem (6.6) is a compact perturbation of the monodromy operator of the system (6.8). We state the following two lemmas :

Lemma 6.4. *We consider $U(t,s) : C_s \to C_t$. Let $\tilde{\mathbf{B}}_\mathbf{2}(s) : (x_1,\phi) \mapsto (x_1, \mathbf{B_2}(s)\phi)$. There exists an independent family of functions $\tilde{\Phi}(s)$ at least C^1 such that $\tilde{\mathbf{B}}_\mathbf{2}(s)\tilde{\Phi}(s) = I_{k+n}$. So we have that $T_D(t,s)(I_{k+n} - \tilde{\Phi}(s)\tilde{\mathbf{B}}_\mathbf{2}(s))$ has the spectrum of $T_D(t,s)$ plus possibly 0 and :*

$$U(t,s) = T_D(t,s)(I_{k+n} - \tilde{\Phi}(s)\tilde{\mathbf{B}}_\mathbf{2}(s)) + K(t,s), \tag{6.13}$$

with $K(t,s)$ compact operator from C_s to C_t for all t,s real such that $t \geq s$.

Proof. The existence of an independent family of functions such that $\Phi(s)$ is a matrix of size $n \times n$ at least C^1 and $\mathbf{B_2}(s)\Phi(s) = I_n$ is obvious. So we define the matrix $\tilde{\Phi}(s)$ as :

$$\tilde{\Phi}(s) := \begin{bmatrix} I_k & 0_{k,n} \\ 0_{n,k} & \Phi(s) \end{bmatrix} \tag{6.14}$$

and we have $\tilde{\mathbf{B}}_{\mathbf{2}}(s)\tilde{\Phi}(s) = I_{k+n}$. To see that the spectrum of $T_D(t,s)(I_{k+n} - \tilde{\Phi}(s)\tilde{\mathbf{B}}_{\mathbf{2}}(s))$ has the spectrum of $T_D(t,s)$ plus possibly 0 we have just to remark that we have the following decomposition :

$$C_s = \tilde{C}_s \oplus W(s), \tag{6.15}$$

where $W(s) := \{(x, \Phi(s)A_2(s)x) | x \in \mathbb{R}^k\}$ because of $\tilde{C}_s$ is the kernel of the operator $\tilde{\mathbf{B}}_{\mathbf{2}}(s)$. As $T_D(t,s)(I_{k+n} - \tilde{\Phi}(s)\tilde{\mathbf{B}}_{\mathbf{2}}(s))$ is zero on $W(s)$ and is equal to $T_D(t,s)$ on $\tilde{C}_s$, we have the result. Let $(x_0, \varphi_0) \in C_s$. We have that :

$$K(t,s)(x_0, \varphi_0) = (x(t), w_t), \tag{6.16}$$

where w satisfies the equation :

$$\begin{cases} w_s = \Phi(s)A_2(s)x_0 \\ w(t) = \sum_{i=1}^{N} B_i^2(t)w(t-\tau_i) + A_2(t)x(t), \text{ for } t > s, \end{cases} \tag{6.17}$$

and $x(t)$ is the solution of the equation (6.6) with initial data (x_0, φ_0). We have that $x(\cdot)$ is a continuously differentiable function, and since we assumed that $A_2(t)$, $B_i^2(t)$ for all $i = 1, \cdots, N$ and $\Phi(s)$ are at least C^1, we have that $w(\cdot)$ is a lipschitzian function on each compact. So by compact imbedding, we have that $K(t,s)$ is a compact operator on C_s to C_t. □

Lemma 6.5. *We have for all $p \in [1, \infty]$:*

$$U_p(t,s) = T_{D,p}(t,s)P_p(t,s) + K_p(t,s), \ t \geq s \tag{6.18}$$

with P_p the canonical projection from L_s^p in $\tilde{L}_t^p$ and $K_p(t,s)$ compact operator from L_s^p to L_t^p for all t and s real with $t \geq s$.

Proof. It is the same than the proof of Lemma 6.4. □

Remark 6.6. *To prove Lemma 6.4 and Lemma 6.5, we used the smoothness of the matrices in the system (6.6). But the result would be true with just continuous regularity assumption on the matrices in this system but the proof would be more technical, involving Gronwall Lemma, recursive arguments and Arzela-Ascoli (or Kolmogorov-Riesz-Fréchet) theorem as for the neutral differential case (Chapter 2).*

Until now we dealt with a tremendous number of spaces, ie the spaces of continuous and L^p functions for the systems (6.6) and (6.8). However in term of the exponential stability of the origin of these systems, all these spaces are equivalent. Reciting the result of the Chapter 5, we have that the L^p exponential stability for one $p \in [|1, \infty|]$ for the system (6.8) is equivalent to the L^q exponential stability for all $q \in [|1, \infty|]$ and also the C^0 stability.

Proposition 6.7. *We have the following equivalence :*

i. The system (6.8) is L^p exponentially stable for one $p \in [1, +\infty]$.

ii. The system (6.8) is L^p exponentially stable for all $p \in [1, +\infty]$.

iii. The system (6.8) is C^0 exponentially stable.

Assuming that the system (6.8) is L^p, for one $p \in [1, \infty]$, or C^0 exponentially stable, then Lemma 6.4, Lemma 6.5 and Proposition 6.7 leads to the the fact that it is equivalent for the system (6.6) to be L^p for one $p \in [1, +\infty]$ or C^0 exponentially stable.

Theorem 6.8. *Assuming the system (6.8) is L^p for one $p \in [1, +\infty]$ or C^0 exponentially stable. Then the monodromy operators $V_p(s)$ for all $p \in [1, \infty]$ and $V(s)$ have the same finite number of eigenvalues outside a disk with radius strictly less than one. We note $\zeta_1, \cdots, \zeta_N$ the eigenvalues. Moreover we have the following equivalence :*

i. The system (6.6) is L^p exponentially stable for one $p \in [1, \infty]$.

ii. The system (6.6) is L^p exponentially stable for all $p \in [1, \infty]$.

iii. The system (6.6) is C^0 exponentially stable.

Proof. Since the system (6.8) is exponentially stable, we have by Lemma 6.4, Lemma 6.5, and the theory of compact perturbation ([Kat95]), that $V_p(s)$ for $p \in [1, \infty]$ and $V(s)$ have a finite number of eigenvalues outside a disk with a radius stricly less than one.

We have by definition that i implies ii. Let $p \in [1, \infty]$ and $q \in [1, \infty]$. If $q \leq p$ we have that L^q is dense in L^p , and then since the restriction of $V_p(s)$ to L^q is stable and equal to $V_q(s)$, we have that the spectrum of $V_p(s)$ is included in the spectrum of $V_q(s)$. Since $V_q(s)$ and $V_p(s)$ have a finite number of eigenvalues outside a disk with a radius stricly less than one then $V_q(s)$ and $V_p(s)$ have exactly the same spectrum outside a disk with a radius strictly then one. The same reasonning would prove that the result holds for $q \geq p$. If we assume that the system (6.6) is L^p exponentially stable, we deduce from Proposition 6.3 that the monodromy operator $V_p(s)$ has not an eigenvalue outside the open unit disk and so the spectrum of $V_q(s)$ satisfies the same property and we have that the system (6.6) is L^q exponentially stable. We proved that ii implies i. We omit the proof of the fact that iii is equivalent to i because the proof is similar to the fact that i is equivalent to ii. □

Remark 6.9. *In Theorem 6.8, we assumed that the system (6.8) is exponentially stable. It should be possible to remove this assumption and to have the same result even when the system (6.6) is time-varying (and not just time periodic). However, the arguments would be of the same type that for the proof of Proposition 6.7 that we proved in Chapter 5.*

Theorem 6.8 is useful because it permits to work on the spaces of square integrable function to conclude with the C^0 stability. From now, we always assume :

Assumption 6.10. *The periodic difference delay system (6.8) is C^0 exponentially stable.*

6.1.2 Variation of constant formulas

We consider the following system :

$$\begin{cases} \frac{dx(t)}{dt} = A_1(t)x(t) + \mathbf{B_1}(t)y_t + u_1(t) \\ \mathbf{B_2}(t)y_t = A_2(t)x(t) + u_2(t), \text{for } t \geq s. \end{cases} \tag{6.19}$$

Until the statement of the theorem, we keep ambiguous the spaces where belong the functions x, y, u_1 and u_2, but we have to keep in mind that the functions are continuous or square integrable.

We want to express the function x and y in function of the solution operators U_2 or U_∞. The formula that we search is called the variation of the constant formula and it is well known for delay system and neutral system (see [Hen74] and [HVL93]), and for the time constant hybrid delay system (see [HH95]). Here we give a variation of constant formula for hybrid periodic delay system. For all $s \in \mathbb{R}$ and $t \geq s$, we introduce the fundamental solution $X(t,s)$ which is a block matrix :

$$X(t,s) := \begin{bmatrix} X(t,s)_{1,1} & X(t,s)_{1,2} \\ X(t,s)_{2,1} & X(t,s)_{2,2} \end{bmatrix}, \tag{6.20}$$

where $X(t,s)_{1,1}$, $X(t,s)_{1,2}$, $X(t,s)_{2,1}$ and $X(t,s)_{2,2}$ are $k \times k$, $k \times n$, $n \times k$ and $n \times n$ matrix respectively which satisfy the following equations :

$$\begin{cases} \frac{dX(t,s)_{1,2}}{dt} = A_1(t)X(t,s)_{1,2} + \sum_{i=0}^{N} B_i^1(t)X(t-\tau_i,s)_{2,2} \\ X(t,s)_{2,2} = I_n + \sum_{i=1}^{N} B_i^2(t)X(t-\tau_i,s)_{2,2} + A_2(t)X(t,s)_{1,2} \end{cases} \tag{6.21}$$

with initial data $X(t,s)_{1,2} = 0_{k,n}$, $X(t,s)_{2,2} = 0_{n,n}$ for $t < s$ and $X(s,s)_{1,2} = 0_{k,n}$, $X(s,s)_{2,2} = I_n$, and :

$$\begin{cases} \frac{dX(t,s)_{1,1}}{dt} = A_1(t)X(t,s)_{1,1} + \sum_{i=0}^{N} B_i^1(t)X(t-\tau_i,s)_{2,1} \\ X(t,s)_{2,1} = \sum_{i=1}^{N} B_i^2(t)X(t-\tau_i,s)_{2,1} + A_2(t)X(t,s)_{1,1}, \end{cases} \tag{6.22}$$

with initial data $X(t,s)_{1,1} = 0_{k,k}$, $X(t,s)_{2,1} = 0_{n,k}$ for $t < s$ and $X(s,s)_{1,1} = I_k$, $X(s,s)_{2,1} = A_2(s)I_k$.

Define :

$$\begin{bmatrix} \int_{s^-}^{t^+} X(t,\alpha)_{1,1}u_1(\alpha)d\alpha + \int_{s^-}^{t^+} d_\alpha X(t,\alpha)_{1,2}u_2(\alpha) \\ \int_{s^-}^{t+\theta^+} X(t+\theta,\alpha)_{2,1}u_1(\alpha)d\alpha + \int_{s^-}^{t+\theta^+} d_\alpha X(t+\theta,\alpha)_{2,2}u_2(\alpha) \end{bmatrix} := \int_{s^-}^{t+\theta^+} d_\alpha \tilde{X}(t+\theta,\alpha)u(\alpha),$$

where the notation $+$ (resp. $-$) denotes the integral taken with its upper (resp. lower) value, $u = (u_1, u_2)$ is a column vector and $\theta \in [-\tau_N, 0]$.

Theorem 6.11 (Variation of constant formulas)**.** *For all $s \in \mathbb{R}$,*

- *if $x(\cdot)$, $y(\cdot)$, $u_1(\cdot)$ and $u_2(\cdot)$ belong to $C^0([s,+\infty[,\mathbb{R}^k)$, $C^0([s-\tau_N,+\infty[,\mathbb{R}^n)$, $C^0([s,+\infty[,\mathbb{R}^k)$ and $C^0([s,+\infty[,\mathbb{R}^n)$ respectively and which verify the equation (6.19) for all $t \geq s$, we have for all $t \geq s$:*

$$(x(t), y_t) = U_\infty(t,s)(x(s), \tilde{\phi}) + \int_{s^-}^{t^+} d_\alpha K(t,\alpha)u(\alpha), \tag{6.23}$$

where the previous integral is understood in Bochner sense, $\tilde{\phi}(\theta) = y_s(\theta)$ for $\theta \in [-\tau_N, 0[$ and $\tilde{\phi}(0) = \sum_{i=1}^{N} B_i^2(s)y(s-\tau_i) + A_2(s)x(s)$, and

$$K(t,\alpha)(\theta) = \int_s^{\alpha^+} d_v \tilde{X}(t+\theta, v). \tag{6.24}$$

- *if* $x(\cdot)$, $y(\cdot)$, $u_1(\cdot)$ *and* $u_2(\cdot)$ *belong to* $L^2_{loc}([s,+\infty[,\mathbb{R}^k)$, $L^2_{loc}([s-\tau_N,+\infty[,\mathbb{R}^n)$, $L^2_{loc}([s,+\infty[,\mathbb{R}^k)$ *and* $L^2_{loc}([s,+\infty[,\mathbb{R}^n)$ *respectively and which verify the equation* (6.19) *for almost all* $t \geq s$, *we have for all* $t \geq s$:

$$(x(t), y_t) = U_2(t,s)(x(s), y_s) + \int_{s^-}^{t^+} d_\alpha K(t,\alpha)u(\alpha), \tag{6.25}$$

where the previous integral is understood in Bochner sense and $K(\cdot,\cdot)$ *is the operator defined in* (6.24).

