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Dag Prawitz's *theory of grounds*

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Résumé

Dans la récente théorie des grounds, Prawitz développe ses investigations sémantiques dans la direction d'une analyse, à la fois philosophique et formelle, de l'origine et de la nature du pouvoir que les inférences valides, ainsi que les démonstrations où ces inférences figurent, exercent sur des agents engagés dans l'activité déductive ; à savoir, le pouvoir d'obliger épistémiquement à accepter les conclusions, si l'on en a accepté les prémisses ou les hypothèses. Il s'agit de la plus ancienne des questions à laquelle s'intéresse la logique, depuis sa naissance avec Aristote - on pourrait dire, presque de la raison d'être de cette discipline.

La notion de base est celle de ground. Un ground, *grosso modo*, est ce dont on est en possession lorsqu'on est justifié à affirmer un certain énoncé. Les grounds peuvent être construits en accomplissant des opérations qui permettent le passage d'un état de justification à un autre ; il s'agit, donc, d'objets abstraits mais épistémiques, qui contiennent des opérations abstraites mais calculables. Un acte d'inférence consiste à l'application d'une opération des grounds pour les prémisses aux grounds pour la conclusion, acte qui sera valide si l'opération accomplie produit effectivement des grounds pour la conclusion quand appliquée aux grounds pour les prémisses. Finalement, une démonstration est une concaténation d'inférences valides.

Relativement à son objectif de fond, la théorie des grounds présente des avancements indubitables par rapport à la précédente approche de Prawitz, la *proof-theoretic semantics*. En particulier, la théorie des grounds offre une définition du concept d'inférence valide en vertu de laquelle il devient possible de faire dépendre la contrainte épistémique des démonstrations de celle des inférences valides dont ces démonstrations se composent. Dans la *proof-theoretic semantics*, la notion d'inférence valide dépend de celle de démonstration, alors que la caractérisation suggérée apparaît difficile - sinon impossible.

Mais la théorie des grounds et la *proof-theoretic semantics* partagent un problème ; dans l'une comme dans l'autre, inférences valides et démonstrations pourraient être telles qu'il est impossible, pour des agents qui les

utilisent, de reconnaître le fait qu'elles justifient leur conclusion. Si une inférence valide ne peut pas être reconnue comme valide, et si une démonstrations ne peut pas être reconnue comme une démonstration, aucun agent qui les utilisent ne sera obligé à accepter la conclusion par le seul fait de les avoir accomplies. Il doit aussi reconnaître que ce qu'il a fait sert à fonder épistémiquement les résultats auxquels il vise.

Nous allons développer le cadre formel de la proposition de Prawitz en introduisant, d'un côté, un « univers » de grounds et opérations sur grounds et, de l'autre, des langages formels de grounding dont les termes dénotent grounds ou opérations sur grounds. Tout langage de grounding doit être indéfiniment ouvert à l'ajout des nouvelles ressources expressives. De plus, en raison du théorème d'incomplétude de Gödel, il ne peut pas exister un langage de grounding clôt capable de décrire tous les grounds ou toutes les opérations sur grounds. Par conséquent, nous allons également introduire une notion d'expansion de langage de grounding, ce qui nous permettra de créer une hiérarchie de langages et de fonctions de dénotation. Ainsi, il deviendra possible de décrire propriétés et résultats dont les langages de grounding jouissent, à la fois singulièrement et par rapport à leurs expansions.

À côté des langages de grounding, nous proposerons aussi des systèmes de grounding, à l'aide desquels démontrer des propriétés significatives des termes des langages de grounding (par exemple le fait qu'un terme dénote un ground ou une opération sur grounds, ou le fait que deux termes dénotent le même ground ou la même opération sur grounds), ou des composantes syntaxiques qui figurent dans ces termes (par exemple le fait qu'un certain symbol opérationnel est défini de façon qu'il dénote une opération sur grounds avec un certain domaine et un certain co-domaine, ou qu'il est traduisible dans les symboles opérationnels d'un sous-langage du langage auquel il appartient).

Finalement, nous allons aborder deux questions concernant langages et systèmes. Tout d'abord, celle de la complétude de la logique intuitionniste par rapport à la théorie des grounds ; en d'autres mots, nous discuterons la conjecture de Prawitz dans le cadre formel que nous avons proposé. En second lieu, nous poursuivrons une analyse du problème de reconnaissabilité déjà évoqué, à la lumière des acquisitions formelles permises par langages et systèmes de grounding.

Riassunto

Nella recente teoria dei grounds, Prawitz sviluppa le sue indagini semantiche nella direzione di un'analisi, al contempo filosofica e formale, dell'origine e della natura di quella speciale forza che le inferenze valide, e le dimostrazioni in cui tali inferenze sono coinvolte, esercitano su agenti impegnati nell'attività deduttiva: la forza di costringere epistemicamente ad accettare le conclusioni dell'inferenza o della dimostrazione, se se ne sono accettate le premesse o le ipotesi. Si tratta della più antica delle questioni di cui la logica si interessa fin dai tempi della sua nascita con Aristotele - diremmo quasi della *raison d'être* di tale disciplina.

La nozione di fondo è quella di ground. Un ground è, *grosso modo*, ciò di cui si è in possesso quando si è giustificati nell'asserire un certo enunciato. I grounds possono essere costruiti compiendo operazioni che consentano il passaggio da uno stato di giustificazione all'altro; si tratta, perciò, di oggetti astratti ma epistemici, in cui sono coinvolte operazioni astratte ma computabili. Un atto inferenziale consiste nell'applicazione di un'operazione dai grounds per le premesse ai grounds per la conclusione, atto che risulterà legittimo quando l'operazione compiuta è effettivamente tale da produrre grounds per la conclusione quando applicata a grounds per le premesse. Una dimostrazione, infine, è una concatenazione di inferenze valide.

Relativamente al suo obiettivo di fondo, la teoria dei grounds presenta indubbi avanzamenti rispetto al precedente approccio di Prawitz, la *proof-theoretic semantics*. In particolare, la teoria dei grounds offre una definizione della nozione di inferenza valida in virtù della quale diventa possibile far dipendere la costrizione epistemica esercitata delle dimostrazioni da quella esercitata dalle inferenze valide di cui le dimostrazioni si compongono. Nella *proof-theoretic semantics*, al contrario, la nozione di inferenza valida dipende da quella di dimostrazione, sicché la caratterizzazione suggerita risulta difficile - se non impossibile.

Ma teoria dei grounds e *proof-theoretic semantics* condividono un problema; nell'una come nell'altra, inferenze valide e dimostrazioni potrebbero essere tali da risultare impossibile, ad agenti che ne facciano uso nella conc-

reta pratica deduttiva, un riconoscimento del fatto che esse giustificano la loro conclusione. Se un'inferenza valida non può essere riconosciuta come valida, e se una dimostrazione non può essere riconosciuta come dimostrazione, nessun agente che ne faccia uso si sentirà costretto ad accettare la conclusione per il solo fatto di averle compiute. Egli deve anche riconoscere che quanto fatto serve a sostanziare epistemicamente i risultati cui egli mira.

Il quadro formale della proposta di Prawitz sarà da noi articolato introducendo un "universo" di grounds ed operazioni su grounds e, poi, linguaggi formali di grounding i cui termini denotano grounds od operazioni su grounds. Ogni linguaggio di grounding deve essere indefinitamente aperto all'aggiunta di nuove risorse espressive né, a causa dell'incompletezza di Gödel, può esistere un linguaggio di grounding chiuso capace di descrivere tutti i possibili grounds o tutte le possibili operazioni su grounds. Pertanto, introdurremo anche una nozione di espansione di linguaggio di grounding, il che ci consentirà di creare una gerarchia di linguaggi e di funzioni di denotazione. Diventerà così possibile descrivere proprietà e risultati di cui i linguaggi di grounding godono, tanto singolarmente, quanto in relazione alle loro espansioni.

Accanto ai linguaggi di grounding, proporremo anche sistemi di grounding, in cui dimostrare proprietà rilevanti dei termini dei linguaggi di grounding (ad esempio, il fatto che un termine denoti un ground o un'operazione su grounds, o che due termini denotino lo stesso ground o la stessa operazione su grounds), o di alcune componenti sintattiche in essi coinvolte (ad esempio, che un certo simbolo funzionale è definito in modo da denotare un'operazione su grounds con un certo dominio ed un certo co-dominio, o in modo da essere riscrivibile in termini di simboli operazionali di un sotto-linguaggio del linguaggio cui esso appartiene).

Ci occuperemo infine di due questioni relative a linguaggi e sistemi. Innanzitutto, la questione della completezza della logica intuizionista rispetto alla teoria dei grounds; in altre parole, discuteremo una riformulazione della congettura di Prawitz nel quadro formale da noi proposto. In secondo luogo, perseguiremo una disamina del succitato problema di riconoscibilità alla luce delle acquisizioni formali consentite dai linguaggi e dai sistemi di grounding.

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Contents

Introduction

According to a rather widespread interpretation, logic is to be understood as the science of correct reasoning. Far from being a definition, however, this expression is of a mere indicative nature. It raises more questions than the ones it answers. Even leaving out the problematic issue on what we can and should consider as science, it is far from clear what a reasoning is, and even more what a correct reasoning is.

From a general point of view, we can follow a long and well-established tradition according to which a reasoning is a concatenation of passages from certain premises to certain conclusions, called inferences. This position is for example present and aware in Descartes who, in a well-known passage of his *Rules for the direction of the mind*, equates reasoning to

> a continuous and uninterrupted movement of thought in which each individual proposition is clearly intuited. This is similar to the way in which we know that the last link in a long chain is connected to the first: even if we cannot take in at one glance all the intermediate links on which the connection depends, we can have knowledge of the connection provided we survey the links one after the other, and keep in mind that each link from the first to the last is attached to its neighbour. (Descartes 1985, 15)

It is also obvious, however, that this idea can be further declined in many different alternative ways; a more specific determination will vary depending on the point of view adopted on the nature of premises and conclusions, and on the passage itself, as well as on the basis of which of these factors are considered really relevant to the logical investigation.

Also in relation to the narrower notion of correct reasoning, fortunately we have a basic intuition to which we can hold on, even if also such intuition is, in the final analysis, partial and liable to many different ramifications. It can be illustrated with the famous words that Aristotle binds to the key notion of the system – the first in its kind – developed in the *Organon*:

> the syllogism is a discourse in which, certain things being laid down, something follows of necessity from them. (Aristotle 1949, 287)

As a paradigm of correct reasoning, a syllogism therefore has a feature that Aristotle emphasizes: necessity. On the other hand, if the question "what did Aristotle mean by necessity?" concerns the history of logic or the history of philosophy, instead, it is very meaningful for what concerns us the more general question "what kind of necessity do we refer to when we talk about correct reasoning?". We could think that logic, in the cloak of the name of science, has a unanimously shared vision on the nature of necessity. But this is by no means the case.

It would obviously be impossible, in the restricted framework of this introduction, to review if only in general the multiple reflections which, over the centuries, have concerned the notion of necessity. Nor would such review be of any use with respect to the theme of our work: Dag Prawitz's *theory of grounds*. On the other hand, necessity plays in Prawitz, and it will play for us, a decisive role to say the least. The theory of grounds, in fact, assumes the shape of an attempt at the same time philosophical and formal to respond, with sufficient and satisfactory articulation, to what is perhaps the most original among the questions of logic: how and why do some inferences, commonly called deductively correct, have the epistemic power to force us to accept their conclusion, assuming we have epistemically accepted their premises? If we adopt the aforementioned point of view, that an argument is a chain of inferences, to answer this crucial question also means explaining how and why deductively correct reasoning can exert that force that, since very distant times, has made it the main source of conclusive knowledge, and a source of irrefutable epistemic certainty for human beings.

In focusing on the power of epistemic binding of deductively correct inferences and reasoning, Prawitz emphasizes a very particular kind of necessity. In a paper that is in many ways a watershed between his previous research and the theory of grounds, the Swedish logician uses the fitting expression of *necessity of thought*, specifying that with it he means the circumstance in which

> one is committed to holding α true, having accepted the truth of the sentences of Γ; one is compelled to hold α true, given that one holds all the sentences of Γ true; on pain of irrationality, one must accept the truth of α, having accepted the truth of the sentences of Γ. (Prawitz 2005, 677)

This kind of necessity is diametrically different from another, equally well known and perhaps more practiced in logic, which is based on the notion of possible world and according to which necessity means truth in all possible worlds. The fact that something is true in all possible worlds is obviously alien to issues relating to knowledge: it could be a circumstance of which we are simply unaware or, even if we are aware of it, we might not see why it occurs. In the latter case, it is an occurrence that we must accept not by virtue of an epistemic binding but, so to speak, of a mere factual statement.

However problematic, the notion of possible world is often used to substantiate the idea that model-theory, the child of the pioneer works of Bolzano and Tarski, as well as for many years the standard formal semantics of contemporary mathematical logic, actually captures that notion of necessity that characterizes, giving them a decisive modal structure, the concepts of validity and logical consequence. However, many criticisms have been raised against the thesis that model-theory contains modal ingredients of any kind and, given in such general terms, the question is still widely debated. What we can say with certainty, is that the modality captured by model-theory, although present, is undoubtedly *not* of the type Prawitz is interested in, that is, not of an epistemic type.

It is therefore not a case if, starting from some important results in proof theory, Prawitz has developed a formal semantics alternative to model-theory, today known as *proof-theoretic semantics*. Proof-theoretic semantics offers definitions of the concepts of validity and logical consequence that, in accordance with necessity of thought, replaces the notion of truth, as a semantic core, with those of proof or valid argument. The idea would seem obvious: the epistemic binding that one experiences in necessity of thought can occur only when we are in possession of a proof or a valid argument.

Proof-theoretic semantics is therefore based on mathematically rigorous characterizations of the concepts of proof and valid argument, from which to move forward to reach the more general semantic definitions. This happens, it seems to us, by following essentially three lines of research: the BHK semantics for first-order intuitionist logic, which originated with Heyting, some observations by Gentzen on the relationship between introduction and elimination rules for first-order logical constants in natural deduction systems, and Dummett's investigations in theory of meaning, with a particular focus on a verificationist theory of meaning. These three sources are obviously mutually linked, and in turn, each of them, or all three jointly, are bound to reflections of another type, more or less explicitly recognized. By setting itself at the crossroads of so many suggestions, and in a sense harmonizing them all, proof-theoretic semantics is therefore of an interest more widely philosophical, which goes beyond the formal results, albeit fundamental, that it allows to reach.

However, the proof-theoretic approach is not entirely free of problems, of which the main one, related to the issues mentioned above, concerns precisely the possibility of accounting for the epistemic power of deductively correct inferences and reasoning. If we intend to explain how and why a correct reasoning can force epistemically to accept its conclusions, thus providing justification towards them, there seems to be no other choice but to make this force depend on that, of analogous nature, enjoyed by the inferential passages occurring in the reasoning itself. To make this explanation work, however, the notion of correct inference must conceptually take priority over that of correct reasoning, thus following the same explanatory order that makes reasoning a chain of inferences. In proof-theoretic semantics, though, the notion of correct inference is defined in terms of proofs or valid arguments, by stating that an inference is correct when it preserves provability, or the validity of the structure in which it occurs. It is in this sense not surprising that, with the theory of grounds, Prawitz renounces a characterization of this type, returning to the correct inferences their pivotal role.

This inversion of the natural relationship among the notions of correct inference, proof and valid argument, refers to another point on which the theory of the grounds offers, in our opinion, a doubtless progress. To be forced to validate the correctness of judgments or assertions that convey or express knowledge is something we experience when, for example, we follow the steps made by someone who is proving something, or when we personally perform a proof act. The epistemic binding is something we "feel", and of which we are aware; in this "feeling", in this being aware, however, we are not in a condition of passivity, but, on the contrary, in order to accomplish this experience we must *do* something, carry out appropriate acts. In the words of Cozzo (Cozzo 2015), necessity of thought has a *phenomenal character*, and assumes for us the form of an *active experience*.

Strictly speaking, therefore, correct reasoning *leads* to a state of epistemic justification, but it is not itself, as such, an epistemic justification; the proof act is what by virtue of which the binding manifests itself, acting upon us, but it is the result of this act, what the act leads to, that qualifies as the condition of epistemic success. Subtle, but crucial in the reconstruction of the epistemic force of correct inferences and reasoning, this distinction can be summed up as a dichotomy between *proof-objects*, on the one hand, and *proof-acts*, on the other (see mainly Sundholm 1998). In this regard, proof-theoretic semantics seems ambiguous, since it deals with proofs and valid arguments at the same time as objects and as acts, while much more precise is the theory of grounds, in which Prawitz distinguishes between *states of justification* – reifying, he actually speaks of *objects*, called precisely grounds, of which we are in possession when we are justified in judging or asserting – and *acts* that enable us to enter a state of justification – namely, proofs that produce grounds.

In the theory of grounds, as just mentioned, "ground" is the expression used by Prawitz to indicate what we have when we are epistemically justified. Gounds are objects that reify states of epistemic success, and are obtained by performing inferential acts – single, or concatenated so as to form a proof. The most original trait of the theory of grounds is perhaps the reconstruction that Prawitz offers of *how* an inferential act can produce grounds and, therefore, of *what* an inferential act actually is. In the commonly accepted meaning, which we have also referred to at the beginning of this introduction, an inference is simply identified by certain premises and certain conclusions, since it appears as a passage from the ones to the others. In the light of convincing arguments, however, Prawitz believes that this reconstruction is far too poor to allow an adequate explanation of the phenomenon of epistemic compulsion. He adds to it the crucial idea that to make an inference means applying an operation that transforms constructively grounds for the premises into grounds for the conclusion, and that therefore an inference, in addition to premises and conclusions, must also involve an operation of this kind. A deductively correct inference is therefore an inference the operation of which actually returns grounds for the conclusion, when applied to grounds for the premises.

Joining the aforementioned distinction between proof-objects and proof-acts, and merging into the notion of deductively correct inference, this innovative way to characterize inferential acts makes the theory of grounds a solid apparatus for a rigorous explanation, not circular and philosophically meaningful, of the relationship between correct inferences and correct reasoning, as well as of the epistemic power of both. Both proof-theoretic semantics and the theory of grounds attribute a crucial role to the so-called *canonical* cases, primitive in contrast to the *non-canonical* ones, that on the contrary need justification. However, while in proof-theoretic semantics the objects themselves, as not distinguished from the acts, can be canonical or non-canonical, in the theory of grounds the distinction applies only to the acts, whereas the grounds are specified solely by virtue of primitive operations, by simple induction on the complexity of the formulas for which they are grounds. From this perspective, the fact that the deductive correctness of the inferential acts is explained on the base of objects of this type, and not in relation to the acts in which these inferences occur, allows to overcome some problems of circularity, from which analogous attempts in the setup of proof-theoretic semantics suffer.

Understanding the inferential acts as applications of operations on grounds has also, in our opinion, an important role in relation to a problem that, this time, the theory of grounds shares in full with proof-theoretic semantics. If a proof or valid argument must give justification, it seems we cannot help but request that it be *recognizable* that they have this power. In the same way, if the possession of a ground coincides with being in a state of justification, we need to be able to *recognize* that the inference that gives possession of that ground is deductively correct. Otherwise, the fact that we have made correct deductions would correspond to the possession of abstract objects, so that we could be totally unaware of their epistemic weight; no justification, let alone any epistemic binding, seems to be possible under such circumstances. Similar issues are often raised in relation to the clauses for the implication and the universal quantification in BHK semantics. As regards the possibility itself of arguing that certain constructs, ultimately formal, are capable of respecting the wishes concerning epistemic justification and binding, the point is obviously crucial also and above all in Prawitz. Unfortunately, a precise delineation of *how* this recognizability can occur, assuming that it actually can, remains decidedly elusive. A good starting point could be to clarify *what* the recognition in question is, but already here positions are disparate and discordant, going from "strong" readings, in terms of decidability, to more "weak" readings, that involve pragmatic elements.

In the two frameworks, proof-theoretic semantics and theory of grounds, the problem of recognizability arises for the non-canonical cases; since they are not primitive in the explanation of meaning, namely not primitive in the determination of what counts as justification for judgments or assertions about propositions or sentences of different logical form, they must be justified. And we need to be able to recognize that the justification given works, fulfills the task requested. In this perspective, the ground-theoretic idea of inferences as applications of operations on grounds allows a minimum, albeit limited, progress. In certain specific circumstances – when the inference is performed starting from premises for which we already have grounds – what an agent is in possession of at the end of the inference is not something that can be canonical or non-canonical, but an object defined only by primitive operations. All this because to perform the inference means for the agent applying an operation, that is "to compute it" on the grounds of which he/she is already in possession, so as to get a ground for the conclusion. The same cannot be said of proofs or valid arguments in proof-theoretic semantics, where to make an inference only means to ensure that a conclusion follows certain premises, and where the structure resulting from the deductive activity is therefore susceptible to the canonical/non-canonical distinction; when the structure is non-canonical, the fact that it is valid will be visible only after the subsequent application of justifications that show how to reduce it to a canonical structure. Nor, due to structural reasons, can the application of such justifications be understood as simultaneous to the completion of the inferential passages, as instead happens in the theory of grounds.

However, the theory of grounds still suffers from a problem of recognizability of the good position of certain equations. Since inferences are intended as applications of operations on grounds, non-canonical inferences are to be understood as applications of non-primitive operations on grounds. The latter are in turn defined, not only by a domain and a codomain, but also by an equation that shows how the operation behaves. The equation provides a method of "computation", or "transformation", through which, when executed, it is constructively possible to pass from grounds for the premises – arguments of the operations – to grounds for the conclusion – the value of the operation on those arguments. The equations are a "functional" version of the reduction or justification procedures of non-canonical inferences in proof-theoretic semantics. In this sense, we could say that the theory of grounds proposes a sort of "internalization" of such procedures to the process of construction of the argument structure. In appropriate circumstances, to prove means "computing" reductions of non-canonical structures.

