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Hao Zhang

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Elliptic Cocycle for $\mathrm{GL}_N(\mathbb{Z})$ and Hecke Operators

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M. Henri Darmon (Co-supervisor)

To my parents

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Introduction

It is a classical result of Shimura and Manin that the map that assigns to a cusp form f its period polynomial r_f is a Hecke equivariant map. This thesis aims to generalize this property to dimension N. In the general setting, the period polynomial attached to f is replaced by a family of rational functions in N variables. For this purpose, we develop a theory of Hecke operators for the elliptic cocycle recently introduced by Charollois. It is an $(N-1)$-cocycle for $\mathrm{GL}_N(\mathbb{Z})$ valued in meromorphic functions on $\mathbb{C}^N \times \mathbb{C}^N$. We start by recalling the classical setting of modular forms and Hecke operators.

Period polynomial

Let τ be a point in the upper half-plane $\mathbb{H}$ and $q = \exp(2\pi i\tau)$. Let f be a cusp form of weight k on $\mathrm{SL}_2(\mathbb{Z})$ with Fourier expansion $f(\tau) = \sum_{n>0} a_f(n)q^n$. The associated L-function is given by

$$L(f,s) = \sum_{n>0} a_f(n)n^{-s}. \tag{1}$$

It is well-known that the L-function $L(f,s)$ converges absolutely for $\mathrm{Re}(s) \gg 0$ and has an analytic continuation to the whole plane. The period polynomial $r_f(x)$ of f is given by

$$r_f(x) = \int_0^{i\infty} f(\tau)(\tau - x)^{k-2}\, d\tau. \tag{2}$$

It is equivalent to

$$r_f(x) = -\sum_{n=0}^{k-2} \frac{(k-2)!L(f,n+1)}{(k-2-n)!(2\pi i)^{n+1}} x^{k-2-n}. \tag{3}$$

Let $\mathbf{V}_{k-2}$ be the vector space of polynomials of degree $\leq k-2$ in one variable x with coefficients in $\mathbb{C}$. The space $\mathbf{V}_{k-2}$ is equipped with an action of $\mathrm{GL}_2(\mathbb{C})$ by

$$(P|\gamma)(x) = (cx+d)^{k-2} P\left(\frac{ax+b}{cx+d}\right), \tag{4}$$

where $P(x) \in \mathbf{V}_{k-2}, \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{C})$. The element $\epsilon = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ acts by $(P|\epsilon)(x) = P(-x)$ and splits $\mathbf{V}_{k-2}$ into the direct sum of the spaces $\mathbf{V}_{k-2}^+$ and $\mathbf{V}_{k-2}^-$ of even and odd polynomials respectively. It is clear that $r_f(x) \in \mathbf{V}_{k-2}$ and satisfies the following cocycle relation:

$$r_f|(\mathrm{Id} + S) = r_f|(\mathrm{Id} + U + U^2) = 0, \tag{5}$$

where $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $U = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$. Let

$$\mathbf{W}_{k-2} = \{P \in \mathbf{V}_{k-2} \,|\, P + P|S = P + P|U + P|U^2 = 0\}.$$

The equation (5) shows that the period polynomial r_f belongs to $\mathbf{W}_{k-2}$. $\mathbf{W}_{k-2}$ splits into two parts: $\mathbf{W}_{k-2} = \mathbf{W}_{k-2}^+ \oplus \mathbf{W}_{k-2}^-$ where $\mathbf{W}_{k-2}^\pm = \mathbf{W}_{k-2} \cap \mathbf{V}_{k-2}^\pm$. Similarly, let r_f^+, r_f^- to be the even and odd part of r_f respectively.

Hecke operators

Let N, m be two positive integers. Let $M_N(m)$ be the set of all $N \times N$ integral matrices of determinant m. Given a modular form f of weight k, the action of the m-th Hecke operator is given by

$$T_m f(\tau) = m^{\frac{k}{2}-1} \sum_{\gamma \in \mathrm{SL}_2(\mathbb{Z}) \backslash M_2(m)} (f|_k\gamma)(\tau), \tag{6}$$

where $f|_k\gamma$ is the slash operator given by

$$(f|_k\gamma)(\tau) = \frac{\det(\gamma)^{\frac{k}{2}}}{(c\tau+d)^k} f\left(\frac{a\tau+b}{c\tau+d}\right). \tag{7}$$

The action of the Hecke operator on a modular symbol can be defined in a similar

way. Let A be a $\mathrm{GL}_2(\mathbb{Q})$-module. An A-valued modular symbol is a function:

$$\begin{aligned} r : \mathbb{Z}^2\backslash 0 \times \mathbb{Z}^2\backslash 0 &\to A \\ (\alpha, \beta) &\mapsto r\{\alpha, \beta\} \end{aligned} \tag{8}$$

satisfying

1. $r\{\alpha, \beta\} = 0$ for all $\alpha \in \mathbb{Q}\beta$,
2. $r\{\alpha, \beta\} + r\{\beta, \delta\} = r\{\alpha, \delta\}$ for all $\alpha, \beta, \delta \in \mathbb{Z}^2\backslash 0$.

Here 0 means the zero vector in $\mathbb{Z}^2$. A modular symbol $r\{\alpha, \beta\}$ is said to be homogeneous if

$$r\{g\alpha, g\beta\} = g \cdot r\{\alpha, \beta\} \text{ for all } g \in \mathrm{SL}_2(\mathbb{Z}) \text{ and } \alpha, \beta \in \mathbb{Z}^2\backslash 0. \tag{9}$$

Then the action of the Hecke operator on a homogeneous modular symbol is given by:

$$\mathbb{T}_m r\{\alpha, \beta\} = \sum_{\gamma \in M_2(m)/\mathrm{SL}_2(\mathbb{Z})} \gamma \cdot r\{m\gamma^{-1}(\alpha, \beta)\}. \tag{10}$$

Here to distinguish the Hecke operator on modular forms, we use the notation $\mathbb{T}_m$.

With the help of relation (5), the period polynomial can be extended to a homogeneous modular symbol. Let $X = (X_1, X_2)$ be a vector and let

$$r_f\{\mathbf{e}_1, \mathbf{e}_2\}(X) = X_2^{k-2} r_f\left(\frac{X_1}{X_2}\right), \tag{11}$$

where $\mathbf{e}_1 = (1, 0)^t, \mathbf{e}_2 = (0, 1)^t$. Let $\alpha, \beta \in \mathbb{Z}^2_{\mathrm{prim}}$ be two column vectors, then due to the cocycle relations (5), $r_f\{\alpha, \beta\}$ is totally determined by the following conditions:

1. $r_f\{\alpha, \beta\} = 0$ if $\alpha \in \mathbb{Q}\beta$,
2. $r_f\{\alpha, \beta\} + r_f\{\beta, \delta\} = r_f\{\alpha, \delta\}$ for any $\alpha, \beta, \delta \in \mathbb{Z}^2_{\mathrm{prim}}$,
3. $r_f\{g\alpha, g\beta\}(X) = r_f\{\alpha, \beta\}(Xg)$ for any $g \in \mathrm{SL}_2(\mathbb{Z})$.

By the discussion above, r induces two maps:

$$r^{\pm} : S_k \to \mathbf{W}^{\pm}_{k-2}, \tag{12}$$

where S_k is the space of cusp forms of weight k. The basic result of Eichler-Shimura [**Shi59**] is the following:

Theorem 0.0.1 (Eichler-Shimura)**.** *The map $r^- : S_k \to \boldsymbol{W}_{k-2}^-$ is an isomorphism. The map $r^+ : S_k \to \boldsymbol{W}_{k-2}^+$ is an isomorphism onto a subspace $\boldsymbol{W}'_{k-2}$ of $\boldsymbol{W}_{k-2}^+$ of codimension* 1*.*

Moreover, Manin [**Man73**] proved that the map r is Hecke equivariant (see also Merel [**Mer94**]):

Theorem 0.0.2. *Let f be a cusp form of weight k. Then for any vectors $\alpha, \beta \in \mathbb{Z}^2_{\text{prim}}$, we have*

$$\mathbb{T}_m r_f\{\alpha, \beta\} = r_{T_m f}\{\alpha, \beta\}. \tag{13}$$

In particular, if $f = q + a_f(2)q^2 + a_f(3)q^3 + \cdots$ is a Hecke eigenform, then $r_f\{\alpha, \beta\}$ is an eigenvector of the Hecke operator $\mathbb{T}_m$ with eigenvalue $a_f(m)$.

Example 0.0.3. *When $k = 12$, the space S_{12} is* 1*-dimensional generated by the Ramanujan Delta function:*

$$\Delta(\tau) = q \prod_{n \geq 1} (1 - q^n)^{24} = q - 24q^2 + 252q^3 - 1472q^4 + 4830q^5 + \cdots. \tag{14}$$

The space $\boldsymbol{W}_{10}^+$ is spanned by

$$P_0(x) = x^{10} - 1, \quad P_1(x) = x^8 - 3x^6 + 3x^4 - x^2. \tag{15}$$

The space $\boldsymbol{W}_{10}^-$ is spanned by

$$P_2(x) = 4x^9 - 25x^7 + 42x^5 - 25x^3 + 4x. \tag{16}$$

*On the other hand, the period polynomial $r_\Delta(x)$ was calculated by Shimura [**Shi59**]:*

$$\begin{aligned} r_\Delta(x) =& 0.005958\ldots Ix^{10} + 0.0370771\ldots x^9 - 0.1143790\ldots Ix^8 - 0.2317319\ldots x^7 \\ &+ 0.343137\ldots Ix^6 + 0.389309\ldots x^5 - 0.343137\ldots Ix^4 - 0.2317319\ldots x^3 \\ &+ 0.114379\ldots Ix^2 + 0.037077\ldots x - 0.005958\ldots I \end{aligned}$$

If we set $\omega^+ = 0.0643382\ldots I$ and $\omega^- = 0.0092692\ldots$, we see that

$$r_\Delta^+(x) = \omega^+(36P_0(x) - 691P_1(x)), \quad r_\Delta^-(X) = \omega^- P_2(x). \tag{17}$$

Elliptic cocycle

In this thesis, we will generalize Theorem 0.0.2 to higher dimension. According to the terminology introduced in [**BCG20**], Theorem 0.0.2 can be viewed as a

compatibility identity attached to the reductive dual pair $(\mathrm{GL}_2(\mathbb{Z}); \mathrm{GL}_2(\mathbb{Z}))$. The first factor corresponds to the operator $\mathbb{T}_m$ and the second factor corresponds to the operator T_m. We will generalize this to the pair $(\mathrm{GL}_N(\mathbb{Z}); \mathrm{GL}_2(\mathbb{Z}))$. To motivate the definition, we recall a result of Zagier [**Zag91**].

The definition of period polynomial can be extended to Eisenstein series. Let $k \geq 4$ be an even integer and

$$G_k(\tau) = -\frac{B_k}{2k} + \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n. \tag{18}$$

For $\mathrm{Re}(s) \gg 0$, let

$$L^*(G_k, s) := \int_0^{\infty} \left(G_k(iy) + \frac{B_k}{2k} \right) dy = (2\pi)^{-s} \Gamma(s) L(G_k, s), \tag{19}$$

where $L(G_k, s) = \sum_{n=1}^{\infty} \sigma_{k-1}(n) n^{-s}$. Then r_{G_k} is given by

$$r_{G_k}(x) = \frac{-B_k}{2k(k-1)} (x^{k-1} + x^{-1}) + \sum_{n=0}^{k-2} i^{1-n} \binom{k-2}{n} L^*(G_k, n+1) x^{k-2-n}. \tag{20}$$

In this case, $r_{G_k}(x)$ is no longer a polynomial, but a rational function. Now let

$$c_k(\tau, x, y) = \sum_{\substack{f \in M_k \\ \text{eigenform}}} \frac{(r_f(x) r_f(y))^-}{(2i)^{k-3}(f, f)} f(\tau), \tag{21}$$

where the summation is taken over all weight k normalized eigenforms, both Eisenstein series and Hecke eigenforms and

$$(r_f(x) r_f(y))^- = \frac{1}{2} \left(r_f(x) r_f(y) - r_f(-x) r_f(-y) \right). \tag{22}$$

Zagier [**Zag91**] considered the following generating function that encodes all these period polynomials:

$$C(\tau, x, y, T) = \frac{(xy-1)(x+y)}{x^2 y^2} (2\pi i T)^{-2} + \sum_{k=2}^{\infty} c_k(\tau, x, y) \frac{(2\pi i T)^{k-2}}{(k-2)!}. \tag{23}$$

He found a closed formula for it:

Theorem 0.0.4 (Zagier)**.** *The function $C(\tau, x, y, T)$ is given by*

$$(2\pi i)^2 C(\tau, x, y, T) = \mathcal{K}(\tau, xT, yT) \mathcal{K}(\tau, -xyT, T), \tag{24}$$

where $\mathcal{K}(\tau, x, y)$ is the Kronecker theta function

$$\mathcal{K}(\tau, x, y) = \frac{\theta'_\tau(0)\theta_\tau(x+y)}{\theta_\tau(x)\theta_\tau(y)}, \tag{25}$$

and $\theta_\tau(z)$ is the Jacobi theta function:

$$\theta_\tau(z) = \sum_{n\in\mathbb{Z}} (-1)^n q^{\frac{1}{2}(n+\frac{1}{2})^2} \boldsymbol{e}\left(\left(n+\frac{1}{2}\right)z\right). \tag{26}$$

From the relation (5), one deduces that the function $C(\tau, x, y, T)$ satisfies the following linear relations:

$$C(\tau, x, y, T) + C\left(\tau, -\frac{1}{x}, y, xT\right) = 0, \tag{27}$$

$$C(\tau, x, y, T) + C\left(\tau, 1-\frac{1}{x}, y, xT\right) + C\left(\tau, \frac{1}{1-x}, y, (1-x)T\right) = 0. \tag{28}$$

Such relations are the main 1-cocycle relations satisfied by products of two Kronecker theta functions. More generally, we will construct an $(N-1)$-cocycle $\mathscr{E}(\tau, \sigma, x, x')$ for $\mathrm{GL}_N(\mathbb{Z})$ valued in functions on $\mathbb{H} \times \mathbb{C}^N \times \mathbb{C}^N$ that involves the products of N Kronecker theta functions. Here σ is an element of $M_N(\mathbb{Z})$ and x, x' are vectors in $\mathbb{C}^N$. The elliptic cocycle $\mathscr{E}$ was first introduced by Charollois [**Cha1**], [**CS16**][section 3]. We will further introduce the action of two kinds of Hecke operators T_m and $\mathbb{T}_m$ on the elliptic cocycle $\mathscr{E}(\tau, \sigma, x, x')$. The operator T_m corresponds to the action of $\mathrm{GL}_2(\mathbb{Z})$ on the parameter τ in analogy with the Hecke operator on modular forms. The operator $\mathbb{T}_m$ corresponds to the action of $\mathrm{GL}_N(\mathbb{Z})$ on the parameter σ and it is the analog of the Hecke operator $\mathbb{T}_m$ on the modular symbol as in (10).

The idea to consider such Hecke operators on the elliptic cocycle $\mathscr{E}$ originally comes from the recent paper [**BCG20**], where Bergeron-Charollois-Garcia introduced a differential form E_ψ that realizes an Eisenstein theta correspondence for the dual pair $(\mathrm{GL}_N; \mathrm{GL}_2)$.

Main results

As the definition of the Hecke operator on modular symbols, we can generalize the definition of the Hecke operator to higher dimension directly. Let A be a $\mathrm{GL}_N(\mathbb{Q})$-module and $\mathbb{Z}^N_{\mathrm{prim}}$ be the set of primitive vectors in $\mathbb{Z}^N$. Let $\mathcal{P}_h(A)$ be the set of

functions

$$\begin{aligned} r : (\mathbb{Z}^N_{\text{prim}})^N &\to A \\ (\sigma_1, \cdots, \sigma_N) &\mapsto r\{\sigma_1, \cdots, \sigma_N\} \end{aligned} \tag{29}$$

satisfying the conditions

1. $r\{\sigma_1, \cdots, \sigma_N\} = 0$ if $\det(\sigma_1, \cdots, \sigma_N) = 0$,
2. $\sum_{j=0}^{N} (-1)^j r\{\sigma_0, \cdots, \hat{\sigma}_j, \cdots, \sigma_N\} = 0$ for all $\sigma_0, \cdots, \sigma_N \in \mathbb{Z}^N_{\text{prim}}$,
3. $r\{g\sigma_1, \cdots, g\sigma_N\} = g \cdot (r\{\sigma_1, \cdots, \sigma_N\})$ for all $g \in \mathrm{SL}_N(\mathbb{Z})$.

We define the operator $\mathbb{T}_m$ on the set $\mathcal{P}_h(A)$ by:

$$\mathbb{T}_m r\{\sigma_1, \cdots, \sigma_N\} = \sum_{\gamma \in M_N(m)/\mathrm{SL}_N(\mathbb{Z})} \gamma \cdot \left(r\{m\gamma^{-1}\sigma_1, \cdots, m\gamma^{-1}\sigma_N\}\right), \tag{30}$$

Then we will show that the elliptic cocycle $\mathscr{E}(\tau, \sigma, x, x')$ belongs to $\mathcal{P}_h(\mathcal{F}(\mathbb{H} \times \mathbb{C}^N \times \mathbb{C}^N))$, where $\mathcal{F}(\mathbb{H} \times \mathbb{C}^N \times \mathbb{C}^N)$ is the set of meromorphic functions on $\mathbb{H} \times \mathbb{C}^N \times \mathbb{C}^N$. In Chapter 3, we prove the following theorem:

Theorem 0.0.5. *Let m, N be two positive integers. Then for any $x, x' \in \mathbb{C}^N$ and $\sigma \in M_N(\mathbb{Z})$, we have the formula*

$$\mathbb{T}_m \mathscr{E}(\tau, \sigma, x, x') = \sum_{d|m} A(N, d) T_{\frac{m}{d}} \mathscr{E}(\tau, \sigma, x, dx'), \tag{31}$$

where $\{A(N, d) \,|\, d = 1, 2, \cdots\}$ is a certain sequence of integers. The exact definition is given in 3.3.23. We also give a table of the first $A(N, d)$ in the appendix.

One part of the proof of this Theorem was inspired by the method of Borisov-Gunnells used in [**BG02**], where they proved the Hecke stability of the space of the products of Eisenstein series under the operator T_m. They also proved a kind of Hecke equivariance for the products of two Eisenstein series.

With the help of Theorem 0.0.5, we are able to generalize the Theorem 0.0.4 and Theorem 0.0.2. Let $X = (X_1, \ldots, X_N)$, $y \in \mathbb{C}$, $\sigma \in M_N(\mathbb{Z})$ and $M \in M_N(\mathbb{C})$ with $M + M^t = 0$, we define

$$\mathcal{B}(\tau, \sigma, M, X, y, T) := \frac{1}{(2\pi i)^N} \mathscr{E}(\tau, \sigma, XT, MX^t yT). \tag{32}$$

Such function can be viewed as a generalization of $C(\tau, x, y, T)$ since when $N = 2$,

we have

$$\mathcal{B}(\tau, \mathrm{Id}, S, (X_1, X_2), y, T) = C(\tau, X_2/X_1, y, X_1 T). \tag{33}$$

Just like the function $C(\tau, x, y, T)$, we will obtain the Laurent expansion of the function $\mathcal{B}(\tau, \sigma, M, X, y, T)$ in T. The proof will be given in Theorem 5.2.6.

Theorem 0.0.6. *The function $\mathcal{B}(\tau, \sigma, M, X, y, T)$ has the following Laurent expansion in T:*

$$\begin{aligned}\mathcal{B}(\tau, \sigma, M, X, y, T) =& P_{-N}(\sigma, M, X, y)(2\pi i T)^{-N} \\ &+ \sum_{k \geq 4} \sum_{\substack{f \text{ eigenform} \\ \text{weight } k}} P_f(\sigma, M, X, y) f(\tau) \frac{(2\pi i T)^{k-N}}{(k-N)!}.\end{aligned}$$

where P_{-N} and P_f are certain rational functions in X and P_f/P_{-N} are polynomials.

Theorem 0.0.6 provides the definition of P_f for a normalized eigenform f. We can extend it to any modular form by linear combination. Combining with the Theorem 0.0.5, we can study the action of Hecke operators on the function $P_f(\sigma, M, X, y)$. The following theorem generalizes the Hecke equivariance to higher dimension.

Theorem 0.0.7. *For any modular form $f(\tau)$, we have the following formula:*

$$\mathbb{T}_m P_f(\sigma, M, X, y) = \sum_{d|m} A(N, d) P_{T_{\frac{m}{d}} f}(\sigma, M, X, dy). \tag{34}$$

In particular, if we write the Laurent expansion

$$P_f(\sigma, M, X, y) = \sum_{t \geq -N} P_f^{(t)}(\sigma, M, X) y^t, \tag{35}$$

and take $f = a_f(0) + q + a_f(2)q^2 + \cdots$ to be a normalized eigenform, then if the rational function $P_f^{(t)}(\sigma, M, X)$ is non-zero, it is an eigenvector of $\mathbb{T}_m$ with eigenvalue $\sum_{d|m} A(N, d) a_f\left(\frac{m}{d}\right) d^t$, i.e.

$$\mathbb{T}_m P_f^{(t)}(\sigma, M, X) = \left(\sum_{d|m} A(N, d) a_f\left(\frac{m}{d}\right) d^t \right) P_f^{(t)}(\sigma, M, X). \tag{36}$$

We will also show that there are many non-zero rational functions $P_f^{(t)}(\sigma, M, X)$. In fact, the space generated by

$$\langle P_f^{(t)}(\mathrm{Id}, M, X) \mid M + M^t = 0 \rangle,$$

is an eigenspace for $\mathbb{T}_m$ with eigenvalue $\sum_{d|m} A(N,d)a_f\left(\frac{m}{d}\right)d^t$, which is infinite dimensional in most of the cases.

As an example, when $N=2$, $P_f^{(t)}(\mathrm{Id}, M, X)$ is exactly the odd or even part of $r_f\{\mathbf{e}_1, \mathbf{e}_2\}$ up to the parity of t. When $N=3$, we will also give some examples for $f = G_k$ and the Ramanujan Delta function in the section 5.4.

For each rational function $P_f^{(t)}(\sigma, M, X)$, it corresponds a family of eigenvalues, hence naturally we can consider the corresponding L-series :

$$L_f^{(t)}(s) = \sum_{m\geq 1}\left(\sum_{d|m} A(N,d)a_f\left(\frac{m}{d}\right)d^t\right)m^{-s}. \tag{37}$$

We have the following theorem

Theorem 0.0.8. *Let $f(\tau)$ be an eigenform of weight k. Then for* $\mathrm{Re}(s) > \max\{k, N+t\}$, *$L_f^{(t)}(s)$ converges absolutely. It has a meromorphic continuation to the whole plane. Moreover, we have the decomposition*

$$L_f^{(t)}(s) = L(f,s)\prod_{j=1}^{N-2}\zeta(s-j-t),$$

where $L(f,s)$ is the L-function associated to the eigenform f.

Outline of the thesis

In chapter 1, we recall the Kronecker theta function and its basic properties. Then we will give a relation between the Kronecker theta function and Eisenstein series. The Kronecker theta function can be written as a generating series of Eisenstein series.

In chapter 2, we will introduce the elliptic cocycle $\mathscr{E}(\tau, \sigma, x, x')$ and prove its cocycle relation in dimension 2 and 3. We first prove a special case by considering the poles of elliptic cocycle, and then extend it to the general case by using our general extension theorem about cocycles for GL_N when $N = 2, 3$.

In chapter 3, we consider the action of two kinds of Hecke operators T_m and $\mathbb{T}_m$ on the elliptic cocycle $\mathscr{E}(\tau, \sigma, x, x')$. The main result in this chapter is the relation between these two kinds of Hecke operators given in Theorem 3.3.23. The basic technique is translating the summation over Hecke operator T_m to the counting of number of lattices as shown in Theorem 3.3.21.

In chapter 4, we will introduce the Eisenstein cocycle that consists of products

of Eisenstein series. Such Eisenstein cocycle is obtained from the elliptic cocycle $\mathscr{E}$. Also, we will do the smoothing for the Eisenstein cocycle at some prime ℓ. This allows us only to keep the main term. Then we recall Sczech's cocycle on the Bianchi group. By comparing our cocycle with Sczech's cocycle, we show that they coincide when we restrict to the group $\Gamma_0(\ell)$ after smoothing at the prime ℓ. At last, we study the algebraicity of the value of elliptic cocycle and Eisenstein cocycle when τ is a CM point.

In chapter 5, attached to each modular form f, we construct a family of rational functions P_f that are the generalization of the modular symbol attached to a cusp form. We will prove the Hecke equivariance of these rational functions in Section 5.2. In particular, if we take f to be a normalized eigenform, then we can associate a finite family of L-series to the function P_f. We calculate this $L_f^{(t)}(s)$ in Section 5.3 and establish that it possesses an Euler product and meromorphic continuation to the whole s-plane. In Section 5.4, we will give some numerical examples when f is Eisenstein series and the Ramanujan Delta function.

Finally, since the function P_f is a rational function in most cases, for a future research, we look for a polynomial analog of P_f. Because the space of polynomial solutions is finite dimensional, we are able to provide an exhaustive list of such polynomial analogs Q_f in low weight. We found a basis of eigenvectors for the Hecke operator $\mathbb{T}_m$ for small m, compute the eigenvalues and identify the related L-functions. The numerical evidences and open questions are displayed in Section 5.5.

Chapter

1 Kronecker theta function

In this chapter, we will recall the Kronecker theta function. This is the basic 1-dimensional object that we need to get us started for higher dimension later. The Kronecker theta function occurs in different work with different forms. For example, in [**Weil76**], Weil refered the Kronecker theta function from the Kronecker double series. In [**Zag91**], Zagier proved some basic properties of Kronecker theta function. In this thesis, we will follow the idea of Weil, but with some normalization to adapt to other cases.

1.1 Definition

We begin with the definition of Kronecker theta function. We follow the definition of [**Weil76**][Chap. VIII]:

Definition 1.1.1. *Let x, x' be two complex numbers and τ a point of the Poincaré half plane $\mathbb{H}$. We further assume that $|q| < |\boldsymbol{e}(x')| < 1$ and $x \notin \Lambda = \mathbb{Z} + \mathbb{Z}\tau$ where $q = \boldsymbol{e}(\tau)$ and $\boldsymbol{e}(x) = \exp(2\pi i x)$. Then we define*

$$\mathcal{K}(\tau, x, x') = \sum_{m \in \mathbb{Z}} \frac{2\pi i \boldsymbol{e}(mx')}{q^m \boldsymbol{e}(x) - 1}. \tag{1.1}$$

Under the condition on x', the series is convergent for all x not in Λ. If we restrict x to $|q| < |\mathbf{e}(x)| < |q|^{-1}$ and $x \notin \mathbb{Z}$, then the Kronecker theta function has the symmetric expansion:

$$\begin{aligned}\mathcal{K}(\tau, x, x') =& 2\pi i \Bigg(1 - \frac{1}{1 - \mathbf{e}(x)} - \frac{1}{1 - \mathbf{e}(x')} \\ &- \sum_{m,n \geq 1} q^{mn} \big(\mathbf{e}(nx + mx') - \mathbf{e}(-nx - mx')\big)\Bigg).\end{aligned} \tag{1.2}$$

In the following, we will give the meromorphic continuation of $\mathcal{K}(\tau, x, x')$.

Proposition 1.1.2. *The Kronecker theta function $\mathcal{K}(\tau, x, x')$ has a meromorphic continuation to $\mathbb{C}^2$ with simple poles at $x, x' \in \Lambda$. Moreover, the residue of $\mathcal{K}(\tau, x, x')$ at $x = n + m\tau$ equals $\boldsymbol{e}(-mx')$, the residue of $\mathcal{K}(\tau, x, x')$ at $x' = n + m\tau$ equals $\boldsymbol{e}(-mx)$.*

Proof. We choose a positive integer N and divide the last term of (1.2) into two parts:

$$\begin{aligned}
&\sum_{\substack{m\geq 1\\ 1\leq n<N}} q^{mn}(\mathbf{e}(nx+mx')-\mathbf{e}(-nx-mx'))+\sum_{\substack{m\geq 1\\ n\geq N}} q^{mn}(\mathbf{e}(nx+mx')-\mathbf{e}(-nx-mx'))\\
&=\sum_{1\leq n<N}\sum_{m\geq 1} q^{mn}(\mathbf{e}(nx+mx')-\mathbf{e}(-nx-mx'))\\
&\quad+\sum_{m\geq 1}\left(\sum_{n\geq 0} q^{mn}(\mathbf{e}(nx+mx'+Nx)-\mathbf{e}(-nx-mx'-Nx))\right)q^{Nm}\\
&=\sum_{1\leq n<N}\left(\frac{q^n\mathbf{e}(nx+x')}{1-q^n\mathbf{e}(x')}-\frac{q^n\mathbf{e}(-nx-x')}{1-q^n\mathbf{e}(-x')}\right)\\
&\quad+\sum_{m\geq 1}\left(\frac{\mathbf{e}(Nx+mx')}{1-q^m\mathbf{e}(x)}-\frac{\mathbf{e}(-Nx-mx')}{1-q^m\mathbf{e}(-x)}\right)q^{Nm}.
\end{aligned} \tag{1.3}$$

Now for any $x' \in \mathbb{C}\backslash\Lambda$, we choose N large enough such that $q^N < |\mathbf{e}(x')| < q^{-N}$. Then the series above is convergent absolutely. Thus we get the meromorphic continuation of $\mathcal{K}(\tau, x, x')$.

Now we consider the residue of $\mathcal{K}$. By the symmetric property of $\mathcal{K}$, we only need to consider the pole $x = n + m\tau$. If $m = 0$, the residue of $-\frac{2\pi i}{1-\mathbf{e}(x)}$ equals to 1. If $m > 0$, only the term $-\frac{\mathbf{e}(-Nx-mx')}{1-q^m\mathbf{e}(-x)}q^{Nm}$ has a simple pole at $x = n + m\tau$, and it is easy to see that the residue equals to $\mathbf{e}(-mx')$. The case $m < 0$ is similar. □

It is easy to see the symmetry $\mathcal{K}(\tau, x, x') = \mathcal{K}(\tau, x', x)$ when $|q| < |\mathbf{e}(x)|, |\mathbf{e}(x')| < 1$ and $x, x' \notin \Lambda$ from equation (1.2). Then we can extend it to $x, x' \in \mathbb{C}\backslash\Lambda$ by the meromorphic continuation.

1.2 Basic properties

In this section, we prove some properties of the Kronecker theta function. These properties we will often use later. Some of them also appear in [**Cha1**].

Proposition 1.2.1. *The Kronecker theta function $\mathcal{K}$ satisfies the ellipticity property:*

$$\mathcal{K}(\tau, x+k+l\tau, x') = \boldsymbol{e}(-lx')\mathcal{K}(\tau, x, x') \quad \text{for any } k, l \in \mathbb{Z}, \tag{1.4}$$

and the distribution relation

$$\mathcal{K}(D\tau, Dx, x') = \frac{1}{D}\sum_{j=0}^{D-1} \mathcal{K}\left(\tau, x+\frac{j}{D}, x'\right). \tag{1.5}$$

where D is a positive integer.

Proof. These two properties follow directly from the definition. In fact, for the elliptic property, we have

$$\begin{aligned}\mathcal{K}(\tau, x+k+l\tau, x') &= \sum_{m\in\mathbb{Z}} \frac{2\pi i\mathbf{e}(mx')}{q^m\mathbf{e}(x+k+l\tau)-1} \\ &= \sum_{m\in\mathbb{Z}} \frac{2\pi i\mathbf{e}((m-l)x')}{q^m\mathbf{e}(x)-1} \\ &= \mathbf{e}(-lx')\mathcal{K}(\tau, x, x').\end{aligned} \tag{1.6}$$

For the distribution relation,

$$\begin{aligned}&\frac{1}{D}\sum_{j=0}^{D-1} \mathcal{K}\left(\tau, x+\frac{j}{D}, x'\right) \\ =&\frac{1}{D}\sum_{j=0}^{D-1}\sum_{m\in\mathbb{Z}} \frac{2\pi i\mathbf{e}(mx')}{q^m\mathbf{e}\left(x+\frac{j}{D}\right)-1} \\ =&\frac{1}{D}\sum_{m\in\mathbb{Z}}\sum_{j=0}^{D-1}\sum_{k=0}^{D-1} \frac{2\pi i q^{km}\mathbf{e}\left(mx'+kx+\frac{kj}{D}\right)}{q^{Dm}\mathbf{e}(Dx)-1}.\end{aligned} \tag{1.7}$$

We note that

$$\frac{1}{D}\sum_{j=0}^{D-1}\mathbf{e}\left(-\frac{kj}{D}\right) = \begin{cases} 0 & \text{if } D \nmid k, \\ 1 & \text{if } D|k. \end{cases} \tag{1.8}$$

Hence the sum (1.7) gives

$$\sum_{m\in\mathbb{Z}} \frac{2\pi i\mathbf{e}\left(mx'\right)}{q^{Dm}\mathbf{e}(Dx)-1},$$

which is exactly $\mathcal{K}(D\tau, Dx, x')$. This completes the proof. □

More generally, we have the following distribution relation over $\Lambda/D\Lambda$:

Proposition 1.2.2. *Let D be a positive integer. Then we have*

$$\frac{1}{D}\sum_{y,y'=0}^{D-1} \boldsymbol{e}(xy)\mathcal{K}\left(\tau, Dx, \frac{x'+y\tau+y'}{D}\right) = \mathcal{K}(\tau, x, x'). \tag{1.9}$$

Proof. Applying the distribution relation to y', the left hand side of (1.9) equals

$$\sum_{y=0}^{D-1} \mathbf{e}(xy)\mathcal{K}(D\tau, Dx, x'+y\tau). \tag{1.10}$$

And applying the distribution relation again to the first variable, we have

$$\frac{1}{D}\sum_{y,j=0}^{D-1} \mathbf{e}(xy)\mathcal{K}\left(\tau, x+\frac{j}{D}, x'+y\tau\right). \tag{1.11}$$

Then applying the elliptic property of $\mathcal{K}$, we get

$$\frac{1}{D}\sum_{y,j=0}^{D-1} \mathbf{e}\left(-\frac{jy}{D}\right)\mathcal{K}(\tau, x+\frac{j}{D}, x'). \tag{1.12}$$

By using the formula (1.8), we complete the proof. □

Now we are able to give the relation between the Kronecker theta function and the classical theta function. First of all, we recall the definition of theta function in [**Cha1**]:

Definition 1.2.3. *Let x be a complex number and $\tau \in \mathbb{H}$. Then we define the theta function by*

$$\theta(\tau, x) = 2q^{\frac{1}{8}}\sin(\pi x)\prod_{n\geq 1}(1-q^n\boldsymbol{e}(x))(1-q^n\boldsymbol{e}(-x)). \tag{1.13}$$

From the definition, it is easy to see that

$$\theta(\tau, x+\tau) = -q^{-\frac{1}{2}}\mathbf{e}(-x)\theta(\tau, x). \tag{1.14}$$

Now we can deduce Charollois' definition of $\mathcal{K}(\tau, x, x')$:

Proposition 1.2.4. *For any $x, x' \in \mathbb{C}\backslash\Lambda$, we have*

$$\mathcal{K}(\tau, x, x') = \frac{\theta'(\tau, 0)\theta(\tau, x+x')}{\theta(\tau, x)\theta(\tau, x')}. \tag{1.15}$$

Proof. We consider the function

$$F(\tau, x, x') = \frac{\mathcal{K}(\tau, x, x')\theta(\tau, x)\theta(\tau, x')}{\theta(\tau, x + x')}.$$

We note that the theta function $\theta(\tau, x)$ has a zero of order 1 at $x \in \Lambda$ and no poles or zeros elsewhere. Moreover, since $\mathcal{K}(\tau, x, -x) = \mathcal{K}(\tau, -x, x) = -\mathcal{K}(\tau, x, -x)$, we have $\mathcal{K}(\tau, x, x') = 0$ when $x + x' \in \Lambda$. Hence by Proposition 1.1.2, $F(\tau, x, x')$ is holomorphic everywhere. With the help of elliptic property of $\mathcal{K}$ and the transformation property (1.14) of $\theta(\tau, x)$, we see that $F(\tau, x, x')$ is τ-periodic. It is obvious that $F(\tau, x, x')$ is 1-periodic. Thus $F(\tau, x, x')$ is a holomorphic elliptic function in x. By the symmetry property, it is also an elliptic function in x'. So $F(\tau, x, x')$ is a constant up to τ. To calculate this constant, we take the limit $x \to 0$. By Proposition 1.1.2, the residue of $\mathcal{K}(\tau, x, x')$ at $x = 0$ is 1, hence

$$\lim_{x \to 0} F(\tau, x, x') = \theta'(\tau, 0).$$

This completes he proof. □

Remark 1.2.5. *If we use the Jacobi theta function given by*

$$\theta_\tau(x) = \sum_{n \in \mathbb{Z}} (-1)^n q^{\frac{1}{2}\left(n+\frac{1}{2}\right)^2} \boldsymbol{e}\left(\left(n + \frac{1}{2}\right) x\right), \tag{1.16}$$

*then by the same method as above, we get the formula proved by Zagier in [**Zag91**][p.456]*

$$\mathcal{K}(\tau, x, x') = \frac{\theta'_\tau(0)\theta_\tau(x + x')}{\theta_\tau(x)\theta_\tau(x')}. \tag{1.17}$$

Proposition 1.2.6. *For any $x, x' \in \mathbb{C}$, we have the exponential formula:*

$$\mathcal{K}(\tau, x, x') = \frac{x + x'}{xx'} \exp\left(\sum_{\substack{k \geq 2 \\ k \text{ even}}} 2(x^k + x'^k - (x + x')^k) G_k(\tau) \frac{(2\pi i)^k}{k!}\right), \tag{1.18}$$

where

$$G_k(\tau) = -\frac{B_k}{2k} + \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n. \tag{1.19}$$

Proof. It is well known that the theta function $\theta_\tau(x)$ satisfies the following triple Jacobi identity:

$$\theta_\tau(x) = q^{\frac{1}{8}} \sin(\pi x) \prod_{n \geq 1} (1 - q^n)(1 - q^n \mathbf{e}(x))(1 - q^n \mathbf{e}(-x)). \tag{1.20}$$

After dividing by $x\theta'_\tau(0)$ and taking the logarithm, we get

$$\begin{aligned}\log\left(\frac{\theta_\tau(x)}{x\theta'_\tau(0)}\right) &= \log\left(\frac{\sin(\pi x)}{\pi x}\right) + \sum_{n\geq 1}\log(1-q^n\mathbf{e}(x)) + \log(1-q^n\mathbf{e}(-x)) - 2\log(1-q^n)\\ &= -\sum_{k\geq 1}\frac{\zeta(2k)}{k}x^{2k} - \sum_{n,m\geq 1}\frac{2q^{mn}(\cos(2\pi x)-1)}{m}\\ &= -\sum_{k\geq 1}\frac{\zeta(2k)}{k}x^{2k} - \sum_{k\geq 1}\sum_{n,m\geq 1}2m^{2k-1}q^{mn}\frac{(2\pi i x)^{2k}}{(2k)!}\\ &= -\sum_{k\geq 1}2G_{2k}(\tau)\frac{(2\pi i x)^{2k}}{(2k)!}.\end{aligned} \tag{1.21}$$

Then this proposition follows immediately from the equation 1.17. □

1.3 Relation with Eisenstein series

In this section, we connect the Kronecker theta function with the Eisenstein series as in [**Weil76**].