Proof. The proof deals with the continuous case. The proof for the L^2 case is similar and is omitted. With no loss of generality, we can assume $s = 0$. For $t = 0$, we have by the definition of the fundamental solution that :

$$U_\infty(0,0)(x(0),\tilde{\phi}) + \int_{0^-}^{0^+} d_\alpha K(t,\alpha)u(\alpha) = (x(0), y_0). \tag{6.26}$$

Let $\theta \in [-\tau_N, 0]$ and $t > 0$ such that $t+\theta > 0$, We prove that the formula (6.23) verifies the equation (6.19) in $t+\theta$. We note $(\tilde{x}(t), \tilde{y}_t) = U_\infty(t,s)(x(s),\tilde{\phi})$.

- We have :

$$\begin{aligned} y_t(\theta) = y(t+\theta) &= \tilde{y}(t+\theta) + \int_{0^-}^{t+\theta^+} X(t+\theta,\alpha)_{2,1}u_1(\alpha)d\alpha + \int_{0^-}^{t+\theta^+} d_\alpha X(t+\theta,\alpha)_{2,2}u_2(\alpha) \\ &= \sum_{i=1}^{N} B_i^2(t+\theta)y(t+\theta-\tau_i) + A_2(t+\theta)x(t+\theta) + u_2(t+\theta). \end{aligned}$$

- We have :

$$\begin{aligned} \frac{dx(t)}{dt} = \frac{d\tilde{x}(t)}{dt} + u_1(t) + \int_0^t \frac{d}{dt}X(t,\alpha)_{1,1}u_1(\alpha)d\alpha + \int_0^t d_\alpha\left[\frac{d}{dt}X(t,\alpha)_{1,2}\right]u_2(\alpha) \\ = A_1(t)x(t) + \sum_{i=0}^{N} B_i^1(t)y(t-\tau_i) + u_1(t). \end{aligned}$$

□

Proposition 6.12 (Fundamental solution)**.** *Let* s *and* $t \geq s$ *two real. Introducing the set :*

$$\Sigma_r := \{k \in \mathbb{R},\ k = k_1\tau_1 + \cdots + k_n\tau_n \text{ for } (k_1,\cdots,k_n) \in \mathbb{N}^n\ \}, \tag{6.27}$$

we have that the distribution $d_s\tilde{X}(t,s)$ *can be written as :*

$$d_s\tilde{X}(t,s) = h(t,s) + \sum_{k\in\Sigma_r} a_k(s)\delta_k(t-s). \tag{6.28}$$

The $a_k(s) := \tilde{X}(s+k^+,s) - \tilde{X}(s+k^-,s)$ *are the jump when* $\tilde{X}(\cdot,s)$ *pass through the discontinuities* $s+k$ *and are* C^2. *The function* $h(t,s)$ *is* C^2 *on the domains* $\{(t,s) \in I\!R^2,\ t-s \notin \Sigma_r\}$. *The set* $s+\Sigma_r$ *represents the localization where the distribution* $d_s\tilde{X}(\cdot,s)$ *has its jumps.*

Proposition 6.13. *For all τ real, the distribution-valued map $t \longmapsto d_s\tilde{X}(\cdot, \cdot - \tau)$ is T-periodic and two times continuously differentiable. Moreover for all t real there exists $K, \gamma > 0$ such that :*

$$\int_0^\tau \left\| \frac{\partial^j}{\partial t^j} d_s\tilde{X}(t, t-s) \right\| \leq Ke^{\gamma\tau}, \text{ for } j = \{0, 1, 2\}. \tag{6.29}$$

Proof.

- We prove by induction that there exists $K, \gamma > 0$ such that :

$$\|[X(t,s)_{1,1}, X(t,s)_{2,1}]\| \leq Ke^{\gamma(t-s)}, \tag{6.30}$$

and by abuse of notation we note $||\cdot||$ the norm induced by the euclidean norm on $\mathbb{R}^k$, $\mathbb{R}^n$ and $\mathbb{R}^k \times \mathbb{R}^n$. In fact if $0 \leq t < \tau_1$, we can find a K and a γ which verify the equation (6.30). Assume that the majoration (6.30) holds for $(k-1)\tau_1 \leq t < k\tau_1$ for an integer k. Putting the second equation in the first in the equation (6.22), we have for $k\tau_1 \leq t < (k+1)\tau_1$:

$$\begin{aligned} \frac{dX(t,s)_{1,1}}{dt} = A_1(t)X(t,s)_{1,1} + B_0^1(t)\left(\sum_{i=1}^N B_i^2(t)X(t-\tau_i,s)_{2,1} + A_2(t)X(t,s)_{1,1}\right) \\ + \sum_{i=1}^N B_i^1(t)X(t-\tau_i,s)_{2,1}. \end{aligned} \tag{6.31}$$

Since the system is periodic and continuous, we deduce from the equations (6.31) and (6.22), the variation of constant formula for the ordinary differential equations (Theorem 1.38, Chapter 1) and the recurrence assumption that there exists K' real positive and $0 < \gamma_1 < \gamma$ such that :

$$||X(t,s)_{1,1}|| \leq K'e^{\gamma_1(t-s)} + \frac{K'K}{\gamma - \gamma_1}e^{\gamma(t-\tau_1-s)} \tag{6.32}$$

and

$$||X(t,s)_{2,1}|| \leq K'Ke^{\gamma(t-s)}e^{-\gamma\tau_1} + K'[K'e^{\gamma_1(t-s)} + \frac{K'K}{\gamma - \gamma_1}e^{\gamma(t-\tau_1-s)}]. \tag{6.33}$$

Take γ enough large in the equations (6.32) and (6.33) achieves the recurrence.

- For $j = 1, 2$ and t and s real, we define for $t \geq s$:

$$Y(t,s)_{j,2} = \int_s^t d_u X(t,u)_{j,2}. \tag{6.34}$$

We have that $Y(t,s) = [Y(t,s)_{1,2}, Y(t,s)_{2,2}]$ is a L^∞ function on each compact of $\mathbb{R}^2$ and verifies the equation (6.22). Then we can apply the same reasonning that we did for $[X(t,s)_{1,1}, X(t,s)_{2,1}]$ and we conclude that there exists K_4, γ_4 such that:

$$||Y(t,s)|| \leq K_4 e^{\gamma_4(t-s)}. \tag{6.35}$$

.

- To sum up, we proved :

$$\int_0^\tau \left\| d_s\tilde{X}(t, t-s) \right\| \leq Ke^{\gamma\tau}. \tag{6.36}$$

Proposition 6.12 stated that the distribution $t \longmapsto d_s\tilde{X}(t, t-\tau)$ is C^2 when τ does not belong to Σ_r. When τ is in Σ_r we can see that, since $h(t, t-\tau)$ has an analytic expression which depends in t and τ then it is also C^2. Then the distribution $t \longmapsto d_s\tilde{X}(\cdot, \cdot-\tau)$ is C^2 everywhere. The same kind of Gronwall reasoning would prove that there exists $K, \gamma > 0$ such that:

$$\int_0^\tau \left\| \frac{\partial^j}{\partial t^j} d_s\tilde{X}(t, t-s) \right\| \leq Ke^{\gamma\tau}, \text{ for } j = \{1, 2\}. \tag{6.37}$$

□

6.1.3 Local stability result

We search now to prove the same local stability results that we have for the ordinary differential equation (Proposition 1.7 of Chapter 1) or the scalar neutral differential equations (Theorem 2.3 of Chapter 2) in the case of the nonlinear hybrid delay system (6.1) endowed with the C^0 norm. The idea is always the same. We exploit the fact that the linearised system arround the periodic trajectory approximate locally the nonlinear system because of the regularity of the nonlinearities. We apply then the variation of constant formula (6.23) that we obtained and the fact that the fundamental solution is exponentially stable when the linearised system (6.8) is C^0 exponentially stable. Putting togethers, these arguments lead to the following theorem :

Theorem 6.14. *Assuming that* $\sup_{\theta\in[-\tau_N,0]} ||U(t,s)(\theta)|| \leq Ke^{-\alpha(t-s)}$ *for* K *and* γ *positive, then there exists* $\delta > 0$ *such that :*

$$\sup_{\theta\in[-\tau_N,0]} ||(\overline{x}(0), \overline{y}_0(\theta)) - (x(0), y_0(\theta))|| \leq \delta \Rightarrow \sup_{\theta\in[-\tau_N,0]} ||(\overline{x}(t), \overline{y}_t(\theta)) - (x(t), y_t(\theta))|| \leq Ke^{-\alpha(t-s)},$$

for all $t \geq s$*, where* $(x(t), y_t)$ *is the solution of (6.1) starting from* $(x(0), y_0)$*.*

Proof. We have that $[X(t,s)_{1,1}, X(t,s)_{2,1}]$ is a L^∞ function of $\mathbb{R}^2$ on each bounded compact set. So we deduce from the fact that $||U(t,s)|| \leq Ke^{-\alpha(t-s)}$ and Theorem 6.8 :

$$||[X(t,s)_{1,1}, X(t,s)_{2,1}]|| \leq Ke^{-\alpha(t-s)}. \tag{6.38}$$

Let $Y(t,s) = [Y(t,s)_{1,2}Y(t,s)_{2,2}]$ where the functions $Y(t,s)_{1,2}$ and $Y(t,s)_{2,2}$ are defined in the equation (6.34), we have that $Y(t,s)$ is a L^∞ function of $\mathbb{R}^2$ on each compact which verifies the equation (6.6) and always the use of Theorem 6.8, we have :

$$||Y(t,s)|| \leq Ke^{-\alpha(t-s)}. \tag{6.39}$$

The use of a straightforward generalization of Lemma 2.9, based on the mean value theorem in Chapter 2, we have that for $\eta > 0$, there exists $\delta_1 > 0$ such that if for $S > 0$ we have $||(x(t) - \overline{x}(t), y_t - \overline{y}_t)|| \leq \delta_1$ for all $t \in [s, S+s]$ and for all functions $y, \overline{y}$ (resp. $x, \overline{x}$) continuous function defined on $[-r+s, S+s]$ (resp. $[s, S+s]$), then :

$$\begin{aligned} ||f(t, x(t), y(t), y(t-\tau_1), \cdots, y(t-\tau_N)) - f(t, \overline{x}(t), \overline{y}(t), \overline{y}(t-\tau_1), \cdots, \overline{y}(t-\tau_N))|| \\ \leq (N+2)\eta \sup_{\theta\in[-\tau_N,0]} ||(x(t), y_t(\theta)) - (\overline{x}(t), \overline{y}_t(\theta))|| \end{aligned}$$

and

$$\begin{aligned} ||g(t, y(t), y(t-\tau_1), \cdots, y(t-\tau_N), x(t)) - g(t, \overline{y}(t), \overline{y}(t-\tau_1), \cdots, \overline{y}(t-\tau_N), \overline{x}(t))|| \\ \leq (N+2)\eta \sup_{\theta\in[-\tau_N,0]} ||(x(t), y_t(\theta)) - (\overline{x}(t), \overline{y}_t(\theta))||. \end{aligned}$$

The use of the variation of constant formula (6.11), Theorem 6.8 to have the exponential decreasing of the operator U_∞ and following exactly the same proof than Theorem 2.3 in the Chapter 2, we obtain the result. □

6.1.4 Regularity periodic solution

We assumed that the periodic solution of the system (6.1) was C^3. However, with the Assumption 6.10 and the regularity assumption of g and f then all continuous periodic solution to the system (6.1) is C^3. In practice, this result is a generalization in the case of nonlinear hybrid system admitting a periodic solution of the Lemma 7.1 p291 Hale [HVL93] stated for the time-invariant neutral system.