Obviously, in his writings on the theory of grounds, Prawitz does not limit himself to state the ideas that, in a very general line, we have been listing so far. On the contrary, he indicates their formal articulation, which in turn seems to go in two distinct directions, albeit connected. The first consists of a more accurate characterization of grounds and operations on grounds as objects typed on formulaa of a background language. The typing establishes a link between the object and the judgment or assertion for which that object constitutes justification. The second aims at the development of a formal language, that includes terms denoting the aforementioned objects, and formulas that indicate their main properties. The resulting picture seems to come close to approaches of similar intuitionistic or, more generally, constructivist inspiration, such as the Kreisel-Goodman theory of constructions (Kreisel 1962, 1965; Goodman 1968, 1970, 1973) or Martin-Löf intuitionistic type theory (Martin-Löf 1984). It is not surprising, then, that both the typing of objects, and the formal languages the terms of which denote such objects, come near the analogues in the typed λ-calculus; the general directives of the theory of grounds can therefore be taken into account also, and perhaps mainly, in the perspective of the *formulas-as-types conception*, a cornerstone of the Curry-Howard isomorphism (Howard 1982).

Despite these precious suggestions, however, the more formal side is, in the writings that Prawitz has so far dedicated to the subject, in an only embryonic state. On the other hand, the importance of the demand and of the basic objectives of the theory of grounds, the wide range of theories, traditions and hints to which it is connected, both in the philosophical field and with respect to contemporary mathematical logic, and last but not least, the progress it allows in many respects, are, in our opinion, more than enough reasons to reveal the need for an in-depth development of the "technical" area of the theory. Of course, this in-depth analysis does not have only a purpose of systematization, since it allows also to better understand some of the basic assumptions of the theory itself, to enlighten some of its not secondary philosophical aspects, to draw non-trivial consequences, and to indicate further possible theorizations.

In this context, our proposal will move along two main directives. First, the delineation of a class of formal languages of grounding, and the related definition of a denotation function that allows to associate terms with grounds and operations on grounds. Secondly, the development of a class of formal systems of grounding, which allow to establish, deductively, relevant properties, expressed by means of appropriate formulas, of the terms and of some of the symbols of the languages of grounding. In either case, these primary intentions will be accompanied by a certain refinement of the analysis, with the aim of perfecting general definitions, by introducing concepts and proving results that allow a more specific application of the definitions themselves. Far from claiming to be exhaustive, our contribution is to be understood as the first draft of a fully "mathematized" theory of grounds, so as to highlight fundamental characteristics which, in our opinion, should apply in a complete formalization. Starting from this core, it will also become clearer how and where further advancements could differ, depending on the different needs, in specific choices or alternative characterizations.

The work is divided into three parts. The first illustrates the theoretical background, with its results and its open problems, on which the theory of grounds is based, or to which, more or less directly, it is linked. The second part, which corresponds entirely to the fourth chapter, aims at a reconstruction of the theory of grounds as it has so far been presented by Prawitz, showing the progresses it allows, the problems shared with the background illustrated in the first part, and the points liable to further refinement. These points are finally taken into account in the third part, with a view to a first attempt at formalization, as systematic and pregnant as possible.

The first part is in turn divided into three chapters. The first raises the fundamental problem of the theory of grounds, namely the explanation of the power of epistemic compulsion of deductively correct inferences and reasoning, providing a clarification of the fundamental concepts involved – i.e. inference, proof, premises/conclusion, and so on. The second chapter, starting from the consolidated bond that, in contemporary mathematical logic, usually exists between the concepts of (logically) valid inference and (logical) consequence, explains the main reasons for the inadequacy of model-theory in capturing a notion of necessity epistemically understood. After that, we will introduce the alternative Prawitz's proof-theoretic semantics, following the different configurations it has taken over the years, and focus on some of its weak points from which it seems to suffer compared to a satisfactory explanation of the epistemic force of deduction. The third Chapter of the first part offers a general description of two theories conceptually and formally similar to the theory of grounds – Kreisel-Goodman theory of constructions (Kreisel 1962, 1965; Goodman 1968, 1970, 1973) and Martin-Löf's intuitionistic type theory (Martin-Löf 1984) - and discusses some of Prawitz's observations on these theories, concerning issues that then will be crucial in the theory of grounds.

The third part is again divided into three chapters, going from the fifth to the seventh. In the fifth Chapter we introduce, related to the notions of first-order logical language and atomic base on such a language, a class of languages of grounding that include only terms; the class is developed by expansions of a *core* language which contains operational symbols corresponding to primitive operations on grounds, and the expansions are classified, according to different properties, in relation to a denotation function that associates grounds or operations on grounds to the terms of the language. In the following chapter, languages of grounding are enriched with predicates that allow us to construct formulas which in turn allow to express properties of the terms and of some of the symbols of the alphabet. The provability of these properties is reached then by means of a class of formal systems of grounding, each equipped with a set of rules identified starting from the same principles that had led to the characterization of the denotation functions. Finally, in the seventh and final Chapter two questions are discussed: first, the completeness of first-order intuitionist logic with respect to the apparatus developed in the two previous chapters, and this in the form of a conjecture - in a "weak" and "strong" version – which transposes the conjecture elaborated by Prawitz (Prawitz 1973) for his proof-theoretic semantics; the second discusses the aforementioned problem of recognizability, as it appears in the theory of grounds, in relation to the theme of the general form of the equations that set the behaviour of non-primitive operations on grounds, so as to define them.

Part I

Theoretical Background

Chapter 1

Inferences and proofs

1.1 Nature of inferences

Our mental activity is characterized by a series of processes and acts through which, by elaborating information, knowledge, thoughts and beliefs, we pass to other information, other knowledge, other thoughts and other beliefs. These operations, as well as their purposes, can concern several levels of awareness and voluntariness, varying degrees of complexity, a different use of time, different memory resources, thus implying a greater or lesser force in the results obtained. Frequently, an absolute unconsciousness is accompanied by unintentionality and automatism, and immediacy, rapidity and relative simplicity produce/generate uncertain or fallible acquisitions. At the opposite extreme, the completion of operations can be totally conscious and voluntary, often complex, long and tiring, and lead to a state of which the epistemic content seems conclusive and not refutable. Obviously, between these two poles there are many intermediate stages, in which various elements are combined in a partial and heterogeneous way.

Linguistic practices, on the other hand, are largely, if not essentially, aimed at the exchange of information and knowledge. The linguistic heritage, which a more or less extensive community uses for communication purposes, is, it seems, closely linked to the mental activity of each speaker; the two levels, only apparently distinct in public and private, intersect each other in such a narrow way as to be, from certain points of view, hardly, or at least not significantly, separable. When we are willing and able to do it, we express what we think, sometimes remarking that what has been said depends on something else, from previous elements in our possession, and maybe that we have arrived there through a certain path. Our interlocutor can share, take with him/her what we have made known, in turn communicate to us; but also, he can criticize us, ask for further explanations, and indicate errors we had not noticed. This urges us, or should urge us, as it were, on retracing our steps, reviewing our line of thought, modifying it, by expanding our intellectual baggage or rejecting its elements no longer sustainable. Dialectics, then, can continue, in an overall activity of which it is probably impossible (and in any case not required here) to give a precise picture. In this context, a role of primary importance is played by those that, using a widespread term of technical literature, we call *inferences*.

The notion of inference seems to rest on at least three indisputable points. First, an inference involves a *passage* from certain data, generally called *premises*, to another datum (or more), the *conclusion* (or the conclusions). Secondly, inferences must be connected to *reasoning*. In fact, they can be described as minimum units which can be reached by breaking a reasoning down into progressively simpler parts; in turn, a reasoning can be understood as a chain of inferences. Finally, inferences are *important to logic*. According to a fairly widespread reading, in fact, the latter is to be understood as the science of correct reasoning. It must therefore deal with inferences and reasoning in general, providing tools by which to establish *which* inferences are valid and *which* pieces of reasoning are correct, as well as a further analysis of the notions of validity and correctness themselves, that is to say, *when* and *why* an inference can be said valid, and a reasoning correct. The choice of the theoretical armament, in this sense, will have to be adequate with regard to informal *desiderata*.

From this point of view, their minimal character clearly does not exclude that the inferential units can be further analyzed. However, this circumstance could collide with the intention of finding a notion of inference that does not change depending on the historical and scientific context. Similarly, if the idea that a reasoning is a chain is not accompanied by the requirement that its steps have a minimal complexity, an enormously complicated reasoning can be reduced to an inference having as premises its hypothesis, and as conclusion its conclusion. This, for its part, could not go together with the epistemic *desiderata* of an analysis that looks at inferences as acts carried out consciously and voluntarily by agents with limited time and resources. In fact, some transitions could be too difficult, so that agents of the type described may be never willing to carry them out.

In both cases, the description of the nature of inferences will influence the description of the nature of reasoning and, if we are too generous in our way of looking at inferences, the link could be affected. If we accept that some inferences may be unconscious, involuntary and even automatic steps, we should also be willing to accept that steps of this type do *not* occur in reasoning, or be equally generous in reasoning, so as to authorize unconscious, involuntary and automatic components in it. More generally, the interdependence between inferences and reasoning makes it impossible to adopt the above description as a definition of the concepts involved: one cannot, on pain of vicious circles, postulate that an inference is the minimum unit of a reasoning, and at the same time that a reasoning is a chain of inferences. The aforementioned intertwinement is therefore a result to aspire to, and not a given from which to start. We will have the opportunity to discuss more extensively these problems later.

Cesare Cozzo has argued that different conceptions about the nature of inferences are not necessarily incompatible, since they can rather serve different purposes. In particular,

> the question "what conception of inference ought we to adopt?" thus leads to the question "what is the problem?". (Cozzo 2014, 165)

Given the three previous points, the answer to the question on the nature of inferences, therefore, cannot and should not be univocal. How to articulate it, then? Cozzo himself indicates, in a commendable way, seven relevant factors.

The *first* one concerns the nature of premises and conclusion. If some authors maintain that the description given above, according to which premises and conclusion are mere data, is satisfying, others consider it too permissive. Premises and conclusion are truth-bearers, that is, entities that are liable to be true or false. And in turn, this can be understood in at least three manners; truth-bearers can have an abstract nature, in which case one usually speaks of propositions, or a linguistic nature, being therefore sentences, or, finally, they can assume the form of mental states, or beliefs. However, the range of possible answers does not end here. Following yet another approach, we could in fact require that

> premises and conclusions are not objects or states in which we happen to find ourselves, but responsible *acts* or *actions*, which we *do*. (Cozzo 2014, 162)

Such acts or actions can, once again, stand out on a mental level, as judgments of various kinds, or on a linguistic level, as in the case of assertions (but also, perhaps, of questions or commands). But, more precisely, what are propositions, sentences, beliefs, judgments and assertions? Here too, the answers are heterogeneous, and so we have numerous, further subramifications. The *second* factor is given by the inferential agent. As in the previous case, we can reason in a more or less exclusive way:

> if you think that only a person can make an inference, you have a narrow conception of the subject of inference. If you believe that not only a person, but also a machine, or a non-personal biological entity can infer, then you have a broad conception of the subject. (Cozzo 2014, 162 - 163)

The *third* ingredient involves the relation between the agent and the set of premises and conclusion, and depends, or it is expected to depend, on the way in which the first and the second factor have been settled. Thus, we could say that

> the data X are stored in S; S is in the representational state X; S is in the neural state X; S performs the act X, *etc.* (Cozzo 2014, 163)

Probably, the most important factor, no doubt central to the logical inquiry, is what Cozzo labels as the *fourth*, and concerns the relation between premises and conclusion. Here, we move from an extremely inclusive response, according to which an inference is simply a pair where the first element is the set of premises and the second the conclusion, to answers instead insisting on the fact that, in an inference, premises and conclusion cannot be completely untied, for some sort of connection must occur among them. What this connection is, however, is anything but unquestionable. Is it an abstract relation, or maybe a causal relation dependent on a psychological, possibly unaware, involuntary and automatic transition? Perhaps, none of these two things, but in a more epistemic sense,

> a *conscious and deliberate act* on the part of the subject. (Cozzo 2014, 163 - 164)

The *fifth* element is the stability of the premises-conclusion relation. This relation in fact can be completely aleatory, or substantial but refutable according to future circumstances. However, some inferences, commonly called *deductive*, seem to be such that

> the connection between premises and conclusion is stable and can never be subverted by a new piece of information. (Cozzo 2014, 164)

At the *sixth* point, we find the more or less public character of inferences. If inferences are conceived as subpersonal psychological transitions, it will be very difficult to think of them as something publicly communicable. According to many, however, this view must be rejected: inferences, and the pieces of reasoning in which they are involved, must be able to materialize in practices accessible to all members of the community. Finally, the *seventh* factor concerns what Cozzo defines

> the context in which premises and conclusion are placed. (Cozzo 2014, 165)

When we describe an inference, shall we limit ourselves to describing only its premises, conclusion, agent, and their relations? Should we not, perhaps, take into account also the broader context in which the inference is accomplished? Maybe other inferences, or pieces of reasoning, to which it is connected? Or the whole information or knowledge in possession of the agent? Or even the whole set of co-agents able to perform and accept inferences?

1.2 Valid inferences

The seven factors outlined by Cozzo offer a general, neutral grid in which to frame different conceptions on the nature of inferences. However, as far as we are concerned, if the adoption of a determinate conception depends on the problem we intend to solve, what we have just affirmed must be reported to the object of this investigation, namely, Dag Prawitz's *theory of grounds*, and to the fundamental question the latter aims to answer: in what sense and why do some inferences - often called *valid* - seem endowed with a power of epistemic compulsion?

1.2.1 Epistemic compulsion

Epistemic compulsion is something we often experience in everyday life. The bill for a dinner at the restaurant amounts to 45 euros, we have given the waiter a banknote of 50 and expect to receive 5 euros change, and so the waiter will have to do. An inference has been made, the premises of which concern the following circumstances: the bill is of 45 euros, we have given the waiter a banknote of 50, and 5 is what we get by subtracting 45 from 50 (being subtraction the relevant operation). Of course, we could feel magnanimous, or have made a mistake, and ask for a lower change. In the same way, the waiter might have been mixed up, or could refuse to give us his due, perhaps claiming to follow a strange arithmetic in which 45 and 50 indicate the same quantity. Except the details and the secondary aspects, however, the inference compels us to accept its conclusion: we feel authorized to claim

5 euros, and the waiter should feel obliged to give us exactly that sum. Again, if we are reasonably convinced that A implies B, and we have ascertained A, we have to conclude B. Without this having consequences of any kind, we are obviously free to refuse the conclusion. There is though a clear sense in which anyone would maintain that such a behaviour would be wrong, or even irrational. Being free to do what one wants seems to fail if the correctness of reasoning or the sustainability of its conclusions are at stake: we are forced because we *feel* forced.

We are, in other words, in the presence of a very particular phenomenon, which originates from and depends on

> a special force, which is neither the threat of violence, nor the charm of a seductive persuader: it is simply the sober force of reasoning. (Cozzo 2014, 165)

Reasons and modalities of epistemic compulsion have often been at the center of the reflection of philosophers and logicians. Precisely to this issue Prawitz has mainly and explicitly dedicated some of his latest works. In them, the Swedish logician proposes and develops the aforementioned theory of grounds, an answer, at the same time philosophical and formal, that passes through an innovative characterization of the nature of inferences and of their validity, of proofs, and of the conceptual content of such, interconnected, notions. From this perspective, the theory of grounds is, so to speak, intrinsically worthy of interest. However, its importance derives also from the solutions and advances that it is able to offer with respect to similar approaches, including the one that Prawitz himself developed in the past. After all, and as we shall see, the epistemic relevance of valid inferences and proofs has always been one of the pivotal points of Prawitz's semantic investigations; and the theory of grounds constitutes a significant change of perspective with respect to these previous conceptions.

Focusing on the problem of epistemic compulsion seems to impose some forced choices on the type of inferences to be considered, a quite binding selection with respect to the general framework provided by the seven factors identified by Cozzo. First, as regards the stability of the premises-conclusion relation, we are only interested in deductive inferences. Of course, we could also feel epistemically forced towards conclusions drawn on the base of inferences, the strength of which might decrease, or even vanish, in view of future occurrences. However, in this case, the inference accomplished does not provide a truly conclusive support, and inferences of this type may be of interest for logic only in relation to a notion of valid inference, as cases that do not respect the conditions of the corresponding definition.

As for the nature of premises and conclusions, it seems reasonable to argue that their description in terms of data is decidedly unsatisfactory, or at the very least too general. There are certainly data for which it makes sense to speak of epistemic compulsion. It is a datum for those who accept the rules of usual arithmetic that every multiple of 3 is divisible by 3, and that $6 = 2 \cdot 3$, and it is therefore a datum to which one is epistemically compelled that 6 is divisible by 3. However, there are also data for which this discourse does not seem to be valid. Let us assume we are monitoring the different configurations that the iris of an observer O takes according to different colors projected on a screen; when the red color, the datum-premise, appears on the screen, the iris of O will assume a certain configuration, the datum-conclusion. From a certain point of view, O is forced, so to speak, to draw a certain conclusion, but it would be strained to speak of an epistemic compulsion. The constraint, in fact, is not generated by the sober force of reasoning, nor is O free, if he/she wants, to oppose it. On the contrary, the compulsion acts in an unconscious, involuntary and automatic way (the awareness that the red color has been projected arrives at a later moment of reflection). Premises and conclusion should therefore be, at the very least, truth-bearers, that is to say, propositions, sentences or beliefs. Of course, it is compatible with our purposes even the stronger circumstance that premises and conclusion are judgments or assertions.

In the light of the above, it does not even seem promising to claim that the agent of an epistemically compelling inference can be a generic biological entity; propositions, sentences, beliefs, judgments and assertions are objects or acts concerning an abstract sphere, conceptual or linguistic, which only with an extreme forcing we could attribute, for example, to a jellyfish. A human being is certainly more suitable, but what about the famous Chrysippus' dog? Following its master, which is far and not visible, it arrives at a crossroads, sniffs one of the possible branches and, not smelling its master, confident and without sniffing takes the other path. Similarly, can the agent of an epistemically compelling inference be a machine? As Prawitz rightly remarks, the central point is, here, that epistemic compulsion involves a reflective activity, lacking in animals and machines. In fact, it seems to go with a reflection through which the performed activity can be appropriately generalized, and understood as deductively reliable:

> when we deliberate over an issue or are epistemically vigilant in general, we are conscious about our assumptions and are careful about the inference steps that we take, anxious to get good reasons for the conclusions we draw. [...] taking for granted the truth of a disjunction 'A or B', and getting evidence for the truth of not-

> A, we start to behave as if we held B true without noticing that we have made an inference [...] the Babylonian mathematicians were quite advanced, for instance knowing Pythagora's theorem in some way, but, as far as we know, they never tried to prove theorems deductively. (Prawitz 2015, 67)

Essentially the same reasons that have guided the previous choices should at this point induce to demand that, between the agent of an epistemically compelling inference and the set of premises and conclusion of such an inference, there is something more than having stored certain information, or finding oneself in a neural state. The agent must be in a sense conscious of the content of premises and conclusions, if we consider them as propositions or sentences, or, in the case of a reading in terms of beliefs, find him/herself towards them in an intentional state, or, finally, when premises and conclusions are conceived as judgments or assertions, perform the acts to which they correspond. This, one would say, also favors a reading of the premises-conclusion relation in terms of a conscious and deliberate act on the part of the agent. The conclusion is (or binds to) the prefixed goal, which the agent shows to aim to achieve on the basis of some sort of support provided by the premises. In any case, such an act could be accomplished at a later stage, after the achieved awareness of a link between premises and conclusion. Therefore, it is not possible to exclude *a priori* that the premises-conclusion relation can be of an abstract type.

Finally, the question relating to epistemic compulsion is undoubtedly compatible with a public view of inferences, and of the reasoning in which they occur. In fact, the reflective character of the inferences in question seems to imply that we can give of them, and of the reasoning in which they are used, a testimony, and this without denying that the phenomenon can involve - also, or mainly - mental dynamics. Certainly, there is a long tradition, dating back to at least René Descartes (Descartes 1985), according to which some acts, essentially intuitive and private, induce an epistemic compulsion. A discussion on the plausibility of this thesis, or on the way it can be articulated, would obviously take us too far; therefore, we will restrict ourselves here to emphasize that the compelling character of such acts must in any case refer to certain propositions or sentences, and ultimately depend on the meaning attributed to them. But then, the analytical character of such supposed intuitive knowledge is something that can manifest itself in practices that testify the acceptance and sharing of meanings. This observation leads us to the final case of the context in which inferences are accomplished, or located. In this case, however, the possible alternatives are different, and all mutually compatible.

1.2.2 Justification

That some inferences are commonly held to have a power of epistemic compulsion is certainly a fact. On the other hand, the existence of valid inferences can be a matter of doubt. The deductive ability to support conclusively propositions, sentences, beliefs, judgments or assertions, might be nothing but an illusion, and some forms of scepticism seem to be oriented exactly towards this thesis. Prawitz himself observes, however, that

> to justify deduction in order to dispel sceptical doubts about the phenomenon is not likely to succeed, since such an attempt can hardly avoid using deductive inference, the very thing that is to be justified. This should not prevent us from trying to explain why and how deductive inference is able to attain its aims. Deduction should thus be explained rather than justified. (Prawitz 2015, 65 - 66)

If we successfully fulfill the task of explaining how and why some inferences force epistemically, we are also in possession of a weapon against the sceptic. On the other hand, in order to express more than a mere opinion, the sceptic should show that no plausible conception on the nature of inferences, and of their validity, is such as to attribute to valid inferences something that can count as a power of epistemic compulsion. Even assuming that this does not go through the use of inferences that the sceptic treats as epistemically compelling, it should still result in something very similar to what Prawitz, although with opposite objectives, suggests to do. Therefore, the basic question of the theory of grounds is, it seems to us, important even for the sceptic. Be that as it may, Prawitz does not give too much credit to possible sceptical positions, arguing rather that

> we take for granted that some inferences have such a power, and there is no reason to doubt that they have. But what gives them this power? This should be explained. (Prawitz 2015, 73)

Far more important, especially for the development of our reasoning, is instead another consideration. The questions related to what we are here calling epistemic compulsion, in fact, have often been formulated as questions related to the ability that valid inferences have to justify who performs them with respect to their conclusion. It is often said that valid inferences preserve, or transmit, justification, in the sense that, if one is justified with regard to the premises, the same will be true for the conclusion. What "to be justified with regard to ..." means obviously depends on the adopted conception of inference, but it is clear that the justification we are talking about is of the strongest possible type: a *conclusive* and *non-refutable* justification, a *definitive* reason for the truth of propositions, sentences or beliefs, or for the correctness of judgments or assertions. The idea seems to be implicit, or at the very least relevant, even when valid inferences are conceived in the most abstract sense of necessary truth-preservation: it cannot occur that the premises are true, and the conclusion is false. The manner in which, from the necessary preservation of truth one passes to the preservation or transmission of justification becomes then, in general, a central problem in the frameworks, philosophical as well as formal, which adopt such an approach.

