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# Frequency Design of Passive Electronic Filters: a Modern System Approach

Arthur Perodou

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# Frequency Design of Passive Electronic Filters: a Modern System Approach

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par Arthur PERODOU

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Présidente	Mme. Isabelle QUEINNEC	Directrice de Recherche CNRS, LAAS
Rapporteur	M. Yann LE GORREC	Professeur des Universités, ENSMM
Rapporteur	M. Paolo Rapisarda	Full Professor, University of Southampton (R-U)
Examinateur	M. Eric KERHERVÉ	Professeur des Universités, ENSEIRB-MATMECA
Directeur de thèse	M. Ian O'CONNOR	Professeur des Universités, Ecole Centrale de Lyon
co-Directeur (invité)	M. Gérard SCORLETTI	Professeur des Universités, Ecole Centrale de Lyon
Encadrant	M. Anton KORNIENKO	Maître de Conférences, Ecole Centrale de Lyon
Encadrant (invité)	M. Mykhailo ZARUDNIEV	Docteur, Ingénieur-Chercheur, CEA LETI
Encadrant (invité)	M. Jean-Baptiste DAVID	Ingénieur-Chercheur, CEA LETI



*“The answer, of course, is to have a society based on human values other than buying and selling. To arrive at this society, we need a good deal of planning and a good deal of struggle, which, if the best comes to the best, may be on the plane of ideas, and otherwise - who knows?”*

Norbert Wiener  
in *Cybernetics, or Control and Communication in the Animal and the Machine (1948)*



UNIVERSITÉ DE LYON

## *Abstract*

Ecole Centrale de Lyon  
 Institut des Nanotechnologies de Lyon (INL) - Laboratoire Ampère

### **Frequency Design of Passive Electronic Filters: a Modern System Approach**

by Arthur PERODOU

The current explosion of communicating devices (smartphones, drones, IoT ...), along with the ever-growing data to be transmitted, produces an exponential growth of the radiofrequency bands. All solutions devised to handle this increasing demand, such as carrier aggregation, require to synthesise frequency filters with stringent industrial requirements (performance, energy consumption, cost ...). While the technology of acoustic wave (AW) resonators, that seem to be the only passive micro-electronic components available to fulfil these requirements, is mature, the associate design problem becomes dramatically complex. Traditional design methods, based on the intuition of designers and the use of generic optimisation algorithms, appear very limited to face this complexity. Thus, systematic and efficient design methods need to be developed.

The design problem of AW filters happens to be an instance of the more general design problem of passive electronic filters, that played an important role in the early development of Linear Control and System theory. Systematic design methods were developed in particular cases, such as for LC-ladder filters, but do not enable to tackle the case of AW filters. Our aim is then to revisit and generalise these methods using a modern System approach, in order to develop systematic and efficient design methods of passive electronic filters, with a special focus on AW filters.

To achieve this, the paradigm of convex optimisation, and especially the sub-class of Linear Matrix Inequality (LMI) optimisation, appears for us a natural candidate. It is a powerful framework, endowed with efficient solvers, able to optimally solve a large variety of engineering problems in a low computational time. In order to link the design problem with this framework, it is proposed to use modern tools such as the Linear Fractional Transformation (LFT) representation and a mathematical characterisation coming from Dissipative System theory.

Reviewing the different design methods, two design approaches stand out. The first approach consists in directly tuning the characteristic values of the components until the frequency requirements are satisfied. While very flexible and close to the original problem, this typically leads to a complex optimisation problem with important convergence issues. Our first main contribution is to make explicit the sources of this complexity and to significantly reduce it, by introducing an original representation resulting from the combination of the LFT and the Port-Hamiltonian Systems (PHS) formalism. A sequential algorithm based on LMI relaxations is then proposed, having a decent convergence rate when a suitable initial point is available.

The second approach consists of two steps. First, a transfer function is synthesised such that it satisfies the frequency requirements. This step is a classical problem in

Control and Signal Processing and can be efficiently solved using LMI optimisation. Second, this transfer function is realised as a passive circuit in a given topology. To this end, the transfer function needs to satisfy some conditions, namely realisation conditions. The issue is to get them with a convex formulation, in order to keep efficient algorithms. As this is generally not possible, an idea is to relax the problem by including common practices of designers. This leads to solve some instances of a general problem denoted as frequency LFT filter synthesis. Our second main contribution is to provide efficient synthesis methods, based on LMI optimisation, for solving these instances. This is achieved by especially generalising the spectral factorisation technique with extended versions of the so-called KYP Lemma.

For particular electronic passive filters, such as bandpass LC-ladder filters, this second approach allows to efficiently solve the design problem. More generally, it provides an initial point to the first approach, as illustrated on the design of a particular AW filter.

**Keywords** Frequency Filter Design, LFT systems, Passive elements, Dissipativity, KYP Lemma, Spectral Factorisation, LMI optimisation, Port-Hamiltonian systems, Differential-Algebraic Equations (DAE), Acoustic Wave (AW) resonators.

UNIVERSITÉ DE LYON

*Résumé*Ecole Centrale de Lyon  
Institut des Nanotechnologies de Lyon (INL) - Laboratoire Ampère**Synthèse Fréquentielle de Filtres Electroniques Passifs :  
une Approche Systémique Moderne**

par Arthur PERODOU

L'explosion actuelle du nombre d'appareils connectés (smartphones, drones, IoT, ...) et du volume des données à transmettre engendre une croissance exponentielle du nombre de bandes radiofréquences. Toutes les solutions élaborées pour faire face à cette demande croissante, telle que l'agrégation de porteuses, impliquent de concevoir des filtres fréquentiels satisfaisant des contraintes (performance, consommation d'énergie, coût, ...) toujours plus strictes. Les filtres passifs AW, pour acoustic wave (AW) en anglais, semblent être les seuls pouvant satisfaire ces contraintes. Cependant, face à l'augmentation drastique de la complexité de leur problème de conception, les méthodes traditionnelles de conception apparaissent limitées. Il devient donc nécessaire de développer de nouvelles méthodes, qui soient systématiques et efficaces d'un point de vue algorithmique.

Le problème de synthèse des filtres AW est un cas particulier du problème de synthèse de filtres électroniques passifs, intrinsèquement lié aux origines de la théorie des Systèmes linéaires et de l'Automatique. Des méthodes systématiques ont été développées pour des cas particuliers, tels que les filtres LC-échelle, mais n'incluent pas les filtres AW. Notre but est donc de les revisiter et de les généraliser en utilisant une approche systémique moderne, afin d'obtenir une méthodologie systématique et efficace de conception de filtres électroniques passifs, avec un intérêt particulier pour les filtres AW.

Pour ce faire, le paradigme de l'optimisation convexe, et particulièrement la sous-classe nommée optimisation LMI, nous paraît être un candidat naturel. Doté de solveurs efficaces, il permet de résoudre un large éventail de problèmes d'ingénierie en un faible temps de calcul. Afin de relier notre problème de conception à cet environnement, il est proposé d'utiliser des outils modernes tels que la représentation LFT et la caractérisation mathématique dite de dissipativité.

Historiquement, deux approches de conception se sont opposées. La première consiste à faire varier les valeurs caractéristiques des composants jusqu'à satisfaction du gabarit fréquentiel. Bien que flexible et proche de la formulation originelle du problème, cette approche aboutit typiquement à un problème d'optimisation complexe. Notre première contribution est d'avoir révélé les sources de cette complexité ainsi que de les avoir considérablement réduites, en introduisant une représentation originale qui résulte de la combinaison de l'outil LFT et du formalisme des Systèmes Hamiltoniens à Ports. Un algorithme résolvant séquentiellement des problèmes LMIs est proposé, possédant un taux de convergence raisonnable si le point initial est bien choisi.



La seconde approche se compose de deux étapes. Une fonction de transfert est d'abord synthétisée de façon à satisfaire le gabarit fréquentiel. Cette étape correspond à un problème classique d'Automatique et de Traitement du Signal qui peut être efficacement résolu via l'optimisation LMI. Puis, cette fonction de transfert est réalisée comme un circuit avec une topologie donnée. Pour cela, elle doit satisfaire des conditions de réalisation. Ces dernières ne peuvent pas toutes être incluses dans la première étape, et nous formalisons certaines pratiques courantes pour en considérer le plus possible. Cela nous mène à résoudre le problème général de synthèse fréquentielle de filtres LFT. Notre seconde contribution est d'avoir fourni des méthodes de synthèse efficaces, à base d'optimisation LMI, pour résoudre certains sous-problèmes. Cela est accompli en généralisant la technique de la factorisation spectrale conjointement avec l'utilisation des extensions du Lemme KYP.

Pour certains filtres électroniques passifs, comme les filtres LC-échelle passe-bande, la seconde approche permet de résoudre efficacement le problème de conception associé. Plus généralement, elle procure un point initial à la première approche, comme illustré dans le cas d'un filtre AW.

**Mots-Clés** Synthèse Filtres Fréquentiels, Systèmes LFT, Eléments Passifs, Dissipativité, Lemme KYP, Factorisation Spectrale, Optimisation LMI, Systèmes Hamiltoniens à Ports, Equations Différentielles-Algébriques (EDA), résonateurs à onde acoustique.

Note : un résumé étendu en français, d'une longueur équivalente à environ 25% de ce manuscrit, est disponible en Annexe C (p. 209).

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# Notations

## Acronyms

<b>(B/S)AW</b>	<b>(Bulk/Surface) Acoustic Wave</b>
<b>BMI</b>	<b>Bilinear Matrix Inequality</b>
<b>CAD software</b>	<b>Computer-Aided Design software</b>
<b>(D)AE</b>	<b>(Differential) Algebraic Equations</b>
<b>FIR filter</b>	<b>Finite Impulse Response digital filter</b>
<b>IIR filter</b>	<b>Infinite Impulse Response digital filter</b>
<b>KYP lemma</b>	<b>Kalman-Yakubovich-Popov lemma</b>
<b>LFT</b>	<b>Linear Fractional Transformation</b>
<b>LFR</b>	<b>Linear Fractional Representation</b>
<b>LMI</b>	<b>Linear Matrix Inequality</b>
<b>LTI system</b>	<b>Linear Time Invariant system</b>
<b>MIMO system</b>	<b>Multiple Input Multiple Output system</b>
<b>PHS</b>	<b>Port-Hamiltonian Systems</b>
<b>QC</b>	<b>Quadratic Constraint</b>
<b>SISO system</b>	<b>Single Input Single Output system</b>

## Symbols

$:=$	Definition of symbol or variable
$\geq$	Greater than or equal to
$>$	Greater than
$\lceil x \rceil$	Nearest integer greater than or equal to $x$
$\mathbb{C}$	The set of complex numbers
$\mathbb{C}^{n \times m}$	The set of complex matrices with $n$ rows and $m$ columns
$\mathbb{C}^-, \mathbb{C}^0, \mathbb{C}^+$	The set of complex numbers with negative/zero/positive real part
$\mathcal{D}(\mathbb{R}^{n \times n})$	The set of diagonal matrices of $\mathbb{R}^{n \times n}$
$\mathbb{N}$	The set of natural numbers
$\mathbb{R}$	The set of real numbers
$\mathbb{R}^{n \times m}$	The set of real matrices with $n$ rows and $m$ columns
$s$	Laplace variable
$\Re/\Im m$	Real/Imaginary part
$\mathcal{R}_n[s]$	Real polynomial functions of degree $n$
$\mathcal{R}_n \mathcal{H}_\infty(s, d(s))$	Proper, stable, real rational functions of order $n$ having $d(s)$ as denominator
$\star$	Redheffer Star Product
$\otimes$	Kronecker Product
$A^T, A^*$	Transpose/Conjugate Transpose of the matrix $A$
$\Lambda(A)$	Set of eigenvalues of $A$



## Chapter 1

# General Introduction

### 1.1 Motivation and purpose of this thesis

#### On the need for systematic and efficient design methods of radiofrequency filters

Nowadays, we are witnessing an increase of the importance of mobile communicating devices in our daily life. According to a Cisco's report [Cis19], more than 600 million mobile devices and connections were added in 2017. At the time, there were up to 8.6 billions of such devices, and this number is expected to reach 12.3 billions in 2022 (Figure 1.1 (A)). While the relative contribution of smartphones seems to stagnate, the main source of this growth comes from the spread of machine-to-machine (M2M) connections, that is the basis of the Internet of Things (IoT), with for instance wearable devices, connected vehicles or medical applications.

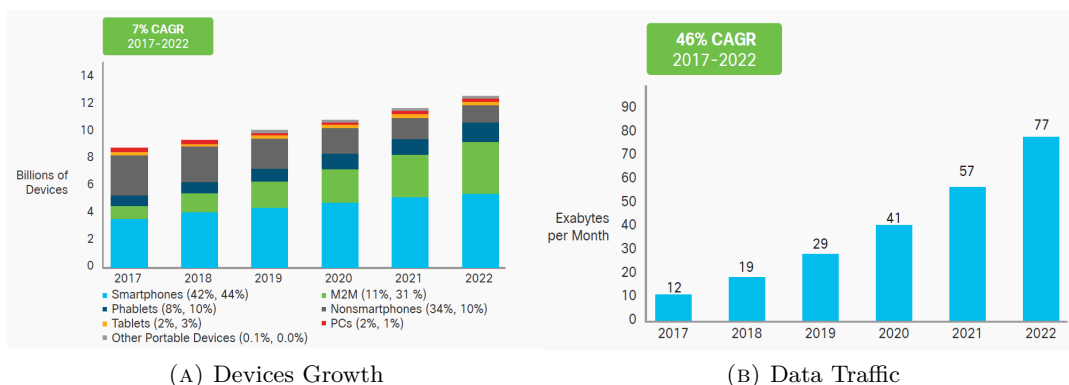


FIGURE 1.1: Mobile Devices Growth<sup>1</sup> and Data Traffic, 2017 to 2022.

Source: Cisco VNI Mobile [Cis19]

Although this rise of devices appears approximatively constant over years, there is an associated explosion of data traffic (Figure 1.1 (B)). Indeed, more devices naturally produce more data. But these new devices also propose new services and user applications that tend to multiply the volume of these data. In particular, videos generate much of this data traffic [Cis19]. Solutions are then developed by digital companies to handle this exponential growth, while keeping pushing forward higher data transmission speed.

For mobile devices, the communication channel is the radiofrequency spectrum, suitable for wave propagation in the atmosphere. To prevent interferences, this spectrum is split into frequency bands, which are then allocated to certain applications or services (mobile communications, radar, radio broadcasting, amateur radio,...). In

<sup>1</sup>Figures in parentheses refer to (2017, 2022) share.

order to cope with this massive arrival of data, while still increasing the transmission speed, a natural choice has been to increase the number of the available frequency bands for mobile phones. This coarsely explains the differences between the generations of telecommunication standards (2G, 3G, 4G, 5G) [GSM17]. However, the radiofrequency spectrum is finite (30 kHz - 300 GHz) and has to be shared between the different applications. This led to an overcrowded and segmented spectrum, where transition spaces between bands are viewed as wasted spectrum and become then tinier over time. On Figure 1.2, an example coming from [Mil15a] illustrates the coexistence of bands allocated for mobile phones and U.S. public safety bands on the spectrum portion (746 MHz – 806 MHz).



FIGURE 1.2: Portion example of the segmented spectrum with tiny separation bands. Source [Mil15a].

When an input signal is received by the antenna of a mobile device communicating through an allocated frequency band, it is needed to only keep the useful part of this signal, i.e. the frequency components of its spectral power density included in the frequency band. On the other hand, when it is aimed to transmit a signal through the device antenna, one has to ensure that the spectral density of the signal is null outside of the allocated frequency band in order to prevent interferences. Both operations are based on a process called *frequency filtering*, which consists in removing unwanted frequency components from a given signal and are achieved by a so-called *frequency filter*. An example of the typical frequency response<sup>2</sup> of a radiofrequency filter is provided on Figure 1.3.

Usually, a filter is exclusively designed for a specific band. Therefore, the explosion of number of bands has led to a similar explosion in terms of radiofrequency filters per devices. Moreover, the limited size of the radiofrequency spectrum requires the development of several advanced techniques of spectrum management, which generally leads to the addition of extra filters. An important example is the frequency division duplexing which consists in splitting a frequency band in two parts, one for transmission and the other for reception, leading to twice as much filters as frequency bands [Mil15b].

Therefore, the number of filters per devices considerably increases, adding significant space and cost pressures to the already stringent industrial requirements [Mah17]. Furthermore, tinier transition bands require filters to be very selective. As an example, the filter represented on Figure 1.3 can not be associated to Band 13 DL of Figure 1.2. Indeed, it does not attenuate the public safety band (769 MHz – 775 MHz).

<sup>2</sup>actually the squared magnitude of the frequency response

<sup>3</sup><https://www.qorvo.com/products/d/da003047>

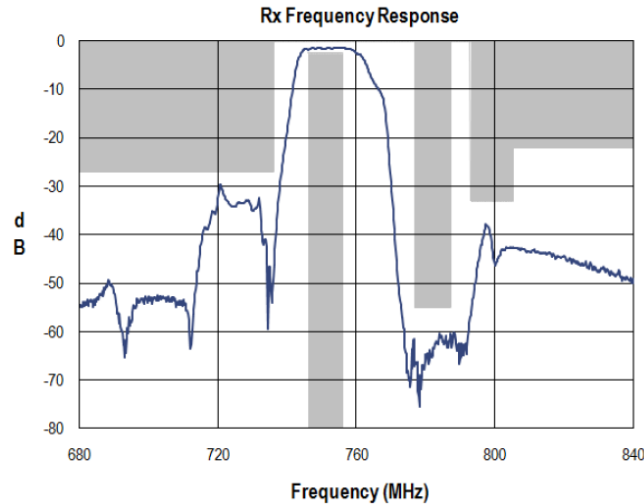


FIGURE 1.3: Example of a typical radiofrequency filter response.  
Source: TriQuint 856879 datasheet<sup>3</sup>.

In cases where no filter solution is found, the utility of the band drastically reduces [Mil15a]. All of these tight constraints, coupled with new techniques such as carrier aggregation [Mil16] that improve the data transmission speed but requires to jointly design a multitude of filters [FVAJ+16; Mah17], and other strict industrial requirements such as energy consumption, lead to a truly challenging filter design problem.

To face this challenge, a consensus appears on the necessary use of acoustic-wave (AW) filters [FVAJ+16; Mah17; PKK+09]. Roughly, these filters are the interconnection of acoustic-wave resonators, which are *passive* micro-electronic components. These technologies, usually split into surface acoustic wave (SAW) and bulk acoustic wave (BAW) resonators, are recognised for their high selectivity and their abilities to be adaptable to strict industrial requirements (size, performance, energy consumption, cost,...) as they have grown simultaneously with the development of mobile telecommunications in the last three decades [Mil15b]. Their realisation technologies are mature and have then well-characterised models, from equivalent *RLC* circuits to more sophisticated electromechanical models [Has09].

When it comes to the design of *AW* filters, different approaches may be applied [WL15; GB16]. A first approach is to use global optimisation techniques. While this may eventually solve the design problem, it results generally in a very time-consuming task. Another approach is to start with an already existing filter, and to modify it using the experience of the designer or some local optimisation algorithms provided by the software. This approach is usually forced by the time-to-market pressure. It may work in some cases, but with the ever-increasing complexity of the design problem, this approach appears too much time-consuming, or even endless.

A more structured approach, comprehensively detailed in [WL15], is more relevant in practice to tackle this complexity. This consists in an iterative process, where the global design problem is decomposed into sub-problems of lower complexity with appropriate design purposes. The first step tackles the design sub-problem of the equivalent circuit of the filter. The components are modelled using linear *RLC*-models, so-called BVD and mBVD model [LIBWR00; Has09], and the filter results from their ideal electrical interconnection. This step is almost free from technological realisation constraints, and only the frequency requirements are considered. The designer is required to provide an initial point, i.e. characteristic values of the models,

which is fed into a routine. If the resulting solution satisfies the requirements, one goes to the next step. If not, one should change the initial point and re-run the algorithm. In the second step, more sophisticated models are used, such as electromechanical models, and simulations are made to check if it satisfies the associated requirements. In the third step, simulations of even more complex models, that fully include the realisation technology and the packaging, are finally made.

Surprisingly enough, while the first step is crucial in this approach as it determines the rest of the iterative process, it is mainly based on the designer ability of finding a *suitable* initial point and on a general routine. While the inclusion of intuition from seasoned designer is interesting in the objective of reducing even more the problem complexity, the absence of a rigorous methodology is limiting. Indeed, for instance, if the solution given by the algorithm does not satisfy the requirements, does it mean that there is not such a solution or the algorithm has not found it? And then, how to change the initial point? By changing the values of the *RLC* models of the components or the topology, i.e. their interconnection? Or perhaps the number of components is too low? One may increase it, but can not be sure to do not have lost a solution with a lower number.

Last considerations lead important questions that are complicated to tackle in practice. As the first step is the most repeated in the design process, it appears that *systematic* and *efficient* design methods are required. And indeed some have appeared [SPJK05; Shi06; GB16; GVS18]. These methods are based on the direct extension of standard analytical methods developed in the first part of the 20<sup>th</sup> century. However, in the author's view, while providing *AW* filters of practical use and apparently promising, these methods may struggle to face more complex problems. Before introducing the inherent reasons of such a claim, and in order to understand the approach developed in this work, let us briefly go through the historical development of methods for the broader design problem of passive electronic filters.

### A bit of history

The topic of passive electronic filtering has a venerable history at the engineering scale. The first filter appeared during World War I and was the result of the development of Circuit theory in the 19<sup>th</sup> century [Dar84]. Derived from Electromagnetic theory, Circuit theory has its origins in the famous Ohm's law (1827) for which

a resistor is [now] considered as a 2-terminal *black box* defined by the relation  $v = Ri$ , rather than a physical [component] made of metal or carbon [Bel62].

The modelling of the electrical interconnection followed shortly after with the equally famous Kirchhoff's circuit laws (1845). In the second half of the 19<sup>th</sup> century, fundamental concepts and tools were developed, with Heaviside as one of the main contributor [Nah83; Kal10]. In particular, Heaviside developed operational calculus, in which complex derivative and integral operations in the time-domain are replaced by simple algebraic operations on the Laplace variable  $s$ . This technique will be extensively used in the present work.

The fundamental problems raised by the practical need of a design methodology for filters, accompanied the early development of research fields between the interwar period [Bel62; Zad62; Dar84]. The needs for more sophisticated electronic components pushed the development of Electronics. Fundamental questions such as checking the stability of a system, or establishing the link between a circuit realisation and a transfer function, accompanied Control and System theory. The generality of

some questions even led to the emergence of whole theory, such as the Realisation theory [Kal10].

The result is that, at end of World War II, technology<sup>3</sup> and design methodology for filtering applications were well established. There were still unsolved problems but the research interest in new methods started to decline. Two main reasons appear to the author in order to explain this. First, there were no practical needs for further development [Dar84]. There were systematic design methods for  $LC$  ladder filters, i.e. filters made of inductances  $L$  and capacitances  $C$  in a particular topology named *ladder*, that were rather efficient, if one considers that calculations were analytically made. The realisation technology was mastered, relatively cheap, and there were little reasons to change. In addition,  $LC$  ladder filters were significantly robust to component values variations [TO73]. Indeed, still nowadays, digital filters aim to imitate this nice robustness property [VM19]. Second, in the post-war period the attention was focused on the invention of the transistor (1947). This led to the rise of digital filters, which can achieve unreachable performance for analog filters, such as exact linear phase or structural stability, and can be reconfigured online [PB87]. More importantly, discrete transfer function can always be realised as a circuit made of sum, gain and delay blocks. While some digital filters were designed using simple conversion from analog to digital domain, all of this required new methods that were developed along with the rise of computers and their computational power.

In parallel, the previous sub-branches of Circuit theory, such as Electronics, Control and System Theory, Signal Processing, became established field of research with their own topics of interest and success. An important transversal tool for these disciplines emerged at the end of last century. This is convex optimisation. Combined with the improvement of computational power, it has enabled to solve complex engineering problems and has provided systematic and efficient methods for analysing, controlling and designing systems [BTN01]. In particular, this has led to a great success for the efficient design of FIR digital filters [Dav10].

Meanwhile, the interest in acoustic-wave resonators grew suddenly with the development of the second generation of mobile phones [PKK+09]. Rapidly, these components happen to have unbeatable filtering performance, high energy efficiency, small size and low manufacturing cost. Although the design methods of  $LC$  filters were inapplicable for their  $(R)LC$  equivalent circuits, these nice properties have enabled to use synthesis techniques based on intuition and simulations.

### On the use of modern Control and System methods

Let us return, after this digression, to the development of systematic and efficient design methods. Instead of extending analytical-based methods and to implement them on a computer, with a strong risk of numerical issues, we find better to develop methods having as initial requirement to be computationally efficient. And to achieve this, convex optimisation appears to be a suitable framework. Moreover, we propose to use some concepts and tools from Control and System theory that provide a natural path to convex optimisation, and more specifically to the subclass of Linear Matrix Inequality (LMI) optimisation [BBFG94]. This is also motivated by previous success in designing micro-electronic systems with such an approach [Kor11; Zar13; KSCB16].

The purpose of this work is then to develop systematic and efficient design methods of passive electronics filters, with a special focus on  $AW$  filters, using a modern System approach.

---

<sup>3</sup>That sometimes consisted in simple common objects such as for the realisation of crystal radios: [https://en.wikipedia.org/wiki/Crystal\\_radio](https://en.wikipedia.org/wiki/Crystal_radio).



## 1.2 Scope and contributions

Our aim is then to develop systematic and efficient methods for the design of passive electronic filters. To achieve this, Control and System tools are used as a path to the framework of convex optimisation. However, this requires to choose tools adapted to the two inherent features of passive electronic filters.

The first feature of electronic filters is their *hierarchical* modelling. Indeed, they are viewed as the interconnection of subsystems, the components, that are themselves interconnection of simpler subsystems, i.e. resistances, inductances and capacitances. The filter model is then structurally constrained at two interconnection levels. In fact, structure constraints appear on the global interconnection to stand for the so-called *topology* of the filter, and at local interconnections to match with the models of the components. Thus, we propose to use a representation based on the Linear Fractional Transformation (LFT) tool. Coming from Robust Control [DPZ91; ZDG96], this mathematical representation unifies and generalises both transfer function and state-space representation, and allows to distinctly exhibit the structure of each interconnection level.

The second important feature to consider is the passivity of components. This implies particular properties on their models that can be propagated to the behaviour of their interconnection [AV73]. To consider this, the paradigm of Dissipative Systems theory appears suitable [Wil72b; Moy14]. This provides a mathematical characterisation that enables to both express components passivity and the global behaviour of the filter. Moreover, this paradigm tends to provide LMI-based methods to solve standard problems of Linear Control and System theory.

The combination of both tools, LFT representation and dissipative characterisation, has led to the development of several systematic and efficient methods, based on LMI optimisation, for controlling hierarchical systems [Sco97; SD01; Kor11; Zar13; ZKSV13; KSCB16]. The key idea of these methods consists in revisiting efficient methods of standard problems in such a way that they can be efficiently extended to problems of interest.

In this thesis, it is proposed to follow this approach. This leads us to revisit the different methods developed for the design of passive electronic filters. From a system perspective, these methods can be sorted following if the mathematical representation to be synthesised is a state-space representation or a transfer function. The structural constraints imposed by the local and global interconnection result then either on constraints on the state-space matrices or on the coefficients of a transfer function.

The problem of structured state-space synthesis is first tackled. Unfortunately, combining state-space representation and structured synthesis generally go hand in hand with non-convexity [BT97]. And this claim will be corroborated in this manuscript. We focus then on reducing as most as possible this *degree* of non-convexity. To achieve this, a representation originating from the Port-Hamiltonian Systems (PHS) theory appears for us especially suitable [Sch13]. The PHS theory is a multi-physics, energy-based framework especially adapted for modelling, analysing and controlling interconnected passive systems [SM03; DMSB09; Sch17]. Curiously, while it is intrinsically linked with passive electronic *circuits* [SJ14; MSB95], it seems that the frequency design problem of passive electronic *filters* has not been tackled within this framework. A particular feature of this framework is to provide from any circuit a structured state-space representation where the components values, i.e. our design variables, linearly and distinctly appear. Therefore, our first contribution is to develop a synthesis method which is adapted to this particular representation. This

leads to solve a non-convex optimisation problem, as expected, but of a particular form and with a *limited* degree of non-convexity. Furthermore, in order to reduce this degree even more, we combine the PHS representation with the LFT tool, leading to an original representation. Our second contribution is then to adapt our synthesis method to this new representation. This results in solving an optimisation problem with a *low* degree of non-convexity.

Then, the synthesis problem of a structured transfer function is tackled. Unlike the state-space case, this problem admits some instances with convex synthesis methods, leading for example to the so-called Butterworth and Chebyshev filters. These cases are mainly due to a particular global interconnection, named ladder topology, and the simplicity of the subsystems, being either inductances and capacitances, that lead to convex structural constraints. Another of our contribution is to convexly extend these structural constraints to the ladder interconnection of a broader family of passive components, that is interesting for our applications. This is achieved by interpreting the Laplace variable  $s$  as being the impedance of a unitary inductance, or similarly the admittance of a unitary capacitance, and thus by introducing some transfer functions to encapsulate the local subsystems interconnection, i.e. the models of the components. This leads us to frequency synthesise a transfer function that is expressed as rational function of transfer functions, that is to tackle the synthesis problem of frequency LFT filters.

This problem is the extension of the standard synthesis problem of frequency filters, for which there exists an efficient resolution method. This method consists of two successive steps. First step is concerned with the frequency synthesis of a *magnitude* function and is achieved by solving an LMI optimisation problem. This step is mainly based on the Kalman-Yakubovich-Popov (KYP) Lemma. This lemma, which is a generalisation of the so-called Positive-Real lemma and Bounded-Real lemma that are historically linked to passive circuit synthesis [AV73], is a fundamental and transversal tool in Linear Control and System theory [Wil72b; TW91; BGF94; Ran96]. It admits then several generalised versions [IMF00a; Din05; PV11; PIH13] that appear especially useful for filter design [RSF03; IH05; Dav10], either digital or analog. Moreover, this lemma is even compatible with the LFT representation and the dissipative characterisation [Sco97; SD01; Zar13]. The second step, named *spectral factorisation*, consists in extracting a stable transfer function from the synthesised magnitude. The spectral factorisation problem is also linked with passive circuit synthesis, has been extensively studied for filtering and other applications, and has efficient resolution methods [You61; Wil72b; AV73; ZDG96; TR99; WBV99]. Unfortunately, the LFT spectral factorisation problem does not seem to have been solved in the literature. Therefore, by detailing and completing the approach of [Zar13], we tackle the frequency LFT filter synthesis problem from scratch, by revisiting the usual efficient synthesis methods of frequency filters with the LFT representation and the dissipative characterisation. Our contribution is then to extend this approach for particular LFT systems, that is interesting for designing certain electronic passive filters. However, for more general LFT systems, we show that the approach is not extendible, as a factorisation error appears and is inherent to our approach. Therefore, one of the main contribution of this thesis is to generalise this synthesis approach by merging the magnitude synthesis and the spectral factorisation steps into a single step, which is achieved by solving an LMI optimisation problem. This enables to take into account the spectral factorisation error from the outset. Furthermore, this generalised approach also leads us to contribute to the 2D LFT case by providing an LMI optimisation problem. While it has an inherent conservatism, the resulting method tends to provide solutions in our cases of design interest.

Finally, putting all this together will lay the foundations for the development of systematic and efficient design methods. In particular, this will lead to a systematic, efficient and near-optimal methodology for the design of a family of passive electronic filters, that includes  $LC$  ladder and  $LC$ -resonator ladder filters. Furthermore, while not fully systematic, a promising approach based on efficient synthesis stages will be provided for the design of  $AW$ -resonator ladder filters.

### What will not be found in this thesis

In this thesis, we aim to develop an efficient and systematic methodology for the frequency design of passive electronic filters, with a system-based approach.

Therefore, this thesis is **not** about the development of design methods of an  $AW$  filter realised in a given technology associated to a given frequency band. We modestly focus on the very first stage of the design flow, i.e. to obtain a circuit, with the  $RLC$  models of the components, that satisfies some frequency requirements. Indeed, developing systematic and efficient methods for this purpose is already a non-trivial task. Thus, no technological realisation words such as FBAR, SMR or  $L_iNO_3$ , will be found in this manuscript. However, the design methods are developed with the aim of being flexible, and allowing to include the most technological constraints as possible. Indeed, it is known that some technological considerations such as the surface of the top electrode or the thickness of the piezoelectric layer can be lifted to the  $RLC$  model [CBG+13]. Therefore, a high-end method for the design of  $AW$  filters will not be found in this manuscript, but it is aimed to prepare the ground before facing current and near-future challenging problems.

Moreover, while heavily relying on convex optimisation, this thesis is **not** about developing new optimisation techniques or algorithms. Convex optimisation is viewed as a mature and ready-to-use tool. This is besides one of the main benefits of the approach developed in this work.

## 1.3 Structure

The core of this thesis is organised as follows.

**Chapter 2** We begin by introducing some key notions that will be at the heart of this work. Starting from a brief discussion on the increasing design complexity of systems for modern engineering applications, it is first concluded that a relevant use of computers power should be made, and thus design methods should be developed in that respect. In particular, this motivates the use of convex optimisation, and the particular subclass of Linear Matrix Inequality (LMI) optimisation, and the introduction of concepts and tools originating from Control and System theory. Second, modern systems are typically viewed as the interconnection of subsystems having complex models. Advanced tools called LFT representation and  $\{x, y, z\}$ -dissipative characterisation are then presented to mathematically represent and characterise the subsystems and their interconnection. The KYP Lemma and its diverse extensions finally enable to link these tools with convex optimisation and design requirements.

**Chapter 3** The systems to be designed in this work, i.e. passive electronic filters, are then presented and modelled. Defining the notion of passivity, and its lossless subcase, standard results on stability and  $\{x, y, z\}$ -dissipative characterisation of impedance and scattering of electrically-interconnected (lossless) passive elements are provided. A particular interconnection, called ladder topology, is also introduced.

Having this material, the design problem that is tackled in this work is explicitly stated. Reviewing the systematic filter design methods with a historical perspective, it appears that two design approaches stand out.

**Chapter 4** Both approaches are then revisited within our framework, in the particular case of lossless passive elements. First design approach, that we denote as *elements-value tuning*, consists in setting the elements in a given topology and tuning their characteristic values until the design requirements are satisfied. While traditionally achieved through the interface of a CAD software and running an underlying algorithm, we explicitly display a mathematical representation of the circuit and the associated optimisation problem. This is especially achieved thanks to the introduction of a particular representation coming from the Port-Hamiltonian System theory, and its associated modelling procedure. Coupled with the LFT representation, a BMI optimisation problem, being non-convex but of moderate complexity, is obtained to solve our design problem. An algorithm, based on sequential LMI relaxations, is then presented which requires a *good* initial point as starting point. Second design approach, denoted as *realisable filter synthesis*, consists in synthesising a transfer function such that it satisfies the frequency requirements *and* such that it corresponds to the scattering parameter  $s_{21}$  of a passive electronic filter. There exists a classical case, named *LC* ladder filters, for which there is an efficient synthesis method with convex realisation constraints. We convexly extend these realisation constraints for a larger class of filters that include *LC* ladder filters and certain *LC*-resonator ladder filters. This especially leads the resulting scattering parameter  $s_{21}$  to be a rational function of a given transfer function  $T(s)$ . Moreover, using common practices of designers, we provide realisations constraints that are partly convex for particular *AW*-resonators ladder filters, leading  $s_{21}$  to be a rational function of two transfer functions  $T_s(s)$  and  $T_p(s)$ . However, both types of realisation constraints lead to solve a general problem named frequency LFT filter synthesis.

**Chapter 5** The frequency LFT filter synthesis problem is then tackled. Using the particular connection between the LFT operation and rational functions, the usual filter synthesis problem, i.e. the synthesis of a transfer function such that it satisfies upper and lower bounds on the magnitude of its frequency response, is reformulated as the synthesis of an LFT in  $\frac{1}{s}$ . This leads to revisit the efficient synthesis method of frequency filters, which is usually based on two steps: the magnitude synthesis step, typically leading to an LMI feasibility problem, and the spectral factorisation step, solved using a certain Algebraic Riccati equation. It is then aimed to extend this method to the frequency synthesis of LFT in  $T(s)$ . Based on the  $\{x, y, z\}$ -dissipative characterisation, two cases appear. If  $T(s)$  is lossless  $\{x, y, z\}$ -dissipative, the previous efficient synthesis approach can be directly extended, up to a loop-shifting. If  $T(s)$  is simply  $\{x, y, z\}$ -dissipative, the spectral factorisation step can not be independently achieved from the magnitude synthesis step. Therefore, we develop a generalised approach that enables to efficiently tackle all these cases by solving an LMI optimisation problem. Finally, the synthesis problem of LFT in two lossless  $\{0, 1, 0\}$ -dissipative transfer functions  $T_s(s)$  and  $T_p(s)$  is tackled. Using a similar approach than previously, an LMI optimisation problem is proposed as sufficient condition to solve this problem. While being *conservative*, the resulting method will enable to find solutions in our cases of interest.

**Chapter 6** In the last chapter, two design examples of passive electronic filters are solved. The first example deals with the design of *LC*-resonators interconnected in a

ladder topology, for a bandpass application. Based on the convex realisation conditions found in Chapter 4 and the efficient synthesis approach developed in Chapter 5, the design example is optimally and efficiently solved. In particular, the optimality of the approach is compared with more traditional synthesis methods. The second example deals with the design of *AW*-resonators interconnected in a ladder topology, for a bandpass application. In order to solve this design example, it is proposed to combine both design approaches given in Chapter 4, consisting of three steps. First, the synthesis method of Chapter 5 for LFT filters in two lossless  $\{0, 1, 0\}$ -dissipative  $T_s(s)$  and  $T_p(s)$  is applied such that it includes the convex part of the associated realisation constraints given in Chapter 4. Second, as the resulting solution satisfies the frequency requirements but is not realisable as an *AW*-resonators ladder filter, an approximation is achieved, leading to a realisable filter but that does not necessary satisfies all the frequency requirements. Third, as this filter is close to be a solution, it is fed as initial point of the algorithm developed for the elements-values tuning design approach. This leads to solve the design example for a relatively low computational time. While both design examples are academics, the performance of these approaches is promising and is meant to be applied on practical design problems.

## 1.4 How to read this thesis

This manuscript is addressed to readers with different interests, such as filters design for mobile communications, frequency design of passive circuits, or simply system design. It is tried to meet these expectations by allowing different levels of reading. Due to our approach, some advanced concepts and tools of Control and System theory are involved. Even though they are mainly based on widespread Linear Algebra, the reader may not be familiar with them, and numerous illustrative and numerical examples are then provided all along this thesis with the aim of emphasising the main underlying ideas. However, a reader not interested in technical details may skip them and only read the first and last sections of each chapter, in which the main ideas are introduced and summarised. If a method draw one's attention, step-by-step procedures are also provided to directly implement the proposed methods. The resulting optimisation problems may be implemented using the Matlab software, and especially the LMI solvers<sup>4</sup> of the Robust Control Toolbox.

Nonetheless, for a reader interested in very particular points of this thesis, we distinguish four presumed points of interest, detailed below.

**Frequency design of passive circuits** Due to the traditional fashion in Electronics to model passive components using ideal resistances, inductances and capacitances, engineers may be led to design equivalent *RLC* circuits. One with such interest should refer to Section 4.2 of Chapter 4.

**Frequency (LFT) filter synthesis** The frequency filter synthesis is a standard problem with particular applications in Signal Processing, for frequency filtering, and Control theory, for the computation of  $H_\infty$ -weights. A reader may then be interested in the synthesis method given in Section 5.3 of Chapter 5 that is efficient and optimal, i.e. leading to a transfer function of minimum order. In Section 5.4 of the same chapter, one will find an efficient synthesis method for transfer functions expressed as LFTs in a repeated transfer function  $T(s)$ . Note that this requires to have a  $\{x, y, z\}$ -dissipative characterisation of  $T(s)$  (cf Section 2.5 of Chapter 2). Finally,

<sup>4</sup><https://mathworks.com/help/robust/lmis.html>

in Section 5.5 is provided a conservative but efficient method for the synthesis of 2D LFT filters.

### **Optimal and efficient synthesis of $LC$ , and $LC$ -resonator ladder filters**

These filters appear of little use for today's mobile applications, but may be still used, such as for instance in some low-power application [CRV11]. A reader interested in a step-by-step approach for a synthesis method that is efficient and optimal, i.e. with minimum number of elements, is invited to refer to Section 6.1 of Chapter 6.

### **Synthesis of $AW$ -resonator ladder filters**

A reader interested in a synthesis procedure of reduced complexity for  $AW$ -resonator ladder filters, but which requires an initial point, should apply Algorithm 4.1 (p. 84). To achieve this, one will especially needs to apply the modelling procedure of Subsection 4.2.2.2 (p. 61), in which a modelling example of  $AW$ -resonator ladder filters is provided. Furthermore, in Section 6.2 of Chapter 6 is proposed an approach to efficiently synthesise a suitable initial point.

## 1.5 Publications

The next papers are available on the following link

[https://hal.archives-ouvertes.fr/search/index/q/\\*/authIdHal\\_s/arthur-perodou](https://hal.archives-ouvertes.fr/search/index/q/*/authIdHal_s/arthur-perodou)

### **Extended abstract for a poster session**

- **A. Perodou**, A. Korniienko, M. Zarudniev, G. Scorletti, I. O'Connor, J-B. David. "On Motivations for Designing Analog Filters Under LFT Framework". Extended abstract. *20th IFAC World Congress*, Jul 2017, Toulouse, France.

### **Peer-reviewed conference papers**

- [PKSO18] **A. Perodou**, A. Korniienko, G. Scorletti, and I. O'Connor. "Systematic Design Method of Passive Ladder Filters using a Generalised Variable". In: *2018 Conference on Design of Circuits and Integrated Systems (DCIS)*. Nov 2018, Lyon, France.
- [PKZS18] **A. Perodou**, A. Korniienko, M. Zarudniev, and G. Scorletti. "Frequency Design of Interconnected Dissipative Systems: A Unified LMI Approach". In: *2018 IEEE Conference on Decision and Control (CDC)*. Dec 2018, Miami, FL, United States.



## Chapter 2

# A Modern System Design Framework

### 2.1 Introduction

Before stating and tackling the design problem of passive electronic filters, we introduce several concepts and tools used in this work. These concepts and tools mainly come from Control and System Theory, and are traditionally gathered into a generic framework for the analysis and the control of systems. As passive electronic filters are typically viewed *open-loop* systems, i.e. without the external intervention of a *controller*, the associated design problem does not straightly come under the scope of this framework. The objective of this chapter is then to provide an *adapted* design framework, which will be developed all along this manuscript.

This chapter starts with a brief explanation, in our view, of why the design of systems for modern engineering applications becomes increasingly complex (Section 2.2). This especially motivates the introduction of advanced tools thereafter. Indeed, it is first concluded that a relevant use of the computer power should be made, and thus design methods need to be developed in that respect. In the light of this, convex optimisation seems for us particularly suitable (Section 2.3). Second, *modern* systems are typically viewed as the interconnection of subsystems having *complex* models. Considering this feature, appropriate tools to mathematically represent (Section 2.4) and characterise (Section 2.5) the models of these subsystems and their interconnection is introduced. In particular, the characterisation of the subsystems leads to simple criteria to check the stability of their interconnection. In addition, this even enables to provide a simple procedure for the design a stable interconnection of homogeneous subsystems. Finally, a link between these tools, convex optimisation and the design requirements is made explicit (Section 2.6) through the introduction of a unifying tool, namely the KYP Lemma, converted from an analysis tool to a design tool for the occasion.

### 2.2 On the complexity of developing modern system design methods

As we will see throughout this manuscript, developing design methods for modern engineering applications is complex. In our view, two main reasons appear to justify this complexity: the design problem of *modern* systems is itself *complex*, while the design methods must be easily used by design engineers, and thus be also *modern*.

#### On the complexity of designing modern systems

Systems can be represented by models of different complexity, especially depending



on the physical phenomena taken into account. For passive electronic filters, simplest models generally consider components as the ideal electrical interconnection of ideal inductances, capacitances and resistances. The complexity of the models increases when complex physical phenomena are included, such as electromagnetism or electromechanics, or technological constraints, for instance the dimensions of the components. Some differences between a model and the physical system can not *a priori* be known, such as the manufacturing dispersion, and need to be added using *uncertain* parameters, increasing the complexity of the model even further. Usual design flows start with simple models and increase their complexity step-by-step [WL15]. There is generally a trade-off between the precision of the model and the time-efficiency of the design methods. Thus, from a certain step the design problem becomes too complex to be solved in a decent time. In practice, to tackle this issue, engineers add margins upstream to the requirements with the hope of including all the differences between a system and its model.

Using models allows then to address the design problem by tackling sub-problems of lower complexity. However, even lower, this complexity may be still important. As an example, consider the problem of designing a filter as a circuit made of inductors and capacitors, such that it removes undesirable frequency components from an input signal. The simplest associated design sub-problem is obtained by modelling inductors and capacitors by respectively ideal inductances and capacitances, their interconnection by the ideal electrical interconnection, and the objective as lower and upper bounds on the magnitude of the frequency-response of the filter. This simple form raises some important questions.

1. Does there exist a transfer function which satisfies the frequency requirements?
2. If so, do there exist ideal inductances and capacitances such that their interconnection corresponds to this transfer function?

These key questions have been extensively studied. There are particular cases having positive answers to these questions, as will be illustrated in this manuscript. However, in general, and for most of interesting applications, only part of these questions have an answer. In addition, this answer may produce constraints that are too complex to be included in the design process. Therefore, the problem of *system design*, even for simple systems, is inherently complex.

Moreover, another source of complexity is the increasing sophistication of modern systems, which tend to provide increasingly complex models. This leads to increase the number and the complexity of steps to validate in the design process. The design flow is then longer. To illustrate this idea, take the same example as above but with other components, such as acoustic wave resonators, instead of inductors and capacitors. As usual in Electronics, these components can be modelled by a particular interconnection of inductances and capacitances. The wanted filter is then modelled as the interconnection of interconnected inductances and capacitances. Hence, the resulting design problem is even more complex than previously, as the model of the filter is not only required to satisfy structural constraints on the global stage of interconnection but also on the local interconnection of the inductances and capacitances.

### Easy-to-use design methods

An important point that should not be overlooked when developing design methods is the target audience: here, the design engineers. Indeed, to be accepted by seasoned designers accustomed to use certain methods, a new design method needs to be easy-to-use in practice. More importantly, this method should be able to include the good

practices and the intuition of seasoned designers accumulated over the years, in order to reduce the complexity of the design problem.

To reach this objective, an indispensable modern tool is the Computer-Aided Design (CAD) software. Using already-implemented methods, it improves the design flow by preventing the user to make tedious calculations, potentially source of errors. Furthermore, it enables to get the most from computer power, which is essential to deal with the complexity of the design problem. For example, many design simulations can be achieved, and the designer may have an intuition to obtain a solution.

In order to be user-friendly, most of CAD softwares contain an interface between the user and the implemented design methods. This interface is usually in the form of a graphical representation. Graphical representation is a type of qualitative model used everywhere in Science. It is a simple, synthetic, pedagogical, abstract and visual mean to represent and interpret complex phenomena and concepts. When it comes to Engineering, these drawings take the form of a *block diagram*, i.e. a diagram where some *blocks* are connected by lines. Indeed, the blocks represent the subsystems while the lines enable to represent their mathematical relations. Moreover, its modularity property is especially useful for simulation software.

Therefore, a new design method needs to be able to be easily included into a CAD software, user-friendly and computationally efficient.

### Design framework

In the sequel, we detail the framework that appears for us to be the most appropriate for the scope of this work. Starting from the aim of developing CAD methods, a particular class of optimisation problem is exhibited. Mathematical tools to represent and characterise *modern* systems are then developed in order to bring the design problem into this class.

## 2.3 Efficient use of computer power: convex optimisation

Among all requirements mentioned in the last section, the requirement to be computer friendly is the limiting factor when developing design methods. Indeed, it would be of no use to develop advanced design methods, if they can not be easily included in a CAD software and *efficiently* implemented. The last point is particularly important. For centuries, mathematical problems have been analytically and graphically solved, using paper and pencil, and solving methods have been developed for this purpose. The advent of computers has provided a tool with a dramatic calculation power, but requires a transformation of the problem from the mathematical language into the computer language. For decades, the efficiency of this transformation has not been a major concern for engineers, as the computational power has regularly increased and seemed to be virtually unlimited. Understandably, the paradigm was to get the suitable computer rather than numerical methods with low computational cost. However, with the end of Moore's law coupled with the increase of complexity of engineering problems, going along with social concern for energy saving, it is becoming more and more crucial to formulate these problems under a *suitable* form for computers.

To understand what is meant by suitable, let us consider the following solving scheme. Assume that one has a mathematical formulation associated with an engineering problem. The first step is to transform it into an *optimisation problem*. We

define here an optimisation problem as follows:

$$\begin{array}{ll} \min_{\mathbf{x} \in \mathbb{R}^m} & f(\mathbf{x}) \\ \text{such that} & \mathbf{x} \in \mathcal{C} \end{array}$$

where  $\mathbf{x}$  is the vector of the optimisation variables,  $f$  is a real-valued function called *objective function* and  $\mathcal{C}$  the *constraints set*. In particular, to test if the set  $\mathcal{C}$  is not empty, i.e. if there exists  $\mathbf{x} \in \mathcal{C}$ , is called a *feasibility problem*. This step is generally not simple. Indeed, there are potentially many optimisation problems associated with a mathematical problem. A choice is required and usually depends on the next step.

The second step is about running an algorithm in order to solve the resulting optimisation problem. A *naive* approach would be to develop an algorithm for each problem. In addition to require some non-trivial skills in order to get a numerically stable algorithm which converges in a *decent* computational time to a solution close to the optimum, if it exists, this approach significantly increases the design flow, as there is a large diversity of engineering problems.

Instead, classes of optimisation problems have been formulated. The idea is pretty simple. If a mathematical problem can be transformed as an instance of a class of optimisation problems, one can use with little programming effort the provided resolution algorithm, for which all the parameters have been suitably chosen. The first objective was to develop a generic algorithm which solves the widest possible class of optimisation problems. However, it appeared that optimisation problems may be of different *complexity* [GJ79]. Thus, a generic algorithm may be time-consuming to solve problems of moderate complexity. Therefore, a modern approach has emerged and consists in making a trade-off between the generality of the class and the time-efficiency of the associated resolution algorithm. While one may find algorithms with less computational time for a given optimisation problem, this approach appears more adequate for the system design researcher aiming to develop prototype numerical methods for different engineering problems.

For many engineering problems, including the one of this work, the most interesting class appears for us to be the class of *convex* problems [BTN01]. From a mathematical point of view, convex problems are optimisation problems for which  $\mathcal{C}$  is a convex set and  $f$  a convex function on  $\mathcal{C}$ . Its most striking property is that any local minimum is also a global minimum. Furthermore, while this class is not endowed with generic algorithms that are time-efficient, it includes several sub-classes with such property (Figure 2.1). Indeed, convex optimisation is the generalisation of Linear Programming. Linear Programming has been introduced and heavily studied in the second-half of the 20<sup>th</sup> century, and resolution algorithms have been developed for this class of optimisation problems. This has permitted to solve rather simple engineering problems, but this class is too small to be used for most applications in practice. At end of the 80s, the increase of the computational power coupled with the development of modern algorithms, and especially the generalisation of the so-called *interior-point* methods, provided *generic* and *efficient* methods to solve Quadratic Programming problems and the broad subclass of Semidefinite Programming problems [BTN01]. By efficient, it is meant that algorithms have worst-case polynomial complexity, corresponding to a computational time from less than a second to few minutes for average-size engineering problems. For some applications, real-time computation is even possible [MB10].

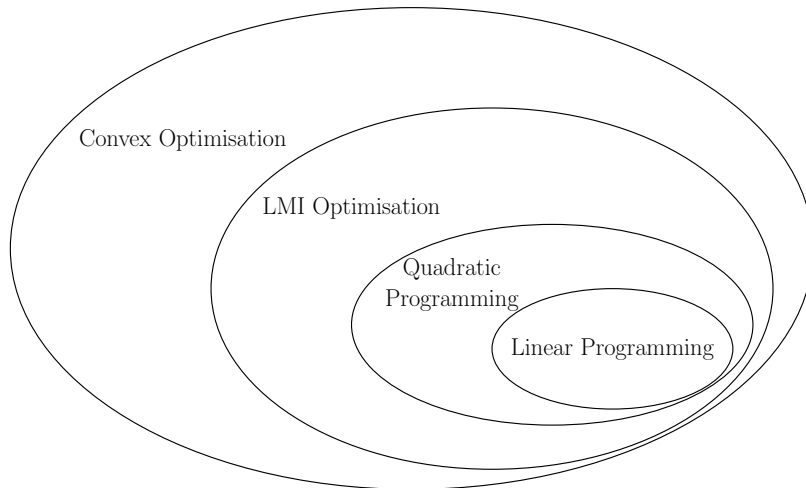


FIGURE 2.1: Convex optimisation and its subclasses

Semidefinite Programming is then a particular subclass of convex problems for which the constraints set is defined by positive semi-definite inequalities<sup>1</sup>

$$\mathcal{C} = \{x \in \mathbb{R}^m \mid F(x) \geq 0\}, \quad F(x) := F_0 + \sum_{i=1}^m x_i F_i$$

where  $x = [x_1 \ \dots \ x_m]^T$  is the vector of decision variables and  $F_i = F_i^T \in \mathbb{R}^{n \times n}$ ,  $i = 1, \dots, m$ , are given symmetric matrices.

Semidefinite Programming is often referred to as Linear Matrix Inequality (LMI) optimisation, as this will be the case in this manuscript, as  $\mathcal{C}$  is usually formulated in terms of a linear inequality<sup>2</sup> in an optimisation matrix variable [BTN01]. In particular, when the objective function is linear, i.e  $f(x) = c^T x$  for a given vector  $c \in \mathbb{R}^m$ , the resulting optimisation problem is denoted as a linear minimisation problem over LMI constraints.

Really few engineering problems have *naturally* a mathematical formulation which can be directly transformed into an LMI optimisation problem. There exist some standard techniques to reduce some particular constraints to LMIs [Sco06], but this is not sufficient for a straightforward transformation. Therefore, this paradigm has moved the difficulty from the level of algorithms to the level of mathematical reformulation of problems. This is especially beneficial as it allows a more explicit link with the original engineering problem than parameters of a generic algorithm, which may help to reduce the complexity of the resulting optimisation problem, as will be illustrated in this manuscript.

In Control and System theory, people have been historically used to juggle with mathematical formulations in order to find appropriate ones. In the beginning of the 90s, LMI optimisation has encountered a great success when it was illustrated that, among all these formulations, LMIs frequently appear [BGFB94]. The enthusiasm has reached a few years latter the Signal Processing community, and in particular those working on digital filters design [Dav10].

Therefore, in the remaining sections of this chapter we provide and adapt mathematical tools coming from Control and System theory, with the aim of formulating

<sup>1</sup>or positive definite inequalities  $\mathcal{C} = \{x \in \mathbb{R}^m \mid F(x) > 0\}$

<sup>2</sup>actually affine

the design problem considered in this work under a framework leading to LMI optimisation problems.

## 2.4 Modern systems representation via LFT

In order to forge a design framework adapted to modern systems, the first tool to consider is the mathematical representation of their models. In this work, we are interested in finite-dimension, Linear Time-Invariant (LTI) models of dynamical systems, generally abbreviated in LTI systems. Roughly, this means that, under mere assumptions, models can be represented in the time-domain, involving linear differential equations with constant coefficients, or equivalently in the complex-domain, involving algebraic equations in the Laplace variable  $s$ . While this may seem to be restrictive, this class of models is sufficiently large within our context, as systems will be modelled by ideal electrical interconnection of ideal inductances, capacitances and resistances.

There exist several mathematical representations of an LTI system. In the sequel, two traditional representations of Control theory, namely the *transfer function* and the *state-space representation*, are first briefly presented. However, while suitable for representing the subsystems, both representations appear limited when dealing with their interconnection. A generalisation called linear fractional representation is then introduced.

Historically, the input-output approach, where a system is viewed as a black-box receiving information from the environment thanks to some *inputs* and acting on it using *outputs*, has been the first successful paradigm to emerge. In this approach, a system is represented using transfer functions, which is very convenient from the engineering point of view as it allows to easily express the performance specifications. Assuming that a given LTI system ( $\Sigma$ ) has only a single input  $w$  and a single output  $z$  (SISO), it may be represented using a transfer function  $H$  as follows:

$$(\Sigma) : \left\{ z(s) = H(s)w(s), \quad H(s) = \frac{b_0 + b_1s + \dots + b_ms^m}{a_0 + a_1s + \dots + a_ns^n} \right. \quad (2.1)$$

In Control theory, another way of representation emerged in the 50s, with notable success, named state-space representation. Using this representation, the *internal* relations of a system are made explicit using the so-called state-space equations. This representation has the important advantage to be computer friendly, and a significant number of control problems can be formulated as LMI optimisation problems [BBFG94]. For some given real matrices  $A, B, C, D$ , a state-space representation of ( $\Sigma$ ), expressed in the complex domain, is of the form:

$$(\Sigma) : \begin{cases} p(s) = \left(\frac{1}{s} \cdot I\right) q(s) \\ q(s) = Ap(s) + Bw(s) \\ z(s) = Cp(s) + Dw(s) \end{cases} \quad (2.2)$$

Both representations are known to be equivalent under mere assumptions. Indeed, to any state-space representation (2.2) is associated a transfer matrix  $H(s)$  given by:

$$H(s) = D + C \left(\frac{1}{s} \cdot I\right) \left(I - A \left(\frac{1}{s} \cdot I\right)\right)^{-1} B \quad (2.3)$$

Reciprocally, to any *proper*  $H(s)$ , i.e.  $m \leq n$ , can be associated a state-space representation. For example, assuming without loss of generality  $a_n = 1$  and defining  $b_{m+1} = \dots = b_n = 0$  if  $m < n$ , the following matrices leads to a state-space representing admitting  $H(s)$  of (2.1) as transfer function:

$$A = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -a_0 & -a_1 & -a_2 & \dots & -a_{n-1} \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \quad C = \begin{bmatrix} b_0 - a_0 b_n \\ b_1 - a_1 b_n \\ \vdots \\ b_{n-2} - a_{n-2} b_n \\ b_{n-1} - a_{n-1} b_n \end{bmatrix}^T \quad D = b_n$$

An important feature of both representations for design perspectives is their link with block diagrams. Indeed, a transfer function and a state-space representation are realisable as the ideal interconnection of three types of block. These are the *sum* block, which sums its inputs, the *gain* block, which multiplies the input by a constant value, and the *integrator* block, which multiplies (complex) inputs by  $\frac{1}{s}$ . Using previous notations, an example of realisation of (2.2) with  $n = 2$  is provided on Figure 2.2 (a).

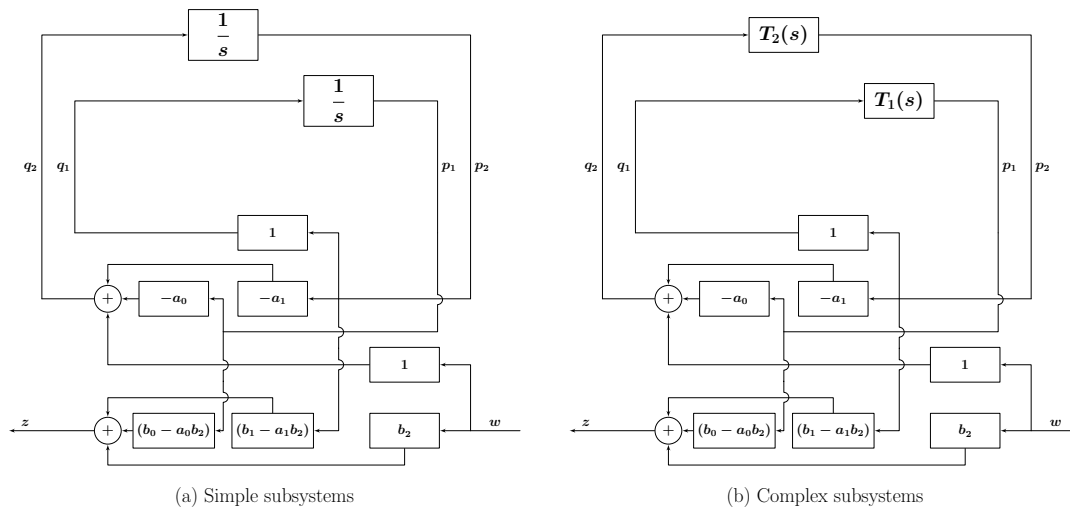


FIGURE 2.2: Block diagram representations

Therefore, both representations can be synthesised as the interconnection of subsystems having identical transfer function  $\frac{1}{s}$ . In our view, these subsystems are *simple*, as they have transfer functions of least order, excepted constant gains. From an Electronics interpretation, the integrator block may be viewed as a unitary inductance or a unitary capacitance, as they both have  $\frac{1}{s}$  as a representative transfer function, either the admittance or the impedance, and the gain block as a resistance.

However, an important feature of *modern* systems is their interpretation as the interconnection of *complex* subsystems, i.e. having a transfer function  $T(s)$  of order higher than one. An example of such subsystem, that will play an important role in this work, is the electronic component named acoustic wave resonator, which has an impedance proportional to the transfer function  $T(s) = \frac{s^2 + \omega_r^2}{s(s^2 + \omega_a^2)}$ . A block diagram illustration for two subsystems having respectively transfer functions  $T_1(s)$  and  $T_2(s)$  is provided on Figure 2.2 (b).

From a design perspective, the previous representations appear limited when considering modern systems. Indeed, it seems difficult to link these representations to

the realisation of interconnected subsystems with a different transfer function than  $\frac{1}{s}$ . Therefore, we introduce another representation, based on the mathematical function Linear Fractional Transformation (LFT), which generalises both transfer function and state-space representation and will appear more suitable for the design of modern systems. A definition adapted from [ZDG96, Chap.10] for the use of this work is given below.

**Definition 2.1 (Linear Fractional Representation/Transformation).**

A Linear Fractional Representation (LFR) of a system is a set of equations

$$\begin{cases} p = \Delta q \\ q = Ap + Bw \\ z = Cp + Dw \end{cases}$$

with  $A \in \mathbb{R}^{n \times n}$ ,  $B \in \mathbb{R}^{n \times n_w}$ ,  $C \in \mathbb{R}^{n_z \times n}$ ,  $D \in \mathbb{R}^{n_z \times n_w}$  are real matrices,  $\Delta$  is a  $n \times n$  transfer matrix, and  $q, p, z, w$  are complex signals of appropriate dimensions. Provided that  $(I - A\Delta(s))^{-1}$  exists, the associated Linear Fractional Transformation (LFT) is defined as:

$$\Delta(s) \star \left[ \begin{array}{c|c} A & B \\ \hline C & D \end{array} \right] := D + C\Delta(s)(I - A\Delta(s))^{-1}B$$

and links  $z$  and  $w$  by:  $z = \left( \Delta(s) \star \left[ \begin{array}{c|c} A & B \\ \hline C & D \end{array} \right] \right) w$ .

*Remark 2.1.* In the literature, the last definition corresponds to *upper* LFR/LFT, and the matrices  $A, B, C$ , and  $D$  are complex [ZDG96, Chap.10].

The LFT tool provides a framework which generalises the state-space representation and the transfer function tools. In particular, when  $\Delta(s) = \frac{1}{s} \cdot I$ , the LFT provides the transfer function of (2.1) and the state-space representation of (2.2) is found back with the LFR. From a system perspective, the state-space representation is viewed as the interconnection of a constant gain  $\left[ \begin{array}{c|c} A & B \\ \hline C & D \end{array} \right]$  and homogeneous subsystems with transfer function  $\frac{1}{s}$ . Notice that, while the state-space representation only permits to generate proper transfer functions, the LFR allows to consider also non-proper transfer functions, with for instance  $\Delta(s) = s \cdot I$ . As mentioned, without any further assumption,  $\Delta = \frac{1}{s} \cdot I$  and  $\Delta = s \cdot I$  represent *simple* subsystems. The LFR allows to consider more complex subsystems. For instance, by setting  $\Delta(s) = T(s) \cdot I$ , the resulting LFR represent the interconnection of homogeneous subsystems having  $T(s)$  as transfer function. Another example is the case of  $\Delta(s) = \begin{bmatrix} T_1(s) & 0 \\ 0 & T_2(s) \end{bmatrix}$  stands for the interconnection of two subsystems  $T_1(s)$  and  $T_2(s)$ . Therefore, the LFT is a very flexible tool to synthetically represent the interconnection of complex subsystems.

Moreover, an important property of the LFT is that the interconnection of LFTs is again an LFT. Most common interconnections (serial, parallel, cascade, feedback,...), and the formulas to calculate the new LFT, are provided in [ZDG96, Chap.10] and are based on the Redheffer star product  $\star$  (cf Appendix A, p. 195), of which the LFT is a special case.

Furthermore, the LFT tool is also appropriate for dealing with certain traditional mathematical objects. Indeed, any polynomial or rational function can be represented by an LFT [ZDG96, Chap.10]. Transfer functions, as rational functions of the Laplace

variable  $s$ , are obviously included in the LFT framework. But the LFT framework also includes rational functions of  $T(s)$ , or even of multiple  $T_1(s), \dots, T_k(s)$ . This nice property will help us to encompass the local structure of the subsystems.

Last but not least, the LFT framework contains a dedicated block diagram representation (Figure 2.3 (a)). It can be linked with the interconnection of complex subsystems. As an example, the system of Figure 2.2 (b) corresponds to Figure 2.3 (b). While this diagram representation is somewhat different from usual electrical block diagrams, it synthetically represents the interconnection of systems and make a direct link with the underlying mathematical representations. This property is again very interesting for CAD methods as it allows a graphical construction of interconnected systems.

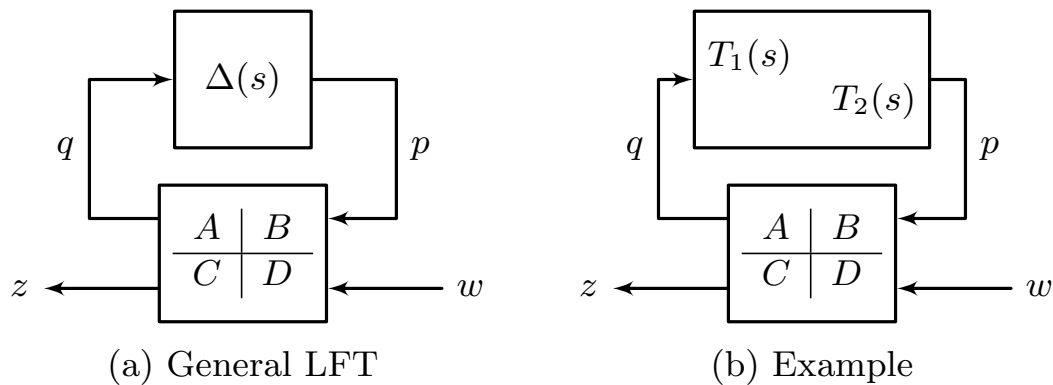


FIGURE 2.3: LFT block diagram representation

To sum up, the LFT tool enables to represent the interconnection of complex systems. It is a generalisation of the transfer function and the state-space representation, and enables to use the advantages of both approaches. Moreover, it can synthetically represent interconnection of complex systems. The LFT tool is also a very generic mathematical tool. Different mathematical objects, such as rational functions of generalised variables, are special cases of LFT. Finally, a block diagram representation is associated to the LFT, which visually represents mathematical operations. The LFT seems then an appropriate candidate for representing systems and their interconnection in a CAD software.

## 2.5 Dissipative systems characterisation

The LFT function provides a suitable representation for modern systems, viewed as the interconnection of complex subsystems. The objective now is to provide a mathematical characterisation of these subsystems and their interconnection to be included in our design framework.

To achieve this, a characterisation coming from the Dissipative Systems theory, originally introduced in [Wil72a; Wil72b], appears appropriate. Indeed, this characterisation, namely  $\{X, Y, Z\}$ -dissipative characterisation, essentially leads to a frequency-domain quadratic constraint on the transfer function of the considered system. This is beneficial as this type of constraints can be typically transformed into an LMI optimisation problem, as detailed in the next section. Moreover, using the dissipative characterisation, stability of interconnected subsystems can be established by essentially checking the satisfaction of another frequency-domain quadratic constraint, leading then also to an LMI problem. In particular, when the subsystems



are homogeneous, i.e. with same transfer function, this constraint appears especially simple to check, and even enables to provide a simple process to compute a stable interconnection.

Before introducing the dissipative characterisation, let us define the concepts of *stability* and *marginal stability*. Roughly, a system is said to be marginally stable if its output and its internal signals associated with non-zero initial conditions remain bounded over time. It is said to be stable if in addition these signals decay to zero, and if they remain bounded when a bounded input is applied. For LTI systems, it is known that both definitions are equivalent to criteria on the matrices of a state-space representation (refer for instance to [AV73, Chap. 3]). For the sake of simplicity, these criteria will be used in this work as definitions. Remarkably, these criteria can be transformed into LMI feasibility problems using the so-called Lyapunov Lemma (cf Theorem A.6 and Theorem A.7, p. 194 of Appendix A).

**Working Definition 2.2 (Marginal Stability).**

A system is said to be *marginally stable* if it admits a state-space representation such as in (2.2) where the eigenvalues of  $A$  have non-positive real part  $\Re(\Lambda(A)) \leq 0$ , and pure imaginary eigenvalues occur only in  $1 \times 1$  blocks in the Jordan form<sup>3</sup> of  $A$ .

**Working Definition 2.3 (Stability).**

A system is said to be *stable* if it admits a state-space representation such as in (2.2) where the eigenvalues of  $A$  have negative real part  $\Re(\Lambda(A)) < 0$ .

These criteria can be propagated from a state-space representation to the associated transfer function, under some assumptions. These assumptions are known as *controllability* and *observability* and are defined below. Once again, while both concepts have a meaningful interpretation in Control and System theory, we are only interested in their equivalent algebraic criteria [AV73, Chap. 3] for the purpose of this work.

**Working Definition 2.4 (Controllability/Observability).**

A pair  $(A, B)$  of matrices  $A \in \mathbb{R}^{n \times n}$  and  $B \in \mathbb{R}^{n \times n_w}$  is *controllable* if it satisfies<sup>4</sup>:

$$\text{rank} \begin{bmatrix} B & AB & A^2B & \dots & A^{n-1}B \end{bmatrix} = n$$

A pair  $(A, C)$  of matrices  $A \in \mathbb{R}^{n \times n}$  and  $C \in \mathbb{R}^{n_z \times n}$  is *observable* if the pair  $(A^T, C^T)$  is controllable.

In particular, when a state-space representation as (2.2) is such that  $(A, B)$  is controllable and  $(A, C)$  is observable, the representation is said to be *minimal*. Using this property, one can check the stability and marginal stability on the poles of the associated transfer matrix [AV73, Chap. 3].

**Property 2.1.**

Let (2.2) be a minimal state-space representation of a given system  $(\Sigma)$ . Define  $H(s)$  as the transfer matrix resulting from (2.3). It comes then:

1. If the poles of  $H(s)$  have non-positive real part, and purely imaginary poles are all distinct from one another, then  $(\Sigma)$  is marginally stable.

By extension, the transfer matrix  $H(s)$  will be said to be marginally stable.

<sup>3</sup>Cf the definition of the Jordan form given in Theorem A.2 (p. 191) of Appendix A.

<sup>4</sup>Rigorously, the displayed formula is correct only if  $n \geq 3$ . If  $n = 1$  or  $n = 2$ , this formula respectively reduces to  $\text{rank } B = 1$  and  $\text{rank} \begin{bmatrix} B & AB \end{bmatrix} = 2$ .

2. If the poles of  $H(s)$  have negative real part then  $(\Sigma)$  is stable.  
By extension, the transfer matrix  $H(s)$  will be said to be stable.

The  $\{X, Y, Z\}$ -dissipative characterisation is now defined. As for stability, we use for simplicity an equivalent characterisation of the usual concept of dissipativeness as definition. Compared to the literature, this characterisation is closely connected to the definition used in [TW91] and can be viewed as a weaker version<sup>5</sup> of the  $(Q, S, R)$ -dissipativity given in [MH78; Moy14].

**Working Definition 2.5** ( $\{X, Y, Z\}$ -Dissipativity).

Let  $X = X^T \in \mathbb{R}^{n_z \times n_z}$ ,  $Y \in \mathbb{R}^{n_z \times n_w}$ ,  $Z = Z^T \in \mathbb{R}^{n_w \times n_w}$  be real matrices such that

$$\begin{bmatrix} X & Y \\ Y^T & Z \end{bmatrix}$$

is full rank. A marginally stable transfer matrix  $T(s)$  is  $\{X, Y, Z\}$ -dissipative if:

$$\forall \omega \in \mathbb{R}, j\omega \text{ not a pole of } T(s), \quad \begin{bmatrix} T(j\omega) \\ I \end{bmatrix}^* \begin{bmatrix} X & Y \\ Y^T & Z \end{bmatrix} \begin{bmatrix} T(j\omega) \\ I \end{bmatrix} \geq 0 \quad (2.4)$$

This transfer matrix  $T(s)$  is *lossless*  $\{X, Y, Z\}$ -dissipative if:

$$\forall \omega \in \mathbb{R}, j\omega \text{ not a pole of } T(s), \quad \begin{bmatrix} T(j\omega) \\ I \end{bmatrix}^* \begin{bmatrix} X & Y \\ Y^T & Z \end{bmatrix} \begin{bmatrix} T(j\omega) \\ I \end{bmatrix} = 0 \quad (2.5)$$

This transfer matrix  $T(s)$  is *strictly*  $\{X, Y, Z\}$ -dissipative if:

$$\forall \omega \in \mathbb{R}, j\omega \text{ not a pole of } T(s), \quad \begin{bmatrix} T(j\omega) \\ I \end{bmatrix}^* \begin{bmatrix} X & Y \\ Y^T & Z \end{bmatrix} \begin{bmatrix} T(j\omega) \\ I \end{bmatrix} > 0 \quad (2.6)$$

As a matter of illustration, the impedance  $z(s) = R$  of a non-negative resistance  $R \geq 0$  is  $\{X, Y, Z\}$ -dissipative with  $X = Z = 0$  and  $Y = 1$  as:

$$\begin{bmatrix} z(j\omega) \\ 1 \end{bmatrix}^* \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} z(j\omega) \\ 1 \end{bmatrix} = z(j\omega)^* + z(j\omega) = R \geq 0$$

Similarly, the impedance  $z(s) = L \cdot s$  of an inductance  $L$  and the impedance  $z(s) = \frac{1}{C \cdot s}$  of a capacitance  $C$  are both lossless  $\{0, 1, 0\}$ -dissipative as:

$$\begin{bmatrix} z(j\omega) \\ 1 \end{bmatrix}^* \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} z(j\omega) \\ 1 \end{bmatrix} = z(j\omega)^* + z(j\omega) = 0$$

These particular dissipative characterisations will be linked in the next chapter to two concepts that will play an important role in this work: *passivity* and *lossless passivity*.

From a geometric interpretation, constraint (2.4) imposes  $T(j\omega)$  to belong to a subset of  $\mathbb{C}^{n_z \times n_w}$ , defined by the matrices  $X, Y, Z$ . For sake of graphical illustration, assume that the system is SISO, i.e.  $n_w = n_z = 1$ , and  $T(s)$  is  $\{x, y, z\}$ -dissipative, with  $x, y, z \in \mathbb{R}$  three scalars. Different cases appear following the sign of  $x$ . Indeed, if  $x = 0$  then (2.4) is equivalent to:

$$\forall \omega \in \mathbb{R}, \quad y(T(j\omega) + T(j\omega)^*) + z \geq 0$$

<sup>5</sup>named as *cyclodissipativity*

Hence for all  $\omega \in \mathbb{R}$ ,  $T(j\omega)$  belongs to a half-plane of  $\mathbb{C}$ . If  $x < 0$ , then (2.4) is equivalent to:

$$\forall \omega \in \mathbb{R}, \quad \left( T(j\omega) + \frac{y}{x} \right)^* \left( T(j\omega) + \frac{y}{x} \right) \leq \frac{y^2}{x^2} - \frac{z}{x}$$

Similarly, for all  $\omega \in \mathbb{R}$ ,  $T(j\omega)$  belongs to a disk of centre  $c$  and radius  $r$  with

$$c := -\frac{y}{x} \quad r := \sqrt{\frac{y^2}{x^2} - \frac{z}{x}}$$

An example of such a disk is provided on Figure 2.4. The last case  $x > 0$  leads  $T(j\omega)$  to belong to the complement of the previous disk in the complex plane  $\mathbb{C}$ . This can be generalised in the multidimensional case, where  $T(j\omega)$  is constrained by (2.4) to be inside an ellipsoid, into an hyperplane, or outside an ellipsoid. In the lossless case,  $T(j\omega)$  is simply constrained to belong to the boundary of the set generated by constraint (2.4). In the SISO case, when  $\geq$  is replaced by  $=$ , the (2.4) constrains  $T(j\omega)$  to belong to a vertical line in  $-\frac{z}{y}$  when  $x = 0$ , and to a circle of centre  $c$  and radius  $r$  when  $x \neq 0$ , in the complex plane  $\mathbb{C}$ .

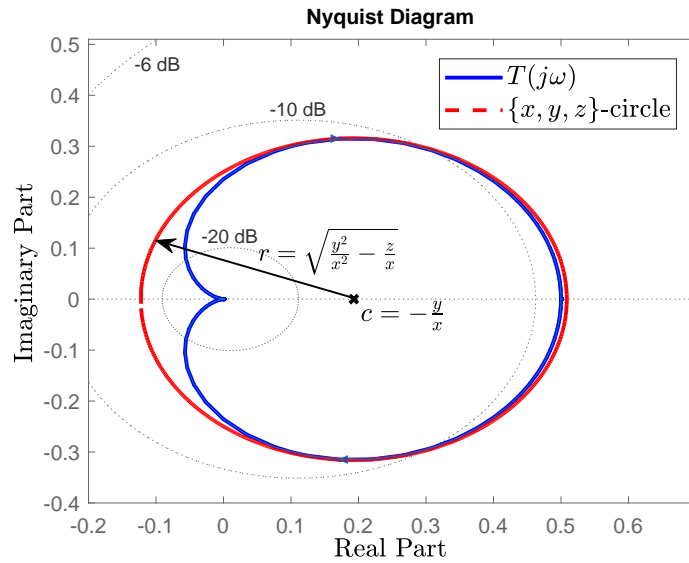


FIGURE 2.4: Example - Geometrical interpretation of the  $\{x, y, z\}$ -dissipativity in the Nyquist plot

*Remark 2.2.* In most cases, we will add the constraints  $x \leq 0$ , for  $T(j\omega)$  to belong to a convex subset of  $\mathbb{C}$ , and  $z \geq 0$  to ensure that  $\frac{y^2}{x^2} - \frac{z}{x} \geq 0$ , and thus the associate radius  $r$  exists.

Obviously, there is not a unique  $\{X, Y, Z\}$  parametrisation associated with a transfer matrix  $T(s)$ . First, constraints (2.4) can be multiplied by any positive scalar  $\alpha > 0$ , and thus  $T$  is also  $\{\alpha X, \alpha Y, \alpha Z\}$  dissipative. In the particular case where  $\{X, Y, Z\} = \{xI, yI, zI\}$  and  $x, y, z$  are scalars, one can multiply (2.4) by any positive-definite matrix  $P = P^T > 0$  and then  $T(s)$  is also  $\{xP, yP, zP\}$ -dissipative. Notice that, in the lossless case, this matrix  $P$  is only required to be *invertible*. The geometrical set generated by these constraints remains the same. Notice that both  $\alpha$  and  $P$  are denoted as *multipliers* and will play an important role in the next section.

Second, a transfer function  $T(s)$  may be  $\{x_1, y_1, z_1\}$ - and  $\{x_2, y_2, z_2\}$ -dissipative, where these parametrisation are not linked as previously. Geometrically, this means that the Nyquist plot of  $T(s)$  belongs to different disks. However, there exists a disk of minimal radius. A choice of the  $\{x, y, z\}$  parametrisation of this minimum-radius disk can be computed by solving an LMI optimisation problem, as illustrated in the next section.

Reversely, several transfer functions may satisfy the same  $\{x, y, z\}$ -dissipative characterisation. A  $\{x, y, z\}$  parametrisation defines then a family of systems. For example, one can easily check that  $T_1(s) = \frac{1}{s}$  and  $T_2(s) = \frac{s}{s^2+1}$  are both lossless  $\{0, 1, 0\}$ -dissipative. This feature, especially used in Robust Control, will be of importance in this work. For instance, as  $\frac{1}{s}$  is lossless  $\{0, 1, 0\}$ -dissipative, it will be used to directly extend several classical results involving state-space representation to LFT representations in a repeated, lossless  $\{0, 1, 0\}$ -dissipative, transfer function  $T(s)$ .

We finish this section by providing a sufficient condition to ensure the stability of interconnected subsystems using the  $\{X, Y, Z\}$ -dissipative characterisation. This result is adapted from [Moy14] and may be viewed as a slight modification of the so-called theorem of separation of graph [SD01]. Roughly, it can be stated as follows: a system, defined as the interconnection of subsystems, is stable if the subsystems are  $\{X, Y, Z\}$ -dissipative and the interconnection is *strictly*  $\{-Z, -Y^T, -X\}$ -dissipative. In the particular case of an LFT representation, where the interconnection is only made of constant gains and sum blocks, this directly leads to an LMI.

**Theorem 2.1** (adapted from [Moy14]).

Let  $(\Sigma)$  be a system represented by the LFT

$$\Delta(s) \star \left[ \begin{array}{c|c} A & B \\ \hline C & D \end{array} \right]$$

where  $A, B, C, D$  are real, constant matrices of appropriate dimensions, and  $\Delta(s)$  is a given  $\{X, Y, Z\}$ -dissipative transfer matrix defined by

$$\Delta(s) = \left( \frac{1}{s} \cdot I \right) \star \left[ \begin{array}{c|c} A_\Delta & B_\Delta \\ \hline C_\Delta & D_\Delta \end{array} \right]$$

where  $A_\Delta, B_\Delta, C_\Delta, D_\Delta$  are real, constant matrices of appropriate dimensions. Assume there exists a real matrix  $K_\Delta$  such that  $(I - D_\Delta K_\Delta)$  is invertible and

$$\begin{bmatrix} K_\Delta \\ I \end{bmatrix}^T \begin{bmatrix} -Z & -Y^T \\ -Y & -X \end{bmatrix} \begin{bmatrix} K_\Delta \\ I \end{bmatrix} > 0 \quad (2.7)$$

$$\Lambda \left( A_\Delta + B_\Delta K_\Delta (I - D_\Delta K_\Delta)^{-1} C_\Delta \right) \subset \mathbb{C}^- \quad (2.8)$$

If  $A$  satisfies the inequality

$$\begin{bmatrix} A \\ I \end{bmatrix}^T \begin{bmatrix} -Z & -Y^T \\ -Y & -X \end{bmatrix} \begin{bmatrix} A \\ I \end{bmatrix} > 0 \quad (2.9)$$

Then the system  $(\Sigma)$  is stable.

Constraints (2.7) and (2.8) may appear uselessly convoluted at first sight. They are actually here to reduce the set of multipliers to those which are *well-posed*. As already mentioned, our characterisation of  $\{X, Y, Z\}$ -dissipativeness is weaker than the one of [Moy14]. As this stability result only holds for the latest, these constraints

enable to bridge the gap between both characterisations. From a Control point of view, they can be interpreted as having any  $\{X, Y, Z\}$ -dissipative system being stable when it is closed on an output feedback gain  $K$ .

*Remark 2.3* (Link with the Lyapunov Lemma). Consider the particular case where

$$\Delta(s) = \frac{1}{s} \cdot I \quad \text{and} \quad \left[ \begin{array}{c|c} A_\Delta & B_\Delta \\ \hline C_\Delta & D_\Delta \end{array} \right] = \left[ \begin{array}{c|c} 0 & I \\ \hline I & 0 \end{array} \right]$$

Then,  $\Delta(s)$  is lossless  $\{0, P, 0\}$ -dissipative where  $P = P^T$  is any nonsingular, real symmetric matrix. Constraints (2.7) and (2.8) of last theorem require that:

$$K_\Delta^T(-P) + (-P)K_\Delta > 0 \quad \text{and} \quad \Lambda(K_\Delta) \subset \mathbb{C}^-$$

This enforces<sup>6</sup> the matrix  $P$  to be positive-definite  $P > 0$ . Combined with the sufficient condition (2.9) for stability, this leads to the traditional condition of the Lyapunov Lemma (Theorem A.6, p. 194):

$$\exists P = P^T > 0, \quad A^T(-P) + (-P)A > 0$$

The main feature of Theorem 2.1 is that it provides a very simple test, in the LMI form, to check stability. This can be used in two ways. First, one can easily check the stability of some interconnected subsystems with a given LFT representation. Second, given the matrix interconnection  $A$ , this LMI feasibility problem can be used to determine admissible values for  $\{X, Y, Z\}$ , and then obtaining an admissible family of subsystems such their interconnection is stable.

Based on Theorem 2.1, a simple criteria on the eigenvalues of the interconnection matrix  $A$  is given below for a stable interconnection of homogeneous subsystems.

**Corollary 2.2.**

*Assume that  $T(s)$  is a given  $\{x, y, z\}$ -dissipative transfer function with  $x \leq 0$  and  $z \geq 0$ , associated with a state-space representation (2.2) where  $A, B, C, D$  are given real matrices of appropriate dimensions. Assume there exists a real matrix  $K_\Delta$  such that  $(I - D_\Delta K)$  is invertible and both (2.7) and (2.8) hold.*

*If the following inequalities hold*

$$\forall i \in 1, \dots, n, \quad -z(\lambda_i^* \lambda_i) - y(\lambda_i^* + \lambda_i) - x > 0 \quad (2.10)$$

*where  $\lambda_1, \lambda_2, \dots, \lambda_n$  are the eigenvalues of  $A$ , then the transfer matrix given by*

$$(T(s) \cdot I) \star \left[ \begin{array}{c|c} A & B \\ \hline C & D \end{array} \right]$$

*is stable.*

*Proof.* By applying the Theorem 2.2 of [CG96], it comes that (2.10) is equivalent to:

$$\exists P = P^T > 0, \quad \begin{bmatrix} A \\ I \end{bmatrix}^T \begin{bmatrix} -zP & -yP \\ -yP & -xP \end{bmatrix} \begin{bmatrix} A \\ I \end{bmatrix} > 0$$

<sup>6</sup>As  $P$  is real symmetric, its eigenvalues are real. Assume there is one which is non-positive  $\lambda \leq 0$ , and associate an eigenvector  $x$ . Then, last inequality leads to  $(-\lambda)x^T (K_\Delta^T + K_\Delta) x > 0$ . This is a contradiction as  $\Lambda(K_\Delta) \subset \mathbb{C}^-$ .

As  $T(s)$  is  $\{x, y, z\}$ -dissipative and  $P > 0$ , its repetition  $T(s) \cdot I$  is in particular  $\{xP, yP, zP\}$ -dissipative. Applying Theorem 2.1 leads to the conclusion.  $\square$

*Remark 2.4.* When  $T(s) = \frac{1}{s}$ , which is lossless  $\{0, 1, 0\}$ -dissipative, last corollary leads the eigenvalues of  $A$  to be with negative real part, i.e. the criteria that was used for defining stability (Working Definition 2.3).

This provides a convenient way to compute a stable LFT in a repeated  $T(s)$ . Indeed, it is sufficient to compute an interconnection matrix  $A$  with eigenvalues belonging to the set generated by the constraint (2.10), to get a stable transfer matrix. The synthesis of a matrix  $A$  with pre-defined eigenvalues is a standard problem in Control theory called *pole placement* [ZDG96, Chap. 3]. This is interpreted as computing a state-feedback gain  $K$  such that  $A = A_0 + B_0K$  has the willing eigenvalues, where  $(A_0, B_0)$  is any controllable pair. The computation of such matrix  $K$  can be achieved by using the *Matlab* function *place*<sup>7</sup>. A straightforward procedure for the computation of an interconnection matrix  $A$  such that the associated LFT is stable is then provided in Algorithm 2.1.

---

**Algorithm 2.1:** Computation of a stable interconnection matrix  $A$

---

**Input:**  $x, y, z \in \mathbb{R}$ ,  $n \in \mathbb{N}$

**Output:**  $A$

- 1 Define  $n$  complex-conjugate eigenvalues  $\lambda_i$ ,  $i = 1, \dots, n$ , such that (2.10).
  - 2 Define  $A_0 \in \mathbb{R}^{n \times n}$ ,  $B_0 \in \mathbb{R}^{n \times 1}$  such that  $(A_0, B_0)$  is controllable.
  - 3 Compute  $K \in \mathbb{R}^{1 \times n}$  such that  $A_0 + B_0K$  has  $\lambda_i$ ,  $i = 1, \dots, n$  for eigenvalues.
  - 4  $A \leftarrow A_0 + B_0K$
- 

*Remark 2.5.* Assuming that  $\lambda_i \neq 0$ , one may note that (2.10) is equivalent to

$$-x \left( \lambda_i^{-*} \lambda_i^{-1} \right) - y \left( \lambda_i^{-*} + \lambda_i^{-1} \right) - z > 0$$

Therefore,  $\lambda_i^{-1}$  belongs to the complement in the complex plane of the set generated by the  $\{x, y, z\}$ -dissipative parametrisation. For instance, if this set is a disk, the  $\lambda_i^{-1}s$  only need to be chosen outside this disk to get a stable interconnection. While this provides a direct and nice graphical criteria, an adapted version of Algorithm 2.1 would require to achieve a matrix inversion, which may involve numerical errors.

## 2.6 A unifying tool: the KYP Lemma

In the last two sections, the LFT tool has been introduced to represent complex systems and their interconnection, while the  $\{X, Y, Z\}$ -dissipative characterisation parametrises them using a frequency-domain quadratic constraint. Now, the following *analysis* problem is tackled:

Given  $\{X, Y, Z\}$ -dissipative subsystems and an LFT representation of their interconnection, does the resulting system satisfy some given design criteria?

The design criteria are typically split in two types: *stability* and *performance*. Last section has provided an LMI feasibility problem as sufficient condition for stability. In addition, this has lead to a simple method to design stable interconnection of identical SISO subsystems. Likewise, the objective of this section is to provide methods for

<sup>7</sup><https://mathworks.com/help/control/ref/place.html>

solving the above analytical problem using the LMI framework, in such a way that they can be generalised for design purposes.

In this work, similarly to the dissipative characterisation, performance criteria will be expressed using frequency-domain quadratic constraints. For instance, it may be required to synthesise a system of which the squared magnitude of its frequency response has an upper bound  $|H(j\omega)|^2 \leq U^2$  on the frequency interval  $[0, \omega_U]$ . This design criterion can be rewritten as:

$$\forall \omega \in [0, \omega_U], \quad \begin{bmatrix} H(j\omega) \\ 1 \end{bmatrix}^* \begin{bmatrix} -1 & 0 \\ 0 & U^2 \end{bmatrix} \begin{bmatrix} H(j\omega) \\ 1 \end{bmatrix} \geq 0$$

In order to fulfil our objective, the so-called KYP Lemma is introduced and generalised in the sequel.

The Kalman–Yakubovich–Popov (KYP) Lemma transforms a frequency-domain quadratic constraint, or an integral quadratic constraint in the time-domain [IHF05], of an LFT system into an LMI feasibility problem. Historically, the KYP Lemma was stated in a linear matrix equality form, and for two particular cases known as positive-real and bounded-real lemma. The book [AV73] gives a comprehensive study on these two cases, and provides an impressive number of different proofs. The LMI version was then given in the pioneer paper [Wil71]. The modern form of the KYP lemma has then been developed, using dissipative system theory [Wil72b; TW91] or convexity theory [Ran96]. It is stated in the following theorem with large inequalities. The strict inequality and the equality forms can be obtained by exchanging the symbol  $\geq$  by respectively  $>$  and  $=$ .

**Theorem 2.3 (KYP Lemma [TW91]).**

Define a transfer matrix

$$\mathcal{B}(s) := \left( \frac{1}{s} \cdot I \right) \star \left[ \begin{array}{c|c} A & B \\ \hline C & D \end{array} \right]$$

with  $A \in \mathbb{R}^{n \times n}$ ,  $B \in \mathbb{R}^{n \times n_w}$ ,  $C \in \mathbb{R}^{n_z \times n}$ ,  $D \in \mathbb{R}^{n_z \times n_w}$  and  $(A, B)$  controllable. Let  $X = X^T \in \mathbb{R}^{n_z \times n_z}$  be a real symmetric matrix. The following two statements are equivalent.

(i) The following constraint holds

$$\forall \omega \in \mathbb{R} \text{ such that } j\omega \notin \Lambda(A), \quad \mathcal{B}(j\omega)^* X \mathcal{B}(j\omega) \geq 0 \quad (2.11)$$

(ii) There exists a symmetric matrix  $\mathbf{P} = \mathbf{P}^T \in \mathbb{R}^{n \times n}$  such that:

$$\begin{bmatrix} A & B \\ I & 0 \end{bmatrix}^T \begin{bmatrix} 0 & -\mathbf{P} \\ -\mathbf{P} & 0 \end{bmatrix} \begin{bmatrix} A & B \\ I & 0 \end{bmatrix} + \begin{bmatrix} C^T \\ D^T \end{bmatrix} X \begin{bmatrix} C & D \end{bmatrix} \geq 0 \quad (2.12)$$

*Remark 2.6.* For the sake of compactness, (2.12) will sometimes be written in the equivalent form:

$$\begin{bmatrix} A & B \\ I & 0 \\ C & D \end{bmatrix}^T \begin{bmatrix} 0 & -\mathbf{P} & 0 \\ -\mathbf{P} & 0 & 0 \\ 0 & 0 & X \end{bmatrix} \begin{bmatrix} A & B \\ I & 0 \\ C & D \end{bmatrix} \geq 0$$

*Remark 2.7.* A stronger version of Theorem 2.3, for which (2.11) is replaced by an inequality in the closed right-half complex plane  $\mathbb{C}^+ \cup \mathbb{C}^0$  and  $\mathbf{P}$  is required to be

positive semi-definite  $\mathbf{P} \geq 0$ , is sometimes referred to as the KYP Lemma (cf Theorem A.9 of Appendix A, p. 196).

The frequency-domain inequality (2.11) is complicated to solve. One may solve it analytically for simple problems. However, this constraint depends on the continuous variable  $\omega$ , which implies that there is an infinite number of inequalities, and no algorithm can guarantee in a finite-time that this constraint is satisfied. On the opposite, the equivalent statement (ii) leads to an LMI feasibility problem, with all the previously-mentioned numerical advantages, for which a feasible solution guarantees that (2.11) holds. Notice that this is achieved by introducing the optimisation symmetric-matrix variable  $\mathbf{P}$ , i.e.  $\frac{n(n+1)}{2}$  optimisation variables.

Despite these benefits, the KYP Lemma under this form is of little use to tackle our analysis and design problems. Indeed, (2.11) is stated for all  $\omega$  in  $\mathbb{R}$  and for systems represented as LFT in repeated  $\frac{1}{s}$ . Therefore, it needs to be generalised to consider constraints expressed on a frequency interval and to LFT representations with more complex subsystems. This is achieved in the remaining of this section using an interpretation of the KYP Lemma based on dissipative characterisation and the role of multipliers. A reader not interested in the technical details may skip this part and only retain the following take home messages.

1. The KYP Lemma transforms a performance criterion into an equivalent LMI feasibility problem.
2. There exist generalised versions of the KYP Lemma for different systems and performance criteria of our interest, also leading to a LMI feasibility problem, but may have a *conservative* formulation<sup>8</sup>.
3. If the design problem can be suitably reformulated, efficient design methods can be developed using the KYP Lemma and its generalised versions.

### Interpretation of the KYP Lemma

Inequality (2.12) can be decomposed as the sum of two terms. The term

$$\begin{bmatrix} A & B \\ I & 0 \end{bmatrix}^T \begin{bmatrix} 0 & -\mathbf{P} \\ -\mathbf{P} & 0 \end{bmatrix} \begin{bmatrix} A & B \\ I & 0 \end{bmatrix}$$

results from the dissipative characterisation of the identical subsystems  $\frac{1}{s} \cdot I$ , while

$$\begin{bmatrix} C^T \\ D^T \end{bmatrix} X \begin{bmatrix} C & D \end{bmatrix}$$

originates from (2.11) and ensures that the system satisfies the performance criterion. Mathematically, this clearly appears by pre- and post-multiplying (2.12) by respectively  $\begin{bmatrix} \left(\frac{1}{j\omega} \cdot I\right) (I - A \left(\frac{1}{j\omega} \cdot I\right)^{-1} B)^* \\ I \end{bmatrix}$  and  $\begin{bmatrix} \left(\frac{1}{j\omega} \cdot I\right) (I - A \left(\frac{1}{j\omega} \cdot I\right)^{-1} B) \\ I \end{bmatrix}$ , as it comes:

$$F(j\omega)^* \begin{bmatrix} I \\ \frac{1}{j\omega} \cdot I \end{bmatrix}^* \begin{bmatrix} 0 & -\mathbf{P} \\ -\mathbf{P} & 0 \end{bmatrix} \begin{bmatrix} I \\ \frac{1}{j\omega} \cdot I \end{bmatrix} F(j\omega) + \mathcal{B}(j\omega)^* X \mathcal{B}(j\omega) \geq 0$$

<sup>8</sup>The equivalence feature is lost: the LMI conditions is only sufficient.



with  $F(j\omega) := A\left(\frac{1}{j\omega} \cdot I\right)(I - A\left(\frac{1}{j\omega} \cdot I\right))^{-1}B + B$ . Now, as each subsystem  $\frac{1}{s}$  is lossless  $\{0, 1, 0\}$ -dissipative, one can easily check that:

$$\begin{bmatrix} I \\ \frac{1}{j\omega} \cdot I \end{bmatrix}^* \begin{bmatrix} 0 & -\mathbf{P} \\ -\mathbf{P} & 0 \end{bmatrix} \begin{bmatrix} I \\ \frac{1}{j\omega} \cdot I \end{bmatrix} = - \begin{bmatrix} \frac{1}{j\omega} \cdot I \\ I \end{bmatrix}^* \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{j\omega} \cdot I \\ I \end{bmatrix} \mathbf{P} = 0$$

Hence the global performance (2.11) holds.

Therefore, the matrix  $\begin{bmatrix} 0 & \mathbf{P} \\ \mathbf{P} & 0 \end{bmatrix}$  matches with the combination of  $\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$ , which stands for the lossless  $\{0, I, 0\}$ -dissipative characterisation of  $\frac{1}{s} \cdot I$ , and the multiplier matrix  $\mathbf{P}$ . The latter matrix does not influence the dissipative characterisation of the subsystems, as a  $\{0, I, 0\}$ -dissipative system is also  $\{0, \mathbf{P}, 0\}$ -dissipative. However, it changes the resolvability of (2.12). Indeed,  $\mathbf{P}$  can be viewed as an additional *degree of freedom*, as it provides additional variables to optimise on. In concrete terms, there may exist a solution  $\mathbf{P}$ , while (2.12) may not hold when  $\mathbf{P} = I$ . This matrix  $\mathbf{P}$  can be interpreted as searching in all dissipative characterisation equivalent to the lossless  $\{0, I, 0\}$ -dissipativity.

### Frequency-interval KYP Lemma

The classical version of the KYP Lemma appears to be limited for tackling practical requirements. In practice, requirements are more often expressed in frequency intervals. Research have then been undertaken to extend the KYP Lemma to frequency ranges. This was achieved in [IH05], going along with a nice geometrical interpretation. Several extensions and generalised versions were developed afterwards. In particular, we are interested in the version proposed in [PV11], treating the particular case of real LFT matrices, which reduces the number of optimisation variables.

Using the previous interpretation of the KYP Lemma, the fact that  $\omega$  belongs to a frequency interval  $\Omega$  is viewed as requiring each lossless  $\{0, 1, 0\}$ -dissipative subsystem  $\frac{1}{s}$  to satisfy an additional frequency-domain quadratic constraints. Indeed, one may verify that the set of  $\omega \in \mathbb{R}$  such that:

$$\begin{bmatrix} \frac{1}{j\omega} \\ 1 \end{bmatrix}^* \begin{bmatrix} x_\Omega & y_\Omega \\ y_\Omega^* & z_\Omega \end{bmatrix} \begin{bmatrix} \frac{1}{j\omega} \\ 1 \end{bmatrix} \geq 0$$

determines a frequency interval  $\Omega$  when

$$\begin{bmatrix} x_\Omega & y_\Omega \\ y_\Omega^* & z_\Omega \end{bmatrix} = \begin{bmatrix} \omega_0^2 & 0 \\ 0 & -1 \end{bmatrix} \quad \text{for } \Omega = [0, \omega_0] \quad (2.13)$$

$$\begin{bmatrix} x_\Omega & y_\Omega \\ y_\Omega^* & z_\Omega \end{bmatrix} = \begin{bmatrix} -\omega_1\omega_2 & j\frac{\omega_1+\omega_2}{2} \\ -j\frac{\omega_1+\omega_2}{2} & -1 \end{bmatrix} \quad \text{for } \Omega = [\omega_1, \omega_2] \quad (2.14)$$

$$\begin{bmatrix} x_\Omega & y_\Omega \\ y_\Omega^* & z_\Omega \end{bmatrix} = \begin{bmatrix} -\omega_0^2 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{for } \Omega = [\omega_0, +\infty) \quad (2.15)$$

Therefore, the LMI feasible problem of Theorem 2.3 is adapted to take into account this quadratic constraint, viewed as a local performance of the subsystems. This leads to the following theorem.

**Theorem 2.4 (KYP Lemma on a frequency interval)** (adapted from [PV11]).  
Assume that a frequency interval  $\Omega$  is given and define  $x_\Omega, y_\Omega, z_\Omega$  as in (2.13)-(2.15).

Define a transfer matrix

$$\mathcal{B}(s) := \left( \frac{1}{s} \cdot I \right) \star \left[ \begin{array}{c|c} A & B \\ \hline C & D \end{array} \right]$$

with  $A \in \mathbb{R}^{n \times n}$ ,  $B \in \mathbb{R}^{n \times n_w}$ ,  $C \in \mathbb{R}^{n_z \times n}$ ,  $D \in \mathbb{R}^{n_z \times n_w}$  and  $(A, B)$  controllable. Let  $X = X^T \in \mathbb{R}^{n_z \times n_z}$  be a real symmetric matrix. The following statements are equivalent.

(i) The following inequality holds

$$\forall \omega \in \Omega \text{ such that } j\omega \notin \Lambda(A), \quad \mathcal{B}(j\omega)^* X \mathcal{B}(j\omega) \geq 0$$

(ii) For all  $\omega \in \mathbb{R}$  such that  $j\omega \notin \Lambda(A)$  and

$$\begin{bmatrix} \frac{1}{j\omega} \cdot I \\ I \end{bmatrix}^* \begin{bmatrix} x_\Omega \cdot I & y_\Omega \cdot I \\ y_\Omega^* \cdot I & z_\Omega \cdot I \end{bmatrix} \begin{bmatrix} \frac{1}{j\omega} \cdot I \\ I \end{bmatrix} \geq 0$$

the following inequality holds

$$\mathcal{B}(j\omega)^* X \mathcal{B}(j\omega) \geq 0$$

(iii) There exists symmetric matrices  $\mathbf{P} = \mathbf{P}^T \in \mathbb{R}^{n \times n}$  and  $\mathbf{D} = \mathbf{D}^T \in \mathbb{R}^{n \times n}$ , such that:

$$\mathbf{D} > 0$$

$$\begin{bmatrix} A & B \\ I & 0 \end{bmatrix}^T \left( \begin{bmatrix} 0 & -\mathbf{P} \\ -\mathbf{P} & 0 \end{bmatrix} + \begin{bmatrix} -z_\Omega \mathbf{D} & -y_\Omega^* \mathbf{D} \\ -y_\Omega \mathbf{D} & -x_\Omega \mathbf{D} \end{bmatrix} \right) \begin{bmatrix} A & B \\ I & 0 \end{bmatrix} + \begin{bmatrix} C^T \\ D^T \end{bmatrix} X \begin{bmatrix} C & D \end{bmatrix} \geq 0$$

The LMI feasibility problem of statement (iii) of last theorem can be interpreted as for the usual KYP Lemma. Indeed, consider for instance that  $\Omega = [0, \omega_0]$ . By using same reasoning as previously, one gets:

$$F(j\omega)^* \begin{bmatrix} I \\ \frac{1}{j\omega} \cdot I \end{bmatrix}^* \begin{bmatrix} \mathbf{D} & -\mathbf{P} \\ -\mathbf{P} & -\omega_0^2 \mathbf{D} \end{bmatrix} \begin{bmatrix} I \\ \frac{1}{j\omega} \cdot I \end{bmatrix} F(j\omega) + \mathcal{B}(j\omega)^* X \mathcal{B}(j\omega) \geq 0 \quad (2.16)$$

Moreover for all  $\omega \in [0, \omega_0]$ ,

$$\begin{bmatrix} I \\ \frac{1}{j\omega} \cdot I \end{bmatrix}^* \begin{bmatrix} 0 & -\mathbf{P} \\ -\mathbf{P} & 0 \end{bmatrix} \begin{bmatrix} I \\ \frac{1}{j\omega} \cdot I \end{bmatrix} = - \begin{bmatrix} \frac{1}{j\omega} \cdot I \\ I \end{bmatrix}^* \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{j\omega} \cdot I \\ I \end{bmatrix} \mathbf{P} = 0 \quad (2.17)$$

$$\begin{bmatrix} I \\ \frac{1}{j\omega} \cdot I \end{bmatrix}^* \begin{bmatrix} \mathbf{D} & 0 \\ 0 & -\omega_0^2 \mathbf{D} \end{bmatrix} \begin{bmatrix} I \\ \frac{1}{j\omega} \cdot I \end{bmatrix} = - \begin{bmatrix} \frac{1}{j\omega} \cdot I \\ I \end{bmatrix}^* \begin{bmatrix} \omega_0^2 I & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} \frac{1}{j\omega} \cdot I \\ I \end{bmatrix} \mathbf{D} \leq 0 \quad (2.18)$$

Therefore, any solution of statement (iii) implies that (2.16) holds and then for all  $\omega$  such that (2.17) and (2.18), the constraints  $\mathcal{B}(j\omega)^* X \mathcal{B}(j\omega) \geq 0$  is verified. Again,

$$\begin{bmatrix} \mathbf{D} & -\mathbf{P} \\ -\mathbf{P} & -\omega_0^2 \mathbf{D} \end{bmatrix}$$

can be interpreted in terms of multipliers. Indeed, the introduction of  $\mathbf{P}$  and  $\mathbf{D} > 0$  permits to search in respectively all equivalent quadratic constraints of (2.17), resulting from the lossless  $\{0, I, 0\}$ -dissipative of the subsystems  $\frac{1}{s} \cdot I$ , and all equivalent quadratic constraints of (2.18), resulting from the propagation of  $\omega \in \Omega$  to a local

performance criterion of these subsystems. Graphically, the satisfaction of both (2.17) and (2.18) is viewed as the intersection in the complex plane of the imaginary axis and the complement of the disk of centre  $c = 0$  and radius  $r := \frac{1}{\omega_0}$  (cf discussion of the previous section or [IH05] for more details).

### Generalised KYP Lemma and the research of suitable multipliers

Now the KYP Lemma and its frequency-interval extension are generalised from systems represented by an LFT in  $\frac{1}{s} \cdot I$  to those represented by an LFT in a transfer matrix  $\Delta(s)$ . While this new version, stated below, is too general for the sake of this work, it allows to make explicit the underlying process to decline the different versions given in Section A.5 of Appendix A (p. 196), and that will be used in the next chapters.

**Theorem 2.5 (Generalised KYP Lemma)** (adapted from [Din05]).

Let  $\Delta(s)$  be a given transfer matrix. Let  $X = X^T \in \mathbb{R}^{n_z \times n_z}$  be a real symmetric matrix and define an LFT transfer matrix  $\mathcal{B}(\Delta(s))$  as:

$$\mathcal{B}(\Delta(s)) := \Delta(s) \star \left[ \begin{array}{c|c} A & B \\ \hline C & D \end{array} \right]$$

with  $A \in \mathbb{R}^{n \times n}$ ,  $B \in \mathbb{R}^{n \times n_w}$ ,  $C \in \mathbb{R}^{n_z \times n}$ ,  $D \in \mathbb{R}^{n_z \times n_w}$  and  $(A, B)$  controllable<sup>9</sup>.

Define  $\mathbb{D} = \{\omega \in \mathbb{R}, (I - A\Delta(j\omega)) \text{ is invertible}\}$ . Let  $\Omega$  be a frequency interval and define the sets  $\mathbb{M}$  and  $\mathbb{M}_\Omega$  as:

$$\begin{aligned} \mathbb{M} &:= \left\{ M = M^T \mid \forall \omega \in \mathbb{R} \cap \mathbb{D}, \begin{bmatrix} \Delta(j\omega) \\ I \end{bmatrix}^* M \begin{bmatrix} \Delta(j\omega) \\ I \end{bmatrix} \geq 0 \right\} \\ \mathbb{M}_\Omega &:= \left\{ M_\Omega = M_\Omega^T \mid \forall \omega \in \Omega \cap \mathbb{D}, \begin{bmatrix} \Delta(j\omega) \\ I \end{bmatrix}^* M_\Omega \begin{bmatrix} \Delta(j\omega) \\ I \end{bmatrix} \geq 0 \right\} \end{aligned}$$

The following two statements are equivalent.

(i) The following inequality is satisfied

$$\forall \omega \in \Omega \cap \mathbb{D}, \quad \mathcal{B}(\Delta(j\omega))^* X \mathcal{B}(\Delta(j\omega)) \geq 0$$

(ii) There exists  $M \in \mathbb{M}$  and  $M_\Omega \in \mathbb{M}_\Omega$  such that:

$$\begin{bmatrix} A & B \\ I & 0 \end{bmatrix}^T \left( \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} (-M - M_\Omega) \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \right) \begin{bmatrix} A & B \\ I & 0 \end{bmatrix} + \begin{bmatrix} C^T \\ D^T \end{bmatrix} X \begin{bmatrix} C & D \end{bmatrix} \geq 0$$

Despite its apparent formulation, statement (ii) is not an LMI feasibility problem. Indeed,  $\mathbb{M}$  and  $\mathbb{M}_\Omega$  are not necessarily convex sets, and more precisely do not necessarily admit an equivalent affine parametrisation. It happens that  $\mathbb{M}$  and  $\mathbb{M}_\Omega$  admit such parametrisations, as this is in the case in the KYP Lemma and its extended version on a frequency interval, where  $\Delta(s) = \frac{1}{s} \cdot I$ . However, the conditions to get these equivalent affine parametrisations [IMF00b; Din05] suggest that these are very particular cases. Instead, one is reduced to look for affine-parametrised subsets  $\mathbb{M}_{ap}$  and  $\mathbb{M}_{\Omega_{ap}}$ :

$$\mathbb{M}_{ap} \subset \mathbb{M} \quad \mathbb{M}_{\Omega_{ap}} \subset \mathbb{M}_\Omega$$

<sup>9</sup>One may remind the, somewhat abusive, notation of Working Definition 2.4.

As a result, in the previous lemma, the equivalence lowers into an implication: if there exists a solution to (ii), there is one to (i) but the reverse is not true anymore:

$$(ii) \Rightarrow (i)$$

Therefore, there is a mismatch between resolving the LMI feasibility problem and the satisfaction of the mathematical performance criterion. The LMI problem may be infeasible while the criterion is actually satisfied. This is called a *conservative* formulation. In order to reduce this conservatism, one may expect that the affine-parametrised subsets  $\mathbb{M}_{ap}$  and  $\mathbb{M}_{\Omega_{ap}}$  should be as large as possible. In the rest of this part, such subsets in the cases of interest of this work are displayed and discussed.

A natural candidate for  $\mathbb{M}_{ap}$  is the  $\{X, Y, Z\}$ -characterisation. Assume for instance that  $\Delta(s) = T(s) \cdot I$  where  $T(s)$  is a  $\{x, y, z\}$ -dissipative transfer function. Then a possible  $\mathbb{M}_{ap}$  is given by:

$$\mathbb{M}_{ap} = \left\{ \begin{bmatrix} xP & yP \\ yP & zP \end{bmatrix}, P = P^T > 0 \right\}$$

Once again, the introduction of the multiplier  $P > 0$  allows to search in all equivalent  $\{x, y, z\}$ -parametrisations and reduces the conservatism. Intuitively, this conservatism can be even more reduced by taking the finest dissipative parametrisation. From a graphical interpretation, this means finding the disk of smallest radius that includes  $T(j\omega)$  for all  $\omega \in \mathbb{R}$ . At the end of this section, an LMI optimisation problem is provided to compute a  $\{x, y, z\}$ -parametrisation corresponding to this disk.

In the first three lines of Table 2.1 are summarised some affine parametrisations  $\mathbb{M}_{ap}$  when  $T(s) = \frac{1}{s}$ , when  $T(s)$  is lossless  $\{x, y, z\}$ -dissipative and when  $T(s)$  is  $\{x, y, z\}$ -dissipative. Using Theorem 2.5, the last two cases respectively lead to the lossless dissipative KYP Lemma and the dissipative KYP Lemma that are given in p. 196 of Appendix A, and were previously obtained in [Sco97; Zar13].

In this work, we will be interested in the interconnection of heterogeneous subsystems that are of two types, i.e. when the subsystems are represented by either a transfer function  $T_1(s)$  or a transfer function  $T_2(s)$ . In particular,  $T_1(s)$  and  $T_2(s)$  will be known to be respectively lossless  $\{0, 1, 0\}$ -dissipative. In Lemma 2.1, we *propose* an affine parametrisation  $\mathbb{M}_{ap}$  for this case, that enables to obtain the associated version of the KYP Lemma (Theorem A.12, p. 197).

**Lemma 2.1.**

Assume that  $T_1(s)$  and  $T_2(s)$  are two lossless  $\{0, 1, 0\}$ -dissipative transfer functions. Define the sets  $\mathbb{M}$  and  $\mathbb{D}$  as in Theorem 2.5 where  $\Delta(s)$  is defined by

$$\Delta(s) := \begin{bmatrix} T_1(s) \cdot I_{n_1} & 0 \\ 0 & T_2(s) \cdot I_{n_2} \end{bmatrix}$$

Then an affine-parametrised subset  $\mathbb{M}_{ap} \subseteq \mathbb{M}$  is given by:

$$\mathbb{M}_{ap} = \left\{ \begin{bmatrix} 0 & P \\ P & 0 \end{bmatrix} \left| \exists P_1 = P_1^T \in \mathbb{R}^{n_1 \times n_1}, \exists P_2 = P_2^T \in \mathbb{R}^{n_2 \times n_2}, P = \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} \right. \right\}$$

TABLE 2.1: Proposed affine-parametrised subset  $\mathbb{M}_{ap}$  of  $\mathbb{M}$ 

$\Delta(s)$	$\mathbb{M}_{ap}$
$\frac{1}{s} \cdot I$	$\left\{ \begin{bmatrix} 0 & P \\ P & 0 \end{bmatrix}, P = P^T \right\}$
$T(s) \cdot I$ $T(s)$ lossless $\{x, y, z\}$ -dissipative	$\left\{ \begin{bmatrix} xP & yP \\ yP & zP \end{bmatrix}, P = P^T \right\}$
$T(s) \cdot I$ $T(s)$ $\{x, y, z\}$ -dissipative	$\left\{ \begin{bmatrix} xP & yP \\ yP & zP \end{bmatrix}, P = P^T > 0 \right\}$
$\begin{pmatrix} T_1(s) \cdot I & 0 \\ 0 & T_2(s) \cdot I \end{pmatrix}$ $T_1(s), T_2(s)$ lossless $\{0, 1, 0\}$ -dissipative	$\left\{ \begin{bmatrix} 0 & P \\ P & 0 \end{bmatrix} \mid \begin{array}{l} \exists P_1 = P_1^T, \exists P_2 = P_2^T, \\ P = \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} \end{array} \right\}$
$\begin{pmatrix} s \cdot I & 0 \\ 0 & \frac{1}{s} \cdot I \end{pmatrix}$	$\left\{ \begin{bmatrix} P_{int} & P \\ P & P_{int} \end{bmatrix} \mid \begin{array}{l} \exists P_1 = P_1^T, \exists P_2 = P_2^T, \exists P_{12}, \\ P = \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix}, P_{int} = \begin{bmatrix} 0 & P_{12} \\ P_{12}^T & 0 \end{bmatrix} \end{array} \right\}$

*Proof.* The set  $\mathbb{M}_{ap}$  is clearly affine in the multipliers matrices  $P_1$  and  $P_2$ . Moreover, one may easily verify that the following constraint holds:  $\forall \omega \in \mathbb{R} \cap \mathbb{D}$ ,

$$\begin{bmatrix} T_1(j\omega) \cdot I_{n_1} & 0 \\ 0 & T_2(j\omega) \cdot I_{n_2} \\ I & 0 \\ 0 & I \end{bmatrix}^* \begin{bmatrix} 0 & 0 & P_1 & 0 \\ 0 & 0 & 0 & P_2 \\ P_1 & 0 & 0 & 0 \\ 0 & P_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} T_1(j\omega) \cdot I_{n_1} & 0 \\ 0 & T_2(j\omega) \cdot I_{n_2} \\ I & 0 \\ 0 & I \end{bmatrix} = 0$$

□

Moreover, a particular case appears when  $T_1(s) = s$  and  $T_2(s) = \frac{1}{s}$ . As we are interested in the interconnection of inductances with impedance  $z_L(s) = L \cdot s$  and capacitances with impedance  $z_C(s) = \frac{1}{C \cdot s}$ , this case will naturally occur in this work. While  $T_1(s)$  and  $T_2(s)$  are both lossless  $\{0, 1, 0\}$ -dissipative, and thus the parametrisation of Lemma 2.1 can be used, we provide below a *finest* parametrisation. This parametrisation complete the table of affine-parametrised subset  $\mathbb{M}_{ap}$  that will be used in this work (Table 2.1).