Let x, x' be two complex numbers and $\tau \in \mathbb{H}$. We denote $\Lambda = \mathbb{Z} + \mathbb{Z}\tau$. For each x' we may associate a character function ψ of Λ. We define

$$\psi_\Lambda(\lambda, x') = \exp\left(\frac{\lambda\bar{x'} - x'\bar{\lambda}}{A(\Lambda)}\right), \tag{1.22}$$

where $\lambda \in \Lambda$, $A(\Lambda) = \frac{\text{Area}(\Lambda)}{\pi}$ and $\text{Area}(\Lambda)$ is the area of the fundamental domain of Λ.

We consider the following series:

$$\sum_{\lambda\in\Lambda}\frac{\psi_\Lambda(\lambda, x')}{x+\lambda}, \tag{1.23}$$

where we assume $x \notin \Lambda$. However we see that such series don't converge absolutely. Hence we may consider the Eisenstein summation given by

$$\lim_{M\to\infty}\sum_{m=-M}^{M}\left(\lim_{N\to\infty}\sum_{n=-N}^{N}\frac{\psi_\Lambda(n+m\tau, x')}{x+n+m\tau}\right). \tag{1.24}$$

In [**Weil76**][p.70], Weil proved

Lemma 1.3.1. *Let $x, x' \in \mathbb{C}\backslash\Lambda$. We write $x' = \alpha + \beta\tau$. If we further assume that $\beta \notin \mathbb{Z}$, then we have*

$$\sum_{\lambda\in\Lambda} \frac{\psi_\Lambda(\lambda, x')}{x+\lambda} = \boldsymbol{e}(\beta x)\mathcal{K}(\tau, x, x'). \tag{1.25}$$

Proof. First we observe that

$$\psi_\Lambda(n + m\tau, x') = \mathbf{e}(m\alpha - n\beta).$$

By Proposition 1.2.1, we see that both two sides of formula (1.25) are 1-periodic in β, hence we may assume that $\beta \in (0, 1)$.

By using the formula

$$\sum_{n\in\mathbb{Z}} \frac{\mathbf{e}(-n\beta)}{x+n} = 2\pi i \frac{\mathbf{e}(x\beta)}{\mathbf{e}(x)-1} \quad \text{for } \beta \in (0,1), \tag{1.26}$$

the Eisenstein summation (1.24) gives

$$2\pi i \mathbf{e}(x\beta) \sum_{m\in\mathbb{Z}} \frac{\mathbf{e}(mx')}{q^m \mathbf{e}(x) - 1}, \tag{1.27}$$

which is $\mathbf{e}(x\beta)\mathcal{K}(\tau, x, x')$.

□

Now we define the Eisenstein series of weight k by:

Definition 1.3.2. *Let k be a positive integer and $x' = \alpha + \beta\tau \in \mathbb{C}\backslash\Lambda$ be a complex number with $\beta \notin \mathbb{Z}$. We define*

$$E_k(x') = \sum_{\lambda\in\Lambda}{}' \frac{\psi_\Lambda(\lambda, x')}{\lambda^k}. \tag{1.28}$$

where $\sum'$ means the sum taken over all $\lambda \in \Lambda$ except 0. When $k = 1$ or 2, the series is defined by Eisenstein summation.

Following the formula (1.25), the Kronecker theta function gives the generating series of E_k:

Proposition 1.3.3. *Let $x, x' \in \mathbb{C}\backslash\Lambda$ be two complex numbers. We write $x' = \alpha + \beta\tau$*

with $\beta \notin \mathbb{Z}$. Then we have

$$\boldsymbol{e}(x\beta)\mathcal{K}(\tau, x, x') = \frac{1}{x} + \sum_{k\geq 0} E_{k+1}(x')(-x)^k. \tag{1.29}$$

Example 1.3.4. *We can also deduce the Fourier expansion of E_k from (1.2). For example, if we write $x = \alpha + \beta\tau$ with $\beta \in (0,1)$, then the Fourier expansion of $E_1(x)$ is*

$$2\pi i\left(\frac{1}{2} + \beta\right) - \frac{2\pi i}{1 - \boldsymbol{e}(x)} - 2\pi i \sum_{m,n\geq 1} q^{mn}(\boldsymbol{e}(mx) - \boldsymbol{e}(-mx)). \tag{1.30}$$

1.4 Eisenstein-Kronecker function

In this section, we want to extend the Eisenstein series $E_k(x)$ to all points in $\mathbb{C}$. To do that, we recall the Eisenstein-Kronecker function. One can find more details in [**Weil76**][Chapter VIII] or [**Ban06**][Section 1].

Definition 1.4.1. *Let Λ be a lattice in $\mathbb{C}$. Let a be a non-negative integer and $x, x' \in \mathbb{C}$. For $\mathrm{Re}(s) > \frac{a}{2} + 1$, we define the Eisenstein-Kronecker function to be the following series:*

$$K_a(x, x', s; \Lambda) = \sum_{\lambda\in\Lambda}{}' \frac{(\bar{x}_0 + \bar{\lambda})^a}{|x + \lambda|^{2s}} \psi_\Lambda(\lambda, x'). \tag{1.31}$$

where $\sum'$ means the sum taken over all $\lambda \in \Lambda$ if $x \notin \Lambda$, and except $-x$ if $x \in \Lambda$.

We note that the Eisenstein-Kronecker function has an meromorphic continuation to the whole complex s-plane. More precisely, the following proposition holds:

Proposition 1.4.2. *The function $K_a(x, x', s; \Lambda)$ has a meromorphic continuation to the whole s-plane with possible poles at $s = 0, 1$. If $a = 0$ and $x \in \Lambda$, $K_a(x, x', s; \Lambda)$ has a pole at $s = 0$ with residue $-\psi_\Lambda(x, -x')$. If $a = 0$ and $x' \in \Lambda$, $K_a(x, x', s; \Lambda)$ has a pole at $s = 1$ with residue $A(\Lambda)^{-1}$. It satisfies the functional equation:*

$$\Gamma(s)K_a(x, x', s; \Lambda) = A(\Lambda)^{a+1-2s}\Gamma(a+1-s)K_a(x', x, a+1-s; \Lambda)\psi_\Lambda(x, -x'). \tag{1.32}$$

Proof. See [**Weil76**][Chapter VIII §13]. □

Now we can extend the definition of $E_k(x)$ to all $x \in \mathbb{C}$ using the Eisenstein-Kronecker function $K_a(x, x', s; \Lambda)$:

We take $a = s = 1$, $x = 0$ and $\Lambda = \mathbb{Z} + \mathbb{Z}\tau$ for some $\tau \in \mathbb{H}$. Then the functional equation (1.32) gives

$$K_1(x, 0, 1; \Lambda) = K_1(0, x, 1; \Lambda).$$

As in [**MS90**], we can calculate the Fourier expansion of $K_1(x, 0, 1; \Lambda)$:

$$\begin{aligned} K_1(x, 0, 1; \Lambda) =& 2\pi i\beta + \sum_{n\in\mathbb{Z}} \pi \cot \pi(x + n\tau) \\ =& 2\pi i\beta + \pi i \frac{\mathbf{e}(x) + 1}{\mathbf{e}(x) - 1} - 2\pi i \sum_{m,n>0} q^{mn}(\mathbf{e}(mx) - \mathbf{e}(-mx)), \end{aligned} \tag{1.33}$$

where $x = \alpha + \beta\tau$. We see that this coincides with the Fourier expansion (1.30) of $E_1(x)$. By the definition of $K_a(x, x', s; \Lambda)$, we see that

$$\frac{\partial}{\partial \bar{x}} K_2(x, 0, 1; \Lambda) = K_1(x, 0, 1; \Lambda).$$

Hence it is not hard to show that

$$K_2(0, x, 2; \Lambda) = E_2(x). \tag{1.34}$$

If $k \geq 3$, we always have $K_k(0, x, k; \Lambda) = E_k(x)$ since $E_k(x)$ converges absolutely. We note that $K_k(0, x, k; \Lambda)$ is well-defined for all $x \in \mathbb{C}$ although it is not even continuous at lattice points. So we can extend Definition 1.3.2:

Definition 1.4.3. *Let k be a positive integer and $x \in \mathbb{C}$. Then we define*

$$E_k(x) = K_k(0, x, k; \Lambda). \tag{1.35}$$

Remark 1.4.4. *The continuation of $E_k(x')$ allows us to extend the Proposition 1.3.3 to all $x, x' \in \mathbb{C}\backslash\Lambda$ since both sides of (1.29) are continuous in $x' \in \mathbb{C}\backslash\Lambda$.*

By using the Eisenstein-Kronecker function, we are able to extend the distribution relation of $E_k(x)$ to the following case:

Proposition 1.4.5. *Let k be a positive integer and $x \in \mathbb{C}$, then for any $c \in \mathcal{O}_\Lambda = \{c \in \mathbb{C} \,|\, c\Lambda \subset \Lambda\}$, we have*

$$\sum_{r\in\Lambda/c\Lambda} E_k\left(\frac{x + r}{c}\right) = \frac{c}{\bar{c}^{k-1}} E_k(x). \tag{1.36}$$

Proof. First we note that

$$\begin{aligned}\sum_{r\in\Lambda/c\Lambda} K_a\left(\frac{x+r}{c},0,s;\Lambda\right) &= \sum_{r\in\Lambda/c\Lambda}\sum_{\lambda\in\Lambda}{}' \frac{\overline{\left(\frac{x+r}{c}+\lambda\right)}^a}{\left|\frac{x+r}{c}+\lambda\right|^{2s}} \\ =\frac{|c|^{2s}}{\bar{c}^a}\sum_{r\in\Lambda/c\Lambda}\sum_{\lambda\in\Lambda}{}' \frac{\overline{(x+r+c\lambda)}^a}{|x+r+c\lambda|^{2s}} &= \frac{|c|^{2s}}{\bar{c}^a}K_a(x,0,s;\Lambda).\end{aligned} \tag{1.37}$$

By using the functional equation (1.32) of the Eisenstein-Kronecker function, we have

$$\sum_{r\in\Lambda/c\Lambda} K_a\left(0,\frac{x+r}{c},s;\Lambda\right) = \frac{|c|^{2(a+1-s)}}{\bar{c}^a}K_a(0,x,s;\Lambda). \tag{1.38}$$

Now by taking $a=s=k$, we complete the proof. □

We give some connections with modular forms now. Let l be a positive integer. Let $\Lambda[\frac{1}{l}]=\{\alpha\in\mathbb{C}\,|\,l\alpha\in\Lambda\}/\Lambda$ be the group of l-torsion points modulo Λ and

$$\Gamma(l)=\left\{\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in \mathrm{SL}_2(\mathbb{Z})\,\middle|\,\begin{pmatrix}a & b\\ c & d\end{pmatrix}\equiv\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\pmod{l}\right\}. \tag{1.39}$$

Then we have the following:

Theorem 1.4.6. *Let k be a positive integer, and $x_0\in\Lambda[\frac{1}{l}]$ be a non-zero torsion point. Then the function $E_k(x_0)$ in τ is a modular form of weight k for the congruence subgroup $\Gamma(l)$.*

Proof. The functional equation (1.32) gives

$$\begin{aligned}E_k(x_0)=K_k(0,x_0,k,\Lambda) &= \left(\frac{\mathrm{Im}(\tau)}{\pi}\right)^{1-k}K_k(x_0,0,1;\Lambda)\\ &= \frac{1}{\Gamma(k)}\left(\frac{\mathrm{Im}(\tau)}{\pi}\right)^{1-k}\sum_{\lambda\in\mathbb{Z}+\tau\mathbb{Z}}\frac{1}{(x_0+\lambda)^k|x_0+\lambda|^{2s}}\Bigg|_{s=1-k}.\end{aligned} \tag{1.40}$$

The equation 9.3a of [**Shi07**] shows the invariance property of $E_k(x_0)$. Then this theorem follows directly from [**Shi07**][Theorem 9.6]. □

Remark 1.4.7. *Let $x_0=\frac{p+q\tau}{l}\in\Lambda[\frac{1}{l}]$ be a non-zero l-torsion point, the Fourier*

*expansion of $E_k(x_0)$ is also given in [**Shi07**][Section 9.5]:*

$$\frac{l}{(-2i)^k\pi}E_k\left(\frac{p+q\tau}{l}\right)$$
$$= -\frac{l^{k-1}B_k(p/l)}{k} + \sum_{\substack{0<m\in p+l\mathbb{Z}\\ n\geq 1}} m^{k-1}\boldsymbol{e}\left(\frac{n(m\tau+q)}{l}\right) - \sum_{\substack{0<m\in -p+l\mathbb{Z}\\ n\geq 1}} (-m)^{k-1}\boldsymbol{e}\left(\frac{n(m\tau-q)}{l}\right). \tag{1.41}$$

At last, we prove a lemma that involves the products of Eisenstein series. One can find a similar property in [**LS**][Lemma 1.5].

Lemma 1.4.8. *Let $N \geq 1$ be an integer, $\Lambda = \mathbb{Z} + \tau\mathbb{Z}$. Let $x_0, x_1, \ldots, x_N \in \mathbb{C}\backslash\Lambda$ satisfy*

$$\sum_{j=0}^{N} x_j \in \mathbb{Z}. \tag{1.42}$$

Then the coefficient of t^N in

$$\prod_{j=0}^{N}\left(1 + \sum_{k\geq 1} E_k(x_j)(-t)^k\right) \tag{1.43}$$

equals 0.

Proof. We define

$$F(t) := \prod_{j=0}^{N} \mathcal{K}(\tau, t, x_j). \tag{1.44}$$

It is clear that $F(t+1) = F(t)$. By Proposition 1.2.1, we see that

$$F(t+\tau) = \prod_{j=0}^{N} \mathcal{K}(\tau, t+\tau, x_j) = \mathbf{e}\left(\sum_{j=0}^{N} x_j\right)\prod_{j=0}^{N}\mathcal{K}(\tau, t, x_j) = F(t). \tag{1.45}$$

Hence $F(t)$ is an elliptic function for the lattice Λ. Moreover, by Proposition 1.1.2, $F(t)$ has only one pole in the fundamental domain of $\mathbb{C}/\Lambda$ at $t = 0$ of order $N+1$. So the residue of $F(t)$ at $t = 0$ is 0. By Proposition 1.3.3, we see that the residue of $F(t)$ at $t = 0$ is exactly the coefficient of t^n in

$$\prod_{j=0}^{N}\mathbf{e}(t\beta_j)\left(1 + \sum_{k\geq 1} E_k(x_j)(-t)^k\right) \tag{1.46}$$

where $x_j = \alpha_j + \tau\beta_j$. Since $\sum_j x_j \in \mathbb{Z}$, we have $\sum_j \beta_j = 0$. This proves the Lemma. $\square$

Corollary 1.4.9. *If we put $N = 2$ in the Lemma above, then we have*

$$E_1(x_1)E_1(x_2) + E_1(x_1)E_1(x_3) + E_1(x_2)E_1(x_3) = E_2(x_1) + E_2(x_2) + E_2(x_3), \quad (1.47)$$

where $x_1 + x_2 + x_3 = 0$. If we put $N = 3$, then we have

$$\begin{aligned} &E_1(x_1)E_1(x_2)E_1(x_3) + E_1(x_1)E_1(x_2)E_1(x_4) + E_1(x_1)E_1(x_3)E_1(x_4) \\ &+ E_1(x_2)E_1(x_3)E_1(x_4) = \sum_{i \neq j} E_1(x_i)E_2(x_j) - \sum_{j=1}^{4} E_3(x_j) \end{aligned} \quad (1.48)$$

where $x_1 + x_2 + x_3 + x_4 = 0$.

*Identity (1.47) also occurs in [**BG02**][Prop. 3.7]. In Chapter 4, we will see that these identities give some naive cocycle relations of Eisenstein series.*

Chapter

2 Elliptic Cocycle

In this chapter, we will introduce the elliptic cocycle $\mathscr{E}(\tau, \sigma, x, x')$. This is an $(N-1)$-cocycle on the group $\mathrm{GL}_N(\mathbb{Z})$ valued in meromorphic functions on $\mathbb{H} \times \mathbb{C}^N \times \mathbb{C}^N$ that was recently introduced by Charollois in [**Cha1**]. Explicit formulas for the elliptic cocycle $\mathscr{E}$ consist of N-fold products of the Kronecker theta function defined in Chapter 1. The main results in this chapter are Theorem 2.2.3 and Theorem 2.2.6, which state the cocycle relation of $\mathscr{E}(\tau, \sigma, x, x')$ in dimensions 2 and 3 respectively. Charollois proved the general cocycle relation by taking the summation of the trigonometric cocycle introduced by Sczech (see [**CS16**]). We will give an independent proof in the cases $N = 2, 3$ directly from the definition. In Chapter 4 and Chapter 5, we will specialize the elliptic cocycle $\mathscr{E}$ to obtain other related cocycles. They have their own merits and will be studied in different ways.

2.1 Definition and basic properties of the elliptic cocycle

First we give the definition of the elliptic cocycle $\mathscr{E}$. Let $N > 1$ be an integer. Fix two vectors $x = (x_1, \ldots, x_N) \in \mathbb{C}^N$, $x' = (x'_1, \ldots, x'_N)^t \in \mathbb{C}^N$, let

$$\mathscr{K}(\tau, x, x') = \prod_{i=1}^{N} \mathcal{K}(\tau, x_i, x'_i) \tag{2.1}$$

be the multivariable Kronecker theta function.

Definition 2.1.1. *Let $\sigma \in M_N(\mathbb{Z})$, $x, x' \in \mathbb{C}^N$ and $\tau \in \mathbb{H}$. If $\det(\sigma) \neq 0$,*

$$\mathscr{E}(\tau, \sigma, x, x') = \frac{1}{\det \sigma} \sum_{y, y' \in \mathbb{Z}^N / \sigma \mathbb{Z}^N} \boldsymbol{e}(x \cdot y) \mathscr{K}(\tau, x\sigma, \sigma^{-1}(x' + y\tau + y')). \tag{2.2}$$

And if $\det(\sigma) = 0$,

$$\mathscr{E}(\tau, \sigma, x, x') = 0.$$

Remark 2.1.2. *In this thesis, we always assume that the columns of $\sigma \in M_N(\mathbb{Z})$ are non-zero.*

Proposition 2.1.3. *For any $\sigma \in M_N(\mathbb{Z})$ and $g_0 \in \mathrm{GL}_N(\mathbb{Z})$, the following homogeneous relation holds:*

$$\mathscr{E}(\tau, g_0\sigma, xg_0^{-1}, g_0x') = \det(g_0)\mathscr{E}(\tau, \sigma, x, x'). \tag{2.3}$$

Proof. If $\det(\sigma) = 0$, the homogeneous relation is obvious since both of two sides equal to 0. Hence we assume that $\det(\sigma) \neq 0$. By changing the variables $y = g_0^{-1}z$ and $y' = g_0^{-1}z'$, we have

$$\begin{aligned}
&\mathscr{E}(\tau, g_0\sigma, xg_0^{-1}, g_0x') \\
=&\frac{1}{\det(g_0\sigma)} \sum_{z,z' \in \mathbb{Z}^N/g_0\sigma\mathbb{Z}^N} \mathbf{e}(xg_0^{-1} \cdot z)\mathscr{K}\left(\tau, x\sigma, (g_0\sigma)^{-1}(g_0x' + z\tau + z')\right) \\
=&\frac{\det(g_0)}{\det(\sigma)} \sum_{y,y' \in \mathbb{Z}^N/\sigma\mathbb{Z}^N} \mathbf{e}(x \cdot y)\mathscr{K}\left(\tau, x\sigma, \sigma^{-1}(x' + y\tau + y')\right) \\
=&\det(g_0)\mathscr{E}(\tau, \sigma, x, x').
\end{aligned}$$

This completes the proof. □

Following [**Cha1**], we give a more symmetric equivalent definition of $\mathscr{E}(\tau, \sigma, x, x')$.

Proposition 2.1.4. *Let $\sigma \in M_N(\mathbb{Z})$ with $D := \det(\sigma) \neq 0$. Then for any $x, x' \in \mathbb{C}^N$, we have*

$$\mathscr{E}(\tau, \sigma, x, x') = \frac{\mathrm{sign}(D)}{|D|^N} \sum_{\substack{z \in \mathbb{Z}^N/\mathbb{Z}^N|D|\sigma^{-1} \\ z' \in \mathbb{Z}^N/\sigma\mathbb{Z}^N}} \mathscr{K}\left(\frac{\tau}{|D|}, \frac{x+z}{|D|}\sigma, \sigma^{-1}(x' + z')\right). \tag{2.4}$$

Proof. First we note that by applying the Proposition 2.1.3 to any matrix g_0 of determinant -1 if $D < 0$, it reduces to the case of $D > 0$. Hence we may assume that $D > 0$. By using the distribution relation of Kronecker theta function N times, we have

$$\begin{aligned}
\mathscr{E}(\tau, \sigma, x, x') =& \frac{1}{D} \sum_{z,z' \in \mathbb{Z}^N/\sigma\mathbb{Z}^N} \mathbf{e}(x \cdot y)\mathscr{K}(\tau, x\sigma, \sigma^{-1}(x' + z\tau + z')) \\
=& \frac{1}{D^{N+1}} \sum_{\mathbf{j} \in \mathbb{Z}^N/D\mathbb{Z}^N} \sum_{z,z' \in \mathbb{Z}^N/\sigma\mathbb{Z}^N} \mathbf{e}(x \cdot z)\mathscr{K}\left(\frac{\tau}{D}, \frac{x\sigma + \mathbf{j}}{D}, \sigma^{-1}(x' + z\tau + z')\right).
\end{aligned}$$

Since the entries of $D\sigma^{-1}$ are integers and $z \in \mathbb{Z}^N/\sigma\mathbb{Z}^N$, we may apply Proposition 1.2.1 to $\mathscr{K}$:

$$\begin{aligned}
&\mathscr{K}\left(\frac{\tau}{D}, \frac{x\sigma+\mathbf{j}}{D}, \sigma^{-1}(x'+z\tau+z')\right)\\
=&\mathscr{K}\left(\frac{\tau}{D}, \frac{x\sigma+\mathbf{j}}{D}, D\sigma^{-1}z\frac{\tau}{D}+\sigma^{-1}(x'+z')\right)\\
=&\mathbf{e}\left(-\frac{x\sigma+\mathbf{j}}{D}\cdot D\sigma^{-1}z\right)\mathscr{K}\left(\frac{\tau}{D}, \frac{x\sigma+\mathbf{j}}{D}, \sigma^{-1}(x'+z')\right).
\end{aligned}$$

Thus $\mathscr{E}(\tau,\sigma,x,x')$ equals

$$\frac{1}{D^{N+1}}\sum_{\mathbf{j}\in\mathbb{Z}^N/D\mathbb{Z}^N}\sum_{z,z'\in\mathbb{Z}^N/\sigma\mathbb{Z}^N}\mathbf{e}(-\mathbf{j}\cdot\sigma^{-1}z)\mathscr{K}\left(\frac{\tau}{D}, \frac{x\sigma+\mathbf{j}}{D}, \sigma^{-1}(x'+z')\right).$$

We note that the summation over $z \in \mathbb{Z}^N/\sigma\mathbb{Z}^N$ is given by:

$$\frac{1}{D}\sum_{z\in\mathbb{Z}^N/\sigma\mathbb{Z}^N}\mathbf{e}\left(\mathbf{j}\cdot\sigma^{-1}z\right)=\begin{cases}1 & \text{if } \mathbf{j}\in\mathbb{Z}^N\sigma/D\mathbb{Z}^N,\\ 0 & \text{else.}\end{cases}$$

This implies that

$$\mathscr{E}(\tau,\sigma,x,x')=\frac{1}{D^N}\sum_{\mathbf{j}\in\mathbb{Z}^N\sigma/D\mathbb{Z}^N}\sum_{z'\in\mathbb{Z}^N/\sigma\mathbb{Z}^N}\mathscr{K}\left(\frac{\tau}{D}, \frac{x\sigma+\mathbf{j}}{D}, \sigma^{-1}(x'+z')\right).$$

Now we write $\mathbf{j} = y\sigma$ where y runs through $\mathbb{Z}^N/D\sigma^{-1}\mathbb{Z}^N$. This completes the proof. □

Now we define an action of $\mathrm{GL}_N(\mathbb{Q})$ on $\mathscr{E}$. Let $g_0 \in \mathrm{GL}_N(\mathbb{Q})$, and choose the smallest positive integers $\lambda_1, \cdots, \lambda_N$ such that $g = g_0\lambda \in M_N(\mathbb{Z})$ where λ is the diagonal matrix $\mathrm{diag}\{\lambda_1, \cdots, \lambda_N\}$. Then

$$(g_0\cdot\mathscr{E})(\tau,\sigma,x,x') := \frac{1}{\det(g)}\sum_{z,z'\in\mathbb{Z}^N/g\mathbb{Z}^N}\mathscr{E}\left(\tau,\sigma,xg,g^{-1}(x'+z\tau+z')\right)\mathbf{e}(x\cdot z).$$

Then we will extend Proposition 2.1.3 to any matrix in $\mathrm{GL}_N(\mathbb{Q})$. We first prove a simple and useful lemma (see [**CD14**][lemma 2.9]).

Lemma 2.1.5. *Let $g_1, g_2 \in M_N(\mathbb{Z})$ be two matrices and $\det(g_1), \det(g_2) \neq 0$. L is a lattice in $\mathbb{C}$. If we fix sets of representatives $\{y\}$ and $\{z\}$ for L^N/g_1L^N and*

L^N/g_2L^N respectively, then the following map is a bijection

$$\begin{aligned} L^N/g_1L^N \times L^N/g_2L^N &\to L^N/g_2g_1L^N \\ (y,z) &\mapsto z+g_2y. \end{aligned}$$

Proof. If $z+g_2y = z'+g_2y' \pmod{g_2g_1L^N}$ for some $y,y' \in L^N/g_1L^N$ and $z_1,z_2 \in L^N/g_1L^N$, then we have

$$z-z' \in g_2g_1L^N + g_2(y'-y) \subset g_2L^N.$$

Since we have already fixed the set of representatives, then we have $z=z'$. The same reason implies that $y=y'$. So this map is injective. Moreover, the cardinality of the two sides equal to $\det(g_1g_2)^2$. Hence it is a bijection. □

Remark 2.1.6. *[**CD14**][Remark 2.10] We need to note that such map is not a group homomorphism. Such a correspondence depends on the choice of the set of representatives.*

Proposition 2.1.7. *Let $\tau \in \mathbb{H}$ and $x,x' \in \mathbb{C}^N$. Let $g_0 \in \mathrm{GL}_N(\mathbb{Q})$ and $g = g_0\lambda \in M_N(\mathbb{Z})$ for some diagonal matrix λ. Then we have*

$$(g_0 \cdot \mathscr{E})(\tau,\sigma,x,x') = \mathscr{E}(\tau,g\sigma,x,x'). \tag{2.5}$$

In particular, let λ be an invertible diagonal matrix. Then for any $\sigma \in M_N(\mathbb{Z})$, we have

$$\mathscr{E}(\tau,\sigma\lambda,x,x') = \mathscr{E}(\tau,\sigma,x,x'). \tag{2.6}$$

Proof. By definition of $\mathscr{E}$, we have

$$\begin{aligned} (g_0 \cdot \mathscr{E})(\tau,\sigma,x,x') &= \frac{1}{\det(g)} \sum_{z,z' \in \mathbb{Z}^N/g\mathbb{Z}^N} \mathscr{E}\left(\tau,\sigma,xg,g^{-1}(x'+z\tau+z')\right)\mathbf{e}(xz) \\ &= \frac{1}{\det(g\sigma)} \sum_{\substack{z,z' \in \mathbb{Z}^N/g\mathbb{Z}^N \\ y,y' \in \mathbb{Z}^N/\sigma\mathbb{Z}^N}} \mathscr{K}\left(\tau,xg\sigma,\sigma^{-1}g^{-1}(x'+(z+gy)\tau+z'+gy')\right)\mathbf{e}(x(z+gy)). \end{aligned} \tag{2.7}$$

By Lemma 2.1.5, we know that the sets $\{z+gy\}$ and $\{z'+gy'\}$ give representatives of $\mathbb{Z}^N/g\sigma\mathbb{Z}^N$. Hence the summation above equals

$$\frac{1}{\det(g\sigma)} \sum_{w,w' \in \mathbb{Z}^N/g\sigma\mathbb{Z}^N} \mathscr{K}\left(\tau,xg\sigma,\sigma^{-1}g^{-1}(x'+w\tau+w')\right)\mathbf{e}(xw)$$

which is exactly $\mathscr{E}(\tau, g\sigma, x, x')$. In particular, if we take λ to be the diagonal matrix $\mathrm{diag}\{a_1, \cdots, a_N\}$, then by applying Proposition 1.2.2 N times, we have

$$\begin{aligned}
&\mathscr{E}(\tau, \sigma\lambda, x, x') \\
=&\frac{1}{\det(\sigma\lambda)} \sum_{\substack{z,z' \in \mathbb{Z}^N/\sigma\mathbb{Z}^N \\ y,y' \in \mathbb{Z}^N/\lambda\mathbb{Z}^N}} \mathscr{K}\left(\tau, x\sigma\lambda, \lambda^{-1}\sigma^{-1}(x' + (z+\sigma y)\tau + z' + \sigma y')\right) \mathbf{e}(x(z+\sigma y)). \\
=&\frac{1}{\det(\sigma)} \sum_{z,z' \in \mathbb{Z}^N/\sigma\mathbb{Z}^N} \mathscr{K}\left(\tau, x\sigma, \sigma^{-1}(x' + z\tau + z')\right) \mathbf{e}(xz).
\end{aligned} \tag{2.8}$$

This proves that

$$\mathscr{E}(\tau, \sigma\lambda, x, x') = \mathscr{E}(\tau, \sigma, x, x'), \tag{2.9}$$

which completes the proof.

□

Remark 2.1.8. *From the equation (2.6), we can assume that the column vectors of σ are primitive. Hence we always make this assumption in the following.*

2.2 Cocycle relation

In this section, we will prove the cocycle relation of $\mathscr{E}$ in dimension 2 and 3. The proof would divide into two parts: the first part is proving the cocycle relation for some special cases, i.e. the Lemma 2.2.4 for dimension 2 and the Lemma 2.2.7, 2.2.8 for dimension 3. The second part is proving the extension theorem, i.e. the Theorem 2.2.5 for dimension 2 and the Theorem 2.2.9 for dimension 3. These two extension theorems are purely algebraic statements which holds for any $\mathrm{GL}_2(\mathbb{Z})$ and $\mathrm{GL}_3(\mathbb{Z})$ module. The extension theorems allow us to extend the cocycle relation to general cases.

We note that the cocycle relation of $\mathscr{E}$ is in fact an identity for meromorphic functions of all variables x, x'. Our strategy of the main Lemma 2.2.4 is using the fact that the residue of elliptic function for the lattice $\mathbb{Z} + \tau\mathbb{Z}$ must be 0. The case of dimension 3 uses the same strategy and reduce to the case of dimension 2.

To start the proof, we first introduce a parameter t to $\mathscr{E}$.

Definition 2.2.1. *Let $\tau \in \mathbb{H}$, $\sigma \in M_N(\mathbb{Z})$ and $x, x' \in \mathbb{C}^N$. Fix a non-zero vector $\nu \in \mathbb{Z}^N$, for any $t \in \mathbb{C}$, we define*

$$\mathscr{E}_\nu(\tau, \sigma, t, x, x') = \mathscr{E}(\tau, \sigma, x + t\nu, x').$$

Lemma 2.2.2. *Fix τ, σ, x, x', ν, the function $t \mapsto \mathscr{E}_\nu(\tau, \sigma, t, x, x')\mathcal{K}(\tau, t, x'')$ is $\mathbb{Z}+\mathbb{Z}\tau$-periodic, where $x'' = -\nu \cdot x'$.*

Proof. Since $\mathcal{K}(\tau, x, x')$ is 1-periodic in x and $\nu \in \mathbb{Z}^N$, it is easy to see that

$$\mathscr{E}_\nu(\tau, \sigma, t+1, x, x')\mathcal{K}(\tau, t+1, x'') = \mathscr{E}_\nu(\tau, \sigma, t, x, x')\mathcal{K}(\tau, t, x'').$$

For the part of τ-periodic, by the Proposition 1.2.1, we have:

$$\mathcal{K}(\tau, t+\tau, -\nu \cdot x') = \mathbf{e}(\nu \cdot x')\mathcal{K}(\tau, t, -\nu \cdot x'),$$

and

$$\begin{aligned}
&\mathscr{E}_\nu(\tau, \sigma, t+\tau, x, x') \\
=&\frac{1}{\det \sigma} \sum_{y, y' \in \mathbb{Z}^N/\sigma\mathbb{Z}^N} \mathbf{e}((x + t\nu + \tau\nu) \cdot y)\mathscr{K}(\tau, (x + t\nu + \tau\nu)\sigma, \sigma^{-1}(x' + y\tau + y')) \\
=&\frac{1}{\det \sigma} \sum_{y, y' \in \mathbb{Z}^N/\sigma\mathbb{Z}^N} \mathbf{e}((x + t\nu + \tau\nu) \cdot y)\mathbf{e}(-\nu \cdot (x' + y\tau + y'))\cdot \\
&\qquad\qquad \mathscr{K}(\tau, (x + t\nu)\sigma, \sigma^{-1}(x' + y\tau + y')) \\
=&\frac{1}{\det \sigma} \sum_{y, y' \in \mathbb{Z}^N/\sigma\mathbb{Z}^N} \mathbf{e}((x + t\nu) \cdot y)\mathbf{e}(-\nu \cdot x')\mathscr{K}(\tau, (x + t\nu)\sigma, \sigma^{-1}(x' + y\tau + y')) \\
=&\mathbf{e}(-\nu \cdot x')\mathscr{E}_\nu(\tau, \sigma, t, x, x').
\end{aligned}$$

Hence we see that $\mathscr{E}_\nu(\sigma, t, x, x')\mathcal{K}(\tau, t, x'')$ is τ-periodic. This completes the proof. □

2.2.1 Dimension 2

In this subsection, we prove the following theorem:

Theorem 2.2.3. *Let $\tau \in \mathbb{H}$, and $x, x' \in \mathbb{C}^2$. Then for any vectors $\sigma_0, \sigma_1, \sigma_2 \in \mathbb{Z}^2_{\text{prim}}$, we have the following cocycle relation:*

$$\mathscr{E}(\tau, (\sigma_0, \sigma_1), x, x') - \mathscr{E}(\tau, (\sigma_0, \sigma_2), x, x') + \mathscr{E}(\tau, (\sigma_1, \sigma_2), x, x') = 0. \tag{2.10}$$

We first prove a special case:

Lemma 2.2.4. *The cocycle relation holds for (2.10) for $\sigma_0 = (1, 0)^t$, $\sigma_1 = (0, 1)^t$, $\sigma_2 = (r, s)^t$ where $0 \le r < s$. Namely, we have the identity:*

$$\mathscr{E}\left(\tau, M_0, x, x'\right) - \mathscr{E}\left(\tau, M_1, x, x'\right) + \mathscr{E}\left(\tau, M_2, x, x'\right) = 0$$

where $M_0 = \begin{pmatrix} 0 & r \\ 1 & s \end{pmatrix}$, $M_1 = \begin{pmatrix} 1 & r \\ 0 & s \end{pmatrix}$, $M_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Proof. For fixed σ_i, x, x', we choose a vector $\nu = (1, v) \in \mathbb{Z}^2$ such that $\frac{x\sigma_i}{\nu\sigma_i}$ are different for all i. For such vector ν, we introduce a function:

$$F(t) = \left(\mathscr{E}_\nu(\tau, M_0, t, x, x') - \mathscr{E}_\nu(\tau, M_1, t, x, x') + \mathscr{E}_\nu(\tau, M_2, t, x, x')\right) \mathcal{K}(\tau, t, x''),$$

where $x'' = -\nu \cdot x'$. By writing the function $F(t)$ in terms of the Kronecker theta function, we have

$$\begin{aligned} F(t) = &\Bigg(\mathscr{K}(\tau, x + t\nu, x') - \frac{1}{s} \sum_{y,y' \in \mathbb{Z}^2/M_1\mathbb{Z}^2} \mathscr{K}\left(\tau, (x + t\nu)M_1, M_1^{-1}(x' + y\tau + y')\right) \\ &- \frac{1}{r} \sum_{y,y' \in \mathbb{Z}^2/M_2\mathbb{Z}^2} \mathscr{K}\left(\tau, (x + t\nu)M_2, M_2^{-1}(x' + y\tau + y')\right) \Bigg) \mathcal{K}(\tau, t, x_0'). \end{aligned} \tag{2.11}$$

Since $\mathcal{K}(\tau, x, x_0)$ has a simple pole at $x = 0$ with residue 1 and by the assumption on ν, the function $F(t)$ has a simple pole at $t = 0, -x_1, -\frac{x_2}{v}, -\frac{rx_1+sx_2}{r+vs}$.

Next we calculate the residue at each pole. For the pole $-x_1$, the residue of the first term of (2.11) equals to $\mathcal{K}(\tau, x_2 - vx_1, x_2')\mathcal{K}(\tau, -x_1, x_0')$. The residue of the second term of (2.11) equals to:

$$\frac{1}{s} \sum_{y_2, y_2' = 1}^{s} \mathbf{e}(y_2(x_2 - vx_1)) \mathcal{K}\left(\tau, s(x_2 - vx_1), \frac{x_2' + y_2\tau + y_2'}{s}\right) \mathcal{K}(\tau, -x_1, x_0'). \tag{2.12}$$

By the Proposition 1.2.2, we see that the summation equals to

$$\mathcal{K}(\tau, x_2 - vx_1, x_2')\mathcal{K}(\tau, -x_1, x_0').$$

Since the third term of (2.11) is holomorphic at $t = -x_1$, hence we have

$$\mathrm{Res}_{t=-x_1} F(t) = 0.$$

Similarly, the residue of $F(t)$ at $t = -\frac{x_2}{v}$ and $-\frac{rx_1+sx_2}{r+vs}$ are 0. This means that $F(t)$ has only a simple pole at $t = 0$. However, by the Lemma 2.2.2, $F(t)$ is an elliptic function, it forces $\mathrm{Res}_{t=0} F(t) = 0$, which gives exactly the cocycle relation:

$$\mathrm{Res}_{t=0} F(t) = \mathscr{E}\left(\tau, M_0, x, x'\right) - \mathscr{E}\left(\tau, M_1, x, x'\right) + \mathscr{E}\left(\tau, M_2, x, x'\right) = 0.$$

□

To prove the Theorem 2.2.3, we need the following general extension theorem. The extension theorem allows us to prove the general cocycle relation from special cases as in Lemma 2.2.4. In the Theorem 2.2.5 and Theorem 2.2.9, we always assume that the columns of matrix $\sigma \in M_2(\mathbb{Z})$ or $M_3(\mathbb{Z})$ are non-zero.

Theorem 2.2.5. *Denote $M_2^1(\mathbb{Z}) = \{\sigma \in M_2(\mathbb{Z}) \mid \det(\sigma) = 0 \text{ or } 1\}$. Let L be a left $\mathrm{SL}_2(\mathbb{Z})$-module. Let $\Phi : M_2^1(\mathbb{Z}) \to L$ be a map satisfing the following conditions:*

1. *$\Phi(\sigma) = 0$ if $\det(\sigma) = 0$,*
2. *The following two equations hold:*

$$\Phi\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + \Phi\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} = 0, \tag{2.13}$$

$$\Phi\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + \Phi\begin{pmatrix}0 & -1\\ 1 & 1\end{pmatrix} = \Phi\begin{pmatrix}1 & -1\\ 0 & 1\end{pmatrix}, \tag{2.14}$$

3. *$\Phi(g\sigma) = g\Phi(\sigma)$ for any $g \in \mathrm{SL}_2(\mathbb{Z})$ and $\sigma \in M_2^1(\mathbb{Z})$.*

Then there exists a unique extension of Φ to $\hat{\Phi} : M_2(\mathbb{Z}) \to L$ such that the following conditions are satisfied:

(i) $\hat{\Phi}|_{M_2^1(\mathbb{Z})} = \Phi$,

(ii) $\Phi(\sigma_0, \sigma_1) - \Phi(\sigma_0, \sigma_2) + \Phi(\sigma_1, \sigma_2) = 0$ *for any* $(\sigma_0, \sigma_1), (\sigma_0, \sigma_2), (\sigma_1, \sigma_2) \in M_2(\mathbb{Z})$,

(iii) $\Phi(g\sigma) = g\Phi(\sigma)$ *for any* $g \in \mathrm{SL}_2(\mathbb{Z})$ *and* $\sigma \in M_2(\mathbb{Z})$.