As Cozzo further points out (Cozzo 2014), epistemic compulsion and preservation or transmission of justification are in any case two sides of the same coin. The general argument in favor of this position is based on a

> plausible assumption concerning the relation between the notion of justification and the notion of argumentative context with rational disputants, the idea that justification is something publicly acknowledgeable by rational subjects: if a person is justified in asserting a sentence, or an inference has the power to transmit justification, then both facts must be acknowledged by the disputants involved, if they are rational. (Cozzo 2014, 165)

From this assumption it derives the equivalence between the circumstance that someone is forced or justified in accepting something, and the circumstance that each of his/her interlocutors is equally forced on pain of irrationality, or justified to do the same. Therefore, given two interlocutors A and B, let us suppose that A is justified in accepting the premises in a set Γ of an inference I from Γ to the conclusion α that preserves or transmits justification, and let us suppose furthermore that A performs I. According to the public nature of justification, B recognizes that A is justified in accepting the premises in Γ and, according to the public nature of the preservation or transmission of justification, B also recognizes that A is justified in accepting α. Therefore, on pain of irrationality, B is forced to accept α. Vice versa, let us suppose that I is epistemically compelling and, again, that A is justified in accepting the premises in Γ, and that he/she performs I. According to the public nature of justification, B recognizes that A is justified in accepting the premises in Γ, and therefore he/she is him/herself obliged to accept them; but then, B will be forced, on pain of irrationality, to accept α. It follows that, for the above equivalence, A will be justified in accepting α.

Although the capacity to force epistemically and the capability to preserve or convey justification are equivalent concepts, we will have the opportunity to see how, in the specific framework of Prawitz's theory of grounds, the formulation in terms of epistemic compulsion is nevertheless significantly informative of a precise, and in many respects innovative, way of looking at inferences and analyzing their validity.

The acquisition of knowledge is the ultimate goal, as well as the main result, of deductive activity. The means by which this end is achieved is, of course, that particular form of reasoning called proof. But what is a proof? And what is the connection between proofs and valid inferences?

1.2.3 Proofs

Inferences can be valid, but they can also fail. They can indeed force, in the epistemic way described above, to accept a certain conclusion, and really preserve or transmit justification towards it. But they can also only give the illusion of doing it. This happens frequently, and becomes particularly evident within the framework of disputes among subjects who aim to achieve truth or knowledge. Some perform premises-conclusion steps of which they are firmly convinced, and therefore feel epistemically forced, or justified, to accept the conclusions they have reached. However, they have made some mistakes, and if the other interlocutors are careful enough to realize it, they will point out that something has gone wrong and that for this reason the conclusions must be withdrawn. Therefore, those who, at first, had made demands on the cogency of their results will be willing to admit they were wrong.

The same argument applies to the reasoning to which the inferences are linked; it can be wrong or, similarly to valid inferences, force epistemically, and conclusively justify the acceptance of its conclusions. The term commonly used in the latter case is that of *proof*. A proof is a reasoning crowned by epistemic success, and therefore it does not make sense to speak, for it, of error. From a conceptual point of view, a wrong proof is a *contradictio in terminis*.

Proofs play a major role in deductive activity that has been involving human beings since the dawn of exact sciences. They are the most secure source, the most reliable instrument in the pursuit of truth and of the knowledge of what one is intellectually committed to, or intends to find out. Therefore, as early as its birth with Aristotle, logic has already been questioning the nature and reasons of the mysterious force that proofs exert on us, and the conditions that a reasoning must satisfy to be said correct. Over the centuries, the dual objective of justifying and explaining this link has been understood in very different ways. However, contemporary logic is the child of a tradition dating back to the late nineteenth century, and to the first half of the twentieth, when the researches of Gottlob Frege, Bertrand Russell, David Hilbert and Alfred Tarski, in particular, marked a point of no return compared to previous authors.

Through a reflection on the rules and universal principles of deductive practice, and an axiomatization of the same, the formalization process has allowed a structural definition of proof as a derivation in a system. The adequacy of purely syntactic configurations was then connected to, and made to depend on, a rigorous delineation of semantic concepts. From this point of view, the completeness theorem for predicative logic, and the incompleteness theorem for Peano's first-oder arithmetic, both proven by Kurt Gödel, have been the watershed for all the subsequent work. They have especially influenced a reflection on the notion of proof as such, which led to what we now call *proof-theory*. The origins of this theory can be traced back to the same Hilbert, who

> hoped to establish the consistency of mathematics or, more generally, to obtain a reduction of mathematics to a certain constructive part of it. Hence, the study of proofs was here only a tool to obtain this reduction, and it could thus not use principles that were more advanced than those contained in the constructive part of mathematics to which all mathematics was to be reduced. We may call such a study *reductive proof theory*. (Prawitz 1973, 225)

However, a disciple of Hilbert's, Gerhard Gentzen, had already taken his first steps towards a more comprehensive approach, with the so-called *sequent calculi*, and the fundamental *Cut*-elimination theorem, and with the *natural deduction calculi*. Gentzen's legacy was largely collected, among others, by Prawitz, who, first with the normalization theorems for the natural deduction calculi, and then with his proof-theoretic semantics, laid the foundations for what he himself defines as *general proof-theory*. Here,

> we are - in contrast - interested in understanding the very proofs in themselves, i.e., in understanding not only *what* deductive connections hold but also *how* they are established, and we do not impose any special restrictions on the means that may be used in the study of these phenomena. (Prawitz 1973, 225)

Over the years, and in a constant comparison with suggestions such as Michael Dummett's researches in theory of meaning and with the intuitionistic tradition coming from Luitzen Brouwer and Arend Heyting, Prawitz's theories have evolved. Thus, the importance of the underlying question of the current theory of grounds is mainly due to the link between valid inferences and proofs: to explain how and why valid inferences can be epistemically compelling means also to shed light on proofs, on their strength and reliability. But how to connect valid inferences with proofs?

As we have already affirmed, some inferences are involved in reasoning. We can look at the first as minimum units of the latter, or give the latter the meaning of inferential chains. However, in both cases the relation between the two notions becomes relevant for the type of inferences we must take into account, if we intend to deal with epistemic compulsion. Deduction, in particular, does not seem to be a generic data processing; it is a conscious activity, in which one deliberately aims to highlight dependency relations among propositions or sentences, to validate beliefs, to support judgments or assertions. It is accomplished by intelligent agents, who reflect on what they are doing, and can recognize the content of the information in their possession, have access to intentional states, make moves in thinking and in language, and communicate with one another within the framework of a more or less broad context.

As a correct reasoning, a proof has a particular link with valid inferences. From an informal point of view, the common idea seems to be quite precise here: the notion of valid inference is prior compared to the notion of proof. And this seems, for example, the concept Descartes illustrates in a famous passage:

> a continuous and uninterrupted movement of thought in which each individual proposition is clearly intuited. This is similar to the way in which we know that the last link in a long chain is connected to the first: even if we cannot take in at one glance all the intermediate links on which the connection depends, we can have knowledge of the connection provided we survey the links one after the other, and keep in mind that each link from the first to the last is attached to its neighbour. (Descartes 1985, 15)

This line of reasoning contains two important intuitions: 1) *all* the premises-conclusion steps of a proof are epistemically compelling, and, 2) proofs are something we do consciously, acts carried out deliberately and voluntarily. As we will see, not always more formal descriptions of the notion of proof have followed these guidelines. On the contrary, the theory of grounds seems to look at such a way of proceeding as the only promising strategy towards an adequate description of the epistemic power of valid inferences, as well as of the epistemic power of proofs.

1.3 The fundamental task

How should the underlying question of the theory of grounds be more precisely formulated? Is it possible to identify a programmatic setting of a satisfactory explanation of the strength of epistemic compulsion of which valid inferences are endowed? In the articles *Inference and knowledge* (Prawitz 2009) and *The epistemic significance of valid inference* (Prawitz 2012a), Prawitz outlines a general framework that may perhaps serve that purpose. Given the premises

a) I is a valid inference from the premises Γ to the conclusion α;

b) A is in possession of a ground for each of the premises in Γ,

which further condition c) must be added to a) and b) in order to get

d) A has a ground for α?

First of all, it should be noted that Prawitz, in this formulation, uses the expression "ground". As it is easy to imagine, the notion of ground will receive, in its resulting theory, a particular connotation as well as a formal expression. However, the question is sufficiently clear, so that it can be guessed without knowing the theory of grounds. "Ground" can for the moment be understood in its ordinary sense, as the proof of the truth of something, or the reason to believe in it, or the guarantee for the correctness of a judgment or an assertion.

It is manifestly clear, however, that the identification of the further condition c), and the derivation of d) from it and from a) and b), will require an extension, a deepening and a more rigorous characterization of the notion of ground. But even before doing this, two other goals must be achieved: to provide an adequate definition of the notion of valid inference and, then, to identify an appropriate relation between A and I. Instead, at the conclusion of this section, it is perhaps appropriate to point out with Prawitz that

> the problem posed is not how an agent is to show that she is in possession of a ground [...]. Our problem is to account for how, thanks to a valid deductive argument or inference, an agent gets a ground [...], in other words to say under what conditions she obtains a ground. When the conditions are satisfied and the agent is thus in possession of a ground [...], she is thereby justified [...] without having to show anything. But the philosopher who claims to have given an account of how grounds are obtained has

> of course to verify that the stated conditions are sufficient, and should therefore be able to derive from them that the agent has in fact a ground. (Prawitz 2012a, 892 - 893)

In *Explaining deductive inference* (Prawitz 2015) (where the point is put in less general terms, which is why we have preferred here the previous formulations), Prawitz also points out how, in trying to derive d) from the further condition c),

> one may very well use a generic inference of the same kind as the one that is shown to be legitimate in this way. The point is not to convince a sceptic who doubts that the inference is legitimate, but to explain why it is legitimate, and to this end there can be no objection to use this very inference. (Prawitz 2015, 75)

Chapter 2

From models to proofs

2.1 Inference and consequence

According to a rather widespread conception, the notion of valid inference must be defined through the notion of consequence: an inference is valid if, and only if, its conclusion is a consequence of its premises. The fact that the notions of valid inference and consequence are closely related emerges clearly both from a conceptual examination of what, informally, we mean by them, and from the historical development of mathematical logic.

When we say that something is a consequence of something else, generally we mean that what follows has, in what it follows from, foundation, reason, guarantee, and this by virtue of a bond that, with absolute cogency, applies on the basis of unquestionable and universal properties or laws. If the bond can be captured by thought, so expressing in the form of reasoning the properties or laws that determine it, what is obtained is, or it is expected to be, a proof of what we get on the basis of the hypothesis from which we started. Sometimes, the proof consists in a single, simple passage; in other cases, we need to perform many, complex intermediate transitions. But whatever the length and difficulty of the journey, we will have to deal with valid inferences, which exert a power, from an epistemic point of view, by preserving or transmitting justification. On the other hand, if we have achieved some conclusions by proof starting from certain hypotheses, we are also led to believe that between hypotheses and conclusion there is a special bond, and that the two poles are therefore united with the same force of which the corresponding reasoning is endowed.

Thus, valid inference and consequence refer to each other and, since the famous Aristotelian definition of syllogisms, they have been approaching to the point to become often indistinguishable. Since proofs highlight universal connections of consequentiality, in dealing with the former, logic will have to question also (and perhaps mainly) on the latter. However, between the two notions there is a substantial and often ignored difference, the relevance of which is evident when the analysis of the notion of valid inference privileges an epistemic point of view.

To deal with the power of epistemic compulsion of deductively valid inferences requires looking at the latter as *acts* through which intelligent agents *pass*, in a conscious and voluntary way, from certain premises to a certain conclusion. Premises and conclusion can be truth-bearers, that is propositions, sentences or beliefs, or consist, in turn, of acts, namely judgments or assertions; moreover, between premises and conclusions there must be a link, which, however abstract, is in some way known to the agent. Consequence is, on the contrary, a *relation* between truth-bearers, and specifically between propositions or sentences; in mathematical logic we speak, more precisely, of *logical consequence*, where, according to Prawitz (Prawitz 2005) and Cozzo (Cozzo 2015), this last notion can be unanimously understood as a particular instance of a more general relation of deductive consequence:

(CD) α is a deductive consequence of Γ if, and only if, necessarily, if all the elements in Γ are true, α is true;

(CL) α is a logical consequence of Γ if, and only if, α is a deductive consequence of Γ, and this holds only by virtue of the logical form of α and of the logical form of the elements in Γ.

Deductive consequence is, as indicated by the expression "necessarily" occurring in (CD), a *modal* relation; as a restriction of deductive consequence, logical consequence inherits this modality, but (CL) informs us that it, in this case, depends solely on the *logical form* of propositions or sentences under consideration. What kind of modality is that? And what is the logical form?

The second question can be answered in a way that, though often criticized and subjected to major or minor revisions, is nevertheless shared, or at least assumed, by most of the scholars in this area. It is a line of thought dating back to Bernard Bolzano, corrected and expanded by Tarski, and finally generalized by model-theory. The underlying idea is that, within a more or less formalized language, one must distinguish between constant symbols and variable symbols, whereas the former include, at least, those that are usually called logical symbols, "not", "and", "or" (in the inclusive sense), "if ... then", "all", "some" (which will come later formally indicated with the respective symbols, $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\forall$ and $\exists$). The fact that the logical consequence relation depends solely on the logical form of α and on the logical form of the elements in Γ, can at this point be understood in the most rigorous sense that it depends solely on logical symbols, taken as constants, regardless of the specific content of the non-logical ones, considered instead as variables.

With this in mind, Bolzano (Bolzano 1837) proposed a so-called *substitutional* approach. Leaving aside secondary details, given a proposition or sentence α, we call *substitution for* α a proposition or sentence α^{Σ} obtained from α by replacing the non-logical symbols with other non-logical symbols (not necessarily different) of the reference language. Given a set of propositions or sentences Γ, we indicate with Γ^{Σ} a set of substitutions for every $\beta \in \Gamma$. Then

(Bol) α is a logical consequence of Γ if, and only if, for every α^{Σ} and Γ^{Σ}, if all the elements in Γ^{Σ} are true, α^{Σ} is true.

However, Tarski noted how (Bol) constitutes a necessary but not sufficient condition for a plausible characterization of the notion of logical consequence. In particular, it violates what John Etchemendy (Etchemendy 1990) called the *persistence principle*: if α is a logical consequence of Γ in a language L, α must be a logical consequence of Γ in every expansion of L. In fact, if the reference language L to which (Bol) refers does not contain a sufficiently large number of non-logical symbols, it might be the case that, despite all the possible substitutions with respect to L are such as to meet the conditions required by (Bol), there is an expansion of L obtained by adding to the latter non-logical symbols of the appropriate type and such that, by virtue of the new substitutions available, the relation of logical consequence ceases to be valid. (Bol) could therefore be validated simply by virtue of expressive resources, essentially not logical, of the reference language. As we will see in Section 2.3, Tarski, and the model-theory arising from his research, solve the problem with a much more refined approach than the substitutional one. Nevertheless, the overall intuition remains the same: the fact that the relation of logical consequence depends solely on logical symbols, regardless of the content of the non-logical symbols, must be understood as invariance of consequentiality under variation of this content. However, by far more difficult is the question related to modality. How to understand the expression "necessarily" in (CD)?

A first, famous reading concerns the so-called *possible worlds*, and postulates that α is declared as a deductive consequence of Γ if, and only if, α is true in every possible world where all the elements in Γ are true. This setting, however, faces multiple difficulties: in fact it is not clear how the notion of possible world can be articulated without giving rise to circular explanations (Cozzo 2015) and, even if one appeals to model-theory, by saying that a possible world is nothing more than one of the models described in this semantics, there is some doubt about the actual plausibility of the equation between models and possible worlds (Etchemendy 1990, Prawitz 2005). Anyway, there are some authors who defend such approach. Stewart Shapiro, for example, considers it functional to a combined vision of (CD) and (CL), which leads to a limitation from the analytically possible worlds to the sole logically possible worlds (Shapiro 2005). It is therefore a controversial issue, and probably far from a definitive answer. What we can certainly say, however, is that the interpretation in terms of possible worlds is averse to the epistemic interests of the kind of analysis we intend to carry out.

On the contrary, the idea proposed by Prawitz in his *Logical consequence from a constructivist point of view* (Prawitz 2005) is much more suitable. Here, the fact that α is a deductive consequence of Γ is delineated through what Prawitz himself calls *necessity of thought*:

(NT) α is deductive consequence of Γ if, and only if, the truth of α follows by necessity of thought from the truth of all the elements in Γ,

where, with necessity of thought, one wants to say that

> one is committed to holding α true, having accepted the truth of the sentences of Γ; one is compelled to hold α true, given that one holds all the sentences of Γ true; on pain of irrationality, one must accept the truth of α, having accepted the truth of the sentences of Γ. (Prawitz 2005, 677)

Anticipated in *Remarks on some approach to the concept of logical consequence* (Prawitz 1985), *Logical consequence from a constructivist point of view* marks, in a sense, a turning point in Prawitz's semantic investigation; the reference to the necessity of thought, in fact, means that the attention focuses on primarily inferential aspects, so to have as its final consequence, as we shall see, a reversal of the order of explanation of the notions of consequence and valid inference: the first is to be explained in terms of the second, rather than vice versa. In addition, the notion of necessity of thought anticipates many of the aspects of what will be then, in the theory of grounds, epistemic compulsion. It is no coincidence that Cozzo emphasizes how

> the key feature of the relation between Γ and α is the *compulsion of the inference* from Γ to α. One cannot be compelled if one does not feel compelled. Inferential compulsion is a power that acts upon us only in so far as we are *aware* of its force. Therefore, the necessity of though is an *epistemic* necessity: by making the inference a person *recognizes* a guarantee of the truth of α given a recognition of the truth of the members of Γ. (Cozzo 2015, 104)

An epistemic understanding of the modality involved in (CD) seems, therefore, a promising way to achieve our target. On the other hand, we certainly cannot be content with the informal framework proposed by (NT). Then, the question is how to give a precise content to Prawitz's idea of necessity of thought.

2.2 Syntactic approach

An attempt in all appearances correct may consist in describing the relation of deductive consequence through that of derivability in an *appropriate formal system*. As a matter of fact, that is a path followed by all those who, as Tarski affirms,

> believed that they had succeeded, by means of a relatively meagre stock of concepts, in grasping almost exactly the content of the common concept of consequence, or rather in defining a new concept which coincided in extent with the common one. Such a belief could easily arise amidst the new achievements of the methodology of deductive science. Thanks to the progress of mathematical logic we have learnt [...] how to present mathematical disciplines in the shape of formalized deductive theories. In these theories, as is well known, the proof of every theorem reduces to single or repeated application of some simple rules of inferences. (Tarski 1956a, 409 - 410)

By formal system we mean an ordered pair $\langle L, \Re \rangle$, where L is a formal language and $\Re$ a finite set of inference rules for formulas of L. In turn, a formal language is determined by an explicitly and strictly declared alphabet, with precise clauses specifying a (possible) set of terms and a set of formulas. On the basis of $\Re$, we can then inductively define a set of derivations in $\langle L, \Re \rangle$, and, consequently, a derivability relation among the formulas of L: α is derivable from Γ in $\langle L, \Re \rangle$ if, and only if, there exists a derivation in $\langle L, \Re \rangle$ of α from Γ. Therefore the idea is that, for an appropriate formal system $\langle L^*, \Re^* \rangle$, α is a deductive consequence of Γ if, and only if, α is derivable from Γ in $\langle L^*, \Re^* \rangle$.

It soon became clear, however, that this approach was doomed to failure. In fact, above all on the basis of Gödel's incompleteness theorems, it emerged that no sufficiently rich formal system was able to capture, in the form of a derivability relation relative to it, a notion of deductive consequence sufficiently extensive, and therefore relevant. The reference to Gödel's results is at the core of Tarski's thesis, which begins by emphasizing the existence of a system S in which, for each natural number n and for a determined property P, it is possible to derive

(A_n) n enjoys the property P

but not that

($A_\forall$) every n enjoys the property P.