Lemma 6.15. *Let $\overline{x}(t)$ a continuous periodic solution of the system :*

$$\overline{x}(t) - H(t, \overline{x}(t-\tau_1), \overline{x}(t-\tau_2), \cdots, \overline{x}(t-\tau_N)) = h(t), \tag{6.40}$$

where H is C^2 and h is C^1. Assume that the system

$$x(t) - D_1(t)x(t-\tau_1) - \cdots - D_N(t)x(t-\tau_N) = 0 \tag{6.41}$$

is C^0 exponentially stable, then the periodic solution $\overline{x}$ is C^1.

Proof. With no loss of generality, we can consider the system (6.40) with one single delay :

$$\overline{x}(t) - H(t, \overline{x}(t-\tau_1)) = h(t), \tag{6.42}$$

We can rewrite the system (6.42) :

$$\overline{x}(t) - D_1(t)\overline{x}(t-\tau_1) := h(t) + \tilde{H}(t),$$

where $\tilde{H}(t) = -D_1(t)\overline{x}(t-\tau_1) + H(t, \overline{x}(t-\tau_1))$. Since the majoration in [HVL93, ch. 9, thm 3.5] holds for the periodic difference delay system which are exponentially stable, we have for all $t > \sigma$ and for all τ and s real the existence of $K > 0$ and $a < 0$ such that :

$$\left|\frac{\overline{x}_{t+\tau} - \overline{x}_t}{\tau}\right| \leq Ke^{a(t-\sigma)}\left[\left|\frac{\overline{x}_{\sigma+\tau} - \overline{x}_\sigma}{\tau}\right|\right] + K\sup_{\sigma\leq u\leq t}\left|\frac{h(u+\tau)-h(u)}{\tau}\right| + K\sup_{\sigma\leq u\leq t}\left|\frac{\tilde{H}(u+\tau)-\tilde{H}(u)}{\tau}\right|$$

When σ tends to $-\infty$., we obtain :

$$\left|\frac{\overline{x}_{t+\tau} - \overline{x}_t}{\tau}\right| \leq K\sup_{-\infty\leq u\leq t}\left|\frac{h(u+\tau)-h(u)}{\tau}\right| + K\sup_{-\infty\leq u\leq t}\left|\frac{\tilde{H}(u+\tau)-\tilde{H}(u)}{\tau}\right|$$

Applying two times the mean value theorem, there exists $\epsilon > 0$ enough small and $\delta > 0$ such that for all u real and all $|s, \tau| < \delta$ we have :

$$\left|\frac{\overline{x}_{t+\tau} - \overline{x}_t}{\tau}\right| \leq K\sup_{-\infty\leq u\leq t}\left|\frac{h(u+\tau)-h(u)}{\tau}\right| + K\sup_{-\infty\leq u\leq t}\left|\frac{H(u+\tau, \overline{x}(u-\tau_1) - H(u, \overline{x}(u-\tau_1)}{\tau}\right| + \epsilon K\sup_{-\infty\leq u\leq t}\left|\frac{\overline{x}_{u+\tau} - \overline{x}_u}{\tau}\right|$$

Since the right member of the previous equation is increasing in t, we obtain :

$$(1-\epsilon K)\sup_{-\infty\leq u\leq t}\left|\frac{\overline{x}_{u+\tau} - \overline{x}_u}{\tau}\right| \leq K\sup_{-\infty\leq u\leq t}\left|\int_0^1 \dot{h}(u+v\tau)dv\right| + K\sup_{-\infty\leq u\leq t}\left|\int_0^1 [\partial_1 H(u+v\tau, \overline{x}(u-\tau_1)]dv\right|$$

Since the right member tends uniformly to 0, we obtain that $\overline{x}$ is derivable and then C^1.

□

Corollary 6.16. *Under Assumption 6.10 and the assumption the regularity assumption of g and f then all the continuous periodic solutions of the system (6.1) are C^3.*

Proof. The proof is a direct consequence of Lemma 6.15. □

6.1.5 Harmonic Transfer Function

This subsection gives the link between the spectrum of the monodromy operator associated to the periodic linear hybrid delay system (6.3) and the singularities of the harmonic transfer function. The proof is similar to the one given for the neutral differential equation in Chapter 2 and the slight changes come from the fact that we have to handle different functional spaces and to apply the variation of constant formula (6.25). Contrary to the example given in Chapter 2, we need to assume that we verify some detectability and stabilizability assumptions to ensure that we do not lose some poles in the harmonic transfer function.

We consider an input-output system for the hybrid delay system (6.3). Let $u \in L^2_{loc}([0,+\infty),\mathbb{R})$ the input and let the $z(t) \in \mathbb{R}$ the output at the time $t \geq 0$, more precisely :

$$\begin{cases} \frac{dx(t)}{dt} = A_1(t)x(t) + \mathbf{B_1}(t)y_t + C_1(t)u(t) \\ \mathbf{B_2}(t)y_t = A_2(t)x(t) + C_2(t)u(t) \\ z(t) = \mathbf{F}(t)(x(t), y_t) + C_3(t)u(t),\ t \geq 0, \end{cases} \tag{6.43}$$

where $C_3(\cdot)$ is a scalar T-periodic function and two times continuously differentiable and :

$$\mathbf{F}(t)(x(t), y_t) = \sum_{i=0}^{N} B_i^3(t)y(t-\tau_i) + A_3(t)x(t), \tag{6.44}$$

with $A_3(\cdot)$ a $1 \times k$ T-periodic matrix and two times continuously differentiable. The $(B_i^3)_{i\in[|0,N|]}(t)$ represent a family of matrices of size $1\times n$ T-periodic matrix and two times continuously differentiable. We assume that the system is zero before the time zero, ie $x(t)$, $y(t)$ and $z(t)$ are zero for $t < 0$. For an input-output system we have that the output $z(t)$ is a convolution of the input $u(t)$. The kernel of this operator is called the impulsional response to the system. In the case of periodic ordinary differential equation, the structure of the the kernel is well known through the variation of constant formula that's why we generalize this idea in the case of hybrid periodic system in the following theorem :

Theorem 6.17. *We have :*

$$z(t) = \int_0^{t^+} d_\alpha \tilde{K}(t,\alpha)u(\alpha),\ t \geq 0, \tag{6.45}$$

where :

$$d_\alpha \tilde{K}(t,\alpha) = \mathbf{F}(t)d_\alpha K(t,\alpha)(C_1(\alpha), C_2(\alpha)) + C_3(t)\delta_0(t-\alpha), \tag{6.46}$$

where $K(t,\alpha)$ is defined in the equation (6.24) and $(C_1(\alpha), C_2(\alpha))$ is understood as a column vector.

Proof. Putting the variation constant formula (6.25) in the equation (6.43) leads to the result. □

Theorem 6.18. *For all τ real, the distribution $t \longmapsto d_s\tilde{K}(\cdot, \cdot - \tau)$ is T-periodic and two times continuously differentiable. Moreover for all t real there exists $K, \gamma > 0$ such that :*

$$\int_0^\tau \left\| \frac{\partial^j}{\partial t^j} d_s\tilde{K}(t, t-s) \right\| \leq Ke^{\gamma\tau},\ \textit{for } j = \{0,1,2\}. \tag{6.47}$$

Proof. The result is obtained by combination of Proposition 6.13 and Theorem 6.17. □

Definition 6.19 (ITF)**.** The function :

$$G(s,t) = \int_0^{+\infty} d_\tau \tilde{K}(t, t-\tau)e^{-s\tau}, \tag{6.48}$$

for $s \in \mathbb{C}$ is called the instantaneous transfer function.

Proposition 6.20. *The function $t \mapsto G(t,s)$ is C^2 and T periodic.*

Proof. It is just a corollary of Theorem 6.18. □

The properties of the ITF are similar to the classical transfer function. We list here the most important (see [Lou14]) :

- The response to an exponential complex function $e^{i\omega t}$ is :

$$y(t) = G(t, i\omega)e^{i\omega t} \tag{6.49}$$

- If we consider zero as an initial data, we have :

$$y(t) = \mathcal{L}^{-1}\{G(s,t)U(s)\}, \tag{6.50}$$

  where $\mathcal{L}^{-1}$ is the inverse of the Laplace transform.

Developing in Fourier series the ITF :

$$G(s,t) = \sum G_n(s)e^{i\omega_0 nt}, \tag{6.51}$$

it leads to the concept of the harmonic transfer function. The harmonic transfer function is an infinite matrix which relate the output at each frequency $s + \frac{2i\pi k}{T}$ for a fixed $k \in \mathbb{Z}$ with the input at all frequency $s + \frac{2i\pi n}{T}$ for all $n \in \mathbb{Z}$. More precisely :

Definition 6.21 (HTF)**.** The infinite matrix $H(s)$ defined by $H_{m,n}(s) := G_{m-n}(s - \frac{2i\pi n}{T})$ for $s \in \mathbb{C}$ is called the harmonic transfer function.

Proposition 6.22. *There exists $K(\Re(s))$ real positive which depend only of the real part enough large of s complex such that :*

$$|G_n(s)| \leq \frac{K(\Re(s))}{1+n^2}. \tag{6.52}$$

Moreover, letting $U = \mathcal{L}\{u\}$ and $Z = \mathcal{L}\{z\}$, we have :

$$Z(s + in\omega_0) = \sum_{n\in\mathbb{Z}} H_{m,n}(s)U(s + in\omega_0).$$

Proof. The first part of the proposition is implied by Proposition 6.20 and the second part come from the first part of the proposition. □

In the view to bring out the monodromy operator, we discretise the continuous dynamical system in a discrete dynamical one. For that we consider the input and the output on interval of length T. Put :

$$\widetilde{u}_k := u(kT + t) \text{ for } t \in [0,T] \text{ and } k \in \mathbb{Z}$$
$$\widetilde{z}_k := z(kT + t) \text{ for } t \in [0,T] \text{ and } k \in \mathbb{Z},$$

Thus we have the following discrete system :

Theorem 6.23. *We have for all integer positive k :*

$$\begin{cases} (\tilde{x},\tilde{y})_{(k+1)T} &= \widetilde{A}(\tilde{x},\tilde{y})_{kT} + \widetilde{B}\widetilde{u}_k \\ \widetilde{z}_k &= \widetilde{C}(\tilde{x},\tilde{y})_{(k+1)T} + \widetilde{D}\widetilde{u}_k \end{cases} \tag{6.53}$$

where :

- $\begin{array}{rl} \widetilde{A} : \mathbb{R}^k \times L^2([-\tau_N, 0], \mathbb{R}^n) & \longrightarrow \mathbb{R}^k \times L^2([-\tau_N, 0], \mathbb{R}^n) \\ v & \longmapsto U_2(T, 0)v. \end{array}$

- $\begin{array}{rl} \widetilde{B} : L^2([0, T], \mathbb{R}) & \longrightarrow \mathbb{R}^k \times L^2([-\tau_N, 0], \mathbb{R}^n) \\ v & \longmapsto \int_0^{T+\cdot} d[\tilde{K}(T, \alpha)]v(\alpha), \end{array}$

- $\begin{array}{rl} \widetilde{C} : \mathbb{R}^k \times L^2([-\tau_N, 0], \mathbb{R}^n) & \longrightarrow L^2([0, T], \mathbb{R}) \\ v & \longmapsto \mathbf{F}(\cdot)U_2(\cdot, 0)v \end{array}$

- $\begin{array}{rl} \widetilde{D} : L^2([0, T], \mathbb{R}) & \longrightarrow L^2([0, T], \mathbb{R}) \\ v & \longmapsto \int_0^{\cdot} d_\alpha \tilde{K}(\cdot, \tau)v(\tau)d\tau, \end{array}$

- $(\tilde{x}, \tilde{y})_0 = 0$

Proof. The use of the variation of constant formula (6.25) in the system (6.43), we obtain for $t \in \mathbb{R}$:

$$\begin{cases} (\tilde{x}, \tilde{y}_t)_{(k+1)T} & = U_2((k+1)T, kT)(\tilde{x}, \tilde{y}_t)_{kT} + \int_{kT}^{(k+1)T} d_\alpha \tilde{K}((k+1)T, \alpha)u(\alpha), \\ z(t + kT) & = \mathbf{F}(t + kT)\left[U_2(t, 0)(\tilde{x}, \tilde{y}_t)_{kT} + \int_0^{t^+} d_\alpha \tilde{K}(t, \alpha)u(\alpha)\right] + \int_0^{t^+} \delta_0(t - \tau)C_3(\tau)u_k(\tau)d\tau. \end{cases}$$

By periodicity we have the result. □

For a sequence of operator $a := (a_n)_{n\in\mathbb{N}}$ we define the $p - transform$ as:

$$\hat{a} := \sum_{n\in\mathbb{N}} a_n p^{-n}, \tag{6.54}$$

when for $p \in \mathbb{C}$ this series exists. Taking the $p - transform$ in the equations (6.53), we have :

$$\widehat{Z} = [\widehat{C}(pId - \widehat{A})^{-1}\widehat{B} + \widehat{D}]\widehat{U}. \tag{6.55}$$

On in the other side, we can compute the impulsional response of the discrete system and we have :

$$\widetilde{z}_n(t) = \sum_{k=0}^{n} H_{[k]}\widetilde{u}_{n-k}(t), \tag{6.56}$$

where :

$$H_{[k]}v(t) = \int_0^T d_\tau \tilde{K}(kT + t, \tau)v(\tau) \tag{6.57}$$

is the discrete impulsional response of the discretised dynamical system. Taking the $p - transform$ in the equation (6.56) for p with great modulus (which is licit because of Proposition 6.18), we have :

$$\widehat{Z}(p) = \widehat{H}(p)\widehat{U}(p). \tag{6.58}$$

We have that $\tilde{A} = U_2(T, 0)$ is a compact perturbation of the operator $T_{D,2}(T, 0)$. We note a the spectral radius of the operator $T_D(T, 0)$ and $(\zeta_1, \cdots, \zeta_n)$ the eigenvalues of $U_2(T, 0)$ outside the disk we radius a and centrum 0. We introduce the set $P_a := \{z \in \mathbb{C} || z| > a\}$

Theorem 6.24. *For all $p \in P_a/\{\zeta_1, \cdots, \zeta_n\}$, we have :*

$$\widehat{H}(p) = \widetilde{C}(pId - \widetilde{A})^{-1}\widetilde{B} + \widetilde{D}, \tag{6.59}$$

and ζ_i is possibly a pole of $\widehat{H}(p)$ for all $i \in \{1, \cdots, n\}$.