Proof. The idea of the proof relies on the Hermite normal form of σ. Because of the condition (3), we could always reduce to the case of Hermite normal form.

To prove the existence, we give the construction of $\hat{\Phi}$ by induction. Let $\sigma = \begin{pmatrix}a & b\\ c & d\end{pmatrix} \in M_2(\mathbb{Z})$. We first assume that the colums of σ are primitive. If $\det(\sigma) = 0$ or 1, then we define

$$\hat{\Phi}(\sigma) = \Phi(\sigma).$$

If $\det(\sigma) > 1$, we know that there exists a unique $g \in \mathrm{SL}_2(\mathbb{Z})$ such that $\sigma = g\begin{pmatrix}1 & r\\ 0 & s\end{pmatrix}$ with $0 \le r < s$. Then we define

$$\hat{\Phi}(\sigma) = g \cdot \left(\hat{\Phi}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} - \hat{\Phi}\begin{pmatrix}r & 0\\ s & 1\end{pmatrix}\right). \tag{2.15}$$

Since the determinant of $\begin{pmatrix} r & 0 \\ s & 1 \end{pmatrix}$ is less than s, $\hat{\Phi}\begin{pmatrix} r & 0 \\ s & 1 \end{pmatrix}$ is defined by induction assumption. If $\det(\sigma) < 0$, let

$$\hat{\Phi}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = -\hat{\Phi}\begin{pmatrix} b & a \\ d & c \end{pmatrix}. \tag{2.16}$$

Finally, if the columns of σ are not primitive, let

$$\hat{\Phi}(\sigma) = \hat{\Phi}\begin{pmatrix} a/\gcd(a,c) & b/\gcd(b,d) \\ c/\gcd(a,c) & d/\gcd(b,d) \end{pmatrix}. \tag{2.17}$$

Next we check that $\hat{\Phi}$ satisfies the 3 conditions. The conditions (i) and (iii) follow by construction. So we only need to check the condition (ii). We note that the equation (2.17) allows us to reduce to the case that σ_i are primitive. Let $k = \max\{|\det(\sigma_0,\sigma_1)|, |\det(\sigma_0,\sigma_2)|, |\det(\sigma_1,\sigma_2)|\}$. We prove the condition (ii) by induction on k.

When $k = 1$. Without loss of generality, we may assume that $\det(\sigma_0,\sigma_1) = 1$. Then multiplying by $(\sigma_0,\sigma_1)^{-1}$, the condition (ii) is equivalent to

$$\hat{\Phi}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \hat{\Phi}\begin{pmatrix} 0 & c_1 \\ 1 & c_2 \end{pmatrix} = \hat{\Phi}\begin{pmatrix} 1 & c_1 \\ 0 & c_2 \end{pmatrix}, \tag{2.18}$$

for some $c_1, c_2 \in \mathbb{Z}$. By our assumption, c_1, c_2 must be 0 or ± 1. We will show the cocycle relation for $(c_1, c_2) = (1,1)$, the other cases are easily deduced in the same way. By taking the action of $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ on the both sides of (2.14), we obtain

$$\hat{\Phi}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + \hat{\Phi}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \hat{\Phi}\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}. \tag{2.19}$$

By the definition (2.16), and by acting the matrix $\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$ on the equation (2.13), we have

$$\hat{\Phi}\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} = \hat{\Phi}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \hat{\Phi}\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}. \tag{2.20}$$

Combining with (2.19), we complete the proof of the (2.18) for $(c_1, c_2) = (1,1)$.

When $k > 1$. Without loss of generality, we assume that $\det(\sigma_0,\sigma_1) = k$. By

multiplying a suitable matrix in $\mathrm{SL}_2(\mathbb{Z})$ on the left, we may assume that

$$(\sigma_0, \sigma_1, \sigma_2) = \begin{pmatrix} 1 & s & c_1 \\ 0 & k & c_2 \end{pmatrix},$$

where $0 \le s < k$ and $c_1, c_2 \in \mathbb{Z}$. The assumption $k = \max\{|\det(\sigma_0, \sigma_1)|, |\det(\sigma_0, \sigma_2)|, |\det(\sigma_1, \sigma_2)|\}$ implies that $-k \le c_2 \le k$ and $|sc_2 - kc_1| \le k$. The negative case of c_2 is similar, hence we may assume that $0 \le c_2 \le k$.

We note that $|c_1| \le k$, otherwise $|kc_1 - sc_2| \ge |kc_1| - |sc_2| \ge k(k+1) - k(k-1) > k$. And if $|c_1| = k$, we must have $c_2 = k, s = k-1$. But such case would not occur since $\sigma_2 = (c_1, c_2)^t$ is assumed to be primitive. This means $|c_1| \le k-1$. Moreover, since c_2, s are non-negative, c_1 must be non-negative.

By (2.15), we have

$$\hat{\Phi}\begin{pmatrix} 1 & s \\ 0 & k \end{pmatrix} = \hat{\Phi}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \hat{\Phi}\begin{pmatrix} s & 0 \\ k & 1 \end{pmatrix}. \tag{2.21}$$

If $c_2 = k$, then $0 < c_1 < c_2$. Hence the definition (2.15) gives:

$$\hat{\Phi}\begin{pmatrix} 1 & c_1 \\ 0 & c_2 \end{pmatrix} = \hat{\Phi}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \hat{\Phi}\begin{pmatrix} c_1 & 0 \\ c_2 & 1 \end{pmatrix}. \tag{2.22}$$

If $c_2 < k$, the induction assumption also gives the same equation.

If $|sc_2 - rc_1| < k$, then by using the induction assumption to $\begin{pmatrix} s & c_1 & 0 \\ k & c_2 & 1 \end{pmatrix}$, we have

$$\hat{\Phi}\begin{pmatrix} s & c_1 \\ k & c_2 \end{pmatrix} - \hat{\Phi}\begin{pmatrix} s & 0 \\ k & 1 \end{pmatrix} + \hat{\Phi}\begin{pmatrix} c_1 & 0 \\ c_2 & 1 \end{pmatrix} = 0. \tag{2.23}$$

Combining equations (2.21), (2.22), (2.23), we prove the condition (ii).

If $sc_2 - kc_1 = k$, then since s is coprime to k, it forces $c_2 = k$. Hence $s - c_1 = 1$. We can find $g_1 \in \mathrm{SL}_2(\mathbb{Z})$ such that $\begin{pmatrix} s & c_1 \\ k & k \end{pmatrix} = g_1 \begin{pmatrix} 1 & s_1 \\ 0 & k \end{pmatrix}$ with $0 \le s_1 < k$. So by the definition of $\hat{\Phi}\begin{pmatrix} s & c_1 \\ k & k \end{pmatrix}$, we have

$$\hat{\Phi}\begin{pmatrix} s & c_1 \\ k & k \end{pmatrix} + \hat{\Phi}\begin{pmatrix} c_1 & (-ss_1 + c_1)/k \\ k & 1 - s_1 \end{pmatrix} = \hat{\Phi}\begin{pmatrix} s & (-ss_1 + c_1)/k \\ k & 1 - s_1 \end{pmatrix}. \tag{2.24}$$

Applying the induction assumption to

$$\begin{pmatrix} s & (-ss_1+c_1)/k & 0 \\ k & 1-s_1 & 1 \end{pmatrix}$$

and

$$\begin{pmatrix} s & (-ss_1+c_1)/k & 0 \\ k & 1-s_1 & 1 \end{pmatrix},$$

we have

$$\hat{\Phi}\begin{pmatrix} s & (-ss_1+c_1)/k \\ k & 1-s_1 \end{pmatrix} + \hat{\Phi}\begin{pmatrix} (-ss_1+c_1)/k & 0 \\ 1-s_1 & 1 \end{pmatrix} = \hat{\Phi}\begin{pmatrix} s & 0 \\ k & 1 \end{pmatrix}, \tag{2.25}$$

$$\hat{\Phi}\begin{pmatrix} (-ss_1+c_1)/k & 0 \\ 1-s_1 & 1 \end{pmatrix} + \hat{\Phi}\begin{pmatrix} c_1 & (-ss_1+c_1)/k \\ k & 1-s_1 \end{pmatrix} = \hat{\Phi}\begin{pmatrix} c_1 & 0 \\ k & 1 \end{pmatrix}. \tag{2.26}$$

So combining equations (2.21), (2.22), (2.24), (2.25), (2.26), we have

$$\begin{aligned}
&\hat{\Phi}\begin{pmatrix} 1 & s \\ 0 & k \end{pmatrix} + \hat{\Phi}\begin{pmatrix} s & c_1 \\ k & k \end{pmatrix} \\
=&\hat{\Phi}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \hat{\Phi}\begin{pmatrix} s & 0 \\ k & 1 \end{pmatrix} + \hat{\Phi}\begin{pmatrix} s & (-ss_1+c_1)/k \\ k & 1-s_1 \end{pmatrix} - \hat{\Phi}\begin{pmatrix} c_1 & (-ss_1+c_1)/k \\ c_2 & 1-s_1 \end{pmatrix} \\
=&\hat{\Phi}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \hat{\Phi}\begin{pmatrix} (-ss_1+c_1)/k & 0 \\ 1-s_1 & 1 \end{pmatrix} - \hat{\Phi}\begin{pmatrix} c_1 & (-ss_1+c_1)/k \\ k & 1-s_1 \end{pmatrix} \\
=&\hat{\Phi}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \hat{\Phi}\begin{pmatrix} c_1 & 0 \\ k & 1 \end{pmatrix} = \hat{\Phi}\begin{pmatrix} 1 & c_1 \\ 0 & k \end{pmatrix}.
\end{aligned}$$

If $sc_2 - kc_1 = -k$, then it forces that $c_2 = k$ and $s - c_1 = -1$. The proof is the same as the case $sc_2 - kc_1 = k$.

The uniqueness is clear from the condition (ii). This completes the proof. □

Proof of Theorem 2.2.3. As what we did in Lemma 2.2.4, for fixed σ_i, x, x', we choose a vector $\nu = (1, v) \in \mathbb{Z}^2$ such that $\frac{x\sigma_i}{\nu\sigma_i}$ are different for all i. For such vector ν, we introduce a function:

$$F(t) = \left(\mathscr{E}_\nu(\tau, (\sigma_0, \sigma_1), t, x, x') - \mathscr{E}_\nu(\tau, (\sigma_0, \sigma_2), t, x, x') + \mathscr{E}_\nu(\tau, (\sigma_1, \sigma_2), t, x, x')\right) \mathcal{K}(\tau, t, x'')$$

where $x'' = -\nu \cdot x'$. By Lemma 2.2.2, $F(t)$ is an elliptic function for the lattice $\mathbb{Z} + \mathbb{Z}\tau$.

We apply Theorem 2.2.5 to the function $\mathscr{E}(\tau,-,x,x')|_{M_2^1(\mathbb{Z})}$. We take $r,s\in\{\pm1,0\}$ in Lemma 2.2.4. It implies that $\mathscr{E}(\tau,-,x,x')|_{M_2^1(\mathbb{Z})}$ satisfies the condition (2) of Theorem 2.2.5. The condition (1) is clear from definition. The condition (3) follows from Proposition 2.1.3. Then Theorem 2.2.5 shows that there exists a unique extension $\widehat{\mathscr{E}(\tau,-,x,x')}|_{M_2^1(\mathbb{Z})}$. Moreover, Lemma 2.2.4 also implies that $\mathscr{E}(\tau,-,x,x')$ coincides with the extension $\widehat{\mathscr{E}(\tau,-,x,x')}|_{M_2^1(\mathbb{Z})}$. Hence according to Theorem 2.2.5, the extension $\widehat{\mathscr{E}(\tau,-,x,x')}|_{M_2^1(\mathbb{Z})}$ must satisfy the cocycle relation (2.10) for any σ_i. □

2.2.2 Dimension 3

In this subsection, we prove the cocycle relation in dimension 3. The strategy is similar to the case of dimension 2. We prove some special cases in Lemma 2.2.7 and 2.2.8. Then by using the extension theorem 2.2.9 to complete the proof.

Theorem 2.2.6. *Let $\tau\in\mathbb{H}$ and $x,x'\in\mathbb{C}^3$. Let $\sigma_0,\sigma_1,\sigma_2,\sigma_3\in\mathbb{Z}^3_{\text{prim}}$. For $i=0,1,2,3$, we denote M_i the matrix consists of $\sigma_0,\sigma_1,\sigma_2,\sigma_3$ which omits σ_i. Then we have the following cocycle relation:*

$$\sum_{i=0}^{3}(-1)^i\mathscr{E}(\tau,M_i,x,x')=0. \tag{2.27}$$

Proof. By using the same argument as for Theorem 2.2.3, we only need to check the following 2 special cases and the extension Theorem 2.2.9.

□

Lemma 2.2.7. *Let $\sigma_0=(1,0,0)^t$, $\sigma_1=(0,1,0)^t$, $\sigma_2=(d,e,0)^t$, $\sigma_3=(a,b,c)^t$ with $0\le d<e$, $0\le a,b<c$ and $1<e\le c$. Let $x,x'\in\mathbb{C}^3$. Then we have the following cocycle relation*

$$\sum_{i=0}^{3}(-1)^i\mathscr{E}(\tau,M_i,x,x')=0.$$

Proof. We note that $\det(M_3)=0$, hence $\mathscr{E}(\tau,M_3,x,x')=0$. By the definition, $\mathscr{E}(\tau,M_2,x,x')$ gives

$$\sum_{z_3\in\Lambda/c\Lambda}\mathcal{K}(\tau,x_1,x_1'-\frac{a}{c}(x_3'+z_3))\mathcal{K}(\tau,x_2,x_2'-\frac{b}{c}(x_3'+z_3))\mathcal{K}(\tau,x_3,\frac{1}{c}(x_3'+z_3))\mathbf{e}(x_3y_3),$$

where we write $z_3=y_3\tau+y_3'$. For fixed $z_3\in\Lambda/c\Lambda$, we denote $x_1''=x_1'-\frac{a}{c}(x_3'+z_3)$

and $x_2'' = x_2' - \frac{a}{c}(x_3' + z_3)$. We observe that

$$\mathscr{E}(\tau, M_2, x, x') = \sum_{z_3 \in \Lambda/c\Lambda} \mathscr{E}\left(\tau, M_2', (x_1, x_2), (x_1'', x_2'')^t\right) \mathcal{K}(\tau, x_3, \frac{1}{c}(x_3' + z_3))\mathbf{e}(x_3 y_3)$$

where $M_2' = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Similarly, we have

$$\mathscr{E}(\tau, M_1, x, x') = \sum_{z_3 \in \Lambda/c\Lambda} \mathscr{E}\left(\tau, M_1', (x_1, x_2), (x_1'', x_2'')^t\right) \mathcal{K}(\tau, x_3, \frac{1}{c}(x_3' + z_3))\mathbf{e}(x_3 y_3),$$

$$\mathscr{E}(\tau, M_1, x, x') = \sum_{z_3 \in \Lambda/c\Lambda} \mathscr{E}\left(\tau, M_0', (x_1, x_2), (x_1'', x_2'')^t\right) \mathcal{K}(\tau, x_3, \frac{1}{c}(x_3' + z_3))\mathbf{e}(x_3 y_3),$$

where $M_1' = \begin{pmatrix} 1 & d \\ 0 & e \end{pmatrix}$ and $M_0' = \begin{pmatrix} 0 & d \\ 1 & e \end{pmatrix}$. Hence by applying the Theorem 2.2.3 to M_0', M_1', M_2', we complete the proof.

□

Lemma 2.2.8. *Let $\sigma_0 = (1,0,0)^t$, $\sigma_1 = (0,1,0)^t$, $\sigma_2 = (0,0,1)^t$, $\sigma_3 = (a,b,c)^t$ with $0 \le a, b < c$. Let $x, x' \in \mathbb{C}^3$. Then we have the following cocycle relation*

$$\sum_{i=0}^{3} (-1)^i \mathscr{E}(\tau, M_i, x, x') = 0.$$

Proof. The proof is very similar to Lemma 2.2.7. So we omit the details.

□

To complete the proof of Theorem 2.2.6, we need the following extension theorem. In the case of dimension 2, we prove the extension theorem by induction on the maximum value of the determinant of 3 matrices $|\det(\sigma_0, \sigma_1)|, |\det(\sigma_1, \sigma_2)|, |\det(\sigma_0, \sigma_2)|$. However, in dimension 3, such induction becomes combinatorially difficult. Inspired by the method of Bykovskii [**Byk03**], we modify the proof by induction on the determinant of the first matrix $|\det(\sigma_0, \sigma_1, \sigma_2)|$. To simplify the notation, for any function F on $M_3(\mathbb{Z})$, we denote

$$\partial F(\sigma_0, \sigma_1, \sigma_2, \sigma_3) = F(\sigma_0, \sigma_1, \sigma_2) - F(\sigma_0, \sigma_1, \sigma_3) + F(\sigma_0, \sigma_2, \sigma_3) - F(\sigma_1, \sigma_2, \sigma_3).$$

Theorem 2.2.9. *Denote $M_3^1(\mathbb{Z}) = \{\gamma \in M_3(\mathbb{Z}) \mid \det(\gamma) = 0 \text{ or } 1\}$. Let L be a left $\mathrm{SL}_3(\mathbb{Z})$-module. Let $\Phi : M_3^1(\mathbb{Z}) \to L$ be a map satisfies the following conditions:*

1. $\Phi(\sigma) = 0$ *if* $\det(\sigma) = 0$,

2. *For all* $(a, b, c) = (1, 0, 0)$, $(0, -1, 0)$, $(1, -1, 0)$, $(1, 0, 1)$, $(0, -1, 1)$, $(1, -1, 1)$, *we have*
$$\partial\Phi \begin{pmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \end{pmatrix} = 0, \tag{2.28}$$

3. $\Phi(g\sigma) = g\Phi(\sigma)$ *for any* $g \in \mathrm{SL}_3(\mathbb{Z})$ *and* $\sigma \in M_3^1(\mathbb{Z})$.

Then there exists a unique extension $\hat{\Phi}$ *of* Φ *to* $M_3(\mathbb{Z})$ *such that the following conditions are satisfied:*

(i) $\hat{\Phi}|_{M_3^1(\mathbb{Z})} = \Phi$,

(ii) $\partial\hat{\Phi}(\sigma_0, \sigma_1, \sigma_2, \sigma_3) = 0$ *for any non-zero vectors* $\sigma_0, \sigma_1, \sigma_2, \sigma_3 \in \mathbb{Z}^3$;

(iii) $\hat{\Phi}(g\sigma) = g\hat{\Phi}(\sigma)$ *for any* $g \in \mathrm{SL}_3(\mathbb{Z})$ *and* $\sigma \in M_3(\mathbb{Z})$.

Proof. The idea of this theorem is similar to the case of dimension 2. We reduce to the case of Hermite normal form by the condition (3). First of all, we define the extension $\hat{\Phi}$ by induction on the determinant of matrix. Let $\sigma = (\sigma_{ij}) \in M_3(\mathbb{Z})$, we first assume that all the columns of σ are primitive. If $\det(\sigma) > 0$, then there exist a unique $g \in \mathrm{SL}_3(\mathbb{Z})$ such that

$$\sigma = g \begin{pmatrix} 1 & d & a \\ 0 & e & b \\ 0 & 0 & c \end{pmatrix},$$

with $0 \le d < e$, $0 \le a, b < c$ and $1 \le e \le c$. If $e = 1$, we define $\hat{\Phi}$ by

$$\hat{\Phi}(\sigma) = g \cdot \hat{\Phi} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + g \cdot \hat{\Phi} \begin{pmatrix} 1 & 0 & a \\ 0 & 0 & b \\ 0 & 1 & c \end{pmatrix} - g \cdot \hat{\Phi} \begin{pmatrix} 0 & 0 & a \\ 1 & 0 & b \\ 0 & 1 & c \end{pmatrix}.$$

If $e > 1$, we define it by

$$\hat{\Phi}(\sigma) = g \cdot \hat{\Phi} \begin{pmatrix} 0 & d & a \\ 1 & e & b \\ 0 & 0 & c \end{pmatrix} + g \cdot \hat{\Phi} \begin{pmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & c \end{pmatrix}.$$

We note that the determinants of the two matrices on the right hind side are strictly less than $\det(\sigma)$, hence they are well-defined by induction assumption. If $\det(\sigma) < 0$,

then let $\hat{\Phi}(\sigma) = \hat{\Phi}(\sigma')$ where σ' is obtained by exchanging the first and second column of σ.

If some columns of σ are not primitive, we write $\sigma = (\sigma_1, \sigma_2, \sigma_3)$, and define $\hat{\Phi}$ by

$$\hat{\Phi}(\sigma) = \hat{\Phi}(\sigma_1/\gcd(\sigma_1), \sigma_2/\gcd(\sigma_2), \sigma_3/\gcd(\sigma_3)), \tag{2.29}$$

where $\gcd(\sigma_i)$ is the gcd of each elements of σ_i.

The conditions (i) and (iii) are straight from definition. Now we check the condition (ii). In the following, we always assume that σ_i is primitive due to the equation (2.29).

Lemma 2.2.10. *Let $\sigma_0, \ldots, \sigma_4 \in \mathbb{Z}^3_{\mathrm{prim}}$. For any function F on $M_3(\mathbb{Z})$, if*

$$\partial F(\sigma_0, \ldots, \hat{\sigma}_i, \ldots, \sigma_4) = 0$$

for all $i = 1, 2, 3, 4$, then $\partial F(\sigma_1, \sigma_2, \sigma_3, \sigma_4) = 0$.

Proof. We can check it by direct calculation, so we omit the proof. □

Lemma 2.2.11. *Let $e_1 = (1, 0, 0)^t, e_2 = (0, 1, 0)^t, e_3 = (0, 0, 1)^t, \sigma_0 = (a, b, c)^t \in \mathbb{Z}^3_{\mathrm{prim}}$, then we have*

$$\partial\hat{\Phi}(e_1, e_2, e_3, \sigma_0) = 0. \tag{2.30}$$

Proof. We prove it by induction on $k = \max\{|a|, |b|, |c|\}$. When $k = 1$, then $a, b, c = 0$ or ± 1. We show that the equation (2.30) holds for $\sigma_0 = (1, 1, 1)^t$. The other cases can be deduced in the same way.

Multiplying by $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$ on the left hand side of equation (2.28) for $(a, b, c) = (1, -1, 1)$, we have

$$\hat{\Phi}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} - \hat{\Phi}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix} + \hat{\Phi}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} - \hat{\Phi}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ -1 & 0 & 1 \end{pmatrix} = 0. \tag{2.31}$$

Multiplying by the same matrix on the left hand side of equation (2.28) for $(a, b, c) = (0, -1, 0)$, we have

$$\hat{\Phi}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} + \hat{\Phi}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 0. \tag{2.32}$$

Similarly, we can prove that

$$\hat{\Phi}\begin{pmatrix}1 & 0 & 1\\ 0 & 0 & 1\\ 0 & 1 & 1\end{pmatrix} = \hat{\Phi}\begin{pmatrix}1 & 0 & 1\\ 0 & 0 & 1\\ 0 & -1 & 1\end{pmatrix}, \quad \hat{\Phi}\begin{pmatrix}0 & 0 & 1\\ 0 & 1 & 1\\ -1 & 0 & 1\end{pmatrix} = \hat{\Phi}\begin{pmatrix}0 & 0 & 1\\ 1 & 0 & 1\\ 0 & 1 & 1\end{pmatrix}. \tag{2.33}$$

Combining the equations (2.31), (2.32), (2.33), we complete the proof of equation (2.30) for $(a, b, c) = (1, 1, 1)$.

Now we consider the case $k > 1$. Without loss of generality, we assume that c is positive and maximal. Then by multiplying a suitable matrix on the left, we only need to prove

$$\partial\hat{\Phi}(e_1, e_2, e_3', \sigma_0') = 0,$$

where $\sigma_0' = (a', b', c)$ with $0 \le a', b' < c$, $e_3' = (\epsilon_1, \epsilon_2, 1)$ with $\epsilon_1 = \frac{a'-a}{c}, \epsilon_2 = \frac{b'-b}{c}$. Note that $\epsilon_1, \epsilon_2 = 0$ or ± 1. Hence by the condition (2), we have $\partial\hat{\Phi}(e_1, e_2, e_3', e_3) = 0$. By the definition of $\hat{\Psi}$, we have $\partial\hat{\Phi}(e_1, e_2, e_3, \sigma_0') = 0$. We note that $0 \le a', b' < c$, then by the induction assumption, we have

$$\partial\hat{\Phi}(e_1, e_3, e_3', \sigma_0') = \partial\hat{\Phi}(e_2, e_3, e_3', \sigma_0') = 0.$$

This completes the proof. □

Now we back to the theorem. We will prove it by induction on the determinant of $d = |\det(\sigma_0, \sigma_1, \sigma_2)|$.

If $d = 0$, if $\operatorname{rank}(\sigma_0, \sigma_1, \sigma_2) = 1$, it is clear that $\partial\hat{\Phi}(\sigma_0, \sigma_1, \sigma_2, \sigma_3) = 0$ since every term is 0.

If $\operatorname{rank}(\sigma_0, \sigma_1, \sigma_2) = 2$, by multiplying a suitable element in $\mathrm{GL}_3(\mathbb{Z})$, we may assume that $0 \le a, b < c$. We fix such $\sigma_3 = (a, b, c)^t$ and introduce a new cocycle by

$$\Psi_{\sigma_3}\begin{pmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix} := \hat{\Psi}\begin{pmatrix}a_{11} & a_{12} & a\\ a_{21} & a_{22} & b\\ 0 & 0 & c\end{pmatrix}. \tag{2.34}$$

Here we assume that $\begin{pmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix} \in M_2^1(\mathbb{Z})$, We see that Ψ_{σ_3} satisfies the 3 conditions of Theorem 2.2.5. In fact, we only need to check the condition (2), by introducing e_3, then we can deduce it from Lemma 2.2.10 and Lemma 2.2.11. Hence according to Theorem 2.2.5, it has a unique extension $\hat{\Psi}_{\sigma_3}$. Moreover, in the construction of the extension, it coincides with the definition of $\hat{\Psi}$. Hence (2.34) can be extended to all matrix in $M_2(\mathbb{Z})$. Then $\partial\hat{\Phi}(\sigma_0, \sigma_1, \sigma_2, \sigma_3) = 0$ follows from Theorem 2.2.5.

If $d = 1$, then it follows from Lemma 2.2.11.

Now for $d > 1$, by the Hermite normal form, we only need to consider the case $\sigma_0 = e_1, \sigma_1 = (a_{12}, a_{22}, 0)^t, \sigma_2 = (a_{13}, a_{23}, a_{33})^t$ with $0 \leq a_{12} < a_{22}$ and $0 \leq a_{13}, a_{23} < a_{33}$. We introduce an auxiliary vector e_i, here $i = 3$ if $a_{22} = 1$ and $i = 2$ if $a_{22} \neq 1$. Then by definition, we see that $\partial\hat{\Phi}(\sigma_0, \sigma_1, \sigma_2, e_i) = 0$. By the induction assumption, we have

$$\partial\hat{\Phi}(\sigma_0, \sigma_1, e_i, \sigma_3) = \partial\hat{\Phi}(\sigma_0, e_i\sigma_2, \sigma_3) = \partial\hat{\Phi}(e_i, \sigma_1, \sigma_2, \sigma_3) = 0.$$

Thus by Lemma 2.2.10, we have $\partial\hat{\Phi}(\sigma_0, \sigma_1, \sigma_2, \sigma_3) = 0$. This completes the proof. □

Remark 2.2.12. *These two extension theorems remain true for* $\mathrm{GL}_2(\mathbb{Z})$ *and* $\mathrm{GL}_3(\mathbb{Z})$ *if we assume* L *is a* $\mathrm{GL}_3(\mathbb{Z})$*-module, and modify the condition* (2) *in Theorem 2.2.9 to* $\partial\Phi\begin{pmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \end{pmatrix} = 0$ *for all* $a, b, c \in \{0, \pm 1\}$ *and* $(a, b, c) \neq (0, 0, 0)$ *and assume the condition* (3) *holds for all* $g \in \mathrm{GL}_3(\mathbb{Z})$*. The case of dimension* 2 *is similar.*

At last, we want to state the general cocycle relation of $\mathscr{E}$. Charollois [**Cha1**] proved it by taking the summation of trigonometric cocycle introduced by Sczech (see [**CS16**]).

Theorem 2.2.13. *Let* $\tau \in \mathbb{H}$ *and* $x, x' \in \mathbb{C}^N$*. Let* $\mathbb{Z}^N_{\mathrm{prim}}$ *be the set of primitive vectors in* $\mathbb{Z}^N$*. Then for any* $\sigma_0, \sigma_1, \ldots, \sigma_N$ *in* $\mathbb{Z}^N_{\mathrm{prim}}$*, we have*

$$\sum_{i=0}^{N} (-1)^i \mathscr{E}(\tau, (\sigma_0, \ldots, \hat{\sigma}_i, \ldots, \sigma_N), x, x') = 0. \tag{2.35}$$

Chapter

3 Hecke Operators

In this chapter, we will introduce the action of two kinds of Hecke operators T_m and $\mathbb{T}_m$ on the elliptic cocycle $\mathscr{E}(\tau, \sigma, x, x')$ which was defined in Chapter 2. The operator T_m corresponds to the action of $\mathrm{GL}_2(\mathbb{Z})$ on the parameter τ. Such operator is very similar to the classical Hecke operator on modular forms. The operator $\mathbb{T}_m$ corresponds to the action of $\mathrm{GL}_N(\mathbb{Z})$ on the parameter σ. The idea to introduce such Hecke operators stems from the recent paper [**BCG20**], where Bergeron-Charollois-Garcia considered a differential form E_ψ that realizes an Eisenstein theta correspondence for the dual pair $(\mathrm{GL}_N; \mathrm{GL}_2)$. One can find more details in [**BCG20**][section 13]. The main result of this chapter is a precise relation between the Hecke operators T_m and $\mathbb{T}_m$ stated in Theorem 3.3.23. To prove the relation, we will define a function $f_{L,C_1,C_2}(\tau, x, x')$ associated to a certain lattice L and cone C. Such function is very similar to the toric modular form defined by Borisov and Gunnells in [**BG01**]. Borisov and Gunnells proved the stability of the vector space of toric modular forms under the action of the Hecke operator T_m. We use a similar method to prove the relation between these two Hecke operators on $f_{L,C_1,C_2}(\tau, x, x')$ in Theorem 3.3.16 and Theorem 3.3.21. Finally, we generalize this relation to the elliptic cocycle $\mathscr{E}$.

3.1 The operators T_m and $\mathbb{T}_m$

We give the definition of the first kind of Hecke operator.

Definition 3.1.1. *Let m, N be two positive integers. Fix two vectors $x, x' \in \mathbb{C}^N$. Then we define T_m by*

$$T_m\mathscr{E}(\tau, \sigma, x, x') = m^{N-1} \sum_{\substack{a,d>0 \\ ad=m}} \frac{1}{d^N} \sum_{b=0}^{d-1} \mathscr{E}\left(\frac{a\tau + b}{d}, \sigma, ax, ax'\right). \tag{3.1}$$

In particular, if p is a prime number, then the Hecke operator T_p is given by

$$T_p\mathscr{E}(\tau,\sigma,x,x') = \frac{1}{p}\sum_{b=0}^{p-1}\mathscr{E}\left(\frac{\tau+b}{p},\sigma,x,x'\right) + p^{N-1}\mathscr{E}(p\tau,\sigma,px,px'). \tag{3.2}$$

Proposition 3.1.2. *Let $k \geq 2$ be a positive integer and p be a prime number, then for any $x, x' \in \mathbb{C}^N$ and $\sigma \in M_N(\mathbb{Z})$, we have the formula:*

$$T_{p^k}\mathscr{E}(\tau,\sigma,x,x') = T_pT_{p^{k-1}}\mathscr{E}(\tau,\sigma,x,x') - p^{N-1}T_{p^{k-2}}\mathscr{E}(\tau,\sigma,px,px').$$

Proof. By definition,

$$\begin{aligned}
T_pT_{p^{k-1}}\mathscr{E}(\tau,\sigma,x,x') &= T_p\left(\sum_{t=0}^{k-1}\sum_{b=0}^{p^{k-1-t}-1}\frac{p^{(k-1)(N-1)}}{p^{N(k-1-t)}}\mathscr{E}\left(\frac{p^t\tau+b}{p^{k-1-t}},\sigma,p^tx,p^tx'\right)\right)\\
=&\sum_{t=0}^{k-1}\sum_{b=0}^{p^{k-1-t}-1}\sum_{r=0}^{p-1}\frac{p^{(k-1)(N-1)}}{p^{N(k-1-t)+1}}\mathscr{E}\left(\frac{p^t\tau+p^{k-1-t}r+b}{p^{k-t}},\sigma,p^tx,p^tx'\right)\\
&+\sum_{t=0}^{k-1}\sum_{b=0}^{p^{k-1-t}-1}\frac{p^{k(N-1)}}{p^{N(k-1-t)}}\mathscr{E}\left(\frac{p^t\tau+b}{p^{k-2-t}},\sigma,p^{t+1}x,p^{t+1}x'\right).
\end{aligned}$$

For the first summation, we note that as r, b run through $\mathbb{Z}/p\mathbb{Z}$ and $\mathbb{Z}/p^{k-1-t}\mathbb{Z}$ respectively, $p^{k-1-t}r + b$ runs through $\mathbb{Z}/p^{k-t}\mathbb{Z}$, hence the first summation gives

$$\sum_{t=0}^{k-1}\sum_{b=0}^{p^{k-t}-1}\frac{p^{(k-1)(N-1)}}{p^{N(k-1-t)+1}}\mathscr{E}\left(\frac{p^t\tau+b}{p^{k-t}},\sigma,p^tx,p^tx'\right).$$

As for the second summation, if $t \neq k-1$, then we write $b = p^{k-2-t}b_1 + b_2$ where $b_1 \in \mathbb{Z}/p\mathbb{Z}$ and $b_2 \in \mathbb{Z}/p^{k-2-t}\mathbb{Z}$. We note that $\mathscr{E}(\tau,\sigma,x,x')$ is $\mathbb{Z}$-periodic in τ, hence the second summation gives

$$\sum_{t=0}^{k-2}\sum_{b_2=0}^{p^{k-2-t}-1}\frac{p^{k(N-1)+1}}{p^{N(k-1-t)}}\mathscr{E}\left(\frac{p^t\tau+b_2}{p^{k-2-t}},\sigma,p^{t+1}x,p^{t+1}x'\right) + p^{k(N-1)}\mathscr{E}(p^k\tau,\sigma,p^kx,p^kx').$$

We observe that this equals to

$$p^{N-1}T_{p^{k-2}}\mathscr{E}(\tau,\sigma,px,px') + p^{k(N-1)}\mathscr{E}(p^k\tau,\sigma,p^kx,p^kx').$$

Combining all the calculation, we see that

$$\begin{aligned}
&T_p T_{p^{k-1}} \mathscr{E}(\tau, \sigma, x, x') \\
=&\sum_{t=0}^{k-1} \sum_{b=0}^{p^{k-t}-1} \frac{p^{(k-1)(N-1)}}{p^{N(k-1-t)+1}} \mathscr{E}\left(\frac{p^t\tau + b}{p^{k-t}}, \sigma, p^t x, p^t x'\right) + p^{k(N-1)} \mathscr{E}(p^k\tau, \sigma, p^k x, p^k x') \\
&+ p^{N-1} T_{p^{k-2}} \mathscr{E}(\tau, \sigma, px, px') \\
=&T_{p^k} \mathscr{E}(\tau, \sigma, x, x') + p^{N-1} T_{p^{k-2}} \mathscr{E}(\tau, \sigma, px, px').
\end{aligned}$$

□

The second kind of Hecke operator can be defined in more general setting. To do that, we introduce some concepts:

Definition 3.1.3. *Let A be a $\mathrm{GL}_N(\mathbb{Q})$-module. Let $\mathcal{P}(A)$ be the set of functions*

$$\begin{aligned} r : \mathbb{Z}^N \backslash 0 \times \cdots \times \mathbb{Z}^N \backslash 0 &\to A \\ (\sigma_1, \cdots, \sigma_N) &\mapsto r\{\sigma_1, \cdots, \sigma_N\} \end{aligned} \tag{3.3}$$

satisfying the conditions

1. $r\{\sigma_1, \cdots, \sigma_N\} = 0$ *if* $\det(\sigma_1, \cdots, \sigma_N) = 0$,
2. $\sum_{j=0}^{N} (-1)^j r\{\sigma_0, \cdots, \hat{\sigma_j}, \cdots, \sigma_N\} = 0$ *for all* $\sigma_0, \cdots, \sigma_N \in \mathbb{Z}^N \backslash 0$.

An element of $\mathcal{P}(A)$ is said to be homogeneous if for all $g \in \mathrm{SL}_N(\mathbb{Z})$ and $\sigma_1, \cdots, \sigma_N \in \mathbb{Z}^N \backslash 0$, we have

$$r\{g\sigma_1, \cdots, g\sigma_N\} = g \cdot (r\{\sigma_1, \cdots, \sigma_N\}). \tag{3.4}$$

Let $\mathcal{P}_h(A)$ be the subset of homogeneous elements of $\mathcal{P}(A)$.