The situation is clearly unpleasant: from an intuitive point of view ($A_\forall$) is in every respect a deductive consequence of the set of the various (A_n), on the standard model of natural numbers. In order to solve the impasse, one might think of expanding S by adding to it a rule that authorizes the passage from the infinite premises (A_n) to the conclusion ($A_\forall$). However, such a rule would result in inferences of a nature radically different from the rules of S, where the number of premises is, on the contrary, always *finite*. Since it is reasonable to require that the desired expansion have the same logical structure as S, this line of thought does not seem feasible. Nor, for the same reasons, can we accept an expansion of S obtained by means of a rule which authorizes the passage from a premise B that fixes the derivability in S of all the (A_n), to the conclusion ($A_\forall$); B, in fact, is not a formula of the language which S is referred to, since it rather belongs to a meta-language that speaks of properties related to S itself. The latter case, however, allows a further move. Indeed, Tarski notes that now it is possible to

> restrict consideration to those deductive theories in which the arithmetic of natural numbers can be developed, and observe that in every such theory all the concepts and sentences of the corresponding metatheory can be interpreted. [...] We can replace in the rule discussed the sentence B by the sentence B', which is the arithmetical interpretation of B. (Tarski 1956a, 412)

The expansion of S generates a new, more extensive relation of derivability. We could therefore aim at repeating the expanding strategy by making similar moves, until the notion of deductive consequence fits together with the notion of derivability in an opportune, and sufficiently powerful, expansion of S. The desired goal, however, will never be reached; in fact, Gödel's incompleteness applies exactly to the formal systems we are appealing to, and establishes that, in them and in any expansion of theirs, there will always be a set of formulas Γ and a formula α such that α is intuitively a deductive consequence of Γ, although there is no correspondent derivation in the reference system. Therefore, Tarski's conclusion is that

> in order to obtain the proper concept of consequence, which is close in essentials to the common concept, we must resort to quite different methods and apply quite different conceptual apparatus in defining it. (Tarski 1956a, 413)

Anyway, there are so many reasons not related to Gödel's theorems as to believe that a merely syntactic approach is inadequate. If we just observe (CD) and (NT), we can realize that they involve semantic notions totally unrelated to derivability in a formal system. In saying that a formula α derives from a set of formulas Γ, in fact, we make no reference to the truth of α, or to the modal dependence of this truth from that of the elements in Γ. In particular, the rules of a system could be fixed in a totally arbitrary way, and without any justification of the choice accomplished. On the other hand, this makes it difficult, if not impossible, to say when, or that, a given system is *appropriate*, and any attempt to get round that difficulty ends up resulting in the inevitable adoption of a semantic perspective. As rightly pointed out by Etchemendy,

> systems of deduction require *external* proofs of their extensional adequacy (or inadequacy, as the case may be). To be sure, with careful selection of our rules of proof, it is fairly easy to guarantee that only valid arguments are provable in a given system. But our assurance that *all* valid arguments are provable in the system - if such an assurance is to be had - must come from somewhere other than the deductive system itself. We need outside evidence that our system is "complete". (Etchemendy 1990, 3)

2.3 Model-theory

In some articles of the first half of the twentieth century, Tarski laid the foundations for the formal semantics, known today as *model-theory*. Generalizing or expanding the suggestions of the Polish logician, and often incorporating them with approaches and results from different backgrounds, model-theory soon became the standard in the definition of semantic notions suitable to go beyond, and, in a sense, to explain, the syntactic framework of deductive theories. Even today, in fact, correctness and completeness are generally evaluated with regard to model-theoretic procedures and, when reached, considered as a guarantee of *adequacy* for the deductive apparatus under examination. This demonstrates how model-theory is not considered as a mere approach among many others, but, on the contrary, as the most correct and meaningful one. The reasons for this situation are many, and probably too many to be reviewed. Undoubtedly, model-theory has had, and has, much merit of a technical as well as conceptual nature, since it offers a theoretical framework in which to demonstrate results of profound relevance, and articulate in a clear and elegant way a conception of meaning based on assumptions of considerable philosophical depth. However, the success of the model-theoretic notion of consequence does not imply, obviously, that it suits to the epistemic purposes we intend to pursue. Therefore it must be considered through the necessity of thought, and through the subsequent question concerning the power of compulsion exerted by valid inferences.

In the famous *The concept of truth in formalized languages* (Tarski 1956b), Tarski aims at a *materially adequate* and *formally correct* definition of the notion of truth in a language. Formal correctness demands a limitation of the analysis to formal languages, and to a development of the discourse in formal meta-languages in which to define a truth-predicate T applicable to names $\ulcorner\alpha\urcorner$ for formulas α of object languages. Besides offering evident advantages in terms of clarity and precision, formal languages

> possess no terms belonging to the theory of language, i.e. no expressions which denote signs and expressions of the same or another language or which describe the structural connections between them; (Tarski 1956b, 167)

therefore, they are distinguished, in a radical way, from natural languages, since they avoid that *semantic closure* that Tarski identifies as the main responsible in the occurrence of antinomies. The condition of material adequacy, instead, will be satisfied if the definition of T is such that it can be derived as a theorem, for each formula α of the reference language, the schema

$$T(\ulcorner\alpha\urcorner) \text{ if, and only if, } \mathbf{p},$$

where $\mathbf{p}$ is a structural description of α. The techniques and the results of *The concept of truth in formalized languages* will converge later in *On the concept of logical consequence* (Tarski 1956a). Here, starting from the problem of an appropriate characterization of the notion of consequence, Tarski enunciates his well-known definition of logical consequence. Except for some negligible differences, model-theory faithfully adopts the Tarskian line of thought.

As in the rest of the present survey, the model-theoretic notion of logical consequence will be discussed here with reference to a first-order language L. The alphabet of L will include individual variables x, y (if necessary with indexes), individual symbols c (if necessary with indexes), functional symbols f^h (if necessary with indexes), predicative symbols P^m (if necessary with indexes) and logical symbols $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\forall$, $\exists$. We will use the meta-variables $t, u, ...$ (if necessary with indexes) to indicate generic terms, meta-variables $\alpha, \beta, ...$ (if necessary with indexes) to indicate generic formulas, and meta-variables $\Gamma, \Delta, ...$ (if necessary with indexes) to indicate generic sets of formulas. With expressions of the type $t[u_1, \dots, u_n/x_1, \dots, x_n]$ and $\alpha[t_1, \dots, t_n/x_1, \dots, x_n]$, we will indicate, respectively, the result of a function that replaces in t the variable x_i with the term u_i, and the result of a function that replaces in α the variable x_i with the term t_i. With the expression $\Gamma[t_1, \dots, t_n/x_1, \dots, x_n]$ we will indicate the set obtained from Γ by substituting, in every $\alpha \in \Gamma$, the variable x_i with the term t_i $(i \leq n)$. Constants and quantifiers determine a distinction between open and closed terms, and between open formulas and closed formulas: in the open terms and in the open formulas there will be variables that occur free, that is to say out of the scope of $\forall$ and $\exists$, while in the closed terms and in the closed formulas the variables, if any, occur bound, that is to say in the scope of $\forall$ or $\exists$. The set of the free (bound) variables of t and α will be respectively indicated with $FV(t)$ and $FV(\alpha)$ ($BV(t)$ and $BV(\alpha)$). Obviously, $FV(\Gamma)$ ($BV(\Gamma)$) indicates the union set of $FV(\alpha)$ ($BV(\alpha)$) for $\alpha \in \Gamma$.

Model-theory involves interpretations $\mathtt{A}_D$ of some elements of the alphabet of L on sets D (usually understood according to the Zermelo-Fraenkel set theory with the axiom of choice, or ZFC): elements $e \in D$ for individual symbols c (it is usual to extend L by introducing a distinct individual symbol for each distinct element of D); functions $f \colon D^h \rightarrow D$ for functional symbols f^h; relations $R^m \subseteq D^m$ for predicative symbols P^m. On this basis, $\mathtt{A}_D$ interprets then closed terms on D, and closed formulas on the set $\{1, 0\}$ where 1 stands for "true" and 0 for "false". Finally, an open formula α will be interpreted by $\mathtt{A}_D$ on 1 or 0, depending on whether or not it is interpreted by $\mathtt{A}_D$ on 1 or 0 its universal closure $CL(\alpha)$, which is the formula obtained from α by universally quantifying all the variables that occur free in α. With $CL(\Gamma)$ we indicate instead the set obtained from Γ by closing universally every $\alpha \in \Gamma$. So $\mathtt{A}_D$ is called *model* of α if, and only if, $\mathtt{A}_D$ interprets α on 1, that is to say if α is true in D under $\mathtt{A}_D$. α is said to be *logical consequence* of Γ - indicated with $\Gamma \models \alpha$ - if, and only if, assuming that $FV(\Gamma \cup \alpha) = \{x_1, ..., x_n\}$, for each $\mathtt{A}_D$, for each $c_1, \dots, c_n$, if $\mathtt{A}_D$ is a model of $\Gamma[c_1, ..., c_n/x_1, ..., x_n]$, $\mathtt{A}_D$ is a model of $\alpha[c_1, ..., c_n/x_1, ..., x_n]$; for $\Gamma = \emptyset$, α is said to be *logically valid* - indicated with $\models \alpha$. As we can see, if $FV(\Gamma \cup \alpha) = \emptyset$, $\Gamma \models \alpha$ if, and only if, every model of Γ is a model of α, that is to say if, and only if, for each $\mathtt{A}_D$, α is true in D under $\mathtt{A}_D$ if all the formulas of Γ are true in D under $\mathtt{A}_D$. Likewise, if $FV(\alpha) = \emptyset$, $\models \alpha$ if, and only if, each $\mathtt{A}_D$ is a model of α, that is to say, if, and only if, for each $\mathtt{A}_D$, α is true in D under $\mathtt{A}_D$. The fact that such a characterization concerns the notion of logical consequence depends essentially on its taking into account a totality of interpretations over a totality of sets and, above all, on the way in which such interpretations are defined on such sets. In particular, the non-logical symbols are variable, in so far as they are from time to time associated with elements, functions and relations in or on the reference domain. On the contrary, when establishing the truth conditions of the formulas, as we shall see in detail in the next section, the role of logical symbols is intended to remain constant.

According to the informal framework offered by (CD) and (CL) in Section 2.1, logical consequence is a particular instance of a more general notion of deductive consequence, and, since the latter is a modal relation, the same the relation of logical consequence must also be. Can we can say this requirement is respected by the model-theoretic definition? Even before concentrating, as suggested, on an epistemic understanding of modality in terms of necessity of thought, the answer seems to be no: in model-theoretic semantics it holds what Etchemendy (Etchemendy 1990) called the *reduction principle*, that is to say

(RP) if the second-order universal closure $\forall v_1, ..., v_n\ \alpha^*$ of a formula α^* obtained from a closed formula α by replacing the variable symbols with variables of the appropriate type is true, every instance of $\forall v_1, ..., v_n\ \alpha^*$ is logically valid.

A while ago, we have introduced a first-oder notion of universal closure of an open formula. The transition to the second order is achieved by enriching a first-order language with functional variables ϕ and predicative variables X, and with quantifiers that act on these variables. Given a closed formula (for example, in an arithmetic language α could be $\forall x \exists y(2^x < y)$), we replace the non-logical symbols of this formula with variables of an appropriate type and take into account the universal closure $\forall v_1, ..., v_n\ \alpha^*$ of the open formula α^* thus obtained (in our example, α^* is $\forall x \exists y X(\phi(z, x), y)$ and $\forall v_1, ..., v_n\ \alpha^*$ is $\forall X \forall \phi \forall z(\forall x \exists y X(\phi(z, x), y)))$. The reduction principle tells us then that the truth of $\forall v_1, ..., v_n\ \alpha^*$ implies the logical validity of each $\alpha^*[\varepsilon_1, ..., \varepsilon_n / v_1, ..., v_n]$, where $\varepsilon_1, ..., \varepsilon_n$ are constants of the appropriate type (continuing with the example, the truth of $\forall X \forall \phi \forall z(\forall x \exists y X(\phi(z, x), y)$ implies the logical validity of $(\forall x \exists y X(\phi(z, x), y)[P^2/X, \mathtt{f}/\phi, \overline{a}/z]$ for each $R^2 \subseteq D^2$, $\mathtt{f}\colon D^2 \to D$ and $a \in D$, for an appropriate D). Since the converse of (RP) is trivially valid, the truth of $\forall v_1, ..., v_n\ \alpha^*$ will be equivalent to the logical validity of each of its instances. Therefore, if we maintain that the model-theoretic notion of logical consequence has modal ingredients, on the other hand we must admit that, in some specific circumstances, it is equivalent to the notion of truth, and this is problematic in so far as the latter notion does not seem to involve any kind of modality.

Prawitz, in actual fact, had already pointed out the question in *Remarks on some approaches to the concept of logical consequence* (Prawitz 1985). Here, he makes a distinction between *logical* and *factual* sentences, the latter containing non-logical constants in which the former are, on the contrary, wanting. From this point of view, the reduction principle is valid in an even stronger formulation: the logical validity of a logical sentence $\forall v_1, ..., v_n \ \alpha^*$ is equivalent to the truth of $\forall v_1, ..., v_n \ \alpha^*$ itself. So the problem is that

> Tarski also has an analysis of truth, to be sure, and hence, of what is meant by [the truth of $\forall v_1, ..., v_n \ \alpha^*$]. But the analysis makes no distinction between logical sentences [...] and factual sentences. The effect is that a logical sentences is understood as logically true just in case it is true in the same sense as factual sentences are true. (Prawitz 1985, 154 - 155)

The reduction principle seems to offer a serious argument against the presence, in the model-theoretic notion of logical consequence, of modal ingredients *whatever* the type of modality one intends to capture. A detailed treatment of this subject matter, in particular of the possible objections to the reduction principle, is outside the scope of this work, whereas, instead, much closer to our goal is the question related to the possibility of attributing to the model-theoretic approach the epistemic type of modality involved in the necessity of thought. In fact, as early as 1985, Prawitz emphasizes how what one should ask oneself is

> what is the *ground* for a universal truth like [$\forall v_1, ..., v_n \ \alpha^*$], or how can we come to *know*, even with certainty, that a logical sentence is true in all domains. (Prawitz 1985, 155)

In order to approach the problem, it is worth reflecting on the fact that logical consequence is a modal relation, *since* it is a particular instance of a more general relation of deductive consequence. Modality, understood as necessity of thought, must therefore be referred primarily to deductive consequence. Now, in his *On the concept of logical consequence*, although starting from general considerations on the notion of consequence as such, Tarski actually deals exclusively with the logical case of this notion. But, what is deductive consequence in Bolzano, Tarski and in model-theory?

Taken as a whole, (CD) and (CL) suggest, in a certain, plausible interpretation, to look at logical consequence in terms of invariance of deductive consequentiality under variation of the content of the non-logical symbols. This seems, in fact, the manner in which Bolzano, Tarski, and model-theory conceive the idea of dependence on the sole logical form. When adopting a substitutional perspective, this would mean that the presence of logical consequence is nothing more than the presence of deductive consequence within each substitution. In an interpretational approach, instead, the fact that α is a logical consequence of Γ simply means that, for each $\mathtt{A}_D$, the interpretation of α on D under $\mathtt{A}_D$ is a deductive consequence of the interpretration of Γ on D under $\mathtt{A}_D$. The question is, therefore, what the consequence is in a substitution, or on a set under an interpretation. On closer inspection, however, neither Bolzano, nor Tarski, nor model-theory really answer the question. The only available information is that, if all the elements in Γ are true, also α is true, but this is far too little to obtain modal links of an epistemic type. Thus, Tarski's thesis (Tarski 1956a), according to which, on the basis of his definition, every consequence of true sentences *must* be true, may perhaps be valid when understanding modality in terms of possible worlds, namely with a reference to objective structures independent of our knowledge, but it is certainly wrong if we are engaged in the characterization of the necessity of thought. To put it with Prawitz,

> if we stay within the framework of Bolzano and Tarski, [the distinction between deductive and logical consequence] becomes pointless, because the notion of [deductive] consequence will then collapse into that of truth of the corresponding material implication. (Prawitz 2013, 185)

At this point, it becomes important to observe how, with the distinction between deductive consequence and logical consequence, it is possible to match an analogous distinction between valid inferences and *logically* valid inferences, the former being those in which the conclusion is a deductive consequence of the premises, whereas in the latter this relation persists under every variation of the content of the non-logical symbols, that is to say, in the presence of logical consequentiality between premises and conclusion. But then, the approach in question is also incapable of attributing to valid inferences a power, justificatory or of epistemic compulsion. In fact, if, on the one hand, we are not able to distinguish significantly between deductive consequence and logical consequence, we will not even be able to distinguish significantly between valid inferences and logically valid inferences, and this is problematic insofar as the survey focuses on a notion of validity that is independent, at least in the first instance, from the role that logical constants play in the link between premises and conclusion. On the other hand, the fact that the deductive consequence collapses on the truth of correspondent material implications has a clear generalization to the inferential case, so that

> it would only remain to say that an inference is valid if either one of the premises is false or the conclusion is true, but clearly no one is interested in equating the validity of an inference with such a relation between the truth-values of the involved sentences. [...] Although [this] property is certainly *relevant* for the question whether the inference has the power to justify a belief in the conclusion (being a necessary condition for that), it is clearly not sufficient for the inference to have this power. (Prawitz 2013, 185 - 186)

It must however be said that the formulation just discussed constitutes only one of the possible ways of looking at deductive consequence in the Bolzano-Tarski-model-theoretic framework. Perhaps, the notion of deductive consequence should not be inferred from that of logical consequence, by leaving out the reference to the totality of substitutions or structures on which the content of non-logical symbols is intended to vary. On the contrary, it could be argued that the reference to invariance by logical form is to be understood as *consequence with respect to the set of logical symbols* $\{\neg, \wedge, \vee, \rightarrow, \forall, \exists\}$, where deductive consequence instead should be described as a consequence with respect to an appropriate set of symbols $\mathbb{K} \supset \{\neg, \wedge, \vee, \rightarrow, \forall, \exists\}$. In such a perspective, deductive consequence would therefore be defined in terms of substitutions for symbols not belonging to $\mathbb{K}$ or through interpretation functions that make the symbols of $\mathbb{K}$ constant. Even this option, however, suffers from many difficulties. First of all, and in a way analogous to that of Tarski (Tarski 1956a), when the latter refers to the precise demarcation between logical and non-logical symbols, we could ask ourselves what the elements of the supposed set $\mathbb{K}$ should be. But other than that, it remains essentially unexplained how, from the assertion of certain circumstances in certain objective structures, a bond of epistemic modality can be generated for those who intend to establish that a given truth-bearer, possibly believed or judged to be true, or asserted, is valid on the basis of a support provided by other truth-bearers, or by possible beliefs, judgments or assertions related to them. From this perspective, we can also add the shareable observation advanced by Cozzo:

> there are infinitely many pairs $\langle \Gamma, \alpha \rangle$ such that all models of Γ are models of α but we fully ignore that they are. If one is fully unaware that this relation obtains, one will not (or in any case

> not legitimately) take any responsibility for a support that the premises Γ provide for the conclusion α. (Cozzo 2015, 104)

The notion of consequence so far discussed does not therefore capture the necessity of thought, and is therefore unsatisfactory with regard to the problem of validity of inferences. Is there another, more appropriate, way of articulating the idea that an inference is valid if, and only if, its conclusion follows from its premises?

2.4 Meaning: from truth to evidence

As we have already said, the notions of epistemic compulsion and of necessity of thought have many things in common. Both of them involve the idea of a link that exerts its strength on agents engaged in the activity of deducing, in a justified manner, conclusions from premises. Therefore, a link resulting from necessity of thought must be something which who performs the corresponding inferential passage is aware of, something of which one can experience. On the other hand, it is precisely by virtue of an accomplished awareness that the passage must be, and in a sense is already carried out. As Cozzo affirms, necessity of thought has

> a phenomenal character [...]. We have an *experience* of necessity. But this experience is at the same time the experience of *performing the act of making an inference*. Therefore it is an *active experience*. (Cozzo 2015, 108)

The central question then becomes how a bond that induces necessity of thought can become known, manifesting itself in the deductive activity. And the most natural answer seems just the one Prawitz gives, not surprisingly, in *Logical consequence from a constructivist point of view*, namely, through proofs or valid arguments.

In the 2005 article we are referring to, Prawitz hints that a valid argument for α from Γ is the linguistic expression of a reasoning which takes as hypothesis the elements in Γ and as a conclusion α, whereas a proof of α from Γ can instead be seen as what a valid argument for α from Γ expresses, that is to say the reasoning itself. In both cases, however, we are in the presence of something

> such that, when we know of it, we are compelled to hold α true, given that we hold the sentences of Γ true. (Prawitz 2015, 678)

The idea then is to further develop (NT) in Section 2.1, by translating it as

(PT) α is a deductive consequence of Γ if, and only if, it exists a proof or a valid argument for α from Γ.

Without a doubt, (PT) constitutes a remarkable refinement of the intuition contained in (NT). However, the reference to valid proofs and arguments does not seem satisfactory if we are not also able to say, more precisely, what valid proofs and arguments are, and, moreover, if we are not able to show that, on the basis of their description, proofs and valid arguments are such as to make experience those who perform them the phenomenon of the necessity of thought. However, even before turning to a formal definition of the notions of proof and valid argument, a correct orientation of the analysis requires clarifying on what basis proofs and valid arguments *can* have an epistemic value. Again, the wisest path seems also the most obvious, since

> it is difficult to think of any answer that does not bring in the meaning of the sentences [occurring in proofs or valid arguments]. In the end it must be because of the meaning of the expressions involved that we get committed to holding one sentence true, given the truth of some other sentences. (Prawitz 2015, 678)

The meaning of propositions or sentences will depend, plausibly, on the meaning of the simplest non-propositional or non-sentential components of which they are made. But what is the latter meaning? How should it be determined, and on which notions is it based? Perhaps, as a preliminary simplification, we could begin by focusing on the sole logical constant, and pass, at a later time, from such logically relevant expressions to those having a more particular nature.

As a matter of fact, the question of the meaning of the logical constants has always been at the center of the attention of mathematical logic and of philosophy of logic. A guiding intuition, in this sense generally accepted, goes back to the Fregean *context principle* (Frege 1884), and can be understood here in the terms of Dummett's interpretation (Dummett 1973, 1978a, 1993b): the meaning of a non-propositional or non-sentential expression E is given by the contribution of E to the determination of the meaning of the propositions or of the sentences in which E occurs. Another influential thesis, again due to Frege (Frege 1893) and to Ludwig Wittgenstein (Wittgenstein 1921), is the *principle of truth-conditionality*: the meaning of a proposition or sentence is given by the necessary and sufficient conditions under which it is true. Therefore, the two intuitions so combined correspond to the idea that the meaning of a constant logical C is given by the contribution of C to the determination of the necessary and sufficient conditions for the truth of the propositions or sentences in which C occurs as the main logical constant. From this point of view, the Tarskian-model-theoretic definition of the truth predicate fits perfectly with this setting, and permits to derive the following clauses

($\wedge_T$) $\alpha \wedge \beta$ is true if, and only if, α is true and β is true;

($\vee_T$) $\alpha \vee \beta$ is true if, and only if, α is true or β is true;

($\rightarrow_T$) $\alpha \rightarrow \beta$ is true if, and only if, if α is true, β is true;

($\forall_T$) $\forall x \alpha(x)$ is true if, and only if, for every c, $\alpha(c)$ is true;

($\exists_T$) $\exists x \alpha(x)$ is true if, and only if, it exists c such that $\alpha(c)$ is true.