Proof. The equation (6.59) hold for p with a huge modulus and by the analytic continuation, we have the result. □

Lemma 6.25. *We have :*

$$G(s,t) = e^{-st}\widehat{H}(e^{sT})e^{st} \tag{6.60}$$

Proof. We have, by the definition of the $p-transform$ and the equation (6.57), for $v \in \mathbb{L}^2([0,T],\mathbb{R})$ and $t \in [0,T]$ that:

$$\begin{aligned}
\widehat{H}(z)v(t) &= \sum_{k=0}^{+\infty} H_{[k]}v(t)p^{-k} && (6.61)\\
&= \sum_{k=-\infty}^{+\infty} p^{-k}\int_0^T d_\tau \tilde{K}(kT+t,\tau)v(\tau). && (6.62)
\end{aligned}$$

and on the other side the equation (6.48) where we performed an integration by substition $(t \to t-\tau)$ and using the periodicity of the system leads to :

$$\begin{aligned}
G(s,t) &= \int_{-\infty}^{t} d_\tau \tilde{K}(t,\tau)e^{s(\tau-t)} && (6.63)\\
&= \int_{-\infty}^{+\infty} d_\tau \tilde{K}(t,\tau)e^{s(\tau-t)} && (6.64)\\
&= \sum_{k=-\infty}^{+\infty}\int_0^T d_\tau \tilde{K}(t,\tau-kT)e^{s(\tau-t-kT)} && (6.65)\\
&= \sum_{k=0}^{+\infty} e^{-s(t+kT)}\int_0^T d_\tau \tilde{K}(t+kT,\tau)e^{s\tau} && (6.66)
\end{aligned}$$

and thus we have the result with $p = e^{sT}$. □

We introduce the sets $\tilde{P}_a := \{z \in \mathbb{C} | \Re(z) \geq ln(|a|)/T\}$, $E_i := ker(\zeta_i id - \tilde{A})$ and $\tilde{E}_i := \{\phi \in L^2([0,T],\mathbb{R}) | \tilde{B}\phi \in E_i\}$ for $i = 1, \cdots, n$. We make the assumption :

Assumption 6.26.
- *The set $\tilde{E}_i$ is non empty for all $i = 1, \cdots, n$.*
- *$\tilde{C}\tilde{B}\tilde{E}_i$ is a non-zero operator for all $i = 1, \cdots, n$.*

.

The first and the second items of the assumption translate stabizability and detectability of the discrete system (6.53) respectively.

Theorem 6.27. *We have that the harmonic transfer function $H(p)$ verifies the following properties:*

- *the singularities in P_a are possibly poles of the form $\frac{ln(\zeta_j)+2ik\pi}{T}$ for $k \in \mathbb{Z}$ and $j \in \{1, \cdots, n\}$ for a certain integer n and define a continuous operator on l^2 to l^2 elsewhere.*

- *under the assumption (6.53), for all $j \in \{1, \cdots, n\}$, there exists a $k \in \mathbb{Z}$ such that $\frac{ln(\zeta_j)+2ik\pi}{T}$ is a pole of the harmonic transfer function and there exists an element of the infinite matrices which possesses the pole too. Moreover if the harmonic transfer function is analytic in the right half plane then the periodic solution $(\overline{x}(\cdot), \overline{y}(\cdot))$ of the system (6.1) is locally exponentially stable with the C^0 norm; i.e. all solution which starts relatively close to the periodic solution of the system (6.1) tends exponentially to this periodic solution.*

proof. Fix $s \in \mathbb{C}$ with real part enough large. Let

$$\Lambda(s) : L^2([0,T],\mathbb{R}) \to L^2([0,T],\mathbb{R}), \tag{6.67}$$

where $[\Lambda(s)\phi](t) = e^{-st}[\hat{H}(e^{sT})e^{s\cdot}\phi(\cdot)](t)$ for all $\phi \in L^2([0,T],\mathbb{R})$. We have that the Fourier development of $\Lambda(s)\phi(t)$ is :

$$\Lambda(s)\phi(t) = \sum_{n\in\mathbb{Z}} \left(\sum_{k\in\mathbb{Z}} a_k G_{n-k}(s + 2i\pi k/T) \right) e^{2i\pi n/T}, \tag{6.68}$$

with $(a_k)_{k\in\mathbb{Z}}$ the Fourier coefficients of ϕ. In fact, when $\phi \in C^2$, we deduce from Lemma 6.25, Proposition 6.22, the regularity of the Fourier coefficient of ϕ that the equation (6.68) holds. By a density argument, the equation (6.68) holds as well for a function $\phi \in L^2([0,T],\mathbb{R})$.

By Parseval's identity :

$$\left.\begin{array}{r}\text{The HTF } H(s) \text{ is} \\ \text{a bounded operator on } l^2(\mathbb{Z})\end{array}\right\} \Leftrightarrow \Lambda(s) \text{ is a bounded operator on } L^2([0,T],\mathbb{R}).$$

$\Lambda(s)$ is a bounded operator on L^2 if and only if e^{sT} is not in the spectrum of the monodromy operator $U_2(T,0)$, we have that $H(s)$ admits an analytic continuation on the domain P_a except on the $\frac{ln\zeta+2i\pi k}{T}$ where ζ is an element of the spectrum of the monodromy operator of $U_2(T,0)$. Put ζ_i for some i in $\{1,\cdots,n\}$ an isolated eigenvalue of $U_2(T,0)$ and $\tilde{E}_i$ its kernel. Under the Assumption 6.26, $\tilde{E}_i$ is not empty. For all $\phi \in \tilde{E}_i$ we have by the representation of the resolvent of $\tilde{A}$:

$$\hat{H}(z)\phi = \sum_{k=0}^{p} \frac{\tilde{C}N^k\tilde{B}\phi}{(z-\zeta_i)^{k+1}}, \tag{6.69}$$

where N a nilpotent operator, for all z complex in a neighborhood of ζ_i and p is an integer. Put s_0 such that $e^{s_0T} = \zeta_i$, s is a complex in the neighborhood of s_0 and $\Psi_s(\cdot) = e^{-s\cdot}\phi(\cdot)$. Thus :

$$\lim_{s\to s_0} \left(e^{sT} - e^{s_0T}\right)^{p+1} [\Lambda(s)\phi](t) = e^{-s_0t}\left[\tilde{C}\tilde{B}\Psi_{s_0}(\cdot)e^{s_0\cdot}\right](t). \tag{6.70}$$

Since the discrete system is detectable *via* the Assumption 6.26, the right hand side of the equation (6.70) is not identically equal to zero. We deduce that there exists n' and k' such that $\lim_{s\to s_0} \left(e^{sT} - e^{s_0T}\right)^{p+1} G_{n-k}(s + 2i\pi k/T)$ is non-zero. We deduce the result from the Theorem 6.8 and Theorem 6.14. □

6.2 Circuit containing lossless transmission lines

In this section we apply the results of Section 6.1 in the case of the circuit containing lossless transmission line. We consider a circuit made of :

- resistors, diodes, periodic generator, transistors, capacitors and inductors with the nonlinearities at least C^4,
- lossless transmissions lines.

This kind of circuit leads to a hybrid differential nonlinear delay equation (equation (3.17) in Chapter 3) which is an equation like the equation (6.1). We assume that there exists a continuous periodic solution to this system.

The main result (Theorem 6.36) of the section is the fact that the harmonic transfer function associated to a linearised circuit containing transmission lines around a periodic solution, and made of dissipative multiport at high frequency, have just isolated poles in the closed right half plan; moreover if the harmonic transfer function is holomorphic in the closed right half plan then the periodic solution is locally exponentially stable for the uniform norm (under some controllability and observability assumptions). Before the statement of our main result in Section 6.2.3, we start by defining properly the concept of the dissipativity at high frequency and the concept of realistic electrical components in the view to apply the results of Chapter 5. The results of the sections 6.2.1 and 6.2.2 can be found in [BCC+18] in the frequency domain for the time-invariant systems, however we adapt freely the definitions and propositions in the time domain (periodic system) for our purposes.

6.2.1 Dissipativity at high frequency for linearised components

We consider the components of Chapter 1, *i.e.* capacitors, diodes, periodic generator, inductors, resistors and transistors which are linearised around a periodic trajectory. Putting together these components, we can form a multiport like we did in the chapter 3. We consider in this section the multiport linearised around a periodic trajectory. As we already saw in Chapter 3, formally speaking, a multiport is a directed graph with labeled vertices (called junction nodes), and edges (called branches). Branches correspond to the components and nodes to terminals thereof.

To each junction node j is associated a potential v_j, and to each edge k an electric current i_k. One of the junction nodes, say v_n, is the ground (its potential is 0 by convention). *We always assume that the graph associated with a multiport is connected.* The currents $i_m := (i_{k_1}, \cdots, i_{k_m})$ are plugged in between the ground and the junction nodes $(k_1, \cdots, k_m)$ of the circuit containing the components (see figure 6.1 below), where m is an integer which denotes the number of output of the multiport.

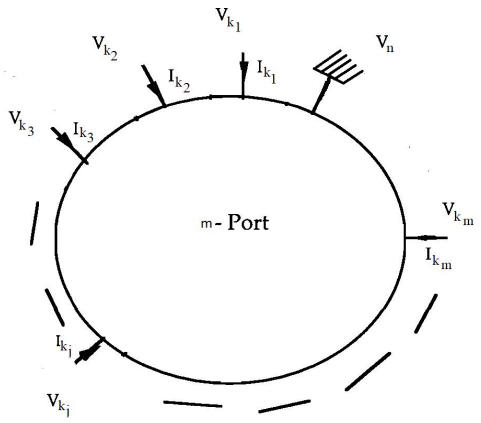


Figure 6.1 : A m multiport

To give the equations which govern the behavior of a such multiport, we assume that the voltage of such multiport $v_m := (v_{k_1}, \cdots, v_{k_m})$ can be expressed with the voltage of the capacitors, the current of the inductors and i_m the a current of the multiport thus by the Brayton-Moser theorem (see [BM64]), we have that the equation of the multiport is :

$$\begin{cases} C_m \frac{dv_{c_m}}{dt} = A_{c_m}(t)v_{c_m}(t) + B_{c_m}(t)i_{l_m}(t) + C_{c_m}(t)i_m \\ L_m \frac{di_{l_m}}{dt} = A_{l_m}(t)v_{c_m}(t) + B_{l_m}(t)i_{m_p}(t) + C_{l_m}(t)i_m, \\ v_m = A_m(t)v_{c_m}(t) + B_m(t)i_{l_m}(t) + C_m(t)i_m, \end{cases} \tag{6.71}$$

where v_{c_m} (resp. i_{l_m}) denotes the voltage (resp. current) of the capacitors (resp. inductors) contained in the multiport and the matrices involved in the equation (6.71) are T-periodic, two times continuously differentiable and with the size required to give a meaning to the equation. It has to be noted that the equation (6.71) is just the equation (3.2) linearised around a periodic trajectory.

We now introduce the concept of the dissipativity at high frequency. In fact, all electrical components physically built dissipate energy when it operates at high frequency. For us, the behavior at high frequency of a multiport is the behavior of the multiport where the capacitors becom wires and the inductors become open switches, and thus there is just the resistive effects at high frequency. The multiport linearised around a periodic trajectory has negative resistors which come from the linearisation of the nonlinearies. The idea of the dissipativity at high frequency means that the negative resistors do not play any role at high frequency because their effects are reduced by the capacitive and inductive effects.