**Lemma 2.2.**

Define the sets  $\mathbb{M}$  and  $\mathbb{D}$  as in Theorem 2.5 where  $\Delta(s)$  is defined by

$$\Delta(s) := \begin{bmatrix} s \cdot I_{n_1} & 0 \\ 0 & \frac{1}{s} \cdot I_{n_2} \end{bmatrix}$$

Then an affine-parametrised subset  $\mathbb{M}_{ap} \subseteq \mathbb{M}$  is given by:

$$\mathbb{M}_{ap} = \left\{ \begin{bmatrix} P_{int} & P \\ P & P_{int} \end{bmatrix} \mid \begin{array}{l} \exists P_1 = P_1^T \in \mathbb{R}^{n_1 \times n_1}, \exists P_2 = P_2^T \in \mathbb{R}^{n_2 \times n_2}, \exists P_{12} \in \mathbb{R}^{n_1 \times n_2}, \\ P = \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix}, P_{int} = \begin{bmatrix} 0 & P_{12} \\ P_{12}^T & 0 \end{bmatrix} \end{array} \right\}$$

*Proof.* The set  $\mathbb{M}_{ap}$  is clearly affine in the multipliers matrices  $P_1$ ,  $P_2$  and  $P_{12}$ . Moreover, from the proof of Lemma 2.1 and noticing that

$$(j\omega)^* \times \frac{1}{j\omega} + 1 = 0$$

one may easily verify that the following constraint holds:  $\forall \omega \in \mathbb{R} \cap \mathbb{D}$ ,

$$\begin{bmatrix} j\omega \cdot I_{n_1} & 0 \\ 0 & \frac{1}{j\omega} \cdot I_{n_2} \\ I & 0 \\ 0 & I \end{bmatrix}^* \begin{bmatrix} 0 & P_{12} & P_1 & 0 \\ P_{12}^T & 0 & 0 & P_2 \\ P_1 & 0 & 0 & P_{12} \\ 0 & P_2 & P_{12}^T & 0 \end{bmatrix} \begin{bmatrix} j\omega \cdot I_{n_1} & 0 \\ 0 & \frac{1}{j\omega} \cdot I_{n_2} \\ I & 0 \\ 0 & I \end{bmatrix} = 0$$

□

The research of a parametrised subset  $\mathbb{M}_{\Omega_{ap}}$  is more tedious. Using a graphical interpretation when  $\Delta(s) = T(s) \cdot I$  for instance, this may be viewed as willing to characterise some parts of curves using the intersection of disks and half-planes. One may find an appropriate subset but at the price of a lengthy research and an important increase of the number of multipliers, simultaneously increasing the computational cost. Therefore, we will generally use adapted formulation of the performance criteria, and only use the frequency-interval KYP Lemma, to avoid such a parametrisation. Nonetheless, a noticeable exception is provided in next lemma.

**Lemma 2.3.**

Define the sets  $\mathbb{M}_{\Omega}$  and  $\mathbb{D}$  as in Theorem 2.5 where  $\Delta(s)$  is defined by

$$\Delta(s) := \begin{bmatrix} s \cdot I_{n_1} & 0 \\ 0 & \frac{1}{s} \cdot I_{n_2} \end{bmatrix}$$

Then an affine-parametrised subset  $\mathbb{M}_{\Omega_{ap}} \subseteq \mathbb{M}_{\Omega}$  is given by:

$$\mathbb{M}_{\Omega_{ap}} = \left\{ \begin{bmatrix} x_{\Omega} D_1 & 0 & y_{\Omega} D_1 & 0 \\ 0 & z_{\Omega} D_2 & 0 & y_{\Omega}^* D_2 \\ y_{\Omega}^* D_1 & 0 & z_{\Omega} D_1 & 0 \\ 0 & y_{\Omega} D_2 & 0 & x_{\Omega} D_2 \end{bmatrix}, D_1 = D_1^T > 0 \in \mathbb{R}^{n_2 \times n_2}, D_2 = D_2^T > 0 \in \mathbb{R}^{n_2 \times n_2} \right\}$$

*Proof.* The set  $\mathbb{M}_{ap}$  is clearly affine in the multipliers matrices  $D_1$  and  $D_2$ . Moreover, one may easily verify that the following constraint holds:  $\forall \omega \in \Omega \cap \mathbb{D}$ ,

$$\begin{bmatrix} j\omega \cdot I_{n_1} & 0 \\ 0 & \frac{1}{j\omega} \cdot I_{n_2} \\ I & 0 \\ 0 & I \end{bmatrix}^* \begin{bmatrix} x_{\Omega} D_1 & 0 & y_{\Omega} D_1 & 0 \\ 0 & z_{\Omega} D_2 & 0 & y_{\Omega}^* D_2 \\ y_{\Omega}^* D_1 & 0 & z_{\Omega} D_1 & 0 \\ 0 & y_{\Omega} D_2 & 0 & x_{\Omega} D_2 \end{bmatrix} \begin{bmatrix} j\omega \cdot I_{n_1} & 0 \\ 0 & \frac{1}{j\omega} \cdot I_{n_2} \\ I & 0 \\ 0 & I \end{bmatrix} \geq 0$$

□

**From analysis to design**

The KYP Lemma and its various generalisations have been introduced until now as tools to verify if a frequency-domain quadratic constraint holds, or not, using an LMI feasible problem. It turns out that they can be used in a design perspective. In fact, as it will be shown in the Chapter 5, filter design problems can be formulated as the search of a symmetric matrix  $\mathbf{X} = \mathbf{X}^T$  such that it satisfies frequency-domain

quadratic constraints, as for example:

$$\forall \omega \in \mathbb{R}, \quad \mathcal{B}(j\omega)^* \mathbf{X} \mathcal{B}(j\omega) \geq 0$$

Using the KYP Lemma, this leads to find if there exists  $\mathbf{P} = \mathbf{P}^T$  such that:

$$\begin{bmatrix} A & B \\ I & 0 \end{bmatrix}^T \begin{bmatrix} 0 & -\mathbf{P} \\ -\mathbf{P} & 0 \end{bmatrix} \begin{bmatrix} A & B \\ I & 0 \end{bmatrix} + \begin{bmatrix} C^T \\ D^T \end{bmatrix} \mathbf{X} \begin{bmatrix} C & D \end{bmatrix} \geq 0$$

As one can easily check, last inequality is an LMI in the optimisation variables  $\mathbf{P}$  and  $\mathbf{X}$ . Therefore, the KYP Lemma is an efficient analysis tool that can be extended for design perspectives, if the design problem can be suitably reformulated.

To illustrate the design potential of the KYP Lemma on a simple example, let us search an  $\{x, y, z\}$ -dissipative parametrisation corresponding to the smallest disk which contains the Nyquist plot of a SISO transfer function  $T(s)$ , say for example

$$T(s) = \frac{1}{(s+1)(s+2)}$$

As there is infinite number of possible  $\{x, y, z\}$  associated with this disk, one can fix the value of  $x$  to any negative number, for instance  $x = -1$ , without loss of generality.

The problem is then: minimise  $\mathbf{r}^2$ , where  $\mathbf{r} := \sqrt{\frac{y^2}{x^2} - \frac{z}{x}}$ , such that

$$\forall \omega \in \mathbb{R}, \quad \begin{bmatrix} T(j\omega) \\ 1 \end{bmatrix}^* \begin{bmatrix} x & \mathbf{y} \\ \mathbf{y} & z \end{bmatrix} \begin{bmatrix} T(j\omega) \\ 1 \end{bmatrix} \geq 0 \quad (2.19)$$

holds. As the function to be minimised is non-linear into the optimisation variables, introduce the *slack* variable  $\lambda$  such that:

$$\lambda \geq \frac{y^2}{x^2} - \frac{z}{x}$$

Applying the Schur lemma (Lemma A.3, p. 195), last inequality is equivalent to the following LMI:

$$\begin{bmatrix} \lambda + \frac{z}{x} & \frac{y}{x} \\ \frac{y}{x} & 1 \end{bmatrix} \geq 0$$

Now, define

$$\mathcal{B}(s) := \begin{bmatrix} T(s) \\ 1 \end{bmatrix} = \left( \frac{1}{s} \cdot I \right) \star \left[ \begin{array}{c|c} A & B \\ \hline C & D \end{array} \right] := \left( \frac{1}{s} \cdot I_2 \right) \star \left[ \begin{array}{cc|c} 0 & 1 & 0 \\ -2 & -3 & 1 \\ \hline 1 & 0 & 0 \\ 0 & 0 & 1 \end{array} \right]$$

Constraint (2.19) can be written as  $\mathcal{B}(j\omega)^* \begin{bmatrix} x & \mathbf{y} \\ \mathbf{y} & z \end{bmatrix} \mathcal{B}(j\omega) \geq 0$ . Finally, by applying the KYP Lemma, the following LMI optimisation problem is found:

$$\begin{aligned} & \min_{\substack{\lambda \\ \mathbf{y}, z, \lambda \in \mathbb{R} \\ \mathbf{P} = \mathbf{P}^T \in \mathbb{R}^{n \times n}}} \lambda \\ \text{such that} & \begin{bmatrix} \lambda + \frac{z}{x} & \frac{\mathbf{y}}{x} \\ \frac{\mathbf{y}}{x} & 1 \end{bmatrix} \geq 0 \\ & \begin{bmatrix} A & B \\ I & 0 \end{bmatrix}^T \begin{bmatrix} 0 & -\mathbf{P} \\ -\mathbf{P} & 0 \end{bmatrix} \begin{bmatrix} A & B \\ I & 0 \end{bmatrix} + \begin{bmatrix} C^T \\ D^T \end{bmatrix} \begin{bmatrix} x & \mathbf{y} \\ \mathbf{y} & z \end{bmatrix} \begin{bmatrix} C & D \end{bmatrix} \geq 0 \end{aligned}$$

By solving last LMI problem using Matlab, we find  $\begin{bmatrix} x & \mathbf{y} \\ \mathbf{y} & z \end{bmatrix} = \begin{bmatrix} -1 & 0.193 \\ 0.193 & 0.0624 \end{bmatrix}$ .

## 2.7 Conclusion

Developping design methods for modern engineering applications is not an easy task. The design problems are becoming increasingly complex, while the design methods should be flexible, user-friendly and computationally efficient. The framework exposed in this chapter appears for us to be a good candidate to develop such methods. Although originally forged for analysing and controlling systems, the assumption underlying this work is that this framework also enables to tackle the design challenge. While it is not claimed that efficient design methods for any engineering problems can be developed, our aim is to develop a generic approach which can be adapted by the design engineers. Hopefully, this will provide a good starting point and significantly improve the design flow.





## Chapter 3

# Passive Electronic Filters: Description and Design Problem

### 3.1 Introduction

In Chapter 2, it is explained that developing systematic and efficient design methods for modern systems is a complex task. Some tools are provided and merged into a design framework, in order to tackle this challenging problem. In this chapter, we introduce systems to be designed in this work, namely passive electronic filters, their associated design problem and a literature review of traditional design methods.

First, passive electronic filters are presented and modelled (Section 3.2). In a system fashion manner, the subsystems, i.e. the components, are first modelled. The ubiquitous concept of *passivity* is especially introduced. A passive electronic filter is then defined as the electrical interconnection of passive components, and transfer matrices are provided for describing its behaviour. In particular, the role of the passive feature of the components on the behaviour of the filter is exhibited.

Then, a general formulation of the design problem of passive electronic filters is given (Section 3.3). A focus is made on *lossless* passive filters, as they will be the main topic in the next chapters.

Finally, systematic design methods are reviewed with an historical perspective (Section 3.4). This reveals two design approaches, that will be revisited in Chapter 4.

### 3.2 Passive electronic filters: description and properties

#### 3.2.1 Components modelling

All along this manuscript, components are described by mathematical models. To highlight the difference between a model of component and its physical realisation, the term *element* is introduced and refers to a model of a physical component. An electrical element, carried by a current  $i(s)$  and admitting a voltage  $v(s)$  between its ports, can be described by an *impedance* transfer function  $z(s)$  (3.1) or its inverse, the *admittance* transfer function  $y(s) = \frac{1}{z(s)}$ .

$$v(s) = z(s) \cdot i(s) \quad (3.1)$$

Traditionally, in Electronics, components are modelled as the electrical interconnection of resistances  $R$ , inductances  $L$  and capacitances  $C$ . Their circuit models are represented on Figure 3.1, and their associated impedances are given in (3.2). These three specific models are denoted *building elements* in the sequel, as they are the building blocks of the elements.

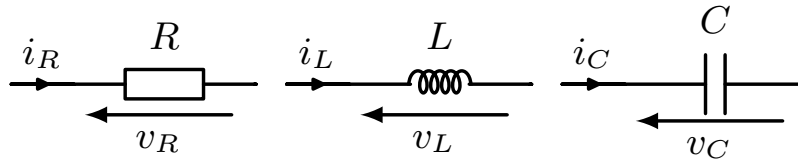


FIGURE 3.1: Building elements: resistance  $R$ , inductance  $L$  and capacitance  $C$

$$z_R(s) = R \quad z_L(s) = L \cdot s \quad z_C(s) = \frac{1}{C \cdot s} \quad (3.2)$$

The reader should notice that a component may have several models, and then be represented by different interconnections of building elements. Reversely, a model may also represent different components, independently from the technology of realisation.

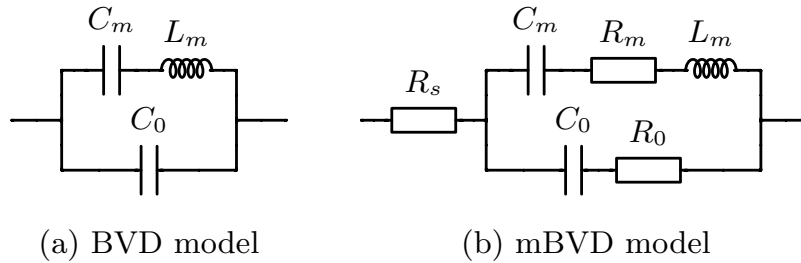


FIGURE 3.2: Acoustic wave resonator represented by two elements

Some components of particular interest in this work are the acoustic wave (AW) resonators. An AW resonator is typically viewed as a piezoelectric material positioned between two electrodes. It has different models (1D electric model, Mason, 2.5D, 3D finite element model,...), depending on the level of precision required and physical phenomena taken into account [CBG+13]. For design purpose, as we focus on LTI models, we only consider 1D electric models. These are made of the Butterworth Van Dike (BVD) model (Figure 3.2 (a)), which is a lossless model, and the modified BVD (mBVD) model (Figure 3.2 (b)), which takes into account various losses [LIBWR00]. The BVD model is the most used in this work. Its associated impedance  $z_{BVD}(s)$  can be calculated as:

$$z_{BVD}(s) := \frac{1}{C_0 \cdot s} \cdot \frac{s^2 + \omega_r^2}{s^2 + \omega_a^2}$$

with

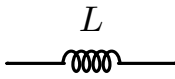
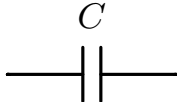
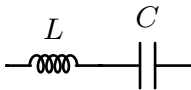
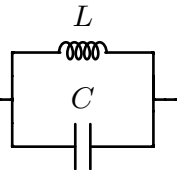
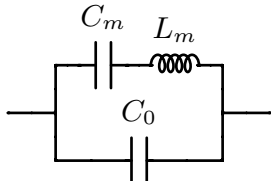
$$\omega_r^2 = \frac{1}{L_m C_m} \quad \omega_a^2 = \frac{C_0 + C_m}{L_m C_m C_0} = \omega_r^2 \left(1 + \frac{C_m}{C_0}\right)$$

Physically,  $L_m$  and  $C_m$  constitute the motion arm while  $C_0$  is called the natural capacity of the resonator. The frequencies  $\omega_r$  and  $\omega_a$  are respectively named *resonant* and *anti-resonant* frequencies.

Usually, in the design problem of an electronic filter, the type of components that can be used is pre-defined. Consequently, only some types of elements, i.e. particular interconnections of building elements, are available. We denote a type of elements as a *family*  $\mathcal{F}$  of elements. Some examples of families are provided in Table 3.1.

In this work, we are especially interested in *passive* and *lossless passive* elements. Intuitively, an element is called passive if it can dissipate, store and release energy, but can not release more energy than was externally supplied to it. This qualitative

TABLE 3.1: Examples of families of elements

Family	Elements	
$\mathcal{F}_{LC}$		
$\mathcal{F}_{LCr}$		
$\mathcal{F}_{AWr}$		

property happens to be transformable into a mathematical characterisation of the impedance, or admittance, of an element [AV73]. This characterisation is used here as definition.

**Working Definition 3.1 ((lossless) passive element).**

Let  $e$  be an element with impedance  $z_e(s)$ . Assume that  $z_e(s)$  is marginally stable.  $e$  is said to be a *passive* element if  $z_e(s)$  is such that

$$\forall s \in \mathbb{C}^+, \quad z_e(s)^* + z_e(s) \geq 0$$

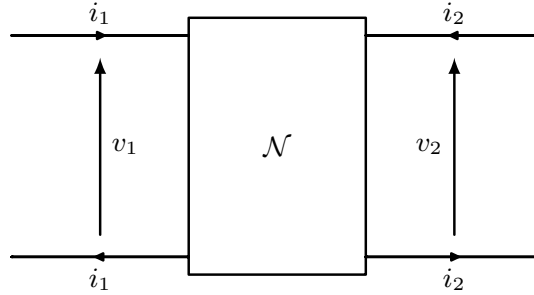
If, in addition,  $z_e(s)$  is lossless  $\{0, 1, 0\}$ -dissipative,  $e$  is said to be *lossless passive*.

*Remark 3.1.* In Network theory [AV73],  $z_e(s)$  is generally referred to be *positive-real*.

One may note from (3.2) that a resistance  $R$ , an inductance  $L$  or a capacitance  $C$  are required to have positive characteristic values  $R > 0$ ,  $L > 0$ ,  $C > 0$  to be passive. In addition, these positive-valued inductance  $L > 0$  and capacitance  $C > 0$  are also lossless passive elements.

*Remark 3.2.* One should be cautious with this definition of passivity. While it may be seen obvious that the interconnection of passive elements provides another passive element, the reverse is false. Indeed, a passive element may be made of non-passive building elements. As an example, the serial interconnection of a passive resistance  $R_1 > 0$  and a non-passive resistance  $R_2 < 0$  is passive if  $R_1 + R_2 > 0$ .

The next subsection presents electronic filters. In order to describe the behaviour of this filter, two transfer matrices are used. It is especially exhibited that, when an electronic filter is only made of passive elements, the resulting description matrices satisfy certain dissipative properties.

FIGURE 3.3: Electronic two-port  $\mathcal{N}$ 

### 3.2.2 Impedance and scattering descriptions

The electrical interconnection of electronic components is typically modelled as an electronic two-port [You71]. An electronic two-port<sup>1</sup>  $\mathcal{N}$  (Figure 3.3), referred as a *two-port* in the sequel, is an electronic system linking currents  $i_1$ ,  $i_2$  and voltages  $v_1$ ,  $v_2$ . An electronic filter is generally viewed as a two-port  $\mathcal{N}$  terminated on a resistive voltage source ( $e_g, R_g$ ) and a resistive load  $R_l$  (Figure 3.4). The complex signals  $i_1(s)$ ,  $v_1(s)$ ,  $i_2(s)$ ,  $v_2(s)$  are then linked by the relations:

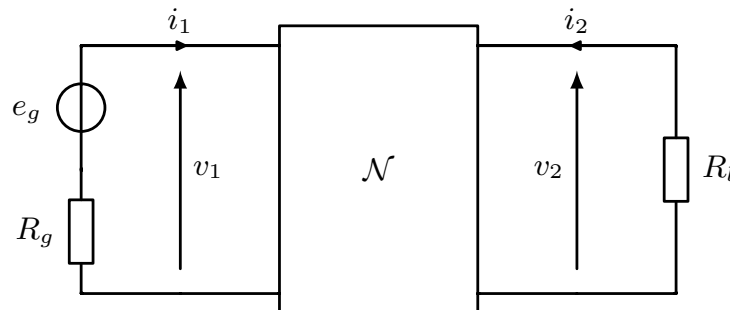
$$v_1(s) = e_g(s) - R_g i_1(s) \quad (3.3)$$

$$v_2(s) = -R_l i_2(s) \quad (3.4)$$

Two matrix descriptions are typically used to represent the behaviour of electronic filters: the impedance matrix  $Z(s)$  and the scattering matrix  $S(s)$  [Bah84]. The impedance matrix  $Z(s)$  is a transfer matrix admitting complex currents  $i_1(s)$  and  $i_2(s)$  as inputs and complex voltages  $v_1(s)$  and  $v_2(s)$  as outputs (Equation 3.5). Its derivation is based on physical laws and can be directly computed, when it exists, from an electronic circuit. The impedance matrix  $Z(s)$  is used in this work as an intermediary between a circuit diagram and the scattering matrix  $S$ .

$$\forall s \in \mathbb{C}, \quad \begin{bmatrix} v_1(s) \\ v_2(s) \end{bmatrix} = \underbrace{\begin{bmatrix} z_{11}(s) & z_{12}(s) \\ z_{21}(s) & z_{22}(s) \end{bmatrix}}_{Z(s)} \begin{bmatrix} i_1(s) \\ i_2(s) \end{bmatrix} \quad (3.5)$$

Derived from the impedance matrix  $Z(s)$ , the *input impedance*  $z_{in}(s)$  models the be-

FIGURE 3.4: Electronic filter  $\mathcal{N}$ 

<sup>1</sup>The terminology *two-port network* also appears in the literature. As the word *network* seems to be ambiguous, sometimes referring to a system [Dar84] and sometimes to the model an electrical circuit [Bel62], it will not be used in this manuscript.

haviour of the two-port  $\mathcal{N}$  terminated on a resistive load  $R_l$ . It is the transfer function between complex current  $i_1(s)$  and complex voltage  $v_1(s)$  (Equation 3.6). The input impedance  $z_{in}(s)$  is of interest when designing filters with a specific interconnection, as will be detailed in Subsection 4.3.3.

$$\forall s \in \mathbb{C}, \quad v_1(s) = z_{in}(s)i_1(s) \quad (3.6)$$

The scattering matrix  $S$  models the power transfer between the terminals of an electronic filter. The electronic filter design problem is usually formulated on the scattering matrix  $S$ , as filter requirements are expressed as power frequency constraints [You71]. The scattering matrix  $S$  is defined as:

$$\forall s \in \mathbb{C}, \quad \begin{bmatrix} \beta_1(s) \\ \beta_2(s) \end{bmatrix} = \underbrace{\begin{bmatrix} s_{11}(s) & s_{12}(s) \\ s_{21}(s) & s_{22}(s) \end{bmatrix}}_{S(s)} \begin{bmatrix} \alpha_1(s) \\ \alpha_2(s) \end{bmatrix} \quad (3.7)$$

with

$$\begin{aligned} \beta_1(s) &:= \frac{v_1(s) - R_g i_1(s)}{2\sqrt{R_g}} & \alpha_1(s) &:= \frac{v_1(s) + R_g i_1(s)}{2\sqrt{R_g}} \\ \beta_2(s) &:= \frac{v_2(s) - R_l i_2(s)}{2\sqrt{R_l}} & \alpha_2(s) &:= \frac{v_2(s) + R_l i_2(s)}{2\sqrt{R_l}} \end{aligned}$$

The complex signals  $\alpha_1(s)$ ,  $\alpha_2(s)$ ,  $\beta_1(s)$ ,  $\beta_2(s)$  are homogeneous to square roots of power signals, and can be physically interpreted as *power waves*.  $\alpha_1$  and  $\alpha_2$  are considered as *incident* power waves and  $\beta_1$  and  $\beta_2$  as *reflected* power waves. The entries of the scattering matrix  $S$  can then be viewed as ratios of power waves which characterise the power transfers between terminals of a two-port. In particular, the transfer function  $s_{21}(s)$  characterises the power transfer from the source to the load. Indeed, using (3.3), (3.4) and (3.7),  $s_{21}(s)$  can be calculated as:

$$s_{21}(s) = \frac{v_2(s)/\sqrt{R_l}}{e_g(s)/2\sqrt{R_g}}$$

Additionally, define the power gain  $G$  as the ratio of  $P_l(\omega^2) = |v_2(j\omega)|^2 / R_l$ , the average power delivered to the load  $R_l$ , to  $P_g(\omega^2) = |e_g(j\omega)|^2 / 4R_g$  the available generator power [You71]:

$$\forall \omega \in \mathbb{R}, \quad G(\omega^2) = \frac{P_l(\omega^2)}{P_g(\omega^2)} = \frac{|v_2(j\omega)|^2 / R_l}{|e_g(j\omega)|^2 / 4R_g}$$

Therefore, from an electrical point of view, the so-called *insertion-loss*  $|s_{21}(j\omega)|^2$  characterises the power transfer from the generator to the load:

$$\forall \omega \in \mathbb{R}, \quad G(\omega^2) = |s_{21}(j\omega)|^2$$

This interpretation leads to the definition of particular zeros of  $s_{21}$  and  $s_{11}$ , namely *transmission zeros* and *reflected zeros* [Bah84]. A transmission zero  $j\omega_{t_0}$  is a complex number such that  $|s_{21}(j\omega_{t_0})|^2 = 0$ . As a result, the reflected power wave  $\beta_2$  is null and no power is dissipated into the load. Similarly, a reflection zero  $j\omega_{r_0}$  is a complex number such that  $|s_{11}(j\omega_{r_0})|^2 = 0$ . The reflected power wave  $\beta_1$  is null and the power is then dissipated into the resistances of  $\mathcal{N}$  and the load.

When  $\mathcal{N}$  is only made of lossless elements, i.e. without resistances, a reflection zero also corresponds to  $|s_{21}(j\omega_{r_0})|^2 = 1$ , as will be detailed in Subsection 3.3.2. Thus, all the available generator power is delivered to the load. The power dissipated

into the load is then  $P_l = \frac{E^2 R_l}{(R_l + R_g)^2}$ , where  $E$  is the voltage source amplitude. This power is maximum when the resistance  $R_l$  of the load equals the resistance  $R_g$  of the source  $R_l = R_g$  [You15, Chap 10], as illustrated on Figure 3.5. This is called *impedance matching*<sup>2</sup>. In practice, the source and load resistances are then generally chosen to be equal to a same value. For calculation simplification, this value is often normalised to  $1 \Omega$  in the design methods. Impedance denormalisation can then be achieved to any level of impedance [Bah84, Chap. 10].

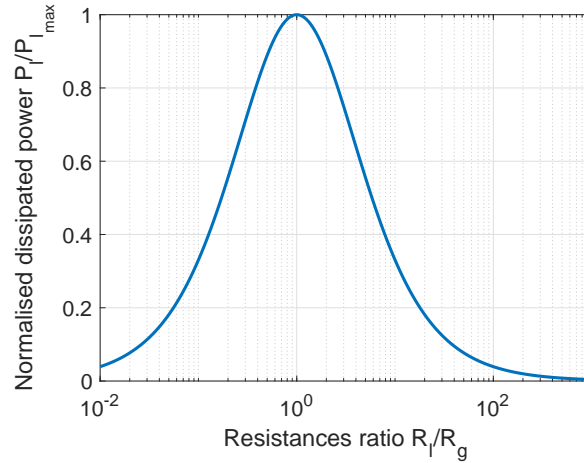


FIGURE 3.5: Impedance matching illustration - Normalised dissipated power  $\frac{P_l}{P_{l_{max}}}$  in the load ( $P_{l_{max}} = \frac{E^2}{4R_g}$ ) against resistances ratio  $\frac{R_l}{R_g}$

There exist standard relations between the scattering matrix  $S$ , the impedance matrix  $Z$ , and the input impedance  $z_{in}$ . These are provided in Properties 3.1 and 3.2.

**Property 3.1.**

Define  $\mathcal{R} := \begin{bmatrix} R_g & 0 \\ 0 & R_l \end{bmatrix}$  and assume that  $(\mathcal{R}^{-1/2} Z \mathcal{R}^{-1/2} + I)^{-1}$  exists. Then the impedance matrix  $Z$  and the scattering matrix  $S$  of an electronic filter  $\mathcal{N}$  are linked by the relation:

$$S = (\mathcal{R}^{-1/2} Z \mathcal{R}^{-1/2} - I) (\mathcal{R}^{-1/2} Z \mathcal{R}^{-1/2} + I)^{-1}$$

This relation can be written using the LFT framework as:

$$S = (\mathcal{R}^{-1/2} Z \mathcal{R}^{-1/2}) \star \left[ \begin{array}{c|c} -I & \sqrt{2}I \\ \hline \sqrt{2}I & -I \end{array} \right]$$

*Proof.* Cf Appendix B (p. 199). □

<sup>2</sup>More generally, when the source and the load have complex impedance  $Z_g$  and  $Z_l$ , impedance matching happens when  $Z_l = Z_g^*$ . In this case, the resulting dissipated power is  $P_{l_{max}} = \frac{E^2}{4R_g}$ . As the maximum power provided by the unloaded voltage source is  $\frac{E^2}{2R_g}$ , the power efficiency is 50%, meaning that there is as much power dissipated in the load resistance than in the source resistance. Impedance adaptation is then not truly satisfactory from an energy perspective, but acceptable for low-power applications that we are interested in. For a summary on the difference between power transfer and power efficiency, refer to [https://en.wikipedia.org/wiki/Maximum\\_power\\_transfer\\_theorem](https://en.wikipedia.org/wiki/Maximum_power_transfer_theorem).

**Property 3.2.** Similarly,  $z_{in}$  and  $s_{11}$  are linked by the following relations:

$$s_{11} = \frac{z_{in} - R_g}{z_{in} + R_g} \quad z_{in} = R_g \cdot \frac{1 - s_{11}}{1 + s_{11}}$$

We finish this subsection by exhibiting how the (lossless) passivity property of elements is propagated to their electrical interconnection (Property 3.3). By electrical interconnection, it is meant serial, parallel and cascade interconnection, as it usually the case in Network theory [AV73].

**Property 3.3.**

Let  $\mathcal{N}$  be an electronic filter with impedance  $Z(s)$  and scattering  $S(s)$  matrices.

1. If  $\mathcal{N}$  is the electrical interconnection of passive elements, then  $Z(s)$  is marginally stable and  $\{0, I_2, 0\}$ -dissipative and  $S(s)$  is stable and  $\{-I_2, 0, I_2\}$ -dissipative.
2. If  $\mathcal{N}$  is the electrical interconnection of lossless passive elements, then  $Z(s)$  is marginally stable and lossless  $\{0, I_2, 0\}$ -dissipative and  $S(s)$  is stable and lossless  $\{-I_2, 0, I_2\}$ -dissipative.

$\mathcal{N}$  is said to be respectively a passive and a lossless passive electronic filter.

*Proof.* This is a very standard result of Network theory. An interested reader may refer for instance to the proofs of [New66; AV73; Bah84].  $\square$

For design purposes, Property 3.3 will be used in the following sense: a given transfer matrix  $S(s)$  must be stable and (lossless)  $\{-I_2, 0, I_2\}$ -dissipative to be the scattering matrix of a (lossless) passive electronic filter.

### 3.2.3 A particular electrical interconnection: the ladder topology

An electronic filter is then defined as the electrical interconnection of electronic elements, set between a resistive source and a resistive load. A focus is now made on the interconnection.

In Property 3.3, it is stated how some element properties ((lossless) passivity) induce some filter dissipative properties ((lossless)  $\{-I_2, 0, I_2\}$ -dissipativity of  $S$ ) without explicitly specifying the electrical interconnection. Such implication is not usual. In general, this is the combination between elements dissipative properties and the type of interconnection which provides the global behaviour of the system.

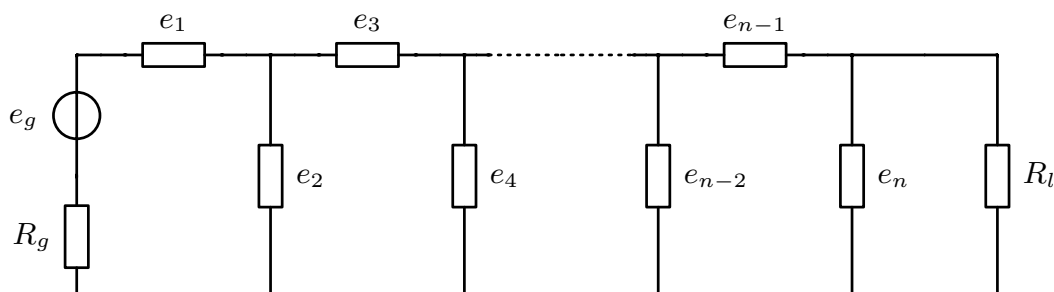


FIGURE 3.6: Electronic filter with a ladder topology

In the case of an electronic filter, the interconnection is called the *topology*. Given electronic elements, the choice of the topology influences the global filter properties. An important constraint of this topology is that it should be *physically implementable*.

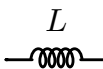
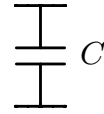
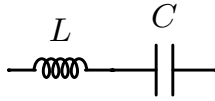
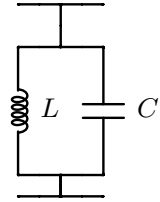
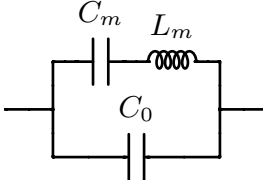
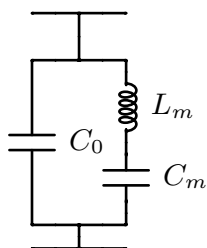


There exist several physically implementable topologies for electronic filters: ladder, lattice, mid-shunt, mid-series, to name a few [GB16; CBG+13; Bah84]. However, it appears that the ladder topology outshines the others. This is for two main reasons. First, the use of the ladder topology, associated with particular families of elements, significantly reduces the complexity of the filter design problem, as will be detailed in Chapter 4 (Section 4.3). Second, ladder filters with lossless passive elements are known to have low sensitivity to component value variations [TO73]. As a result, many *active* and *digital* filters are designed in a ladder form, with the objective of imitating this robustness property [Bah84; VM19]. Hence, we focus on the ladder topology in this document. It consists in consecutively setting serial and parallel elements, as illustrated on Figure 3.6. Notice that it is equivalent to start with a serial or a parallel element. Both configurations will be denoted as ladder topology.

Moreover, the ladder topology connects transmission zeros and reflection zeros with the local behaviour of elements [You15, Chap. 5]. Indeed, when a serial element  $e_i$  with impedance  $z_i$  behaves as an open-circuit ( $z_i(j\omega_{t_0}) \rightarrow \infty$ ) or a parallel element  $e_k$  with admittance  $y_k$  as a short-circuit ( $y_k(j\omega_{t_0}) \rightarrow \infty$ ) at a complex number  $j\omega_{t_0}$ , no power is transmitted through the filter and  $j\omega_{t_0}$  is a transmission zero  $|s_{21}(j\omega_{t_0})|^2 = 0$ . More restrictively, when *all* serial elements behave as short-circuits ( $z_i(j\omega_{r_0}) = 0$ ) and parallel elements as open-circuits ( $y_k(j\omega_{r_0}) = 0$ ) at a complex number  $j\omega_{r_0}$ , all power is transmitted through the filter and  $j\omega_{r_0}$  is a reflection zero  $|s_{21}(j\omega_{r_0})|^2 = 1$ . In practice, these connections help to fix some design variables, as will be illustrated in Chapter 4 (Section 4.3) and Chapter 6.

Finally, we define in Table 3.2 some filters with a ladder topology, associated with particular families of elements. These filters will be of importance throughout this manuscript for application purposes.

TABLE 3.2: Definition of particular filters with ladder topology

Name	Serial elements	Parallel elements
LC ladder		
LC-resonator ladder		
AW-resonator ladder		

### 3.3 Design problem

#### 3.3.1 Problem formulation

In this manuscript, we are interested in designing electronic filters such that the scattering matrix  $S(s)$  satisfies given magnitude constraints in the frequency-domain. Frequency filtering constraints are typically set on the squared modulus of the frequency-response of  $s_{21}$ . These constraints will be referred to as *spectral mask* in the sequel. More specifically, we are interested in constant upper and lower bounds defined on frequency intervals.

**Definition 3.2** (Spectral Mask  $\mathcal{SM}$ ). Consider  $N_U$  constant *upper bounds*, defined by  $N_U$  positive real numbers  $U_u$  associated with  $N_U$  frequency intervals  $\Omega_u^U \subseteq \mathbb{R}$ . Consider  $N_L$  constant *lower bounds*, defined by  $N_L$  positive real numbers  $L_l$  associated with  $N_L$  frequency intervals  $\Omega_l^L \subseteq \mathbb{R}$ . Define the *spectral mask*  $\mathcal{SM}$  as the union of these upper and lower bounds. A transfer function  $W$  will be said to satisfy  $\mathcal{SM}$  if the constraints defined by:

$$\begin{aligned} \forall u \in \{1, \dots, N_U\}, \quad \forall \omega \in \Omega_u^U, \quad |W(j\omega)|^2 &\leq U_u^2 \\ \forall l \in \{1, \dots, N_L\}, \quad \forall \omega \in \Omega_l^L, \quad |W(j\omega)|^2 &\geq L_l^2 \end{aligned}$$

hold.

**Example 3.1.** A typical bandpass spectral mask for passive electronic filters is provided on Figure 3.7 and constrains the scattering parameter  $s_{21}$  to satisfy:

$$\begin{aligned} \forall \omega \in \mathbb{R}, \quad |s_{21}(j\omega)|^2 &\leq 1 \\ \forall \omega \in [0; \omega_{U_1}], \quad |s_{21}(j\omega)|^2 &\leq U_1^2 \\ \forall \omega \in [\omega_{L_1}; \omega_{L_2}], \quad |s_{21}(j\omega)|^2 &\geq L^2 \\ \forall \omega \in [\omega_{U_2}, \infty[, \quad |s_{21}(j\omega)|^2 &\leq U_2^2 \end{aligned}$$

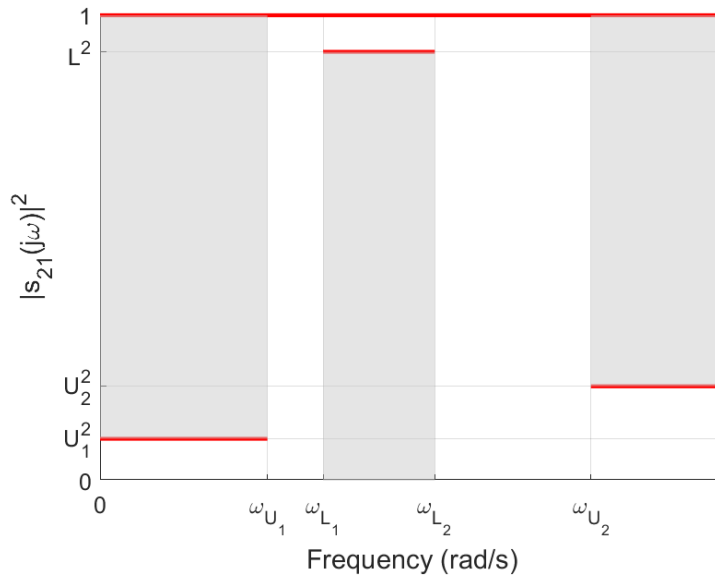


FIGURE 3.7: Typical bandpass spectral mask

The next problem (Problem 3.1) provides a general formulation of the design problem of passive electronic filters.

**Problem 3.1 (Passive Electronic Filter Design).**

**GIVEN** a family of passive elements  $\mathcal{F}$ ,  
a topology  $\mathcal{T}$ ,  
a source resistance  $R_g$  and a load resistance  $R_l$ ,  
 $N_U$  upper bounds  $U_u$  associated with  $N_U$  real intervals  $\Omega_u^U \subseteq \mathbb{R}$ ,  
 $N_L$  lower bounds  $L_l$  associated with  $N_L$  real intervals  $\Omega_l^L \subseteq \mathbb{R}$ ,  
**FIND** if there exists an electronic filter, defined as the interconnection of elements from  $\mathcal{F}$  in the topology  $\mathcal{T}$  between a resistive voltage source with source resistance  $R_g$  and a resistive load  $R_l$ , such that its scattering parameter  $s_{21}$  is stable and satisfies the following spectral mask  $\mathcal{SM}$ :

$$\forall u \in \{1, \dots, N_U\}, \quad \forall \omega \in \Omega_u^U, \quad |s_{21}(j\omega)|^2 \leq U_u^2 \quad (3.8)$$

$$\forall l \in \{1, \dots, N_L\}, \quad \forall \omega \in \Omega_l^L, \quad |s_{21}(j\omega)|^2 \geq L_l^2 \quad (3.9)$$

If such electronic filter exists, CALCULER le.

Problem 3.1 is stated for a general topology  $\mathcal{T}$  and a general family of passive element  $\mathcal{F}$ . In practice, most of the illustrative design examples will concern  $LC$  ladder,  $LC$ -resonator ladder and AW-resonator ladder filters (Table 3.2).  $LC$  ladder and  $LC$ -resonator ladder filters are interesting for two reasons. First, electronic filters were historically made of coils and capacitors, which can be modelled by inductances and capacitances. Thus, their design methodology is comprehensive and detailed [Bah84]. Second, the associated design problems appear to be relatively simple and is a good reference to validate new design methods. AW-resonator ladder filters are interesting for Micro-Electronics application purposes, especially for mobile communication [Has09; HKM+15; GVS18]. In addition, the resulting design problem is more complex, providing then a discriminatory mean to test new design methods.

### 3.3.2 Problem simplification for lossless passive filters

Important properties on the scattering matrix  $S$  of interconnected lossless passive elements can be deduced from Property 3.3 (p. 45). This especially imposes strong constraints on  $S$  and leads to the so-called Belevitch Representation theorem [You71]. Additionally, the interconnection of lossless building elements  $L$  and  $C$  satisfies a structural property, namely *reciprocity* property [AV73]. This enables to sharpen last representation in order to obtain the Reciprocal Belevitch Representation theorem (Theorem 3.1). In particular, the only knowledge of  $s_{21}$  will be sufficient to *build* a scattering matrix  $S$ .

**Theorem 3.1** (Reciprocal Belevitch Representation [You71]).

Let  $\mathcal{N}$  be a lossless passive filter. Then, its scattering matrix  $S(s)$  has the following description:

$$S(s) = \begin{bmatrix} s_{11}(s) & s_{12}(s) \\ s_{21}(s) & s_{22}(s) \end{bmatrix} = \frac{1}{g(s)} \begin{bmatrix} h(s) & f(s) \\ f(s) & \mu h(-s) \end{bmatrix}$$

where

1.  $g$ ,  $h$ , and  $f$  are real polynomials of the Laplace variable  $s$
2.  $f$  is either even or odd and  $\mu := -\frac{f(-s)}{f(s)}$

3.  $g$  is strictly Hurwitz<sup>3</sup>

$$4. \forall s \in \mathbb{C}, h(s)h(-s) + f(s)f(-s) = g(s)g(-s)$$

Therefore, for an electronic filter  $\mathcal{N}$  made of interconnected lossless passive elements, the only knowledge of the transfer function  $s_{21}$  is sufficient to synthesise an entire scattering matrix  $S$ . Equivalently, this enables to characterise the entire impedance matrix  $Z$ . For synthesis purpose, last statement is reversely used. The knowledge of a  $\{-1, 0, 1\}$ -dissipative transfer function  $s_{21} := \frac{f}{g}$ , where  $f$  is either an even or an odd polynomial and  $g$  a Hurwitz polynomial, permits to *build* such a scattering matrix  $S$ .

In addition, Theorem 3.1 has for consequence the next important property.

**Property 3.4.** *Let  $\mathcal{N}$  be a lossless passive filter. Let  $S$  be its scattering matrix. Then:*

$$\forall \omega \in \mathbb{R}, \quad |s_{11}(j\omega)|^2 + |s_{21}(j\omega)|^2 = 1 \quad (3.10)$$

The squared magnitude  $|s_{11}(j\omega)|^2$  characterises the ratio of the *reflected* power by  $\mathcal{N}$  over the available generator power, and  $|s_{21}(j\omega)|^2$  the power transmitted through  $\mathcal{N}$ . A physical interpretation of (3.10) is that no power is dissipated into  $\mathcal{N}$ .

Furthermore, Property 3.4 leads us to provide an alternative formulation of the Passive Electronic Filter Design problem (Problem 3.1). Indeed, while the design problem of lossless passive electronic filters is a particular sub-problem of Problem 3.1, where  $\mathcal{F}$  is chosen to be a family of *lossless* passive elements, an alternative version will be needed in the design approach 1 of Chapter 4. In this approach, the design variable is the frequency response  $s_{21}(j\omega)$ . Unfortunately, the spectral mask  $\mathcal{SM}$  of Problem 3.1 is non-convex in this design variable. In fact, upper and lower bounds on  $|s_{21}(j\omega)|^2$  are involved in  $\mathcal{SM}$ . However, the set of transfer functions  $W$  satisfying lower bounds is not convex. Geometrically, an upper bound is interpreted as demanding the frequency-response  $W(j\omega)$  to belong to a disk in the complex plane, which is a convex set. On the opposite, a lower bound constrains  $W(j\omega)$  to belong to the complement of a disk in the complex plane, and is not convex. Notice that this issue will not appear in the design approach 2, as the design variable will be the squared magnitude  $|s_{21}(j\omega)|^2$ , as comprehensively discussed in Chapter 5 (Subsection 5.3.1).

Nonetheless, from Property 3.4, it appears to be equivalent to express a spectral mask either on  $s_{11}$  or  $s_{21}$  for lossless passive filters. The lower bound (3.9) on  $|s_{21}(j\omega)|^2$  is then equivalent to an upper bound on  $|s_{11}(j\omega)|^2$ . This leads to the following alternative version for lossless passive electronic filter.

**Problem 3.2 (Alternative Lossless Passive Electronic Filter Design).**

*Given* a family of lossless passive elements  $\mathcal{F}$ ,

a topology  $\mathcal{T}$ ,

a source resistance  $R_g$  and a load resistance  $R_l$ ,

$N_U$  upper bounds  $U_u$  associated with  $N_U$  real intervals  $\Omega_u^U \subseteq \mathbb{R}$ ,

$N_L$  lower bounds  $L_l$  associated with  $N_L$  real intervals  $\Omega_l^L \subseteq \mathbb{R}$ ,

*Find* if there exists an electronic filter, defined as the interconnection of elements from  $\mathcal{F}$  in the topology  $\mathcal{T}$  between a resistive voltage source with source resistance  $R_g$  and a resistive load  $R_l$ , such that its scattering parameters  $s_{11}$  and  $s_{21}$  are stable and satisfy the spectral mask  $\mathcal{SM}$ , given by (3.8) and:

$$\forall l \in \{1, \dots, N_L\}, \quad \forall \omega \in \Omega_l^L, \quad |s_{11}(j\omega)|^2 \leq 1 - L_l^2$$

<sup>3</sup>A Hurwitz polynomial is a polynomial whose roots are located in the left-half plane of the complex plane.

If such electronic filter exists, *Compute* it.

One may notice that, using this alternative formulation, the scattering parameters  $s_{11}$  and  $s_{21}$  need to satisfy the constraints of the Reciprocal Belevitch theorem (Theorem 3.1) in order to be able to build a whole scattering matrix  $S$ .

### 3.4 Systematic filter design methods: a historical perspective

Problem 3.1 has two facets. First facet is the requirement to synthesise an electronic filter which is the interconnection in a given topology  $\mathcal{T}$  of passive elements belonging to a given family  $\mathcal{F}$ . Second facet is the requirement that the resulting filter should have a stable scattering parameter  $s_{21}$  which satisfies a spectral mask  $\mathcal{SM}$ . Tackling simultaneously both facets is very challenging. Therefore, two types of design methods have historically competed. First type of design methods tackles primarily the first facet. This approach is close to the practical problem and the resulting filter is ensured to be physically realisable. We will refer to it as the *elements-value tuning* approach, as this approach leads to tune the characteristic values of the elements. Second type of design methods is more concerned with the second facet. These methods have generally a low computational cost and the resulting filter has a relatively-low number of elements. Likewise, it is referred as the *realisable filter synthesis* approach, as it consists in synthesising a transfer function such that it satisfies realisability conditions. As systematic methods of both approaches have successively being successful, depending on the application and the available computational power, they are presented below in a chronological order.

The first systematic filter design method belongs to the elements-value tuning approach and is named *image-parameter* method. Developed during the inter-war period, it was especially used until the end of the Second World War [Bel62; Dar84]. This method relies on the impedance matching concept and consists in cascading simple filters sections, which impedances are two-by-two adapted, until the resulting scattering parameter  $s_{21}$  satisfies the spectral mask  $\mathcal{SM}$ . The image-parameter method is intuition-based and help to design filters which meets practical requirements. It especially gives a direct link between the values of the components and filters properties. Another advantage is its ability to design high-order filters without requiring computer power. However, this method assumes that the terminations are the image impedances of the filter, and not resistive terminations. It is also inherently restricted by the required cancellation of the zeros and the poles of the open- and short- circuit impedance. Therefore, the resulting filter may have average filter performance or unnecessary high number of elements.

To overcome these difficulties, the *insertion-loss* method was developed in parallel [Bel62; You71; Dar84]. While this method enjoyed significant success, it was not generally accepted until some years later, after the Second World War. This is due to its competition with the image-parameter method, used by highly-skilled filter designers. Filters made by insertion-loss method perform better, but required lengthy calculations. The advent of computers widened its use among designers.

The *insertion-loss* design method belongs to the *realisable filter synthesis* approach. It is made of two steps [You71; Bah84]. First, a transfer function  $s_{21}$  is calculated such that it is stable and satisfies the spectral mask  $\mathcal{SM}$ . Earliest methods to compute such  $s_{21}$  are the famous Butterworth and Chebyshev approximation methods, respectively developed by Butterworth and Cauer around 1930. Second step is the *realisation step*. It consists of synthesising a filter with a prescribed insertion

loss  $s_{21}$ . This is not always possible and realisation conditions have to be found. In general, these conditions are complex to find, leading to the development of a new research discipline named *realisation theory* [Kal10].

This discipline started around 1925 with the Foster reactance theorem and the Foster and Cauer synthesis of  $LC$  one-ports. It flourished in 1931 with the Brune's synthesis theorem, proving that any positive real impedance matrix is realisable by a two-port made of passive elements. This led to an adapted version for filter synthesis, the so-called Darlington Synthesis (1938), which showed that any positive real function can be realised as the input impedance of a lossless reciprocal two-port terminated in a resistance. Both Brune and Darlington synthesis techniques were systematic and generated two-ports with minimum number of elements, but they had a main drawback: transformers were required. While transformers could be used in communication applications of that time, post-war applications made these components to be avoided. Lots of research were undertaken to avoid transformers. The most striking result in that sense is the Bott-Duffin synthesis (1949), which enables to realise any positive real impedance matrix as a two-ports only made of resistances, inductances and capacitances. However, this method is truly wasteful in elements, and is generally impractical. Despite a lot of research from different communities, refer for instance to [New66] for an input-output approach<sup>4</sup> and to [AV73] for its state-space counterpart, there were no new practical results developed in this discipline.

While there is renewed interest for mechanical applications [Smi02; CPS+09], realisation theory for passive two-ports is generally admitted to have died as field of research in the 1970's [Kal10]. For electronic filter interest, this may be explained by two main reasons. First, there generally exist physically implementable electronic filters made of inductances  $L$  and capacitances  $C$  which realise the scattering parameters  $s_{21}$  synthesised by usual approximation methods. From a practical point of view, this was then not necessary to have realisation conditions for  $LC$  filters. Second, the year 1948 saw the invention of the transistor. This led to the rise of digital filters, for which there always exists a physical implementation associated with a given transfer function [PB87]. In addition, digital filters have the particularly nice property to be reprogrammable, and having characteristics not reachable by analog passive filters such as *exact* linear phase.

Analog passive filters are still of importance for electronic applications, such as those requiring low power consumption [CRV11]. In RadioFrequency (RF) filtering applications, analog passive filters, based on acoustic wave resonators (SAW/BAW), are especially appreciated for their performance, low-power consumption and high-quality factor [Has09; HKM+15]. However, as the insertion-loss method provides satisfactory results only for  $LC$  filters, designers have developed practical methods based on their experience, which belong to the elements-value tuning approach. While this usually enables to produce satisfactory filters, the recent huge increase in the design problem complexity requires systematic methods with a solid theoretical basis. Therefore, new methodologies have been developed to extend the insertion-loss method to components of interest, in order to improve and make more efficient the design of passive electronic filters [SPJK05; Shi06; GB16; GVS18]. Unfortunately, these methods generally lead to add external components in order to get physical realisability, and tend to get numerical issues when implemented on a computer.

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<sup>4</sup>or for an idle curiosity of reading *poetry* about Network theory.

### 3.5 Summary

In this chapter, passive electronic filters have been first presented. To achieve this, the term *element* has been introduced to differentiate a model of a component from its physical realisation. The term *building element* has also been introduced to denote the models of inductances, capacitances and resistances, as electronic components are traditionally modelled using them. The (lossless) passive property of elements has especially been mathematically defined, while a passive electronic filter is an electrical interconnection of passive elements, terminated on a resistive voltage source and a resistive load (Figure 3.4, p 42).

The behaviour of passive electronic filters is described using the impedance  $Z$  and the scattering  $S$  matrices. In this work, the impedance matrix  $Z$  is used to provide a link between a circuit block diagram and a mathematical model, while the design requirements are expressed on the scattering matrix  $S$ . An important property on (lossless) passive electronic filters is that  $Z$  is (lossless)  $\{0, I_2, 0\}$ -dissipative and  $S$  is (lossless)  $\{-I_2, 0, I_2\}$ -dissipative (Property 3.3, p. 45).

Then, the Passive Electronic Filter Design problem (Problem 3.1, p. 48) states the design problem as finding an electronic filter, with a pre-defined topology and a family of elements, such that its scattering parameter  $s_{21}$  satisfies some frequency requirements, denoted spectral mask, on the magnitude of its frequency response. The formulation is very general and the associated design problem is equally complex. Therefore, the design methods developed in the sequel tackle more specific form. In particular, our illustrative design examples will mainly concern  $LC$  ladder,  $LC$ -resonator ladder and AW-resonator ladder filters (Table 3.2, p 46).

Finally, the review of systematic filter design methods (Section 3.4) reveals that two design approaches have historically competed. First approach, namely *elements-value tuning*, consists in setting the elements in ready-to-implement form and to tune the characteristic values of the elements until the spectral mask is fulfilled. Second approach, called *realisable filter synthesis*, consists in synthesising a transfer function such that it satisfies the spectral mask *and* realisability conditions.

Both approaches have their own benefits in terms of physically realisability or efficiency, and have then been successively successful, depending on the application and the available computational power. Our aim is then to revisit them under our framework. This is the matter of Chapter 4.

## Chapter 4

# Design Approaches of Passive Electronic Filters

### 4.1 Introduction

In Chapter 3, the design problem of passive electronic filters, i.e. the synthesis of passive elements from a given family interconnected in a given topology such that a certain scattering parameter satisfies a given spectral mask, has been introduced. While substantially complex, it appears that systematic design methods have been developed and met some success in solving this design problem for particular families and topologies. Reviewing these methods, it appears that two design approaches are distinguishable. In this chapter, our aim is to revisit both approaches under our design framework.

First approach, named elements-values tuning in this work, consists in setting the elements from a given family into a given topology, and to tune their characteristic values until the spectral mask is fulfilled. In practice, this is achieved by using a circuit diagram and a CAD software. In Section 4.2, we make explicit the underlying mathematical representation and develop an appropriate synthesis method. This method is generic, close to the design problem and allows to include some technological constraints. The only assumption, that could be removed in a further work, is that the elements are *lossless* passive.

Second approach, denoted as realisable filter synthesis, consists in synthesising a scattering parameter such that it satisfies a spectral mask  $SM$  and realisation constraints. This approach is traditionally applied for particular electronic filters, such as  $LC$  ladder and  $LC$ -resonator ladder filters (cf Table 3.2, p. 46). In Section 4.3, the usual design method is reformulated within our framework. Based on this, a *part* of this method, that is the search of realisation constraints, is extended to more general ladder filters. The remaining part will be extended in Chapter 5.

### 4.2 Design approach 1: elements-values tuning

#### 4.2.1 Introduction

One of the main difficulty in the design of passive electronic filters is to find an appropriate mathematical representation. This representation should satisfy two requirements: to enable an efficient design and to represent a physical implementation. A trade-off is usually made between these requirements. In Section 4.3 and Chapter 5, it will be shown that the frequency filter synthesis problem, of finding a transfer function  $s_{21}$  such that it satisfies a given spectral mask  $SM$ , can be formulated as a convex optimisation problem. Then, there exist efficient solving methods. However, there is generally not a physical implementation associated with the resulting  $s_{21}$ . This leads



to the search of realisation constraints on  $s_{21}$ . Additionally, these constraints need to have a specific form in order to preserve a convex formulation.

In practice, when considering methods used by designers, the balance tips in favour of physical implementation. Using a block diagram model of the electronic filter and a CAD software, the elements are interconnected in a given topology. Characteristic values of these elements are then tuned until the resulting transfer function  $s_{21}$  satisfies  $\mathcal{SM}$ . The tuning methods are generally based on the intuition of seasoned designers in order to find an initial guess, which is then optimised with the aid of generic solvers [WL15]. This typically results in a non-convex problem. The methodology is non-systematic, and the design time may be important, depending on the initial guess. Nevertheless, it ends up with an implementable filter and allows to take into account numerous practical constraints. Therefore, it is aimed in this section to revisit this design methodology, and to make explicit the underlying optimisation problem. The motivation is to benefit from its implementable property, while reducing to the minimum the *degree*<sup>1</sup> of the optimisation problem.

When a block diagram representation of a physical circuit is made by a designer, a mathematical model is implicitly built. The state-space representation seems *a priori* suitable to provide a direct mean to make explicit this model. By an adequate choice of the state vector, the dynamics of the building elements should naturally be displayed, while the topology of the filter may appear through algebraic constraints on the state-space matrices, leading to a so-called *structured* state-space representation. In addition, these constraints should be simpler than constraints on the coefficients of a transfer function<sup>2</sup>. However, the structured state-space synthesis problem has typically a non-convex formulation [BT97]. This naturally leads to the following two questions:

- Q1. How to obtain a state-space representation from a circuit diagram?
- Q2. How to reduce the degree of non-convexity of the design problem? In particular,
  - a) Among the infinite number of state-space representations, does there exist one which reduces this degree?
  - b) Is there a synthesis method able to take into account the structure of this state-space representation, such that this degree is even more reduced?

The Port-Hamiltonian Systems (PHS) framework seems appropriate for tackling Q1 and Q2 a) for three reasons (Subsection 4.2.2). First, it provides a straightforward *systematic* modelling procedure. Actually, the result of the modelling procedure is generally *not* a state-space representation such as introduced in Chapter 2. Indeed, as this procedure aims to stay close to the laws of Physics, algebraic constraints between the state-variables commonly arise and prevent from directly obtaining such a representation. As a consequence, for these cases, the procedure ends up with an extension of the state-space representation, namely the Differential-Algebraic Equations (DAE) representation. Second, the resulting representation is particularly simple from a design perspective. In fact, the design variables, i.e. the elements values, distinctly, compactly and *linearly* appear as the entries of a *diagonal* matrix  $Q$ . Intuitively, these features should reduce the complexity of the design problem, especially when compared to representations with products between the design variables (cf Example 4.1 below). Third, stability and lossless  $\{-I_2, 0, I_2\}$ -dissipativity of the scattering

<sup>1</sup>While not strictly defined in this manuscript, the use of the term *degree*, to measure the non-convexity a problem, will be justified in this subsection by the discussion around Example 4.2.

<sup>2</sup>Some will be given in Subsection 4.3.3.

matrix  $S$  can be checked in a very simple manner. Indeed, the sign condition  $Q > 0$  will be sufficient to ensure these properties.

**Example 4.1.** Consider the  $LC$  ladder filter of Figure 4.1 with  $n = 2$  elements. Assume that  $R_g = R_l = 1 \Omega$ . In Section 4.3, it will be shown that the scattering

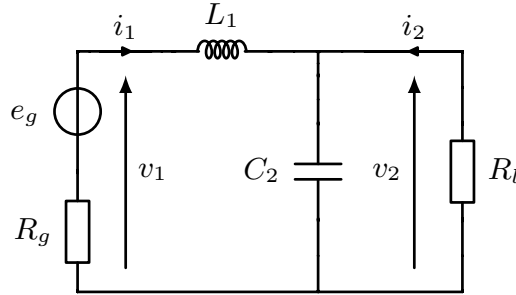


FIGURE 4.1: Two-elements example of a  $LC$  ladder filter

parameter  $s_{21}(s)$  is given by  $s_{21}(s) = \frac{2q_1q_2}{2q_1q_2 + (q_1+q_2) \cdot s + s^2}$  with  $q_1 := \frac{1}{L_1}$  and  $q_2 := \frac{1}{C_2}$ . A space-space realisation of  $s_{21}$  in the controllable canonical form [ZDG96, Chap. 3] can then be computed as:

$$\begin{cases} p_{can} = \left(\frac{1}{s} \cdot I_2\right) q_{can} \\ q_{can} = \begin{bmatrix} 0 & 1 \\ -2q_1q_2 & -(q_1+q_2) \end{bmatrix} p_{can} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w_1 \\ z_2 = \begin{bmatrix} 2q_1q_2 & 0 \end{bmatrix} p_{can} \end{cases}$$

where  $w_1$  and  $z_2$  are respectively the input and the output vectors, i.e.  $z_2 = s_{21}(s)w_1$ , and  $q_{can}$  and  $p_{can}$  are suitable vectors of appropriate dimensions. Now, using the PHS modelling procedure that will be provided in the next subsection, it appears that no algebraic constraint between state variables occurs for this example and the following state-space representation is obtained:

$$\begin{cases} p = \left(\frac{1}{s} \cdot I_2\right) q \\ q = (J - BB^T)Qp + \sqrt{2}B_1w_1 \\ z_2 = \sqrt{2}B_2^T Qp \end{cases} \quad (4.1)$$

with  $q := \frac{1}{\sqrt{2}} \begin{bmatrix} v_{L_1} \\ i_{C_2} \end{bmatrix}$ ,  $p := \frac{1}{\sqrt{2}} Q^{-1} \begin{bmatrix} i_{L_1} \\ v_{C_2} \end{bmatrix}$ ,  $w_1 := \frac{v_1 + R_g i_1}{2\sqrt{R_g}}$ ,  $z_2 := \frac{v_2 - R_l i_2}{2\sqrt{R_l}}$  and

$$J := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad Q := \begin{bmatrix} q_1 & 0 \\ 0 & q_2 \end{bmatrix} \quad B := \begin{bmatrix} B_1 & | & B_2 \end{bmatrix} := \begin{bmatrix} 1 & | & 0 \\ 0 & | & 1 \end{bmatrix}$$

This representation is linear into the diagonal matrix  $Q$ , which has the design variables as entries. It looks then more simple than the controllable canonical form, for which these variables non-compactly come out and are multiplied together. In addition, it is also closer to the circuit diagram of Figure 4.1. Indeed,  $p$  and  $q$  are directly linked with the local voltages and currents of  $L_1$  and  $C_2$ , and their values, while it seems *a priori* difficult to establish such a link with  $p_{can}$  and  $q_{can}$ .

While the PHS framework provides a simple mathematical representation where the decision variables linearly appear, the design problem remains complex and the synthesis method must be wisely chosen. Indeed, a direct application of the KYP Lemmas, for instance, leads to an optimisation problem with an important number of non-linear products between the optimisation variables, as illustrated below in Example 4.2. These products make here the problem non-convex, and prevent from using efficient algorithm to solve the design problem. Their number may then provide an indicator of the degree of non-convexity of the problem, and our aim is to reduce it to the minimum.

**Example 4.2** (Example 4.1 continued). Consider the following synthesis problem: Find  $\mathbf{Q} > 0$  such that

$$s_{21}(s) = \left( \frac{1}{s} \cdot I_2 \right) \star \left[ \begin{array}{c|c} (J - BB^T)\mathbf{Q} & \sqrt{2}B_1 \\ \hline \sqrt{2}B_2^T\mathbf{Q} & 0 \end{array} \right]$$

satisfies the following upper bound:

$$\forall \omega \in [0, \omega_0], \quad |s_{21}(j\omega)|^2 \leq U^2$$

Applying the suitable KYP Lemma (Theorem 2.4, p. 30) leads to check the existence of  $\mathbf{P} = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \in \mathbb{R}^{2 \times 2}$ ,  $\mathbf{D} = \begin{bmatrix} d_{11} & d_{12} \\ d_{12} & d_{22} \end{bmatrix} \in \mathbb{R}^{2 \times 2}$  such that:  $\mathbf{Q} > 0$ ,  $\mathbf{D} > 0$  and

$$\begin{bmatrix} (J - BB^T)\mathbf{Q} & \sqrt{2}B_1 \\ I & 0 \\ \sqrt{2}B_2^T\mathbf{Q} & 0 \\ 0 & I \end{bmatrix}^T \begin{bmatrix} \mathbf{D} & \mathbf{P} & 0 & 0 \\ \mathbf{P} & -\omega_0^2\mathbf{D} & 0 & 0 \\ 0 & 0 & -I & 0 \\ 0 & 0 & 0 & U^2I \end{bmatrix} \begin{bmatrix} (J - BB^T)\mathbf{Q} & \sqrt{2}B_1 \\ I & 0 \\ \sqrt{2}B_2^T\mathbf{Q} & 0 \\ 0 & I \end{bmatrix} \geq 0$$

Pre- and post-multiplying by  $\begin{bmatrix} \mathbf{Q}^{-1} & 0 \\ 0 & I \end{bmatrix}$  leads to the equivalent inequality:

$$\begin{bmatrix} J - BB^T & \sqrt{2}B_1 \\ I & 0 \\ \sqrt{2}B_2^T & 0 \\ 0 & I \end{bmatrix}^T \begin{bmatrix} \mathbf{D} & \mathbf{P}\mathbf{Q}^{-1} & 0 & 0 \\ \mathbf{Q}^{-1}\mathbf{P} & -\omega_0^2\mathbf{Q}^{-1}\mathbf{D}\mathbf{Q}^{-1} & 0 & 0 \\ 0 & 0 & -I & 0 \\ 0 & 0 & 0 & U^2I \end{bmatrix} \begin{bmatrix} J - BB^T & \sqrt{2}B_1 \\ I & 0 \\ \sqrt{2}B_2^T & 0 \\ 0 & I \end{bmatrix} \geq 0$$

Defining  $\mathbf{Q}^{inv} := \mathbf{Q}^{-1}$ ,  $\mathbf{q}_1^{inv} = (\mathbf{q}_1)^{-1}$  and  $\mathbf{q}_2^{inv} = (\mathbf{q}_2)^{-1}$ , the non-linear matrix terms are given by:

$$\mathbf{Q}^{inv}\mathbf{D}\mathbf{Q}^{inv} = \begin{bmatrix} d_{11} (\mathbf{q}_1^{inv})^2 & d_{12} (\mathbf{q}_2^{inv})^2 \\ d_{12} (\mathbf{q}_1^{inv})^2 & d_{22} (\mathbf{q}_2^{inv})^2 \end{bmatrix} \quad \mathbf{P}\mathbf{Q}^{inv} = \begin{bmatrix} p_{11}\mathbf{q}_1^{inv} & p_{12}\mathbf{q}_2^{inv} \\ p_{12}\mathbf{q}_1^{inv} & p_{22}\mathbf{q}_2^{inv} \end{bmatrix}$$

There are then  $2 \times \frac{n(n+1)}{2} = 6$  different non-linear products between the optimisation variables.

In Example 4.2, the non-linear matrix terms may be distinguished in two types: those such as  $\mathbf{P}\mathbf{Q}^{inv}$ , denoted as *bilinear* matrix terms, involving two-by-two variable products called *bilinearities*, and those such as  $\mathbf{Q}^{inv}\mathbf{D}\mathbf{Q}^{inv}$ , involving variable products of higher order. In the sequel, we make the following qualitative assumption: the simpler the type of variable products is, and the less numerous these products

are, the less complex is the optimisation problem. Our aim is then to avoid variable products of higher order than two, while reducing their number to the minimum.

To achieve this, we develop a synthesis method for PHS DAE representations (Subsection 4.2.3). This synthesis method, unusual in the literature of system design or in the PHS framework, enables to obtain an optimisation problem where the non-linear matrix terms are solely bilinear. Furthermore, we reveal how to significantly reduce the number of bilinearities, by making some algebraic operations on the representation matrices.

While still non-convex, the particular form of the resulting optimisation problem, called Bilinear Matrix Inequality (BMI) problem, enables to use dedicated algorithms of the literature (Subsection 4.2.4). These algorithms typically converge to a solution with limited convergence issues, under the assumption that a *good enough* initial point is available. One of these algorithms, based on the sequential resolution of LMI optimisation problems, is then detailed. Finally, our developed synthesis method is applied on a simple design example (Subsection 4.2.5).

## 4.2.2 Port-Hamiltonian Systems representations, modelling, analysis

The Port-Hamiltonian Systems (PHS) approach is a multi-physics framework for modelling, analysing and controlling dissipative systems. It especially aims to provide mathematical models close to the equations used in Physics, with a specific focus on the notion of energy. In the sequel, we adapt the PHS framework, and in particular the part dedicated to the Differential-Algebraic Equations (DAE) representation, for the study of lossless passive electronic filters. This especially provides a **systematic modelling** procedure, where the **design parameters of the building elements** appears distinctly and **linearly**. In addition, it provides an easy-to-check criterion for the scattering matrix  $S$  to be stable and lossless  $\{-I_2, 0, I_2\}$ -dissipative.

### Note on references

In this subsection, the material is mainly adapted from [Sch17; Sch13; SJ14]. For complementary materials on the PHS approach, the references [SM03; DMSB09] are also to be noted. One may also be interested in [AV73] for a state-space approach of modelling and analysis in Network theory. Furthermore, while we use the DAE representation as an intermediary tool for modelling and to simplify the design problem, there are whole theories behind for modelling, simulating, analysis and synthesis systems. In addition to the above references on the PHS approach, one may also be interested in the *linear implicit system* [Apl91], the *linear differential system* [Rap98], and the *behavioral* [Wil07] approaches.

### 4.2.2.1 PHS DAE representations

When one aims to get a mathematical model from a circuit diagram, a natural strategy seems to list all the characteristic electric signals and of the building elements, to exhibit their dynamical equations, to write down the interconnection scheme using Kirchhoff's circuit laws, and finally to state their relations with some pre-defined inputs and outputs. From the resulting set of equations, the closest mathematical

representation is *a priori* a state-space representation, such as in (4.2).

$$\begin{cases} p = \left(\frac{1}{s} \cdot I\right) q & (a) \\ q = Ap + Bu & (b) \\ y = Cp + Du & (c) \end{cases} \quad (4.2)$$

Indeed, the dynamical equations are gathered in (a), while the building elements interconnection appears in (b), and the link between the input  $u$ , the so-called state vector  $p$  and the output  $y$  is given in (c).

However, there are two main reasons for which the state-space representation appears generally unsuitable for this task. First, if one computes from (4.2) the associate transfer matrix, the entries of this matrix will necessarily be proper transfer function. This is problematic as the impedance  $Z$  and the admittance  $Y$  matrices of a circuit may have non-proper entries, and do not have then a corresponding state-space representation. In Network theory, this issue is sometimes tackled using a third matrix description, namely Hybrid matrix [AV73]. Another option is to do not *explicitly* label some variables as inputs and outputs. Indeed, the input-output labelling is not crucial for the modelling task, and may even be explicitly unwanted<sup>3</sup>. In this case, equations (b) and (c) of (4.2) are respectively replaced by:

$$\begin{aligned} q &= Ap + B_u u + B_y y \\ 0 &= Cp + D_u u + D_y y \end{aligned}$$

The second reason is that algebraic constraints between the state-space variables commonly arise, due to the interconnection laws, which can not be included in (4.2):

$$0 = E_{AEP}$$

Therefore, the state-space representation has to be extended. Several extension have been studied in the literature and are known over different names, such as *differential-algebraic*, *implicit* or *behavioral* representations [Sch13; Apl91; Wil07; Rap98]. These representations are too general for the purpose of this work. Instead, we use the Differential-Algebraic Equations (DAE) representation adapted to the PHS framework of next definition. This representation will be especially suitable for the modelling of lossless passive electronic filters.

**Definition 4.1 (PHS DAE representation** (Adapted from [Sch13])).

Consider matrices  $E, F \in \mathbb{R}^{(n+n_e) \times n}$  and  $E_e, F_e \in \mathbb{R}^{(n+n_e) \times n_e}$  such that:

$$EF^T + FE^T + E_e F_e^T + F_e E_e^T = 0 \quad (4.3)$$

$$\text{rank} \begin{bmatrix} F & E & F_e & E_e \end{bmatrix} = n + n_e \quad (4.4)$$

Then a Port-Hamiltonian Systems Differential-Algebraic Equations (PHS DAE) representation is defined as:

$$\begin{cases} p = \left(\frac{1}{s} \cdot I_n\right) q & (a) \\ Fq = EQp + F_e f_e + E_e e_e & (b) \end{cases} \quad (4.5)$$

<sup>3</sup>Refer to section *Cause and Effect* of [Wil07] or the discussion on computational causality of [DMSB09, Chap. 1] for more details.

where  $Q = Q^T > 0 \in \mathbb{R}^{n \times n}$  is a positive-definite matrix,  $q, p \in \mathbb{R}^n$  are denoted as the *internal* variables, and  $f_e, e_e \in \mathbb{R}^{n_e}$  the *external* variables.

In the context of lossless electronic filters, the internal variables  $q$  and  $p$  contain the characteristic electrical quantities of the building elements, while the external variables  $e_e$  and  $f_e$  are the electrical quantities linking the circuit with the source and the load, i.e. with conventions of Figure 3.4 (p. 42):

$$e_e = \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} \quad f_e = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \quad (4.6)$$

The positive definite  $Q$  will be *diagonal* with the inverse of the element values as diagonal entries. This important feature of the matrix  $Q$  will help to significantly reduce the complexity of the design problem (Subsection 4.2.3).

Definition 4.1 of a PHS representation is especially useful for modelling. Indeed, as will be detailed in Subsection 4.2.2.2, electronic filters are *naturally* modelled in this form. The dynamics of the building elements is represented by (a) of (4.5) and (b) provides a compact form of the equations coming from the Kirchhoff's laws, while (4.4) ensures that there is exactly the same number of these equations than building elements. Finally (4.3) express the lossless passivity property of the building elements. It is usually interpreted as a power preserving equation.

For design purposes, the following *scattering* PHS DAE representation is introduced. It will represent the scattering matrix  $S$  and provides then a direct link with the design criteria. This definition is slightly modified from the one given in [Sch17].

**Definition 4.2 (Scattering PHS DAE representation).**

Consider matrices  $E, F \in \mathbb{R}^{(n+n_e) \times n}$ ,  $E_e, F_e \in \mathbb{R}^{(n+n_e) \times n_e}$  and  $Q = Q^T > 0 \in \mathbb{R}^{n \times n}$ , and vectors  $q, p \in \mathbb{R}^n$ ,  $e_e, f_e \in \mathbb{R}^{n_e}$  such as in Definition 4.1.

Given a diagonal, positive-definite matrix  $\mathcal{R} > 0 \in \mathcal{D}(\mathbb{R}^{n_e \times n_e})$ , a *scattering* PHS DAE representation associated to  $\mathcal{R}$  is defined as:

$$\begin{cases} p = \left( \frac{1}{s} \cdot I_n \right) q \\ Fq = EQp + Gz + Hw \end{cases} \quad (4.7)$$

where  $z, w \in \mathbb{R}^{n_e}$  and  $G, H \in \mathbb{R}^{(n+n_e) \times n_e}$  are given by:

$$\begin{aligned} z &= \frac{1}{2} \left( \mathcal{R}^{-\frac{1}{2}} f_e - \mathcal{R}^{\frac{1}{2}} e_e \right) & G &= F_e \mathcal{R}^{\frac{1}{2}} - E_e \mathcal{R}^{-\frac{1}{2}} \\ w &= \frac{1}{2} \left( \mathcal{R}^{-\frac{1}{2}} f_e + \mathcal{R}^{\frac{1}{2}} e_e \right) & H &= F_e \mathcal{R}^{\frac{1}{2}} + E_e \mathcal{R}^{-\frac{1}{2}} \end{aligned} \quad (4.8)$$

The variables  $z$  and  $w$  are also denoted as external variables.

For this representation, the matrix constraints (4.3) and (4.4) become then:

$$EF^T + FE^T + \frac{1}{2} (HH^T - GG^T) = 0 \quad (4.9)$$

$$\text{rank} \begin{bmatrix} F & E & G & H \end{bmatrix} = n + n_e \quad (4.10)$$

When  $e_e$  and  $f_e$  are chosen as in (4.6), and  $\mathcal{R}$  is defined as  $\mathcal{R} := \begin{bmatrix} R_g & 0 \\ 0 & R_l \end{bmatrix}$ , the signals  $w$  and  $z$  are the input and the output of the scattering matrix  $S$  of an electronic

filter with source resistance  $R_g$  and load resistance  $R_l$  (cf p. 43):

$$w := \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} \quad z := \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} \quad (4.11)$$

Therefore, the scattering PHS DAE representation enables to provide a different representation of the link between the input and output of the scattering matrix  $S$ . Unlike the impedance  $Z(s)$  or the admittance  $Y(s)$  matrices, the scattering matrix  $S(s)$  of a passive electronic filter always has proper transfer functions as entries [AV73]. Therefore, it admits a state-space representation. Unfortunately, this does not imply that this representation can be set into a scattering PHS representation, that will be required to reduce the complexity of the design problem. Nonetheless, this implies that the external variables of (4.7) are *explicitly* separable into inputs  $w$  and outputs  $z$ , as stated in next property.

**Property 4.1.** *Consider a scattering PHS DAE representation as in Definition 4.2. Then (4.7) can be re-written as follows:*

$$\begin{cases} p = \left(\frac{1}{s} \cdot I_n\right) q \\ F_1 q = E_1 Q p + H_1 w \\ z = E_2 Q p + H_2 w \end{cases} \quad (4.12)$$

In addition, denote by  $r$  the rank of  $F_1$ ,  $r = \text{rank}(F_1)$ , and  $n_{AE}$  such that  $r + n_{AE} = n$ . Then, (4.12) can be expanded as follows:

$$\begin{cases} p = \left(\frac{1}{s} \cdot I_n\right) q \\ F_{1,r} q = E_{1,r} Q p + H_{1,r} w \\ 0 = E_{1_{AE}} Q p \\ z = E_2 Q p + H_2 w \end{cases} \quad (4.13)$$

where  $F_{1,r} \in \mathbb{R}^{r \times n}$  and  $E_{1_{AE}} \in \mathbb{R}^{n_{AE} \times n}$  are full row-rank.

*Proof.* Refer to Appendix B (p. 200). □

The representation (4.12) is said to be in the *descriptor form* [Apl91]. This form straightly appears in the modelling procedure of next subsection, after some basic algebraic manipulations on the representation matrices. It especially allows to compute a similar representation of the scattering parameter  $s_{11}$  and  $s_{21}$ , as shown in next lemma.

**Lemma 4.1.** *Assume that a scattering PHS DAE representation of a lossless passive electronic filter, with source resistance  $R_g$  and load resistance  $R_l$ , is given in the descriptor form of (4.12), where  $w$  and  $z$  are defined by (4.11). In addition, define  $E_{z_1}, E_{z_2} \in \mathbb{R}^{1 \times n}$ ,  $H_{w_1}, H_{w_2} \in \mathbb{R}^{n \times 1}$  and  $H_{s_{11}}, H_{s_{12}}, H_{s_{21}}, H_{s_{22}} \in \mathbb{R}$  such as:*

$$E_2 = \begin{bmatrix} E_{z_1} \\ E_{z_2} \end{bmatrix} \quad H_1 = \begin{bmatrix} H_{w_1} & H_{w_2} \end{bmatrix} \quad H_2 = \begin{bmatrix} H_{s_{11}} & H_{s_{12}} \\ H_{s_{21}} & H_{s_{22}} \end{bmatrix}$$

Then a representation of the scattering parameter  $s_{11}$  is provided by

$$\begin{cases} p = \left(\frac{1}{s} \cdot I_n\right) q \\ F_1 q = E_1 Q p + H_{w_1} w_1 \\ z_1 = E_{z_1} Q p + H_{s_{11}} w_1 \end{cases} \quad (4.14)$$

and a representation of the scattering parameter  $s_{21}$  is given by:

$$\begin{cases} p = \left(\frac{1}{s} \cdot I_n\right) q \\ F_1 q = E_1 Q p + H_{w_1} w_1 \\ z_2 = E_{z_2} Q p + H_{s_{21}} w_1 \end{cases} \quad (4.15)$$

*Proof.* From the relation (3.4) (p. 42), it appears that the incident power wave  $\alpha_2$  is always null for electronic filters, meaning that no power is generated by the load. The scattering parameters  $s_{11}$  and  $s_{21}$  are then the transfer functions from the input  $w_1$  to respectively the output  $z_1$  and the output  $z_2$ . This leads to (4.14) and (4.15).  $\square$

Finally, an important subclass of the PHS DAE representations, and its scattering version, is the class of PHS state-space representations, also referred to as *input-state-output* PHS representations. These representations require the external variables to be split into inputs and outputs and the absence of algebraic constraints between the state variables [SJ14]. Indeed, when there are not such algebraic equations in the developed descriptor form of (4.13), one gets  $n_{AE} = 0$  and thus  $r = n$ , i.e.  $F_{1,r}$  is invertible. The following PHS state-space representation can then be found (cf Appendix B, p. 204):

$$\begin{cases} \tilde{p} = \left(\frac{1}{s} \cdot I_n\right) \tilde{q} \\ \tilde{q} = (J - BB^T) Q \tilde{p} + \sqrt{2} B w \\ z = -\sqrt{2} DB^T Q \tilde{p} + D w \end{cases}$$

where  $\tilde{p} := \frac{1}{\sqrt{2}} p$ ,  $\tilde{q} := \frac{1}{\sqrt{2}} q$ , and the matrices  $J$  and  $D$  are such that:

$$J = -J^T \quad D^T D = I$$

In the synthesis method of Subsection 4.2.3, the optimisation problem associated with the PHS state-space representation will provide the least number of bilinearities.

#### 4.2.2.2 Modelling procedure and illustrations

An attracting benefit of the PHS framework is its systematic modelling procedure<sup>4</sup> [SJ14; DMSB09]. This procedure is adapted for the modelling of lossless passive electronic filters. The aim here is to obtain, from a circuit diagram, a scattering PHS DAE description in the descriptor form with a diagonal matrix  $Q$ . When it is possible, a PHS state-space representation with a diagonal matrix  $Q$  is also obtained.

<sup>4</sup>Rigorously, this is not a *modelling* procedure, as a circuit diagram is already a model of a physical system. This procedure enables to shift from the level of circuit diagrams to the level of mathematical equations. However, for sake of simplicity, the word *modelling* is preferred in this context.



**Modelling procedure** This consists of four steps. An optional step is also provided for the PHS state-space case, when applicable.

*Assumption:* a circuit diagram of a lossless passive electronic filter with source resistance  $R_g$  and load resistance  $R_l$  is provided.

1. The equations resulting from the Kirchhoff's circuit laws are written:

$$\sum_k i_k = 0 \quad \text{for each node} \quad (\text{Kirchhoff's Current Law})$$

$$\sum_k v_k = 0 \quad \text{for each loop} \quad (\text{Kirchhoff's Voltage Law})$$

2. These equations are written in the compact form:

$$\begin{bmatrix} F & E & F_e & E_e \end{bmatrix} \begin{bmatrix} f \\ e \\ f_e \\ e_e \end{bmatrix} = 0$$

where  $f_e := \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$  and  $e_e := \begin{bmatrix} i_1 \\ i_2 \end{bmatrix}$  and the vectors  $f$  and  $e$  are defined such that: for the  $k^{\text{th}}$  building element,

$$\begin{aligned} f_k &= -v_{L_k} & e_k &= i_{L_k} & \text{for an inductance } L_k \\ f_k &= -i_{C_k} & e_k &= v_{C_k} & \text{for a capacitance } C_k \end{aligned}$$

3. The relations between  $f$  and  $e$  of the building elements are incorporated. To achieve this, intermediary variables are introduced such as: for an inductance  $L_k$ :

$$f_k = -\Phi_{L_k} \cdot s \quad e_k = \frac{\Phi_{L_k}}{L_k} \quad (4.16)$$

for a capacitance  $C_k$ :

$$f_k = -Q_{C_k} \cdot s \quad e_k = \frac{Q_{C_k}}{C_k} \quad (4.17)$$

The quantity  $\Phi_{L_k}$  can be interpreted as the complex current density of inductance  $L_k$ , and  $Q_{C_k}$  as the complex charge density of capacitance  $C_k$ . Let us define now the matrix  $Q$  as the diagonal concatenation of the inverse of the values of the building elements, i.e. for the  $k^{\text{th}}$  building element,

$$\begin{aligned} Q(k, k) &= \frac{1}{L_k} & \text{for an inductance } L_k \\ Q(k, k) &= \frac{1}{C_k} & \text{for a capacitance } C_k \end{aligned}$$

By defining  $q$  and  $p$  such that:

$$q = -f \quad Qp = e$$

a PHS DAE presentation (Definition 4.1) is obtained:

$$\begin{cases} p = \left(\frac{1}{s} \cdot I_n\right) q \\ Fq = EQp + F_e f_e + E_e e_e \end{cases}$$

4. By defining  $z$ ,  $w$ ,  $H$  and  $G$  as in (4.8), a scattering PHS DAE representation (Definition 4.2) associated to  $\mathcal{R} := \begin{bmatrix} R_g & 0 \\ 0 & R_l \end{bmatrix}$  is obtained:

$$\begin{cases} p = \left(\frac{1}{s} \cdot I_n\right) q \\ Fq = EQp + Gz + Hw \end{cases}$$

Compute an invertible matrix  $V \in \mathbb{R}^{(n+n_e) \times (n+n_e)}$  (cf proof of Property 4.1 for the existence and the computation of such matrix) such that:

$$V \begin{bmatrix} F & E & G & H \end{bmatrix} = \begin{bmatrix} F_1 & E_1 & 0_{n \times n_e} & H_1 \\ 0_{n_e \times n} & E_2 & -I_{n_e} & H_2 \end{bmatrix}$$

Finally, a descriptor form of the above PHS DAE representation is obtained:

$$\begin{cases} p = \left(\frac{1}{s} \cdot I_n\right) q \\ F_1 q = E_1 Q p + H_1 w \\ z = E_2 Q p + H_2 w \end{cases}$$

5. (Optional) If  $F_1$  is invertible, then a PHS state-space representation is given by (cf p. 204 of Appendix B):

$$\begin{cases} \tilde{p} = \left(\frac{1}{s} \cdot I_n\right) \tilde{q} \\ \tilde{q} = (J - BB^T) Q \tilde{p} + \sqrt{2} B w \\ z = -\sqrt{2} D B^T Q \tilde{p} + D w \end{cases}$$

with  $\tilde{p} := \frac{1}{\sqrt{2}} p$ ,  $\tilde{q} := \frac{1}{\sqrt{2}} q$ , and

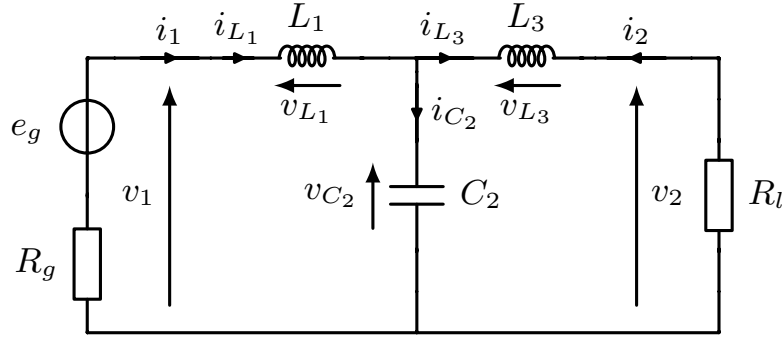
$$J := \frac{1}{2} (F_1^{-1} E_1 - E_1^T F_1^{-T}) \quad B := \frac{1}{2} F_1^{-1} H_1 \quad D := H_2$$

Additionally,  $J$  is skew-symmetric  $J = -J^T$  and  $D$  is a unitary matrix  $DD^T = I$ .

### Modelling examples

The PHS modelling procedure is now illustrated on two examples. Example 4.3 is concerned with the modelling of a three-elements,  $LC$  ladder filter. Example 4.4 is concerned with the modelling of a three-elements, AW-resonator ladder filter.

**Example 4.3.** Consider the filter of Figure 4.2, made of  $L$  and  $C$  elements with a ladder topology, terminated on a resistive generator ( $e_g, R_g$ ) and a resistive load  $R_l$ . For simplicity, the resistances  $R_g$  and  $R_l$  are assumed to be unitary:  $R_g = R_l = 1\Omega$ .

FIGURE 4.2: Three-elements example of a  $LC$  ladder filter

1. The equations resulting from the Kirchhoff's circuit laws are written:

$$\left. \begin{aligned} i_1 - i_{L_1} &= 0 \\ v_1 - v_{L_1} - v_{C_2} &= 0 \\ i_{L_1} - i_{C_2} - i_{L_3} &= 0 \\ v_{C_2} - v_{L_3} - v_2 &= 0 \\ i_2 + i_{L_3} &= 0 \end{aligned} \right\}$$

2. Define

$$f := \begin{bmatrix} -v_{L_1} \\ -i_{C_2} \\ -v_{L_3} \end{bmatrix} \quad e := \begin{bmatrix} i_{L_1} \\ v_{C_2} \\ i_{L_3} \end{bmatrix} \quad f_e := \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \quad e_e := \begin{bmatrix} i_1 \\ i_2 \end{bmatrix}$$

Re-write the Kirchhoff's laws in the following compact form:

$$\begin{bmatrix} F & E & F_e & E_e \end{bmatrix} \begin{bmatrix} f \\ e \\ f_e \\ e_e \end{bmatrix} = 0$$

with

$$\left[ F \mid E \mid F_e \mid E_e \right] := \left[ \begin{array}{ccc|ccc|cc|cc} 1 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array} \right]$$

3. Define the matrix  $Q$  as:

$$Q := \begin{bmatrix} \frac{1}{L_1} & 0 & 0 \\ 0 & \frac{1}{C_2} & 0 \\ 0 & 0 & \frac{1}{L_3} \end{bmatrix}$$

and the vectors  $q$  and  $p$  such that:

$$q = -f \quad Qp = e$$

a PHS DAE presentation (Definition 4.1) is obtained:

$$\begin{cases} p = \left(\frac{1}{s} \cdot I_3\right) q \\ Fq = EQp + F_e f_e + E_e e_e \end{cases}$$

One may verify that conditions (4.3) and (4.4) of Definition 4.1 are fulfilled.

4. By defining  $z$ ,  $w$ ,  $H$  and  $G$  as in (4.8), a scattering PHS DAE representation (Definition 4.2) associated to  $\mathcal{R} := \begin{bmatrix} R_g & 0 \\ 0 & R_l \end{bmatrix}$  is obtained:

$$\begin{cases} p = \left(\frac{1}{s} \cdot I_3\right) q \\ Fq = EQp + Gz + Hw \end{cases}$$

where  $G = F_e \mathcal{R}^{\frac{1}{2}} - E_e \mathcal{R}^{-\frac{1}{2}}$  and  $H = F_e \mathcal{R}^{\frac{1}{2}} + E_e \mathcal{R}^{-\frac{1}{2}}$  are defined by:

$$G := \begin{bmatrix} G_1 \\ G_2 \end{bmatrix} := \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & -1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix} \quad H := \begin{bmatrix} H_1 \\ H_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

By defining the non-singular matrix  $V := \begin{bmatrix} I_3 & -G_1 G_2^{-1} \\ 0_{2 \times 3} & -G_2^{-1} \end{bmatrix}$ , one gets:

$$\begin{aligned} V \left[ F \mid E \mid G \mid H \right] &= \begin{bmatrix} F_1 & E_1 & 0_{3 \times 2} & H_1 - G_1 G_2^{-1} H_2 \\ 0_{2 \times 3} & E_2 & -I_2 & -G_2^{-1} H_2 \end{bmatrix} \\ &:= \begin{bmatrix} 1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & -1 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 1 \end{bmatrix} \end{aligned}$$

Denoting  $w$  and  $z$  respectively as input and output, and defining  $\tilde{q} := q/\sqrt{2}$  and  $\tilde{p} := p/\sqrt{2}$ , a state-space PHS representation of the scattering matrix  $S$  is obtained as:

$$\begin{cases} \tilde{p} = \left(\frac{1}{s} \cdot I_3\right) \tilde{q} \\ \tilde{q} = \left( \underbrace{\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}}_{=J} - \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{=BB^T} \right) Q\tilde{p} + \sqrt{2} \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & -1 \end{bmatrix}}_{=B} w \\ z = \sqrt{2} \underbrace{\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{=-\sqrt{2}DB^T} Q\tilde{p} + \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{=D} w \end{cases}$$

A noticeable point is that the matrix  $Q$  is diagonal with the inverse of the element characteristic values as diagonal entries. The PHS representation makes explicit, and in a compact way, the design parameters  $L_1, C_2, L_3$ .

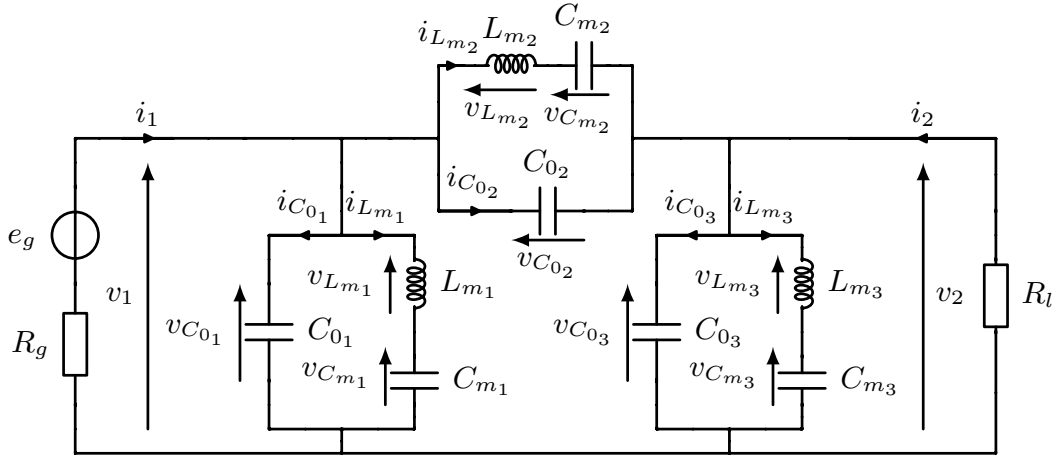


FIGURE 4.3: Three-elements example of an AW-resonator ladder filter

**Example 4.4.** Consider the two-port of Figure 4.3, made of AW elements with a ladder topology, terminated on a resistive generator ( $e_g, R_g$ ) and a resistive load  $R_l$ . For simplicity, the resistances  $R_g$  and  $R_l$  are assumed to be unitary:  $R_g = R_l = 1 \Omega$ .

1. The equations resulting from the Kirchhoff's circuit laws are written. First, one can note that for each AW element  $k, k \in \{1, 2, 3\}$ , the following relations hold:

$$\left. \begin{aligned} i_{L_{m_k}} - i_{C_{m_k}} &= 0 \\ v_{C_{0_k}} - v_{L_{m_k}} - v_{C_{m_k}} &= 0 \end{aligned} \right\}$$

Then, the laws corresponding to the elements interconnection are provided by:

$$\left. \begin{aligned} v_1 - v_{C_{0_1}} &= 0 \\ i_1 - i_{C_{0_1}} - i_{L_{m_1}} - i_{C_{0_2}} - i_{L_{m_2}} &= 0 \\ v_{C_{0_1}} - v_{C_{0_2}} - v_{C_{0_3}} &= 0 \\ i_2 + i_{C_{0_2}} + i_{L_{m_2}} - i_{C_{0_3}} - i_{L_{m_3}} &= 0 \\ v_2 - v_{C_{0_3}} &= 0 \end{aligned} \right\} \quad (4.18)$$

2. Define  $f_e := [v_1 \ v_2]^T$ ,  $e_e := [i_1 \ i_2]^T$  and

$$\begin{aligned} f &:= - [i_{C_{0_1}} \ v_{L_{m_1}} \ i_{C_{m_1}} \ i_{C_{0_2}} \ v_{L_{m_2}} \ i_{C_{m_2}} \ i_{C_{0_3}} \ v_{L_{m_3}} \ i_{C_{m_3}}]^T \\ e &:= [v_{C_{0_1}} \ i_{L_{m_1}} \ v_{C_{m_1}} \ v_{C_{0_2}} \ i_{L_{m_2}} \ v_{C_{m_2}} \ v_{C_{0_3}} \ i_{L_{m_3}} \ v_{C_{m_3}}]^T \end{aligned}$$

Re-write the Kirchhoff's laws in the following compact form:

$$\begin{bmatrix} F & E & F_e & E_e \end{bmatrix} \begin{bmatrix} f \\ e \\ f_e \\ e_e \end{bmatrix} = 0$$





PHS DAE representation such that these constraints are satisfied, a scattering matrix will then be *implicitly* synthesised such that it satisfies the spectral mask.

Hence, it is aimed now to make explicit the link between the scattering representation and the scattering matrix  $\mathcal{S}$ . In the next property, it is shown how to compute the scattering matrix  $S$  from the scattering representation. This is basically achieved by eliminating the algebraic constraints on the state variable  $p$  in order to obtain a state-space representation.

**Property 4.2.** *Assume that a scattering PHS DAE representation of a lossless passive electronic filter is given in a developed descriptor form as follows:*

$$\begin{cases} p = \left(\frac{1}{s} \cdot I_n\right) q \\ F_{1_r} q = E_{1_r} Q p + H_{1_r} w \\ 0 = E_{1_{AE}} Q p \\ z = E_2 Q p + H_2 w \end{cases}$$

where  $F_{1_r} \in \mathbb{R}^{r \times n}$  and  $E_{1_{AE}} \in \mathbb{R}^{n_{AE} \times n}$  are full row-rank. Using a singular-value decomposition (cf Theorem A.4, p. 192), factorise  $E_{1_{AE}}$  as:

$$E_{1_{AE}} = V^T \begin{bmatrix} 0 & \Sigma_{n_{AE}} \end{bmatrix} U$$

where in particular  $U \in \mathbb{R}^{n \times n}$  is a unitary matrix  $U^T U = I$ . Define:

$$\begin{aligned} \begin{bmatrix} \tilde{p}_1 \\ \tilde{p}_2 \end{bmatrix} &:= U p & \begin{bmatrix} \tilde{q}_1 \\ \tilde{q}_2 \end{bmatrix} &:= U q & \begin{bmatrix} \tilde{Q}_{11} & \tilde{Q}_{12} \\ \tilde{Q}_{12}^T & \tilde{Q}_{22} \end{bmatrix} &:= U Q U^T \\ \begin{bmatrix} \tilde{F}_{11_r} & \tilde{F}_{12_r} \end{bmatrix} &:= F_{1_r} U^T & \begin{bmatrix} \tilde{E}_{11_r} & \tilde{E}_{12_r} \end{bmatrix} &:= E_{1_r} U^T & \begin{bmatrix} \tilde{E}_{21} & \tilde{E}_{22} \end{bmatrix} &:= E_2 U^T \end{aligned}$$

Then the scattering representation can be reduced to the following state-space representation

$$\begin{cases} \tilde{p}_1 = \left(\frac{1}{s} \cdot I_r\right) \tilde{q}_1 \\ \tilde{q}_1 = \tilde{F}_{11_r}^{-1} \tilde{E}_{11_r} \tilde{Q} \tilde{p}_1 + \tilde{F}_{11_r}^{-1} H_{1_r} w \\ z = \tilde{E}_{21} \tilde{Q} \tilde{p}_1 + H_2 w \end{cases}$$

where the matrix  $\hat{Q}$  is given by  $\hat{Q} := \tilde{Q}_{11} - \tilde{Q}_{12} \tilde{Q}_{22}^{-1} \tilde{Q}_{12}^T$ .

The resulting scattering matrix  $S$  of the electronic filter is then computed as:

$$S(s) := \left(\frac{1}{s} \cdot I_r\right) \star \left[ \begin{array}{c|c} \tilde{F}_{11_r}^{-1} \tilde{E}_{11_r} \hat{Q} & \tilde{F}_{11_r}^{-1} H_{1_r} \\ \hline \tilde{E}_{21} \hat{Q} & H_2 \end{array} \right]$$

*Proof.* Cf Appendix B (p. 205). □

The constraints (4.9) and (4.10) of the scattering PHS DAE representation can be adapted to the obtained state-space representation of Property 4.2. Then, this state-space representation can be re-written as a *PHS* state-space representation using the procedure given in Appendix B (p. 204). In this case, one may notice that the matrix  $\hat{Q}$  is positive definite as  $U Q U^T > 0$  is positive definite and by the Schur



Lemma (Lemma A.3, p. 195):

$$\widehat{Q} := \widetilde{Q}_{11} - \widetilde{Q}_{12}\widetilde{Q}_{22}^{-1}\widetilde{Q}_{21} > 0 \Leftrightarrow \begin{bmatrix} \widetilde{Q}_{11} & \widetilde{Q}_{12} \\ \widetilde{Q}_{21} & \widetilde{Q}_{22} \end{bmatrix} > 0$$

This enables to find an alternative proof of the result that the scattering matrix  $S$  of a lossless passive electronic filter is stable and lossless  $\{-I, 0, I\}$  dissipative (Property 3.3, p. 45).

**Property 4.3.** *Consider a scattering PHS DAE representation of a lossless passive electronic filter as in Definition 4.2. Then, the resulting scattering matrix  $S$  is stable and lossless  $\{-I, 0, I\}$ -dissipative.*

*Proof.* The proof is usually based on energy considerations [SM03; DMSB09; Sch13]. One may also compute a PHS state-space representation from Property 4.2. The matrix  $\widehat{Q}$  leads to a Lyapunov candidate function and its positivity leads to the stability results, while the adaptation of (4.9) to this representation implies the lossless  $\{-I, 0, I\}$ -dissipativity result [Sch08].

In order to provide the underlying idea, let us consider the following scattering PHS *minimal* state-space representation of a lossless passive electronic filters:

$$\begin{cases} p = \left(\frac{1}{s} \cdot I_n\right) q \\ q = (J - BB^T) Qp + \sqrt{2}Bw \\ z = -\sqrt{2}DB^T Qp + Dw \end{cases}$$

with  $Q = Q^T > 0$ ,  $J = -J^T$  and  $D^T D = I$ . The associate scattering matrix  $S$  is given by

$$S(s) = \left(\frac{1}{s} \cdot I_n\right) \star \left[ \begin{array}{c|c} (J - BB^T) Q & \sqrt{2}B \\ \hline -\sqrt{2}DB^T Q & D \end{array} \right]$$

Then, one can note:

$$\left((J - BB^T)Q\right)^T Q + Q \left((J - BB^T)Q\right) = Q \left(-2BB^T\right) Q$$

In addition, as

$$\left(-\sqrt{2}DB^T Q\right)^T \left(-\sqrt{2}DB^T Q\right) = 2QBB^T Q$$

by the Lyapunov Lemma (Theorem A.6, p. 194) and by minimality of the state-space representation, it comes that the eigenvalues of  $J - BB^T$  have negative real parts, and thus  $S$  is stable (cf Chapter 2). Moreover, from the KYP Lemma (Theorem 2.3, p. 28),  $S$  is lossless  $\{-I, 0, I\}$ -dissipative if and only if there exists a matrix  $P = P^T$  such that:

$$\left[ \begin{array}{cc} Q(J - BB^T)^T(-P) + (-P)(J - BB^T)Q - 2QBDD^T B^T Q & \sqrt{2}(-P)B + \sqrt{2}QBDD^T D \\ \sqrt{2}B^T(-P) + \sqrt{2}D^T DB^T Q & I - D^T D \end{array} \right] = 0$$

Choose  $P = Q$ . Then, as  $D^T D = I$ , little calculation provides the result.  $\square$

Therefore, when designing a scattering PHS DAE representations, it will be sufficient to get the design variables positive  $Q > 0$  for ensuring stability and lossless  $\{-I, 0, I\}$ -dissipativity of the resulting scattering matrix  $S(s)$ . Similarly, the scattering parameters  $s_{11}$  and  $s_{21}$  will be stable and  $\{-I, 0, I\}$ -dissipative.

### 4.2.3 Design of electronic filters with a DAE PHS representation

In the introductory example of this section (Example 4.2, p. 56), it is illustrated how a direct application of the KYP lemma leads to a non-convex optimisation problem. The main source of non-convexity comes from non-linear matrix products between the multipliers, in the example  $P$  and  $D$ , and the matrix  $Q$ . These products are distinguished into bilinear matrix terms and higher-order matrix terms, presumed to lead to a more complex problem. The degree of non-convexity is then qualitatively measured by the order of these non-linear *matrix* products and the number of the resulting *variable* products. Hence, our aim is, first, to only get bilinear matrix terms, and second, to reduce their number to the minimum.

To achieve this, we develop now a synthesis method adapted to the scattering PHS DAE representation. First, Theorem 4.1 (Subsection 4.2.3.1) is established in order to split the products between the multipliers and the matrix  $Q$ . The resulting optimisation problem solely involves inequality with bilinear matrix products, and is then a so-called Bilinear Matrix Inequality (BMI) problem. In addition, when  $Q$  is diagonal, the number of different bilinearities is at worst  $\frac{n(n+1)}{2}$ , which is also an improvement compared to the number of non-linearities appearing in the application of the KYP Lemma (cf Example 4.2).

Second, it is observed that the BMI problem has a particularly simple form for the state-space case, as only  $n$  bilinearities occur. Based on this observation, and on some algebraic operations leading to the introduction of the LFT representation, it is revealed how to get such a low number for the PHS DAE case (Subsection 4.2.3.2).

#### 4.2.3.1 PHS DAE representation synthesis as a BMI problem

The synthesis method is now presented. It is based on the application of Theorem 4.1, stated and proved below. This theorem provides sufficient conditions, expressed as a BMI feasibility problem, for the resolution of the design problem. Notice that the alternative formulation of the design problem for lossless passive electronic filters is used, as this method directly synthesises the frequency response of  $S$  (cf Problem 3.2, p. 49, and the discussion around it).

##### Theorem 4.1.

Given a positive integer  $n \in \mathbb{N}^*$ ,  
 a source resistance  $R_g$  and a load resistance  $R_l$ ,  
 a scattering PHS DAE representation with  $\mathcal{R} = \begin{bmatrix} R_g & 0 \\ 0 & R_l \end{bmatrix} > 0$  and  $n_e = 2$ ,  
 associated with a family of lossless passive elements  $\mathcal{F}$  and a topology  $\mathcal{T}$ ,  
 $N_U$  upper bounds  $U_u$  associated with  $N_U$  real intervals  $\Omega_u^U \subseteq \mathbb{R}$ ,  
 $N_L$  lower bounds  $L_l$  associated with  $N_L$  real intervals  $\Omega_l^L \subseteq \mathbb{R}$ ,  
 Define the resulting representation matrices  $F_1, E_1 \in \mathbb{R}^{n \times n}$ ,  $E_{z_1}, E_{z_2} \in \mathbb{R}^{1 \times n}$ ,  
 $H_{w_1} \in \mathbb{R}^{n \times 1}$  and  $H_{s_{11}}, H_{s_{21}} \in \mathbb{R}$ , as in Lemma 4.1 (p. 60).

Then (i) implies (ii).

(i) There exists a solution to the following feasibility problem:

$$\begin{aligned} & \exists Q^{inv} \in \mathcal{D}(\mathbb{R}^{n \times n}), \\ & \forall u \in \{1, \dots, N_U\}, \exists P_u = P_u^T \in \mathbb{R}^{n \times n}, \exists D_u = D_u^T \in \mathbb{R}^{n \times n}, \exists \tau_u \in \mathbb{R}, \\ & \forall l \in \{1, \dots, N_L\}, \exists P_l = P_l^T \in \mathbb{R}^{n \times n}, \exists D_l = D_l^T \in \mathbb{R}^{n \times n}, \exists \tau_l \in \mathbb{R}, \end{aligned}$$

$$Q^{inv} > 0$$

$$D_u > 0 \quad \tau_u > 0$$

$$\begin{bmatrix} -z_{\Omega_u^U} \mathbf{D}_u & -y_{\Omega_u^U}^* \mathbf{D}_u - \mathbf{P}_u & 0 & 0 \\ -y_{\Omega_u^U} \mathbf{D}_u - \mathbf{P}_u & -x_{\Omega_u^U} \mathbf{D}_u & 0 & 0 \\ 0 & 0 & -\tau_u & 0 \\ 0 & 0 & 0 & U_u^2 \tau_u \end{bmatrix} + \begin{bmatrix} \mathbf{Q}^{inv} (F_1^T F_1) \mathbf{Q}^{inv} & -\mathbf{Q}^{inv} F_1^T E_1 & 0 & -\mathbf{Q}^{inv} F_1^T H_{w_1} \\ -E_1^T F_1 \mathbf{Q}^{inv} & E_1^T E_1 + E_{z_2}^T E_{z_2} & -E_{z_2}^T & E_1^T H_{w_1} + E_{z_2}^T H_{s_{21}} \\ 0 & -E_{z_2} & 1 & -H_{s_{21}} \\ -H_{w_1}^T F_1 \mathbf{Q}^{inv} & H_{w_1}^T E_1 + H_{s_{21}}^T E_{z_2} & -H_{s_{21}}^T & H_{w_1}^T H_{w_1} + H_{s_{21}}^T H_{s_{21}} \end{bmatrix} \geq 0 \quad (4.19)$$

$$\mathbf{D}_l > 0 \quad \tau_l > 0$$

$$\begin{bmatrix} -z_{\Omega_l^L} \mathbf{D}_l & -y_{\Omega_l^L}^* \mathbf{D}_l - \mathbf{P}_l & 0 & 0 \\ -y_{\Omega_l^L} \mathbf{D}_l - \mathbf{P}_l & -x_{\Omega_l^L} \mathbf{D}_l & 0 & 0 \\ 0 & 0 & -\tau_l & 0 \\ 0 & 0 & 0 & (1 - L_l^2) \tau_l \end{bmatrix} + \begin{bmatrix} \mathbf{Q}^{inv} (F_1^T F_1) \mathbf{Q}^{inv} & -\mathbf{Q}^{inv} F_1^T E_1 & 0 & -\mathbf{Q}^{inv} F_1^T H_{w_1} \\ -E_1^T F_1 \mathbf{Q}^{inv} & E_1^T E_1 + E_{z_1}^T E_{z_1} & -E_{z_1}^T & E_1^T H_{w_1} + E_{z_1}^T H_{s_{11}} \\ 0 & -E_{z_1} & 1 & -H_{s_{11}} \\ -H_{w_1}^T F_1 \mathbf{Q}^{inv} & H_{w_1}^T E_1 + H_{s_{11}}^T E_{z_1} & -H_{s_{11}}^T & H_{w_1}^T H_{w_1} + H_{s_{11}}^T H_{s_{11}} \end{bmatrix} \geq 0 \quad (4.20)$$

where  $x_{\Omega_u^U}$ ,  $y_{\Omega_u^U}$ ,  $z_{\Omega_u^U}$  and  $x_{\Omega_l^L}$ ,  $y_{\Omega_l^L}$ ,  $z_{\Omega_l^L}$  are respectively determined by the frequency intervals  $\Omega_u^U$  and  $\Omega_l^L$  and are defined as in (2.13)-(2.15) (p. 30 of Chapter 2).

(ii) There exists a solution to the Alternative Lossless Passive Electronic Filter Design problem (Problem 3.2, p 49).

Indeed, define  $Q = (Q^{inv})^{-1}$ . Then, there exists an electronic filter, defined as the interconnection of elements from  $\mathcal{F}$  in the topology  $\mathcal{T}$  between a resistive voltage source with source resistance  $R_g$  and a resistive load  $R_l$ , such that its scattering parameters  $s_{11}$  and  $s_{21}$  are stable,  $\{-1, 0, 1\}$ -dissipative, and satisfy the spectral mask SM:

$$\forall u \in \{1, \dots, N_U\}, \quad \forall \omega \in \Omega_u^U, \quad |s_{21}(j\omega)|^2 \leq U_u^2 \quad (4.21)$$

$$\forall l \in \{1, \dots, N_L\}, \quad \forall \omega \in \Omega_l^L, \quad |s_{11}(j\omega)|^2 \leq 1 - L_l^2 \quad (4.22)$$

*Proof.* By Property 4.3 (p. 70), the condition  $Q^{inv} > 0$ , equivalent to  $Q > 0$ , implies that the scattering matrix  $S$  is stable and lossless- $\{-I, 0, I\}$ -dissipative. Consequently,  $s_{11}$  and  $s_{21}$  are stable and  $\{-1, 0, 1\}$ -dissipative.

The proof is based on the use of the S-procedure lemma. This provides sufficient conditions for the satisfaction of the constraints (4.21) and (4.22) but requires to formulate the problem as quadratic inequalities and quadratic equalities on signals.

Let  $q$ ,  $p$ ,  $z_2$ ,  $w_1$  be complex signals such that:

$$\begin{bmatrix} -F_1 & E_1 Q & 0 & H_{w_1} \\ 0 & E_{z_2} Q & -1 & H_{s_{21}} \end{bmatrix} \begin{bmatrix} q \\ p \\ z_2 \\ w_1 \end{bmatrix} = 0$$

This is equivalent to the quadratic form:

$$\begin{bmatrix} q \\ p \\ z_2 \\ w_1 \end{bmatrix}^* \begin{bmatrix} -F_1^T & 0 \\ Q E_1^T & Q E_{z_2}^T \\ 0 & -1 \\ H_{w_1}^T & H_{s_{21}}^T \end{bmatrix} \begin{bmatrix} -F_1 & E_1 Q & 0 & H_{w_1} \\ 0 & E_{z_2} Q & -1 & H_{s_{21}} \end{bmatrix} \begin{bmatrix} q \\ p \\ z_2 \\ w_1 \end{bmatrix} = 0 \quad (4.23)$$

As in Chapter 2, constraint (4.21) can be expressed using two quadratic constraints: one on  $\frac{1}{j\omega}$ , to ensure that  $\omega \in \Omega_u^U$ , and one on  $s_{21}(j\omega)$ , to ensure that its magnitude is upper bounded. Using the relations  $p = \left(\frac{1}{j\omega} \cdot I_n\right) q$  and  $z_2 = s_{21}(j\omega) w_1$ , these quadratic constraints can be equivalently expressed as quadratic constraints in the signals  $p$ ,  $q$ ,  $w_1$  and  $z_2$  [Sco97]. Indeed, the first constraint can be stated as follows:

there exist  $\tilde{P}_u = \tilde{P}_u^T \in \mathbb{R}^{n \times n}$ ,  $\tilde{D}_u = \tilde{D}_u^T \in \mathbb{R}^{n \times n}$ , such that  $\tilde{D}_u > 0$  and

$$\begin{bmatrix} p \\ q \end{bmatrix}^* \begin{bmatrix} x_{\Omega_u^U} \tilde{D}_u & y_{\Omega_u^U} \tilde{D}_u + \tilde{P}_u \\ y_{\Omega_u^U}^* \tilde{D}_u + \tilde{P}_u & z_{\Omega_u^U} \tilde{D}_u \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix} \geq 0 \quad (4.24)$$

Likewise, the second constraint is equivalently stated as:

$$\begin{bmatrix} z_2 \\ w_1 \end{bmatrix}^* \begin{bmatrix} -I & 0 \\ 0 & U_u^2 I \end{bmatrix} \begin{bmatrix} z_2 \\ w_1 \end{bmatrix} \geq 0 \quad (4.25)$$

Constraint (4.21) can then be formulated as follows: Find if there exist complex signals  $q$ ,  $p$ ,  $z_2$ ,  $w_1$  that satisfy (4.25) such that (4.23) and (4.24) hold. The S-procedure lemma (Lemma A.1, p. 194 of Appendix A) allows to transform these constraints on signals into a matrix inequality. Applying the S-procedure lemma, (4.21) is then implied by:  $\exists \tilde{\beta}_u > 0$ ,  $\exists \tilde{\tau}_u \in \mathbb{R}$ ,

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -I & 0 \\ 0 & 0 & 0 & U_u^2 I \end{bmatrix} + \tilde{\beta}_u \begin{bmatrix} -z_{\Omega_u^U} \tilde{D}_u & -y_{\Omega_u^U} \tilde{D}_u - \tilde{P}_u & 0 & 0 \\ -y_{\Omega_u^U}^* \tilde{D}_u - \tilde{P}_u & -x_{\Omega_u^U} \tilde{D}_u & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \tilde{\tau}_u \begin{bmatrix} -F_1^T & 0 \\ Q E_1^T & E_{z_2}^T \\ 0 & -1 \\ H_{w_1}^T & H_{s_{21}}^T \end{bmatrix} \begin{bmatrix} -F_1 & E_1 Q & 0 & H_{w_1} \\ 0 & E_{z_2} & -1 & H_{s_{21}} \end{bmatrix} \geq 0$$

Without loss of generality, one can impose  $\tilde{\tau}_u > 0$ , as the term behind  $\tilde{\tau}_u$  is semi-definite positive. By defining  $\hat{D}_u = \frac{\tilde{\beta}_u}{\tilde{\tau}_u} \tilde{D}_u$ ,  $\hat{P}_u = \frac{\tilde{\beta}_u}{\tilde{\tau}_u} \tilde{P}_u$  and  $\tau_u = \frac{1}{\tilde{\tau}_u} > 0$ , last inequality is equivalent to:

$$\begin{bmatrix} -z_{\Omega_u^U} \hat{D}_u & -y_{\Omega_u^U} \hat{D}_u - \hat{P}_u & 0 & 0 \\ -y_{\Omega_u^U}^* \hat{D}_u - \hat{P}_u & -x_{\Omega_u^U} \hat{D}_u & 0 & 0 \\ 0 & 0 & -\tau_u & 0 \\ 0 & 0 & 0 & U_u^2 \tau_u \end{bmatrix} + \begin{bmatrix} F_1^T F_1 & -F_1^T E_1 Q & 0 & -F_1^T H_{w_1} \\ -Q E_1^T F_1 & Q (E_1^T E_1 + E_{z_2}^T E_{z_2}) Q & -Q E_{z_2}^T & Q (E_1^T H_{w_1} + E_{z_2}^T H_{s_{21}}) \\ 0 & E_{z_2} Q & 1 & -H_{s_{21}} \\ -H_{w_1}^T F_1 & (H_{w_1}^T E_1 + H_{s_{21}}^T E_{z_2}) Q & -H_{s_{21}}^T & H_{w_1}^T H_{w_1} + H_{s_{21}}^T H_{s_{21}} \end{bmatrix} \geq 0$$

Pre- and post-multiplying last inequality by  $\begin{bmatrix} Q^{-1} & & & \\ & Q^{-1} & & \\ & & I & \\ & & & I \end{bmatrix}$  leads to:

$$\begin{bmatrix} -z_{\Omega_u^U} D_u & -y_{\Omega_u^U} D_u - P_u & 0 & 0 \\ -y_{\Omega_u^U}^* D_u - P_u & -x_{\Omega_u^U} D_u & 0 & 0 \\ 0 & 0 & -\tau_u & 0 \\ 0 & 0 & 0 & U_u^2 \tau_u \end{bmatrix} + \begin{bmatrix} Q^{-1} (F_1^T F_1) Q^{-1} & -Q^{-1} F_1^T E_1 & 0 & -Q^{-1} F_1^T H_{w_1} \\ -E_1^T F_1 Q^{-1} & E_1^T E_1 + E_{z_2}^T E_{z_2} & -E_{z_2}^T & E_1^T H_{w_1} + E_{z_2}^T H_{s_{21}} \\ 0 & -E_{z_2} & 1 & -H_{s_{21}} \\ -H_{w_1}^T F_1 Q^{-1} & H_{w_1}^T E_1 + H_{s_{21}}^T E_{z_2} & -H_{s_{21}}^T & H_{w_1}^T H_{w_1} + H_{s_{21}}^T H_{s_{21}} \end{bmatrix} \geq 0$$

where  $P_u = Q^{-1} \hat{P}_u Q^{-1}$  and  $D_u = Q^{-1} \hat{D}_u Q^{-1} > 0$ .