Definition 3.1.4. *Let m be a positive integer. Then we can define the second kind of Hecke operator on $\mathcal{P}_h(A)$ by:*

$$\mathbb{T}_m r\{\sigma_1, \cdots, \sigma_N\} = \sum_{\gamma \in \Gamma_N(m)} \gamma \cdot \left(r\{m\gamma^{-1}\sigma_1, \cdots, m\gamma^{-1}\sigma_N\}\right), \tag{3.5}$$

where $\Gamma_N(m)$ is a finite set of representatives of $M_N(m)/\mathrm{SL}_N(\mathbb{Z})$ and

$$M_N(m) = \{\gamma \in M_N(\mathbb{Z}) \mid \det(\gamma) = m\}.$$

Lemma 3.1.5. *The operator $\mathbb{T}_m$ is well-defined, i.e. it doesn't depend on choice of the set of representatives and it maps $\mathcal{P}_h(A)$ to $\mathcal{P}_h(A)$.*

Proof. Let $\gamma' = \gamma g_\gamma$ for some $g_\gamma \in \mathrm{SL}_N(\mathbb{Z})$. Then the homogeneous property of r gives

$$\begin{aligned}
&\sum_{\gamma' \in \Gamma_N(m)} \gamma' \cdot \left(r\{m\gamma'^{-1}\sigma_1, \cdots, m\gamma'^{-1}\sigma_N\}\right) \\
=&\sum_{\gamma \in \Gamma_N(m)} \gamma g_\gamma \cdot \left(r\{m g_\gamma^{-1}\gamma^{-1}\sigma_1, \cdots, m g_\gamma^{-1}\gamma^{-1}\sigma_N\}\right) \\
=&\sum_{\gamma \in \Gamma_N(m)} \gamma \cdot \left(r\{m\gamma^{-1}\sigma_1, \cdots, m\gamma^{-1}\sigma_N\}\right).
\end{aligned} \tag{3.6}$$

This shows that the definition of $\mathbb{T}_m$ doesn't depend on the choice of the set of representativs. The cocycle property of $\mathbb{T}_m r$ can be checked directly:

$$\begin{aligned}
&\sum_{j=0}^{N} \mathbb{T}_m r\{\sigma_0, \cdots, \hat{\sigma_j}, \cdots, \sigma_N\} \\
=&\sum_{\gamma \in \Gamma_N(m)} \sum_{j=0}^{N} \gamma \cdot (r\{m\gamma^{-1}\sigma_0, \cdots, \widehat{m\gamma^{-1}\sigma_j}, \cdots, m\gamma^{-1}\sigma_N\}) \\
=&0.
\end{aligned} \tag{3.7}$$

As for the homogeneous property, let $g \in \mathrm{SL}_N(\mathbb{Z})$, then

$$\begin{aligned}
&\mathbb{T}_m r\{g\sigma_1, \cdots, g\sigma_N\} \\
=&\sum_{\gamma \in \Gamma_N(m)} \gamma \cdot \left(r\{m\gamma^{-1}g\sigma_1, \cdots, m\gamma^{-1}g\sigma_N\}\right) \\
=&\sum_{\gamma \in \Gamma_N(m)} \gamma \cdot \left(r\{mg(g^{-1}\gamma g)^{-1}\sigma_1, \cdots, mg(g^{-1}\gamma g)^{-1}\sigma_N\}\right) \\
=&\sum_{\gamma \in \Gamma_N(m)} g(g^{-1}\gamma g) \cdot \left(r\{m(g^{-1}\gamma g)^{-1}\sigma_1, \cdots, m(g^{-1}\gamma g)^{-1}\sigma_N\}\right) \\
=&g \cdot (\mathbb{T}_m r\{\sigma_1, \cdots, \sigma_N\}).
\end{aligned} \tag{3.8}$$

The last equality holds since $\{g^{-1}\gamma g \,|\, \gamma \in \Gamma_N(m)\}$ is again a set of representatives of $\Gamma_N(m)$. This completes the proof. □

Proposition 3.1.6. *Let m, n be two positive integers with no common factor. Then we have*

$$\mathbb{T}_{mn} = \mathbb{T}_n \mathbb{T}_m.$$

In particular, $\mathbb{T}_m$ commutes with $\mathbb{T}_n$.

Proof. By definition, it is enough to prove that $\Gamma_N(mn) = \Gamma_N(m)\Gamma_N(n)$. We note

that we can choose the following representatives of $\Gamma_N(n)$:

$$\begin{pmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ a_{21} & a_{22} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{N1} & a_{N2} & a_{N3} & \cdots & a_{NN} \end{pmatrix}$$

where $n = a_{11}a_{22}\cdots a_{NN}$ and $0 \leq a_{ji} < a_{jj}$ for all $1 \leq i < j \leq N$. It is easy to see that the cardinal of $\Gamma_N(n)$ equals to

$$\sum a_{22}a_{33}^2 \cdots a_{NN}^{N-1}, \tag{3.9}$$

where the sum is over all the sequences $a_{11}, a_{22}, \cdots, a_{NN}$ such that

$$a_{11}a_{22}\cdots a_{NN} = n. \tag{3.10}$$

Hence if m, n have no common factor, then

$$|\Gamma_N(mn)| = |\Gamma_N(m)||\Gamma_N(n)|. \tag{3.11}$$

On the other hand, we consider the map

$$\begin{aligned} \phi : \Gamma_N(m) \times \Gamma_N(n) &\to \Gamma_N(mn) \\ (\gamma, \gamma') &\mapsto \gamma\gamma' \end{aligned}$$

We check that this map is injective. In fact, if $\gamma_1\gamma_1' = \gamma_2\gamma_2' g$ for some $\gamma_i \in \Gamma_N(m)$, $\gamma_i' \in \Gamma_N(n)$ and $g \in \mathrm{SL}_N(\mathbb{Z})$, then we claim that

$$\gamma_2^{-1}\gamma_1 = \gamma_2' g \gamma_1'^{-1} \in \mathrm{SL}_N(\mathbb{Z}).$$

It is clear that $\gamma_2' g \gamma_1'^{-1} \in \mathrm{SL}_N(\mathbb{Q})$. If some of the entries of $\gamma_2' g \gamma_1'^{-1}$ doesn't belong to $\mathbb{Z}$, the denominator must divides n. However, the denominators of entries of $\gamma_2^{-1}\gamma_1$ divide m and $\gcd(m, n) = 1$. Hence this forces that $\gamma_2' g \gamma_1'^{-1} \in \mathrm{SL}_N(\mathbb{Z})$. This means γ_1 and γ_2 belong to the same class. Hence $\gamma_1 = \gamma_2$ since we have already fixed a set of representatives. So γ_1' and γ_2' also belong to the same class. This implies the map ϕ is injective. But the cardinal of both sides of ϕ are the same, so ϕ is a bijection. This proves that $\mathbb{T}_{mn} = \mathbb{T}_m\mathbb{T}_n$. □

Remark 3.1.7. *The discussion in Chapter 2 shows that $\mathscr{E}(-, \sigma, -, -)$ belongs to $\mathcal{P}_h(\mathcal{F}(\mathbb{H} \times \mathbb{C}^N \times \mathbb{C}^N))$ where $\mathcal{F}(\mathbb{H} \times \mathbb{C}^N \times \mathbb{C}^N)$ is the set of meromorphic functions*

on $\mathbb{H} \times \mathbb{C}^N \times \mathbb{C}^N$ *equipped with the action of* $\mathrm{GL}_N(\mathbb{Q})$ *defined by*

$$g \cdot f(\tau, x, x') = f(\tau, xg, \det(g)g^{-1}x'), \tag{3.12}$$

where $g \in \mathrm{GL}_N(\mathbb{Q})$ *and* $f(\tau, x, x') \in \mathcal{F}(\mathbb{H} \times \mathbb{C}^N \times \mathbb{C}^N)$. *We note that instead of writing* N *columns, we write a matrix* σ *in* $\mathscr{E}(\tau, \sigma, x, x')$. *Then the action of* $\mathbb{T}_m$ *on* $\mathscr{E}(\tau, \sigma, x, x')$ *is given by*

$$\mathbb{T}_m \mathscr{E}(\tau, \sigma, x, x') = \sum_{\gamma \in \Gamma_N(m)} \mathscr{E}(\tau, m\gamma^{-1}\sigma, x\gamma, m\gamma^{-1}x'). \tag{3.13}$$

So far, these two Hecke operators are well-defined on $\mathscr{E}(\tau, \sigma, x, x')$. The following proposition shows the commutation of $\mathbb{T}_m$ and T_n on $\mathscr{E}(\tau, \sigma, x, x')$.

Proposition 3.1.8. *Let* m, n *be two positive integers. Fix* $\tau \in \mathbb{H}, \sigma \in M_N(\mathbb{Z})$ *and* $x, x' \in \mathbb{C}^N$. *Then we have*

$$\mathbb{T}_n T_m \mathscr{E}(\tau, \sigma, x, x') = T_m \mathbb{T}_n \mathscr{E}(\tau, \sigma, x, x'). \tag{3.14}$$

Proof. By the Definition 3.1.1 and the equation (3.13), we have

$$\begin{aligned}
\mathbb{T}_n(T_m\mathscr{E})(\tau, \sigma, x, x') &= \sum_{\gamma \in \Gamma_N(n)} T_m \mathscr{E}(\tau, n\gamma^{-1}\sigma, x\gamma, n\gamma^{-1}x') \\
&= \sum_{\gamma \in \Gamma_N(n)} \sum_{\substack{a,d>0 \\ ad=m}} \frac{m^{N-1}}{d^N} \mathscr{E}\left(\frac{a\tau + b}{d}, n\gamma^{-1}\sigma, ax\gamma, an\gamma^{-1}x'\right) \\
&= \sum_{\substack{a,d>0 \\ ad=m}} \frac{m^{N-1}}{d^N} \mathbb{T}_n \mathscr{E}\left(\frac{a\tau + b}{d}, \sigma, ax, ax'\right) \\
&= T_m(\mathbb{T}_n\mathscr{E})(\tau, \sigma, x, x').
\end{aligned} \tag{3.15}$$

This completes the proof. □

The Hecke operator T_m also shares a similar property as in Proposition 3.1.6.

Proposition 3.1.9. *Let* m, n *be two positive integers with no common factor. Fix* $\tau \in \mathbb{H}, \sigma \in M_N(\mathbb{Z})$ *and* $x, x' \in \mathbb{C}^N$. *Then we have*

$$T_{mn}\mathscr{E}(\tau, \sigma, x, x') = T_m T_n \mathscr{E}(\tau, \sigma, x, x') = T_n T_m \mathscr{E}(\tau, \sigma, x, x'). \tag{3.16}$$

Proof. Similar to the proof of Proposition 3.1.6, we have the following bijection:

$$\left\{\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \middle| ad = m, a, d > 0, b \pmod d\right\} \times \left\{\begin{pmatrix} a' & b' \\ 0 & d' \end{pmatrix} \middle| a'd' = n, a', d' > 0, b' \pmod{d'}\right\}$$
$$\to \left\{\begin{pmatrix} a'' & b'' \\ 0 & d'' \end{pmatrix} \middle| a''d'' = mn, a'', d'' > 0, b'' \pmod{d''}\right\}. \tag{3.17}$$

Hence by the Definition 3.1.1, we have

$$\begin{aligned} T_m T_n \mathscr{E}(\tau, \sigma, x, x') &= \sum_{\substack{a,d>0 \\ ad=m}} \sum_{b=0}^{d-1} \frac{m^{N-1}}{d^N} T_n \mathscr{E}\left(\frac{a\tau + b}{d}, \sigma, ax, ax'\right) \\ &= \sum_{\substack{a,d>0 \\ ad=m}} \sum_{b=0}^{d-1} \frac{m^{N-1}}{d^N} \sum_{\substack{a',d'>0 \\ a'd'=n}} \sum_{b'=0}^{d'-1} \frac{(n)^{N-1}}{d^N d'^N} \mathscr{E}\left(\frac{aa'\tau + a'b + b'd}{dd'}, \sigma, aa'x, aa'x'\right) \\ &= \sum_{\substack{a'',d''>0 \\ a''d''=mn}} \sum_{b''=0}^{d''-1} \frac{(mn)^{N-1}}{d''^N} \mathscr{E}\left(\frac{a''\tau + b''}{d''}, \sigma, a''x, a''x'\right) \\ &= T_{mn} \mathscr{E}(\tau, \sigma, x, x'). \end{aligned} \tag{3.18}$$

This completes the proof. □

3.2 Dimension 1

In this section, we discuss the case of dimension 1. The case of dimension 1 is much easier since all the results can be deduced from trigomometric summation. In this case, the cocycle reduced to the Kronecker theta function. Similar to the Definition 3.1.1, we can define the Hecke operator T_m on the Kronecker theta function by

$$T_m \mathcal{K}(\tau, x_0, x_0') = \sum_{\substack{a,d>0 \\ ad=m}} \frac{1}{d} \sum_{b=0}^{d-1} \mathcal{K}\left(\frac{a\tau + b}{d}, ax_0, ax_0'\right). \tag{3.19}$$

Theorem 3.2.1. *Let p be a prime number, $x_0, x_0' \in \mathbb{C}$. Then we have*

$$T_p \mathcal{K}(\tau, x_0, x_0') = \mathcal{K}(\tau, x_0, px_0') + \mathcal{K}(\tau, px_0, x_0'). \tag{3.20}$$

Proof. In fact, by using the Fourier expansion (1.2) of Kronecker function, we have

$$\begin{aligned}
&\frac{1}{2p\pi i}\sum_{r=1}^{p}\mathcal{K}\left(\frac{\tau+r}{p},x_0,x_0'\right)=1-\frac{1}{1-\mathbf{e}(x_0)}-\frac{1}{1-\mathbf{e}(x_0')}\\
&-\frac{1}{p}\sum_{r=1}^{p}\sum_{m,n\geq 1}\mathbf{e}\left(mn\frac{\tau+r}{p}\right)\left(\mathbf{e}(nx_0+mx_0')-\mathbf{e}(-nx_0-mx_0')\right)\\
=&1-\frac{1}{1-\mathbf{e}(x_0)}-\frac{1}{1-\mathbf{e}(x_0')}-\sum_{\substack{m,n\geq 1\\ p|mn}}q^{\frac{mn}{p}}\left(\mathbf{e}(nx_0+mx_0')-\mathbf{e}(-nx_0-mx_0')\right).
\end{aligned}$$

We divide the summation into three parts:

$$\begin{aligned}
&\sum_{\substack{m,n\geq 1\\ p|mn}}q^{\frac{mn}{p}}\left(\mathbf{e}(nx_0+mx_0')-\mathbf{e}(-nx_0-mx_0')\right)\\
=&\sum_{p|m}+\sum_{p|n}-\sum_{p|m,p|n}q^{\frac{mn}{p}}\left(\mathbf{e}(nx_0+mx_0')-\mathbf{e}(-nx_0-mx_0')\right).
\end{aligned}$$

This gives the formula (3.20). □

Theorem 3.2.2. *Let m be a positive integer and $x_0, x_0' \in \mathbb{C}$, then we have*

$$T_m\mathcal{K}(\tau,x_0,x_0')=\sum_{d|m}\mathcal{K}\left(\tau,dx_0,\frac{m}{d}x_0'\right). \tag{3.21}$$

Proof. We first consider the case $m = p^k$ where p is a prime number and k is a positive integer. We prove it by induction on k. When $k = 1$, this follows from Theorem 3.2.1.

Now we assume that $k > 1$. By definition, we have

$$\begin{aligned}
T_{p^k}\mathcal{K}(\tau,x_0,x_0')&=\frac{1}{p^k}\sum_{b=0}^{p^k-1}\mathcal{K}\left(\frac{\tau+b}{p^k},x_0,x_0'\right)+\sum_{\substack{a'd=p^{k-1}\\ a',d>0}}\frac{1}{d}\sum_{b=0}^{d-1}\mathcal{K}\left(\frac{a'p\tau+b}{d},a'px_0,a'px_0'\right)\\
&=\frac{1}{p^k}\sum_{b=0}^{p^k-1}\mathcal{K}\left(\frac{\tau+b}{p^k},x_0,x_0'\right)+T_{p^{k-1}}\mathcal{K}(p\tau,px_0,px_0').
\end{aligned}$$

By the induction assumption, the second term equals to

$$\sum_{j=0}^{k-1}\mathcal{K}(p\tau,p^{j+1}x_0,p^{k-j}x_0').$$

So we consider the first summation next:

$$\frac{1}{p^k}\sum_{b=0}^{p^k-1}\mathcal{K}\left(\frac{\tau+b}{p^k},x_0,x_0'\right)=1-\frac{1}{1-\mathbf{e}(x_0)}-\frac{1}{1-\mathbf{e}(x_0')}-\sum_{\substack{m,n\\p^k|mn}}q^{\frac{mn}{p^k}}\mathbf{e}(nx_0+mx_0')$$

$$=1-\frac{1}{1-\mathbf{e}(x_0)}-\frac{1}{1-\mathbf{e}(x_0')}\sum_{p\nmid m,p^k|n}q^{\frac{mn}{p^k}}\mathbf{e}(nx_0+mx_0')+\sum_{\substack{m,n\\p^{k-1}|mn}}q^{\frac{mn}{p^{k-1}}}\mathbf{e}(nx_0+mpx_0')$$

$$=\mathcal{K}(\tau,p^kx_0,x_0')-\mathcal{K}(p\tau,p^kx_0,px_0')+1-\frac{1}{1-\mathbf{e}(x_0)}-\frac{1}{1-\mathbf{e}(px_0')}$$

$$+\sum_{\substack{m,n\\p^{k-1}|mn}}q^{\frac{mn}{p^{k-1}}}\mathbf{e}(nx_0+mpx_0')$$

$$=\cdots=\sum_{j=0}^{k}\mathcal{K}(\tau,p^jx_0,p^{k-j}x_0')-\sum_{j=0}^{k-1}\mathcal{K}(p\tau,p^{j+1}x_0,p^{k-j}x_0').$$

This proves the formula (3.21) for $m=p^k$. For the general case, the same argument as the Proposition 3.1.9 shows that

$$T_{p_1^{k_1}\cdots p_s^{k_s}}=T_{p_1^{k_1}}\cdots T_{p_s^{k_s}},\tag{3.22}$$

where $p_1,\cdots,p_s$ are different primes. Combining the result above, we complete the proof. □

3.3 Dimension N

In this section, we will consider the case of dimension N. Inspired by the case of dimension 1, we consider the Fourier expansion of the elliptic cocycle $\mathscr{E}$. Since the elliptic cocycle $\mathscr{E}$ is a product of N times Kronecker theta function, by expanding the product, we see that it consists of the series:

$$\sum_{\substack{m\in S_1\\n\in S_2}}q^{m\cdot n}\mathbf{e}(x\cdot n+m\cdot x')\tag{3.23}$$

where the summation over certain subsets $S_1,S_2\subset\mathbb{Z}^N$. Before giving the precise definition, we present some basic concepts.

3.3.1 Preliminary

The first concept we need is the lattice in N-dimensional Euclidean space where we follows the standard reference [**Cas71**].

Definition 3.3.1. *A discrete subset L of $\mathbb{R}^N$ is called a lattice of rank N if it is the $\mathbb{Z}$-span of N vectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_N$ which are linear independent over $\mathbb{R}$.*

Definition 3.3.2. *Let L be a lattice in $\mathbb{R}^N$. A subset $L' \subset L$ is called a sublattice of L when it is a subgroup of L of finite index. We denote the index of L' in L by $[L : L']$.*

In this chapter, we will need to know how many sublattices there are with fixed index. In fact, we establish the following formula:

Lemma 3.3.3. *Let L be a lattice of rank N. Let p be a prime number and $k \geq 0$ be an integer. Then the number of sublattices of L of index p^k is*

$$H(N, p^k) := \prod_{j=1}^{k} \frac{p^{N+j-1} - 1}{p^j - 1}. \tag{3.24}$$

Proof. We first translate the question into counting the number of certain matrices. In fact, every sublattice L' of L of index p^k corresponds to a matrix of the form

$$M = \begin{pmatrix} p^{a_1} & 0 & \ldots & 0 \\ b_{21} & p^{a_2} & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ b_{N1} & b_{N2} & \ldots & p^{a_N} \end{pmatrix}$$

where $0 \leq b_{21} < p^{a_2}, \ldots, 0 \leq b_{N1}, \ldots, b_{N,N-1} < p^{a_N}$ and $a_1 + \cdots + a_N = k$. To count this number, we deduce a recursion formula for $H(N, p^k)$. If $N = 1$, it is not hard to see that $H(1, p^k) = 1$ for any k. If $N \geq 2$, then for each $a_N \leq k$, $b_{N1}, \ldots, b_{N,N-1}$ have p^{a_N} choices. The number of submatrices

$$\begin{pmatrix} p^{a_1} & 0 & \ldots & 0 \\ b_{21} & p^{a_2} & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ b_{N-1,1} & b_{N-12} & \ldots & p^{a_{N-1}} \end{pmatrix}$$

is exactly $H(N-1, p^{k-a_N})$. Hence we have the formula

$$H(N, p^k) = \sum_{l=0}^{k} p^{l(N-1)} H(N-1, p^{k-l}). \tag{3.25}$$

Then for $k \geq 1$, we have

$$\begin{aligned} H(N, p^k) =& H(N-1, p^k) + p^{N-1} \sum_{l=0}^{k-1} p^{l(N-1)} H(N-1, k-1-l) \\ =& H(N-1, p^k) + p^{N-1} H(N, p^{k-1}). \end{aligned} \tag{3.26}$$

Moreover if $k = 0$, then for any $N \geq 1$, we have

$$H(N, 1) = H(N-1, 1) = \cdots = H(1, 1) = 1.$$

We observe that $H(N, p^k)$ is totally uniquely determined by the recursion formula (3.26) and the value $H(N, 1)$ and $H(1, p^k)$. On the other hand, if we denote the right hand side of (3.24) by $F(N, p^k)$. It is easy to check that $F(1, p^k) = 1$ for all $k \geq 0$ and $F(N, 1) = 1$ for all $N \geq 1$. Moreover, it satisfies the same recursion formula

$$\begin{aligned} F(N-1, p^k) + p^{N-1} F(N, p^{k-1}) =& \prod_{j=1}^{k} \frac{p^{N+j-2} - 1}{p^j - 1} + p^{N-1} \prod_{j=1}^{k-1} \frac{p^{N+j-1} - 1}{p^j - 1} \\ =& \left(\frac{p^{N-1} - 1}{p^{N+k-1} - 1} + p^{N-1} \frac{p^k - 1}{p^{N+k-1} - 1} \right) \prod_{j=1}^{k} \frac{p^{N+j-1} - 1}{p^j - 1} \\ =& F(N, p^k). \end{aligned}$$

Hence $F(N, p^k)$ must coincide with $H(N, p^k)$. This completes the proof.

□

Remark 3.3.4. *We also have a formula for arbitrary index. Let d be a positive integer with prime factorization $d = p_1^{r_1} \dots p_s^{r_s}$. Then the number of sublattices of L of index d is*

$$H(N, d) := \prod_{i=1}^{s} \prod_{j=1}^{r_i} \frac{p_i^{N+j-1} - 1}{p_i^j - 1}. \tag{3.27}$$

We can use the similar method of Proposition 3.1.6 to prove that

$$H(N, mn) = H(N, m) H(N, n),$$

where $\gcd(m,n)=1$.

Lemma 3.3.5. *Let L be a lattice of rank N. Let p be a prime number and $j \leq k$ be two non-negative integers. Let ν be a vector contained in p^jL but not contained in $p^{j+1}L$. Then the number of sublattices of L of index p^k which contain ν is*

$$\sum_{l=0}^{j} p^{l(N-1)} H(N-1, p^{k-l}).$$

Proof. Since ν is contained in p^jL but not contained in $p^{j+1}L$, by multiplying a suitable element in $\mathrm{SL}_N(\mathbb{Z})$, we may assume that $\nu = (0,\ldots,0,ap^j)^t$ with $p \nmid a$. With the same idea as the lemma above, it is equivalent to count the number of matrices

$$M = \begin{pmatrix} p^{a_1} & 0 & \ldots & 0 \\ b_{21} & p^{a_2} & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ b_{N1} & b_{N2} & \ldots & p^{a_N} \end{pmatrix}$$

such that $b_{ij} \pmod{p^{a_i}}$ and $(0,\ldots,0,ap^j)^t \in M\mathbb{Z}^N$. It is easy to see that the condition $(0,\ldots,0,ap^j)^t \in M\mathbb{Z}^N$ is equivalent to $a_N \leq j$. For each $a_N \leq j$, the $(b_{N1},\ldots,b_{N,N-1})$ has $p^{(N-1)a_N}$ choices, and we have $H(N-1,p^{k-a_N})$ choices for the upper-left submatrix. Hence the number of matrices M equals to

$$\sum_{l=0}^{j} p^{l(N-1)} H(N-1, p^{k-l}).$$

□

Beyond the lattice L, we also need its dual L^* given by

Definition 3.3.6.

$$L^* = \{x \in (\mathbb{R}^N)^* \mid x \cdot y \in \mathbb{Z} \text{ for all } y \in L\}.$$

Remark 3.3.7. *In this thesis, we always fix a basis $v_1,\ldots,v_N$ of L such that $L = v_1\mathbb{Z} + \cdots + v_N\mathbb{Z}$. We write $y \in L$ as a $N \times 1$ matrix under this basis.*

The next concept we need is that of a cone. We follow the standard reference [**Ful93**].

Definition 3.3.8. *A cone is a closed subset of $\mathbb{R}^N$ of the form*

$$C = \left\{ \sum_{i=1}^{s} r_i v_i \,|\, r_i \in \mathbb{R}_{\geq 0}, i = 1, \cdots, s \right\},$$

where $v_1, \ldots, v_s$ are fixed vectors in $\mathbb{R}^N$. We call s the rank of the cone C. The vectors $v_1, \ldots, v_s$ are called the generators of the cone C. Moreover, given a lattice L in $\mathbb{R}^N$, if the generators are belonging to L, then we call C rational cone with respect to L.

As the dual lattice, to each cone, we associate a dual cone given by

$$C^* = \{x \in (\mathbb{R}^N)^* \,|\, x \cdot y \geq 0 \text{ for all } y \in C\}.$$

Here the meaning of $x \cdot y$ is the same as what in dual lattice.

It is easy to check that C^* is a cone. Moreover, if C is a rational cone with respect to lattice L, then C^* is a rational cone with respect to L^*.

Definition 3.3.9. *Let C be a cone, and let $x \in C^*$, then the intersection of C and the hyperplane*

$$\{y \in C \,|\, x \cdot y = 0\}$$

is called a face of C.

We state some properties of cone and its faces; one can also find them in [**Ful93**][section 1.2]:

Proposition 3.3.10. *Let C be a cone, then*

1. *C is a face of itself.*
2. *Every face F of C is a cone.*
3. *Every intersection of faces of C is a face of C.*
4. *Every face of a face is a face.*

Definition 3.3.11. *The relative interior of a cone C is the topological interior of the cone C in the space $\mathbb{R} \cdot C$ generated by C. We denote it by C°.*

3.3.2 Relation between T_p and $\mathbb{T}_p$

In this subsection, we will give a relation between T_p and $\mathbb{T}_p$. The main part of the proof is inspired by Borisov-Gunnells [**BG01**]. They proved the Hecke stability

of the space of toric modular form under T_p. They also proved a kind of Hecke equivariance for products of two Eisenstein series. We further introduce the Hecke operator $\mathbb{T}_p$ for $\mathrm{GL}_N(\mathbb{Z})$ and then deduce the Hecke equivariance for the products of N Eisenstein series in the next chapter. To do this, we first give a precise definition of the series (3.23).

Definition 3.3.12. *Let L be a lattice in $\mathbb{R}^N$ and C a rational cone respect to L generated by N linearly independent vectors. Let C_1 be a face of C^* and C_2 be a face of C. Let $S = \{x \in \mathbb{C}^N \mid \mathrm{Im}(x \cdot n) > 0$ for all $n \in L \cap C - \{0\}\}$ and $S' = \{x' \in \mathbb{C}^N \mid \mathrm{Im}(m \cdot x') > 0$ for all $m \in L^* \cap C^* - \{0\}\}$. Then for any $x \in S$ and $x' \in S'$, we define the following series:*

$$f_{L,C_1,C_2}(\tau, x, x') = \sum_{\substack{m \in L^* \cap C_1^\circ \\ n \in L \cap C_2^\circ}} q^{m \cdot n} \boldsymbol{e}(x \cdot n + m \cdot x'). \tag{3.28}$$

Lemma 3.3.13. *The series (3.28) is absolutely convergent for all $x \in S$ and $x' \in S'$.*

Proof. By assumption, we can choose linearly independent generators $v_1, \ldots, v_N$ and their duals $v_1^*, \ldots, v_N^*$ such that

$$L \cap C = v_1 \mathbb{Z}_{\geq 0} + \cdots + v_N \mathbb{Z}_{\geq 0},$$

and

$$L^* \cap C^* = v_1^* \mathbb{Z}_{\geq 0} + \cdots + v_N^* \mathbb{Z}_{\geq 0}.$$

We write $x = (x_1, \ldots, x_N)$ and $x' = (x_1', \ldots, x_N')$ under these bases respectively. Then $x \in S$ and $x' \in S'$ means that $\mathrm{Im}(x_i), \mathrm{Im}(x_i') > 0$ for all $i = 1, \ldots, N$. Hence

$$\left| \sum_{\substack{m \in L^* \cap C_1^\circ \\ n \in L \cap C_2^\circ}} q^{m \cdot n} \mathbf{e}(x \cdot n + m \cdot x') \right| < \prod_{i=1}^{N} \sum_{m_i, n_i > 0} |\mathbf{e}(x_i)|^{n_i} |\mathbf{e}(x_i')|^{m_i}. \tag{3.29}$$

So the series (3.28) is absolutely convergent. □

Example 3.3.14. *Suppose $L = \mathbb{Z}$, $C = \mathbb{R}_{\geq 0}$. Then $L^* = \mathbb{Z}$ and $C^* = \mathbb{R}_{\geq 0}$. If we take $C_1 = C^*$ and $C_2 = C$, then we have*

$$f_{L,C_1,C_2}(\tau, x, x') = \sum_{m,n > 0} q^{mn} \boldsymbol{e}(nx + mx').$$

Example 3.3.15. *Suppose $L = \mathbb{Z}^2$, C is generated by the vectors $\boldsymbol{e}_1 = (1,0)^t$ and $\boldsymbol{e}_2 = (0,1)^t$. Then L^* and C^* coincide with L and C respectively. If we take $C_1 = C^*$*

and $C_2 = \mathbb{R}_{>0}\boldsymbol{e}_1$*, then we have*

$$f_{L,C_1,C_2}(\tau, x, x') = \sum_{m_1,m_2,n>0} q^{m_1 n}\mathbf{e}(nx_1 + m_1x_1' + m_2x_2').$$

Similar to the case of $\mathscr{E}$, we define the action of Hecke operator T_m on f_{L,C_1,C_2} by

$$T_m f_{L,C_1,C_2}(\tau, x, x') = m^{N-1}\sum_{\substack{a,d>0\\ ad=m}} \frac{1}{d^N}\sum_{b=0}^{d-1} f_{L,C_1,C_2}\left(\frac{a\tau+b}{d}, ax, ax'\right).$$

Theorem 3.3.16. *Let p be a prime number. Let L, C_1, C_2, x, x' be as in Definition 3.3.12. Then we have the following formula*

$$T_p f_{L,C_1,C_2}(\tau, x, x') = \sum_S f_{S,C_1,C_2}(\tau, px, x') - \frac{p^{N-1}-p}{p-1} f_{L,C_1,C_2}(\tau, x, px'), \tag{3.30}$$

where S runs through the lattices such that $L \subset S \subset \frac{1}{p}L$ and $[S : L] = p^{N-1}$.

Proof. We first calculate the left hand side of (3.30)

$$\begin{aligned}
&T_p f_{L,C_1,C_2}(\tau, x, x')\\
=&\frac{1}{p}\sum_{r=1}^{p}\sum_{\substack{m\in L^*\cap C_1^\circ\\ n\in L\cap C_2^\circ}} \mathbf{e}\left(m\cdot n\frac{\tau+r}{p} + x\cdot n + m\cdot x'\right) + p^{N-1} f_{L,C_1,C_2}(p\tau, px, px')\\
=&\sum_{m\in L^*\cap C_1^\circ}\sum_{\substack{n\in L\cap C_2^\circ\\ p\mid m\cdot n}} q^{\frac{m\cdot n}{p}}\mathbf{e}(x\cdot n + m\cdot x') + p^{N-1} f_{L,C_1,C_2}(p\tau, px, px').
\end{aligned} \tag{3.31}$$

On the other hand, we calculate the first term of right hand side of (3.30)

$$\sum_S f_{S,C_1,C_2}(\tau, px, x').$$

By definition, it equals to

$$\sum_S \sum_{\substack{m\in L^*\cap C_1^\circ\\ m\cdot S\subset\mathbb{Z}}} \sideset{}{'}\sum_{n\in S\cap C_2^\circ} q^{m\cdot n}\mathbf{e}(px\cdot n + m\cdot x').$$

By multiplying p and changing the variable, it becomes

$$\sum_R \sum_{\substack{m\in L^*\cap C_1^\circ\\ m\cdot R\subset p\mathbb{Z}}} \sideset{}{'}\sum_{n\in R\cap C_2^\circ} q^{\frac{m\cdot n}{p}}\mathbf{e}(x\cdot n + m\cdot x'), \tag{3.32}$$

where R runs through all sublattices of L of index p and contain pL.

For each $m \in L^*$, we consider the series:

$$\sum_{\substack{R \\ m\cdot R\subset p\mathbb{Z}}} \sum_{n\in R\cap C_2^\circ}{}' q^{\frac{m\cdot n}{p}} \mathbf{e}(x\cdot n + m\cdot x'). \tag{3.33}$$

If $m \notin pL^*$, let $R_m = \{n \in L \,|\, m\cdot n \equiv 0 \pmod p\}$. We note that R_m is a sublattice of L of index p. In fact, the map:

$$\begin{aligned} L/R_m &\to \mathbb{F}_p \\ n &\mapsto m\cdot n \pmod p \end{aligned}$$

is an isomorphism. It is easy to see that $pL \subset R_m$. Hence R_m is the only sublattice that occurring in the summation. This means that the summation (3.33) equals to

$$\sum_{\substack{n\in L\cap C_2^\circ \\ p|m\cdot n}} q^{\frac{m\cdot n}{p}} \mathbf{e}(x\cdot n + m\cdot x').$$

If $m \in pL^*$, then for any R, we always have $m\cdot R \subset p\mathbb{Z}$. If $n \in pL\cap C_2^\circ$, then every R contains n, so such n occurs $1+p+\cdots+p^{N-1}$ times in the summation (3.33). If $n \notin pL\cap C_2^\circ$, then there exist $1+p+\cdots+p^{N-2}$ R containing n, this means (3.33) equals to

$$\begin{aligned} &\frac{p^{N-1}-1}{p-1} \sum_{n\in(L-pL)\cap C_2^\circ} q^{\frac{m\cdot n}{p}} \mathbf{e}(x\cdot n + m\cdot x') + \frac{p^N-1}{p-1} \sum_{n\in pL\cap C_2^\circ} q^{\frac{m\cdot n}{p}} \mathbf{e}(x\cdot n + m\cdot x') \\ =&\frac{p^{N-1}-1}{p-1} \sum_{n\in L\cap C_2^\circ} q^{\frac{m\cdot n}{p}} \mathbf{e}(x\cdot n + m\cdot x') + p^{N-1} \sum_{n\in pL\cap C_2^\circ} q^{\frac{m\cdot n}{p}} \mathbf{e}(x\cdot n + m\cdot x') \\ =&\frac{p^{N-1}-1}{p-1} \sum_{n\in L\cap C_2^\circ} q^{\frac{m\cdot n}{p}} \mathbf{e}(x\cdot n + m\cdot x') + p^{N-1} \sum_{n\in L\cap C_2^\circ} q^{m\cdot n} \mathbf{e}(px\cdot n + m\cdot x'). \end{aligned}$$

Combining the calculation above, we see that the series (3.32) equals to

$$\begin{aligned}
&\sum_{\substack{m\in L^*\cap C_1^\circ\\ m\notin pL^*}}\sum_{\substack{n\in L\cap C_2^\circ\\ p|m\cdot n}} q^{\frac{m\cdot n}{p}}\mathbf{e}(x\cdot n+m\cdot x')\\
&+\sum_{\substack{m\in L^*\cap C_1^\circ\\ m\in pL^*}}\left(\frac{p^{N-1}-1}{p-1}\sum_{n\in L\cap C_2^\circ} q^{\frac{m\cdot n}{p}}\mathbf{e}(x\cdot n+m\cdot x')+p^{N-1}\sum_{n\in L\cap C_2^\circ} q^{m\cdot n}\mathbf{e}(px\cdot n+m\cdot x')\right)\\
=&\sum_{m\in L^*\cap C_1^\circ}\sum_{\substack{n\in L\cap C_2^\circ\\ p|m\cdot n}} q^{\frac{m\cdot n}{p}}\mathbf{e}(x\cdot n+m\cdot x')\\
&+\frac{p^{N-1}-p}{p-1}\sum_{\substack{m\in L^*\cap C_1^\circ\\ n\in L\cap C_2^\circ}} q^{m\cdot n}\mathbf{e}(x\cdot n+pm\cdot x')+p^{N-1}\sum_{\substack{m\in L^*\cap C_1^\circ\\ n\in L\cap C_2^\circ}} q^{pm\cdot n}\mathbf{e}(px\cdot n+pm\cdot x').
\end{aligned}$$

Comparing with (3.31), we complete the proof. □

As with the elliptic cocycle $\mathscr{E}(\tau,\sigma,x,x')$, we introduce a parameter $\sigma\in M_N(\mathbb{Z})$ to f_{L,C_1,C_2}. We will mimic the equivalent definition introduced in Proposition 2.1.4 instead of the original definition of $\mathscr{E}$.

Definition 3.3.17. *Let $\sigma\in M_N(\mathbb{Z})$. If $D=\det(\sigma)\neq 0$,*

$$f_{L,C_1,C_2}(\tau,\sigma,x,x'):=\frac{\operatorname{sign}(D)}{|D|^N}\sum_{\substack{z\in L^*/L^*D\sigma^{-1}\\ z'\in L/\sigma L}} f_{L,C_1,C_2}\left(\frac{\tau}{|D|},\frac{(x+z)\sigma}{|D|},\sigma^{-1}(x'+z')\right).$$

If $D=0$,

$$f_{L,C_1,C_2}(\tau,\sigma,x,x')=0.$$

Lemma 3.3.18. *If $\det(\sigma)\neq 0$, then we have*

$$f_{L,C_1,C_2}(\tau,\sigma,x,x')=f_{\sigma^{-1}L,C_1,C_2}(\tau,x\sigma,\sigma^{-1}x').$$

Proof. We denote $D=\det(\sigma)$. First we note two summation formulas:

$$\frac{1}{D^{n-1}}\sum_{z\in L^*/L^*D\sigma^{-1}}\mathbf{e}\left(\frac{z\sigma n}{D}\right)=\begin{cases}1 & \text{if } n\in D\sigma^{-1}L,\\ 0 & \text{else,}\end{cases}$$

and

$$\frac{1}{D}\sum_{z'\in L/\sigma L}\mathbf{e}\left(m\sigma^{-1}z'\right)=\begin{cases}1 & \text{if } m\in L^*\sigma,\\ 0 & \text{else.}\end{cases}$$

Then by definition and using the formulas above, we have

$$\begin{aligned}
f_{L,C_1,C_2}(\tau,\sigma,x,x') =& \frac{1}{D^N} \sum_{\substack{z\in L^*/L^*D\sigma^{-1}\\ z'\in L/\sigma L}} \sum_{\substack{m\in L^*\cap C_1^\circ\\ n\in L\cap C_2^\circ}} q^{\frac{m\cdot n}{D}} \mathbf{e}\left(\frac{(x+z)\sigma n}{D} + m\sigma^{-1}(x'+z')\right)\\
=& \sum_{\substack{m\in L^*\sigma\cap C_1^\circ\\ n\in D\sigma^{-1}L\cap C_2^\circ}} q^{\frac{m\cdot n}{D}} \mathbf{e}\left(\frac{x\sigma n}{D} + m\sigma^{-1}x'\right)\\
=& \sum_{\substack{m\in L^*\sigma\cap C_1^\circ\\ n\in \sigma^{-1}L\cap C_2^\circ}} q^{m\cdot n}\mathbf{e}(x\sigma n + m\sigma^{-1}x')\\
=& f_{\sigma^{-1}L,C_1,C_2}(\tau, x\sigma, \sigma^{-1}x').
\end{aligned}$$

□

Corollary 3.3.19. *Let p be a prime number. Then we have*

$$\begin{aligned}
&T_p f_{L,C_1,C_2}\left(\tau,\sigma,x,x'\right) + \frac{p^{N-1}-p}{p-1} f_{L,C_1,C_2}(\tau,\sigma,x,px')\\
=& \sum_{\gamma\in\Gamma_N(p)} f_{L,C_1,C_2}(\tau, p\gamma^{-1}\sigma, x\gamma, p\gamma^{-1}x').
\end{aligned}$$

where the action of T_p on $f_{L,C_1,C_2}\left(\tau,\sigma,x,x'\right)$ follows from $f_{L,C_1,C_2}\left(\tau,x,x'\right)$ which is given by

$$T_p f_{L,C_1,C_2}\left(\tau,\sigma,x,x'\right) = \frac{1}{p}\sum_{r=0}^{p-1} f_{L,C_1,C_2}\left(\frac{r+\tau}{p},\sigma,x,x'\right) + p^{N-1} f_{L,C_1,C_2}(p\tau,\sigma,px,px').$$

Proof. We note that all the sublattices S of L/p of index p which contain L are given by $\frac{1}{p}\gamma L$ where $\gamma\in\Gamma_N(p)$. Hence by Theorem 3.3.16 and Lemma 3.3.18, we have

$$\begin{aligned}
&T_p f_{L,C_1,C_2}(\tau,\sigma,x,x') + \frac{p^{N-1}-p}{p-1} f_{L,C_1,C_2}(\tau,\sigma,x,px')\\
=& T_p f_{\sigma^{-1}L,C_1,C_2}(\tau,x\sigma,\sigma^{-1}x') + \frac{p^{N-1}-p}{p-1} f_{\sigma^{-1}L,C_1,C_2}(\tau,x\sigma,p\sigma^{-1}x')\\
=& \sum_S f_{\sigma^{-1}S,C_1,C_2}(\tau,px\sigma,\sigma^{-1}x')\\
=& \sum_{\gamma\in\Gamma_N(p)} f_{\frac{1}{p}\sigma^{-1}\gamma L,C_1,C_2}(\tau,px\sigma,\sigma^{-1}x').
\end{aligned}$$

Moreover, we note that the entries of $p\gamma^{-1}$ are integers, then we can apply the

Lemma 3.3.18 to $p\gamma^{-1}\sigma$ again,

$$\sum_{\gamma\in\Gamma_N(p)} f_{\frac{1}{p}\sigma^{-1}\gamma L,C_1,C_2}(\tau, px\sigma, \sigma^{-1}x') = \sum_{\gamma\in\Gamma_N(p)} f_{L,C_1,C_2}(\tau, p\gamma^{-1}\sigma, x\gamma, p\gamma^{-1}x').$$

This proves the corollary. □

Now we can give the relation between the Hecke operators T_p and $\mathbb{T}_p$.