From a mathematical viewpoint, the behavior at high frequency of the multiport is obtained when we replace the voltage v_{c_m} and the current i_{l_m} by zero in the equation (6.71). Then the equation of the multiport at high frequency is :

$$v_m = C_m(t)i_m, \tag{6.72}$$

and the concept of the dissipativity at high frequency is introduced in Definition 6.28 below.

Definition 6.28. We say that a multiport is dissipative at high frequency if there exists $\alpha > 0$ independent of the time $t \in \mathbb{R}$ such that :

$$C_m(t) + C_m(t)^* \geq \alpha I_m. \tag{6.73}$$

We introduce the following classic lemma 6.29 which simply means that a real matrix with positive definite real part is invertible and the inverse has a positive definite symmetric part.

Lemma 6.29. *If a multiport is dissipative at high frequency then C_m is invertible and there exists $\tilde{\alpha}$ independent of the time $t \in \mathbb{R}$ such that :*

$$C_m^{-1}(t) + (C_m^{-1}(t))^* \geq \tilde{\alpha} I_m. \tag{6.74}$$

Proof. Let t a real. We can find in [Joh70] that the symmetric matrix $C_m(t) + C_m(t)^*$ is definite positive if and only if $x^* C_m(t) x > 0$ for all real non-zero vector x. We deduce that $C_m(t)$ is invertible. Since for all real non-zero vector x there exists a real vector non-zero vector y such that $y = C_m^{-1}(t)x$, we have :

$$x^* C_m^{-1}(t) x = y^* C_m(t) y > 0, \tag{6.75}$$

and then we have that $\mathrm{C}_m^{-1}(t) + (C_m^{-1}(t))^*$ is a definite positive matrix, and by periodicity and continuity we deduce the equation (6.74). □

We have the following nice property which says that the composition of multiport dissipative at high frequency is dissipative at high frequency. This result is well known by the electrical engineers but it is difficult to find a proper proof of this fact that is why we follow the proof in [BCC+18] (thank you Sylvain Chevillard for writing this clear proof) :

Proposition 6.30. *A multiport which is composed of dissipative multiports at high frequency is dissipative at high frequency.*

Proof. We consider the behavior of the multiport at high frequency. To express the effect of $(i_{k_1}, \cdots, i_{k_m})$ on the potential $(v_{k_1}, \cdots, v_{k_m})$, we use nodal analysis which is a classical method to derive voltages at the nodes of a circuit in terms of the branch currents [CC98, sect.2.9]. Specifically, we denote by $V = (v_1, \ldots, v_{n-1})^*$ the vector of all node voltages (except v_n, the reference ground voltage) and by $I = (i_1, \ldots, i_p)^*$ the vector of all currents in the branches. The *(node-branch) incidence matrix* of the circuit, say $A = (A_{ij})$, has $n-1$ rows corresponding to the nodes (except the ground) and p columns corresponding the branches. It is defined by the rule:

$$\begin{cases} A_{ij} = 1 \text{ if edge } e_j \text{ is incident away from node } i, \\ A_{ij} = -1 \text{ if edge } e_j \text{ is incident towards node } i, \\ A_{ij} = 0 \text{ otherwise.} \end{cases}$$

Since the graph is connected, A has full row rank $(n-1)$ [Che91, Th. 2.1]. Now, because we plug in $(i_1, \cdots, i_m)$ at nodes $(k_1, \cdots, k_m)$, Kirchhoff's law gives us

$$A\,I = \begin{pmatrix} 0 & \cdots & i_{k_1} & 0 & \ldots & 0 & i_{k_m} & 0 & \ldots & 0 \end{pmatrix}^*. \tag{6.76}$$

Next, we substitute currents with voltages using the relations given by the equations of the multiport at high frequency. For this, we form the *branch admittance matrix*, a block diagonal matrix $\mathbf{Y}_b(t) = \mathrm{diag}(C_{m_1}^{-1}(t), C_{m_2}^{-1}(t), \ldots, C_{m_p}^{-1}(t))$ where the $C_{m_j}^{-1}(t)$ is the inverse matrix of the multiport j and thanks to Lemma 6.29 satisfies :

$$C_{m_j}^{-1}(t) + (C_{m_j}^{-1}(t))^* \geq \alpha_j I_{m_j}, \tag{6.77}$$

for $\alpha_j > 0$ and for all $j = 1, \cdots, p$. With a convenient ordering of nodes and edges, it holds that $I = \mathbf{Y}_b A^* V$, and (6.76) yields

$$\mathbf{Y}\,V = \begin{pmatrix} 0 & \cdots & i_{k_1} & 0 & \ldots & 0 & i_{k_m} & 0 & \ldots & 0 \end{pmatrix}^*. \tag{6.78}$$

where $\mathbf{Y}(s)$ is a $(n-1) \times (n-1)$ matrix, called *nodal admittance matrix*, which is related to the branch admittance matrix through ([CC98] eq. (2.9.8))

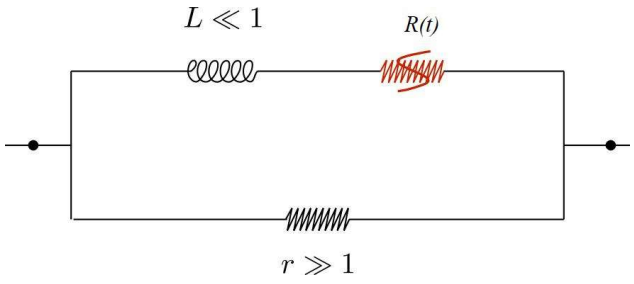
$$\mathbf{Y} = A\mathbf{Y}_b A^*. \tag{6.79}$$

Because of the equation (6.77), we have that $\mathbf{Y}_b + \mathbf{Y}_b^* \geq \alpha I_m$ for one $\alpha > 0$ and since the incident matrix A has full rank, by the assumed connectivity of the graph of the multiport, we have that there exists $\tilde{\alpha} > 0$ such that $\mathbf{Y} + \mathbf{Y}^* \geq \tilde{\alpha} I_m$. Thus $\mathbf{Y}^{-1} + (\mathbf{Y}^{-1})^* \geq \tilde{\alpha} I_m$ is a symmetric definite positive matrix by Lemma 6.29. Thus the symmetric real part of the matrix which links $(v_{k_1}, \cdots, v_{k_m})$ and $(i_{k_1}, \cdots, i_{k_m})$ is a principal submatrix of a definite positive matrix and so is definite positive. Then we have that the multiport is dissipative at high frequency. □

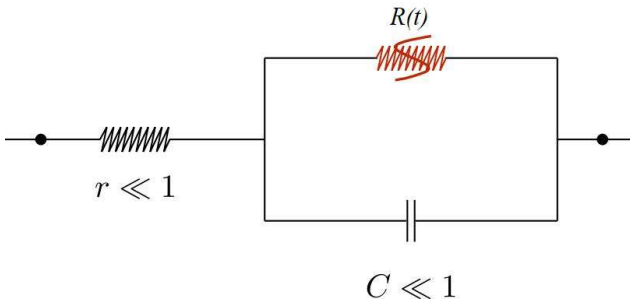
6.2.2 Realistic models of linearised active components

The only elementary electronic elements which fail to be dissipative at high frequency are obviously the linearised diode (equation (1.3) Chapter 1) and the linearised transistor (equation (1.6) Chapter 1). In fact these models represent the behaviour of an ideal diode and transistor and do not take account the dissipativity and the stability of a diode or a transistor when the circuit occurs at high frequency. An equivalent way to say the same thing is that the models of the diodes and transistors do not take account the capacitive or inductive effects. We propose below models which are more realistic than those in Chapter 1 because they verify the dissipative at high frequency property. The models proposed below can be found in [BCC+18] .

- Concerning the diodes, a realistic models would be those in Figure 6.2. The model (a) in Figure 6.2 represents a resistor $(r \gg 1)$ in parallel of an inductor $(L \ll)$ and the linearised ideal diode $(R(t))$. The model (b) in the Figure 6.2 represents a small resistor $(r \ll 1)$ coupled with a capacitor $(C \ll 1)$ and the linearised diode $(R(t))$ in parallel.



(a) With inductive effect and high resistance (e.g., of the air around the diode)



(b) With capacitive effect and small resistance (e.g., of the wire)

Figure 6.2 : Two realistic models of a linearised diode

Proposition 6.31. *The models of the diodes in Figure 6.2 are dissipative at high frequency.*

Proof. It is obvious that the behavior at high frequency of the model (a) and (b) are $v_m = ri_m$ and then we have the result. □

- Transistors can also be modeled in a realistic way to account for the fact that actual devices have no gain anymore at very high frequencies. It has to be noted that the model the Figure 6.3, we have that $g_m(t)$ and $g_d(t)$ are strictly positive.

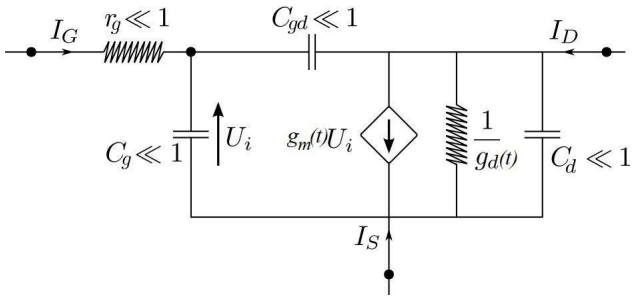


Figure 6.3 : Intrinsic model of a linearised transistor

Proposition 6.32. *The model presented in Figure 6.3 is dissipative at high frequency.*

Proof. The Kirchhoff laws give that the matrix which at high frequency link (I_G, I_D) and (V_{GS}, V_{GD}) is :

$$Y = \begin{pmatrix} \frac{1}{r_g} & -\frac{1}{r_g} \\ -\frac{1}{r_g} & g_d(t) + g_m(t) + \frac{1}{r_g} \end{pmatrix}. \tag{6.80}$$

We see that this matrix has a definite positive real part and then the inverse of this matrix has a positive definite real part. □

6.2.3 Structure of the Harmonic Transfer Function for dissipative circuit at high frequency

We consider a circuit $\mathcal{C}$ made of resistors, diodes, transistors, periodic generator, capacitors, inductors and lossless transmission lines where the nonlinearities are at least C^4. Since Chapter 3 is far away we resume this chapter without the technical details and we rewrite the key equations which have been obtained. We interpreted a such circuit as a graph where the nodes were multiports (resistors, diodes, periodic generator, transistors, capacitors and inductors) interconnected with the edges representating the lossless transmission lines. Under the Brayton-Moser assumption, we arrived to a nonlinear hybrid delay equation :

$$\begin{cases} 0 = \check{h}_1\left(t, \begin{pmatrix} x_1(t) \\ \vdots \\ x_N(t) \\ y_1(t) \\ \vdots \\ y_N(t) \end{pmatrix}, \begin{pmatrix} x_1(t-\tau_1) \\ \vdots \\ x_N(t-\tau_N) \\ y_1(t-\tau_1) \\ \vdots \\ y_N(t-\tau_N) \end{pmatrix}, z(t)\right), \\ \frac{dz(t)}{dt} = \check{h}_2\left(t, \begin{pmatrix} x_1(t) \\ \vdots \\ x_N(t) \\ y_1(t) \\ \vdots \\ y_N(t) \end{pmatrix}, \begin{pmatrix} x_1(t-\tau_1) \\ \vdots \\ x_N(t-\tau_N) \\ y_1(t-\tau_1) \\ \vdots \\ y_N(t-\tau_N) \end{pmatrix}, z(t)\right). \end{cases} \tag{6.81}$$

We assume that the system (6.81) admits a periodic solution which is forced by the presence of a periodic generator. We are then able to linearise the components of the circuit around the periodic trajectory. From now we do the following dissipativity assumption :

Assumption 6.33. *The circuit is made of dissipative at high frequency linearised components.*