Similar arguments apply for (4.22) with the associate representation of  $s_{11}$ . Optimisation problem of statement (i) is then obtained by defining  $Q^{inv} := Q^{-1}$ .  $\square$

Theorem 4.1 provides sufficient conditions, in the form of an optimisation problem, for the design of scattering PHS DAE representations satisfying a given spectral mask  $\mathcal{SM}$ . When the matrices involved in the scattering PHS representations (4.14) and (4.15) correspond to a given circuit, and the decision matrix  $Q$  has the associated structure, then a solution of the optimisation problem provides then an electronic filter, with appropriate elements and topology, that satisfies the design requirements. This is an obvious advantage of this approach.

This approach is also doubly beneficial for practical concerns. First, it is the same approach that is used by designers, i.e. tuning characteristic values of the elements of a physically implementable filter such that the spectral mask is fulfilled. But it is

formalised into a mathematical framework, where the model is made explicit. Therefore, methods used by designers in practice to reduce the problem complexity, such as fixing some (ratio of) elements value, can be directly incorporated. This approach may also be straightforwardly included in the global design flow. Second, this approach is flexible and may include technological constraints. For instance, a lower bound  $Q_L$  and an upper bound  $Q_U$  on the decision matrix  $Q$ , which represents the inverse of characteristic values of the elements, leads respectively to an upper bound  $Q_U^{inv}$  and a lower bound  $Q_L^{inv}$  on the optimisation variable  $Q^{inv}$  (Equation 4.26), and can be added without significantly increasing the complexity of the optimisation problem.

$$Q_L^{inv} \leq Q^{inv} \leq Q_U^{inv} \quad (4.26)$$

*Remark 4.1.* When designing AW filters, a common constraint imposed by the technological realisation is to fix the ratio  $\omega_a/\omega_r$  of the anti-resonant frequency  $\omega_a$  over the resonant frequency  $\omega_r$  of each AW element. From the lossless model given in page 40, this is equivalent to fix the ratio  $r := C_m/C_0$ . Recalling that the matrix  $Q$  has the

diagonal structure  $Q := \begin{bmatrix} Q_1 & & & \\ & Q_2 & & \\ & & \ddots & \\ & & & Q_n \end{bmatrix}$ , with  $Q_k := \begin{bmatrix} \frac{1}{C_{0k}} & & \\ & \frac{1}{L_{mk}} & \\ & & \frac{1}{C_{mk}} \end{bmatrix}$  for the

$k^{\text{th}}$  AW element, last constraint can be convexly added by defining the optimisation

variable  $Q^{inv} := \begin{bmatrix} Q_1^{inv} & & & \\ & Q_2^{inv} & & \\ & & \ddots & \\ & & & Q_n^{inv} \end{bmatrix}$  such as  $Q_k^{inv} := \begin{bmatrix} q_{k1}^{inv} & & \\ & q_{k2}^{inv} & \\ & & r q_{k1}^{inv} \end{bmatrix}$ .

The main drawback of the sufficient conditions of Theorem 4.1 is that there is currently no efficient algorithm available to solve the optimisation problem. Inequalities (4.19) and (4.20) involve the bilinear matrix product  $Q^{inv} (F_1^T F_1) Q^{inv}$  and are called Bilinear Matrix Inequalities (BMIs). Optimisation problems with BMI constraints are known to be generally complex to solve (cf Subsection 4.2.4). This complexity reduces when the number of bilinear products decreases. For this purpose, the optimisation problem of Theorem 4.1 is formulated using the optimisation variable  $Q^{inv}$  instead of  $Q$ , for which two bilinear matrix products  $Q (E_1^T E_1 + E_{z_2}^T E_{z_2}) Q$  and  $Q (E_1^T E_1 + E_{z_1}^T E_{z_1}) Q$  would be involved instead of one (cf proof of Theorem 4.1).

In the worst case, i.e. when the matrix  $F_1^T F_1$  is full,  $\frac{n(n+1)}{2}$  different bilinear *variable* products are involved in the matrix product  $Q^{inv} (F_1^T F_1) Q^{inv}$ , when  $Q^{inv}$  is diagonal. The next subsection shows how this number can even more be decreased.

#### 4.2.3.2 Getting a particularly simple BMI form

We reveal now how to get an alternative BMI feasibility problem of statement (i) of Theorem 4.1, for which *only*  $n$  bilinear products between design variables appear. First, the particular case of the scattering PHS *state-space* representation is considered. This case is then generalised by using the tools of our design framework, introduced in Chapter 2.

### A particular case: the scattering PHS state-space representation

When the scattering PHS representation is in the state-space form, the complexity of the optimisation problem of Theorem 4.1 significantly reduces. Indeed, in the state-space form, the matrix  $F_1$  is the identity matrix:  $F_1 = I$ . The bilinear matrix term becomes then  $Q^{inv} Q^{inv}$ . Because of the diagonal structure of  $Q^{inv}$ , this implies that only  $n$  bilinearities appear:

$$Q^{inv} Q^{inv} = \begin{bmatrix} (q_1^{inv})^2 & & & & \\ & (q_2^{inv})^2 & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & (q_n^{inv})^2 \end{bmatrix}$$

Therefore, the state-space form reduces the number of bilinearities from  $\frac{n(n+1)}{2}$ , when  $F_1^T F_1$  is full, to  $n$ , i.e. the number of design variables.

Overall, our synthesis method significantly reduces the complexity of the optimisation problem compared to a direct application of the KYP Lemma, as illustrated in the next example.

**Example 4.5** (Example 4.2 revisited). Consider the PHS state-space representation (4.1) (p. 55) of the scattering parameter  $s_{21}$  and define the following matrices:

$$F_1 := I_2 \quad \begin{bmatrix} E_1 \\ E_{z_2} \end{bmatrix} := \begin{bmatrix} J - BB^T \\ -\sqrt{2}B_2^T \end{bmatrix} \quad \begin{bmatrix} H_{w_1} \\ H_{s_{21}} \end{bmatrix} := \begin{bmatrix} \sqrt{2}B_1 \\ 0 \end{bmatrix}$$

Applying Theorem 4.1 leads to check for the existence of  $Q^{inv} = \begin{bmatrix} q_1^{inv} & 0 \\ 0 & q_2^{inv} \end{bmatrix} \in \mathbb{R}^{2 \times 2}$ ,

$P = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \in \mathbb{R}^{2 \times 2}$ ,  $D = \begin{bmatrix} d_{11} & d_{12} \\ d_{12} & d_{22} \end{bmatrix} \in \mathbb{R}^{2 \times 2}$ ,  $\tau \in \mathbb{R}$  such that:

$$Q^{inv} > 0 \quad D > 0 \quad \tau > 0$$

$$\begin{bmatrix} d_{11} + (q_1^{inv})^2 & d_{12} & p_{11} + q_1^{inv} & p_{12} + q_1^{inv} & 0 & -\sqrt{2}q_1^{inv} \\ d_{12} & d_{22} + (q_2^{inv})^2 & p_{12} - q_2^{inv} & p_{22} + q_2^{inv} & 0 & 0 \\ p_{11} + q_1^{inv} & p_{12} - q_2^{inv} & 2 - \omega_0^2 d_{11} & -\omega_0^2 d_{12} & 0 & -\sqrt{2} \\ p_{12} + q_1^{inv} & p_{22} + q_2^{inv} & -\omega_0^2 d_{12} & 4 - \omega_0^2 d_{22} & -\sqrt{2} & -\sqrt{2} \\ 0 & 0 & 0 & -\sqrt{2} & 1 - \tau & 0 \\ -\sqrt{2}q_1^{inv} & 0 & -\sqrt{2} & -\sqrt{2} & 0 & 2 + U^2 \tau \end{bmatrix} \geq 0$$

This inequality involves  $n = 2$  bilinearities. Compared to the type and the number of non-linear products, between optimisation variables, of the solution proposed in Example 4.2, this substantially reduces the complexity of the optimisation problem.

### Generalisation to the scattering PHS LFT representation

The optimisation problem of Theorem 4.1 have then a limited number of bilinearities for electronic filters with a scattering PHS *state-space* representation and a *diagonal* matrix  $Q$ . While this particular representation seems to naturally appear in the modelling of some circuits, such as *LC* ladder filters (Example 4.3), a circuit

may not have such representation in general, such as for AW-resonator ladder filters (Example 4.4), and there are additional bilinearities in the optimisation problem.

One may then be tempted to reduce a given scattering PHS DAE representation into a state-space form. This is achieved by eliminating algebraic constraints on the state variables (cf Subsection 4.2.2.3). However, this process breaks the diagonal structure of the matrix  $Q$ , which is essential to limit the number of bilinearities. In addition, it loses the particular connection between the representation and the circuit diagram.

Instead, we generalise in the next lemma the state-space feature that allows to get  $n$  bilinearities, to other PHS representations. This is achieved by rewriting the equation  $p = \left(\frac{1}{s} \cdot I_n\right) \cdot q$  under the form  $\begin{bmatrix} p_1 \\ q_2 \end{bmatrix} = \begin{pmatrix} \frac{1}{s} \cdot I_r & 0 \\ 0 & s \cdot I_{n_{AE}} \end{pmatrix} \begin{bmatrix} q_1 \\ p_2 \end{bmatrix}$ . The result may be viewed as the introduction of the scattering PHS LFT representation.

**Lemma 4.2.** *Assume that a scattering PHS DAE representation is given as:*

$$\begin{cases} p = \left(\frac{1}{s} \cdot I_n\right) \cdot q \\ F_{1_r} q = E_{1_r} Q p + H_{1_r} w \\ 0 = E_{1_{AE}} Q p \\ z = E_2 Q p + H_2 w \end{cases}$$

where the matrix  $F_{1_r} \in \mathbb{R}^{r \times n}$  is full row-rank, define  $n_{AE}$  thsuch that:

$$n = r + n_{AE}$$

Select  $r$  independent columns of  $F_{1_r}$  and group them into a matrix  $F_{11_r}$ . Write, possibly after permutations,  $F_{1_r} = \begin{bmatrix} F_{11_r} & F_{12_r} \end{bmatrix}$  and correspondingly

$$\begin{aligned} q &= \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} & Q &= \begin{bmatrix} Q_1 & 0 \\ 0 & Q_2 \end{bmatrix} & p &= \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} \\ E_{1_r} &= \begin{bmatrix} E_{11_r} & E_{12_r} \end{bmatrix} & E_{1_{AE}} &= \begin{bmatrix} E_{11_{AE}} & E_{12_{AE}} \end{bmatrix} & E_2 &= \begin{bmatrix} E_{21} & E_{22} \end{bmatrix} \end{aligned}$$

Then the matrix  $\begin{bmatrix} F_{11_r} & E_{12_r} \\ 0 & E_{12_{AE}} \end{bmatrix} \in \mathbb{R}^{n \times n}$  is invertible. Furthermore, an equivalent representation is provided by:

$$\begin{cases} \check{p} = \begin{pmatrix} \frac{1}{s} \cdot I_r & 0 \\ 0 & s \cdot I_{n_{AE}} \end{pmatrix} \cdot \check{q} \\ \check{q} = (J - BB^T) \check{Q} \check{p} + \sqrt{2} B w \\ z = -\sqrt{2} D B^T \check{Q} \check{p} + D w \end{cases} \quad (4.27)$$

with  $\check{q} = \frac{1}{\sqrt{2}} \begin{bmatrix} q_1 \\ p_2 \end{bmatrix}$ ,  $\check{p} = \frac{1}{\sqrt{2}} \begin{bmatrix} p_1 \\ q_2 \end{bmatrix}$ ,  $\check{Q} = \begin{bmatrix} Q_1 & 0 \\ 0 & Q_2^{-1} \end{bmatrix}$ , and  $J$  and  $D$  are such that:

$$J = -J^T \quad D D^T = I$$

The representation (4.27) is denoted as a PHS LFT representation in the sequel.

*Proof.* The proof is based on the ideas for establishing the hybrid input-output representation [SJ14] in the PHS framework (refer to [Gol02, App. B] for the proof).

Let us first demonstrate that  $\begin{bmatrix} F_{11_r} & E_{12_r} \\ 0 & E_{12_{AE}} \end{bmatrix}$  is invertible. As  $F_{11_r} \in \mathbb{R}^{r \times r}$  is full column-rank, it is invertible. The condition  $E_{12_{AE}} \in \mathbb{R}^{n_{AE} \times n_{AE}}$  invertible is then required to get the result.

Using the algebraic matrix constraint (4.9) (p. 59) on matrices for a PHS DAE representation, it comes the following equations:

$$E_{11_r} F_{11_r}^T + F_{11_r} E_{11_r}^T + E_{12_r} F_{12_r}^T + F_{12_r} E_{12_r}^T + \frac{1}{2} H_{1_r} H_{1_r}^T = 0 \quad (4.28a)$$

$$E_{11_{AE}} F_{11_r}^T + E_{12_{AE}} F_{12_r}^T = 0 \quad (4.28b)$$

$$E_{21} F_{11_r}^T + E_{22} F_{12_r}^T + \frac{1}{2} H_2 H_{1_r}^T = 0 \quad (4.28c)$$

$$H_2 H_2^T - I = 0 \quad (4.28d)$$

Moreover, using the rank constraint (4.10) (p. 59) of the scattering PHS DAE representation, one gets  $\text{rank} \begin{bmatrix} E_{11_{AE}} & E_{12_{AE}} \end{bmatrix} = n_{AE}$  as:

$$\text{rank} \begin{bmatrix} F_{11_r} & F_{12_r} & E_{11_r} & E_{12_r} & 0 & H_{1_r} \\ 0 & 0 & E_{11_{AE}} & E_{12_{AE}} & 0 & 0 \\ 0 & 0 & E_{21} & E_{22} & -I & H_2 \end{bmatrix} = r + n_{AE} + n_e$$

In addition, one gets  $E_{11_{AE}} = -E_{12_{AE}} F_{12_r}^T F_{11_r}^{-T}$  by (4.28b). Then, by Property B.1 (p. 200, App. B), it comes that

$$\text{rank} \begin{bmatrix} E_{11_{AE}} & E_{12_{AE}} \end{bmatrix} = \text{rank} \left( E_{12_{AE}} \begin{bmatrix} -F_{12_r}^T F_{11_r}^{-T} & I \end{bmatrix} \right) = \text{rank} (E_{12_{AE}}) = n_{AE}$$

Therefore,  $E_{12_{AE}} \in \mathbb{R}^{n_{AE} \times n_{AE}}$  is invertible, and so is  $\begin{bmatrix} F_{11_r} & E_{12_r} \\ 0 & E_{12_{AE}} \end{bmatrix}$ .

Furthermore, the scattering PHS DAE representation can be re-written in the following compact form:

$$\left\{ \begin{array}{l} \begin{bmatrix} p_1 \\ q_2 \end{bmatrix} = \begin{pmatrix} \frac{1}{s} \cdot I_r & 0 \\ 0 & s \cdot I_{n_{AE}} \end{pmatrix} \begin{bmatrix} q_1 \\ p_2 \end{bmatrix} \\ \left[ \begin{array}{cc|cc|c|c} F_{11_r} & -E_{12_r} & E_{11_r} Q_1 & -F_{12_r} Q_2^{-1} & 0 & H_{1_r} \\ 0 & -E_{12_{AE}} & E_{11_{AE}} Q_1 & 0 & 0 & 0 \\ \hline 0 & -E_{22} & E_{21} Q_1 & 0 & -I & H_2 \end{array} \right] \begin{bmatrix} q_1 \\ p_2 \\ p_1 \\ q_2 \\ z \\ w \end{bmatrix} = 0 \end{array} \right.$$

By pre-multiplying the second equation by  $\begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & -E_{22} E_{12_{AE}}^{-1} & I \end{bmatrix}$ , one gets:

$$\left[ \begin{array}{cc|cc|c|c} F_{11_r} & -E_{12_r} & E_{11_r} Q_1 & -F_{12_r} Q_2^{-1} & 0 & H_{1_r} \\ 0 & -E_{12_{AE}} & E_{11_{AE}} Q_1 & 0 & 0 & 0 \\ \hline 0 & 0 & (E_{21} - E_{22} E_{12_{AE}}^{-1} E_{11_{AE}}) Q_1 & 0 & -I & H_2 \end{array} \right] \begin{bmatrix} q_1 \\ p_2 \\ p_1 \\ q_2 \\ z \\ w \end{bmatrix} = 0$$



And then we find the following representation:

$$\begin{cases} \begin{bmatrix} p_1 \\ q_2 \end{bmatrix} = \begin{pmatrix} \frac{1}{s} \cdot I_r & 0 \\ 0 & s \cdot I_{n_{AE}} \end{pmatrix} \cdot \begin{bmatrix} q_1 \\ p_2 \end{bmatrix} \\ \begin{bmatrix} q_1 \\ p_2 \end{bmatrix} = A\check{Q} \begin{bmatrix} p_1 \\ q_2 \end{bmatrix} + Bw \\ z = C\check{Q} \begin{bmatrix} p_1 \\ q_2 \end{bmatrix} + Dw \end{cases}$$

with

$$\begin{aligned} A &:= \begin{bmatrix} F_{11_r} & -E_{12_r} \\ 0 & -E_{12_{AE}} \end{bmatrix}^{-1} \begin{bmatrix} E_{11_r} & -F_{12_r} \\ E_{11_{AE}} & 0 \end{bmatrix} & \hat{B} &:= \begin{bmatrix} F_{11_r} & -E_{12_r} \\ 0 & -E_{12_{AE}} \end{bmatrix}^{-1} \begin{bmatrix} H_{1_r} \\ 0 \end{bmatrix} \\ C &:= \begin{bmatrix} E_{21} - E_{22}E_{12_{AE}}^{-1}E_{11_{AE}} & 0 \end{bmatrix} & D &:= H_2 \end{aligned}$$

Finally, let us denote  $J := \frac{1}{2}(A - A^T)$  the skew-symmetric part of  $A$ , and compute its symmetric part:

$$\begin{aligned} A + A^T &= \begin{bmatrix} F_{11_r} & -E_{12_r} \\ 0 & -E_{12_{AE}} \end{bmatrix}^{-1} \left( \begin{bmatrix} E_{11_r} & -F_{12_r} \\ E_{11_{AE}} & 0 \end{bmatrix} \begin{bmatrix} F_{11_r} & -E_{12_r} \\ 0 & -E_{12_{AE}} \end{bmatrix}^T + \begin{bmatrix} F_{11_r} & -E_{12_r} \\ 0 & -E_{12_{AE}} \end{bmatrix} \begin{bmatrix} E_{11_r} & -F_{12_r} \\ E_{11_{AE}} & 0 \end{bmatrix}^T \right) \begin{bmatrix} F_{11_r} & -E_{12_r} \\ 0 & -E_{12_{AE}} \end{bmatrix}^{-T} \\ &= \begin{bmatrix} F_{11_r} & -E_{12_r} \\ 0 & -E_{12_{AE}} \end{bmatrix}^{-1} \begin{bmatrix} E_{11_r}F_{11_r}^T + F_{11_r}E_{11_r}^T + E_{12_r}F_{12_r}^T + F_{12_r}E_{12_r}^T & E_{11_{AE}}F_{11_r}^T + E_{12_{AE}}F_{12_r}^T \\ F_{11_r}E_{11_{AE}}^T + F_{12_r}E_{12_{AE}}^T & 0 \end{bmatrix} \begin{bmatrix} F_{11_r} & -E_{12_r} \\ 0 & -E_{12_{AE}} \end{bmatrix}^{-T} \end{aligned}$$

By (4.28a) and (4.28b), this leads to:

$$A + A^T = \begin{bmatrix} F_{11_r} & -E_{12_r} \\ 0 & -E_{12_{AE}} \end{bmatrix}^{-1} \left( -\frac{1}{2} \begin{bmatrix} H_{1_r}H_{1_r}^T & 0 \\ 0 & 0 \end{bmatrix} \right) \begin{bmatrix} F_{11_r} & -E_{12_r} \\ 0 & -E_{12_{AE}} \end{bmatrix}^{-T} = -\frac{1}{2}\hat{B}\hat{B}^T$$

And thus:

$$A = J - \frac{1}{4}\hat{B}\hat{B}^T$$

Moreover, one can notice that

$$\begin{bmatrix} F_{11_r} & -E_{12_r} \\ 0 & -E_{12_{AE}} \end{bmatrix}^{-1} = \begin{bmatrix} F_{11_r}^{-1} & -F_{11_r}^{-1}E_{12_r}E_{12_{AE}}^{-1} \\ 0 & -E_{12_{AE}}^{-1} \end{bmatrix}$$

Hence, by successively using (4.28c) and (4.28b), one gets:

$$\begin{aligned} -\frac{1}{2}DB^T &= \begin{bmatrix} -\frac{1}{2}H_2H_{1_r}^T F_{11_r}^{-T} & 0 \end{bmatrix} = \begin{bmatrix} E_{21} + E_{22}F_{12_r}^T F_{11_r}^{-T} & 0 \end{bmatrix} \\ &= \begin{bmatrix} E_{21} - E_{22}E_{12_{AE}}^{-1}E_{11_{AE}} & 0 \end{bmatrix} = C \end{aligned}$$

The final constraint  $DD^T = I$  comes directly from (4.28d). The resulting representation is then obtained by defining  $B := \frac{\hat{B}}{2}$ .  $\square$

The PHS LFT representation (4.27) is interesting for several reasons. First, the representation matrices are still linear in a diagonal matrix  $\check{Q}$ , having the design variables as entries. Second, there is not a singular matrix before the signal  $\check{q}$  in the second equation of (4.27). Similar to the state-space representation, but unlike the DAE representation, this feature allows to get the simple bilinear matrix term  $\check{Q}\check{Q}$

as the unique non-linear matrix term in the optimisation problem, as will be shown in the next theorem. Coupling with the diagonal property of  $\check{Q}$ , this leads to only  $n$  different bilinearities. Third, while the first equation of (4.27) is unusual and may appear complex to deal with, it can actually be straightly taken into account using our framework. In fact, this leads to find some multiplier matrices in order to get the closest affine parametrisation, as discussed in Chapter 2. Hence, Theorem 4.1, and its proof, are straightly adaptable. Finally, representation (4.27) enables to easily compute the associated scattering matrix  $S(s)$ :

$$S(s) = \begin{pmatrix} \frac{1}{s} \cdot I_r & 0 \\ 0 & s \cdot I_{n_{AE}} \end{pmatrix} \star \left[ \begin{array}{c|c} (J - BB^T)\check{Q} & \sqrt{2}B \\ \hline -\sqrt{2}DB^T\check{Q} & D \end{array} \right]$$

Before introducing the associated synthesis theorem, we define for the sake of clearness the following matrices:

$$\left[ \begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array} \right] := \left[ \begin{array}{c|c} J - BB^T & \sqrt{2}B \\ \hline -\sqrt{2}DB^T & D \end{array} \right] \quad (4.29)$$

In particular, this allows to compute the scattering parameters  $s_{11}$  and  $s_{21}$  such as:

$$s_{11}(s) = \begin{pmatrix} \frac{1}{s} \cdot I_r & 0 \\ 0 & s \cdot I_{n_{AE}} \end{pmatrix} \star \left[ \begin{array}{c|c} A\check{Q} & B_1 \\ \hline C_1\check{Q} & D_{11} \end{array} \right] \quad (4.30)$$

$$s_{21}(s) = \begin{pmatrix} \frac{1}{s} \cdot I_r & 0 \\ 0 & s \cdot I_{n_{AE}} \end{pmatrix} \star \left[ \begin{array}{c|c} A\check{Q} & B_1 \\ \hline C_2\check{Q} & D_{21} \end{array} \right] \quad (4.31)$$

Next theorem provides the design optimisation problem corresponding to this type of LFT scattering PHS representations.

**Theorem 4.2.**

Given a positive integer  $n \in \mathbb{N}^*$ ,  
a source resistance  $R_g$  and a load resistance  $R_l$ ,  
a scattering PHS DAE representation with  $\mathcal{R} = \begin{bmatrix} R_g & 0 \\ 0 & R_l \end{bmatrix} > 0$  and  $n_e = 2$ ,  
associated with a family of lossless passive elements  $\mathcal{F}$  and a topology  $\mathcal{T}$ ,  
 $N_U$  upper bounds  $U_u$  associated with  $N_U$  real intervals  $\Omega_u^U \subseteq \mathbb{R}$ ,  
 $N_L$  lower bounds  $L_l$  associated with  $N_L$  real intervals  $\Omega_l^L \subseteq \mathbb{R}$ ,  
Define the resulting PHS LFT representation (4.27) of Lemma 4.2,  
 $A \in \mathbb{R}^{n \times n}$ ,  $B_1 \in \mathbb{R}^{n \times 1}$ ,  $C_1, C_2 \in \mathbb{R}^{1 \times n}$ ,  $D_{11}, D_{21} \in \mathbb{R}$ , as in (4.29).

Then (i) implies (ii).

(i) There exists a solution to the following feasibility problem:

$$\begin{aligned} & \exists \check{Q}^{inv} \in \mathcal{D}(\mathbb{R}^{n \times n}), \\ & \forall u \in \{1, \dots, N_U\}, \exists \mathbf{P}_u \in \mathbb{M}, \exists \mathbf{P}_{u_{int}} \in \mathbb{M}_{int}, \exists \mathbf{D}_u \in \mathbb{M}, \exists \boldsymbol{\tau}_u \in \mathbb{R}, \\ & \forall l \in \{1, \dots, N_L\}, \exists \mathbf{P}_l \in \mathbb{M}, \exists \mathbf{P}_{l_{int}} \in \mathbb{M}_{int}, \exists \mathbf{D}_l \in \mathbb{M}, \exists \boldsymbol{\tau}_l \in \mathbb{R}, \end{aligned}$$

$$\check{Q}^{inv} > 0$$

$$\mathbf{D}_u > 0$$

$$\boldsymbol{\tau}_u > 0$$

$$\begin{bmatrix} -\check{z}_{\Omega_u^U} D_u - P_{u_{int}} & -\check{y}_{\Omega_u^U}^* D_u - P_u & 0 & 0 \\ -\check{y}_{\Omega_u^U} D_u - P_u & -\check{x}_{\Omega_u^U} D_u - P_{u_{int}} & 0 & 0 \\ 0 & 0 & -\tau_u & 0 \\ 0 & 0 & 0 & U_u^2 \tau_u \end{bmatrix} + \begin{bmatrix} \check{Q}^{inv} \check{Q}^{inv} & -\check{Q}^{inv} A & 0 & -\check{Q}^{inv} B_1 \\ -A^T \check{Q}^{inv} & A^T A + C_2^T C_2 & -C_2^T & A^T B_1 + C_2^T D_{21} \\ 0 & -C_2 & 1 & -D_{21} \\ -B_1^T \check{Q}^{inv} & B_1^T A + D_{21}^T C_2 & -D_{21}^T & B_1^T B_1 + D_{21}^T D_{21} \end{bmatrix} \geq 0 \quad (4.32)$$

$$D_l > 0 \quad \tau_l > 0$$

$$\begin{bmatrix} -\check{z}_{\Omega_l^L} D_l - P_{l_{int}} & -\check{y}_{\Omega_l^L}^* D_l - P_l & 0 & 0 \\ -\check{y}_{\Omega_l^L} D_l - P_l & -\check{x}_{\Omega_l^L} D_l - P_{l_{int}} & 0 & 0 \\ 0 & 0 & -\tau_l & 0 \\ 0 & 0 & 0 & (1 - L_l^2) \tau_l \end{bmatrix} + \begin{bmatrix} \check{Q}^{inv} \check{Q}^{inv} & -\check{Q}^{inv} A & 0 & -\check{Q}^{inv} B_1 \\ -A^T \check{Q}^{inv} & A^T A + C_1^T C_1 & -C_1^T & A^T B_1 + C_1^T D_{11} \\ 0 & -C_1 & 1 & -D_{11} \\ -B_1^T \check{Q}^{inv} & B_1^T A + D_{11}^T C_1 & -D_{11}^T & B_1^T B_1 + D_{11}^T D_{11} \end{bmatrix} \geq 0 \quad (4.33)$$

where  $\check{x}_{\Omega_u^U}$ ,  $\check{y}_{\Omega_u^U}$ ,  $\check{z}_{\Omega_u^U}$  and  $\check{x}_{\Omega_l^L}$ ,  $\check{y}_{\Omega_l^L}$ ,  $\check{z}_{\Omega_l^L}$  are defined as:

$$\left[ \begin{array}{c|c} \check{x}_{\Omega} & \check{y}_{\Omega} \\ \check{y}_{\Omega}^* & \check{z}_{\Omega} \end{array} \right] := \left[ \begin{array}{cc|cc} x_{\Omega} \cdot I_r & 0 & y_{\Omega} \cdot I_r & 0 \\ 0 & z_{\Omega} \cdot I_{n_{AE}} & 0 & y_{\Omega}^* \cdot I_{n_{AE}} \\ \hline y_{\Omega}^* \cdot I_r & 0 & z_{\Omega} \cdot I_r & 0 \\ 0 & y_{\Omega} \cdot I_{n_{AE}} & 0 & x_{\Omega} \cdot I_{n_{AE}} \end{array} \right]$$

where  $x_{\Omega}$ ,  $y_{\Omega}$ ,  $z_{\Omega}$  are determined by the frequency interval and are defined as in (2.13)-(2.15) (p. 30 of Chapter 2), and the sets  $\mathbb{M}$  and  $\mathbb{M}_{int}$  are defined by:

$$\mathbb{M} := \left\{ M = M^T \mid \exists M_m \in \mathbb{R}^{r \times r}, \exists M_{AE} \in \mathbb{R}^{n_{AE} \times n_{AE}}, M = \begin{bmatrix} M_m & 0 \\ 0 & M_{AE} \end{bmatrix} \right\}$$

$$\mathbb{M}_{int} := \left\{ M_{int} = M_{int}^T \mid \exists M_{mAE} \in \mathbb{R}^{r \times n_{AE}}, M_{int} = \begin{bmatrix} 0 & M_{mAE} \\ M_{mAE}^T & 0 \end{bmatrix} \right\}$$

(ii) There exists a solution to the Alternative Lossless Passive Electronic Filter Design problem (Problem 3.2, p 49).

Indeed, define  $\check{Q} = (\check{Q}^{inv})^{-1}$ . Then, there exists an electronic filter, defined as the interconnection of elements from  $\mathcal{F}$  in the topology  $\mathcal{T}$  between a resistive voltage source with source resistance  $R_g$  and a resistive load  $R_l$ , such that its scattering parameters  $s_{11}$  and  $s_{21}$ , defined by (4.30) and (4.31), are stable,  $\{-1, 0, 1\}$ -dissipative, and satisfy the spectral mask  $\mathcal{SM}$ :

$$\forall u \in \{1, \dots, N_U\}, \quad \forall \omega \in \Omega_u^U, \quad |s_{21}(j\omega)|^2 \leq U_u^2$$

$$\forall l \in \{1, \dots, N_L\}, \quad \forall \omega \in \Omega_l^L, \quad |s_{11}(j\omega)|^2 \leq 1 - L_l^2$$

*Proof.* First, one may observe that  $\check{Q} > 0$  implies that  $Q > 0$ , where  $Q$  comes from the original PHS DAE scattering representation (Lemma 4.2). Thus, by Property 4.3 (p. 70), the scattering matrix  $S$  is stable and lossless- $\{-I_2, 0, I_2\}$ -dissipative, and  $s_{11}$  and  $s_{21}$  are stable and  $\{-1, 0, 1\}$ -dissipative.

Then, the remaining of the proof is similar in all aspects to the proof of Theorem 4.1. The main difference is the appropriate choice of multipliers, that were given in Chapter 2. For instance, the upper bound constraint  $U_u$  is formulated as follows: Find if there exist complex signals  $q$ ,  $p$ ,  $z_2$ ,  $w_1$  that satisfy the dissipative inequality

$$\begin{bmatrix} z_2 \\ w_1 \end{bmatrix}^* \begin{bmatrix} -I & 0 \\ 0 & U_u^2 I \end{bmatrix} \begin{bmatrix} z_2 \\ w_1 \end{bmatrix} \geq 0$$

such that

$$\begin{bmatrix} q \\ p \\ z_2 \\ w_1 \end{bmatrix}^* \begin{bmatrix} -I & 0 \\ \check{Q}A^T & \check{Q}C_2^T \\ 0 & -1 \\ B_1^T & D_{21}^T \end{bmatrix} \begin{bmatrix} -I & A\check{Q} & 0 & B_1 \\ 0 & C_2\check{Q} & -1 & D_{21} \end{bmatrix} \begin{bmatrix} q \\ p \\ z_2 \\ w_1 \end{bmatrix} = 0$$

and such that there exist  $\tilde{P}_u \in \mathbb{M}$ ,  $\tilde{P}_{u_{int}} \in \mathbb{M}_{int}$ ,  $\tilde{D}_u \in \mathbb{M}$ , such that  $\tilde{D}_u > 0$  and

$$\begin{bmatrix} p \\ q \end{bmatrix}^* \begin{bmatrix} \check{x}_{\Omega_u} \tilde{D}_u + \tilde{P}_{u_{int}} & \check{y}_{\Omega_u} \tilde{D}_u + \tilde{P}_u \\ \check{y}_{\Omega_u}^* \tilde{D}_u + \tilde{P}_u & \check{z}_{\Omega_u} \tilde{D}_u + \tilde{P}_{u_{int}} \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix} \geq 0$$

The S-procedure lemma (Lemma A.1, p. 194) and a suitable change of variable provide the result.  $\square$

Theorem 4.2 provides a synthesis method to solve the design problem by the resolution of a BMI feasible problem. The bilinear term, when it appears, is the same in all inequalities, i.e.  $\check{Q}\check{Q}$ . Therefore, only  $n$  different bilinearities are involved. From this view, it outshines Theorem 4.1, for which bilinearities appear up to the number of  $\frac{n(n+1)}{2}$  in the optimisation problem.

However, this is at the price of some conservatism. Indeed, because of the block  $\begin{pmatrix} \frac{1}{s} \cdot I_r & 0 \\ 0 & s \cdot I_{n_{AE}} \end{pmatrix}$ , the multiplier matrices happen to be structured, for instance  $\mathbf{D}_u \in \mathbb{M}$ . In practice, this means that less optimisation variables are available to find a solution. Indeed,  $\mathbf{D}_u$  is block diagonal in Theorem 4.2 while it is full symmetric in Theorem 4.1. Therefore, the set of solution of the optimisation problem of Theorem 4.2 is intuitively smaller than the corresponding one of Theorem 4.1.

Overall, the combination of the PHS representations with the development of an adapted synthesis method and the help of the LFT tool provides an optimisation problem with a very limited number of bilinearities. However, this optimisation problem still does not belong to the LMI framework, and no generic efficient algorithms may be used. Instead, different algorithms have been proposed to solve that kind of optimisation problem. In the next subsection, some of them are reviewed.

#### 4.2.4 Resolution of BMI optimisation problems

Feasibility optimisation problem (ii) of Theorem 4.2 involves two-by-two products of optimisation variables. It belongs to a specific class of optimisation problem called Bilinear (or Bi-affine) Matrix Inequalities (BMI) optimisation problem. For the purpose of this work, BMI optimisation problems are defined as the minimisation of a linear objective under a BMI constraint:

$$\begin{aligned} & \min_{\mathbf{x} \in \mathbb{R}^n} && c^T \mathbf{x} \\ & \text{such that} && \rho(\mathbf{x}, \mathbf{x}\mathbf{x}^T) \geq 0 \end{aligned} \tag{4.34}$$

where  $\mathbf{x} \in \mathbb{R}^n$  is the vector of the optimisation variables,  $c \in \mathbb{R}^n$  defines the linear objective and  $\rho$  is a linear matrix-valued function expressed as:

$$\rho(\mathbf{x}, \mathbf{x}\mathbf{x}^T) = F_0 + \sum_{i=1}^n \mathbf{x}_i F_i + \sum_{j,k=1}^n \mathbf{x}_j \mathbf{x}_k G_{jk}$$

where  $F_i = F_i^T \in \mathbb{R}^{m \times m}$ ,  $G_{jk} = G_{jk}^T \in \mathbb{R}^{m \times m}$  are symmetric matrices.

A BMI optimisation problem is an extension of an LMI optimisation problem. It is a simple but flexible framework for Control Design problems. It is able to handle specifications not available for the LMI framework such as controller structure or controller order. Many linear and robust control design problems are formulated as BMI optimisation problems, such as reduced-order output-feedback control, decentralised control,  $\mu$ -synthesis [SHL94; GSL96].

However, BMI optimisation problems are generally non-convex [GSP94], implying two theoretical barriers. First, they do not generally have the local-global property, i.e a local minimum is necessary a global minimum. Rigorously, convexity is not a necessary condition for the local-global property, but BMI problems are typically known to have multiple extrema [GTS+94]. Second, there are not generic efficient, i.e polynomial-time, algorithms of resolution. Indeed, it can be even shown that equivalent BMI optimisation problems of some basic linear control design problems are NP-hard [BT97].

Despite these theoretical barriers, many approaches of resolution of BMI problems have been developed. First, some specific BMI problems can be equivalently formulated under the LMI framework using standard techniques [Sco06, Sec. 3.6], [BTN01, Chap. 4], [BGFB94]. Unfortunately, these techniques fail to perform such a reformulation in our case. Then, approaches are usually split into two types, depending if the aim is to find a global minimum (Global Optimisation) or a local minimum (Local Optimisation). Those belonging to the first type, such as Branch-and-Bound [BB93], are computationally prohibitive for problems with more than a few optimisation variable. In addition, as the condition (ii) of Theorem 4.2 is a feasible optimisation problem, a local minimum may be sufficient. Therefore, we focus here on Local Optimisation approaches.

A popular approach of local optimisation is to solve a sequence of convex relaxations of the BMI problem until a satisfactory solution is obtained [KZM18]. Two criteria may be used to differentiate algorithms of this approach: the feasibility of the initial point and the type of the convex relaxation. First criterion is the requirement or not of algorithms to have a feasible solution of the BMI problem as initial point. As we are interested in a feasibility optimisation problem, we only consider those that do *not* require such initial point. Second criterion is about finding the appropriate convex relaxation. To understand this, let write the BMI problem given in (4.34) in the following equivalent formulation:

$$\begin{array}{ll} \min_{\mathbf{x} \in \mathbb{R}^m, \mathbf{X} \in \mathbb{R}^{m \times m}} & c^T \mathbf{x} \\ \text{such that} & \rho(\mathbf{x}, \mathbf{X}) \geq 0 \\ & \mathbf{X} = \mathbf{x}\mathbf{x}^T \end{array}$$

This transforms the BMI problem into an LMI problem with an additional equality constraint which captures the non-convexity. A convex relaxation of  $\mathbf{X} = \mathbf{x}\mathbf{x}^T$  is given by the inequality:

$$\mathbf{X} \geq \mathbf{x}\mathbf{x}^T$$

By Schur Lemma (Lemma A.3, p.195), this is equivalent to the LMI:

$$\begin{bmatrix} \mathbf{X} & \mathbf{x} \\ \mathbf{x}^T & 1 \end{bmatrix} \geq 0 \quad (4.35)$$

The Schur Complement Lemma (Lemma A.2, p.195) implies that if (4.35) and

$$\text{rank} \left( \begin{bmatrix} \mathbf{X} & \mathbf{x} \\ \mathbf{x}^T & 1 \end{bmatrix} \right) = 1$$

hold then  $\mathbf{X} = \mathbf{x}\mathbf{x}^T$ . The rank function is discontinuous and non-convex, and rank-constrained optimisation problems are then NP-hard. Most of the methods minimise then a relaxed convex approximation of the rank function over a convex set, which is then computationally tractable [SD17; KZM18].

A popular heuristic method for the minimisation of the rank is the minimisation of the nuclear norm of  $\begin{bmatrix} \mathbf{X} & \mathbf{x} \\ \mathbf{x}^T & 1 \end{bmatrix}$ , here equivalent to minimise its trace. Indeed, it appears that the nuclear norm is the convex envelope of the function rank on the set of matrices with spectral norm less than or equal to one [RFP10]. This heuristic tends to provide good results in practice, and many algorithms are based on it (cf [DV16; KZM18; KZAM18] and the references therein). The main issue is that the relaxed function cannot represent the rank function and the gap between both functions prevents from guaranteeing to satisfy the original BMI problem.

The performance of these algorithms is subject to a critical point: the choice of the initial point. Indeed, if this initial point is not *close* to the BMI feasible set, these algorithms will not converge to a satisfactory result, or without convergence issues.

In the sequel, we adapt an algorithm proposed in [KZAM18], that will be applied on a simple example in next subsection, and will be used in Chapter 6. Starting from an arbitrary initial point, it proceeds by solving a sequence of LMI relaxations of the BMI optimisation problem. An important benefit of this algorithm is its ability to guarantee to produce feasible points of the original BMI problem, when the initial point is *sufficiently close* to the BMI feasible set, as proved in [KZM18].

### Sequential LMI relaxations algorithm

Consider notations of Theorem 4.2. Before introducing the relaxation, define the matrix-valued linear functions  $\rho_u$  and  $\rho_l$  as follows:

$$\begin{aligned} \rho_u \left( \check{Q}^{inv}, X, P_u, P_{u_{int}}, D_u, \tau_u \right) = & \\ \begin{bmatrix} -\check{z}_{\Omega_u^y} D_u - P_{u_{int}} & -\check{y}_{\Omega_u^y}^* D_u - P_u & 0 & 0 \\ -\check{y}_{\Omega_u^y} D_u - P_u & -\check{x}_{\Omega_u^y} D_u - P_{u_{int}} & 0 & 0 \\ 0 & 0 & -\tau_u & 0 \\ 0 & 0 & 0 & U_u^2 \tau_u \end{bmatrix} + & \begin{bmatrix} X & -\check{Q}^{inv} A & 0 & -\check{Q}^{inv} B_1 \\ -A^T \check{Q}^{inv} & A^T A + C_2^T C_2 & -C_2^T & A^T B_1 + C_2^T D_{21} \\ 0 & -C_2 & 1 & -D_{21} \\ -B_1^T \check{Q}^{inv} & B_1^T A + D_{21}^T C_2 & -D_{21}^T & B_1^T B_1 + D_{21}^T D_{21} \end{bmatrix} \\ \rho_l \left( \check{Q}^{inv}, X, P_l, P_{l_{int}}, D_l, \tau_l \right) = & \\ \begin{bmatrix} -\check{z}_{\Omega_l^l} D_l - P_{l_{int}} & -\check{y}_{\Omega_l^l}^* D_l - P_l & 0 & 0 \\ -\check{y}_{\Omega_l^l} D_l - P_l & -\check{x}_{\Omega_l^l} D_l - P_{l_{int}} & 0 & 0 \\ 0 & 0 & -\tau_l & 0 \\ 0 & 0 & 0 & (1 - L_l^2) \tau_l \end{bmatrix} + & \begin{bmatrix} X & -\check{Q}^{inv} A & 0 & -\check{Q}^{inv} B_1 \\ -A^T \check{Q}^{inv} & A^T A + C_1^T C_1 & -C_1^T & A^T B_1 + C_1^T D_{11} \\ 0 & -C_1 & 1 & -D_{11} \\ -B_1^T \check{Q}^{inv} & B_1^T A + D_{11}^T C_1 & -D_{11}^T & B_1^T B_1 + D_{11}^T D_{11} \end{bmatrix} \end{aligned}$$

Note that  $\rho_u \left( \check{Q}^{inv}, (\check{Q}^{inv})^2, P_u, P_{u_{int}}, D_u, \tau_u \right)$  provides the left-hand term of (4.32) and  $\rho_l \left( \check{Q}^{inv}, (\check{Q}^{inv})^2, P_l, P_{l_{int}}, D_l, \tau_l \right)$  the left-hand term of (4.33).

Algorithm 4.1 is adapted from the one proposed in [KZAM18]. This algorithm proceeds iteratively until a feasible solution of the original BMI problem is obtained *or* the maximum number of round *maxRound* is reached. At each iteration  $k$ , the LMI optimisation problem (4.36) is solved. Then, the LMI feasibility problem (4.37)

checks if the resulting solution  $\check{Q}_{\{k\}}^{inv}$  is a solution to the original BMI problem or not. Finally, the line 7 of Algorithm 4.1 is called the *Nestorov's acceleration* and is generally observed to improve the convergence of the algorithm [KZAM18], when the iteration-dependant parameter  $\beta_k$  is *suitably* chosen. Some approaches have been proposed to interpret  $\beta_k$  and find an optimal choice of such parameter, with Control theory concepts [LRP16; HL17a; HL17b]. Nevertheless, we arbitrarily choose it as  $\beta_k = 1$ .

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**Algorithm 4.1:** Sequential LMI relaxation
 

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**Input:**  $maxRound \in \mathbb{N}$ ,  $\check{Q}_{init}^{inv}$   
**Output:**  $\check{Q}^{inv}$

- 1  $\check{Q}_{\{0\}}^{inv} \leftarrow \check{Q}_{init}^{inv}$
- 2  $t_{\{0\}} \leftarrow 1$
- 3  $k \leftarrow 1$
- 4 **while**  $t_{\{k-1\}} \geq 0 \parallel k \leq maxRound$  **do**
- 5      $\check{Q}_{\{k\}}^{inv} \leftarrow \text{Solve (4.36)}$
- 6      $t_{\{k\}} \leftarrow \text{Solve (4.37)}$
- 7      $\check{Q}_{init}^{inv} \leftarrow \check{Q}_{\{k\}}^{inv} + \beta_k (\check{Q}_{\{k\}}^{inv} - \check{Q}_{\{k-1\}}^{inv})$
- 8      $k \leftarrow k + 1$
- 9  $\check{Q}^{inv} \leftarrow \check{Q}_{\{k-1\}}^{inv}$

---

$$\left. \begin{array}{l}
 \min_{\substack{\check{Q}^{inv} > 0 \in \mathcal{D}(\mathbb{R}^{n \times n}), \mathbf{X} > 0 \in \mathcal{D}(\mathbb{R}^{n \times n}), \\
 \{P_u \in \mathbb{M}, P_{u_{int}} \in \mathbb{M}_{int}, D_u > 0 \in \mathbb{M}, \tau_u > 0\}_{u=1}^{N_U}, \\
 \{P_l \in \mathbb{M}, P_{l_{int}} \in \mathbb{M}_{int}, D_l > 0 \in \mathbb{M}, \tau_l > 0\}_{l=1}^{N_L}}} \\
 \text{trace} \left( \mathbf{X} - 2\check{Q}_{init}^{inv} \check{Q}^{inv} + \check{Q}_{init}^{inv} \check{Q}_{init}^{inv} \right) \\
 \text{such that} \\
 \begin{bmatrix} \mathbf{X} & \check{Q}^{inv} \\ \check{Q}^{inv} & I \end{bmatrix} \geq 0 \\
 \forall u \in \{1, \dots, N_U\}, \quad \rho_u \left( \check{Q}^{inv}, \mathbf{X}, P_u, P_{u_{int}}, D_u, \tau_u \right) \geq 0 \\
 \forall l \in \{1, \dots, N_L\}, \quad \rho_l \left( \check{Q}^{inv}, \mathbf{X}, P_l, P_{l_{int}}, D_l, \tau_l \right) \geq 0
 \end{array} \right\} \quad (4.36)$$

$$\left. \begin{array}{l}
 \min_{t \in \mathbb{R}} t \\
 \{P_u \in \mathbb{M}, P_{u_{int}} \in \mathbb{M}_{int}, D_u > 0 \in \mathbb{M}, \tau_u > 0\}_{u=1}^{N_U}, \\
 \{P_l \in \mathbb{M}, P_{l_{int}} \in \mathbb{M}_{int}, D_l > 0 \in \mathbb{M}, \tau_l > 0\}_{l=1}^{N_L}, \\
 \text{such that} \\
 \forall u \in \{1, \dots, N_U\}, \quad \rho_u \left( \check{Q}_{\{k\}}^{inv}, \left( \check{Q}_{\{k\}}^{inv} \right)^2, P_u, P_{u_{int}}, D_u, \tau_u \right) \geq -t \\
 \forall l \in \{1, \dots, N_L\}, \quad \rho_l \left( \check{Q}_{\{k\}}^{inv}, \left( \check{Q}_{\{k\}}^{inv} \right)^2, P_l, P_{l_{int}}, D_l, \tau_l \right) \geq -t
 \end{array} \right\} \quad (4.37)$$

#### 4.2.5 Application on a simple example

In order to illustrate this approach, let us consider the following simple design example. The design problem is to find a lossless passive electronic filter, defined as the interconnection of elements from  $\mathcal{F}_{LC}$  (Table 3.1, p. 41) in a ladder topology,

i.e. a  $LC$  ladder filter (Table 3.2, p. 46), with source and load resistances being unitary  $R_g = R_l = 1 \Omega$ , such that its scattering parameter  $s_{21}$  is stable satisfies the following spectral mask  $\mathcal{SM}$ :

$$\begin{aligned} \forall \omega \in [0, \omega_L], & \quad |s_{21}(j\omega)|^2 \geq L^2 \\ \forall \omega \in [\omega_U, +\infty), & \quad |s_{21}(j\omega)|^2 \leq U^2 \end{aligned}$$

with  $\omega_L := 0.8 \text{ rad/s}$ ,  $L^2 := 0.7$ ,  $\omega_U := 1.5 \text{ rad/s}$  and  $U^2 := 0.2$ . As this approach solves an alternative formulation of the design problem (Problem 3.2, p. 49), the lower bound on  $|s_{21}(j\omega)|^2$  of  $\mathcal{SM}$  is exchanged with an equivalent upper bound on  $|s_{11}(j\omega)|^2$  as follows:

$$\forall \omega \in [0, \omega_L], \quad |s_{11}(j\omega)|^2 \leq 1 - L^2$$

From the design approach of next section, or using traditional charts [MYJ63], it is known that a  $n = 2$ -elements  $LC$  ladder filter, known as a Butterworth filter, solves this design problem. Indeed, the filter illustrated on Figure 4.1 (p. 55) with elements values given by:

$$L_{1_{Butter}} := \sqrt{2} \quad C_{2_{Butter}} := \sqrt{2}$$

provides a scattering parameter  $s_{21}$  that satisfies the spectral mask  $\mathcal{SM}$  as illustrated in Figure 4.4. Therefore, the design approach is applied with  $n = 2$ .

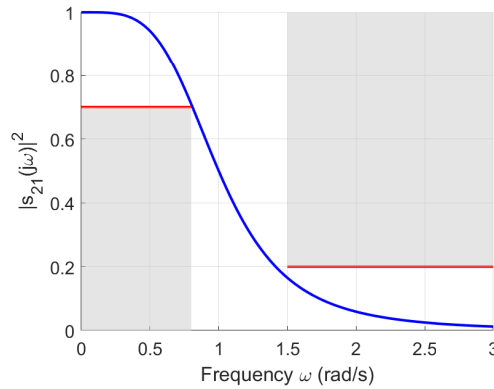


FIGURE 4.4: A Butterworth filter solution for the design example of Subsection 4.2.5

First, a scattering representation of the filter is calculated from Figure 4.1. From the modelling procedure of Subsection 4.2.2.2, a scattering PHS state-space representation is obtained:

$$\begin{cases} \tilde{p} = \left(\frac{1}{s} \cdot I_n\right) \tilde{q} \\ \tilde{q} = (J - BB^T) Q\tilde{p} + \sqrt{2}Bw \\ z = -\sqrt{2}DB^T Q\tilde{p} + Dw \end{cases}$$

where  $\tilde{q} := \frac{1}{\sqrt{2}} [v_{L_1} \quad i_{C_2}]^T$ ,  $\tilde{p} := \frac{1}{\sqrt{2}} [i_{L_1} \quad v_{C_2}]^T$  and

$$Q := \begin{bmatrix} \frac{1}{L_1} & 0 \\ 0 & \frac{1}{C_2} \end{bmatrix} \quad J := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad B := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad D := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Second, an optimisation problem is obtained using Theorem 4.2 (p. 79). This leads to the BMI feasibility problem of the statement (i) of this theorem with  $A \in \mathbb{R}^{n \times n}$ ,



$B_1 \in \mathbb{R}^{n \times 1}$ ,  $C_1, C_2 \in \mathbb{R}^{1 \times n}$ ,  $D_{11}, D_{21} \in \mathbb{R}$ , defined by (4.29) (p. 79),  $r = n$ ,  $n_{AE} = 0$ , and

$$\left[ \begin{array}{c|c} \check{x}_{\Omega_1^L} & \check{y}_{\Omega_1^L} \\ \check{y}_{\Omega_1^L}^* & \check{z}_{\Omega_1^L} \end{array} \right] := \left[ \begin{array}{c|c} \omega_L^2 & 0 \\ 0 & -1 \end{array} \right] \quad \left[ \begin{array}{c|c} \check{x}_{\Omega_1^U} & \check{y}_{\Omega_1^U} \\ \check{y}_{\Omega_1^U}^* & \check{z}_{\Omega_1^U} \end{array} \right] := \left[ \begin{array}{c|c} -\omega_U^2 & 0 \\ 0 & 1 \end{array} \right]$$

Third, Algorithm 4.1 is applied. In order to have an *intuitive* idea of the algorithm performance, it is run  $n_{loop} = 1000$  times, where each loop corresponds to a different initial point. The initial points are randomly generated from the Butterworth solution such as:

$$\check{Q}_{init}^{inv} = Q_{Butter}^{inv} (1 + \Delta)$$

where  $Q_{Butter}^{inv} := \begin{bmatrix} L_{1Butter} & 0 \\ 0 & C_{2Butter} \end{bmatrix}$ ,  $\Delta := \begin{bmatrix} \delta_1 & 0 \\ 0 & \delta_2 \end{bmatrix}$  and  $\delta_1, \delta_2$  are random variables with a uniform distribution in the interval  $(-d, d)$ . The real number  $d$  is then the maximum deviation allowed for the initial point  $\check{Q}_{init}^{inv}$  compared to the Butterworth solution  $Q_{Butter}^{inv}$ . We choose here  $d = 0.5$ . Finally, the maximum number of iterations is arbitrarily set to  $maxRound = 500$ .

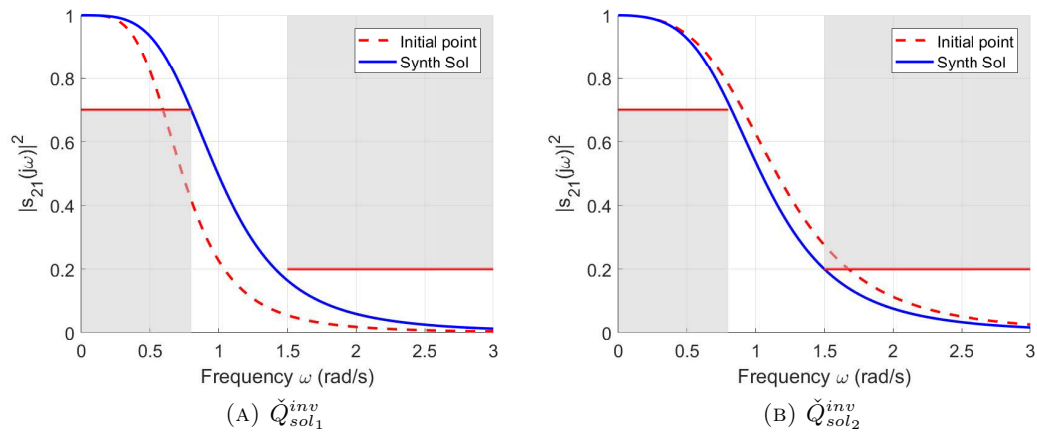


FIGURE 4.5: Two illustrative solutions provided by Algorithm 4.1 for the design example of Subsection 4.2.5

On Table 4.1 are given the convergence rate, i.e. the average number of solutions of the original BMI problem founded by the algorithm divided by  $n_{loop}$ , the average number of iterations per loop and the average computational time for a loop. Algo-

TABLE 4.1: Average performance of the algorithm for  $n_{loop} = 1000$  tests

Convergence rate	Average number of iterations	Average computational time
92.9 %	264.7	4.9 s

gorithm 4.1 seems to perform well as 92.9% of the tests have converged to a solution for an average computational time of 4.9 s, with a deviation of 50% to a feasible point. Two illustrations of the resulting solutions are provided in (4.38) and plotted in Figure 4.5, with their associated initial points.

$$\check{Q}_{sol_1}^{inv} = \begin{bmatrix} 1.238 & 0 \\ 0 & 1.609 \end{bmatrix} \quad \check{Q}_{sol_2}^{inv} = \begin{bmatrix} 1.721 & 0 \\ 0 & 0.995 \end{bmatrix} \quad (4.38)$$

The performance of Algorithm 4.1 is promising but must be proportioned to the simplicity of the design example. For more complex design problems, i.e. for a higher number of building elements, the computational cost of each iteration increases. As one might expect, if the initial point happens to be far from the BMI feasibility set, the number of iteration raises, resulting in a lengthy computation. Therefore, it is important to provide an initial point *sufficiently* close to the feasibility set. In Chapter 6, it will be proposed an efficient method to find such initial point for the design of *AW*-resonator ladder filters (Table 3.2, p. 46). This method is mainly based on the realisable filter synthesis approach, developed in the next section.

## 4.3 Design approach 2: realisable filter synthesis

### 4.3.1 Introduction

In Section 4.2, the design approach consists in setting the elements in an ready-to-implement form, and then to tune the characteristic values of the elements such that the resulting scattering parameter  $s_{21}$  satisfies a given spectral mask. This leads to solve a non-convex optimisation problem.

In this section, an alternative design approach is presented. This is typically split in two steps. First, a transfer function  $W$  is synthesised such that it satisfies a spectral mask. In the sequel, this is referred to as the *frequency filter synthesis* step. This leads to a *convex* optimisation problem which can be efficiently solved. Second, an electronic circuit is computed as the interconnection of elements from a given family  $\mathcal{F}$  in a given topology  $\mathcal{T}$ , such that  $s_{21}(s) = W(s)$ . This is referred to as the *circuit synthesis* step. There does not exist such a circuit for all transfer functions. Constraints on  $W$  such that there exists this circuit, namely *realisation constraints*, are then needed. These constraints are required to have a *suitable* formulation in order to be convexly added into the frequency filter synthesis step. This is the main difficulty of this approach.

In the sequel, the frequency filter synthesis step is first introduced. The problem is presented, and *traditional* synthesis methods are reviewed. A more recent approach, using tools from our design framework, will be detailed in Chapter 5. Then, the problem of finding convex realisation conditions is tackled. It appears that realisation conditions with a convex formulation have only been established for *LC* ladder filters (Table 3.2, p. 46). These conditions are stated and *convexly* extended for a more general class of ladder filters, namely *T*-ladder filter, which includes *LC* ladder and *particular LC*-resonator ladder filters. This is our main contribution of this section, and this was published in [PKSO18]. To achieve this, inspired by the traditional design method of *LC* ladder filters, a transfer function  $T(s)$  is introduced. Using a similar scheme, two transfer functions  $T_s(s)$  and  $T_p(s)$  are introduced, based on common design practice of *AW* ladder filters, leading to the definition of  $T_s, T_p$ -ladder filters. This enables to significantly reduce the complexity of the realisation conditions, but no convex formulation are found for these filters. The introduction of  $T(s)$  and  $T_s(s), T_p(s)$  requires to extend the frequency filter synthesis step from transfer functions, i.e. LFT in  $\frac{1}{s}$ , to respectively LFT in  $T(s)$  and LFT in  $T_s(s)$  and  $T_p(s)$ . These new problems will be tackled in Chapter 5.

**Literature comments** In our opinion, among all the literature of standard design methods for *LC* ladder filters, the books [PB87] and [Bah84] stand out from others. Despite its name, [PB87] provides a comprehensive study of analytical methods for

*analog* frequency filter synthesis, as digital filters can be synthesised from analog filters by using an analog-to-digital conversion. The synthesis problem and resolution steps are clearly identified, the four classical filters are presented in a synthetic and unified manner, and their optimal features are discussed. While both synthesis steps are treated, the main contribution of [Bah84] is its approach on the circuit synthesis step. The realisation theory for filter applications is studied, all filter mathematical descriptions (impedance, input impedance, scattering, transmission, etc.) are linked and their properties for interconnection of passive elements are detailed. Furthermore, while most of the standard literature implicitly assumes that transfer functions resulting from the previous step are realisable, as this is generally the case for the four classical filters, it explicitly exhibits the notion of realisation constraints. Finally, realisation constraints for *LC* ladder filters are expressed on the input impedance  $z_{in}$  and the scattering parameter  $s_{21}$ , and are proved.

### 4.3.2 Frequency filter synthesis

Problem 4.1 states the general formulation of the frequency filter synthesis problem. This consists in synthesising a filter, whose frequency-response is required to satisfy constant upper and lower bounds on its modulus.

#### Problem 4.1 (Frequency Filter Synthesis).

Given a positive integer  $n \in \mathbb{N}^*$ ,

Given  $N_U$  positive real numbers  $U_u$  associated with  $N_U$  real intervals  $\Omega_u^U \subseteq \mathbb{R}$ ,

Given  $N_L$  positive real numbers  $L_l$  associated with  $N_L$  real intervals  $\Omega_l^L \subseteq \mathbb{R}$ ,

Find if there exist real matrices  $\mathbf{A} \in \mathbb{R}^{n \times n}$ ,  $\mathbf{B} \in \mathbb{R}^{n \times 1}$ ,  $\mathbf{C} \in \mathbb{R}^{1 \times n}$  and  $\mathbf{D} \in \mathbb{R}^{1 \times 1}$  such

that  $W(s) := \left(\frac{1}{s} \cdot I_n\right) \star \left[ \begin{array}{c|c} \mathbf{A} & \mathbf{B} \\ \hline \mathbf{C} & \mathbf{D} \end{array} \right]$  is stable and satisfies the spectral mask  $\mathcal{SM}$ :

$$\begin{aligned} \forall u \in \{1, \dots, N_U\}, \quad \forall \omega \in \Omega_u^U, \quad |W(j\omega)|^2 &\leq U_u^2 \\ \forall l \in \{1, \dots, N_L\}, \quad \forall \omega \in \Omega_l^L, \quad |W(j\omega)|^2 &\geq L_l^2 \end{aligned}$$

and *Compute* them.

*Remark 4.2.* Problem 4.1 is traditionally stated in the transfer function form, as follows: Find if there exist  $\mathbf{b}_0, \dots, \mathbf{b}_m$  and  $\mathbf{a}_0, \dots, \mathbf{a}_n$  (with  $m \leq n$ ) such that

$$W(s) = \frac{\mathbf{b}_0 + \mathbf{b}_1 \cdot s + \dots + \mathbf{b}_m \cdot s^m}{\mathbf{a}_0 + \mathbf{a}_1 \cdot s + \dots + \mathbf{a}_n \cdot s^n}$$

is stable and satisfies  $\mathcal{SM}$ .

*Convex* synthesis methods typically operate with the same scheme. First, the squared modulus  $|W(j\omega)|^2$  is synthesised such that it satisfies the spectral mask  $\mathcal{SM}$ , referred to as the *magnitude synthesis* step. Second, a stable  $W(s)$  is extracted, having  $|W(j\omega)|^2$  as squared modulus. This is called the *spectral factorisation* step. In [RSF03], it is shown that Problem 4.1 can be equivalently formulated as a finite-dimensional LMI optimisation problem, and is then optimally and efficiently solvable. This approach is comprehensively presented in Chapter 5. In this section, certain traditional synthesis methods are given. These methods are non-optimal, in the sense that the order  $n$  of the resulting filter  $W(s)$  is not necessarily minimal. However, there exists an electronic filter, made of inductances and capacitances, having  $W(s)$  as scattering parameter  $s_{21}(s) = W(s)$ . This property is highly beneficial for solving our design problem (Problem 3.1, p. 48), and is displayed in Subsection 4.3.3.1.

Traditional synthesis methods operates as follows [PB87; Bah84]. First, a low-pass prototype filter with a normalised cut-off frequency is calculated. Second, four types of filter (low-pass, high-pass, band-pass, band-stop) are obtained using specific changes of variable.

The first stage consists in approximating a desired complex function  $W_d(s)$  by a stable transfer function  $W(s)$ . When evaluated on the imaginary axis  $s = j\omega$ , the desired function  $W_d$  behaves as an ideal low-pass filter with a cut-off frequency normalised to 1 *rad/s* (Figure 4.6). As the synthesis problem is formulated in terms of squared magnitude, the approximation process is formulated in terms of the squared magnitude of  $W(s)$ . To achieve this, an error criterion between the desired function

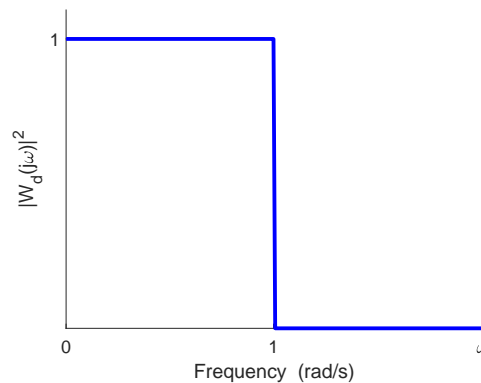


FIGURE 4.6: Ideal normalised low-pass filter

and the approximation is minimised. There are four usual approximations named Butterworth, Chebyshev I, Chebyshev II and Elliptic/Cauer. Each approximation can be generated by a combination of two criteria, namely Taylor series and min-max criterion. Taylor series criterion consists in a Taylor series approximation achieved at  $\omega = 0$  or  $\omega = \infty$ . Filters which satisfy this criterion tends to have flat responses. The min-max criterion<sup>5</sup> results in minimising the maximum error over pass-band or stop-band. The satisfaction of this criterion produces filters with equi-ripple responses. The different combinations of both criteria and the corresponding filters are given in Table 4.2. An important remark is that each approximation is optimal, in the sense that they are the optimal solutions of the approximation problem defined by the choice of the error criteria. For instance, the Butterworth filter is the best Taylor series approximation to the ideal low-pass filter magnitude at both  $\omega = 0$  and  $\omega = \infty$ , while Chebyshev I filter gives the smallest maximum magnitude error over the entire pass-band  $[0, 1]$  over filters that are also Taylor series approximation at  $\omega = \infty$ .

TABLE 4.2: Combinations of criteria and corresponding usual filters

	Taylor Series ( $\omega = 0$ )	Min-Max (pass-band)
Taylor Series ( $\omega = \infty$ )	Butterworth	Chebyshev I
Min-Max (stop-band)	Chebyshev II	Elliptic/Cauer

Interestingly, the optimal solutions of the four approximation problems can be analytically calculated. The resulting squared magnitudes of the frequency-response are provided in (4.39)-(4.42), where  $n$  is the order of the filter,  $\epsilon$  is the parameter

<sup>5</sup>sometimes called the Chebyshev criterion

that controls the ripple size,  $C_n$  is an  $n$ -order *Chebyshev polynomial* and  $G$  a *Chebyshev rational function*.  $|W(j\omega)|^2$  is plotted on Figure 4.7 for each approximation of order  $n = 5$ . It is required a lower-bound ripple of  $R_p = 0.89$  in the passband for the Chebyshev I and the Elliptic filters, and an upper ripple of  $R_s = 0.99$  for the Chebyshev II and the Elliptic filters.

$$\text{Butterworth} \quad |W(j\omega)|^2 = \frac{1}{1 + \omega^{2n}} \quad (4.39)$$

$$\text{Chebyshev I} \quad |W(j\omega)|^2 = \frac{1}{1 + \epsilon^2 C_n(\omega)^2} \quad (4.40)$$

$$\text{Chebyshev II} \quad |W(j\omega)|^2 = \frac{\epsilon^2 C_n(\frac{1}{\omega})^2}{1 + \epsilon^2 C_n(\frac{1}{\omega})^2} \quad (4.41)$$

$$\text{Elliptic/Cauer} \quad |W(j\omega)|^2 = \frac{1}{1 + \epsilon^2 G(\omega)^2} \quad (4.42)$$

A stable transfer function  $W(s)$  is then factorised from  $|W(j\omega)|^2$ . This is also done analytically. For more information on  $C_n$ ,  $G$  or the factorisation procedure, the reader may refer to the comprehensive book [PB87].

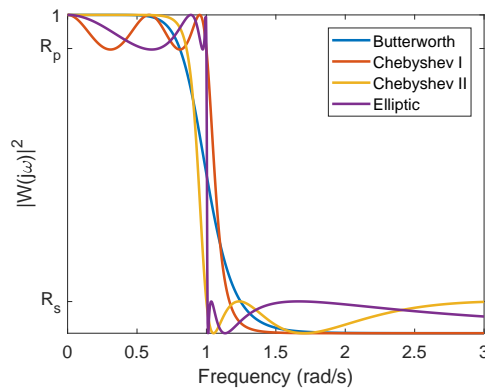


FIGURE 4.7: Squared Magnitude Plot of 5<sup>th</sup>-order usual filters

In addition to the low-pass normalised filter  $W(s)$ , other types of filter are needed in practice. Four classical versions can be obtained from  $W(s)$ . These are the low-pass filter (cut-off frequency  $\omega_0$ ), high-pass filter (cut-off frequency  $\omega_0$ ), band-pass filter (passband  $[\omega_1, \omega_2]$ ) and band-stop (stopband  $[\omega_1, \omega_2]$ ). These are synthesised by replacing the complex Laplace variable  $s$  by a new variable expressed as a rational function of  $s$ . These changes of variable are given in Table 4.3.

TABLE 4.3: Frequency transformations from prototype low-pass filters to other common versions

From prototype to	Low-Pass	High-Pass	Band-Pass	Band-Stop
	$\omega_0$	$\omega_0$	$\omega_0 = \sqrt{\omega_1 \omega_2}$	$\omega_0 = \sqrt{\omega_1 \omega_2}$
Exchange $s$ with	$\frac{s}{\omega_0}$	$\frac{\omega_0}{s}$	$\frac{s^2 + \omega_0^2}{(\omega_2 - \omega_1)s}$	$\frac{(\omega_2 - \omega_1)s}{s^2 + \omega_0^2}$

From a frequency filter synthesis point of view, the main benefit of this approach is its analytical calculation process. Another advantage is that the designer knows the behaviour of the phase of the resulting filter. Remind that the magnitude and

the phase can not generally be independently synthesised. Using the Butterworth approximation, for instance, ensures that the resulting filter will approximately have a linear phase in the passband, which can be critical for some applications.

An obvious drawback of these methods is that they do not solve the original problem, made of upper and lower bounds, but a problem consisting in approximating an ideal curve. The problem formulation is artificially constrained, and an important number of filters satisfying the magnitude constraints can not be computed. As a result, the order of the filter may be uselessly high, especially in the Butterworth case.

Another disadvantage of this approach is its lack of flexibility. This is mainly due to the changes of variables to obtain the common filters, which only enable to synthesise a restricted number of different filters. For instance, only band-pass filter with symmetric transition bandwidth can be achieved. For Radiofrequency applications, many band-pass filters are required to have a transition bandwidth significantly shorter than the other, as illustrated for instance on Figure 1.3 (p. 3).

### 4.3.3 Circuit synthesis

Once the frequency filter  $W(s)$  has been synthesised, one would like to synthesise a circuit such that the resulting scattering parameter  $s_{21}(s)$  is *exactly*  $W(s)$ , i.e.  $s_{21}(s) = W(s)$ . Given a general transfer function, there may no exist such a circuit. An approach coming from the *realisation theory* has provided necessary and sufficient realisation conditions on  $W(s)$  for the existence of a circuit as the interconnection of passive elements [New66; AV73]. These conditions are simple and convex, but the resulting circuits generally appear of little practical use, especially because of unrealistic type of interconnection.

As mentioned in Chapter 3, to prevent last issue, we focus on this work on the ladder topology. In addition, this topology tends to provide rather simple realisation conditions when associated with some particular families  $\mathcal{F}$  of element, as will be detailed in the sequel. A general ladder filter is represented on Figure 3.6 (p. 45). From a mathematical perspective, this is equivalent to get the input impedance  $z_{in}$  having the following decomposition:

$$z_{in} = z_1 + \frac{1}{y_2 + \frac{1}{z_3 + \frac{1}{y_4 + \frac{1}{\ddots + \frac{1}{y_{n-2} + \frac{1}{z_{n-1} + \frac{1}{y_n + \frac{1}{R_l}}}}}}}} \quad (4.43)$$

where  $z_i$  is the impedance of the serial element  $e_i$  and  $y_k$  is the admittance of the parallel element  $e_k$ .

By choosing the family of elements used and by making explicit the impedance  $z_i$ 's and the admittance  $y_k$ 's, one can find realisation conditions expressed on the coefficients of  $z_{in}$ . They can then be expressed on the scattering matrix  $S$ , using standard relations (Property 3.2, p. 44). These conditions are generally complex and can not be easily, or at least convexly, included in the synthesis process of  $W$ . A noticeable exception are *LC* ladder filters for which the realisation conditions are convex.

In the sequel, the realisation conditions for  $LC$  ladder filters are first stated. Then, generalisation of these conditions to other ladder filters are considered. First, inspired by the traditional design method of  $LC$  ladder and *particular*  $LC$  resonator filters, the  $T$ -ladder filter is introduced. Convex realisation conditions for this type of filters are stated and proved. This provides our main contribution of the section. Then, using a similar scheme, two transfer functions  $T_s(s)$  and  $T_p(s)$  are introduced, based on common design practice of AW-resonator ladder filters, leading to the definition of  $T_s, T_p$ -ladder filters. This enables to significantly reduce the complexity of the realisation conditions for AW-resonator ladder filters, but no convex formulation are found for these filters. Both realisation conditions, for  $T$ -ladder and  $T_s, T_p$ -ladder filters, leads to the introduction of the synthesis problem of frequency LFT filters that will be considered in Chapter 5.

#### 4.3.3.1 Realisation conditions of $LC$ ladder filters

A  $LC$  ladder filter (Table 3.2, p. 46) is a ladder filter for which the serial elements are inductances  $z_i(s) = L_i \cdot s$  and the parallel elements are capacitances  $y_j(s) = C_j \cdot s$ . As far as the author knows,  $LC$  ladder filters are the only passive electronic filters proved to have convex conditions of realisation with minimal number of dynamic building elements, i.e. equalled to the order of the resulting scattering parameter  $s_{21}$ .

In the sequel, the realisation conditions are first stated on the input impedance  $z_{in}$ . It is used for a historical reason, namely the Darlington synthesis [Bah84], but also because of the simplicity of the realisation conditions when expressed on  $z_{in}$ . Then, using the link between  $z_{in}$ ,  $s_{11}$  and  $s_{21}$  for lossless passive filters, the realisation conditions are expressed on  $s_{21}$ , which appear to be similarly simple.

The decomposition of (4.43) applied to  $LC$  ladder filters is referred to as a *continued fractional expansion* [You15]:

$$z_{in}(s) = \alpha_1 \cdot s + \frac{1}{\alpha_2 \cdot s + \frac{1}{\alpha_3 \cdot s + \frac{1}{\ddots \frac{1}{\alpha_{n-1} \cdot s + \frac{1}{\alpha_n \cdot s + \frac{1}{R_l}}}}}} \quad (4.44)$$

where  $\alpha_1 = L_1 > 0$ ,  $\alpha_2 = C_2 > 0$ ,  $\alpha_3 = L_3 > 0$ ,  $\dots$ ,  $\alpha_{n-1} = L_{n-1} > 0$ ,  $\alpha_n = C_n > 0$ . Therefore, a transfer function  $z_{in}$  is the input impedance of an  $LC$  ladder filter if and only if it has a continued fractional expansion with positive  $\alpha_i$ 's.

The continued fractional expansion condition is complex to directly use in the design process. Indeed, it is not trivial to establish an equivalent decomposition of (4.44) for  $s_{21}$ . Instead, the conditions of Theorem 4.3 are introduced.

**Theorem 4.3** ([Bah84]).

Given a stable  $\{0, 1, 0\}$ -dissipative transfer function  $z_{in}(s)$  of order  $n$  written as:

$$z_{in}(s) := \frac{N(s)}{D(s)}$$

where  $N(s)$  and  $D(s)$  are real polynomials of  $s$  of order  $n$ .

If  $z_{in}(s)$  satisfies the following condition:

$$\Re(z_{in}(j\omega)) = \frac{K}{|D(j\omega)|^2}$$

where  $K > 0$  is a positive constant,

Then  $z_{in}(s)$  admits a continued fraction expansion as in (4.44), with  $\alpha_1 > 0$ ,  $\alpha_2 > 0, \dots, \alpha_n > 0$ , and  $R_l > 0$ .

As a result,  $z_{in}(s)$  is the input impedance of an LC ladder filter terminated on a resistance  $R_l$ , given by  $R_l := \Re(z_{in}(0)) = \frac{K}{D(0)^2}$ .

In other words, Theorem 4.3 states that if the real part of the frequency response of the  $\{0, 1, 0\}$ -dissipative transfer function  $z_{in}(s)$  has a constant and positive numerator, then  $z_{in}(s)$  is the input impedance of a LC ladder filter terminated on a resistance  $R_l$ . The relations between  $z_{in}(s)$  and  $s_{11}(s)$  (Property 3.2, p. 44) and between  $s_{11}(s) = \frac{h(s)}{g(s)}$  and  $s_{21} = \frac{f(s)}{g(s)}$  (p. 48), imply a special link between the numerator of the real part of  $z_{in}(j\omega)$  and the numerator of  $|s_{21}(j\omega)|^2$ :

$$\Re(z_{in}(j\omega)) = R_g \frac{|s_{21}(j\omega)|^2}{|1 + s_{11}(j\omega)|^2} = R_g \frac{|f(j\omega)|^2}{|g(j\omega) + h(j\omega)|^2}$$

Using this relation, simple realisation conditions can be found for the scattering parameter  $s_{21}$ , as shown in next corollary.

**Corollary 4.4** ([Bah84]).

Given a source resistance  $R_g > 0$ ,

a stable  $\{-1, 0, 1\}$ -dissipative transfer function  $W(s)$  of order  $n$ .

If  $W(s)$  satisfies the following condition:

$$|W(j\omega)|^2 = \frac{1}{|g(j\omega)|^2} \quad (4.45)$$

where  $g(s)$  is a real polynomial of  $s$  of degree  $n$ .

Then there exists a  $n$ -elements, LC ladder filter, operating between a resistive voltage source with source resistance  $R_g$  and a resistive load  $R_l$ , which admits  $W(s)$  as scattering parameter:

$$s_{21}(s) = W(s)$$

Moreover, denoting  $g(s) := \sum_{i=0}^n a_i \cdot s^i$ , the resistive load  $R_l > 0$  admits two possible values, given by  $R_l := \frac{R_g}{a_0^2 \left(1 \pm \sqrt{a_0^2 - 1}\right)^2}$

The condition (4.45) admits a physical interpretation. As mentioned in Subsection 3.2.3, no power is transmitted through a ladder filter when the serial elements behave as open-circuits  $z_i(j\omega_0) = \infty$  or the parallel elements behave as short-circuits  $y_i(j\omega_0) = \infty$ . These special complex numbers  $j\omega_0$  matches with the transmission zeros. For an LC ladder filter, this only happens at  $\omega_0 = \infty$ . Therefore  $|W(j\omega)|^2$  should have a constant numerator, i.e. (4.45).

Corollary 4.4 explains the particular interest of Butterworth and Chebyshev I filters. Indeed, one can easily check on (4.39) and (4.40) (p. 90) that the squared magnitude of the frequency response of Butterworth and Chebyshev I approximations, inherently satisfy the realisation conditions. In other words, to any Butterworth



and Chebyshev I filter corresponds an  $LC$  ladder filter. This not the case when considering the two other approximations (Chebyshev II, Elliptic/Cauer) or other topologies. There exists a topology named mid-shunt/mid-series ladder, derived from  $LC$  ladder, which admits equivalent realisation conditions expressed on  $z_{in}$  [You71; Bah84]. However, these conditions have a complex formulation and are difficult to include in the design process. Elliptic/Cauer filters often satisfy these conditions, but not systemically. These conditions are then not further investigated in this work.

*Remark 4.3.* As mentioned in Chapter 3, impedance matching is a desired property for electronic filters. This happens when the resistances of the source and the load to be equal:  $R_g = R_l$ . With notations of Corollary 4.4, impedance matching requires then  $a_0 := \pm 1$ . The resulting  $W$  needs then to be structured as follows:

$$W(s) = \pm \frac{1}{1 + a_1 \cdot s + a_2 \cdot s^2 + \dots + a_n s^n}$$

Butterworth and odd-order Chebyshev I satisfy this property. However for even-order Chebyshev I filters, the latest condition is not verified and  $R_l \neq R_g$ . There is then impedance mismatching. This mismatch is directly linked with the ripple in the pass band. In practice, two options appear. One can increase the order of the filter by 1, i.e. add another element, but this increases the complexity of the filter. Otherwise, if the ripples are low enough, the impedance mismatching may be accepted.

Another illustration of the special connection between Butterworth and Chebyshev I approximations and  $LC$  ladder filters is the  $L, C$  transformations associated with the change of variable of Table 4.3 (p. 90). Indeed, in the frequency filter synthesis methodology discussed in the previous subsection, a prototype filter is first synthesised and then the four usual filters (low-pass, high-pass, band-pass, band-stop) are obtained using frequency transformations. Table 4.4 shows that there is an element transformation associated with these frequency transformations. Therefore, the circuit associated with the four usual version of the Butterworth and Chebyshev I filters can be straightly obtained.

TABLE 4.4: Building element transformations from a prototype filter to common versions

Prototype	Low-Pass $\omega_0$	High-Pass $\omega_0$	Band-Pass $\omega_0 = \sqrt{\omega_1 \omega_2}$	Band-Stop $\omega_0 = \sqrt{\omega_1 \omega_2}$

In Chapter 5, a *convex* synthesis method for solving the frequency filter synthesis

problem will be given. A remarkable aspect of the realisation conditions of Corollary 4.4 is that they can be added into this convex optimisation problem without changing the convexity property. The  $\{-1, 0, 1\}$ -dissipativeness condition of  $W$  is inherently convex, while the structural condition (4.45) can be straightly included in the synthesis method. Notice that the predefined value of  $R_l$  imposes a constraint on  $a_0$  that can also be straightly included. Therefore, this enables to provide a convex optimisation problem to solve the passive electronic filter design problem (Problem 3.1, p. 48) for the ladder topology and the family  $\mathcal{F}_{LC}$  (Table 3.1, p. 41). The resulting design problem can then be efficiently solved.

We finish this subsection by illustrating the link between the realisation conditions on  $W$  and the associated constraints on its coefficients.

**Link between realisation conditions and coefficients of  $W$**  Conditions of Corollary 4.4 can be mathematically interpreted as constraints on the coefficients of  $W$ . Let us illustrate this idea on a simple  $LC$  ladder filter example.

**Example 4.6.** Consider the two-element,  $LC$  ladder filter of Figure 4.1 (p. 55), with  $R_g = R_l = 1\Omega$ . The input impedance  $z_{in}$  can be directly computed using (4.44) as:

$$z_{in} = L_1 \cdot s + \frac{1}{C_2 \cdot s + \frac{1}{1}} = \frac{1 + L_1 \cdot s + L_1 C_2 \cdot s^2}{1 + C_2 \cdot s}$$

Then,  $s_{11}(s)$  and  $s_{21}(-s)s_{21}(s)$  are obtained using respectively Property 3.2 (p. 44) and Property 3.4 (p. 49). Finally  $s_{21}(s)$  is factorised, leading to:

$$s_{21}(s) = \frac{2}{2 + (C_1 + L_2) \cdot s + C_1 L_2 \cdot s^2}$$

Assume now that we are given the following second-order transfer function  $W(s)$ :

$$W(s) = \frac{b_0 + b_1 \cdot s + b_2 \cdot s^2}{a_0 + a_1 \cdot s + a_2 \cdot s^2}$$

For this example, the realisation conditions for  $W(s)$  to be the scattering parameter of a two-elements,  $LC$  ladder filter are then: there exist  $C_1 > 0$  and  $L_2 > 0$  such that

$$b_2 = b_1 = 0 \tag{4.46}$$

$$b_0 = a_0 = 2 \tag{4.47}$$

$$a_1 = C_1 + L_2 \tag{4.48}$$

$$a_2 = C_1 L_2 \tag{4.49}$$

Constraints (4.48) and (4.49) are equivalent to  $C_1$  and  $L_2$  being solutions of the quadratic equation  $x^2 - a_1 x + a_2 = 0$ . This equation admits two *real* solutions if and only if:

$$a_1^2 - 4a_2 \geq 0 \tag{4.50}$$

In addition, the positivity of the solutions requires:

$$a_1 > 0 \quad a_2 > 0 \tag{4.51}$$

The realisation conditions are then given by (4.46), (4.47), (4.50) and (4.51). Now consider the conditions of Corollary 4.4 applied to  $W(s)$ . Condition (4.45) is naturally equivalent to (4.46) and (4.47). Stability of  $W$  is equivalent to (4.51). Furthermore,

$\{-1, 0, 1\}$ -dissipativeness of  $W$  requires that:

$$\forall \omega \in \mathbb{R}, \quad |W(j\omega)|^2 = \frac{4}{4 + (a_1^2 - 4a_2) \cdot \omega^2 + a_2^2 \cdot \omega^4} \leq 1$$

This is equivalent to the condition

$$\forall \omega \in \mathbb{R}, \quad (a_1^2 - 4a_2 + a_2^2 \cdot \omega^2) \cdot \omega^2 \geq 0$$

which holds if and only if (4.50) is satisfied.

Therefore, realisations conditions of Corollary 4.4 are equivalent to constraints on the coefficient of  $W(s)$ . However, those of Corollary 4.4 are easier to satisfy, as they can be included in a convex optimisation problem, than complex non-linear conditions on the coefficients of  $W(s)$ .

#### 4.3.3.2 Realisation conditions of $T$ -ladder filters

Previous subsection provides compact realisation conditions for  $LC$  ladder filters. These conditions are rather simple and can be convexly introduced in the synthesis problem of a frequency filter  $W$ . Now, it is aimed to extend these realisation conditions to other ladder filters.

A first natural extension of  $LC$  ladder filters are the  $LC$ -resonator ladder filters (Table 3.2, p. 46), for which a two-elements illustration is provided on Figure 4.8. From above Table 4.4, they appear to be typically used for bandpass applications.

We start first by finding the realisation conditions on the coefficients of a transfer function  $W(s)$  for a simple example. As they appear to be complex and no equivalent convex formulation straightly appear on  $|W(j\omega)|^2$ , we define the class of  $T$ -ladder filters, which include  $LC$  ladder and *particular*  $LC$ -resonator ladder filters. We establish a convex formulation of the realisation constraints for  $T$ -ladder filters, similar in all aspects as those provided in Theorem 4.3 and Corollary 4.4.

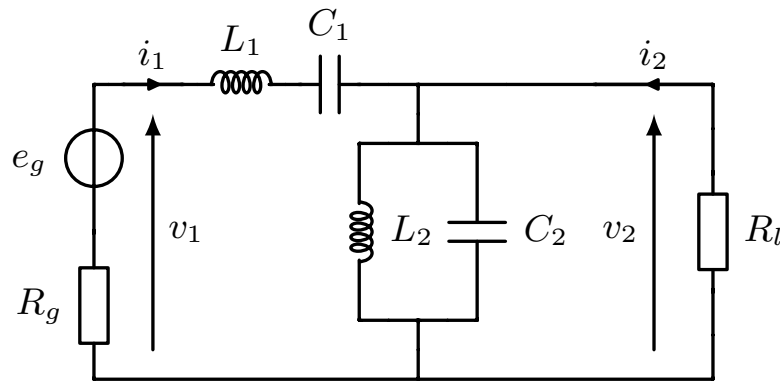


FIGURE 4.8: Two-elements example of a  $LC$ -resonator ladder filter

#### Example of realisation conditions of $LC$ -resonator ladder filters

**Example 4.7.** Consider the two-elements,  $LC$ -resonator ladder filter (Figure 4.8). Assume that  $R_g = R_l = 1\Omega$ . Using the same procedure than in Example 4.6, the scattering parameter  $s_{21}$  can be calculated as:

$$s_{21}(s) := \frac{2C_1L_2s^2}{1 + (C_1 + L_2)s + (C_1L_1 + 2C_1L_2 + C_2L_2)s^2 + C_1L_2(L_1 + C_2)s^3 + C_1C_2L_1L_2s^4}$$

Consider now the transfer function  $W(s)$ :

$$W(s) := \frac{b_2 \cdot s^2}{1 + a_1 \cdot s + a_2 \cdot s^2 + a_3 \cdot s^3 + a_4 \cdot s^4}$$

The realisation conditions consists in finding when there exist  $L_1 > 0$ ,  $C_1 > 0$ ,  $L_2 > 0$  and  $C_2 > 0$  such that:

$$\begin{aligned} b_2 &= 2C_1L_2 \\ a_1 &= C_1 + L_2 \\ a_2 &= L_1C_1 + 2C_1L_2 + L_2C_2 \\ a_3 &= C_1L_2(L_1 + C_2) \\ a_4 &= L_1C_1L_2C_2 \end{aligned}$$

Or equivalently:

$$b_2 = 2C_1L_2 \quad (4.52)$$

$$a_1 = C_1 + L_2 \quad (4.53)$$

$$a_2 = L_1C_1 + b_2 + L_2C_2 \quad (4.54)$$

$$a_3 = \frac{b_2}{2}(L_1 + C_2) \quad (4.55)$$

$$a_4 = \frac{b_2}{2}L_1C_2 \quad (4.56)$$

First, let us consider (4.52) and (4.53) independently from the other constraints. Similarly as in Example 4.6, there exists real, positive  $C_1 > 0$ ,  $L_2 > 0$  such that both constraints hold if and only if:

$$a_1 > 0 \quad b_2 > 0 \quad a_1^2 - 2b_2 \geq 0 \quad (4.57)$$

If this holds, one can choose  $C_1 = \frac{1}{2} \left( a_1 + \sqrt{a_1^2 - 2b_2} \right)$  and  $L_2 = \frac{1}{2} \left( a_1 - \sqrt{a_1^2 - 2b_2} \right)$ .

Likewise, (4.55) and (4.56) admit two real, positive solutions  $L_1 > 0$  and  $C_2 > 0$  if and only if:

$$a_3 > 0 \quad a_4b_2 > 0 \quad a_3^2 - 2a_4b_2 \geq 0 \quad (4.58)$$

And one can then choose  $L_1 = \frac{1}{b_2} \left( a_3 + \sqrt{a_3^2 - 2a_4b_2} \right)$  and  $C_2 = \frac{1}{b_2} \left( a_3 - \sqrt{a_3^2 - 2a_4b_2} \right)$ .

By injecting the values of  $L_1$ ,  $C_1$ ,  $L_2$ ,  $C_2$  in (4.54), one gets the final condition:

$$a_2 = b_2 + \frac{1}{2b_2} \left( a_1a_3 + \sqrt{(a_1^2 - 2b_2)(a_3^2 - 2a_4b_2)} \right) \quad (4.59)$$

Therefore (4.57), (4.58) and (4.59) provide the coefficients realisation conditions. Consider now the square modulus of the frequency response of  $W$ :

$$|W(j\omega)|^2 = \frac{b_2^2 \cdot \omega^4}{1 - (a_1^2 - 2a_2) \cdot \omega^2 + (a_2^2 + 2a_4 - 2a_1a_3) \cdot \omega^4 - (a_3^2 - 2a_2a_4) \cdot \omega^6 + a_4^2 \cdot \omega^8}$$

There is not an apparent connection between the realisation conditions and  $|W(j\omega)|^2$ , and an equivalent convex formulation, if it exists, seems complex to find.

### On the introduction of $T$ -ladder filter

The interpretation of the realisation conditions for  $LC$ -resonator ladder filter, in terms of constraint between coefficients of the scattering parameter  $s_{21}$ , does not allow to get a convex formulation of these conditions.

In the classic synthesis method, particular  $LC$ -resonator ladder filters are designed using element transformations from an  $LC$  ladder filter. This transformation, given in Table 4.4, especially implies that, for an  $n$ -elements  $LC$ -resonator ladder filter, the characteristic values of the elements are constrained as follows:

$$\frac{1}{L_1 C_1} = \frac{1}{L_2 C_2} = \dots = \frac{1}{L_n C_n} = \omega_0^2$$

where  $\omega_0^2 = \omega_1 \omega_2$ , where  $\omega_1$  and  $\omega_2$  are the limit frequencies of the pass-band  $[\omega_1; \omega_2]$ .

*Remark 4.4.* Recalling that the zeros of reflection of ladder filters match with the short-circuit and the open-circuit frequencies of respectively the serial and parallel arms, last assumption implies that  $|s_{11}(j\omega_0)| = 0$ , or equivalently  $|s_{21}(j\omega_0)| = 1$ .

Restated, this may be viewed (Figure 4.9) as a ladder filter for which each serial element  $e_i$  has an admittance  $y_i$  and each parallel element  $e_k$  an impedance  $z_k$  such that:

$$y_i(T(s)) = \frac{1}{L_i} \cdot T(s) \quad z_k(T(s)) = \frac{1}{C_k} \cdot T(s)$$

with

$$T(s) = \frac{s}{s^2 + \omega_0^2}$$

We define this type of filters as  $T$ -ladder filters.

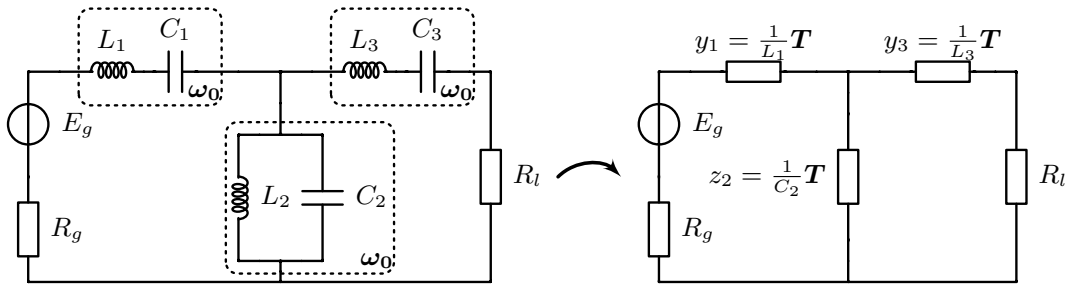


FIGURE 4.9: Conversion of a particular  $LC$ -resonator ladder filter into a  $T$ -ladder filter

**Definition 4.3.** Let  $T(s)$  be a transfer function. A  $T$ -ladder filter refers to a ladder filter where each serial element  $e_i$  has an admittance of the form  $y_i(T(s)) = \beta_i \cdot T(s)$  and each parallel element  $e_k$  an impedance  $z_k(T(s)) = \beta_k \cdot T(s)$ , with  $\beta_i > 0$  and  $\beta_k > 0$ .

Notice that a  $LC$  ladder filter is a  $T$ -ladder filters with  $T(s) = \frac{1}{s}$ . In the sequel, we generalise the realisation conditions of  $LC$  ladder filters to  $T$ -ladder filters, when  $T(s)$  satisfies a certain property. For sake of consistency, as Theorem 4.3 and Corollary 4.4 are stated with polynomial functions of the variable  $s$ , the realisation conditions will be expressed on  $T^{-1}(s)$  in the sequel. Before stating and proving such conditions, the following examples illustrates how the introduction of the transfer function  $T(s)$  simplifies the realisation conditions.

**Example 4.8** (Example 4.7 continued). Assume now that a real number  $\omega_0$  is given such that:

$$\frac{1}{L_1 C_1} = \frac{1}{L_2 C_2} = \omega_0^2$$

Then, the resulting filter is a  $T$ -ladder filter, with  $T(s) = \frac{s}{s^2 + \omega_0^2}$ . The calculation of  $s_{21}(T(s))$  leads then to:

$$s_{21}(T(s)) = \frac{2 \cdot (T(s))^2}{L_1 C_2 + (L_1 + C_2) \cdot T(s) + 2 \cdot (T(s))^2}$$

or equivalently in  $T^{-1}(s)$

$$s_{21}(T^{-1}(s)) = \frac{2}{2 + (L_1 + C_2) \cdot T^{-1}(s) + L_1 C_2 \cdot (T^{-1}(s))^2}$$

Consider the transfer function

$$W(T^{-1}(s)) = \frac{b_0 + b_1 \cdot T^{-1}(s) + b_2 \cdot (T^{-1}(s))^2}{a_0 + a_1 \cdot T^{-1}(s) + a_2 \cdot (T^{-1}(s))^2}$$

In addition to be simpler than in Example 4.7, the realisation conditions expressed on the coefficients of  $W(T^{-1}(s))$  are the same than those of Example 4.6. They have then a convex formulation.

#### Convex realisation conditions for $T$ -ladder filters

In the literature, usual proofs of Theorem 4.3 and Corollary 4.4 are based on the properties that  $s^* + s > 0$  when  $s \in \mathbb{C}^+$  and  $s^* + s = 0$  when  $s = j\omega$  [Bah84; NHDD96]. Therefore, it is proposed here to extend these proofs to any  $T^{-1}(s)$  such that  $T^{-1}(s)^* + T^{-1}(s) > 0$  when  $s \in \mathbb{C}^+$  and  $T^{-1}(s)^* + T^{-1}(s) = 0$  when  $s = j\omega$ .

#### Theorem 4.5.

Given a lossless  $\{0, 1, 0\}$ -dissipative transfer function  $T^{-1}(s)$  such that

$$\forall s \in \mathbb{C}^+, \quad T^{-1}(s)^* + T^{-1}(s) > 0 \quad (4.60)$$

a stable  $\{0, 1, 0\}$ -dissipative transfer function  $z_{in}(T^{-1}(s))$  written as:

$$z_{in}(T^{-1}(s)) := \frac{N(T^{-1}(s))}{D(T^{-1}(s))}$$

where  $N(T^{-1})$  and  $D(T^{-1})$  are real polynomials of  $T^{-1}$  with respective degrees  $n_N$  and  $n_D$ .

If  $z_{in}(T^{-1}(s))$  satisfies the following condition:

$$\Re \left( z_{in}(T^{-1}(j\omega)) \right) = \frac{K}{|D(T^{-1}(j\omega))|^2} \quad (4.61)$$

where  $K > 0$  is a positive constant,

Then  $z_{in}(T^{-1}(s))$  admits the following continued fraction expansion in  $T^{-1}(s)$ :

$$z_{in}(T^{-1}(s)) = \alpha_1 \cdot T^{-1}(s) + \frac{1}{\alpha_2 \cdot T^{-1}(s) + \frac{1}{\ddots \cdot \frac{1}{\alpha_n \cdot T^{-1}(s) + \frac{1}{R_l}}}} \quad (\text{or } R_l)$$

with  $\alpha_1 > 0, \alpha_2 > 0, \dots, \alpha_n > 0, n := \max(n_N, n_D)$  and  $R_l > 0$ .

As a result,  $z_{in}(T^{-1}(s))$  is the input impedance of a  $T$ -ladder filter terminated on a resistance  $R_l$ , given by  $R_l := \Re(z_{in}(0)) = \frac{K}{D(0)^2}$ .

The proof of Theorem 4.5 is based on the following lemma.

**Lemma 4.3.**

Let  $T^{-1}(s)$  be marginally stable, transfer function such that (4.60). Let  $z_{in}(T^{-1})$  be a rational function of  $T^{-1}$ . Assume that  $z_{in}(T^{-1}(s))$  is a stable,  $\{0, 1, 0\}$ -dissipative transfer function such that (4.61). Define  $T^{inv}$  as a complex variable with definition domain given by

$$\mathcal{D}_{T^{inv}} := \{T^{-1}(s) \mid s \in \mathbb{C}^+\}$$

Then, if it exists, a pole at infinity of  $z_{in}(T^{inv})$ , i.e. such that  $z_{in}(0) = 0$ , is simple and have real positive residue.

*Proof.* Consider a pole at infinity of order  $r$  of  $z_{in}(T^{inv})$ . In the neighbourhood of this pole, the rational function  $z_{in}(T^{inv})$  has a Laurent expansion of the form:

$$z_{in}(T^{inv}) = \sum_{k=1}^r \alpha_k (T^{inv})^k + \sum_{l=0}^{\infty} \frac{\beta_l}{(T^{inv})^l}$$

where the  $\alpha_k$  and  $\beta_k$  are called *residues*. The dominant term of this expansion is  $\alpha_r (T^{inv})^r$ . Using a polar form, write  $\alpha_r = |\alpha_r|e^{j\theta_r}$  and  $T^{inv} = |T^{inv}|e^{j\phi(T^{inv})}$ . As  $\Re(T^{inv}) > 0$ , by definition of  $\mathcal{D}_{T^{inv}}$  and (4.60), one gets  $\phi(T^{inv}) \in (-\frac{\pi}{2}; \frac{\pi}{2})$ . Then the real part of the dominant term of  $z_{in}(T^{inv})$  is given by:

$$\Re(\alpha_r \cdot (T^{inv})^r) = |\alpha_r| |T^{inv}|^r \cos(\theta_r + r\phi(T^{inv}))$$

Moreover, by (4.61) with  $K > 0$  and stability of  $z_{in}(T^{-1}(s))$ , the application of the *minimum real part* theorem (App. A, p. 195) leads to  $\Re(z_{in}(T^{inv}))$  being positive for all  $T^{inv}$  in  $\mathcal{D}_{T^{inv}}$ . As a result,  $\Re(\alpha_r \cdot (T^{inv})^r)$  is positive for all  $\phi(T^{inv}) \in (-\frac{\pi}{2}; \frac{\pi}{2})$ . This implies that  $\theta_r = 0$  and  $r = 1$ . Therefore, a pole at infinity is at most simple and  $\alpha_r > 0$ .  $\square$

Using this lemma, we can prove Theorem 4.5.

*Proof of Theorem 4.5.* Without loss of generality, assume that the degree  $n_N$  of the numerator  $N(T^{-1})$  is greater or equal to the degree  $n_D$  of the denominator  $D(T^{-1})$ :  $n_N \geq n_D$ . By Lemma 4.3, a pole at infinity of  $z_{in}(T^{inv})$ , if it exists, must be simple and then  $n_N$  and  $n_D$  may differs by at most 1:  $|n_N - n_D| \leq 1$ . Furthermore, by (4.61),  $N(T^{-1})$  and  $D(T^{-1})$  cannot have same degree. This leads to:

$$n_N = n_D + 1$$

Then, using the partial fractional expansion technique leads to:

$$z_{in}(T^{-1}(s)) = \alpha_1 \cdot T^{-1}(s) + z_{in_2}(T^{-1}(s))$$

with the residue  $\alpha_1$  positive:  $\alpha_1 > 0$ . Moreover, as  $T^{-1}(s)$  is lossless  $\{0, 1, 0\}$ -dissipative, one has

$$\forall \omega \in \mathbb{R}, \quad \Re e(z_{in}(T^{-1}(j\omega))) = \Re e(z_{in_2}(T^{-1}(j\omega)))$$

By Lemma 4.3,  $y_{in_2}(T^{inv}) = \frac{1}{z_{in_2}(T^{inv})}$  has also a pole at infinity and one can write:  $y_{in_2}(T^{-1}(s)) = \alpha_2 \cdot T^{-1}(s) + y_{in_3}(T^{-1}(s))$  with  $\alpha_2 > 0$ . The process is iterated until the value of the resistance  $R_l := \Re e(z_{in}(0)) = \frac{K}{D(0)^2}$  is obtained. This occurs after a number of infinity pole extractions equal to the degree in  $T(s)$  of the input impedance  $z_{in}(T^{-1}(s))$ . □

Similarly as in last subsection, the realisation conditions for the input impedance of Theorem 4.5 can be transformed into realisation conditions for the scattering parameter.

**Corollary 4.6.**

Given a source resistance  $R_g > 0$ ,

a lossless  $\{0, 1, 0\}$ -dissipative transfer function  $T^{-1}(s)$  such that (4.60),

a stable  $\{-1, 0, 1\}$ -dissipative transfer function  $W(T^{-1}(s))$ .

If  $W(T^{-1}(s))$  satisfies the following condition:

$$|W(T^{-1}(j\omega))|^2 = \frac{1}{|g(T^{-1}(j\omega))|^2}$$

where  $g(T^{-1})$  is a real polynomial of  $T^{-1}$  of degree  $n$ .

Then there exists a  $n$ -elements,  $T$ -ladder filter, operating between a resistive voltage source with source resistance  $R_g$  and a resistive load  $R_l$ , which admits  $W(T(s))$  as scattering parameter:

$$s_{21}(T^{-1}(s)) = W(T^{-1}(s))$$

Moreover, denoting  $g(T(s)) := \sum_{i=0}^n a_i \cdot T(s)^i$ , the resistive load  $R_l > 0$

admits two possible values, given by  $R_l := \frac{R_g}{a_0^2 \left(1 \pm \sqrt{a_0^2 - 1}\right)^2}$

*Proof.* As  $W(T^{-1}(s))$  is a stable  $\{-1, 0, 1\}$ -dissipative transfer function, it will be demonstrated in Chapter 5 (cf Subsection 5.4.3.1) that there exists a stable transfer function  $V(T^{-1}(s)) := \frac{h(T^{-1}(s))}{g(T^{-1}(s))}$  such that:

$$\forall \omega \in \mathbb{R}, \quad |V(T^{-1}(j\omega))|^2 = 1 - |W(T^{-1}(j\omega))|^2$$

Define the transfer function  $z_{in}(T^{-1}(s))$  as:

$$z_{in}(T(s)) := R_g \cdot \frac{1 - V(T(s))}{1 + V(T(s))}$$

Then

$$\forall \omega \in \mathbb{R}, \quad \Re e(z_{in}(T(j\omega))) = R_g \cdot \frac{1 - |V(T(j\omega))|^2}{|1 + V(T(j\omega))|^2} = R_g \cdot \frac{|W(T(j\omega))|^2}{|1 + V(T(j\omega))|^2}$$



Then,  $z_{in}(T^{-1}(s))$  is a  $\{0, 1, 0\}$ -dissipative transfer function. Moreover, last equality also implies that:

$$\forall \omega \in \mathbb{R}, \quad \Re(z_{in}(T(j\omega))) = R_g \cdot \frac{1}{|g(T^{-1}(j\omega)) + h(T^{-1}(j\omega))|^2}$$

Therefore, by Theorem 4.5,  $z_{in}(T(s))$  is the input impedance of an  $n$ -elements ladder filter terminated on a resistive  $R_l = \Re(z_{in}(0))$ , which admits two possible values by definition of  $V(T(s))$ . Finally, using Property 3.2 (p. 44) and Property 3.4 (p. 49), and by definition of  $V(T(s))$  and  $z_{in}(T(s))$ , there exists then a filter that admits  $W(T(s))$  as scattering parameter  $s_{21}(T(s)) = W(T(s))$ , with terminated resistances  $R_g$  and  $R_l$ .  $\square$

*Remark 4.5.* Analogically to Remark 4.3 (p.94), in order to get impedance matching, it is sufficient for  $s_{21}(T^{-1}(s))$  to satisfy the following condition:

$$s_{21}(T^{-1}(s)) = \frac{a_0}{a_0 + a_1 \cdot T^{-1}(s) + a_2 \cdot (T^{-1}(s))^2 + \dots + a_n (T^{-1}(s))^n}$$

Now that we have simple realisation conditions for  $T$ -ladder filters, it is aimed to synthesise a  $W(T^{-1}(s))$  which satisfies the spectral mask and these conditions. Using the usual filter design methods of Subsection 4.3.2, this is not possible. The design method given in Chapter 5 will be enough flexible to be able to include these constraints in the optimisation problem.

#### 4.3.3.3 Realisation conditions of $T_s, T_p$ -ladder filters

A *natural* extension of  $LC$ -resonator ladder filters are  $AW$ -resonator ladder filters (Table 3.2, p. 46), i.e. a ladder filter made of  $AW$  elements. A 3-elements examples is given in Figure 4.3 (p. 66). Recall that the impedance of an  $AW$  element is given by  $z(s) = \frac{1}{C_0 \cdot s} \cdot \frac{s^2 + \omega_r^2}{s^2 + \omega_a^2}$ , as introduced in Chapter 3.

In order to find realisation conditions for such filters, one may be tempted to find constraint on the coefficient of the scattering parameter  $s_{21}$ . However, as illustrated previously for  $LC$ -resonator ladder filters, these conditions are complex and a convex formulation may not be found. Because of the importance of  $AW$  filters in radiofrequency applications, there have been some attempts to extend the approximation methods of Subsection 4.3.2, such as pseudo-elliptic or generalized Chebyshev approximations [Shi06; GB16]. However, the realisation conditions can not all be taken into account in the frequency filter synthesis step, leading to add undesirable elements, in particular extra inductances, to ensure physical realisability of the elements.

Instead of these approaches, we go back to the idea which lead to the introduction of  $T$ -ladder filters. In the last subsection, the transfer function  $T(s)$  has been introduced to take explicitly into account a common simplification of the design problem for particular  $LC$ -resonator ladder filters. Using a similar idea, we aim to take into account simplification used in practice for the design of  $AW$  filters.

An interpretation of the introduction of  $T(s)$  is to fix a reflection zero at  $j\omega_0$ . Similarly, common practices for  $AW$ -resonator ladder filters are based on the localisation of reflection and transmission zeros. Indeed, the resonant and anti-resonant frequencies of the serial elements are typically constrained as follows:

$$\omega_{r_1} = \omega_{r_3} = \dots = \omega_0 \quad \omega_{a_1} = \omega_{a_3} = \dots = \omega_{a_s}$$

while the resonant and anti-resonant frequencies of the parallel elements satisfy:

$$\omega_{a_2} = \omega_{a_4} = \dots = \omega_0 \quad \omega_{r_2} = \omega_{r_4} = \dots = \omega_{r_p}$$

where  $\omega_{r_l} = \frac{1}{\sqrt{L_{m_l} C_{m_l}}}$  and  $\omega_{a_l} = \sqrt{\frac{C_{0_l} + C_{m_l}}{L_{m_l} C_{m_l} C_{0_l}}}$  for an element  $e_l$ . Again, the simplification again comes from the matching between zeros of transmission and reflection, and the elements behaviour as open- and short-circuits. Indeed, serial anti-resonant frequencies  $\omega_{a_1}, \omega_{a_3}, \dots$ , are all zeros of transmission. In addition, by constraining these frequencies to be all equal to  $\omega_{a_s}$ , one gets steeper slopes in the transition bandwidths, which is a crucial point for nowadays applications. Similarly, parallel resonant frequencies  $\omega_{r_2}, \omega_{r_4}, \dots$ , are constrained to be equal to  $\omega_{r_p}$ . Moreover, setting serial resonant  $\omega_{r_1}, \omega_{r_3}, \dots$ , and parallel anti-resonant frequencies  $\omega_{a_2}, \omega_{a_4}, \dots$  to be equal to the same value  $\omega_0$  is the only possibility to have a zero of reflection. As the transmission and reflection zeros are reasonably simple to choose in practice, this reduces the number of design variables, and thus the complexity, of the design problem.

In order to take explicitly, and synthetically, these constraints into account, we introduce two transfer functions  $T_s$  and  $T_p$  defined by:

$$T_s(s) = \frac{s^2 + \omega_0^2}{s(s^2 + \omega_{a_s}^2)} \quad T_p(s) = \frac{s^2 + \omega_{r_p}^2}{s(s^2 + \omega_0^2)}$$

As illustrated on Figure 4.10, serial element  $e_i$  and parallel element  $e_k$  have impedances

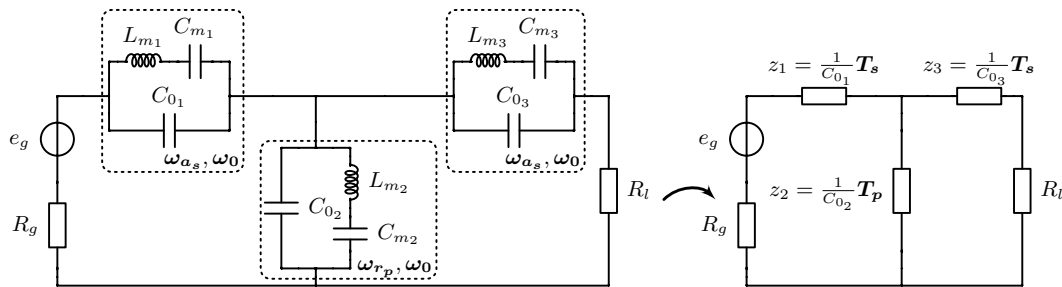


FIGURE 4.10: Conversion of a particular AW-resonator ladder filter into a  $T_s, T_p$ -ladder filter

given by:

$$z_i(s) = \frac{1}{C_{0_i}} \cdot T_s(s) \quad z_k(s) = \frac{1}{C_{0_k}} \cdot T_p(s)$$

We define this particular type of filters as  $T_s, T_p$ -ladder filters.

**Definition 4.4.** Let  $T_s$  and  $T_p$  be two transfer functions. A  $T_s, T_p$ -ladder filter refers to a ladder filter where each serial element  $e_i$  has an impedance  $z_i(T_s(s)) = \beta_i \cdot T_s(s)$  and each parallel element  $e_k$  an impedance  $z_k = \beta_k \cdot T_p(s)$ , with  $\beta_i > 0$  and  $\beta_k > 0$ .

A  $T$ -ladder filter is then a particular  $T_s, T_p$ -ladder filters with  $T_s(s) = T^{-1}(s)$  and  $T_p(s) = T(s)$ .

Unfortunately, the introduction of  $T_s$  and  $T_p$  is not sufficient to get a convex formulation of the realisation conditions. However, it considerably simplifies their expression, as illustrated in the next example.

**Example 4.9.** Consider a two-elements  $T_s, T_p$ -ladder filter (remove third arm of Figure 4.10). The associated scattering parameter  $s_{21}(T_s, T_p)$  is given by:

$$s_{21}(T_s(s), T_p(s)) := \frac{2\alpha_2 \cdot T_p(s)}{1 + \alpha_1 \cdot T_s(s) + 2\alpha_2 \cdot T_p(s) + \alpha_1\alpha_2 T_s(s)T_p(s)}$$

with  $\alpha_1 := \frac{1}{C_{01}}$  and  $\alpha_2 := \frac{1}{C_{02}}$ . Consider now the structured transfer function:

$$W(T_s(s), T_p(s)) := \frac{a_p \cdot T_p(s)}{1 + a_s \cdot T_s(s) + a_p \cdot T_p(s) + a_{sp} \cdot T_s(s)T_p(s)}$$

The realisation conditions are then: there exist  $\alpha_1 > 0$  and  $\alpha_2 > 0$  such that

$$\begin{aligned} a_s &:= \alpha_1 \\ a_p &:= 2\alpha_2 \\ a_{sp} &:= \alpha_1\alpha_2 \end{aligned}$$

This is equivalent to the following conditions:

$$a_s > 0 \quad a_p > 0 \quad a_{sp} = \frac{a_s a_p}{2}$$

Consider now the square modulus of the frequency response of  $W(T_s, T_p)$ :

$$|W(T_s(j\omega), T_p(j\omega))|^2 = \frac{a_p^2 \cdot |T_p(j\omega)|^2}{1 + 2(a_s a_p - a_{sp}) \cdot |T_s(j\omega)T_p(j\omega)| + a_s^2 \cdot |T_s(j\omega)|^2 + a_p^2 \cdot |T_p(j\omega)|^2 + a_{sp}^2 \cdot |T_s(j\omega)T_p(j\omega)|^2}$$

The constraint  $a_{sp} = \frac{a_s a_p}{2}$  prevents from having a suitable formulation of the realisation constraints that can be convexly added into the synthesis of  $|W(T_s(j\omega), T_p(j\omega))|^2$ . However, these constraints are less complex when expressed on the coefficients of a rational function of  $T_s$  and  $T_p$ , than if they were expressed on the coefficients of a rational function in  $s$ .

## 4.4 Summary and conclusion

In this chapter, we have revisited two approaches for the design of lossless passive electronic filters.

First approach (Section 4.2) consists in setting the elements in ready-to-implement form and to tune the characteristic values of the elements. Mathematically, this means to synthesise a structured state-space representation, leading to *non-convex* optimisation problem. Tackling directly the design problem with this approach leads to a highly *non-convex* optimisation problem. In order to reduce the *degree* of non-convexity of the optimisation problem, a mathematical representation belonging the Port-Hamiltonian Systems (PHS) framework, namely the PHS DAE representation, has been first introduced to model the behaviour of lossless passive electronic filters. The modelling procedure, endowed in the PHS framework, allows to move from a circuit diagram into such representation. This representation appears then linear into a certain diagonal matrix  $Q$ , which has the design variables as diagonal entries. In order to take advantage of this feature, an adapted synthesis method has then been developed. It provides a mean to provide a solution to the design problem by solving a particular non-convex problem, namely a BMI problem. Furthermore, by coupling this synthesis method with additional tools from our design framework, this

BMI problem is transformed into a very particular form, reducing significantly the complexity. A sequential algorithm based on LMI relaxations and requiring an initial point has then been provided, and illustrated on a simple example. This approach is doubly beneficial. First, it is very flexible and allows to tackle the design problem of any electronic filters in an unifying manner, while numerous practical constraints can be easily added. Second, it is the same approach that is used by designers in practice, i.e. finding an initial guess and then tuning characteristic values of the elements of a physically implementable filter. It can then be directly included in the design flow, and the experience of seasoned designers to find a suitable initial point can be used. The main drawback is the necessity of finding this suitable initial point. This initial point should be *close enough* to a solution in order to ensure the convergence of the algorithm. This is generally a truly challenging task.