What we should at this point ask ourselves is if the mere ruling of ($\wedge_T$) - ($\exists_T$) is sufficient. As a reply, we could reasonably expect that a satisfactory truth-conditional theory also clarifies which notion of truth is really at stake. Already proposed by Dummett (Dummett 1973, 1978e), the assumption is after all expressed by Prawitz himself:

> questions about the meaning of the logical consequence thus seem to have a straightforward answer in terms of [truth conditions]. However, the substance of this answer depends on what we take truth to be. (Prawitz 2005, 679 - 680)

In a sense, it could be argued that Tarski's formal semantics and the resulting model-theory aim *exactly*, according to our wishes, at a delineation of the notion of truth. This can be admitted, as long as we observe that the definition of the truth *predicate* does not exhaust, but rather does imply, the corresponding *notion*. As Gabriele Usberti (Usberti 2016) points out, for example, Tarski (1956b) relates explicitly to the Aristotelian theory of *truth as correspondence*, or even, and perhaps more significantly, proves that every closed formula is true or false by resorting, in an essential way, to a generalized *bivalence principle*, according to which each truth-bearer is determinately either true or false. The central point, however, is that such a description implies that the meaning of the logical constants is *known*. Tarski (Tarski 1956b), not without reason, takes the latter for acquired, by proving the material adequacy of his approach through examples based on the possibility of matching names $\ulcorner \alpha \urcorner$ of the meta-language with meta-linguistic translations of formulas α of the object language. If the meaning of α were unknown, these examples would have no probative value.

Therefore, the abovementioned clauses identify the truth only on the condition of a prior understanding of the meaning of the logical constants; hence, they alone will not bel able to offer a notion on which to base the complete clarification of this meaning. But even assuming that this is not the case here, and even if the notion of truth were completely independent, there are many reasons for doubting that the clauses are able to justify *simultaneously* both truth and meaning. From a general point of view, in fact, formalization is confined to circumscribe the extension of a meta-linguistic predicate for names of formulas of a given object language. The fact that the predicate at stake is *actually* a truth-predicate can be recognized only if one has already some underlying, or primitive, idea of truth. Once again, in line with Dummett (Dummett 1973, 1978e) Prawitz observes in this regard that

> if we have defined a set S of sentences by saying that it is the least set of sentences containing certain atomic formulas and satisfying certain equivalences, such as $A \wedge B$ belongs to S if and only if both A and B belong to S, then obviously we get no information about the meaning of the logical constants by being told again that these equivalences hold. [...] We must conclude that truth conditions can serve as meaning explanations only if we already have a grasp of truth. (Prawitz 2005, 680)

Therefore, the survey must go, so to speak, deeper than $(\wedge_T)$ - $(\exists_T)$ permit to do. In other words, it is necessary to question the most fundamental notion of truth by which the mathematical apparatus itself is inspired. A formal semantics, after all, is the rigorous report of the overall articulation of a language, the structural counterpart of a larger theory of meaning based on general, and often implicit, concepts.

From this point of view, the aforementioned principle of bivalence is adopted, more or less tacitly, by the overwhelming majority of the proponents of truth-conditionality. Is this a plausible option? The idea that every truth-bearer is always determinately true or false is at the core of Dummett's criticism. In fact, he has dedicated celebrated arguments to this perspective, which he described as *realistic*, aiming at showing its unsustainability (Dummett 1978c, 1978e, 1993b). However, the discussion on these themes goes beyond the aims of the present work. It will suffice to note how bivalence implies that truth or falsity are given by virtue of facts totally unrelated to our *epistemic* abilities. The semantic status of propositions or sentences will then be independent of the possibility to *know* the circumstances that make them true or false, and the conditions of truth will transcend the conditions of correctness for *judgements* or *assertions* (see Cozzo 1994b, 2008). Thus, even regardless of the general reasons on the basis of which Dummett refused such a setting, it is easy to see that it does not even seem, as a matter of principle, to conform to the objective to pursue. If the meaning of propositions or sentences is averse to what it means for us to use them appropriately in the deductive practice, we are not able to see how, on account of this meaning, they can endow with an epistemic strength the proofs or valid arguments in which they are involved.

Therefore, in the view of these difficulties, Prawitz proposes to turn to an *epistemic* conception in which

> truth is instead determined in terms of what it is for us to acquire knowledge, and sentences are true in virtue of the potential existence of evidence for them. (Prawitz 2005, 681)

This move obviously makes the notion of *evidence* the core of the discussion, to the detriment of the notion of truth, transforming the original truth-conditional theory into a theory based on conditions of correct *judicability* or *assertability* - it is only when in possession of an appropriate evidence, that judgements and assertions can be said to be correct. The old principle of Frege and Wittgenstein can still be reused, because now

> we may go back to the idea that meaning is determined by truth conditions. A more profound way of accounting for the meaning of a sentence is now opened up, namely, in terms of what counts as evidence for the sentence. Knowing what counts as evidence for the sentence, one also knows the truth conditions of the sentence. (Prawitz 2005, 681)

Through the previous reflections, we have brought into focus some important topics. However, the term "evidence" is obviously too vague. How can we make it more precise? Prawitz's *proof-theoretic semantics* aims at answering this question.

2.5 Prawitz's proof-theoretic semantics

In order to better understand what we will say later, it is necessary to make a brief digression. As anticipated, in fact, Prawitz's proof-theoretic semantics has two main sources of inspiration: on the one hand, some pioneering research by Gentzen; on the other, and with a specific reference to the formal apparatus of such research, the fundamental results that Prawitz himself has proven. Along with other ideas coming from the related intuitionistic tradition, these two suggestions merge in almost all Prawitz's semantic investigation, including the contemporary theory of grounds.

Therefore, the next Section is dedicated to the introduction of concepts and results, the technical nature of which, although different from the type of topic so far dealt with, is, in our opinion, an indispensable ingredient for the correct understanding of the following exposition.

2.5.1 Gentzen's systems and Prawitz's normalization

In the well-known *Untersuchungen über das logische Schließen* (Gentzen 1934 - 1935), Gentzen outlines two types of formal systems for a first-order language L: a calculus of sequents, and a calculus of natural deduction. The latter, to which we will devote our exclusive attention, is based on the idea of associating to every logical constant C a rule (of inference) of *introduction*, in which C occurs as principal logical constant in the conclusion, and a rule (of inference) of *elimination*, in which C occurs as principal logical constant in one of the premises (called *major premise*, whereas the others, if any, are called *minor premises*).

$$\frac{\alpha \qquad \beta}{\alpha \wedge \beta}\,(\wedge_I) \qquad \frac{\alpha_1 \wedge \alpha_2}{\alpha_i}\,(\wedge_{E,i}),\ i = 1, 2$$

$$\frac{\alpha_i}{\alpha_1 \vee \alpha_2}\,(\vee_I),\ i = 1, 2 \qquad \frac{\alpha \vee \beta \qquad \begin{matrix}[\alpha] \\ \vdots \\ \gamma\end{matrix} \qquad \begin{matrix}[\beta] \\ \vdots \\ \gamma\end{matrix}}{\gamma}\,(\vee_E)$$

$$\frac{\begin{matrix}[\alpha] \\ \vdots \\ \beta\end{matrix}}{\alpha \to \beta}\,(\to_I) \qquad \frac{\alpha \to \beta \qquad \alpha}{\beta}\,(\to_E)$$

$$\frac{\alpha(x)}{\forall y \alpha(y/x)}\,(\forall_I) \qquad \frac{\forall x \alpha(x)}{\alpha(t/x)}\,(\forall_E)$$

$$\frac{\alpha(t/x)}{\exists x \alpha(x)}\ (\exists_I) \qquad \frac{\exists y \alpha(y/x) \qquad \begin{matrix}[\alpha(x)]\\ \vdots \\ \beta\end{matrix}}{\beta}\ (\exists_E)$$

In the rules $(\vee_E)$, $(\rightarrow_I)$ and $(\exists_E)$, the vertical dots between two formulas indicate that the second one is understood as obtained in dependence of the first; the square brackets instead indicate that the formula put into them can be discharged, in the sense that the conclusion no longer depends on this formula. In the rules for $\forall$ and $\exists$, $\alpha(y/x)$ and $\alpha(t/x)$ are notations of convenience for $\alpha(x)[y/x]$ and $\alpha(x)[t/x]$. In $(\forall_I)$, x must not occur free in any undischarged formula on which $\alpha(x)$ depends; in the same way, in $(\exists_E)$ x must not occur free either in β, or in any undischarged formula on which β depends, other than $\alpha(x)$. In both cases, x is called the *proper variable* of the inference and, if $y \neq x$, y must not occur free in $\alpha(x)$ and must be free for x in $\alpha(x)$. In the rules $(\forall_E)$ and $(\exists_I)$, t must be free for x in $\alpha(x)$.

So, on the whole, we have introduced a formal system `ML` called *minimal logic*. If we add a constant symbol $\perp$ for absurdum to the so-called atomic formulas of the language of reference - a subset of the set of formulas, the elements of which consist of applications of relational constants to terms - and put

$$\neg\alpha \stackrel{def}{=} \alpha \rightarrow \perp$$

by adding to `ML` the rule

$$\frac{\perp}{\alpha}\ (\perp)$$

we obtain a formal system `IL` for *intuitionistic logic*. Finally, by adding to `IL` one of the rules

$$\frac{}{\alpha \vee \neg\alpha}\ \text{EM} \qquad \frac{\neg\neg\alpha}{\alpha}\ \text{DN} \qquad \frac{\begin{matrix}[\neg\alpha]\\ \vdots \\ \perp\end{matrix}}{\alpha}\ \text{RAA} \qquad \frac{}{((\alpha \rightarrow \beta) \rightarrow \alpha) \rightarrow \alpha}\ \text{PRC}$$

we obtain a formal system `CL` for *classical logic*. The rules permit to define a set of derivations, which are represented as tree structures whose nodes are formulas; the initial nodes are the *assumptions* of the derivation, while the final node is the *conclusion*. Here we will take into account only `IL`, on a first-order reference language L with set of terms $\texttt{TERM}_L$ and set formulas $\texttt{FORM}_L$ ($\texttt{ATOM}_L$ for the atomic formulas).

Definition 1. The set $\mathtt{DER}_{\mathtt{IL}}$ of the *derivations* of IL is the smallest set X such that

- the single node $\alpha \in X$ for every $\alpha \in \mathtt{FORM}_L$
- $\begin{array}{c}\Delta_1\\ \alpha\end{array}$ and $\begin{array}{c}\Delta_2\\ \beta\end{array} \in X \Rightarrow$

$$\dfrac{\begin{array}{c}\Delta_1\\ \alpha\end{array} \qquad \begin{array}{c}\Delta_2\\ \beta\end{array}}{\alpha \wedge \beta}\,(\wedge_I) \qquad \in X$$

- $\begin{array}{c}\Delta\\ \alpha_1 \wedge \alpha_2\end{array} \in X \Rightarrow$

$$\dfrac{\begin{array}{c}\Delta\\ \alpha_1 \wedge \alpha_2\end{array}}{\alpha_i}\,(\wedge_{E,i}),\ i = 1,2 \qquad \in X$$

- $\begin{array}{c}\Delta\\ \alpha_i\end{array} \in X$ with $i = 1,2 \Rightarrow$

$$\dfrac{\begin{array}{c}\Delta\\ \alpha_i\end{array}}{\alpha_1 \vee \alpha_2}\,(\vee_I),\ i = 1,2 \qquad \in X$$

- $\begin{array}{c}\Delta_1\\ \alpha \vee \beta\end{array}$, $\begin{array}{c}\alpha\\ \Delta_2\\ \gamma\end{array}$ and $\begin{array}{c}\beta\\ \Delta_3\\ \gamma\end{array} \in X \Rightarrow$

$$\dfrac{\begin{array}{c}\Delta_1\\ \alpha \vee \beta\end{array} \qquad \begin{array}{c}[\alpha]\\ \Delta_2\\ \gamma\end{array} \qquad \begin{array}{c}[\beta]\\ \Delta_3\\ \gamma\end{array}}{\gamma}\,(\vee_E) \qquad \in X$$

- $\begin{array}{c}\alpha\\ \Delta\\ \beta\end{array} \in X \Rightarrow$

$$\dfrac{\begin{array}{c}[\alpha]\\ \Delta\\ \beta\end{array}}{\alpha \to \beta}\,(\to_I) \qquad \in X$$

- $\begin{array}{c}\Delta_1\\ \alpha \to \beta\end{array}$ and $\begin{array}{c}\Delta_2\\ \alpha\end{array}$ $\in X \Rightarrow$

$$\dfrac{\begin{array}{cc}\Delta_1 & \Delta_2\\ \alpha \to \beta & \alpha\end{array}}{\beta}\,(\to_E) \qquad \in X$$

- $\begin{array}{c}\Delta\\ \bot\end{array}$ $\in X \Rightarrow$

$$\dfrac{\begin{array}{c}\Delta\\ \bot\end{array}}{\alpha}\,(\bot) \qquad \in X$$

- $\begin{array}{c}\Delta\\ \alpha(x)\end{array}$ $\in X$ in compliance with the restriction on $(\forall_I) \Rightarrow$

$$\dfrac{\begin{array}{c}\Delta\\ \alpha(x)\end{array}}{\forall y\alpha(y/x)}\,(\forall_I) \qquad \in X$$

- $\begin{array}{c}\Delta\\ \forall x\alpha(x)\end{array}$ $\in X \Rightarrow$

$$\dfrac{\begin{array}{c}\Delta\\ \forall x\alpha(x)\end{array}}{\alpha(t/x)}\,(\forall_E) \qquad \in X$$

  in compliance with the restriction on $(\forall_E)$

- $\begin{array}{c}\Delta\\ \alpha(t/x)\end{array}$ $\in X$ in compliance with the restriction on $(\exists_I) \Rightarrow$

$$\dfrac{\begin{array}{c}\Delta\\ \alpha(t/x)\end{array}}{\exists x\alpha(x)}\,(\exists_I) \qquad \in X$$

- $\begin{array}{c}\Delta_1\\ \exists y\alpha(y/x)\end{array}$ and $\begin{array}{c}\alpha(x)\\ \Delta_2\\ \beta\end{array}$ $\in X$ in compliance with the restriction on $(\exists_E)$ $\Rightarrow$

$$\dfrac{\begin{matrix}\Delta_1\\ \exists y\alpha(y/x)\end{matrix} \qquad \begin{matrix}[\alpha(x)]\\ \Delta_2\\ \beta\end{matrix}}{\beta}\,(\exists_E) \qquad \in X$$

α is *derivable* from Γ in IL, indicated $\Gamma \vdash_{\texttt{IL}} \alpha$, if, and only if, there is $\Delta \in \texttt{DER}_{\texttt{IL}}$ with set of undischarged assumptions Γ and conclusion α.

Prawitz (Prawitz 1971, 2006), as we have already said, has obtained a fundamental normalization theorem for IL, and hence, trivially, for its subsystem ML (as well as for other systems, including CL itself). Just like the *Cut*-elimination, proven by Gentzen himself (Gentzen 1934 - 1935) for sequents calculi, this theorem permits to eliminate the *detours* inside the derivations, and reduces the latter to structures having interesting properties and far-reaching consequences. Leaving aside inessential details, we will limit ourselves here to introduce simple definitions and to enunciate only the most relevant results.

Definition 2. A *maximal formula* of Δ is an occurrence of a formula in Δ which is consequence of an application of an introduction rule or of $(\bot)$, and major premise of an application of an elimination rule.

Δ is said to be in *normal form* if, and only if, it does not contain maximal formulas.

For rules other than $(\bot)$, maximal formulas can be eliminated through appropriate reductions.

$$\dfrac{\dfrac{\begin{matrix}\Delta_1\\ \alpha_1\end{matrix}\qquad \begin{matrix}\Delta_2\\ \alpha_2\end{matrix}}{\boldsymbol{\alpha_1\wedge\alpha_2}}\,(\wedge_I)}{\alpha_i}\,(\wedge_{E,i}),\ i=1,2 \qquad \wedge_i\text{-rid} \qquad \begin{matrix}\Delta_i\\ \alpha_i\end{matrix}$$

$$\dfrac{\dfrac{\begin{matrix}\Delta\\ \alpha_i\end{matrix}}{\boldsymbol{\alpha_1\vee\alpha_2}}\,(\vee_I),\ i=1,2 \qquad \begin{matrix}[\alpha_1]\\ \Delta_1\\ \beta\end{matrix}\qquad \begin{matrix}[\alpha_2]\\ \Delta_2\\ \beta\end{matrix}}{\beta}\,(\vee_E) \qquad \vee\text{-rid} \qquad \begin{matrix}\Delta\\ [\alpha_i]\\ \Delta_i\\ \beta\end{matrix}$$

$$\dfrac{\dfrac{\begin{matrix}[\alpha]\\ \Delta_1\\ \beta\end{matrix}}{\boldsymbol{\alpha\to\beta}}\,(\to_I) \qquad \begin{matrix}\Delta_2\\ \alpha\end{matrix}}{\beta}\,(\to_E) \qquad \to\text{-rid} \qquad \begin{matrix}\Delta_2\\ [\alpha]\\ \Delta_1\\ \beta\end{matrix}$$

$$\dfrac{\dfrac{\begin{array}{c}\Delta(x)\\ \alpha(x)\end{array}}{\boldsymbol{\forall y\alpha(y/x)}}\,(\forall_I)}{\alpha(t/x)}\,(\forall_E) \qquad \forall\text{-rid} \qquad \begin{array}{c}\Delta(t/x)\\ \alpha(t/x)\end{array}$$

$$\dfrac{\dfrac{\begin{array}{c}\Delta\\ \alpha(t/x)\end{array}}{\boldsymbol{\exists y\alpha(y/x)}}\,(\exists_I) \qquad \begin{array}{c}[\alpha(x)]\\ \Delta_1(x)\\ \beta\end{array}}{\beta}\,(\exists_E) \qquad \exists\text{-rid} \qquad \begin{array}{c}\Delta\\ [\alpha(t/x)]\\ \Delta_1(t/x)\\ \beta\end{array}$$

where $\Delta(t/x)$ indicates the substitution in $\Delta(x)$ with t of every free occurrence of x in occurrences of formulas in $\Delta(x)$. One could reasonably expect that the application of one among $\wedge_i$-rid, $\vee$-rid, $\rightarrow$-rid, $\forall$-rid and $\exists$-rid to Δ with undischarged assumptions Γ and conclusion α generates a new derivation with undischarged assumptions $\Gamma^* \subseteq \Gamma$ and conclusion α. We can achieve this by adopting a simple convention that, by virtue of some theorems of which it is not advisable to go into details (see, for example, Van Dalen 1994), does not cause any loss of generality.

Convention 3. In each Δ (1) free and bound variables are all distinct - property (FB) - and (2) proper and non-proper variables are all distinct, and each proper variable is used in at most only one application of $(\forall_I)$ or $(\exists_E)$ - property (PN).

As regards $(\bot)$, it will be sufficient to recall the result according to which, if $\Gamma \vdash_{\texttt{IL}} \alpha$, it exists Δ with set of undischarged assumptions Γ and conclusion α such that, for each occurrence of β in Δ, if the occurrence of β is the conclusion of an application of $(\bot)$, then $\beta \in \texttt{ATOM}_L$. So we can, in a sense, leave aside the maximal formulas that are the consequences of the applications of $(\bot)$, and focus only on those to which it is associated an appropriate reduction.

Theorem 4. If $\Gamma \vdash_{\texttt{IL}} \alpha$, it exists Δ in normal form with undischarged assumptions $\Gamma^* \subseteq \Gamma$ and conclusion α.

Since ML is a subsystem of IL, theorem 4 extends trivially to minimal logic. In both cases, the basic strategy for the proof consists, as we can imagine, in progressively eliminating all the maximal formulas of a derivation (and any new maximal formulas resulting from the elimination itself) through reiterated applications of $\wedge_i$-rid, $\vee$-rid, $\rightarrow$-rid, $\forall$-rid and $\exists$-rid. The latter, as we will see more extensively and more generally in Section 2.5.2.1, induce, for their part, a reducibility relation among derivations, by virtue of which theorem 4 can be expressed more vigorously by saying that each derivation of `IL` (and therefore of `ML`) reduces to a derivation in a normal form.

Beyond an intrinsic technical interest, the normalization result has a considerable philosophical importance, and lends itself, in a natural way, to semantic discussions. This is consistent with what Gentzen already seems to suggest, in a passage by now become famous:

> the introductions represent, as it were, the "definitions" of the symbol concerned, and the eliminations are no more, in the final analysis, than the consequences of these definitions. This fact may be expressed as follows: in eliminating a symbol, we may use the formula with whose terminal symbol we are dealing only "in the sense afforded it by the introduction of that symbol". [...] By making these ideas more precise it should be possible to display the [elimination inferences] as unique functions of their corresponding [introduction inferences]. (Gentzen 1934 - 1935, 192)

Gentzen's words can be interpreted in two different, although closely linked ways. If we are focused on the idea that elimination inferences are nothing but univocal functions of the corresponding introduction inferences, with the latter, on the contrary, "defining" the symbols occurring as principal in their conclusions, we arrive at an *inversion principle* which, introduced for the first time by Lorenzen (Lorenzen 1950, 1955), was formulated by Prawitz with the following words:

> let **a** be an application of an elimination rule that has β as consequence. Then, deductions that satisfy the sufficient condition [...] for deriving the major premise of **a**, when combined with deductions of the minor premisses of **a** (if any), already "contain" a deduction of β; the deduction of β is thus obtainable directly from the given deductions without the addition of **a**. (Prawitz 2006, 33)

As can easily be seen, $\wedge_i$-rid, $\vee$-rid, $\rightarrow$-rid, $\forall$-rid and $\exists$-rid instantiate the inversion principle, in the formulation of Prawitz, on, respectively, $(\wedge_{E,i})$, $(\vee_E)$, $(\rightarrow_E)$, $(\forall_E)$ and $(\exists_E)$. Instead, if we are focused on the request that the elimination of a symbol occurring in the major premise is consistent with the way in which this symbol is "defined" in the introduction concerning it, Gentzen's thesis can be inserted in a semantic perspective. The introduction rules fix the *meaning* of the symbols that occur as principal in the conclusion, and the elimination rules must be, so to speak, in harmony with this meaning. As we will see later, in the theories of meaning based on the notion of evidence, and, in particular, in proof-theoretic semantics, the concept of harmony receives a well precise connotation. What we can already observe now, however, is that, from a semantic point of view, $\wedge_i$-rid, $\vee$-rid, $\rightarrow$-rid, $\forall$-rid and $\exists$-rid can be seen as *justifications* of, respectively, $(\wedge_{E,i})$, $(\vee_E)$, $(\rightarrow_E)$, $(\forall_E)$ and $(\exists_E)$. In other words, since the introduction rules fix the meaning, it will be considered "acceptable" any derivation that ends with the application of an introduction rule to "acceptable" derivations. Reductions, then, by showing how to transform derivations with *detours* into derivations without *detours*, show also the "acceptability" of a derivation that ends with the application of an elimination rule to "acceptable" derivations - where the derivation of the major premise must obviously satisfy the meaning conditions of the logical constant of reference. Therefore, the semantic reading is not at all incompatible with the setting that takes to the normalization theorems; on the contrary, it is a faithful generalization of it, a generalization in which the inversion principle becomes a requisite that every legitimate elimination rule must fulfill. Not without reason, Peter Schroeder-Heister emphasized what he calls the *fundamental corollary* of theorem 4: by repeatedly applying $\wedge_i$-rid, $\vee$-rid, $\rightarrow$-rid, $\forall$-rid or $\exists$-rid to Δ with assumptions $\Gamma = \emptyset$, we arrive at a derivation that ends with the application of an introduction rule. This result is

> philosophically interpreted by *requiring* that a valid closed derivation be reducible to one using an introduction inference in the last step. (Schroeder-Heister 2006, 531)

To conclude, we feel we need to introduce the notion of *first-order atomic system*. As we will see in the next section, proof-theoretic semantics relativizes its key notions to bases consisting of derivations in systems of this type, and an analogous approach can be adopted for the theory of grounds too - and indeed, we will adopt it explicitely in this latter case. More specifically, we will characterize the first-order atomic systems through the so-called *Post systems*. Although just one of the many possibilities, this is the *modus operandi* adopted by Prawitz himself.