We can remark that Assumption 6.33 is verified in the particular case where we take the linearised models of the diode and transistor that we can find in Section 6.2.2 (Figure 6.2 and 6.3). Obviously, we could take another diode or transistor modelization as long as it satisfies the Definition 6.28. Since a multiport which is composed of dissipative multiports at high frequency is dissipative at high frequency through Proposition 6.30, we have that each node of the circuit is a multiport dissipative at high frequency. It means that at high frequency, the circuit dissipates the energy in each nodes. After the linearisation of the equation (6.81) around the periodic solution, we obtain the equation :

$$\begin{cases} \begin{pmatrix} x_1(t) \\ \vdots \\ x_N(t) \\ y_1(t) \\ \vdots \\ y_N(t) \end{pmatrix} = -\left(I + \mathbf{A}(t)\,\mathbf{K}\right)^{-1}\left(I - \mathbf{A}(t)\,\mathbf{K}\right) P_2 \begin{pmatrix} x_1(t-\tau_1) \\ \vdots \\ x_N(t-\tau_N) \\ y_1(t-\tau_1) \\ \vdots \\ y_N(t-\tau_N) \end{pmatrix} + \mathbf{A}_{\mathbf{z}}(t) z(t), \\ \frac{dz(t)}{dt} = \mathbf{B}(t)\mathbf{K}\left[\begin{pmatrix} x_1(t) \\ \vdots \\ x_N(t) \\ x_1(t-\tau_1) \\ \vdots \\ x_N(t-\tau_N) \end{pmatrix} + \begin{pmatrix} y_1(t) \\ \vdots \\ y_N(t) \\ y_1(t-\tau_1) \\ \vdots \\ y_N(t-\tau_N) \end{pmatrix}\right] + \mathbf{B}_{\mathbf{z}}(t) z(t). \end{cases} \tag{6.82}$$

Assumption 6.33 implies that the there exists a positive number α, independent of t, such that :

$$\mathbf{A}(t) + \mathbf{A}^*(t) \geq \alpha\, Id, \qquad \alpha > 0 \qquad t \in \mathbb{R}. \tag{6.83}$$

The invertibility of the matrix $I + \mathbf{A}(t)\,\mathbf{K}$ is induced by the equation (6.83) and then the equation (6.82) has always a sense. Putting $z = 0$ we have that the behavior of the circuit at high frequency (equation (6.82)) is a periodic delay system of the form :

$$\begin{pmatrix} x_1(t) \\ \vdots \\ x_N(t) \\ y_1(t) \\ \vdots \\ y_N(t) \end{pmatrix} = -\left(I + \mathbf{A}(t)\,\mathbf{K}\right)^{-1}\left(I - \mathbf{A}(t)\,\mathbf{K}\right) P_2 \begin{pmatrix} x_1(t-\tau_1) \\ \vdots \\ x_N(t-\tau_N) \\ y_1(t-\tau_1) \\ \vdots \\ y_N(t-\tau_N) \end{pmatrix}. \tag{6.84}$$

The dissipativity at high frequency of the linearised components leads to the exponential stability of the high frequency system (6.84) associated to the circuit and that all continuous periodic solution are at least C^4 because the regularity assumption on the nonlinear components of the circuit.

Theorem 6.34. *The periodic delay system (6.84) is exponentially stable with the C^0 norm.*

Proof. The dissipativity at high frequency Assumption 6.33 implies that the equation (6.83) is verified and then we can apply Theorem 5.21 of Chapter 5 to have the result. □

Theorem 6.35. *The continuous periodic solution to the system (6.1) is at least C^4.*

Proof. It is a direct application of Corollary 6.16. □

Disturbing the linearised system (6.82) by a current i_ϵ and observing the voltage u_ϵ we obtain the following input-output system :

$$\begin{cases} \frac{dz(t)}{dt} = \mathbf{B}(t)\mathbf{K}\left[\begin{pmatrix} x_1(t) \\ \vdots \\ x_N(t) \\ x_1(t-\tau_1) \\ \vdots \\ x_N(t-\tau_N) \end{pmatrix} + \begin{pmatrix} y_1(t) \\ \vdots \\ y_N(t) \\ y_1(t-\tau_1) \\ \vdots \\ y_N(t-\tau_N) \end{pmatrix}\right] + \mathbf{B}_{\mathbf{z}}(t)z(t) + \mathbf{B}_\epsilon(t)i_\epsilon(t), \\ \begin{pmatrix} x_1(t) \\ \vdots \\ x_N(t) \\ y_1(t) \\ \vdots \\ y_N(t) \end{pmatrix} = -\left(I + \mathbf{A}(t)\,\mathbf{K}\right)^{-1}\left(I - \mathbf{A}(t)\,\mathbf{K}\right) P_2 \begin{pmatrix} x_1(t-\tau_1) \\ \vdots \\ x_N(t-\tau_N) \\ y_1(t-\tau_1) \\ \vdots \\ y_N(t-\tau_N) \end{pmatrix} + \mathbf{A}_{\mathbf{z}}(t)z(t) + \mathbf{A}_\epsilon(t)i_\epsilon(t), \\ u_\epsilon(t) = \mathbf{C}(t)\mathbf{K}\left[\begin{pmatrix} x_1(t) \\ \vdots \\ x_N(t) \\ x_1(t-\tau_1) \\ \vdots \\ x_N(t-\tau_N) \end{pmatrix} + \begin{pmatrix} y_1(t) \\ \vdots \\ y_N(t) \\ y_1(t-\tau_1) \\ \vdots \\ y_N(t-\tau_N) \end{pmatrix}\right] + \mathbf{C}_{\mathbf{z}}(t)z(t) + \mathbf{C}_\epsilon(t)i_\epsilon(t). \end{cases} \tag{6.85}$$

We are now able to state the structure of the harmonic transfer function associated to the system (6.85) and the link with the local stability of the periodic solution, thanks to Section 6.1. In fact, to sum up, the equations (6.81), (6.82), (6.84) and (6.85) are like the equations (6.1), (6.3), (6.8) and (6.43) under Assumption 6.33. Moreover, we proved under Assumption 6.33 that the system (6.84) is exponentially stable. So the equivalent system (6.8) is C^0 exponentially stable and Assumption 6.10 is fulfilled. The theory developed in Section 6.1 is applicable, and the application of Theorem 6.27 leads to the following theorem :

Theorem 6.36 (Structure HTF and local stability)**.** *Consider a circuit $\mathcal{C}$ made of resistors, diodes, periodic generator, transistors, capacitors, inductors and lossless transmission lines where the nonlinearities are at least C^4. Assume the existence of a continuous periodic solution and the dissipativity at high frequency of the linearised components of the circuit. The linearised circuit around the periodic trajectory which is disturbed by a source current i_ϵ, and which has the output response voltage associated v_ϵ, has a harmonic transfer function $H(s)$ verifying the following properties :*

- *there exists an $a < 0$ such that H is a meromorphic operator $l^2(\mathbb{Z}) \to l^2(\mathbb{Z})$ in the complex domain $\{z \in \mathbb{C} | \Re(z) \geq a\}$ which possibly poles at $\left\{ z_{j,k} = \frac{ln(\zeta_j)+2ik\pi}{T}, j \in \{1...n\}, k \in \mathbb{Z} \right\}$ where $(\zeta_j)_{j\in\{1...n\}}$ is a finite family of complex numbers.*

Moreover, under the stabilizability and detectability Assumption 6.26,

- *for all $j = 1, \cdots, n$ there exists at least one $k \in \mathbb{Z}$ such that $z_{j,k}$ is a pole of a coefficient of the matrix H.*
- *if the harmonic transfer function is analytic in the closed right half plane, then the periodic solution is locally stable; i.e. all solution which starts relatively close to the periodic solution of the system (6.81) tends exponentially to this periodic solution.*

Appendix A

Appendix

A.1 Proof of formula (4.24)

A.1.1 reduction to a Volterra equation

Observe from (4.22) that for $s+\tau_N \le t$, $\alpha \mapsto X(t,\alpha)$ has bounded variation on $[s, s+\tau_j]$ for each j and satisfies

$$d_\alpha X(t,\alpha) = \sum_{j=1}^{N} D_j(t)\, d_\alpha X(t-\tau_j,\alpha) \qquad \text{on} \quad [s, s+\tau_j]. \tag{A.1}$$

Substituting (A.1) in (4.24) yields (6.6), provided that $t \ge s+\tau_N$. Hence, by uniqueness of a solution y to (6.6) satisfying $y(s+\theta) = \phi(\theta)$ for $\theta \in [-\tau_N, 0]$, it is enough to check (4.24) for $s \le t < s+\tau_N$. For this, we adopt the point of view of reference [HVL93], which is to construe delay systems as Stieltjes-Volterra equations upon representing delays by measures. More precisely, we can rewrite (6.6) as a Lebesgue-Stieltjes integral:

$$y(t) = \int_{-\tau_N^-}^{0^-} d\mu(t,\theta) y(t+\theta), \qquad t \ge s, \tag{A.2}$$

with

$$\mu(t,\theta) = \sum_{j=1}^{N} D_j(t) \mathfrak{H}(\theta+\tau_j), \tag{A.3}$$

where $y(\tau)$ is understood to be $\phi(\tau - s)$ when $s-\tau_N \le \tau \le s$ and $\mathfrak{H}(\tau)$ is the Heaviside function which is zero for $\tau \le 0$ and 1 for $\tau > 0$, so that the associated measure on an interval of the form $[0,a]$ or $[0,a)$ is a Dirac delta at 0. Note that $\mathfrak{H}(0) = 0$, which is not the usual convention, but if we defined $\mathfrak{H}$ so that $\mathfrak{H}(0) = 1$ then expanding (A.2) using (A.3) would not give us back (6.6) for the term $D_N(t)y(t-\tau_N)$ would be missing. Observe also, since $\tau_j > 0$ for all j, that the minus sign in the upper bound of the integral in (A.2) is immaterial and could be traded for a plus. For $s \le t \le s+\tau_N$, singling out the initial data in (A.2) yields

$$y(t) = \int_{(s-t)^-}^{0^-} d\mu(t,\theta) y(t+\theta) + f(t) \quad \text{with} \quad f(t) := \int_{-\tau_N^-}^{(s-t)^-} d\mu(t,\theta)\phi(t+\theta-s), \tag{A.4}$$

where we took into account, when separating the integrals, that $\theta \mapsto \mu(t,\theta)$ is left continuous, while the integral over the empty interval is understood to be zero. It will be convenient to study (A.4) for $t \in [s, s+\tau_N]$, even though in the end the values of $y(t)$ only matter to us for $t \in [s, t+\tau_N)$. Define

$$k(t,\tau) := \begin{cases} \mu(t,\tau-t) - \sum_{j=1}^{N} D_j(t) & \text{for } \tau \in [s,t], \\ 0 & \text{for } \tau > t, \end{cases} \qquad t,\tau \in [s, s+\tau_N]. \tag{A.5}$$

Note that $k(t,\tau) = 0$ when $t - \tau < \tau_1$, and $d_\tau k(t,\tau) = d_\tau \mu(t, \tau - t)$ on $[s, s+\tau_N]$ for fixed t. Hence, (A.4) becomes

$$y(t) = \int_{s^-}^{t^-} dk(t,\tau) y(\tau) + f(t), \qquad s \le t \le s + \tau_N. \tag{A.6}$$

Now, (A.6) is the Stieltjes-Volterra equation we shall work with.

It suffices to prove (4.24) for $t \in [s, s+\tau_N)$ under the additional assumption that ϕ and the D_j, which are continuous by hypothesis, also have bounded and locally bounded variation, on $[-\tau_N, 0]$ and $\mathbb{R}$ respectively. Indeed, functions of bounded variation are dense in C_s (resp. $C_0(\mathbb{T}, \mathbb{R}^{d\times d})$), for instance because C^1-functions are, and if ϕ_k converges uniformly to ϕ in C_s while $D_{j,k}$ converges uniformly to D_j in $[s, s+\tau_N]$ as $k \to \infty$, then the solution to (6.6) with initial condition ϕ_k and coefficients $D_{j,k}$ converges uniformly on $[s, s+\tau_N]$ to the solution with initial condition ϕ and coefficients D_j, as is obvious by inspection. Hence, we shall assume without loss of generality that ϕ has bounded variation and the D_j have locally bounded variation. Then, since it follows from (A.3) and (A.4) that

$$f(t) = \sum_{\tau_\ell \in (t-s, \tau_N]} D_\ell(t) \phi(t - s - \tau_\ell), \tag{A.7}$$

it is clear from (A.7) and (4.9) that $f \in BV_l[s, s+\tau_N] \cap BV_r[s, s+\tau_N]$, since it is continuous.

A.1.2 Volterra kernels of type B^∞

Volterra equations for functions of a single variable have studied extensively, see *e.g.* [Bru17, GLS90]. However, the specific assumption that the kernel has bounded variation seems to be treated somewhat tangentially. On the one hand, it is subsumed in the measure-valued case presented in [GLS90, Ch. 10], but no convenient criterion is given there for the existence of a resolvent kernel. On the other hand, [HVL93, Ch. 9, Sec. 1] sketches the main arguments needed to handle kernels of bounded variation, but the exposition has issues[1], which is why we provide a proof in this section.