Second approach (Section 4.3) consists in synthesising a stable transfer function such that it satisfies a spectral mask *and* realisation constraints. The transfer function synthesis is convex. Therefore, if the realisation constraints have a suitable formulation, the resulting optimisation problem is convex and can be efficiently solved. This is the main benefit of this approach. However, the realisation conditions do not generally have such a formulation. *LC* ladder filters are a well-known exception. Based in particular on the lossless  $\{0, 1, 0\}$ -dissipative property, we have extended their *convex* realisation conditions to a new family of lossless passive electronic filters, namely *T*-ladder filters. This family especially contains particular *LC*-resonator filters used in traditional design methods for bandpass applications. Nevertheless, while the realisation conditions are convex, this requires to synthesise a transfer function  $s_{21}(T(s))$  expressed as an LFT of a repeated transfer function  $T(s)$ , which is not possible with usual synthesis methods. Finally, using common practices of designers, we have illustrated how some AW-resonator ladder filters can be viewed as belonging to a particular family, namely  $T_s, T_p$ -ladder filters. This does not allow to have convex realisation conditions but significantly reduces their complexity, by decreasing the number of decision variables. Similarly, the synthesis of a transfer function  $s_{21}(T_s, T_p)$ , as an LFT in a repeated  $T_s$  and a repeated  $T_p$ , is required.

To sum up, while considerably reducing the complexity of the design problem, both approaches enable to efficiently solve it for very particular cases. They both reach a deadlock when tackling problems of practical use. Therefore, we consider a transitional problem in Chapter 5: the synthesis of a transfer function with an LFT representation such that it satisfies a spectral mask. While this may not directly solve the design problem, the aim is to bridge the gap between both approaches in order to tackle problems of practical interest (Chapter 6).



## Chapter 5

# Frequency LFT Filter Synthesis

### 5.1 Introduction

In Chapter 4, two approaches for the design of electronic filters are revisited under our design framework. Unfortunately, excepted in very particular cases, both revisited approaches fail to provide efficient design methods. This chapter is then concluded by exhibiting the interest of solving a transitional problem, namely the frequency LFT filter synthesis, which may be viewed as the extension of a standard problem, called the frequency filter synthesis.

Frequency filtering is one of the main topics of interest of System Theory, especially appearing in Signal Processing, Electronics, Control Theory. The role of frequency filters in communication systems have made their synthesis essential for a wide variety of applications. The generality of the filter synthesis problem makes it even appearing in problems where the final result is not a filter, such as  $H_\infty$ -weights computation [SP07; RSF03; Zar13].

Frequency filter synthesis is about synthesising a stable transfer function, or equivalently a state-space representation, such that the resulting frequency response satisfies some requirements on its modulus or its phase<sup>1</sup>. In the context of this work, only modulus constraints are considered. However, one may note that some particular phase properties may be obtained by imposing structural constraints on the coefficients of transfer function [PB87], or the state-space matrices. In particular, *minimum-phase* filters can be synthesised with the method developed in this chapter.

Originally, frequency filters were analog, as they were realised with inductances and capacitances. This especially led to the development of analytical synthesis methods, enabling fast synthesis, as those given in Chapter 4 (Subsection 4.3.2, p. 88). Digital filters interest appeared in the second half of the 20<sup>th</sup> century, with the invention of the transistor and the ever-growing computational power. There are two types of digital filters, FIR (Finite Impulse Response) and IIR (Infinite Impulse Response) filters. IIR filter synthesis methods are traditionally based on those of analog filters, by the use of approximations such as the bilinear transformation [PB87]. While FIR filters may be viewed as particular IIR filters, these methods do not allow to synthesise them. Instead, dedicated computer-aided methods have been developed, based on efficient algorithms. The maturation of convex optimisation in the 90s enabled to develop new flexible, and still efficient, methods for FIR filters synthesis [Dav10]. These methods were then extended at the beginning of the 21<sup>st</sup> century to optimally and efficiently synthesise analog and IIR filters [RSF03].

In parallel, the increase of complexity of modern systems has led to challenging engineering problems. When systems are interconnected identical subsystems,

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<sup>1</sup>Actually, it is known that the modulus and the phase of a causal and stable system are not independent. Refer to §6.4 of [ZDG96] and the references therein for further details.

a new paradigm has emerged known as the homogeneous multi-agent systems approach. Recently, this approach has been successful in diverse areas of applications such as formation flying [MV09], micro-Electronics [ZKSV13; KSCB16], biological networks [HTI14]. These systems have the common property to be described by an LFT of a repeated transfer function  $T(s)$ . Many useful results have then been extended from state-space systems, i.e systems where  $T(s) = \frac{1}{s}$ , to homogeneous LFT systems [ZKSV13; HTI14; HIT10]. However, there are still extended usual problems to solve and the *frequency LFT filter synthesis*, i.e the synthesis of a frequency filter with an LFT representation, is one of them.

The objective of this chapter is to propose a method to solve the synthesis problem of frequency LFT filters (Section 5.2). To achieve this, the convex synthesis method of traditional analog filters ( $T(s) = \frac{1}{s}$ ) is revisited under the LFT framework (Section 5.3). This method is then extended to LFT filters with a more general  $T(s)$  (Section 5.4). Using the dissipative characterisation of  $T(s)$ , we distinguish two cases. First case is when there exists a *lossless*  $\{x, y, z\}$ -dissipative characterisation of  $T(s)$ . The lossless feature enables to directly extend the usual synthesis approach. The second case treats with general dissipative  $T(s)$ . As the usual approach can not be directly extended, we propose a generalised approach able to solve the synthesis problem. Finally, the 2D LFT filters synthesis problem, i.e. a filter with an LFT representation in two repeated transfer functions  $T_s(s)$  and  $T_p(s)$ , is considered (Section 5.5). Several theoretical barriers prevent from generalising the usual synthesis approach. However, an LMI minimisation problem is still provided, able to find some solutions of the synthesis problem.

In order to get Electronics insight, one can interpret this scheme as follows. The usual case  $T(s) = \frac{1}{s}$  corresponds to the interconnection of inductances  $L$ , with admittance  $y_L(s) = \frac{1}{Ls}$ , and capacitances  $C$ , with impedance  $z_C(s) = \frac{1}{Cs}$ . The extension to any lossless dissipative  $T(s)$  then matches with a move to other elements, such as  $LC$ -resonator elements (Table 3.1, p. 41) where  $T(s) = \frac{s}{s^2 + \omega_0^2}$ . In Chapter 4, the elements were considered lossless passive. One may notice that this assumption is not required anymore as, for instance, the transfer function

$$T(s) = \frac{1}{Ls + R}$$

which may be interpreted as the admittance of a resistive inductance, is lossless  $\{-R, \frac{1}{2}, 0\}$ -dissipative. Furthermore, non-lossless dissipative  $T(s)$  matches with lossy AW elements for which the admittance is proportional to (cf mBVD model of Figure 3.2, p. 40):

$$T(s) = \frac{s}{1 + \frac{s}{\omega_a Q_0}} \cdot \frac{\omega_r^2 + \frac{\omega_r}{Q_m} s + s^2}{\omega_a^2 + \left(\frac{\omega_r}{Q_m} + \frac{\omega_a^2 - \omega_r^2}{\omega_a Q_0}\right) s + s^2}$$

Finally, the 2D LFT case concerns the synthesis of  $AW$ -resonator filters, viewed as the interconnection of two repeated, lossless  $\{0, 1, 0\}$ -dissipative transfer functions  $T_s(s)$  and  $T_p(s)$  (cf Subsection 4.3.3.3) given by:

$$T_s(s) = \frac{s^2 + \omega_{rs}^2}{s(s^2 + \omega_{as}^2)} \quad T_p(s) = \frac{s^2 + \omega_{rp}^2}{s(s^2 + \omega_{ap}^2)}$$

**Literature comments** Most of the literature of frequency filter synthesis focuses on the case where  $T(s) = \frac{1}{s}$ , or its discrete equivalent  $z^{-1}$ . For traditional methods of either analog or digital filters, the reader is invited to refer to the synthetic and easy

reading book of [PB87]. A summary of the convex methods introduced for FIR filter synthesis is provided in [Dav10]. In particular, references [DLS02; IH05] transform the problem into an equivalent LMI optimisation problem. In [RSF01; RSF03], the authors use the same underlying ideas, but with a different formulation, to optimally solve the synthesis problem of analog filters, and to some extent of IIR filters. Again an LMI feasibility problem is obtained. The book [Dum07] is also to notice for its comprehensiveness on efficient synthesis methods for digital filters, and especially methods concerning 2-D FIR filters.

As far as the authors know, the LFT frequency filter case with a *lossless*  $\{x, y, z\}$ -dissipative  $T(s)$  has not been yet considered in the literature. Nonetheless, the book [FCG79] provides significant insights on how extending usual LTI results to LTI systems interconnected with a *positive rational operator*, of which LFT systems of a lossless passive  $T(s)$  are a special case. Furthermore, the clarity of explanation to link passivity, spectral factorisation, and algebraic properties of the solution sets of the Algebraic Riccati Inequality (ARI) and the Algebraic Riccati Equation (ARE), is remarkable.

Finally, the LFT filter synthesis problem with a general  $\{x, y, z\}$ -dissipative  $T(s)$  has been tackled in [Zar13], for  $H_\infty$ -weights computation. Sufficient conditions are obtained under a LMI feasibility problem are obtained, which tend to provide satisfactory results when the mismatch between the  $\{x, y, z\}$ -dissipative characterisation and the graph of  $T(s)$  is not too important. However, this is not generally the case and one may reach a deadlock. In this work, we take over their approach and take explicitly into account this mismatch into the synthesis in order to be able to consider any  $\{x, y, z\}$ -dissipative  $T(s)$  [PKZS18].

## 5.2 Problem formulation for LFT filters in a repeated $T(s)$

Problem 5.1 gives the formulation of the synthesis problem of a filter, expressed as an LFT in a repeated dissipative  $T(s)$ , which is required to satisfy a spectral mask made of constant upper and lower bounds.

**Problem 5.1** (LFT Filter Synthesis).

- Given* a SISO,  $\{x, y, z\}$ -dissipative transfer function  $T(s)$ ,  
 $N_U$  upper bounds  $U_u$  associated with  $N_U$  real intervals  $\Omega_u^U \subseteq \mathbb{R}$ ,  
 $N_L$  lower bounds  $L_l$  associated with  $N_L$  real intervals  $\Omega_l^L \subseteq \mathbb{R}$ ,  
*Find* if there exists a stable LFT  $\mathbf{W}(T(s))$  in a repeated  $T(s)$  that satisfies the spectral mask  $\mathcal{SM}$ :

$$\forall u \in \{1, \dots, N_U\}, \quad \forall \omega \in \Omega_u^U, \quad |\mathbf{W}(T(j\omega))|^2 \leq U_u^2 \quad (5.1)$$

$$\forall l \in \{1, \dots, N_L\}, \quad \forall \omega \in \Omega_l^L, \quad |\mathbf{W}(T(j\omega))|^2 \geq L_l^2 \quad (5.2)$$

If so, *Compute* it.

Three main issues listed below appear in directly tackling Problem 5.1.

1. First, the decision variable  $\mathbf{W}(T(s))$  belongs to an infinite-dimensional space. In fact,  $\mathbf{W}(T(s))$  is rational function of  $T(s)$ , and the vector space of rational functions is of infinite dimension.
2. Second, the formulation of the constraints is non-convex in  $\mathbf{W}(T(j\omega))$ . As already mentioned in Subsection 4.2.3.1, upper bound constraints (5.1) and lower



bound constraints (5.2) can be geometrically interpreted as enforcing  $W(T(j\omega))$  to respectively belong to a disk and the complement of a disk in the complex plane. While the first set is convex, the second is not. The problem is then non-convex and there are no generic efficient resolution algorithms available.

3. Third, the optimisation problem is of infinite dimension. There is indeed an infinite number of inequalities to satisfy, as Inequalities (5.1) and (5.2) depend on the continuous variable  $\omega$  which belongs to an interval. These constraints can then not be directly verified in a finite time.

Despite these issues, a synthesis approach is developed in this chapter to efficiently solve Problem 5.1.

### 5.3 LFT filter synthesis with $T(s) = \frac{1}{s}$

In this section, the usual filter synthesis problem, i.e. Problem 5.1 with  $T(s) = \frac{1}{s}$ , is solved. First, a finite-dimensional convex formulation is provided (Subsection 5.3.1). This leads us to split the resolution in two steps: the magnitude synthesis (Subsection 5.3.2) and the spectral factorisation (Subsection 5.3.3). A synthesis procedure finally summarises step-by-step the approach (Subsection 5.3.4).

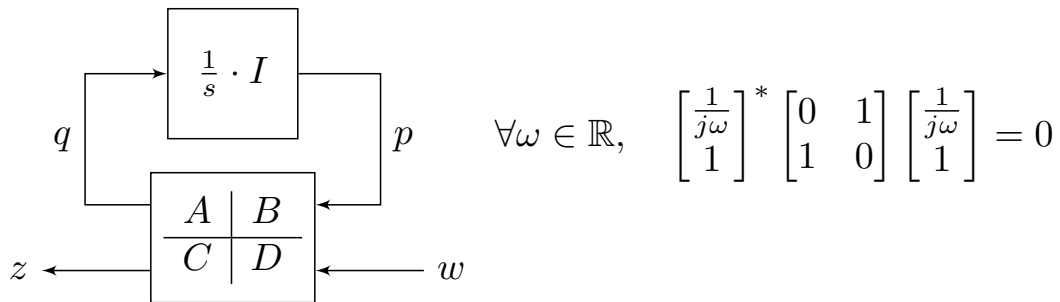


FIGURE 5.1: LFT system in the repeated lossless  $\{0, 1, 0\}$ -dissipative  $\frac{1}{s}$

#### 5.3.1 Finite-dimensional convex formulation

In the literature, systematic procedures to explicitly formulate the frequency filter synthesis as a finite convex optimisation problem were first developed for FIR digital filters [PB87; Dav10]. Transfer functions of FIR filters are *polynomial* functions of a complex variable  $z^{-1}$ . They belong then to a finite-dimensional space, and the first issue mentioned above does not occur. A possible option to solve the convexity issue is to constrain the phase of the filter to be *linear*. The synthesis problem leads then to a linear programming optimisation problem. For a filter with a *non-linear* phase, such as a *minimum-phase* filter, a suitable change of variable is required to get a convex formulation of the synthesis problem. In order to tackle the third issue, a method to obtain *finite* LMI formulation was first given in [WBV96; WBV99]. However, it is based on the discretisation of the frequency intervals  $\Omega_f^L, \Omega_u^U$ , leading to a large-scale optimisation problem. The computational time then tends to be very important. In [DLS02], an extension of the KYP lemma is introduced and used instead of the discretisation. The synthesis problem for FIR filters is *equivalently* transformed into a finite LMI optimisation problem. With similar ideas, the approach developed in [RSF01; RSF03] provides a systematic procedure to optimally and efficiently synthesise minimum-phase *analog* filters. In particular, a finite dimensional

parametrisation is given to tackle issue 1 of Problem 5.1. Likewise minimum-phase FIR filters, a change of variable is made, and an equivalent finite LMI optimisation problem is obtained using extensions of the KYP Lemma.

In the sequel, we revisit under our framework these approaches for the synthesis of analog filters, i.e. when  $T(s) = \frac{1}{s}$ .

### 5.3.1.1 Finite dimensional parametrisation

The decision variable  $\mathbf{W}$  of Problem 5.1, when  $T(s) = \frac{1}{s}$ , belongs to the vector space of rational functions of  $\frac{1}{s}$ , which is infinite dimensional. In order to get a problem where the decision variables belong to a finite dimension space, the order of  $W(s)$  is first set to a finite value  $n \in \mathbb{N}^*$ . Then  $W(s)$  is factorised as follows:

$$W(s) := \frac{W_N(s)}{W_D(s)}$$

There exist several equivalent types of factorisation. For instance,  $W_N(s)$  and  $W_D(s)$  may be chosen to belong to  $\mathcal{R}_n[s]$ , the space of real polynomials of the complex variable  $s$  of order  $n$ . If  $W_N$  and  $W_D$  do not have common zeros, the above factorisation may be interpreted as a *coprime factorisation* [ZDG96, Chap. 5]. Define a real polynomial  $d(s) \in \mathcal{R}_n[s]$ , which is additionally Hurwitz<sup>2</sup>. Another possibility is  $\mathcal{R}_n\mathcal{H}_\infty(s, d(s))$ , the vector space of proper, stable, real rational functions of  $s$  of order  $n$  having  $d(s)$  as denominator. The resulting factorisation may similarly be interpreted as a *rational coprime factorisation* [ZDG96, Chap. 5]. Both  $\mathcal{R}_n[s]$  and  $\mathcal{R}_n\mathcal{H}_\infty(s, d(s))$  are vector spaces of *finite* dimension. We choose  $\mathcal{R}_n\mathcal{H}_\infty(s, d(s))$  as it allows a more direct use of the LFT framework tools<sup>3</sup>, and more broadly the Control Theory tools. As a result,  $W_N(s)$  and  $W_D(s)$  may be decomposed as:

$$W_N(s) = \Gamma_N^T \mathcal{B}(s) \quad W_D(s) = \Gamma_D^T \mathcal{B}(s)$$

where  $\Gamma_N, \Gamma_D \in \mathbb{R}^{n+1}$  and  $\mathcal{B}(s)$  is the column concatenation of the elements of a given basis of  $\mathcal{R}_n\mathcal{H}_\infty(s, d(s))$ . Therefore, the new decision variables  $\mathbf{\Gamma}_N$  and  $\mathbf{\Gamma}_D$  parametrise all the  $W$  defined as an LFT in  $\frac{1}{s} \cdot I_n$ , and belong to a finite dimensional space.

**Example 5.1.** In order to illustrate last parametrisation, assume that  $n = 2$ . Define a Hurwitz real polynomial  $d(s) \in \mathcal{R}_n[s]$ . Set for example  $d(s) := (s + 1)^2$ . A possible choice of a basis  $\mathcal{B}(s)$  of  $\mathcal{R}_n\mathcal{H}_\infty(s, d(s))$  is then:

$$\mathcal{B}(s) = \frac{1}{d(s)} \begin{bmatrix} 1 \\ s \\ s^2 \end{bmatrix}$$

Then any transfer function  $W(s)$  of order  $n$  can be decomposed as:

$$W(s) = \frac{W_N(s)}{W_D(s)} = \frac{\Gamma_N^T \mathcal{B}(s)}{\Gamma_D^T \mathcal{B}(s)} = \frac{\gamma_{N_0} \cdot 1 + \gamma_{N_1} \cdot s + \gamma_{N_2} \cdot s^2}{\gamma_{D_0} \cdot 1 + \gamma_{D_1} \cdot s + \gamma_{D_2} \cdot s^2}$$

where  $\mathbf{\Gamma}_N := [\gamma_{N_0} \quad \gamma_{N_1} \quad \gamma_{N_2}]^T$  and  $\mathbf{\Gamma}_D := [\gamma_{D_0} \quad \gamma_{D_1} \quad \gamma_{D_2}]^T$ .

<sup>2</sup>A Hurwitz polynomial is a polynomial whose roots are located in the left-half plane of the complex plane.

<sup>3</sup>One may recall, as viewed in Chapter 2, that an LFT in  $(\frac{1}{s} \cdot I_n)$  is a proper, rational function of  $s$  of order  $n$ . And *vice versa*.

### 5.3.1.2 Convex formulation

Last parametrisation leads to search  $\mathbf{\Gamma}_N, \mathbf{\Gamma}_D \in \mathbb{R}^{n+1}$  such that constraints (5.1) and (5.2) are satisfied. For instance, (5.1) for a particular  $u$  becomes:

$$\forall \omega \in \Omega_u^U, \quad \mathcal{B}(j\omega)^* \mathbf{\Gamma}_N \mathbf{\Gamma}_N^T \mathcal{B}(j\omega) \leq U_u^2 \left( \mathcal{B}(j\omega)^* \mathbf{\Gamma}_D \mathbf{\Gamma}_D^T \mathcal{B}(j\omega) \right)$$

This constraint is not convex into  $\mathbf{\Gamma}_D$ . Similarly, constraints (5.2) lead to non-convex constraints into  $\mathbf{\Gamma}_N$ . In order to obtain a convex formulation, the following change of variables is achieved:

$$X_N := \mathbf{\Gamma}_N \mathbf{\Gamma}_N^T \quad X_D := \mathbf{\Gamma}_D \mathbf{\Gamma}_D^T \quad (5.3)$$

with  $X_N = X_N^T, X_D = X_D^T \in \mathbb{R}^{(n+1) \times (n+1)}$ . The above constraint becomes then:

$$\forall \omega \in \Omega_u^U, \quad \mathcal{B}(j\omega)^* \mathbf{X}_N \mathcal{B}(j\omega) \leq U_u^2 \left( \mathcal{B}(j\omega)^* \mathbf{X}_D \mathcal{B}(j\omega) \right)$$

This constraint is linear in the decision variables  $\mathbf{X}_N$  and  $\mathbf{X}_D$  and is then convex. Likewise, constraints (5.1) and (5.2) lead to convex constraints into  $\mathbf{X}_N$  and  $\mathbf{X}_D$ .

The change of variables of (5.3) leads then to a convex problem. Now, the issue is to make the reverse change of variables, i.e. to move for instance from  $X_N$  to  $\mathbf{\Gamma}_N$ . This is possible if  $\text{rank}(X_N) = 1$ . As discussed in Subsection 4.2.4 (p. 81), this is a very hard constraint to satisfy and can not be added into the convex problem.

To solve this issue, one can note that if a solution  $X_N$  of the last convex problem is given, a matrix  $\tilde{X}_N$  can be computed such that  $X_N + \tilde{X}_N$  is also a solution of this problem. The aim is then to find an appropriate  $\tilde{X}_N$  which degenerates the rank of  $X_N + \tilde{X}_N$ . That is to say to find  $\tilde{X}_N$  such that

$$\begin{aligned} \forall \omega \in \mathbb{R}, \quad \mathcal{B}(j\omega)^* \left( X_N + \tilde{X}_N \right) \mathcal{B}(j\omega) &= \mathcal{B}(j\omega)^* X_N \mathcal{B}(j\omega) \\ \text{rank}(X_N + \tilde{X}_N) &= 1 \end{aligned}$$

First constraint is easy to satisfy as there is potentially an infinite number of such matrix  $\tilde{X}_N$ , as illustrated in the next example.

**Example 5.2.** Assume that  $n = 2$  and  $\mathcal{B}(s)$  is defined as in Example 5.1. Assume that a symmetric matrix  $X = X^T \in \mathbb{R}^{3 \times 3}$  is given such that:

$$\begin{aligned} \mathcal{B}(j\omega)^* X \mathcal{B}(j\omega) &:= \frac{1}{d(j\omega)^*} \begin{bmatrix} 1 \\ j\omega \\ (j\omega)^2 \end{bmatrix}^* \begin{bmatrix} X_{11} & 0 & X_{13} \\ 0 & X_{22} & 0 \\ X_{13} & 0 & X_{33} \end{bmatrix} \begin{bmatrix} 1 \\ j\omega \\ (j\omega)^2 \end{bmatrix} \frac{1}{d(j\omega)} \\ &= \frac{X_{11} + (X_{22} - 2X_{13})\omega^2 + X_{33}\omega^4}{|d(j\omega)|^2} \end{aligned}$$

Then, for example, any matrix  $\tilde{X} := \begin{bmatrix} 0 & \tilde{X}_{12} & 0 \\ \tilde{X}_{12} & 0 & \tilde{X}_{23} \\ 0 & \tilde{X}_{23} & 0 \end{bmatrix}$  provides:

$$\begin{aligned} \mathcal{B}(j\omega)^* (X + \tilde{X}) \mathcal{B}(j\omega) &:= \frac{1}{d(j\omega)^*} \begin{bmatrix} 1 \\ j\omega \\ (j\omega)^2 \end{bmatrix}^* \begin{bmatrix} X_{11} & \tilde{X}_{12} & X_{13} \\ \tilde{X}_{12} & X_{22} & \tilde{X}_{23} \\ X_{13} & \tilde{X}_{23} & X_{33} \end{bmatrix} \begin{bmatrix} 1 \\ j\omega \\ (j\omega)^2 \end{bmatrix} \frac{1}{d(j\omega)} \\ &= \frac{X_{11} + (X_{22} - 2X_{13})\omega^2 + X_{33}\omega^4}{|d(j\omega)|^2} \end{aligned}$$

The main difficulty is then to find such a  $\tilde{X}_N$  such that  $\text{rank}(X_N + \tilde{X}_N) = 1$  or, similarly, such that:

$$X_N + \tilde{X}_N = \Gamma_N \Gamma_N^T$$

This implies the necessary condition  $X_N + \tilde{X}_N \geq 0$ . However, it is clearly not a sufficient condition to ensure the existence of such a  $\tilde{X}_N$ . In the literature<sup>4</sup>, it is known that a necessary and sufficient condition is the positivity constraint:

$$\forall \omega \in \mathbb{R}, \quad \mathcal{B}(j\omega)^* X_N \mathcal{B}(j\omega) \geq 0 \quad (5.4)$$

To understand last constraint, let us interpret the change of variables of (5.3) as moving from the LFT  $W_N(s)$  to the squared magnitude  $M_N(j\omega)$ . Indeed, define

$$M_N(j\omega) := \mathcal{B}(j\omega)^* X_N \mathcal{B}(j\omega)$$

By (5.3), it comes that:  $M_N(j\omega) = \mathcal{B}(j\omega)^* (\Gamma_N \Gamma_N^T) \mathcal{B}(j\omega) = |W_N(j\omega)|^2$ . The squared magnitude  $M_N(j\omega)$  needs then to satisfy the positivity condition. Reversely, the change of variables is well-posed if there exists a stable  $W_N(s)$  such that:

$$\forall \omega \in \mathbb{R}, \quad |W_N(j\omega)|^2 = M_N(j\omega)$$

The positivity constraint appears then to be necessary and sufficient for the existence of such  $W_N(s)$ , i.e. for the change of variables of (5.3) to be reversible.

The usual key approach is made of two steps. First, the squared magnitudes  $M_N(j\omega)$  and  $M_D(j\omega)$  are synthesised, i.e.  $X_N$  and  $X_D$  are computed, such that they satisfy the previous positivity condition and

$$M(j\omega) := \frac{M_N(j\omega)}{M_D(j\omega)}$$

satisfies the spectral mask  $\mathcal{SM}$ . Second,  $W_N$  and  $W_D$  are factorised, i.e.  $\Gamma_N$  and  $\Gamma_D$  are computed via the search of appropriate  $\tilde{X}_N$  and  $\tilde{X}_D$ , such that

$$|W_N(j\omega)|^2 = M_N(j\omega) \quad |W_D(j\omega)|^2 = M_D(j\omega)$$

and  $W(s) = \frac{W_N(s)}{W_D(s)}$  is stable. These two steps will respectively be referred to as *Magnitude Synthesis* and *Spectral Factorisation* in the sequel.

**Magnitude synthesis** Last considerations lead to solve the following magnitude synthesis problem

<sup>4</sup>cf Subsection 5.3.3 for more details.

**Problem 5.2** (Magnitude Synthesis).

- Given* a positive integer  $n \in \mathbb{N}^*$ ,  
a basis  $\mathcal{B}(s)$  of  $\mathcal{R}_n \mathcal{H}_\infty[s, d(s)]$ ,  
 $N_U$  upper bounds  $U_u$  associated with  $N_U$  real intervals  $\Omega_u^U \subseteq \mathbb{R}$ ,  
 $N_L$  lower bounds  $L_l$  associated with  $N_L$  real intervals  $\Omega_l^L \subseteq \mathbb{R}$ ,  
*Find* if there exist real symmetric matrices  $\mathbf{X}_N = (\mathbf{X}_N)^T \in \mathbb{R}^{(n+1) \times (n+1)}$  and  
 $\mathbf{X}_D = (\mathbf{X}_D)^T \in \mathbb{R}^{(n+1) \times (n+1)}$  such that:

$$\begin{aligned} \forall \omega \in \mathbb{R}, \quad & \mathcal{B}(j\omega)^* \mathbf{X}_N \mathcal{B}(j\omega) \geq 0 \\ \forall \omega \in \mathbb{R}, \quad & \mathcal{B}(j\omega)^* \mathbf{X}_D \mathcal{B}(j\omega) > 0 \\ \forall u \in \{1, \dots, N_U\}, \quad \forall \omega \in \Omega_u^U, \quad & \frac{\mathcal{B}(j\omega)^* \mathbf{X}_N \mathcal{B}(j\omega)}{\mathcal{B}(j\omega)^* \mathbf{X}_D \mathcal{B}(j\omega)} \leq U_u^2 \\ \forall l \in \{1, \dots, N_L\}, \quad \forall \omega \in \Omega_l^L, \quad & \frac{\mathcal{B}(j\omega)^* \mathbf{X}_N \mathcal{B}(j\omega)}{\mathcal{B}(j\omega)^* \mathbf{X}_D \mathcal{B}(j\omega)} \geq L_l^2 \end{aligned}$$

If so, *Compute* them.

Problem 5.2 has a convex formulation and the decision variables  $\mathbf{X}_N$  and  $\mathbf{X}_D$  belong to a finite-dimensional space. However, there is still an infinite number of constraints. Applying the KYP Lemma and its extensions (Section 2.6), an equivalent finite LMI feasibility problem is obtained.

**Spectral factorisation** Once the magnitudes  $M_N(j\omega)$  and  $M_D(j\omega)$  have been synthesised, the reverse change of variable, i.e. the computation of the stable spectral factors  $W_N(s)$  and  $W_D(s)$  such that:

$$\forall \omega \in \mathbb{R}, \quad |W_N(j\omega)|^2 = M_N(j\omega) \quad |W_D(j\omega)|^2 = M_D(j\omega)$$

needs to be achieved. This problem is denoted as **spectral factorisation**. For sake of generality, and as there is not significant change compared to the SISO case, it is solved in the MIMO case.

**Problem 5.3** (Spectral Factorisation).

- Given* positive integers  $n, n_w, n_z \in \mathbb{N}^*$ ,  
two symmetric matrices  $X_N = X_N^T \in \mathbb{R}^{n_z \times n_z}$  and  $X_D = X_D^T \in \mathbb{R}^{n_z \times n_z}$ ,  
a stable LFT  $\mathcal{B}(s)$  in  $\frac{1}{s}$  such as:

$$\mathcal{B}(s) = \left( \frac{1}{s} \cdot I_n \right) \star \left[ \begin{array}{c|c} A & B \\ \hline C & D \end{array} \right]$$

with  $A \in \mathbb{R}^{n \times n}$ ,  $B \in \mathbb{R}^{n \times n_w}$ ,  $C \in \mathbb{R}^{n_z \times n}$ ,  $D \in \mathbb{R}^{n_z \times n_w}$ .

- Find* if there exist two stable LFT  $\mathbf{W}_N$  and  $\mathbf{W}_D$  in  $\left( \frac{1}{s} \cdot I_n \right)$  such that:

$$\begin{aligned} \forall \omega \in \mathbb{R}, \quad & \mathcal{B}(j\omega)^* X_N \mathcal{B}(j\omega) = \mathbf{W}_N(j\omega)^* \mathbf{W}_N(j\omega) \\ \forall \omega \in \mathbb{R}, \quad & \mathcal{B}(j\omega)^* X_D \mathcal{B}(j\omega) = \mathbf{W}_D(j\omega)^* \mathbf{W}_D(j\omega) \end{aligned}$$

where  $\mathbf{W}_N$  and  $\mathbf{W}_D$  are defined by:

$$\mathbf{W}_N(s) := \left(\frac{1}{s} \cdot I_n\right) \star \left[ \begin{array}{c|c} \mathbf{A}_N & \mathbf{B}_N \\ \hline \mathbf{C}_N & \mathbf{D}_N \end{array} \right]$$

$$\mathbf{W}_D(s) := \left(\frac{1}{s} \cdot I_n\right) \star \left[ \begin{array}{c|c} \mathbf{A}_D & \mathbf{B}_D \\ \hline \mathbf{C}_D & \mathbf{D}_D \end{array} \right]$$

with  $\mathbf{A}_N, \mathbf{A}_D \in \mathbb{R}^{n \times n}$ ,  $\mathbf{B}_N, \mathbf{B}_D \in \mathbb{R}^{n \times n_w}$ ,  $\mathbf{C}_N, \mathbf{C}_D \in \mathbb{R}^{n_w \times n}$ ,  $\mathbf{D}_N, \mathbf{D}_D \in \mathbb{R}^{n_w \times n_w}$ .  
If so, Compute them.

*Remark 5.1.* It is required that  $\mathbf{C}_N, \mathbf{C}_D \in \mathbb{R}^{k \times n}$  and  $\mathbf{D}_N, \mathbf{D}_D \in \mathbb{R}^{k \times n_w}$  with  $k = n_w$  in order to have  $W_N$  and  $W_D$  of minimal size, as will be shown in Subsection 5.3.3.

One may analytically compute such spectral factors as this comes to sort zeros and poles, depending on whether they belong to the left-half plane or the right-half plane. In practice, this is not convenient and one may prefer using a numerical method based on the resolution of a certain quadratic algebraic equality [TW91] for which there exist efficient solvers.

### 5.3.2 Magnitude synthesis

The following theorem gives an equivalent formulation of the magnitude synthesis problem (Problem 5.2). This formulation is a feasibility optimisation problem under LMI constraints. The magnitude synthesis problem is then *optimally* and *efficiently* solvable. By optimally, it is meant that, given  $n$ , if there exists a squared magnitude of order  $2n$  which satisfies the spectral mask  $\mathcal{SM}$ , we will find it by solving the equivalent LMI feasibility problem. Note that this was not true for approximation methods of Subsection 4.3.2 (p. 88).

**Theorem 5.1.** Consider  $\mathcal{B}(s)$  a stable LFT in  $\frac{1}{s} \cdot I_n$  defined by:

$$\mathcal{B}(s) := \left(\frac{1}{s} \cdot I_n\right) \star \left[ \begin{array}{c|c} A & B \\ \hline C & D \end{array} \right]$$

with  $A \in \mathbb{R}^{n \times n}$ ,  $B \in \mathbb{R}^{n \times 1}$ ,  $C \in \mathbb{R}^{(n+1) \times n}$  and  $D \in \mathbb{R}^{(n+1) \times 1}$ . Assume that  $(A, B)$  is controllable. Then problem (ii) admits a solution if and only if the optimisation problem defined in (i) is feasible.

- (i)  $\exists \mathbf{X}_N = (\mathbf{X}_N)^T \in \mathbb{R}^{(n+1) \times (n+1)}$ ,  $\exists \mathbf{X}_D = (\mathbf{X}_D)^T \in \mathbb{R}^{(n+1) \times (n+1)}$ ,  
 $\exists \mathbf{P}_N = (\mathbf{P}_N)^T \in \mathbb{R}^{n \times n}$ ,  $\exists \mathbf{P}_D = (\mathbf{P}_D)^T \in \mathbb{R}^{n \times n}$ ,  
 $\forall u \in \{1, \dots, N_U\}$ ,  $\exists \mathbf{D}_u = (\mathbf{D}_u)^T \in \mathbb{R}^{n \times n}$ ,  $\exists \mathbf{P}_u = (\mathbf{P}_u)^T \in \mathbb{R}^{n \times n}$ ,  
 $\forall l \in \{1, \dots, N_L\}$ ,  $\exists \mathbf{D}_l = (\mathbf{D}_l)^T \in \mathbb{R}^{n \times n}$ ,  $\exists \mathbf{P}_l = (\mathbf{P}_l)^T \in \mathbb{R}^{n \times n}$ ,

$$\begin{bmatrix} A & B \\ I & 0 \end{bmatrix}^T \begin{bmatrix} 0 & -\mathbf{P}_N \\ -\mathbf{P}_N & 0 \end{bmatrix} \begin{bmatrix} A & B \\ I & 0 \end{bmatrix} + \begin{bmatrix} C^T \\ D^T \end{bmatrix} \mathbf{X}_N \begin{bmatrix} C & D \end{bmatrix} \geq 0 \quad (5.5)$$

$$\begin{bmatrix} A & B \\ I & 0 \end{bmatrix}^T \begin{bmatrix} 0 & -\mathbf{P}_D \\ -\mathbf{P}_D & 0 \end{bmatrix} \begin{bmatrix} A & B \\ I & 0 \end{bmatrix} + \begin{bmatrix} C^T \\ D^T \end{bmatrix} \mathbf{X}_D \begin{bmatrix} C & D \end{bmatrix} > 0 \quad (5.6)$$

$$\forall u \in \{1, \dots, N_U\},$$

$$\mathbf{D}_u > 0$$

$$\begin{bmatrix} A & B \\ I & 0 \end{bmatrix}^T \begin{bmatrix} -z_{\Omega_u^U} \mathbf{D}_u & -y_{\Omega_u^U}^* \mathbf{D}_u - \mathbf{P}_u \\ -y_{\Omega_u^U} \mathbf{D}_u - \mathbf{P}_u & -x_{\Omega_u^U} \mathbf{D}_u \end{bmatrix} \begin{bmatrix} A & B \\ I & 0 \end{bmatrix} + \begin{bmatrix} C^T \\ D^T \end{bmatrix} \left( U_u^2 \mathbf{X}_D - \mathbf{X}_N \right) \begin{bmatrix} C & D \end{bmatrix} \geq 0$$

$$\forall l \in \{1, \dots, N_L\}, \quad \mathbf{D}_l > 0$$

$$\begin{bmatrix} A & B \\ I & 0 \end{bmatrix}^T \begin{bmatrix} -z_{\Omega_l^L} \mathbf{D}_l & -y_{\Omega_l^L}^* \mathbf{D}_l - \mathbf{P}_l \\ -y_{\Omega_l^L}^* \mathbf{D}_l - \mathbf{P}_l & -x_{\Omega_l^L} \mathbf{D}_l \end{bmatrix} \begin{bmatrix} A & B \\ I & 0 \end{bmatrix} + \begin{bmatrix} C^T \\ D^T \end{bmatrix} (\mathbf{X}_N - L_l^2 \mathbf{X}_D) \begin{bmatrix} C & D \end{bmatrix} \geq 0$$

where  $x_\Omega, y_\Omega, z_\Omega$  depend on  $\Omega$  and are defined as in Subsection 2.6 (p. 27).

(ii) There exists a solution to Problem 5.2:

$$\exists \mathbf{X}_N = (\mathbf{X}_N)^T \in \mathbb{R}^{(n+1) \times (n+1)}, \quad \exists \mathbf{X}_D = (\mathbf{X}_D)^T \in \mathbb{R}^{(n+1) \times (n+1)},$$

$$\forall \omega \in \mathbb{R}, \quad \mathcal{B}(j\omega)^* \mathbf{X}_N \mathcal{B}(j\omega) \geq 0 \quad (5.7)$$

$$\forall \omega \in \mathbb{R}, \quad \mathcal{B}(j\omega)^* \mathbf{X}_D \mathcal{B}(j\omega) > 0 \quad (5.8)$$

$$\forall u \in \{1, \dots, N_U\}, \quad \forall \omega \in \Omega_u^U, \quad \frac{\mathcal{B}(j\omega)^* \mathbf{X}_N \mathcal{B}(j\omega)}{\mathcal{B}(j\omega)^* \mathbf{X}_D \mathcal{B}(j\omega)} \leq U_u^2 \quad (5.9)$$

$$\forall l \in \{1, \dots, N_L\}, \quad \forall \omega \in \Omega_l^L, \quad \frac{\mathcal{B}(j\omega)^* \mathbf{X}_N \mathcal{B}(j\omega)}{\mathcal{B}(j\omega)^* \mathbf{X}_D \mathcal{B}(j\omega)} \geq L_l^2 \quad (5.10)$$

*Proof.* Applying the KYP lemma (Theorem 2.3, p. 28), (5.7) and (5.8) are respectively equivalent to (5.5) and (5.6). Moreover, (5.9) and (5.10) are equivalent to:

$$\forall u \in \{1, \dots, N_U\}, \quad \forall \omega \in \Omega_u^U, \quad \mathcal{B}(j\omega)^* (U_u^2 \mathbf{X}_D - \mathbf{X}_N) \mathcal{B}(j\omega) \geq 0$$

$$\forall l \in \{1, \dots, N_L\}, \quad \forall \omega \in \Omega_l^L, \quad \mathcal{B}(j\omega)^* (\mathbf{X}_N - L_l^2 \mathbf{X}_D) \mathcal{B}(j\omega) \geq 0$$

By the frequency interval version of the KYP Lemma (Theorem 2.4, p. 30), these constraints are equivalent to the remaining LMIs of optimisation problem (i).  $\square$

Non-negativity constraint (5.7) and positivity constraint (5.8) are necessary for  $M_N(j\omega) = \mathcal{B}(j\omega)^* \mathbf{X}_N \mathcal{B}(j\omega)$  and  $M_D(j\omega) = \mathcal{B}(j\omega)^* \mathbf{X}_D \mathcal{B}(j\omega)$  to be squared magnitudes. In the next subsection, it is shown that their LMI counterparts (5.5) and (5.6) always permit to compute spectral factors  $W_N$  and  $W_D$ . These conditions are therefore also sufficient. Moreover, the positivity constraint (5.8) will ensure the stability of  $W(s) = \frac{W_N(s)}{W_D(s)}$ .

### 5.3.3 Spectral factorisation

Theorem 5.1 gives a way to compute a real-valued rational function  $M(j\omega)$  which is positive and satisfies a given spectral mask  $\mathcal{SM}$ . The problem is now to find a stable transfer function  $H$  such that:

$$\forall \omega \in \mathbb{R}, \quad H(j\omega)^* H(j\omega) = M(j\omega) \quad (5.11)$$

This problem is called *spectral factorisation*. There was historically a lot of research into this problem, as it is linked with many applications in linear Control Theory and Signal Processing such as time series prediction [Wie49], stochastic realisation [And69; FCG79], network synthesis [AV73].

Generally, the spectral factorisation problem arises in a MIMO form: to compute, if it exists, a stable, real-rational  $n_w \times n_w$  matrix  $H$  such that (5.11) holds, with  $M(j\omega)$  a real rational  $n_w \times n_w$  matrix. Any such  $H$  is called a *spectral factor*. There may exist many different spectral factors of size  $k \times n_w$ . Among them, only those with *minimal*  $k$ , i.e.  $k = n_w$ , are of our interest. We define then a *minimal* spectral factor as a spectral factor with  $k = n_w$ . Two questions arise then:

Q1. Does there always exist a stable, minimal spectral factor?

Q2. If so, how to compute one?

From (5.11), a necessary condition for the existence of a stable, minimal spectral factor  $H$  is that  $M(j\omega)$  is positive semi-definite for all  $\omega \in \mathbb{R}$ . It appears that this condition is also sufficient [You61; FCG79; ZDG96, chap.13]. Matter of Q1 is then resolved. An obvious answer of Q2 is to analytically calculate a solution. This is typically achieved for analog frequency filter synthesis, where the squared magnitude is of a specific form  $M(j\omega)$  [PB87, chap.7]. However, it generally results to a lengthy and laborious work. Instead, some efficient computational methods have been developed. Most popular methods do not tackle the problem in the transfer matrix form, but using a minimal state-space realisation. The predominant method is based on the resolution of the *so-called* Algebraic Riccati Equation (ARE). Over the past half century, this equation appeared to be ubiquitous in Modern Control Theory, leading to the development of dedicated algorithms able to efficiently solve it [BLW91].

Theorem 5.2 is generally referred to as Spectral Factorisation Theorem. Applied separately on  $\mathcal{B}(j\omega)^* X_N \mathcal{B}(j\omega)$  and  $\mathcal{B}(j\omega)^* X_D \mathcal{B}(j\omega)$ , it solves Problem 5.2.

**Theorem 5.2** (Spectral Factorisation). *Let  $X = X^T \in \mathbb{R}^{n_z \times n_z}$  be a given real symmetric matrix, and  $\mathcal{B}(s)$  a stable,  $n_z \times n_w$  transfer matrix defined by:*

$$\mathcal{B}(s) = \left( \frac{1}{s} \cdot I_n \right) \star \left[ \begin{array}{c|c} A & B \\ \hline C & D \end{array} \right]$$

with  $A \in \mathbb{R}^{n \times n}$ ,  $B \in \mathbb{R}^{n \times n_w}$ ,  $C \in \mathbb{R}^{n_z \times n}$ ,  $D \in \mathbb{R}^{n_z \times n_w}$ , and  $(A, B)$  is controllable. Assume that:  $D^T X D > 0$  and

$$\forall \omega \in \mathbb{R}, \quad \mathcal{B}(j\omega)^* X \mathcal{B}(j\omega) \geq 0 \quad (5.12)$$

Then (i)  $\Rightarrow$  (ii).

(i) *There exists a symmetric matrix  $\mathbf{P} = \mathbf{P}^T \in \mathbb{R}^{n \times n}$  solution of the (ARE):*

$$R(\mathbf{P}) = 0 \quad (\text{ARE})$$

where

$$R(\mathbf{P}) := A^T(-\mathbf{P}) + (-\mathbf{P})A + C^T X C - ((-\mathbf{P})B + C^T X D) (D^T X D)^{-1} (B^T(-\mathbf{P}) + D^T X C) \quad (5.13)$$

(ii) *There exist  $\mathbf{C}_H \in \mathbb{R}^{n_w \times n}$  and  $\mathbf{D}_H \in \mathbb{R}^{n_w \times n_w}$  such that:*

$$\forall \omega \in \mathbb{R}, \quad \mathcal{B}(j\omega)^* X \mathcal{B}(j\omega) = \mathbf{H}(j\omega)^* \mathbf{H}(j\omega) \quad (5.14)$$

where  $\mathbf{H}$  is a stable LFT in  $\frac{1}{s}$  defined by:

$$\mathbf{H}(s) = \left( \frac{1}{s} \cdot I_n \right) \star \left[ \begin{array}{c|c} A & B \\ \hline \mathbf{C}_H & \mathbf{D}_H \end{array} \right] \quad (5.15)$$

*Proof.* By applying the KYP lemma, (5.12) is equivalent to:  $\exists \mathbf{P} = \mathbf{P}^T \in \mathbb{R}^{n \times n}$ ,

$$Q(\mathbf{P}) \geq 0$$



with

$$Q(\mathbf{P}) = \begin{bmatrix} A & B \\ I & 0 \end{bmatrix}^T \begin{bmatrix} 0 & -\mathbf{P} \\ -\mathbf{P} & 0 \end{bmatrix} \begin{bmatrix} A & B \\ I & 0 \end{bmatrix} + \begin{bmatrix} C^T \\ D^T \end{bmatrix} X \begin{bmatrix} C & D \end{bmatrix}$$

Applying the Schur complement lemma (p. 195), as  $D^T X D > 0$ , a congruent relation (Definition A.1, p. 191) can be established for  $Q(\mathbf{P})$ :

$$Q(\mathbf{P}) = \Pi^T \begin{bmatrix} R(\mathbf{P}) & 0 \\ 0 & D^T X D \end{bmatrix} \Pi$$

where  $R(\mathbf{P})$  is defined in (5.13). Therefore, if  $\mathbf{P}$  is solution to (ARE), by congruency (Theorem A.1, p. 191) one has:

$$\text{rank}(Q(\mathbf{P})) = \text{rank}(D^T X D) = n_w$$

Moreover, as  $Q(\mathbf{P}) \geq 0$ , there exist  $\mathbf{C}_H \in \mathbb{R}^{n_w \times n}$  and  $\mathbf{D}_H \in \mathbb{R}^{n_w \times n_w}$  such that:

$$Q(\mathbf{P}) = \begin{bmatrix} \mathbf{C}_H^T \\ \mathbf{D}_H^T \end{bmatrix} \begin{bmatrix} \mathbf{C}_H & \mathbf{D}_H \end{bmatrix}$$

By pre- and post-multiplying last equation by  $\begin{bmatrix} \left(\frac{1}{-s} \cdot I\right) (I - A \left(\frac{1}{-s} \cdot I\right))^{-1} B \\ I \end{bmatrix}^T$  and  $\begin{bmatrix} \left(\frac{1}{s} \cdot I\right) (I - A \left(\frac{1}{s} \cdot I\right))^{-1} B \\ I \end{bmatrix}$ , and by defining  $F$  as

$$F(s) := A \left(\frac{1}{s} \cdot I\right) (I - A \left(\frac{1}{s} \cdot I\right))^{-1} B + B = (I - A \left(\frac{1}{s} \cdot I\right))^{-1} B$$

straightforward computation gives:

$$F(-s)^T \begin{bmatrix} I \\ \frac{1}{-s} \cdot I \end{bmatrix}^T \begin{bmatrix} 0 & -\mathbf{P} \\ -\mathbf{P} & 0 \end{bmatrix} \begin{bmatrix} I \\ \frac{1}{s} \cdot I \end{bmatrix} F(s) + \mathcal{B}(-s)^T X \mathcal{B}(s) = \mathbf{H}(-s)^T \mathbf{H}(s) \quad (5.16)$$

where  $\mathbf{H}(s)$  is defined as in (5.15). In addition,  $\mathbf{H}(s)$  is stable as  $A$  is strictly Hurwitz by stability of  $\mathcal{B}(s)$ . Finally, as  $\frac{1}{s}$  is lossless passive, one can easily check that:

$$\forall \omega \in \mathbb{R}, \quad \begin{bmatrix} I \\ \frac{1}{j\omega} \cdot I \end{bmatrix}^* \begin{bmatrix} 0 & -\mathbf{P} \\ -\mathbf{P} & 0 \end{bmatrix} \begin{bmatrix} I \\ \frac{1}{j\omega} \cdot I \end{bmatrix} = - \begin{bmatrix} \frac{1}{j\omega} \cdot I \\ I \end{bmatrix}^* \begin{bmatrix} 0 & \mathbf{P} \\ \mathbf{P} & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{j\omega} \cdot I \\ I \end{bmatrix} = 0$$

holds and then (5.14) is verified.  $\square$

An important remark is that the last theorem does not allow to choose the state-space representation matrices  $A$  and  $B$ . From a transfer function point of view, this basically means that the poles of  $H(s)$  are fixed by those of  $\mathcal{B}(s)$ , and only the zeros of  $H(s)$  can be chosen. One may note that this is not limiting using our parametrisation, as the decision variables are precisely the zeros of  $W_N(s)$  and  $W_D(s)$ , which have the same pre-defined denominator  $d(s)$ .

Theorem 5.2 raises the following question: when is there a solution to the ARE? As detailed in the next subsection, the conditions of regularity  $D^T X D > 0$ , of non-negativity (5.12) and of controllability of  $(A, B)$  are sufficient to always ensure the existence of a solution. But before that, let us finish this part with two points. The

first point concerns the computation of  $C_H$  and  $D_H$ . The second point makes explicit the link between Theorem 5.2 and the initial change of variable.

**Computation of  $C_H$  and  $D_H$**  In the proof of Theorem 5.2, it is demonstrated that if the matrix  $P$  is a solution of (ARE) (p. 117), then there exist  $C_H \in \mathbb{R}^{n_w \times n}$  and  $D_H \in \mathbb{R}^{n_w \times n_w}$  such that

$$Q(P) = \begin{bmatrix} C_H^T \\ D_H^T \end{bmatrix} \begin{bmatrix} C_H & D_H \end{bmatrix}$$

These matrices can be simply calculated. Indeed, as  $\text{rank}(Q(P)) = n_w$ , there exists  $X_H = X_H^T \in \mathbb{R}^{(n+n_w) \times (n+n_w)}$  such that  $Q(P) = X_H$ . Moreover, as  $Q(P) \geq 0$ , the singular value decomposition of  $X_H$  can be expressed (Theorem A.5, p. 193) as follows:

$$X_H = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix}^T \begin{bmatrix} \Sigma_{n_w} & 0_{n_w \times n} \\ 0_{n \times n_w} & 0_{n \times n} \end{bmatrix} \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix}$$

where  $\Sigma_{n_w} > 0 \in \mathcal{D}(\mathbb{R}^{n_w \times n_w})$  is a positive-definite, diagonal matrix, and the matrix  $U := \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix} \in \mathbb{R}^{(n+n_w) \times (n+n_w)}$  is unitary  $UU^T = I$ .

Therefore, defining  $\Sigma_{n_w}^{1/2} \in \mathbb{R}^{n_w \times n_w}$  such that  $(\Sigma_{n_w}^{1/2})^T \cdot \Sigma_{n_w}^{1/2} = \Sigma_{n_w}$  and

$$C_W := \Sigma_{n_w}^{1/2} \cdot U_{11} \quad D_W := \Sigma_{n_w}^{1/2} \cdot U_{12}$$

leads to the result. One may note that all possible  $\Sigma_{n_w}^{1/2}$  are linked by the pre-multiplication of a unitary matrix  $V$  ( $V^*V = I$ ), and then all the resulting spectral factors  $H$  are unique up to the pre-multiplication by such matrix  $V$ .

**Explicit link with the initial change of variable** Let us now make an explicit link with the discussion of p. 112, concerning the change of variable  $X = \Gamma\Gamma^T$ . For this, assume that the basis  $\mathcal{B}(s)$  is such as:

$$\mathcal{B}(s) := \frac{1}{d(s)} \begin{bmatrix} s & s^2 & \dots & s^n & d(s) \end{bmatrix}^T = \begin{bmatrix} \frac{s}{d(s)} & \frac{s^2}{d(s)} & \dots & \frac{s^n}{d(s)} & 1 \end{bmatrix}^T$$

with for example  $d(s) = a_0 + a_1s + \dots + a_{n-1}s^{n-1} + s^n := (s+1)^n$ . Choose the following LFT representation:

$$\mathcal{B}(s) = \left( \frac{1}{s} \cdot I_n \right) \star \left[ \begin{array}{c|c} A & B \\ \hline C & D \end{array} \right] := \left[ \begin{array}{ccccc|c} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 1 & 0 \\ -a_0 & -a_1 & -a_2 & \dots & -a_{n-1} & 1 \\ \hline 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & 1 & 0 & 0 \\ 0 & 0 & \dots & 0 & 1 & 0 \\ 0 & 0 & \dots & 0 & 0 & 1 \end{array} \right]$$

Then, noticing that  $\begin{bmatrix} C & D \end{bmatrix} = I_{n+1}$ ,  $Q(P)$  defined in the last proof (p. 117) becomes:

$$Q(P) = \tilde{X} + X$$

where  $\tilde{X}$  is defined as  $\tilde{X} := \begin{bmatrix} A & B \\ I & 0 \end{bmatrix}^T \begin{bmatrix} 0 & -P \\ -P & 0 \end{bmatrix} \begin{bmatrix} A & B \\ I & 0 \end{bmatrix}$ . Therefore, when one calculates a solution of the ARE, and lowers the rank of  $Q(P)$ , one implicitly calculates a  $\tilde{X}$  such that  $\text{rank}(X + \tilde{X}) = 1$ , i.e.  $X + \tilde{X} = \Gamma\Gamma^T$  where  $\Gamma \in \mathbb{R}^{n+1}$ . As far as we know, this is the only non-trivial case where there exists an efficient algorithm for solving a rank constrained problem.

### 5.3.3.1 ARE, ARI and spectral factorisation

Over the past 300 years, the Algebraic Riccati Equation (ARE) has been of the most studied nonlinear matrix equations arising in mathematics and engineering [Jun17]. An ARE is basically any algebraic equation involving quadratic matrices, as in (5.13). Its ubiquity in Modern Control Theory, with applications in Linear Quadratic Regulator (LQR) problem,  $\mathcal{H}_2$  and  $\mathcal{H}_\infty$  control to name a few [ZDG96], or estimation, prediction and filtering, such as in the Kalman filter [AM79], is remarkable.

Its important role in such diverse areas lead to the development of efficient and numerically-robust algorithms of resolution. The most common algorithm, implemented for example in Matlab<sup>5</sup>, is based on the computation of the invariant subspaces of a certain *Hamiltonian* matrix [AL84; ZDG96, Chap.13].

The ARE is inherently linked with the KYP lemma, through the Algebraic Riccati Inequality (ARI). This link is detailed in the next paragraph. This especially leads to a resolution algorithm able to find particular solutions of the ARE by solving a linear minimisation problem under LMI constraints. The connection between these solutions and the so-called minimum-phase and maximum-phase factors is detailed and illustrated on a simple example.

**Link between ARE, ARI and strict ARI** First properties were established between the ARE and the matrix inequality of the so-called positive-real lemma for different applications: stochastic realisation [And69], LQR optimal control [Wil71], temporal series filtering and hyperstability analysis [FCG79]. Most of these properties were extended to the matrix inequality involved in the KYP lemma, and are summarised under the dissipative systems framework in [TW91].

**Theorem 5.3** (Adapted from [TW91]). *Let  $A \in \mathbb{R}^{n \times n}$ ,  $B \in \mathbb{R}^{n \times n_w}$ ,  $C \in \mathbb{R}^{n_z \times n}$  and  $D \in \mathbb{R}^{n_z \times n_w}$  be given real matrices. Assume that the pair  $(A, B)$  is controllable. Let  $X = X^T \in \mathbb{R}^{n_z \times n_z}$  be a real symmetric matrix. Define the matrix-valued function  $R$  as in (5.13) of Theorem 5.2. Furthermore, assume that the following regularity assumption holds:*

$$D^T X D > 0$$

*The following statements are equivalent:*

S1. Let  $\mathcal{B}(s) = \left(\frac{1}{s} \cdot I_n\right) \star \left[ \begin{array}{c|c} A & B \\ \hline C & D \end{array} \right]$  be an LFT in  $\frac{1}{s}$ . Then:

$$\forall \omega \in \mathbb{R}, j\omega \notin \Lambda(A), \quad \mathcal{B}(j\omega)^* X \mathcal{B}(j\omega) \geq 0$$

<sup>5</sup>[www.mathworks.com/help/control/ref/care.html](http://www.mathworks.com/help/control/ref/care.html)

S2.  $\exists P = P^T \in \mathbb{R}^{n \times n}$  of the Algebraic Riccati Inequality (ARI), defined by

$$R(P) \geq 0 \quad (\text{ARI})$$

S3.  $\exists P = P^T \in \mathbb{R}^{n \times n}$  of the (ARE)

$$R(P) = 0$$

Assume that one of these statements holds. Then there exist symmetric solutions  $P_{min} = P_{min}^T$  and  $P_{max} = P_{max}^T$  of the (ARE) such that any symmetric solution  $P = P^T$  of the (ARI) satisfies:

$$P_{min} \leq P \leq P_{max} \quad (5.17)$$

$P_{min}$  and  $P_{max}$  are respectively called minimal and maximal solutions of the (ARI).

*Remark 5.2.* Statements S2 and S3 establish existence properties between the solutions of the ARE and the ARI. There is no reference to the Laplace variable  $s$ , or the frequency variable  $\omega$ . This will be used in the next sections, to establish the spectral factorisation theorems for LFT in repeated  $T(s)$ , and in repeated  $T_s(s)$  and  $T_p(s)$ .

The equivalence between statements S1 and S2 is simply the KYP lemma in a hidden form. Indeed, applying the Schur Lemma (Lemma A.3, p. 195), as  $D^T X D > 0$ , S2 is equivalent to:

$$\exists P = P^T \in \mathbb{R}^{n \times n}, \quad \begin{bmatrix} A^T(-P) + (-P)A + C^T X C & (-P)B + C^T X D \\ B^T(-P) + D^T X C & D^T X D \end{bmatrix} \geq 0$$

Moreover, the implication S3  $\Rightarrow$  S2 is trivial as it is clear that:

$$\{P = P^T, R(P) = 0\} \subseteq \{P = P^T, R(P) \geq 0\}$$

Then, it should be clear that (5.17) also holds for any symmetric matrix  $P$  solution of the (ARE).

Historically, specific solutions of the ARI were computed using the ARE. With the development of efficient numerical algorithms to solve LMI optimisation problems, the reverse is also possible: to compute special solutions,  $P_{min}$  and  $P_{max}$ , using the LMI framework. While this method may not be as efficient as algorithms dedicated to ARE, it is more flexible. It especially permits to solve coupled or non-standard AREs (see [RZ00] for example). We will use this method in Section 5.4 and Section 5.5 to convexly add ARE constraints into a convex optimisation problem.

For instance,  $P_{min}$  is found by solving the following linear minimisation problem under a LMI constraint:

$$\left. \begin{array}{l} \min_{P=P^T} \quad \text{Trace}(P) \\ \text{subject to} \quad \begin{bmatrix} A^T(-P) + (-P)A + C^T X C & (-P)B + C^T X D \\ B^T(-P) + D^T X C & D^T X D \end{bmatrix} \geq 0 \end{array} \right\} \quad (5.18)$$

The proof that there exists a unique optimal solution  $P_{opt}$  of optimisation problem (5.18) and that  $P_{opt} = P_{min}$  is straightforward using Theorem 5.3.  $P_{min}$  is clearly a solution, as  $\mathcal{R}(P_{min}) = 0$ , and for all  $P = P^T$  such that  $R(P) \geq 0$ , we have  $P \geq P_{min}$ . The uniqueness comes from the resulting constraints between  $P_{min}$  and a

hypothetically different  $P_{opt}$ :

$$\begin{aligned} \text{Trace}(P_{opt} - P_{min}) &= 0 \\ P_{opt} - P_{min} &\geq 0 \end{aligned}$$

yields  $P_{opt} = P_{min}$ .

Theorem 5.3 and the underlying consequences can be extended to the case with strict inequalities [Sch90; ZDG96, Chap.13]. These are summed up in the following theorem. The differences with Theorem 5.3 will ensure the filter  $W = W_N(W_D)^{-1}$  to be stable.

**Theorem 5.4.** *Let  $A \in \mathbb{R}^{n \times n}$ ,  $B \in \mathbb{R}^{n \times n_w}$ ,  $C \in \mathbb{R}^{n_z \times n}$  and  $D \in \mathbb{R}^{n_z \times n_w}$  be given real matrices, with the pair  $(A, B)$  controllable. Let  $X = X^T \in \mathbb{R}^{n_z \times n_z}$  be a real symmetric matrix. Define the matrix-valued function  $R$  as in (5.13) of Theorem 5.2. Furthermore, assume that the following regularity assumption holds:*

$$D^T X D > 0$$

Then statements S1. and S2. are equivalent:

S1. Let  $\mathcal{B}(s) = \left(\frac{1}{s} \cdot I_n\right) \star \left[ \begin{array}{c|c} A & B \\ \hline C & D \end{array} \right]$  be an LFT in  $\frac{1}{s}$  and denote  $\Lambda(A)$  as the spectrum of  $A$  (Definition A.2, p. 191). Then:

$$\forall \omega \in \mathbb{R}, j\omega \notin \Lambda(A), \quad \mathcal{B}(j\omega)^* X \mathcal{B}(j\omega) > 0$$

S2.  $\exists P = P^T \in \mathbb{R}^{n \times n}$  of the (strict ARI), defined by:

$$R(P) > 0 \quad (\text{strict ARI})$$

Assume that statement S1. or statement S2. holds. Then, statement S3. holds.

S3.  $\exists P = P^T \in \mathbb{R}^{n \times n}$  of the (ARE)

$$R(P) = 0$$

Furthermore, there exist symmetric solutions  $P_{min} = P_{min}^T$  and  $P_{max} = P_{max}^T$  of the (ARE) such that any symmetric solution  $P = P^T$  of the (strict ARI) satisfies:

$$P_{min} < P < P_{max}$$

$P_{min}$  and  $P_{max}$  are respectively called minimal and maximal solutions.

**Link with spectral factorisation** Theorem 5.2 makes explicit an already mentioned statement: any hermitian, real-rational matrix  $M(j\omega)$  which is positive semi-definite over  $\mathbb{R}$  admits a stable transfer function  $H$  as minimal spectral factor. Therefore, constraints (5.7) and (5.8) of Theorem 5.1, which were necessary to synthesise square magnitudes, are also sufficient to enable spectral factorisation.

In addition, Theorem 5.2 states that any solution of the ARE leads to a stable minimal spectral factor. Indeed, such a solution leads to the following decomposition:

$$\begin{bmatrix} A^T(-P) + (-P)A + C^T X C & (-P)B + C^T X D \\ B^T(-P) + D^T X C & D^T X D \end{bmatrix} = \begin{bmatrix} C_H^T \\ D_H^T \end{bmatrix} \begin{bmatrix} C_H & D_H \end{bmatrix}$$

where  $C_H$  and  $D_H$  are of minimal size, and gives the associate stable minimal spectral factor  $H(s) = \left(\frac{1}{s} \cdot I_n\right) \star \left[ \begin{array}{c|c} A & B \\ \hline C_H & D_H \end{array} \right]$ .

The influence of the choice of the solution of the ARE on the resulting spectral factor  $H$  is now investigated. More precisely, the choice of  $P_{min}$  leads to a *minimum-phase* factor, i.e. all its zeros are in the left-half complex plane, while the choice of  $P_{max}$  leads to a *maximum-phase* factor, having all its zeros in the right-half complex plane. Indeed, using the inversion formula of LFT (p. 195, Appendix A),  $H^{-1}$  can be computed as:

$$H^{-1}(s) = \left(\frac{1}{s} \cdot I_n\right) \star \left[ \begin{array}{c|c} A - BD_H^{-1}C_H & BD_H^{-1} \\ \hline D_H^{-1}C_H & D_H^{-1} \end{array} \right] \quad (5.19)$$

The zeros of  $H(s)$  are the poles of  $H^{-1}(s)$  and leads to the study of the eigenvalues of  $\hat{A} := A - BD_H^{-1}C_H$ . By definition of  $C_H$  and  $D_H$ , this matrix can be expressed in terms of  $P$  as follows:

$$\hat{A} = A - B(D_H^T D_H)^{-1} D_H^T C_H = A - B(D^T X D)^{-1} (B^T (-P) + D^T X C)$$

This leads to the following results.

**Property 5.1** ([Wil71; TW91]). *Define*

$$\begin{aligned} \hat{A}_{min} &:= A - B(D^T X D)^{-1} (B^T (-P_{min}) + D^T X C) \\ \hat{A}_{max} &:= A - B(D^T X D)^{-1} (B^T (-P_{max}) + D^T X C) \end{aligned}$$

Then the spectrum of  $\hat{A}_{min}$  and of  $\hat{A}_{max}$  are such that:

$$\Lambda(\hat{A}_{min}) \subset \mathbb{C}^- \cup \mathbb{C}^0 \quad \text{and} \quad \Lambda(\hat{A}_{max}) \subset \mathbb{C}^+ \cup \mathbb{C}^0$$

In addition, if there exists a solution to (strict ARI), then

$$\Lambda(\hat{A}_{min}) \subset \mathbb{C}^- \quad \text{and} \quad \Lambda(\hat{A}_{max}) \subset \mathbb{C}^+$$

From the spectral factorisation point of view, the choice of the solution of the (ARE) results in the choice of the zeros of the spectral factor from the squared magnitude. Choosing  $P_{min}$  results in choosing all the zeros with negative real part,  $P_{max}$  in the zeros with positive real part, while any other solution  $P$  leads to both zeros with negative and positive real part.

*Remark 5.3.* In the LQ theory [ZDG96, Chap.14], a solution  $P$  of the ARE is directly linked with the gain of the constant controller. Choosing  $P_{min}$  leads to a stable closed-loop system while  $P_{max}$  leads to an unstable closed-loop system. Therefore, in the literature,  $P_{min}$  is sometimes called *stabilising* solution and  $P_{max}$  *anti-stabilising* solution of the ARE.

**Example 5.3.** To illustrate the last results simply, assume that we have to factorise the following squared magnitude:

$$M(j\omega) = \mathcal{B}(j\omega)^* X \mathcal{B}(j\omega) := \frac{1}{-j\omega + 1} \begin{bmatrix} 1 \\ j\omega \end{bmatrix}^* \begin{bmatrix} 9 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ j\omega \end{bmatrix} \frac{1}{j\omega + 1} = \frac{\omega^2 + 9}{\omega^2 + 1}$$

An LFT realisation of  $\mathcal{B}(j\omega)$  is given by:

$$\mathcal{B}(j\omega) = \left(\frac{1}{j\omega} \cdot I\right) \star \left[ \begin{array}{c|c} A & B \\ \hline C & D \end{array} \right] := \left(\frac{1}{j\omega} \cdot 1\right) \star \left[ \begin{array}{c|c} -1 & 1 \\ \hline 1 & 0 \\ -1 & 1 \end{array} \right]$$

Clearly,  $M(j\omega)$  is positive along the  $j\omega$ -axis:  $\forall \omega \in \mathbb{R}, M(j\omega) \geq 0$ . Therefore, by the KYP lemma, there exists  $P \in \mathbb{R}$  such that:

$$\underbrace{\begin{bmatrix} A^T(-P) + (-P)A + C^T X C & B^T(-P) + C^T X D \\ (-P)B + D^T X C & D^T X D \end{bmatrix}}_{\begin{bmatrix} 2P + 6 & -P + 1 \\ -P + 1 & 1 \end{bmatrix}} \geq 0 \quad (5.20)$$

By Schur lemma, as  $D^T X D = 1 > 0$ , finding a  $P$  such that the left-hand term of last inequality is of minimum-rank is equivalent solving the ARE:

$$2P + 6 - (-P + 1)(1)^{-1}(-P + 1) = -P^2 + 4P + 5 = 0$$

Solving last equation leads to two solutions:  $P_{min} = -1$  et  $P_{max} = 5$ . The set of solutions of Inequality (5.20) is then the closed interval  $[-1; 5]$ . To any solution  $P$  belonging to this interval there corresponds a spectral factor, but which is not necessary minimal. For instance, if  $P = 1 \in [-1; 5]$  then

$$\begin{bmatrix} 2P + 6 & -P + 1 \\ -P + 1 & 1 \end{bmatrix} = \begin{bmatrix} 8 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \sqrt{8} & 0 \\ 0 & 1 \end{bmatrix}^T \begin{bmatrix} \sqrt{8} & 0 \\ 0 & 1 \end{bmatrix}$$

which leads to the stable but non-minimal spectral factor:  $H(s) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} \sqrt{8} \\ 0 \end{bmatrix} \frac{1}{s+1}$ .

Finally, if one chooses  $P = P_{min}$  and  $P = P_{max}$ , one obtains two stable minimal spectral factors which are respectively minimum-phase and maximum-phase:

$$\begin{aligned} H^{min}(s) &= 1 + 2 \cdot \frac{1}{s+1} \cdot 1 = \frac{s+3}{s+1} \\ H^{max}(s) &= 1 - 4 \cdot \frac{1}{s+1} \cdot 1 = \frac{s-3}{s+1} \end{aligned}$$

### 5.3.4 Synthesis procedure

#### 5.3.4.1 Reverse parametrisation

Subsection 5.3.2 and Subsection 5.3.3 detail how to obtain  $W_N$  and  $W_D$  such that  $W = W_N(W_D)^{-1}$  is stable and satisfies a given spectral mask  $\mathcal{SM}$ . In order to fully solve Problem 5.1 in the case where  $T(s) = \frac{1}{s}$ , the resulting  $W$  should be expressed as an LFT, i.e. the reverse parametrisation from  $W_N, W_D$  should be achieved. As this is generally a tough task, we provide here simple and analytical formulas in the next property. It is provided for any LFT in a transfer matrix  $\Delta(s)$ , and not necessary for  $\frac{1}{s} \cdot I$ , as this result will be used in the next sections. The proof, based on calculations of LFT, is given in Appendix A (p. 206). A reader interested in an elegant proof, based on the interpretation of the process in terms of a feedback loop, should refer to [Zar13, Chap.5].

**Property 5.2.** Let  $W_N(\Delta(s))$  and  $W_D(\Delta(s))$  be two LFT in a transfer matrix  $\Delta(s)$  such that  $W_D(\Delta(s))$  is invertible and defined by:

$$W_N(\Delta(s)) := \Delta(s) \star \left[ \begin{array}{c|c} A & B \\ \hline C_N & D_N \end{array} \right] \quad W_D(\Delta(s)) := \Delta(s) \star \left[ \begin{array}{c|c} A & B \\ \hline C_D & D_D \end{array} \right]$$

with  $A \in \mathbb{R}^{n \times n}$ ,  $B \in \mathbb{R}^{n \times n_w}$ ,  $C_N, C_D \in \mathbb{R}^{n_z \times n}$  and  $D_N, D_D \in \mathbb{R}^{n_z \times n}$ . Then the ratio  $W(\Delta(s)) := W_N(\Delta(s))W_D^{-1}(\Delta(s))$  is also an LFT in  $\Delta(s)$  and is given by:

$$W(\Delta(s)) = \Delta(s) \star \left[ \begin{array}{c|c} A - BD_D^{-1}C_D & BD_D^{-1} \\ \hline C_N - D_N D_D^{-1}C_D & D_N D_D^{-1} \end{array} \right] \quad (5.21)$$

By (5.21) with  $\Delta(s) = \frac{1}{s} \cdot I_n$ , the stability of  $W(s)$  leads to the study of the eigenvalues of the matrix  $A_W := A - BD_D^{-1}C_D$ . If one chooses the minimal solution of the ARE associated with the spectral factorisation of  $W_D(s)$ , by (5.19) and Property 5.1, it is clear that  $\Lambda(A_W) \subset \mathbb{C}^-$ . In other words, we find again the result that having  $W_D(s)$  minimum-phase leads  $W(s)$  to be stable. Similarly, one can check by inverting  $W(s)$  that taking  $W_N(s)$  minimum-phase leads  $W(s)$  to be minimum-phase.

#### 5.3.4.2 On the choice of the representation matrices of $\mathcal{B}(s)$

The column  $\mathcal{B}(s)$  is the column concatenation of the elements of any basis of the given space  $\mathcal{R}_n \mathcal{H}_\infty(s, d(s))$ . The choice of the basis does not affect our approach. However, the choice of the matrices  $A, B, C, D$  of the LFT representation of  $\mathcal{B}(s)$  does have an impact on our approach. While there is an infinite number of representation that are *mathematically* equivalent, they are not *computationally* equivalent. Indeed, it is known that, among all these representations, the *balanced* representation provides numerical results that are more accurate (see for instance [Thi86] in the context of frequency filter realisation). Therefore, we propose to use it in the numerical implementation of our synthesis approach.

#### 5.3.4.3 Explicit synthesis procedure

The overall procedure to solve Problem 5.1, with  $T(s) = \frac{1}{s}$ , is summarised step-by-step below.

1. Choose the order  $n$ .
2. Define a stable basis  $\mathcal{B}(s)$  of order  $n$  under an LFT form:

$$\mathcal{B}(s) = \left( \frac{1}{s} \cdot I_n \right) \star \left[ \begin{array}{c|c} A & B \\ \hline C & D \end{array} \right]$$

3. Solve feasibility problem (i) defined in Theorem 5.1 (p. 115).  
If it is feasible, there exist symmetric matrices  $X_N = X_N^T$  and  $X_D = X_D^T$  such that the magnitude defined by

$$M(j\omega) = \frac{\mathcal{B}(j\omega)^* X_N \mathcal{B}(j\omega)}{\mathcal{B}(j\omega)^* X_D \mathcal{B}(j\omega)}$$

is positive and satisfies the spectral mask  $\mathcal{SM}$ .

If it is not feasible, then increment  $n$ :  $n \leftarrow n + 1$ , and back to step 2.



4. Apply independently Theorem 5.2 (p. 117) on  $\mathcal{B}(j\omega)^* X_N \mathcal{B}(j\omega)$  and  $\mathcal{B}(j\omega)^* X_D \mathcal{B}(j\omega)$  to get two stable minimal  $W_N$  and  $W_D$  such that:

$$\begin{aligned} W_N(j\omega)^* W_N(j\omega) &= \mathcal{B}(j\omega)^* X_N \mathcal{B}(j\omega) \\ W_D(j\omega)^* W_D(j\omega) &= \mathcal{B}(j\omega)^* X_D \mathcal{B}(j\omega) \end{aligned}$$

This requires to independently solve two AREs such as (ARE) (p. 117). Choose the minimal solution  $P = P_{D_{min}}$  for  $W_D$  to be minimum-phase. Compute the associate  $C_N, C_D \in \mathbb{R}^{1 \times n}$  and  $D_N, D_D \in \mathbb{R}$ .  $W_N$  and  $W_D$  are then two LFTs in  $\frac{1}{s} \cdot I_n$  given by:

$$\begin{aligned} W_N(s) &:= \left( \frac{1}{s} \cdot I_n \right) \star \left[ \begin{array}{c|c} A & B \\ \hline C_N & D_N \end{array} \right] \\ W_D(s) &:= \left( \frac{1}{s} \cdot I_n \right) \star \left[ \begin{array}{c|c} A & B \\ \hline C_D & D_D \end{array} \right] \end{aligned}$$

5. Use (5.21) to get a stable  $W$  of order  $n$  which satisfies the spectral mask  $\mathcal{SM}$  and which can be expressed as the LFT:

$$W(s) = \left( \frac{1}{s} \cdot I_n \right) \star \left[ \begin{array}{c|c} A - BD_D^{-1}C_D & BD_D^{-1} \\ \hline C_N - D_N D_D^{-1}C_D & D_N D_D^{-1} \end{array} \right]$$

## 5.4 LFT filter synthesis with dissipative $T(s)$

In Section 5.3, a method was provided to synthesise a stable  $W(s)$  as an LFT in  $\frac{1}{s} \cdot I$ , which satisfies a spectral mask. This method is now generalised in order to solve Problem 5.1, i.e. for the synthesis of a stable  $W(T(s))$  as an LFT in  $T(s) \cdot I$  with  $T(s)$  a  $\{x, y, z\}$ -dissipative transfer function (Figure 5.2). To achieve this, we discuss step-by-step the extension of this method.

The procedure of Subsection 5.3.1 to obtain a finite-dimensional convex formulation is first directly extended for transfer functions expressed as LFT in  $T(s) \cdot I$ , where  $T(s)$  is a dissipative transfer function (Subsection 5.4.1). Again, this leads us to split the synthesis problem into the magnitude synthesis and the spectral factorisation sub-problems. In Section 5.3, both problems are efficiently solved by using only one property of the transfer function  $\frac{1}{s}$ , namely its lossless  $\{0, 1, 0\}$ -dissipativeness:

$$\forall \omega \in \mathbb{R}, \quad \left[ \begin{array}{c} \frac{1}{j\omega} \\ 1 \end{array} \right]^* \left[ \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right] \left[ \begin{array}{c} \frac{1}{j\omega} \\ 1 \end{array} \right] = 0$$

Indeed, in the magnitude synthesis (Theorem 5.1, p. 115), this property appears in the choice of the appropriate version of the KYP lemma, while in the spectral factorisation (Theorem 5.2, p. 117), it enables to obtain a spectral factor from (5.16). Therefore, this synthesis method can be directly extended from  $\frac{1}{s}$  to any lossless  $\{0, 1, 0\}$ -dissipative  $T(s)$ .

In this section, we go one step further and extend this method to any lossless dissipative  $T(s)$ :

$$\forall \omega \in \mathbb{R}, \quad \begin{bmatrix} T(j\omega) \\ 1 \end{bmatrix}^* \begin{bmatrix} x & y \\ y & z \end{bmatrix} \begin{bmatrix} T(j\omega) \\ 1 \end{bmatrix} = 0, \quad \text{with } \begin{cases} x \leq 0 \\ \begin{bmatrix} x & y \\ y & z \end{bmatrix} \text{ full rank} \end{cases}$$

The magnitude synthesis problem is transformed into an LMI problem by using a suitable version of the KYP lemma (Subsection 5.4.2), while the resulting magnitude is spectrally factorised by solving AREs thanks to loop-shifting operations (Subsection 5.4.3.1).

Furthermore, we aim to generalise the synthesis approach to any SISO dissipative transfer function  $T(s)$ :

$$\forall \omega \in \mathbb{R}, \quad \begin{bmatrix} T(j\omega) \\ 1 \end{bmatrix}^* \begin{bmatrix} x & y \\ y & z \end{bmatrix} \begin{bmatrix} T(j\omega) \\ 1 \end{bmatrix} \geq 0, \quad \text{with } \begin{cases} x \leq 0 \\ \begin{bmatrix} x & y \\ y & z \end{bmatrix} \text{ full rank} \end{cases}$$

Unfortunately, the previous approach can not be directly extended. While one can synthesise a magnitude function which satisfies the spectral mask (Subsection 5.4.2), a mismatch appears between this function and the spectral factor computed using the traditional technique (Subsection 5.4.3.2). Hence, we propose a generalised synthesis approach in Subsection 5.4.4 in order to solve Problem 5.1.

Finally, both synthesis approaches, when  $T(s)$  is either lossless *or* lossy dissipative, are summarised and numerically illustrated in Subsection 5.4.5.

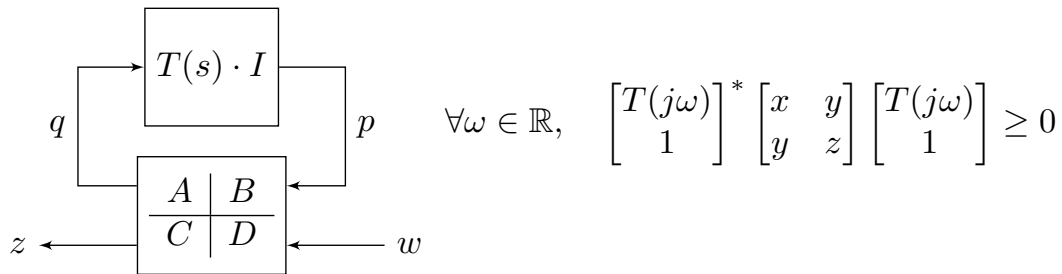


FIGURE 5.2: LFT system in the repeated  $\{x, y, z\}$ -dissipative  $T(s)$

### 5.4.1 Finite-dimensional convex formulation

In this subsection, using similar arguments that in Subsection 5.3.1, a finite-dimensional convex formulation of Problem 5.1 is established.

First, a **finite dimensional parametrisation** for all LFT  $W(T(s))$  of  $T(s)$  is required. As in the usual case, the order of  $W(T(s))$  can be set to a finite value  $n$  and the following factorisation holds:

$$W(T(s)) := \frac{W_N(T(s))}{W_D(T(s))}$$

where  $W_N(T(s))$  and  $W_D(T(s))$  can be chosen to belong to  $\mathcal{R}_n \mathcal{H}_\infty(T(s), d(T(s)))$ , the vector space of stable, proper, real rational functions of  $T(s)$  of order  $n$  having  $d(T(s))$  as denominator, where  $d(T(s))$  is a polynomial in  $T(s)$ . As this vector space is finite,

a basis exists and one can write  $W_N(T(s))$  and  $W_D(T(s))$  as:

$$W_N(T(s)) = \Gamma_N^T \mathcal{B}(T(s)) \quad W_D(T(s)) = \Gamma_D^T \mathcal{B}(T(s))$$

where  $\Gamma_N, \Gamma_D \in \mathbb{R}^{n+1}$  and  $\mathcal{B}(T(s))$  is the column concatenation of the elements of a given basis.

Second, the procedure to get a **convex formulation** can also be straightforwardly extended. Indeed, using same change of variable as in (5.3), a convex formulation of constraints (5.1) and (5.2) is obtained. Likewise, this leads to solve the **LFT magnitude synthesis** problem, explicitly stated below.

**Problem 5.4** (LFT Magnitude Synthesis).

*Given* a SISO,  $\{x, y, z\}$ -dissipative transfer function  $T(s)$ ,  
a positive integer  $n \in \mathbb{N}^*$ ,  
a basis  $\mathcal{B}(T(s))$  of  $\mathcal{R}_n \mathcal{H}_\infty[T(s), d(s)]$ ,

$N_U$  upper bounds  $U_u$  associated with  $N_U$  real intervals  $\Omega_u^U \subseteq \mathbb{R}$ ,

$N_L$  lower bounds  $L_l$  associated with  $N_L$  real intervals  $\Omega_l^L \subseteq \mathbb{R}$ ,

*Find* if there exist real symmetric matrices  $\mathbf{X}_N = (\mathbf{X}_N)^T \in \mathbb{R}^{(n+1) \times (n+1)}$  and  $\mathbf{X}_D = (\mathbf{X}_D)^T \in \mathbb{R}^{(n+1) \times (n+1)}$  such that:

$$\begin{aligned} \forall \omega \in \mathbb{R}, \quad & \mathcal{B}(T(j\omega))^* \mathbf{X}_N \mathcal{B}(T(j\omega)) \geq 0 \\ \forall \omega \in \mathbb{R}, \quad & \mathcal{B}(T(j\omega))^* \mathbf{X}_D \mathcal{B}(T(j\omega)) > 0 \\ \forall u \in \{1, \dots, N_U\}, \quad \forall \omega \in \Omega_u^U, \quad & \frac{\mathcal{B}(T(j\omega))^* \mathbf{X}_N \mathcal{B}(T(j\omega))}{\mathcal{B}(T(j\omega))^* \mathbf{X}_D \mathcal{B}(T(j\omega))} \leq U_u^2 \\ \forall l \in \{1, \dots, N_L\}, \quad \forall \omega \in \Omega_l^L, \quad & \frac{\mathcal{B}(T(j\omega))^* \mathbf{X}_N \mathcal{B}(T(j\omega))}{\mathcal{B}(T(j\omega))^* \mathbf{X}_D \mathcal{B}(T(j\omega))} \geq L_l^2 \end{aligned}$$

If so, *Compute* them.

Problem 5.4 is a convex problem, where the decision variables  $\mathbf{X}_N$  and  $\mathbf{X}_D$  belong to a finite-dimensional space. However, the optimisation problem is of infinite dimension, as the constraints depend on the continuous variable  $\omega$ , and is then generally difficult to solve. Nonetheless, using our framework, this comes to find the suitable generalisation of the KYP Lemma. Indeed, a finite LMI feasibility problem is obtained, and this problem can be efficiently solved.

Once the magnitudes  $M_N(T(j\omega))$  and  $M_D(T(j\omega))$  have been synthesised, the reverse change of variable, i.e. the computation of the stable spectral factors  $W_N(T(s))$  and  $W_D(T(s))$ , needs then to be achieved. This problem is denoted as **LFT spectral factorisation**. As previously, this problem is stated and solved in the MIMO case.

**Problem 5.5** (LFT Spectral Factorisation).

*Given* a SISO,  $\{x, y, z\}$ -dissipative transfer function  $T(s)$ ,  
 positive integers  $n, n_w, n_z \in \mathbb{N}^*$ ,  
 two symmetric matrices  $X_N = X_N^T \in \mathbb{R}^{n_z \times n_z}$  and  $X_D = X_D^T \in \mathbb{R}^{n_z \times n_z}$ ,  
 a stable LFT  $\mathcal{B}(T(s))$  in  $T(s)$  such as:

$$\mathcal{B}(T(s)) = (T(s) \cdot I_n) \star \left[ \begin{array}{c|c} A & B \\ \hline C & D \end{array} \right]$$

with  $A \in \mathbb{R}^{n \times n}$ ,  $B \in \mathbb{R}^{n \times n_w}$ ,  $C \in \mathbb{R}^{n_z \times n}$ ,  $D \in \mathbb{R}^{n_z \times n_w}$ .

*Find* if there exist two stable LFT  $\mathbf{W}_N(T(s))$  and  $\mathbf{W}_D(T(s))$  in  $T(s)$  such that:

$$\begin{aligned} \forall \omega \in \mathbb{R}, \quad & \mathcal{B}(T(j\omega))^* X_N \mathcal{B}(T(j\omega)) = \mathbf{W}_N(T(j\omega))^* \mathbf{W}_N(T(j\omega)) \\ \forall \omega \in \mathbb{R}, \quad & \mathcal{B}(T(j\omega))^* X_D \mathcal{B}(T(j\omega)) = \mathbf{W}_D(T(j\omega))^* \mathbf{W}_D(T(j\omega)) \end{aligned}$$

where  $\mathbf{W}_N(T(s))$  and  $\mathbf{W}_D(T(s))$  are defined by:

$$\begin{aligned} \mathbf{W}_N(T(s)) &:= (T(s) \cdot I_n) \star \left[ \begin{array}{c|c} \mathbf{A}_N & \mathbf{B}_N \\ \hline \mathbf{C}_N & \mathbf{D}_N \end{array} \right] \\ \mathbf{W}_D(T(s)) &:= (T(s) \cdot I_n) \star \left[ \begin{array}{c|c} \mathbf{A}_D & \mathbf{B}_D \\ \hline \mathbf{C}_D & \mathbf{D}_D \end{array} \right] \end{aligned}$$

with  $\mathbf{A}_N, \mathbf{A}_D \in \mathbb{R}^{n \times n}$ ,  $\mathbf{B}_N, \mathbf{B}_D \in \mathbb{R}^{n \times n_w}$ ,  $\mathbf{C}_N, \mathbf{C}_D \in \mathbb{R}^{n_z \times n}$ ,  $\mathbf{D}_N, \mathbf{D}_D \in \mathbb{R}^{n_z \times n_w}$ .  
 If so, *Compute* them.

### 5.4.2 Magnitude synthesis

There are two types of constraints involved in the magnitude design problem: the positivity constraints and the spectral mask constraints. These constraints can be transformed into LMIs using the KYP Lemma and its extensions. Nonetheless, several versions of the KYP Lemma can be used, depending if the basis  $\mathcal{B}(T(s))$  is viewed as an LFT in  $T(s) \cdot I_n$  or in  $\frac{1}{s} \cdot I_{n_s}$ . The positivity constraints enable to achieve spectral factorisation. As the resulting spectral factor is required to be an LFT in  $T(s) \cdot I_n$ , the associated KYP lemma is used for these constraints. For spectral mask constraints, one gets the choice between the  $T(s)$ -version, leading to conservative LMIs conditions but with less optimisation variables, and the  $\frac{1}{s}$ -version, with equivalent LMIs conditions but of bigger size, of the KYP lemma. As the optimisation problems involved in this work are of moderate size, we choose the second option.

Hence, it is required to express the basis  $\mathcal{B}(T(s))$  as an LFT in  $T(s)$  and as an LFT in  $\frac{1}{s}$ . Suppose that  $\mathcal{B}(T(s))$  is provided as an LFT in  $T(s)$ :

$$\mathcal{B}(T(s)) = (T(s) \cdot I_n) \star \left[ \begin{array}{c|c} A & B \\ \hline C & D \end{array} \right]$$

Assume that  $T(s)$  is given by the following LFT in  $\frac{1}{s}$ :

$$T(s) := \left( \frac{1}{s} \cdot I_{n_T} \right) \star \left[ \begin{array}{c|c} A_T & B_T \\ \hline C_T & D_T \end{array} \right]$$

Then, using the LFT framework (cf Figure 5.3 for a graphical illustration), and the convenient notation of the Kronecker product (Definition A.5, p. 193),  $\mathcal{B}(T(s))$  can

be expressed as an LFT in  $\frac{1}{s}$  as:

$$\mathcal{B}(T(s)) = \left( \frac{1}{s} \cdot I_{n_s} \right) \star \left[ \begin{array}{c|c} A_s & B_s \\ \hline C_s & D_s \end{array} \right]$$

where  $n_s = n \times n_T$  and

$$\left[ \begin{array}{c|c} A_s & B_s \\ \hline C_s & D_s \end{array} \right] := \left[ \begin{array}{c|c} I_n \otimes A_T & I_n \otimes B_T \\ \hline I_n \otimes C_T & I_n \otimes D_T \end{array} \right] \star \left[ \begin{array}{c|c} A & B \\ \hline C & D \end{array} \right] \quad (5.22)$$

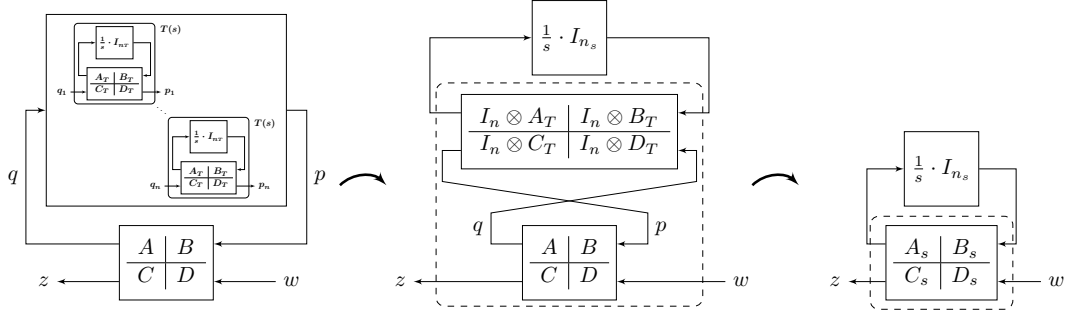


FIGURE 5.3: Expansion of LFT

Theorem 5.5 provides a sufficient condition, formulated as an LMI feasibility problem, for solving the magnitude synthesis problem (Problem 5.4).

**Theorem 5.5.** *Let  $T(s)$  be a  $\{x, y, z\}$ -dissipative transfer function. Consider a given transfer matrix  $\mathcal{B}(T(s))$  equivalently defined by an LFT of  $T(s)$  and an LFT of  $\frac{1}{s}$ :*

$$\mathcal{B}(T(s)) := (T(s) \cdot I_n) \star \left[ \begin{array}{c|c} A & B \\ \hline C & D \end{array} \right] := \left( \frac{1}{s} \cdot I_{n_s} \right) \star \left[ \begin{array}{c|c} A_s & B_s \\ \hline C_s & D_s \end{array} \right]$$

with  $A \in \mathbb{R}^{n \times n}$ ,  $B \in \mathbb{R}^{n \times 1}$ ,  $C \in \mathbb{R}^{(n+1) \times n}$ ,  $D \in \mathbb{R}^{(n+1) \times 1}$ , and  $A_s \in \mathbb{R}^{n_s \times n_s}$ ,  $B_s \in \mathbb{R}^{n_s \times 1}$ ,  $C_s \in \mathbb{R}^{(n+1) \times n_s}$ ,  $D_s \in \mathbb{R}^{(n+1) \times 1}$ . Assume that  $(A, B)$  and  $(A_s, B_s)$  are controllable<sup>6</sup>. Then problem (ii) admits a solution if the optimisation problem defined in (i) is feasible.

- (i)  $\exists \mathbf{X}_N = (\mathbf{X}_N)^T \in \mathbb{R}^{(n+1) \times (n+1)}$ ,  $\exists \mathbf{X}_D = (\mathbf{X}_D)^T \in \mathbb{R}^{(n+1) \times (n+1)}$ ,  
 $\exists \mathbf{P}_N = (\mathbf{P}_N)^T \in \mathbb{R}^{n \times n}$ ,  $\exists \mathbf{P}_D = (\mathbf{P}_D)^T \in \mathbb{R}^{n \times n}$ ,  
 $\forall u \in \{1, \dots, N_U\}$ ,  $\exists \mathbf{D}_u = (\mathbf{D}_u)^T \in \mathbb{R}^{n_s \times n_s}$ ,  $\exists \mathbf{P}_u = (\mathbf{P}_u)^T \in \mathbb{R}^{n_s \times n_s}$ ,  
 $\forall l \in \{1, \dots, N_L\}$ ,  $\exists \mathbf{D}_l = (\mathbf{D}_l)^T \in \mathbb{R}^{n_s \times n_s}$ ,  $\exists \mathbf{P}_l = (\mathbf{P}_l)^T \in \mathbb{R}^{n_s \times n_s}$ ,

$$\mathbf{P}_N > 0 \quad \mathbf{P}_D > 0 \quad (5.23)$$

$$\begin{bmatrix} A & B \\ I & 0 \end{bmatrix}^T \begin{bmatrix} -z\mathbf{P}_N & -y\mathbf{P}_N \\ -y\mathbf{P}_N & -x\mathbf{P}_N \end{bmatrix} \begin{bmatrix} A & B \\ I & 0 \end{bmatrix} + \begin{bmatrix} C^T \\ D^T \end{bmatrix} \mathbf{X}_N \begin{bmatrix} C & D \end{bmatrix} \geq 0 \quad (5.24)$$

$$\begin{bmatrix} A & B \\ I & 0 \end{bmatrix}^T \begin{bmatrix} -z\mathbf{P}_D & -y\mathbf{P}_D \\ -y\mathbf{P}_D & -x\mathbf{P}_D \end{bmatrix} \begin{bmatrix} A & B \\ I & 0 \end{bmatrix} + \begin{bmatrix} C^T \\ D^T \end{bmatrix} \mathbf{X}_D \begin{bmatrix} C & D \end{bmatrix} > 0 \quad (5.25)$$

<sup>6</sup>One may recall that we abusively denote a pair  $(A, B)$  to be controllable if the matrix  $\begin{bmatrix} B & AB & A^2B & \dots & A^{n-1}B \end{bmatrix}$  has full row-rank.

$$\forall u \in \{1, \dots, N_U\}, \quad \mathbf{D}_u > 0$$

$$\begin{bmatrix} A_s & B_s \\ I & 0 \end{bmatrix}^T \begin{bmatrix} -z_{\Omega_u^U} \mathbf{D}_u & -y_{\Omega_u^U}^* \mathbf{D}_u - \mathbf{P}_u \\ -y_{\Omega_u^U} \mathbf{D}_u - \mathbf{P}_u & -x_{\Omega_u^U} \mathbf{D}_u \end{bmatrix} \begin{bmatrix} A_s & B_s \\ I & 0 \end{bmatrix} + \begin{bmatrix} C_s^T \\ D_s^T \end{bmatrix} (U_u^2 \mathbf{X}_D - \mathbf{X}_N) \begin{bmatrix} C_s & D_s \end{bmatrix} \geq 0$$

$$\forall l \in \{1, \dots, N_L\}, \quad \mathbf{D}_l > 0$$

$$\begin{bmatrix} A_s & B_s \\ I & 0 \end{bmatrix}^T \begin{bmatrix} -z_{\Omega_l^L} \mathbf{D}_l & -y_{\Omega_l^L}^* \mathbf{D}_l - \mathbf{P}_l \\ -y_{\Omega_l^L} \mathbf{D}_l - \mathbf{P}_l & -x_{\Omega_l^L} \mathbf{D}_l \end{bmatrix} \begin{bmatrix} A_s & B_s \\ I & 0 \end{bmatrix} + \begin{bmatrix} C_s^T \\ D_s^T \end{bmatrix} (\mathbf{X}_N - L_l^2 \mathbf{X}_D) \begin{bmatrix} C_s & D_s \end{bmatrix} \geq 0$$

where  $x_\Omega, y_\Omega, z_\Omega$  depend on  $\Omega$  and are defined as in Subsection 2.6 (p. 27).

In particular, if  $T(s)$  is lossless  $\{x, y, z\}$ -dissipative, (5.23) can be deleted.

(ii) There exists a solution to Problem 5.4, i.e.

$$\exists \mathbf{X}_N = (\mathbf{X}_N)^T \in \mathbb{R}^{(n+1) \times (n+1)}, \quad \exists \mathbf{X}_D = (\mathbf{X}_D)^T \in \mathbb{R}^{(n+1) \times (n+1)},$$

$$\forall \omega \in \mathbb{R}, \quad \mathcal{B}(T(j\omega))^* \mathbf{X}_N \mathcal{B}(T(j\omega)) \geq 0 \quad (5.26)$$

$$\forall \omega \in \mathbb{R}, \quad \mathcal{B}(T(j\omega))^* \mathbf{X}_D \mathcal{B}(T(j\omega)) > 0 \quad (5.27)$$

$$\forall u \in \{1, \dots, N_U\}, \quad \forall \omega \in \Omega_u^U, \quad \frac{\mathcal{B}(T(j\omega))^* \mathbf{X}_N \mathcal{B}(T(j\omega))}{\mathcal{B}(T(j\omega))^* \mathbf{X}_D \mathcal{B}(T(j\omega))} \leq U_u^2 \quad (5.28)$$

$$\forall l \in \{1, \dots, N_L\}, \quad \forall \omega \in \Omega_l^L, \quad \frac{\mathcal{B}(T(j\omega))^* \mathbf{X}_N \mathcal{B}(T(j\omega))}{\mathcal{B}(T(j\omega))^* \mathbf{X}_D \mathcal{B}(T(j\omega))} \geq L_l^2 \quad (5.29)$$

*Proof.* Applying the dissipative versions of the KYP lemma (cf Subsection 2.6), (5.24) and (5.25) imply (5.26) and (5.27). Moreover, (5.28) and (5.29) are equivalent to:

$$\forall u \in \{1, \dots, N_U\}, \quad \forall \omega \in \Omega_u^U, \quad \mathcal{B}(T(j\omega))^* (U_u^2 \mathbf{X}_D - \mathbf{X}_N) \mathcal{B}(T(j\omega)) \geq 0$$

$$\forall l \in \{1, \dots, N_L\}, \quad \forall \omega \in \Omega_l^L, \quad \mathcal{B}(T(j\omega))^* (\mathbf{X}_N - L_l^2 \mathbf{X}_D) \mathcal{B}(T(j\omega)) \geq 0$$

Applying the frequency-interval version of the KYP lemma, these constraints are equivalent to the remaining LMIs of optimisation problem (i).  $\square$

### 5.4.3 Spectral factorisation

Once the magnitude has been synthesised, the next step is to extract a spectral factor. When  $T(s)$  is lossless dissipative, the procedure to compute such factor is similar to the usual case, up to some loop transformations. However, when  $T(s)$  is dissipative, a term generally appears between the magnitude and the squared modulus of the spectral factor. The spectral factorisation is then said to be *lossy*.

#### 5.4.3.1 Spectral factorisation with lossless dissipative $T(s)$

The usual spectral factorisation technique, when considering a lossless dissipative  $T(s)$ , needs to be adapted. Indeed, the direct application of this approach leads to rank constraints on the left-hand terms of (5.24) and (5.25), which, by the Schur Complement Lemma (Lemma A.2, p. 195), lead to solve algebraic equations of the type:

$$zA^T(-\mathbf{P})A + y(A^T(-\mathbf{P}) + (-\mathbf{P})A) + x(-\mathbf{P}) + C^T X C$$

$$= (y(-\mathbf{P})B + zA^T(-\mathbf{P})B + C^T X D) (D^T X D + zB^T(-\mathbf{P})B)^{-1} (yB^T(-\mathbf{P}) + zB^T(-\mathbf{P})A + D^T X C)$$

This equation is a generalised form of the (ARE), where  $\{x, y, z\} = \{0, 1, 0\}$ , and the so-called discrete ARE, where  $\{x, y, z\} = \{-1, 0, 1\}$ . Unfortunately, there are not algorithms currently available to directly solve it. Instead, the idea here is to use congruency relations in order to transform this equation into an ARE. Indeed, for an appropriate choice of  $\alpha, \beta, \gamma$  one can note that:

$$\begin{bmatrix} z(-P) & y(-P) \\ y(-P) & x(-P) \end{bmatrix} = \left( \frac{1}{\alpha\gamma - \beta^2} \begin{bmatrix} \gamma I & -\beta I \\ -\beta I & \alpha I \end{bmatrix} \right)^T \begin{bmatrix} 0 & -P \\ -P & 0 \end{bmatrix} \left( \frac{1}{\alpha\gamma - \beta^2} \begin{bmatrix} \gamma I & -\beta I \\ -\beta I & \alpha I \end{bmatrix} \right)$$

Therefore, by propagating this congruency relation, this leads to have left-hand terms of (5.24) and (5.25) congruent to a matrix comparable to the left-hand terms of the usual KYP lemma, which is then congruent to an ARE. Lemma 5.1 details last statement for the most common case. For the general case, and the proof, refer to p. 207 of Appendix B.

**Lemma 5.1.** Assume that  $\begin{bmatrix} x & y \\ y & z \end{bmatrix}$  is a full-rank matrix with  $x < 0$ . Define  $r$  and  $c$  such that:

$$c := -\frac{y}{x} \quad r^2 := \frac{y^2 - xz}{x^2}$$

Define:

$$\begin{aligned} Q(P) &:= \begin{bmatrix} A & B \\ I & 0 \end{bmatrix}^T \begin{bmatrix} -zP & -yP \\ -yP & -xP \end{bmatrix} \begin{bmatrix} A & B \\ I & 0 \end{bmatrix} + \begin{bmatrix} C^T \\ D^T \end{bmatrix} X \begin{bmatrix} C & D \end{bmatrix} \\ \hat{Q}(P) &:= \begin{bmatrix} \hat{A} & \hat{B} \\ I & 0 \end{bmatrix}^T \begin{bmatrix} 0 & -P \\ -P & 0 \end{bmatrix} \begin{bmatrix} \hat{A} & \hat{B} \\ I & 0 \end{bmatrix} + \begin{bmatrix} \hat{C}^T \\ \hat{D}^T \end{bmatrix} X \begin{bmatrix} \hat{C} & \hat{D} \end{bmatrix} \end{aligned}$$

with

$$\left[ \begin{array}{c|c} \hat{A} & \hat{B} \\ \hline \hat{C} & \hat{D} \end{array} \right] := \left[ \begin{array}{c|c} ((c^2 - r^2)A - (c+r)I)(I - (c+r)A)^{-1} & -r\sqrt{-2x(c+r)}(I - (c+r)A)^{-1}B \\ \hline \sqrt{\frac{2(c+r)}{-x}}C(I - (c+r)A)^{-1} & D + (c+r)C(I - (c+r)A)^{-1}B \end{array} \right]$$

Then  $Q(P)$  and  $\hat{Q}(P)$  are congruent:

$$\Pi^T Q(P) \Pi = \hat{Q}(P)$$

with

$$\Pi = \begin{bmatrix} \sqrt{\frac{2(c+r)}{-x}}C(I - (c+r)A)^{-1} & D + (c+r)C(I - (c+r)A)^{-1}B \\ 0 & I \end{bmatrix}$$

*Remark 5.4.* As explained in [Zar13], from an LFT perspective, last lemma is equivalent of a process called *loop-shifting* and  $\left[ \begin{array}{c|c} \hat{A} & \hat{B} \\ \hline \hat{C} & \hat{D} \end{array} \right]$  can be calculated as:

$$\left[ \begin{array}{c|c} \hat{A} & \hat{B} \\ \hline \hat{C} & \hat{D} \end{array} \right] = \left[ \begin{array}{c|c} -(c+r)I & -r\sqrt{-2x(c+r)}I \\ \hline \sqrt{\frac{2(c+r)}{-x}}I & (c+r)I \end{array} \right] \star \left[ \begin{array}{c|c} A & B \\ \hline C & D \end{array} \right]$$

This enables to extend the spectral factorisation technique for transfer function expressed as LFT of repeated, lossless dissipative  $T(s)$ .

**Theorem 5.6** (Lossless Spectral Factorisation). *Let  $T(s)$  be a lossless  $\{x, y, z\}$ -dissipative transfer function. Let  $X = X^T \in \mathbb{R}^{n_z \times n_z}$  be a given real symmetric matrix, and  $\mathcal{B}(T(s))$  a stable,  $n_z \times n_w$  transfer matrix defined by:*

$$\mathcal{B}(T(s)) = (T(s) \cdot I_n) \star \left[ \begin{array}{c|c} A & B \\ \hline C & D \end{array} \right]$$

with  $A \in \mathbb{R}^{n \times n}$ ,  $B \in \mathbb{R}^{n \times n_w}$ ,  $C \in \mathbb{R}^{n_z \times n}$  and  $D \in \mathbb{R}^{n_z \times n_w}$ . Define the matrices  $\hat{A} \in \mathbb{R}^{n \times n}$ ,  $\hat{B} \in \mathbb{R}^{n \times n_w}$ ,  $\hat{C} \in \mathbb{R}^{n_z \times n}$ ,  $\hat{D} \in \mathbb{R}^{n_z \times n_w}$  as in Lemma 5.1. Assume that the pairs  $(A, B)$  and  $(\hat{A}, \hat{B})$  are controllable. In addition, assume that  $\hat{D}^T X \hat{D} > 0$  and there exists a symmetric matrix  $P = P^T$  such that

$$Q(P) := \begin{bmatrix} A & B \\ I & 0 \end{bmatrix}^T \begin{bmatrix} -zP & -yP \\ -yP & -xP \end{bmatrix} \begin{bmatrix} A & B \\ I & 0 \end{bmatrix} + \begin{bmatrix} C^T \\ D^T \end{bmatrix} X \begin{bmatrix} C & D \end{bmatrix} \geq 0$$

Then (i)  $\Rightarrow$  (ii).

(i) There exists a symmetric matrix  $\mathbf{P} = \mathbf{P}^T \in \mathbb{R}^{n \times n}$  solution of the (ARE):

$$\hat{R}(\mathbf{P}) = 0$$

where

$$\hat{R}(\mathbf{P}) := \hat{A}^T(-\mathbf{P}) + (-\mathbf{P})\hat{A} + \hat{C}^T X \hat{C} - ((-\mathbf{P})\hat{B} + \hat{C}^T X \hat{D}) (\hat{D}^T X \hat{D})^{-1} (\hat{B}^T(-\mathbf{P}) + \hat{D}^T X \hat{C}) \quad (5.30)$$

(ii) There exist  $\mathbf{C}_H \in \mathbb{R}^{n_w \times n}$  and  $\mathbf{D}_H \in \mathbb{R}^{n_w \times n_w}$  such that:

$$\forall \omega \in \mathbb{R}, \quad \mathcal{B}(T(j\omega))^* X \mathcal{B}(T(j\omega)) = \mathbf{H}(T(j\omega))^* \mathbf{H}(T(j\omega)) \quad (5.31)$$

where  $\mathbf{H}(T(s))$  is a stable transfer function, defined by the LFT in  $T(s)$ :

$$\mathbf{H}(T(s)) = (T(s) \cdot I_n) \star \left[ \begin{array}{c|c} A & B \\ \hline \mathbf{C}_H & \mathbf{D}_H \end{array} \right] \quad (5.32)$$

*Proof.* By Lemma 5.1, the following congruency relation for  $Q(P)$  is found:

$$Q(P) = \hat{\Pi}^T \left( \begin{bmatrix} \hat{A} & \hat{B} \\ I & 0 \end{bmatrix}^T \begin{bmatrix} 0 & -P \\ -P & 0 \end{bmatrix} \begin{bmatrix} \hat{A} & \hat{B} \\ I & 0 \end{bmatrix} + \begin{bmatrix} \hat{C}^T \\ \hat{D}^T \end{bmatrix} X \begin{bmatrix} \hat{C} & \hat{D} \end{bmatrix} \right) \hat{\Pi}$$

Applying the Schur complement lemma (Lemma A.2, p. 195), as  $\hat{D}^T X \hat{D} > 0$ , the following congruency relation is established:

$$Q(P) = \hat{\Pi}^T \Pi^T \begin{bmatrix} \hat{R}(P) & 0 \\ 0 & \hat{D}^T X \hat{D} \end{bmatrix} \Pi \hat{\Pi}$$

Therefore, if  $\mathbf{P}$  is a solution to the ARE  $\hat{R}(\mathbf{P}) = 0$ , by congruency one has:

$$\text{rank}(Q(\mathbf{P})) = \text{rank}(\hat{D}^T X \hat{D}) = n_w$$



Moreover, as  $Q(\mathbf{P}) \geq 0$ , there exist  $\mathbf{C}_H \in \mathbb{R}^{n_w \times n}$  and  $\mathbf{D}_H \in \mathbb{R}^{n_w \times n_w}$  such that:

$$Q(\mathbf{P}) = \begin{bmatrix} \mathbf{C}_H^T \\ \mathbf{D}_H^T \end{bmatrix} \begin{bmatrix} \mathbf{C}_H & \mathbf{D}_H \end{bmatrix}$$

Furthermore, pre-multiplying by  $\begin{bmatrix} (T(-s) \cdot I) (I - A(T(-s) \cdot I))^{-1} B \\ I \end{bmatrix}^T$  and post-multiplying by  $\begin{bmatrix} (T(s) \cdot I) (I - A(T(s) \cdot I))^{-1} B \\ I \end{bmatrix}$  last equation, and defining  $F(T(s))$  as:

$$F(T(s)) := A(T(s) \cdot I) (I - A(T(s) \cdot I))^{-1} B + B = (I - A(T(s) \cdot I))^{-1} B \quad (5.33)$$

little calculation gives:

$$\begin{aligned} F(T(-s))^T \begin{bmatrix} I \\ T(-s) \cdot I \end{bmatrix}^T \begin{bmatrix} -z\mathbf{P} & -y\mathbf{P} \\ -y\mathbf{P} & -x\mathbf{P} \end{bmatrix} \begin{bmatrix} I \\ T(s) \cdot I \end{bmatrix} F(T(s)) + \mathcal{B}(T(-s))^T X \mathcal{B}(T(s)) \\ = \mathbf{H}(T(-s))^T \mathbf{H}(T(s)) \end{aligned}$$

where  $\mathbf{H}(T(s))$  is defined as in (5.32), and is stable by stability of  $\mathcal{B}(T(s))$ . Finally, as  $T$  is lossless dissipative, one can easily check that:

$$\forall \omega \in \mathbb{R}, \quad \begin{bmatrix} I \\ T(j\omega) \cdot I \end{bmatrix}^* \begin{bmatrix} -z\mathbf{P} & -y\mathbf{P} \\ -y\mathbf{P} & -x\mathbf{P} \end{bmatrix} \begin{bmatrix} I \\ T(j\omega) \cdot I \end{bmatrix} = - \begin{bmatrix} T(j\omega) \cdot I \\ I \end{bmatrix}^* \begin{bmatrix} x\mathbf{P} & y\mathbf{P} \\ y\mathbf{P} & z\mathbf{P} \end{bmatrix} \begin{bmatrix} T(j\omega) \cdot I \\ I \end{bmatrix} = 0$$

holds and then (5.31) is verified.  $\square$

#### 5.4.3.2 Lossy spectral factorisation with dissipative $T(s)$

Theorem 5.6 can not be directly extended when  $T(s)$  is *not* lossless dissipative. Indeed, using same reasoning than in the proof of Theorem 5.6 leads to:

$$\forall \omega \in \mathbb{R}, \quad -E_P(T(j\omega)) + \mathcal{B}(T(j\omega))^* X \mathcal{B}(T(j\omega)) = H(T(j\omega))^* H(T(j\omega))$$

where

$$E_P(T(j\omega)) := F(T(j\omega))^* \begin{bmatrix} T(j\omega) \cdot I \\ I \end{bmatrix}^* \begin{bmatrix} x\mathbf{P} & y\mathbf{P} \\ y\mathbf{P} & z\mathbf{P} \end{bmatrix} \begin{bmatrix} T(j\omega) \cdot I \\ I \end{bmatrix} F(T(j\omega))$$

Thus, if  $T$  is not lossless dissipative, the term

$$\begin{bmatrix} T(j\omega) \cdot I \\ I \end{bmatrix}^* \begin{bmatrix} x\mathbf{P} & y\mathbf{P} \\ y\mathbf{P} & z\mathbf{P} \end{bmatrix} \begin{bmatrix} T(j\omega) \cdot I \\ I \end{bmatrix} \neq 0$$

is not null and there is a mismatch  $-E_P(T(j\omega))$  between  $\mathcal{B}(T(j\omega))^* X \mathcal{B}(T(j\omega))$  and  $H(T(j\omega))^* H(T(j\omega))$ . The transfer function  $W(T(s)) = \frac{W_N(T(s))}{W_D(T(s))}$ , given by the resulting spectral factors  $W_N(T(s))$  and  $W_D(T(s))$ , has then the following squared magnitude:

$$W(T(j\omega))^* W(T(j\omega)) = \frac{-E_{P_N}(T(j\omega)) + \mathcal{B}(T(j\omega))^* X_N \mathcal{B}(T(j\omega))}{-E_{P_D}(T(j\omega)) + \mathcal{B}(T(j\omega))^* X_D \mathcal{B}(T(j\omega))}$$

and is not guaranteed to satisfy the spectral mask  $\mathcal{SM}$ . The terms  $-E_{P_N}(T(j\omega))$  and  $-E_{P_D}(T(j\omega))$  are considered as *factorisation errors* and are directly linked with the lossy dissipative characterisation of  $T(s)$ . In [Zar13, Chap.5], the synthesis approach is directly extended, without taking explicitly into account these errors. In the particular case they consider, these errors are not too important and the resulting  $W(T(s))$  still satisfies the spectral mask  $\mathcal{SM}$ . However, in a more general case,  $\mathcal{SM}$  may not be satisfied. Therefore, in the sequel, we provide a more general synthesis approach, able to tackle the LFT filter synthesis problem, for any dissipative  $T(s)$ .

#### 5.4.4 Generalised synthesis with factorisation error management

As the usual spectral factorisation technique can not be directly extended to the dissipative case, a generalisation of the usual procedure is required. An idea is to explicitly take into account the factorisation errors into the magnitude synthesis step. However, this requires to have the knowledge of these errors. Both steps, magnitude synthesis and spectral factorisation, need then to be simultaneously tackled. This is achieved in Theorem 5.7.

**Theorem 5.7.** *Let  $T(s)$  be a  $\{x, y, z\}$ -dissipative transfer function. Consider a stable transfer vector  $\mathcal{B}(T(s))$  defined by an LFT of  $T(s)$ :*

$$\mathcal{B}(T(s)) := (T(s) \cdot I_n) \star \left[ \begin{array}{c|c} A & B \\ \hline C & D \end{array} \right]$$

with  $A \in \mathbb{R}^{n \times n}$ ,  $B \in \mathbb{R}^{n \times 1}$ ,  $C \in \mathbb{R}^{(n+1) \times n}$  and  $D \in \mathbb{R}^{(n+1) \times 1}$ . Define the matrices  $\hat{A} \in \mathbb{R}^{n \times n}$ ,  $\hat{B} \in \mathbb{R}^{n \times 1}$ ,  $\hat{C} \in \mathbb{R}^{(n+1) \times n}$ ,  $\hat{D} \in \mathbb{R}^{(n+1) \times 1}$  as in Lemma 5.1. Moreover, define the matrix-valued function  $\hat{R}(\cdot, \cdot)$  as:

$$\hat{R}(P, X) := \hat{A}^T(-P) + (-P)\hat{A} + \hat{C}^T X \hat{C} - ((-P)\hat{B} + \hat{C}^T X \hat{D}) (\hat{D}^T X \hat{D})^{-1} (\hat{B}^T(-P) + \hat{D}^T X \hat{C})$$

Finally, define  $F(T(s))$  as in (5.33) and the stable transfer vector  $\bar{\mathcal{B}}(s)$  as follows:

$$\bar{\mathcal{B}}(s) := \begin{bmatrix} T(s)F(T(s)) \\ F(T(s)) \\ \mathcal{B}(T(s)) \end{bmatrix} = \left( \frac{1}{s} \cdot I_{n_s} \right) \star \left[ \begin{array}{c|c} \bar{A}_s & \bar{B}_s \\ \hline \bar{C}_s & \bar{D}_s \end{array} \right]$$

with  $\bar{A}_s \in \mathbb{R}^{n_s \times n_s}$ ,  $\bar{B}_s \in \mathbb{R}^{n_s \times 1}$ ,  $\bar{C}_s \in \mathbb{R}^{(n+1) \times n_s}$ ,  $\bar{D}_s \in \mathbb{R}^{(n+1) \times 1}$ .