Theorem 3.3.20. *Let p be a prime number. Fix two vectors $x, x' \in \mathbb{C}^N$. Let $\sigma \in M_N(\mathbb{Z})$ be any matrix. Then we have*

$$T_p\mathscr{E}(\tau,\sigma,x,x') = \mathbb{T}_p\mathscr{E}(\tau,\sigma,x,x') - \frac{p^{N-1}-p}{p-1}\mathscr{E}(\tau,\sigma,x,px'). \tag{3.34}$$

Proof. If $\det(\sigma) = 0$, the result is trivial. Hence we only need to prove the case $\det(\sigma) \neq 0$. First we consider the case $\sigma = \mathrm{Id}$. In this case,

$$\mathscr{E}(\tau, \mathrm{Id}, x, x') = \prod_{i=1}^{N} \mathcal{K}(\tau, x_i, x_i').$$

We use the Fourier expansion (1.2) of the Kronecker theta function. Then $\mathscr{E}(\tau, \mathrm{Id}, x, x')$ gives

$$\prod_{i=1}^{N}\left(1 - \frac{1}{1-\mathbf{e}(x_i)} - \frac{1}{1-\mathbf{e}(x_i')} - \sum_{m,n\geq 1} q^{mn}\left(\mathbf{e}(nx_i+mx_i') - \mathbf{e}(-nx_i-mx_i')\right)\right). \tag{3.35}$$

We note that the Fourier expansion of the Kronecker theta function $\mathcal{K}(\tau, x_0, x_0')$ converges absolutely for $|q| < |\mathbf{e}(x_0)|, |\mathbf{e}(x_0')| < 1$. Hence the equation (3.35) converges absolutely in the domain $|q| < |\mathbf{e}(x_i)|, |\mathbf{e}(x_i')| < 1$ for all i. Then we have

$$1 - \frac{1}{1-\mathbf{e}(x_i)} - \frac{1}{1-\mathbf{e}(x_i')} = -1 - \sum_{n\geq 1}\mathbf{e}(nx_i) - \sum_{m\geq 1}\mathbf{e}(mx_i').$$

Then the equation (3.35) gives

$$\prod_{i=1}^{N} - \left(1 + \sum_{n\geq 1}\mathbf{e}(nx_i) + \sum_{m\geq 1}\mathbf{e}(mx_i') + \sum_{m,n\geq 1} q^{mn}\left(\mathbf{e}(nx_i+mx_i') - \mathbf{e}(-nx_i-mx_i')\right)\right). \tag{3.36}$$

Now we expand this product. Let I_1, I_2, I_3, I_4 be four subsets of $\{1, 2, \ldots, N\}$ which

don't intersect pairwise. if $i \notin I_1 \cup I_2 \cup I_3 \cup I_4$, then the term

$$1 + \sum_{n\geq 1} \mathbf{e}(nx_i) + \sum_{m\geq 1} \mathbf{e}(mx_i') + \sum_{m,n\geq 1} q^{mn} \left(\mathbf{e}(nx_i + mx_i') - \mathbf{e}(-nx_i - mx_i')\right) \quad (3.37)$$

contributes 1. If $i \in I_1$, then the term (3.37) contributes $\sum_{n\geq 1} \mathbf{e}(nx_i)$. If $i \in I_2$, then the term (3.37) contributes $\sum_{m\geq 1} \mathbf{e}(mx_i')$. If $i \in I_3$, then the term (3.37) contributes $\sum_{m,n\geq 1} q^{mn}\mathbf{e}(nx_i + mx_i')$, and finally if $i \in I_4$, then the term (3.37) contributes $\sum_{m,n\geq 1} -q^{mn}\mathbf{e}(-nx_i - mx_i')$. For each quadruple (I_1, I_2, I_3, I_4), we denote

$$\begin{aligned} f_{I_1,I_2,I_3,I_4}(\tau, x, x') = &\prod_{i_1\in I_1} \sum_{n_{i_1}\geq 1} \mathbf{e}(n_{i_1}x_{i_1}) \prod_{i_2\in I_2} \sum_{m_{i_2}\geq 1} \mathbf{e}(m_{i_2}x_{i_2}') \\ &\prod_{i_3\in I_3} \sum_{m_{i_3},n_{i_3}\geq 1} q^{m_{i_3}n_{i_3}}\mathbf{e}(n_{i_3}x_{i_3} + m_{i_3}x_{i_3}') \prod_{i_4\in I_4} \sum_{m_{i_4},n_{i_4}\geq 1} -q^{m_{i_4}n_{i_4}}\mathbf{e}(-n_{i_4}x_{i_4} - m_{i_4}x_{i_4}') \end{aligned} \quad (3.38)$$

Then we have

$$\mathscr{E}(\tau, \mathrm{Id}, x, x') = (-1)^N \sum_{I_1,I_2,I_3,I_4} f_{I_1,I_2,I_3,I_4}(\tau, x, x'), \quad (3.39)$$

where I_1, I_2, I_3, I_4 run through all the subsets of tand not intersect pairwise. Now we want to apply the Corollory 3.3.19. So we set $L = e_1\mathbb{Z} + \cdots + e_N\mathbb{Z}$ where $e_1, \ldots, e_N$ is the standard basis of $\mathbb{R}^N$. Let $e_1^*, \ldots, e_N^*$ be the dual basis of $e_1, \ldots, e_N$. Let

$$C = \bigoplus_{i\in I_1\cup I_2\cup I_3} e_i\mathbb{R}_{\geq 0} \oplus \bigoplus_{i\in I_4} e_i\mathbb{R}_{\leq 0}. \quad (3.40)$$

Then the dual cone of C is given by

$$C^* = \bigoplus_{i\in I_1\cup I_2\cup I_3} e_i^*\mathbb{R}_{\geq 0} \oplus \bigoplus_{i\in I_4} e_i^*\mathbb{R}_{\leq 0}. \quad (3.41)$$

We denote

$$C_1 = \bigoplus_{i\in I_2\cup I_3} e_i^*\mathbb{R}_{\geq 0} \oplus \bigoplus_{i\in I_4} e_i^*\mathbb{R}_{\leq 0}, \quad C_2 = \bigoplus_{i\in I_1\cup I_3} e_i\mathbb{R}_{\geq 0} \oplus \bigoplus_{i\in I_4} e_i\mathbb{R}_{\leq 0}, \quad (3.42)$$

Then C_1 is a face of C^* and C_2 is a face of C. Hence by Definition 3.3.12, we have

$$f_{\mathbb{Z}^N,C_1,C_2}(\tau, x, x') = (-1)^{|I_4|} f_{I_1,I_2,I_3,I_4}(\tau, x, x'). \quad (3.43)$$

Combining with (3.39), we have

$$\mathscr{E}(\tau, \mathrm{Id}, x, x') = \sum_{C,(C_1,C_2)} \pm f_{\mathbb{Z}^N,C_1,C_2}(\tau, x, x'). \tag{3.44}$$

Here the summation over the cone C determined by (3.40) and pairs (C_1, C_2) determined by (3.42). Now we consider any matrix σ with $D = \det(\sigma) \neq 0$. We use the equivalent definition in Proposition 2.1.4 of $\mathscr{E}$:

$$\begin{aligned}
\mathscr{E}(\tau, \sigma, x, x') &= \frac{\mathrm{sign}(D)}{|D|^N} \sum_{\substack{z \in \mathbb{Z}^N/\mathbb{Z}^N|D|\sigma^{-1} \\ z' \in \mathbb{Z}^N/\sigma\mathbb{Z}^N}} \mathscr{E}\left(\frac{\tau}{|D|}, \mathrm{Id}, \frac{(x+z)\sigma}{|D|}, \sigma^{-1}(x'+z')\right) \\
&= \frac{\mathrm{sign}(D)}{|D|^N} \sum_{\substack{z \in \mathbb{Z}^N/\mathbb{Z}^N|D|\sigma^{-1} \\ z' \in \mathbb{Z}^N/\sigma\mathbb{Z}^N}} \sum_{C,(C_1,C_2)} f_{\mathbb{Z}^N,C_1,C_2}\left(\frac{\tau}{|D|}, \frac{(x+z)\sigma}{|D|}, \sigma^{-1}(x'+z')\right) \\
&= \sum_{C,(C_1,C_2)} f_{\mathbb{Z}^N,C_1,C_2}(\tau, \sigma, x, x')
\end{aligned}$$

where the summation is over the same pairs (C_1, C_2) and C as in equation (3.44). Thus the Theorem 3.3.20 follows from Corollary 3.3.19.

Finally, we have already established the identity (3.34) between meromorphic functions on $\mathbb{C}^N \times \mathbb{C}^N$ on an open subset, hence it holds everywhere. □

3.3.3 The general case

In this subsection, we will generalize the Theorem 3.3.20 to the relation between $\mathbb{T}_m$ and T_m for any integer m. To start, we first consider the case of $m = p^k$ where p is a prime number.

Theorem 3.3.21. *Let p be a prime number and k be a positive integer, $x \in S$ and $x' \in S'$ where S, S' are defined in Definition 3.3.12. Let $a(N, p^t)$ be a family of integers given by $a(N, 1) = 1$ and*

$$a(N, p^t) = H(N-1, p^t) - H(N-1, p^{t-1}) \text{ for } t = 1, 2, \ldots, \tag{3.45}$$

where the number $H(N, p^t)$ is defined in the Lemma 3.3.3. Then we have the following formula

$$\sum_S f_{S,C_1,C_2}(\tau, p^k x, x') = \sum_{t=0}^{k} a(N, p^t) T_{p^{k-t}} f_{L,C_1,C_2}(\tau, x, p^t x'), \tag{3.46}$$

where S runs through the lattices such that $L \subset S \subset \frac{1}{p^k}L$ and $[S : L] = p^{k(N-1)}$.

Proof. By definition, the left hand side of (3.46) equals to

$$\sum_{m \in L^* \cap C_1^\circ} \sum_{\substack{R \\ m \cdot R \subset p^k \mathbb{Z}}} \sum_{n \in R \cap C_2^\circ} q^{\frac{m \cdot n}{p^k}} \mathbf{e}(x \cdot n + m \cdot x'), \tag{3.47}$$

where the summation over all lattices $p^k L \subset R \subset L$ with $[L : R] = p^k$.

Let j be an integer with $0 \leq j \leq k$. We consider the element $m \in L^* \cap C_1^\circ$ with $m \in p^j L^*$, but $m \notin p^{j+1} L^*$. Here if $j = k$, then we don't need the condition $m \notin p^{j+1} L^*$. For such m we consider the following series:

$$\sum_{\substack{R \\ m \cdot R \subset p^k \mathbb{Z}}} \sum_{n \in R \cap C_2^\circ} q^{\frac{m \cdot n}{p^k}} \mathbf{e}(x \cdot n + m \cdot x'). \tag{3.48}$$

We consider the subset of L:

$$R_m^{(j)} = \left\{ n \in L \,\middle|\, \frac{m \cdot n}{p^j} \equiv 0 \pmod{p^{k-j}} \right\}.$$

It is not hard to see that $R_m^{(j)}$ is a sublattice of L of index p^{k-j}. We see that if a sublattice R occurs in the series (3.48), then it must contained in $R_m^{(j)}$. Hence only the sublattices of $R_m^{(j)}$ of index p^j occurs in the summation (3.48). Let l be a non-negative integer with $0 \leq l \leq j-1$, if $n \in p^l R_m^{(j)}$ but $n \notin p^{l+1} R_m^{(j)}$, then by Lemma 3.3.5, there exists $\sum_{i=0}^{l} p^{i(N-1)} H(N-1, p^{j-i})$ sublattices of $R_m^{(j)}$ of index p^j containing n. If $n \in p^j R_m^{(j)}$, then all the sublattices of $R_m^{(j)}$ of index p^j containing n. Hence by Lemma 3.3.3, there exists $H(N, p^j)$ such sublattices. Thus the series (3.48) equals

$$\sum_{l=0}^{j} p^{l(N-1)} H(N-1, p^{j-l}) \sum_{n \in p^l R_m^{(j)} \cap C_2^\circ} q^{\frac{m \cdot n}{p^k}} \mathbf{e}(x \cdot n + m \cdot x').$$

Combining the calculation above, we see that the series (3.47) equals

$$\begin{aligned} &\sum_{j=0}^{k-1} \sum_{p^j \| m} \sum_{l=0}^{j} \sum_{n \in p^l R_m^{(j)} \cap C_2^\circ} p^{l(N-1)} H(N-1, p^{j-l}) q^{\frac{m \cdot n}{p^k}} \mathbf{e}(x \cdot n + m \cdot x') \\ &+ \sum_{m \in p^k L^* \cap C_1^\circ} \sum_{l=0}^{k} \sum_{n \in p^l L \cap C_2^\circ} p^{l(N-1)} H(N-1, p^{k-l}) q^{\frac{m \cdot n}{p^k}} \mathbf{e}(x \cdot n + m \cdot x'). \end{aligned} \tag{3.49}$$

Here the notation $p^j||m$ means that $m \in p^j L^* \cap C_1^\circ$, but $m \notin p^{j+1} L^* \cap C_1^\circ$.

We note that

$$\begin{aligned}&\sum_{m\in p^j L^*\cap C_1^\circ}\ \sum_{n\in p^l R_m^{(j)}\cap C_2^\circ} q^{\frac{m\cdot n}{p^k}}\,\mathbf{e}(x\cdot n+m\cdot x')\\ =&\sum_{m\in L^*\cap C_1^\circ}\ \sum_{\substack{n\in L\cap C_2^\circ\\ p^{k-j}|m\cdot n}} q^{\frac{m\cdot n}{p^{k-l-j}}}\,\mathbf{e}(p^l x\cdot n+p^j m\cdot x').\end{aligned} \tag{3.50}$$

To simplify the calculation, we denote the series (3.50) by $b(l,j)$. Hence the summation (3.49) equals to

$$\begin{aligned}&\sum_{j=0}^{k-1}\sum_{l=0}^{j} p^{l(N-1)}H(N-1,p^{j-l})\left(b(l,j)-b(l,j+1)\right)+\sum_{l=0}^{k} p^{l(N-1)}H(N-1,p^{k-l})b(l,k)\\ =&\sum_{l=0}^{k} p^{l(N-1)}b(l,l)+\sum_{j=1}^{k}\sum_{l=0}^{j-1} p^{l(N-1)}\left(H(N-1,p^{j-l})-H(N-1,p^{j-1-l})\right)b(l,j).\end{aligned}$$

By taking $t=j-l$, it gives

$$\sum_{l=0}^{k} p^{l(N-1)}b(l,l)+\sum_{t=1}^{k}\sum_{l=0}^{k-t} p^{l(N-1)}\left(H(N-1,p^t)-H(N-1,p^{t-1})\right)b(l,l+t). \tag{3.51}$$

On the other hand, we consider the Hecke operator $T_{p^{k-t}}$. By definition, we have

$$\begin{aligned}&T_{p^{k-t}}f_{L,C_1,C_2}(\tau,x,p^t x')\\ =&p^{(k-t)(N-1)}\sum_{l=0}^{k-t}\frac{1}{p^{N(k-t-l)}}\sum_{b=0}^{p^{k-t-l}-1} f_{L,C_1,C_2}\left(\frac{p^l\tau+b}{p^{k-t-l}},p^l x,p^{l+t}x'\right)\\ =&\sum_{\substack{m\in L^*\cap C_1^\circ\\ n\in L\cap C_2^\circ}}\sum_{l=0}^{k-t}\frac{p^{lN}}{p^{k-t}}\sum_{b=0}^{p^{k-j}-1}\mathbf{e}\left(m\cdot n\frac{p^l\tau+b}{p^{k-t-l}}+p^l x\cdot n+p^{l+t}m\cdot x'\right)\\ =&\sum_{j=0}^{k} p^{l(N-1)}\sum_{m\in L^*\cap C_1^\circ}\ \sum_{\substack{n\in L\cap C_2^\circ\\ p^{k-t-l}|m\cdot n}} q^{\frac{m\cdot n}{p^{k-t-2l}}}\,\mathbf{e}(p^l x\cdot n+p^{l+t}m\cdot x').\end{aligned}$$

Hence the summation (3.51) equals to

$$T_{p^k}f_{L,C_1,C_2}(\tau,x,x')+\sum_{t=1}^{k}(H(N-1,p^t)-H(N-1,p^{t-1}))T_{p^{k-t}}f_{L,C_1,C_2}(\tau,x,p^t x').$$

This completes the proof. □

Theorem 3.3.22. *Let k be a positive integer and p be a prime number. Then for any $x, x' \in \mathbb{C}^N$ and $\sigma \in M_N(\mathbb{Z})$, we have the formula:*

$$\mathbb{T}_{p^k}\mathscr{E}(\tau,\sigma,x,x') = \sum_{t=0}^{k} a(N,p^t)T_{p^{k-t}}\mathscr{E}(\tau,\sigma,x,p^t x').$$

where $a(N,p^t)$ is the same as the Theorem 3.3.21.

Proof. Similar to Theorem 3.3.20. □

Now we are able to give the general relation between the Hecke operators $\mathbb{T}_m$ and T_m.

Theorem 3.3.23. *Let m be a positive integer with the prime factorization $m = p_1^{k_1}\cdots p_s^{k_s}$. Then for any $x, x' \in \mathbb{C}^N$ and $\sigma \in M_N(\mathbb{Z})$, we have the formula:*

$$\mathbb{T}_m\mathscr{E}(\tau,\sigma,x,x') = \sum_{d|m} A(N,d)T_{\frac{m}{d}}\mathscr{E}(\tau,\sigma,x,dx'),$$

where $d = p_1^{k_1'}\cdots p_s^{k_s'}$,

$$A(N,d) = \prod_{i=1}^{s} a(N,p_i^{k_i'}), \tag{3.52}$$

and $a(N,p^t)$ is given in the Theorem 3.3.21.

Proof. By Proposition 3.1.6, and applying Theorem 3.3.22 to $\mathbb{T}_{p_s^{k_s}}$ we have

$$\mathbb{T}_m\mathscr{E}(\tau,\sigma,x,x') = \sum_{t_s=0}^{k_s} \mathbb{T}_{p_1^{k_1}}\cdots\mathbb{T}_{p_{s-1}^{k_{s-1}}} a(N,p^{t_s})T_{p_s^{k_s-t_s}}\mathscr{E}(\tau,\sigma,x,p_s^{t_s}x'). \tag{3.53}$$

By Proposition 3.1.8, the Hecke operator T_m commutes with $\mathbb{T}_n$. Hence (3.53) equals to

$$\mathbb{T}_m\mathscr{E}(\tau,\sigma,x,x') = \sum_{t_s=0}^{k_s} a(N,p^{t_s})T_{p_s^{k_s-t_s}}\mathbb{T}_{p_1^{k_1}}\cdots\mathbb{T}_{p_{s-1}^{k_{s-1}}}\mathscr{E}(\tau,\sigma,x,p_s^{t_s}x'). \tag{3.54}$$

Repeating this process $s-1$ times, we obtain

$$\mathbb{T}_m\mathscr{E}(\tau,\sigma,x,x') = \sum_{t_1=0}^{k_1}\cdots\sum_{t_s=0}^{k_s} a(N,p_s^{t_s})T_{p_s^{k_s-t_s}}\cdots a(N,p_s^{t_1})T_{p_1^{k_1-t_1}}\mathscr{E}(\tau,\sigma,x,p_1^{t_1}\cdots p_s^{t_s}x') \tag{3.55}$$

At last, by applying Proposition 3.1.9, we have

$$\begin{aligned}
&\mathbb{T}_m\mathscr{E}(\tau,\sigma,x,x') \\
=&\sum_{t_1=0}^{k_1}\cdots\sum_{t_s=0}^{k_s} a(N,p_1^{t_1})\cdots a(N,p_s^{t_s})T_{p_1^{k_1-t_1}\cdots p_s^{k_s-t_s}}\mathscr{E}(\tau,\sigma,x,p_1^{t_1}\cdots p_s^{t_s}x') \\
=&\sum_{d|m} A(N,d)T_{\frac{m}{d}}\mathscr{E}(\tau,\sigma,x,dx').
\end{aligned} \tag{3.56}$$

This completes the proof. □

Corollary 3.3.24. *If $N=2$, then for any positive integer m, the two Hecke operators $\mathbb{T}_m$ and T_m coincide on $\mathscr{E}(\tau,\sigma,x,x')$, i.e.*

$$T_m\mathscr{E}(\tau,\sigma,x,x') = \mathbb{T}_m\mathscr{E}(\tau,\sigma,x,x').$$

Proof. In fact, we note that for any positive integer l,

$$H(1,l)=1.$$

Hence for any integer $k\geq 1$ and prime number p, we have $a(2,p^k)=0$. Hence the definition (3.52) of $A(N,d)$ shows that $A(2,d)=0$ for any $d>1$. Then this corollary follows directly from Theorem 3.3.23. □

Chapter

4 Eisenstein Cocycle

We recall that Corollary 1.4.9 gives relation of Eisenstein series:

$$\begin{aligned} &E_1(x_1)E_1(x_2) + E_1(x_1)E_1(-x_1-x_2) + E_1(-x_1-x_2)E_1(x_2) \\ =&E_2(x_1) + E_2(x_2) + E_2(-x_1-x_2), \end{aligned} \tag{4.1}$$

where $x_1, x_2 \in \mathbb{C}\backslash(\mathbb{Z}+\tau\mathbb{Z})$ and $x_1 + x_2 \notin \mathbb{Z}+\tau\mathbb{Z}$. In section 4.1, we will see that this relation is an instance of a 1-cocycle relation for $\mathrm{GL}_2(\mathbb{Q})$ valued in functions on $\mathbb{C}^2 \times \mathbb{C}^2$ as shown in Example 4.1.4. We will generalize this relation to an $(N-1)$-cocycle relation for $\mathrm{GL}_N(\mathbb{Q})$ valued in functions on $\mathbb{C}^N \times \mathbb{C}^N$ in Theorem 4.1.2. Our goal in section 4.1 is to define a cocycle which only depends on one variable in $\mathbb{C}^N$. We do it by two steps. First we specialize the elliptic cocycle $\mathscr{E}$ defined in Chapter 2 at $x = 0$ as given in Theorem 4.1.2. Then following [**CD14**], we do the smoothing at prime number ℓ to get rid of the parameter x as shown in Theorem 4.1.6.

In section 4.2, we first recall the Sczech's cocycle on the Bianchi group, and then we do the same smoothing process for such Sczech's cocycle. As an aside, we compare in Theorem 4.2.3 the smoothed Eisenstein cocycle to the Sczech cocycle on the Bianchi group.

In section 4.3, we study the algebraicity of the Eisenstein cocycle $\Psi^{(N)}$, the smoothed Eisenstein cocycle $\Psi_\ell^{(N)}$ and the elliptic cocycle $\mathscr{E}$ at the CM points.

At last, we will construct a variant $\mathcal{E}$ of the elliptic cocycle $\mathscr{E}$ which consists of the Eisenstein-Kronecker function defined in Chapter 1. Such cocycle only depends on the lattice in $\mathbb{C}$ instead of the generators. We will prove a relation between the Hecke operator T_m and $\mathbb{T}_m$ for $\mathcal{E}$ in Theorem 4.4.6.

4.1 The smoothed Eisenstein cocycle $\Psi_\ell^{(N)}$

In this section, we introduce the Eisenstein cocycle which consists of the product of Eisenstein series which was defined in Chapter 1. We want to deduce a new cocycle valued in functions depending only on x' by specializing the elliptic cocycle $\mathscr{E}$ of

Chapter 2 at $x = 0$. To do that, we consider the Laurent expansion of $\mathscr{E}(\tau, \sigma, xT, x')$ in T and take the constant term to be our Eisenstein cocycle $\Psi^{(N)}$. The exact definition is given in 4.1.1. However, because of the pole at $x = 0$, the result still depends rationally on x, see the equation (4.2). Following [**CD14**][Section 2], we introduce the concept of smoothed cocycle. It enables us to eliminate the dependence on x and simultaneously cancel the term of Eisenstein series of higher weight.

4.1.1 Definition of the Eisenstein cocycle $\Psi^{(N)}$

Definition 4.1.1. *Let $\mathcal{A} = (A_1, \ldots, A_N) \in \mathrm{GL}_N(\mathbb{Q})^N$. We denote $\sigma(\mathcal{A}) = (\sigma_j)$ where σ_j is the product of a certain integer λ and first column of A_j such that σ_j is a primitive vector in $\mathbb{Z}^N$. Let $x, x' \in \mathbb{C}^N$ such that the components of $x\sigma(\mathcal{A})$ are non-zero. Then we define the Eisenstein cocycle by:*

If $\det(\sigma(\mathcal{A})) \neq 0$,

$$\Psi^{(N)}(\mathcal{A})(\tau, x, x') = \frac{1}{\det(\sigma(\mathcal{A}))} \sum_{z' \in \sigma(\mathcal{A})^{-1}\Lambda^N/\Lambda^N} \Bigg(E_1\left(w'_1 + z'_1\right) \ldots E_1\left(w'_N + z'_N\right) + \sum_{m=1}^{N-1} \sum_{s_i, k_i} \prod_{s \in \{s_i\}^\circ} w_s^{-1} \prod_{i=1}^{m} (-w_{s_i})^{k_i - 1} E_{k_i}(w'_{s_i} + z'_{s_i}) \Bigg), \tag{4.2}$$

where $w = x\sigma(\mathcal{A})$, $w' = \sigma(\mathcal{A})^{-1}x'$, the summation over all subsets $\{s_1, \ldots, s_m\} \subset \{1, 2, \ldots, N\}$ and non-negative integers k_i such that $\sum k_i = N$, and $\{s_i\}^\circ$ denote the set $\{1, \ldots, N\} \backslash \{s_1, \ldots, s_m\}$. Here to simplify the formula, we take $E_0(x) = 1$ for all x. Furthermore, we need to assume that all the components $w'_i + z'_i \notin \Lambda = \mathbb{Z} + \tau\mathbb{Z}$.

If $\det(\sigma(\mathcal{A})) = 0$,

$$\Psi^{(N)}(\mathcal{A})(\tau, x, x') = 0. \tag{4.3}$$

Now we define an action of $\mathrm{GL}_N(\mathbb{Q})$ on $\Psi^{(N)}$. Let $g_0 \in \mathrm{GL}_N(\mathbb{Q})$, we choose the smallest integer λ such that $g = \lambda g_0 \in M_N(\mathbb{Z})$. Then we define

$$(g_0 \cdot \Psi^{(N)})(\mathcal{A})(\tau, x, x') = \frac{1}{\det(g)} \sum_{z, z' \in \mathbb{Z}^N/g\mathbb{Z}^N} \Psi^{(N)}(\mathcal{A})\left(\tau, xg, g^{-1}(x' + z\tau + z')\right). \tag{4.4}$$

Theorem 4.1.2. *The Eisenstein cocycle $\Psi^{(N)}$ is a homogeneous $(N-1)$-cocycle for $\mathrm{GL}_N(\mathbb{Q})$ valued in the vector space $\mathcal{F}(\mathbb{C}^N \times \mathbb{C}^N)$ of functions on $\mathbb{C}^N \times \mathbb{C}^N$:*

$$\Psi^{(N)} \in H^{N-1}(\mathrm{GL}_N(\mathbb{Q}), \mathcal{F}(\mathbb{C}^N \times \mathbb{C}^N)). \tag{4.5}$$

Proof. We begin with the elliptic cocycle $\mathscr{E}(\tau, \sigma(\mathcal{A}), xT, x')$. Here we multiple x by an independent element T. We write $x' = \alpha + \beta\tau$ where $\alpha, \beta \in \mathbb{R}^N$. Then

$$\begin{aligned}
&\mathbf{e}(x \cdot \beta T)\mathscr{E}(\tau, \sigma(\mathcal{A}), xT, x') \\
=&\frac{1}{\det(\sigma(\mathcal{A}))} \sum_{y,y' \in \mathbb{Z}^N/\sigma(\mathcal{A})\mathbb{Z}^N} \mathbf{e}(x \cdot (\beta + y)T)\mathscr{K}(\tau, x\sigma(\mathcal{A})T, \sigma(\mathcal{A})^{-1}(x' + y\tau + y')) \\
=&\frac{1}{\det(\sigma(\mathcal{A}))} \sum_{z,z' \in \sigma(\mathcal{A})^{-1}\mathbb{Z}^N/\mathbb{Z}^N} \mathbf{e}(w \cdot (\beta' + z)T)\mathscr{K}(\tau, wt, w' + z\tau + z'),
\end{aligned} \tag{4.6}$$

where $w = x\sigma(\mathcal{A}), w' = \sigma(\mathcal{A})^{-1}x'$, $\beta' = \sigma(\mathcal{A})^{-1}\beta$. Now we use the Laurant expansion (1.29) of Kronecker theta function, we see that $\mathbf{e}(x \cdot \beta T)\mathscr{E}(\tau, \sigma(\mathcal{A}), xT, x')$ gives

$$\frac{1}{\det(\sigma(\mathcal{A}))} \sum_{z,z' \in \sigma(\mathcal{A})^{-1}\mathbb{Z}^N/\mathbb{Z}^N} \prod_{i=1}^{N} \left(\frac{1}{w_i t} + \sum_{k \geq 0} (-w_i)^k E_{k+1}(w_i' + z_i')T^k \right). \tag{4.7}$$

Expanding the product and taking the constant term, we get the Eisenstein cocycle $\Psi^{(N)}$. □

When $N = 2$, the Eisenstein cocycle $\Psi^{(2)}$ has a simpler formula. In this case, we consider the inhomogeneous cocycle, i.e. we always assume $A_1 = \mathrm{Id}$ and $A_2 \in \mathrm{GL}_2(\mathbb{Q})$.

Corollary 4.1.3. *Let* $A = \begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Q})$, *and* $(a, c)^t$ *be a primitive vector in* $\mathbb{Z}^2$ *which is a scalar of* $(a_0, c_0)^t$. *Let* $x, x' \in \mathbb{C}^2$ *such that* $x_1, ax_1 + cx_2 \neq 0$ *and* $x_2', cx_1' - ax_2' \notin \Lambda$. *Then if* $c \neq 0$,

$$\begin{aligned}
\Psi^{(2)}(A)(\tau, x, x') =& \frac{1}{c} \sum_{r \in \Lambda/c\Lambda} E_1\left(\frac{cx_1' - ax_2' - ar}{c}\right) E_1\left(\frac{x_2' + r}{c}\right) \\
&- \frac{x_1}{c(ax_1 + cx_2)} E_2(cx_1' - ax_2') - \frac{ax_1 + cx_2}{cx_1} E_2(x_2'),
\end{aligned} \tag{4.8}$$

if $c = 0$,

$$\Psi^{(2)}(A)(\tau, x, x') = 0. \tag{4.9}$$

Proof. We only need to consider the case of $c \neq 0$. We take

$$\sigma = \begin{pmatrix} 1 & a \\ 0 & c \end{pmatrix}.$$

Then by the Theorem 4.1.2, we have

$$\Psi^{(2)}(A)(\tau,x,x') = \frac{1}{\det\sigma}\sum_{z'\in\sigma^{-1}\Lambda^2/\Lambda^2}\Big(E_1(w_1'+z_1')E_1(w_2'+z_2') - \frac{w_1}{w_2}E_2(w_1'+z_1') - \frac{w_2}{w_1}E_2(w_2'+z_2')\Big). \tag{4.10}$$

Now we take $\{(-ar/c, r/c)^t \mid r\in\Lambda/c\Lambda\}$ to be the set of representatives of $(\sigma^{-1}\Lambda^2/\Lambda^2)$. Moreover, we note that $\gcd(a,c)=1$, hence $-ar$ runs through the set of representatives of $\Lambda/c\Lambda$ as r runs through the set of representatives of $\Lambda/c\Lambda$. Then by the distribution relation of E_2, the last two terms of (4.10) equal to

$$\begin{aligned}&\sum_{r\in\Lambda/c\Lambda}\frac{x_1}{c(ax_1+cx_2)}E_2\left(\frac{cx_1'-ax_2'-ar}{c}\right)+\sum_{r\in\Lambda/c\Lambda}\frac{ax_1+cx_2}{cx_1}E_2\left(\frac{x_2'+r}{c}\right)\\ =&\frac{x_1}{c(ax_1+cx_2)}E_2(cx_1'-ax_2')+\frac{ax_1+cx_2}{cx_1}E_2(x_2').\end{aligned} \tag{4.11}$$

This completes the proof. □

Example 4.1.4. *We will show how to recover the equation (4.1) from our Eisenstein cocycle $\Psi^{(2)}$. Let*

$$A_1=\begin{pmatrix}1&0\\0&1\end{pmatrix},\quad A_2=\begin{pmatrix}0&-1\\1&1\end{pmatrix},\quad A_3=\begin{pmatrix}-1&-1\\1&0\end{pmatrix}.$$

Then $\sigma((A_1,A_2))=\begin{pmatrix}1&0\\0&1\end{pmatrix}$ and by Definition 4.1.1, we have

$$\Psi^{(2)}(A_1,A_2)(\tau,x,x') = E_1(x_1')E_1(x_2') - \frac{x_1}{x_2}E_2(x_1') - \frac{x_2}{x_1}E_2(x_2'). \tag{4.12}$$

Similarly, we have

$$\Psi^{(2)}(A_1,A_3)(\tau,x,x') = E_1(x_1'+x_2')E_1(x_2') - \frac{x_1}{x_2-x_1}E_2(x_1'+x_2') - \frac{x_2-x_1}{x_1}E_2(x_2'),$$

$$\Psi^{(2)}(A_2,A_3)(\tau,x,x') = E_1(x_1'+x_2')E_1(-x_1') - \frac{x_2}{x_2-x_1}E_2(x_1'+x_2') - \frac{x_2-x_1}{x_2}E_2(-x_1').$$

Hence the cocycle relation

$$\Psi^{(2)}(A_1,A_2)(\tau,x,x') - \Psi^{(2)}(A_1,A_3)(\tau,x,x') + \Psi^{(2)}(A_2,A_3)(\tau,x,x') = 0,$$

gives

$$\begin{aligned} &E_1(x_1')E_1(x_2') - E_1(x_1'+x_2')E_1(x_2') - E_1(x_1'+x_2')E_1(x_1') \\ =&E_2(x_1') + E_2(x_2') + E_2(x_1'+x_2'). \end{aligned} \tag{4.13}$$

This is exactly the equation (4.1). Here we note that $E_k(x) = (-1)^k E_k(-x)$.

If we carry Definition 3.1.1 of the Hecke operator to the Hecke operator on $\Psi^{(N)}$, then we can deduce the following result from Theorem 3.3.23:

Corollary 4.1.5. *Let m be a positive integer, then for any $\sigma \in M_N(\mathbb{Z})$ and $x, x' \in \mathbb{C}^N$ satisfying the conditions in Definition 4.1.1, we have*

$$\sum_{\gamma\in\Gamma_N(m)} \Psi^{(N)}(m\gamma^{-1}\mathcal{A})(\tau, x\gamma, m\gamma^{-1}x') = \sum_{d|m} A(N,d) T_{\frac{m}{d}} \Psi^{(N)}(\mathcal{A})(\tau, x, x'), \tag{4.14}$$

where $A(N,d)$ is the same as in Theorem 3.3.23.

4.1.2 The smoothed Eisenstein cocycle $\Psi_\ell^{(N)}$

In this subsection, we will smooth the cocycle $\Psi^{(N)}$ at a prime ℓ which gives a new cocycle $\Psi_\ell^{(N)}$ defined on a certain congruence subgroup.

Let ℓ be a prime number and let $\mathbb{Z}_{(\ell)} = \mathbb{Z}[1/p, p \neq \ell]$ be the localization of $\mathbb{Z}$ at the prime ideal (ℓ). Let Γ_ℓ denote the congruence subgroup

$$\Gamma_\ell := \Gamma_0(\ell\mathbb{Z}_{(\ell)}) = \left\{ A \in \mathrm{SL}_N(\mathbb{Z}_{(\ell)}) \,|\, A \equiv \begin{pmatrix} * & * & \cdots & * \\ 0 & * & \cdots & * \\ \vdots & \vdots & & \vdots \\ 0 & * & \cdots & * \end{pmatrix} \pmod{\ell} \right\}. \tag{4.15}$$

Let π_ℓ be the following diagonal matrix

$$\pi_\ell = \begin{pmatrix} \ell & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}. \tag{4.16}$$

For $\mathcal{A} = (A_1, \ldots, A_N) \in \Gamma_\ell^N$, we define

$$\mathcal{A}' = \pi_\ell \mathcal{A} \pi_\ell^{-1} = (\pi_\ell A_1 \pi_\ell^{-1}, \ldots, \pi_\ell A_N \pi_\ell^{-1}) \in \mathrm{GL}_N(\mathbb{Z}_{(\ell)})^N. \tag{4.17}$$

It is easy to see that $\ell\pi_\ell^{-1}\sigma(\mathcal{A}') = \sigma(\mathcal{A})$.

Now we define the smoothed cocycle $\Psi_\ell^{(N)}$ of $\Psi^{(N)}$:

$$\Psi_\ell^{(N)}(\mathcal{A})(\tau, x, x') = \Psi^{(N)}(\mathcal{A}')(\tau, \ell x\pi_\ell^{-1}, \pi_\ell x') - \ell\Psi^{(N)}(\mathcal{A})(\tau, x, x'). \tag{4.18}$$

The following theorem shows that the smoothed Eisenstein cocycle is in fact independent of x.