We define a Stieltjes-Volterra kernel of type B^∞ on $[a,b] \times [a,b]$ as a measurable function $\kappa : [a,b] \times [a,b] \to \mathbb{R}^{d\times d}$, with $\kappa(t,\tau) = 0$ for $\tau \ge t$, such that the partial maps $\kappa(t,.)$ lie in $BV_l[a,b]$ and $\|\kappa(t,.)\|_{BV([a,b])}$ is uniformly bounded with respect to $t \in [a,b]$. In addition, we require that $\lim_{\tau \to t^-} W_{[\tau,t)}(\kappa(t,.)) = 0$ uniformly with respect to t; *i.e.*, to every $\varepsilon > 0$, there exists $\eta > 0$ such that $W_{[\tau,t)}(\kappa(t,.)) < \varepsilon$ as soon as $0 < t - \tau < \eta$. Note that $W_{[\tau,t)}(\kappa(t,.)) \to 0$ for fixed t as $\tau \to t^-$ whenever $\kappa(t,.)$ has bounded variation on $[a,b]$, by the very definition (4.8); so, the assumption really is that the convergence is uniform with respect to t. Hereafter, we drop the qualifier "of type B^∞" and simply speak of Stieltjes-Volterra kernels on $[a,b]$. We endow the space $\mathcal{K}_{[a,b]}$ of such kernels with the norm $\|\kappa\|_{[a,b]} := \sup_{t\in[a,b]} \|\kappa(t,.)\|_{BV[a,b]}$. If κ_k is a Cauchy sequence in $\mathcal{K}_{[a,b]}$, then κ_k converges uniformly on $[a,b]\times[a,b]$ to a $\mathbb{R}^{d\times d}$-valued function κ because

$$|||\kappa_k(t,\tau) - \kappa_l(t,\tau)||| = |||(\kappa_k(t,\tau) - \kappa_l(t,\tau)) - (\kappa_k(t,t) - \kappa_l(t,t))||| \le \|\kappa_k(t,.) - \kappa_l(t,.)\|_{BV[a,b]}.$$

Clearly, κ is measurable and left continuous for fixed t with $\kappa(t,\tau) = 0$ for $\tau \ge t$. Also, if m is so large that $\|\kappa_k - \kappa_l\|_{[a,b]} < \varepsilon$ for $k,l \ge m$ and $\eta > 0$ so small that $W_{[\tau,t)}(\kappa_m) < \varepsilon$ when $t - \tau < \eta$, we get that $W_{[\tau,t)}(\kappa_l) \le W_{[\tau,t)}(\kappa_m) + W_{[\tau,t)}(\kappa_m - \kappa_l) < 2\varepsilon$, and letting $l \to \infty$ we get from [Łoj88, thm. 1.3.5] that $W_{[\tau,t)}(\kappa) \le 2\varepsilon$. Furthermore, the same reference implies that $\|\kappa\|_{[a,b]} \le \sup_k \|\kappa_k\|_{[a,b]}$ so that $\kappa \in \mathcal{K}_{[a,b]}$. Finally, writing that $W_{[a,b]}(\kappa_k(t,.) - \kappa_l(t,.)) < \varepsilon$ and passing to the limit as $l \to \infty$, we see that $\lim_k \|\kappa_k - \kappa\|_{[a,b]} = 0$, whence $\mathcal{K}_{[a,b]}$ is a Banach space. Note that a Stieltjes-Volterra kernel κ is necessarily bounded with $\sup_{[a,b]\times[a,b]} |||\kappa(t,\tau)||| \le \|\kappa\|_{[a,b]}$. As $\mathfrak{H}$ is left continuous and the D_j

[1] For example, the equation satisfied by $\widetilde{\rho}(t,s)$ at top of page 258 is not right.

are bounded, on $[s, s+\tau_N]$, it is easy to check that $k(t,\tau)$ defined in (A.5) is a Stieltjes-Volterra kernel on $[s, s+\tau_N]\times[s, s+\tau_N]$.

A *resolvent* for the Stieltjes-Volterra kernel κ on $[a,b]\times[a,b]$ is a Stieltjes-Volterra kernel ρ on $[a,b]\times[a,b]$ satisfying

$$\rho(t,\beta) = -\kappa(t,\beta) + \int_{\beta^-}^{t^-} d\kappa(t,\tau)\rho(\tau,\beta), \qquad a \le t, \beta \le b. \tag{A.8}$$

Lemma A.1. *If κ is a Stieltjes-Volterra kernel on $[a,b]\times[a,b]$, a resolvent for κ uniquely exists.*

Proof. Pick $r>0$ to be adjusted later, and for $\Psi \in \mathcal{K}_{[a,b]}$ let us define

$$F_r(\Psi)(t,\beta) := \int_{\beta^-}^{t^-} e^{-r(t-\tau)} d\kappa(t,\tau)\Psi(\tau,\beta), \qquad a \le \beta, t \le b.$$

Then, $F_r(\Psi)(t,\beta)=0$ for $\beta \ge t$, and for $a \le \beta_1 < \beta_2 < t$ we have that

$$F_r(\Psi)(t,\beta_2) - F_r(\Psi)(t,\beta_1) = \int_{\beta_2^-}^{t^-} e^{-r(t-\tau)} d\kappa(t,\tau)\left(\Psi(\tau,\beta_2)-\Psi(\tau,\beta_1)\right) - \int_{\beta_1^-}^{\beta_2^-} e^{-r(t-\tau)} d\kappa(t,\tau)\Psi(\tau,\beta_1),$$

where we used that $\kappa(t,.)$ is left continuous to assign the lower (resp. upper) bound β_2^- to the first (resp. second) integral in the above right hand side. Now, the first integral goes to 0 as $\beta_1 \to \beta_2$ by dominated convergence, since $\Psi(t,.)$ is left-continuous; the second integral also goes to 0, because $|\nu_{\kappa(t,.)}|([\beta_1,\beta_2)) \to 0$ when $\beta_1 \to \beta_2$, by standard properties of finite measures. Altogether, we see that $F_r(\Psi)(t,.)$ is left-continuous. Moreover, for $[c,d]\subset[a,t)$ and $c=\beta_0<\beta_1<\cdots<\beta_N=d$,

$$\begin{aligned}
\sum_{i=1}^N |||F_r(\Psi)(t,\beta_i) - F_r(\Psi)(t,\beta_{i-1})||| &\le \sum_{i=1}^N \left|\left|\left| \int_{\beta_i^-}^{t^-} e^{-r(t-\tau)} d\kappa(t,\tau)(\Psi(\tau,\beta_i)-\Psi(\tau,\beta_{i-1})) \right|\right|\right| \\
&\quad + \sum_{i=1}^N \left|\left|\left| \int_{\beta_{i-1}^-}^{\beta_i^-} e^{-r(t-\tau)} d\kappa(t,\tau)\Psi(\tau,\beta_{i-1}) \right|\right|\right| \\
&\le \sum_{i=1}^N \int_{\beta_i^-}^{t^-} e^{-r(t-\tau)} d|\nu_{\kappa(t,.)}|\, |||(\Psi(\tau,\beta_i)-\Psi(\tau,\beta_{i-1}))||| + \sum_{i=1}^N \int_{\beta_{i-1}^-}^{\beta_i^-} e^{-r(t-\tau)} d|\nu_{\kappa(t,.)}|\, |||\Psi(\tau,\beta_{i-1})||| \\
&\le \int_{d^-}^{t^-} d|\nu_{\kappa(t,.)}| \sum_{i=1}^N |||(\Psi(\tau,\beta_i)-\Psi(\tau,\beta_{i-1}))||| + e^{-r(t-d)} \int_{c^-}^{d^-} d|\nu_{\kappa(t,.)}| \sum_{i=1}^N |||(\Psi(\tau,\beta_i)-\Psi(\tau,\beta_{i-1}))||| \\
&\quad + \sup_{[a,t]\times[a,t]} |||\Psi||| \int_{c^-}^{d^-} e^{-r(t-\tau)} d|\nu_{\kappa(t,.)}| \\
&\le 2W_{[d,t)}(\kappa(t,.)) \sup_{\tau\in[d,t)} W_{[c,d]}(\Psi(\tau,.)) + 2e^{-r(t-d)} W_{[c,d)}(\kappa(t,.)) \sup_{\tau\in[c,d)} W_{[c,d]}(\Psi(\tau,.)) \\
&\quad + 2e^{-r(t-d)} \sup_{[a,t]\times[a,t]} |||\Psi|||\; W_{[c,d)}(\kappa(t,.)).
\end{aligned}$$

When $d=t$, the same inequality holds but then $W_{[d,t)}(\kappa(t,.))$ is zero. Setting $c=a$ and $d=t$, we get from the above majorization that $W_{[a,t]}(F_r(\Psi)(t,.)) \le 4\|\kappa\|_{[a,b]}\|\Psi\|_{[a,b]}$, and since $F_r(\Psi)(t,\tau)=0$ for $\tau \ge t$ we deduce that $W_{[a,b]}(F_r(\Psi)(t,.)) = W_{[a,t]}(F_r(\Psi)(t,.))$ is bounded, uniformly with respect to t. Next, if we fix $\varepsilon>0$ and pick $\eta>0$ so small that $W_{[\tau,t)}(\kappa(t,.)) \le \varepsilon$ as soon as $t-\tau \le \eta$ (this is possible because $\kappa \in \mathcal{K}_{[a,b]}$), the same estimate yields

$$W_{[c,t)}(F_r(\Psi)(t,.)) \le 4W_{[c,t)}(\kappa(t,.))\|\Psi\|_{[a,b]} \le 4\varepsilon\|\Psi\|_{[a,b]}, \qquad t-c \le \eta. \tag{A.9}$$

Altogether, we just showed that $F_r(\Psi) \in \mathcal{K}_{[a,b]}$. Moreover, if we take r so large that $e^{-r\eta} < \varepsilon$, then either $t - a \leq \eta$ and then (A.9) with $c = a$ gives us $W_{[a,t)}(F_r(\Psi)(t,.) \leq 4\varepsilon\|\Psi\|_{[a,b]}$, or else $t - \eta > a$ in which case (A.9) with $c = t - \eta$, together with our initial estimate when $c = a$ and $d = t - \eta$, team up to produce:

$$\begin{aligned} W_{[a,t)]}(F_r(\Psi))(t,.) = W_{[a,t-\eta]}(F_r(\Psi)(t,.)) + W_{[t-\eta,t)]}(F_r(\Psi)(t,.)) \leq 2\varepsilon \sup_{\tau\in[t-\eta,t)} W_{[a,t-\eta]}(\Psi(\tau,.)) \\ +2\varepsilon W_{[a,t-\eta)}(\kappa(t,.)) \sup_{\tau\in[a,t-\tau)} W_{[a,t-\eta]}(\Psi(\tau,.)) + 2\varepsilon \sup_{[a,t]\times[a,t]} |||\Psi|||W_{[a,t-\eta)}(\kappa(t,.)) + 4\varepsilon\|\Psi\|_{[a,b]} \\ \leq 2\varepsilon\|\Psi\|_{[a,b]} \left(3 + 2\|\kappa\|_{[a,b]}\right). \end{aligned}$$

Consequently, as $W_{[a,t)]}(F_r(\Psi)(t,.)) = W_{[a,t]}(F_r(\Psi)(t,.))$ by the left continuity of $F_r(\Psi)(t,.)$, we can ensure upon choosing r sufficiently large that the operator $F_r : \mathcal{K}_{[a,b]} \to \mathcal{K}_{[a,b]}$ has arbitrary small norm. Hereafter, we fix r so that $|||F_r||| < \lambda < 1$.