Assume that the pairs  $(A, B)$ ,  $(\hat{A}, \hat{B})$  and  $(\bar{A}_s, \bar{B}_s)$  are controllable.

If the optimisation problem (i) is feasible, then problem (ii) admits a solution.

- (i)  $\exists \mathbf{X}_N = (\mathbf{X}_N)^T \in \mathbb{R}^{(n+1) \times (n+1)}$ ,  $\exists \mathbf{X}_D = (\mathbf{X}_D)^T \in \mathbb{R}^{(n+1) \times (n+1)}$ ,  
 $\exists \mathbf{P}_N = (\mathbf{P}_N)^T \in \mathbb{R}^{n \times n}$ ,  $\exists \mathbf{P}_D = (\mathbf{P}_D)^T \in \mathbb{R}^{n \times n}$ ,  
 $\forall u \in \{1, \dots, N_U\}$ ,  $\exists \mathbf{D}_u = (\mathbf{D}_u)^T \in \mathbb{R}^{n_s \times n_s}$ ,  $\exists \mathbf{P}_u = (\mathbf{P}_u)^T \in \mathbb{R}^{n_s \times n_s}$ ,  
 $\forall l \in \{1, \dots, N_L\}$ ,  $\exists \mathbf{D}_l = (\mathbf{D}_l)^T \in \mathbb{R}^{n_s \times n_s}$ ,  $\exists \mathbf{P}_l = (\mathbf{P}_l)^T \in \mathbb{R}^{n_s \times n_s}$ ,

$$\hat{D}^T \mathbf{X}_N \hat{D} > 0 \quad \hat{D}^T \mathbf{X}_D \hat{D} > 0 \quad (5.34)$$

$$\hat{R}(\mathbf{P}_N, \mathbf{X}_N) = 0 \quad \hat{R}(\mathbf{P}_D, \mathbf{X}_D) = 0 \quad (5.35)$$

$$\begin{bmatrix} \bar{A}_s & \bar{B}_s \\ I & 0 \end{bmatrix}^T \begin{bmatrix} 0 & -\mathbf{P}_D^{pos} \\ -\mathbf{P}_D^{pos} & 0 \end{bmatrix} \begin{bmatrix} \bar{A}_s & \bar{B}_s \\ I & 0 \end{bmatrix} + \begin{bmatrix} \bar{C}_s^T \\ \bar{D}_s^T \end{bmatrix} \bar{\mathbf{X}}_D \begin{bmatrix} \bar{C}_s & \bar{D}_s \end{bmatrix} > 0 \quad (5.36)$$

$$\begin{aligned} \forall u \in \{1, \dots, N_U\}, \quad & \mathbf{D}_u > 0 \\ \begin{bmatrix} \bar{A}_s & \bar{B}_s \\ I & 0 \end{bmatrix}^T \begin{bmatrix} -z_{\Omega_u^U} \mathbf{D}_u & -y_{\Omega_u^U}^* \mathbf{D}_u - \mathbf{P}_u \\ -y_{\Omega_u^U} \mathbf{D}_u - \mathbf{P}_u & -x_{\Omega_u^U} \mathbf{D}_u \end{bmatrix} \begin{bmatrix} \bar{A}_s & \bar{B}_s \\ I & 0 \end{bmatrix} + \begin{bmatrix} \bar{C}_s^T \\ \bar{D}_s^T \end{bmatrix} (U_u^2 \bar{\mathbf{X}}_D - \bar{\mathbf{X}}_N) \begin{bmatrix} \bar{C}_s & \bar{D}_s \end{bmatrix} \geq 0 \end{aligned} \quad (5.37)$$

$$\begin{aligned} \forall l \in \{1, \dots, N_L\}, \quad & \mathbf{D}_l > 0 \\ \begin{bmatrix} \bar{A}_s & \bar{B}_s \\ I & 0 \end{bmatrix}^T \begin{bmatrix} -z_{\Omega_l^L} \mathbf{D}_l & -y_{\Omega_l^L}^* \mathbf{D}_l - \mathbf{P}_l \\ -y_{\Omega_l^L} \mathbf{D}_l - \mathbf{P}_l & -x_{\Omega_l^L} \mathbf{D}_l \end{bmatrix} \begin{bmatrix} \bar{A}_s & \bar{B}_s \\ I & 0 \end{bmatrix} + \begin{bmatrix} \bar{C}_s^T \\ \bar{D}_s^T \end{bmatrix} (\bar{\mathbf{X}}_N - L_l^2 \bar{\mathbf{X}}_D) \begin{bmatrix} \bar{C}_s & \bar{D}_s \end{bmatrix} \geq 0 \end{aligned} \quad (5.38)$$

where

$$\bar{\mathbf{X}}_N := \begin{bmatrix} -x\mathbf{P}_N & -y\mathbf{P}_N & 0 \\ -y\mathbf{P}_N & -z\mathbf{P}_N & 0 \\ 0 & 0 & \mathbf{X}_N \end{bmatrix} \quad \bar{\mathbf{X}}_D := \begin{bmatrix} -x\mathbf{P}_D & -y\mathbf{P}_D & 0 \\ -y\mathbf{P}_D & -z\mathbf{P}_D & 0 \\ 0 & 0 & \mathbf{X}_D \end{bmatrix}$$

and  $x_\Omega, y_\Omega, z_\Omega$  depend on  $\Omega$  and are defined as in Subsection 2.6 (p. 27).

(ii) There exist stable transfer functions  $\mathbf{W}_N(T(s))$  and  $\mathbf{W}_D(T(s))$  defined by:

$$\mathbf{W}_N(T(s)) := (T(s) \cdot I_n) \star \left[ \begin{array}{c|c} A & B \\ \hline \mathbf{C}_N & \mathbf{D}_N \end{array} \right] \quad \mathbf{W}_D(T(s)) := (T(s) \cdot I_n) \star \left[ \begin{array}{c|c} A & B \\ \hline \mathbf{C}_D & \mathbf{D}_D \end{array} \right]$$

with  $[\mathbf{C}_N \ \mathbf{D}_N] \in \mathbb{R}^{1 \times (n+1)}$  and  $[\mathbf{C}_D \ \mathbf{D}_D] \in \mathbb{R}^{1 \times (n+1)}$ , such that

$$\forall u \in \{1, \dots, N_U\}, \quad \forall \omega \in \Omega_u^U, \quad \left| \frac{\mathbf{W}_N(T(j\omega))}{\mathbf{W}_D(T(j\omega))} \right|^2 \leq U_u^2 \quad (5.39)$$

$$\forall l \in \{1, \dots, N_L\}, \quad \forall \omega \in \Omega_l^L, \quad \left| \frac{\mathbf{W}_N(T(j\omega))}{\mathbf{W}_D(T(j\omega))} \right|^2 \geq L_l^2 \quad (5.40)$$

hold.

*Proof.* From Lemma 5.1 and constraints (5.34)-(5.35), it comes that there exist  $\mathbf{C}_N, \mathbf{C}_D \in \mathbb{R}^{1 \times n}$  and  $\mathbf{D}_N, \mathbf{D}_D \in \mathbb{R}$  such that:

$$\begin{aligned} \begin{bmatrix} A & B \\ I & 0 \end{bmatrix}^T \begin{bmatrix} -z\mathbf{P}_N & -y\mathbf{P}_N \\ -y\mathbf{P}_N & -x\mathbf{P}_N \end{bmatrix} \begin{bmatrix} A & B \\ I & 0 \end{bmatrix} + \begin{bmatrix} \mathbf{C}_N^T \\ \mathbf{D}_N^T \end{bmatrix} \mathbf{X}_N \begin{bmatrix} C & D \end{bmatrix} &= \begin{bmatrix} \mathbf{C}_N^T \\ \mathbf{D}_N^T \end{bmatrix} \begin{bmatrix} \mathbf{C}_N & \mathbf{D}_N \end{bmatrix} \\ \begin{bmatrix} A & B \\ I & 0 \end{bmatrix}^T \begin{bmatrix} -z\mathbf{P}_D & -y\mathbf{P}_D \\ -y\mathbf{P}_D & -x\mathbf{P}_D \end{bmatrix} \begin{bmatrix} A & B \\ I & 0 \end{bmatrix} + \begin{bmatrix} \mathbf{C}_D^T \\ \mathbf{D}_D^T \end{bmatrix} \mathbf{X}_D \begin{bmatrix} C & D \end{bmatrix} &= \begin{bmatrix} \mathbf{C}_D^T \\ \mathbf{D}_D^T \end{bmatrix} \begin{bmatrix} \mathbf{C}_D & \mathbf{D}_D \end{bmatrix} \end{aligned}$$

Pre- and post-multiplying by  $\begin{bmatrix} T(-s)(I - AT(-s))^{-1}B \\ I \end{bmatrix}^T$  and  $\begin{bmatrix} T(s)(I - AT(s))^{-1}B \\ I \end{bmatrix}$  leads to:

$$\begin{aligned} \bar{\mathbf{B}}(T(-s))^T \bar{\mathbf{X}}_N \bar{\mathbf{B}}(T(s)) &= \mathbf{W}_N(T(-s))^T \mathbf{W}_N(T(s)) \\ \bar{\mathbf{B}}(T(-s))^T \bar{\mathbf{X}}_D \bar{\mathbf{B}}(T(s)) &= \mathbf{W}_D(T(-s))^T \mathbf{W}_D(T(s)) \end{aligned}$$

By the KYP lemma and its dissipative extensions, it comes that (5.36) implies that

$$\forall \omega \in \mathbb{R}, \quad |\mathbf{W}_D(T(j\omega))|^2 > 0$$

holds, while (5.37) and (5.38) respectively are equivalent to (5.39) and (5.40).  $\square$

Optimisation problem (i) of last theorem is not convex. This is due to the AREs (5.35). Nonetheless, it has been previously shown (p. 121) how to obtain a particular solution of the ARE using a linear minimisation problem under an LMI constraint. Indeed, minimal solutions can be obtained by minimising the trace of  $\mathbf{P}_N$  and the trace of  $\mathbf{P}_D$  such that:

$$\begin{bmatrix} A & B \\ I & 0 \end{bmatrix}^T \begin{bmatrix} -z\mathbf{P}_N & -y\mathbf{P}_N \\ -y\mathbf{P}_N & -x\mathbf{P}_N \end{bmatrix} \begin{bmatrix} A & B \\ I & 0 \end{bmatrix} + \begin{bmatrix} C^T \\ D^T \end{bmatrix} \mathbf{X}_N \begin{bmatrix} C & D \end{bmatrix} \geq 0 \quad (5.41)$$

$$\begin{bmatrix} A & B \\ I & 0 \end{bmatrix}^T \begin{bmatrix} -z\mathbf{P}_D & -y\mathbf{P}_D \\ -y\mathbf{P}_D & -x\mathbf{P}_D \end{bmatrix} \begin{bmatrix} A & B \\ I & 0 \end{bmatrix} + \begin{bmatrix} C^T \\ D^T \end{bmatrix} \mathbf{X}_D \begin{bmatrix} C & D \end{bmatrix} \geq 0 \quad (5.42)$$

Hence, we propose to exchange the resolution of the AREs by the resolution of such linear minimisation LMI problem, leading to the following linear minimisation LMI problem:

$$\left. \begin{array}{l} \min \quad \mathbf{X}_N, \mathbf{X}_D, \mathbf{P}_N, \mathbf{P}_D, \\ \quad \mathbf{P}_D^{pos}, P_u, D_u > 0, P_l, D_l > 0 \\ \text{subject to} \end{array} \quad \left. \begin{array}{l} \text{Trace}(\mathbf{P}_N) + \text{Trace}(\mathbf{P}_D) \\ (5.41), (5.42), (5.34), (5.36), (5.37), (5.38) \end{array} \right\} \quad (5.43)$$

One may notice that is just a proposition and it is *not* claimed that (5.43) is equivalent to the feasibility problem of statement (i) of Theorem 5.7. Indeed, (5.37) and (5.38) induce a coupling between two AREs. As a result, the optimal solutions  $P_N^*$  and  $P_D^*$  of (5.43) may not be solutions of (5.35). This point needs to be made clearer, perhaps using the approach developed in [RG96; Ram97] for solving coupled AREs using LMIs.

Assuming that the solution of this optimisation exists and is found, and that the resulting optimal argument  $P_N^*$  and  $P_D^*$  are the minimal solutions of the associated AREs. The resulting spectral factors  $W_N(T(s))$  and  $W_D(T(s))$  are then inversely stable. Hence, the transfer function  $W(T(s)) = \frac{W_N(T(s))}{W_D(T(s))}$  is stable and inversely stable. As a result, such a solution of (5.43) enables to solve Problem 5.1, with the particularity that  $W(T(s))$  is minimum-phase.

**Remark on implementation** One may find challenging to efficiently solve optimisation problem (5.43) by implementing it in this form. Indeed, the solutions  $P_N^*$  and  $P_D^*$  to be found are precisely the extreme solutions of (5.41) and (5.42). This typically leads to numerical issues.

A major source of these numerical issues is the *homogeneity* feature of the constraints (5.41), (5.42), (5.34), (5.36), (5.37), (5.38). This means that if a set of optimisation variables  $x := (X_N, X_D, P_N, P_D, P_D^{pos}, P_u, D_u, P_l, D_l)$  satisfies these constraints, then any  $\alpha \cdot x$  with  $\alpha > 0$  will also satisfy them. In order to remove this feature, one can freely fix an entry of either  $X_N$  or  $X_D$ . Indeed, recalling that  $X_N$  and  $X_D$  are directly linked with the coefficients of  $W_N(T(s))$  and  $W_D(T(s))$ , one may note that for instance:

$$W(T(s)) = \frac{W_N(T(s))}{W_D(T(s))} := \frac{b_0 + b_1 \cdot T(s) + \dots + b_n \cdot T(s)}{a_0 + a_1 \cdot T(s) + \dots + a_n \cdot T(s)} = \frac{\tilde{b}_0 + \tilde{b}_1 \cdot T(s) + \dots + \tilde{b}_n \cdot T(s)}{1 + \tilde{a}_1 \cdot T(s) + \dots + \tilde{a}_n \cdot T(s)}$$

with  $\tilde{b}_i = \frac{b_i}{a_0}$  and  $\tilde{a}_i = \frac{a_i}{a_0}$ . Therefore, fixing such value *breaks* the homogeneity feature without modifying the initial problem, and prevents some numerical issues.

### 5.4.5 Synthesis procedures and numerical examples

In this subsection, the two synthesis procedures, when  $T(s)$  is *lossless* dissipative or *lossy* dissipative, are summarised and numerically illustrated. Even though the second procedure can be applied to the first case, the first procedure is preferred as generally leading to less numerical issues.

#### 5.4.5.1 Lossless dissipative $T(s)$

The overall procedure to solve Problem 5.1, with  $T(s)$  lossless dissipative, is summarised and illustrated step-by-step below. For the illustration,  $T(s)$  is taken as:

$$T(s) := \frac{1}{s+1} = \left(\frac{1}{s} \cdot I_1\right) \star \left[ \begin{array}{c|c} A_T & B_T \\ \hline C_T & D_T \end{array} \right] := \left(\frac{1}{s} \cdot I_1\right) \star \left[ \begin{array}{c|c} -1 & 1 \\ \hline 1 & 0 \end{array} \right]$$

and is lossless dissipative with, for example,  $\{x, y, z\} = \{-1, 0.5, 0\}$ . As shown in Figure 5.4, the Nyquist plot of  $T(s)$  is a circle with centre  $c := \frac{-y}{x} = 0.5$  and radius  $r := \sqrt{\frac{y^2 - xz}{x^2}} = 0.5$ . The considered spectral mask  $\mathcal{SM}$  is the following:

$$\begin{aligned} \forall \omega \in \Omega_L = [0, \omega_L], & \quad |W(T(j\omega))|^2 \geq L^2 \\ \forall \omega \in \Omega_U = [\omega_U, +\infty), & \quad |W(T(j\omega))|^2 \leq U^2 \end{aligned}$$

with  $\omega_L = 1$  rad/s,  $L = 10$  dB,  $\omega_U = 10$  rad/s and  $U = -10$  dB.

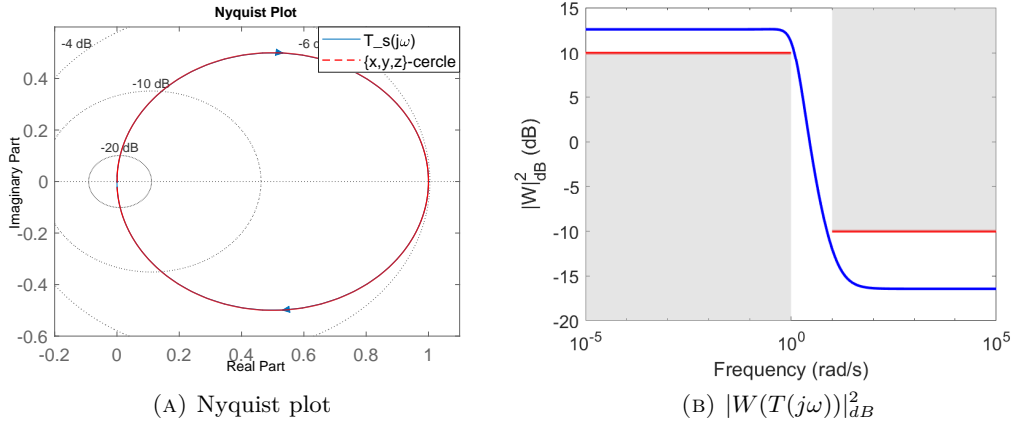


FIGURE 5.4: Illustrative example -  $T(s)$  lossless dissipative

1. Choose the order  $n$ .
2. Define a stable basis  $\mathcal{B}(T(s))$  of order  $n$  under the LFT form:

$$\mathcal{B}(T(s)) = (T(s) \cdot I_n) \star \left[ \begin{array}{c|c} A & B \\ \hline C & D \end{array} \right] = \left(\frac{1}{s} \cdot I_{n_s}\right) \star \left[ \begin{array}{c|c} A_s & B_s \\ \hline C_s & D_s \end{array} \right]$$

For the illustration, choose:

$$\mathcal{B}(T(s)) = \frac{1}{(T(s) + 1)^n} \cdot \begin{bmatrix} 1 \\ T(s) \\ \vdots \\ T(s)^n \end{bmatrix} = (T(s) \cdot I_n) \star \left[ \begin{array}{cccc|c} 0 & 1 & & & 0 \\ & \ddots & \ddots & & \vdots \\ & & 0 & 1 & 0 \\ -a_0 & \dots & -a_{n-2} & -a_{n-1} & 1 \\ -a_0 & \dots & -a_{n-2} & -a_{n-1} & 1 \\ & & & 1 & 0 \\ & & & & 0 \\ & \ddots & & & \vdots \\ 1 & & & & 0 \end{array} \right] \quad (5.44)$$

where  $a_0, \dots, a_{n-1}$  are such that

$$(T(s) + 1)^n = a_0 + a_1 \cdot T(s) + \dots + a_{n-1} \cdot T(s)^{n-1} + T(s)^n$$

Define  $A_s$ ,  $B_s$ ,  $C_s$  and  $D_s$  as in (5.22) (p. 130).

3. Solve feasibility problem (i) defined in Theorem 5.5 (p.130).

If it is feasible, there exist symmetric matrices  $X_N = X_N^T$  and  $X_D = X_D^T$  such that the magnitude defined by

$$M(T(j\omega)) = \frac{\mathcal{B}(T(j\omega))^* X_N \mathcal{B}(T(j\omega))}{\mathcal{B}(T(j\omega))^* X_D \mathcal{B}(T(j\omega))}$$

is positive and satisfies the spectral mask  $\mathcal{SM}$ .

If it is not feasible, then increment  $n$ :  $n \leftarrow n + 1$ , and back to step 2.

For our illustration, the minimal order  $n$  is  $n = 2$ , and the resulting matrices are given by:

$$X_N = \begin{bmatrix} 1.02 & 57.25 & 52.58 \\ 57.25 & 156.61 & 289.06 \\ 52.58 & 289.06 & 628.49 \end{bmatrix} \quad X_D = \begin{bmatrix} 44.89 & -48.35 & 12.29 \\ -48.35 & 32.06 & 16.08 \\ 12.29 & 16.08 & 49.76 \end{bmatrix}$$

4. Apply independently Theorem 5.6 (p. 132) on the magnitudes  $\mathcal{B}(T(j\omega))^* X_N \mathcal{B}(T(j\omega))$  and  $\mathcal{B}(T(j\omega))^* X_D \mathcal{B}(T(j\omega))$  to get two stable, minimal spectral factors  $W_N(T(s))$  and  $W_D(T(s))$  such that:

$$\begin{aligned} |W_N(T(j\omega))|^2 &= \mathcal{B}(T(j\omega))^* X_N \mathcal{B}(T(j\omega)) \\ |W_D(T(j\omega))|^2 &= \mathcal{B}(T(j\omega))^* X_D \mathcal{B}(T(j\omega)) \end{aligned}$$

This requires to independently solve two AREs defined in Theorem 5.6. Choose the minimal solution  $P = P_{D_{min}}$  for  $W_D$  to be minimum-phase.  $W_N$  and  $W_D$  are then two LFTs of the form:

$$\begin{aligned} W_N(T(s)) &:= (T(s) \cdot I_n) \star \left[ \begin{array}{c|c} A & B \\ \hline C_N & D_N \end{array} \right] \\ W_D(T(s)) &:= (T(s) \cdot I_n) \star \left[ \begin{array}{c|c} A & B \\ \hline C_D & D_D \end{array} \right] \end{aligned}$$

For our illustration, where  $W_N$  is also chosen to be minimum-phase,  $C_N$ ,  $D_N$ ,  $C_D$  and  $D_D$  are computed as:

$$\left[ C_N \mid D_N \right] = \left[ -24.04 \quad -11.72 \mid -1.01 \right] \quad \left[ C_D \mid D_D \right] = \left[ -1.88 \quad -15.60 \mid 6.7 \right]$$

5. Use (5.21) (p. 125) to get a stable transfer function  $W(T(s))$  of order  $n$  which satisfies the spectral mask  $\mathcal{SM}$  and which can be expressed as an LFT in  $T(s)$ :

$$W(T(s)) = (T(s) \cdot I_n) \star \left[ \begin{array}{c|c} A - BD_D^{-1}C_D & BD_D^{-1} \\ \hline C_N - D_N D_D^{-1}C_D & D_N D_D^{-1} \end{array} \right]$$

For our illustration, the following  $W(T(s))$  is obtained:

$$W(T(s)) = (T(s) \cdot I_2) \star \left[ \begin{array}{cc|c} 0 & 1.00 & 0 \\ -0.72 & 0.33 & 0.15 \\ \hline -24.32 & -14.07 & -0.15 \end{array} \right]$$

One can check that  $W(T(s))$  is stable, using Corollary 2.2 (p. 26), and  $W(T(s))$  satisfies the spectral mask on Figure 5.4.

#### 5.4.5.2 General dissipative $T(s)$

The overall procedure to solve Problem 5.1, with  $T(s)$   $\{x, y, z\}$ -dissipative, is summarised and illustrated step-by-step below. For the illustration,  $T(s)$  is taken as:

$$T(s) := \frac{1}{(s+1)(s+2)} = \left( \frac{1}{s} \cdot I_2 \right) \star \left[ \begin{array}{c|c} A_T & B_T \\ \hline C_T & D_T \end{array} \right] := \left( \frac{1}{s} \cdot I_2 \right) \star \left[ \begin{array}{cc|c} 0 & 1 & 0 \\ -2 & -3 & 1 \\ \hline 1 & 0 & 0 \end{array} \right]$$

and is lossless dissipative with, for example,  $\{x, y, z\} = \{-1, 0.193, 0.0624\}$ . As shown in Figure 5.5, the Nyquist plot of  $T(s)$  is a circle with centre  $c := \frac{-y}{x} = 0.193$  and radius  $r := \sqrt{\frac{y^2 - xz}{x^2}} = 0.3157$ . The considered spectral mask  $\mathcal{SM}$  is the same as in the previous illustration.

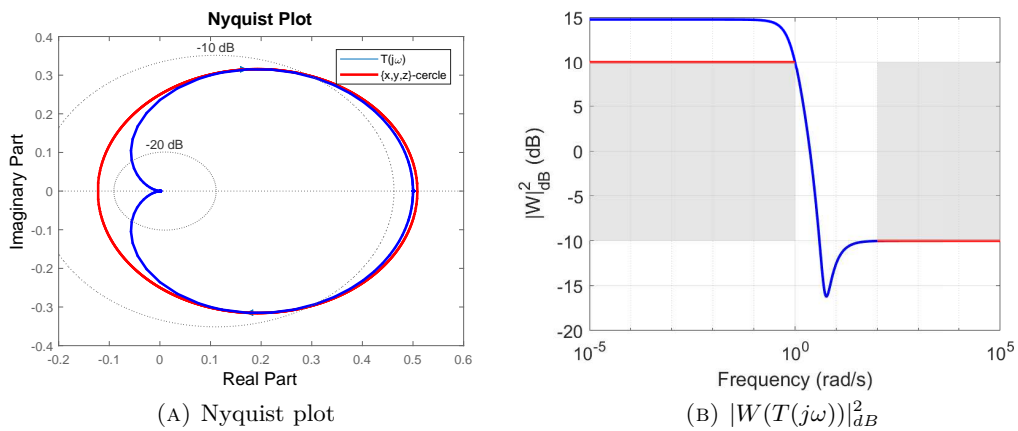


FIGURE 5.5: Illustrative example -  $T(s)$  dissipative

1. Choose the order  $n$ .

2. Define a stable basis  $\mathcal{B}(T(s))$  of order  $n$  under the LFT form:

$$\mathcal{B}(T(s)) = (T(s) \cdot I_n) \star \left[ \begin{array}{c|c} A & B \\ \hline C & D \end{array} \right] = \left( \frac{1}{s} \cdot I_{n_s} \right) \star \left[ \begin{array}{c|c} A_s & B_s \\ \hline C_s & D_s \end{array} \right]$$

For the illustration, choose  $\mathcal{B}(T(s))$  as in (5.44). Define  $A_s$ ,  $B_s$ ,  $C_s$  and  $D_s$  as in (5.22) (p. 130).

3. Solve LMI minimisation problem (5.43) resulting from Theorem 5.7 (p. 135). If it is feasible, there exist symmetric matrices  $X_N = X_N^T$ ,  $X_D = X_D^T$ ,  $P_N = P_N^T$ ,  $P_D = P_D^T$  such that the magnitude

$$M(T(j\omega)) = \frac{-E_{P_N}(T(j\omega)) + \mathcal{B}(T(j\omega))^* X_N \mathcal{B}(T(j\omega))}{-E_{P_D}(T(j\omega)) + \mathcal{B}(T(j\omega))^* X_D \mathcal{B}(T(j\omega))}$$

is positive and satisfies the spectral mask  $\mathcal{SM}$ .

If it is not feasible, then increment  $n$ :  $n \leftarrow n + 1$ , and back to step 2.

For our illustration, the minimal order  $n$  is  $n = 2$ , and the resulting matrices are given by:

$$\begin{aligned} X_N &= \begin{bmatrix} 0.2764 & 1.4236 & 0.5319 \\ 1.4236 & 9.7916 & -0.0779 \\ 0.5319 & -0.0779 & 9.9709 \end{bmatrix} & X_D &= \begin{bmatrix} -0.4971 & -2.2682 & -0.2737 \\ -2.2682 & 9.5099 & -0.0591 \\ -0.2737 & -0.0591 & 9.9929 \end{bmatrix} \\ P_N &= \begin{bmatrix} -8.9649 & 2.0599 \\ 2.0599 & 4.1420 \end{bmatrix} & P_D &= \begin{bmatrix} -9.9466 & -1.8900 \\ -1.8900 & -10.8414 \end{bmatrix} \end{aligned}$$

4. Compute  $C_N$ ,  $D_N$ ,  $C_D$  and  $D_D$  in order to get

$$\begin{aligned} W_N(T(s)) &:= (T(s) \cdot I_n) \star \left[ \begin{array}{c|c} A & B \\ \hline C_N & D_N \end{array} \right] \\ W_D(T(s)) &:= (T(s) \cdot I_n) \star \left[ \begin{array}{c|c} A & B \\ \hline C_D & D_D \end{array} \right] \end{aligned}$$

$W_N(T(s))$  and  $W_D(T(s))$  are then two stable and inversely-stable, minimal spectral factors such that:

$$\begin{aligned} |W_N(T(j\omega))|^2 &= -E_{P_N}(T(j\omega)) + \mathcal{B}(T(j\omega))^* X_N \mathcal{B}(T(j\omega)) \\ |W_D(T(j\omega))|^2 &= -E_{P_D}(T(j\omega)) + \mathcal{B}(T(j\omega))^* X_D \mathcal{B}(T(j\omega)) \end{aligned}$$

For our illustration,  $C_N$ ,  $D_N$ ,  $C_D$  and  $D_D$  are computed as:

$$\begin{aligned} \left[ \begin{array}{c|c} C_N & D_N \end{array} \right] &= \left[ \begin{array}{cc|c} -0.8691 & -3.4332 & -0.1339 \end{array} \right] \\ \left[ \begin{array}{c|c} C_D & D_D \end{array} \right] &= \left[ \begin{array}{cc|c} -0.2084 & -0.9838 & 0.4235 \end{array} \right] \end{aligned}$$

5. Use (5.21) (p. 125) to get a stable and inversely-stable  $W(T(s))$  of order  $n$  which satisfies the spectral mask  $\mathcal{SM}$  and which can be expressed as an LFT in  $T(s)$ :

$$W(T(s)) = (T(s) \cdot I_n) \star \left[ \begin{array}{c|c} A - BD_D^{-1}C_D & BD_D^{-1} \\ \hline C_N - D_N D_D^{-1}C_D & D_N D_D^{-1} \end{array} \right]$$



For our illustration, the following  $W(T(s))$  is obtained:

$$W(T(s)) = (T(s) \cdot I_2) \star \left[ \begin{array}{cc|c} 0 & 1.00 & 0 \\ -0.5079 & 0.3229 & 2.3611 \\ \hline -0.9350 & -3.7443 & -0.3162 \end{array} \right]$$

One can check that  $W(T(s))$  is stable, using Corollary 2.2 (p. 26), and  $W(T(s))$  satisfies the spectral mask on Figure 5.5.

## 5.5 On the extension to 2D LFT filters synthesis

We intend now to extend the usual synthesis approach to 2D LFT filters. In this manuscript, a 2D LFT system is referred to as a system having the following LFT representation:

$$\begin{cases} p = \Delta(T_s(s), T_p(s))q \\ q = Ap + Bw \\ z = Cp + Dw \end{cases}$$

where  $T_s(s)$  and  $T_p(s)$  are two stable transfer functions and

$$\Delta(T_s(s), T_p(s)) := \begin{pmatrix} T_s(s) \cdot I_{n_{T_s}} & 0 \\ 0 & T_p(s) \cdot I_{n_{T_p}} \end{pmatrix}$$

The resulting transfer function  $W(T_s(s), T_p(s))$  is then provided by the LFT:

$$W(T_s(s), T_p(s)) = \Delta(T_s(s), T_p(s)) \star \left[ \begin{array}{c|c} A & B \\ \hline C & D \end{array} \right]$$

In addition, as this is the case of interest in this work,  $T_s(s)$  and  $T_p(s)$  are considered lossless  $\{0, 1, 0\}$ -dissipative:

$$\forall \omega \in \mathbb{R}, \quad \begin{bmatrix} T_s(j\omega) \\ 1 \end{bmatrix}^* \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} T_s(j\omega) \\ 1 \end{bmatrix} = 0 \quad \begin{bmatrix} T_p(j\omega) \\ 1 \end{bmatrix}^* \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} T_p(j\omega) \\ 1 \end{bmatrix} = 0$$

Henceforth, for the sake of clarity, the following notation is used for any matrix function  $f(T_s(s), T_p(s))$ :

$$f_{T_s T_p}(s) := f(T_s(s), T_p(s))$$

The 2D LFT filters synthesis problem is then stated as follows.

### Problem 5.6 (2D LFT Filters Synthesis).

*Given* two SISO,  $\{0, 1, 0\}$ -dissipative transfer functions  $T_s(s)$ ,  $T_p(s)$ ,  
 $N_U$  upper bounds  $U_u$  associated with  $N_U$  real intervals  $\Omega_u^U \subseteq \mathbb{R}$ ,  
 $N_L$  lower bounds  $L_l$  associated with  $N_L$  real intervals  $\Omega_l^L \subseteq \mathbb{R}$ ,  
*Find* if there exists a stable  $\mathbf{W}(T_s, T_p)$  expressed as an LFT in  $T_s(s)$ ,  $T_p(s)$  such that it satisfies the spectral mask  $\mathcal{SM}$ :

$$\begin{aligned} \forall u \in \{1, \dots, N_U\}, \quad \forall \omega \in \Omega_u^U, \quad & |\mathbf{W}_{T_s T_p}(j\omega)|^2 \leq U_u^2 \\ \forall l \in \{1, \dots, N_L\}, \quad \forall \omega \in \Omega_l^L, \quad & |\mathbf{W}_{T_s T_p}(j\omega)|^2 \geq L_l^2 \end{aligned}$$

If so, *Compute* it.

As far as we know, Problem 5.6 has not yet been tackled in the literature. The closest related work is the design of *linear-phase* 2D digital filters [Dum07; RDV07]. The *exact* linear-phase allows to directly design the *amplitude* of the filter, which is decomposed into the basis of trigonometric polynomials. However, such approach can not be used here, as *exact* linear-phase is not achievable for analog filters.

In Subsection 5.5.1, the extension of the usual synthesis approach to 2D LFT filters is discussed step-by-step. Two main issues prevent from a direct extension. First, a one-to-one correspondence between 2D LFT systems and 2D rational functions is not possible. Therefore, the decomposition in a basis of 2D polynomials becomes *conservative* and basis-dependent. Second, there may exist 2D rational functions which satisfy the spectral mask (Subsection 5.5.2), but do not admit spectral factors. Furthermore, even though a spectral factor may exist, the ARE-based method of the last sections may not be able to compute it (Subsection 5.5.3). As a consequence, the spectral factorisation of 2D systems remains an open-problem. Some algorithms have been proposed, refer to [Bas00; AT08] for instance, but they generally lead to *non-minimal* spectral factors.

Because of these crucial issues, in the sequel, we do not claim to solve Problem 5.6. Instead, we provide an adaptation of the usual synthesis approach (Subsection 5.5.4) which, if it is successful, provide a solution to Problem 5.6. Finally, the resulting approach is summarised and numerically illustrated (Subsection 5.5.5).

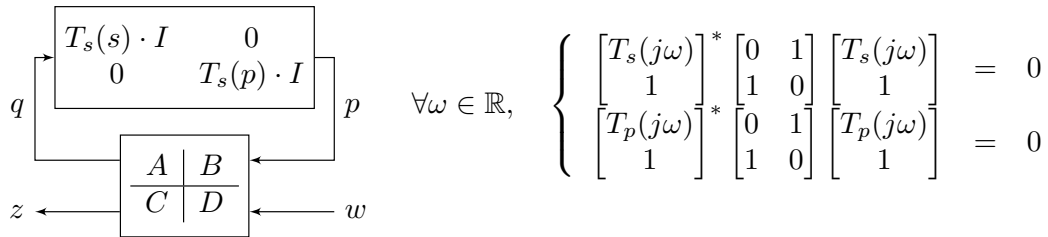


FIGURE 5.6: LFT system in the repeated lossless  $\{0, 1, 0\}$ -dissipative  $T_s(s)$  and  $T_p(s)$

## 5.5.1 Conservative finite-dimensional convex formulation

### 5.5.1.1 Finite-dimensional parametrisation

In the usual procedure, and its extension of the last section, the first stage consists in finding an *equivalent* finite-dimensional parametrisation of LFT  $W(T)$  in a repeated  $T$ . Unfortunately, this can not be extended to 2D LFT  $W(T_s, T_p)$  in repeated  $T_s$  and  $T_p$ . Indeed, one may find a similar finite-dimensional parametrisation, as illustrated below, but only for sub-classes of 2D LFT systems. The parametrisation is said to be *conservative*. As this stage is required to solve the problems of magnitude synthesis and spectral factorisation with the efficient methods considered in this chapter, we adapt it in the sequel, while identifying its limitations.

The finite-dimensional parametrisation of LFT  $W(T)$  in a repeated  $T$  is based on the equality between the set of LFT with  $n$  repeated  $T$  and the set of  $n^{\text{th}}$ -order rational functions of  $T$  having in  $T = 0$  a finite real value<sup>7</sup>, i.e.

$$\left\{ (T \cdot I_n) \star M, M \in \mathbb{R}^{(n+1) \times (n+1)} \right\} = \left\{ H(T) = \frac{b_0 + b_1 \cdot T + \dots + b_n \cdot T^n}{1 + a_1 \cdot T + \dots + a_n \cdot T^n} \right\}$$

<sup>7</sup>Or equivalently, the set of *proper*  $n^{\text{th}}$ -order rational functions of  $T^{-1}$ .

This one-to-one correspondence is not true when considering 2D LFT systems. Indeed, the set  $\mathbb{S}_1$  of 2D LFT with  $n_{T_s}$  repeated  $T_s$  and  $n_{T_p}$  repeated  $T_p$  is included into the set  $\mathbb{S}_2$  of  $(n_{T_s}, n_{T_p})$ -order rational functions of  $(T_s, T_p)$  having in  $(T_s, T_p) = (0, 0)$  a finite real value:

$$\mathbb{S}_1 \subset \mathbb{S}_2$$

with

$$\mathbb{S}_1 := \left\{ \Delta(T_s, T_p) \star M, M \in \mathbb{R}^{(n+1) \times (n+1)}, n = n_{T_s} + n_{T_p} \right\}$$

$$\mathbb{S}_2 := \left\{ H(T_s, T_p) = \frac{b_{0,0} + \dots + b_{n_{T_s}, n_{T_p}} \cdot T_s^{n_{T_s}} T_p^{n_{T_p}}}{1 + a_{1,0} \cdot T_s + a_{0,1} \cdot T_p + a_{1,1} \cdot T_s T_p + \dots + a_{n_{T_s}, n_{T_p}} \cdot T_s^{n_{T_s}} T_p^{n_{T_p}}} \right\}$$

This inclusion is strict, as shown in the next example.

**Example 5.4.** Consider the following  $(1, 1)$ -rational function of  $(T_s, T_p)$ :

$$H(T_s, T_p) = \frac{T_s - T_p}{1 + T_s + T_p + 2T_s T_p}$$

This can not be written as an LFT in  $T_s$  and  $T_p$  with  $n_{T_s} = n_{T_p} = 1$ . Indeed, defining

$$A := \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad B := \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \quad C := \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \quad D := d$$

leads to  $W(T_s, T_p) := \Delta_{T_s T_p} \star \left[ \begin{array}{c|c} A & B \\ \hline C & D \end{array} \right]$  being as:

$$W(T_s, T_p) := d + \frac{b_1 c_1 \cdot T_s + b_2 c_2 \cdot T_p + (c_1(a_{12} b_2 - a_{22} b_1) + c_2(a_{21} b_1 - a_{11} b_2)) \cdot T_s T_p}{1 - a_{11} \cdot T_s - a_{22} \cdot T_p + (a_{11} a_{22} - a_{12} a_{21}) \cdot T_s T_p}$$

Therefore, the equality  $W(T_s, T_p) = H(T_s, T_p)$  leads to the equations:

$$\begin{aligned} a_{11} &= -1 & a_{22} &= -1 & a_{12} a_{21} &= -1 \\ d &= 0 & b_1 c_1 &= 1 & b_2 c_2 &= -1 \\ c_1(a_{12} b_2 - a_{22} b_1) + c_2(a_{21} b_1 - a_{11} b_2) &= 0 \end{aligned}$$

Injecting the two first lines of constraint in the last one leads to the constraint:

$$a_{12} \frac{b_2}{b_1} - a_{21} \frac{b_1}{b_2} = 0$$

which do not admits real solutions when  $a_{12} a_{21} \leq 0$ . Therefore  $H$  cannot be represented as a 2D LFT with  $n_{T_s} = n_{T_p} = 1$ .

For a given order, the class of rational functions of two variables is then larger than the class of 2D LFT systems. Therefore, the use of rational functions does not appropriately represent the interconnection of heterogeneous subsystems, and one should work with the LFT representation. However, the adapted synthesis procedure requires to factorise the 2D LFT system  $W(T_s, T_p)$  as  $W(T_s, T_p) := \frac{W_N(T_s, T_p)}{W_D(T_s, T_p)}$ , with  $W_N(T_s, T_p)$  and  $W_D(T_s, T_p)$  being 2D LFT systems decomposed as:

$$W_N(T_s, T_p) := \Gamma_N^T \mathcal{B}(T_s, T_p) \quad W_D(T_s, T_p) := \Gamma_D^T \mathcal{B}(T_s, T_p) \quad (5.45)$$

with  $\Gamma_N, \Gamma_D \in \mathbb{R}^{(n+1) \times 1}$ , and  $\mathcal{B}(T_s, T_p)$  a 2D LFT system given by:

$$\mathcal{B}(T_s, T_p) := \Delta(T_s, T_p) \star \left[ \begin{array}{c|c} A & B \\ \hline C & D \end{array} \right]$$

with  $n = n_{T_s} + n_{T_p}$ . In the 1D case, the transfer vector  $\mathcal{B}(T_s(s), T_p(s))$  is the column concatenation of the basis elements of a vector space, under mere assumptions on the LFT matrices. This property does not generalise to the 2D cases, and the choice of the LFT matrices influences the class of 2D sub-systems that will be synthesised. This is illustrated in the next example.

**Example 5.5.** Consider  $n_{T_s} := 1$ ,  $n_{T_p} := 1$ ,  $A_1 := \begin{bmatrix} -1 & -1 \\ 0 & -1 \end{bmatrix}$ ,  $A_2 := \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$  and  $B := \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ . Then the following LFTs:

$$W_1(T_s, T_p) = \Delta_{T_s T_p} \star \left[ \begin{array}{cc|c} A_1 & & B \\ \hline c_1 & c_2 & d_1 \end{array} \right] = \frac{d + (c_1 + d) \cdot T_s + (c_2 + d) \cdot T_p + (c_2 + d) \cdot T_s T_p}{1 + T_s + T_p + T_s T_p}$$

$$W_2(T_s, T_p) = \Delta_{T_s T_p} \star \left[ \begin{array}{cc|c} A_2 & & B \\ \hline c_1 & c_2 & d_1 \end{array} \right] = \frac{d + (c_1 + d) \cdot T_s + (c_2 + d) \cdot T_p + (c_1 + c_2 + d) \cdot T_s T_p}{1 + T_s + T_p + T_s T_p}$$

belong to two different subclasses of 2D LFT systems.

Therefore, the parametrisation (5.45) is conservative and depends on the choice of  $\mathcal{B}(T_s, T_p)$ .

### 5.5.1.2 Convex formulation

Once a finite parametrisation has been obtained, the next step is to find a convex formulation of the problem. In the usual case, this leads to make the change of variable from a stable  $H(s)$  to the square magnitude  $M(j\omega) = |H(j\omega)|^2$ . This change of variable is well-posed, i.e. given  $M(j\omega)$  there exists a stable  $H(s)$  with this magnitude, if  $M(j\omega) \geq 0$  all along the  $j\omega$ -axis.

When considering 2D systems, and more generally nD systems, this statement is not true anymore. Indeed, there are *positive* 2D rational functions which do not admit spectral factors. This statement is illustrated in the next example in the context of polynomials.

**Example 5.6.** Define

$$M(\omega_1, \omega_2) := 1 + \omega_1^2 + (1 - \omega_1 \omega_2)^2$$

Then  $M(\omega_1, \omega_2) \geq 0$  for all  $\omega_1, \omega_2 \in \mathbb{R}$ . However, there is no spectral factor associated to  $M(\omega_1, \omega_2)$ . Indeed, setting  $H(s_1, s_2) := a + bs_1 + cs_2 + ds_1 s_2$  and equalling  $M(\omega_1, \omega_2) = H(-j\omega_1, -j\omega_2)H(j\omega_1, j\omega_2)$  leads to a contradiction.

In the last example,  $M(\omega_1, \omega_2)$  is the sum of the square polynomials  $1$ ,  $\omega_1^2$  and  $(1 - \omega_1 \omega_2)^2$ . Multivariate polynomials admitting such a decomposition are denoted as Sum-of-Square (SOS) polynomials. This class obviously includes spectral factors, and is a particular subclass of positive polynomials, as illustrated in the next example.

**Example 5.7.** The following two-variables polynomials  $P_1$  and  $P_2$  are positives but can not be expressed as the SOS of real polynomial (refer to [Bas00] and the references

therein):

$$P_1(\omega_1, \omega_2) = \omega_1^2 \omega_2^2 (\omega_1^2 + \omega_2^2 - 1) + 1 \quad P_2(\omega_1, \omega_2) = \omega_1^2 \omega_2^4 + \omega_1^2 (\omega_1^2 - 3) \omega_2^2 + 1$$

Therefore, we have the strict inclusion of sets for polynomials:

$$\{\text{Spectral Factor}\} \subset \{\text{SOS}\} \subset \{\text{Positive}\}$$

For single-variable polynomials these sets are equal, and checking for a polynomial to belong to a given set is equivalent to check if it belongs to one of the other sets. The representation is then chosen depending on the problem.

The conditions of existence of minimal spectral factors and their computations remain open problems in the literature, related to the Hilbert's 17<sup>th</sup> problem (see for example [Pfi76] and the discussion in [PW02, §9.2]), and we do not claim to solve them. Instead, we develop in the sequel a procedure which, if it is successful, provides a 2D LFT filter satisfying a given spectral mask.

### 5.5.2 Positive rational function synthesis

The synthesis of a positive rational function is given in Theorem 5.8. Before introducing this theorem, it is shown below how to express a 2D LFT transfer vector  $B_{T_s T_p}(s)$  as an LFT in  $\frac{1}{s}$ . To achieve this, assume that  $\mathcal{B}(T_s, T_p)$  is given as:

$$\mathcal{B}(T_s, T_p) := \Delta(T_s, T_p) \star \left[ \begin{array}{c|c} A & B \\ \hline C & D \end{array} \right] \quad (5.46)$$

with  $n := n_{T_s} + n_{T_p}$  and

$$T_s(s) := \left( \frac{1}{s} \cdot I_{n_{T_s}} \right) \star \left[ \begin{array}{c|c} A_{T_s} & B_{T_s} \\ \hline C_{T_s} & D_{T_s} \end{array} \right] \quad T_p(s) := \left( \frac{1}{s} \cdot I_{n_{T_p}} \right) \star \left[ \begin{array}{c|c} A_{T_p} & B_{T_p} \\ \hline C_{T_p} & D_{T_p} \end{array} \right]$$

Then,  $\mathcal{B}_{T_s T_p}(s)$  can be expressed as an LFT in  $\frac{1}{s}$  as:

$$\mathcal{B}_{T_s T_p}(s) := \left( \frac{1}{s} \cdot I_{n_s} \right) \star \left[ \begin{array}{c|c} A_s & B_s \\ \hline C_s & D_s \end{array} \right] \quad (5.47)$$

with  $n_s := n_{T_s} n_{T_s} + n_{T_p} n_{T_p}$  and

$$\left[ \begin{array}{c|c} A_s & B_s \\ \hline C_s & D_s \end{array} \right] := \left[ \begin{array}{cc|cc} I_{n_{T_s}} \otimes A_{T_s} & 0 & I_{n_{T_s}} \otimes B_{T_s} & 0 \\ 0 & I_{n_{T_p}} \otimes A_{T_p} & 0 & I_{n_{T_p}} \otimes B_{T_p} \\ \hline I_{n_{T_s}} \otimes C_{T_s} & 0 & I_{n_{T_s}} \otimes D_{T_s} & 0 \\ 0 & I_{n_{T_p}} \otimes C_{T_p} & 0 & I_{n_{T_p}} \otimes D_{T_p} \end{array} \right] \star \left[ \begin{array}{c|c} A & B \\ \hline C & D \end{array} \right]$$

**Theorem 5.8.** *Let  $T_s(s)$ ,  $T_p(s)$  be two lossless  $\{0, 1, 0\}$ -dissipative transfer functions. Consider a given transfer matrix  $\mathcal{B}(T_s, T_p)$  equivalently defined by (5.46) and (5.47) with  $A \in \mathbb{R}^{n \times n}$ ,  $B \in \mathbb{R}^{n \times 1}$ ,  $C \in \mathbb{R}^{(n+1) \times n}$ ,  $D \in \mathbb{R}^{(n+1) \times 1}$ , and  $A_s \in \mathbb{R}^{n_s \times n_s}$ ,  $B_s \in \mathbb{R}^{n_s \times 1}$ ,  $C_s \in \mathbb{R}^{(n+1) \times n_s}$ ,  $D_s \in \mathbb{R}^{(n+1) \times 1}$ .*

*Assume that  $(A, B)$  and  $(A_s, B_s)$  are controllable.*

*If the optimisation problem defined in (i) is feasible then the problem defined in (ii) admits a solution.*

$$(i) \quad \exists \mathbf{X}_N = (\mathbf{X}_N)^T \in \mathbb{R}^{(n+1) \times (n+1)}, \exists \mathbf{X}_D = (\mathbf{X}_D)^T \in \mathbb{R}^{(n+1) \times (n+1)}, \\ \exists \mathbf{P}_N \in \mathbb{M}, \exists \mathbf{P}_D \in \mathbb{M},$$

$$\begin{aligned} \forall u \in \{1, \dots, N_U\}, \exists \mathbf{D}_u = (\mathbf{D}_u)^T \in \mathbb{R}^{n_s \times n_s}, \exists \mathbf{P}_u = (\mathbf{P}_u)^T \in \mathbb{R}^{n_s \times n_s}, \\ \forall l \in \{1, \dots, N_L\}, \exists \mathbf{D}_l = (\mathbf{D}_l)^T \in \mathbb{R}^{n_s \times n_s}, \exists \mathbf{P}_l = (\mathbf{P}_l)^T \in \mathbb{R}^{n_s \times n_s}, \end{aligned}$$

$$\begin{bmatrix} A & B \\ I & 0 \end{bmatrix}^T \begin{bmatrix} 0 & -\mathbf{P}_N \\ -\mathbf{P}_N & 0 \end{bmatrix} \begin{bmatrix} A & B \\ I & 0 \end{bmatrix} + \begin{bmatrix} C^T \\ D^T \end{bmatrix} \mathbf{X}_N \begin{bmatrix} C & D \end{bmatrix} \geq 0 \quad (5.48)$$

$$\begin{bmatrix} A & B \\ I & 0 \end{bmatrix}^T \begin{bmatrix} 0 & -\mathbf{P}_D \\ -\mathbf{P}_D & 0 \end{bmatrix} \begin{bmatrix} A & B \\ I & 0 \end{bmatrix} + \begin{bmatrix} C^T \\ D^T \end{bmatrix} \mathbf{X}_D \begin{bmatrix} C & D \end{bmatrix} > 0 \quad (5.49)$$

$$\forall u \in \{1, \dots, N_U\},$$

$$\mathbf{D}_u > 0$$

$$\begin{bmatrix} A_s & B_s \\ I & 0 \end{bmatrix}^T \begin{bmatrix} -z_{\Omega_u^U} \mathbf{D}_u & -y_{\Omega_u^U}^* \mathbf{D}_u - \mathbf{P}_u \\ -y_{\Omega_u^U} \mathbf{D}_u - \mathbf{P}_u & -x_{\Omega_u^U} \mathbf{D}_u \end{bmatrix} \begin{bmatrix} A_s & B_s \\ I & 0 \end{bmatrix} + \begin{bmatrix} C_s^T \\ D_s^T \end{bmatrix} (U_u^2 \mathbf{X}_D - \mathbf{X}_N) \begin{bmatrix} C_s & D_s \end{bmatrix} \geq 0$$

$$\forall l \in \{1, \dots, N_L\},$$

$$\mathbf{D}_l > 0$$

$$\begin{bmatrix} A_s & B_s \\ I & 0 \end{bmatrix}^T \begin{bmatrix} -z_{\Omega_l^L} \mathbf{D}_l & -y_{\Omega_l^L}^* \mathbf{D}_l - \mathbf{P}_l \\ -y_{\Omega_l^L} \mathbf{D}_l - \mathbf{P}_l & -x_{\Omega_l^L} \mathbf{D}_l \end{bmatrix} \begin{bmatrix} A_s & B_s \\ I & 0 \end{bmatrix} + \begin{bmatrix} C_s^T \\ D_s^T \end{bmatrix} (\mathbf{X}_N - L_l^2 \mathbf{X}_D) \begin{bmatrix} C_s & D_s \end{bmatrix} \geq 0$$

where  $x_\Omega$ ,  $y_\Omega$ ,  $z_\Omega$  depend on  $\Omega$  and are defined as in Subsection 2.6 (p. 27), and the set  $\mathbb{M}$  is defined by:

$$\mathbb{M} := \left\{ M = M^T \mid \exists M_{T_s} = M_{T_s}^T \in \mathbb{R}^{n_{T_s} \times n_{T_s}}, \exists M_{T_p} = M_{T_p}^T \in \mathbb{R}^{n_{T_p} \times n_{T_p}}, M = \begin{bmatrix} M_{T_s} & 0 \\ 0 & M_{T_p} \end{bmatrix} \right\}$$

(ii) There exist two matrices  $\mathbf{X}_N = (\mathbf{X}_N)^T$ ,  $\mathbf{X}_D = (\mathbf{X}_D)^T \in \mathbb{R}^{(n+1) \times (n+1)}$  such that:

$$\forall \omega \in \mathbb{R}, \quad \mathcal{B}_{T_s T_p}(j\omega)^* \mathbf{X}_N \mathcal{B}_{T_s T_p}(j\omega) \geq 0 \quad (5.50)$$

$$\forall \omega \in \mathbb{R}, \quad \mathcal{B}_{T_s T_p}(j\omega)^* \mathbf{X}_D \mathcal{B}_{T_s T_p}(j\omega) > 0 \quad (5.51)$$

$$\forall u \in \{1, \dots, N_U\}, \quad \forall \omega \in \Omega_u^U, \quad \frac{\mathcal{B}_{T_s T_p}(j\omega)^* \mathbf{X}_N \mathcal{B}_{T_s T_p}(j\omega)}{\mathcal{B}_{T_s T_p}(j\omega)^* \mathbf{X}_D \mathcal{B}_{T_s T_p}(j\omega)} \leq U_u^2$$

$$\forall l \in \{1, \dots, N_L\}, \quad \forall \omega \in \Omega_l^L, \quad \frac{\mathcal{B}_{T_s T_p}(j\omega)^* \mathbf{X}_N \mathcal{B}_{T_s T_p}(j\omega)}{\mathcal{B}_{T_s T_p}(j\omega)^* \mathbf{X}_D \mathcal{B}_{T_s T_p}(j\omega)} \geq L_l^2$$

At first sight, Theorem 5.8 provides sufficient LMI conditions for the rational function  $M_{T_s T_p}(j\omega) := \frac{\mathcal{B}_{T_s T_p}(j\omega)^* \mathbf{X}_N \mathcal{B}_{T_s T_p}(j\omega)}{\mathcal{B}_{T_s T_p}(j\omega)^* \mathbf{X}_D \mathcal{B}_{T_s T_p}(j\omega)}$  to be positive and to satisfy upper and lower bounds on some frequency intervals. Upon close inspection, it also provides sufficient conditions for  $M_{T_s T_p}(j\omega)$  to be the fraction of two SOS polynomials. This is due to the use of the adapted KYP Lemma.

Indeed, as an example, applying the KYP Lemma leads the LMI (5.48) to be a sufficient condition for the check of the positivity constraint (5.50). Rewrite (5.48) into the equality form

$$\begin{bmatrix} A & B \\ I & 0 \end{bmatrix}^T \begin{bmatrix} 0 & -\mathbf{P}_N \\ -\mathbf{P}_N & 0 \end{bmatrix} \begin{bmatrix} A & B \\ I & 0 \end{bmatrix} + \begin{bmatrix} C^T \\ D^T \end{bmatrix} \mathbf{X}_N \begin{bmatrix} C & D \end{bmatrix} = \begin{bmatrix} C_N^T \\ D_N^T \end{bmatrix} \begin{bmatrix} C_N & D_N \end{bmatrix}$$

where  $\begin{bmatrix} C_N & D_N \end{bmatrix} \in \mathbb{R}^{k \times (n+1)}$ . By pre- and post-multiplying 1st equation by respectively  $\begin{bmatrix} \Delta_{T_s T_p}(j\omega) (I - A\Delta_{T_s T_p}(j\omega))^{-1} B \\ I \end{bmatrix}^*$  and  $\begin{bmatrix} \Delta_{T_s T_p}(j\omega) (I - A\Delta_{T_s T_p}(j\omega))^{-1} B \\ I \end{bmatrix}$  leads to:

$$0 + \mathcal{B}_{T_s T_p}(j\omega)^* X_N \mathcal{B}_{T_s T_p}(j\omega) = \frac{N_{T_s T_p}(j\omega)^* N_{T_s T_p}(j\omega)}{d_{T_s T_p}(j\omega)^* d_{T_s T_p}(j\omega)}$$

with  $N(T_s, T_p)$  is a polynomial matrix and  $d(T_s, T_p)$  a polynomial such that:

$$\frac{N(T_s, T_p)}{d(T_s, T_p)} = \begin{bmatrix} C_N & D_N \end{bmatrix} \begin{bmatrix} \Delta(T_s, T_p) (I - A\Delta(T_s, T_p))^{-1} B \\ I \end{bmatrix}$$

Finally, as  $\mathcal{B}_{T_s T_p}(j\omega)^* X_N \mathcal{B}_{T_s T_p}(j\omega)$  is scalar, one can write:

$$\mathcal{B}_{T_s T_p}(j\omega)^* X_N \mathcal{B}_{T_s T_p}(j\omega) = \frac{\Re(N_{T_s T_p}(j\omega))^T \Re(N_{T_s T_p}(j\omega)) + \Im(N_{T_s T_p}(j\omega))^T \Im(N_{T_s T_p}(j\omega))}{\Re(d_{T_s T_p}(j\omega))^2 + \Im(d_{T_s T_p}(j\omega))^2}$$

Then,  $\mathcal{B}_{T_s T_p}(j\omega)^* X_N \mathcal{B}_{T_s T_p}(j\omega)$  is the rational fraction of two SOS polynomials. Using similar reasoning for  $\mathcal{B}_{T_s T_p}(j\omega)^* X_D \mathcal{B}_{T_s T_p}(j\omega)$  leads  $M_{T_s T_p}(j\omega)$  to be a fraction of two SOS polynomials. Therefore, despite its statement, the KYP lemma tests a SOS condition and not positivity. While equivalent for 1D systems, both conditions are not the same for 2D systems, as mentioned previously.

In the next subsection, an attempt is made to compute a spectral factor from the positive rational functions resulting from Theorem 5.8.

### 5.5.3 Lossy spectral factorisation

In the usual procedure, spectral factors are computed from (5.48) and (5.49) by solving an ARE of the form

$$A^T(-\mathbf{P}) + (-\mathbf{P})A + C^T X C - ((-\mathbf{P})B + C^T X D) (D^T X D)^{-1} (B^T(-\mathbf{P}) + D^T X C) = 0$$

Applying this idea here leads to solve similar AREs but with the additional structural constraint  $\mathbf{P} \in \mathbb{M}$ , i.e.  $\mathbf{P}$  is block diagonal. This is an issue for two reasons. First, there may not exist such a structured solution to the AREs. Second, even if it exists, there is no efficient algorithm available to compute it.

Instead, define two symmetric matrices  $P_N^c$  and  $P_D^c$  such that  $P_N + P_N^c$  and  $P_D + P_D^c$  are *full* and symmetric matrices. In a sense,  $P_N^c$  and  $P_D^c$  respectively *complete*  $P_N$  and  $P_D$ . There exist then solutions to the associated AREs, and are easily computable. However, these solutions imply a lossy spectral factorisation. For instance, replacing  $P_N$  by  $P_N + P_N^c$  in (5.48) and assuming that  $P_N + P_N^c$  is a solution to the associated ARE leads to the factorisation:

$$\begin{aligned} F_{T_s T_p}(j\omega)^* \begin{bmatrix} I \\ \Delta_{T_s T_p}(j\omega) \end{bmatrix}^* \begin{bmatrix} 0 & -P_N^c \\ -P_N^c & 0 \end{bmatrix} \begin{bmatrix} I \\ \Delta_{T_s T_p}(j\omega) \end{bmatrix} F_{T_s T_p}(j\omega) + \mathcal{B}_{T_s T_p}(j\omega)^* X_N \mathcal{B}_{T_s T_p}(j\omega) \\ = W_{N_{T_s T_p}}(j\omega)^* W_{N_{T_s T_p}}(j\omega) \end{aligned}$$

with

$$F_{T_s T_p}(s) := (I - A\Delta_{T_s T_p}(s))^{-1} B = B + A\Delta_{T_s T_p}(s)(I - A\Delta_{T_s T_p}(s))^{-1} B \quad (5.52)$$

There is then a mismatch between  $|W_{T_s T_p}(j\omega)|^2$  and  $\mathcal{B}_{T_s T_p}(j\omega)^* X_N \mathcal{B}_{T_s T_p}(j\omega)$ . Thus,  $W_N(T_s, T_p)$  may not satisfy the spectral mask.

In the next subsection, we develop another approach, closely connected to the synthesis approach of dissipative LFT filter (Subsection 5.4.4), in order to synthesise a magnitude function which admits a spectral factor.

#### 5.5.4 Synthesis with factorisation error management

In Theorem 5.9, the factorisation error, due to the introduction of the completion matrices  $P_N^c$  and  $P_D^c$ , is directly managed into the magnitude synthesis.

**Theorem 5.9.** *Let  $T_s(s)$ ,  $T_p(s)$  be two lossless  $\{0, 1, 0\}$ -dissipative transfer functions. Consider a given transfer matrix  $\mathcal{B}(T_s, T_p)$  defined by:*

$$\mathcal{B}(T_s, T_p) := \Delta(T_s, T_p) \star \left[ \begin{array}{c|c} A & B \\ \hline C & D \end{array} \right]$$

with  $n = n_{T_s} + n_{T_p}$ ,  $A \in \mathbb{R}^{n \times n}$ ,  $B \in \mathbb{R}^{n \times 1}$ ,  $C \in \mathbb{R}^{(n+1) \times n}$  and  $D \in \mathbb{R}^{(n+1) \times 1}$ . Define  $F_{T_s, T_p}(s)$  as in (5.52). Consider  $\bar{\mathcal{B}}_{T_s, T_p}(s)$  as follows:

$$\bar{\mathcal{B}}_{T_s, T_p}(s) := \begin{bmatrix} \Delta_{T_s T_p}(s) F_{T_s T_p}(s) \\ F_{T_s T_p}(s) \\ \mathcal{B}_{T_s T_p}(s) \end{bmatrix} = \left( \frac{1}{s} \cdot I_{n_s} \right) \star \left[ \begin{array}{c|c} \bar{A}_s & \bar{B}_s \\ \hline \bar{C}_s & \bar{D}_s \end{array} \right]$$

with  $\bar{A}_s \in \mathbb{R}^{n_s \times n_s}$ ,  $\bar{B}_s \in \mathbb{R}^{n_s \times 1}$ ,  $\bar{C}_s \in \mathbb{R}^{(n+1) \times n_s}$ ,  $\bar{D}_s \in \mathbb{R}^{(n+1) \times 1}$ . Moreover, define the matrix-valued function  $R(\cdot, \cdot)$  as:

$$R(P, X) := A^T(-P) + (-P)A + C^T X C - ((-P)B + C^T X D) (D^T X D)^{-1} (B^T(-P) + D^T X C)$$

Assume that  $(A, B)$  and  $(\bar{A}_s, \bar{B}_s)$  are controllable.

If the optimisation problem defined in (i) is feasible, then problem (ii) admits a solution.

$$\begin{aligned} (i) \quad & \exists \mathbf{X}_N = (\mathbf{X}_N)^T \in \mathbb{R}^{(n+1) \times (n+1)}, \exists \mathbf{X}_D = (\mathbf{X}_D)^T \in \mathbb{R}^{(n+1) \times (n+1)}, \\ & \exists \mathbf{P}_N \in \mathbb{M}, \exists \mathbf{P}_N^c \in \mathbb{M}_c, \exists \mathbf{P}_D \in \mathbb{M}, \exists \mathbf{P}_D^c \in \mathbb{M}_c, \\ & \forall u \in \{1, \dots, N_U\}, \exists \mathbf{D}_u = (\mathbf{D}_u)^T \in \mathbb{R}^{n_s \times n_s}, \exists \mathbf{P}_u = (\mathbf{P}_u)^T \in \mathbb{R}^{n_s \times n_s}, \\ & \forall l \in \{1, \dots, N_L\}, \exists \mathbf{D}_l = (\mathbf{D}_l)^T \in \mathbb{R}^{n_s \times n_s}, \exists \mathbf{P}_l = (\mathbf{P}_l)^T \in \mathbb{R}^{n_s \times n_s}, \end{aligned}$$

$$D^T \mathbf{X}_N D > 0 \quad D^T \mathbf{X}_D D > 0 \quad (5.53)$$

$$R(\mathbf{P}_N + \mathbf{P}_N^c, \mathbf{X}_N) = 0 \quad R(\mathbf{P}_D + \mathbf{P}_D^c, \mathbf{X}_D) = 0 \quad (5.54)$$

$$\begin{bmatrix} \bar{A}_s & \bar{B}_s \\ I & 0 \end{bmatrix}^T \begin{bmatrix} 0 & -\mathbf{P}_D^{pos} \\ -\mathbf{P}_D^{pos} & 0 \end{bmatrix} \begin{bmatrix} \bar{A}_s & \bar{B}_s \\ I & 0 \end{bmatrix} + \begin{bmatrix} \bar{C}_s^T \\ \bar{D}_s^T \end{bmatrix} \bar{\mathbf{X}}_D \begin{bmatrix} \bar{C}_s & \bar{D}_s \end{bmatrix} > 0 \quad (5.55)$$

$$\forall u \in \{1, \dots, N_U\}, \quad \mathbf{D}_u > 0 \quad (5.56)$$

$$\begin{bmatrix} \bar{A}_s & \bar{B}_s \\ I & 0 \end{bmatrix}^T \begin{bmatrix} -z_{\Omega_u^U} \mathbf{D}_u & -y_{\Omega_u^U}^* \mathbf{D}_u - \mathbf{P}_u \\ -y_{\Omega_u^U} \mathbf{D}_u - \mathbf{P}_u & -x_{\Omega_u^U} \mathbf{D}_u \end{bmatrix} \begin{bmatrix} \bar{A}_s & \bar{B}_s \\ I & 0 \end{bmatrix} + \begin{bmatrix} \bar{C}_s^T \\ \bar{D}_s^T \end{bmatrix} (U_u^2 \bar{\mathbf{X}}_D - \bar{\mathbf{X}}_N) \begin{bmatrix} \bar{C}_s & \bar{D}_s \end{bmatrix} \geq 0 \quad (5.57)$$



$$\forall l \in \{1, \dots, N_L\}, \quad \mathbf{D}_l > 0 \quad (5.58)$$

$$\begin{bmatrix} \bar{A}_s & \bar{B}_s \\ I & 0 \end{bmatrix}^T \begin{bmatrix} -z_{\Omega_l^L} \mathbf{D}_l & -y_{\Omega_l^L}^* \mathbf{D}_l - \mathbf{P}_l \\ -y_{\Omega_l^L} \mathbf{D}_l - \mathbf{P}_l & -x_{\Omega_l^L} \mathbf{D}_l \end{bmatrix} \begin{bmatrix} \bar{A}_s & \bar{B}_s \\ I & 0 \end{bmatrix} + \begin{bmatrix} \bar{C}_s^T \\ \bar{D}_s^T \end{bmatrix} (\bar{\mathbf{X}}_N - L_l^2 \bar{\mathbf{X}}_D) \begin{bmatrix} \bar{C}_s & \bar{D}_s \end{bmatrix} \geq 0 \quad (5.59)$$

where

$$\bar{\mathbf{X}}_N := \begin{bmatrix} 0 & -\mathbf{P}_N^c & 0 \\ -\mathbf{P}_N^c & 0 & 0 \\ 0 & 0 & \mathbf{X}_N \end{bmatrix} \quad \bar{\mathbf{X}}_D := \begin{bmatrix} 0 & -\mathbf{P}_D^c & 0 \\ -\mathbf{P}_D^c & 0 & 0 \\ 0 & 0 & \mathbf{X}_D \end{bmatrix}$$

and  $x_\Omega, y_\Omega, z_\Omega$  depend on  $\Omega$  and are defined as in Subsection 2.6 (p. 27), and the sets  $\mathbb{M}$  and  $\mathbb{M}_c$  are defined by:

$$\mathbb{M} := \left\{ M = M^T \mid \exists M_{T_s} = M_{T_s}^T \in \mathbb{R}^{n_{T_s} \times n_{T_s}}, \exists M_{T_p} = M_{T_p}^T \in \mathbb{R}^{n_{T_p} \times n_{T_p}}, M = \begin{bmatrix} M_{T_s} & 0 \\ 0 & M_{T_p} \end{bmatrix} \right\}$$

$$\mathbb{M}_c := \left\{ M_c = M_c^T \mid \exists M_{T_s T_p} \in \mathbb{R}^{n_{T_s} \times n_{T_p}}, M_c = \begin{bmatrix} 0 & M_{T_s T_p} \\ M_{T_s T_p}^T & 0 \end{bmatrix} \right\}$$

(ii) There exist two stable 2D LFT  $\mathbf{W}_N(T_s, T_p)$  and  $\mathbf{W}_D(T_s, T_p)$  defined by

$$\mathbf{W}_N(T_s, T_p) := \Delta(T_s, T_p) \star \left[ \begin{array}{c|c} A & B \\ \hline C_N & D_N \end{array} \right] \quad \mathbf{W}_D(T_s, T_p) := \Delta(T_s, T_p) \star \left[ \begin{array}{c|c} A & B \\ \hline C_D & D_D \end{array} \right]$$

with  $C_N, C_D \in \mathbb{R}^{1 \times n}$  and  $D_N, D_D \in \mathbb{R}$ , such that  $\mathbf{W}(T_s, T_p) = \frac{\mathbf{W}_N(T_s, T_p)}{\mathbf{W}_D(T_s, T_p)}$  satisfies

$$\forall u \in \{1, \dots, N_U\}, \quad \forall \omega \in \Omega_u^U, \quad |\mathbf{W}_{T_s T_p}(j\omega)|^2 \leq U_u^2 \quad (5.60)$$

$$\forall l \in \{1, \dots, N_L\}, \quad \forall \omega \in \Omega_l^L, \quad |\mathbf{W}_{T_s T_p}(j\omega)|^2 \geq L_l^2 \quad (5.61)$$

*Proof.* From constraints (5.53)-(5.54), it comes that there exist  $\mathbf{C}_N, \mathbf{C}_D \in \mathbb{R}^{1 \times n}$  and  $\mathbf{D}_N, \mathbf{D}_D \in \mathbb{R}$  such that:

$$\begin{bmatrix} A & B \\ I & 0 \end{bmatrix}^T \begin{bmatrix} 0 & -\mathbf{P}_N - \mathbf{P}_N^c \\ -\mathbf{P}_N - \mathbf{P}_N^c & 0 \end{bmatrix} \begin{bmatrix} A & B \\ I & 0 \end{bmatrix} + \begin{bmatrix} C^T \\ D^T \end{bmatrix} \mathbf{X}_N \begin{bmatrix} C & D \end{bmatrix} = \begin{bmatrix} \mathbf{C}_N^T \\ \mathbf{D}_N^T \end{bmatrix} \begin{bmatrix} \mathbf{C}_N & \mathbf{D}_N \end{bmatrix} \quad (5.62)$$

$$\begin{bmatrix} A & B \\ I & 0 \end{bmatrix}^T \begin{bmatrix} 0 & -\mathbf{P}_D - \mathbf{P}_D^c \\ -\mathbf{P}_D - \mathbf{P}_D^c & 0 \end{bmatrix} \begin{bmatrix} A & B \\ I & 0 \end{bmatrix} + \begin{bmatrix} C^T \\ D^T \end{bmatrix} \mathbf{X}_D \begin{bmatrix} C & D \end{bmatrix} = \begin{bmatrix} \mathbf{C}_D^T \\ \mathbf{D}_D^T \end{bmatrix} \begin{bmatrix} \mathbf{C}_D & \mathbf{D}_D \end{bmatrix} \quad (5.63)$$

Pre- and post-multiplying last equations by  $\begin{bmatrix} \Delta_{T_s T_p}(j\omega)(I - A\Delta_{T_s T_p}(j\omega))^{-1}B \\ I \end{bmatrix}^*$  and

$\begin{bmatrix} \Delta_{T_s T_p}(j\omega)(I - A\Delta_{T_s T_p}(j\omega))^{-1}B \\ I \end{bmatrix}$  leads to:

$$\bar{\mathbf{B}}_{T_s T_p}(j\omega)^* \bar{\mathbf{X}}_N \bar{\mathbf{B}}_{T_s T_p}(j\omega) = \mathbf{W}_{N_{T_s T_p}}(j\omega)^* \mathbf{W}_{N_{T_s T_p}}(j\omega)$$

$$\bar{\mathbf{B}}_{T_s T_p}(j\omega)^* \bar{\mathbf{X}}_D \bar{\mathbf{B}}_{T_s T_p}(j\omega) = \mathbf{W}_{D_{T_s T_p}}(j\omega)^* \mathbf{W}_{D_{T_s T_p}}(j\omega)$$

By the KYP lemma and its dissipative extensions, it comes that (5.55) implies that

$$\forall \omega \in \mathbb{R}, \quad |\mathbf{W}_{D_{T_s T_p}}(j\omega)|^2 > 0$$

holds, while (5.56)-(5.59) are equivalent to (5.60)-(5.61).  $\square$

Similarly as in Theorem 5.7 (p. 135), optimisation problem (i) is not convex, due to the AREs (5.54). Again, minimal solutions of these AREs can be obtained by solving LMI minimisation problem. This is achieved by minimising the trace of  $\mathbf{P}_N + \mathbf{P}_N^c$ , equalled to trace of  $\mathbf{P}_N$ , and the trace of  $\mathbf{P}_D + \mathbf{P}_D^c$ , equalled to trace of  $\mathbf{P}_D$ , such that:

$$\begin{bmatrix} A & B \\ I & 0 \end{bmatrix}^T \begin{bmatrix} 0 & -\mathbf{P}_N - \mathbf{P}_N^c \\ -\mathbf{P}_N - \mathbf{P}_N^c & 0 \end{bmatrix} \begin{bmatrix} A & B \\ I & 0 \end{bmatrix} + \begin{bmatrix} C^T \\ D^T \end{bmatrix} \mathbf{X}_N \begin{bmatrix} C & D \end{bmatrix} \geq 0 \quad (5.64)$$

$$\begin{bmatrix} A & B \\ I & 0 \end{bmatrix}^T \begin{bmatrix} 0 & -\mathbf{P}_D - \mathbf{P}_D^c \\ -\mathbf{P}_D - \mathbf{P}_D^c & 0 \end{bmatrix} \begin{bmatrix} A & B \\ I & 0 \end{bmatrix} + \begin{bmatrix} C^T \\ D^T \end{bmatrix} \mathbf{X}_D \begin{bmatrix} C & D \end{bmatrix} \geq 0 \quad (5.65)$$

We propose then to exchange these AREs by the resolution of such LMI minimisation problems, leading to the following minimisation problem:

$$\left. \begin{array}{l} \min_{\mathbf{X}_N, \mathbf{X}_D, \mathbf{P}_N, \mathbf{P}_N^c, \mathbf{P}_D, \mathbf{P}_D^c,} \\ \quad \mathbf{P}_D^{pos}, \mathbf{P}_u, \mathbf{D}_u > 0, \mathbf{P}_l, \mathbf{D}_l > 0} \\ \text{subject to} \end{array} \right\} \begin{array}{l} \text{Trace}(\mathbf{P}_N) + \text{Trace}(\mathbf{P}_D) \\ (5.64), (5.65), (5.53), (5.55), (5.57), (5.59) \end{array} \quad (5.66)$$

Likewise the discussion of p. 137, the influence of the coupling of the AREs of (5.54), by the constraints (5.57) and (5.59), needs to be made clearer. However, if the optimal solutions  $\mathbf{P}_N^*$  and  $\mathbf{P}_D^*$  of (5.66) are also solutions of the AREs of (5.54), then minimal, stable and inversely-stable spectral factors  $W_N(T_s, T_p)$  and  $W_D(T_s, T_p)$  can be computed. The resulting transfer function  $W(T_s, T_p) = \frac{W_N(T_s, T_p)}{W_D(T_s, T_p)}$  is then stable and inversely-stable. As a result, such a solution of (5.66) enables to solve Problem 5.6, with the particularity that  $W(T_s(s), T_p(s))$  is minimum-phase.

### 5.5.5 Synthesis procedure and numerical example

The overall procedure to find solutions of Problem 5.6, is summarised and illustrated step-by-step below. For the illustration,  $T_s(s)$  and  $T_p(s)$  are taken as:

$$T_s(s) := \frac{s^2 + \omega_{r_s}^2}{s(s^2 + \omega_{a_s}^2)} = \left(\frac{1}{s} \cdot I_3\right) \star \left[ \begin{array}{c|c} A_{T_s} & B_{T_s} \\ \hline C_{T_s} & D_{T_s} \end{array} \right] := \left(\frac{1}{s} \cdot I_3\right) \star \left[ \begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -\omega_{a_s}^2 & 0 & 1 \\ \hline \omega_{r_s}^2 & 0 & 1 & 0 \end{array} \right]$$

$$T_p(s) := \frac{s^2 + \omega_{r_p}^2}{s(s^2 + \omega_{a_p}^2)} = \left(\frac{1}{s} \cdot I_3\right) \star \left[ \begin{array}{c|c} A_{T_p} & B_{T_p} \\ \hline C_{T_p} & D_{T_p} \end{array} \right] := \left(\frac{1}{s} \cdot I_3\right) \star \left[ \begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -\omega_{a_p}^2 & 0 & 1 \\ \hline \omega_{r_p}^2 & 0 & 1 & 0 \end{array} \right]$$

with  $\omega_{r_s} := 1.80$  rad/s,  $\omega_{a_s} := 3.06$  rad/s,  $\omega_{r_p} := 1.00$  rad/s and  $\omega_{a_p} := 1.70$  rad/s.

The following spectral mask  $\mathcal{SM}$  is considered:

$$\begin{aligned} \forall \omega \in \mathbb{R}, & \quad |W_{T_s T_p}(j\omega)|^2 \leq 1 \\ \forall \omega \in \Omega_{U_1} = [0, \omega_{U_1}], & \quad |W_{T_s T_p}(j\omega)|^2 \leq U_1^2 \\ \forall \omega \in \Omega_L = [\omega_{L_1}, \omega_{L_2}], & \quad |W_{T_s T_p}(j\omega)|^2 \geq L^2 \\ \forall \omega \in \Omega_{U_2} = [\omega_{U_2}, +\infty), & \quad |W_{T_s T_p}(j\omega)|^2 \leq U_2^2 \end{aligned}$$

with  $\omega_{U_1} = 0.8$  rad/s,  $U_1 = 0.7$ ,  $\omega_{L_1} = 1.5$  rad/s,  $\omega_{L_2} = 2.5$  rad/s,  $L = 0.65$ , and  $\omega_{U_2} = 3.5$  rad/s,  $U_2 = 0.5$ .

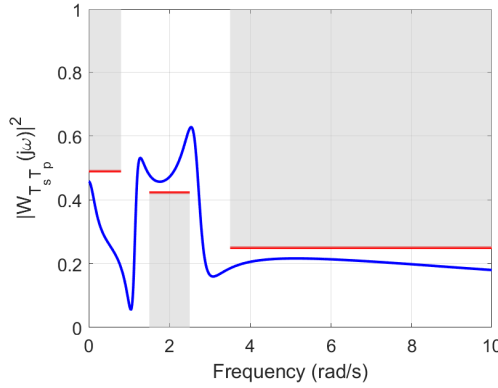


FIGURE 5.7: Illustrative example -  $T_s(s)$ ,  $T_p(s)$  lossless  $\{0, 1, 0\}$ -dissipative

1. Choose the orders  $n_{T_s}$  and  $n_{T_p}$ . Define  $n := n_{T_s} + n_{T_p}$ .
2. Define a transfer matrix  $\mathcal{B}(T_s, T_p)$  of dimension  $(n+1) \times 1$  under the LFT form:

$$\mathcal{B}(T_s, T_p) = \Delta(T_s, T_p) \star \left[ \begin{array}{c|c} A & B \\ \hline C & D \end{array} \right]$$

Recall that the choice of  $\Delta(T_s, T_p)$  and of the matrices  $A$ ,  $B$ ,  $C$ , and  $D$  influences the subclass of spectral factors that will be synthesised. Define then  $\bar{\mathcal{B}}_{T_s, T_p}(s)$  as follows:

$$\bar{\mathcal{B}}_{T_s T_p}(s) = \begin{bmatrix} \Delta_{T_s T_p}(s) F_{T_s T_p}(s) \\ F_{T_s T_p}(s) \\ \mathcal{B}_{T_s T_p}(s) \end{bmatrix} = \left( \frac{1}{s} \cdot I_{n_s} \right) \star \left[ \begin{array}{c|c} \bar{A}_s & \bar{B}_s \\ \hline \bar{C}_s & \bar{D}_s \end{array} \right]$$

with

$$\left[ \begin{array}{c|c} \bar{A}_s & \bar{B}_s \\ \hline \bar{C}_s & \bar{D}_s \end{array} \right] := \left[ \begin{array}{cc|cc} I_{n_{T_s}} \otimes A_{T_s} & 0 & I_{n_{T_s}} \otimes B_{T_s} & 0 \\ 0 & I_{n_{T_p}} \otimes A_{T_p} & 0 & I_{n_{T_p}} \otimes B_{T_p} \\ \hline I_{n_{T_s}} \otimes C_{T_s} & 0 & I_{n_{T_s}} \otimes D_{T_s} & 0 \\ 0 & I_{n_{T_p}} \otimes C_{T_p} & 0 & I_{n_{T_p}} \otimes D_{T_p} \end{array} \right] \star \left[ \begin{array}{c|c} A & B \\ \hline I & 0 \\ A & B \\ \hline C & D \end{array} \right]$$

For the illustration,  $n_{T_s} = n_{T_p} = 1$  provides a solution, and  $\mathcal{B}(T_s, T_p)$  is arbitrarily chosen as:

$$\mathcal{B}(T_s, T_p) := \Delta(T_s, T_p) \star \left[ \begin{array}{cc|c} -1 & -1 & 1 \\ 0 & -1 & 1 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -1 & 1 \end{array} \right] = \frac{1}{(T_s + 1)(T_p + 1)} \begin{bmatrix} T_s & \\ T_p + T_s T_p & \\ & 1 \end{bmatrix}$$

3. Solve LMI minimisation problem (5.66) resulting from Theorem 5.9.

If it is feasible, there exist symmetric matrices  $X_N = X_N^T$ ,  $X_D = X_D^T$ ,  $P_N^c = (P_N^c)^T$ ,  $P_D^c = (P_D^c)^T$  such that the magnitude

$$M_{T_s T_p}(j\omega) = \frac{\overline{\mathcal{B}}_{T_s T_p}(j\omega)^* \overline{X}_N \overline{\mathcal{B}}_{T_s T_p}(j\omega)}{\overline{\mathcal{B}}_{T_s T_p}(j\omega)^* \overline{X}_D \overline{\mathcal{B}}_{T_s T_p}(j\omega)}$$

is positive and satisfies the spectral mask  $\mathcal{SM}$ .

If it is not feasible, then either:

- Go back to step 2 and define new matrices  $A$ ,  $B$ ,  $C$ ,  $D$  and/or  $\Delta(T_s, T_p)$ .
- Increment  $n_{T_s}$  ( $n_{T_s} \leftarrow n_{T_s} + 1$ ) and/or  $n_{T_p}$  ( $n_{T_p} \leftarrow n_{T_p} + 1$ ), and go back to step 2.

For our illustration, the value of  $X_D(n+1, n+1)$  is fixed  $X_D(n+1, n+1) = 1$  in order to avoid numerical issues. The resulting matrices are given by:

$$X_N = \begin{bmatrix} 68.6535 & -180.2546 & -17.7489 \\ -180.2546 & 22.4888 & -152.7877 \\ -17.7489 & -152.7877 & 0.0708 \end{bmatrix} \quad X_D = \begin{bmatrix} 50.8798 & -49.9923 & -69.7345 \\ -49.9923 & 49.1201 & -70.9726 \\ -69.7345 & -70.9726 & 1.0000 \end{bmatrix}$$

$$P_N^c = \begin{bmatrix} 0 & 33.2296 \\ 33.2296 & 0 \end{bmatrix} \quad P_D^c = \begin{bmatrix} 0 & 10.4218 \\ 10.4218 & 0 \end{bmatrix}$$

4. Compute  $C_N$ ,  $D_N$ ,  $C_D$  and  $D_D$  such as in (5.62) and (5.63) in order to get

$$W_N(T_s, T_p) := \Delta(T_s, T_p) \star \left[ \begin{array}{c|c} A & B \\ \hline C_N & D_N \end{array} \right]$$

$$W_D(T_s, T_p) := \Delta(T_s, T_p) \star \left[ \begin{array}{c|c} A & B \\ \hline C_D & D_D \end{array} \right]$$

$W_N(T_s, T_p)$  and  $W_D(T_s, T_p)$  are then two stable and inversely-stable, minimal spectral factors such that:

$$|W_{N_{T_s T_p}}(j\omega)|^2 = \overline{\mathcal{B}}_{T_s T_p}(j\omega)^* \overline{X}_N \overline{\mathcal{B}}_{T_s T_p}(j\omega)$$

$$|W_{D_{T_s T_p}}(j\omega)|^2 = \overline{\mathcal{B}}_{T_s T_p}(j\omega)^* \overline{X}_D \overline{\mathcal{B}}_{T_s T_p}(j\omega)$$

For our illustration,  $C_N$ ,  $D_N$ ,  $C_D$  and  $D_D$  are computed as:

$$\left[ \begin{array}{c|c} C_N & D_N \end{array} \right] = \left[ \begin{array}{cc|c} -1.2152 & -4.4761 & -0.2661 \end{array} \right]$$

$$\left[ \begin{array}{c|c} C_D & D_D \end{array} \right] = \left[ \begin{array}{cc|c} -4.4805 & -6.0086 & -1.0000 \end{array} \right]$$

5. Use (5.21) (p. 125) to get a stable and inversely-stable  $W_{T_s T_p}$  of order  $n$  which satisfies the spectral mask  $\mathcal{SM}$  and which can be expressed as an LFT in  $\Delta_{T_s T_p}$ :

$$W(T_s, T_p) = \Delta_{T_s T_p} \star \left[ \begin{array}{c|c} A - BD_D^{-1}C_D & BD_D^{-1} \\ \hline C_N - D_N D_D^{-1}C_D & D_N D_D^{-1} \end{array} \right]$$

For our illustration, the following  $W(T(s))$  is obtained:

$$W(T_s, T_p) = \Delta_{T_s T_p} \star \left[ \begin{array}{cc|c} -5.4805 & -7.0086 & -1.0000 \\ -4.4805 & -7.0086 & -1.0000 \\ \hline -0.0229 & -2.8772 & 0.2661 \end{array} \right]$$

One can check that  $W_{T_s T_p}(s)$  is stable, using Theorem 2.1 (p. 25), and  $W_{T_s T_p}(s)$  satisfies the spectral mask on Figure 5.7.

## 5.6 Summary

In this chapter, we have revisited the usual approach for frequency filter synthesis under the LFT framework. First, a finite-dimensional parametrisation of all the transfer functions  $W(s)$  is made, using the connection between transfer functions and rational functions. Then, a convex formulation is obtained thanks to a suitable change of variable. This typically leads to solve two sub-problems, namely *magnitude synthesis* and *spectral factorisation*.

In the typical state-space case, i.e. when the filter is an LFT in  $\frac{1}{s} \cdot I$ , the synthesis problem is optimally and efficiently solved. Indeed, an equivalent finite-dimensional parametrisation of all the proper transfer functions exists, the magnitude synthesis problem is equivalent to an LMI feasibility problem, and the spectral factorisation is achieved by solving two AREs. These AREs certainly admit solutions, and there exist efficient algorithms to solve them.

Our first contribution is to have directly extended the synthesis approach to transfer functions  $W(T(s))$  expressed as an LFT in  $T(s) \cdot I$ , where  $T(s)$  is a *lossless*  $\{x, y, z\}$ -dissipative transfer function. Each step can be directly extended, only requiring algebraic manipulations on the LFT matrices.

Moreover, we have shown that this approach fails to tackle the case of  $T(s)$  being  $\{x, y, z\}$ -dissipative, and not necessarily lossless. In fact, the spectral factorisation technique becomes *lossy*, and one can not certify that the resulting filter  $W(T(s))$  satisfies the spectral mask. Our second contribution is to have developed a generalised synthesis approach in order to cope with last issue. This requires to merge both magnitude synthesis and spectral factorisation sub-problems into a single one, leading to solve a linear minimisation problem over LMI constraints.

Furthermore, the synthesis problem of 2D LFT filters, i.e. filters  $W(T_s, T_p)$  expressed as an LFT in  $\begin{pmatrix} T_s \cdot I & 0 \\ 0 & T_p \cdot I \end{pmatrix}$ , is tackled when  $T_s$  and  $T_p$  are lossless  $\{0, 1, 0\}$ -dissipative, as this is the case of interest in this work. The extension of the usual approach leads to a *conservative* approach, induced by both the finite-dimensional parametrisation and the change of variable. After detailing the sources of this conservatism, they appear to be inherent to this approach. Still, a linear minimisation problem over LMI constraints is given which, if a solution is obtained, enables to find solutions to the 2D LFT filters synthesis problem, leading to our third contribution.

All of these efficient synthesis methods can be included in the design approach 2 of Chapter 4. The method given in Section 5.3, when  $T(s) = \frac{1}{s}$ , coupled with convex realisation constraints of Subsection 4.3.3.1, leads to an efficient method for designing *LC* ladder filters. Likewise, the extension given in Section 5.4, when  $T(s)$  is lossless dissipative, can be coupled with convex realisation constraints of Subsection 4.3.3.2 for the design of some *LC*-resonator ladder filters. This will be illustrated in Chapter 6. Similarly, it will also be illustrated how the convex part of the realisation constraints for some *AW*-ladder filters (cf Subsection 4.3.3.3), can be included in the 2D LFT filter synthesis approach of Section 5.5.

However, these methods can achieve even better. Indeed, as illustrated in the introduction, the synthesis approach for LFT filters in a repeated dissipative  $T(s)$ , whether lossless or not, enables to consider interconnection of passive elements, and not uniquely lossless passive elements. In addition, it can even synthesise interconnection of dissipative elements, i.e. elements with an impedance that satisfies a more general dissipative characterisation. The remaining issue is to find realisation constraints, associated with these elements and a topology, with a convex formulation. Nevertheless, even if only a part of these realisation constraints have a convex formulation and can then be considered, the result may provide an initial point to an adapted version of design approach 1 of Chapter 4. This idea is illustrated in Chapter 6 for the design of *AW*-resonator ladder filters.



## Chapter 6

# Design Examples

In this chapter, it is aimed to apply the methods developed in this work on two design examples of passive electronic filters.

First example is concerned with the design of a *LC*-resonator ladder filter (Section 6.1) by applying the design approach 2 of Chapter 4. This mainly consists in synthesising a frequency LFT filter  $W(T(s))$  in a repeated dissipative  $T(s)$ , using the efficient method developed in Section 5.4, combined with the convex realisation constraints provided in Subsection 4.3.3.2. In order to provide a fully-complete, efficient design method, an algorithm is adapted to extract the element values from  $W(T(s))$ .

Second example tackles the design problem of an *AW*-resonator ladder filter (Section 6.2) by combining design approaches 1 and 2. The underlying idea is to get a sufficiently-good initial point, required in design approach 1, by applying design approach 2. Indeed, first a frequency LFT filter  $W(T_s(s), T_p(s))$  in repeated dissipative  $T_s(s)$  and  $T_p(s)$  is synthesised, by applying the method given in Section 5.5, such that it satisfies the *convex part* of the realisation constraints for  $T_s, T_p$ -ladder filters (cf Section 4.3.3.3). As  $W(T_s(s), T_p(s))$  does not satisfy all the realisation constraints, this does not provide an initial point for design approach 1. Then, in order to bridge this gap,  $W(T_s(s), T_p(s))$  is approximated by the scattering parameter  $s_{21}^{\{0\}}(s)$  of a  $T_s, T_p$ -ladder filter, by solving an LMI optimisation problem. Finally,  $s_{21}^{\{0\}}(s)$  is used as initial point of the algorithm given in Subsection 4.2.4 to obtain an *AW*-resonator ladder filter having a scattering parameter  $s_{21}(s)$  that satisfies the design requirements.

### 6.1 Design example of an *LC*-resonator ladder filter

In this section, it is aimed to illustrate the realisable filter synthesis approach, i.e. the design approach 2 of Chapter 4, for the synthesis of a *LC*-resonator ladder filter (Table 3.2, p. 46), with source and load resistances being equal  $R_g = R_l = R$ , with  $R := 50 \Omega$ , such that its scattering parameter  $s_{21}$  satisfies a given spectral mask  $\mathcal{SM}$ . In this example, the spectral mask  $\mathcal{SM}$  is defined by:

$$\begin{aligned} \forall \omega \in [0, \omega_{U_1}], & \quad |s_{21}(j\omega)|^2 \leq U_1^2 \\ \forall \omega \in [\omega_{L_1}, \omega_{L_2}], & \quad |s_{21}(j\omega)|^2 \geq L^2 \\ \forall \omega \in [\omega_{U_2}, +\infty), & \quad |s_{21}(j\omega)|^2 \leq U_2^2 \end{aligned}$$

with  $\omega_{U_1} := 1.6\pi$  rad/s,  $U_1|_{dB} := 20$ ,  $\omega_{L_1} := 2\pi$  rad/s,  $\omega_{L_2} := 3\pi$  rad/s,  $L|_{dB} := 0.28$  and  $\omega_{U_2} := 4.4\pi$  rad/s,  $U_2|_{dB} := 12$ , where the following notation is used:

$$K|_{dB} := -20 \log_{10}(K)$$



In this example, the characteristic values  $(L_k, C_k)$  of any element  $k$  are constrained to be such as:

$$\frac{1}{L_k C_k} = \omega_0^2$$

where  $\omega_0^2 := \omega_{L_1} \omega_{L_2}$ . As explained in Subsection 4.3.3.2 of Chapter 4, this permits to transform the electronic filter as a  $T$ -ladder filter where  $T$  is defined by:

$$T(s) = \frac{s}{s^2 + \omega_0^2}$$

As  $T(s)$  is lossless  $\{0, 1, 0\}$ -dissipative,  $T^{-1}(s)$  is also lossless  $\{0, 1, 0\}$ -dissipative, Corollary 4.6 (p. 101) can be applied and provides convex realisation constraints. Therefore, in the following, we detail the systematic approach for the design of such filters, and apply it on the example. First, a transfer function  $W(T(s))$  satisfying the realisation constraints is synthesised (Subsection 6.1.1). Second, an  $LC$ -resonator ladder electronic filter with unitary source and load resistances is synthesised (Subsection 6.1.2), such that its scattering parameter  $s_{21}(s)$  is  $W(T(s))$ :  $s_{21}(s) = W(T(s))$ . Using impedance scaling formulas, the source and load resistances are brought back to  $R = 50 \Omega$ . Finally, a comparison with the Butterworth and Chebyshev I filters of the usual design approach is drawn (Subsection 6.1.3).

### 6.1.1 Synthesis of a realisable $W(T(s))$

A transfer function  $W(T(s))$ , expressed as an LFT in a repeated  $T(s)$ , is now synthesised such that it is stable and satisfies the spectral mask  $\mathcal{SM}$  and the realisation constraints of Corollary 4.6 for  $T$ -ladder filters. To achieve this, the synthesis approach of Chapter 5 is used. As  $T(s)$  is *lossless* dissipative, the approach is composed of two distinct steps. First, the magnitude synthesis step is adapted in order to include the convex realisation constraints. This still leads to an LMI feasibility problem. Second, the spectral factorisation step is achieved, and is adapted in anticipation of the circuit synthesis stage of Subsection 6.1.2.

#### 6.1.1.1 Magnitude synthesis under realisation constraints

In the synthesis approach of Section 5.4 (Chapter 5), the finite-dimensional parametrisation of the transfer function  $W(T(s))$ , expressed as an LFT in  $T(s) \cdot I_n$ , leads the squared magnitude  $M(T(j\omega)) := |W(T(j\omega))|^2$  to be written as:

$$M(T(j\omega)) = \frac{\mathcal{B}(T(j\omega))^* X_N \mathcal{B}(T(j\omega))}{\mathcal{B}(T(j\omega))^* X_D \mathcal{B}(T(j\omega))}$$

where  $\mathcal{B}(T(s))$  is a  $(n+1) \times 1$  transfer vector, as  $\mathcal{B}(T(s))$  is the concatenation of the elements of a basis of  $\mathcal{R}_n \mathcal{H}_\infty(T(s), d(T(s)))$ , and  $X_N = X_N^T$ ,  $X_D = X_D^T$  are two symmetric matrices of suitable size. For this example,  $\mathcal{B}(T(s))$  is chosen as follows:

$$\mathcal{B}(T(s)) = \frac{1}{(T(s) + 1)^n} \cdot \begin{bmatrix} 1 \\ T(s) \\ \vdots \\ T(s)^n \end{bmatrix} = (T(s) \cdot I_n) \star \left[ \begin{array}{c|c} A & B \\ \hline C & D \end{array} \right] = \left( \frac{1}{s} \cdot I_{n_s} \right) \star \left[ \begin{array}{c|c} A_s & B_s \\ \hline C_s & D_s \end{array} \right]$$

where  $A, B, C$  and  $D$  are defined in (5.44) (p. 139), and  $A_s, B_s, C_s$  and  $D_s$  are given in (5.22) (p. 130). Note that (5.22) requires to get an LFT decomposition of  $T(s)$

in  $\frac{1}{s} \cdot I_{n_T}$ . We arbitrarily choose:

$$T(s) = \left(\frac{1}{s} \cdot I_{n_T}\right) \star \left[ \begin{array}{c|c} A_T & B_T \\ \hline C_T & D_T \end{array} \right] := \left(\frac{1}{s} \cdot I_2\right) \star \left[ \begin{array}{cc|c} 0 & 1 & 0 \\ -\omega_0^2 & 0 & 1 \\ \hline 0 & 1 & 0 \end{array} \right] \quad (6.1)$$

In order to ensure the existence of a stable spectral factor  $W(T(s))$ ,  $X_N$  and  $X_D$  should satisfy the constraints:

$$\forall \omega \in \mathbb{R}, \quad \mathcal{B}(T(j\omega))^* X_N \mathcal{B}(T(j\omega)) \geq 0 \quad (6.2)$$

$$\forall \omega \in \mathbb{R}, \quad \mathcal{B}(T(j\omega))^* X_D \mathcal{B}(T(j\omega)) > 0 \quad (6.3)$$

The realisation constraints of Corollary 4.6 for  $T$ -ladder filters are threefold: two structural constraints and a dissipative constraint. First structural constraint implies the squared magnitude  $M(T(j\omega))$  to be structured such as:

$$M(T(j\omega)) = \frac{1}{d_0 + d_2 \cdot |T^{-1}(j\omega)|^2 + \dots + d_{2(n-1)} \cdot |T^{-1}(j\omega)|^{2(n-1)} + d_{2n} \cdot |T^{-1}(j\omega)|^{2n}}$$

or equivalently

$$M(T(j\omega)) = \frac{|T(j\omega)|^{2n}}{d_0 \cdot |T(j\omega)|^{2n} + d_2 \cdot |T(j\omega)|^{2(n-1)} + \dots + d_{2(n-1)} \cdot |T(j\omega)|^2 + d_{2n}}$$

Second, because  $R_l$  is pre-defined and is equal to  $R_g$ ,  $R_l = R_g$ , there is the additional constraint:

$$d_0 = 1$$

These structural constraints are taken into account in the magnitude synthesis step by forcing  $X_N$  and  $X_D$  to be such as:

$$X_N = \begin{bmatrix} 0 & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & 0 \\ 0 & \dots & 0 & 1 \end{bmatrix} \quad (6.4)$$

$$X_D = \begin{bmatrix} X_{D_{1,1}} & \dots & X_{D_{1,n}} & X_{D_{1,n+1}} \\ \vdots & \ddots & \vdots & \vdots \\ X_{D_{1,n}} & \dots & X_{D_{n,n}} & X_{D_{n,n+1}} \\ X_{D_{1,n+1}} & \dots & X_{D_{n,n+1}} & 1 \end{bmatrix} \quad (6.5)$$

Notice that an important consequence of the imposed structure of  $X_N$  is the satisfaction of the constraint (6.2).

The third realisation constraints requires  $W(T(s))$  to be  $\{-1, 0, 1\}$ -dissipative:

$$\forall \omega \in \mathbb{R}, \quad \frac{\mathcal{B}(T(j\omega))^* X_N \mathcal{B}(T(j\omega))}{\mathcal{B}(T(j\omega))^* X_D \mathcal{B}(T(j\omega))} \leq U_{pas}^2 \quad (6.6)$$

with  $U_{pas}^2 = 1$ .

Therefore, the magnitude synthesis under realisation constraints is equivalent to search for the existence of  $\mathbf{X}_D = (\mathbf{X}_D)^T \in \mathbb{R}^{(n+1) \times (n+1)}$  such that (6.3), (6.5), (6.6),

and

$$\begin{aligned} \forall \omega \in [0, \omega_{U_1}], \quad & \frac{\mathcal{B}(T(j\omega))^* X_N \mathcal{B}(T(j\omega))}{\mathcal{B}(T(j\omega))^* \mathbf{X}_D \mathcal{B}(T(j\omega))} \leq U_1^2 \\ \forall \omega \in [\omega_{L_1}, \omega_{L_2}], \quad & \frac{\mathcal{B}(T(j\omega))^* X_N \mathcal{B}(T(j\omega))}{\mathcal{B}(T(j\omega))^* \mathbf{X}_D \mathcal{B}(T(j\omega))} \geq L_l^2 \\ \forall \omega \in [0, \omega_{U_2}], \quad & \frac{\mathcal{B}(T(j\omega))^* X_N \mathcal{B}(T(j\omega))}{\mathcal{B}(T(j\omega))^* \mathbf{X}_D \mathcal{B}(T(j\omega))} \geq U_2^2 \end{aligned}$$

hold.

Applying Theorem 5.5 (Chap. 5, p. 130), for the magnitude synthesis of LFT in a repeated dissipative  $T(s)$ , leads to solve the LMI feasibility problem (6.7). This feasibility problem is written in the form of an LMI minimisation problem. In other words, the problem is feasible if the resulting  $t$  is negative  $t < 0$ .

$$\begin{aligned} & \min_{t \in \mathbb{R}} t \\ & \mathbf{X}_D = (\mathbf{X}_D)^T \in \mathbb{R}^{(n+1) \times (n+1)}, \mathbf{P}_D = (\mathbf{P}_D)^T \in \mathbb{R}^{n \times n}, \\ & \mathbf{P}_D^{pas} = (\mathbf{P}_D^{pas})^T, \mathbf{P}_{U_1} = (\mathbf{P}_{U_1})^T, \mathbf{P}_L = (\mathbf{P}_L)^T, \mathbf{P}_{U_2} = (\mathbf{P}_{U_2})^T \in \mathbb{R}^{n_s \times n_s} \\ & \mathbf{D}_{U_1} = (\mathbf{D}_{U_1})^T > 0, \mathbf{D}_L = (\mathbf{D}_L)^T > 0, \mathbf{D}_{U_2} = (\mathbf{D}_{U_2})^T > 0 \in \mathbb{R}^{n_s \times n_s} \end{aligned} \quad \left. \vphantom{\min} \right\} t \tag{6.5}$$

such that

$$\begin{aligned} M^T \begin{bmatrix} 0 & -\mathbf{P}_D & 0 \\ -\mathbf{P}_D & 0 & 0 \\ 0 & 0 & \mathbf{X}_D \end{bmatrix} M &> -t \\ M_s^T \begin{bmatrix} 0 & -\mathbf{P}_D^{pas} & 0 \\ -\mathbf{P}_D^{pas} & 0 & 0 \\ 0 & 0 & U_{pas}^2 \mathbf{X}_D - X_N \end{bmatrix} M_s &\geq -t \\ M_s^T \begin{bmatrix} \mathbf{D}_{U_1} & -\mathbf{P}_{U_1} & 0 \\ -\mathbf{P}_{U_1} & -\omega_{U_1}^2 \mathbf{D}_{U_1} & 0 \\ 0 & 0 & U_1^2 \mathbf{X}_D - X_N \end{bmatrix} M_s &\geq -t \\ M_s^T \begin{bmatrix} \mathbf{D}_L & -j\omega_c \mathbf{D}_L - \mathbf{P}_L & 0 \\ j\omega_c \mathbf{D}_L - \mathbf{P}_L & \omega_{L_1} \omega_{L_2} \mathbf{D}_L & 0 \\ 0 & 0 & X_N - L^2 \mathbf{X}_D \end{bmatrix} M_s &\geq -t \\ M_s^T \begin{bmatrix} -\mathbf{D}_{U_2} & -\mathbf{P}_{U_2} & 0 \\ -\mathbf{P}_{U_2} & \omega_{U_2}^2 \mathbf{D}_{U_2} & 0 \\ 0 & 0 & U_2^2 \mathbf{X}_D - X_N \end{bmatrix} M_s &\geq -t \end{aligned} \quad \left. \vphantom{M^T} \right\} \tag{6.7}$$

with  $\omega_c := \frac{\omega_{L_1} + \omega_{L_2}}{2}$  and

$$M := \begin{bmatrix} A & B \\ I & 0 \\ C & D \end{bmatrix} \quad M_s := \begin{bmatrix} A_s & B_s \\ I & 0 \\ C_s & D_s \end{bmatrix} \tag{6.8}$$

In order to find the minimum  $n$  such that (6.7) is feasible, the following iterative algorithm (Algorithm 6.1) is used.

**Comments on implementation** Before providing the obtained result for this example, let us comment the implementation of Algorithm 6.1.

**Algorithm 6.1:** Magnitude synthesis with minimum order  $n$ 


---

**Input:**  $\mathcal{SM}$   
**Output:**  $n, X_D$

- 1 Define  $A_T, B_T, C_T$  and  $D_T$  as in (6.1)
- 2  $\omega_c \leftarrow \frac{\omega_{L_1} + \omega_{L_2}}{2}$
- 3  $n \leftarrow 0$
- 4  $t \leftarrow 1$
- 5 **while**  $t \geq 0$  **do**
- 6      $n \leftarrow n + 1$
- 7     Define  $A, B, C$  and  $D$  as in (5.44) (p. 139)
- 8     Define  $A_s, B_s, C_s$  and  $D_s$  as given in (5.22) (p.130)
- 9     Define  $M$  and  $M_s$  as in (6.8)
- 10    Define  $X_N$  as in (6.4)
- 11     $t, X_D \leftarrow$  Solve (6.7)

---

First, the fourth constraint of (6.7) contains complex matrices. While some particular interfaces of LMI solvers accept such constraints, this is not the case for all of them which only admit LMIs with real matrices. Using Property A.1 (p. 193) of Appendix A, this constraint is transformed into the *real*, lifted form:

$$\begin{bmatrix} M_s & 0 \\ 0 & M_s \end{bmatrix}^T \begin{bmatrix} \Re(\mathbf{G}_L) & \Im(\mathbf{G}_L) \\ -\Im(\mathbf{G}_L) & \Re(\mathbf{G}_L) \end{bmatrix} \begin{bmatrix} M_s & 0 \\ 0 & M_s \end{bmatrix} \geq 0$$

where  $\mathbf{G}_L := \begin{bmatrix} \mathbf{D}_L & -j\omega_c \mathbf{D}_L - \mathbf{P}_L & 0 \\ j\omega_c \mathbf{D}_L - \mathbf{P}_L & \omega_{L_1} \omega_{L_2} \mathbf{D}_L & 0 \\ 0 & 0 & X_N - L^2 \mathbf{X}_D \end{bmatrix}$

Second, because of its structure, the magnitude synthesis  $M(T(j\omega))$  is such that:

$$M(T(j\omega_0)) = 1$$

Combined with the constraint (6.6) this may create numerical issues, as some solvers are only able to deal with strict inequalities<sup>1</sup>. In order to manage this issue, we suggest to use  $U_{pas}^2 = 1 + \epsilon$  in (6.6), where  $\epsilon$  is a small quantity to be suitably chosen.

**Result for the example** We use Algorithm 6.1 with the numerical values of the spectral mask of our example and with a chosen value of  $\epsilon = 10^{-6}$ . The algorithm outputs  $n = 4$  and

$$X_D = \begin{bmatrix} 1.4458 \cdot 10^{-4} & -1.5001 \cdot 10^{-4} & 6.8020 \cdot 10^{-2} & -1.6985 \cdot 10^{-5} & 3.6702 \cdot 10^{-1} \\ -1.5001 \cdot 10^{-4} & 1.3368 \cdot 10^{-1} & -1.7974 \cdot 10^{-5} & 3.9894 \cdot 10^{-1} & 1.4493 \cdot 10^{-5} \\ 6.8020 \cdot 10^{-2} & -1.7974 \cdot 10^{-5} & 7.3251 \cdot 10^{-2} & 7.9567 \cdot 10^{-5} & -3.1169 \\ -1.6985 \cdot 10^{-5} & 3.9894 \cdot 10^{-1} & 7.9567 \cdot 10^{-5} & -6.2309 & -4.5613 \cdot 10^{-5} \\ 3.6702 \cdot 10^{-1} & 1.4493 \cdot 10^{-5} & -3.1169 & -4.5613 \cdot 10^{-5} & 1.0000 \end{bmatrix}$$

<sup>1</sup>cf for instance <https://mathworks.com/help/robust/ug/lmi-solvers.html>

### 6.1.1.2 Spectral factorisation

The second step is the spectral factorisation. This consists in synthesising stable transfer functions  $W_N(T(s))$  and  $W_D(T(s))$ , expressed as LFT in  $T(s) \cdot n$ , such that:

$$\forall \omega \in \mathbb{R}, \quad |W_N(T(j\omega))|^2 = \mathcal{B}(T(j\omega))^* X_N \mathcal{B}(T(j\omega)) \quad (6.9)$$

$$\forall \omega \in \mathbb{R}, \quad |W_D(T(j\omega))|^2 = \mathcal{B}(T(j\omega))^* X_D \mathcal{B}(T(j\omega)) \quad (6.10)$$

From the structure of  $X_N$  in (6.4), the spectral factorisation of  $W_N(T(s))$  is directly computable. Indeed, noticing that

$$X_N = \Gamma_N \Gamma_N^T$$

with  $\Gamma_N \in \mathbb{R}^{(n+1) \times 1}$  given by:

$$\Gamma_N = \begin{bmatrix} 0_{n \times 1} \\ 1 \end{bmatrix}$$

leads to (6.9) with  $W_N(T(s))$  given by:

$$W_N(T(s)) = \Gamma_N^T \mathcal{B}(T(s)) = (T(s) \cdot I_n) \star \left[ \begin{array}{c|c} A & B \\ \hline C_N & D_N \end{array} \right]$$

with

$$C_N = \Gamma_N^T C = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \quad D_N = \Gamma_N^T D = 0$$

The spectral factorisation of  $W_D(T(s))$  needs the application of the Lossless Spectral Factorisation theorem (Chap 5, Theorem 5.6, p 132). Note that the underlying loop-shifting is not required, as  $T(s)$  is (lossless)  $\{0, 1, 0\}$ -dissipative. This theorem leads to solve the ARE:

$$A^T(-P) + (-P)A + C^T X_D C - ((-P)B + C^T X_D D) (D^T X_D D)^{-1} (B^T(-P) + D^T X_D C) = 0$$

By taking the minimal solution  $P_D^{min} = \begin{bmatrix} -0.8024 & -3.4373 & -0.0731 & 0.3550 \\ -3.4373 & -0.1839 & -0.0147 & -0.0097 \\ -0.0731 & -0.0147 & -0.0138 & 0.0642 \\ 0.3550 & -0.0097 & 0.0642 & -0.0010 \end{bmatrix}$ ,

one ends up with the factorisation:

$$\begin{bmatrix} A & B \\ I & 0 \end{bmatrix}^T \begin{bmatrix} 0 & -P_D^{min} \\ -P_D^{min} & 0 \end{bmatrix} \begin{bmatrix} A & B \\ I & 0 \end{bmatrix} + \begin{bmatrix} C^T \\ D^T \end{bmatrix} X_D \begin{bmatrix} C & D \end{bmatrix} = \begin{bmatrix} C_D^T \\ D_D^T \end{bmatrix} \begin{bmatrix} C_D & D_D \end{bmatrix}$$

with

$$C_D = \begin{bmatrix} -0.9880 & -0.7542 & -0.2483 & -0.0250 \end{bmatrix} \quad D_D = -0.0120$$

Then,  $W_D(T(s))$  is a stable and inversely-stable spectral factor such that (6.10) holds, and is given by

$$W_D(T(s)) = (T(s) \cdot I_n) \star \left[ \begin{array}{c|c} A & B \\ \hline C_D & D_D \end{array} \right]$$

The resulting transfer function  $W(T(s)) := \frac{W_N(T(s))}{W_D(T(s))}$  is then computed as an LFT in  $T(s) \cdot I_n$  using Property 5.2 (Chap.5, p. 124) as:

$$\begin{aligned} W(T(s)) &= (T(s) \cdot I_n) \star \left[ \begin{array}{c|c} A_W & B_W \\ \hline C_W & D_W \end{array} \right] \\ &= (T(s) \cdot I_n) \star \left[ \begin{array}{cccc|c} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -83.17 & -66.727 & -26.65 & -6.082 & -83.17 \\ \hline 1 & 0 & 0 & 0 & 0 \end{array} \right] \\ &= \frac{-83.17 \cdot T(s)^4}{1 + 6.082 \cdot T(s) + 26.65 \cdot T(s)^2 + 66.73 \cdot T(s)^3 + 83.17 \cdot T(s)^4} \end{aligned}$$

One may verify that both  $A_W$  have eigenvalues in the open left-half complex plane  $\mathbb{C}^-$ . Hence, by Corollary 2.2 (p. 26),  $W(T(s))$  is stable. In addition, by developing last expression, the numerator results in being  $-83.17 \cdot s^4$ . Therefore,  $W(T(s))$  is also minimum-phase [You15].

On Figure 6.1 is plotted the squared magnitude  $|W(T(j\omega))|^2$  in the blue solid line. It can be checked that  $W(T(s))$  satisfies then the spectral mask  $\mathcal{SM}$ .

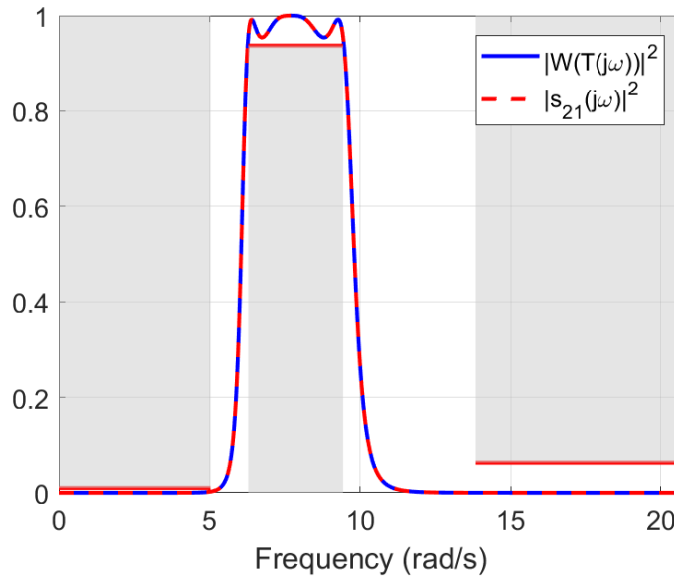


FIGURE 6.1: Plot of  $|W(T(j\omega))|^2$  and of  $|s_{21}(j\omega)|^2$  of the synthesised LC-resonator ladder filter

Finally, in anticipation of the next subsection, the circuit synthesis stage, a transfer function  $V(T(s))$  is also spectrally factorised from

$$|V(T(j\omega))|^2 = 1 - |W(T(j\omega))|^2 = \frac{\mathcal{B}(T(j\omega))^*(X_D - X_N)\mathcal{B}(T(j\omega))}{\mathcal{B}(T(j\omega))^*X_D\mathcal{B}(T(j\omega))}$$

One can note that decomposing  $V(T(s)) = \frac{V_N(T(s))}{V_D(T(s))}$  leads to  $V_D(T(s)) = W_D(T(s))$ . The computation of the numerator  $V_N(T(s))$  leads to the computation of the solutions

of the ARE associated to  $X_D - X_N$ . Taking for instance the minimal solution  $P_V^{min}$ :

$$P_V^{min} = \begin{bmatrix} -4.5638 \cdot 10^{-5} & -3.1169 & 1.4489 \cdot 10^{-5} & 3.6702 \cdot 10^{-1} \\ -3.1169 & -5.3546 \cdot 10^{-3} & 3.1390 \cdot 10^{-2} & -6.5388 \cdot 10^{-4} \\ 1.4489 \cdot 10^{-5} & 3.1390 \cdot 10^{-2} & -3.9613 \cdot 10^{-4} & 6.6790 \cdot 10^{-2} \\ 3.6702 \cdot 10^{-1} & -6.5388 \cdot 10^{-4} & 6.6790 \cdot 10^{-2} & -2.7129 \cdot 10^{-4} \end{bmatrix}$$

leads to the computation of a stable and inversely stable

$$V_N(T(s)) = (T(s) \cdot I_n) \star \left[ \begin{array}{c|c} A & B \\ \hline C_{V_N} & D_{V_N} \end{array} \right]$$

with

$$C_{V_N} = \begin{bmatrix} -0.0120 & 0.0049 & 0.0302 & -0.0380 \end{bmatrix} \quad D_{V_N} = 0.0120$$

which in turn leads to the computation of  $V(T(s))$ :

$$V(T(s)) = (T(s) \cdot I_n) \star \left[ \begin{array}{cccc|c} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \hline -83.17 & -66.73 & -26.65 & -6.082 & -83.17 \\ -1.00 & -0.75 & -0.22 & -0.06 & -1.00 \end{array} \right]$$

$$= \frac{-1 - 0.8388 \cdot T(s) - 8.510 \cdot T(s)^2 - 4.405 \cdot T(s)^3 - 4.007 \cdot 10^{-5} \cdot T(s)^4}{1 + 6.082 \cdot T(s) + 26.65 \cdot T(s)^2 + 66.73 \cdot T(s)^3 + 83.17 \cdot T(s)^4}$$

### 6.1.2 Elements value extraction

Once  $W(T(s))$  has been synthesised such that it satisfies realisation constraints, the final step is to exhibit a circuit having  $W(T(s))$  as scattering parameter

$$s_{21}(s) = W(T(s))$$

The realisation constraints provide the topology and the elements of this circuit. It is then only required to extract from  $W(T(s))$  the characteristic value of the elements.