Theorem 4.1.6. *The cocycle $\Psi_\ell^{(N)}$ is a linear combination of the product of N Eisenstein series and independent of x. Thus we will omit the variable x later. More precisely, if $\det(\sigma(\mathcal{A})) \neq 0$, we have*

$$\begin{aligned}\Psi_\ell^{(N)}(\mathcal{A})(\tau, x') =& \frac{1}{\det(\sigma(\mathcal{A}'))} \sum_{z' \in \sigma(\mathcal{A}')^{-1}\Lambda^N/\Lambda^N} E_1\left(\ell w_1' - z_1'\right) \dots E_1\left(\ell w_N' + z_N'\right) \\ &- \frac{1}{\det(\sigma(\mathcal{A}'))} \sum_{z \in \sigma(\mathcal{A})^{-1}\Lambda^N/\Lambda^N} E_1\left(w_1' - z_1\right) \dots E_1\left(w_N' + z_N\right),\end{aligned} \tag{4.19}$$

where $w' = \sigma(\mathcal{A})^{-1}x'$.

Proof. By multiplying a suitable matrix in $\mathrm{SL}_N(\mathbb{Z})$, we may assume that the first column of $\sigma(\mathcal{A})$ is $(1, 0, \dots, 0)^t$. We write

$$\sigma(\mathcal{A}) = \begin{pmatrix} 1 & \mathbf{a} \\ 0 & M(\mathcal{A}) \end{pmatrix},$$

where $\mathbf{a} = (a_2, \dots, a_N)$ and $M(\mathcal{A}) \in M_{N-1}(\mathbb{Z})$. Then $M(\mathcal{A}) = \ell M(\mathcal{A}')$. We only need to consider the case of $\det(M(\mathcal{A})) \neq 0$ below.

We fix a set of representatives $\{z' = (z_2', \dots, z_N')^t\}$ of $M(\mathcal{A}')^{-1}\Lambda^{N-1}/\Lambda^{N-1}$ and a set of representative $\{\mathbf{r} = (r_2, \dots, r_N)^t\}$ of $\Lambda^{N-1}/\ell\Lambda^{N-1}$, then $\{\frac{\mathbf{r}+z'}{\ell}\}$ gives a set of representatives of $M(\mathcal{A})^{-1}\Lambda^{N-1}/\Lambda^{N-1}$. Moreover, if we put $z_1' = \mathbf{a}\cdot z'$ and $r_1 = \mathbf{a}\cdot\mathbf{r}$, then

$$\{(z_1, z_2, \dots, z_N)^t\},$$

and

$$\left\{\left(\frac{z_1 + r_1}{\ell}, \dots, \frac{z_N + r_N}{\ell}\right)^t\right\},$$

give the sets of representatives of $\sigma(\mathcal{A}')^{-1}\Lambda^N/\Lambda^N$ and $\sigma(\mathcal{A})^{-1}\Lambda^N/\Lambda^N$ respectively.

To prove this theorem, we want to find the cancellation between the terms of $\Psi^{(N)}(\mathcal{A}')(\tau, \ell x\pi_\ell^{-1}, \pi_\ell x')$ and of $\ell\Psi^{(N)}(\mathcal{A})(\tau, x, x')$. We consider the following term

which occurs in $\Psi^{(N)}(\mathcal{A})(\tau, x, x')$

$$\frac{1}{\det(\sigma(\mathcal{A}))} \sum_{\substack{z' \in \sigma(\mathcal{A}')^{-1}\Lambda^N/\Lambda^N \\ \mathbf{r} \in \Lambda^N/\ell\pi_\ell^{-1}\Lambda^N}} \prod_{s \in \{s_i\}^\circ} w_s^{-1} \prod_{\substack{i=1 \\ \sum k_i = N}}^{m} w_{s_i}^{k_i-1} E_{k_i}\left(w'_{s_i} + \frac{z'_{s_i} + r_{s_i}}{\ell}\right), \tag{4.20}$$

where $m < N$. We want to cancel the corresponding term in $\Psi^{(N)}(\mathcal{A}')(\tau, \ell x \pi_\ell^{-1}, \pi_\ell x')$.

If $s_i \neq 1$ for all i, then by applying the distribution relation of Eisenstein series to each E_{k_i}, the term (4.20) equals to

$$\frac{1}{\ell \det(\sigma(\mathcal{A}'))} \sum_{z' \in \sigma(\mathcal{A}')^{-1}\Lambda^N/\Lambda^N} \prod_{s \in \{s_i\}^\circ} w_s^{-1} \prod_{\substack{i=1 \\ \sum k_i = N}}^{m} w_{s_i}^{k_i-1} E_{k_i}\left(\ell w'_{s_i} + z'_{s_i}\right). \tag{4.21}$$

If $s_i = 1$ for some i, we assume $s_1 = 1$. Since $m < N$, there exists an integer $2 \leq s \leq N$ such that $s \neq s_i$ for all i. Since $\ell \nmid a_j$ for all j, we fix all other k_i and apply the distribution relation to $E_{k_1}\left(w'_1 + \frac{z'_1 + r_1}{\ell}\right)$. It gives

$$\sum_{r_k \in \Lambda/\ell\Lambda} E_{k_1}\left(w'_1 + \frac{z'_1 + r_1}{\ell}\right) = \ell^{2-k_1} E_{k_1}\left(\ell w'_1 + z'_1 + \sum_{j \neq s} a_j r_j\right) = \ell^{2-k_1} E_{k_1}(\ell w'_1 + z'_1). \tag{4.22}$$

The second equality holds since the Eisenstein series is $\mathbb{Z}$-periodic. Then we can apply distribution relation to other k_i separably, which also gives

$$\frac{1}{\ell \det(\sigma(\mathcal{A}'))} \sum_{z' \in \sigma(\mathcal{A}')^{-1}\Lambda^N/\Lambda^N} \prod_{s \in \{s_i\}^\circ} w_s^{-1} \prod_{\substack{i=1 \\ \sum k_i = N}}^{m} w_{s_i}^{k_i-1} E_{k_i}\left(\ell w'_{s_i} + z'_{s_i}\right). \tag{4.23}$$

On the other hand, by definition, $\ell x \pi_\ell \sigma(\mathcal{A}')^{-1} = x$ and $\sigma(\mathcal{A}')^{-1}\pi_\ell x' = \ell\sigma(\mathcal{A})x' = \ell w'$. Hence we have

$$\begin{aligned}\Psi^{(N)}(\mathcal{A}')(\tau, \ell x \pi_\ell^{-1}, \pi_\ell x') = &\frac{1}{\det(\sigma(\mathcal{A}'))} \sum_{z' \in \sigma(\mathcal{A}')^{-1}\Lambda^N/\Lambda^N} \Big(E_1(\ell w'_1 + z'_1) \dots E_1(\ell w'_N + z'_N) \\ &+ \sum_{m=1}^{N-1} \sum_{s_i, k_i} \prod_{s \in \{s_i\}^\circ} w_s^{-1} \prod_{i=1}^{m} w_{s_i}^{k_i-1} E_{k_i}(\ell w'_{s_i} + z'_{s_i}) \Big).\end{aligned} \tag{4.24}$$

By comparing the terms of higher weight, we see that they are cancelled. This proves the theorem. □

Let m be an integer coprime to ℓ. By taking the following set of representatives of $\Gamma_N(m)$, we can get a similar result of Theorem 3.3.23 for $\Psi_\ell^{(N)}$: we denote $\Gamma_N(m,\ell)$ the set of matrices of the form

$$\begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ \ell a_{21} & a_{22} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ \ell a_{N1} & a_{N2} & \cdots & a_{NN} \end{pmatrix},$$

where

$$a_{11}a_{22}\dots a_{NN} = m,$$

and for all $1 \le j < i \le N$,

$$0 \le a_{ij} \le a_{ii} - 1.$$

Corollary 4.1.7. *Let m be a positive integer coprime to ℓ. Then for any $\mathcal{A} \in \Gamma_\ell^N$ and $x' \in \mathbb{C}^N$ satisfying the condition in Definition 4.1.1, we have*

$$\sum_{\gamma\in\Gamma_N(m,\ell)} \Psi_\ell^{(N)}(m\gamma^{-1}\mathcal{A})(\tau, m\gamma^{-1}x') = \sum_{d|m} A(N,d)T_{\frac{m}{d}}\Psi_\ell^{(N)}(\mathcal{A})(\tau,x') \tag{4.25}$$

where $A(N,d)$ is the same as in Theorem 3.3.23

Proof. By the definition, we have

$$\begin{aligned}
&\sum_{\gamma\in\Gamma_N(m,\ell)} \Psi_\ell^{(N)}(m\gamma^{-1}\mathcal{A})(\tau, m\gamma^{-1}x') \\
=&\sum_{\gamma\in\Gamma_N(m,\ell)} \Psi^{(N)}(m\pi_\ell\gamma^{-1}\mathcal{A}\pi_\ell^{-1})(\tau, \ell x\gamma\pi_\ell^{-1}, m\pi_\ell\gamma^{-1}x') - \ell\Psi^{(N)}(m\gamma^{-1}\mathcal{A})(\tau, x\gamma, m\gamma^{-1}x') \\
=&\sum_{\gamma\in\Gamma_N(m)} \Psi^{(N)}(m\gamma^{-1}\pi_\ell\mathcal{A}\pi_\ell^{-1})(\tau, \ell x\pi_\ell^{-1}\gamma, m\gamma^{-1}\pi_\ell x') - \sum_{\gamma\in\Gamma_N(m,\ell)} \ell\Psi^{(N)}(m\gamma^{-1}\mathcal{A})(\tau, x\gamma, m\gamma^{-1}x').
\end{aligned}$$

But we note that since m is coprime to ℓ, the set $\Gamma_N(m,\ell)$ also gives a representative of $\Gamma_N(m)$, hence by Corollary 4.1.5, the summation above equals to

$$\begin{aligned}
&\sum_{d|m} A(N,d)T_{\frac{m}{d}}\Psi^{(N)}(\pi_\ell\mathcal{A}\pi_\ell^{-1})(\tau, \ell x\pi_\ell^{-1}, \pi_\ell x') - \sum_{d|m} A(N,d)T_{\frac{m}{d}}\ell\Psi^{(N)}(\mathcal{A})(\tau,x,x') \\
=&\sum_{d|m} A(N,d)T_{\frac{m}{d}}\Psi_\ell^{(N)}(\mathcal{A})(\tau,x,x').
\end{aligned} \tag{4.26}$$

This completes the proof. □

4.2 Relation with Sczech's cocycle on the Bianchi group

In this section, we will recall the Sczech's cocycle on the Bianchi group and then compare it with our Eisenstein cocycle.

4.2.1 Sczech's cocycle on the Bianchi group

Let $\tau \in \mathbb{H}$ such that $F = \mathbb{Q}(\tau)$ is an imaginary quadratic field. Let $\Lambda = \mathbb{Z} + \tau\mathbb{Z}$. Let $\mathcal{O}_\Lambda$ be a subring of $\mathbb{C}$:

$$\mathcal{O}_\Lambda = \{z \in \mathbb{C} \,|\, z\Lambda \subset \Lambda\}.$$

For any non-negative integer k, we denote

$$G_k(x) = K_k(x, 0, k; \Lambda), \tag{4.27}$$

$$G(x) = K_2(x, 0, 1; \Lambda), \tag{4.28}$$

where $K_k(x, x', s; \Lambda)$ is the Kronecker-Eisenstein function defined in Chapter 1. We note that

$$G_0(x) = \begin{cases} -1 & x \in \Lambda, \\ 0 & x \notin \Lambda. \end{cases}. \tag{4.29}$$

Definition 4.2.1. *Let* $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathcal{O}_\Lambda)$*, and* $x \in \mathbb{C}^2$*. If* $c \neq 0$*, we define*

$$\begin{aligned} \Psi_{Scz}(A)(x) =& \overline{\left(\frac{a}{c}\right)} G(x_1) + \overline{\left(\frac{d}{c}\right)} G(x_1^*) + \frac{a}{c} G_0(x_1) G_2(x_2) + \frac{d}{c} G_0(x_1^*) G_2(x_2^*) \\ &+ \frac{1}{c} \sum_{r \in \Lambda/c\Lambda} G_1\left(\frac{ar + ax_1 + cx_2}{c}\right) G_1\left(\frac{r + x_1}{c}\right), \end{aligned} \tag{4.30}$$

and if $c = 0$*, we define*

$$\Psi_{Scz}(A)(x) = \overline{\left(\frac{b}{d}\right)} G(x_1) + \frac{b}{d} G_0(x_1) G_2(x_2), \tag{4.31}$$

where $x^* = xA$*.*

Sczech proved the theorem:

Theorem 4.2.2. *Ψ is a 1-cocycle, i.e. we have the cocycle relation:*

$$\Psi_{Scz}(AB)(x) = \Psi_{Scz}(A)(x) + \Psi_{Scz}(B)(xA). \tag{4.32}$$

Proof. See [**Ito87**] or [**Scz84**]. □

4.2.2 Relation between the smoothed cocycles

We note that the cocycle $\Psi^{(2)}(A)$ only involve the first column of the matrix A but with two parameters x and x'. $\Psi_{Scz}(A)$ involves all the entries of A but has only one parameter. So there is little hope to have a relation directly between these two cocycles. However, we have already seen that we can get rid of the parameter x by smoothing the cocycle $\Psi^{(2)}(A)$ at the prime number ℓ. Moreover, if we do the smoothing for the Sczech's cocycle, we will get rid of the second column of A. We are wondering there is a relation between these two smoothed cocycles. Now we are going to do this.

Let $\Gamma_\ell(\mathcal{O}_\Lambda)$ to be the congruence subgroup

$$\Gamma_\ell(\mathcal{O}_\Lambda) = \left\{ A \in \mathrm{SL}_2(\mathcal{O}_\Lambda) \,\middle|\, A \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \pmod{\ell} \right\}.$$

We define the smoothed Sczech's cocycle on the Bianchi group in the same way. For any $A \in \Gamma_\ell(\mathcal{O}_\Lambda)$, let

$$\Psi_{Scz,\ell}(A)(x) = \Psi_{Scz}(\pi_\ell A \pi_\ell^{-1})(\ell x \pi_\ell^{-1}) - \ell \Psi_{Scz}(A)(x). \tag{4.33}$$

Moreover, if we assume that $x_1, ax_1 + cx_2 \notin \Lambda$, then $G_0(x_1) = G_0(ax_1 + cx_2) = 0$. Hence

$$\Psi_{Scz,\ell}\left(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}\right)(x) = \overline{\left(\frac{\ell b}{d}\right)} G(x_1) - \ell \overline{\left(\frac{b}{d}\right)} G(x_1) = 0, \tag{4.34}$$

and if $c \neq 0$,

$$\begin{aligned} \Psi_{Scz,\ell}\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)(x) = \frac{\ell}{c} \sum_{z \in \Lambda/(c/\ell)\Lambda} G_1\left(\frac{\ell(ar + ax_1 + cx_2)}{c}\right) G_1\left(\frac{\ell(r + x_1)}{c}\right) \\ - \frac{\ell}{c} \sum_{z \in \Lambda/c\Lambda} G_1\left(\frac{ar + ax_1 + cx_2}{c}\right) G_1\left(\frac{r + x_1}{c}\right). \end{aligned} \tag{4.35}$$

Let

$$\Gamma_0(\ell) = \left\{ A \in \mathrm{SL}_2(\mathcal{O}_\Lambda) \,|\, A \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \pmod{\ell} \right\}.$$

With the help of the explicit expression of $\Psi_{Scz,\ell}$, by restricting A to the congruence subgroup $\Gamma_0(\ell)$, we have the following theorem:

Theorem 4.2.3. *Let* $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(\ell)$. *Then for any* $x_1, x_2 \in \mathbb{C}$ *with* $x_1, ax_1 + cx_2 \notin \Lambda$, *we have*

$$\Psi_\ell^{(2)}(A)((x_1, x_2)^t) = \Psi_{Scz,\ell}(A)((-x_2, x_1)). \tag{4.36}$$

Proof. Note that by the functional equation (1.32), we have $G_1(x) = E_1(x)$. Then this theorem is clear by comparing the explicit expression (4.19) of $\Psi_\ell^{(2)}$ and (4.34) of $\Psi_{Scz,\ell}$. □

This means that $\Psi_\ell^{(2)}(A)(x)$ can be extended to a cocycle for the group $\Gamma_\ell(\mathcal{O}_\Lambda)$. Then the natural question is can we extend the cocycle $\Psi_\ell^{(N)}(\mathcal{A})(x)$ to a larger group? And can we extend the cocycle $\Psi^{(N)}(\mathcal{A})(x, x')$ to a larger group? By abusing the notation, we denote

$$\Gamma_\ell(\mathcal{O}_\Lambda) = \left\{ A \in \mathrm{SL}_N(\mathcal{O}_\Lambda) \,|\, A \equiv \begin{pmatrix} * & * & \cdots & * \\ 0 & * & \cdots & * \\ \vdots & \vdots & & \vdots \\ 0 & * & \cdots & * \end{pmatrix} \pmod{\ell} \right\}.$$

According to some numerical check, we can not extend the cocycle $\Psi^{(N)}(\mathcal{A})(x, x')$ to $\Gamma_\ell(\mathcal{O}_\Lambda)$, even for $N = 2$. However, for the cocycle $\Psi_\ell^{(N)}(\mathcal{A})(x)$, the answer is not clear yet.

4.3 Algebraicity

In this section, we study the algebraicity of the values of $\Psi^{(N)}$ and $\mathscr{E}$ at CM points.

Let $\tau \in \mathbb{H}$ such that $F = \mathbb{Q}(\tau)$ is an imaginary quadratic field. We denote $\mathcal{O}_F$ the ring of integers of F. We take a basis $(1, \tau_0)$ for $\mathcal{O}_F$ with $\tau_0 \in \mathbb{H}$. We associate it a number

$$\lambda_0 = 2\pi |\eta(\tau_0)|^2,$$

where $\eta(\tau_0)$ is the Dedekind eta function given by

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n).$$

Then we recall a theorem which originally due to Damerell [**Dam71**]:

Theorem 4.3.1. *Let τ be a CM point in $\mathbb{H}$ and $F = \mathbb{Q}(\tau)$. Let k, b be two integers such that $0 < b \leq k$. We denote $\Lambda = \mathbb{Z} + \mathbb{Z}\tau$. Then for any $x_0, x_0' \in F$, the value*

$$\pi^{k-b} \lambda_0^{-k} K_k(x_0, x_0', b; \Lambda) \tag{4.37}$$

is algebraic over $\mathbb{Q}$. In particular, if we put $k = b$, then

$$\lambda_0^{-k} E_k(x_0) \tag{4.38}$$

is algebraic over $\mathbb{Q}$.

Proof. See [**Weil76**][Chapter VIII, §15]. □

Theorem 4.3.2. *Let $\mathcal{A} \in \mathrm{GL}_N(\mathbb{Q})^N$, $x \in \bar{\mathbb{Q}}^N$ and $x' \in F^N$ satisfying the condition in Definition 4.1.1. Then we have*

$$\lambda_0^{-N} \Psi^{(N)}(\mathcal{A})(\tau, x, x') \in \bar{\mathbb{Q}}. \tag{4.39}$$

and

$$\lambda_0^{-N} \Psi_\ell^{(N)}(\mathcal{A})(\tau, x') \in \bar{\mathbb{Q}}. \tag{4.40}$$

Proof. The Theorem 4.3.1 shows that

$$\prod_{i=1}^{m} E_{k_i}(x_i') \in \lambda_0^N \bar{\mathbb{Q}}, \tag{4.41}$$

where $\sum_{i=1}^m k_i = N$. The Definition 4.1.1 shows that $\Psi^{(N)}(\mathcal{A})(\tau, x, x')$ is a linear combination of the form (4.41) with coefficient in $\bar{\mathbb{Q}}$. This implies that

$$\lambda_0^{-N} \Psi^{(N)}(\mathcal{A})(\tau, x, x') \in \bar{\mathbb{Q}}. \tag{4.42}$$

The Theorem 4.1.6 shows that $\Psi_\ell^{(N)}(\mathcal{A})(\tau, x')$ is a linear combination of the form (4.41) with $k_i = 1$ for all i. Hence we also have

$$\lambda_0^{-N} \Psi_\ell^{(N)}(\mathcal{A})(\tau, x') \in \bar{\mathbb{Q}}. \tag{4.43}$$

□

We know that the Eisenstein cocycle $\Psi^{(N)}(\mathcal{A})(\tau, x, x')$ is deduced from $\mathscr{E}(\tau, \sigma, x, x')$. Hence we have a similar algebraicity property for the elliptic cocycle $\mathscr{E}(\tau, \sigma, x, x')$.

Theorem 4.3.3. *Let $\sigma \in M_N(\mathbb{Z})$. Let $x, x' \in F^N$ be two vectors such that at least one of them belongs to $\mathbb{Q}^N$. Assume $\mathscr{E}(\tau, \sigma, x, x')$ doesn't have a pole at such x, x', then*

$$\lambda_0^{-N} \mathscr{E}(\tau, \sigma, x, x') \tag{4.44}$$

is algebraic over $\mathbb{Q}$.

Proof. Let $x_0, x_0' \in F \backslash \Lambda$ We note that by Lemma 1.3.1, if $\mathrm{Im}(x_0') \notin \mathbb{Z}$, we have

$$K_1(x_0, x_0', 1; \Lambda) = \mathbf{e}(x_0\beta)\mathcal{K}(\tau, x_0, x_0'), \tag{4.45}$$

where $\beta = \mathrm{Im}(x_0')/\mathrm{Im}(\tau) \in \mathbb{Q}$. But the functions $K_1(x_0, x_0', 1; \Lambda)$ and $\mathcal{K}(\tau, x_0, x_0')$ are continuous away from the lattice points, hence the equation (4.45) holds for any $x_0' \notin \Lambda$. If x_0' is real, then $\mathbf{e}(x_0\beta) = 1$. If x_0 is real, then $\mathbf{e}(x_0\beta)$ is a root of unity since $\beta \in \mathbb{Q}$. Combining the Theorem 4.3.1, in both of the two cases, we see that

$$\lambda_0^{-1} \mathcal{K}(\tau, x_0, x_0') \tag{4.46}$$

is algebraic over $\mathbb{Q}$. Then for any $x, x' \in F^N$ and at least one of them belong to $\mathbb{Q}^N$, the value

$$\lambda_0^N \mathscr{K}(\tau, x, x') = \lambda_0^N \prod_{i=1}^{N} \mathcal{K}(\tau, x_i, x_i') \tag{4.47}$$

is algebraic over $\mathbb{Q}$. By using the equivalent definition of $\mathscr{E}$ in Proposition 2.1.4, we see that $\mathscr{E}(\tau, \sigma, x, x')$ is a linear combination of multivariable Kronecker theta function with coefficient in $\mathbb{Q}$:

$$\mathscr{E}(\tau, \sigma, x, x') = \frac{\mathrm{sign}(D)}{|D|^N} \sum_{\substack{z \in \mathbb{Z}^N/\mathbb{Z}^N|D|\sigma^{-1} \\ z' \in \mathbb{Z}^N/\sigma\mathbb{Z}^N}} \mathscr{K}\left(\frac{\tau}{|D|}, \frac{x+z}{|D|}\sigma, \sigma^{-1}(x'+z')\right). \tag{4.48}$$

If $x \in \mathbb{Q}^N$, then $(x+z)\sigma \in \mathbb{Q}^N$ for any $z \in \mathbb{Z}^N/\mathbb{Z}^N \det(\sigma)\sigma^{-1}$. If $x' \in \mathbb{Q}^N$, then $\sigma^{-1}(x'+z') \in \mathbb{Q}^N$ for any $z \in \mathbb{Z}^N/\sigma\mathbb{Z}^N$. Hence the value

$$\lambda_0^{-N} \mathscr{E}(\tau, \sigma, x, x')$$

is algebraic over $\mathbb{Q}$. □

4.4 The cocycle $\mathcal{E}$

In this section, we construct a new cocycle $\mathcal{E}$ from the Eisenstein-Kronecker function. Such cocycle doesn't depend on the choice of generators, it only depends on the lattice Λ in $\mathbb{C}$. We will prove some similar properties as the elliptic cocycle $\mathscr{E}$.

4.4.1 Definition

We first define the second kind of Kronecker theta function. Such Kronecker theta function is the same as the Kronecker theta function $\mathcal{K}$ defined in Chapter 1 up to a factor when $\Lambda = \mathbb{Z} + \mathbb{Z}\tau$. The advantage of this function is that it doesn't depend on the choice of the generators of the lattice Λ.

Definition 4.4.1. *Let $x_0, x_0' \in \mathbb{C}$ be two complex numbers, we define the second kind of Kronecker theta function by*

$$\Theta(x_0, x_0'; \Lambda) = \exp\left(\frac{x_0\bar{x}_0'}{A(\Lambda)}\right) K_1(x_0, x_0', 1; \Lambda), \tag{4.49}$$

where $A(\Lambda) = \mathrm{Area}(\Lambda)/\pi$ and $\mathrm{Area}(\Lambda)$ is the area of the fundamental domain of Λ. Let $y_0, y_0' \in \mathbb{C}$, we define the translation Kronecker theta function by:

$$\Theta_{y_0,y_0'}(x_0, x_0'; \Lambda) := \exp\left(-\frac{y_0\bar{y}_0' + x_0\bar{y}_0' + x_0'\bar{y}_0}{A(\Lambda)}\right)\Theta(x_0 + y_0, x_0' + y_0'; \Lambda). \tag{4.50}$$

Definition 4.4.2. *Let $x, x', y, y' \in \mathbb{C}^N$, and Λ be a lattice in $\mathbb{C}$, then we define the multivariable theta function by*

$$\mathscr{T}_{y,y'}(x, x'; \Lambda) := \prod_{i=1}^{N} \Theta_{y_i,y_i'}(x_i, x_i'; \Lambda). \tag{4.51}$$

Let $\sigma \in M_N(\mathbb{Z})$ with determinant D, if $D \neq 0$, we define

$$\mathcal{E}(\sigma, x, x'; \Lambda) := \frac{1}{D} \sum_{y \in \sigma^{-1}\Lambda^N/\Lambda^N} \mathscr{T}_{0,y}\left(x\sigma, \sigma^{-1}x'; \Lambda\right). \tag{4.52}$$

If $D = 0$,

$$\mathcal{E}(\sigma, x, x'; \Lambda) = 0. \tag{4.53}$$

The cocycle relation of $\mathcal{E}$ is not obvious from the definition. To prove the cocycle relation of $\mathcal{E}$, we will give the relation between the two cocycles $\mathcal{E}$ and $\mathscr{E}$ first, and then deduce the cocycle relation from $\mathscr{E}$.

Lemma 4.4.3. *If $\Lambda = \mathbb{Z} \oplus \tau\mathbb{Z}$ for some $\tau \in \mathbb{H}$, then the new cocycle coincides with $\mathscr{E}$ up to a factor. More precisely, we have*

$$\mathcal{E}(\sigma, x, x'; \Lambda) = \exp\left(\frac{xx'}{A(\Lambda)}\right) \mathscr{E}(\tau, \sigma, x, x'). \tag{4.54}$$

Proof. We note that in the case $\Lambda = \mathbb{Z} \oplus \tau\mathbb{Z}$, the relation between the Kronecker function $\mathcal{K}$ and Θ is given in Lemma 1.3.1:

$$\Theta(x, x'; \Lambda) = \exp\left(\frac{xx'}{A(\Lambda)}\right) \mathcal{K}(\tau, x, x'). \tag{4.55}$$

Hence for any $z, z' \in \sigma^{-1}\mathbb{Z}^N/\mathbb{Z}^N$ and $x, x' \in \mathbb{C}^N$, we have

$$\begin{aligned}
&\mathscr{T}_{0,z\tau+z'}(x, x'; \Lambda) = \prod_{i=1}^{N} \Theta_{0,z_i\tau+z_i'}(x_i, x_i'; \Lambda) \\
=&\exp\left(-\frac{x(z\bar{\tau}+z')}{A(\Lambda)}\right) \exp\left(\frac{x(x'+z\tau+z')}{A(\Lambda)}\right) \mathscr{K}(\tau, x, x'+z\tau+z') \\
=&\exp\left(\frac{xx'}{A(\Lambda)}\right) \mathbf{e}(xz) \mathscr{K}(\tau, x, x'+z\tau+z').
\end{aligned} \tag{4.56}$$

Then by definition, we have

$$\begin{aligned}
&\mathcal{E}(\sigma, x, x'; \Lambda) = \frac{1}{D} \sum_{y \in \sigma^{-1}\Lambda^N/\Lambda^N} \mathscr{T}_{0,y}\left(x\sigma, \sigma^{-1}x'; \Lambda\right) \\
=&\frac{1}{D} \sum_{z,z' \in \sigma^{-1}\mathbb{Z}^N/\mathbb{Z}^N} \mathscr{T}_{0,z\tau+z'}\left(x\sigma, \sigma^{-1}x'; \Lambda\right) \\
=&\frac{1}{D} \sum_{z,z' \in \sigma^{-1}\mathbb{Z}^N/\mathbb{Z}^N} \exp\left(\frac{xx'}{A(\Lambda)}\right) \mathbf{e}(x\sigma z) \mathscr{K}(\tau, x\sigma, \sigma^{-1}x'+z\tau+z') \\
=&\exp\left(\frac{xx'}{A(\Lambda)}\right) \mathscr{E}(\tau, \sigma, x, x').
\end{aligned} \tag{4.57}$$

This proves the lemma. □

Lemma 4.4.4. *Let u be a non-zero complex number, $\sigma \in M_N(\mathbb{Z})$ and $x, x' \in \mathbb{C}^N$. Let Λ be any lattice in $\mathbb{C}$. Then we have*

$$\mathcal{E}(\sigma, ux, ux'; u\Lambda) = \frac{1}{u^N} \mathcal{E}(\sigma, x, x'; \Lambda). \tag{4.58}$$

Proof. The proof is directly from definition. If $\det(\sigma) = 0$, both of the two sides

equal to 0. So we only need to consider the case of $\det(\sigma) \neq 0$. We note that

$$A(u\Lambda) = |u|^2 A(\Lambda). \tag{4.59}$$

Then for any $\lambda, x_0' \in \mathbb{C}$, we have

$$\psi_{u\Lambda}(u\lambda, ux_0') = \psi_\Lambda(\lambda, x_0'). \tag{4.60}$$

Hence for any $x_0, x_0' \in \mathbb{C}$,

$$\begin{aligned} K_a(ux_0, ux_0', s; u\Lambda) &= \sum_{\lambda \in u\Lambda}{}' \frac{(\overline{ux_0} + \bar{\lambda})^a}{|ux_0 + \lambda|^{2s}} \psi_{u\Lambda}(\lambda, ux_0') \\ &= \sum_{\lambda \in \Lambda}{}' \frac{(\overline{ux_0} + \overline{u\lambda})^a}{|ux_0 + u\lambda|^{2s}} \psi_{u\Lambda}(u\lambda, ux_0') \\ &= \frac{\bar{u}^a}{|u|^{2s}} K_a(x_0, x_0', s; \Lambda). \end{aligned} \tag{4.61}$$

This implies that

$$\Theta_{uy_0, uy_0'}(ux_0, ux_0'; u\Lambda) = \frac{1}{u} \Theta_{y_0, y_0'}(x_0, x_0'; \Lambda). \tag{4.62}$$

Hence

$$\begin{aligned} \mathcal{E}(\sigma, ux, ux'; u\Lambda) &= \frac{1}{D} \sum_{y \in \sigma^{-1}(u\Lambda)^N/(u\Lambda)^N} \mathscr{T}_{0,y}\left(ux\sigma, u\sigma^{-1}x'; u\Lambda\right) \\ &= \frac{1}{Du^N} \sum_{y \in \sigma^{-1}\Lambda^N/\Lambda^N} \mathscr{T}_{0,y}\left(x\sigma, \sigma^{-1}x'; \Lambda\right) \\ &= \frac{1}{u^N} \mathcal{E}(\sigma, x, x'; \Lambda). \end{aligned} \tag{4.63}$$

□

Theorem 4.4.5. *Let $\sigma_0, \sigma_1, \ldots, \sigma_N \in \mathbb{Z}_{\text{prim}}$. Then $\mathcal{E}$ satisfies the following cocycle relation:*

$$\sum_{i=0}^{N} (-1)^i \mathcal{E}((\sigma_0, \ldots, \hat{\sigma}_i, \cdots, \sigma_N), x, x'; \Lambda) = 0. \tag{4.64}$$

Proof. First we assume that $\Lambda = \mathbb{Z} \oplus \tau\mathbb{Z}$ for some $\tau \in \mathbb{H}$. By Lemma 4.4.3, we have

$$\mathcal{E}(\sigma, x, x'; \Lambda) = \exp\left(\frac{xx'}{A(\Lambda)}\right) \mathscr{E}(\tau, \sigma, x, x'). \tag{4.65}$$

We note that the factor $\exp\left(\frac{xx'}{A(\Lambda)}\right)$ does not depend on σ, hence the cocycle property of $\mathcal{E}$ follows from $\mathscr{E}$.

In general, we choose generators u, v of Λ, i.e $\Lambda = u\mathbb{Z} \oplus v\mathbb{Z}$ with $\mathrm{Im}\left(\frac{v}{u}\right) > 0$. Let $\tau = \frac{v}{u}$, and we denote $\Lambda_\tau = \mathbb{Z} \oplus \tau\mathbb{Z} = \frac{1}{u}\Lambda$. Then by applying the Lemma 4.4.4, we have

$$\begin{aligned}
&\sum_{i=0}^{N}(-1)^i\mathcal{E}((\sigma_0,\ldots,\hat{\sigma}_i,\cdots,\sigma_N),x,x';\Lambda)\\
=&\frac{1}{u^N}\sum_{i=0}^{N}(-1)^i\mathcal{E}\left((\sigma_0,\ldots,\hat{\sigma}_i,\cdots,\sigma_N),\frac{x}{u},\frac{x'}{u};\Lambda_\tau\right)\\
=&0.
\end{aligned} \tag{4.66}$$

This completes the proof. □

4.4.2 Hecke operators

Now we want to generalize Theorem 3.3.23 to the cocycle $\mathcal{E}$. Similar to the elliptic cocycle $\mathscr{E}$, we can define the Hecke operator $\mathbb{T}_m$ by

$$\mathbb{T}_m\mathcal{E}(\sigma,x,x';\Lambda) = \sum_{\gamma\in\Gamma_N(m)}\mathcal{E}(m\gamma^{-1}\sigma,x\gamma,m\gamma^{-1}x';\Lambda). \tag{4.67}$$

The Hecke operator T_m can be defined in a more symmetric way:

$$T_m\mathcal{E}(\sigma,x,x';\Lambda) = \frac{1}{m}\sum_{\Lambda'}\mathcal{E}(\sigma,x,x';\Lambda'), \tag{4.68}$$

where the Λ' runs through all sublattices of $\frac{1}{m}\Lambda$ of index m that contain Λ.

Theorem 4.4.6. *Let m be a positive integer and Λ be a lattice in $\mathbb{C}$. Then for any $x, x' \in \mathbb{C}^N$ and $\sigma \in M_N(\mathbb{Z})$, we have*

$$\mathbb{T}_m\mathcal{E}(\sigma,x,x';\Lambda) = \sum_{d|m}A(N,d)T_{\frac{m}{d}}\mathcal{E}(\sigma,x,dx;\Lambda), \tag{4.69}$$

where $A(N,d)$ is the same in Theorem 3.3.23.

Proof. We first assume that $\Lambda = \mathbb{Z} \oplus \tau\mathbb{Z}$ for some $\tau \in \mathbb{H}$. Then we note that the Hecke operator acting on $\mathcal{E}$ coincides with the Hecke operator acting on $\mathscr{E}$ up to a

scalar. In fact, all the sublattices of $\frac{1}{m}\Lambda$ of index m that contain Λ are given by

$$\Lambda(a,b) := \frac{1}{a}\mathbb{Z} \oplus \frac{a\tau+b}{m}\mathbb{Z},$$

where a runs through all the factors of m and $b = 0, 1, \ldots, d-1$ with $d = \frac{m}{a}$. Then by the definition of Hecke operator and Lemma 4.4.3, we have

$$\begin{aligned} T_m\mathcal{E}(\sigma, x, x'; \Lambda) &= \frac{1}{m}\sum_{a|m}\sum_{b=0}^{d-1}\mathcal{E}(\sigma, x, x'; \Lambda(a,b)) \\ =&\frac{1}{m}\sum_{a|m}\sum_{b=0}^{d-1} a^N \exp\left(\frac{a^2xx'}{A(a\Lambda(a,b))}\right)\mathscr{E}\left(\frac{a\tau+b}{d}, \sigma, ax, ax'\right) \\ =&\exp\left(\frac{mxx'}{A(\Lambda)}\right)T_m\mathscr{E}(\tau, \sigma, x, x'). \end{aligned} \tag{4.70}$$

Here we note that $A(a\Lambda(a,b)) = \frac{a}{d}A(\Lambda)$.

On the other hand, we apply Lemma 4.4.3 again,

$$\begin{aligned} &\sum_{\gamma\in\Gamma_N(m)}\mathcal{E}(m\gamma^{-1}\sigma, x\gamma, m\gamma^{-1}x'; \Lambda) \\ =&\exp\left(\frac{mxx'}{A(\Lambda)}\right)\sum_{\gamma\in\Gamma_N(m)}\mathscr{E}(\tau, m\gamma^{-1}\sigma, x\gamma, m\gamma^{-1}x'). \end{aligned} \tag{4.71}$$

Thus the formula follows from Theorem 3.3.20.

Now we consider the general case. We choose generators u, v of Λ, i.e $\Lambda = u\mathbb{Z}\oplus v\mathbb{Z}$ with $\mathrm{Im}(v/u) > 0$. Let $\tau = v/u$, and we write $\Lambda_\tau = \mathbb{Z}\oplus\tau\mathbb{Z} = \frac{1}{u}\Lambda$. Then we apply Lemma 3.3.20, we have

$$T_m\mathcal{E}(\sigma, x, x'; \Lambda) = \frac{1}{m}\sum_{\Lambda'}\mathcal{E}(\sigma, x, x'; \Lambda') = \frac{1}{u^N}T_m\mathcal{E}\left(\sigma, \frac{x}{u}, \frac{x'}{u}; \Lambda_\tau\right), \tag{4.72}$$

and

$$\mathbb{T}_m\mathcal{E}(\sigma, x, x; \Lambda) = \sum_{\gamma\in\Gamma_N(m)}\mathcal{E}(m\gamma^{-1}\sigma, x\gamma, m\gamma^{-1}x'; \Lambda) = \frac{1}{u^N}\mathbb{T}_m\mathscr{E}\left(\sigma, \frac{x}{u}, \frac{x'}{u}; \Lambda_\tau\right). \tag{4.73}$$

Then the theorem follows from the special case for Λ_τ. □

Chapter

5 Relation With Modular Forms

In this chapter, we will define a function $\mathcal{B}(\tau, \sigma, M, X, y, T)$ which is shown to be an $(N-1)$-cocycle for $\mathrm{GL}_N(\mathbb{Z})$ in Proposition 5.2.4, where $\sigma \in M_N(\mathbb{Z})$, M is an anti-symmetric matrix, $X = (X_1, \cdots, X_N)$ and $y, T \in \mathbb{C}$. We define it in terms of the elliptic cocycle $\mathscr{E}$ introduced in Chapter 2. Our function $\mathcal{B}$ in dimension 2 is a homogeneous analogue of $C(\tau, X, Y, T)$ defined by Zagier [**Zag91**]. One of the main result in this chapter is Theorem 5.2.6 where we give the Laurent expansion of $\mathcal{B}$ in T. Moreover, the coefficients of T^m are linear combinations of Hecke eigenforms with coefficients that are rational functions $P_f(\sigma, M, X, y)$. Then we define an operator $\mathbb{T}_m$ on $P_f(\sigma, M, X, y)$ that is siimilar to the operator $\mathbb{T}_m$ defined in Chapter 3. By applying Theorem 3.3.23, we get another main result of this chapter: Theorem 5.2.10 shows that the coefficients of the rational function $P_f(\sigma, M, X, y)$ in the Laurent expansion at $y = 0$ are eigenvectors with respect to the operator $\mathbb{T}_m$. We can make up an L-series from these eigenvalues. We give an explicit formula for this L-series in terms of the Riemann ζ function and the L-functions associated to modular forms in Theorem 5.3.2. At last, we give some examples. We calculate the explicit formula of $P_f(\sigma, M, X, y)$ for the Eisenstein series and the Ramanujan Delta function in some cases. For future research, we expect to find a polynomial analog $Q_f$of P_f. We explore this question numerically in low weight and present some computational results in Section 5.5. By contrast, we compare the associated L-functions for P_f and Q_f.