Now, let $\widetilde{\rho}_0 = 0$ and define inductively:

$$\widetilde{\rho}_{k+1}(t,\beta) = -e^{-rt}\kappa(t,\beta) + F_r(\widetilde{\rho}_k)(t,\beta).$$

Clearly $(t,\beta) \mapsto e^{-rt}\kappa(t,\beta)$ lies in $\mathcal{K}_{[a,b]}$, so that $\widetilde{\rho}_k \in \mathcal{K}_{[a,b]}$ for all k. Moreover, we get from what precedes that $\|\widetilde{\rho}_{k+1} - \widetilde{\rho}_k\|_{[a,b]} \leq \lambda\|\widetilde{\rho}_k - \widetilde{\rho}_{k-1}\|_{[a,b]}$. Thus, by the shrinking lemma, $\widetilde{\rho}_k$ converges in $\mathcal{K}_{[a,b]}$ to the unique $\widetilde{\rho} \in \mathcal{K}_{[a,b]}$ such that

$$\widetilde{\rho}(t,\beta) = -e^{-rt}\kappa(t,\beta) + F_r(\widetilde{\rho})(t,\beta) = -e^{-rt}\kappa(t,\beta) + \int_{\beta^-}^{t^-} e^{-r(t-\tau)}d\kappa(t,\tau)\widetilde{\rho}(\tau,\beta), \qquad a \leq t,\beta \leq b. \tag{A.10}$$

Putting $\rho(t,\beta) := e^{rt}\widetilde{\rho}(t,\beta)$, one can see that ρ lies in $\mathcal{K}_{[a,b]}$ if and only if $\widetilde{\rho}$ does, and that (A.10) is equivalent to (A.8). This achieves the proof. □

□

Lemma A.2. *Let κ be a Stieltjes-Volterra kernel on $[a,b]\times[a,b]$ and ρ its resolvent. For each $\mathbb{R}^d$-valued function $g \in BV_r([a,b])$, the unique bounded measurable solution to the equation*

$$y(t) = \int_{a^-}^{t^-} d\kappa(t,\tau)y(\tau) + g(t), \qquad a \leq t \leq b, \tag{A.11}$$

is given by

$$y(t) = g(t) - \int_{a^-}^{t^-} d\rho(t,\alpha)g(\alpha), \qquad a \leq t \leq b. \tag{A.12}$$

Proof. Define y through (A.12) so that $y(a) = g(a)$, by inspection. Since $g \in BV_r[a,b]$ and $\rho(t,\cdot)$, $k(t,.)$ lie in $BV_l[a,b]$, integrating by parts [Łoj88, thm. 7.5.9][2] while using (A.8) along with Fubini's

[2]This reference deals with open intervals only, and we stick to this case at the cost of a slightly lengthier computation

theorem and the relations $\kappa(t,\alpha)=\rho(t,\alpha)=0$ for $\alpha\geq t$, we get that

$$\begin{aligned}
&\textstyle\int_{a^-}^{t^-} d\kappa(t,\alpha)y(\alpha) = (\kappa(t,a^+)-\kappa(t,a))y(a)+\int_{a^+}^{t^-} d\kappa(t,\alpha)y(\alpha)\\
&= (\kappa(t,a^+)-\kappa(t,a))g(a)+\textstyle\int_{a^+}^{t^-} d\kappa(t,\alpha)g(\alpha)-\int_{a^+}^{t^-} d\kappa(t,\alpha)\int_{a^-}^{\alpha^-} d\rho(\alpha,\beta)g(\beta)\\
&= (\kappa(t,a^+)-\kappa(t,a))g(a)+\textstyle\int_{a^+}^{t^-} d\kappa(t,\alpha)g(\alpha)-\int_{a^+}^{t^-} d\kappa(t,\alpha)\int_{a^+}^{\alpha^-} d\rho(\alpha,\beta)g(\beta)\\
&\qquad\qquad -\textstyle\int_{a^+}^{t^-} d\kappa(t,\alpha)(\rho(\alpha,a^+)-\rho(\alpha,a))g(a)\\
&= (\kappa(t,a^+)-\kappa(t,a))g(a)+\textstyle\int_{a^+}^{t^-} d\kappa(t,\alpha)g(\alpha)+\int_{a^+}^{t^-} d\kappa(t,\alpha)\int_{a^+}^{\alpha^-} \rho(\alpha,\beta)dg(\beta)-\int_{a^+}^{t^-} d\kappa(t,\alpha)\left[\rho(\alpha,\beta)g(\beta)\right]_{\beta=a^+}^{\beta=\alpha^-}\\
&\qquad\qquad -\textstyle\int_{a^+}^{t^-} d\kappa(t,\alpha)(\rho(\alpha,a^+)-\rho(\alpha,a))g(a)\\
&= (\kappa(t,a^+)-\kappa(t,a))g(a)+\textstyle\int_{a^+}^{t^-} d\kappa(t,\alpha)g(\alpha)+\int_{a^+}^{t^-}\left(\int_{\beta^-}^{t^-} d\kappa(t,\alpha)\rho(\alpha,\beta)\right)dg(\beta)+\int_{a^+}^{t^-} d\kappa(t,\alpha)\rho(\alpha,a^+)g(a)\\
&\qquad\qquad -\textstyle\int_{a^+}^{t^-} d\kappa(t,\alpha)(\rho(\alpha,a^+)-\rho(\alpha,a))g(a)\\
&= (\kappa(t,a^+)-\kappa(t,a))g(a)+\textstyle\int_{a^+}^{t^-} d\kappa(t,\alpha)g(\alpha)+\int_{a^+}^{t^-}(\rho(t,\beta)+\kappa(t,\beta))dg(\beta)+\int_{a^+}^{t^-} d\kappa(t,\alpha)\rho(\alpha,a)g(a)\\
&= (\kappa(t,a^+)-\kappa(t,a))g(a)+\left[\kappa(t,\alpha)g(\alpha)\right]_{\alpha=a^+}^{\alpha=t^-}+\textstyle\int_{a^+}^{t^-}\rho(t,\beta)dg(\beta)+\int_{a^+}^{t^-} d\kappa(t,\alpha)\rho(\alpha,a)g(a)\\
&= -\kappa(t,a)g(a)+\left[\rho(t,\beta)g(\beta)\right]_{\beta=a^+}^{\beta=t^-}-\textstyle\int_{a^+}^{t^-} d\rho(t,\beta)g(\beta)+\int_{a^-}^{t^-} d\kappa(t,\alpha)\rho(\alpha,a)g(a)\\
&= -\kappa(t,a)g(a)-\rho(t,a^+)g(a)-\textstyle\int_{a^+}^{t^-} d\rho(t,\beta)g(\beta)+(\kappa(t,a)+\rho(t,a))g(a)=-\int_{a^-}^{t^-} d\rho(t,\beta)g(\beta)=y(t)-g(t).
\end{aligned}$$

Thus, y is a solution to (A.11). Clearly, it is measurable, and it is also bounded since $\|\rho(t,.)\|_{BV([a,b])}$ is bounded independently of t and g is bounded. If $\widetilde{y}$ is another solution to (A.11) then $\widetilde{y}(a)=y(a)=g(a)$ by inspection, so that $z:=y-\widetilde{y}$ is a bounded measurable solution to the homogeneous equation:

$$z(t)=\int_{a^+}^{t^-} d\kappa(t,\tau)z(\tau),\qquad a\leq t\leq b.$$

Pick $r>0$ to be adjusted momentarily, and set $\widetilde{z}(t):=e^{-rt}z(t)$ so that

$$\widetilde{z}(t)=\int_{a^+}^{t^-} e^{-r(t-\tau)}d\kappa(t,\tau)\widetilde{z}(\tau). \tag{A.13}$$

Let $\eta>0$ be so small that $W_{[\tau,t)}(\kappa(t,.))\leq 1/4$ as soon as $t-\tau\leq\eta$; this is possible because $\kappa\in\mathcal{K}_{[a,b]}$. Then, it follows from (A.13) that for $t-\eta>a$:

$$\begin{aligned}
|\widetilde{z}(t)| &\leq \left|\int_{a^+}^{(t-\eta)^+} e^{-r(t-\tau)^+}d\kappa(t,\tau)\widetilde{z}(\tau)\right|+\left|\int_{(t-\eta)^+}^{t^-} e^{-r(t-\tau)^-}d\kappa(t,\tau)\widetilde{z}(\tau)\right|\\
&\leq 2e^{-r\eta}W_{(a,t-\eta]}(\kappa(t,\cdot))\sup_{(a,t-\eta]}|\widetilde{z}|+\frac{1}{2}\sup_{(t-\eta,t)}|\widetilde{z}|\leq\sup_{(a,t)}|\widetilde{z}|\big(2e^{-r\eta}\|\kappa\|_{[a,b]}+\frac{1}{2}\big),
\end{aligned}$$

while for $t-\eta\leq a$ we simply get $|\widetilde{z}(t)|\leq\sup_{(a,t)}|\widetilde{z}|/2$. Hence, choosing r large enough, we may assume that $|\widetilde{z}(t)|\leq\lambda\sup_{(a,t)}|\widetilde{z}|$ for some $\lambda<1$ and all $t\in[a,b]$. Thus, if we choose $\lambda'\in(\lambda,1)$ and $t_0\in(a,b]$, we can find $t_1\in(a,t_0)$ such that $|\widetilde{z}(t_1)|\geq(1/\lambda')|\widetilde{z}(t_0)|$, and proceeding inductively we construct a sequence (t_n) in $(a,t_0]$ with $|\widetilde{z}(t_n)|\geq(1/\lambda')^n|\widetilde{z}(t_0)|$. If we had $|\widetilde{z}(t_0)|>0$, this would contradict the boundedness of $\widetilde{z}$, therefore $\widetilde{z}\equiv 0$ on $(a,b]$, whence $z\equiv 0$ so that $y=\widetilde{y}$. □

□

A.1.3 proof of formula (4.24).

Let ρ denote the resolvent of the Stieltjes-Volterra kernel k on $[s,s+\tau_N]\times[s,s+\tau_N]$ defined in (A.5). As f defined in (A.4) lies in $BV_r[s,s+\tau_N]$ (it is even continuous), the solution y to (A.6) is given, in view of Lemma A.2, by

$$y(t)=f(t)-\int_{s^-}^{t^-} d\rho(t,\alpha)f(\alpha),\qquad s\leq t\leq s+\tau_N. \tag{A.14}$$

Since $\rho(t,\alpha)=0$ when $\alpha \geq t$, the integral $\int_{s^-}^{t^-}$ can be replaced by $\int_{s^-}^{(s+\tau_N)^+}$ in (A.14), and setting $\widetilde{X}(t,\alpha) := I_d\mathcal{H}(t-\alpha)+\rho(t,\alpha)$ where $\mathcal{H}(\tau)$ is the "standard" Heaviside function which is 0 for $\tau<0$ and 1 for $\tau \geq 0$, we deduce from (A.7), since $d_\alpha \mathcal{H}(t-\alpha) = -\delta_t$ on $[s, s+\tau_N]$ for $s \leq t < s+\tau_N$, that

$$y(t) = -\int_{s^-}^{(s+\tau_N)^+} d\widetilde{X}(t,\alpha) f(\alpha) = -\int_{s^-}^{(s+\tau_N)^+} d\widetilde{X}(t,\alpha) \left(\sum_{\tau_\ell \in (\alpha-s,\tau_N]} D_\ell(\alpha)\phi(\alpha-s-\tau_\ell) \right), \quad s \leq t < s+\tau_N.$$

Rearranging, we get that

$$y(t) = -\sum_{j=1}^{N} \int_{s^-}^{(s+\tau_j)^-} d\widetilde{X}(t,\alpha) D_j(\alpha)\phi(\alpha-s-\tau_j), \qquad s \leq t < s+\tau_N,$$

which is what we want (namely: formula (4.24) for $s \leq t < s+\tau_N$) if only we can show that $\widetilde{X}(t,\alpha)$ coincides with $X(t,\alpha)$ when $\alpha \in [s, s+\tau_j)$ for each j and every $t \in [s, s+\tau_N)$; here, $X(t,\alpha)$ is defined by (4.22) where we set $s=\alpha$.

For this, we first observe that $X(t,\alpha) = \widetilde{X}(t,\alpha) = 0$ when $\alpha > t$ and that $X(t,t) = \widetilde{X}(t,t) = Id$. Hence, we need only consider the case where $\alpha \in [s,t)$ with $s < t < s+\tau_N$. For $s \leq \alpha < t$, we get that

$$-k(t,\alpha) = k(t,t^-) - k(t,\alpha) = \int_{\alpha^-}^{t^-} dk(t,\tau).$$

Thus, (A.8) (where $\kappa = k$) in concert with the definition of $\widetilde{X}(t,\alpha)$ imply that

$$\begin{aligned} \widetilde{X}(t,\alpha) &= I_d\mathcal{H}(t-\alpha) - k(t,\alpha) + \int_{\alpha^-}^{t^-} dk(t,\tau)\rho(\tau,\alpha) \\ &= I_d + \int_{\alpha^-}^{t^-} dk(t,\tau)(I_d+\rho(\tau,\alpha)) = I_d + \int_{\alpha^-}^{t^-} dk(t,\tau)\widetilde{X}(\tau,\alpha). \end{aligned}$$

Now, on $[\alpha,t)$, we compute from (A.3) and (A.5) that $d_\tau k(t,\tau) = \sum_{t-\tau_j \geq \alpha} D_j(t)\delta_{t-\tau_j}$ and hence, since $\widetilde{X}(t-\tau_j,\alpha) = 0$ when $\alpha > t-\tau_j$, the previous equation becomes:

$$\widetilde{X}(t,\alpha) = I_d + \sum_{j=1}^{N} D_j(t)\widetilde{X}(t-\tau_j,\alpha) \qquad \text{for } s \leq \alpha < t \text{ and } s \leq t < s+\tau_N. \tag{A.15}$$

Comparing (A.15) and (4.22), we see that $\widetilde{X}(t,\alpha)$ and $X(t,\alpha)$ coincide on $[s,s+\tau_N)\times[s,s+\tau_N)$, thereby achieving the proof.

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