A first possibility is to achieve a continued fractional expansion (cf p. 92) on the input impedance  $z_{in}$ . This analytical method is suitable for low-order systems but appears more limited for higher order and is not adapted for a computer implementation. For some particular filters, such as Butterworth and Chebyshev prototypes, explicit formulas of the elements value can be obtained from the zeros and the poles of the scattering parameter  $s_{21}$  [Bah84, Chap. 10]. However, this is limited to few filters as these formulas become much more complex for other filters.

Instead, the method proposed in [NHDD96, Chap. 6], and rediscovered in [SEN11], is used for its generality and is easily implementable on a computer. This method is based on the factorisation of a certain *transfer scattering* matrix. For sake of simplicity, this method is presented in the sequel only for *LC*-resonator ladder filters, and the factorisation is adapted from the transfer scattering matrix to the scattering matrix.

Notice that the corresponding circuit is first synthesised for unitary source and load resistances. These resistances are then brought to  $R = 50 \Omega$  by using standard impedance scaling formulas.

### 6.1.2.1 Elements value extraction

In order to synthesise an electronic filter in a ladder topology with a prescribed scattering matrix  $S$ , a new matrix description, namely the *transfer scattering* matrix  $\Theta$ , is introduced. Its definition is derived from the definition of the scattering matrix  $S$  given in (3.7) (p. 43):

$$\begin{bmatrix} \beta_1(s) \\ \alpha_1(s) \end{bmatrix} = \Theta(s) \begin{bmatrix} \alpha_2(s) \\ \beta_2(s) \end{bmatrix}$$

The transfer scattering matrix  $\Theta$  is then directly linked to the scattering matrix  $S(s)$ , and can be computed as follows:

$$\Theta(s) = \frac{1}{s_{21}(s)} \begin{bmatrix} -\det(S(s)) & s_{11}(s) \\ -s_{22}(s) & 1 \end{bmatrix}$$

An important property of this matrix description is that the transfer scattering matrix  $\Theta$  of the cascade of a two-port  $a$  with a two-port  $b$  is the products of their transfer scattering matrices:

$$\Theta(s) = \Theta_a(s) \cdot \Theta_b(s)$$

Recalling that the ladder topology consists in cascading serial and parallel elements, the transfer matrix appears then to be suitable for extracting elements values for an electronic filter in a ladder topology. Indeed, if one wants to compute the remaining  $\Theta_b$  after having extracted an element  $a$ , one has simply to compute:

$$\Theta_b(s) = \Theta_a^{-1}(s) \cdot \Theta(s) \quad (6.11)$$

When considering lossless passive filters, as this is the case in this example, the scattering matrix has the particular form of Theorem 3.1 (Chap. 3, p. 48). This leads to the following description:

$$\Theta(s) = \frac{1}{f(s)} \begin{bmatrix} \mu g(-s) & h(s) \\ \mu h(-s) & g(s) \end{bmatrix}$$

This enables to develop and simplify (6.11) as it comes:

$$h_b(s) = \frac{h(s)g_a(s) - g(s)h_a(s)}{\mu_a f_a(s)f_a(-s)} \quad g_b(s) = \frac{g(s)g_a(-s) - h(s)h_a(-s)}{f_a(s)f_a(-s)} \quad f_b(s) = \frac{f(s)}{f_a(s)} \quad (6.12)$$

For simplicity, the two-ports are usually extracted for building element by building element. In this example, there are then four possible  $\Theta_a$  to extract, depending if the extracted element is a capacitance  $C$  or an inductances  $L$  and if it is a serial or a parallel element. This leads to:

$$\text{Serial } L \quad h_a(s) = \frac{L}{2} \cdot s \quad g_a(s) = \frac{L}{2} \cdot s + 1 \quad f_a(s) = 1 \quad (6.13)$$

$$\text{Parallel } L \quad h_a(s) = -\frac{1}{2L} \quad g_a(s) = s + \frac{1}{2L} \quad f_a(s) = s \quad (6.14)$$

$$\text{Serial } C \quad h_a(s) = \frac{1}{2C} \quad g_a(s) = s + \frac{1}{2C} \quad f_a(s) = s \quad (6.15)$$

$$\text{Parallel } C \quad h_a(s) = -\frac{C}{2} \cdot s \quad g_a(s) = \frac{C}{2} \cdot s + 1 \quad f_a(s) = 1 \quad (6.16)$$



Finally, let us denote the polynomial functions  $g(s)$  and  $h(s)$  of the transfer scattering matrix  $\Theta(s)$  as follows:

$$g(s) := g_0 + g_1 \cdot s + g_2 \cdot s^2 + \dots + g_{n_s} \cdot s^{n_s}$$

$$h(s) := h_0 + h_1 \cdot s + h_2 \cdot s^2 + \dots + h_{n_s} \cdot s^{n_s}$$

Then, the sign of

$$\alpha = \frac{h_0}{g_0}$$

will determine if the first element is a serial *or* a parallel *LC*-resonator element. The step-by-step process is summarised in Algorithm 6.2.

---

**Algorithm 6.2:** Circuit synthesis of *LC*-resonator ladder filters

---

**Input:**  $h, g, f, n_s$   
**Output:**  $L, C, \alpha_0$

- 1  $\alpha_0 \leftarrow \frac{h_0}{g_0};$
- 2  $n_e \leftarrow \frac{n_s}{2};$
- 3 **for**  $i = 1, \dots, n_e$  **do**
- 4      $\alpha_i \leftarrow \frac{h_0}{g_0};$
- 5     **if**  $\alpha_i = +1$  **then**
- 6          $L_i \leftarrow \frac{g_{n_s} + \alpha_i h_{n_s}}{g_{n_s-1} - \alpha_i h_{n_s-1}}$
- 7         Compute  $h_a, g_a, f_a$  with (6.13)
- 8         Compute  $h_b, g_b, f_b$  with (6.12)
- 9          $h \leftarrow h_b; g \leftarrow g_b; f \leftarrow f_b;$
- 10          $n_s \leftarrow n_s - 1;$
- 11          $C_i \leftarrow \frac{g_1 - \alpha_i h_1}{g_0 + \alpha_i h_0}$
- 12         Compute  $h_a, g_a, f_a$  with (6.15)
- 13         Compute  $h_b, g_b, f_b$  with (6.12)
- 14          $h \leftarrow h_b; g \leftarrow g_b; f \leftarrow f_b;$
- 15          $n_s \leftarrow n_s - 1;$
- 16     **else**
- 17          $L_i \leftarrow \frac{g_1 - \alpha_i h_1}{g_0 + \alpha_i h_0}$
- 18         Compute  $h_a, g_a, f_a$  with (6.14)
- 19         Compute  $h_b, g_b, f_b$  with (6.12)
- 20          $h \leftarrow h_b; g \leftarrow g_b; f \leftarrow f_b;$
- 21          $n_s \leftarrow n_s - 1;$
- 22          $C_i \leftarrow \frac{g_{n_s} + \alpha_i h_{n_s}}{g_{n_s-1} - \alpha_i h_{n_s-1}}$
- 23         Compute  $h_a, g_a, f_a$  with (6.16)
- 24         Compute  $h_b, g_b, f_b$  with (6.12)
- 25          $h \leftarrow h_b; g \leftarrow g_b; f \leftarrow f_b;$
- 26          $n_s \leftarrow n_s - 1;$

---

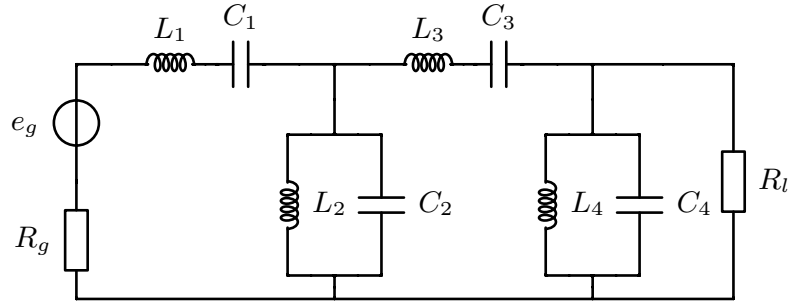


FIGURE 6.2: A four-elements, LC-resonator ladder filter

### 6.1.2.2 Application to the example

The application of Algorithm 6.2 with  $n_s = n_T \cdot n = 8$  and  $f, g, h$  such that:

$$\frac{f(s)}{g(s)} := W(T(s)) \quad \frac{h(s)}{g(s)} := V(T(s))$$

leads to a four-elements, LC-resonator ladder filter of Figure 6.2 with the elements values given in Table 6.1 and the source and the load resistances being unitary  $R_g = R_l = 1\Omega$ . One can verify that  $\forall i \in \{1, 2, 3, 4\}$ ,  $L_i C_i = \frac{1}{\omega_0^2}$ .

TABLE 6.1: Building element values for unitary source and load resistances  $R_g = R_l = 1\Omega$

Element	1	2	3	4
Inductance $L$ (mH)	44.27	460.4	35.64	289.0
Capacitance $C$ (mF)	381.5	36.68	473.9	58.43

Furthermore, when the source and load resistances are brought to  $R = 50\Omega$ , standard impedance scaling transformation [Bah84, Chap. 10], to get the new appropriate element values, say  $L_R$  and  $C_R$ . These transformations for an inductance  $L_i$  and a capacitance  $C_i$  are such that:

$$L_{R_i} = R \cdot L_i \quad C_{R_i} = \frac{C_i}{R} \quad (6.17)$$

One may note that this scaling keeps the feature that  $L_{R_i} C_{R_i} = \frac{1}{\omega_0^2}$ . The scaled element values are then provided in Table 6.2. Finally, the squared magnitude of the

TABLE 6.2: Building element values for  $R_g = R_l = R = 50\Omega$

Element	1	2	3	4
Inductance $L_R$ (H)	2.214	23.01	1.782	14.45
Capacitance $C_R$ (mF)	7.629	0.7336	9.477	1.169

frequency response of the resulting scattering parameter  $s_{21}$  is plotted in Figure 6.1, in the red dashed line. One may verify that it matches with  $|W(T(j\omega))|^2$  and satisfies then the spectral mask  $\mathcal{SM}$ . Therefore, the design problem is solved.

### 6.1.3 Order comparison with Butterworth and Chebyshev I filters

The resulting filter of our design approach is now compared with the Butterworth and Chebyshev filters of the traditional approach. In order to draw such comparison, the order of the scattering parameter of the filter, or equivalently the number of elements, is taken as indicator for discrimination.

For the design of *LC*-resonator ladder filters, the traditional design approach processes as follows:

1. The minimal orders  $n_{Butter}$  and  $n_{Cheby}$  of the *prototype* low-pass filters are calculated. This may be obtained using the *Matlab* functions *buttord* and *cheb1ord*.
2. The *prototype* low-pass transfer functions  $W_{Butter}^p(s)$  and  $W_{Cheby}^p(s)$  are computed using these orders. The Chebyshev approximation additionally requires the maximum authorised ripples in the pass-band, which is here given by  $1 - L^2$ . The transfer functions can be computed using the *Matlab* functions *buttapp* and *cheblap*.
3. If  $W_{Butter}^p(s)$  and  $W_{Cheby}^p(s)$  satisfy the *LC* ladder realisation constraints of Corollary 4.4 (p. 93), the *prototype LC* ladder filters are synthesised such that  $s_{21Butter}^p = W_{Butter}^p(s)$  and  $s_{21Cheby}^p = W_{Cheby}^p(s)$ . To achieve this, an algorithm similar to Algorithm 6.2 is provided in [NHDD96; §EN11].
4. If required, the frequency transformations of Table 4.3 (p. 90) are used to obtain the bandpass filters  $W_{Butter}(s)$  and  $W_{Cheby}(s)$  of order  $n_{sButter} = 2 \cdot n_{Butter}$  and  $n_{sCheby} = 2 \cdot n_{Cheby}$ . In particular, the *Matlab* function *lp2bp* achieves this transformation.
5. Finally, the associated building element transformations of Table 4.4 (p. 94) are achieved. Structurally, the resulting *LC*-resonator ladder filters will have scattering parameters such that  $s_{21Butter} = W_{Butter}(s)$  and  $s_{21Cheby} = W_{Cheby}(s)$ .

Applied on this design example, steps 1 and 2 lead to the minimum orders  $n_{Butter} = 5$ :  $n_{Cheby} = 4$ , and

$$W_{Butter}^p(s) := \frac{1}{1 + 3.236 \cdot s + 5.236 \cdot s^2 + 5.236 \cdot s^3 + 3.236 \cdot s^4 + s^5}$$

$$W_{Cheby}^p(s) := \frac{0.4844}{0.5002 + 1.32 \cdot s + 1.993 \cdot s^2 + 1.409 \cdot s^3 + s^4}$$

The prototype Butterworth  $W_{Butter}^p(s)$  satisfies the *LC* ladder realisation constraints and can then be realised as this. Steps 4 and 5 will then lead to a *LC*-resonator ladder filters with  $n_{sButter} = 10$  building elements which solve the design problem example.

However, as pointed out in Remark 4.3 (p. 94), prototype Chebyshev filters of even order do not fulfil all the realisation constraints of Corollary 4.4. In this example for instance, as  $R_l$  is predefined to be as  $R_l = R_g$ ,  $W_{Cheby}^p(s)$  is required to be such that  $W_{Cheby}^p(0) = 1$ , which is not the case here. In practice, one has three options. First, one may modify the load resistance in order to get  $W_{Cheby}^p(s)$  satisfying the realisation constraints of Corollary 4.4. However, this does not solve the design problem, and one may have issues with impedance mismatching (cf discussion of p. 44). Second, a *LC*-resonator ladder filter with  $n_{Cheby}$  building elements may be synthesised using the algorithm in [NHDD96; §EN11]. However, there will be a mismatch between the resulting scattering parameter  $s_{21Cheby}^p$  and  $W_{Cheby}^p(s)$ . At the end of the procedure, this mismatch may be important and  $s_{21Cheby}$  may not satisfy the

spectral mask  $\mathcal{SM}$ , as illustrated on Figure 6.3. Third, one may increment the order of the prototype, as odd-order Chebyshev filters structurally satisfy the realisation constraints [Bah84]. The resulting filter will then be a *LC*-resonator ladder filters with  $n_{s_{Cheby}} = 10$  building elements, and will solve the design problem example.

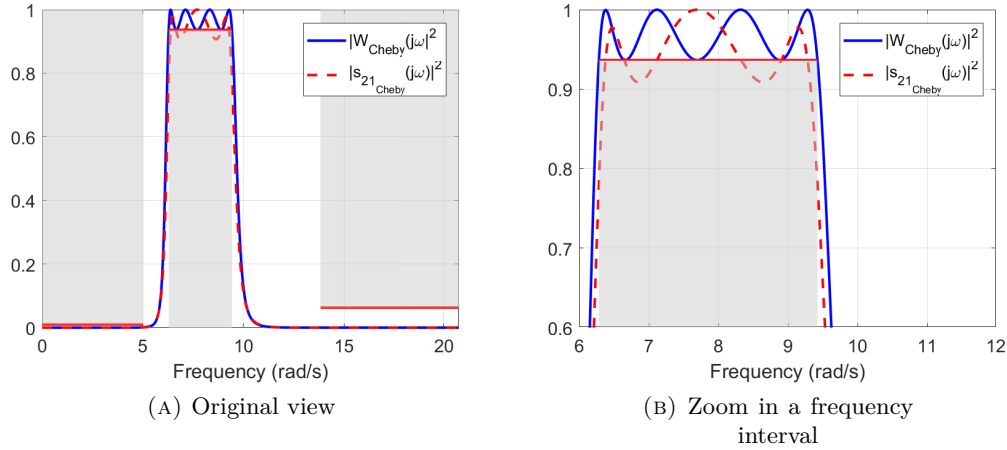


FIGURE 6.3: Typical mismatch appearing between  $|W_{Cheby}(j\omega)|^2$  and  $|s_{21_{Cheby}}(j\omega)|^2$  for even-order Chebyshev filters

Therefore, both Butterworth and Chebyshev filters will require two additional building elements compared to our approach. While this may not seem significant on this relatively-simple design example, the property of our approach to provide minimal-order filters, or equivalently minimal number of building elements, may be a determinant for more complex design applications.

## 6.2 Design example of an *AW*-resonator ladder filter

In this section, it is aimed to solve the design example of an *AW*-resonator ladder filter (Table 3.2, p. 46), i.e. for the ladder interconnection of *AW* elements. As a remainder, the  $k^{\text{th}}$  element  $e_k$  is then represented by an impedance  $z_k(s)$  given by:

$$z_k(s) = \frac{1}{C_{0_k} \cdot s} \cdot \frac{s^2 + \omega_{r_k}^2}{s^2 + \omega_{a_k}^2}$$

where  $\omega_{r_k}$  and  $\omega_{a_k}$  are respectively called resonant and anti-resonant frequencies and are linked to the characteristic element values ( $C_{0_k}, L_{m_k}, C_{m_k}$ ) by:

$$\omega_{r_k}^2 = \frac{1}{L_{m_k} C_{m_k}} \quad \omega_{a_k}^2 = \frac{C_{0_k} + C_{m_k}}{L_{m_k} C_{m_k} C_{0_k}} = \omega_{r_k}^2 \left(1 + \frac{C_{m_k}}{C_{0_k}}\right)$$

For this example, the source and load resistances are equal  $R_g = R_l = R$  and unitary<sup>2</sup>  $R := 1\Omega$ , and the spectral mask  $\mathcal{SM}$  is defined as follows:

$$\begin{aligned} \forall \omega \in [0, \omega_{U_1}], \quad & |s_{21}(j\omega)|^2 \leq U_1^2 \\ \forall \omega \in [\omega_{L_1}, \omega_{L_2}], \quad & |s_{21}(j\omega)|^2 \geq L^2 \\ \forall \omega \in [\omega_{U_2}, +\infty), \quad & |s_{21}(j\omega)|^2 \leq U_2^2 \end{aligned}$$

<sup>2</sup>As in the last section, one may use the impedance scaling transformation of (6.17) for another value of  $R$ .

with  $\omega_{U_1} := 1.0$  rad/s,  $U_1 := 0.5$ ,  $\omega_{L_1} := 1.5$  rad/s,  $\omega_{L_2} := 2.5$  rad/s,  $L := 0.85$  and  $\omega_{U_2} := 3.2$  rad/s,  $U_2 := 0.4$ .

In order to tackle this problem, both design approaches of Chapter 4 are combined. The underlying idea is to get a sufficiently-good initial point, required in design approach 1, by applying design approach 2. To achieve this, the following scheme is used. First (Subsection 6.2.1) a frequency LFT filter  $W(T_s(s), T_p(s))$  in repeated dissipative  $T_s(s)$  and  $T_p(s)$  is synthesised, by applying the method given in Section 5.5, such that it satisfies the *convex part* of the realisation constraints for  $T_s, T_p$ -ladder filters of Section 4.3.3.3. As  $W(T_s(s), T_p(s))$  does not satisfy all the realisation constraints, this does not provide an initial point for design approach 1. Then, in order to bridge this gap,  $W(T_s(s), T_p(s))$  is approximated by the scattering parameter  $s_{21}^{\{0\}}(s)$  of a  $T_s, T_p$ -ladder filter, by solving an LMI optimisation problem (Subsection 6.2.2). Finally,  $s_{21}^{\{0\}}(s)$  is used as initial point of the algorithm given in Subsection 4.2.4 in order to obtain an *AW*-resonator ladder filter having a scattering parameter  $s_{21}(s)$  that satisfies the spectral mask  $\mathcal{SM}$  (Subsection 6.2.3).

### Preliminary

The starting point of the proposed method is a  $T_s, T_p$ -ladder filter. From Subsection 4.3.3.3 of Chapter 4, an *AW*-ladder filter can be transformed into a  $T_s, T_p$ -ladder filter by requiring the resonant and anti-resonant frequencies of the serial and parallel elements to be such as:

$$\begin{aligned} \omega_{r_1} = \omega_{r_3} = \dots = \omega_0 & & \omega_{a_1} = \omega_{a_3} = \dots = \omega_{a_s} \\ \omega_{r_2} = \omega_{r_4} = \dots = \omega_{r_p} & & \omega_{a_2} = \omega_{a_4} = \dots = \omega_0 \end{aligned}$$

One may recall that this implies that  $j\omega_0$  is a zero of reflection while  $j\omega_{a_s}$  and  $j\omega_{r_p}$  are two zeros of transmission. In this example, the values of these zeros are arbitrarily fixed as follows:

$$\omega_{r_p} = 1.00 \text{ rad/s} \quad \omega_0 = 1.75 \text{ rad/s} \quad \omega_{a_s} = 3.0625 \text{ rad/s}$$

Notice that for each *AW* element  $k$ , the ratio of the anti-resonant frequency over the resonant frequency is the same:  $\frac{\omega_0}{\omega_{r_p}} = \frac{\omega_{a_s}}{\omega_0} = 1.75$ . The transfer functions  $T_s(s)$  and  $T_p(s)$  are finally defined as:

$$T_s(s) := \frac{s^2 + \omega_0^2}{s(s^2 + \omega_{a_s}^2)} \quad T_p(s) := \frac{s^2 + \omega_{r_p}^2}{s(s^2 + \omega_0^2)}$$

One may verify that  $T_s$  and  $T_p$  are both lossless  $\{0, 1, 0\}$ -dissipative.

## 6.2.1 2D LFT filter synthesis with some realisation constraints

### 6.2.1.1 Synthesis with factorisation error management

The first stage consists in synthesising a 2D LFT filter  $W(T_s, T_p)$  such that it satisfies *some* realisation constraints of  $T_s, T_p$ -ladder filters and the spectral mask  $\mathcal{SM}$ , using the method developed in Section 5.5 of Chapter 5. Indeed, from the discussion of Subsection 4.3.3.3 of Chapter 4, it appears that there is not a convex formulation for all the realisation constraints. Therefore, only those having a convex formulation are taken into account.

This requires primarily to express the realisation conditions, and to extract those that can be convexly included in the synthesis of  $|W_{T_s, T_p}(j\omega)|^2$ . In this example, it

will turn out that these convex realisation constraints consist in  $|W_{T_s, T_p}(j\omega)|^2$  having a simple numerator, directly factorisable, and some equality conditions between the coefficients of its numerator and its denominator. Using the notations of Chapter 5,

$$f_{T_s T_p}(j\omega) := f(T_s(j\omega), T_p(j\omega))$$

$$\Delta(T_s, T_p) := \begin{pmatrix} T_s \cdot I_{n_{T_s}} & 0 \\ 0 & T_p \cdot I_{n_{T_p}} \end{pmatrix}$$

it follows the following factorisation:

$$\forall \omega \in \mathbb{R}, \quad |W_{T_s, T_p}(j\omega)|^2 := \frac{|W_{N_{T_s, T_p}}(j\omega)|^2}{|W_{D_{T_s, T_p}}(j\omega)|^2} := \frac{\mathcal{B}_{T_s, T_p}(j\omega)^* X_N \mathcal{B}_{T_s, T_p}(j\omega)}{E_{D_{T_s, T_p}}(j\omega) + \mathcal{B}_{T_s, T_p}(j\omega)^* X_D \mathcal{B}_{T_s, T_p}(j\omega)}$$

where  $\mathcal{B}(T_s, T_p) = \Delta(T_s, T_p) \star \left[ \begin{array}{c|c} A & B \\ \hline C & D \end{array} \right]$  will be a given transfer matrix and

$$E_{D_{T_s T_p}}(j\omega) := ((I - A\Delta_{T_s T_p}(j\omega))^{-1}B)^* \begin{bmatrix} I \\ \Delta_{T_s T_p}(j\omega) \end{bmatrix}^* \begin{bmatrix} 0 & -P_D^c \\ -P_D^c & 0 \end{bmatrix} \begin{bmatrix} I \\ \Delta_{T_s T_p}(j\omega) \end{bmatrix} ((I - A\Delta_{T_s T_p}(j\omega))^{-1}B)$$

Indeed, in the sequel, it will be displayed that the numerator is directly factorisable if  $X_N$  is non-negative

$$X_N \geq 0$$

which implies there is not a factorisation error  $E_{N_{T_s T_p}}(j\omega)$ . The equality conditions on the coefficients will involve linear equality conditions between the entries of  $X_N$ ,  $X_D$  and  $P_D^c$ , and lead then to structural constraints on  $X_N$  and  $X_D$ .

From Property 3.3 (p. 45), the last necessary convex realisation condition is the  $\{0, 1, 0\}$ -dissipativeness of  $W(T_s, T_p)$ . Similarly to Section 6.1, we relax this constraint to prevent numerical issues as follows:

$$\forall \omega \in \mathbb{R}, \quad \frac{\mathcal{B}_{T_s, T_p}(j\omega)^* X_N \mathcal{B}_{T_s, T_p}(j\omega)}{E_{D_{T_s, T_p}}(j\omega) + \mathcal{B}_{T_s, T_p}(j\omega)^* X_D \mathcal{B}_{T_s, T_p}(j\omega)} \leq U_{pas}^2$$

where  $U_{pas}^2 = 1 + \epsilon$ . For this example, it is chosen  $\epsilon = 0.01$ .

Applying Theorem 5.9 (p. 149), and the discussion afterwards, leads to solve linear minimisation problem over LMI constraints of (6.18).

$$\begin{array}{l}
 \min_{t \in \mathbb{R}} \quad \text{trace}(\mathbf{P}_D) + t \\
 \mathbf{X}_N = (\mathbf{X}_N)^T, \mathbf{X}_D = (\mathbf{X}_D)^T \in \mathbb{R}^{(n+1) \times (n+1)}, \\
 \mathbf{P}_D = (\mathbf{P}_D)^T, \mathbf{P}_D^c = (\mathbf{P}_D^c)^T \in \mathbb{R}^{n \times n}, \\
 \mathbf{P}_D^{pas} = (\mathbf{P}_D^{pas})^T, \mathbf{P}_{U_1} = (\mathbf{P}_{U_1})^T, \mathbf{P}_L = (\mathbf{P}_L)^T, \mathbf{P}_{U_2} = (\mathbf{P}_{U_2})^T \in \mathbb{R}^{n_s \times n_s} \\
 \mathbf{D}_{U_1} = (\mathbf{D}_{U_1})^T > 0, \mathbf{D}_l = (\mathbf{D}_l)^T > 0, \mathbf{D}_{U_2} = (\mathbf{D}_{U_2})^T > 0 \in \mathbb{R}^{n_s \times n_s}
 \end{array}
 \left. \vphantom{\begin{array}{l} \min \\ \mathbf{X}_N \\ \mathbf{X}_D \\ \mathbf{P}_D \\ \mathbf{P}_D^c \\ \mathbf{P}_D^{pas} \\ \mathbf{P}_{U_1} \\ \mathbf{P}_L \\ \mathbf{P}_{U_2} \\ \mathbf{D}_{U_1} \\ \mathbf{D}_l \\ \mathbf{D}_{U_2} \end{array}} \right\}$$

such that

$$\begin{array}{l}
 \mathbf{X}_N \geq 0 \\
 \mathbf{M}^T \begin{bmatrix} 0 & -\mathbf{P}_D - \mathbf{P}_D^c & 0 \\ -\mathbf{P}_D - \mathbf{P}_D^c & 0 & 0 \\ 0 & 0 & \mathbf{X}_D \end{bmatrix} \mathbf{M} > -t \\
 \overline{\mathbf{M}}_s^T \begin{bmatrix} 0 & -\mathbf{P}_D^{pas} & 0 \\ -\mathbf{P}_D^{pas} & 0 & 0 \\ 0 & 0 & U_{pas}^2 \overline{\mathbf{X}}_D - \overline{\mathbf{X}}_N \end{bmatrix} \overline{\mathbf{M}}_s \geq -t \\
 \overline{\mathbf{M}}_s^T \begin{bmatrix} \mathbf{D}_{U_1} & -\mathbf{P}_{U_1} & 0 \\ -\mathbf{P}_{U_1} & -\omega_{U_1}^2 \mathbf{D}_{U_1} & 0 \\ 0 & 0 & U_1^2 \overline{\mathbf{X}}_D - \overline{\mathbf{X}}_N \end{bmatrix} \overline{\mathbf{M}}_s \geq -t \\
 \overline{\mathbf{M}}_s^T \begin{bmatrix} \mathbf{D}_L & -j\omega_c \mathbf{D}_L - \mathbf{P}_L & 0 \\ j\omega_c \mathbf{D}_L - \mathbf{P}_L & \omega_{L_1} \omega_{L_2} \mathbf{D}_L & 0 \\ 0 & 0 & \overline{\mathbf{X}}_N - L^2 \overline{\mathbf{X}}_D \end{bmatrix} \overline{\mathbf{M}}_s \geq -t \\
 \overline{\mathbf{M}}_s^T \begin{bmatrix} -\mathbf{D}_{U_2} & -\mathbf{P}_{U_2} & 0 \\ -\mathbf{P}_{U_2} & \omega_{U_2}^2 \mathbf{D}_{U_2} & 0 \\ 0 & 0 & U_2^2 \overline{\mathbf{X}}_D - \overline{\mathbf{X}}_N \end{bmatrix} \overline{\mathbf{M}}_s \geq -t
 \end{array}
 \quad (6.18)$$

with  $\omega_c := \frac{\omega_{L_1} + \omega_{L_2}}{2}$  and

$$\mathbf{M} := \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{I} & \mathbf{0} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} \quad \overline{\mathbf{M}}_s := \begin{bmatrix} \overline{\mathbf{A}}_s & \overline{\mathbf{B}}_s \\ \mathbf{I} & \mathbf{0} \\ \overline{\mathbf{C}}_s & \overline{\mathbf{D}}_s \end{bmatrix} \quad \overline{\mathbf{X}}_N := \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \mathbf{X}_N \end{bmatrix} \quad \overline{\mathbf{X}}_D := \begin{bmatrix} 0 & -\mathbf{P}_D^c & 0 \\ -\mathbf{P}_D^c & 0 & 0 \\ 0 & 0 & \mathbf{X}_D \end{bmatrix}$$

and  $\overline{\mathbf{A}}_s, \overline{\mathbf{B}}_s, \overline{\mathbf{C}}_s$  and  $\overline{\mathbf{D}}_s$ ) are defined as in Theorem 5.9 (p. 149).

In Algorithm 6.3 is proposed a procedure, that is applied in the sequel, in order to synthesise a 2D LFT filter which satisfies the spectral mask  $\mathcal{SM}$  and the previous convex realisation constraints. One may notice that the number  $n_{T_s}$  of serial elements and the number  $n_{T_p}$  of parallel elements of a  $T_s, T_p$ -ladder filter differ by at most 1, i.e.  $|n_{T_s} - n_{T_p}| \leq 1$ .

### Case $n = 2$ ( $n_{T_s} = n_{T_p} = 1$ )

First iteration of Algorithm 6.3 is the case of  $n = 2$ , i.e.  $n_{T_s} = n_{T_p} = 1$ , of Figure 6.4 (a). From Example 4.9 (p. 103), the squared magnitude of the scattering parameter  $s_{21}(T_s, T_p)$  is given by:

$$|s_{21_{T_s T_p}}(j\omega)|^2 = \frac{4q_2^2 \cdot |T_p(j\omega)|^2}{1 + q_1^2 \cdot |T_s(j\omega)|^2 + 4q_2^2 \cdot |T_p(j\omega)|^2 + 2q_1 q_2 \cdot |T_s(j\omega) T_p(j\omega)| + q_1^2 q_2^2 \cdot |T_s(j\omega) T_p(j\omega)|^2}$$

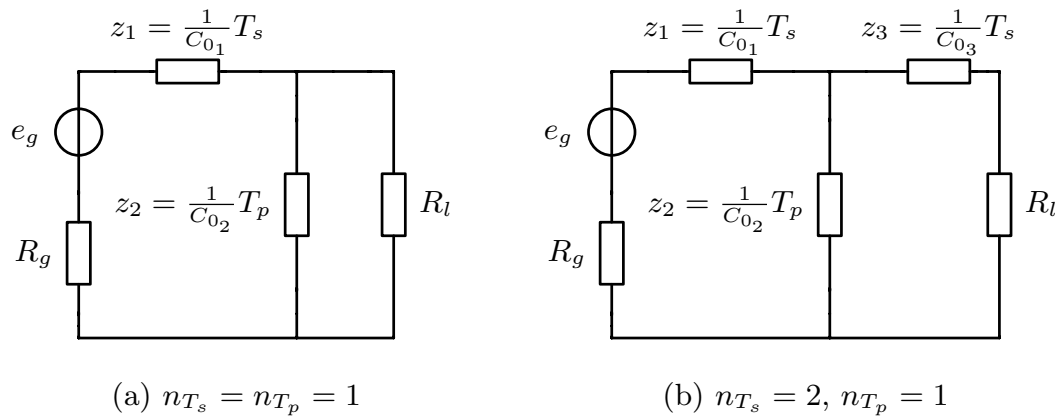
where  $q_1 := \frac{1}{C_{0_1}}$ ,  $q_2 := \frac{1}{C_{0_2}}$ . The convex realisation constraints here are the structure of the numerator and the coefficient equality between the numerator and the

**Algorithm 6.3:** 2D LFT filters Synthesis with convex realisation constraints**Input:**  $\mathcal{SM}$ **Output:**  $n_{T_s}, n_{T_p}, X_N, X_D, P_D^c$ 

```

1  $n \leftarrow 0$ 
2  $t \leftarrow 1$ 
3 while  $t \geq 0$  do
4    $n \leftarrow n + 1$ 
5    $n_{T_s} \leftarrow \lceil \frac{n}{2} \rceil$ 
6    $n_{T_p} \leftarrow n - n_{T_s}$ 
7   while  $t \geq 0$  &  $|n_{T_s} - n_{T_p}| \leq 1$  do
8     Exhibit the convex part of realisation constraints
9     Define  $A \in \mathbb{R}^{n \times n}$ ,  $B \in \mathbb{R}^{n \times 1}$ ,  $C \in \mathbb{R}^{1 \times n}$  and  $D \in \mathbb{R}$ 
10    Determine the associated structure of  $X_N$  and  $X_D$ 
11    Define  $\bar{A}_s, \bar{B}_s, \bar{C}_s$  and  $\bar{D}_s$  as in Theorem 5.9 (p. 149)
12     $t, X_N, X_D, P_D^c \leftarrow$  Solve (6.18)
13     $n_{T_s} \leftarrow n_{T_s} - 1$ 
14     $n_{T_p} \leftarrow n_{T_p} + 1$ 
15  $n_{T_s} \leftarrow n_{T_s} + 1$ 
16  $n_{T_p} \leftarrow n_{T_p} - 1$ 

```

FIGURE 6.4:  $T_s, T_p$ -ladder filter examples with  $n = 2$  and  $n = 3$  elements

denominator. It is then aimed to synthesise a  $W(T_s, T_p)$  such that:

$$|W_{T_s T_p}(j\omega)|^2 = \frac{a_{2p} \cdot |T_p(j\omega)|^2}{1 + a_{2s} \cdot |T_s(j\omega)|^2 + a_{2p} \cdot |T_p(j\omega)|^2 + a_{1s1p} \cdot |T_s(j\omega)T_p(j\omega)| + a_{2s2p} \cdot |T_s(j\omega)T_p(j\omega)|^2}$$

To express these constraints, define for instance the following matrices:

$$A := \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix} \quad B := \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad C := \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \quad D := \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$



Then the transfer matrix  $\mathcal{B}(T_s, T_p)$  can be computed as:

$$\mathcal{B}(T_s, T_p) = \begin{pmatrix} T_s \cdot I_1 & 0 \\ 0 & T_p \cdot I_1 \end{pmatrix} \star \left[ \begin{array}{c|c} A & B \\ \hline C & D \end{array} \right] = \frac{1}{(T_s + 1)(T_p + 1)} \begin{bmatrix} T_s(T_p + 1) & \\ & T_p \\ 1 + T_s + 2T_p + T_s T_p & \end{bmatrix}$$

For a given  $P_D^c = \begin{bmatrix} 0 & p_D^c \\ p_D^c & 0 \end{bmatrix}$ , the factorisation error of the denominator will then be:

$$E_{D_{T_s T_p}}(j\omega) = \frac{-2p_D^c (T_p(j\omega)T_s(j\omega) + |T_p(j\omega)|^2)}{|(T_s(j\omega) + 1)|^2 |(T_p(j\omega) + 1)|^2}$$

Therefore, in order to impose previous convex realisation conditions,  $X_N$  and  $X_D$  are structured as follows:

$$X_N = \begin{bmatrix} 0 & 0 & 0 \\ 0 & x_{22} - 2p_D^c & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad X_D = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{12} & x_{22} & x_{23} \\ x_{13} & x_{23} & 1 \end{bmatrix}$$

By running the LMI optimisation problem (6.18), it appears to be infeasible.

### Case $n = 3$ ( $n_{T_s} = 2$ , $n_{T_p} = 1$ )

Consider now the case of  $n = 3$ , with  $n_{T_s} = 2$ ,  $n_{T_p} = 1$ , of Figure 6.4 (b). In order to obtain the realisation conditions on  $|s_{T_s T_p}(j\omega)|^2$ , the input impedance is first computed using (4.43) (p. 91):

$$z_{in}(T_s, T_p) = \frac{q_1 \cdot T_s + q_2 \cdot T_p + q_2(q_1 + q_3) \cdot T_s T_p + q_1 q_3 \cdot T_s^2}{1 + q_2 \cdot T_p + q_3 \cdot T_s}$$

where

$$q_1 := \frac{1}{C_{0_1}} \quad q_2 := \frac{1}{C_{0_2}} \quad q_3 := \frac{1}{C_{0_3}}$$

Using Property 3.2 (p. 44), one can compute  $s_{11}(T_s, T_p)$  as:

$$s_{11}(T_s, T_p) = \frac{-1 + (q_1 - q_3) \cdot T_s + q_2(q_1 + q_3) \cdot T_s T_p + q_1 q_3 \cdot T_s^2}{1 + (q_1 + q_3) \cdot T_s + 2q_2 \cdot T_p + q_2(q_1 + q_3) \cdot T_s T_p + q_1 q_3 \cdot T_s^2} \quad (6.19)$$

Hence, by Property 3.4 (p. 49), the squared magnitude of the scattering parameter  $s_{21}(T_s, T_p)$  is given by:

$$|s_{21_{T_s T_p}}(j\omega)|^2 = \frac{|N_{T_s T_p}(j\omega)|^2}{|D_{T_s T_p}(j\omega)|^2} \quad (6.20)$$

where

$$\begin{aligned} |N_{T_s T_p}(j\omega)|^2 &= 4q_2^2 \cdot |T_p(j\omega)|^2 \\ |D_{T_s T_p}(j\omega)|^2 &= 1 + (q_1^2 + q_3^2) \cdot |T_s(j\omega)|^2 + 4q_2^2 \cdot |T_p(j\omega)|^2 + 2q_2(q_1 + q_3) \cdot |T_s(j\omega)T_p(j\omega)| \\ &\quad + q_1^2 q_3^2 \cdot |T_s(j\omega)|^4 + 2q_1 q_2 q_3 (q_1 + q_3) \cdot |T_s(j\omega)|^3 |T_p(j\omega)| + q_2^2 (q_1 + q_3)^2 \cdot |T_s(j\omega)|^2 |T_p(j\omega)|^2 \end{aligned}$$

It is then aimed to synthesise a  $W(T_s, T_p)$  such that:

$$|W_{T_s T_p}(j\omega)|^2 = \frac{a_{2p}|T_p(j\omega)|^2}{1 + a_{2s}|T_s(j\omega)|^2 + a_{2p}|T_p(j\omega)|^2 + a_{1s1p}|T_s(j\omega)T_p(j\omega)| + a_{4s}|T_s(j\omega)|^4 + a_{3s1p}|T_s(j\omega)^3 T_p(j\omega)| + a_{2s2p}|T_s(j\omega)|^2 |T_p(j\omega)|^2}$$

To express these constraints, define for instance the following matrices:

$$A := \begin{bmatrix} -1 & 0 & 0 \\ -1 & -1 & 0 \\ 0 & -1 & -1 \end{bmatrix} \quad B := \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \quad C := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & -1 \end{bmatrix} \quad D := \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Then,

$$\mathcal{B}(T_s, T_p) = \begin{pmatrix} T_s \cdot I_2 & 0 \\ 0 & T_p \cdot I_1 \end{pmatrix} \star \left[ \begin{array}{c|c} A & B \\ \hline C & D \end{array} \right] = \frac{1}{(T_s + 1)^2 (T_p + 1)} \begin{bmatrix} T_s(T_s + 1)(T_p + 1) \\ T_s(T_s + 2)(T_p + 1) \\ T_p \\ 1 \end{bmatrix}$$

For a given  $P_D^c = \begin{bmatrix} 0 & 0 & p_{D_1}^c \\ 0 & 0 & p_{D_2}^c \\ p_{D_1}^c & p_{D_2}^c & 0 \end{bmatrix}$ , the factorisation error will then be:

$$E_{D_{T_s T_p}}(j\omega) = \frac{(2p_{D_1}^c + 2p_{D_2}^c) |T_s(j\omega)|^2 + (-2p_{D_1}^c - 4p_{D_2}^c) |T_p(j\omega)|^2 + 2p_{D_2}^c \cdot |T_s(j\omega)T_p(j\omega)|}{|T_s(j\omega) + 1|^4 |T_p(j\omega) + 1|^2}$$

Therefore, in order to impose previous convex realisation conditions,  $X_N$  and  $X_D$  are structured as follows:

$$X_N = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & x_{33} - 2p_{D_1}^c - 4p_{D_2}^c & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad X_D = \begin{bmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{12} & x_{22} & x_{23} & x_{24} \\ x_{13} & x_{23} & x_{33} & x_{34} \\ x_{14} & x_{24} & x_{34} & 1 \end{bmatrix}$$

By running the LMI optimisation problem (6.18), it appears to be feasible. The following matrices are then obtained:

$$X_N = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 4.9978 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad P_D^c = \begin{bmatrix} 0 & 0 & 2.6929 \\ 0 & 0 & 10.533 \\ 2.693 & 10.533 & 0 \end{bmatrix}$$

$$X_D = \begin{bmatrix} 65.144 & -29.412 & 11.993 & 28.316 \\ -29.412 & -3.547 & -48.303 & -53.457 \\ 11.993 & -48.303 & 52.518 & -67.144 \\ 28.316 & -53.457 & -67.144 & 1.0000 \end{bmatrix}$$

### 6.2.1.2 Spectral factorisation

Now that a 2D magnitude synthesis has been synthesised, the aim is to spectrally factorise it. This consists in synthesising stable LFT  $W_N(T_s, T_p)$  and  $W_D(T_s, T_p)$  such

that:

$$\begin{aligned}\forall \omega \in \mathbb{R}, \quad |W_{N_{T_s, T_p}}(j\omega)|^2 &= \mathcal{B}_{T_s, T_p}(j\omega)^* X_N \mathcal{B}_{T_s, T_p}(j\omega) \\ \forall \omega \in \mathbb{R}, \quad |W_{D_{T_s, T_p}}(j\omega)|^2 &= \bar{\mathcal{B}}_{T_s, T_p}(j\omega)^* \bar{X}_D \bar{\mathcal{B}}_{T_s, T_p}(j\omega)\end{aligned}$$

From the structure of  $X_N$  and the non-negativity constraint  $X_N \geq 0$ , it comes that:

$$X_N = \Gamma_N \Gamma_N^T$$

with  $\Gamma_N \in \mathbb{R}^{n+1}$  given by:

$$\Gamma_N = \begin{bmatrix} 0 \\ 0 \\ \sqrt{x_{33} - 2p_{D_1}^c - 4p_{D_2}^c} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2.2356 \\ 0 \end{bmatrix}$$

And then  $W_N(T_s, T_p)$  is given by:

$$W_N(T_s, T_p) = \gamma_N^T \mathcal{B}(T_s, T_p) = \begin{pmatrix} T_s \cdot I_2 & 0 \\ 0 & T_p \cdot I_1 \end{pmatrix} \star \left[ \begin{array}{c|c} A & B \\ \hline C_N & D_N \end{array} \right]$$

with

$$C_N = \Gamma_N^T C = \begin{bmatrix} 0 & 0 & 2.2356 \end{bmatrix} \quad D_N = \Gamma_N^T D = 0$$

From Theorem 5.9 (p. 149) and the minimisation of the trace of  $P_D$ , it comes directly that:

$$\begin{bmatrix} A & B \\ I & 0 \\ C & D \end{bmatrix}^T \begin{bmatrix} 0 & -P_D - P_D^c & 0 \\ -P_D - P_D^c & 0 & 0 \\ 0 & 0 & X_D \end{bmatrix} \begin{bmatrix} A & B \\ I & 0 \\ C & D \end{bmatrix} = \begin{bmatrix} C_D^T \\ D_D^T \end{bmatrix} \begin{bmatrix} C_D & D_D \end{bmatrix}$$

with  $\begin{bmatrix} C_D & D_D \end{bmatrix} \in \mathbb{R}^{1 \times (n+1)}$  and

$$C_D = \begin{bmatrix} -0.3266 & 0.9915 & 1.2356 \end{bmatrix} \quad D_D = 1.0000$$

Then  $W_D(T_s, T_p)$  is a stable and inversely-stable spectral factor given by:

$$W_D(T_s, T_p) = \begin{pmatrix} T_s \cdot I_2 & 0 \\ 0 & T_p \cdot I_1 \end{pmatrix} \star \left[ \begin{array}{c|c} A & B \\ \hline C_D & D_D \end{array} \right]$$

The resulting transfer function  $W(T_s, T_p) := \frac{W_N(T_s, T_p)}{W_D(T_s, T_p)}$  is then computed using Property 5.2 (Chap. 5, p. 124) as:

$$\begin{aligned} W(T_s, T_p) &= (T(s) \cdot I_n) \star \left[ \begin{array}{c|c} A_W & B_W \\ \hline C_W & D_W \end{array} \right] \\ &= \begin{pmatrix} T_s \cdot I_2 & 0 \\ 0 & T_p \cdot I_1 \end{pmatrix} \star \left[ \begin{array}{ccc|c} -0.6734 & -0.9915 & -1.2356 & 1 \\ -0.3468 & -2.9831 & -2.4711 & 2 \\ 0.3266 & -1.9915 & -2.2356 & 1 \\ \hline 0 & 0 & 2.2356 & 0 \end{array} \right] \\ &= \frac{2.2356 \cdot T_p}{1 + 3.6566 \cdot T_s + 2.2356 \cdot T_p + 3.6566 \cdot T_s T_p + 1.6649 \cdot T_s^2 + 1.6649 \cdot T_s^2 T_p} \end{aligned}$$

One may verify that  $A_W$  have its eigenvalues in the open left-half complex plane  $\mathbb{C}^-$ . Hence, by Corollary 2.2 (Chap. 2, p. 26),  $W(T_s, T_p)$  is stable. In addition, developing last expression leads the numerator to be given by  $2.2356 \cdot s^2(s^2 + \omega_{r_p}^2)(s^2 + \omega_{a_s}^2)^2$ . Therefore,  $W_{T_s T_p}(s)$  is also minimum-phase [You15].

On Figure 6.5 is plotted the squared magnitude  $|W_{T_s T_p}(j\omega)|^2$  in the blue solid line. One can check that it satisfies the spectral mask  $\mathcal{SM}$ .

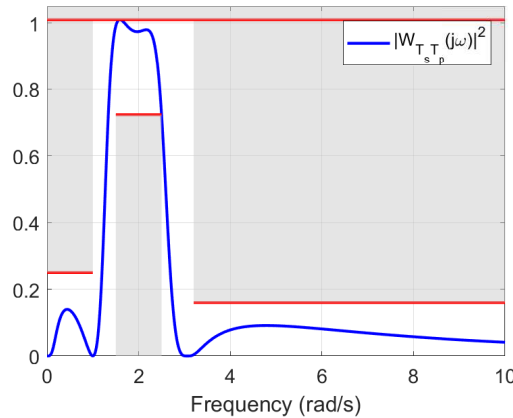


FIGURE 6.5: Plot of the 2D LFT filter  $|W_{T_s T_p}(j\omega)|^2$  including convex part of realisation constraints

### 6.2.2 $T_s, T_p$ -ladder approximation

While the synthesised  $W(T_s, T_p)$  satisfies the spectral mask  $\mathcal{SM}$  and the convex part of the realisation constraints of  $T_s, T_p$ -ladder filters, it does not satisfy all of them. Indeed, let us factorise  $s_{21}(T_s, T_p)$  from (6.19) and (6.20) as follows:

$$s_{21}(T_s, T_p) = \frac{2q_2 \cdot T_p}{1 + (q_1 + q_3) \cdot T_s + 2q_2 \cdot T_p + q_2(q_1 + q_3) \cdot T_s T_p + q_1 q_3 \cdot T_s^2}$$

One may note the mismatch between  $s_{21}(T_s, T_p)$  and  $W(T_s, T_p)$ , which is of the form:

$$W(T_s, T_p) := \frac{a_{1p} \cdot T_p}{1 + a_{1s} \cdot T_s + a_{1p} \cdot T_p + a_{2s} \cdot T_s^2 + a_{2s1p} \cdot T_s^2 T_p}$$

In order to provide an initial point to the design approach 1, it is proposed to look for  $\mathbf{q}_1 > 0$ ,  $\mathbf{q}_2 > 0$  and  $\mathbf{q}_3 > 0$  such that the transfer function  $\mathbf{s}_{21}^{\{0\}}(T_s, T_p)$  defined by

$$\mathbf{s}_{21}^{\{0\}}(T_s, T_p) := \frac{2\mathbf{q}_2 \cdot T_p}{1 + (\mathbf{q}_1 + \mathbf{q}_3) \cdot T_s + 2\mathbf{q}_2 \cdot T_p + \mathbf{q}_2(\mathbf{q}_1 + \mathbf{q}_3) \cdot T_s T_p + \mathbf{q}_1 \mathbf{q}_3 \cdot T_s^2}$$

approximates  $W(T_s, T_p)$  in the sense that they minimise the relative error given below, which is based on the  $\mathcal{H}_\infty$  norm [DPZ91].

$$\left\| 1 - \frac{W(T_s, T_p)}{\mathbf{s}_{21}^{\{0\}}(T_s, T_p)} \right\|_\infty^2 \quad (6.21)$$

It is shown below that this approximation problem admits an equivalent LMI optimisation formulation if  $\mathbf{s}_{21}^{\{0\}}(T_s, T_p)$  has a constant numerator and the decision variables are the coefficients of its denominator. Therefore, the value of  $\mathbf{q}_2$  is fixed as

$$\mathbf{q}_2 = \frac{a_{1p}}{2} = 1.1178$$

and the following change of variables is achieved

$$\mathbf{b}_1 := \mathbf{q}_1 + \mathbf{q}_3 \quad \mathbf{b}_2 := \mathbf{q}_1 \mathbf{q}_3$$

Hence, the decision variables are now  $\mathbf{b}_1 > 0$  and  $\mathbf{b}_2 > 0$  and the expression of  $\mathbf{s}_{21}^{\{0\}}(T_s, T_p)$  reduces to:

$$\mathbf{s}_{21}^{\{0\}}(T_s, T_p) = \frac{2\mathbf{q}_2 \cdot T_p}{1 + \mathbf{b}_1 \cdot T_s + 2\mathbf{q}_2 \cdot T_p + \mathbf{b}_2 \cdot T_s^2 + \mathbf{b}_1 \mathbf{q}_2 \cdot T_s T_p}$$

### Approximation using an LMI optimisation problem

In order to minimise the relative error given in (6.21), introduce the *slack* variable  $\lambda$  and rewrite the problem as follows: minimise  $\lambda$  such that

$$\forall \omega \in \mathbb{R}, \quad \lambda \geq \left( 1 - \frac{W_{T_s T_p}(j\omega)}{\mathbf{s}_{21 T_s T_p}^{\{0\}}(j\omega)} \right)^* \left( 1 - \frac{W_{T_s T_p}(j\omega)}{\mathbf{s}_{21 T_s T_p}^{\{0\}}(j\omega)} \right)$$

By Schur Lemma (Lemma A.3, p. 195), this constraint is equivalent to:

$$\forall \omega \in \mathbb{R}, \quad \begin{bmatrix} 1 - \frac{W_{T_s T_p}(j\omega)}{\mathbf{s}_{21 T_s T_p}^{\{0\}}(j\omega)} \\ \mathbf{s}_{21 T_s T_p}^{\{0\}}(j\omega) \\ 1 \end{bmatrix}^* \begin{bmatrix} -1 & 0 \\ 0 & \lambda \end{bmatrix} \begin{bmatrix} 1 - \frac{W_{T_s T_p}(j\omega)}{\mathbf{s}_{21 T_s T_p}^{\{0\}}(j\omega)} \\ \mathbf{s}_{21 T_s T_p}^{\{0\}}(j\omega) \\ 1 \end{bmatrix} \geq 0$$

As  $\mathbf{s}_{21 T_s T_p}^{\{0\}}(j\omega)$  and  $W_{T_s T_p}(j\omega)$  have same numerator, one can write:

$$1 - \frac{W_{T_s T_p}(j\omega)}{\mathbf{s}_{21 T_s T_p}^{\{0\}}(j\omega)} = \left( \frac{1}{j\omega} \cdot I \right) \star \left[ \begin{array}{c|c} A_{re} & B_{re} \\ \hline C_{re} & D_{re} \end{array} \right]$$

Applying the KYP Lemma (Theorem 2.3, p. 28) leads to solve the following equivalent feasibility optimisation problem is found:  $\exists \mathbf{P} = \mathbf{P}^T$ ,

$$\begin{bmatrix} A & B \\ I & 0 \end{bmatrix}^T \begin{bmatrix} 0 & -\mathbf{P} \\ -\mathbf{P} & 0 \end{bmatrix} \begin{bmatrix} A & B \\ I & 0 \end{bmatrix} + \begin{bmatrix} C & D \\ 0 & I \end{bmatrix}^T \begin{bmatrix} -1 & 0 \\ 0 & \lambda \end{bmatrix} \begin{bmatrix} C & D \\ 0 & I \end{bmatrix} \geq 0$$

Finally, in order to get an LMI, apply again the Schur lemma to last inequality to obtain:

$$\left[ \begin{array}{c|c} \begin{bmatrix} A_{re} & B_{re} \\ I & 0 \end{bmatrix}^T \begin{bmatrix} 0 & -\mathbf{P} \\ -\mathbf{P} & 0 \end{bmatrix} \begin{bmatrix} A_{re} & B_{re} \\ I & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \lambda \end{bmatrix} & \begin{bmatrix} C_{re}^T \\ D_{re}^T \end{bmatrix} \\ \hline \begin{bmatrix} C_{re} & D_{re} \end{bmatrix} & I \end{array} \right] \geq 0 \quad (6.22)$$

Finally, the following linear minimisation problem over LMI constraint is obtained:

$$\left. \begin{array}{l} \min \\ \lambda, \mathbf{b}_1, \mathbf{b}_2 \in \mathbb{R}, \\ \mathbf{P} = \mathbf{P}^T \in \mathbb{R}^{n \times n}, \\ \text{such that} \quad (6.22) \\ \mathbf{b}_1 > 0 \\ \mathbf{b}_2 > 0 \end{array} \right\} \lambda \quad (6.23)$$

### Application to the example

Solving (6.23) leads to

$$b_1 = 3.3751 \quad b_2 = 1.6262$$

from which the values of  $q_1$  and  $q_3$  can be deduced:

$$q_1 = 2.7928 \quad q_3 = 0.5823$$

Therefore, the  $T_s, T_p$ -ladder filter of Figure 6.4 (b) is obtained with  $C_{0_1}$ ,  $C_{0_2}$  and  $C_{0_3}$  given in Table 6.3. On Figure 6.6, the resulting  $|s_{21_{T_s, T_p}}^{\{0\}}(j\omega)|^2$  is plotted in the red dashed line, and compared with the original  $|W_{T_s, T_p}^{\{0\}}(j\omega)|^2$ , plotted in the blue solid line. As one may observe, the scattering parameter of this  $T_s, T_p$ -ladder approximation does not satisfy the spectral mask  $\mathcal{SM}$ . However, it should be sufficiently close to a solution to be used as an initial point for design approach 1.

For this purpose, the  $T_s, T_p$ -ladder filter is transformed into the AW-resonator filter of Figure 6.7, where the characteristic values are given in Table 6.3.

TABLE 6.3: Building element values of the AW-resonator ladder approximation

Element	1	2	3
$C_{0_k}^{\{0\}}$ (F)	0.3581	0.8946	1.7174
$L_{m_k}^{\{0\}}$ (H)	0.4561	0.5914	0.0951
$C_{m_k}^{\{0\}}$ (F)	0.6767	1.6908	3.2458

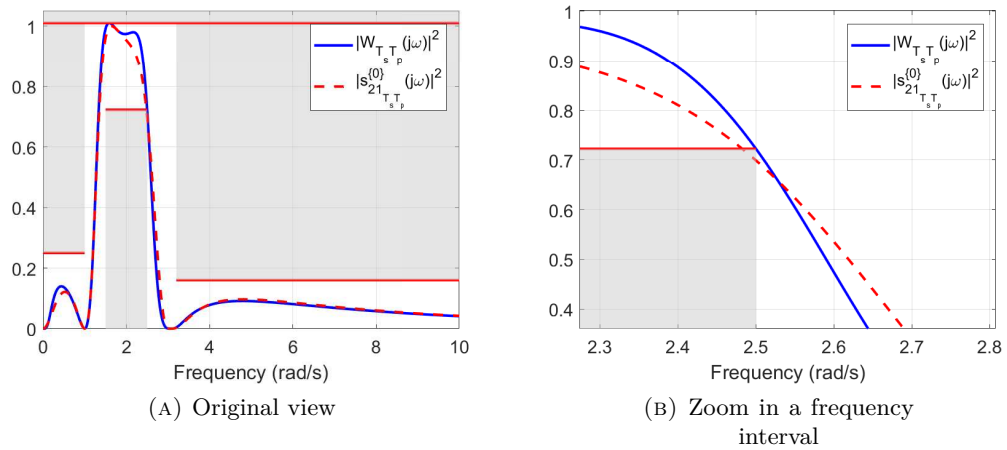


FIGURE 6.6: Plot of the scattering parameter  $|s_{21}^{(0)}(j\omega)|^2$  of the  $AW$ -resonator ladder approximation of  $|W_{T_s T_p}(j\omega)|^2$

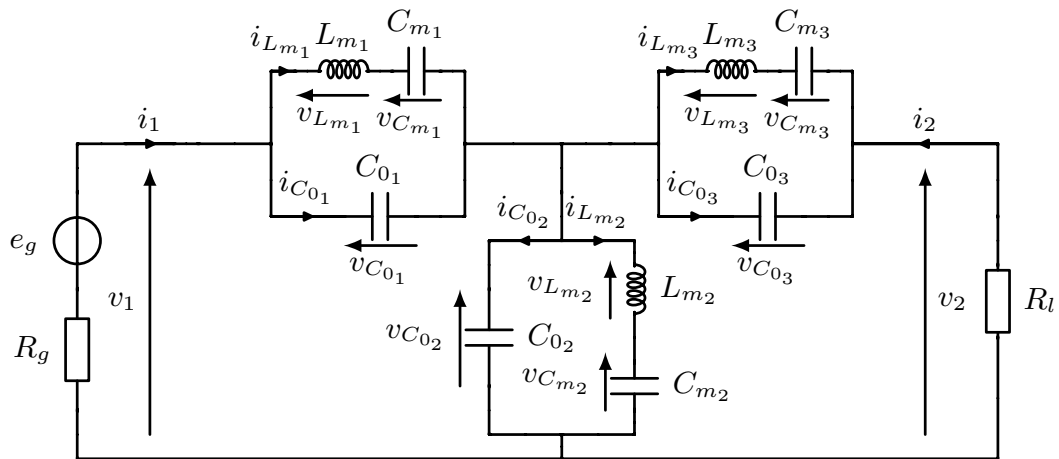


FIGURE 6.7: A three-elements,  $AW$ -resonator ladder filter

### 6.2.3 PHS synthesis algorithm

Now, as an  $AW$ -resonator ladder filter has been synthesised and does not satisfy the spectral mask  $SM$ , but is close to, the aim is to provide it as initial point to the design approach 1 of Chapter 4.

To achieve this, a scattering PHS representation is first obtained using the modelling procedure given in Subsection 4.2.2.2 (p. 61). This leads to the following representation of the scattering matrix  $S(s)$ :

$$\begin{cases} p = \left(\frac{1}{s} \cdot I_n\right) q \\ q = (J - BB^T) Qp + \sqrt{2}Bw \\ z = -\sqrt{2}DB^T Qp + Dw \end{cases}$$

with  $n = 9$  and

$$J := \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \quad B := \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & -1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \quad D := \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$Q := \begin{bmatrix} Q_1 & 0 & 0 \\ 0 & Q_2 & 0 \\ 0 & 0 & Q_3 \end{bmatrix} \quad \text{with} \quad Q_k := \begin{bmatrix} \frac{1}{C_{0_k}} & 0 & 0 \\ 0 & \frac{1}{L_{m_k}} & 0 \\ 0 & 0 & \frac{1}{C_{m_k}} \end{bmatrix}$$

Define now similarly the matrix  $\check{Q}_{init}^{inv}$  using the values of Table 6.3 as follows

$$\check{Q}_{init}^{inv} := \begin{bmatrix} \check{Q}_{1_{init}}^{inv} & 0 & 0 \\ 0 & \check{Q}_{2_{init}}^{inv} & 0 \\ 0 & 0 & \check{Q}_{3_{init}}^{inv} \end{bmatrix} \quad \text{with} \quad \check{Q}_{k_{init}}^{inv} := \begin{bmatrix} C_{0_k}^{\{0\}} & 0 & 0 \\ 0 & L_{m_k}^{\{0\}} & 0 \\ 0 & 0 & C_{m_k}^{\{0\}} \end{bmatrix}$$

Algorithm 4.1 of the design approach 1 (p. 84) is then run using  $\check{Q}_{init}^{inv}$  as initial point and  $maxRound = 500$ . After 26 iterations for a computational time of 321.6 s, the algorithm converged to a solution. The building element values of the resulting AW-resonator ladder filter are given in Table 6.4, while the squared magnitude of the associate scattering parameter  $s_{21}^{sol}$  is plotted on Figure 6.8.

Therefore, the combination of both approaches 1 and 2 has allowed to solve the design problem of an AW-resonator ladder filter on this example.

TABLE 6.4: Building element values of the resulting AW-resonator ladder filter

Element	1	2	3
$C_{0_k}^{sol}$ (F)	0.3571	0.8934	1.7174
$L_{m_k}^{sol}$ (H)	0.4535	0.5906	0.0985
$C_{m_k}^{sol}$ (F)	0.6756	1.6908	3.2459



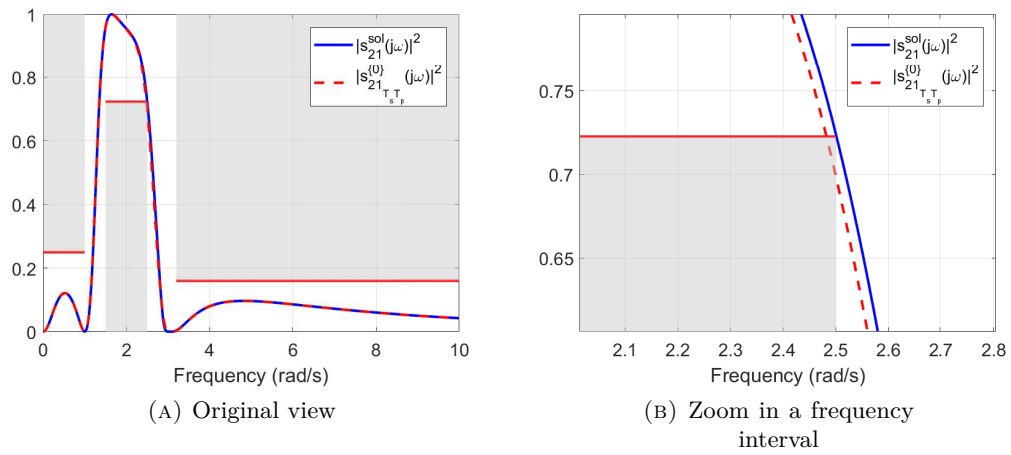


FIGURE 6.8: Plot of the scattering parameter  $|s_{21}^{sol}(j\omega)|^2$  of the resulting AW-resonator ladder filter

## Chapter 7

# Conclusions and Perspectives

### 7.1 Conclusions

The purpose of this work was to develop systematic and efficient methods for the design of passive electronic filters. To achieve this, it was proposed to use a modern System approach. Concretely, this implied to establish a design framework based on advanced tools of Control and System theory, such as LFT and dissipativity, to represent and characterise electronic elements and their interconnection, in order to clear a path towards convex optimisation (Chapter 2).

Based on a review with an historical perspective of the design methods, two design approaches stood out (Chapter 3). It was then proposed to revisit these approaches under our design framework (Chapter 4).

The first design approach, that we have denoted as elements-values tuning, is very flexible and usually preferred by designers in practice, as it is visual, close to the design problem and allows to include some technological constraints. This basically consists in placing the elements in a given topology, by means of the graphical interface of a CAD software, and tuning their characteristic values until the frequency requirements are fulfilled, using an optimisation solver. In addition to have illustrated how a direct synthesis approach may involve an optimisation problem of high complexity, and to have made it explicit, our contribution is to have obtained an optimisation problem of reduced complexity. While still non-convex, as being a BMI feasibility problem, this optimisation problem appeared of *moderate* complexity, thanks to the use of a representation combining features of the PHS and the LFT representations. An algorithm, based on sequential LMI relaxations, was then provided. However, in order to decrease the number of iterations, and then the computational time, an initial point relatively close to the original BMI feasibility set is required.

The second design approach, denoted as realisable filter synthesis, is historically linked with *LC* ladder filters and the traditional analytical methods of Butterworth and Chebyshev synthesis. It consists in synthesising a frequency filter, i.e. a transfer function fulfilling some frequency requirements, such that it satisfies certain realisation constraints. While efficient methods exist for frequency filter synthesis, the difficulty of the approach lies in finding realisation constraints without altering the efficiency of these methods, i.e. that can be convexly added. After having exhibited convex realisation constraints for *LC* ladder filters, another contribution is to have *convexly* extended them for a family of ladder filters, that we called *T*-ladder filters. This family especially includes *LC* ladder filters and certain *LC*-resonator ladder filters that are typically used for band-pass applications. Based on the introduction of common practices used by designers for *AW*-resonator ladder filters, a similar idea led us to introduce a larger family, namely  $T_s, T_p$ -ladder. While the resulting realisation constraints are simpler than for general *AW*-resonator ladder filters, only part of them was found to be convex.

This drove us to tackle the synthesis problem of frequency LFT filters (Chapter 5). This problem may be viewed as a generalised formulation of the usual frequency filter synthesis problem. The standard efficient approach to solve last problem is typically made of two steps: the magnitude synthesis step and the spectral factorisation step. As ever, we have revisited this two-steps approach under our framework, viewing a transfer function as an LFT in the repeated, lossless  $\{0, 1, 0\}$ -dissipative, transfer function  $\frac{1}{s}$ . By extending each step, we obtained an efficient synthesis approach for transfer functions expressed as an LFT in a repeated, lossless  $\{x, y, z\}$ -dissipative  $T(s)$ . While the main ideas came from [Zar13], our contribution is to have detailed and completed their approach. Furthermore, it was shown that the spectral factorisation can not be extended to any  $\{x, y, z\}$ -dissipative  $T(s)$ , as a factorisation error appears and is inherent to the approach. Therefore, one of the main contribution of this thesis is to have generalised this synthesis approach by merging the magnitude synthesis and the spectral factorisation steps into a single step, which is achieved by solving an LMI optimisation problem. This was made possible by considering the spectral factorisation error from the outset. Finally, we have clarified the issues emerging when trying to tackle the case of LFT in two transfer functions  $T_s(s)$  and  $T_p(s)$ . Nonetheless, we have developed an approach, also based on the resolution of a linear minimisation problem over LMI constraints, when  $T_s(s)$  and  $T_p(s)$  are both lossless  $\{0, 1, 0\}$ -dissipative, as this is our case of interest. While it has an inherent conservatism, this approach tends to find solutions in practice.

Putting all this together enables to get design methods for passive electronic filters. To illustrate this, two design examples were detailed (Chapter 6). First, a systematic and efficient design method for  $T$ -ladder filters, with  $T(s)$  being lossless  $\{0, 1, 0\}$ -dissipative, was obtained by coupling the associated realisation constraints, which are convex, and the synthesis method of frequency filters expressed as LFTs in the repeated  $T(s)$ . In particular, this includes the design of  $LC$  and  $LC$ -resonator ladder filters. Moreover, this methods tends to provide optimal results, in the sense that the resulting number of elements is minimal. Second, a design method was developed for  $T_s, T_p$ -ladder filters, where  $T_s(s)$  and  $T_p(s)$  are two lossless  $\{0, 1, 0\}$ -dissipative transfer functions, based on three stages. First stage consisted in the coupling of the convex part of the associated realisation constraints with the synthesis method of frequency filters expressed as LFTs in the repeated  $T_s(s)$  and  $T_p(s)$ . The resulting solution was then approximated by the scattering parameter of a  $T_s, T_p$ -ladder filter in the second stage. This led to an initial point that was fed in the third stage to the algorithm given in the elements-values tuning approach. Strictly, we are unable to claim that a *fully* systematic and efficient design method was obtained for  $AW$ -resonator ladder filters. However, the part to systematise is clearly identified. Moreover, the results obtained on the design example are promising and the developed method deserves to be applied on design problems with spectral masks that arise in practice.

## 7.2 Perspectives

In our quest to develop systematic and efficient design methods, several problems stood in our way. Some of them were solved, others partly solved while still others were left out. In the sequel, in order to adapt, complete or improve our methods to solve the remaining problems, some further works are discussed (Subsection 7.2.1). It has especially been tried to gather these points depending on the presumed reader interest. Afterwards, some new lines of research that was revealed to us are provided (Subsection 7.2.2). This essentially leads to the need of new collaborations between

Electronics and Control and Systems Theory, in order to tackle the current and future challenging problems raised by the design of Radiofrequency filters.

### 7.2.1 Further works

#### On multipliers and conservatism

Multipliers have a significant impact on the efficiency of our design methods. Indeed, as discussed in Chapter 2, an affine parametrisation  $\mathbb{M}_{ap} \subseteq \mathbb{M}$  of a certain set  $\mathbb{M}$ , arising from the design criteria, is achieved to get LMI constraints. This parametrisation, essentially resulting from the research of multipliers, needs to be the finest possible in order to avoid conservatism.

Intuitively, we think that the **affine parametrisation** given in Chapter 2 for the case of a repeated, **lossless  $\{x, y, z\}$ -dissipative**  $T(s)$ , is **exact**, i.e.  $\mathbb{M}_{ap} = \mathbb{M}$ . Using the conditions given in [Din05; IMF00b], it would be interesting to prove or disprove this intuition. If this is true, this would allow the statements of the lossless dissipative KYP Lemma (Theorem A.10, p. 196) to be equivalent. As a result, this would also prove that the realisable filter synthesis approach is optimal, i.e. leading to the minimum number of elements, for the design of  $T$ -ladder filters when  $T(s)$  is lossless  $\{0, 1, 0\}$ -dissipative.

Similarly, we have also the intuition that the multipliers given in Chapter 2 for **LFTs in**  $\begin{bmatrix} s \cdot I_{n_1} & 0 \\ 0 & \frac{1}{s} \cdot I_{n_2} \end{bmatrix}$  leads also to an **exact affine parametrisation**. This intuition is based on the fact that the *size* of the parametrisation, i.e. the number of multiplier *scalar* variables, is the same as for the affine parametrisation of LFTs in  $\frac{1}{s} \cdot I_n$ , with  $n = n_1 + n_2$ , which is known to be exact. If it is true, this would lead to get a *lossless* S-procedure [IMF00b] and get an equivalent optimisation problem in the elements-values tuning approach.

Finally, as a more open problem, it would be also interesting to **investigate** for **finer multipliers** for LFTs in  $\begin{bmatrix} T_s(s) \cdot I & 0 \\ 0 & T_p(s) \cdot I \end{bmatrix}$  when

$$T_s(s) = \frac{s^2 + \omega_0^2}{s(s^2 + \omega_{a_s}^2)} \quad T_p(s) = \frac{s^2 + \omega_{r_p}^2}{s(s^2 + \omega_0^2)}$$

Indeed, both transfer functions share a somewhat similar *structure*. Additionally, this *feeling* is enhanced when the additional constraint  $\frac{\omega_0}{\omega_{r_p}} = \frac{\omega_{a_s}}{\omega_0}$  is considered, such as in Chapter 6.

#### On the frequency LFT filter synthesis

One of our main contribution is the development of efficient methods for the synthesis of frequency LFT filters. However, there are still some matters to clarify and some further generalisation to consider, stated below.

First, while the synthesis method for lossless dissipative  $T(s)$  is comprehensive, up to the previously-mentioned equivalence of the lossless dissipative KYP lemma, its generalisation to any dissipative  $T(s)$  requires to be completed. Indeed, our generalisation involves to merge the magnitude synthesis step with the spectral factorisation step, which in turns lead to solve a feasibility problem that **couples** some **AREs** and some **LMIs**. In order to end up with only a single LMI problem, it was proposed to exchange each ARE by a linear minimisation problem over an LMI constraint. While this can always be achieved when the AREs are considered independently, this

point needs **to be made clearer** when there is such a coupling. To achieve this, a possibility is to refer to the approach developed in [RG96; Ram97] for solving certain coupled AREs using LMIs.

This problem is also involved in the proposed synthesis method for LFT in  $T_s(s)$  and  $T_p(s)$ . Moreover, while this synthesis method can be extended for  $T_s(s)$  and  $T_p(s)$  being  $\{x_s, y_s, z_s\}$ - and  $\{x_p, y_p, z_p\}$ -dissipative, it also raises some problems that remain unsolved. In particular, the mismatch between 2D LFT systems, i.e. the interconnection of systems of two kinds, and rational functions in  $T_s(s)$  and  $T_p(s)$ , prevents from an efficient extension of the previous methods.

A first point to be considered is the role of the transfer matrix  $B(T_s, T_p)$ . It was illustrated that different  $B(T_s, T_p)$  lead to different  $W(T_s, T_p)$ . It would be of interest to investigate this link and **search** for a **systematic** and **efficient** way of **choosing suitable  $B(T_s, T_p)$** . At a more general level, as LFT appears more suitable to represent the interconnection of heterogeneous subsystems than rational functions, one may find some benefits to **fully revisit** the **approach** in terms of **LFTs**. While the magnitude synthesis and the spectral factorisation are directly formulated in terms of LFTs, this mainly implies to extend the separation between numerator and denominator, perhaps using the so-called coprime factorisation [ZDG96], and the notion of *basis*.

Furthermore, the **extension** to **MIMO** systems appears rather **straightforward**. Indeed, the basic tools that are the coprime factorisation [ZDG96], the KYP Lemma and the spectral factorisation technique are transparent to SISO or MIMO systems. However, this required to extend the design requirements from magnitude constraints to minimum and maximum singular values [SF09].

Finally, **polynomial** and **discrete spectral masks**, interesting for  $H_\infty$ -weights generation, appear to be **reachable** using our approach [RSF03; Zar13; PKZS18].

### Improvements for practical use of the design approaches

There are places to improve our design approaches for a direct application on practical problems, especially when both approaches are combined. Some are stated below.

While the **elements-values tuning approach** can be directly included in the current design flow, where an initial guess is required by the designer, its performance is determined by the computational cost of the resolution algorithm (Algorithm 4.1, p. 84). As this algorithm is based on sequential LMI relaxations, the computational cost mainly results from the number of iterations. In this work, to reduce this number, we focused on the research of a reasonably-good initial point. However, one may also act on the algorithm itself. First, one may find a better choice for the iteration-dependent parameter  $\beta_k$  that is involved in the **Nesterov's acceleration**. To achieve this, this parameter can be *manually* tuned on simple examples, or chosen using one of the methods given in [LRP16; HL17a; HL17b]. Second, it would be interesting to adapt the algorithm to take into account the particular structure of the bilinear matrix term. Indeed, as  $\check{Q}^{inv}$  is diagonal, the **bilinear matrix term  $\check{Q}^{inv}\check{Q}^{inv}$**  is also **diagonal**. Intuitively, adapting the algorithm specially for this case should result in a lower computational cost.

Most of the improvements of the **realisable filter synthesis** approach are found in the **LFT frequency filter** step, already discussed. In particular, in the synthesis of  $T_s, T_p$ -ladder filters, the choice of  $B(T_s, T_p)$  needs to be clarified, as it has a direct consequence on the *order* of the 2D LFT  $W(T_s, T_p)$ , and thus the number of elements.

Finally, when **both approaches** are **combined**, the **transition** step needs to be improved. In the particular case of the approximation by a  $T_s, T_p$ -ladder filter such as in Section 6.2 of Chapter 6, the process should be **systematised**. This may also

be improved by achieving the approximation only on the frequency bands of interest, as this is not necessary on the transition bandwidth for instance.

### Lossy passive elements

In Chapter 4, both design approaches were developed for lossless passive elements. A more precise modelling of components, such as for the mBVD model for AW elements, implies the elements to be *lossy* passive, i.e. to be the interconnection of inductances, capacitances *and* resistances. Hence, it would be interesting to adapt or extend these approaches to lossy passive elements.

For both approaches, the first step consists in obtaining an **explicit formula** linking the entries of the scattering matrix  $S$ , and more specifically **between  $s_{11}$  and  $s_{21}$** . In other terms, the Reciprocal Belevitch Representation (Theorem 3.1, p. 48) needs to be extended to lossy passive filters. In particular, this would allow to get an extended version of the alternative design problem (Problem 3.2, p. 49). In addition, this would link the input impedance  $z_{in}$  and the scattering parameter  $s_{21}$ , that is required in the search of realisation conditions.

For the **elements-values tuning approach**, the **more general PHS DAE representation** given in [Sch17, Chap. 6], that includes resistance elements, should be used. The proofs of Property 4.1 (p. 60), to obtain the **descriptor form**, and of Lemma 4.2 (p. 76), to obtain the **LFT form**, have then to be **adapted** to this representation. In order to illustrate how to include the resistance elements in the design method, consider the following scattering PHS, state-space representation:

$$\begin{cases} \tilde{p} = \left(\frac{1}{s} \cdot I_n\right) \tilde{q} \\ \tilde{q} = \left(J - (R + BB^T)\right) Q\tilde{p} + \sqrt{2}Bw \\ z = -\sqrt{2}DB^T Q\tilde{p} + Dw \end{cases}$$

The main difference with the representation used in this manuscript (cf p. 61) is the positive semi-definite matrix  $R = R^T \geq 0$ , which includes the contributions of all the resistances. Applying the same reasoning that led Theorem 4.2 (p. 79) leads to check the feasibility of BMIs of the following form:

$$\mathcal{A}(D, P, \tau) + \begin{bmatrix} Q^{inv} Q^{inv} & -Q^{inv} (J - (R + BB^T)) & 0 & -Q^{inv} \sqrt{2}B \\ -(J - (R + BB^T))^T Q^{inv} & (J - (R + BB^T))^T (J - (R + BB^T)) + 2BB^T & \sqrt{2}BD^T & (J - (R + BB^T) - I)^T \sqrt{2}B \\ 0 & \sqrt{2}DB^T & 1 & -D \\ -\sqrt{2}B^T Q^{inv} & \sqrt{2}B^T (J - (R + BB^T) - I) & -D^T & 2B^T B + D^T D \end{bmatrix} \geq 0$$

where  $\mathcal{A}(D, P, \tau)$  is an affine matrix in the multipliers  $D, P, \tau$ . Two cases appear then. If the values of the resistances are known, i.e. if  $R$  is *not* a decision variable, the resulting optimisation problem is similar in all aspects as the BMI problem obtained in Chapter 4. If  $R$  is a decision variable, this leads to a much more complex optimisation problem, that involve several bilinear matrix products.

When, it comes to **realisable filter synthesis** approach, the move to lossy elements is even more simple. Indeed, the **extension** of the first step has **already** been **achieved**, as the synthesis method for **frequency LFT filters** that we have developed enables to consider such cases. Much of the effort have then to be concentrated on the realisation constraints, and their convex formulations.

### Uncertain elements values: analysis and synthesis

As mentioned in Chapter 2, the framework developed in this manuscript is not limited

to systems design. Indeed, the LFT tool and its link with LMI originally played an important role for robust analysis [DPZ91]. **Robust analysis** consists in checking if a system satisfies or not stability or performance criteria in the presence of uncertainties. In particular, this enables to take into account some mismatches between a model and a physical system by considering uncertainties on the parameters of the model.

In [FKL+18] was proposed an efficient method for the robust analysis of circuits made of uncertain elements, i.e. with uncertainties on the elements values. Based on the systematic derivation of a suitable LFT representation from any circuit diagram [FKS+15], the application of an appropriate version of the KYP Lemma enables to check if the given frequency requirements, made of upper and lower bounds, are fulfilled by solving an LMI feasibility problem. Unlike the Monte Carlo approach, which consists in randomly picking some realisations of the uncertain circuit and simulating them, this method checks if the requirements are satisfied in *all* the uncertain set and tend to be dramatically faster<sup>1</sup>. Therefore, the **developed method is systematic, efficient, transparent to the user**, as the starting point is a circuit diagram, **directly applicable** and provides a **higher level of guarantee**. Thus, it would be particularly interesting to include it into a CAD software.

Surprisingly, addressing the problem of robust synthesis to parametric uncertainty, i.e. the synthesis of the nominal values of *uncertain* elements such that the resulting filter satisfies the design requirements for the whole uncertainty set, appears of similar complexity than the nominal case of the elements-values tuning approach. Indeed, assume for instance that each building element value  $\tilde{q}_i$  can be written as

$$\tilde{q}_i = q_i(1 + \delta_i)$$

where  $q_i$  is the nominal value and  $\delta_i$  a random real number belonging to a given interval. As a result the matrix  $\tilde{Q}$  of the PHS representation is such that:

$$\tilde{Q} = Q(I + \Delta)$$

with  $\tilde{Q}$ ,  $Q$  and  $\Delta$  diagonal matrices. Applying the same reasoning as in the proof of Theorem 4.2 (p. 79) leads then to

$$\mathcal{A}(\mathbf{D}, \mathbf{P}, \boldsymbol{\tau}) + \begin{bmatrix} (I + \Delta)^{-1} \mathbf{Q}^{inv} \mathbf{Q}^{inv} (I + \Delta)^{-1} & \mathcal{L}(\mathbf{Q}^{inv})(I + \Delta)^{-1} \\ (\mathcal{L}(\mathbf{Q}^{inv})(I + \Delta)^{-1})^T & \mathcal{C} \end{bmatrix} \geq 0$$

where  $\mathcal{A}(\mathbf{D}, \mathbf{P}, \boldsymbol{\tau})$  is an affine matrix in some multipliers,  $\mathcal{L}(\mathbf{Q}^{inv})$  is a linear matrix in  $\mathbf{Q}^{inv}$  and  $\mathcal{C}$  a *constant* matrix. Thus, this can be re-written as:

$$\begin{bmatrix} (I + \Delta)^{-1} \\ I \end{bmatrix}^T \begin{bmatrix} \mathbf{Q}^{inv} \mathbf{Q}^{inv} & \mathcal{L}(\mathbf{Q}^{inv}) \\ \mathcal{L}(\mathbf{Q}^{inv})^T & \mathcal{A}(\mathbf{D}, \mathbf{P}, \boldsymbol{\tau}) + \mathcal{C} \end{bmatrix} \begin{bmatrix} (I + \Delta)^{-1} \\ I \end{bmatrix} \geq 0$$

Defining real matrices  $A$ ,  $B$ ,  $C$  and  $D$  such

$$\Delta \star \left[ \begin{array}{c|c} A & B \\ \hline C & D \end{array} \right] := \begin{bmatrix} (I + \Delta)^{-1} \\ I \end{bmatrix}$$

<sup>1</sup>On an example, the authors of [FKL+18] find that the Monte Carlo method requires a computational time a thousand times higher compared to their method.

the application of an adapted KYP Lemma [FKS+15] leads to search for *structured* matrices  $\mathbf{D}_\Delta = \mathbf{D}_\Delta^T > 0$  and  $\mathbf{G}_\Delta = -\mathbf{G}_\Delta^T$  such that:

$$\begin{bmatrix} A & B \\ I & 0 \end{bmatrix}^T \begin{bmatrix} \mathbf{D}_\Delta & \mathbf{G}_\Delta \\ -\mathbf{G}_\Delta & -\mathbf{D}_\Delta \end{bmatrix} \begin{bmatrix} A & B \\ I & 0 \end{bmatrix} + \begin{bmatrix} C^T \\ D^T \end{bmatrix} \begin{bmatrix} \mathbf{Q}^{inv} \mathbf{Q}^{inv} & \mathcal{L}(\mathbf{Q}^{inv}) \\ \mathcal{L}(\mathbf{Q}^{inv})^T & \mathcal{A}(\mathbf{D}, \mathbf{P}, \boldsymbol{\tau}) + \mathcal{C} \end{bmatrix} \begin{bmatrix} C & D \end{bmatrix} \geq 0$$

While this BMI feasibility problem has a certain degree of conservatism, due to the structure of the multipliers  $\mathbf{D}_\Delta$  and  $\mathbf{G}_\Delta$ , the only non-linear term happens to be the same as in the nominal case, that is  $\mathbf{Q}^{inv} \mathbf{Q}^{inv}$ . Therefore, by adapting the algorithm provided in the elements-values tuning approach, one might achieve a synthesis robust to elements-values variations.

### 7.2.2 Future researches

#### An open problem: realisation-robust synthesis

In the design approach 2, only the *exact* realisation problem was considered. This means that, given a transfer function  $W$  that satisfies a spectral mask  $\mathcal{SM}$ , a circuit was synthesised such that  $W$  was exactly the scattering parameter  $s_{21}$ , i.e.  $s_{21} = W$ . This implied the search of realisation constraints that could be convexly added in the synthesis of  $W$ . Generally, only a part of these realisations constraints was found convex and was considered in the synthesis step. In Chapter 6, this led us to approximate  $W$  by a function  $\tilde{W}$  that is realisable as the scattering parameter  $s_{21} = \tilde{W}$  of a circuit, but is not guaranteed to satisfy the spectral mask  $\mathcal{SM}$ . This brings us to the following problem:

*Synthesise a frequency filter  $W$  that is robust to a given realisation process.*

A first and natural solution is to simply add some extra safe-bands to the spectral mask  $\mathcal{SM}$ . Unfortunately, this heuristic-based approach makes even more complex a problem that tends to be increasingly complex. A second intuitive solution is to maximise the *distance* between the squared magnitude of the filter response and the spectral mask, in the Bode magnitude plot. However, this comes to approximate an ideal curve and it has been illustrated in Chapter 6 that the approximation tends to be better on some frequency intervals than others. This then also leads to uselessly over-constrain the synthesis problem.

Instead, a more skilled approach seems needed to overcome these issues. The approximation step of  $W$  by  $\tilde{W}$  mainly consists in approximating the coefficients of the transfer function  $W$ , or equivalently the matrices of its LFT representation. An option might be to consider the realisation approximation as uncertainties on these coefficients, or matrices, leading to the problem of robust synthesis.

This problem is beyond the scope of solely the synthesis of passive electronics filters. Indeed, it also appears in digital filter synthesis, where the quantification process prevents from exactly implementing a given (discrete) transfer function [PB87; VM19]. Even more general, while the realisation problem appears truly complex [Kal10], this is the realisation problem *toward* a precise objective that is here at stake, and that we think should be preferably tackled.

#### Future collaboration for the design of acoustic wave filters

As detailed in the very first chapter of this manuscript, the overcrowded spectrum coupled with the demand for ever-increasing data-transmission rate has led to a challenging design problem. This has significant impact on the complexity at each step of the global design process.



First, at the equivalent circuit level, while it was decided in this thesis to first obtain efficient and systematic design methods for a single filter, the co-design problem needs to be tackled in a near future. The main difficulty of this problem is that, either using switches or a multiplexing architecture [Mah17; FVAJ+16], a capacitive coupling appears between filters. This needs to be considered in the design of the filters as it impacts their performance. However, as the filters are jointly designed, the coupling effect is constantly changing. Hence, a sequential design would not be recommended. In addition, this coupling also influences the impedance matching property that is required for a suitable power transmission. Currently, this leads to add matching components [FVAJ+16], which in turn changes again the performance of the filters. Therefore, design methods for stand-alone filters appear limited to tackle this problem and sophisticated co-design methods need then to be developed.

Second, the market pressure for miniaturisation along with more stringent performance requirement leads to develop new technologies for which the mismatch between measurements and simulations with  $RLC$  models tend to increase [Mah17]. Some sources of this mismatch between the model and the measurement may be viewed as parametric uncertainties on the models, such as the temperature. Thus, efficient analysis methods for parametric variations can be used for a robust design of electronic systems [Fil10; FKS+15; FKL+18]. However, more complex models are required in general to fit measurements and lead to consider the family of non-linear models [WL15]. While there exist analytical non-linear models of acoustic wave components and filters [Fel09; IHI+11], most of the simulations are based on the finite-element analysis method, leading to lengthy computations. As an important part of Control and System theory is dedicated to the study of passive non-linear system, see for instance [Kha02; Sch17], it would be interesting to investigate if some of these techniques can be applied for developing efficient methods adapted for the analysis and simulation of acoustic-wave filters.

Putting this all together leads to a boosting and interdisciplinary research topic with practical interests at the interface of Electronics and System theory.

## Appendix A

# Miscellaneous on Linear Algebra and Control and System Theory

### A.1 Linear Algebra

#### Congruence

**Definition A.1 (Congruence [HJ12]).**

Let  $A, B \in \mathbb{C}^{n \times n}$  be given.  $A$  and  $B$  are said to be *congruent* if there exists a non-singular matrix  $\Pi \in \mathbb{C}^{n \times n}$  such that:

$$A = \Pi^* B \Pi$$

**Theorem A.1 ([HJ12]).**

Let  $A, B \in \mathbb{C}^{n \times n}$  be two congruent matrices. Then:

$$\text{rank}(A) = \text{rank}(B)$$

#### Eigenvalues, Jordan Form and Singular Value Decomposition

**Definition A.2 (Eigenvalues, Eigenvectors and Spectrum [HJ12]).**

Let  $A \in \mathbb{C}^{n \times n}$ . If a scalar  $\lambda \in \mathbb{C}$  and a non-zero vector  $x \in \mathbb{C}^n \setminus \{0\}$  satisfy the equation:

$$Ax = \lambda x$$

then  $\lambda$  is called an *eigenvalue* of  $A$  and  $x$  an *eigenvector* of  $A$  associated with  $\lambda$ .

The *spectrum* of  $A$  is the set of all  $\lambda$  that are eigenvalues of  $A$ . We denote this set by  $\Lambda(A)$ .

**Theorem A.2 (Jordan Form [ZDG96]).**

For any square complex matrix  $A \in \mathbb{C}^{n \times n}$ , there exists a nonsingular matrix  $T$  such that  $A$  has the following Jordan Form:

$$A = T J T^{-1}$$

where

$$J := \begin{bmatrix} J_1 & & 0 \\ & \ddots & \\ 0 & & J_l \end{bmatrix}$$

$$J_i := \begin{bmatrix} J_{i1} & & 0 \\ & \ddots & \\ 0 & & J_{im_i} \end{bmatrix}$$

$$J_{ij} := \begin{bmatrix} \lambda_i & 1 & & & \\ & \lambda_i & 1 & & \\ & & \ddots & \ddots & \\ & & & \lambda_i & 1 \\ & & & & \lambda_i \end{bmatrix} \in \mathbb{C}^{n_{ij} \times n_{ij}}$$

with  $\sum_{i=1}^l \sum_{j=1}^{m_i} n_{ij} = n$  and with  $\{\lambda_i, i = 1, \dots, l\}$  the distinct eigenvalues of  $A$ . Each matrix  $J_{ij}$  is denoted as a block of the Jordan form of  $A$ .

**Theorem A.3** (Generalised Eigenvectors [Ber09]).

Let  $A \in \mathbb{C}^{n \times n}$ . There exist  $v_1, \dots, v_n \in \mathbb{C}^n$  such that the following statements hold:

1.  $v_1, \dots, v_n \in \mathbb{C}^n$  are linearly independent.
2. For each  $k_i \times k_i$  Jordan block of  $A$  associated with  $\lambda_i \in \Lambda(A)$ , there exist  $v_{i_1}, \dots, v_{i_{k_i}}$  such that

$$\begin{aligned} Av_{i_1} &= \lambda_i v_{i_1} \\ Av_{i_2} &= v_{i_1} + \lambda_i v_{i_2} \\ &\vdots \\ Av_{i_{k_i}} &= v_{i_{k_i-1}} + \lambda_i v_{i_{k_i}} \end{aligned}$$

$v_1, \dots, v_n \in \mathbb{C}^n$  are denoted as the generalised eigenvectors of  $A$ .

**Theorem A.4** (Singular Value Decomposition [HJ12]).

Let  $A \in \mathbb{C}^{n \times m}$  and assume that  $\text{rank}(A) = r$  ( $\leq \min(m, n)$ ).

1. There exist square matrices  $V \in \mathbb{C}^{n \times n}$  and  $U \in \mathbb{C}^{m \times m}$  that are unitary, i.e.  $V^*V = I_n$  and  $U^*U = I_m$ , and a diagonal matrix  $\Sigma_r \in \mathcal{D}(\mathbb{R}^{r \times r})$  defined as:

$$\Sigma_r := \begin{bmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_r \end{bmatrix}$$

such that  $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_r > 0$  and

$$A = V^* \Sigma U$$

in which

$$\Sigma := \begin{bmatrix} \Sigma_r & 0_{r \times (m-r)} \\ 0_{(n-r) \times r} & 0_{(n-r) \times (m-r)} \end{bmatrix}$$

2. The singular values  $\sigma_1, \dots, \sigma_r$  are the positive square roots of the decreasingly ordered nonzero eigenvalues of  $AA^*$  (equivalently of  $A^*A$ ).
3. If  $A \in \mathbb{R}^{n \times m}$  is real, then  $V \in \mathbb{R}^{n \times n}$  and  $U \in \mathbb{R}^{m \times m}$  are also real.

### Positive (Semi-)Definite Matrices and Particular Decomposition

**Definition A.3** (Hermitian and Symmetric matrices [HJ12]).

Let  $A \in \mathbb{C}^{n \times n}$ . If  $A = A^*$ ,  $A$  is said to be *Hermitian*. If, in addition,  $A \in \mathbb{R}^{n \times n}$ , i.e.  $A = A^T$ ,  $A$  is said to be *symmetric*.

**Definition A.4** (Positive (Semi-)Definite matrices [HJ12]).

A Hermitian matrix  $A = A^* \in \mathbb{C}^{n \times n}$  is *positive semi-definite*, denoted by  $A \geq 0$  if

$$\forall x \in \mathbb{C}^n \setminus \{0\}, \quad x^* A x \geq 0$$

and is *positive definite*, denoted by  $A > 0$ , if

$$\forall x \in \mathbb{C}^n \setminus \{0\}, \quad x^* A x > 0$$

**Theorem A.5** ([HJ12]).

Let  $A \in \mathbb{C}^{n \times n}$  be a positive semi-definite matrix  $A = A^* \geq 0$ . Then the singular value decomposition of  $A$  can be reduced to:

$$A = U^* \Sigma U$$

If, in addition,  $A > 0$  is positive definite, then  $\Sigma > 0$ .

### A convenient notation: the Kronecker Product

**Definition A.5** (Kronecker product [ZDG96]).

Let  $A \in \mathbb{C}^{m \times n}$  be such that

$$A := \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

and  $B \in \mathbb{C}^{p \times q}$ . The *Kronecker product* of  $A$  and  $B$  is defined as:

$$A \otimes B := \begin{bmatrix} a_{11}B & a_{12}B & \dots & a_{1n}B \\ a_{21}B & a_{22}B & \dots & a_{2n}B \\ \vdots & \vdots & & \vdots \\ a_{m1}B & a_{m2}B & \dots & a_{mn}B \end{bmatrix} \in \mathbb{C}^{mp \times nq}$$

## A.2 Transformation of a complex LMI into a real LMI

**Property A.1.**

Let  $A = A^* \in \mathbb{C}^{n \times n}$  be a Hermitian matrix given by  $A := A_r + jA_i$  where  $A_r = A_r^T \in \mathbb{R}^{n \times n}$  and  $A_i = -A_i^T \in \mathbb{R}^{n \times n}$ . Then the following equivalence holds:

$$A \geq 0 \quad \Leftrightarrow \quad \begin{bmatrix} A_r & A_i \\ -A_i & A_r \end{bmatrix} \geq 0$$

*Proof.* By definition,

$$\begin{aligned} A \geq 0 &\Leftrightarrow \forall x_r, x_i \in \mathbb{R}^{n \times 1} \setminus \{0\}, \quad (x_r + jx_i)^*(A_r + jA_i)(x_r + jx_i) \geq 0 \\ \Leftrightarrow x_r^T A_r x_r - x_r^T A_i x_i + x_i^T A_r x_i + x_i^T A_i x_r + j(x_r^T A_r x_i + x_r^T A_i x_r - x_i^T A_r x_r + x_i^T A_i x_i) &\geq 0 \end{aligned}$$

In addition, as  $x_r^T A_r x_i$ ,  $x_r^T A_i x_r$  and  $x_i^T A_i x_i$  are scalars,  $A_r$  is symmetric and  $A_i$  is skew-symmetric, it comes:

$$\begin{aligned} x_r^T A_r x_i &= (x_r^T A_r x_i)^T = x_i^T A_r x_r \\ x_r^T A_i x_r &= (x_r^T A_i x_r)^T = -x_r^T A_i x_r \\ x_i^T A_i x_i &= (x_i^T A_i x_i)^T = -x_i^T A_i x_i \end{aligned}$$

leading to:

$$\begin{aligned} x_r^T A_r x_i - x_i^T A_r x_r &= 0 \\ x_r^T A_i x_r &= 0 \\ x_i^T A_i x_i &= 0 \end{aligned}$$

Hence:

$$\begin{aligned} A \geq 0 &\Leftrightarrow \forall x_r, x_i \in \mathbb{R}^{n \times 1} \setminus \{0\}, \quad x_r^T A_r x_r - x_r^T A_i x_i + x_i^T A_r x_i + x_i^T A_i x_r \geq 0 \\ &\Leftrightarrow \forall x_r, x_i \in \mathbb{R}^{n \times 1} \setminus \{0\}, \quad \begin{bmatrix} x_r \\ -x_i \end{bmatrix}^T \begin{bmatrix} A_r & A_i \\ -A_i & A_r \end{bmatrix} \begin{bmatrix} x_r \\ -x_i \end{bmatrix} \geq 0 \\ &\Leftrightarrow \begin{bmatrix} A_r & A_i \\ -A_i & A_r \end{bmatrix} \geq 0 \end{aligned}$$

□

### A.3 Useful theorem and lemmas

#### Lemma of Lyapunov

**Theorem A.6. Lemma of Lyapunov** (Adapted from [AV73, Chap.3])

Let  $A \in \mathbb{R}^{n \times n}$  be a real matrix. Then  $\Re(\Lambda(A)) < 0$  if and only if for any real matrix  $C$  such that the pair  $(A, C)$  is observable, there exists a unique positive definite symmetric  $P = P^T > 0$  such that:

$$A^T P + P A = -C^T C$$

In particular,  $\Re(\Lambda(A)) < 0$  if there exists a positive definite matrix  $P = P^T > 0$  such that:

$$A^T P + P A < 0$$

**Theorem A.7.** (Adapted from [AV73, Chap.3])

Let  $A \in \mathbb{R}^{n \times n}$  be a real matrix. If there exists a positive definite matrix  $P = P^T > 0$  such that:

$$A^T P + P A \leq 0$$

then  $\Re(\Lambda(A)) \leq 0$  and the Jordan Form of  $A$  has no blocks of size greater than  $1 \times 1$  with pure imaginary diagonal elements.

#### S-procedure Lemma with equalities

**Lemma A.1** (S-Procedure with equalities). ([Sco97])

Given  $\Phi = \Phi^* \in \mathbb{C}^{n \times n}$ ,  $\Theta = \Theta^* \in \mathbb{C}^{n \times n}$ ,  $M = M^* \in \mathbb{C}^{n \times n}$  hermitian matrices, we have the implication:

$$\begin{aligned} \xi^* \Phi \xi \geq 0 \quad \forall \xi \in \mathbb{C}^n \quad &\begin{cases} \xi^* M \xi = 0 \\ \xi^* \Theta \xi \leq 0 \end{cases} \\ \Leftrightarrow \exists \beta > 0, \exists \tau \in \mathbb{R}, \quad &\Phi + \beta \Theta + \tau M \geq 0 \end{aligned}$$

#### Schur Complement Lemma and Schur Lemma

**Lemma A.2** (Schur Complement Lemma). ([HJ12]) Let  $Q = Q^T \in \mathbb{R}^{n \times n}$ ,  $S \in \mathbb{R}^{n \times m}$  and  $R = R^T \in \mathbb{R}^{m \times m}$  be real matrices, with  $Q$  and  $R$  symmetric. Assume that  $R$  is invertible. Then:

$$\begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} = \begin{bmatrix} R^{-1}S^T & I \\ I & 0 \end{bmatrix}^T \begin{bmatrix} R & 0 \\ 0 & Q - SR^{-1}S^T \end{bmatrix} \begin{bmatrix} R^{-1}S^T & I \\ I & 0 \end{bmatrix}$$

In other words,  $\begin{bmatrix} Q & S \\ S^T & R \end{bmatrix}$  and  $\begin{bmatrix} R & 0 \\ 0 & Q - SR^{-1}S^T \end{bmatrix}$  are congruent.

**Lemma A.3** (Schur Lemma). ([BBFG94]) Let  $Q = Q^T \in \mathbb{R}^{n \times n}$ ,  $S \in \mathbb{R}^{n \times m}$  and  $R = R^T \in \mathbb{R}^{m \times m}$  be real matrices, with  $Q$  and  $R$  symmetric. Assume that  $R$  is invertible. Then:

$$\begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \geq 0 \quad \Leftrightarrow \quad \begin{cases} R > 0 \\ Q - SR^{-1}S^T \geq 0 \end{cases}$$

### Minimum-Real Part Theorem

**Theorem A.8** (Minimum-Real Part Theorem [Bah84]).

The real part of an analytic function  $f(s)$  of a complex variable  $s$  attains its minimum value on the boundary of the domain of analyticity, not at any interior point.

## A.4 Redheffer star product and LFT calculation

### Redheffer star product

**Definition A.6** (Redheffer star product [ZDG96]). Suppose  $P$  and  $K$  are compatibly partitioned matrices

$$P := \left[ \begin{array}{c|c} P_{11} & P_{12} \\ \hline P_{21} & P_{22} \end{array} \right] \quad K := \left[ \begin{array}{c|c} K_{11} & K_{12} \\ \hline K_{21} & K_{22} \end{array} \right]$$

such that the matrix product  $P_{22}K_{11}$  is well defined and square, and assume further that  $I - P_{22}K_{11}$  is invertible. Then the Redheffer star product of  $P$  and  $K$  with respect to this partition is defined as

$$P \star K := \left[ \begin{array}{c|c} \frac{P_{11} + P_{12}K_{11}(I - P_{22}K_{11})^{-1}P_{21}}{K_{21}(I - P_{22}K_{11})^{-1}P_{21}} & \frac{P_{12}(I - K_{11}P_{22})^{-1}K_{12}}{K_{22} + K_{21}P_{22}(I - K_{11}P_{22})^{-1}P_{12}} \\ \hline & \end{array} \right]$$

**LFT inversion formula** LFT inversion formula [ZDG96]:

$$\left( \Delta \star \left[ \begin{array}{c|c} A & B \\ \hline C & D \end{array} \right] \right)^{-1} = \Delta \star \left[ \begin{array}{c|c} A - BD^{-1}C & -BD^{-1} \\ \hline D^{-1}C & D^{-1} \end{array} \right]$$

**LFT product formula** Cascade of system LFT product formula [ZDG96]:

$$\left( \Delta_1 \star \left[ \begin{array}{c|c} A_1 & B_1 \\ \hline C_1 & D_1 \end{array} \right] \right) \left( \Delta_2 \star \left[ \begin{array}{c|c} A_2 & B_2 \\ \hline C_2 & D_2 \end{array} \right] \right) = \begin{bmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{bmatrix} \star \left[ \begin{array}{c|c|c} A_1 & B_1C_2 & C_1D_2 \\ \hline 0 & A_2 & B_2 \\ \hline C_1 & D_1C_2 & D_1D_2 \end{array} \right]$$

## A.5 Alternative and generalised versions of the KYP Lemma

### Alternative version of the KYP Lemma

**Theorem A.9 (KYP Lemma - Alternative version [TW91]).**

Define a transfer matrix

$$\mathcal{B}(s) := \left( \frac{1}{s} \cdot I \right) \star \left[ \begin{array}{c|c} A & B \\ \hline C & D \end{array} \right]$$

with  $A \in \mathbb{R}^{n \times n}$ ,  $B \in \mathbb{R}^{n \times n_w}$ ,  $C \in \mathbb{R}^{n_z \times n}$ ,  $D \in \mathbb{R}^{n_z \times n_w}$  with  $(A, B)$  controllable. Let  $X = X^T \in \mathbb{R}^{n_z \times n_z}$  be a real symmetric matrix. The following two statements are equivalent.

(i) The following constraint holds

$$\forall s \in \mathbb{C}^+ \cup \mathbb{C}^0 \text{ with } s \notin \Lambda(A), \quad \begin{bmatrix} \mathcal{B}(s) \\ I \end{bmatrix}^* \begin{bmatrix} X & Y \\ Y^T & Z \end{bmatrix} \begin{bmatrix} \mathcal{B}(s) \\ I \end{bmatrix} \geq 0$$

(ii) There exists a positive semi-definite matrix  $\mathbf{P} = \mathbf{P}^T \geq 0 \in \mathbb{R}^{n \times n}$  such that:

$$\begin{bmatrix} A & B \\ I & 0 \end{bmatrix}^T \begin{bmatrix} 0 & -\mathbf{P} \\ -\mathbf{P} & 0 \end{bmatrix} \begin{bmatrix} A & B \\ I & 0 \end{bmatrix} + \begin{bmatrix} C & D \\ 0 & I \end{bmatrix}^T \begin{bmatrix} X & Y \\ Y^T & Z \end{bmatrix} \begin{bmatrix} C & D \\ 0 & I \end{bmatrix} \geq 0$$

### Lossless Dissipative KYP Lemma

**Theorem A.10 (Lossless dissipative KYP Lemma).**

Assume that  $T(s)$  is a given lossless  $\{x, y, z\}$ -dissipative transfer function.

Let  $X = X^T \in \mathbb{R}^{n_z \times n_z}$  be a real symmetric matrix and define a transfer matrix  $\mathcal{B}(T(s))$  as an LFT in repeated  $T(s)$ :

$$\mathcal{B}(T(s)) := (T(s) \cdot I) \star \left[ \begin{array}{c|c} A & B \\ \hline C & D \end{array} \right]$$

with  $A \in \mathbb{R}^{n \times n}$ ,  $B \in \mathbb{R}^{n \times n_w}$ ,  $C \in \mathbb{R}^{n_z \times n}$ ,  $D \in \mathbb{R}^{n_z \times n_w}$  and  $(A, B)$  controllable.

Define  $\mathbb{D} = \{\omega, (I - AT(j\omega)) \text{ is invertible}\}$ . Then (ii)  $\Rightarrow$  (i).

(i) The following inequality holds

$$\forall \omega \in \mathbb{R} \cap \mathbb{D}, \quad \mathcal{B}(T(j\omega))^* X \mathcal{B}(T(j\omega)) \geq 0$$

(ii) There exists a symmetric matrix  $\mathbf{P} = \mathbf{P}^T \in \mathbb{R}^{n \times n}$  such that:

$$\begin{bmatrix} A & B \\ I & 0 \end{bmatrix}^T \begin{bmatrix} -z\mathbf{P} & -y\mathbf{P} \\ -y\mathbf{P} & -x\mathbf{P} \end{bmatrix} \begin{bmatrix} A & B \\ I & 0 \end{bmatrix} + \begin{bmatrix} C^T \\ D^T \end{bmatrix} X \begin{bmatrix} C & D \end{bmatrix} \geq 0$$

### Dissipative KYP Lemma

**Theorem A.11 (Dissipative KYP Lemma).**

Assume that  $T(s)$  is a given  $\{x, y, z\}$ -dissipative transfer function. Let  $X = X^T \in$

$\mathbb{R}^{n_z \times n_z}$  be a real symmetric matrix and define a transfer matrix  $\mathcal{B}(T(s))$  as an LFT in repeated  $T(s)$ :

$$\mathcal{B}(T(s)) := (T(s) \cdot I) \star \left[ \begin{array}{c|c} A & B \\ \hline C & D \end{array} \right]$$

with  $A \in \mathbb{R}^{n \times n}$ ,  $B \in \mathbb{R}^{n \times n_w}$ ,  $C \in \mathbb{R}^{n_z \times n}$ ,  $D \in \mathbb{R}^{n_z \times n_w}$  and  $(A, B)$  controllable. Define  $\mathbb{D} = \{\omega, (I - AT(j\omega)) \text{ is invertible}\}$ . Then (ii)  $\Rightarrow$  (i).

(i) The following inequality holds

$$\forall \omega \in \mathbb{R} \cap \mathbb{D}, \quad \mathcal{B}(T(j\omega))^* X \mathcal{B}(T(j\omega)) \geq 0$$

(ii) There exists a positive semi-definite matrix  $\mathbf{P} = \mathbf{P}^T \geq 0 \in \mathbb{R}^{n \times n}$  such that:

$$\begin{bmatrix} A & B \\ I & 0 \end{bmatrix}^T \begin{bmatrix} -z\mathbf{P} & -y\mathbf{P} \\ -y\mathbf{P} & -x\mathbf{P} \end{bmatrix} \begin{bmatrix} A & B \\ I & 0 \end{bmatrix} + \begin{bmatrix} C^T \\ D^T \end{bmatrix} X \begin{bmatrix} C & D \end{bmatrix} \geq 0$$

### KYP Lemma with $T_1, T_2$ lossless $\{0, 1, 0\}$ -dissipative

**Theorem A.12 (KYP Lemma with  $T_1, T_2$  lossless  $\{0, 1, 0\}$ -dissipative).**

Assume that  $T_1(s)$  and  $T_2(s)$  are given  $\{0, 1, 0\}$ -dissipative transfer functions. Let  $X = X^T \in \mathbb{R}^{n_z \times n_z}$  be a real symmetric matrix and define a transfer matrix  $\mathcal{B}(T_1(s), T_2(s))$  as an LFT in repeated  $T_1(s)$  and  $T_2(s)$ :

$$\mathcal{B}(T_1(s), T_2(s)) := \begin{pmatrix} T_1(s) \cdot I_{n_1} & 0 \\ 0 & T_2(s) \cdot I_{n_2} \end{pmatrix} \star \left[ \begin{array}{c|c} A & B \\ \hline C & D \end{array} \right]$$

with  $A \in \mathbb{R}^{n \times n}$ ,  $B \in \mathbb{R}^{n \times n_w}$ ,  $C \in \mathbb{R}^{n_z \times n}$ ,  $D \in \mathbb{R}^{n_z \times n_w}$  and  $(A, B)$  controllable. Define

$$\mathbb{D} = \left\{ \omega, \left( I - A \begin{pmatrix} T_1(j\omega) \cdot I_{n_1} & 0 \\ 0 & T_2(j\omega) \cdot I_{n_2} \end{pmatrix} \right) \text{ is invertible} \right\}$$

Then (ii)  $\Rightarrow$  (i).

(i) The following inequality holds

$$\forall \omega \in \mathbb{R} \cap \mathbb{D}, \quad \mathcal{B}(T_1(j\omega), T_2(j\omega))^* X \mathcal{B}(T_1(j\omega), T_2(j\omega)) \geq 0$$

(ii) There exists symmetric matrices  $\mathbf{P}_1 = \mathbf{P}_1^T \in \mathbb{R}^{n_1 \times n_1}$  and  $\mathbf{P}_2 = \mathbf{P}_2^T \in \mathbb{R}^{n_2 \times n_2}$  such that:

$$\begin{bmatrix} A & B \\ I & 0 \end{bmatrix}^T \begin{bmatrix} 0 & 0 & -\mathbf{P}_1 & 0 \\ 0 & 0 & 0 & -\mathbf{P}_2 \\ -\mathbf{P}_1 & 0 & -0 & 0 \\ 0 & -\mathbf{P}_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} A & B \\ I & 0 \end{bmatrix} + \begin{bmatrix} C^T \\ D^T \end{bmatrix} X \begin{bmatrix} C & D \end{bmatrix} \geq 0$$





## Appendix B

# Proofs

### B.1 Chapter 3

#### Link between impedance matrix and scattering matrix

Proof of Property 3.1 (p. 44)

*Proof.* (3.7) can be written as:

$$\frac{1}{2}\mathcal{R}^{-1/2} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} - \frac{1}{2}\mathcal{R}^{1/2} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \mathcal{S} \left( \frac{1}{2}\mathcal{R}^{-1/2} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} + \frac{1}{2}\mathcal{R}^{1/2} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} \right) \quad (\text{B.1})$$

Moreover, by using (3.5):

$$\frac{1}{2} \left( \mathcal{R}^{-1/2}\mathcal{Z} + \mathcal{R}^{1/2} \right) \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \frac{1}{2}\mathcal{R}^{-1/2} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} + \frac{1}{2}\mathcal{R}^{1/2} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$

Thus, assuming  $\left( \mathcal{R}^{-1/2}\mathcal{Z} + \mathcal{R}^{1/2} \right)^{-1}$  exists, this leads to:

$$\frac{1}{2} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \left( \mathcal{R}^{-1/2}\mathcal{Z} + \mathcal{R}^{1/2} \right)^{-1} \left( \frac{1}{2}\mathcal{R}^{-1/2} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} + \frac{1}{2}\mathcal{R}^{1/2} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} \right)$$

Therefore,

$$\begin{aligned} \underbrace{\frac{1}{2} \left( \mathcal{R}^{-1/2}\mathcal{Z} - \mathcal{R}^{1/2} \right) \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}}_{= \frac{1}{2}\mathcal{R}^{-1/2} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} - \frac{1}{2}\mathcal{R}^{1/2} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}} &= \left( \mathcal{R}^{-1/2}\mathcal{Z} - \mathcal{R}^{1/2} \right) \left( \mathcal{R}^{-1/2}\mathcal{Z} + \mathcal{R}^{1/2} \right)^{-1} \left( \frac{1}{2}\mathcal{R}^{-1/2} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} + \frac{1}{2}\mathcal{R}^{1/2} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} \right) \\ &= \left( \mathcal{R}^{-1/2}\mathcal{Z}\mathcal{R}^{-1/2} - I \right) \left( \mathcal{R}^{-1/2}\mathcal{Z}\mathcal{R}^{-1/2} + I \right)^{-1} \left( \frac{1}{2}\mathcal{R}^{-1/2} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} + \frac{1}{2}\mathcal{R}^{1/2} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} \right) \end{aligned}$$

Then, by identifying with (B.1):

$$\begin{aligned} \mathcal{S} &= \left( \mathcal{R}^{-1/2}\mathcal{Z}\mathcal{R}^{-1/2} - I \right) \left( \mathcal{R}^{-1/2}\mathcal{Z}\mathcal{R}^{-1/2} + I \right)^{-1} \\ &= -I + 2\mathcal{R}^{-1/2}\mathcal{Z}\mathcal{R}^{-1/2} \left( \mathcal{R}^{-1/2}\mathcal{Z}\mathcal{R}^{-1/2} + I \right)^{-1} \\ &= \left( \mathcal{R}^{-1/2}\mathcal{Z}\mathcal{R}^{-1/2} \right) \star \left[ \begin{array}{c|c} -I & \sqrt{2}I \\ \hline \sqrt{2}I & -I \end{array} \right] \end{aligned}$$

□

## B.2 Chapter 4

The following proofs extensively use the next property.