Given a first-order language L with predicative, functional and individual symbols, a Post system is then a pair $\langle L, \Re \rangle$, with $\Re$ finite set of rules

$$\frac{\alpha_1 \quad \dots \quad \alpha_n}{\beta}$$

relative to predicative, functional or individual symbols of L and such that:

- $n \geq 0$ (when $n = 0$, the rule is an axiom);
- for every $i \leq n$, $\alpha_i \in \mathtt{ATOM}_L$ and $\alpha_i \neq \bot$;
- $\beta \in \mathtt{ATOM}_L$ and, if x occurs free in β and $n > 0$, it exists $i \leq n$ such that x occurs free in α_i.

As an example for what will follow, let $\mathbb{N}$ be the set of natural numbers, $\doteq$ the usual equality relation in $\mathbb{N}^2$, s the successor function $\mathbb{N} \to \mathbb{N}$, and $+$ and $\cdot$ the functions of, respectively, addition and multiplication $\mathbb{N}^2 \to \mathbb{N}$. Suppose that the predicative, functional and individual symbols of L are identical to the correspondent ones on $\mathbb{N}$. Let us take into account the following rules:

$$\frac{}{t \doteq t}\ (\doteq_R) \qquad \frac{t \doteq u}{u \doteq t}\ (\doteq_S) \qquad \frac{t \doteq u \qquad u \doteq z}{t \doteq z}\ (\doteq_T)$$

$$\frac{0 \doteq s(t)}{\bot}\ (s_1) \qquad \frac{s(t) \doteq s(u)}{t \doteq u}\ (s_2)$$

$$\frac{}{t + 0 \doteq t}\ (+_1) \qquad \frac{}{t + s(u) \doteq s(t + u)}\ (+_2)$$

$$\frac{}{t \cdot 0 \doteq 0}\ (\cdot_1) \qquad \frac{}{t \cdot s(u) \doteq (t \cdot u) + t}\ (\cdot_2)$$

As can be seen, $(\doteq_R)$, $(\doteq_S)$ and $(\doteq_T)$ express the usual properties of reflexivity, symmetry and transitivity of the equality - we will indicate the set of these rules with $\mathtt{EQ}$; (s_1), (s_2), $(+_1)$, $(+_2)$, $(\cdot_1)$ and $(\cdot_2)$, together with the first-order induction rule

$$\frac{\alpha(0) \qquad \begin{matrix}[\alpha(x)] \\ \vdots \\ \alpha(s(x)/x)\end{matrix}}{\alpha(t/x)}\ \mathrm{IND}$$

express instead the so-called *Peano axioms for first-order arithmetic* - we will indicate the set of these rules with $\mathtt{SAM} \cup \{\mathrm{IND}\}$. Taking into account also intuitionistic logic, the system $\mathtt{EQ} \cup \mathtt{SAM} \cup \{\mathrm{IND}\} \cup \mathtt{IL}$ is called *Heyting first-order arithmetic* - we will indicate this system with $\mathtt{HA}$. Obviously $\mathtt{EQ} \cup \mathtt{SAM}$ is a Post system.

2.5.2 Valid arguments and proofs

Since the 1970s, Prawitz has concentrated on semantic investigations. In particular, proof-theoretic semantics was for the first time systematized, along different lines of thought, in the 1971 article *Ideas and results in proof theory*, in the 1973 article *Towards a foundation of a general proof theory* and in the 1977 article *Meaning and proofs: on the conflict between classical and intuitionistic logic*.

From the beginning, Prawitz proves to be aware of the existence of at least two possible approaches. In the 1973 article, for example, he affirms that

> one could try to give a direct characterization of different kinds of proofs, where a *proof* is understood as the abstract process by which a proposition is established, and then study how proofs are represented syntactically by derivation. Or alternatively, one could start on a more concrete level and study the verbal arguments that are intended to convince us of some state of affairs and then attempt to single out those arguments that are valid and which thus represent proofs. (Prawitz 1973, 227)

In the context of a critical-widening discussion of Dummett's theories, *Meaning and proof: on the conflict between classical and intuitionistic logic* presents general considerations on proofs. *Ideas and results in proof theory* and *Towards a foundation of a general proof theory*, on the other hand, introduce and articulate the notion of *valid argument*. Although not always with equal intensity (see, among others, Tranchini 2014a, where it is rightly said that the setting in terms of valid argument is at the beginning predominant, and that the situation is reversed precisely with the theory of grounds - and, we can add, with the articles that anticipate it), both paths remain, over the years, at the center of Prawitz's attention, until they end up merging, as already mentioned, in *Logical consequence from a constructivist point of view*.

In the following paragraphs, valid arguments will be defined as in *Towards a foundation of a general proof theory*, which differs from *Ideas and results in proof theory* essentially for the role attributed to Post systems. We will discuss proofs starting from *Meaning and proofs: on the conflict between classical and intuitionistic logic*, by integrating the reflections therein contained with others, innovative, coming precisely from *Logical consequence from a constructivist point of view*.

In order to approach proof-theoretic semantics, it is worth resuming the discussion at the end of Section 2.4. There, our initial problem had been to explain by virtue of what valid arguments and proofs were such as to induce, on those in possession of it, the epistemic constraint of the necessity of thought. Since any answer to such a question cannot ignore what propositions or sentences mean, we have therefore examined the nature of this meaning. After detecting the inadequacy of an explanation in terms of bivalent truth-conditions, we have then appealed to Prawitz's proposal to focus on the notion of evidence. What we are going to do now is to define the notion of evidence through rigorous definitions of the notions of valid argument and proof. Could we not affirm, with good reason, that we have fallen into a circular explanation? In considering this plausible objection, Prawitz suggests also an escape route:

> there seem to be two clashing intuitions at work here, which also occur in more general discussion concerning the relation between the meaning and the use of a term [...] what counts as a proof of a sentence is one feature of the use of the sentence. (Prawitz 2005, 682)

In this sense, while it is true that proof-theoretic semantics originates from Gentzen's research, and from the normalization theory related to it, it is equally true that this semantics fits perfectly with the thesis, dating back to Wittgenstein (Wittgenstein 1953), according to which *meaning is use*; or rather, more precisely, with a certain interpretation of this thesis.

Wittgenstein's slogan has indeed received all kinds of interpretations. However, among the most influential, and moreover essential for the identification of the *desiderata* that an adequate theory of meaning must satisfy, there is no doubt Dummett's well-known work. Against bivalent truth-conditionality, and in favor of a verificationist framework, Dummett (Dummett 1978c, 1993b) has in fact proposed, among others, arguments based on a *manifestability requirement*, according to which knowledge, or understanding, of meaning must be able to manifest itself through the use of propositions or sentences within the assertive practice. And the way in which Prawitz conceives the idea that meaning is use is affected by Dummett's influence. This is for example evident in the aforementioned *Meaning and proofs: on the conflict between classical and intuitionistic logic*, where the point is discussed in relation to, and framed within the theories of meaning of the English philosopher:

> use is to be taken in its broadest possible sense, i.e., as total use in all its aspects. This is not to say that the meaning is causally determined by the use, because, conversely, it is equally reasonable to hold that use is determined by meaning; nor should it be

> concluded from this that meaning is identical with use. What is claimed is only that if two expressions are used in the same way, then they have the same meaning, or if two persons agree completely about the use of an expression, then they should also agree about its meaning. The principle could be expressed in another way by saying that the meaning of a sentence must be fully manifest in some way in its use. [...] The principle that meaning is determined by use does not preclude the possibility that there is some central feature of sentences other than their total use that determines or constitutes their meaning; it only demands that a feature that constitutes meaning is determined by use. (Prawitz 1977, 3 - 5)

In line with these observations, Prawitz therefore bypasses the difficulty previously highlighted by affirming that

> if someone asks why $3+1=4$, a natural answer is that this is what "4" means, or that this is how "4" is defined and used. Similarly, what can we answer someone who questions the drawing of the conclusion $\alpha \to \beta$, given a proof of β from α, except that this is how $\alpha \to \beta$ is used, it is a part of what $\alpha \to \beta$ means? But a similar answer to the question why $2+2=4$ or why we infer $\alpha \to \beta$ from $\neg\beta \to \neg\alpha$ seems inadequate. That $2+2=4$ or that we infer $\alpha \to \beta$ from $\neg\beta \to \neg\alpha$ is not reasonably looked upon as a usage that can be equated with the meaning of the expressions involved, but rather is something that is to be justified in terms of what the expressions mean. To answer *all* doubts about a certain usage of language by saying that this is how the terms are used, or that this is a part of their meaning, would be a ludicrously conservative way of meeting demands for justification. But for *some* such doubts the reference to common usage is very reasonable and may be the only thing to resort to. (Prawitz 2005, 682)

What are, then, valid arguments and proofs? As was the case for the notion of consequence, an at first sight satisfactory answer could be formulated in purely syntactic terms: valid arguments and proofs are nothing but derivations in an appropriate, sufficiently powerful, formal system. Also in this case, however, Gödel's incompleteness theorems reveal the inadequacy of such a conception. However, over and above that, the key reason is the fact that an arbitrary choice of axioms and rules would prevent from linking, as required, the notions of valid argument and proof to the meaning of propositions or sentences, and from justifying valid arguments and derivations based on this meaning:

> a deductive system is [...] an attempt to codify proofs within a given language, but when setting up such a system, one does not ordinarily try to analyze what makes something a proof. Nor does proof theory ordinarily try to justify a deductive system except for trying to prove its consistency. (Prawitz 2005, 683)

2.5.2.1 Valid arguments (in 1973)

As stated in Section 2.5.1, the derivations of IL are tree structures; the nodes correspond to occurrences of formulas of a first-order language, and the branches reflect the application of $(\wedge_I)$, $(\wedge_{E,i})$, $(\vee_I)$, $(\vee_E)$, $(\rightarrow_I)$, $(\rightarrow_E)$, $(\forall_I)$, $(\forall_E)$, $(\exists_I)$, $(\exists_E)$ or $(\bot)$. However, the argumentative practice is not obviously limited to any fixed set of rules. For this reason, when developing his notion of valid argument, Prawitz places no restriction on the type of inferences used in deduction.

The departure notion is thus a generic notion of argument. An *argument* will still be a tree structure having as its nodes the occurrences of formulas of a first-order language L; *inferences*, however, should be understood now as whatever kind of figures of the type

$$\frac{\Delta_1 \qquad \dots \qquad \Delta_n}{\beta}\; \alpha_1, ..., \alpha_m, x_1, ..., x_s$$

where $\Delta_1, ..., \Delta_n$ are in turn arguments, $\alpha_1, ..., \alpha_m$ occurrences of assumptions discharged by the inference, and $x_1, ..., x_s$ occurrences of free variables bound by it - as usual called *proper variables*. As a convention, we will admit that an inference that binds variables is always such that its proper variables do not occur free either in the conclusion of the inference, or in any of the undischarged assumptions on which such a conclusion depends. By *rule of inference* we mean simply a set of inferences, called *applications* of the rule; two rules of inference are said to be *disjoint* if, and only if, they have empty intersection.

The initial nodes of an argument are its *assumptions*, the final node being instead its *conclusion*; an argument will be called *open* when it contains undischarged assumptions or occurrences of free variables not later bound, *closed* otherwise. As a further request, we will take for granted that all arguments enjoy the properties (FB) and (PN) according to convention 3. An *example* of an open argument Δ is an argument obtained from Δ through a substitution σ of free, not later bound, variables with terms, and of undischarged assumptions with arguments for such assumptions. Given an argument Δ ending in an inference of the kind shown above, we will say that each Δ_i $(i \leq n)$ is an *immediate subargument* of Δ; a *subargument* of Δ is an initial segment of Δ, namely an element of the reflexive and transitive closure with respect to Δ of the relation "Δ_1 is an immediate subargument of Δ_2".

The notion of valid argument is relativized by Prawitz to atomic bases and justifications. An *atomic base* B is a pair $\langle\{\mathtt{K}, \mathtt{F}, \mathtt{R}\}, \mathtt{S}\rangle$ with: $\mathtt{K}$ set of individual symbols, $\mathtt{F}$ set of functional symbols, and $\mathtt{R}$ set of predicative symbols - all three such to contain the individual functional and predicative symbols of L; $\mathtt{S}$ Post system with rules relative to the elements of $\mathtt{K}$, $\mathtt{F}$ and $\mathtt{R}$ - it should be noted that $\mathtt{S}$ determines also a corresponding set of *atomic derivations*. With *B-term* and *B-derivation* we will indicate, respectively, a term consisting of elements of $\mathtt{K}$, $\mathtt{F}$ and $\mathtt{R}$, and a derivation in $\mathtt{S}$. Of course, if $\mathtt{K}$, $\mathtt{F}$ and $\mathtt{R}$ strictly contain the relative individual functional and predicative symbols of L, the notion of valid argument will be developed with regard to an expansion of L. However, hereafter we will assume that the individual functional and predicative symbols are always all and solely those of L. A base $\langle\{\mathtt{K}, \mathtt{F}, \mathtt{R}\}, \mathtt{S}\rangle$ is said to be *consistent* if, and only if, $\nvdash_{\mathtt{S}} \bot$.

As we have seen, in a semantic reading of Gentzen's words quoted in Section 2.5.1, the introduction rules for $\mathtt{IL}$ can be understood as determining, or, in a weaker sense, as mirroring the meaning of the logical constants to which they refer. In fact, in the next Section we will be able to realize that such rules go hand in hand with a description of the meaning in terms of provability conditions. In view of this, an argument ending with an application of an introduction rule to valid arguments of an appropriate type, can be said

> valid by the very meaning of the logical constants when understood constructively [...]. And conversely, it must be possible to bring a valid argument for a compound formula into one of these forms. (Prawitz 1973, 232)

What shall we say then of the (applications of) non-introductory rules, and of arguments ending with them, or in which they occur? Again in Section 2.5.1, it has been argued that reduction operations can be seen as justifications for the elimination rules of $\mathtt{IL}$. The idea behind the relativization of the notion of valid argument to justifications is exactly that of considering the latter as a sort of generalization of the reductions associated with $(\wedge_{E,i})$, $(\vee_E)$, $(\rightarrow_E)$, $(\forall_E)$ and $(\exists_E)$. More in particular, a justification $\mathtt{J}$ of a set of inference rules

$\Re$, each one disjoint from every introduction rule, is a set of constructive functions f, each of which associated with a single element R of $\Re$ and such that: (1) f is defined on some set of arguments ending with applications of R; (2) if f is defined on Δ with assumptions Γ and conclusion α, $f(\Delta)$ is an argument with assumptions $\Gamma^* \subseteq \Gamma$ and conclusion α; (3) if f is defined on Δ, and $\sigma(\Delta)$ is an example of Δ, f is defined on $\sigma(\Delta)$ and is linear with respect to σ, namely, $f(\sigma(\Delta)) = \sigma(f(\Delta))$. A *consistent extension* $\mathtt{J}^+$ of $\mathtt{J}$ is a justification of a set of rules of inference $\Re^+$, such that: (1) $\Re \subseteq \Re^+$; (2) $(\Re^+ - \Re) \cap \Re = \emptyset$.

We can assume as sufficiently clear a notion of substitution of a subargument Δ_1 of Δ with an argument Δ_2, in symbols $\Delta[\Delta_2/\Delta_1]$. Given a justification $\mathtt{J}$ of a set of inference rules $\Re$, Δ_1 is said to *immediately reduce* to Δ_2 *with respect to* $\mathtt{J}$ if, and only if, it exists a subargument Δ of Δ_1 such that, for some $f \in \mathtt{J}$, f is defined on Δ and $\Delta_2 = \Delta_1[f(\Delta)/\Delta]$. $\Delta_1, \Delta_2, ...$ is said a *reduction sequence with respect to* $\mathtt{J}$ if, and only if, for every $i \leq n$ where n is the length of the sequence if the latter ends, and for every $i \in \mathbb{N}$ otherwise, Δ_i reduces immediately to Δ_{i+1} with respect to $\mathtt{J}$. Δ_1 *reduces to* Δ_2 *with respect to* $\mathtt{J}$ if, and only if, there is a reduction sequence with respect to $\mathtt{J}$ starting with Δ_1 and ending with Δ_2. These definitions would actually already be sufficient for our immediate purposes; however, it is perhaps appropriate to introduce some others in order to have a better understanding of what we will say later. Thus, Δ_1 is said *in normal form with respect to* $\mathtt{J}$ if, and only if, there is no Δ_2 to which Δ_1 immediately reduces with respect to $\mathtt{J}$. Δ_1 is said to be *normalizable with respect to* $\mathtt{J}$ if, and only if, there exists Δ_2 in normal form with respect to $\mathtt{J}$ such that Δ_1 reduces to Δ_2 with respect to $\mathtt{J}$. Finally, Δ_1 is said to be *strongly normalizable with respect to* $\mathtt{J}$ if, and only if, every reduction sequence with respect to $\mathtt{J}$ beginning with Δ_1 ends with a Δ_2 in normal form with respect to $\mathtt{J}$. At this stage, it is possible to give the definition of a valid argument with respect to a justification $\mathtt{J}$ and to an atomic base B.

Definition 5. Δ is *valid with respect to* $\mathtt{J}$ *and* B - or, more simply, $(\Delta, \mathtt{J})$ is valid with respect to B - if, and only if,

- Δ is a closed argument the conclusion of which is an atomic formula $\Rightarrow$ Δ reduces with respect to $\mathtt{J}$ to a B-derivation, or
- Δ is a closed argument the conclusion of which is not an atomic formula $\Rightarrow$ Δ reduces with respect to $\mathtt{J}$ to an argument ending with the application of an introduction rule, and the immediate subarguments of which are valid with respect to $\mathtt{J}$ and B, or

- Δ is an open argument $\Rightarrow$ every $(\Delta^\sigma, \mathtt{J}^+)$ is valid with respect to B, where $\mathtt{J}^+$ is a consistent extension of $\mathtt{J}$, and Δ^σ a closed example of Δ obtained by first substituting all the occurrences of free variables not later bound in Δ with closed B-terms, so as to get an argument Δ_1, and then all the assumptions not discharged in Δ_1 with closed arguments for such assumptions, valid with respect to $\mathtt{J}^+$ and B.

$(\Delta, \mathtt{J})$ is *valid* if, and only if, for every B, $(\Delta, \mathtt{J})$ is valid with respect to B.[1]

As we can see, definition 5 proceeds by simultaneous recursion. The notion of closed valid argument for non-atomic formulas of complexity k presupposes the notion of open valid argument for formulas of complexity $h < k$, the latter in turn obtained, at the same complexity, in the terms of the notion of valid closed argument. Therefore, the recursive character depends, in a substantial way, on the fact that in an (application of an) introduction rule, the premises are immediate subformulas of the conclusion, so to have a lower complexity than the latter. As a further observation, Prawitz points out how

> the definition of validity is not a reductive explanation of the logical constants. Rather, it defines a property of justified arguments that must apply when the argument represents a proof. (Prawitz 1973, 236)

This does not mean, however, that it is not possible to use definition 5 to derive clauses that show the meaning of logical constants. In particular, a closed argument Δ valid with respect to a justification $\mathtt{J}$ and to an atomic base B will be such that, if its conclusion is atomic, it reduces to a B-derivation, and

[1] In the 1971 article *Ideas and results in proof theory*, Prawitz takes into account extensions of atomic bases: given an open argument Δ, $(\Delta, \mathtt{J})$ is valid with respect to B if and only if each $(\Delta^\sigma, \mathtt{J}^+)$ is valid with respect to B^+, where $\mathtt{J}^+$ is a consistent extension of J, B^+ an extension of B, and Δ^σ a closed example of Δ obtained by substituting first all the occurrences of free variables not later bound in Δ with closed B^+-terms, so as to get an argument Δ_1, and then all the assumptions not discharged into Δ_1 with closed arguments for such assumptions valid with respect to $\mathtt{J}^+$ and B^+. This kind of definition has the advantage of producing a monotonic notion of validity with respect to extensions of the atomic base (see, among the others, Schroeder-Heister 2006 and Tranchini 2014a). However, to authorize a change of an atomic base obviously means to give up the idea that the latter determines the meaning of terms and formulas of the reference language. The contemporary works on proof-theoretic semantics refer alternately both to the 1971 definition, and to that of 1973. In any case, in later works Prawitz seems to have definitely opted for the latter.