5.1 Zagier's results

In this section, we recall a result of Zagier [**Zag91**]. It shows that a product of two Kronecker theta functions encodes all period polynomials of modular forms in the level 1 case.

Let f be a cusp form of weight k on $\mathrm{SL}_2(\mathbb{Z})$. The period polynomial of f is a

polynomial of degree $k-2$ defined by

$$r_f(x) = \int_0^{i\infty} f(\tau)(\tau - x)^{k-2}\, d\tau.$$

If f is a Eisenstein series of weight k on $\mathrm{SL}_2(\mathbb{Z})$, then the integral above diverges, hence we need to modify the definition. Let

$$f(\tau) = \sum_{n=0}^{\infty} a_n q^n.$$

For $\mathrm{Re}(s) \gg 0$, we define

$$L^*(f,s) := \int_0^{\infty} (f(iy) - a_0) y^{s-1}\, dy = (2\pi)^{-s}\Gamma(s)L(f,s),$$

where $L(f,s) = \sum_{n=1}^{\infty} a_n n^{-s}$. Then we define r_f by

$$r_f(x) = \frac{a_0}{k-1}(x^{k-1} + x^{-1}) + \sum_{n=0}^{k-2} i^{1-n}\binom{k-2}{n} L^*(f, n+1) x^{k-2-n}.$$

In this case, $r_f(x)$ is a rational function. For any modular form f, let

$$(r_f(x)r_f(y))^- = \frac{1}{2}(r_f(x)r_f(y) - r_f(-x)r_f(-y)).$$

We define

$$c_k(\tau, x, y) = \sum_{\substack{f \text{weight } k \\ \text{eigenform}}} \frac{(r_f(x)r_f(y))^-}{(2i)^{k-3}(f,f)} f(\tau),$$

where the summation over all normalized eigenforms of weight k, both Eisenstein series and Hecke eigenforms. Here (f,f) is the Petersson product if f is a cusp form, and when $f = G_k$, it means

$$(G_k, G_k) = \frac{\pi(k-1)!}{3(4\pi)^k} \mathrm{Res}_{s=k}\left(\sum_{n\geq 1} \frac{a_{G_k}(n)\overline{a_{G_k}(n)}}{n^s}\right) = \frac{(k-2)!B_k}{2k(4\pi)^k}\zeta(k-1). \quad (5.1)$$

Zagier considered the following generating function:

$$C(\tau, x, y, T) = \frac{(xy-1)(x+y)}{x^2y^2}(2\pi i T)^{-2} + \sum_{k=2}^{\infty} c_k(\tau, x, y)\frac{(2\pi i T)^{k-2}}{(k-2)!}.$$

He obtained the following closed form for it in terms of the Kronecker theta function:

Theorem 5.1.1 (Zagier)**.** *The function $C(\tau, x, y, T)$ is given by*

$$(2\pi i)^2 C(\tau, x, y, T) = \mathcal{K}(\tau, xT, yT)\mathcal{K}(\tau, -xyT, T).$$

By the period relation:

$$r_f|(1+S) = r_f|(1+U+U^2) = 0, \tag{5.2}$$

where $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $U = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$, one deduces the identities:

$$C(\tau, x, y, T) + C\left(\tau, -\frac{1}{x}, y, xT\right) = 0, \tag{5.3}$$

$$C(\tau, x, y, T) + C\left(\tau, 1-\frac{1}{x}, y, xT\right) + C\left(\tau, \frac{1}{1-x}, y, (1-x)T\right) = 0. \tag{5.4}$$

5.2 Generalization to higher dimensions

In this section, we will define a new cocycle $\mathcal{B}$ which generalize the function $C(\tau, x, y, T)$ to dimension N. We will get a cocycle relation in Proposition 5.2.4. In Example 5.2.5, we will see that (5.3) and (5.4) are just special cases of the Proposition 5.2.4. We will give the Laurent expansion of the cocycle $\mathcal{B}$ in Theorem 5.2.6 whose coefficients are certain rational functions $P_f(\sigma, M, X, y)$. Then we will prove the Hecke equivariance of $P_f(\sigma, M, X, y)$ in Theorem 5.2.9. Such theorem is a generalization of the Hecke equivariance of period polynomials introduced in Theorem 0.0.2. At last, we will prove some non-vanishing properties of $P_f(\sigma, M, X, y)$ in subsection 5.2.3.

5.2.1 The cocycle $\mathcal{B}$

We first construct a new cocycle from the elliptic cocycle $\mathscr{E}$ and give the Laurent expansion of it. Let $\mathcal{M}$ be the set of matrices of the following form

$$\begin{pmatrix} 0 & m_{12} & \cdots & m_{1N} \\ m_{21} & 0 & \cdots & m_{2N} \\ \vdots & \vdots & & \vdots \\ m_{N1} & m_{N2} & \cdots & 0 \end{pmatrix},$$

with $m_{ij} + m_{ji} = 0$ for all $1 \le i < j \le N$ and satisfies one of the following conditions

1. M is invertible,

2. The entries m_{ij} $1 \leq i < j \leq N$ are linear independent over $\mathbb{Q}$.

The condition $m_{ij} + m_{ji} = 0$ for all $1 \leq i < j \leq N$ is equivalent to say that M is anti-symmetric, i.e.

$$M + M^t = 0. \tag{5.5}$$

Lemma 5.2.1. *For any $g \in \mathrm{SL}_N(\mathbb{Z})$ and $M \in \mathcal{M}$, we have $gMg^t \in \mathcal{M}$.*

Proof. Since

$$gMg^t + (gMg^t)^t = g(M + M^t)g^t = 0, \tag{5.6}$$

the anti-symmetric property is stable. If M is invertible, then it is clear that gMg^t is invertible for all $g \in \mathrm{SL}_N(\mathbb{Z})$. Hence we only need to check that another condition is stable. We note that $\mathrm{SL}_N(\mathbb{Z})$ is generated by the matrices $\mathrm{Id} + E_{ij}$ $1 \leq i \neq j \leq N$ where E_{ij} is matrix with 1 at the position (i, j) and 0 elsewhere. Thus we only need to check the stability under the generators. We check it for $\mathrm{Id} + E_{12}$ now. It is easy to see that

$$(\mathrm{Id} + E_{12})M(\mathrm{Id} + E_{12})^t = \begin{pmatrix} 0 & m_{12} & m_{13} + m_{23} & \cdots & m_{1N} + m_{2N} \\ -m_{12} & 0 & m_{23} & \cdots & m_{2N} \\ \vdots & \vdots & \vdots & & \vdots \\ -m_{1N} - m_{2N} & -m_{2N} & -m_{3N} & \cdots & 0 \end{pmatrix}. \tag{5.7}$$

Hence the entries of upper triangular are linear independent over $\mathbb{Q}$. This completes the proof. □

Remark 5.2.2. *The conditions on M ensure that all the rows of gMg^t are non-zero for any $g \in \mathrm{SL}_N(\mathbb{Z})$.*

Definition 5.2.3. *Let $X = (X_1, \ldots, X_N)$, $y \in \mathbb{C}$, $\sigma \in M_N(\mathbb{Z})$ and $M \in \mathcal{M}$, we define*

$$\mathcal{B}(\tau, \sigma, M, X, y, T) = \frac{1}{(2\pi i)^N} \mathscr{E}(\tau, \sigma, XT, MX^t yT).$$

Following the cocycle relation of $\mathscr{E}$, we have

Proposition 5.2.4. *The function $\mathcal{B}(\tau, \sigma, M, X, y, T)$ satisfies the following cocycle relation: for any vectors $\sigma_0, \sigma_1, \ldots, \sigma_N \in \mathbb{Z}^N \backslash 0$, we have*

$$\sum_{i=0}^{N} (-1)^i \mathcal{B}(\tau, (\sigma_0, \ldots, \hat{\sigma}_i, \ldots, \sigma_N), M, X, y, T) = 0. \tag{5.8}$$

Proof. This follows directly from Theorem 2.2.13. □

Example 5.2.5. *When $N = 2$, by Theorem 5.1.1, we see that*

$$\mathcal{B}\left(\tau, \mathrm{Id}, S, (X_1, X_2), y, T\right) = C\left(\tau, \frac{X_2}{X_1}, y, X_1 T\right). \tag{5.9}$$

In fact, by the Definition 5.2.3, we have

$$\begin{aligned}\mathcal{B}\left(\tau, \mathrm{Id}, S, (X_1, X_2), y, T\right) &= \frac{1}{(2\pi i)^2}\mathscr{E}(\tau, \mathrm{Id}, (X_1T, X_2T), (-yX_2T, yX_1T)^t)\\ =&\frac{1}{(2\pi i)^2}\mathcal{K}(\tau, X_1T, -yX_2T)\mathcal{K}(\tau, X_2T, yX_1T) = C\left(\tau, \frac{X_2}{X_1}, y, X_1T\right).\end{aligned}$$

Hence we see that $\mathcal{B}(\tau, \mathrm{Id}, S, X, y, T)$ is a homogeneous version of $C(\tau, X, Y, T)$.

Now we could recover the identities (5.3) and (5.4) by the cocycle relation (5.8). Let $\sigma_0 = (1, 0)^t, \sigma_1 = (0, 1)^t, \sigma_2 = (-1, 0)^t$, then the cocycle relation (5.8) gives

$$\begin{aligned}0 &= \mathcal{B}(\tau, \mathrm{Id}, S, (1, X_2), y, T) + \mathcal{B}(\tau, S, S, (1, X_2), y, T)\\ &= \mathcal{B}(\tau, \mathrm{Id}, S, (1, X_2), y, T) + \mathcal{B}(\tau, \mathrm{Id}, S, (-X_2, 1), y, T)\\ &= C(\tau, X_2, y, T) + C\left(\tau, -\frac{1}{X_2}, y, X_2T\right).\end{aligned}$$

The second equality holds since for any $\sigma \in \mathrm{SL}_N(\mathbb{Z})$, we have

$$\mathcal{B}(\tau, \sigma, M, X, y, T) = \mathcal{B}(\tau, \mathrm{Id}, \sigma^{-1}M\sigma^{-t}, X\sigma, y, T),$$

which follows directly from Proposition 2.1.3. As for the identity (5.4), we put $\sigma_0 = (1, 0)^t, \sigma_1 = (0, 1)^t, \sigma_2 = (-1, 1)^t$. Then with the same reasoning as above, we have

$$\begin{aligned}0 &= \mathcal{B}(\tau, \mathrm{Id}, S, (1, X_2), y, T) + \mathcal{B}(\tau, SU^2S, S, (1, X_2), y, T) + \mathcal{B}(\tau, SUS, S, (1, X_2), y, T)\\ &= \mathcal{B}(\tau, \mathrm{Id}, S, (1, X_2), y, T) + \mathcal{B}(\tau, \mathrm{Id}, S, (1 - X_2, X_2), y, T) + \mathcal{B}(\tau, \mathrm{Id}, S, (X_2, X_2 - 1), y, T)\\ &= C(\tau, X_2, y, T) + C\left(\tau, \frac{1}{1 - X_2}, y, (1 - X_2)T\right) + C\left(\tau, \frac{X_2 - 1}{X_2}, y, X_2T\right)\end{aligned}$$

We recall that Proposition 1.2.6 gives the following identity:

$$\mathcal{K}(\tau, xT, x'T) = \frac{x + x'}{xx'T}\exp\left(\sum_{\substack{k\geq 2\\ k \text{ even}}} 2(x^k + x'^k - (x + x')^k)G_k(\tau)\frac{(2\pi iT)^k}{k!}\right).$$

If we write it as a Laurent series of T, then the coefficient of T^{k-1} is of the form

$$\sum_{f \text{ weight } k} P_f(x,x')f(\tau),$$

where the summation over a certain basis of quasimodular forms of weight k and $P(x,x')$ is a rational function. The reason to take the matrix M in $\mathcal{B}(\tau,\sigma,M,X,y,T)$ is to cancel the quasimodular forms. More precisely, we have the following theorem:

Theorem 5.2.6. *The function $\mathcal{B}(\tau,\sigma,M,X,y,T)$ has the following Laurent expansion in T:*

$$\begin{aligned}\mathcal{B}(\tau,\sigma,M,X,y,T) =& P_{-N}(\sigma,M,X,y)(2\pi iT)^{-N} \\ &+ \sum_{k\geq 4}\sum_{\substack{f \text{ eigenform} \\ \text{weight } k}} P_f(\sigma,M,X,y)f(\tau)\frac{(2\pi iT)^{k-N}}{(k-N)!}.\end{aligned} \tag{5.10}$$

where P_{-N} and P_f are rational functions and P_f/P_{-N} are polynomials.

Proof. We first consider the case $\sigma = \mathrm{Id}$. In this case, the function $\mathcal{B}(\tau,\mathrm{Id},M,X,y,T)$ is just a product of N Kronecker theta functions. Hence by the Proposition 1.2.6, we have

$$\begin{aligned}\mathcal{B}(\tau,\mathrm{Id},M,X,y,T) &= \frac{1}{(2\pi i)^N}\prod_{i=1}^N \mathcal{K}(\tau, X_iT, (MX^t)_i yT) \\ =&\frac{1}{(2\pi i)^N}\prod_{i=1}^N \frac{X_i + y(MX^t)_i}{yX_i(MX^t)_iT}\exp\left(\sum_{\substack{k\geq 2\\ k \text{ even}}} 2(X_i^k + y^k(MX^t)_i^k - (X_i+y(MX^t)_i)^k)G_k(\tau)\frac{(2\pi iT)^k}{k!}\right) \\ =&\left(\prod_{i=1}^N \frac{X_i + y(MX^t)_i}{yX_i(MX^t)_i}\right)(2\pi iT)^{-N} \\ &\exp\left(\sum_{\substack{k\geq 2\\ k \text{ even}}} 2\left(\sum_{i=1}^N \left(X_i^k + y^k(MX^t)_i^k - (X_i+y(MX^t)_i)^k\right)\right)G_k(\tau)\frac{(2\pi iT)^k}{k!}\right),\end{aligned} \tag{5.11}$$

where $(MX^t)_i$ means the i-th component of the vector MX^t. To cancel the quasimodular part, we only need to consider the coefficient of T^2 in the exponential. It equals to

$$\sum_{i=1}^N \left(X_i^2 + y^2(MX^t)_i^2 - (X_i + y(MX^t)_i)^2\right)G_2(\tau),$$

up to a scalar. But we note that this equals to

$$\begin{aligned}&(XX^t + (XM^t)(MX^t)y^2 - (X + XM^t y)(X^t + MX^t y))G_2(\tau)\\ =&- X(M + M^t)X^t y G_2(\tau).\end{aligned}$$

Since we require the matrix M to be anti-symmetric, the $G_2(\tau)$ term is cancelled. To complete the proof in the case $\sigma = \mathrm{Id}$, we just need to expand the exponential in T. Now let $\sigma \in \mathrm{GL}_N(\mathbb{Z})$, then by the homogeneous property of $\mathscr{E}$, we have

$$\begin{aligned}&\mathcal{B}(\tau, \sigma, M, X, y, T) = \mathscr{E}(\tau, \sigma, XT, MX^t yT)\\ =&\det(\sigma)\mathscr{E}(\tau, \mathrm{Id}, XT\sigma, \sigma^{-1}MX^t yT)\\ =&\det(\sigma)\mathcal{B}(\tau, \mathrm{Id}, \sigma^{-1}M\sigma^{-t}, X\sigma, y, T).\end{aligned} \tag{5.12}$$

By Lemma 5.2.1, $\sigma^{-1}M\sigma^{-t}$ is still in $\mathcal{M}$. Hence by applying the result for $\sigma = \mathrm{Id}$, we deduce the result for $\sigma \in \mathrm{GL}_N(\mathbb{Z})$.

For the general case, we prove it by induction on $|\det(\sigma)|$. The case $|\det(\sigma)| = 0$ is trivial. The case $|\det(\sigma)| = 1$ has been proven above. By applying Proposition 2.1.7 to a suitable diagonal matrix, we may assume that each column of σ is primitive. Then we can write σ in the form $g\sigma'$ where $g \in \mathrm{GL}_N(\mathbb{Z})$ and $\sigma' = (\sigma'_{ij})$ is a upper triangular matrix with

$$1 \leq \sigma'_{11} \leq \cdots \leq \sigma'_{NN}.$$

With the same reason as (5.12), we have

$$\mathcal{B}(\tau, \sigma, M, X, y, T) = \mathcal{B}(\tau, \sigma', g^{-1}Mg^{-t}, Xg, y, T).$$

Hence we may reduce the problem to the case $\sigma = (\sigma_{ij})$ is an upper triangular matrix with $1 \leq \sigma_{11} \leq \cdots \leq \sigma_{NN}$. We write $\sigma = (\sigma_1, \ldots, \sigma_N)$. Suppose j is the least number such that $\sigma_{jj} > 1$. We set σ_0 to be the vector with 1 at the j-th component and 0 elsewhere. For $0 \leq t \leq N$, let

$$A_t = (\sigma_0, \sigma_1, \cdots, \hat{\sigma}_t, \cdots, \sigma_N).$$

Hence by Proposition 5.2.4, we have

$$\mathcal{B}(\tau, \sigma, M, X, y, T) = \sum_{t=1}^{N}(-1)^{t+1}\mathcal{B}(\tau, A_t, M, X, y, T).$$

Moreover, by the construction, it is easy to see that

$$\det(A_t) = 0 \text{ for } t = 1, \dots, j-1,$$

and

$$|\det(A_t)| \leq \frac{|\det(\sigma)|}{\sigma_{jj}} < |\det(\sigma)| \text{ for } t = j, \dots, N.$$

Hence this theorem follows directly from the induction assumption. □

Remark 5.2.7. *In the following, we always assume that the eigenforms f are normalized. This means if we write*

$$f(\tau) = a_f(0) + a_f(1)q + a_f(2)q^2 + \cdots,$$

we always take $a_f(1) = 1$.

To end this subsection, we extend the definition of P_f to all modular forms by linear extension. We write the modular form f as a linear combination of normalized eigenforms $f = \sum_j c_j f_j$. Then we define P_f by

$$P_f = \sum_j c_j P_{f_j}. \tag{5.13}$$

5.2.2 The Hecke equivariance of P_f

Let $\mathcal{F}(\mathcal{M} \times \mathbb{C}^N \times \mathbb{C})$ be the set of functions on $\mathcal{M} \times \mathbb{C}^N \times \mathbb{C}$. Then $P_f(\sigma, M, X, y) \in \mathcal{F}(\mathcal{M} \times \mathbb{C}^N \times \mathbb{C})$. We can deduce an action of $\mathrm{GL}_N(\mathbb{Q})$ on $\mathcal{F}(\mathcal{M} \times \mathbb{C}^N \times \mathbb{C})$ from the action on $\mathcal{B}(\tau, \sigma, M, X, y, T)$ which is given by

$$g \cdot (P_f(\sigma, M, X, y)) = P_f\left(\sigma, \det(g) g^{-1} M g^{-t}, Xg, y\right). \tag{5.14}$$

With the notation in section 3.1, we have :

Lemma 5.2.8. *For any modular form f, $P_f(\sigma, M, X, y) \in \mathcal{P}_h(\mathcal{F}(\mathcal{M} \times \mathbb{C}^N \times \mathbb{C}))$.*

Proof. This follows from Proposition 2.1.3. In fact, Proposition 2.1.3 tells us for any $g \in \mathrm{SL}_N(\mathbb{Z})$,

$$\mathscr{E}(\tau, g\sigma, xg^{-1}, gx') = \mathscr{E}(\tau, \sigma, x, x'). \tag{5.15}$$

Hence for any $g \in \mathrm{SL}_N(\mathbb{Z})$,

$$\mathcal{B}(\tau, g\sigma, gMg^t, Xg^{-1}, y, T) = \mathcal{B}(\tau, \sigma, M, X, y, T). \tag{5.16}$$

Then the homogeneous property of $P_f(\sigma, M, X, y)$ follows directly when f is a normalized eigenform. For an arbitrary modular form f, the homogeneous property of $P_f(\sigma, M, X, y)$ follows by linearity. $\square$

Now we can prove the Hecke equivariance for the rational function $P_f(\sigma, M, X, y)$:

Theorem 5.2.9. *Let $f(\tau)$ be a modular form. Then for any $\sigma \in M_N(\mathbb{Z})$ and $M \in \mathcal{M}$, we have*

$$\mathbb{T}_m P_f(\sigma, M, X, y) = \sum_{d|m} A(N, d) P_{T_{\frac{m}{d}} f}(\sigma, M, X, dy), \tag{5.17}$$

where $A(N, d)$ is defined in Theorem 3.3.23.

Proof. We first consider the case of normalized eigenform. Similar to the elliptic cocycle $\mathscr{E}$, we can define the two kinds of Hecke operators by

$$T_m \mathcal{B}(\tau, \sigma, M, X, y, T) = m^{N-1} \sum_{\substack{a,d>0 \\ ad=m}} \frac{1}{d^N} \sum_{b=0}^{d-1} \mathcal{B}\left(\frac{a\tau + b}{d}, \sigma, M, aX, y, T\right), \tag{5.18}$$

and

$$\mathbb{T}_m \mathcal{B}(\tau, \sigma, M, X, y, T) = \sum_{\gamma \in \Gamma_N(m)} \mathcal{B}(\tau, m\gamma^{-1}\sigma, m\gamma^{-1}M\gamma^{-t}, X\gamma, y, T). \tag{5.19}$$

The Theorem 3.3.23 gives the formula:

$$\mathbb{T}_m \mathcal{B}(\tau, \sigma, M, X, y, T) = \sum_{d|m} A(N, d) T_{\frac{m}{d}} \mathcal{B}(\tau, \sigma, M, X, dy, T). \tag{5.20}$$

Let $f(\tau)$ be a normalized eigenform with Fourier expansion $\sum_{m \geq 0} a_f(m) q^m$. Then $T_m f(\tau) = a_f(m) f(\tau)$. Hence by Theorem 5.2.6, we have

$$\begin{aligned}
&T_m \mathcal{B}(\tau, \sigma, M, X, y, T) \\
=&\sigma_{N-1}(m) P_{-N}(\sigma, M, X, y)(2\pi i T)^{-N} + \sum_{k \geq 4} \sum_{\substack{f \text{ eigenform} \\ \text{weight } k}} P_f(\sigma, M, X, y) T_m f(\tau) \frac{(2\pi i T)^{k-N}}{(k-N)!} \\
=&\sigma_{N-1}(m) P_{-N}(\sigma, M, X, y)(2\pi i T)^{-N} + \sum_{k \geq 4} \sum_{\substack{f \text{ eigenform} \\ \text{weight } k}} a_f(m) P_f(\sigma, M, X, y) f(\tau) \frac{(2\pi i T)^{k-N}}{(k-N)!}.
\end{aligned} \tag{5.21}$$

On the other hand,

$$\begin{aligned}
&\mathbb{T}_m\mathcal{B}(\tau,\sigma,M,X,y,T)\\
&= \sum_{\gamma\in\Gamma_N(m)} P_{-N}(m\gamma^{-1}\sigma, m\gamma^{-1}M\gamma^{-t}, X\gamma, y)(2\pi iT)^{-N}\\
&\quad + \sum_{k\geq 4}\sum_{\substack{f \text{ eigenform}\\ \text{weight } k}}\sum_{\gamma\in\Gamma_N(m)} P_f(m\gamma^{-1}\sigma, m\gamma^{-1}M\gamma^{-t}, X\gamma, y) f(\tau)\frac{(2\pi iT)^{k-N}}{(k-N)!}.
\end{aligned} \tag{5.22}$$

By comparing the coefficients of T^{k-N} of equations (5.21) and (5.22), we prove the formula (5.17) for normalized eigenform.

The general case for an arbitrary modular form follows by linearity. □

However, when $N \geq 3$, the rational function $P_f(\sigma, M, X, y)$ is not an eigenvector of the operator $\mathbb{T}_m$ when f is an eigenform because of the factor d. Hence we consider the Laurent expansion of $P_f(\sigma, M, X, y)$ in y:

$$P_f(\sigma, M, X, y) = \sum_{t\geq -N} P_f^{(t)}(\sigma, M, X)y^t.$$

Then following Theorem 5.2.9, we have:

Theorem 5.2.10. *Let $f(\tau)$ be a modular form and t be an integer. Then for any $\sigma \in M_N(\mathbb{Z})$ and $M \in \mathcal{M}$, we have*

$$\mathbb{T}_m P_f^{(t)}(\sigma, M, X) = \sum_{d|m} A(N,d)d^t P^{(t)}_{T_{\frac{m}{d}}f}(\sigma, M, X). \tag{5.23}$$

In particular, if f is a normalized eigenform, and if the function $P_f^{(t)}(\sigma, M, X)$ is non-zero, then it is an eigenvector of $\mathbb{T}_m$ with eigenvalue $\sum_{d|m} A(N,d)a_f\left(\frac{m}{d}\right)d^t$, i.e.

$$\mathbb{T}_m P_f^{(t)}(\sigma, M, X) = \left(\sum_{d|m} A(N,d)a_f\left(\frac{m}{d}\right)d^t\right) P_f^{(t)}(\sigma, M, X).$$

5.2.3 Non-vanishing of $P_f^{(t)}(\sigma, M, X)$

A natural question is how many non-zero eigenvectors there are? Hence in this subsection, we will discuss the range of t such that $P_f^{(t)}(\sigma, M, X)$ is non-zero. First of all, the following proposition shows that there are only finitely many integers t such that $P_f^{(t)}(\sigma, M, X)$ is non-zero.

Proposition 5.2.11. *Let $f(\tau)$ be a modular form of weight k. If $f(\tau)$ is an Eisenstein series, then the function $P_f^{(t)}(\sigma, M, X)$ vanishes for $t \geq k$ and for $t \leq -N$. If $f(\tau)$ is a cusp form, then the function $P_f^{(t)}(\sigma, M, X)$ vanishes for $t \geq k-1$ and for $t \leq -N+1$.*

Proof. It is easy to see that we only need to prove the case f is a normalized eigenform. By the same method as Theorem 5.2.6, we reduce to the case $\sigma = \mathrm{Id}$. Let

$$F_l(M, X, y) = \sum_{i=1}^{N} X_i^l + y^l (MX^t)_i^l - (X_i + y(MX^t)_i)^l.$$

Then the equation (5.11) gives

$$\begin{aligned} &\mathcal{B}(\tau, \mathrm{Id}, M, X, y, T) \\ &= \left(\prod_{i=1}^{N} \frac{X_i + y(MX^t)_i}{yX_i(MX^t)_i} \right) (2\pi i T)^{-N} \exp\left(\sum_{\substack{k \geq 2 \\ k \text{ even}}} 2F_k(M, X, y) G_k(\tau) \frac{(2\pi i T)^k}{k!} \right). \end{aligned} \tag{5.24}$$

We expand the exponential, then

$$P_{-N}(\mathrm{Id}, M, X, y) = \prod_{i=1}^{N} \frac{X_i + y(MX^t)_i}{yX_i(MX^t)_i}, \tag{5.25}$$

and $P_f(\mathrm{Id}, M, X, y)/P_{-N}(\mathrm{Id}, M, X, y)$ is a linear combination of the form

$$\prod_{j=1}^{s} F_{l_j}(M, X, y)$$

where $l_1 + \cdots + l_s = k$. If we view $F_l(M, X, y)$ as a polynomial in y, we see that the constant term and the coefficient of y^l vanish. Hence it is of degree $l-1$. On the other hand, $P_{-N}(\mathrm{Id}, M, X, y)$ is a polynomial in $1/y$ of degree N. This implies that $P_f^{(t)}(\sigma, M, X) = 0$ for $t \geq k$ and for $t \leq -N$.

Moreover, if $f(\tau)$ is a cusp form, we note that it always comes from the product of at least 2 Eisenstein series. This means that $s \geq 2$. Hence if we write $\prod_{j=1}^{s} F_{l_j}(M, X, y)$ as a polynomial in y, then the coefficients of y^0, y^1, y^{k-1}, y^k vanish. This completes the proof.

□

The following proposition shows that there exists at least one integer t such

that $P_f^{(t)}(\mathrm{Id}, M, X)$ is non-zero. In the following, we will denote $P_f^{(t)}(\mathrm{Id}, M, X)$ by $P_f^{(t)}(M, X)$.

Proposition 5.2.12. *Let $f(\tau)$ be a normalized eigenform of weight k and $M \in \mathcal{M}$. Then*

1. *When $f(\tau)$ is an Eisenstein series, $P_f^{(t)}(M, X)$ is non-zero for $-1 \leq t \leq k-1$ with t odd and for $t = 0, k-2$.*

2. *When $f(\tau)$ is a cusp form, $P_f^{(t)}(M, X)$ is non-zero for $0 \leq t \leq k-2$, $t \neq \frac{k}{2} - 1$.*

Proof. The idea of this proof is to find at least one term of $P_f^{(t)}(M, X)$ is non-zero. We consider the Laurent expansion of $P_f^{(t)}(M, X)$ to reduce the problem to dimension 2. Then we prove it by showing the non-vanishing of the coefficients of period polynomials.

Since we require that either the upper triangular entries of M are linear independent over $\mathbb{Q}$ or M is invertible, it forces that there exists at least one non-zero entries for each row. Without loss of generality, we assume that $m_{12} \neq 0$. We consider the coefficient of $(X_3 \cdots X_N)^{-1}$ of $B(\tau, \mathrm{Id}, M, X, y, T)$, which equals to

$$\frac{1}{T^{N-2}} \mathcal{K}(\tau, X_1, m_{12} X_2 y T) \mathcal{K}(\tau, X_2, -m_{12} X_1 y T). \tag{5.26}$$

Hence by the Theorem 5.1.1, we see that up to a non-zero scalar, the coefficient of $(X_3 \cdots X_N)^{-1}$ in $P_f(\mathrm{Id}, M, X, y)$ equals to

$$X_1^{k-2} \left(r_f \left(\frac{X_2}{X_1} \right) r_f(-m_{12} y) \right)^{-}. \tag{5.27}$$

Hence when t is an odd number, then

$$P_f^{(t)}(M, X) = \frac{(-m_{12})^t r_f^{(t)} r_f^{+} \left(\frac{X_2}{X_1} \right) X_1^{k-2}}{X_3 \cdots X_N} + \ldots, \tag{5.28}$$

where $r_f^{(t)}$ is the t-th coefficient of $r_f(x)$ and $r_f^{+}(x)$ is the even part of $r_f(x)$. When t is an even number then

$$P_f^{(t)}(M, X) = \frac{(-m_{12})^t r_f^{(t)} r_f^{-} \left(\frac{X_2}{X_1} \right) X_1^{k-2}}{X_3 \cdots X_N} + \ldots, \tag{5.29}$$

where $r_f^{-}(x)$ is the odd part of $r_f(x)$. So the function $P_f^{(t)}(M, X)$ is non-zero when

$r_f^{(t)}$ is non-zero. If $f(\tau) = G_k(\tau)$, we know that

$$r_{G_k} = \omega_{G_k}^- p_k^- + \omega_{G_k}^+ p_k^+,$$

where

$$p_k^+(x) = x^{k-2} - 1, \quad p_k^-(x) = \sum_{\substack{-1 \le n \le k-1 \\ n \text{ odd}}} \frac{B_{n+1} B_{k-n-1}}{(n+1)!(k-n-1)!} x^n,$$

and

$$\omega_{G_k}^- = -\frac{(k-2)!}{2}, \quad \omega_{G_k}^+ = \frac{\zeta(k-1)}{(2\pi i)^{k-1}} \omega_{G_k}^-.$$

Hence the non-vanishing of $P_{G_k}^{(t)}(M, X)$ is clear. If $f(\tau)$ is a normalized Hecke eigenform, it is well-known that the associated L-function $L(f, s)$ doesn't vanish at $s = 1, 2, \cdots, \frac{k}{2} - 1, \frac{k}{2} + 1, \cdots, k - 1$. This means that the s-th coefficient of $r_f(x)$ is non-zero for $s = 0, 1, \cdots, \frac{k}{2} - 2, \frac{k}{2}, \cdots, k - 2$. Then the non-vanishing property of $P_f^{(t)}(M, X)$ is clear.

□

With more elaborate analysis, we can prove more non-zero terms:

Proposition 5.2.13. *Let f be a normalized eigenform of weight k and $M \in \mathcal{M}$, then for any integer $N \ge 3$,*

1. *If $f = G_k$, then $P_{G_k}^{(1-N)}(M, X)$ is non-zero,*

2. *if f is a cusp form, then $P_f^{(2-N)}(M, X)$ is non-zero.*

Proof. The idea of the proof is to find at least one non-zero term of $P_{G_k}^{(t)}(M, X)$. Following the equation (5.25), we see that

$$P_f^{(t)}(M, X) \prod_{i=1}^{N} X_i (M X^t)_i \tag{5.30}$$

is a polynomial in $X_1, \cdots, X_N$. If we view it as a polynomial in X_1, we will prove that the coefficient of X_1^{k-1} is non-zero.

With the notation in Proposition 5.2.11, $P_f(\mathrm{Id}, M, X, y)/P_{-N}(\mathrm{Id}, M, X, y)$ is a linear combination of functions of the form:

$$\prod_{j=1}^{s} F_{l_j}(M, X, y),$$

where $l_1 + \cdots + l_s = k$ and

$$F_l(M, X, y) = \sum_{i=1}^{N} X_i^l + y^l (MX^t)_i^l - (X_i + y(MX^t)_i)^l. \tag{5.31}$$

If $f = G_k$, then only the term $F_k(M, X, y)$ contributes to $P_{G_k}^{(1-N)}(M, X)$. This implies that

$$P_{G_k}^{(1-N)}(M, X) = 2 \prod_i \frac{1}{(MX^t)_i} \frac{F_k^{(1)}(M, X)}{k!}, \tag{5.32}$$

where

$$F_k^{(1)}(M, X) = -k \sum_{j=1}^{N} X_j^{k-1} (MX^t)_j \tag{5.33}$$

is the coefficient of y of $F_k(M, X, y)$. Hence It is easy to see that $P_{G_k}^{(1-N)}(M, X)$ is non-zero.

If f is a cusp form, then we see that only the term of $\prod_{j=1}^{s} F_{l_j}(M, X, y)$ with $s = 2$ contributes to $P_f^{(2-N)}(M, X)$. This implies that

$$P_f^{(2-N)}(M, X) = 2 \prod_i \frac{1}{(MX^t)_i} \sum_{\substack{h=4 \\ even}}^{k-4} F_h^{(1)}(M, X) F_{k-h}^{(1)}(M, X) \frac{(G_h G_{k-h}, f)}{h!(k-h)!}, \tag{5.34}$$

where $(G_h G_{k-h}, f)$ is the Petersson product of $G_h G_{k-h}$ and f. This product was calculated by Rankin in [**Ran82**]:

$$(G_h G_{k-h}, f) = \frac{1}{(2i)^{k-1}} \binom{k-2}{h-1}^{-1} r_f^{(k-2)} r_f^{(h-1)}, \tag{5.35}$$

where $r_f^{(n)}$ is the n-th coefficient of r_f. By the definition (5.31), we see that

$$F_l^{(1)}(M, X) = l(MX^t)_1 X_1^{l-1} + (\text{ lower terms in } X_1). \tag{5.36}$$

Hence we have

$$\begin{aligned} & P_f^{(2-N)}(M, X) \prod_i (MX^t)_i \\ =& 2 \sum_{\substack{h=4 \\ even}}^{k-4} \frac{h(k-h) r_f^{(k-2)} r_f^{(h-1)}}{(2i)^{k-1} h!(k-h)! \binom{k-2}{h-1}} (MX^t)_1^2 X_1^{k-2} + (\text{ lower terms in } X_1) \end{aligned} \tag{5.37}$$

On the other hand, the definition of period polynomial gives

$$r_f^-(x) = -\sum_{\substack{h=1\\ odd}}^{k-3} r_f^{(h)} x^h. \tag{5.38}$$

We note that 1 is a root of $r_f^-(x)$. In fact, the cocycle relation (5.2) implies that

$$r_f(x) + x^{k-2} r_f\left(-\frac{1}{x}\right) = r_f(x) + x^{k-2} r_f\left(\frac{x-1}{x}\right) + (x-1)^{k-2} r_f\left(-\frac{1}{x-1}\right) = 0. \tag{5.39}$$

We take $x = 1$, then it gives

$$r_f(1) + r_f(-1) = r_f(1) + r_f^{(0)} + r_f^{(k-2)} = 0. \tag{5.40}$$

But the functional equation of $L(f,s)$ shows that $r_f^{(0)} + r_f^{(k-2)} = 0$. Hence 1 and -1 are the roots of $r_f(x)$. Then we have

$$\sum_{\substack{h=4\\ even}}^{k-4} \frac{h(k-h) r_f^{(k-2)} r_f^{(h-1)}}{(2i)^{k-1} h!(k-h)! \binom{k-2}{h-1}} = \frac{-2 r_f^{(k-2)} r_f^{(1)}}{(2i)^{k-1}(k-2)!}. \tag{5.41}$$

This number is non-zero since f is an Hecke eigenform. Hence

$$P_f^{(2-N)}(M,X) \prod_i (MX^t)_i \neq 0. \tag{5.42}$$

This proves the non-vanishing of $P_f^{(2-N)}(M,X)$. □

According to Theorem 5.2.10, we see that the vector space generated by $P_f^{(t)}(M,X)$

$$\langle P_f^{(t)}(M,X) | M \in \mathcal{M} \rangle$$

is an eigenspace of $\mathbb{T}_m$ with eigenvalue $\sum_{d|m} A(N,d) a_f\left(\frac{m}{d}\right) d^t$. We denote such space by $V_f^{(t)}$. So a natural question is how large is this eigenspace? In fact, we have the following results:

Proposition 5.2.14. *We suppose $N = 2$. Let $f(\tau)$ be a normalized eigenform of weight k. Then*

1. *When $f = G_k$, then $V_f^{(t)}$ is one dimensional for $t = -1, 1, \cdots, k-1$ and for $t = 0, k-2$. $V_f^{(t)}$ is zero for other t.*

2. *When f is a cusp form, then $V_f^{(t)}$ is one dimensional for $t = 0, 1, \cdots, \frac{k}{2} - 2, \frac{k}{2}, \cdots, k-3, k-2$. $V_f^{(t)}$ is zero for $t > k-2$ and for $t < 0$.*

Proof. This proposition follows directly from Proposition 5.2.12 since the matrix M is unique up to a scalar. □

Proposition 5.2.15. *We suppose $N \geq 3$. Let $f(\tau)$ be a normalized eigenform of weight k. Then*

1. *When $f(\tau)$ is an Eisenstein series, the eigenspace $V_f^{(t)}$ is of infinite dimensional for $-2 \leq t \leq k-2$ with t even and for $t = -1, k-3$;*

2. *When $f(\tau)$ is a cusp form, the eigenspace $V_f^{(t)}$ is of infinite dimensional for $-1 \leq t \leq k-3$ except $t = \frac{k}{2} - 2$.*

Proof. The idea of the proof is to find a family of matrices $M \in \mathcal{M}$ such that $P_f^{(t)}(M, X)$ has exactly one different pole. Then this family of rational function $P_f^{(t)}(M, X)$ must be linearly independent. Let $M = (m_{ij}) \in \mathcal{M}$. We consider the coefficient of $(X_3 \dots X_{N-1}(MX^t)_N)^{-1}$ in $\mathcal{B}(\tau, \mathrm{Id}, M, X, y, T)$. It equals to

$$\frac{1}{yT}\mathcal{K}\left(\tau, X_1T, \frac{m_{1N}}{m_{2N}}(m_{2N}X_N - m_{12}X_1)yT\right)\mathcal{K}\left(\tau, -\frac{m_{1N}}{m_{2N}}X_1T, (m_{2N}X_N - m_{12}X_1)yT\right).$$

By the Theorem 5.1.1, it equals to

$$C\left(\tau, -\frac{m_{1N}}{m_{2N}}, \frac{(m_{2N}X_N - m_{12}X_1)y}{X_1}, X_1T\right). \tag{5.43}$$

Then the relation (5.9) shows that the coefficient of $(X_3 \dots X_{N-1}(MX^t)_N)^{-1}$ in $P_f(\mathrm{Id}, M, X, y)$ equals to

$$\left(r_f\left(-\frac{m_{1N}}{m_{2N}}\right)r_f\left(\frac{(m_{2N}X_N - m_{12}X_1)y}{X_1}\right)\right)^{-}X_1^{k-2}. \tag{5.44}$$

When we fix $m_{2N}, \dots, m_{N-1,N}$ and let m_{1N} runs through $\mathbb{C}$ such that $r_f\left(-\frac{m_{1N}}{m_{2N}}\right) \neq 0$, then $P_f(\mathrm{Id}, M, X, y)$ always has a different pole at $(MX^t)_N = 0$. With the same argument in Proposition 5.2.12, we can show that $P_f^{(t)}(M, X)$ is non-zero for the t showed above. □

5.3 L-function

Since we have families of eigenvalues, it is natural to consider the associated L-functions. Let t be an integer and f be a normalized eigenform. We denote

$$L_f^{(t)}(s) := \sum_{m\geq 1}\left(\sum_{d|m} A(N,d)a_f\left(\frac{m}{d}\right)d^t\right)m^{-s}. \tag{5.45}$$

Before giving properties of $L_f^{(t)}(s)$, we first establish an identity on $H(N,p^j) = \prod_{i=1}^{j}\frac{p^{N+i-1}-1}{p^i-1}$ given in Lemma 3.24.