**Property B.1.** *If  $A$  is a  $n_1 \times n_2$  matrix and  $B$  a  $n_2 \times n_3$  matrix such that*

$$\text{rank}(B) = n_2$$

*Then*

$$\text{rank}(AB) = \text{rank}(A)$$

*Proof.* Using a singular value decomposition of  $B$ , there exist a positive-definite diagonal matrix  $\Sigma_{n_2}$  and non-singular matrices  $U \in \mathbb{R}^{n_3 \times n_3}$  and  $V \in \mathbb{R}^{n_2 \times n_2}$  such that

$$B = V^T \begin{bmatrix} \Sigma_{n_2} & 0_{n_2 \times (n_3 - n_2)} \end{bmatrix} U$$

Therefore, as  $U$  is non-singular, it comes (cf the rank equalities section of [HJ12, Chap. 0]) that:

$$\text{rank}(AB) = \text{rank} \left( AV^T \begin{bmatrix} \Sigma_{n_2} & 0_{n_2 \times (n_3 - n_2)} \end{bmatrix} \right)$$

Similarly, the result is obtained:

$$\text{rank}(AB) = \text{rank} \begin{bmatrix} AV^T \Sigma_{n_2} & 0_{n_2 \times (n_3 - n_2)} \end{bmatrix} = \text{rank}(AV^T \Sigma_{n_2}) = \text{rank}(A)$$

□

### Proof of Property 4.1 (p. 60)

*Proof.*

Assume that a scattering PHS representation associated with a diagonal, positive-definite matrix  $\mathcal{R} > 0$  is given, where the representation matrices  $E, F \in \mathbb{R}^{(n+n_e) \times n}$ ,  $G, H \in \mathbb{R}^{(n+n_e) \times n_e}$  and  $Q = Q^T > 0 \in \mathbb{R}^{n \times n}$ , and vectors  $q, p \in \mathbb{R}^n$ ,  $w, z \in \mathbb{R}^{n_e}$  are such as in Definition 4.2 (p. 59):

$$\begin{cases} p = \left( \frac{1}{s} \cdot I_n \right) q \\ Fq = EQp + Gz + Hw \end{cases}$$

with

$$EF^T + FE^T + \frac{1}{2} (HH^T - GG^T) = 0 \quad (\text{B.2})$$

$$\text{rank} \begin{bmatrix} F & E & G & H \end{bmatrix} = n + n_e \quad (\text{B.3})$$

Denote as  $r$  the rank of the matrix  $F$ :

$$r = \text{rank}(F)$$

One gets easily that  $r \leq n$ . Denote then  $n_{AE}$  the number such that

$$r + n_{AE} = n$$

Using a singular value decomposition, one can write  $F$  as follows:

$$F = V^T \begin{bmatrix} \Sigma_r & 0_{r \times n_{AE}} \\ 0_{n_{AE} \times r} & 0_{n_{AE} \times n_{AE}} \\ 0_{n_e \times r} & 0_{n_e \times n_{AE}} \end{bmatrix} U$$

where  $V \in \mathbb{R}^{(n+n_e) \times (n+n_e)}$  and  $U \in \mathbb{R}^{n \times n}$  are unitary matrices. Up to a permutation, this enables to define the equivalent representation matrices:

$$\left[ VF \mid VE \mid VG \mid VH \right] := \left[ \begin{array}{cc|cc|c|c} F_{11_r} & F_{12_r} & E_{11_r} & E_{12_r} & G_{1_r} & H_{1_r} \\ 0_{n_{AE} \times r} & 0_{n_{AE} \times n_{AE}} & E_{11_{AE}} & E_{12_{AE}} & G_{1_{AE}} & H_{1_{AE}} \\ 0_{n_e \times r} & 0_{n_e \times n_{AE}} & E_{21} & E_{22} & G_2 & H_2 \end{array} \right]$$

where  $F_{11_r} \in \mathbb{R}^{r \times r}$  is invertible. Notice that (B.2) and (B.3) can be adapted to this new representation and become:

$$(VE)(VF)^T + (VF)(VE)^T + \frac{1}{2} \left( (VH)(VH)^T - (VG)(VG)^T \right) = 0 \quad (\text{B.4})$$

$$\text{rank} \begin{bmatrix} VF & VE & VG & VH \end{bmatrix} = n + n_e \quad (\text{B.5})$$

In particular, (B.4) provides the following equations:

$$E_{11_r} F_{11_r}^T + F_{11_r} E_{11_r}^T + E_{12_r} F_{12_r}^T + F_{12_r} E_{12_r}^T + \frac{1}{2} \left( H_{1_r} H_{1_r}^T - G_{1_r} G_{1_r}^T \right) = 0 \quad (\text{B.6})$$

$$E_{11_{AE}} F_{11_r}^T + E_{12_{AE}} F_{12_r}^T + \frac{1}{2} \left( H_{1_{AE}} H_{1_r}^T - G_{1_{AE}} G_{1_r}^T \right) = 0 \quad (\text{B.7})$$

$$E_{21} F_{11_r}^T + E_{22} F_{12_r}^T + \frac{1}{2} \left( H_2 H_{1_r}^T - G_2 G_{1_r}^T \right) = 0 \quad (\text{B.8})$$

$$H_{1_{AE}} H_{1_{AE}}^T - G_{1_{AE}} G_{1_{AE}}^T = 0 \quad (\text{B.9})$$

$$H_2 H_{1_{AE}}^T - G_2 G_{1_{AE}}^T = 0 \quad (\text{B.10})$$

$$H_2 H_2^T - G_2 G_2^T = 0 \quad (\text{B.11})$$

First, let us show that  $\text{rank} \begin{bmatrix} G_{1_{AE}} \\ G_2 \end{bmatrix} = n_e$ , or similarly:

$$\text{rank} \begin{bmatrix} VF & VE & VG & VH \end{bmatrix} = \text{rank} \begin{bmatrix} F_{11_r} & 0_{r \times n_{AE}} & 0_{r \times n_e} \\ 0_{n_{AE} \times r} & E_{12_{AE}} & G_{1_{AE}} \\ 0_{n_e \times r} & E_{22} & G_2 \end{bmatrix}$$

One may note that (B.6)-(B.8) implies that:

$$\begin{bmatrix} E_{11_r} \\ E_{11_{AE}} \\ E_2 \end{bmatrix} = - \begin{bmatrix} F_{11_r} & F_{12_r} & E_{12_r} & G_{1_r} & H_{1_r} \\ 0 & 0 & E_{12_{AE}} & G_{1_{AE}} & H_{1_{AE}} \\ 0 & 0 & E_{22} & G_2 & H_2 \end{bmatrix} \begin{bmatrix} E_{11_r}^T \\ E_{12_r}^T \\ F_{12_r}^T \\ -\frac{1}{2} G_{1_r}^T \\ \frac{1}{2} H_{1_r}^T \end{bmatrix} (F_{11_r})^{-T}$$

From Property B.1 (p. 200), it comes that

$$\text{rank} \begin{bmatrix} VF & VE & VG & VH \end{bmatrix} = \text{rank} \begin{bmatrix} F_{11_r} & F_{12_r} & E_{12_r} & G_{1_r} & H_{1_r} \\ 0 & 0 & E_{12_{AE}} & G_{1_{AE}} & H_{1_{AE}} \\ 0 & 0 & E_{21} & G_2 & H_2 \end{bmatrix}$$

as

$$\begin{bmatrix} VF & VE & VG & VH \end{bmatrix} = \begin{bmatrix} F_{11_r} & F_{12_r} & E_{12_r} & G_{1_r} & H_{1_r} \\ 0 & 0 & E_{12_{AE}} & G_{1_{AE}} & H_{1_{AE}} \\ 0 & 0 & E_{21} & G_2 & H_2 \end{bmatrix} \begin{bmatrix} I & 0 & -E_{11_r}^T F_{11_r}^{-T} & 0 & 0 & 0 \\ 0 & I & -E_{12_r}^T F_{11_r}^{-T} & 0 & 0 & 0 \\ 0 & 0 & -F_{12_r}^T F_{11_r}^{-T} & I & 0 & 0 \\ 0 & 0 & \frac{1}{2} G_{1_r}^T F_{11_r}^{-T} & 0 & I & 0 \\ 0 & 0 & -\frac{1}{2} H_{1_r}^T F_{11_r}^{-T} & 0 & 0 & I \end{bmatrix}$$

Likewise, by noticing that:

$$\begin{bmatrix} F_{11_r} & F_{12_r} & E_{12_r} & G_{1_r} & H_{1_r} \\ 0 & 0 & E_{12_{AE}} & G_{1_{AE}} & H_{1_{AE}} \\ 0 & 0 & E_{21} & G_2 & H_2 \end{bmatrix} = \begin{bmatrix} F_{11_r} & 0 & 0 & 0 \\ 0 & E_{12_{AE}} & G_{1_{AE}} & H_{1_{AE}} \\ 0 & E_{21} & G_2 & H_2 \end{bmatrix} \begin{bmatrix} I & F_{11_r}^{-1} F_{12_r} & F_{11_r}^{-1} E_{12_r} & F_{11_r}^{-1} G_{1_r} & F_{11_r}^{-1} H_{1_r} \\ 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & I \end{bmatrix}$$

it comes that

$$\text{rank} \begin{bmatrix} VF & VE & VG & VH \end{bmatrix} = \text{rank} \begin{bmatrix} F_{11_r} & 0_{r \times n_{AE}} & 0_{r \times n_e} & 0_{r \times n_e} \\ 0_{n_{AE} \times r} & E_{12_{AE}} & G_{1_{AE}} & H_{1_{AE}} \\ 0_{n_e \times r} & E_{21} & G_2 & H_2 \end{bmatrix} = r + n_{AE} + n_e$$

and necessarily  $\text{rank} \begin{bmatrix} G_{1_{AE}} & H_{1_{AE}} \\ G_2 & H_2 \end{bmatrix} \geq n_e$ . Furthermore, using (B.9)-(B.10), one gets:

$$\begin{aligned} \text{rank} \begin{bmatrix} G_{1_{AE}} & H_{1_{AE}} \\ G_2 & H_2 \end{bmatrix} &= \text{rank} \begin{bmatrix} G_{1_{AE}} & H_{1_{AE}} \\ G_2 & H_2 \end{bmatrix} \begin{bmatrix} G_{1_{AE}}^T & G_2^T \\ H_{1_{AE}}^T & H_2^T \end{bmatrix} = \text{rank} \begin{bmatrix} G_{1_{AE}} G_{1_{AE}}^T & G_{1_{AE}} G_2^T \\ G_2 G_{1_{AE}}^T & G_2 G_2^T \end{bmatrix} \\ &= \text{rank} \begin{bmatrix} G_{1_{AE}} \\ G_2 \end{bmatrix} \leq n_e \end{aligned}$$

Therefore, one gets

$$\text{rank} \begin{bmatrix} G_{1_{AE}} & H_{1_{AE}} \\ G_2 & H_2 \end{bmatrix} = \text{rank} \begin{bmatrix} G_{1_{AE}} \\ G_2 \end{bmatrix} = n_e$$

This implies that there exists a row-permutation matrix  $T \in \mathbb{R}^{(n_e + n_{AE}) \times (n_e + n_{AE})}$  such that:

$$T \begin{bmatrix} G_{1_{AE}} \\ G_2 \end{bmatrix} := \begin{bmatrix} \tilde{G}_{1_{AE}} \\ \tilde{G}_2 \end{bmatrix}$$

with  $\tilde{G}_2 \in \mathbb{R}^{n_e \times n_e}$  being invertible. Defining  $\tilde{T} := \begin{bmatrix} I & 0 \\ 0 & T \end{bmatrix}$  leads to the representation matrices:

$$\tilde{T} \left[ VF \mid VE \mid VG \mid VH \right] := \left[ \begin{array}{cc|cc|c|c} F_{11r} & F_{12r} & E_{11r} & E_{12r} & G_{1r} & H_{1r} \\ 0_{n_{AE} \times r} & 0_{n_{AE} \times n_{AE}} & \tilde{E}_{11AE} & \tilde{E}_{12AE} & \tilde{G}_{1AE} & \tilde{H}_{1AE} \\ 0_{n_e \times r} & 0_{n_e \times n_{AE}} & \tilde{E}_{21} & \tilde{E}_{22} & \tilde{G}_2 & \tilde{H}_2 \end{array} \right]$$

Furthermore, by pre-multiplying by  $W := \begin{bmatrix} I & 0 & -\tilde{G}_{1r} \tilde{G}_2^{-1} \\ 0 & I & -\tilde{G}_{1AE} \tilde{G}_2^{-1} \\ 0 & 0 & -\tilde{G}_2^{-1} \end{bmatrix}$ , one gets:

$$W\tilde{T} \left[ VF \mid VE \mid VG \mid VH \right] := \left[ \begin{array}{cc|cc|c|c} F_{11r} & F_{12r} & \hat{E}_{11r} & \hat{E}_{12r} & 0 & \hat{H}_{1r} \\ 0_{n_{AE} \times r} & 0_{n_{AE} \times n_{AE}} & \hat{E}_{11AE} & \hat{E}_{12AE} & 0 & 0 \\ 0_{n_e \times r} & 0_{n_e \times n_{AE}} & \hat{E}_{21} & \hat{E}_{22} & -I & \hat{H}_2 \end{array} \right]$$

In the last equation, the fact that  $\tilde{H}_{1AE} - \tilde{G}_{1AE} \tilde{G}_2^{-1} \tilde{H}_2 = 0$  comes from (B.9)-(B.11). Indeed, these equations can be rewritten in a compact form as:

$$\begin{bmatrix} H_{1AE} \\ H_2 \end{bmatrix} \begin{bmatrix} H_{1AE} \\ H_2 \end{bmatrix}^T - \begin{bmatrix} G_{1AE} \\ G_2 \end{bmatrix} = 0$$

Then, pre- and post-multiplying by respectively  $T$  and  $T^T$ , it comes:

$$\begin{bmatrix} \tilde{H}_{1AE} \\ \tilde{H}_2 \end{bmatrix} \begin{bmatrix} \tilde{H}_{1AE} \\ \tilde{H}_2 \end{bmatrix}^T - \begin{bmatrix} \tilde{G}_{1AE} \\ \tilde{G}_2 \end{bmatrix} \begin{bmatrix} \tilde{G}_{1AE} \\ \tilde{G}_2 \end{bmatrix}^T = \begin{bmatrix} \tilde{H}_{1AE} \tilde{H}_{1AE}^T - \tilde{G}_{1AE} \tilde{G}_{1AE}^T & \tilde{H}_{1AE} \tilde{H}_2^T - \tilde{G}_{1AE} \tilde{G}_2^T \\ \tilde{H}_2 \tilde{H}_{1AE}^T - \tilde{G}_2 \tilde{G}_{1AE}^T & \tilde{H}_2 \tilde{H}_2^T - \tilde{G}_2 \tilde{G}_2^T \end{bmatrix} = 0$$

Hence, as  $\tilde{H}_2 \tilde{H}_2^T - \tilde{G}_2 \tilde{G}_2^T = 0$ ,  $\tilde{H}_2$  is also invertible and:

$$\tilde{H}_{1AE} - \tilde{G}_{1AE} \tilde{G}_2^{-1} \tilde{H}_2 = \tilde{H}_{1AE} - \tilde{G}_{1AE} \tilde{G}_2^T \tilde{H}_2^{-T} = \left( \tilde{H}_{1AE} \tilde{H}_2^T - \tilde{G}_{1AE} \tilde{G}_2^T \right) \tilde{H}_2^{-T} = 0$$

Finally, defining  $\hat{F}_{1r}$ ,  $\hat{E}_{1r}$ ,  $\hat{E}_{1AE}$  and  $\hat{E}_2$  as follows

$$\left[ \begin{array}{c|c} \hat{F}_{1r} & \hat{E}_{1r} \\ \hline 0_{n_{AE} \times n} & \hat{E}_{1AE} \\ 0_{n_e \times n} & \hat{E}_2 \end{array} \right] := \left[ \begin{array}{cc|cc} F_{11r} & F_{12r} & \hat{E}_{11r} & \hat{E}_{12r} \\ 0_{n_{AE} \times r} & 0_{n_{AE} \times n_{AE}} & \hat{E}_{11AE} & \hat{E}_{12AE} \\ 0_{n_e \times r} & 0_{n_e \times n_{AE}} & \hat{E}_{21} & \hat{E}_{22} \end{array} \right]$$

leads then to the following representation in the developed descriptor form:

$$\begin{cases} p = \left( \frac{1}{s} \cdot I_n \right) q \\ \hat{F}_{1r} q = \hat{E}_{1r} Q p + \hat{H}_{1r} w \\ 0 = \hat{E}_{1AE} Q p \\ z = \hat{E}_2 Q p + \hat{H}_2 w \end{cases}$$

where  $\hat{F}_{1r}$  and  $\hat{E}_{1AE}$  are full row-rank.

In turns, by defining  $\hat{F}_1 := \begin{bmatrix} \hat{F}_{1r} \\ 0 \end{bmatrix}$ ,  $\hat{E}_1 := \begin{bmatrix} \hat{E}_{1r} \\ \hat{E}_{1AE} \end{bmatrix}$  and  $\hat{H}_1 := \begin{bmatrix} \hat{H}_{1r} \\ 0 \end{bmatrix}$ , the descriptor

form is obtained:

$$\begin{cases} p = \left(\frac{1}{s} \cdot I_n\right) q \\ \hat{F}_1 q = \hat{E}_1 Q p + \hat{H}_1 w \\ z = \hat{E}_2 Q p + \hat{H}_2 w \end{cases}$$

□

**Computations of the scattering PHS state-space matrices** Consider a scattering DAE PHS representation in the following descriptor form:

$$\begin{cases} p = \left(\frac{1}{s} \cdot I_n\right) q \\ F_1 q = E_1 Q p + H_1 w \\ z = E_2 Q p + H_2 w \end{cases}$$

where

$$\begin{bmatrix} E_1 F_1^T + F_1 E_1^T & F_1 E_2^T \\ E_2 F_1^T & 0 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} H_1 H_1^T & H_1 H_2^T \\ H_2^T H_1 & H_2 H_2^T - I \end{bmatrix} = 0 \quad (\text{B.12})$$

Assume now that  $F_1$  is invertible. This leads to the state-space representation:

$$\begin{cases} p = \left(\frac{1}{s} \cdot I_n\right) q \\ q = A Q p + B w \\ z = C Q p + D w \end{cases}$$

$$\text{with } A := F_1^{-1} E_1 \quad B := F_1^{-1} H_1 \quad C := E_2 \quad D := H_2$$

Pre- and post-multiplying (B.12) by respectively  $\begin{bmatrix} F_1^{-1} & 0 \\ 0 & I \end{bmatrix}$  and  $\begin{bmatrix} F_1^{-T} & 0 \\ 0 & I \end{bmatrix}$  leads to:

$$\begin{bmatrix} A + A^T & C^T \\ C & 0 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} B B^T & B D^T \\ D B^T & D D^T - I \end{bmatrix} = 0 \quad (\text{B.13})$$

This constrains the state-space matrices to be such as:

$$\frac{1}{2} (A + A^T) = -\frac{1}{4} B B^T \quad C = -\frac{1}{2} D B^T \quad D D^T = I \quad (\text{B.14})$$

Now, by defining the matrix  $J$  as the skew-symmetric part of  $A$ :

$$J := \frac{1}{2} (A - A^T)$$

the following state-space representation is obtained:

$$\begin{cases} p = \left(\frac{1}{s} \cdot I_n\right) q \\ q = \left(J - \frac{1}{4} B B^T\right) p + B w \\ z = -\frac{1}{2} D B^T p + D w \end{cases}$$

where  $D$  is such that:  $DD^T = I$ . Finally, by defining  $\tilde{B} := \frac{1}{2}B$ ,  $\tilde{q} := \frac{q}{\sqrt{2}}$  and  $\tilde{p} := \frac{p}{\sqrt{2}}$ , the following state-space representation is obtained:

$$\begin{cases} \tilde{p} = \left(\frac{1}{s} \cdot I_n\right) \\ \tilde{q} = (J - \tilde{B}\tilde{B}^T)\tilde{p} + \sqrt{2}\tilde{B}w \\ z = -\sqrt{2}D\tilde{B}^T\tilde{p} + Dw \end{cases}$$

### Proof of Property 4.2 (p. 69)

*Proof.* Consider the given following scattering PHS DAE representation of a lossless passive electronic filter:

$$\begin{cases} p = \left(\frac{1}{s} \cdot I_n\right) q & (a) \\ F_{1r}q = E_{1r}Qp + H_{1r}w & (b) \\ 0 = E_{1AE}Qp & (c) \\ z = E_2Qp + H_2w & (d) \end{cases} \quad (\text{B.15})$$

where  $F_{1r} \in \mathbb{R}^{r \times n}$  and  $E_{1AE} \in \mathbb{R}^{n_{AE} \times n}$  are full row-rank, and

$$\begin{bmatrix} F_{1r} \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} E_{1r} \\ E_{1AE} \\ E_2 \end{bmatrix}^T + \begin{bmatrix} E_{1r} \\ E_{1AE} \\ E_2 \end{bmatrix} \begin{bmatrix} F_{1r} \\ 0 \\ 0 \end{bmatrix}^T + \frac{1}{2} \left( \begin{bmatrix} H_{1r} \\ 0 \\ H_2 \end{bmatrix} \begin{bmatrix} H_{1r} \\ 0 \\ H_2 \end{bmatrix}^T - \begin{bmatrix} 0 \\ 0 \\ -I \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ -I \end{bmatrix}^T \right) = 0 \quad (\text{B.16})$$

Using a singular-value decomposition, the matrix  $E_{1AE}$  is factorised as:

$$E_{1AE} = V^T \begin{bmatrix} 0 & \Sigma_{n_{AE}} \end{bmatrix} U$$

where  $V \in \mathbb{R}^{n_{AE} \times n_{AE}}$  is a unitary matrix  $V^T V = I$ ,  $\Sigma_{n_{AE}} \in \mathbb{R}^{n_{AE} \times n_{AE}}$  is a positive-definite, diagonal matrix, and  $U \in \mathbb{R}^{n \times n}$  is a unitary matrix  $U^T U = I$ . By left multiplying (c) of (B.15) by  $\Sigma_{n_{AE}}^{-1} V$ , one gets:

$$0 = \begin{bmatrix} 0 & I_{n_{AE}} \end{bmatrix} UQp = \begin{bmatrix} 0 & I_{n_{AE}} \end{bmatrix} (UQU^T) Up$$

Defining  $\begin{bmatrix} \tilde{p}_1 \\ \tilde{p}_2 \end{bmatrix} := Up$  and  $\tilde{Q} := \begin{bmatrix} \tilde{Q}_{11} & \tilde{Q}_{12} \\ \tilde{Q}_{12}^T & \tilde{Q}_{22} \end{bmatrix} := UQU^T$ , last equation becomes

$$0 = \tilde{Q}_{12}^T \tilde{p}_1 + \tilde{Q}_{22} \tilde{p}_2$$

As  $Q > 0$ , it comes that  $UQU^T > 0$  and then  $\tilde{Q}_{22} > 0$ , i.e.  $\tilde{Q}_{22}$  is invertible. Hence:

$$\tilde{p}_2 = -\tilde{Q}_{22}^{-1} \tilde{Q}_{12}^T \tilde{p}_1$$

Defining  $\begin{bmatrix} \tilde{q}_1 \\ \tilde{q}_2 \end{bmatrix} := Uq$ , by (a) of (B.15), it comes similarly  $\tilde{q}_2 = -\tilde{Q}_{22}^{-1} \tilde{Q}_{12}^T \tilde{q}_1$ . Furthermore, by defining:

$$\begin{bmatrix} \tilde{F}_{11r} & \tilde{F}_{12r} \end{bmatrix} := F_{1r} U^T \quad \begin{bmatrix} \tilde{E}_{11r} & \tilde{E}_{12r} \end{bmatrix} := E_{1r} U^T \quad \begin{bmatrix} \tilde{E}_{21} & \tilde{E}_{22} \end{bmatrix} := E_2 U^T$$



One gets:

$$\left\{ \begin{array}{l} \tilde{p}_1 = \left( \frac{1}{s} \cdot I_r \right) \tilde{q}_1 \\ \begin{bmatrix} \tilde{F}_{11_r} & \tilde{F}_{12_r} \end{bmatrix} \begin{bmatrix} I \\ -\tilde{Q}_{22}^{-1} \tilde{Q}_{12}^T \end{bmatrix} \tilde{q}_1 = \begin{bmatrix} \tilde{E}_{11_r} & \tilde{E}_{12_r} \end{bmatrix} \tilde{Q} \begin{bmatrix} I \\ -\tilde{Q}_{22}^{-1} \tilde{Q}_{12}^T \end{bmatrix} \tilde{p}_1 + H_{1_r} w \\ 0 = \begin{bmatrix} 0 & I_{n_{AE}} \end{bmatrix} \tilde{Q} \begin{bmatrix} I \\ -\tilde{Q}_{22}^{-1} \tilde{Q}_{12}^T \end{bmatrix} \tilde{p}_1 \\ z = \begin{bmatrix} \tilde{E}_{21} & \tilde{E}_{22} \end{bmatrix} \tilde{Q} \begin{bmatrix} I \\ -\tilde{Q}_{22}^{-1} \tilde{Q}_{12}^T \end{bmatrix} \tilde{p}_1 + H_2 w \end{array} \right.$$

Moreover, as  $F_{1_r} E_{1_{AE}}^T = 0$  from (B.16) and:

$$F_{1_r} E_{1_{AE}}^T = F_{1_r} U^T \begin{bmatrix} 0 \\ \Sigma_{n_{AE}} \end{bmatrix} V = \begin{bmatrix} \tilde{F}_{11_r} & \tilde{F}_{12_r} \end{bmatrix} \begin{bmatrix} 0 \\ \Sigma_{n_{AE}} \end{bmatrix} V$$

it comes that:

$$\tilde{F}_{12_r} = 0$$

Hence:

$$\text{rank}(\tilde{F}_{11_r}) = \text{rank} \begin{bmatrix} \tilde{F}_{11_r} & \tilde{F}_{12_r} \end{bmatrix} = \text{rank}(F_{1_r}) = r$$

Therefore, the matrix  $\tilde{F}_{11_r} \in \mathbb{R}^{r \times r}$  is invertible. Finally, by noticing that

$$\tilde{Q} \begin{bmatrix} I \\ -\tilde{Q}_{22}^{-1} \tilde{Q}_{12}^T \end{bmatrix} = \begin{bmatrix} \tilde{Q}_{11} & \tilde{Q}_{12} \\ \tilde{Q}_{12}^T & \tilde{Q}_{22} \end{bmatrix} \begin{bmatrix} I \\ -\tilde{Q}_{22}^{-1} \tilde{Q}_{12}^T \end{bmatrix} = \begin{bmatrix} \tilde{Q}_{11} - \tilde{Q}_{12} \tilde{Q}_{22}^{-1} \tilde{Q}_{12}^T \\ 0 \end{bmatrix}$$

and by defining  $\hat{Q} := \tilde{Q}_{11} - \tilde{Q}_{12} \tilde{Q}_{22}^{-1} \tilde{Q}_{12}^T$ , the following state-space representation is obtained:

$$\left\{ \begin{array}{l} \tilde{p}_1 = \left( \frac{1}{s} \cdot I_r \right) \tilde{q}_1 \\ \tilde{q}_1 = \left( \tilde{F}_{11_r} \right)^{-1} \tilde{E}_{11_r} \hat{Q} \tilde{p}_1 + \left( \tilde{F}_{11_r} \right)^{-1} H_{1_r} w \\ z = \tilde{E}_{21} \hat{Q} \tilde{p}_1 + H_2 w \end{array} \right.$$

□

## B.3 Chapter 5

### Proof of Property 5.2

*Proof of Property 5.2 (p.124).* Using the LFT inversion formula,  $W_D^{-1}(\Delta(s))$  can be calculated as:

$$W_D^{-1}(\Delta(s)) = \Delta(s) \star \left[ \begin{array}{c|c} A - BD_D^{-1}C_D & -BD_D^{-1} \\ \hline D_D^{-1}C_D & D_D^{-1} \end{array} \right]$$

Then, the ratio  $W(\Delta(s)) = W_N(\Delta(s))W_D^{-1}(\Delta(s))$  can be calculated using the LFT product formula as:

$$W(\Delta(s)) = W_N(\Delta(s))W_D^{-1}(\Delta(s)) = \begin{pmatrix} \Delta(s) & 0 \\ 0 & \Delta(s) \end{pmatrix} \star \left[ \begin{array}{cc|c} A & BD_D^{-1}C_D & BD_D^{-1} \\ 0 & A - BD_D^{-1}C_D & -BD_D^{-1} \\ \hline C_N & D_N D_D^{-1} C_D & D_N D_D^{-1} \end{array} \right]$$

Finally, by using the developed form of the previous LFT and noticing that:

$$\begin{aligned} & \left( \begin{bmatrix} I_n - A\Delta(s) & -(BD_D^{-1}C_D)\Delta(s) \\ 0 & I - (A - BD_D^{-1}C_D)\Delta(s) \end{bmatrix} \right)^{-1} \\ &= \begin{bmatrix} (I_n - A\Delta(s))^{-1} & (I_n - A\Delta(s))^{-1}(BD_D^{-1}C_D)\Delta(s) \left( I_n - (A - BD_D^{-1}C_D)\Delta(s) \right)^{-1} \\ 0 & \left( I_n - (A - BD_D^{-1}C_D)\Delta(s) \right)^{-1} \end{bmatrix} \end{aligned}$$

one gets:

$$\begin{aligned} W(\Delta(s)) &= D_N D_D^{-1} + (C_N - D_N D_D^{-1} C_D) \Delta(s) \left( I - (A - BD_D^{-1} C_D) \Delta(s) \right)^{-1} B D_D^{-1} \\ &= \Delta(s) \star \left[ \begin{array}{c|c} A - BD_D^{-1} C_D & BD_D^{-1} \\ \hline C_N - D_N D_D^{-1} C_D & D_N D_D^{-1} \end{array} \right] \end{aligned}$$

□

### Congruency relations and proofs

**Lemma B.1.** Assume that  $\begin{bmatrix} x & y \\ y & z \end{bmatrix}$  is a full-rank matrix with  $x \leq 0$ . Define:

$$\begin{aligned} Q(P) &:= \begin{bmatrix} A & B \\ I & 0 \end{bmatrix}^T \begin{bmatrix} -zP & -yP \\ -yP & -xP \end{bmatrix} \begin{bmatrix} A & B \\ I & 0 \end{bmatrix} + \begin{bmatrix} C^T \\ D^T \end{bmatrix} X \begin{bmatrix} C & D \end{bmatrix} \\ \hat{Q}(P) &:= \begin{bmatrix} \hat{A} & \hat{B} \\ I & 0 \end{bmatrix}^T \begin{bmatrix} 0 & -P \\ -P & 0 \end{bmatrix} \begin{bmatrix} \hat{A} & \hat{B} \\ I & 0 \end{bmatrix} + \begin{bmatrix} \hat{C}^T \\ \hat{D}^T \end{bmatrix} X \begin{bmatrix} \hat{C} & \hat{D} \end{bmatrix} \end{aligned}$$

Then  $Q(P)$  and  $\hat{Q}(P)$  are congruent:

$$\Pi^T Q(P) \Pi = \hat{Q}(P)$$

where:

1. If  $x = 0$ , then define  $c := -\frac{z}{2y}$  and:

$$\left[ \begin{array}{c|c} \hat{A} & \hat{B} \\ \hline \hat{C} & \hat{D} \end{array} \right] := \left[ \begin{array}{c|c} y(cI + A^{-1}) & -yA^{-1}B \\ \hline CA^{-1} & D - CA^{-1}B \end{array} \right] \quad \Pi := \begin{bmatrix} A^{-1} & -A^{-1}B \\ 0 & I \end{bmatrix}$$

2. If  $x < 0$ , define  $c := -\frac{y}{x}$  and  $r^2 := \frac{y^2 - xz}{x^2}$  and:

$$\begin{aligned} \left[ \begin{array}{c|c} \hat{A} & \hat{B} \\ \hline \hat{C} & \hat{D} \end{array} \right] &:= \left[ \begin{array}{c|c} ((c^2 - r^2)A - (c+r)I) (I - (c+r)A)^{-1} & -r\sqrt{-2x(c+r)} (I - (c+r)A)^{-1} B \\ \hline \sqrt{\frac{2(c+r)}{-x}} C (I - (c+r)A)^{-1} & D + (c+r)C (I - (c+r)A)^{-1} B \end{array} \right] \\ \Pi &:= \left[ \begin{array}{c|c} \sqrt{\frac{2(c+r)}{-x}} C (I - (c+r)A)^{-1} & D + (c+r)C (I - (c+r)A)^{-1} B \\ \hline 0 & I \end{array} \right] \end{aligned}$$

*Proof.* 1. Case  $x = 0$ . The proof is based on the following steps:

$$\bullet \begin{bmatrix} -z & -y \\ -y & 0 \end{bmatrix} = \begin{bmatrix} \frac{z}{2} & y \\ 1 & 0 \end{bmatrix}^T \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \frac{z}{2} & y \\ 1 & 0 \end{bmatrix}$$

- $\begin{bmatrix} \frac{z}{2}I & yI \\ I & 0 \end{bmatrix} \begin{bmatrix} A & B \\ I & 0 \end{bmatrix} = \begin{bmatrix} \frac{z}{2}A + yI & \frac{z}{2}B \\ A & B \end{bmatrix}$
- $\begin{bmatrix} \frac{z}{2}A + yI & \frac{z}{2}B \\ A & B \end{bmatrix} T = \begin{bmatrix} \hat{A} & \hat{B} \\ I & 0 \end{bmatrix}$
- $\begin{bmatrix} C & D \end{bmatrix} T = \begin{bmatrix} \hat{C} & \hat{D} \end{bmatrix}$

2. Case  $x < 0$ . Define

$$\alpha = \frac{x}{\sqrt{\rho}} \quad \beta = \frac{-(y + \sqrt{y^2 - xz})}{\sqrt{\rho}} \quad \gamma = \frac{z}{\sqrt{\rho}} \quad \rho = 2(y^2 - xz)(y + \sqrt{y^2 - xz})$$

The proof is based on the following steps:

- $\begin{bmatrix} z(-P) & y(-P) \\ y(-P) & x(-P) \end{bmatrix} = \left( \frac{1}{\alpha\gamma - \beta^2} \begin{bmatrix} \gamma I & -\beta I \\ -\beta I & \alpha I \end{bmatrix} \right)^T \begin{bmatrix} 0 & -P \\ -P & 0 \end{bmatrix} \left( \frac{1}{\alpha\gamma - \beta^2} \begin{bmatrix} \gamma I & -\beta I \\ -\beta I & \alpha I \end{bmatrix} \right)$
- $\frac{1}{\alpha\gamma - \beta^2} \begin{bmatrix} \gamma I & -\beta I \\ -\beta I & \alpha I \end{bmatrix} \begin{bmatrix} A & B \\ I & 0 \end{bmatrix} = \frac{1}{\alpha\gamma - \beta^2} \begin{bmatrix} \gamma A - \beta I & \gamma B \\ -\beta A + \alpha I & -\beta B \end{bmatrix}$
- $$\begin{bmatrix} \hat{A} & \hat{B} \\ I & 0 \end{bmatrix} = \frac{1}{\alpha\gamma - \beta^2} \begin{bmatrix} \gamma A - \beta I & \gamma B \\ -\beta A + \alpha I & -\beta B \end{bmatrix} \underbrace{\begin{bmatrix} (\alpha\gamma - \beta^2)(-\beta A + \alpha \cdot I)^{-1} & (-\beta A + \alpha \cdot I)^{-1}\beta B \\ 0 & I \end{bmatrix}}_{\Pi}$$

$$= \begin{bmatrix} (\gamma A - \beta I)(-\beta A + \alpha I)^{-1} & \frac{1}{\alpha\gamma - \beta^2} ((\gamma A - \beta I)(-\beta A + \alpha I)^{-1}\beta + \gamma I) B \\ 0 & I \end{bmatrix}$$

$$= \begin{bmatrix} (\gamma A - \beta I)(-\beta A + \alpha I)^{-1} & (-\beta A + \alpha I)^{-1} B \\ 0 & I \end{bmatrix}$$
- $$\begin{bmatrix} \hat{C} & \hat{D} \end{bmatrix} = \begin{bmatrix} C & D \end{bmatrix} \underbrace{\begin{bmatrix} (\alpha\gamma - \beta^2)(-\beta A + \alpha \cdot I)^{-1} & (-\beta A + \alpha \cdot I)^{-1}\beta B \\ 0 & I \end{bmatrix}}_{\Pi}$$

$$= \begin{bmatrix} (\alpha\gamma - \beta^2)C(-\beta A + \alpha \cdot I)^{-1} & D + C(-\beta A + \alpha \cdot I)^{-1}\beta B \end{bmatrix}$$
- Additional calculation on the expression of  $\hat{A}$ ,  $\hat{B}$ ,  $\hat{C}$ ,  $\hat{D}$ , and  $\Pi$  leads to the result.

□

## Appendix C

# Synthèse Fréquentielle de Filtres Electroniques Passifs : une Approche Systémique Moderne

## C.1 Introduction générale

### C.1.1 Motivation et objectif de cette thèse

#### Sur la nécessité de méthodes de conception systématiques et efficaces pour les filtres radiofréquences

L'augmentation importante du nombre d'appareils mobiles connectés (smartphones, drones, IoT, ...) et du volume des données à transmettre a engendré ces deux dernières décennies une croissance exponentielle du nombre de bandes radiofréquences, vues comme des canaux de communication, et l'apparition de la cinquième génération de standards téléphoniques (5G) ne peut qu'amplifier ce phénomène. Comme chaque bande requiert la conception de deux à trois filtres exclusivement dédiés, cette croissance implique une explosion du nombre de filtres par appareil. Afin de modérer l'augmentation de taille de ces appareils, une première solution a été de miniaturiser les filtres, mais celle-ci semble de plus en plus atteindre ses limites. D'autres solutions sont donc envisagées, comme par exemple le filtre reconfigurable, mais rendent le problème de conception associé beaucoup plus complexe. De plus, du fait de la finitude du spectre radiofréquence et la co-existence avec plusieurs autres applications, les bandes de transition ont tendance à être réduites drastiquement au fil du temps, demandant des filtres de plus en plus sélectifs et donc nécessitant de résoudre des problèmes de conception plus complexes. En couplant ces considérations avec de nouvelles techniques pour augmenter la vitesse de transmission, comme l'agrégation de porteuse, et des contraintes industrielles toujours plus strictes, comme le coût ou la consommation d'énergie, le problème de conception de filtres radiofréquences devient un vrai défi à résoudre.

Pour faire face à cette complexité, un consensus apparaît sur l'utilisation des filtres à ondes acoustiques (*acoustic wave* en anglais, d'où l'acronyme AW). Ces filtres consistent en l'interconnexion de composants AW, qui sont des composants passifs particulièrement utilisés pour application micro-électronique, tels que les composants SAW ou BAW (pour *surface acoustic wave* et *bulk acoustic wave* en anglais). Ils sont notamment reconnus pour leur grande sélectivité et leur habilité à pouvoir satisfaire des contraintes industrielles strictes. De plus, comme ils sont utilisés depuis les années 90 en téléphone mobile, leurs technologies de réalisation sont matures et leurs différents modèles, des circuits équivalents *RLC* à des modèles électromécaniques plus sophistiqués, ainsi que leurs limites sont bien maîtrisées [Has09].

L'approche la plus structurée pour concevoir les filtres AW, détaillée dans [WL15], consiste en un processus itératif, où le problème de conception global est décomposé en sous-problèmes de complexité plus faible. A chaque étape la complexité du modèle du filtre est augmentée et le sous-problème de conception associé, avec des objectifs de conception adaptés, tente d'être résolu en utilisant le résultat de l'étape précédente comme point de départ. La première étape considère la synthèse des paramètres du modèle le plus simple, à savoir le circuit équivalent, et s'attache principalement à la satisfaction du gabarit fréquentiel. Des modèles linéaires *RLC* sont utilisés pour les composants, comme les modèles BVD ou mBVD [LIBWR00; Has09], tandis que leur interconnexion est modélisée comme une interconnexion électrique idéale. De façon surprenante, bien que cette première étape soit déterminante, car elle sert de base pour le reste du processus, celle-ci repose principalement sur l'habileté du concepteur à déterminer un point initial convenable. En effet, comme il n'y a pas d'étape précédente, celui-ci doit fournir un point initial, qui est ensuite donné à un algorithme de résolution, en utilisant typiquement l'interface d'un logiciel de conception assistée par ordinateur (CAO). Le principal problème de cette méthode est que, si l'algorithme ne converge pas vers une solution, le concepteur ne peut s'en remettre qu'à son intuition pour changer les paramètres (valeurs caractéristiques des modèles du composant, nombre de composants, schéma d'interconnexion, ...) de ce point initial.

Bien que cette méthode ait pu être utilisée pour la conception de filtres AW par le passé, elle devient de plus en plus coûteuse en temps de calcul et semble être bien plus limitée face à la complexité des problèmes actuels de conception. Des méthodes systématiques et efficaces semblent donc être nécessaires. Certaines méthodes ont donc été développées en ce sens [SPJK05; Shi06; GB16; GVS18], se basant principalement sur l'extension directe de méthodes analytiques développées dans la première partie du 20<sup>e</sup> siècle. Toutefois, selon l'auteur, tout en fournissant des filtres AW d'utilisation pratique et apparemment prometteurs, ces méthodes peuvent avoir du mal à faire face à des problèmes plus complexes. Avant d'introduire les raisons inhérentes d'une telle affirmation, et afin de comprendre l'approche développée dans ce travail, passons brièvement en revue le développement historique de méthodes pour le problème plus large de la conception des filtres électroniques passifs.

### Un brin d'histoire

Le problème de synthèse de filtres électroniques passifs a un âge vénérable à l'échelle de l'ingénierie. S'appuyant sur le développement de la théorie des Circuits du 19<sup>e</sup> siècle, avec notamment les lois d'Ohm et de Kirchhoff et les travaux d'Heaviside, le premier filtre est apparu pendant la première guerre mondiale [Bel62; Dar84; Nah83; Kal10]. Pendant l'entre-deux guerres, les besoins pratiques d'une méthodologie de conception de filtres ont alors soulevé certains problèmes fondamentaux, qui ont participé au développement de nouvelles thématiques de recherche, telles l'Electronique, l'Automatique, la théorie des Systèmes ou bien le Traitement du Signal [Bel62; Zad62; Dar84]. La résolution d'une partie d'entre eux a alors permis le développement de méthodes systématiques pour la synthèse de certaines familles de filtres, dont particulièrement ceux dits filtres échelle *LC*.

Puis, du fait notamment de l'invention du transistor (1947), les recherches en filtrage se sont focalisées sur la synthèse des filtres digitaux pendant la deuxième moitié du 20<sup>e</sup> siècle. En parallèle, l'Electronique, l'Automatique, la théorie des Systèmes et le Traitement du Signal sont devenus des domaines de recherche à part entière avec leur propre sujets d'intérêts. Un important outil transversal est alors apparu à la fin du siècle dernier : l'optimisation convexe. Combinée avec l'augmentation de la puissance de calcul des ordinateurs, elle a permis de résoudre des problèmes

complexes d'ingénierie en procurant des méthodes systématiques et efficaces pour analyser, commander et concevoir des systèmes [BTN01]. En particulier pour le filtrage, l'optimisation convexe a rencontré un grand succès quand elle s'est attaquée au problème de conception de filtres digitaux à réponse impulsionnelle finie [Dav10]. A la même époque, un soudain regain d'intérêt est apparu pour la conception de filtres passifs avec le développement de la seconde génération de téléphones portables, utilisant des filtres AW [PKK+09]. Rapidement, il s'avère que ces composants ont des performances de filtrage pratiquement imbattables, une haute efficacité énergétique, une petite taille et un faible coût de fabrication. Bien que les méthodes classiques de conception des filtres  $LC$  ne soient pas directement applicables à la synthèse de leurs circuits équivalents  $(R)LC$ , ces bonnes propriétés ont permis d'utiliser des techniques de synthèse basées sur l'intuition et les simulations.

### Sur l'utilisation de méthodes modernes issues de l'Automatique et de la théorie des Systèmes

Revenons, après cette petite digression, sur le développement de méthodes de conception systématiques et efficaces. Au lieu d'étendre des méthodes analytiques et de les implémenter directement sur un ordinateur, avec un fort risque de problèmes numériques, nous trouvons mieux de développer des méthodes ayant pour exigence initiale d'être efficaces sur le plan computationnel. Et pour atteindre cet objectif, l'optimisation convexe nous semble être un cadre approprié. En outre, nous proposons d'utiliser certains concepts et outils de l'Automatique et de la théorie des Systèmes qui tracent naturellement un chemin vers l'optimisation convexe, et plus particulièrement vers la sous-classe de l'optimisation LMI (pour Linear Matrix Inequality en anglais) [BBFG94]. Cette démarche est également motivée par des succès antérieurs d'une telle approche pour la conception de systèmes micro-électroniques [Kor11; Zar13; KSCB16].

Le but de ce travail est donc de développer des méthodes systématiques et efficaces de conception de filtres électroniques passifs, avec un accent tout particulier sur les filtres AW, en utilisant une approche systémique moderne.

#### C.1.2 Champ de recherche et contributions

Notre objectif est de développer des méthodes systématiques et efficaces pour la conception de filtres électroniques passifs. Pour y parvenir, les outils de contrôle et de système sont utilisés comme moyen d'atteindre le cadre de l'optimisation convexe. Cependant, cela nécessite de choisir des outils adaptés aux deux caractéristiques inhérentes des filtres électroniques passifs, discutées ci-dessous.

La première caractéristique des filtres électroniques est leur modélisation hiérarchique. En effet, ils peuvent être considérés comme l'interconnexion de sous-systèmes, les composants, qui sont eux-mêmes l'interconnexion de sous-systèmes plus simples, c'est-à-dire des résistances, des inductances et des capacités modélisant les composants. Le modèle du filtre est alors structurellement contraint à deux niveaux d'interconnexion. En effet, des contraintes structurelles apparaissent sur l'interconnexion globale pour représenter ce qu'on appelle la *topologie* du filtre, et sur les interconnexions locales pour correspondre aux modèles des composants. Nous proposons donc d'utiliser une représentation basée sur l'outil LFT (pour Linear Fractional Transformation en anglais). Issue de la théorie du Contrôle Robuste [DPZ91; ZDG96], cette représentation mathématique unifie et généralise à la fois la fonction de transfert et la représentation d'état, et permet d'exhiber distinctement la structure de chaque niveau d'interconnexion.

La deuxième caractéristique importante à prendre en compte est la passivité des composants. Cela implique des propriétés particulières sur leurs modèles qui peuvent être propagées à leur interconnexion [AV73]. Pour cela, le paradigme de la théorie des Systèmes Dissipatifs nous semble approprié [Wil72b; Moy14]. Elle fournit une caractérisation mathématique qui permet d'exprimer à la fois la passivité des composants et le comportement global du filtre. De plus, ce paradigme tend à fournir des méthodes basées sur l'optimisation LMI pour résoudre des problèmes classiques de l'Automatique et de la théorie des Systèmes.

La combinaison de ces deux outils, la représentation LFT et la caractérisation de dissipativité, a conduit au développement de plusieurs méthodes systématiques et efficaces, basées sur l'optimisation LMI, pour analyser et contrôler les systèmes hiérarchiques [Sco97; SD01; Kor11; Zar13; ZKSV13; KSCB16]. L'idée clé de ces méthodes consiste à revisiter les méthodes efficaces des problèmes classiques de manière à les étendre efficacement à des problèmes d'intérêt plus complexes.

Dans cette thèse, il est proposé de suivre cette approche. Cela nous amène à revisiter les différentes méthodes développées pour la conception de filtres électroniques passifs. D'un point de vue de la théorie des Systèmes, ces méthodes peuvent être classées selon si la représentation mathématique à synthétiser est une représentation d'état ou une fonction de transfert. Les contraintes structurelles imposées par l'interconnexion locale et globale résultent alors soit en tant que contraintes sur les matrices d'état ou sur les coefficients d'une fonction de transfert.

Le problème de la synthèse structurée d'une représentation d'état est d'abord abordé. Malheureusement, représentation d'état et synthèse structurée vont généralement de pair avec des problèmes non-convexes [BT97]. Et cette affirmation sera corroborée dans ce manuscrit. Nous nous efforçons donc de réduire autant que possible, ce que nous appellerons, le degré de non-convexité du problème. Pour y parvenir, une représentation issue de la théorie des systèmes PHS (pour Port-Hamiltonian System en anglais) nous semble particulièrement adaptée [Sch13]. La théorie des systèmes PHS est un cadre multi-physique, basé sur l'énergie, spécialement adapté pour la modélisation, l'analyse et le contrôle des systèmes passifs interconnectés [SM03; DMSB09; Sch17]. Curieusement, alors cette théorie est intrinsèquement lié aux circuits électroniques passifs [SJ14; MSB95], il semble que le problème de la conception fréquentiel des filtres électroniques passifs n'ait pas été abordé via le cadre PHS. Une caractéristique particulière de ce cadre est de fournir à partir de tout circuit une représentation d'état structurée où les valeurs des composants, c'est-à-dire nos variables de décision, apparaissent de manière linéaire et distincte. Notre première contribution est donc de développer une méthode de synthèse adaptée à cette représentation particulière. Cela nous conduit à résoudre un problème d'optimisation non convexe, comme prévu, mais d'une forme particulière et avec un degré de non-convexité limité. De plus, afin de réduire encore plus ce degré, nous combinons la représentation PHS avec l'outil LFT, ce qui permet d'obtenir une représentation originale. Notre deuxième contribution est alors d'adapter notre méthode de synthèse à cette nouvelle représentation. Cela permet de résoudre un problème d'optimisation avec un faible degré de non-convexité.

Puis, nous nous attaquons au problème de synthèse d'une fonction de transfert structurée. Contrairement au cas de la représentation d'état, ce problème admet certains cas pour lesquels il existe des méthodes de synthèse convexes, conduisant par exemple aux filtres dits de Butterworth et de Tchebyshev. Ces cas sont principalement dus à une interconnexion globale particulière, appelée topologie échelle, et à la simplicité des sous-systèmes, qui sont soit des inductances, soit des capacités, conduisant à des contraintes structurelles convexes. Une autre de nos contributions consiste à étendre de manière convexe ces contraintes structurelles à l'interconnexion

en échelle d'une famille plus large de composants passifs, ce qui est intéressant pour nos applications. Ceci est réalisé en interprétant la variable de Laplace  $s$  comme étant l'impédance d'une inductance unitaire, ou de façon similaire l'admittance d'une capacité unitaire, et donc en introduisant certaines fonctions de transfert pour encapsuler l'interconnexion des sous-systèmes locaux, c'est-à-dire les modèles des composants. Cela nous conduit à synthétiser une fonction de transfert qui s'exprime comme une fonction rationnelle de fonctions de transfert, c'est-à-dire à aborder le problème de synthèse fréquentielle des filtres LFT.

Ce problème est l'extension du problème de synthèse classique de filtres fréquents pour lequel il existe une méthode de résolution efficace. Cette méthode consiste en deux étapes successives. La première étape concerne la synthèse du module d'une réponse fréquentielle et est réalisée en résolvant un problème d'optimisation LMI. Cette étape est principalement basée sur le Lemme de Kalman-Yakubovich-Popov (KYP). Ce lemme, qui est une généralisation de ce qu'on appelle le lemme positif-réel et le lemme borné-réel qui sont historiquement liés à la synthèse de circuits passifs [AV73], est un outil fondamental et transversal dans la théorie des Systèmes linéaires [Wil72b; TW91; BGFB94; Ran96]. De plus, il admet plusieurs versions généralisées [IMF00a; Din05; PV11; PIH13], qui semblent particulièrement utiles pour la conception de filtres [RSF03; IH05; Dav10], qu'ils soient numériques ou analogiques. Ce lemme est même compatible avec la représentation LFT et la caractérisation de dissipativité [Sco97; SD01; Zar13]. La deuxième étape, nommée la factorisation spectrale, consiste à extraire une fonction de transfert stable du module synthétisé. Le problème de la factorisation spectrale est également lié à la synthèse des circuits passifs, a fait l'objet de nombreuses études pour le filtrage et d'autres applications, et peut être résolu de manière efficace [You61; Wil72b; AV73; ZDG96; TR99; WBV99]. Malheureusement, le problème de la factorisation spectrale des LFT ne semble pas avoir été résolu dans la littérature. Par conséquent, en détaillant et en complétant l'approche de [Zar13], nous nous attaquons à ce problème en revisitant les méthodes classiques de synthèse de filtre efficace avec la représentation LFT et la caractérisation de dissipativité. Notre contribution consiste alors à étendre cette approche à des systèmes LFT particuliers, ce qui est intéressant pour la conception de certains filtres électroniques passifs. Toutefois, pour des systèmes LFT plus généraux, nous montrons que l'approche ne peut être directement étendue, car une erreur de factorisation apparaît et semble inhérente à l'approche. Par conséquent, l'une des principales contributions de cette thèse est de généraliser cette approche de synthèse en fusionnant la synthèse de module et la factorisation spectrale en une seule étape, qui est réalisée en résolvant un problème d'optimisation LMI. Ceci permet de prendre en compte l'erreur de factorisation spectrale dès le départ. En outre, cette approche généralisée nous amène également à contribuer au problème de la synthèse des filtres LFT 2D, en fournissant de même un problème d'optimisation LMI pour le résoudre. Bien qu'il soit d'un conservatisme inhérent, la méthode qui en découle tend à apporter des solutions dans les cas qui nous intéressent.

Enfin, la mise en commun de tous ces éléments nous permet de poser les fondations en vue du développement de méthodes de conception systématiques et efficaces. En particulier, cela nous conduit à une méthodologie systématique, efficace et quasi optimale pour la conception d'une famille de produits électroniques passifs comprenant les filtres échelle  $LC$  et les filtres en échelle à résonateurs- $LC$ . En outre, bien qu'elle ne soit pas entièrement systématique, une approche prometteuse basée sur des étapes de synthèse efficaces est fournie pour la conception de filtres échelle à résonateur- $AW$ .



### Ce que le lecteur ne trouvera pas dans cette thèse

Dans cette thèse, notre objectif est de développer une méthodologie efficace et systématique pour la conception fréquentielle des filtres électroniques passifs, avec une approche issue de l'Automatique et de la théorie des Systèmes.

Par conséquent, cette thèse ne porte pas sur le développement de méthodes de conception d'un filtre *AW* réalisé dans une technologie donnée associée à une bande de fréquence donnée. Nous nous concentrons plus modestement sur la toute première étape du flux de conception, c'est-à-dire l'obtention d'un circuit équivalent, avec les modèles *RLC* des composants, qui satisfait certaines exigences du gabarit fréquentiel. En effet, la mise au point de méthodes systématiques et efficaces à cette fin est déjà une tâche non-triviale. Ainsi, aucun mot de réalisation technologique tel que FBAR, SMR ou  $L_iNO_3$  ne se trouve dans ce manuscrit. Cependant, les méthodes de conception sont développées avec l'objectif d'être flexible, et de permettre d'inclure le plus possible de contraintes technologiques. En effet, on sait que certaines considérations technologiques telles que la surface de l'électrode supérieure ou l'épaisseur de la couche piézoélectrique peut être propagées aux modèles *RLC* [CBG+13]. Par conséquent, une méthode prête à l'emploi pour la conception des filtres *AW* ne sera pas trouvée dans ce manuscrit, mais nous visons à préparer le terrain pour faire face aux problèmes actuels et aux défis à venir.

De plus, bien que s'appuyant fortement sur l'optimisation convexe, cette thèse ne vise pas à développer de nouvelles techniques ou de nouveaux algorithmes d'optimisation. L'optimisation convexe est considérée comme un outil mature et prêt à l'emploi. C'est d'ailleurs l'un des principaux avantages de l'approche développée dans ce travail.

### C.1.3 Publications

Les papiers suivants sont disponibles en suivant le lien

[https://hal.archives-ouvertes.fr/search/index/q/\\*/authIdHal\\_s/arthur-perodou](https://hal.archives-ouvertes.fr/search/index/q/*/authIdHal_s/arthur-perodou)

### Résumé étendu pour participation à une session poster

- **A. Perodou**, A. Korniienko, M. Zarudniev, G. Scorletti, I. O'Connor, J-B. David. "On Motivations for Designing Analog Filters Under LFT Framework". Extended abstract. *20th IFAC World Congress*, Jul 2017, Toulouse, France.

### Papiers de Conférence Internationales avec comité de lecture

- [PKSO18] **A. Perodou**, A. Korniienko, G. Scorletti, and I. O'Connor. "Systematic Design Method of Passive Ladder Filters using a Generalised Variable". In: *2018 Conference on Design of Circuits and Integrated Systems (DCIS)*. Nov 2018, Lyon, France.
- [PKZS18] **A. Perodou**, A. Korniienko, M. Zarudniev, and G. Scorletti. "Frequency Design of Interconnected Dissipative Systems: A Unified LMI Approach". In: *2018 IEEE Conference on Decision and Control (CDC)*. Dec 2018, Miami, FL, United States.

## C.2 Un cadre moderne pour la conception de systèmes

Avant de formuler concrètement le problème de la conception des filtres électroniques passifs, certains concepts et outils utilisés dans ce travail sont d'abord présentés.

Ces concepts et outils proviennent principalement de l'Automatique et de la théorie des Systèmes, et forment traditionnellement un cadre générique pour l'analyse et le contrôle des systèmes dynamiques. Comme les filtres électroniques passifs sont généralement considérés comme des systèmes en boucle ouverte, c'est-à-dire sans l'intervention extérieure d'un régulateur, le problème de conception associé n'entre pas directement dans le champ d'application de ce cadre. L'objectif du Chapitre 2 est donc de poser les bases d'un cadre de conception adapté à notre problème, qui sera développé tout au long de ce manuscrit.

### **C.2.1 Sur la complexité de conception des systèmes modernes**

Afin de choisir des outils appropriés au problème de conception, il est tout d'abord nécessaire de comprendre d'où provient la complexité pour développer des méthodes de conception des systèmes modernes. A nos yeux, cette complexité semble provenir de deux raisons. Tout d'abord, le problème de conception des systèmes modernes est lui-même complexe. En effet, concevoir des systèmes, même simples, est déjà une tâche compliquée qui soulève des questions théoriques importantes. De plus, la sophistication des systèmes modernes complexifie encore plus ce problème. La deuxième raison est la nécessité des méthodes de pouvoir être directement utilisées par les ingénieurs de conception, et donc être également modernes. Pour cela, un outil moderne indispensable est l'ordinateur. En plus d'éviter de faire des calculs fastidieux à la main, potentiellement source d'erreurs, l'importante puissance de calcul des ordinateurs semble être nécessaire pour faire face à la complexité du problème. Afin de faire l'interface entre l'utilisateur et les calculs réalisés par l'ordinateur, un logiciel de Conception Assistée par Ordinateur (CAO) est généralement utilisé. Cette interface est souvent graphique, sous la forme d'un schéma électrique par exemple pour notre cas. Plus visuelle que des relations mathématiques, elle permet aux concepteurs d'obtenir plus facilement des intuitions pour diminuer la complexité du problème. Par conséquent, une nouvelle méthode de conception doit être numériquement efficace et pouvoir être facilement incluse dans un logiciel CAO.

Dans la suite, nous détaillons le cadre qui nous semble le plus approprié pour ce travail. Partant de l'objectif de développer des méthodes efficaces de conception, une classe particulière de problèmes d'optimisation est exposée. Des outils mathématiques pour représenter et caractériser les systèmes modernes sont ensuite développés afin d'amener le problème de conception dans cette classe.

### **C.2.2 Sur l'utilisation efficace des ordinateurs : l'optimisation convexe**

Parmi les exigences précédemment mentionnées, l'exigence d'efficacité numérique est le facteur limitant lors de l'élaboration des méthodes de conception. En effet, il serait inutile de développer des méthodes avancées de conception, si elles ne peuvent pas être efficacement mises en œuvre. Ce point est particulièrement important. Pendant des siècles, les problèmes mathématiques ont été résolus de manière analytique et graphique, à l'aide de papier et de crayon, et les méthodes de résolution ont été développées à cette fin. L'avènement de l'ordinateur a fourni un outil doté d'une puissance de calcul spectaculaire, mais nécessite une transformation du problème du langage mathématique en langage informatique. Pendant des décennies, l'efficacité de cette transformation n'a pas été une préoccupation majeure pour les ingénieurs, car la puissance de calcul a régulièrement augmenté et semblait être pratiquement illimitée. Il est compréhensible que le paradigme ait été d'obtenir l'ordinateur adéquat plutôt

que des méthodes numériques à faible coût de calcul. Cependant, avec la fin de la loi de Moore, associée à l'augmentation de la complexité des problèmes d'ingénierie, ainsi que des considérations sociales en terme d'économie d'énergie, il devient de plus en plus crucial de formuler ces problèmes sous une forme adaptée aux ordinateurs.

Pour comprendre ce que l'on entend par approprié, considérons le schéma de résolution suivant. Supposons que l'on dispose d'une formulation mathématique associée à un problème d'ingénierie. La première étape consiste à le transformer en un problème d'optimisation. Nous définissons ici un problème d'optimisation comme suit :

$$\begin{array}{ll} \min_{\mathbf{x} \in \mathbb{R}^m} & f(\mathbf{x}) \\ \text{such that} & \mathbf{x} \in \mathcal{C} \end{array}$$

où  $\mathbf{x}$  est le vecteur des variables d'optimisation,  $f$  est une fonction à valeur réelle appelée fonction objective et  $\mathcal{C}$  l'ensemble des contraintes à respecter. En particulier, tester si l'ensemble  $\mathcal{C}$  n'est pas vide, c'est-à-dire s'il existe  $\mathbf{x} \in \mathcal{C}$ , est appelé un problème de faisabilité. Cette étape n'est généralement pas simple. En effet, il existe potentiellement de nombreux problèmes d'optimisation associés à un problème mathématique. Un choix est nécessaire et dépend généralement de l'étape suivante.

La deuxième étape consiste à exécuter un algorithme afin de résoudre le problème d'optimisation qui en résulte. Une approche naïve consisterait à développer un algorithme pour chaque problème. En plus d'exiger des compétences non triviales pour obtenir un algorithme numériquement stable qui converge en un temps de calcul décent vers une solution proche de l'optimum, s'il existe, cette approche augmente considérablement le flux de conception, car il existe une grande diversité de problèmes d'ingénierie.

Au lieu de cela, des classes de problèmes d'optimisation ont été formulées. L'idée est assez simple. Si un problème mathématique peut être transformé en une instance d'une classe de problèmes d'optimisation, on peut utiliser avec peu d'effort de programmation l'algorithme de résolution associé pour lequel tous les paramètres ont été convenablement choisis. Initialement, le premier objectif était de développer un algorithme générique qui résout la plus large classe possible de problèmes d'optimisation. Cependant, il est apparu que les problèmes d'optimisation peuvent être de complexité différente [GJ79]. Ainsi, un algorithme générique peut prendre beaucoup de temps pour résoudre des problèmes de complexité modérée. Par conséquent, une approche moderne a émergé, consistant à faire un compromis entre la généralité de la classe et l'efficacité de l'algorithme de résolution associé. Si l'on peut trouver des algorithmes avec un temps de calcul plus court pour un problème d'optimisation donné, cette approche semble plus adéquate pour le chercheur en conception de systèmes qui cherche à développer des prototypes de méthodes numériques pour différents problèmes d'ingénierie.

Pour de nombreux problèmes d'ingénierie, dont celui de ce travail, la classe la plus intéressante nous semble être celle des problèmes d'optimisation convexe [BTN01]. D'un point de vue mathématique, les problèmes convexes sont des problèmes d'optimisation pour lesquels  $\mathcal{C}$  est un ensemble convexe et  $f$  une fonction convexe sur  $\mathcal{C}$ . Sa propriété la plus remarquable est que tout minimum local est également un minimum global. En outre, bien que cette classe ne soit pas dotée d'algorithmes génériques efficaces, elle comprend plusieurs sous-classes avec une telle propriété (voir par exemple Figure 2.1, p. 17). En effet, l'optimisation convexe peut être vue comme la généralisation de la Programmation linéaire, qui a été introduite et fortement étudiée dans la seconde moitié du 20e siècle, et pour laquelle des algorithmes de résolution efficace ont été développés. Cela a permis de résoudre des problèmes

d'ingénierie assez simples, mais cette classe est trop petite pour être utilisée pour la plupart des applications en pratique. À la fin des années 80, l'augmentation de la puissance de calcul associée au développement d'algorithmes modernes, et en particulier la généralisation des méthodes dites des points intérieurs, a fourni des méthodes génériques et efficaces pour résoudre les problèmes de programmation quadratique et la grande classe des problèmes d'optimisation semi-définie positive [BTN01]. À noter que par efficace, on entend que les algorithmes ont une complexité polynomiale au pire cas, correspondant à un temps de calcul de moins d'une seconde à quelques minutes pour des problèmes d'ingénierie de taille moyenne. Certaines applications en temps-réel sont même possibles [MB10].

L'optimisation semi-définie positive est donc une sous-classe particulière de problèmes convexes pour lesquels les contraintes fixées sont définies par des inégalités positives semi-définies

$$\mathcal{C} = \{x \in \mathbb{R}^m \mid F(x) \geq 0\}, \quad F(x) := F_0 + \sum_{i=1}^m x_i F_i$$

où  $x = [x_1 \ \dots \ x_m]^T$  est le vecteur des variables de décision et  $F_i = F_i^T \in \mathbb{R}^{n \times n}$ ,  $i = 1, \dots, m$ , sont des matrices symétriques données. L'optimisation semi-définie positive est souvent appelée optimisation LMI (pour *Linear Matrix Inequality* en anglais), comme c'est d'ailleurs le cas dans ce manuscrit, car  $\mathcal{C}$  est généralement formulé en termes d'inégalités qui dépendent linéairement de variables matricielle d'optimisation [BTN01]. En particulier, lorsque la fonction objectif est linéaire, c'est-à-dire  $f(x) = c^T x$  pour un vecteur donné  $c \in \mathbb{R}^m$ , le problème d'optimisation résultant est désigné comme un problème de minimisation linéaire sous contraintes LMIs.

Très peu de problèmes d'ingénierie ont naturellement une formulation mathématique qui peut être directement transformée en un problème d'optimisation LMI. Il existe des techniques standards pour réduire certaines contraintes particulières en LMI [Sco06], mais elles ne sont pas suffisantes pour une transformation directe. Par conséquent, ce paradigme a fait passer la difficulté du niveau des algorithmes au niveau de la reformulation mathématique de problèmes. Cela est particulièrement bénéfique car cela permet un lien plus explicite avec le problème d'ingénierie original que des paramètres d'un algorithme générique, ce qui peut permettre de réduire la complexité du problème d'optimisation résultant, comme on le verra dans ce manuscrit.

Les communautés d'Automatique et de théorie des Systèmes ont historiquement l'habitude de jongler avec des formulations mathématiques afin de trouver celles qui sont le plus appropriées. Au début des années 90, l'optimisation des LMI a rencontré un grand succès lorsqu'il a été illustré que, parmi toutes ces formulations, les LMI apparaissent fréquemment [BGFB94]. L'enthousiasme a atteint quelques années plus tard la communauté du Traitement du Signal, et en particulier les chercheurs travaillant sur la conception de filtres numériques [Dav10].

Par conséquent, dans les sections restantes de ce chapitre, nous fournissons et adaptons des outils mathématiques issus de l'Automatique et de la théorie des systèmes, dans le but de formuler le problème de conception considéré dans ce travail via un cadre conduisant à des problèmes d'optimisation LMI.

### C.2.3 Une représentation moderne : la représentation LFT

Afin de forger un cadre de conception adapté aux systèmes modernes, le premier outil considéré est la représentation mathématique de leurs modèles.

Dans ce travail, nous sommes intéressés par les modèles linéaires temps-invariants (LTI) à dimension finie des systèmes dynamiques, généralement abrégés en systèmes LTI. Ces derniers peuvent être mathématiquement représentés de différentes manières. Par exemple, étant donné un système LTI ( $\Sigma$ ) à une seule entrée et une seule sortie, il peut être représenté par une fonction de transfert  $H$  comme suit :

$$(\Sigma) : \begin{cases} z(s) = H(s)w(s), & H(s) = \frac{b_0 + b_1s + \dots + b_ms^m}{a_0 + a_1s + \dots + a_ns^n} \end{cases}$$

La représentation par fonction de transfert est liée à l'approche entrée-sortie, où un système est vu comme une boîte noire recevant de l'information de l'environnement via ses entrées et agissant sur lui grâce à ses sorties. Du point de vue de l'ingénierie, ce mode de représentation est très commode car il permet d'exprimer facilement les spécifications de performance. Un autre type de représentation est la représentation d'état, qui permet de rendre explicite les relations internes d'un système via les équations dites d'état. En particulier, l'utilisation de cette représentation a permis de formuler un nombre important de problèmes d'Automatique sous la forme de problèmes d'optimisation LMI [BBFG94]. Pour des matrices réelles données  $A$ ,  $B$ ,  $C$ ,  $D$ , la représentation d'état de ( $\Sigma$ ), exprimée dans le domaine de Laplace, est de la forme :

$$(\Sigma) : \begin{cases} p(s) = \left(\frac{1}{s} \cdot I\right) q(s) \\ q(s) = Ap(s) + Bw(s) \\ z(s) = Cp(s) + Dw(s) \end{cases}$$

Les deux représentations sont connues pour être équivalentes, sous couvert d'hypothèses simples. En effet, à toute représentation d'état est associée une matrice de transfert  $H(s)$  donnée par

$$H(s) = D + C \left(\frac{1}{s} \cdot I\right) \left(I - A \left(\frac{1}{s} \cdot I\right)\right)^{-1} B$$

Réciproquement, pour chaque fonction de transfert  $H(s)$  propre, i.e.  $m \leq n$ , une représentation d'état peut être associée (voir Section 2.4, p. 18 pour plus de détails).

Bien qu'adaptées pour la représentation de systèmes, ces deux représentations apparaissent plus limitées pour représenter leur interconnexion. En effet, les deux représentations peuvent être vues comme l'interconnexion de sous-systèmes ayant pour fonction de transfert  $\frac{1}{s}$ . De notre point de vue, ces sous-systèmes sont simples, car ils ont des fonctions de transfert d'ordre le plus faible, à l'exception des gains constants. Du point de vue de l'Electronique, ces sous-systèmes peuvent être considérés comme une inductance unitaire ou une capacité unitaire, car elles ont toutes les deux  $\frac{1}{s}$  comme fonction de transfert représentative, soit leur admittance ou leur impédance. Cependant, une caractéristique importante des systèmes modernes est leur interprétation comme l'interconnexion de sous-systèmes complexes, c'est-à-dire ayant une fonction de transfert  $T(s)$  d'ordre strictement supérieur à un.

Nous introduisons donc une autre représentation, basée sur la fonction mathématique LFT (pour Linear Fractional Transformation en anglais), qui est plus adaptée à la conception des systèmes modernes, car elle permet de représenter l'interconnexion de systèmes complexes. Il s'agit d'une généralisation de la fonction de transfert et de la représentation de l'espace d'état, et permet d'utiliser les avantages des deux approches. En outre, elle peut représenter synthétiquement

l'interconnexion de systèmes complexes. L'outil LFT est également un outil mathématique très générique. Différents objets mathématiques, tels que les fonctions de variables généralisées, sont des cas particuliers de LFT. Enfin, un schéma par diagramme par bloc est associée à la représentation LFT (voir Figure 2.2 et Figure 2.3, p. 19-21), qui permet de représenter visuellement les opérations mathématiques. La représentation LFT semble donc être un candidat approprié pour représenter les systèmes et leur interconnexion via un logiciel de CAO. Une définition adaptée de [ZDG96, Chap.10] pour notre utilisation est donnée ci-dessous.

**Définition C.1 (Représentation/Transformation LFT).**

Une représentation LFT (aussi appelée LFR pour Linear Fractional Representation en anglais) d'un système est un ensemble d'équations

$$\begin{cases} p = \Delta q \\ q = Ap + Bw \\ z = Cp + Dw \end{cases}$$

où  $A \in \mathbb{R}^{n \times n}$ ,  $B \in \mathbb{R}^{n \times n_w}$ ,  $C \in \mathbb{R}^{n_z \times n}$ ,  $D \in \mathbb{R}^{n_z \times n_w}$  sont des matrices réelles,  $\Delta$  est une  $n \times n$  matrice de transfert, et  $q, p, z, w$  sont des signaux complexes de dimensions appropriées. Sous réserve d'existence de  $(I - A\Delta(s))^{-1}$ , la transformation LFT associée est définie par :

$$\Delta(s) \star \left[ \begin{array}{c|c} A & B \\ \hline C & D \end{array} \right] := D + C\Delta(s)(I - A\Delta(s))^{-1}B$$

et relie  $z$  et  $w$  par :  $z = \left( \Delta(s) \star \left[ \begin{array}{c|c} A & B \\ \hline C & D \end{array} \right] \right) w$ .

A noter que, lorsque  $\Delta(s) = \frac{1}{s} \cdot I_n$ , on retrouve la représentation d'état et la fonction de transfert.

#### C.2.4 Caractériser les systèmes par la dissipativité

La fonction LFT fournit une représentation appropriée pour les systèmes modernes, considérés comme l'interconnexion de sous-systèmes complexes. L'objectif est maintenant de fournir à notre cadre de conception une caractérisation mathématique de ces sous-systèmes ainsi que de leur interconnexion.

Pour ce faire, une caractérisation provenant de la théorie des Systèmes Dissipatifs, introduit à l'origine dans [Wil72a; Wil72b], semble appropriée. En effet, cette caractérisation, à savoir la caractérisation de  $\{X, Y, Z\}$ -dissipativité, conduit essentiellement à une contrainte quadratique sur la réponse fréquentielle du système. Ceci est bénéfique car ce type de contraintes peut être typiquement transformé en un problème d'optimisation LMI. En outre, en utilisant cette caractérisation de dissipativité, la stabilité de sous-systèmes interconnectés peut être établie en vérifiant essentiellement la satisfaction d'une autre contrainte quadratique sur la réponse fréquentielle, menant également à résoudre un problème LMI.

La caractérisation de  $\{X, Y, Z\}$ -dissipativité qui nous intéresse dans ce travail est définie ci-dessous. Comparée à la littérature, cette caractérisation est étroitement liée à la définition utilisée dans [TW91] et peut être considérée comme une version plus faible de la  $(Q, S, R)$ -dissipativité donnée dans [MH78; Moy14].

**Définition C.2** ( $\{X, Y, Z\}$ -Dissipativité).

Soient les matrices réelles  $X = X^T \in \mathbb{R}^{n_z \times n_z}$ ,  $Y \in \mathbb{R}^{n_z \times n_w}$  et  $Z = Z^T \in \mathbb{R}^{n_w \times n_w}$  telles que la matrice

$$\begin{bmatrix} X & Y \\ Y^T & Z \end{bmatrix}$$

soit de rang plein. Une matrice de transfert  $T(s)$  marginalement stable<sup>1</sup> est dite  $\{X, Y, Z\}$ -dissipative si :

$$\forall \omega \in \mathbb{R}, j\omega \text{ n'étant pas un pôle de } T(s), \quad \begin{bmatrix} T(j\omega) \\ I \end{bmatrix}^* \begin{bmatrix} X & Y \\ Y^T & Z \end{bmatrix} \begin{bmatrix} T(j\omega) \\ I \end{bmatrix} \geq 0$$

Cette matrice de transfert  $T(s)$  est  $\{X, Y, Z\}$ -dissipative sans-perte si :

$$\forall \omega \in \mathbb{R}, j\omega \text{ n'étant pas un pôle de } T(s), \quad \begin{bmatrix} T(j\omega) \\ I \end{bmatrix}^* \begin{bmatrix} X & Y \\ Y^T & Z \end{bmatrix} \begin{bmatrix} T(j\omega) \\ I \end{bmatrix} = 0$$

Cette matrice de transfert  $T(s)$  est strictement  $\{X, Y, Z\}$ -dissipative si :

$$\forall \omega \in \mathbb{R}, j\omega \text{ not a pole of } T(s), \quad \begin{bmatrix} T(j\omega) \\ I \end{bmatrix}^* \begin{bmatrix} X & Y \\ Y^T & Z \end{bmatrix} \begin{bmatrix} T(j\omega) \\ I \end{bmatrix} > 0$$

D'un point de vue géométrique, la contrainte quadratique de dissipativité impose que  $T(j\omega)$  appartienne à un ellipsoïde, un hyperplan ou au complémentaire d'un ellipsoïde dans  $\mathbb{C}^{n_z \times n_w}$ , selon le signe de  $X$ . Lorsque  $T(s)$  est SISO, i.e. lorsque  $n_w = n_z = 1$ , ces trois ensembles géométriques deviennent un disque, un cercle ou au complémentaire d'un disque dans le plan complexe (voir Figure 2.4, p. 24, par exemple).

De toute évidence, il n'y a pas une caractérisation unique  $\{X, Y, Z\}$  associée à une matrice de transfert  $T(s)$ . Premièrement, les contraintes quadratiques de la Définition C.2 peuvent être multipliées par tout scalaire positive  $\alpha > 0$ . Donc, dans ce cas, si  $T(s)$  est  $\{X, Y, Z\}$ -dissipatif,  $T(s)$  est également  $\{\alpha X, \alpha Y, \alpha Z\}$ -dissipatif. De plus, dans le cas particulier où  $\{X, Y, Z\} = \{xI, yI, zI\}$  et  $x, y, z$  sont des scalaires, on peut multiplier la contrainte quadratique associée par n'importe quelle matrice définie positive<sup>2</sup>  $P = P^T > 0$ , et alors  $T(s)$  est également  $\{xP, yP, zP\}$ -dissipatif. L'ensemble géométrique généré par ces contraintes reste le même. Ces deux quantités,  $\alpha$  et  $P$ , seront désignés par le terme *multiplieur* et joueront un rôle important dans ce manuscrit.

Deuxièmement, une fonction de transfert  $T(s)$  peut être  $\{x_1, y_1, z_1\}$ - et  $\{x_2, y_2, z_2\}$ -dissipative. Géométriquement, cela signifie que la réponse fréquentielle de  $T(s)$  appartient à des disques différents dans le plan complexe. Cependant, il existe un disque de rayon minimal, dont les paramètres  $\{x, y, z\}$  peuvent être calculées en résolvant un problème d'optimisation LMI, comme illustré à la fin de la Section 2.6 (p. 27).

Inversement, plusieurs fonctions de transfert peuvent satisfaire la même caractérisation de  $\{x, y, z\}$ -dissipativité. Une paramétrisation  $\{x, y, z\}$  définit alors une famille de systèmes. Par exemple, on peut facilement vérifier que  $T_1(s) = \frac{1}{s}$  et  $T_2(s) = \frac{s}{s^2+1}$  sont toutes les deux  $\{0, 1, 0\}$ -dissipative sans perte. Cette caractéristique, particulièrement utilisée dans le cadre de la commande robuste, sera importante dans ce travail. Par exemple, comme  $\frac{1}{s}$  est  $\{0, 1, 0\}$ -dissipatif sans perte, cela permettra d'étendre

<sup>1</sup>Voir Section 2.5, p. 21, pour les définitions utilisées de stabilité et stabilité marginale.

<sup>2</sup>dans le cas sans perte, cette matrice  $P$  doit seulement être inversible

directement plusieurs résultats classiques sur la représentation d'état à des LFTs en dans une fonction de transfert  $T(s)$  répétée, avec  $T(s)$   $\{0, 1, 0\}$ -dissipative sans perte.

Pour finir, un attrait important de la caractérisation  $\{X, Y, Z\}$ -dissipativité est qu'elle fournit une condition suffisante pour assurer la stabilité de sous-systèmes interconnectés. Le résultat utilisé dans ce travail (cf Theorem 2.1, p. 25, et Corollary 2.2, p. 26) est adapté de [Moy14] et peut être considéré comme une légère modification du théorème de séparation des graphes [SD01]. De manière synthétique, il peut être formulé comme suit : un système, défini comme l'interconnexion de sous-systèmes, est stable si les sous-systèmes sont  $\{X, Y, Z\}$ -dissipatifs et l'interconnexion est strictement  $\{-Z, -YT, -X\}$ -dissipatif. Dans le cas particulier d'une représentation LFT, ce résultat conduit directement à un problème de faisabilité LMI.

### C.2.5 Un outil unificateur : le lemme KYP

Le problème d'analyse suivant est maintenant considéré :

Étant donné des sous-systèmes  $\{X, Y, Z\}$ -dissipatifs et une représentation LFT de leur interconnexion, le système résultant satisfait-il les critères d'un cahier des charges donné ?

On peut différencier nos critères selon deux types : les critères de la stabilité et ceux de performance. La dernière section a fourni un problème de faisabilité LMI comme condition suffisante pour vérifier la stabilité de systèmes dissipatifs interconnectés. En outre, cela a conduit à une méthode simple pour concevoir une interconnexion stable de sous-systèmes SISO identiques. De même, l'objectif de cette section est de fournir des méthodes pour résoudre le problème d'analyse précédent en utilisant le cadre de l'optimisation LMI, de manière à ce qu'elles puissent être généralisées à des fins de conception.

Dans ce travail, comme pour la caractérisation de dissipativité, les critères de performance seront exprimés à l'aide de contraintes quadratiques dans le domaine des fréquences. Par exemple, il peut être nécessaire de synthétiser un système dont le module au carré de sa réponse fréquentielle admet une limite supérieure  $|H(j\omega)|^2 \leq U^2$  pour  $\omega \in \mathbb{R}$ . Ce critère de conception peut alors être réécrit comme suit :

$$\forall \omega \in \mathbb{R}, \quad \begin{bmatrix} H(j\omega) \\ 1 \end{bmatrix}^* \begin{bmatrix} -1 & 0 \\ 0 & U^2 \end{bmatrix} \begin{bmatrix} H(j\omega) \\ 1 \end{bmatrix} \geq 0$$

Afin d'atteindre notre objectif, nous introduisons et généralisons le lemme Kalman-Yakubovich-Popov, appelé lemme KYP par la suite. Ce lemme transforme une contrainte quadratique dans le domaine fréquentiel, ou une contrainte quadratique intégrale dans le domaine temporel, d'un système avec une représentation LFT en un problème de faisabilité LMI. Par exemple, étant donné une représentation LFT  $\left(\frac{1}{s} \cdot I_n\right) \star \left[ \begin{array}{c|c} A & B \\ \hline C & D \end{array} \right]$  de  $H$ , le lemme KYP établit que la satisfaction de la contrainte quadratique précédente est équivalente à la faisabilité du problème LMI suivant :  $\exists \mathbf{P} = \mathbf{P}^T \in \mathbb{R}^{n \times n}$ ,

$$\begin{bmatrix} A & B \\ I & 0 \end{bmatrix}^T \begin{bmatrix} 0 & -\mathbf{P} \\ -\mathbf{P} & 0 \end{bmatrix} \begin{bmatrix} A & B \\ I & 0 \end{bmatrix} + \begin{bmatrix} C^T \\ D^T \end{bmatrix} X \begin{bmatrix} C & D \end{bmatrix} \geq 0$$

Originellement étudié pour des contraintes sur  $\mathbb{R}$  pour des systèmes avec une représentation d'état [AV73; Wil71; Wil72b; TW91; Ran96], ce lemme a été par la suite étendu



à des contraintes sur des intervalles fréquentiels [IHF05], ou des systèmes avec une représentation LFT [Sco97; SD01; Zar13]. En interprétant le lemme KYP avec des arguments issus de la théorie des multiplicateurs et de la dissipativité, nous procurons des généralisations de ce lemme qui vont être utiles pour le type de systèmes LFT et de contraintes considérés dans le cadre de ce travail. En particulier, les multiplicateurs donnés associés aux LFTs en  $\begin{bmatrix} s \cdot I_{n_1} & 0 \\ 0 & \frac{1}{s} \cdot I_{n_2} \end{bmatrix}$  (cf Lemme 2.2 et Lemme 2.3) sont nouveaux et fournissent donc une de nos contributions.

Un lecteur souhaitant en savoir plus sur le lemme KYP, son interprétation en terme de multiplicateurs et de dissipativité ainsi que le moyen d'obtenir ses différentes généralisations est invité à lire la Section 2.6 (p. 27). Sinon, les trois points à retenir sont les suivants :

1. Le Lemme KYP transforme un critère de performance en un problème *équivalent* de faisabilité LMI.
2. Le lemme KYP est généralisable aux différents systèmes et critères de performance qui nous intéressent dans ce travail. Ces généralisations permettent aussi d'obtenir des conditions sous la forme d'un problème de faisabilité LMI. Cependant ces conditions deviennent généralement conservatives, c'est-à-dire qu'elles sont seulement suffisantes.
3. Si le problème de conception peut être formulé de façon adéquat, des méthodes de conception efficaces peuvent alors être développées grâce au lemme KYP et ses généralisations.

## C.2.6 Conclusion

Développer des méthodes de conception pour des applications d'ingénierie modernes n'est pas une tâche facile. Les problèmes de conception deviennent de plus en plus complexes, tandis que les méthodes de conception doivent être flexibles, adaptées à l'utilisateur et efficace en temps de calcul. Le cadre exposé ici nous semble être un bon candidat pour développer de telles méthodes. Bien que forgé à l'origine pour l'analyse et le contrôle des systèmes, l'hypothèse qui sous-tend ce travail est que ce cadre permet également de relever le défi de la conception. Bien que l'on ne prétende pas que des méthodes de conception efficaces pour tout problème d'ingénierie puissent être développées, notre objectif est de mettre au point une approche générale qui puisse être adaptée par les ingénieurs-concepteurs, et permettant d'améliorer le flux de conception.

## C.3 Filtres électroniques passifs : description et problème de conception

Dans ce chapitre, nous présentons les systèmes à concevoir dans le cadre de ce travail, à savoir les filtres électroniques passifs, le problème de conception qui leur est associé et une revue de la littérature sur les méthodes de conception traditionnelles.

### C.3.1 Filtres électroniques passifs : description et propriétés

Les filtres électroniques passifs sont vus dans ce travail comme l'interconnexion électrique idéale d'éléments passifs. Nous définissons un élément comme un modèle linéaire temps-invariant de dimension finie d'un composant. Comme il est d'usage en

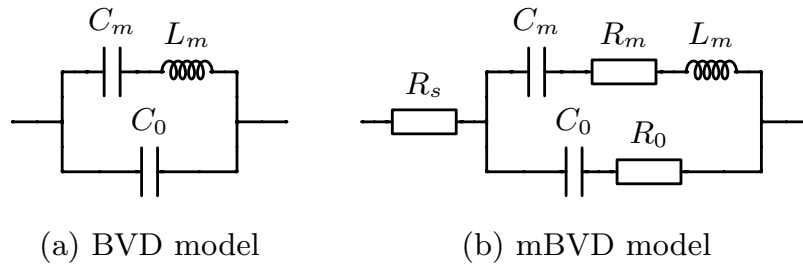


FIGURE C.1: Deux éléments modélisant le résonateur AW

Electronique de modéliser les composants comme l'interconnexion de résistances  $R$ , d'inductances  $L$  et de capacités  $C$ , les éléments seront souvent définis en utilisant des schémas électriques. A titre d'exemple, deux éléments, correspondant aux deux modèles BVD et mBVD du résonateur AW, sont présentés sur la Figure C.1. L'élément de gauche implicitement représente alors l'impédance  $z_{BVD}(s)$  donnée par:

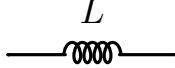
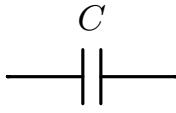
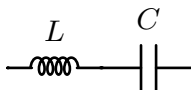
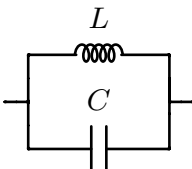
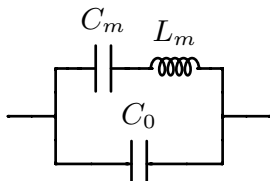
$$z_{BVD}(s) := \frac{1}{C_0 \cdot s} \cdot \frac{s^2 + \omega_r^2}{s^2 + \omega_a^2}$$

avec

$$\omega_r^2 = \frac{1}{L_m C_m} \quad \omega_a^2 = \frac{C_0 + C_m}{L_m C_m C_0} = \omega_r^2 \left(1 + \frac{C_m}{C_0}\right)$$

De plus, la famille d'éléments autorisés est usuellement pré-spécifiée au début du problème de conception. Trois exemples de familles sont présentés sur la Table C.1. Les

TABLE C.1: Exemples de familles d'éléments

Famille	Eléments	
$\mathcal{F}_{LC}$		
$\mathcal{F}_{LC_r}$		
$\mathcal{F}_{AW_r}$		

éléments qui nous intéressent dans ce travail ont la propriété d'être passifs. Intuitivement, un élément est dit passif s'il peut dissiper, emmagasiner ou restituer de l'énergie sans pouvoir restituer plus d'énergie que ce qui lui a été fourni. Plus concrètement, nous utilisons la prochaine définition comme caractérisation de la passivité.

**Définition C.3 (élément passif (sans perte)).**

Soit  $e$  un élément d'impédance  $z_e(s)$ . Supposons que  $z_e(s)$  soit marginalement stable.

$e$  est dit d'être un élément passif si  $z_e(s)$  est tel que

$$\forall s \in \mathbb{C}^+, \quad z_e(s)^* + z_e(s) \geq 0$$

De plus, si  $z_e(s)$  est  $\{0, 1, 0\}$ -dissipatif sans perte,  $e$  est dit passif sans perte.

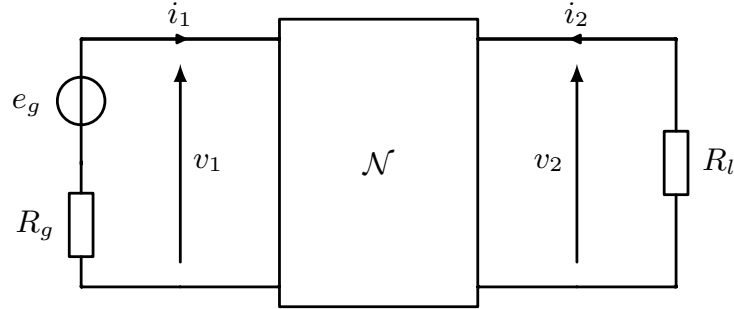


FIGURE C.2: Filtre Electronique  $\mathcal{N}$

Quand des éléments sont interconnectés et placés entre une source de tension résistive ( $e_g, R_g$ ) et une résistance de charge  $R_l$ , ils forment un filtre électronique  $\mathcal{N}$  (Figure C.2). Ce filtre peut alors être décrit par une matrice dite de diffusion  $S$ . C'est sur cette matrice  $S$ , et plus particulièrement son entrée  $s_{21}$ , que sont typiquement exprimés les contraintes du gabarit fréquentiel à satisfaire [You71]. La matrice diffusion  $S$  est définie comme suit :

$$\forall s \in \mathbb{C}, \quad \begin{bmatrix} \beta_1(s) \\ \beta_2(s) \end{bmatrix} = \underbrace{\begin{bmatrix} s_{11}(s) & s_{12}(s) \\ s_{21}(s) & s_{22}(s) \end{bmatrix}}_{S(s)} \begin{bmatrix} \alpha_1(s) \\ \alpha_2(s) \end{bmatrix}$$

avec

$$\begin{aligned} \beta_1(s) &:= \frac{v_1(s) - R_g i_1(s)}{2\sqrt{R_g}} & \alpha_1(s) &:= \frac{v_1(s) + R_g i_1(s)}{2\sqrt{R_g}} \\ \beta_2(s) &:= \frac{v_2(s) - R_l i_2(s)}{2\sqrt{R_l}} & \alpha_2(s) &:= \frac{v_2(s) + R_l i_2(s)}{2\sqrt{R_l}} \end{aligned}$$

Deux propriétés fortes des filtres électroniques passifs, c'est-à-dire composés d'éléments passifs, sont la stabilité et le  $\mathcal{L}_2$ -gain unitaire de sa matrice de répartition  $S$ , peu importe le type d'interconnexion électrique choisie, comme formulée ci-dessous.

**Property C.1.**

*Si  $\mathcal{N}$  est un filtre électronique passif (resp. sans perte)  $\mathcal{N}$ , alors sa matrice de répartition  $S(s)$  est stable et  $\{-I_2, 0, I_2\}$ -dissipative (resp. sans perte).*

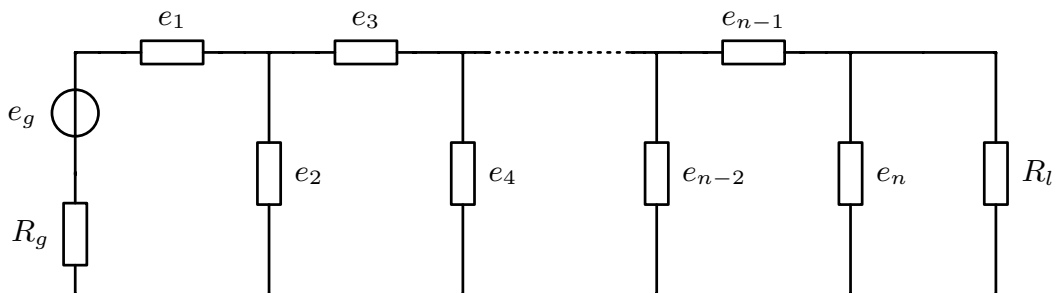
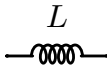
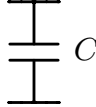
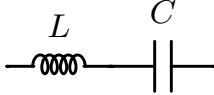
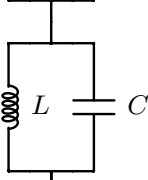
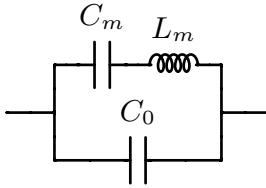
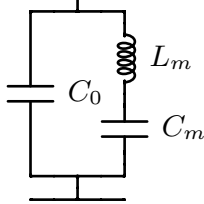


FIGURE C.3: Filtre électronique à topologie échelle

Comme la famille d'éléments, le type d'interconnexion, appelé topologie, est usuellement prédéfinie. Une certaine topologie, appelée topologie échelle et représentée sur la Figure C.3, a un rôle important du fait de son utilisation en pratique et de la simplicité du problème de conception associée. De plus, dans la Table C.2 sont définies trois familles de filtres, associées chacune à une famille d'éléments et à la topologie échelle, qui vont jouer un rôle particulier dans ce manuscrit.

TABLE C.2: Définition de filtres particuliers à topologie échelle

Nom	Eléments séries	Eléments parallèles
<i>LC</i> échelle		
échelle à résonateur- <i>LC</i>		
échelle à résonateur- <i>AW</i>		

### C.3.2 Problème de conception

Maintenant que l'on a défini un filtre électronique passif, nous pouvons formuler son problème de conception. A noter que les gabarits fréquentiels considérés sont uniquement composés de bornes supérieures et inférieures constantes sur un intervalle fréquentiel.

#### Problème C.1 (Conception de filtres électroniques passifs).

ETANT DONNÉS une famille d'éléments passifs  $\mathcal{F}$ ,  
une topologie  $\mathcal{T}$ ,  
une résistance de source de tension  $R_g$  et une résistance de charge  $R_l$ ,  
 $N_U$  bornes supérieures  $U_u$  associées à  $N_U$  intervalles  $\Omega_u^U \subseteq \mathbb{R}$ ,  
 $N_L$  bornes inférieures  $L_l$  associées à  $N_L$  intervalles  $\Omega_l^L \subseteq \mathbb{R}$ ,  
TROUVER s'il existe un filtre électronique, défini comme l'interconnexion d'éléments de  $\mathcal{F}$  dans la topologie  $\mathcal{T}$  entre une source de tension résistive avec résistance de source  $R_g$  et une charge résistive  $R_l$ , tel que son paramètre de diffusion  $s_{21}$  soit stable et satisfasse le gabarit fréquentiel suivant:

$$\begin{aligned} \forall u \in \{1, \dots, N_U\}, \quad \forall \omega \in \Omega_u^U, \quad |s_{21}(j\omega)|^2 &\leq U_u^2 \\ \forall l \in \{1, \dots, N_L\}, \quad \forall \omega \in \Omega_l^L, \quad |s_{21}(j\omega)|^2 &\geq L_l^2 \end{aligned}$$

Si un tel filtre électronique existe, le CALCULER.

### C.3.3 Revue de littérature

Le problème C.1 a deux facettes. La première facette est l'exigence de synthétiser un filtre électronique qui soit l'interconnexion dans une topologie donnée  $\mathcal{T}$  d'éléments passifs appartenant à une famille donnée  $\mathcal{F}$ . La deuxième facette est l'exigence que le filtre résultant ait un paramètre de diffusion stable  $s_{21}$  qui satisfait un gabarit fréquentiel. S'attaquer simultanément aux deux facettes est compliqué. Par conséquent, deux types de méthodes de conception ont rivalisé historiquement.

Le premier type de méthodes de conception s'attaque principalement à la première facette. Cette approche est proche du problème pratique et le filtre résultant est assuré de satisfaire d'être l'interconnexion d'éléments de la famille donnée dans la topologie donnée. Nous l'appellerons l'approche de conception 1, ou encore *synthèse par variation des valeurs des éléments*, car cette approche conduit à synthétiser directement les valeurs caractéristiques des éléments.

Les méthodes de conception du second type s'intéressent davantage à la seconde facette. Ces méthodes ont généralement un faible coût de calcul et le filtre résultant a un nombre d'éléments relativement faible. De même, nous parlerons d'approche de conception 2 ou de l'approche synthèse de filtre réalisable, car elle consiste à synthétiser une fonction de transfert telle qu'elle satisfasse les conditions de réalisation.

Les méthodes systématiques des deux approches ayant successivement fait leurs preuves, en fonction de l'application et de la puissance de calcul disponible, et ayant chacune des avantages qui leur sont propres, elles sont revisitées par la suite à travers notre cadre de conception.

## C.4 Approches de conception des filtres électroniques passifs

### C.4.1 Introduction

L'objectif de ce chapitre est de revisiter les deux approches de conception via notre cadre de notre conception.

La première approche consiste à placer les éléments d'une famille donnée dans une topologie donnée, et à faire varier leurs valeurs caractéristiques jusqu'à ce que le gabarit fréquentiel soit respecté. En pratique, cela est réalisé en représentant le filtre sous la forme d'un schéma électrique via l'interface d'un logiciel CAO. Nous explicitons la représentation mathématique sous-jacente et développons ensuite une méthode de synthèse appropriée. Cette méthode est générique, proche du problème de conception et permet d'inclure certaines contraintes technologiques. La seule hypothèse, qui pourra être supprimée dans un travail ultérieur, est que les éléments soient passifs sans perte.

La seconde approche consiste à synthétiser un paramètre de diffusion de façon à satisfaire un gabarit fréquentiel *et* des contraintes de réalisation. Cette approche est traditionnellement appliquée pour des filtres électroniques particuliers, tels que les filtres  $LC$  et à résonateur  $LC$  en échelle (cf Table C.1). La méthode de conception habituelle est reformulée dans notre cadre. Grâce à cela, une *partie* de cette méthode, à savoir la recherche de contraintes de réalisation, est étendue à des filtres échelles plus généraux. La partie restante sera étendue dans le chapitre suivant.

### C.4.2 Approche de conception 1 : synthèse par variation des valeurs des éléments

Cette approche nécessite tout d'abord d'interconnecter les éléments de la famille donnée dans une topologie donnée. En pratique, cela est fait en représentant le filtre sous la forme d'un schéma électrique grâce à l'interface d'un logiciel de CAO. Les valeurs caractéristiques de ces éléments sont ensuite réglées jusqu'à ce que la fonction de transfert  $s_{21}$  satisfasse le gabarit. Cette seconde partie repose généralement sur l'intuition de concepteurs expérimentés afin de trouver une première estimation des valeurs des éléments, qui est ensuite optimisée à l'aide de solveurs génériques [WL15]. Ceci résulte typiquement en un problème non convexe. L'approche est non systématique, et le temps de conception peut être important, dépendant du choix du point initial. Néanmoins, elle se termine par un filtre réalisable et permet de prendre en compte de nombreuses contraintes pratiques. C'est pourquoi l'objectif de cette section est de revisiter cette approche et de rendre explicite le problème d'optimisation sous-jacent. La motivation est de bénéficier des avantages de l'approche tout en réduisant au minimum le caractère non-convexe du problème d'optimisation.

Lorsqu'un circuit physique est représenté sous la forme d'un schéma électrique par un concepteur, un modèle mathématique est implicitement construit. La représentation d'état semble a priori être appropriée pour fournir un moyen direct de rendre ce modèle explicite. Par un certain choix du vecteur d'état, la dynamique des éléments apparaît naturellement alors que la topologie du filtre intervient à travers des contraintes algébriques sur les matrices de représentation, conduisant à une représentation dite structurée. De plus, ces contraintes sont généralement plus simples que celles sur les coefficients d'une fonction de transfert. Cependant, le problème de la synthèse de représentation d'état structurée a typiquement une formulation non convexe [BT97]. Ces considérations conduisent naturellement aux deux questions :

- Q1. Comment obtenir une représentation d'état à partir d'un schéma électrique ?
- Q2. Comment réduire le caractère non-convexe du problème de conception ? En particulier,
  - a) Parmi le nombre infini de représentations d'état équivalentes, est-ce qu'il en existe une meilleure ?
  - b) Si oui, existe-il une méthode de synthèse adaptée à cette représentation pour réduire au minimum la non-convexité du problème ?

#### C.4.2.1 Sur la représentation PHS DAE

Le cadre des systèmes Hamiltonien à Ports (Port-Hamiltonian System en anglais, d'où l'acronyme PHS) nous semble approprié pour répondre aux questions Q1 et Q2 a) pour trois raisons.

Premièrement, ce cadre fournit une procédure de modélisation systématique à partir d'un schéma électrique. A proprement parler, le résultat de la procédure de modélisation n'est généralement pas une représentation d'état usuelle telle qu'introduite précédemment. En effet, comme cette procédure vise à rester proche des lois de la Physique, des contraintes algébriques entre les variables d'état surviennent couramment et empêchent d'obtenir directement une telle représentation. En conséquence, pour ces cas, la procédure se termine par une extension de la représentation de l'espace d'état connu sous le nom de représentation à équations algébriques-différentielles (DAE pour Differential-Algebraic Equations en anglais). Nous appellerons donc la représentation obtenue représentation PHS DAE. Spécifiée au cas des filtres électroniques

passifs sans perte, la représentation suivante, qui est dite de la forme *descripteur*, de la matrice de répartition  $S$  est obtenue :

$$\begin{cases} p = \left(\frac{1}{s} \cdot I_n\right) q \\ F_1 q = E_1 Q p + H_1 w \\ z = E_2 Q p + H_2 w \end{cases}$$

où  $n$  est le nombre d'inductances et de capacitances,  $w$  et  $z$  sont les signaux d'entrée et de sortie de la matrice de répartition

$$w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} \quad z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}$$

$Q \in \mathbb{R}^{n \times n}$  est une matrice diagonale avec sur sa diagonale l'inverse des valeurs caractéristiques des éléments, et les matrices  $F_1, E_1 \in \mathbb{R}^{n \times n}$ ,  $H_1 \in \mathbb{R}^{n \times 2}$ ,  $E_2 \in \mathbb{R}^{2 \times n}$  et  $H_2 \in \mathbb{R}^{2 \times 2}$  sont telles que:

$$\begin{bmatrix} E_1 \\ E_2 \end{bmatrix} \begin{bmatrix} F_1 \\ 0 \end{bmatrix}^T + \begin{bmatrix} F_1 \\ 0 \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \end{bmatrix}^T + \frac{1}{2} \left( \begin{bmatrix} H_1 \\ H_2 \end{bmatrix} \begin{bmatrix} H_1 \\ H_2 \end{bmatrix}^T - \begin{bmatrix} 0 \\ I \end{bmatrix} \begin{bmatrix} 0 \\ I \end{bmatrix}^T \right) = 0$$

$$\text{rank} \begin{bmatrix} F_1 & E_1 & 0 & H_1 \\ 0 & E_2 & I & H_2 \end{bmatrix} = n + 2$$

Deuxièmement, la représentation qui en résulte est particulièrement simple dans la perspective de la conception. En fait, les variables de conception, c'est-à-dire les valeurs des éléments, apparaissent distinctement, de façon compacte et linéaire comme les entrées de la matrice diagonale  $Q$ . Intuitivement, ces caractéristiques devraient permettre de réduire la complexité du problème de conception.

Troisièmement, les propriétés de stabilité et la  $\{-I_2, 0, I_2\}$ -dissipativité sans perte de la matrice de répartition  $S$  peuvent être assurées d'une façon très simple, en imposant la contrainte de signe  $Q > 0$ .

#### C.4.2.2 Conception par synthèse de représentation PHS DAE

Le but de cette partie est de donner une première réponse à la question Q2 b). Mais, avant cela, il est important de noter que toutes les méthodes de synthèses ne mènent pas à des problèmes d'optimisation de même complexité. Pour qualifier le caractère non-convexe d'un problème, nous utiliserons la règle empirique suivante : plus le problème est *éloigné* d'un problème LMI, plus il est non-convexe. Concrètement, cela se traduira dans un premier temps par réduire l'ordre des produits entre les variables d'optimisation, et dans un second temps par diminuer le nombre de ces produits. Bien sûr, cette règle est empirique et il existe des méthodes afin de reformuler de manière équivalente certains problèmes avec de tels produits en des problèmes d'optimisation LMI (cf par exemple [Sco06]). Cependant, après avoir vérifié que ces méthodes ne s'appliquent pas dans nos cas, cette règle intuitive est généralement vérifiée en pratique.

Pour illustrer ce propos, le lecteur peut remarquer que l'application directe du lemme KYP conduit à un problème avec des produits d'ordre 3 entre les matrices d'optimisation (cf Exemple 4.2, p. 56). De plus, ces produits sont dépendants de chaque contrainte du gabarit. Par conséquent, plus il y a de contraintes dans le gabarit fréquentiel, plus le problème associé à cette méthode est complexe. Par

comparaison, en développant une méthode de synthèse adaptée, nous obtenons le Théorème 4.1 (p. 71) qui amène à résoudre un problème d'optimisation de la forme :

$$\begin{aligned} & \exists \mathbf{Q}^{inv} \in \mathcal{D}(\mathbb{R}^{n \times n}), \\ & \forall u \in \{1, \dots, N_U\}, \exists \mathbf{P}_u = \mathbf{P}_u^T \in \mathbb{R}^{n \times n}, \exists \mathbf{D}_u = \mathbf{D}_u^T \in \mathbb{R}^{n \times n}, \exists \tau_u \in \mathbb{R}, \\ & \forall l \in \{1, \dots, N_L\}, \exists \mathbf{P}_l = \mathbf{P}_l^T \in \mathbb{R}^{n \times n}, \exists \mathbf{D}_l = \mathbf{D}_l^T \in \mathbb{R}^{n \times n}, \exists \tau_l \in \mathbb{R}, \end{aligned}$$

$$\begin{aligned} & \mathbf{Q}^{inv} > 0 \\ & \mathbf{D}_u > 0 \quad \tau_u > 0 \\ & \mathcal{A}_{\Omega_u^U}(\mathbf{D}_u, \mathbf{P}_u, \tau_u) + \begin{bmatrix} \mathbf{Q}^{inv} (F_1^T F_1) \mathbf{Q}^{inv} & \mathcal{A}_U(\mathbf{Q}^{inv}) \\ \mathcal{A}_U(\mathbf{Q}^{inv}) & \mathcal{C}_U \end{bmatrix} \geq 0 \\ & \mathbf{D}_l > 0 \quad \tau_l > 0 \\ & \mathcal{A}_{\Omega_l^L}(\mathbf{D}_l, \mathbf{P}_l, \tau_l) + \begin{bmatrix} \mathbf{Q}^{inv} (F_1^T F_1) \mathbf{Q}^{inv} & \mathcal{A}_L(\mathbf{Q}^{inv}) \\ \mathcal{A}_L(\mathbf{Q}^{inv}) & \mathcal{C}_L \end{bmatrix} \geq 0 \end{aligned}$$

où  $\mathbf{Q}^{inv}$  est l'inverse de la matrice  $\mathbf{Q}$  provenant de la représentation Hamiltonienne à port de la matrice de répartition, i.e.  $\mathbf{Q}^{inv} := \mathbf{Q}^{-1}$ ,  $\mathcal{D}(\mathbb{R}^{n \times n})$  est l'ensemble des matrices diagonales de  $\mathbb{R}^{n \times n}$ ,  $\mathcal{A}_{\Omega_u^U}$ ,  $\mathcal{A}_U$ ,  $\mathcal{A}_{\Omega_l^L}$  et  $\mathcal{A}_L$  sont des fonctions affines à valeur matricielle, et  $\mathcal{C}_U$  et  $\mathcal{C}_L$  sont des matrices constantes.

Ce problème d'optimisation implique des termes matriciels bilinéaires, c'est-à-dire des produits d'ordre 2 entre les matrices d'optimisation, et est communément appelé problème BMI (pour Bilinear Matrix Inequality en anglais). Il est donc moins complexe à résoudre que le problème d'optimisation résultant de l'application directe du lemme KYP. De plus, nous avons fait en sorte que le produit matriciel bilinéaire soit le même pour toutes les contraintes du gabarit, à savoir  $\mathbf{Q}^{inv} (F_1^T F_1) \mathbf{Q}^{inv}$ , ce qui réduit encore plus la complexité du problème. Enfin, il permet de garder les bonnes propriétés de l'approche de conception 1, et permet d'inclure toute considération technologique qui se traduit par des bornes inférieures et/ou supérieures sur la valeur des éléments (voir discussion p. 74).

### C.4.2.3 Conception par synthèse de représentation PHS LFT

L'approche de synthèse développée précédemment a permis d'obtenir un problème d'optimisation dont la source de non-convexité est le terme matriciel bilinéaire  $\mathbf{Q}^{inv} (F_1^T F_1) \mathbf{Q}^{inv}$ .

Du fait du caractère diagonale de  $\mathbf{Q}^{inv}$ , cela implique qu'il y a  $\frac{n(n+1)}{2}$  différents bilinéarités, i.e. termes *scalaires* bilinéaires, dans le pire cas, c'est-à-dire quand  $F_1^T F_1$  est une matrice pleine. Nous nous posons maintenant la question suivante : peut-on réduire ce nombre encore plus ?

On peut déjà remarquer que dans le cas particulier où un filtre électronique admet une représentation d'état PHS *classique*, ce nombre se réduit substantiellement. En effet, la matrice  $F_1$  devient l'identité  $F_1 = I$  et le terme matriciel bilinéaire devient donc  $\mathbf{Q}^{inv} \mathbf{Q}^{inv}$ . Ainsi, la structure diagonale de  $\mathbf{Q}^{inv}$  implique que seulement  $n$



bilinéarités apparaissent :

$$Q^{inv} Q^{inv} = \begin{bmatrix} (q_1^{inv})^2 & & & \\ & (q_2^{inv})^2 & & \\ & & \ddots & \\ & & & (q_n^{inv})^2 \end{bmatrix}$$

Une de nos contributions majeures du Chapitre 4 est de montrer qu'on peut aussi obtenir un tel nombre de bilinéarités pour tous les filtres électroniques passifs sans perte. Pour cela, une représentation équivalente à la représentation PHS DAE est obtenue grâce à l'outil LFT. En effet, nous démontrons notamment dans le Lemme 4.2 (p. 76) qu'une représentation PHS DAE de diffusion est équivalente à une représentation de la forme :

$$\begin{cases} \check{p} = \begin{pmatrix} \frac{1}{s} \cdot I_r & 0 \\ 0 & s \cdot I_{n_{AE}} \end{pmatrix} \cdot \check{q} \\ \check{q} = (J - BB^T) \check{Q} \check{p} + \sqrt{2} B w \\ z = -\sqrt{2} D B^T \check{Q} \check{p} + D w \end{cases}$$

Nous l'appelons représentation PHS LFT. Deux points sont notamment à remarquer. Tout d'abord, la matrice  $\check{Q}$  est directement liée à la matrice  $Q$  et reste diagonale. Elle a sur sa diagonale les valeurs des éléments ou leur inverse. De plus, il n'y a pas de matrices devant le signal  $\check{q}$ . Ces deux points sont ceux qui ont permis d'obtenir un nombre très bas de bilinéarités dans le cas d'une représentation d'état classique PHS et, en adaptant notre méthode de synthèse nous retrouvons un tel nombre pour ce cas plus général par la suite.

Cependant, l'adaptation de notre méthode n'est pas directe. Cette méthode repose notamment sur la caractérisation par des multiplieurs de la relation  $p = \left(\frac{1}{s} \cdot I_n\right) q$  sur un intervalle fréquentiel, et ces multiplieurs sont connus dans la littérature. Par contre, l'originalité de la relation  $\check{p} = \begin{pmatrix} \frac{1}{s} \cdot I_r & 0 \\ 0 & s \cdot I_{n_{AE}} \end{pmatrix} \cdot \check{q}$  fait que, autant que l'auteur sache, il n'existe pas de telles multiplieurs dans cette même littérature. Une autre de nos contributions est donc d'avoir développé de tels multiplieurs, qui ont été fournis dans le Chapitre 2 de ce manuscrit.

Cela nous amène à devoir résoudre le problème d'optimisation de la forme suivante (cf Theorem 4.2, p. 79) :  $\exists \check{Q}^{inv} \in \mathcal{D}(\mathbb{R}^{n \times n})$ ,  
 $\forall u \in \{1, \dots, N_U\}, \exists P_u \in \mathbb{M}, \exists P_{u_{int}} \in \mathbb{M}_{int}, \exists D_u \in \mathbb{M}, \exists \tau_u \in \mathbb{R}$ ,  
 $\forall l \in \{1, \dots, N_L\}, \exists P_l \in \mathbb{M}, \exists P_{l_{int}} \in \mathbb{M}_{int}, \exists D_l \in \mathbb{M}, \exists \tau_l \in \mathbb{R}$ ,

$$\check{Q}^{inv} > 0$$

$$D_u > 0 \quad \tau_u > 0$$

$$\check{A}_{\Omega_u} (D_u, P_u, P_{u_{int}}, \tau_u) + \begin{bmatrix} \check{Q}^{inv} \check{Q}^{inv} & \check{A}_U (\check{Q}^{inv}) \\ \check{A}_U (\check{Q}^{inv}) & \check{C}_U \end{bmatrix} \geq 0$$

$$D_l > 0 \quad \tau_l > 0$$

$$\check{A}_{\Omega_l} (D_l, P_l, P_{l_{int}}, \tau_l) + \begin{bmatrix} \check{Q}^{inv} \check{Q}^{inv} & \check{A}_L (\check{Q}^{inv}) \\ \check{A}_L (\check{Q}^{inv}) & \check{C}_L \end{bmatrix} \geq 0$$

où  $\mathbb{M}$  et  $\mathbb{M}_{int}$  sont des structures de multiplieurs particulières. Grâce à la combinaison des représentations PHS et LFT ainsi que du développement d'une méthode de synthèse appropriée, nous avons donc réussi à donner une condition suffisante pour résoudre notre problème de conception sous la forme d'un problème BMI avec seulement  $n$  bilinéarités, où  $n$  est le nombre de capacités et d'inductances composant le filtre.

#### C.4.2.4 Résolution de problèmes BMI

Après avoir réduit au maximum le caractère non-convexe du problème d'optimisation obtenu, en obtenant un problème BMI avec un nombre remarquablement bas de bilinéarités, nous nous intéressons à le résoudre.

Le classe d'optimisation BMI peut être vu comme l'extension naturelle de l'optimisation LMI. Il s'agit d'un cadre simple mais particulièrement adapté aux problèmes de contrôle et de commande. Il est notamment capable de traiter des spécifications qui ne sont pas accessibles au cadre LMI, telles que la synthèse de régulateur structuré ou à ordre fixé. De nombreux problèmes de synthèse de régulateur linéaire et robuste sont formulés comme des problèmes BMI, tels que la synthèse de régulateur positionné en rétroaction de sortie et d'ordre réduit, de régulateurs décentralisés, la  $\mu$ -synthèse [SHL94; GSL96].

Malheureusement, les problèmes d'optimisation BMI sont généralement non convexes [GSP94], ce qui implique deux barrières théoriques. Premièrement, ils n'ont généralement pas la propriété de minimum locale-globale, c'est-à-dire qu'un minimum local est nécessairement un minimum global. Rigoureusement, la convexité n'est pas une condition nécessaire à la propriété de minimum locale-globale, mais les problèmes BMI sont généralement connus pour avoir de multiples extremums [GTS+94]. Deuxièmement, il n'existe pas d'algorithme efficace, c'est-à-dire en temps polynomial, qui permette de résoudre n'importe quel problème BMI. En effet, on peut même montrer que certains problèmes BMIs sont NP-durs [BT97].

Malgré ces obstacles théoriques, de nombreuses approches de résolution de problèmes BMI ont été développées. Tout d'abord, il est important de noter que certains problèmes BMI spécifiques peuvent être formulés de manière équivalente comme des problèmes LMI, via l'utilisation de certaines techniques standards [Sco06, Sec. 3.6], [BTN01, Chap. 4], [BGFB94]. Malheureusement, ces techniques ne parviennent pas à effectuer une telle reformulation dans notre cas.

Une approche populaire que nous proposons d'utiliser consiste, à partir d'un point initial donné, à résoudre une séquence de relaxations convexes du problème BMI jusqu'à ce qu'une solution satisfaisante soit obtenue [KZM18]. Cette relaxation peut être réalisée de différente manière, et nous proposons d'utiliser l'algorithme fourni dans [KZAM18] car il permet de se ramener, à chaque itération, à un problème de minimisation linéaire sous contraintes LMI. De plus, un avantage important de cet algorithme est sa capacité à garantir la convergence vers l'ensemble de faisabilité du problème BMI original, lorsque le point initial est suffisamment proche de cet ensemble.

Finalement, nous considérons un exemple simple de conception auquel nous appliquons notre approche et cet algorithme (voir sous-Section 4.2.5, p. 84). Ce filtre est un filtre  $LC$ -échelle à deux éléments qui doit satisfaire un gabarit passe-bas. Nous connaissons déjà une solution à ce problème, c'est-à-dire les valeurs de  $L$  et de  $C$  telles que le gabarit soit satisfait. Ces valeurs sont perturbées de plus de 50% et sont données comme point initial à l'algorithme. Ce processus est répété 1000 fois afin d'avoir une idée globale des performances de l'algorithme. Ces performances sont

prometteuses mais doivent être relativisées vis-à-vis de la simplicité de l'exemple. L'algorithme converge dans quasiment tous les cas ( $\approx 93\%$ ) mais pour un nombre moyen d'itération assez élevé ( $> 250$ ). Comme l'exemple est simple, cela se traduit par un temps moyen de calcul faible ( $\approx 5$  s), mais le nombre d'itération laisse songeur pour attaquer des problèmes plus complexes.