($\wedge_A$) the conclusion of Δ is $\alpha \wedge \beta \Rightarrow \Delta$ reduces with respect to $\mathtt{J}$ to a closed argument ending with an application ($\wedge_I$) and whose immediate subarguments are closed arguments for α and β respectively, valid with respect to $\mathtt{J}$ and B;

($\vee_A$) the conclusion of Δ is $\alpha_1 \vee \alpha_2 \Rightarrow \Delta$ reduces with respect to $\mathtt{J}$ to a closed argument ending with an application of ($\vee_I$) and whose immediate subargument is a closed argument for α_i ($i = 1, 2$), valid with respect to $\mathtt{J}$ and B;

($\rightarrow_A$) the conclusion of Δ is $\alpha \rightarrow \beta \Rightarrow \Delta$ reduces with respect to $\mathtt{J}$ to a closed argument ending with an application of ($\rightarrow_I$) and whose immediate subargument is an open argument Δ_1 with undischarged assumption α and conclusion β such that, for every consistent extension $\mathtt{J}^+$ of $\mathtt{J}$, for every closed argument Δ_2 for α valid with respect to $\mathtt{J}^+$ and B, the closed argument $\Delta_1[\Delta_2/\alpha]$ for β is valid with respect to $\mathtt{J}^+$ and B;

($\forall_A$) the conclusion of Δ is $\forall y \alpha(y/x) \Rightarrow \Delta$ reduces with respect to $\mathtt{J}$ to a closed argument ending with an application of ($\forall_I$) and whose immediate subargument is an open argument $\Delta(x)$ with x unbound in $\Delta(x)$ and conclusion $\alpha(x)$ such that, for every closed B-term t, the closed argument $\Delta(t)$ for $\alpha(t/x)$ is valid with respect to $\mathtt{J}$ and B;

($\exists_A$) the conclusion of Δ is $\exists x \alpha(x) \Rightarrow \Delta$ reduces with respect to $\mathtt{J}$ to a closed argument ending with an application of ($\exists_I$) and whose immediate subargument is, for some closed B-term t, a closed argument $\alpha(t/x)$ valid with respect to $\mathtt{J}$ and B.

Here it should also be noted that, if B is consistent, there cannot be a closed argument for $\bot$ valid with respect to some $\mathtt{J}$ and to B. In fact, if there were such an argument, it should reduce with respect to $\mathtt{J}$ to a B-derivation of $\bot$, which contradicts the assumed consistency of B. Therefore, if we assume that the atomic bases are always consistent, we can introduce a further clause that fixes the meaning of the atomic constant $\bot$: there is no closed valid argument for $\bot$ on whatever base.

As is evident from the way they are represented, arguments are chains of inferences. The notion of argument, in other words, depends on that of inference. But when examining the link between validity of arguments and validity of inferences, Prawitz reverses the situation; the notion of valid inference is defined in the terms of the notion of valid argument.

Definition 6. A set of inference rules $\Re$ is *valid with respect to* $\mathtt{J}$ *and* B if, and only if, for every consisten extension $\mathtt{J}^+$ di $\mathtt{J}$, for every application Δ of the type

$$\frac{\Delta_1 \quad \dots \quad \Delta_n}{\beta} \alpha_1, ..., \alpha_m, x_1, ..., x_s$$

of an element of $\Re$, if, for every $i \leq n$, Δ_i is valid with respect to $\mathtt{J}^+$ and B, Δ is valid with respect to $\mathtt{J}^+$ and B. A set of inference rules $\Re$ is *valid with respect to* $\mathtt{J}$ if, and only if, for every B, $\Re$ is valid with respect to $\mathtt{J}$ and B.

In the light of definition 6, we can therefore say that an inference is valid with respect to a justification $\mathtt{J}$ and to an atomic base B in the case that it instantiates an inference rule belonging to a set of inference rules (possibly corresponding to the singleton of this rule) valid with respect to $\mathtt{J}$ and B. Similarly, the inference is valid with respect to $\mathtt{J}$ when it is valid with respect to $\mathtt{J}$ and to every B. To use a figurative language, both the validity of the inference rules and that of the inferences themselves are defined in a "global" way: they depend on the validity of the *entire* arguments, which rules and inferences, respectively, are applied to or occur in.

In connection with his notion of valid argument, Prawitz highlights two problems that will play a central role in our following discussion. The first one, which we can call the *recognizability problem*, is explained with the following words:

> given an argument Δ, we may of course not be able to see whether there is a justification $\mathtt{J}$ such that $(\Delta, \mathtt{J})$ is valid. But even when we are given a valid justified argument $(\Delta, \mathtt{J})$, we may not be able to see that it is valid. (Prawitz 1973, 237)

Being in possession of a closed non-canonical valid argument $(\Delta, \mathtt{J})$ - closed canonical valid arguments are excluded from such a difficulty - means *knowing how* to obtain a closed valid argument $(\Delta_1, \mathtt{J})$, where Δ_1 ends with the application of an introduction rule and has immediate subarguments valid with respect to $\mathtt{J}$ - the effectiveness of the available method is granted by the constructiveness of the functions in $\mathtt{J}$, therefore it is sufficient to apply these functions in some order. However, possession is not the same as *knowing that* $(\Delta, \mathtt{J})$ is a closed valid argument. Such an understanding is in fact given by the additional information obtained by carrying out the method that $(\Delta, \mathtt{J})$ provides. Likewise, being in possession of an open valid argument $(\Delta, \mathtt{J})$ means *knowing how* to obtain closed valid arguments: it is sufficient to replace first the occurrences of the free variables not bound in Δ with closed terms of the atomic base of reference, and then the undischarged assumptions of the argument so obtained with closed arguments for such assumptions, valid with respect to some consistent extension of $\mathtt{J}$ and to the atomic base of reference. Again, this however is not the same as *knowing that* $(\Delta, \mathtt{J})$ is an open valid argument; such an understanding could be achieved by carrying out all the (usually infinite) required substitutions. This situation is clearly uncomfortable: if valid arguments must really represent proofs, they must justify the conclusion under the hypothesis that the assumptions are justified, and for this to be possible, obviously it is not enough to have a method that permits to achieve the goal, since it is also required to recognize that the method has this property. However, it cannot be taken for granted that the recognition can always be obtained immediately, or at least easily. In the case of closed valid arguments, there is no upper limit for the complexity of the reduction procedures, so that agents with limited time and memory resources may not be able to perform thoroughly the method in their possession. As a matter of principle, whenever the resources of time and memory were infinite, its effective character in any case guarantees the possibility of carrying out the method. In this ideal condition, the problem remains for the open valid arguments, since the substitution process of occurrences of free variables and assumptions might never end. Therefore, Prawitz concludes that the valid arguments he defines are not necessarily conclusive, and that

> by a conclusive argument, one may perhaps understand a justified argument $(\Delta, \mathtt{J})$ *together* with an argument showing that $(\Delta, \mathtt{J})$ is valid. This second argument must then again be conclusive, and we would thus be led to an impredicative theory. (Prawitz 1973, 237)

As we will see, an alternative could be that of requiring that validity is decidable, rather than provable, or, in a less strong sense, recognizable. However, even this solution, assuming it is a solution, proves to be problematic in almost every respect from which it is considered.

Another problem, to which we can instead attribute the name of *proofs-as-chains problem*, concerns the idea, intuitively correct, that a proof is nothing more than a concatenation of valid inferences. The question is, then, whether the notion of valid argument permits to derive a result of the type: $(\Delta, \mathtt{J})$ is a valid argument if, and only if, it is composed exclusively of inferences which are applications of inference rules in sets of inference rules valid with respect to $\mathtt{J}$. Unfortunately, Prawitz points out how

> a justified argument schema $(\Delta, \mathtt{J})$ is clearly [...] valid [...] if the inferences in Δ that are not instances of introduction rules [...] are instances of rules in a set $\mathfrak{R}$ which is [...] valid with respect to $\mathtt{J}$ [...]. The converse of this does not hold, however. (Prawitz 1973, 241)

Therefore, not only the result cannot be derived, but it is even false. Suppose we have a closed argument Δ_1 for α valid with respect to a justification $\mathtt{J}$ such that $\wedge_1$-rid $\in \mathtt{J}$, and a closed argument Δ_2 for β which contains an application of an inference rule R such that R is not valid with respect to $\mathtt{J}$: the argument

$$\dfrac{\dfrac{\begin{matrix}\Delta_1 \\ \alpha_1\end{matrix} \qquad \begin{matrix}\Delta_2 \\ \alpha_2\end{matrix}}{\alpha_1 \wedge \alpha_2}\,(\wedge_I)}{\alpha_1}\,(\wedge_{E,1})$$

is valid with respect to $\mathtt{J}$. In fact, by applying $\wedge_1$-rid, it reduces with respect to $\mathtt{J}$ to the argument Δ_1, valid with respect to $\mathtt{J}$ by assumption.

2.5.2.2 Proofs (in 1977 and 2005)

As we have already said, the article *Meaning and proofs: on the conflict between classical and intuitionistic logic*, deals with the notion of proof starting from an analysis of the arguments that Dummett puts forward in support of the thesis according to which

> intuitionistic rather than classical logic describes the correct forms of reasoning within mathematics. (Prawitz 1977, 2)

Although this is not the place for dealing, if only in a generic way, with Dummett's arguments (for which, see Cozzo 1994b, 2008), we have to remember, however, that they are framed in general semantic considerations

> according to which the meaning of a sentence must be understood in terms of the use of the sentence. [...] the meaning of a sentence cannot be treated in isolation from the question of how the truth of the sentence may be established, and in the case of mathematics especially, this means that meaning has somehow to be understood in terms of proofs. (Prawitz 1977, 2 - 3).

As stated above, the connection between meaning and use does not preclude, but rather suggests, the identification of a central aspect of propositions or sentences on which the explanation can be based. More specifically, Dummett makes a distinction between

> two aspects of the use of an assertive sentence: (1) the conditions under which it can be correctly asserted and (2) the commitments made by asserting it. In the case of mathematics, aspect (1) is

expressed in the rules for inferring the sentence, and aspect (2) in the rules for drawing consequences from the sentence. (Prawitz 1977, 7)

On these bases, and dissatisfied with the semantics centered on the notion of bivalent truth, Dummett (Dummett 1978c, 1978e, 1993b) develops a theory that explains meaning in terms of conditions of correct assertability. That recalls to mind what has already been said in Section 2.4: in order to capture the necessity of thought involved in proofs and valid arguments, Prawitz proposes to adopt an epistemic conception of truth, and this implies, as its final result, the attribution of a central role to the notion of evidence. Now, if in this framework we reason more specifically in terms of valid arguments, we can appeal to the clauses ($\wedge_A$) - ($\exists_A$). But how to behave in the case of proofs?

An answer to this question is given by the intuitionistic tradition, and in particular by the idea, inspired by Brouwer's theories, and independently developed by Heyting (Heyting 1956) and Andrej Nikolaevic Kolmogorov (Kolmogorov 1932), to fix the meaning of the logical constants through a specification of the notion of proof by induction on the complexity of the formulas of a first-order logical language - the clauses so obtained are known by the acronym BHK, corresponding to the names of their authors. We will have again an atomic base B with individual, functional and relational symbolsl, and a Post system relative to them. Therefore, a proof with respect to B of an atomic formula is again a B-derivation and, under the usual assumption that there is no proof of $\bot$,

($\wedge_P$) a proof with respect to B of $\alpha \wedge \beta$ is a proof π_1 with respect to B of α and a proof π_2 with respect to B of β;

($\vee_P$) a proof with respect to B of $\alpha \vee \beta$ is a proof π with respect to B of α or of β, with an indication of which of the disjoints π proves;

($\rightarrow_P$) a proof with respect to B of $\alpha \rightarrow \beta$ is a constructive procedure f such that, for every proof π with respect to B of α, $f(\pi)$ is a proof with respect to B of β;

($\forall_P$) a proof with respect to B of $\forall x \alpha(x)$ is a constructive procedure f such that, for every closed B-term t, $f(t)$ is a proof with respect to B of $\alpha(t)$;

($\exists_P$) a proof with respect to B of $\exists x \alpha(x)$ is a proof π with respect to B of $\alpha(t)$ for some B-term t.

As noted by Prawitz himself, and even before by Rosza Peter (Peter 1959), the notion of constructive procedure involved in the clauses ($\rightarrow_P$) and ($\forall_P$) must be assumed here as primitive, since it is not possible to

> define it as a Turing machine that always yields a value when applied to an argument; the quantifier prefix [...] in this definition must then be understood intuitionistically, and this means that to understand the definition we must already know what such a constructive procedure is. (Prawitz 1977, 27)

The first problem with the BHK clauses, however, is that they do not offer an exhaustive description of the conditions of correct assertability. In other words, some structures prove formulas with main logical constant c, although they do not have the form expected with (c_P):

> it is not true even intuitionistically that the condition for asserting a sentence is that we know a proof of it in [the sense of the BHK clauses]. This can be seen even for atomic sentences such as $768+859 = 859+768$, when the given proofs of atomic sentences consist of proofs following the usual rules of computation. The proof of the mentioned equality would then proceed via the calculation of the sums in question. But we are perfectly justified in asserting the equality without knowing these sums and hence without knowing the proof in question; it is sufficient that we know, for instance, a proof of $\forall x \forall y(x + y = y + x)$ and then infer the equality by instantiation. [...] Similarly we may assert even intuitionistically that $\alpha(n) \vee \beta(n)$ for some numeral n without knowing a proof of $\alpha(n)$ or of $\beta(n)$; it would be sufficient, e.g., if we know a proof of $\alpha(0) \vee \beta(0)$ and of $\forall x(\alpha(x) \vee \beta(x) \rightarrow \alpha(x+1) \vee \beta(x+1))$. (Prawitz 1977, 21)

Hence, BHK semantics can be a good starting point, although it is not adequate by itself for the desired objectives. In fact, it defines sufficient but not necessary proof-conditions, and

> this would be quite insufficient for a meaning theory as proposed above, e.g., we would then never be in the position to say that a sentence was incorrectly asserted on a given occasion. (Prawitz 1977, 22)

In order to unravel the difficulty, Prawitz follows Dummett in introducing a distinction between *direct* and *indirect* - or, using a widespread terminology, *canonical* and *non-canonical* - forms of proofs, perfectly symmetrical to the distinction between valid arguments ending with applications of introduction rules, and valid arguments which, ending with applications of non-introductory rules, must be reduced to valid arguments ending with applications of introduction rules:

> the condition for asserting a sentence is that we *either* know a proof of the sentence of the kind mentioned in [the BHK clauses] *or* know a procedure for obtaining such a proof. This procedure may also be called a proof [...] but it is a proof in a secondary sense. (Prawitz 1977, 22)

In this way, we can reuse the BHK characterization by saying that it outlines the notion of canonical proof with respect to the chosen atomic base B, and postulate that a non-canonical proof with respect to B is a constructive procedure to obtain a canonical proof with respect to B - we must remember here that, in this framework, the notion of constructive procedure is primitive. $(\wedge_P)$ - $(\exists_P)$ are now sufficient to explain the meaning of propositions or sentences, and consequently of the logical constants, in terms of conditions of canonical assertability. Non-canonical assertability, on the other hand, is defined in terms of the canonical one, the latter being, therefore, the sole central notion of our theory.

At this point, the proposed framework suggests a natural criterion of acceptability of inferences with respect to B:

> what we should demand is that we know a [...] procedure which, applied to the way in which the conditions for asserting the premisses are satisfied, brings about a situation in which the conclusion can be asserted appropriately. (Prawitz 1977, 23)

As Prawitz himself points out, such a condition will be satisfied if, and only if, we know a constructive procedure f such that, given that $\alpha_1, ..., \alpha_n$ are the premises and β the conclusion of the inference, for each canonical proof π_i with respect to B of α_i, $f(\pi_1, ..., \pi_n)$ produces a canonical proof with respect to B of β. Necessity: if we know canonical proofs $\pi_1, ..., \pi_n$ with respect to B of $\alpha_1, ..., \alpha_n$, respectively, the conditions to assert correctly $\alpha_1, ..., \alpha_n$ are satisfied. If by hypothesis, we have a procedure that, when the proofs to correctly assert $\alpha_1, ..., \alpha_n$ are met, makes it possible to assert correctly β, this means that, said f this procedure, $f(\pi_1, ..., \pi_n)$ produces a canonical proof with respect to B of β (it should be noted that a canonical proof can be trivially understood as a procedure that produces a canonical proof - the identity function). Sufficiency: suppose that the conditions to assert

$\alpha_1, ..., \alpha_n$ are satisfied by virtue of the knowledge of proofs (this time not necessarily canonical) $f_1, ..., f_n$ with respect to B of $\alpha_1, ..., \alpha_n$ respectively; each f_i must be a constructive procedure for obtaining a canonical proof with respect to B of α, so that we can define a constructive procedure ex of execution of each f_i, so that $\texttt{ex}(f_i)$ is a canonical proof with respect to B of α (simply put, $\texttt{ex}(f_i)$ is the reduction procedure of f_i to canonical form). By hypothesis, $f(\texttt{ex}(f_1), ..., \texttt{ex}(f_n))$ produces a canonical proof with respect to B of β, which justifies the assertion of β. We can finally say that an inference (rule) is valid with respect to an atomic base B if, and only if, (for each inference that is an application of this rule) there is a situation in which the inference is acceptable with respect to B. An inference (rule) is valid if, and only if, for each B, it is valid with respect to B. As in the case of the valid arguments, the notions of acceptability of an inference and of validity of an inference (rule) are *globally* defined, so that priority is given to proofs.

However, what we have called the *recognizability* problem and the *proofs-as-chains* problem in the previous Section also hold in relation to the present notion of proof. Actually, the first problem is explicitly raised by Prawitz, who remarks that

> in the case when α is atomic or is built up of atomic sentences by $\wedge$, $\vee$, and $\exists$, the knowledge of a canonical proof [...] can be taken to consist of just the construction of the proof. [...] In the cases when α is an implication or a universal sentence and in the case when we know only a procedure for obtaining a canonical proof, we must require not only a construction or description of an appropriate procedure but also an understanding of this procedure. (Prawitz 1977, 27)

In a completely similar manner to what has been said about the notion of valid argument, the request that a constructive procedure generating proofs of the expected type, or so operating on a class of proofs of a certain type, is not only possessed but also understood as such, derives from the plausible thesis according to which knowing a proof means not only having a mathematical object, but mainly recognizing that the latter enjoys significant epistemic properties. And it is obvious that the mere possession of a proof as understood here is not, from this point of view, sufficient; to *know how* to get a canonical proof does not mean also we *know that*, once carried out, the method will produce the desired result. Likewise, to have a constructive procedure such as the one provided by the clause $(\rightarrow_P)$ or by the clause $(\forall_P)$ is equivalent to *knowing how* to have a certain output on a given input, but not to *knowing that* this will happen whatever the possible values to be computed may be. Again, the absence of an upper limit on the complexity of the computation of the constructive procedures, and the infinity of the substitutions to be made, make it implausible to claim that such procedures contain information sufficient to guarantee the recognition required. In a different way to what Prawitz wrote in 1973, however, the author now seems to reject the idea that recognition can come from

> a description of the procedure together with a proof that the procedure has the property required [...] this would lead to an infinite regress and would defeat the whole project of a theory of meaning as discussed here. (Prawitz 1977, 27)

The alternative to the idea of a meta-proof on the properties of the constructive procedures, an alternative that we have already mentioned and that in the 1977 article is taken into explicit account, consists of asking whether the relation "being a proof of" can be said to be decidable:

> the sentences $\alpha \rightarrow \beta$ and $\forall x \alpha(x)$ can be asserted when we have described and understood certain kinds of procedures, but it is doubtful in what sense, if any, one could decide the question whether this condition obtains in a certain situation. (Prawitz 1977, 29)

Decidability, one could argue, would solve our problem, by implying the existence of a uniform and general constructive procedure to establish if whatever closed construction (without any occurrences of free variables for proofs or for terms not subsequently bound) amounts to a proof for whatever formula of the reference language or, in case the construction is open (with occurrences of free variables for proofs orfor terms not subsequently bound), to a constructive procedure to transform proofs for whatever formulas of the reference language in proofs for whatever formula of the reference language. Be that as it may, Prawitz is sceptical as concerns that possibility, and proposes specific counter-examples in order to show that, in a given interpretation, this path is untenable. We will return later on this matter, after pointing out how, in addition to proof-theoretic semantics based on the notions of valid argument and proof, the question reappears, albeit with a different impact, also in the theory of grounds.

The proofs-as-chains problem, in turn, is more easily visible if we turn to an analysis of the way in which the notion of proof is updated and expanded in *Logical consequence from a constructivist point of view*. In 2005, Prawitz proposes first of all a rewriting the BHK clauses, following an intuition already present, but not made explicit in 1977:

> we should not say simply that a canonical proof of, e.g., $\alpha \wedge \beta$ consists of a canonical proof of α and a canonical proof of β. It is not enough that we have just constructed these two canonical proofs separately to be in the position to assert $\alpha \wedge \beta$ - they entitle us only to assert α and to assert β. [...] we must also be aware of the fact that these two proofs form a sufficient ground to go one step further and assert $\alpha \wedge \beta$. Or, more precisely, one should grant the existence of an operation which yields a canonical proof of $\alpha \wedge \beta$ when performed on canonical proofs of α and β. (Prawitz 1977, 26)

> the general form of a canonical proof for a compound sentence α with the logical constant c as main sign can be written $O_c(\pi)$, where O_c is an operation that stands for the recognition that we have obtained direct evidence for α because of π. (Prawitz 2005, 686)

The clauses, however, are now derivable from a broader framework, similar to that of valid arguments. Let us start with a general notion of *proof-structure* on a first-order language L, understood as a concatenation of constructive procedures, applied to appropriate arguments, which represent inference rules, and inferences, on the formulas of L - the notion of constructive procedure is to be assumed again as primitive. A proof-structure will have some *assumptions*, given by the basic arguments of the composite procedure resulting from the concatenation of procedures to which it corresponds, and a *conclusion*, given by the range of the composite procedure resulting from the concatenation of procedures to which the latter corresponds. Some procedures can discharge assumptions or bind occurrences of free variables, so that a proof-structure will be *open* when it contains undischarged assumptions or occurrences of free variables not subsequently bound, or otherwise *closed*. An *immediate substructure* of a proof-structure π is a proof-structure that appears as an argument of the last procedure of π; a *substructure* of π is an initial segment of π, i.e., an element of the reflexive and transitive closure with respect to π of the relation "π_1 is an immediate substructure of π_2". A proof-structure is *canonical* when it ends with the application of a procedure corresponding to an introduction rule, *non-canonical* otherwise.