Lemma 5.3.1. *For any positive integer N and prime number p, we have the formula:*

$$\prod_{j=0}^{N-1}\frac{1}{1-p^jt} = \sum_{j=0}^{\infty}H(N,p^j)t^j. \tag{5.46}$$

Proof. We prove it by induction on N. When $N=1$, we know that $H(1,p^j)=1$ for any $j\geq 0$. Then the identity (5.46) is clear.

Now we assume that $N\geq 2$. Then by the induction assumption, we have

$$\begin{aligned}\prod_{j=0}^{N-1}\frac{1}{1-p^jt} &= \sum_{j=0}^{\infty}H(N-1,p^j)t^j\sum_{j'=0}^{\infty}p^{(N-1)j'}t^{j'}\\ &= \sum_{j=0}^{\infty}\left(\sum_{j'=0}^{j}p^{(N-1)j'}H(N-1,p^{j-j'})\right)t^j.\end{aligned}$$

By the formula (3.25), we have

$$\sum_{j'=0}^{j}p^{(N-1)j'}H(N-1,p^{j-j'}) = H(N,p^j).$$

This completes the proof. □

Now we prove the following theorem.

Theorem 5.3.2. *Let $f(\tau)$ be an eigenform of weight k. Then for $\mathrm{Re}(s) > \max\{k, N+t\}$, $L_f^{(t)}(s)$ converges absolutely. It has a meromorphic continuation to the whole plane. Moreover, we have the decomposition*

$$L_f^{(t)}(s) = L(f,s)\prod_{j=1}^{N-2}\zeta(s-j-t),$$

where $L(f,s)$ is the L-function associated to the modular form f.

Proof. First we note that $H(N,p^j) = O(p^{Nj})$ as $j \to \infty$. Then by definition

$$a(N,p^j) = H(N-1,p^j) - H(N-1,p^{j-1}) = O(p^{(N-1)j}) \text{ as } j \to \infty. \tag{5.47}$$

If we write $m = p_1^{j_1} \cdots p_l^{j_l}$ where $p_1, \cdots, p_l$ are different prime factors of m, then

$$A(N,m) = a(N,p_1^{j_1}) \cdots a(N,p_l^{j_l}). \tag{5.48}$$

Hence $A(N,m) = O(m^{N-1})$ as $m \to \infty$. Since $f(\tau)$ is a modular form of weight k, then $a_f(m) = O(m^{k-1})$. Hence we have

$$O\left(\sum_{d|m} A(N,d) a_f\left(\frac{m}{d}\right) d^t\right) = O\left(m^{k-1}\sigma_{N+t-k}(m)\right) = O\left(m^{\max\{k-1,N+t-1\}}\right).$$

Hence for any $\mathrm{Re}(s) > \max\{k, N+t\}$, $L_f^{(t)}(s)$ converges absolutely. Now we consider the decomposition. By the equation (5.48), we see that for any positive integers m_1, m_2 with $\gcd(m_1,m_2) = 1$, we have $A(N,m_1m_2) = A(N,m_1)A(N,m_2)$. Hence by the definition (5.45), we have

$$\begin{aligned} L_f^{(t)}(s) =& \sum_{d' \geq 1} \frac{a_f(d')}{d'^s} \sum_{d \geq 1} \frac{A(N,d)}{d^{s-t}} \\ =& L(f,s) \prod_p \left(\sum_{j \geq 0} \frac{a(N,p^j)}{p^{j(s-t)}}\right), \end{aligned}$$

where the product runs through all the prime number p.

By Lemma 5.3.1, we have

$$\begin{aligned} \sum_{j \geq 0} \frac{a(N,p^j)}{p^{j(s-t)}} =& 1 + \sum_{j \geq 1} \frac{H(N-1,p^j)}{p^{j(s-t)}} - \sum_{j \geq 1} \frac{H(N-1,p^{j-1})}{p^{j(s-t)}} \\ =& \prod_{j=0}^{N-2} \frac{1}{1-p^{j+t-s}} - \frac{1}{p^{s-t}} \prod_{j=0}^{N-2} \frac{1}{1-p^{j+t-s}} \\ =& \prod_{j=1}^{N-2} \frac{1}{1-p^{j+t-s}}. \end{aligned}$$

This implies that

$$L_f^{(t)}(s) = L(f,s)\prod_p \prod_{j=1}^{N-2} \frac{1}{1-p^{j+t-s}} = L(f,s)\prod_{j=1}^{N-2}\zeta(s-j-t).$$

The meromorphic continuation immediately follows from the meromorphic continuation of ζ and $L(f,s)$. □

5.4 Examples

5.4.1 $N = 2$

In this subsection, we consider the case $N = 2$.

Example 5.4.1. *In this case, the matrix M is unique up to a scalar. Hence we may fix $M = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. If $\sigma = \mathrm{Id}$, then by Theorem 5.1.1, we have*

$$P_f(\mathrm{Id}, M, (X_1, X_2), y) = \frac{\left(r_f\left(\frac{X_2}{X_1}\right) r_f(y)\right)^-}{(2i)^{k-3}(f,f)} X_1^{k-2}, \tag{5.49}$$

where f is a normalized eigenform of weight k. If we consider the coefficients of the Laurent series in y, then up to a scalar, we have

$$P_f^{(t)}(\mathrm{Id}, M, (X_1, X_2)) = r_f^{\pm}\left(\frac{X_2}{X_1}\right) X_1^{k-2} = r_f^{\pm}\{\boldsymbol{e}_1, \boldsymbol{e}_2\}(X_1, X_2), \tag{5.50}$$

up to the parity of t. Moreover, let $\sigma \in M_2(\mathbb{Z})$, if we put α to be the first column of σ, and β to be the second column of σ. Then the condition of modular symbol $r_f\{\alpha, \beta\}$ introduced in the introduction is equivalent to the cocycle relation of $P_f^{(t)}(\sigma, M, (X_1, X_2))$. This implies that

$$P_f^{(t)}(\sigma, M, (X_1, X_2)) = r_f^{\pm}\{\alpha, \beta\}(X_1, X_2), \tag{5.51}$$

up to a scalar.

5.4.2 Eisenstein series for $N = 3$

Example 5.4.2. *When $N = 3$. Let $\sigma = \mathrm{Id}$ and $M = \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix}$. The Proposition 5.2.12 and Proposition 5.2.13 showed that $P^{(t)}_{G_k}$ is non-zero for $-1 \le t \le k-1$ with t odd and for $t = -2, 0, k-2$. For example:*

$$P^{(-2)}_{G_{12}}(M, X) = \frac{(-aX_2 - bX_3)X_1^{11} + (aX_2^{11} + bX_3^{11})X_1 + (-cX_3X_2^{11} + cX_3^{11}X_2)}{2^7 \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 11(aX_2 + bX_3)(-aX_1 + cX_3)(bX_1 + cX_2)}. \tag{5.52}$$

More generally, we can show that

$$P^{(-2)}_{G_k}(M, X) = \frac{2((-aX_2 - bX_3)X_1^{k-1} + (aX_2^{k-1} + bX_3^{k-1})X_1 + (-cX_3X_2^{k-1} + cX_3^{k-1}X_2))}{(k-1)!(aX_2 + bX_3)(-aX_1 + cX_3)(bX_1 + cX_2)}. \tag{5.53}$$

Let's focus on the case $k = 12$. However, the general theorem above doesn't tell if the terms $P^{(2)}_{G_{12}}, P^{(4)}_{G_{12}}, P^{(6)}_{G_{12}}, P^{(8)}_{G_{12}}$ are zero or not. According to the calculation by PARI/GP, we can check that they are non-zero, here we give an example for $P^{(2)}_{G_{12}}$ and more examples are put in the appendix.

$P^{(2)}_{G_{12}}(M, X) = \frac{1}{2^8 \cdot 3^3 \cdot 5^2 \cdot 691(aX_2+bX_3)(bX_1+cX_2)(aX_1-cX_3)}\Big((13ca^4 - 65c^3a^2 + 13c^5)X_3X_2^{11} + (52cba^3 - 130c^3ba)X_3^2X_2^{10} + (78cb^2a^2 - 65c^3b^2)X_3^3X_2^9 + 52cb^3aX_3^4X_2^8 + 13cb^4X_3^5X_2^7 - 13ca^4X_3^7X_2^5 - 52cba^3X_3^8X_2^4 + (-78cb^2 + 65c^3)a^2X_3^9X_2^3 + (-52cb^3 + 130c^3b)aX_3^{10}X_2^2 + (-13cb^4 + 65c^3b^2 - 13c^5)X_3^{11}X_2 + O(x_1)\Big)$.

5.4.3 The Ramanujan Delta function

In this subsection, we consider some examples for the Ramanujan Delta function:

$$\Delta(\tau) = \sum_{n \ge 1} \tau(n)q^n = q - 24q^2 + 252q^3 - 1472q^4 + 4830q^5 + \cdots. \tag{5.54}$$

Example 5.4.3. *When $N = 3$. Let $\sigma = \mathrm{Id}$ and $M = \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix}$. The Proposition 5.2.11 shows that $P^{(t)}_{\Delta}(M, X) = 0$ for $t > 10$ and for $t < -1$. The Proposition 5.2.12 and Proposition 5.2.13 show that $P^{(t)}_{\Delta}(M, X)$ is non-zero for $t = -1, 0, \cdots, 4, 6, \cdots, 10$. For example, we give the term $P^{(-1)}_{\Delta}(M, X)$ here:*

$P^{(-1)}_{\Delta}(M, X) = \frac{1}{2^5 \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 691(aX_2+bX_3)(bX_1+cX_2)(aX_1-cX_3)}\Big((4a^2X_2^2 + 8baX_3X_2 + 4b^2X_3^2)X_1^{10} + (-25a^2X_2^4 - 25baX_3X_2^3 - 25baX_3^3X_2 - 25b^2X_3^4)X_1^8 + (25caX_3X_2^4 + 25cbX_3^2X_2^3 - 25caX_3^3X_2^2 - 25cbX_3^4X_2)X_1^7 + (42a^2X_2^6 + 42baX_3X_2^5 + 42baX_3^5X_2 + 42b^2X_3^6)X_1^6 + (-42caX_3X_2^6 - 42cbX_3^2X_2^5 + 42caX_3^5X_2^2 + 42cbX_3^6X_2)X_1^5 +$

$(-25a^2X_2^8-25baX_3X_2^7-25baX_3^7X_2-25b^2X_3^8)X_1^4+(25caX_3X_2^8+25cbX_3^2X_2^7-25caX_3^7X_2^2-25cbX_3^8X_2)X_1^3+(4a^2X_2^{10}+25baX_3^3X_2^7-42baX_3^5X_2^5+25baX_3^7X_2^3+4b^2X_3^{10})X_1^2+(-8caX_3X_2^{10}+25caX_3^3X_2^8-25cbX_3^4X_2^7-42caX_3^5X_2^6+42cbX_3^6X_2^5+25caX_3^7X_2^4-25cbX_3^8X_2^3+8cbX_3^{10}X_2)X_1+(4c^2X_3^2X_2^{10}-25c^2X_3^4X_2^8+42c^2X_3^6X_2^6-25c^2X_3^8X_2^4+4c^2X_3^{10}X_2^2)\Big)$.

However, according to our results so far, we still don't know whether the term $P_\Delta^{(5)}(M,X)$ is zero or not. With the help of PARI/GP, we can verify that $P_\Delta^{(5)}(M,X)$ is also non-zero:

$P_\Delta^{(5)}(M,X)=\frac{1}{2^6\cdot 3^3\cdot 5^2\cdot 7\cdot 691X_1X_2X_3(aX_2+bX_3)(bX_1+cX_2)(aX_1-cX_3)}(-252c^7aX_3^2X_2^{13}-252c^7bX_3^3X_2^{12}+4837c^7aX_3^4X_2^{11}+4837c^7bX_3^5X_2^{10}-14511c^7aX_3^6X_2^9-14511c^7bX_3^7X_2^8+14511c^7aX_3^8X_2^7+14511c^7bX_3^9X_2^6-4837c^7aX_3^{10}X_2^5-4837c^7bX_3^{11}X_2^4+252c^7aX_3^{12}X_2^3+252c^7bX_3^{13}X_2^2+O(x_1))$.

More examples are put in the appendix.

So there is a natural question:

Question 5.4.4. *what's the range of t such that $P_f^{(t)}$ is non-zero?*

After checking the examples for all eigenforms of weight ≤ 24 in dimension 3 and 4, we found that the term $P_{G_k}^{(t)}$ is non-zero for all $-N+1\leq t\leq k-1$ and $P_f^{(t)}$ is non-zero for all $-N+2\leq t\leq k-2$ when f is a Hecke eigenform.

5.5 Prospect

We have seen that the rational function $P_f^{(t)}(\sigma,M,X)$ is a kind of generalization of the period polynomial r_f. However, the $P_f^{(t)}(\sigma,M,X)$ is not a polynomial in dimension $N>2$. So a natural question is that if there is a generalization in polynomials? More precisely, we raise the question:

Question 5.5.1. *Let $X=(X_1,\ldots,X_N)$ and $\mathbb{C}_{k,l}[\mathcal{M},X]$ be the vector space of homogeneous polynomials of degree k in X and also a homogeneous polynomial in entries of matrices in $\mathcal{M}$ of degree l, where $\mathcal{M}$ is given in subsection 5.2.1. The space $\mathbb{C}_{k,l}[\mathcal{M},X]$ is equipped with an action of $\mathrm{GL}_N(\mathbb{Q})$ by*

$$g\cdot Q(M,X)=Q\left(\det(g)g^{-1}Mg^{-t},Xg\right). \tag{5.55}$$

We recall a concept defined in Section 3.1. Let $\mathcal{P}_h(\mathbb{C}_{k,l}[\mathcal{M},X])$ be the set of maps

$$\begin{aligned} Q:(\mathbb{Z}^N\backslash 0)^N &\to \mathbb{C}_{k,l}[\mathcal{M},X]\\ (\sigma_1,\cdots,\sigma_N) &\mapsto Q\{\sigma_1,\cdots,\sigma_N\}\end{aligned} \tag{5.56}$$

satisfying the conditions

1. $Q\{\sigma_1,\cdots,\sigma_N\}=0$ *if* $\det(\sigma_1,\cdots,\sigma_N)=0$,

2. $\sum_{j=0}^{N}(-1)^j Q\{\sigma_0, \cdots, \hat{\sigma_j}, \cdots, \sigma_N\} = 0$ *for all* $\sigma_0, \cdots, \sigma_N \in \mathbb{Z}^N \backslash 0$.

3. $Q\{g\sigma_1, \cdots, g\sigma_N\} = g \cdot (Q\{\sigma_1, \cdots, \sigma_N\})$ *for all* $g \in \mathrm{SL}_N(\mathbb{Z})$.

Then is the space $\mathcal{P}_h(\mathbb{C}_{k,l}[\mathcal{M}, X])$ *non-zero? If the space* $\mathcal{P}_h(\mathbb{C}_{k,l}[\mathcal{M}, X])$ *is non-zero, then we can define the Hecke operator on them as introduced in Section 3.1. So can we find the eigenvectors and the eigenvalues for* $\mathbb{T}_m$*?*

For this question, we don't have an answer in general. But we have found some examples when $N = 3$. The Theorem 2.2.9 shows that the map

$$\begin{aligned} \mathcal{P}_h(\mathbb{C}_{k,l}[\mathcal{M}, X]) &\to \mathbb{C}_{k,l}[\mathcal{M}, X] \\ Q(\sigma, M, X) &\mapsto Q(\mathrm{Id}, M, X) \end{aligned}$$

is injective. Hence $\mathcal{P}_h(\mathbb{C}_{k,l}[\mathcal{M}, X])$ is a finite dimensional vector space over $\mathbb{C}$. In the following, we will denote $Q(\mathrm{Id}, M, X)$ by $Q(M, X)$. Then the question 5.5.1 reduces to a purely algebraic problem. According to the extension theorem 2.2.9, it is equivalent to find all the polynomials in $\mathbb{C}_{k,l}[\mathcal{M}, X]$ such that

$$Q(M,X) - \begin{pmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & c \end{pmatrix} \cdot Q(M,X) + \begin{pmatrix} 1 & 0 & a \\ 0 & 0 & b \\ 0 & 1 & c \end{pmatrix} \cdot Q(M,X) - \begin{pmatrix} 0 & 0 & a \\ 1 & 0 & b \\ 0 & 1 & c \end{pmatrix} \cdot Q(M,X) = 0. \tag{5.57}$$

for all non-zero (a, b, c) with $a, b, c \in \{0, \pm 1\}$. Here if the determinant of matrix σ is 0, then we assume that $\sigma \cdot Q(M, X) = 0$. Then it is computable by PARI/GP. We will give some computational results below.

Example 5.5.2. *Charollois found some instances of* $\mathcal{P}_h(\mathbb{C}_{k,l}[X])$ *for small values of* k *and* $l = 0$*. For example, he found that* $\dim(\mathcal{P}_h(\mathbb{C}_{10,0}[X])) = 1$ *and is generated by the polynomial*

$$\begin{aligned} Q_3(X_1, X_2, X_3) =& (-X_2^2 + X_3^2)X_1^8 + (3X_2^4 - 3X_3^4)X_1^6 + (-3X_2^6 + 3X_3^6)X_1^4 \\ &+ (X_2^8 - X_3^8)X_1^2 + (-X_3^2X_2^8 + 3X_3^4X_2^6 - 3X_3^6X_2^4 + X_3^8X_2^2). \end{aligned} \tag{5.58}$$

Moreover, the associated cocycle $Q_3(\sigma, X)$ *is an eigenvector of* $\mathbb{T}_m$ *with eigenvalue as shown in the table below which doesn't depend on* σ*. All the data are computed by PARI/GP.*

Charollois observed that when $m = p$ *is a prime number, then the eigenvalue seems to be* $p\tau(p) + 1$ *where* $\tau(p)$ *is the* p*-th coefficient of the Ramanujan Delta function.*

eigenvalue	$\mathbb{T}_2$	$\mathbb{T}_3$	$\mathbb{T}_4$	$\mathbb{T}_5$	$\mathbb{T}_7$	$\mathbb{T}_8$	$\mathbb{T}_9$	$\mathbb{T}_{11}$
Q_3	-47	757	-5935	24151	-117207	669905	-1022030	5880733

Table 5.1: *Eigenvalues for Q_3*

More general, we also computed some vector spaces $\mathcal{P}_h(\mathbb{C}_{k,l}[\mathcal{M}, X])$ for $l = 2$. Let $M = \begin{pmatrix} 0 & a & -b \\ -a & 0 & c \\ b & -c & 0 \end{pmatrix}$. We found that the vector spaces $\mathcal{P}_h(\mathbb{C}_{4,2}[\mathcal{M}, X]), \mathcal{P}_h(\mathbb{C}_{6,2}[\mathcal{M}, X])$ are of dimension 1 and are generated by the following polynomials respectively:

$$Q_{G_4}(M, X) = (a^2-b^2)X_1^4+(2cbX_2-2caX_3)X_1^3+((b^2-c^2)X_2^2+(-a^2+c^2)X_3^2)X_1^2+(-2cbX_2^3+2caX_3^3)X_1+((-a^2+c^2)X_2^4+2baX_3X_2^3+(a^2-b^2)X_3^2X_2^2-2baX_3^3X_2+(b^2-c^2)X_3^4),$$

$$Q_{G_6}(M, X) = (a^2-b^2)X_1^6+(2cbX_2-2caX_3)X_1^5+(-c^2X_2^2+c^2X_3^2)X_1^4+(b^2X_2^4-a^2X_3^4)X_1^2+(-2cbX_2^5+2caX_3^5)X_1+((-a^2+c^2)X_2^6+2baX_3X_2^5-b^2X_3^2X_2^4+a^2X_3^4X_2^2-2baX_3^5X_2+(b^2-c^2)X_3^6).$$

The spaces $\mathcal{P}_h(\mathbb{C}_{8,2}[\mathcal{M}, X]), \mathcal{P}_h(\mathbb{C}_{10,2}[\mathcal{M}, X])$ are of dimension 2. The generators of $\mathcal{P}_h(\mathbb{C}_{8,2}[\mathcal{M}, X])$ are

$$Q_{G_8}(M, X) = (a^2-b^2)X_1^8+(2cbX_2-2caX_3)X_1^7+(-c^2X_2^2+c^2X_3^2)X_1^6+(b^2X_2^6-a^2X_3^6)X_1^2+(-2cbX_2^7+2caX_3^7)X_1+((-a^2+c^2)X_2^8+2baX_3X_2^7-b^2X_3^2X_2^6+a^2X_3^6X_2^2-2baX_3^7X_2+(b^2-c^2)X_3^8),$$

$$Q_{\Delta}^{(1)}(M, X) = (-a^2+b^2)X_1^8+(16cbX_2-16caX_3)X_1^7+((-18b^2+28c^2)X_2^2+(18a^2-28c^2)X_3^2)X_1^6+(-72cbX_2^3+72caX_3^3)X_1^5+((45b^2-45c^2)X_2^4+(-45a^2+45c^2)X_3^4)X_1^4+(72cbX_2^5-72caX_3^5)X_1^3+((-28b^2+18c^2)X_2^6+(28a^2-18c^2)X_3^6)X_1^2+(-16cbX_2^7+16caX_3^7)X_1+((a^2-c^2)X_2^8+16baX_3X_2^7+(-18a^2+28b^2)X_3^2X_2^6-72baX_3^3X_2^5+(45a^2-45b^2)X_3^4X_2^4+72baX_3^5X_2^3+(-28a^2+18b^2)X_3^6X_2^2-16baX_3^7X_2+(-b^2+c^2)X_3^8).$$

The generators of $\mathcal{P}_h(\mathbb{C}_{10,2}[\mathcal{M}, X])$ are

$$Q_{G_{10}}(M, X) = (a^2-b^2)X_1^{10}+(-2caX_3+2cbX_2)X_1^9+(c^2X_3^2-c^2X_2^2)X_1^8+(-a^2X_3^8+b^2X_2^8)X_1^2+(2caX_3^9-2cbX_2^9)X_1+((b^2-c^2)X_3^{10}-2baX_2X_3^9+a^2X_2^2X_3^8-b^2X_2^8X_3^2+2baX_2^9X_3+(-a^2+c^2)X_2^{10}),$$

$$Q_{\Delta}^{(2)}(M, X) = (-4a^2+4b^2)X_1^{10}+(32cbX_2-32caX_3)X_1^9+((-75b^2-36c^2)X_2^2+(75a^2+36c^2)X_3^2)X_1^8+(-100cbX_2^3+100caX_3^3)X_1^7+((210b^2+175c^2)X_2^4+(-210a^2-175c^2)X_3^4)X_1^6+((-175b^2-210c^2)X_2^6+(175a^2+210c^2)X_3^6)X_1^4+(100cbX_2^7-100caX_3^7)X_1^3+((36b^2+75c^2)X_2^8+(-36a^2-75c^2)X_3^8)X_1^2+(-32cbX_2^9+32caX_3^9)X_1+((4a^2-4c^2)X_2^{10}+32baX_3X_2^9+(-75a^2-36b^2)X_3^2X_2^8-100baX_3^3X_2^7+(210a^2+175b^2)X_3^4X_2^6+(-175a^2-210b^2)X_3^6X_2^4+100baX_3^7X_2^3+(36a^2+75b^2)X_3^8X_2^2-32baX_3^9X_2+(-4b^2+4c^2)X_3^{10}).$$

The space $\mathcal{P}_h(\mathbb{C}_{12,2}[\mathcal{M}, X])$ is of dimension 3 which is generated by

$$Q_{G_{12}}(M,X) = (a^2 - b^2)X_1^{12} + (-2caX_3 + 2cbX_2)X_1^{11} + (c^2X_3^2 - c^2X_2^2)X_1^{10} + (-a^2X_3^{10} + b^2X_2^{10})X_1^2 + (-2caX_3^{11} + 2cbX_2^{11})X_1 + ((b^2-c^2)X_3^{12} - 2baX_2X_3^{11} + a^2X_2^2X_3^{10} - b^2X_2^{10}X_3^2 + 2baX_2^{11}X_3 + (-a^2+c^2)X_2^{12}),$$

$$Q_{\Delta}^{(3)}(M,X) = (-36a^2+36b^2)X_1^{12} + (-72cbX_2 + 72caX_3)X_1^{11} + ((-691b^2+36c^2)X_2^2 + (691a^2 - 36c^2)X_3^2)X_1^{10} + (1382cbX_2^3 - 1382caX_3^3)X_1^9 + ((2073b^2 - 691c^2)X_2^4 + (-2073a^2 + 691c^2)X_3^4)X_1^8 + (-4146cbX_2^5 + 4146caX_3^5)X_1^7 + ((-2073b^2 + 2073c^2)X_2^6 + (2073a^2 - 2073c^2)X_3^6)X_1^6 + (4146cbX_2^7 - 4146caX_3^7)X_1^5 + ((691b^2 - 2073c^2)X_2^8 + (-691a^2 + 2073c^2)X_3^8)X_1^4 + (-1382cbX_2^9 + 1382caX_3^9)X_1^3 + ((-36b^2 + 691c^2)X_2^{10} + (36a^2 - 691c^2)X_3^{10})X_1^2 + (72cbX_2^{11} - 72caX_3^{11})X_1 + ((36a^2 - 36c^2)X_2^{12} - 72baX_3X_2^{11} + (-691a^2 + 36b^2)X_3^2X_2^{10} + 1382baX_3^3X_2^9 + (2073a^2 - 691b^2)X_3^4X_2^8 - 4146baX_3^5X_2^7 + (-2073a^2 + 2073b^2)X_3^6X_2^6 + 4146baX_3^7X_2^5 + (691a^2 - 2073b^2)X_3^8X_2^4 - 1382baX_3^9X_2^3 + (-36a^2 + 691b^2)X_3^{10}X_2^2 + 72baX_3^{11}X_2 + (-36b^2 + 36c^2)X_3^{12}),$$

$$Q_{f_{16}}^{(1)}(M,X) = (2a^2 - 2b^2)X_1^{12} + (-48cbX_2 + 48caX_3)X_1^{11} + ((42b^2 - 132c^2)X_2^2 + (-42a^2 + 132c^2)X_3^2)X_1^{10} + (280cbX_2^3 - 280caX_3^3)X_1^9 + ((-165b^2 + 315c^2)X_2^4 + (165a^2 - 315c^2)X_3^4)X_1^8 + (-528cbX_2^5 + 528caX_3^5)X_1^7 + ((308b^2 - 308c^2)X_2^6 + (-308a^2 + 308c^2)X_3^6)X_1^6 + (528cbX_2^7 - 528caX_3^7)X_1^5 + ((-315b^2 + 165c^2)X_2^8 + (315a^2 - 165c^2)X_3^8)X_1^4 + (-280cbX_2^9 + 280caX_3^9)X_1^3 + ((132b^2 - 42c^2)X_2^{10} + (-132a^2 + 42c^2)X_3^{10})X_1^2 + (48cbX_2^{11} - 48caX_3^{11})X_1 + ((-2a^2 + 2c^2)X_2^{12} - 48baX_3X_2^{11} + (42a^2 - 132b^2)X_3^2X_2^{10} + 280baX_3^3X_2^9 + (-165a^2 + 315b^2)X_3^4X_2^8 - 528baX_3^5X_2^7 + (308a^2 - 308b^2)X_3^6X_2^6 + 528baX_3^7X_2^5 + (-315a^2 + 165b^2)X_3^8X_2^4 - 280baX_3^9X_2^3 + (132a^2 - 42b^2)X_3^{10}X_2^2 + 48baX_3^{11}X_2 + (2b^2 - 2c^2)X_3^{12}).$$

We also computed that $\dim(\mathcal{P}_h(\mathbb{C}_{14,2}[\mathcal{M}, X])) = 3$. In a basis of eigenvectors, one of them corresponds to G_{14}, one of them corresponds to the Hecke eigenform f_{16} of weight 16, and the other one corresponds to the Hecke eigenform f_{18} of weight 18. We checked in the range $m \leq 32$ that all of these polynomials are the eigenvectors of $\mathbb{T}_m$ with eigenvalue as shown in the following table:

eigenvalues $\mathbb{T}_m$ / Q	$\mathbb{T}_2$	$\mathbb{T}_3$	$\mathbb{T}_4$	$\mathbb{T}_5$	$\mathbb{T}_7$	$\mathbb{T}_8$
Q_{G_4}	73	757	4745	15751	117993	304265
Q_{G_6}	265	6589	67913	390751	5765145	17386313
Q_{G_8}	1033	59077	1057865	9765751	282475593	1083254345
$Q_{G_{10}}$	4105	531469	16814153	16814153	244140751	68870771273
$Q_{G_{12}}$	16393	4782997	268582985	6103515751	678223073193	4400463626825
$Q_{\Delta}^{(1)}$	-47	757	-5935	24151	-117207	669905
$Q_{\Delta}^{(2)}$	-95	2269	-23647	120751	-820455	5383073
$Q_{\Delta}^{(3)}$	-191	6805	-94399	603751	-5743191	43159361
$Q_{f_{16}}^{(1)}$	433	-10043	55985	260551	19757193	-32568655

Table 5.2: *Eigenvalues for the polynomials Q in low degrees*

The reason to index these polynomials by modular forms is that the eigenvalues are related to the coefficients of modular forms for $\mathrm{SL}_2(\mathbb{Z})$. Let $\lambda_{G_k}(m)$ be the eigenvalue of $Q_{G_k}(\sigma, M, X)$ respect to $\mathbb{T}_m$ and $\lambda_f^{(t)}(m)$ be the eigenvalue of $Q_f^{(t)}$ respect to $\mathbb{T}_m$ for $t \in \mathbb{Z}$. We computed all the eigenvalues of $\mathbb{T}_m$ for $m \leq 32$, and the following formulas for $\lambda_{G_k}(m)$ and $\lambda_f^{(t)}(m)$ are true for $m \leq 32$:

$$\lambda_{G_k}(m) = \sum_{d|m} d^3 \sigma_{k-1}(d), \tag{5.59}$$

$$\lambda_f^{(t)}(m) = \sum_{d|m} d^t a_f(d). \tag{5.60}$$

More generally, for any even integer $k \geq 4$, the polynomial

$Q_{G_k}(M, X) = (a^2 - b^2)X_1^k + (2cbX_2 - 2caX_3)X_1^{k-1} + (-c^2X_2^2 + c^2X_3^2)X_1^{k-2} + (b^2X_2^{k-2} - a^2X_3^{k-2})X_1^2 + (-2cbX_2^{k-1} + 2caX_3^{k-1})X_1 + ((-a^2 + c^2)X_2^k + 2baX_3X_2^{k-1} - b^2X_3^2X_2^{k-2} + a^2X_3^{k-2}X_2^2 - 2baX_3^{k-1}X_2 + (b^2 - c^2)X_3^k)$,

seems can be extended to a cocycle $Q_{G_k}(\sigma, M, X)$. It seems that the polynomials $Q_{G_k}(\sigma, M, X)$ are eigenvectors of $\mathbb{T}_m$ with eigenvalues given by (5.59).

If the formulas (5.59) and (5.60) are true for any m, then we can associate the eigenvalues $\lambda_f^{(t)}(m)$ an L-function by

$$L_{Q,f}^{(t)}(s) = \sum_{m \geq 1} \frac{\lambda_f^{(t)}(m)}{m^s}. \tag{5.61}$$

We can give a explicit formula for $L_{Q,f}^{(t)}(s)$ in terms of $\zeta(s)$ and $L(f, s)$:

$$L_{Q,f}^{(t)}(s) = \sum_{m \geq 1} \sum_{d|m} \frac{d^t a_f(d)}{m^s} = \sum_{d_1, d_2 \geq 1} \frac{a_f(d_2)}{d_1^s d_2^{s-t}}. \tag{5.62}$$

Hence we have

$$L_{Q,f}^{(t)}(s) = L(f, s - t)\zeta(s). \tag{5.63}$$

For comparison, we recall the L-functions associated to the rational functions obtained in Theorem 5.3.2. In dimension 3,

$$L_f^{(t)}(s) = L(f, s)\zeta(s - t - 1). \tag{5.64}$$

Appendix

The $A(N,d)$

$A(N,d)$ \ d / N	1	2	3	4	5	6	7	8	9	10	11	12
2	1	0	0	0	0	0	0	0	0	0	0	0
3	1	2	3	4	5	6	7	8	9	10	11	12
4	1	6	12	28	30	72	56	120	117	180	132	336
5	1	14	39	140	155	546	399	1240	1170	2170	1463	5460

Table 5.3: *$A(N,d)$ for small N and d*

Examples of $P_{G_{12}}^{(t)}$

When $N=3$. Let $M=\begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix}$. Then

$P_{G_{12}}^{(4)}(M,X)=\frac{1}{2^7\cdot 3^4\cdot 5\cdot 7\cdot 691(aX_2+bX_3)(bX_1+cX_2)(aX_1-cX_3)}\Big((26ca^6-91c^3a^4-91c^5a^2+26c^7)X_3X_2^{11}+(156cba^5-364c^3ba^3-182c^5ba)X_3^2X_2^{10}+(390cb^2a^4-546c^3b^2a^2-91c^5b^2)X_3^3X_2^9+(520cb^3a^3-364c^3b^3a)X_3^4X_2^8+(-26ca^6+390cb^4a^2-91c^3b^4)X_3^5X_2^7+(-156cba^5+156cb^5a)X_3^6X_2^6+((-390cb^2+91c^3)a^4+26cb^6)X_3^7X_2^5+(-520cb^3+364c^3b)a^3X_3^8X_2^4+(-390cb^4+546c^3b^2+91c^5)a^2X_3^9X_2^3+(-156cb^5+364c^3b^3+182c^5b)aX_3^{10}X_2^2+(-26cb^6+91c^3b^4+91c^5b^2-26c^7)X_3^{11}X_2+O(x_1)\Big)$.

$P_{G_{12}}^{(6)}(M,X)=\frac{1}{2^8\cdot 3^4\cdot 5^2\cdot 691(aX_2+bX_3)(bX_1+cX_2)(aX_1-cX_3)}\Big(((39a^9-130b^2a^7-91b^4a^5-130b^6a^3+39b^8a)X_2+(39ba^8-130b^3a^6-91b^5a^4-130b^7a^2+39b^9)X_3)X_1^{11}+\cdots\Big)$.

$P_{G_{12}}^{(8)}(M,X)=\frac{1}{2^9\cdot 3^4\cdot 5^2\cdot 11\cdot 691(aX_2+bX_3)(bX_1+cX_2)(aX_1-cX_3)}\Big(((130a^{11}-429b^2a^9-286b^4a^7-286b^6a^5-429b^8a^3)X_2+(-429b^3a^8-286b^5a^6-286b^7a^4-429b^9a^2+130b^{11})X_3)X_1^{11}+\cdots\Big)$.

Examples of $P_{\Delta}^{(t)}$

When $N = 4$. Let $\sigma = \mathrm{Id}$ and $M = \begin{pmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{pmatrix}$. Proposition 5.2.11 shows that $P_{\Delta}^{(t)}(M, X)$ is zero for $t > 10$ and for $t < -2$. Proposition 5.2.12 and Proposition 5.2.13 show that $P_{\Delta}^{(t)}(M, X)$ is non-zero for $t = -2, 0, 1, \cdots, 10$ except 5. In fact, with the help of PARI, we can verify that $P_{\Delta}^{(-1)}(M, X)$ and $P_{\Delta}^{(5)}(M, X)$ are also non-zero:

$P_{\Delta}^{(-1)}(M, X) = \frac{1}{2^3 \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 691 X_1 X_2 X_3 X_4 (aX_2 + bX_3 + cX_4)(-aX_1 + dX_3 + eX_4)(-bX_1 - dX_2 + fX_4)(-cX_1 - eX_2 - fX_3)} \Big((ca^2 X_3 + X_4 ba^2) X_2^3 + (2cbaX_3^2 + (X_4 a^3 + (2X_4 b^2 + 2X_4 c^2)a) X_3 + 2X_4^2 cba) X_2^2 + (cb^2 X_3^3 + (2X_4 ba^2 + (X_4 b^3 + 2X_4 c^2 b)) X_3^2 + (2X_4^2 ca^2 + (2X_4^2 cb^2 + X_4^2 c^3)) X_3 + X_4^3 c^2 b) X_2 + (X_4 b^2 a X_3^3 + 2X_4^2 cbaX_3^2 + X_4^3 c^2 a X_3) X_1^{12} + \cdots \Big)$.

$P_{\Delta}^{(5)}(M, X) = \frac{1}{2^4 \cdot 3 \cdot 5^2 \cdot 691 X_1 X_2 X_3 X_4 (aX_2 + bX_3 + cX_4)(-aX_1 + dX_3 + eX_4)(-bX_1 - dX_2 + fX_4)(-cX_1 - eX_2 - fX_3)} \Big((-cba^7 X_2^2 + ((-cb^2 a^6 - cb^6 a^2) X_3 + (-X_4 c^2 ba^6 - X_4 c^6 ba^2)) X_2 + (-cb^7 a X_3^2 + (-X_4 c^2 b^6 - X_4 c^6 b^2) a X_3 - X_4^2 c^7 ba)) X_1^{14} + \cdots \Big)$.

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