En résumé, cet exemple légitime le travail effectué auparavant afin de diminuer au minimum la complexité du problème d'optimisation, mais met en exergue la nécessité de fournir un point initial *suffisamment* proche de l'ensemble de faisabilité. Dans le Chapitre 6, il est proposé une méthode efficace afin de trouver un tel point initial dans le cas de la conception de filtre échelles à résonateurs-*AW*. Cette méthode est principalement fondée sur l'application préalable de l'approche de conception 2, développée dans la prochaine sous-section.

### C.4.3 Approche de conception 2 : synthèse de filtre réalisable

La seconde approche, appelée synthèse de filtre réalisable, consiste à synthétiser le paramètre de diffusion  $s_{21}$  de façon à ce qu'il satisfasse un gabarit fréquentiel donné *et* des contraintes de réalisation. Cette approche est traditionnellement appliquée pour des filtres électroniques particuliers, tels que les filtres échelle *LC* et à résonateur-*LC* (cf Table 3.2, p. 46). Dans la Section 4.3, la méthode de conception habituelle est reformulée dans notre cadre. Sur cette base, une *partie* de cette méthode, à savoir la recherche de contraintes de réalisation, est étendue à des filtres échelle plus généraux. La partie restante est étendue dans le Chapitre 5.

Cette approche est généralement divisée en deux étapes. Tout d'abord, une fonction de transfert  $W$  est synthétisée de telle sorte qu'elle satisfasse un gabarit fréquentiel donné. Cela conduit à un problème d'optimisation convexe qui peut être résolu efficacement. Dans la suite, cette étape est nommée synthèse de filtre fréquentiel. Ensuite, un circuit électronique est construit comme l'interconnexion des éléments d'une famille  $\mathcal{F}$  dans une topologie  $\mathcal{T}$  donnée, de sorte que  $s_{21}(s) = W(s)$ . C'est ce qu'on appelle l'étape de synthèse de circuit. Il n'existe pas forcément un tel circuit pour n'importe quelle fonction de transfert. Des contraintes sur  $W$  telles qu'il existe ce circuit, à savoir des contraintes de réalisation, sont alors nécessaires. Ces contraintes doivent avoir une forme appropriée pour pouvoir être ajoutés de manière convexe dans la première étape.

#### C.4.3.1 Synthèse de filtre fréquentiel

Tout d'abord, le problème de synthèse du filtre fréquentiel est présentée. Dans le Chapitre 5, ce problème est résolu efficacement, par la résolution d'un problème de faisabilité LMI, et optimalement, le filtre résultant étant à ordre minimal pour le gabarit donné. Cependant, certaines méthodes convexes classiques sont détaillées, résultant par exemple en les filtres dits de Butterworth et de Chebyshev. Bien que ces méthodes ne permettent de synthétiser qu'un certain type de filtre, et les filtres résultants n'étant pas optimaux selon les critères qui nous intéressent, elles permettent de réaliser la deuxième étape. Les filtres obtenus satisfassent donc certaines contraintes de réalisation, qui sont détaillées dans la prochaine sous-section.

#### C.4.3.2 Synthèse de circuit

Une fois que le filtre fréquentiel  $W(s)$  a été synthétisé, on voudrait synthétiser un circuit, pré-défini par un certain type d'éléments et une certaine topologie, tel que le paramètre de diffusion résultant  $s_{21}(s)$  soit exactement  $W(s)$ , c'est-à-dire  $s_{21}(s) = W(s)$ .

Étant donné une fonction générale de transfert, il se peut qu'un tel circuit n'existe pas. Des contraintes de réalisations doivent donc être rajoutées dans la synthèse de  $W(s)$ . Ces conditions sont généralement complexes et ne peuvent pas être facilement, ou du moins de manière convexe, inclus dans le processus de synthèse de  $W$ . Une exception notable est constituée par les filtres  $LC$ -échelle pour lesquels les conditions de réalisation sont convexes.

Ces conditions sont simples (voir Corollary 4.4, p. 93) : il suffit que  $W$  soit stable,  $\{-1, 0, 1\}$ -dissipative et telle que

$$|W(j\omega)|^2 = \frac{1}{|g(j\omega)|^2}$$

où  $g(s)$  est un polynôme réel de degré  $n$ . Alors on sait qu'il existe un filtre  $LC$ -échelle à  $n$  éléments tel que  $s_{21}(s) = W(s)$ . De plus, en notant  $g(s) := \sum_{i=0}^n a_i \cdot s^i$ , ce filtre pourra être placé entre une source résistive de résistance interne  $R_g$  et une charge résistive  $R_l$ , si la contrainte  $R_l := \frac{R_g}{a_0^2 \left(1 \pm \sqrt{a_0^2 - 1}\right)^2}$  est satisfaite.

Ces contraintes de réalisation ne concernent que les filtres  $LC$ -échelle. Notre contribution est de les avoir étendues, toujours de façon convexe, à une plus grande famille de filtres que nous appellerons les filtres  $T$ -échelle. Ces filtres sont des filtres échelles dont chaque élément série  $i$  a une admittance proportionnelle à une fonction de transfert  $T$ , i.e.  $y_i(T(s)) = \alpha_i \cdot T(s)$ , et chaque élément parallèle  $k$  a une impédance proportionnelle à ce même  $T$ , i.e.  $z_k(T(s)) = \alpha_k \cdot T(s)$ . On constate facilement qu'en posant  $T(s) = \frac{1}{s}$ , on retombe sur les filtres  $LC$ -échelle classique. De plus, cette famille comprend par exemple les filtres échelle à résonateurs  $LC$ , qui sont des filtres utilisés traditionnellement pour des applications passe-bande, en fixant  $T(s) = \frac{s}{s^2 + \omega_0^2}$ .

Dans le cas où  $T$  est  $\{0, 1, 0\}$ -dissipatif sans perte et telle que  $\forall s \in \mathbb{C}^+$ ,  $T^{-1}(s)^* + T^{-1}(s) > 0$ , nous démontrons (cf Corollary 4.6, p. 101) que les contraintes simples, et convexes, suivantes sont suffisantes pour réaliser des filtres  $T$ -échelle : il suffit que  $W(T^{-1}(s))$  soit stable,  $\{-1, 0, 1\}$ -dissipative et telle que

$$|W(T^{-1}(j\omega))|^2 = \frac{1}{|g(T^{-1}(j\omega))|^2}$$

où  $g(T^{-1})$  est un polynôme réel en  $T^{-1}$  de degré  $n$ . De façon similaire au cas  $LC$ -échelle, des contraintes sur les coefficients de  $g(T^{-1})$  sont donnés pour prendre en compte les valeurs de  $R_g$  et  $R_l$ .

Dans un troisième temps, nous cherchons à étendre ces conditions à une famille encore plus grande de filtre, et si possible qui puisse comprendre les filtres échelles à résonateurs  $AW$ . Pour cela, en utilisant une simplification couramment utilisée en pratique pour les concepteurs de filtres  $AW$  pour simplifier le problème de conception, la famille des filtres  $T_s, T_p$ -échelle est introduite. Cette famille concerne les filtres échelles pour lesquels chaque élément série  $i$  a une impédance proportionnelle à une fonction de transfert  $T_s$ , i.e.  $z_i(s) = \alpha_i \cdot T_s(s)$ , et chaque élément parallèle  $k$  a une impédance proportionnelle à une fonction de transfert  $T_p$ , i.e.  $z_k(s) = \alpha_k \cdot T_p(s)$ . A noter que la famille des filtres  $T$ -échelle est obtenue en imposant  $T_s(s) = T^{-1}(s)$  et  $T_p(s) = T(s)$ .

Malheureusement, nous n'avons pas pu obtenir des conditions convexes de réalisation pour cette famille. Seule une partie des conditions a une formulation convexe et peut donc être prise en compte dans la première étape.

#### C.4.4 Conclusion

Dans ce chapitre, nous avons réexaminé deux approches de la conception des filtres électroniques passifs sans perte.

La première approche (Section 4.2) consiste à mettre les éléments sous une forme prête à l'emploi et à régler les valeurs caractéristiques des éléments. Mathématiquement, cela signifie synthétiser une représentation structurée de l'espace d'état, ce qui conduit à un problème d'optimisation *non-convexe*. S'attaquer directement au problème de conception avec cette approche conduit à un problème d'optimisation fortement *non-convexe*. Afin de réduire le *degré* de non-convexité du problème d'optimisation, une représentation mathématique appartenant au cadre des systèmes Hamiltoniens à Port (PHS), à savoir la représentation PHS DAE, a d'abord été introduite pour modéliser le comportement des filtres électroniques passifs sans perte. La procédure de modélisation, issue du cadre PHS, permet de passer d'un schéma de circuit à une telle représentation. Cette représentation apparaît alors linéaire dans une certaine matrice diagonale  $Q$ , qui a les variables de conception comme entrées diagonales. Afin de tirer parti de cette fonctionnalité, une méthode de synthèse adaptée a ensuite été développée. Elle fournit un moyen de fournir une solution au problème de conception en résolvant un problème non convexe particulier, à savoir un problème BMI. De plus, en couplant cette méthode de synthèse avec des outils supplémentaires de notre cadre de conception, ce problème BMI est transformé en une forme très particulière, réduisant ainsi de manière significative la complexité. Un algorithme séquentiel basé sur des relaxations LMI et nécessitant un point initial a ensuite été fourni, et illustré sur un exemple simple. Cette approche est doublement bénéfique. Tout d'abord, elle est très flexible et permet d'aborder le problème de conception de n'importe quel filtre électronique de manière unifiée, tout en ajoutant facilement de nombreuses contraintes pratiques. Deuxièmement, c'est la même approche que celle utilisée par les concepteurs dans la pratique, c'est-à-dire trouver une première estimation et ensuite régler les valeurs caractéristiques des éléments d'un filtre physiquement implémentable. Elle peut alors être directement incluse dans le flux de conception, et l'expérience de concepteurs chevronnés pour trouver un point initial approprié peut être utilisée. Le principal inconvénient est la nécessité de trouver ce point initial approprié. Ce point initial devrait être *suffisamment proche* d'une solution afin d'assurer la convergence de l'algorithme. C'est généralement une tâche complexe.

La seconde approche (Section 4.3) consiste à synthétiser une fonction de transfert stable telle qu'elle satisfasse un masque spectral *et* les contraintes de réalisation. La synthèse de la fonction de transfert est convexe. Par conséquent, si les contraintes de réalisation ont une formulation appropriée, le problème d'optimisation qui en résulte est convexe et peut être résolu efficacement. C'est le principal avantage de cette approche. Cependant, les conditions de réalisation n'ont généralement pas une telle formulation. Les filtres en échelle  $LC$  sont une exception bien connue. En se basant en particulier sur la propriété de  $\{0, 1, 0\}$ -dissipativité sans perte, nous avons étendu leurs conditions de réalisation *convexe* à une nouvelle famille de filtres électroniques passifs sans perte, à savoir les filtres en  $T$ -échelle. Cette famille contient en particulier les filtres échelle à résonateurs- $LC$  utilisés dans les méthodes de conception traditionnelles pour les applications passe-bande. Néanmoins, si les conditions de réalisation sont convexes, cela nécessite de synthétiser une fonction de transfert  $s_{21}(T(s))$  exprimée sous la forme d'une LFT d'une fonction de transfert répétée  $T(s)$ , ce qui n'est pas possible avec les méthodes de synthèse habituelles. Enfin, en utilisant les pratiques courantes des concepteurs, nous avons illustré comment certains filtres en échelle à résonateur  $AW$  peuvent être considérés comme appartenant à une famille particulière,

à savoir les filtres  $T_s, T_p$ -échelle. Cela ne permet pas d'avoir des conditions de réalisation convexes mais réduit significativement leur complexité, en diminuant le nombre de variables de décision. De même, la synthèse d'une fonction de transfert  $s_{21}(T_s, T_p)$ , en tant que LFT en  $T_s$  et  $T_p$ , est nécessaire.

En résumé, tout en réduisant considérablement la complexité du problème de conception, les deux approches permettent de le résoudre efficacement pour des cas très particuliers. Elles aboutissent toutes deux à une impasse lorsqu'il s'agit de résoudre des problèmes d'utilisation pratique. C'est pourquoi nous considérons un problème intermédiaire dans le Chapitre 5 : la synthèse d'une fonction de transfert avec une représentation LFT telle qu'elle satisfasse un gabarit fréquentiel. Bien que cela ne résolve pas directement le problème de conception, l'objectif est de combler le fossé entre les deux approches afin d'aborder des problèmes d'intérêt pratique (Chapitre 6).

## C.5 Synthèse fréquentielle de filtres LFT

### C.5.1 Introduction

Dans le Chapitre 4, les deux approches développées ne fournissent pas de méthodes efficaces pour la conception de filtres électroniques, sauf dans des cas très particuliers. Ce chapitre se termine en montrant l'intérêt de résoudre un problème intermédiaire, à savoir le problème de synthèse fréquentielle des filtres LFT. Ce problème peut être considéré comme l'extension d'un problème classique, à savoir la synthèse de filtres fréquentiels.

Le filtrage fréquentiel est un des principaux sujets d'intérêt de la Théorie des Systèmes, apparaissant notamment dans le Traitement du Signal, l'Electronique et l'Automatique. Le rôle des filtres fréquentiels dans les systèmes de communication a rendu leur synthèse essentielle pour une grande variété d'applications. La généralité du problème de synthèse de filtre le fait même apparaître dans des problèmes où le résultat final n'est pas un filtre, comme par exemple lors du calcul des poids dans la synthèse  $H_\infty$  pondérée [SP07; RSF03; Zar13].

La synthèse de filtre fréquentiel consiste à synthétiser une fonction de transfert stable, ou de manière équivalente une représentation de d'état, de telle sorte que la réponse en fréquence résultante satisfasse un certain gabarit en module ou en phase<sup>3</sup>. Dans le cadre de ce travail, seules les contraintes de module sont prises en compte. Cependant, on peut noter que certaines propriétés de phase particulières peuvent être obtenues en imposant des contraintes structurelles sur les coefficients de la fonction de transfert [PB87], ou sur les matrices de la représentation d'état. En particulier, les filtres à minimum-phase peuvent être synthétisés avec la méthode développée dans ce chapitre.

À l'origine, les filtres de fréquence étaient analogiques, car ils étaient réalisés avec des inductances et des capacités. Cela a conduit en particulier au développement de méthodes de synthèse analytique, permettant une synthèse rapide, comme celles données dans le Chapitre 4 (Sous-section 4.3.2, p. 88). L'intérêt pour les filtres numériques est apparu dans la seconde moitié du 20<sup>e</sup> siècle, avec l'invention du transistor et l'augmentation de la puissance de calcul disponible. Il existe deux types de filtres numériques, les filtres RIF (pour Réponse Impulsionnelle Finie) et RII (pour Réponse Impulsionnelle Infinie). Les méthodes de synthèse des filtres RII sont traditionnellement basées sur celles des filtres analogiques, par l'utilisation d'approximations telles que la transformation bilinéaire [PB87]. Bien que les filtres

<sup>3</sup>En fait, on sait que le module et la phase d'un système causal et stable ne sont pas indépendants. Voir §6.4 de [ZDG96] et les références qui y figurent pour plus de détails.

FIR puissent être considérés comme des filtres RII particuliers, ces méthodes ne permettent pas de les synthétiser. A la place, des méthodes numériques dédiées ont été développées, basées sur des algorithmes efficaces. La maturation de l'optimisation convexe dans les années 90 a permis de développer de nouvelles méthodes flexibles, et toujours efficaces, pour la synthèse des filtres RIF [Dav10]. Ces méthodes ont ensuite été étendues au début du 21<sup>e</sup> siècle pour synthétiser de façon optimale et efficace les filtres analogiques et RII [RSF03].

En parallèle, l'augmentation de la complexité des systèmes modernes a conduit à des problèmes d'ingénierie difficiles. Lorsque les systèmes sont des sous-systèmes identiques interconnectés, un nouveau paradigme est apparu, connu sous le nom de systèmes multi-agents homogènes. Récemment, cette approche a rencontré un certain succès dans divers domaines d'application tels que le vol en formation [MV09], la micro-électronique [ZKSV13; KSCB16] ou encore les réseaux biologiques [HTI14]. Ces systèmes ont la propriété commune d'être décrits par une LFT en fonction de transfert répétée  $T(s)$ . De nombreux résultats utiles ont ensuite été étendus des systèmes à représentation d'état, c'est-à-dire des systèmes où  $T(s) = \frac{1}{s}$ , aux systèmes LFT homogènes [ZKSV13; HTI14; HIT10]. Cependant, il y a encore des extensions de problèmes standards à résoudre et la synthèse fréquentielle de filtre LFT, c'est-à-dire la synthèse d'un filtre fréquentielle avec une représentation LFT, est l'un d'entre eux.

L'objectif du Chapitre 5 est de proposer une méthode pour résoudre le problème de synthèse fréquentielle des filtres LFT (Section 5.2). Pour ce faire, la méthode de synthèse convexe des filtres analogiques traditionnels ( $T(s) = \frac{1}{s}$ ) est revisitée via le cadre LFT (Section 5.3). Cette méthode est ensuite étendue aux filtres LFT avec un  $T(s)$  plus général (Section 5.4). En utilisant la caractérisation dissipative de  $T(s)$ , nous distinguons deux cas. Le premier cas est celui où il existe une caractérisation de  $\{x, y, z\}$  dissipative sans perte de  $T(s)$ . La nature sans perte de la caractérisation permet d'étendre assez directement l'approche de synthèse habituelle. Le second cas traite avec une caractérisation dissipative générale de  $T(s)$ . Comme l'approche habituelle n'est pas directement étendable, nous proposons une approche généralisée capable de résoudre le problème de synthèse. Enfin, le problème de synthèse des filtres LFT 2D, c'est-à-dire un filtre avec une représentation LFT en deux fonctions de transfert répétées  $T_s(s)$  et  $T_p(s)$ , est considéré (Section 5.5). Plusieurs obstacles théoriques empêchent une généralisation efficace de l'approche convexe usuelle. Cependant, nous développons quand même une approche de synthèse qui fournit un problème d'optimisation LMI permettant de trouver certaines solutions au problème de synthèse de filtres LFT 2D.

### C.5.2 Formulation du problème pour les filtres LFT en un $T(s)$ répété

Le problème de synthèse d'un filtre LFT est formulé ci-dessous.

**Problème C.2** (Synthèse de filtre LFT).

ETANT DONNÉS une fonction de transfert  $T(s)$  SISO et  $\{x, y, z\}$ -dissipative,  
 $N_U$  bornes supérieures  $U_u$  associées à  $N_U$  intervalles  $\Omega_u^U \subseteq \mathbb{R}$ ,  
 $N_L$  bornes inférieures  $L_l$  associées à  $N_L$  intervalles  $\Omega_l^L \subseteq \mathbb{R}$ ,  
TROUVER s'il existe une fonction de transfert stable  $\mathbf{W}(T(s))$ , ayant une  
représentation LFT en une  $T(s)$  répétée, qui satisfait le gabarit  
fréquentiel suivant :

$$\forall u \in \{1, \dots, N_U\}, \quad \forall \omega \in \Omega_u^U, \quad |\mathbf{W}(T(j\omega))|^2 \leq U_u^2$$

$$\forall l \in \{1, \dots, N_L\}, \quad \forall \omega \in \Omega_l^I, \quad |\mathbf{W}(T(j\omega))|^2 \geq L_l^2$$

Si une telle LFT existe, la CALCULER.

Trois principaux problèmes, énumérés ci-dessous, apparaissent lorsqu'on cherche à résoudre ce problème.

1. Premièrement, la variable de décision  $\mathbf{W}(T(s))$  appartient à un espace de dimension infinie. En effet,  $\mathbf{W}(T(s))$  est une fonction rationnelle en  $T(s)$ , et l'espace vectoriel des fonctions rationnelles est de dimension infinie.
2. Deuxièmement, la formulation des contraintes est non convexe en  $\mathbf{W}(T(j\omega))$ . Les bornes supérieures et inférieures peuvent être géométriquement interprétées comme imposant  $W(T(j\omega))$  à appartenir respectivement à un disque ou au complément d'un disque dans le plan complexe. Alors que le premier ensemble est convexe, le second ne l'est pas. Le problème est alors non convexe et il n'y a pas d'algorithme générique de résolution efficace disponible.
3. Troisièmement, le problème d'optimisation est de dimension infinie. Il y a en effet un nombre infini d'inégalités à satisfaire, car les contraintes de bornes supérieures et inférieures dépendent de la variable continue  $\omega$  qui appartient à un intervalle. Ces contraintes ne peuvent donc pas être vérifiées directement en temps fini.

Malgré ces problèmes importants, nous arrivons à développer une approche de synthèse efficace pour résoudre Problème C.2.

### C.5.3 Synthèse de filtre LFT avec $T(s) = \frac{1}{s}$

L'approche convexe de synthèse de filtre fréquentiel, c'est-à-dire Problème C.2 avec  $T(s) = \frac{1}{s}$ , est d'abord revisitée sous le cadre LFT de façon à pouvoir l'étendre à des filtres LFT par la suite. Pour cela,  $\frac{1}{s}$  sera considérée comme une fonction de transfert  $\{0, 1, 0\}$ -dissipative (voir Figure C.4).

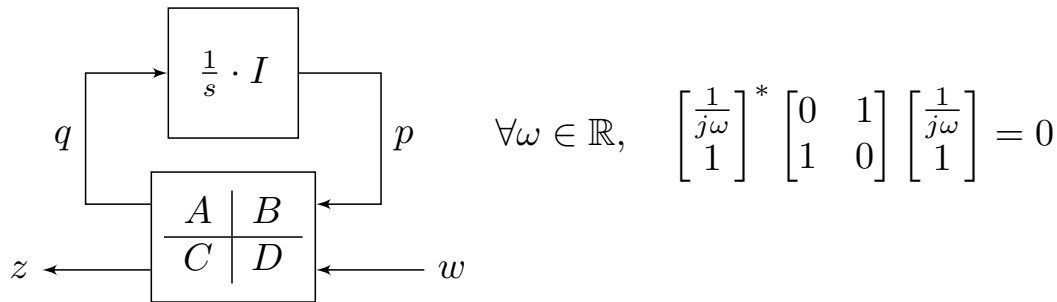


FIGURE C.4: Système avec une représentation LFT en  $\frac{1}{s}$  qui est répété et  $\{0, 1, 0\}$ -dissipatif

#### C.5.3.1 Formulation convexe de dimension finie

En premier lieu, il est nécessaire de reformuler Problème C.2 sous une forme convexe et dimension finie. Pour cela, une paramétrisation de dimension finie de la fonction de transfert  $W(s)$  est tout d'abord obtenue. En effet, au lieu de chercher dans l'espace des fonctions rationnelles, qui est de dimension infinie, l'ordre de  $W(s)$  est fixé à une certaine valeur  $n \in \mathbb{N}^*$  et une paramétrisation est effectuée afin de rechercher dans



l'espace de ses coefficients, qui lui est de dimension finie. Plus concrètement, cela consiste à décomposer  $W(s)$  comme suit :

$$W(s) := \frac{W_N(s)}{W_D(s)} = \frac{\Gamma_N^T \mathcal{B}(s)}{\Gamma_D^T \mathcal{B}(s)}$$

où  $\Gamma_N, \Gamma_D \in \mathbb{R}^{n+1}$  et  $\mathcal{B}(s)$  est le vecteur colonne des éléments d'une base donnée de  $\mathcal{R}_n \mathcal{H}_\infty(s, d(s))$ , l'espace des fonctions rationnelles en  $s$  qui sont propres et d'ordre  $n$  et qui ont  $d(s)$  comme dénominateur<sup>4</sup>, où  $d(s)$  est un polynôme Hurwitz en  $s$  de degré  $n$ . Il vient donc que  $\Gamma_N$  et  $\Gamma_D$  paramètrent toutes les fonction de transfert  $W(s)$  ayant une représentation LFT en  $\frac{1}{s} \cdot I_n$ .

Afin maintenant d'obtenir une formulation convexe, un changement de variable est effectué. Au lieu de synthétiser directement  $W(s)$ , son module au carré  $|W(j\omega)|^2$  est d'abord synthétisé de telle façon qu'il satisfasse le gabarit fréquentiel, puis  $W(s)$  est factorisé de ce module. Nous appellerons la première étape *synthèse du module* tandis que la deuxième est connue sous le nom de *factorisation spectrale*.

En utilisant la caractérisation et le changement de variable précédents, nous obtenons une formulation convexe et de dimension finie de la première étape. Afin d'obtenir le problème d'optimisation de dimension finie, c'est à dire n'ayant plus de contraintes dépendant du la variable  $\omega$ , le lemme KYP et ses généralisations sont utilisés. Un problème de faisabilité LMI est alors obtenu, et peut donc être efficacement résolu. La deuxième étape se ramenera alors à la résolution d'une certaine équation qui, bien que n'ayant pas à proprement parler une formulation convexe, peut être efficacement résolue. Ces deux étapes sont détaillés dans les prochaines sous-sections.

### C.5.3.2 Synthèse du module

Tout d'abord, afin de se ramener au problème de synthèse du module, le changement de variable suivant est effectué :

$$\mathbf{X}_N := \Gamma_N \Gamma_N^T \quad \mathbf{X}_D := \Gamma_D \Gamma_D^T$$

En effet, on peut remarquer alors que :

$$\forall \omega \in \mathbb{R}, \quad |W(j\omega)|^2 = \frac{\mathcal{B}(j\omega)^* \mathbf{X}_N \mathcal{B}(j\omega)}{\mathcal{B}(j\omega)^* \mathbf{X}_D \mathcal{B}(j\omega)}$$

et donc synthétiser  $\mathbf{X}_N$  et  $\mathbf{X}_D$  revient à synthétiser le module au carré  $|W(j\omega)|^2$ . Ce changement de variable est intéressant car il permet de transformer les contraintes du gabarit fréquentiel de Problème C.2 en des inégalités affines, et donc convexes, en  $\mathbf{X}_N$  et  $\mathbf{X}_D$ . Par exemple, une contrainte de borne supérieure devient :

$$\forall \omega \in \Omega_u^U, \quad \mathcal{B}(j\omega)^* (\mathbf{X}_N - U_u^2 \mathbf{X}_D) \mathcal{B}(j\omega) \leq 0$$

L'application d'une des versions du lemme KYP permet alors d'enlever la dépendance de ces contraintes vis-à-vis de  $\omega$ , et donc d'obtenir un problème de faisabilité LMI (voir Théorème 5.1, p. 115), qui peut être efficacement résolu. De plus, il est à noter que le problème de synthèse du module est alors aussi *optimalement* résolu. Par optimal, il est entendu que, étant donné l'ordre  $n$ , s'il existe un module au carré

<sup>4</sup>On aurait pu juste choisir une base des polynômes de degré  $n$ . Cependant, afin d'utiliser plus directement les outils du cadre LFT, et plus généralement de l'Automatique, nous choisissons de diviser le numérateur et le dénominateur de  $W(s)$  par le même  $d(s)$

d'ordre  $2n$  qui satisfait le gabarit fréquentiel, nous pourrions le trouver en résolvant le problème de faisabilité LMI *équivalent*.

### C.5.3.3 Factorisation spectrale

La pertinence du changement variable se limite à cette question : est-il réversible ? C'est à dire, étant donnés  $X_N, X_D \in \mathbb{R}^{n \times n}$ , est-ce qu'il existe  $\Gamma_N, \Gamma_D \in \mathbb{R}^{n \times 1}$  telles que  $\mathbf{W}_N(s) := \Gamma_N^T \mathcal{B}(s)$  et  $\mathbf{W}_D(s) := \Gamma_D^T \mathcal{B}(s)$  soient telles que :

$$|\mathbf{W}_N(j\omega)|^2 = \mathcal{B}(j\omega)^* X_N \mathcal{B}(j\omega) \quad |\mathbf{W}_D(j\omega)|^2 = \mathcal{B}(j\omega)^* X_D \mathcal{B}(j\omega)$$

Ce problème est appelé factorisation spectrale et a été historiquement amplement étudié du fait de ses liens avec beaucoup d'applications en Automatique et Traitement du Signal telles que la prédiction de série temporelle [Wie49], la réalisation de systèmes stochastiques [And69; FCG79] ou encore la synthèse de circuits [AV73]. Il apparaît que, par exemple dans le cas du numérateur, un tel  $\Gamma_N$  existe si et seulement si la condition suivante est respectée [You61; FCG79; ZDG96, chap.13] :

$$\forall \omega \in \mathbb{R}, \quad \mathcal{B}(j\omega)^* X_N \mathcal{B}(j\omega) \geq 0$$

Il suffit donc d'ajouter cette contrainte, et son équivalente pour le dénominateur, au problème de synthèse de module pour s'assurer de l'existence des facteurs spectraux, et donc du bien-posé du changement de variable. Il est à noter que cette contrainte peut être transformée en LMI en appliquant le lemme KYP.

Maintenant qu'un moyen de s'assurer de l'existence des facteurs spectraux, la question de leur calcul se pose. Ceci peut être effectué (voir Théorème 5.2, p. 117) en résolvant une équation algébrique nommée ARE (pour Algebraic Riccati Equation en anglais) qui est de la forme suivante : Trouver  $\mathbf{P}$  tel que

$$A^T(-\mathbf{P}) + (-\mathbf{P})A + C^T X C - ((-\mathbf{P})B + C^T X D) (D^T X D)^{-1} (B^T(-\mathbf{P}) + D^T X C) = 0$$

Au cours de la deuxième moitié du siècle dernier, cette équation est apparue comme omniprésente en Automatique, ce qui a conduit à l'élaboration d'algorithmes spécialisés capables de la résoudre efficacement [BLW91]. L'ARE admet plusieurs solutions, dont chacune est liée à un des facteurs spectraux correspondant à un module au carré. Dans notre cas, une seule solution amène à un facteur spectral stable et doit donc être choisie pour le dénominateur. Pour le numérateur en revanche, il n'y a pas *a priori* de telle contrainte. Le choix de cette même solution implique l'obtention d'un filtre à minimum-phase mais le choix d'une autre solution n'a pas de conséquences sur la résolution de notre problème de filtrage.

### C.5.4 Synthèse de filtre LFT avec $T(s)$ dissipatif

La méthode précédemment décrite est maintenant généralisée étape par étape pour considérer le cas des LFT en  $T(s) \cdot I$  où  $T(s)$  est une fonction de transfert  $\{x, y, z\}$ -dissipative (voir Figure C.5).

La première étape, à savoir l'obtention d'une formulation de dimension finie, est directement étendable. En effet, toute fonction  $W(T(s))$  rationnelle en  $T(s)$  d'ordre  $n$  peut se factoriser sous la forme :

$$W(T(s)) = \frac{W_N(T(s))}{W_D(T(s))} = \frac{\Gamma_N^T \mathcal{B}(T(s))}{\Gamma_D^T \mathcal{B}(T(s))}$$

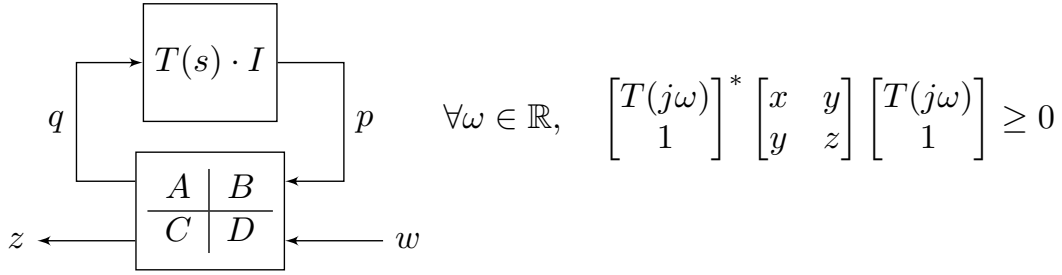


FIGURE C.5: Système avec une représentation LFT en une fonction de transfert  $T(s)$  répétée et  $\{x, y, z\}$ -dissipative

avec  $\Gamma_N, \Gamma_D \in \mathbb{R}^{n+1}$  et  $\mathcal{B}(T(s))$  la concaténation des éléments d'une base de  $\mathcal{R}_n \mathcal{H}_\infty(T(s), d(T(s)))$ .

De plus, les changements de variable  $X_N = \Gamma_N \Gamma_N^T$  et  $X_D = \Gamma_D \Gamma_D^T$  peuvent aussi être réalisés afin d'obtenir une formulation convexe. De façon similaire à précédemment, cela mène à résoudre le problème en deux temps : la synthèse du module puis la factorisation spectrale.

#### C.5.4.1 Synthèse de module de systèmes LFT

Dans un premier temps, la synthèse du module est effectuée, i.e. la synthèse des matrices  $X_N, X_D \in \mathbb{R}^{(n+1) \times (n+1)}$  telles que le module au carré  $\frac{\mathcal{B}(T(j\omega))^* X_N \mathcal{B}(T(j\omega))}{\mathcal{B}(T(j\omega))^* X_D \mathcal{B}(T(j\omega))}$  satisfasse les contraintes fréquentielles. Similairement au cas précédent, l'application du lemme KYP et de ses extensions (cf Appendix A.5, p. 196) nous amène à résoudre un problème de faisabilité LMI (Theorem 5.5, p. 130).

A noter que ce problème LMI est une condition suffisante pour résoudre le problème de synthèse de module, contrairement au cas  $\frac{1}{s}$  où le problème LMI associé était une condition nécessaire et suffisante. Un certain conservatisme apparaît donc ici, dont la source principale est l'utilisation de la caractérisation de  $\{x, y, z\}$ -dissipativité et ses répercussions sur le lemme KYP.

#### C.5.4.2 Factorisation spectrale de systèmes LFT

Dans un second temps, le module est spectralement factorisé, c'est-à-dire on recherche  $\Gamma_N, \Gamma_D \in \mathbb{R}^{(n+1) \times 1}$  telles que :

$$\forall \omega \in \mathbb{R}, \quad \mathcal{B}(T(j\omega))^* \left( \Gamma_N \Gamma_N^T \right) \mathcal{B}(T(j\omega)) = \mathcal{B}(T(j\omega))^* X_N \mathcal{B}(T(j\omega))$$

$$\forall \omega \in \mathbb{R}, \quad \mathcal{B}(T(j\omega))^* \left( \Gamma_D \Gamma_D^T \right) \mathcal{B}(T(j\omega)) = \mathcal{B}(T(j\omega))^* X_D \mathcal{B}(T(j\omega))$$

et telles que filtre résultant  $W(T(s))$  soit stable :

$$W(T(s)) = \frac{W_N(T(s))}{W_D(T(s))} = \frac{\Gamma_N^T \mathcal{B}(T(s))}{\Gamma_D^T \mathcal{B}(T(s))}$$

Pour cela, nous distinguerons deux cas : celui où  $T(s)$  est  $\{x, y, z\}$ -dissipatif sans perte et le cas plus général où  $T(s)$  est simplement  $\{x, y, z\}$ -dissipatif.

**Cas où  $T(s)$  est  $\{x, y, z\}$ -dissipatif sans perte** Pour rappel,  $T(s)$  est  $\{x, y, z\}$ -dissipatif sans perte implique :

$$\forall \omega \in \mathbb{R}, \quad \begin{bmatrix} T(j\omega) \\ 1 \end{bmatrix}^* \begin{bmatrix} x & y \\ y & z \end{bmatrix} \begin{bmatrix} T(j\omega) \\ 1 \end{bmatrix} = 0$$

Dans ce cas, une de nos contributions est de montrer que la méthode du cas  $T(s) = \frac{1}{s}$ , qui peut être interprétée comme étant  $\{0, 1, 0\}$ -dissipatif sans perte, peut s'étendre à tout  $T(s)$   $\{x, y, z\}$ -dissipatif sans perte (Theorem 5.6, p. 132). Cela est notamment possible grâce à une opération dite de *loop-shifting* (voir Lemma 5.1, p. 132, et la remarque le suivant). Une ARE, similaire au cas  $T(s) = \frac{1}{s}$ , doit alors être résolue.

Ainsi, nous avons pu étendre chaque étape, synthèse du module et factorisation spectrale, et nous obtenons donc une méthode efficace pour la synthèse de filtre ayant une représentation LFT en  $T(s) \cdot I_n$ , dans le cas où il existe une caractérisation de  $\{x, y, z\}$ -dissipativité sans perte pour  $T(s)$ .

**Cas où  $T(s)$  est  $\{x, y, z\}$ -dissipatif** Malheureusement, la méthode de factorisation spectrale précédant ne s'étend pas directement à ce cas-là. En effet, en appliquant le même raisonnement que dans le cas sans perte, on obtient, par exemple pour le dénominateur, une factorisation de la forme :

$$\forall \omega \in \mathbb{R}, \quad -E_{P_N}(T(j\omega)) + \mathcal{B}(T(j\omega))^* X_N \mathcal{B}(T(j\omega)) = W_N(T(j\omega))^* W_N(T(j\omega))$$

où  $P_N$  est une solution de l'ARE associée, et

$$E_{P_N}(T(j\omega)) := F(T(j\omega))^* \begin{bmatrix} T(j\omega) \cdot I \\ I \end{bmatrix}^* \begin{bmatrix} xP_N & yP_N \\ yP_N & zP_N \end{bmatrix} \begin{bmatrix} T(j\omega) \cdot I \\ I \end{bmatrix} F(T(j\omega))$$

Par conséquent, dans le cas où  $T(s)$  n'est pas dissipative sans perte, le terme  $E_{P_N}(T(j\omega))$  n'est pas nulle, et une différence apparaît entre le module synthétisé  $\mathcal{B}(T(j\omega))^* X_N \mathcal{B}(T(j\omega))$  et le module de la fonction factorisée  $W_N(T(s))$ . La fonction de transfert résultante  $W(T(s))$  a alors un module au carré de la forme :

$$W(T(j\omega))^* W(T(j\omega)) = \frac{-E_{P_N}(T(j\omega)) + \mathcal{B}(T(j\omega))^* X_N \mathcal{B}(T(j\omega))}{-E_{P_D}(T(j\omega)) + \mathcal{B}(T(j\omega))^* X_D \mathcal{B}(T(j\omega))}$$

Il n'est donc pas garanti que  $W(T(s))$  satisfasse le gabarit fréquentiel.

Les termes  $-E_{P_N}(T(j\omega))$  et  $-E_{P_D}(T(j\omega))$  sont considérées comme des erreurs de factorisation et sont directement liées à la caractérisation dissipative avec perte de  $T(s)$ . Dans [Zar13, Chap.5], l'approche de synthèse est directement étendue, sans tenir compte explicitement de ces erreurs. Dans le cas particulier considéré, ces erreurs ne sont pas trop importantes et le  $W(T(s))$  qui en résulte satisfait toujours le gabarit fréquentiel. Toutefois, de façon plus générale, le gabarit ne sera plus satisfait. C'est pourquoi, dans la suite, nous proposons une approche de synthèse plus générale, capable de s'attaquer au problème de la synthèse de filtre LFT, pour tout  $T(s)$   $\{x, y, z\}$ -dissipatif.

#### C.5.4.3 Approche généralisée avec prise en compte de de l'erreur de factorisation

Comme la technique précédente de factorisation spectrale ne peut pas être directement étendue au cas dissipatif, une généralisation est nécessaire. Notre idée est prendre explicitement en compte les erreurs de factorisation dès l'étape de synthèse de module. Concrètement cela revient à synthétiser  $X_N$ ,  $P_N$ ,  $X_D$ ,  $P_D$  tel que

$$W(T(j\omega))^* W(T(j\omega)) = \frac{-E_{P_N}(T(j\omega)) + \mathcal{B}(T(j\omega))^* X_N \mathcal{B}(T(j\omega))}{-E_{P_D}(T(j\omega)) + \mathcal{B}(T(j\omega))^* X_D \mathcal{B}(T(j\omega))}$$

satisfasse le gabarit fréquentiel. Comme  $P_N$  et  $P_D$  doivent aussi satisfaire une ARE en parallèle, notre idée peut être vue comme la fusion des deux étapes, la synthèse du module et la factorisation spectrale, en une seule et unique étape. Notre contribution est de montrer que cela est possible (voir Theorem 5.7, p. 135) et qu'on obtient donc une généralisation de l'approche à toute fonction de transfert  $T(s)$   $\{x, y, z\}$ -dissipative.

Cependant, le problème d'optimisation obtenu n'est pas convexe. En effet, la fusion des deux étapes revient à résoudre simultanément un problème de faisabilité LMI et deux AREs. Bien qu'il existe des algorithmes efficaces de résolution de l'ARE, du fait de son importance en Automatique et théorie des Systèmes, l'ensemble des solutions de l'ARE n'est pas convexe<sup>5</sup> et donc le problème global ne l'est pas non plus.

Pour surmonter ce problème, nous proposons d'utiliser un résultat de la littérature qui permet d'obtenir une solution particulière de l'ARE par la résolution d'un problème de minimisation d'un coût linéaire sous contrainte LMI. Cela est notamment possible via le lien particulier qu'il existe entre les solutions de l'ARE et l'inégalité algébrique de Riccati (ARI pour Algebraic Riccati Inequality en anglais), détaillé dans la Sous-Section 5.3.3.1 (p. 120). A noter que dans notre cas, les solutions particulières obtenues ne permettront de synthétiser que des facteurs spectraux à minimum-phase, c'est-à-dire d'inverse stable.

Ainsi, en fusionnant les deux étapes et en utilisant le lien particulier entre l'ARE et l'ARI, nous obtenons une condition suffisante sous la forme d'un problème de minimisation d'un coût linéaire sous contraintes LMI. Une de nos contributions majeurs est donc d'avoir développé une méthode efficace pour la résolution du problème de synthèse de filtres LFT.

### C.5.5 Sur l'extension aux filtres LFT 2D

Nous avons maintenant l'intention d'étendre l'approche précédente à la synthèse de filtres LFT 2D. Dans ce manuscrit, un système LFT 2D est défini comme un système ayant la représentation LFT suivante :

$$\begin{cases} p = \Delta(T_s(s), T_p(s))q \\ q = Ap + Bw \\ z = Cp + Dw \end{cases}$$

où  $T_s(s)$  et  $T_p(s)$  sont deux fonctions de transfert stable et

$$\Delta(T_s(s), T_p(s)) := \begin{pmatrix} T_s(s) \cdot I_{n_{T_s}} & 0 \\ 0 & T_p(s) \cdot I_{n_{T_p}} \end{pmatrix}$$

La fonction de transfert résultante  $W(T_s(s), T_p(s))$  peut être calculée en utilisant l'opérateur LFT :

$$W(T_s(s), T_p(s)) = \Delta(T_s(s), T_p(s)) \star \left[ \begin{array}{c|c} A & B \\ \hline C & D \end{array} \right]$$

---

<sup>5</sup>car discret

De plus, comme c'est le cas qui nous intéresse dans ce travail,  $T_s(s)$  et  $T_p(s)$  sont considérées comme étant  $\{0, 1, 0\}$ -dissipatives sans perte :

$$\forall \omega \in \mathbb{R}, \quad \begin{bmatrix} T_s(j\omega) \\ 1 \end{bmatrix}^* \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} T_s(j\omega) \\ 1 \end{bmatrix} = 0 \quad \begin{bmatrix} T_p(j\omega) \\ 1 \end{bmatrix}^* \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} T_p(j\omega) \\ 1 \end{bmatrix} = 0$$

A noter que, désormais, par souci de clarté, la notation suivante est utilisée pour toute fonction  $f(T_s(s), T_p(s))$  :

$$f_{T_s T_p}(s) := f(T_s(s), T_p(s))$$

Le problème de synthèse de filtres LFT 2D est alors formulé comme suit.

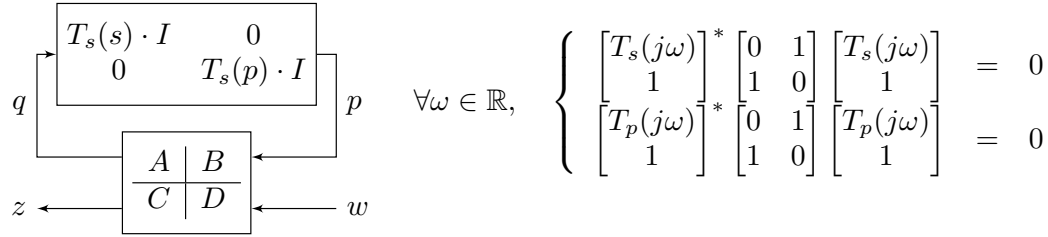


FIGURE C.6: Système LFT 2D en  $T_s(s)$  et  $T_p(s)$   $\{0, 1, 0\}$ -dissipatives sans perte

### Problème C.3 (Synthèse de filtre LFT 2D).

ETANT DONNÉS deux fonctions de transfert  $T_s(s)$ ,  $T_p(s)$   $\{0, 1, 0\}$ -dissipatives,  $N_U$  bornes supérieures  $U_u$  associées à  $N_U$  intervalles  $\Omega_u^U \subseteq \mathbb{R}$ ,  $N_L$  bornes inférieures  $L_l$  associées à  $N_L$  intervalles  $\Omega_l^L \subseteq \mathbb{R}$ ,  
TROUVER s'il existe une fonction de transfert  $\mathbf{W}(T_s, T_p)$  stable, ayant une représentation LFT 2D en  $T_s(s)$ ,  $T_p(s)$ , telle qu'elle satisfasse le gabarit fréquentiel suivante :

$$\begin{aligned} \forall u \in \{1, \dots, N_U\}, \quad \forall \omega \in \Omega_u^U, \quad |\mathbf{W}_{T_s T_p}(j\omega)|^2 &\leq U_u^2 \\ \forall l \in \{1, \dots, N_L\}, \quad \forall \omega \in \Omega_l^L, \quad |\mathbf{W}_{T_s T_p}(j\omega)|^2 &\geq L_l^2 \end{aligned}$$

Si oui, la CALCULER.

Pour autant que nous sachions, le Problème C.3 n'a pas encore été abordé dans la littérature. Le travail le plus proche est la conception de filtres numériques 2D à phase linéaire [Dum07; RDV07]. La phase linéaire exacte permet de concevoir directement l'amplitude du filtre, qui est décomposée en base de polynômes trigonométriques. Cependant, une telle approche ne peut pas être utilisée ici, car la phase linéaire exacte n'est pas réalisable pour les filtres analogiques.

Nous cherchons donc à étendre la méthode de synthèse que nous avons développée pour les filtres LFT aux filtres LFT 2D. Cependant, deux problèmes fondamentaux empêchent une extension directe. Premièrement, la correspondance biunivoque entre les systèmes LFT 2D et les fonctions rationnelles 2D n'est pas possible, contrairement au cas 1D. Par conséquent, la décomposition dans une base de polynômes 2D introduit du conservatisme et ne pas va être équivalent d'une base à une autre. Deuxièmement, il peut exister des fonctions rationnelles 2D qui satisfont le gabarit fréquentiel, mais n'admettent pas de facteurs spectraux. En effet, en 2D, il y a une séparation entre les fonctions rationnelles positives et les facteurs spectraux (voir sous-Section 5.5 pour plus de détails), qui n'existait pas en 1D. Ce problème est notamment relié à

un problème ouvert connu sous le nom de 17<sup>e</sup> problème de Hilbert (voir par exemple [Pfi76] et la discussion dans [PW02, §9.2]). En outre, même si un facteur spectral peut exister, la méthode de l'ARE des dernières sections peut ne pas être en mesure de le calculer (voir sous-Section 5.5.4). En conséquence, la factorisation spectrale des systèmes 2D reste un problème ouvert. Quelques algorithmes ont été proposées, voir par exemple [Bas00; AT08], mais ils conduisent généralement à des facteurs spectraux non minimaux.

En raison de ces problèmes fondamentaux, nous ne prétendons pas résoudre le Problème C.3. Nous proposons plutôt une adaptation de notre approche de synthèse qui apporte une solution au Problème C.3, si elle en trouve une.

Pour cela, après avoir obtenu une formulation convexe de dimension finie mais conservatrice (cf sous-Section 5.5.1, p. 143), nous cherchons à étendre chaque étape, synthèse du module et factorisation spectrale.

Comme dans le cas de LFT en un  $T(s)$  dissipatif, nous montrons (Theorem 5.8, p. 146) que l'extension de la première étape se fait aisément, malgré l'introduction d'un certain conservatisme, grâce à l'utilisation du lemme KYP adapté. Cependant, l'étape de factorisation spectrale nécessite de résoudre une ARE dite *structurée* (voir sous-Section 5.5.3, p. 148), car la solution recherchée  $P$  de l'ARE ne doit plus être une matrice pleine mais une matrice bloc-diagonale, correspondant à la répétition des deux fonctions de transfert  $T_s(s)$  et  $T_p(s)$ . En plus de ne pas pouvoir s'assurer de l'existence d'une telle solution, nous ne disposons pas d'algorithme efficace pour calculer une telle solution. Il est donc proposé de rechercher une matrice pleine comme solution de l'ARE, qui peut donc être calculée efficacement. Cependant, il en découle une erreur lors de la factorisation spectrale. De façon similaire à la méthode générale pour la synthèse de LFT en  $T(s)$  dissipative, nous proposons une méthode qui fusionne les deux étapes et prend en compte l'erreur de factorisation dès le début (Theorem 5.9). Une fois encore, cela est possible grâce au lien particulier entre solutions de l'ARE et de l'ARI.

Malgré un conservatisme inhérent, cette approche nous permet donc d'obtenir efficacement des solutions à un problème qui reste encore largement ouvert.

### C.5.6 Résumé

Dans ce chapitre, nous avons revisité l'approche convexe de synthèse de filtres fréquentiels dans le cadre LFT. Tout d'abord, une paramétrisation en dimension finie de toutes les fonctions de transfert  $W(s)$  est faite, en utilisant le lien biunivoque entre les fonctions de transfert et les fonctions rationnelles. Ensuite, une formulation convexe est obtenue grâce à un changement de variable approprié. Cela conduit typiquement à résoudre deux sous-problèmes, à savoir la synthèse de module et la factorisation spectrale.

Dans le cas classique où les filtres ont une représentation d'état, c'est-à-dire lorsque le filtre admet une représentation LFT en  $\frac{1}{s} \cdot I$ , le problème de synthèse est résolu de manière optimale et efficace. En effet, il existe une paramétrisation finie équivalente de toutes les fonctions de transfert appropriées, le problème de synthèse de magnitude est équivalent à un problème de faisabilité LMI, et la factorisation spectrale est réalisée en résolvant deux ARE. Ces ARE admettent des solutions, et il existe des algorithmes efficaces pour les résoudre.

Notre première contribution est d'avoir directement étendu cette approche aux fonctions de transfert  $W(T(s))$  exprimées sous forme de LFT en  $T(s) \cdot I$ , où  $T(s)$

est une fonction de transfert  $\{x, y, z\}$ -dissipative sans perte. Chaque étape est directement généralisable et ne nécessitant que des manipulations algébriques sur les matrices LFT.

De plus, nous avons montré que cette approche ne permet pas d'aborder le cas plus général où  $T(s)$  est simplement  $\{x, y, z\}$ -dissipatif. En fait, la technique usuelle de factorisation spectrale introduit une erreur de factorisation, et on ne peut pas certifier que le filtre résultant  $W(T(s))$  satisfait le masque spectral. Notre deuxième contribution est d'avoir développé une approche de synthèse généralisée afin de faire face à ce dernier problème. Cela nécessite de fusionner les sous-problèmes de synthèse de module et de factorisation spectrale en un seul, ce qui conduit à résoudre un problème de minimisation linéaire sous contraintes LMI.

De plus, le problème de synthèse des filtres LFT 2D, c'est-à-dire des filtres  $W(T_s, T_p)$  exprimés sous forme de LFT en  $\begin{pmatrix} T_s \cdot I & 0 \\ 0 & T_p \cdot I \end{pmatrix}$ , est abordé lorsque  $T_s$  et  $T_p$  sont  $\{0, 1, 0\}$ -dissipatifs sans perte, comme c'est le cas de l'intérêt de ce travail. L'extension de l'approche habituelle conduit à une approche conservative, due à la fois par la paramétrisation en dimension finie et par le changement de variable. Après avoir détaillé les sources de ce conservatisme, celles-ci semblent inhérentes à cette approche. Néanmoins, un problème de minimisation linéaire sur les contraintes LMI est donné qui, si une solution est obtenue, permet de trouver des solutions au problème de synthèse des filtres LFT 2D, ce qui conduit à notre troisième contribution.

Toutes ces méthodes de synthèse efficaces peuvent être incluses dans l'approche de conception 2 du Chapitre 4. La méthode donnée dans le Chapitre 5.3, quand  $T(s) = \frac{1}{s}$ , couplée aux contraintes de réalisation convexes du Chapitre 4.3.3.1, conduit à une méthode efficace de conception de filtres échelle *LC*. De même, l'extension donnée dans la Section 5.4, lorsque  $T(s)$  est dissipatif sans perte, peut être couplée aux contraintes de réalisation convexes de la Section 4.3.3.2 pour la conception de certains filtres en échelle à résonateur *LC*. Ceci sera illustré dans le Chapitre 6. De même, il sera également illustré comment la partie convexe des contraintes de réalisation de certains filtres *AW-ladder* (cf Sous-section 4.3.3.3), peut être incluse dans l'approche de synthèse des filtres LFT 2D de la Section 5.5.

Cependant, ces méthodes peuvent faire encore mieux. En effet, tel qu'illustrée dans l'introduction, l'approche de synthèse pour les filtres LFT dans un  $T(s)$  dissipatif répété, qu'il soit sans perte ou non, permet de considérer l'interconnexion d'éléments passifs, et non pas uniquement des éléments passifs sans perte. De plus, elle permet même de synthétiser l'interconnexion d'éléments dissipatifs, c'est-à-dire d'éléments ayant une impédance qui satisfait une caractérisation dissipative plus générale. Il reste à trouver des contraintes de réalisation, associées à ces éléments et à une topologie, avec une formulation convexe. Néanmoins, même si une partie seulement de ces contraintes de réalisation ont une formulation convexe et peuvent alors être considérées, le résultat peut fournir un point de départ pour une version adaptée de l'approche de conception 1 du Chapitre 4. Cette idée est illustrée dans le Chapitre 6 pour la conception de filtres en échelle *AW-résonateur*.

## C.6 Exemples de conception

Dans ce dernier chapitre, les méthodes développées dans ce travail sont appliquées sur deux exemples de conception de filtres électroniques passifs.



### C.6.1 Exemple de conception d'un filtre échelle à résonateur-LC

Dans ce premier exemple, l'approche synthèse de filtre réalisable, i.e. l'approche de conception 2 du Chapitre 3, est appliquée à la conception d'un filtre échelle à résonateur-LC et est comparée à l'approche traditionnelle menant aux filtres de Butterworth et de Chebyshev 1. Pour cela, les résistances de source et de charge sont prises égales  $R_g = R_l = R$ , avec  $R = 50\Omega$ , et le gabarit fréquentiel à satisfaire est donné par :

$$\begin{aligned} \forall \omega \in [0, \omega_{U_1}], & \quad |s_{21}(j\omega)|^2 \leq U_1^2 \\ \forall \omega \in [\omega_{L_1}, \omega_{L_2}], & \quad |s_{21}(j\omega)|^2 \geq L^2 \\ \forall \omega \in [\omega_{U_2}, +\infty), & \quad |s_{21}(j\omega)|^2 \leq U_2^2 \end{aligned}$$

avec  $\omega_{U_1} := 1.6\pi$  rad/s,  $U_1|_{dB} := 20$ ,  $\omega_{L_1} := 2\pi$  rad/s,  $\omega_{L_2} := 3\pi$  rad/s,  $L|_{dB} := 0.28$  et  $\omega_{U_2} := 4.4\pi$  rad/s,  $U_2|_{dB} := 12$ , où la notation suivante est utilisée :

$$K|_{dB} := -20 \log_{10}(K)$$

Dans cet exemple, les valeurs caractéristiques  $(L_k, C_k)$  de chaque élément  $k$  sont contraintes d'être telles que :

$$\frac{1}{L_k C_k} = \omega_0^2$$

avec  $\omega_0^2 := \omega_{L_1} \omega_{L_2}$ . Cela permet de transformer le filtre échelle à résonateur-LC en un élément de la classe des filtres  $T$ -échelle, où ici  $T$  est donné par :

$$T(s) = \frac{s}{s^2 + \omega_0^2}$$

On peut alors utiliser les contraintes de réalisation convexes obtenues dans le Chapitre 3 (Corollary 4.6, p. 101). Afin de simplifier les calculs, la valeur des résistances  $R$  est tout d'abord normalisée à  $1 \Omega$ . Elle sera ensuite mise à  $50 \Omega$  en utilisant des formules de mises à l'échelle d'impédance.

#### C.6.1.1 Synthèse d'un $W(T(s))$ réalisable

Une fonction de transfert  $W(T(s))$ , représentée par une LFT en  $T(s)$  répétée, est tout d'abord synthétisée de façon à être stable, à satisfaire le gabarit fréquentiel donné et les contraintes de réalisation des filtres  $T$ -échelle. Pour cela, la méthode de synthèse de filtres LFT avec un  $T(s)$  dissipatif sans perte, développée dans le Chapitre 5, est appliquée. Celle-ci consiste en deux étapes indépendantes.

Premièrement, le module au carré  $|W(T(j\omega))|^2$  est synthétisé de façon à satisfaire le gabarit fréquentiel et les contraintes de réalisation, menant à la résolution d'un problème de faisabilité LMI. Afin de trouver l'ordre minimal  $n$  telle qu'il existe un tel  $|W(T(j\omega))|^2$ , nous appliquons un algorithme itératif qui, partant de  $n = 1$ , incrémente l'ordre  $n$  à chaque itération et résout le problème LMI associé (Algorithm 6.1, p.161). Appliquée à notre exemple, cet algorithme renvoie l'ordre minimal  $n = 4$ .

Deuxièmement, ce module est factorisé spectralement pour extraire  $W(T(s))$ . Cela mène à résoudre une ARE, qui peut être efficacement résolue. On obtient alors

le filtre stable  $W(T(s))$  suivant :

$$\begin{aligned}
 W(T(s)) &= (T(s) \cdot I_n) \star \left[ \begin{array}{cc|cc} A_W & B_W & & \\ \hline C_W & D_W & & \end{array} \right] \\
 &= (T(s) \cdot I_n) \star \left[ \begin{array}{cccc|c} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -83.17 & -66.727 & -26.65 & -6.082 & -83.17 \\ \hline 1 & 0 & 0 & 0 & 0 \end{array} \right] \\
 &= \frac{-83.17 \cdot T(s)^4}{1 + 6.082 \cdot T(s) + 26.65 \cdot T(s)^2 + 66.73 \cdot T(s)^3 + 83.17 \cdot T(s)^4}
 \end{aligned}$$

On peut observer sur la Figure C.7 (courbe bleue pleine), que le module au carré  $|W(T(j\omega))|^2$  de  $W(T(s))$  satisfait bien le gabarit fréquentiel.

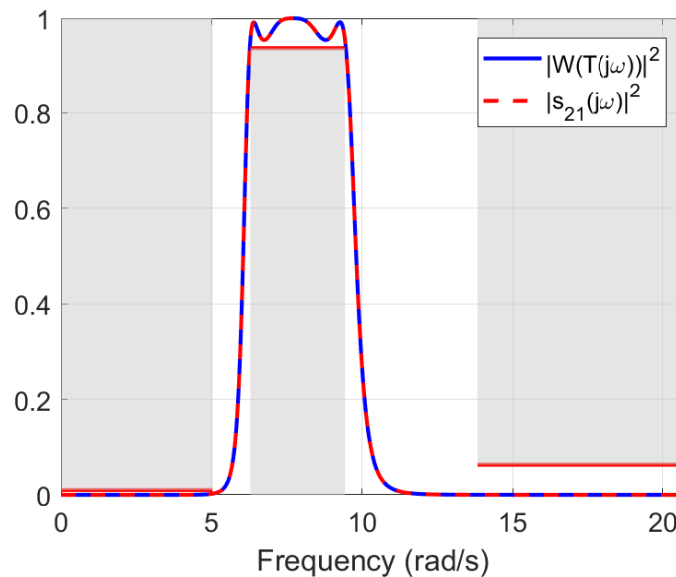


FIGURE C.7: Tracé de  $|W(T(j\omega))|^2$  et de  $|s_{21}(j\omega)|^2$  du filtre échelle à résonateur-LC synthétisé

### C.6.1.2 Extraction de la valeur des éléments

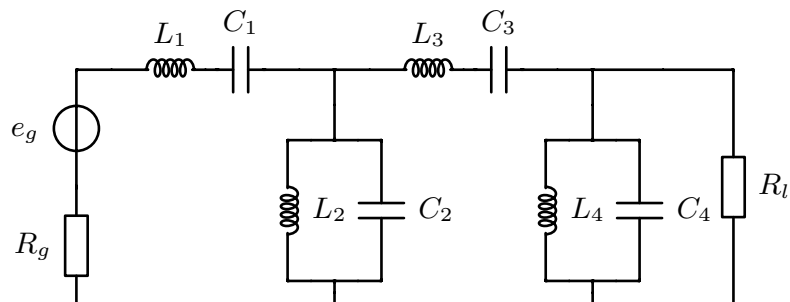


FIGURE C.8: Un filtre échelle à 4 résonateurs LC

Maintenant qu'une fonction de transfert réalisable  $W(T(s))$  a été synthétisée, il faut la réaliser sous la forme d'un filtre échelle avec  $n = 4$  résonateurs- $LC$  de telle sorte que  $s_{21}(T(s)) = W(T(s))$ . Pour cela, un algorithme de la littérature, proposé dans [NHDD96, Chap. 6] et redécouvert dans [SEN11], est adapté et utilisé (Algorithm 6.2, p. 166). Il en résulte le filtre électronique de la Figure C.8 avec les valeurs des éléments données dans la Table C.3, et pour rappel  $R_g = R_l = R = 1\Omega$ . Il est aisé de vérifier que  $\forall i \in \{1, 2, 3, 4\}$ ,  $L_i C_i = \frac{1}{\omega_0^2}$ .

TABLE C.3: Valeurs caractéristiques des éléments pour  $R_g = R_l = 1\Omega$

Element	1	2	3	4
Inductance $L$ (mH)	44.27	460.4	35.64	289.0
Capacitance $C$ (mF)	381.5	36.68	473.9	58.43

Finalement, les formules classiques de mise à l'échelle d'impédance [Bah84, Chap. 10] sont appliquées afin d'obtenir les nouvelles valeurs des éléments, dénotées par  $L_R$  et  $C_R$ , correspondant  $R = 50\Omega$ . Ces formules sont simples et sont données par :

$$L_R = R \cdot L \quad C_R = \frac{C}{R} \quad (\text{C.1})$$

On obtient alors la Table C.4

TABLE C.4: Valeurs caractéristiques des éléments pour  $R_g = R_l = R = 50\Omega$

Element	1	2	3	4
Inductance $L_R$ (H)	2.214	23.01	1.782	14.45
Capacitance $C_R$ (mF)	7.629	0.7336	9.477	1.169

On peut alors vérifier sur la Figure C.7 (courbe rouge en pointillé) que le paramètre  $s_{21}(s)$  résultant est identique à  $W(T(s))$  et satisfait donc le gabarit fréquentiel. Le problème de conception est donc résolu.

### C.6.1.3 Comparaison avec les filtres Butterworth et Chebyshev I

La méthode classique menant aux filtres de Butterworth et de Chebyshev I est maintenant appliquée (voir Subsection 6.1.3, p. 168, pour plus de détails). Tout d'abord, les filtres  $W_{Butter}(s)$  et  $W_{Cheby}(s)$  sont synthétisés en utilisant les méthodes d'approximation dédiées. Il en résulte  $n_{Butter} = 5$  et  $n_{Cheby} = 4$ . Ces filtres sont ensuite réalisés sous la forme de filtre échelle à résonateur- $LC$ . Cela peut être aisément fait pour le filtre de Butterworth, en utilisant par exemple le même algorithme que précédemment (Algorithm 6.2, p. 166), mais le filtre de  $W_{Cheby}(s)$  ne satisfait pas *toutes* les contraintes de réalisation. En effet, la contrainte d'avoir  $R_g = R_l$  n'est pas satisfaite dans ce cas. Trois options sont alors possibles. Premièrement, on peut modifier la résistance de charge afin d'obtenir  $W_{Cheby}(s)$  qui satisfait toutes les contraintes de réalisation. Cependant, cela ne résout pas le problème de la conception, où  $R_g$  et  $R_l$  sont pré-spécifiées, et on peut avoir des problèmes dits de désadaptation d'impédance (cf discussion de la page 44). Deuxièmement, un filtre échelle avec  $n_{Cheby} = 4$  résonateurs- $LC$  peut être synthétisé en utilisant l'algorithme précédent. Cependant, il y aura une différence entre le  $s_{21_{Cheby}}$  résultant et  $W_{Cheby}(s)$ , ce qui

peut impliquée que  $s_{21_{Cheby}}$  ne satisfasse pas les contraintes du gabarit fréquentiel, comme c'est le cas dans notre exemple (voir Figure C.9). Troisièmement, on peut augmenter l'ordre à  $n_{Cheby} = 5$ , car les filtres de Chebyshev d'ordre impair satisfont structurellement les contraintes de réalisation [Bah84]. Le filtre résultant sera alors un filtre échelle à  $n_{Cheby} = 5$  résonateur-LC et résoudra notre problème de conception.

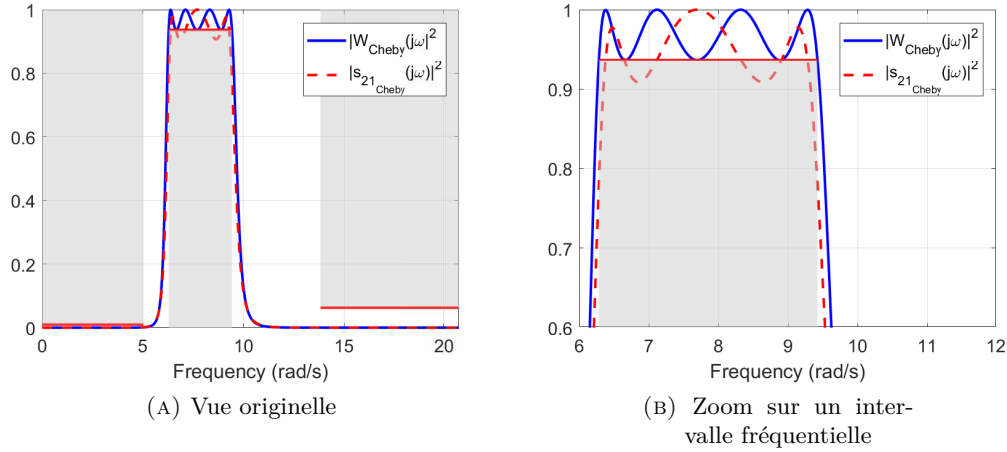


FIGURE C.9: Différence typique apparaissant entre  $|W_{Cheby}(j\omega)|^2$  et  $|s_{21_{Cheby}}(j\omega)|^2$  pour des filtres de Chebyshev d'ordre pair

Par conséquent, les filtres de Butterworth et de Chebyshev nécessiteront un résonateur-LC supplémentaire par rapport à notre approche. Bien que cela ne semble pas significatif dans cet exemple de conception relativement simple, la propriété de notre approche de fournir des filtres d'ordre minimal, ou de façon équivalente un nombre minimal d'éléments, peut être un facteur déterminant pour les applications de conception plus complexes.

### C.6.2 Exemple de conception d'un filtre échelle à résonateur-AW

Dans ce second exemple, on considère la conception d'un filtre échelle à résonateur-AW. Pour rappeler, un résonateur AW  $k$  a une impédance donnée par :

$$z_k(s) = \frac{1}{C_{0_k} \cdot s} \cdot \frac{s^2 + \omega_{r_k}^2}{s^2 + \omega_{a_k}^2}$$

où  $\omega_{r_k}$  et  $\omega_{a_k}$  sont respectivement appelés fréquences de résonance et d'anti-résonance et sont liées aux valeurs caractéristiques des éléments ( $C_{0_k}, L_{m_k}, C_{m_k}$ ) par :

$$\omega_{r_k}^2 = \frac{1}{L_{m_k} C_{m_k}} \quad \omega_{a_k}^2 = \frac{C_{0_k} + C_{m_k}}{L_{m_k} C_{m_k} C_{0_k}} = \omega_{r_k}^2 \left(1 + \frac{C_{m_k}}{C_{0_k}}\right)$$

Pour cet exemple, les résistances de charge et de source sont prises unitaires  $R_g = R_l = R = 1\Omega$  et le gabarit fréquentiel est donné par :

$$\begin{aligned} \forall \omega \in [0, \omega_{U_1}], \quad & |s_{21}(j\omega)|^2 \leq U_1^2 \\ \forall \omega \in [\omega_{L_1}, \omega_{L_2}], \quad & |s_{21}(j\omega)|^2 \geq L^2 \\ \forall \omega \in [\omega_{U_2}, +\infty), \quad & |s_{21}(j\omega)|^2 \leq U_2^2 \end{aligned}$$

où  $\omega_{U_1} := 1.0$  rad/s,  $U_1 := 0.5$ ,  $\omega_{L_1} := 1.5$  rad/s,  $\omega_{L_2} := 2.5$  rad/s,  $L := 0.85$  et  $\omega_{U_2} := 3.2$  rad/s,  $U_2 := 0.4$ .

Afin de résoudre ce problème, les deux approches de conception du Chapitre 4 sont combinées. L'idée sous-jacente est d'obtenir un point initial suffisamment bon, requis pour diminuer le temps d'exécution de l'approche de conception 1, en appliquant l'approche de conception 2. Cette dernière est efficace mais ne peut que considérer certaines contraintes de réalisation. Nous ajoutons donc une étape afin de relier ici les deux approches.

**Préliminaire** Le point de départ de l'approche proposée est un filtre  $T_s, T_p$ -échelle. D'après le Chapitre 4 (Subsection 4.3.3.3, p. 102) un filtre échelle à résonateur  $AW$  peut être transformé en un filtre  $T_s, T_p$ -échelle en imposant les fréquences de résonance et d'anti-résonance des résonateurs séries et parallèles d'être de la sorte :

$$\begin{aligned} \omega_{r_1} = \omega_{r_3} = \dots = \omega_0 & & \omega_{a_1} = \omega_{a_3} = \dots = \omega_{a_s} \\ \omega_{r_2} = \omega_{r_4} = \dots = \omega_{r_p} & & \omega_{a_2} = \omega_{a_4} = \dots = \omega_0 \end{aligned}$$

Ces contraintes sont souvent utilisées en pratique, car les valeurs  $\omega_0$ ,  $\omega_{a_s}$  et  $\omega_{r_p}$  peuvent être reliées aux zéros et aux pôles des paramètres de diffusion  $s_{11}$  et  $s_{21}$ , ce qui permet de simplifier le problème de conception. En effet,  $j\omega_0$  est un zéro dit de réflexion, c'est-à-dire tel que  $s_{11}(j\omega_0) = 0$  et donc  $s_{21}(j\omega_0) = 1$ , tandis que  $j\omega_{a_s}$  et  $j\omega_{r_p}$  sont des zéros dits de transmission, c'est-à-dire tels que  $s_{21}(j\omega_{a_s}) = s_{21}(j\omega_{r_p}) = 0$ . Pour cet exemple, nous choisissons arbitrairement ces valeurs comme suivent :

$$\omega_{r_p} = 1.00 \text{ rad/s} \quad \omega_0 = 1.75 \text{ rad/s} \quad \omega_{a_s} = 3.0625 \text{ rad/s}$$

Les fonctions de transfert  $T_s(s)$  et  $T_p(s)$  associées au filtre  $T_s, T_p$ -échelle sont finalement définies par :

$$T_s(s) := \frac{s^2 + \omega_0^2}{s(s^2 + \omega_{a_s}^2)} \quad T_p(s) := \frac{s^2 + \omega_{r_p}^2}{s(s^2 + \omega_0^2)}$$

On peut facilement vérifier que  $T_s$  et  $T_p$  sont toutes deux  $\{0, 1, 0\}$ -dissipatives sans perte.

### C.6.2.1 Synthèse de filtre LFT 2D avec certaines contraintes de réalisation

La première étape consiste à synthétiser un filtre LFT 2D  $W(T_s, T_p)$  tel qu'il satisfasse certaines contraintes de réalisation des filtres  $T_s, T_p$ -échelle, celles qui sont convexes, et respecte le gabarit fréquentiel, en utilisant la méthode développée dans la Section 5.5 du Chapitre 5.

De façon similaire à l'exemple précédant, un algorithme itératif est appliqué lors de la synthèse du module pour avoir un filtre  $W(T_s, T_p)$  d'ordre minimal (cf Algorithme 6.3, p. 173). Nous trouvons un ordre minimal  $n = 3$  avec les ordres partiels  $n_{T_s} = 2$  et  $n_{T_p} = 1$ . Après factorisation spectrale du module, le filtre LFT 2D suivant

est obtenu :

$$\begin{aligned}
 W(T_s, T_p) &= (T(s) \cdot I_n) \star \left[ \begin{array}{c|c} A_W & B_W \\ \hline C_W & D_W \end{array} \right] \\
 &= \begin{pmatrix} T_s \cdot I_2 & 0 \\ 0 & T_p \cdot I_1 \end{pmatrix} \star \left[ \begin{array}{ccc|c} -0.6734 & -0.9915 & -1.2356 & 1 \\ -0.3468 & -2.9831 & -2.4711 & 2 \\ 0.3266 & -1.9915 & -2.2356 & 1 \\ \hline 0 & 0 & 2.2356 & 0 \end{array} \right] \\
 &= \frac{2.2356 \cdot T_p}{1 + 3.6566 \cdot T_s + 2.2356 \cdot T_p + 3.6566 \cdot T_s T_p + 1.6649 \cdot T_s^2 + 1.6649 \cdot T_s^2 T_p}
 \end{aligned}$$

On peut vérifier aisément sur la Figure C.10, que le filtre  $W(T_s, T_p)$  satisfait bien le gabarit fréquentiel.

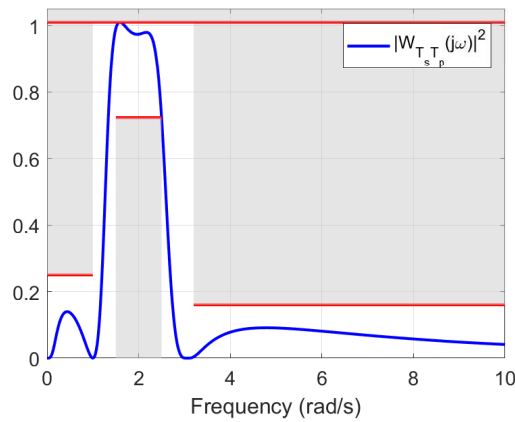


FIGURE C.10: Tracé du filtre LFT 2D  $|W_{T_s T_p}(j\omega)|^2$  qui satisfait la partie convexe des contraintes de réalisation

### C.6.2.2 Approximation par un filtre $T_s, T_p$ -échelle

Bien que le filtre synthétisé  $W(T_s, T_p)$  respecte le gabarit fréquentiel et satisfasse la partie convexe des contraintes de réalisation des filtres  $T_s, T_p$ -échelle, il ne les satisfait pas toutes et on ne peut pas donc réaliser un filtre  $T_s, T_p$ -échelle équivalent tel que  $s_{21}(T_s, T_p) = W(T_s, T_p)$ . Il est donc proposé de synthétiser un filtre  $T_s, T_p$ -échelle de façon à ce que le module au carré de son paramètre de diffusion  $s_{21}$  approxime au mieux  $|W(T_s(j\omega), T_p(j\omega))|^2$ . Nous appellerons cette approximation  $s_{21}^{\{0\}}(T_s, T_p)$  et nous l'utiliserons comme point initial pour la première approche.

Pour réaliser cette approximation, nous proposons de structurer  $s_{21}^{\{0\}}(T_s, T_p)$  et de minimiser l'erreur relative donnée ci-dessous :

$$\left\| 1 - \frac{W(T_s, T_p)}{s_{21}^{\{0\}}(T_s, T_p)} \right\|_{\infty}^2$$

Du fait de la structure particulière de  $s_{21}^{\{0\}}(T_s, T_p)$  et par application du lemme de Schur (Lemma A.3, p. 195), on obtient un problème de minimisation d'un coût linéaire sous contraintes LMI (voir Subsection 6.2.2, p. 177 pour plus de détails). On obtient

alors la fonction suivante :

$$s_{21}^{\{0\}}(T_s, T_p) = \frac{2q_2 \cdot T_p}{1 + (q_1 + q_3) \cdot T_s + 2q_2 \cdot T_p + q_1 q_3 \cdot T_s^2 + q_2(q_1 + q_3) \cdot T_s T_p}$$

où

$$q_1 = 2.7928 \quad q_2 = 1.1178 \quad q_3 = 0.5823$$

Sur la Figure C.11,  $|s_{21}^{\{0\}}(j\omega)|^2$  est représenté par la courbe rouge en pointillés et comparé à  $|W_{T_s T_p}(j\omega)|^2$ , représentée par la courbe bleue pleine. Comme on peut l'observer, le paramètre de diffusion de cette approximation  $T_s, T_p$ -échelle ne satisfait pas complètement le gabarit fréquentiel. Cependant, il semble être suffisamment proche d'une solution pour être utilisé comme point de départ pour l'approche de conception 1.

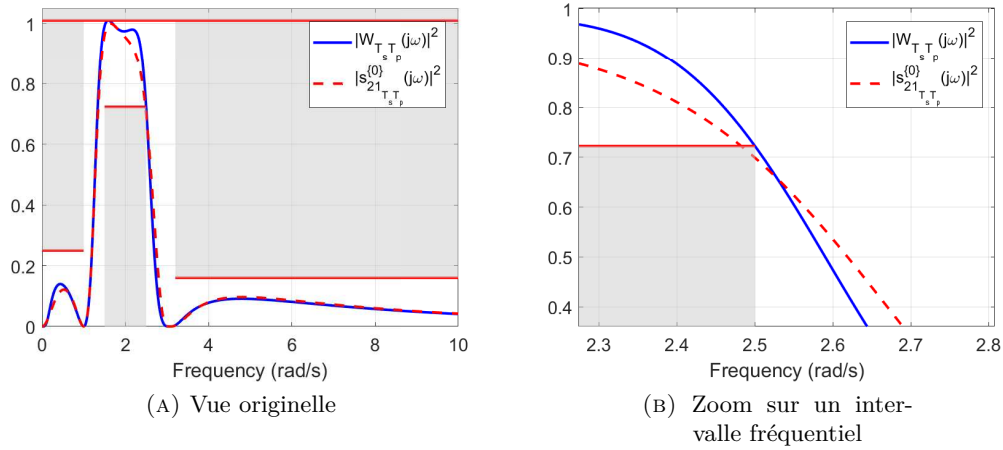


FIGURE C.11: Tracé du paramètre de diffusion  $|s_{21}^{\{0\}}(j\omega)|^2$  de du filtre  $T_s, T_p$ -échelle approximant  $|W_{T_s T_p}(j\omega)|^2$

Dans cette optique, le filtre échelle à résonateur-AW de la Figure C.12 est réalisé avec les valeurs données dans la Table C.5.

TABLE C.5: Valeurs caractéristiques des éléments de l'approximation par un filtre  $T_s, T_p$ -échelle

Element	1	2	3
$C_{0_k}^{\{0\}}$ (F)	0.3581	0.8946	1.7174
$L_{m_k}^{\{0\}}$ (H)	0.4561	0.5914	0.0951
$C_{m_k}^{\{0\}}$ (F)	0.6767	1.6908	3.2458

### C.6.2.3 Synthèse de représentation PHS

Maintenant qu'un filtre échelle à résonateur-AW a été synthétisé et qu'il ne satisfait pas le gabarit fréquentiel, mais en est proche, l'objectif est de le fournir comme point initial à l'approche de conception 1 du Chapitre 4. Pour ce faire, on obtient d'abord une représentation PHS de la matrice de diffusion du filtre en utilisant la procédure de modélisation donnée dans la Sous-Section 4.2.2.2 (p. 61). Cela conduit

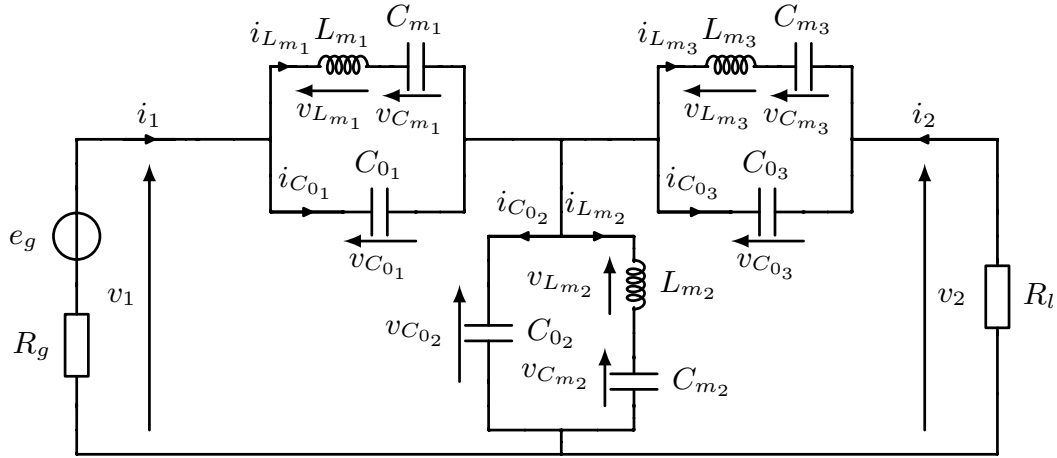


FIGURE C.12: Un filtre échelle à trois résonateurs AW

à la représentation suivante :

$$\begin{cases} p = \left(\frac{1}{s} \cdot I_n\right) q \\ q = (J - BB^T) Qp + \sqrt{2}Bw \\ z = -\sqrt{2}DB^T Qp + Dw \end{cases}$$

avec  $n = 9$  et

$$J := \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \quad B := \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & -1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \quad D := \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$Q := \begin{bmatrix} Q_1 & 0 & 0 \\ 0 & Q_2 & 0 \\ 0 & 0 & Q_3 \end{bmatrix} \quad \text{avec} \quad Q_k := \begin{bmatrix} \frac{1}{C_{0k}} & 0 & 0 \\ 0 & \frac{1}{L_{mk}} & 0 \\ 0 & 0 & \frac{1}{C_{mk}} \end{bmatrix}$$

De plus, la matrice  $\check{Q}_{init}^{inv}$  est définie similairement, en utilisant les valeurs de la Table C.5 comme suit :

$$\check{Q}_{init}^{inv} := \begin{bmatrix} \check{Q}_{1_{init}}^{inv} & 0 & 0 \\ 0 & \check{Q}_{2_{init}}^{inv} & 0 \\ 0 & 0 & \check{Q}_{3_{init}}^{inv} \end{bmatrix} \quad \text{avec} \quad \check{Q}_{k_{init}}^{inv} := \begin{bmatrix} C_{0k}^{\{0\}} & 0 & 0 \\ 0 & L_{mk}^{\{0\}} & 0 \\ 0 & 0 & C_{mk}^{\{0\}} \end{bmatrix}$$

L'algorithme de synthèse de l'approche de conception 1 (Algorithm 4.1, p. 84) est ensuite exécuté en utilisant  $\check{Q}_{init}^{inv}$  comme point initial et  $maxRound = 500$ . Après 26 itérations pour un temps de calcul de 321.6 s, l'algorithme a convergé vers une solution. Les valeurs des éléments du filtre échelle à résonateur AW sont données dans la Table C.6, tandis que le module au carré du paramètre de diffusion  $s_{21}$



associé est représenté sur la Figure C.13, où l'on peut voir que les contraintes du gabarit fréquentiel sont satisfaites.

Par conséquent, la combinaison des deux approches 1 et 2 a permis de résoudre le problème de conception d'un filtre échelle à résonateur-AW sur cet exemple.

TABLE C.6: Valeurs caractéristiques des éléments composants le filtre échelle à résonateurs-AW final

Element	1	2	3
$C_{0_k}^{sol}$ (F)	0.3571	0.8934	1.7174
$L_{m_k}^{sol}$ (H)	0.4535	0.5906	0.0985
$C_{m_k}^{sol}$ (F)	0.6756	1.6908	3.2459

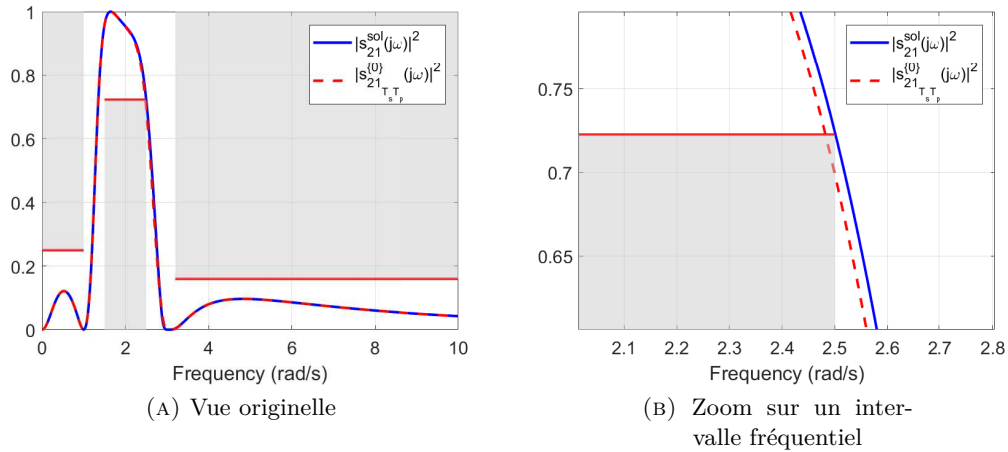


FIGURE C.13: Tracé du paramètre de diffusion  $|s_{21}^{sol}(j\omega)|^2$  du filtre échelle à résonateur-AW final

## C.7 Conclusions et perspectives

### C.7.1 Conclusions

Le but de ce travail était de développer des méthodes systématiques et efficaces pour la conception de filtres électroniques passifs. Pour y parvenir, il a été proposé d'utiliser une approche moderne systémique. Concrètement, cela impliquait d'établir un cadre de conception basé sur des outils avancés de la théorie des Systèmes et l'Automatique, tels que la TFL et la dissipativité, pour représenter et caractériser les éléments électroniques et leur interconnexion, afin d'ouvrir la voie à l'optimisation convexe (Chapitre 2).

Sur la base d'une revue avec une perspective historique des méthodes de conception, deux approches de conception se sont distinguées (Chapitre 3). Il a ensuite été proposé de revoir ces approches dans notre cadre de conception (Chapitre 4).

La première approche de conception, que nous avons appelée synthèse par variation des valeurs des éléments, est très flexible et généralement préférée par les concepteurs dans la pratique, car elle est visuelle, proche du problème de conception et permet d'inclure certaines contraintes technologiques. Elle consiste essentiellement à placer les éléments dans une topologie donnée, au moyen de l'interface graphique

d'un logiciel de CAO, et à accorder leurs valeurs caractéristiques jusqu'à ce que les exigences de fréquence soient satisfaites, en utilisant un solveur d'optimisation. En plus d'avoir illustré comment une approche de synthèse directe peut impliquer un problème d'optimisation de grande complexité, et de l'avoir rendu explicite, notre contribution est d'avoir obtenu un problème d'optimisation de complexité réduite. Bien qu'encore non convexe, car étant un problème BMI, ce problème d'optimisation est apparu de complexité modérée, grâce à l'utilisation d'une représentation combinant les caractéristiques des représentations PHS et LFT. Un algorithme, basé sur des relaxations séquentielles LMI, a ensuite été fourni. Cependant, afin de diminuer le nombre d'itérations, puis le temps de calcul, il faut un point initial relativement proche de l'ensemble de faisabilité du problème BMI original.

La deuxième approche de conception, appelée synthèse de filtre réalisable dans ce travail, est historiquement liée aux filtres échelle  $LC$  et aux méthodes analytiques traditionnelles de du type Butterworth et Chebyshev. Elle consiste à synthétiser un filtre fréquentiel, c'est-à-dire une fonction de transfert remplissant un certain gabarit sur le module de sa réponse fréquentielle, de telle sorte qu'elle satisfasse certaines contraintes de réalisation. S'il existe des méthodes efficaces pour la synthèse de filtres fréquents, la difficulté de l'approche réside dans la recherche de contraintes de réalisation sans altérer l'efficacité de ces méthodes, c'est-à-dire qui peuvent être ajoutées de manière convexe. Après avoir présenté des contraintes de réalisation convexes pour les filtres en échelle  $LC$ , une autre de nos contributions est de les avoir étendues de manière convexe pour une famille de filtres échelle plus large, que nous avons appelée filtres  $T$ -échelle. Cette famille comprend en particulier les filtres échelle  $LC$  et certains filtres échelle à résonateur- $LC$  qui sont typiquement utilisés pour des applications passe-bande. Sur la base de l'introduction de pratiques courantes utilisées par les concepteurs pour les filtres échelle à résonateur- $AW$ , une idée similaire nous a conduit à introduire une famille plus large, à savoir les filtres  $T_s, T_p$ -échelle. Bien que les contraintes de réalisation qui en résultent soient plus simples que pour les filtres généraux échelle à résonateur- $AW$ , seule une partie d'entre eux s'est avérée convexe.

Cela nous a conduit à nous attaquer au problème de synthèse fréquentiel des filtres LFT (Chapitre 5). Ce problème peut être considéré comme une formulation généralisée du problème classique de synthèse des filtres fréquents. L'approche efficace standard pour résoudre ce dernier problème est typiquement faite de deux étapes : l'étape de synthèse de module et l'étape de factorisation spectrale. Nous avons alors réexaminé cette approche à deux étapes dans notre cadre, en considérant une fonction de transfert comme une LFT dans la fonction de transfert répétée  $\frac{1}{s}$ , qui est considérée comme étant  $\{0, 1, 0\}$ -dissipative sans perte. En étendant chaque étape, nous avons obtenu une approche de synthèse efficace pour les fonctions de transfert exprimées sous forme de LFT dans une fonction de transfert répétée  $T(s)$  qui est  $\{x, y, z\}$ -dissipative sans perte. Bien que les idées principales soient venues de [Zar13], notre contribution est d'avoir détaillé et complété leur approche. De plus, il a été montré que la factorisation spectrale ne peut pas être étendue à un  $T(s)$   $\{x, y, z\}$ -dissipatif, car une erreur de factorisation apparaît et est inhérente à l'approche. Par conséquent, une des principales contributions de cette thèse est d'avoir généralisé cette approche de synthèse en fusionnant les étapes de synthèse de magnitude et de factorisation spectrale en une seule étape, ce qui est réalisé en résolvant un problème de minimisation d'un coût linéaire sous contraintes LMI. Ceci a été rendu possible en considérant l'erreur de factorisation spectrale dès le départ. Enfin, nous avons clarifié les questions fondamentales qui se posent lorsque l'on tente d'aborder le cas de LFT dans deux fonctions de transfert  $T_s(s)$  et  $T_p(s)$ . Néanmoins, nous avons élaboré une approche,

également fondée sur la résolution d'un problème de minimisation linéaire sous contraintes LMI, lorsque  $T_s(s)$  et  $T_p(s)$  sont toutes deux  $\{0, 1, 0\}$ -dissipatives sans perte, comme c'est notre cas d'intérêt. Bien qu'elle soit d'un conservatisme inhérent, cette approche tend à trouver des solutions dans la pratique.

L'ensemble de des méthodes développées permet d'obtenir des méthodes de conception de filtres électroniques passifs. Pour illustrer cela, deux exemples de conception ont été détaillés (Chapitre 6). Tout d'abord, une méthode de conception systématique et efficace pour les filtres  $T$ -échelle, avec  $T(s)$   $\{0, 1, 0\}$ -dissipative sans perte, a été obtenue en couplant les contraintes de réalisation associées, qui sont convexes, et la méthode de synthèse des filtres exprimés en LFT dans les  $T(s)$  répétés. En particulier, cela comprend la conception de filtres échelle  $LC$  et à résonateur- $LC$ . De plus, cette méthode tend à donner des résultats optimaux, en ce sens que le nombre d'éléments qui en résulte est minimal. Deuxièmement, une méthode de conception a été développée pour les filtres  $T_s, T_p$ -échelle, où  $T_s(s)$  et  $T_p(s)$  sont deux fonctions de transfert  $\{0, 1, 0\}$ -dissipatives sans perte, basées sur trois étapes. La première étape a consisté à coupler la partie convexe des contraintes de réalisation associées avec la méthode de synthèse des filtres de fréquence exprimés par LFT en  $T_s(s)$  et  $T_p(s)$ . La solution résultante a ensuite été approchée par le paramètre de diffusion  $s_{21}$  d'un filtre  $T_s, T_p$ -échelle dans la deuxième étape. Ceci a conduit à un point initial qui a alimenté, dans la troisième étape, l'algorithme donné pour l'approche de conception 1. Strictement, nous ne pouvons affirmer qu'une méthode de conception systématique et efficace a été complètement obtenue pour les filtres échelle à résonateur- $AW$ . Cependant, la partie à systématiser est clairement identifiée. De plus, les résultats obtenus sur l'exemple de conception sont prometteurs et la méthode développée mérite d'être appliquée à des problèmes de conception avec des contraintes fréquentielles qui se posent en pratique.

## C.7.2 Perspectives

Dans notre quête pour développer des méthodes de conception systématiques et efficaces, plusieurs problèmes se sont posés. Certains ont été résolus, d'autres partiellement, d'autres encore ont été laissés de côté. Dans la suite, afin d'adapter, de compléter ou d'améliorer nos méthodes pour résoudre les problèmes restants, des travaux supplémentaires sont proposés. Il a été spécialement essayé de rassembler ces points en fonction de l'intérêt présumé du lecteur. Ensuite, certaines nouvelles pistes de recherche liées à ce sujet sont fournies. Cela conduit essentiellement à la nécessité de nouvelles collaborations entre les communautés issues de l'Electronique, de la théorie des Systèmes et de l'Automatique, afin de s'attaquer aux problèmes actuels et futurs posés par la conception des filtres de Radiofréquences.

### C.7.2.1 Travaux futurs

#### Sur les multiplieurs et leur conservatisme

Les multiplieurs ont un impact significatif sur l'efficacité de nos méthodes de conception. En effet, comme discutée dans le Chapitre 2, une paramétrisation affine  $\mathbb{M}_{ap} \subseteq \mathbb{M}$  d'un certain ensemble  $\mathbb{M}$ , découlant des critères de conception, est réalisé pour obtenir des contraintes LMI. Cette paramétrisation, résultant essentiellement de la recherche de multiplicateurs, doit être la plus fine possible afin d'éviter tout conservatisme.

Intuitivement, nous pensons que la paramétrisation affine donnée au Chapitre 2 pour le cas d'un  $T(s)$   $\{x, y, z\}$ -dissipatif sans perte, est exacte, c'est-à-dire  $\mathbb{M}_{ap} = \mathbb{M}$ .

En utilisant les conditions données dans [Din05; IMF00b], il serait intéressant de prouver ou de réfuter cette intuition. Si cela est vrai, cela permettrait aux énoncés du lemme KYP dissipatif sans perte (Théorème A.10, p. 196) d'être équivalents. Par conséquent, cela prouverait également que l'approche de synthèse de filtre réalisable est optimale, c'est-à-dire conduisant au nombre minimum d'éléments, pour la conception de filtres  $T$ -échelle lorsque  $T(s)$  est  $\{0, 1, 0\}$ -dissipatif sans perte.

De même, nous avons aussi l'intuition que les multiplieurs donnés dans le Chapitre 2 pour des LFTs en  $\begin{bmatrix} s \cdot I_{n_1} & 0 \\ 0 & \frac{1}{s} \cdot I_{n_2} \end{bmatrix}$  mène aussi à une paramétrisation affine exacte. Cette intuition est basée sur le fait que la taille de la paramétrisation, c'est-à-dire le nombre de variables scalaires liées aux multiplieurs, est la même que pour la paramétrisation affine des LFTs en  $\frac{1}{s} \cdot I_n$ , avec  $n = n_1 + n_2$ , qui est connue pour être exacte. Si c'est vrai, cela conduirait à obtenir une S-procedure sans perte [IMF00b] et à obtenir un problème d'optimisation équivalent dans l'approche de conception 1.

Enfin, comme problème plus ouvert, il serait également intéressant d'investiguer pour des plusieurs multiplieurs plus pour des LFTs dans  $\begin{bmatrix} \mathbf{T}_s(s) \cdot I & 0 \\ 0 & \mathbf{T}_p(s) \cdot I \end{bmatrix}$  quand

$$T_s(s) = \frac{s^2 + \omega_0^2}{s(s^2 + \omega_{a_s}^2)} \quad T_p(s) = \frac{s^2 + \omega_{r_p}^2}{s(s^2 + \omega_0^2)}$$

En effet, les deux fonctions de transfert partagent une structure similaire. De plus, ce sentiment est renforcé lorsque la contrainte supplémentaire  $\frac{\omega_0}{\omega_{r_p}} = \frac{\omega_{a_s}}{\omega_0}$  est considérée, comme dans le Chapitre 6.

### Sur la synthèse de filtres fréquentiels LFT

Une de nos principales contributions est le développement de méthodes efficaces pour la synthèse fréquentielle de filtres LFT. Cependant, il reste quelques points à clarifier et quelques généralisations à envisager, comme indiqué ci-dessous.

Premièrement, si la méthode de synthèse pour le cas d'un  $T(s)$  dissipatif sans perte nous semble complète, exceptée l'équivalence du lemme KYP associé comme mentionnée précédemment, sa généralisation à tout  $T(s)$  dissipative nécessite d'être complétée. En effet, notre généralisation implique de fusionner l'étape de synthèse de module avec l'étape de factorisation spectrale, ce qui conduit à résoudre un problème de faisabilité qui couple la résolution d'ARE et de LMIs. Afin d'aboutir à un seul problème LMI, il a été proposé d'échanger chaque ARE par un problème de minimisation linéaire sous contrainte LMI. Bien que cela puisse toujours être réalisé lorsque les AREs sont considérées indépendamment, ce point nécessite d'être précisé lorsqu'il y a un tel couplage. Pour y parvenir, il est possible de se référer à l'approche développée dans [RG96; Ram97] pour résoudre certains AREs couplées à l'aide de l'optimisation LMI.

Ce problème est également impliqué dans la méthode de synthèse proposée pour des LFT eb  $T_s(s)$  et  $T_p(s)$ . De plus, alors que cette méthode de synthèse peut être étendue pour des  $T_s(s)$  et  $T_p(s)$  respectivement  $\{x_s, y_s, z_s\}$ - et  $\{x_p, y_p, z_p\}$ -dissipatifs, elle soulève également certains problèmes qui ne sont pas encore résolus. En particulier, le décalage entre les systèmes LFT 2D, c'est-à-dire l'interconnexion de systèmes de deux sortes, et les fonctions rationnelles en  $T_s(s)$  et  $T_p(s)$ , empêche une extension efficace des méthodes précédentes.

Un premier point à considérer est le rôle de la matrice de transfert  $B(T_s, T_p)$ . Il a été illustré que différents  $B(T_s, T_p)$  conduisent à différents  $W(T_s, T_p)$ . Il serait intéressant d'étudier ce lien et de chercher pour une façon systématique et efficace de choisir le meilleur  $B(T_s, T_p)$ . A un niveau plus général, comme la représentation LFT semble plus appropriée pour représenter l'interconnexion de sous-systèmes hétérogènes que les fonctions rationnelles, on peut trouver quelques avantages à revisiter entièrement cette approche en termes de LFTs. Alors que la synthèse de module et la factorisation spectrale sont directement formulées en termes de LFTs, cela implique principalement d'étendre la factorisation séparant entre numérateur et dénominateur, peut-être en utilisant la factorisation dite coprime [ZDG96], et la notion de base.

De plus, l'extension aux systèmes MIMO nous semblent plutôt direct. En effet, les outils de base que sont la coprime factorisation [ZDG96], le Lemme KYP et la technique de factorisation spectrale sont transparents pour les systèmes SISO ou MIMO. Cependant, cela nécessite d'étendre les exigences de conception des contraintes de module aux valeurs singulières minimales et maximales de la matrice [SF09].

Enfin, des gabarits polynomiaux et discrets, intéressant pour la génération de poids- $H_\infty$ , nous paraissent clairement accessible en utilisant notre approche [RSF03; Zar13; PKZS18].

### Améliorations pour l'utilisation pratique de nos approches de conception

Il y a encore de la place pour améliorer nos approches de conception en vue d'application directe sur des problèmes pratiques, particulièrement lorsque les deux approches sont combinées. En voici quelques-unes.

Bien que l'approche de conception 1 puisse être directement incluse dans le flux de conception actuel, où un point initial est requis par le concepteur, sa performance est déterminée par le coût de calcul de l'algorithme de résolution (Algorithm 4.1, p. 84). Comme cet algorithme est basé sur des relaxations séquentielles LMI, son coût de calcul résulte principalement du nombre d'itérations. Dans ce travail, pour réduire ce nombre, nous nous sommes concentrés sur la recherche d'un point initial raisonnable. Cependant, on peut aussi agir sur l'algorithme lui-même. Tout d'abord, on peut trouver un meilleur choix pour le paramètre dépendant de l'itération  $\beta_k$  qui est impliqué dans l'accélération de Nesterov. Pour cela, ce paramètre peut être réglé manuellement sur des exemples simples, ou choisi en utilisant une des méthodes données dans [LRP16; HL17a; HL17b]. Deuxièmement, il serait intéressant d'adapter l'algorithme pour prendre en compte la structure particulière du terme matrice bilinéaire. En effet, comme  $\check{Q}^{inv}$  est diagonal, le terme matriciel bilinéaire  $\check{Q}^{inv}\check{Q}^{inv}$  est aussi diagonal. Intuitivement, l'adaptation de l'algorithme spécialement pour ce cas devrait se traduire par un coût de calcul moins élevé.

A nos yeux, la plupart des améliorations de l'approche de conception 2 se trouvent dans l'étape de synthèse fréquentiel de filtre LFT, déjà discutée. En particulier, dans la synthèse de filtre  $T_s, T_p$ -échelle, le choix de  $B(T_s, T_p)$  doit être clarifié, car il a une conséquence directe sur l'ordre de la LFT 2D  $W(T_s, T_p)$ , et donc sur le nombre d'éléments.

Enfin, lorsque les deux approches sont combinées, l'étape transition doit être améliorée. Dans le cas particulier de l'approximation par un filtre  $T_s, T_p$ -échelle comme dans la Section 6.2 du Chapitre 6, le processus devrait être systématisé. On peut également améliorer la situation en réalisant l'approximation uniquement sur les bandes de fréquences concernées, car il n'est pas nécessaire d'être précis sur les bandes de transition par exemple, où il n'y a pas de contraintes fréquentielles.

### Eléments passifs sans perte

Dans le Chapitre 4, les deux approches de conception ont été développées pour les éléments passifs sans perte. Une modélisation plus précise des composants, comme pour le modèle mBVD pour les éléments AW, implique que les éléments soient passifs, c'est-à-dire qu'ils soient l'interconnexion d'inductances, de capacités *et* de résistances. Il serait donc intéressant d'adapter ou d'étendre ces approches aux éléments passifs à perte.

Pour les deux approches, la première étape consiste à obtenir une formule explicite reliant les entrées de la matrice de diffusion  $S$ , et plus précisément entre  $s_{11}$  et  $s_{21}$ . En d'autres termes, la Représentation Réciproque de Belevitch (Theorem 3.1, p. 48) doit être étendue aux filtres passifs avec pertes. En particulier, cela permettrait d'obtenir une version étendue du problème de conception alternatif (Problem 3.2, p. 49). En outre, cela permettrait de lier l'impédance d'entrée  $z_{in}$  et le paramètre de diffusion  $s_{21}$ , nécessaire à la recherche des conditions de réalisation.

Pour l'approche de conception 1, la représentation PHS DAE plus générale donnée dans [Sch17, Chap. 6], qui inclut les éléments résistifs, devrait être utilisée. Les preuves de la Propriété 4.1 (p. 60), pour obtenir la forme dite descripteur, et du Lemma 4.2 (p. 76), pour obtenir la forme LFT, doivent alors être adaptées à cette représentation.

Afin d'illustrer comment inclure les éléments résistifs dans la méthode de conception, considérons la représentation suivante de la représentation d'état PHS de la matrice de diffusion  $S$  :

$$\begin{cases} \tilde{p} = \left( \frac{1}{s} \cdot I_n \right) \tilde{q} \\ \tilde{q} = \left( J - (R + BB^T) \right) Q\tilde{p} + \sqrt{2}Bw \\ z = -\sqrt{2}DB^T Q\tilde{p} + Dw \end{cases}$$

La principale différence avec la représentation utilisée dans ce manuscrit (cf p. 61) est la matrice semi-définie positive  $R = R^T \geq 0$ , qui inclut les contributions de toutes les résistances. En appliquant le même raisonnement que celui qui a conduit au Théorème 4.2 (p. 79) conduit à vérifier la faisabilité de problèmes BMI de la forme suivante :

$$\mathcal{A}(\mathbf{D}, \mathbf{P}, \boldsymbol{\tau}) + \begin{bmatrix} \mathbf{Q}^{inv} \mathbf{Q}^{inv} & -\mathbf{Q}^{inv} (J - (\mathbf{R} + \mathbf{B}\mathbf{B}^T)) & 0 & -\mathbf{Q}^{inv} \sqrt{2}\mathbf{B} \\ -(\mathbf{J} - (\mathbf{R} + \mathbf{B}\mathbf{B}^T))^T \mathbf{Q}^{inv} & (\mathbf{J} - (\mathbf{R} + \mathbf{B}\mathbf{B}^T))^T (\mathbf{J} - (\mathbf{R} + \mathbf{B}\mathbf{B}^T)) + 2\mathbf{B}\mathbf{B}^T & \sqrt{2}\mathbf{B}\mathbf{D}^T & (\mathbf{J} - (\mathbf{R} + \mathbf{B}\mathbf{B}^T) - \mathbf{I})^T \sqrt{2}\mathbf{B} \\ 0 & \sqrt{2}\mathbf{D}\mathbf{B}^T & 1 & -\mathbf{D} \\ -\sqrt{2}\mathbf{B}^T \mathbf{Q}^{inv} & \sqrt{2}\mathbf{B}^T (\mathbf{J} - (\mathbf{R} + \mathbf{B}\mathbf{B}^T) - \mathbf{I}) & -\mathbf{D}^T & 2\mathbf{B}^T \mathbf{B} + \mathbf{D}^T \mathbf{D} \end{bmatrix} \geq 0$$

où  $\mathcal{A}(\mathbf{D}, \mathbf{P}, \boldsymbol{\tau})$  est une matrice affine en les multipliers  $\mathbf{D}, \mathbf{P}, \boldsymbol{\tau}$ . Deux cas apparaissent alors. Si les valeurs des résistances sont connues, c'est-à-dire si  $R$  n'est pas une variable de décision, le problème d'optimisation résultant est similaire sous tous les aspects au problème BMI obtenu au Chapitre 4. Si  $R$  est une variable de décision, cela conduit à un problème d'optimisation plus complexe, qui implique plusieurs produits de matrice bilinéaire.

Lorsqu'il s'agit de l'approche de conception 2, le passage aux éléments avec perte est encore plus simple. En effet, l'extension de la première étape a déjà été réalisée, car la méthode de synthèse fréquentielle pour les filtres LFT que nous avons développée permet de considérer de tels cas. Une grande partie de l'effort doit alors être concentrée sur les contraintes de réalisation, et leurs formulations convexes.

### Eléments incertains : analyse et synthèse

Comme mentionné dans le Chapitre 2, le cadre développé dans ce manuscrit n'est pas limité à la conception de systèmes. En effet, l'outil LFT et son lien avec l'optimisation LMI ont joué à l'origine un rôle important dans l'analyse robuste [DPZ91]. L'analyse robuste consiste à vérifier si un système satisfait ou non des critères de stabilité ou de performance en présence d'incertitude. Cela permet notamment de prendre en compte certaines différences entre un modèle et un système physique en considérant des incertitudes paramétriques dans le modèle.

Dans [FKL+18], il a été proposé une méthode efficace pour l'analyse robuste de circuits électriques constitués d'éléments incertains, c'est-à-dire avec des incertitudes sur les valeurs des éléments. Sur la base de la dérivation systématique d'une représentation LFT appropriée à partir de n'importe quel schéma de circuit [FKS+15], l'application d'une version appropriée du Lemme KYP permet de vérifier si les exigences du gabarit donné, constitué de bornes supérieures et inférieures, sont remplies en résolvant un problème de faisabilité LMI. Contrairement à l'approche de Monte Carlo, qui consiste à choisir de manière aléatoire certaines réalisations du circuit incertain et à les simuler, cette méthode vérifie si les exigences sont satisfaites dans tout l'ensemble incertain et à tendance à être considérablement plus rapide. Sur un exemple, les auteurs de [FKL+18] constatent que la méthode de Monte Carlo nécessite un temps de calcul mille fois plus élevé que leur méthode. Par conséquent, la méthode développée est systématique, efficace, transparente pour l'utilisateur, le point de départ étant un schéma de circuit, directement applicable et fournit un plus haut niveau de garantie. Ainsi, il serait particulièrement intéressant de l'inclure dans un logiciel de CAO.

De façon surprenante, le fait d'aborder le problème de la synthèse robuste à des incertitudes paramétriques, c'est-à-dire la synthèse des valeurs nominales d'éléments incertains de telle sorte que le filtre résultant satisfasse les exigences de conception pour l'ensemble des incertitudes, semble d'une complexité similaire au cas nominal de l'approche de conception 1. En effet, supposons par exemple que la valeur de chaque élément de construction  $\tilde{q}_i$  peut être écrite sous la forme

$$\tilde{q}_i = q_i(1 + \delta_i)$$

où  $q_i$  est la valeur nominale et  $\delta_i$  un nombre réel aléatoire appartenant à un intervalle donné. En conséquence, la matrice  $\tilde{Q}$  de la représentation PHS est telle que :

$$\tilde{Q} = Q(I + \Delta)$$

avec des matrices diagonales  $\tilde{Q}$ ,  $Q$  et  $\Delta$ . En appliquant le même raisonnement que dans la preuve du Théorème 4.2 (p. 79) conduit alors à

$$\mathcal{A}(\mathbf{D}, \mathbf{P}, \boldsymbol{\tau}) + \begin{bmatrix} (I + \Delta)^{-1} \mathbf{Q}^{inv} \mathbf{Q}^{inv} (I + \Delta)^{-1} & \mathcal{L}(\mathbf{Q}^{inv})(I + \Delta)^{-1} \\ (\mathcal{L}(\mathbf{Q}^{inv})(I + \Delta)^{-1})^T & \mathcal{C} \end{bmatrix} \geq 0$$

où  $\mathcal{A}(\mathbf{D}, \mathbf{P}, \boldsymbol{\tau})$  est une matrice affine dans certains multiplicateurs,  $\mathcal{L}(\mathbf{Q}^{inv})$  est une matrice linéaire dans  $\mathbf{Q}^{inv}$  et  $\mathcal{C}$  une matrice constante. Ainsi, ceci peut être réécrit comme :

$$\begin{bmatrix} (I + \Delta)^{-1} \\ I \end{bmatrix}^T \begin{bmatrix} \mathbf{Q}^{inv} \mathbf{Q}^{inv} & \mathcal{L}(\mathbf{Q}^{inv}) \\ \mathcal{L}(\mathbf{Q}^{inv})^T & \mathcal{A}(\mathbf{D}, \mathbf{P}, \boldsymbol{\tau}) + \mathcal{C} \end{bmatrix} \begin{bmatrix} (I + \Delta)^{-1} \\ I \end{bmatrix} \geq 0$$

Définissons maintenant les matrices réelles  $A$ ,  $B$ ,  $C$  et  $D$  telles

$$\Delta \star \left[ \begin{array}{c|c} A & B \\ \hline C & D \end{array} \right] := \left[ \begin{array}{c} (I + \Delta)^{-1} \\ I \end{array} \right]$$

L'application d'un lemme KYP adapté [FKS+15] conduit à rechercher des matrices structurées  $D_\Delta = D_\Delta^T > 0$  et  $G_\Delta = -G_\Delta^T$  telles que :

$$\begin{bmatrix} A & B \\ I & 0 \end{bmatrix}^T \begin{bmatrix} D_\Delta & G_\Delta \\ -G_\Delta & -D_\Delta \end{bmatrix} \begin{bmatrix} A & B \\ I & 0 \end{bmatrix} + \begin{bmatrix} C^T \\ D^T \end{bmatrix} \begin{bmatrix} Q^{inv} Q^{inv} & \mathcal{L}(Q^{inv}) \\ \mathcal{L}(Q^{inv})^T & \mathcal{A}(D, P, \tau) + C \end{bmatrix} \begin{bmatrix} C & D \end{bmatrix} \geq 0$$

Bien que ce problème de faisabilité BMI présente un certain degré de conservatisme, en raison de la structure des multiplieurs  $D_\Delta$  et  $G_\Delta$ , le seul terme non linéaire se trouve être le même que dans le cas nominal, c'est-à-dire  $Q^{inv} Q^{inv}$ . Par conséquent, en adaptant l'algorithme fourni dans l'approche de conception 1, on pourrait obtenir une synthèse robuste aux variations des valeurs des éléments.

### C.7.2.2 Recherches futures

#### Un problème ouvert : la synthèse robuste par rapport à la réalisation

Dans l'approche de conception 2, seul le problème de réalisation exacte a été pris en compte. Cela signifie que, étant donné une fonction de transfert  $W$  qui satisfait un gabarit fréquentiel, un circuit a été synthétisé de telle sorte que  $W$  soit exactement le paramètre de diffusion  $s_{21}$ , c'est-à-dire  $s_{21} = W$ . Cela impliquait la recherche de contraintes de réalisation qui pouvaient être ajoutées de façon convexe dans la synthèse de  $W$ . En général, seule une partie de ces contraintes de réalisation a été trouvée convexe et a été prise en compte dans l'étape de synthèse. Au Chapitre 6, cela nous a conduit à approximer  $W$  par une fonction  $\widetilde{W}$  qui est réalisable comme paramètre de diffusion  $s_{21} = \widetilde{W}$  d'un circuit, mais n'est plus assurée de satisfaire le gabarit fréquentiel. Cela nous amène au problème suivant :

Synthétiser un filtre fréquentiel  $W$  qui soit robuste par rapport à un processus de réalisation donné.

Une première solution naturelle est d'ajouter simplement des bandes de sécurité supplémentaires au gabarit. Malheureusement, cette approche heuristique rend encore plus complexe un problème qui a déjà tendance à être de plus en plus complexe. Une deuxième solution intuitive consiste à maximiser la distance entre le module au carré de la réponse fréquentielle du filtre et le gabarit. Cependant, cela revient à se rapprocher d'une courbe idéale et il a été illustré au Chapitre 6 que l'approximation tend à être meilleure sur certains intervalles de fréquence que sur d'autres. Cela conduit alors à sur-contraindre inutilement le problème de synthèse.

Il semble plutôt qu'une approche plus sophistiquée soit nécessaire pour surmonter ces problèmes. L'étape d'approximation de  $W$  par  $\widetilde{W}$  consiste principalement à approximer les coefficients de la fonction de transfert  $W$ , ou de façon équivalente les matrices de sa représentation LFT. Une option pourrait être de considérer l'approximation de la réalisation comme des incertitudes sur ces coefficients, ou matrices, conduisant au problème de la synthèse robuste.

Ce problème dépasse la portée de la seule synthèse des filtres électroniques passifs. En effet, il apparaît aussi dans la synthèse de filtres numériques, où le processus de quantification empêche de réaliser exactement une fonction de transfert (discrète)



donnée [PB87; VM19]. Plus généralement encore, si le problème de réalisation apparaît vraiment complexe [Kal10], c'est le problème de réalisation en vue d'un objectif précis qui est ici en jeu, et qui nous semble préférable d'aborder.

### **Collaboration future pour la conception de filtres AW**

Comme nous l'avons expliqué dans le tout premier chapitre de ce manuscrit, le spectre surchargé et la demande de vitesse de transmission de données toujours plus élevée ont entraîné un problème de conception difficile. Cela a un impact significatif sur la complexité à chaque étape du processus de conception globale.

Tout d'abord, au niveau du circuit équivalent, alors qu'il a été décidé dans cette thèse d'obtenir d'abord des méthodes de conception efficaces et systématiques pour un seul filtre, le problème de co-conception doit être abordé dans un avenir proche. La principale difficulté de ce problème est que, soit en utilisant des commutateurs soit une architecture de multiplexage [Mah17; FVAJ+16], un couplage capacitif apparaît entre les filtres. Il faut en tenir compte dans la conception des filtres car cela a une incidence sur leurs performances. Cependant, comme les filtres sont conçus conjointement, l'effet de couplage change constamment. Par conséquent, une conception séquentielle n'est pas recommandée. De plus, ce couplage influence également la propriété d'adaptation d'impédance qui est nécessaire pour une transmission de puissance appropriée. Actuellement, cela conduit à ajouter des composants d'adaptation [FVAJ+16], qui à leur tour modifient à nouveau les performances des filtres. Par conséquent, les méthodes qui consistent à concevoir de manière indépendante les filtres semblent limitées pour résoudre ce problème et des méthodes sophistiquées de co-conception doivent alors être développées.

Deuxièmement, la pression du marché pour la miniaturisation ainsi que les exigences de performance plus strictes conduisent à développer de nouvelles technologies pour lesquelles le décalage entre les mesures et les simulations avec les modèles *RLC* a tendance à augmenter [Mah17]. Certaines sources de ce décalage entre le modèle et la mesure peuvent être considérées comme des incertitudes paramétriques sur les modèles, telles que la température. Ainsi, des méthodes efficaces d'analyse des variations paramétriques peuvent être utilisées pour une conception robuste des systèmes électroniques [Fil10; FKS+15; FKL+18]. Cependant, des modèles plus complexes sont nécessaires en général pour ajuster les mesures et conduisent à considérer la famille des modèles non linéaires [WL15]. Bien qu'il existe des modèles analytiques non-linéaires de composants et de filtres AW [Fel09; IHI+11], la plupart des simulations sont basées sur la méthode d'analyse par éléments finis, ce qui conduit à de longs calculs. Comme une partie importante de la théorie des Systèmes et de l'Automatique est consacrée à l'étude des systèmes passifs non linéaires, voir par exemple [Kha02; Sch17], il serait intéressant d'étudier si certaines de ces techniques peuvent être appliquées pour développer des méthodes efficaces adaptées à l'analyse et à la simulation des filtres AW.

L'ensemble de ces éléments conduit à un sujet de recherche stimulant, avec un fort intérêt pratique et interdisciplinaire, à l'interface entre l'Électronique et théorie des Systèmes.

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