In compliance with the idea that the introduction rules fix or, more weakly, reflect the meaning of the logical constant they refer, a canonical proof-structure will be a *canonical proof* assuming that its immediate substructures are such, while a non-canonical proof-structure will be a *categorical proof* (equivalent to the non-canonical proofs of 1977) in the case that it amounts to a constructive procedure to obtain a canonical proof. The notions of canonical proof and of categorical proof, however, must be defined by simultaneous recursion, since, in general, the arguments of the last procedure of a canonical proof could take a non-canonical form. This appears possible because the premises of (an application) of an inference rule are immediate subformulas of the conclusion, and therefore they have a lower complexity than the latter. Though, we also need the additional definitions of *hypothetical proof* and *general proof*. As usual, the whole of these definitions refers to an atomic base B with individual relational and functional symbols, and to a Post system relative to the latter (the derivations of which will be, by assumption, canonical proofs), and with the usual request that there should not be any canonical proof of $\bot$.

Definition 7. A closed canonical proof-structure is a *canonical proof* on B if, and only if, all its immediate substructures are categorical, hypothetical or general proofs on B.

A closed non-canonical proof-structure is a *categorical proof* on B if, and only if, it is a constructive procedure to obtain a canonical proof on B.

An open proof-structure with assumptions $\alpha_1, ..., \alpha_n$ and conclusion β is a *hypothetical proof* on B if, and only if, it is a constructive procedure f such that, for each π_i categorical proof on B of α_i $(i \leq n)$, $f(\pi_1, ..., \pi_n)$ is a categorical proof on B of β.

An open proof-structure with occurrences of free variables not subsequently bound $x_1, ..., x_n$ and conclusion $\alpha(x_1, ..., x_n)$ is a *general proof* on B if, and only if, it is a constructive procedure f such that, for each closed B-term t_i $(i \leq n)$, $f(t_1, ..., t_n)$ is a categorical proof on B of $\alpha(t_1, ..., t_n)$.

A proof-structure is a *proof* if, and only if, for every B, it is a proof on B.

We can now easily obtain the clauses by introducing primitive procedures $O_\wedge$, $O_\vee$, $O_\rightarrow$, $O_\forall$ and $O_\exists$ for the rules $(\wedge_I)$, $(\vee_I)$, $(\rightarrow_I)$, $(\forall_I)$ and $(\exists_I)$ respectively - with the implicit indication of appropriate discharge of assumptions, binding of occurrences of free variables, and restrictions on proper variables.

$(\wedge_P^P)$ a canonical proof of $\alpha \wedge \beta$ is $O_\wedge(\pi_1, \pi_2)$ with π_1 categorical proof of α and π_2 categorical proof of β;

$(\vee_P^P)$ a canonical proof of $\alpha \vee \beta$ is either $O_\vee(\pi_1)$ with π_1 categorical proof of α, or $O_\vee(\pi_2)$ with π_2 categorical proof of β;

$(\rightarrow_P^P)$ a canonical proof of $\alpha \rightarrow \beta$ is $O_\rightarrow(\pi_\beta^\alpha)$ with π_β^α hypothetical proof with assumption α and conclusion β;

($\forall^P_P$) a canonical proof of $\forall x\alpha(x)$ is $O_\forall(\pi(x))$ with $\pi(x)$ general proof with occurrence of free variable not subsequently bound x and conclusion $\alpha(x)$;

($\exists^P_P$) a canonical proof of $\exists x\alpha(x)$ is $O_\exists(t, \pi)$ with t term on the reference base and π categorical proof of $\alpha(t)$.

Three observations. Firstly, as far as canonical proofs are concerned, the adoption of primitive procedures makes it in a sense less urgent the recognizability problem. In fact, it is required that canonical proofs in the critical cases of $\rightarrow$ and $\forall$ consist of applications of procedures which denote the recognition of the related substructures as, respectively, hypothetical and general proofs. Therefore, if we assume that such a recognition has taken place, namely that $O_\rightarrow$ or $O_\forall$ have been applied, and since these operations settle the meaning of, respectively, $\rightarrow$ and $\forall$, the obtained result will automatically be recognized as a canonical proof for $\alpha \rightarrow \beta$ or for $\forall x\alpha(x)$. Secondly, this same adoption also compels to distinguish the primitive procedures from others which, on the contrary, are not primitive; of course, the non-primitive procedures will represent inference rules in a non-introductory form, so that their applications will correspond to inferences of the same type. Prawitz explains the difference already in 1977 with the following words:

> an (or the) essential element of an understanding of [a primitive] operation consists in the ability to carry out the operation so that the respective results are obtained. [...] a primitive operation [is] one whose result has to be conceived in terms of the operation, while the result of a defined operation can be understood independently of the operation. (Prawitz 1977, 28)

In light of this, it then becomes evident how the recognizability problem - which, as can be seen, in the non-canonical case is meaningfully valid - may also be understood as the problem of the type of understanding required so that we know whether a non-primitive procedure produces proofs of a certain type when applied to proofs of a certain type. Finally,

> also in the conditions for asserting conjunctions, disjunctions, and existential statements, there is thus, strictly speaking, a question of understanding certain primitive operations, but these cases are much less complicated. In the case of implications and universal statements, it is not clear that we can decide whether something is a canonical proof or not. (Prawitz 1977, 28 - 29)

In other words, even if primitive procedures are not in themselves problematic, and, indeed, mitigate the problem of recognizability in the canonical case, the fact that in the clauses ($\rightarrow_P^P$) and ($\forall_P^P$) they are applied to procedures for which this problem remains makes it difficult to establish what it means to understand $O_\rightarrow$ and $O_\forall$. In fact, in order to make it clear what the use of the latter consists of, we should make it clear what it means to recognize that a procedure always generates determinate proofs when applied to certain arguments. Therefore, even if the primitive procedures have the effect of making recognizable the canonical proofs under the assumption that they have been applied, namely, under the assumption that certain substructures have been recognized as hypothetical or general proofs, we are not allowed to assume that a canonical proof is always recognizable as such.

We can now return to what we have called the *proofs-as-chains* problem. We must ask ourselves whether the theoretical framework of the years 1977-2005 defines a notion of proof, such that a proof-structure π is a proof if, and only if, all the constructive procedures of which π is composed represent valid inferences (instantiating given inference rules). It is not difficult to realize that, as in the case of valid arguments, the answer is no. Let us assume that: d_1 is a proof (canonical, but in case non-canonical or categorical) for α_1; d_2 a proof-structure with conclusion α_2 containing constructive procedures that represent non-valid inferences (instantiating non-valid inference rules); f a non-primitive constructive procedure defined on elements of the type $O_\wedge(\pi_1, \pi_2)$ by the equation

$$f(O_\wedge(\pi_1, \pi_2)) = \pi_1.$$

As a consequence $f(O_\wedge(d_1, d_2))$ is a non-canonical or categorical proof for α_1. By definition, $f(O_\wedge(d_1, d_2)) = d_1$; by assumption, d_1 is a proof (canonical, but in case non-canonical or categorical) of α_1, or a constructive procedure to get a canonical proof of α_1 (we must remember that if d_1 is a canonical proof of α_1, it is trivially a constructive procedure to obtain a canonical proof of α_1 - the identity function). But then, also $f(O_\wedge(d_1, d_2))$ is a constructive procedure to obtain a canonical proof of α_1 (in fact, f is nothing but a projection on the first element of the pair to which $O_\wedge$ is applied). Therefore, a proof-structure π can be a proof even if some of the constructive procedures it is composed of represent non-valid inferences (instantiating non-valid inference rules).

2.5.3 Problems in proof-theoretic semantics

In Section 2.4 we have outlined two goals that an analysis of the notion of (deductive or logical) consequence in terms of necessity of thought must achieve. First, to define rigorously the notions of valid argument or proof, so as to translate with any precision the link of epistemic modality existing between α and Γ by virtue of which, in case of justification for all the elements in Γ (on a specific or arbitrary base), one is justified in asserting α, or compelled to consider correct its assertion (on a specific or arbitrary base). In the light of what is shown in Section 2.5.2, namely, thanks to the formal apparatus of proof-theoretic semantics in the formulations of 1973, 1977 and 2005, we can consider this point satisfied. In any case, underlying the idea that necessity of thought can be understood in terms of valid arguments and proofs, there is the plausible, already mentioned, thesis according to which what we are asking for is the existence of something that obliges us to hold true α, if we have accepted the truth of the elements of Γ. But then, a second question arises, a question of *adequacy*: given the definitions 5 and 7, can we say that the notions of valid argument and proof that they characterize are such that, if we know a valid open argument or a hypothetical proof for α from Γ, we are compelled to consider α true, in case we have accepted the truth of all the elements in Γ? With this question, Prawitz concludes *Logical consequence from a constructivist point of view*, noting first of all that

> some inferences - namely, the inferences by introduction - become valid by the very meaning of the conclusion of the inference. Because of this meaning, we are compelled to hold the conclusion true when holding the premisses true. [...] An inference in general is compelling when we know a hypothetical proof of its conclusion α from its set of premisses Γ or, alternatively, an open valid argument for α from Γ. [...] knowing such a proof is to be in possession of an effective method which, applied to categorical proofs of the sentences of Γ, yields a categorical proof of α. [...] knowing a valid open argument Δ for α from Γ, we get a valid argument for α by replacing the open assumptions in Δ with closed valid argument for them. (Prawitz 2005, 693)

But, soon thereafter, Prawitz emphasizes significantly how, since the compelling character of open valid arguments and hypothetical proofs is based on their reduction to closed valid arguments and categorical proofs, we should more generally expect that our analysis implies the fact that

> by knowing a proof or valid argument, one gets committed in the way discussed above. (Prawitz 2005, 693)

Now then, one could reasonably argue that valid arguments and proofs induce necessity of thought on an agent who has knowledge of them because, by carrying them out, he/she makes a series of valid inferences. With the transmission of the justification from premises to conclusions, these inferences compel to accept the latter if one accepts the former. This is a *dynamic* vision, in which the epistemic constriction comes from the carrying out of certain *acts*; and it is in agreement with a plausible intuitive point of view, but also with the "phenomenal character" of "active experience" that Cozzo (Cozzo 2015), as we have said, properly attributes to the necessity of thought. However, to choose to follow this path - which seems, after all, the most convincing -

> puts the burden on the notion of valid inference. How is it to be analyzed? If we do not simply say that an inference is valid when its conclusion is a logical consequence of the premisses, which would bring us back to the beginning of this investigation, we have to try to develop some concept of "gapfree" inference. The "gapfree" inferences must then be shown to have a compelling force. (Prawitz 2005, 693)

At a first view, a correct strategy could consist in affirming that an inference is valid if, and only if, it can be reduced to valid "gapfree" inferences. However,

> to define validity in that way would of course have made the whole analysis circular. (Prawitz 2005, 693)

Since the circularity depends on the attribution of validity - what we intend to define - to "gapfree" inferences, an obvious alternative could consist in starting from a definition of valid "gapfree" inference, and then require that a non-"gapfree" inference is valid if, and only if, it can be reduced to valid "gapfree" inferences. Since the introductory inferences are involved in the explanation of the meaning of logical constants, they can be understood as valid "gapfree" by default. Naturally, if they were the only valid "gapfree" inferences, the fact that their premises always have a lower complexity than the conclusion would allow an inductive, and for this reason satisfactory, definition of the notion of valid inference. Obviously, the problem is that we cannot assume every valid inference as reducible only to introduction inferences: there will be valid "gapfree" inferences in a non-introductory form - for example, Gentzen's eliminations. How can we define the validity of the latter? At the conclusion of *Logical consequence from a constructivist point of view*, in examining these themes, Prawitz notes that

> our basic intuition is that of canonical proof or argument [...] this amounts to making inferences by introduction valid - valid by definition, so to say [...] there are inferences other than introductions

> that are gapfree in the sense that they cannot be reasonably be broken down into simpler inferences. An essential ingredient in the analysis proposed here is a way of demonstrating the validity of such inferences [...] by applying reductions that are seen to transform arguments into ones that are known to be valid. (Prawitz 2005, 693 - 694)

In the passage just quoted, Prawitz preserves the explanatory primacy of valid arguments and proofs. This, however, could pose a new problem. If the goal is to show that valid arguments and proofs have a power of epistemic compulsion, since they are composed of epistemically compelling valid inferences, how can we attribute an epistemic power to inferences the validity of which is explained by using the notions of valid argument and proofs? This point already holds, in a significant way, in the case of inferences in an introductory form. Suppose the epistemic force of a closed valid canonical argument

$$\frac{\begin{array}{cc}\Delta_1 & \Delta_2 \\ \alpha & \beta\end{array}}{\alpha \wedge \beta}\ (\wedge_I)$$

or of a canonical proof $O_\wedge(\pi_1, \pi_2)$ are explained in terms of their being composed of sole epistemically compelling valid inferences. If we say now that the last inference is epistemically compelling since it is valid - where valid means that it produces a valid closed canonical argument when applied to Δ_1 and to Δ_2, or a canonical proof when applied to π_1 and to π_2 - we could conclude that this characterization is satisfactory only if we have already accepted as epistemically compelling the starting argument and the starting proof.

To conclude, we would like to point out two things. First, and with reference to the previous example of closed valid canonical argument and of canonical proof, what we have said is clearly not intended to deny that the respective applications of $(\wedge_I)$ and $O_\wedge$ are epistemically compelling; they produce something that the reference semantics treats as evidence - a closed valid canonical argument or a canonical proof, in fact - while the problem is rather to *show* that closed valid canonical arguments and canonical proofs can quite rightly be understood as producing evidence *by virtue of their being made up of valid inferences.* Secondly, the main result of the analysis carried out so far is that, in order to achieve an adequate explanation of the intended relation between the notions of valid argument or proof and of valid inference, there seems to be no other possibility than to define the latter on a basis independent from the former. And this, of course, requires a *local* analysis of valid inferences, as opposed to the global character of that available in proof-theoretic semantics.

2.5.3.1 Recognizability and chains

The transition from *global* to *local* is, as we will see, one of the basic insights of the theory of grounds. However, in the framework under examination, it does not seem, as said, sufficient. The most immediate difficulty already comes from the proofs-as-chains problem: the epistemic power of valid arguments and proofs can reasonably be considered as dependent on the force of the valid inferences involved in them, only to the extent that valid arguments and proofs are composed exclusively of valid inferences - and we have already had the opportunity of saying that this statement is false. However, this does not mean that it may be possible to ascribe an epistemic compulsion to valid arguments and proofs that contain only valid inferences. In fact, the discussion conducted in the previous Section is independent of the type of inferences, whether valid or not, occurring in valid arguments and proofs. There is, therefore, a theoretical difficulty concerning the relation between the definitions of valid argument and proof, on the one hand, and of valid inference, on the other. In the light of this difficulty, we need to define inferential validity in terms of a notion that does not contain any reference to valid arguments and proofs. Obviously, it will serve no purpose even a notion that, similarly to those of valid argument and proof, will return something that is expected to be, or that actually is a concatenation of valid "gapfree" inferences. Moreover, this further circumstance depends, more profoundly, on the indispensable distinction between inferences in an introductory form and inferences in a non-introductory form, between canonical and non-canonical cases.

It is certainly right to consider introduction inferences as valid "gapfree" by default, and then to define as valid the "gapfree" ones non-divisible in valid "gapfree" inferences. Since inferential validity must be defined independently of the notions of valid argument and proof, we could then think of dealing with the case of the "gapfree" inferences in a non-introductory form by saying that they are valid if, and only if, whenever they are applied to chains of valid "gapfree" inferences, the resulting inferential concatenation can be reduced to one ending with an introduction inference applied to a chain of valid "gapfree" inferences. If we adopt this strategy, however, our definition of validity for inferences

$$\frac{\begin{matrix}[\alpha]\\ \Delta\\ \beta\end{matrix}}{\alpha \rightarrow \beta}\ (\rightarrow_I)$$

or represented by proof-structures $O_{\rightarrow}(\pi^{\alpha}_{\beta})$, must request that Δ and π^{α}_{β} are chains of valid "gapfree" inferences; and since we are making a distinction between inferences in an introductory form and inferences in a non-introductory form, between canonical cases and non-canonical cases, we cannot exclude that Δ and π^{α}_{β} contain valid "gapfree" inferences in a non-introductory form, and, consequently, inferences of the same type, having an identical or even greater complexity than that we are defining as valid. Our definition would be once more circular.

What we are saying is emphasized by Usberti (Usberti 2015); after asserting that the recognizability problem for valid arguments dealt with in the 1973 article has as a consequence that the relative notion of closed valid canonical argument cannot establish a theory of meaning *à la* Dummett (though this question, to which we will return later, is still debated - see, for example, Cozzo 2008), Usberti puts forwards a further observation, according to which this problem does not apply to the notion of BHK proof, so that the latter could be suitable for playing the role of central notion in a theory of meaning based on conditions of correct assertability. In particular, the BHK clauses - set out in Section 2.5.2.2 - proceed obviously by induction on the complexity of the formulas and, in the case of $\rightarrow$ and $\forall$, they abstract from the intrinsic complexity of the constructive procedures involved. In the same section, however, it was also said that these clauses set sufficient but not necessary conditions of provability. This means in turn that, although potentially suitable for the explanation of the meaning,

> Heyting's proofs cannot be seen [...] as chains of valid inferential acts. (Usberti 2015, 417)

However, the proposed solution - to distinguish between canonical and non-canonical proofs, and to consider the BHK clauses as a description of the sole notion of canonical proof - transforms the initial induction into a simultaneous recursion. Here, we can no longer abstract from the intrinsic complexity of the constructive procedures, so that the immediate subproofs of proofs for implications and universal quantifications are to be conceived, without any restriction whatsoever, as chains of valid "gapfree" inferences. Then, we could now define as valid introductions of $\rightarrow$ and $\forall$ in connection with hypothetical or general proofs only consisting of valid "gapfree" inferences, but, as we have already affirmed,

> this is just what cannot be done, on pain of being exposed to an objection of circularity. (Usberti 2015, 418)

Naturally, Usberti takes care to remember a passage from Gentzen's *Untersuchungen*, in which this theme is perhaps for the first time focalized:

> in interpreting $\alpha \to \beta$ in this way, I have presupposed that the available proof of β from the assumption α contains merely inferences *already recognized as permissible.* On the other hand, such a proof may contain other $\to$-inferences and then our interpretation *breaks down.* For it is circular to justify the $\to$-inferences on the basis of a $\to$-interpretation which itself already involves the presupposition of the admissibility of the same form of inference. (Gentzen 1935 - 1936, 167; but see also Negri & von Plato 2015)

In relation to Usberti's reconstruction, two additions seems to us necessary. First, it should be remembered that, as shown in Section 2.5.2.2, also the approach of the years 1977 and 2005 in terms of proofs suffers from a problem of recognizability; therefore, if we share the idea that the notion of closed valid canonical argument of 1973 cannot act as a basis for a constructivist explanation of meaning, the same is to be said for the notion of canonical proof obtained from the BHK clauses by introducing the distinction between canonical and non-canonical proofs. Secondly, as we have already seen, the fact that the 1973 notion of valid argument is defined, such as that of proof of the years 1977 and 2005, by simultaneous recursion implies that, also in this case, we cannot pass, on pain of objections of circularity, from this definition to a description of valid arguments as chains of valid "gapfree" inferences. Therefore, the two approaches are basically homologue (in this regard, see also Tranchini 2014a).

The fact that the BHK clauses manage to provide an inductive definition insofar as they abstract from the intrinsic complexity of the constructive procedures, seems to suggest that the only solution to our problem is the following: the notion on which to base the characterization of inferential validity will have to either do without the distinction between canonical and non-canonical cases or, more weakly, refer such a distinction to something that is not to be intended as a chain of valid "gapfree" inferences. On the other hand, we cannot give up the distinction between inferences in an introductory form and inferences in a non-introductory form, and since inferences are to be understood as *acts*, we can legitimately expect that the notion on which inferential validity will be based on will refer to *objects* produced by such acts. From this perspective, it may be useful to reconsider the example referred to at the end of the previous section. Given a closed valid canonical argument and a canonical proof, we can say that the *acts* to which they correspond confer a power of epistemic constraint, because the valid inferences they are composed of are epistemically compelling *acts*, since they produce *objects* that the reference semantics treats as evidence. In particular, the conclusive inferential applications are equipped with an epistemic force, because they allow the construction of the starting closed valid canonical argument or of the starting canonical proof, which are however now to be looked at as the *objects* that the reference semantics treats as evidence. Be that as it may, in order to discuss also the case of closed valid non-canonical arguments (and of non-canonical or categorical proofs), it is necessary to take one further passage; we will return to this point in Section 2.5.3.3.

Thus, the explanation of the epistemic power of valid arguments, proofs and valid inferences in proof-theoretic semantics is made problematic by the interaction between the two roles that valid arguments and proofs play in this framework: on the one hand, as mathematical *objects* aimed at a formal description of the notion of evidence, they act as explicans of meaning; on the other, as *acts* aimed at achieving justification, all their inferences are expected to be valid. In fact, we will affirm later on, that the distinction between objects and acts plays a central role in achieving an adequate characterization, in particular in the theory of grounds. More specifically, through the latter Prawitz seems to validate a distinction ascribable to Göran Sundholm (Sundholm 1998, but see also Sundholm 1983, 1993), between *proof-objects* and *proof-acts*.

2.5.3.2 Validity as independent from inferences

In the recent *The fundamental problem of general proof theory*, Prawitz highlights a further weak point of the notion of valid argument:

> the validity of an argument relative to a base B and a set of reductions $\mathtt{J}$ may depend essentially on the reductions in $\mathtt{J}$ and not at all on the inferences that make up the argument. In contrast, the justification of Gentzen's elimination rules discussed in [section 2.5.1] depends on reductions of a much more restricted kind. When applying such a reduction to a deduction Δ, the result is actually [...] "contained" in the deductions of the premisses of the last inference of Δ. (Prawitz 2018a, 8)

In this context, the Swedish logician proposes a new notion of validity. With the preliminary definitions of Section 2.5.2.1 unchanged, we will say that an argument Δ_1 is *immediately extracted* from a set of arguments Ω if, and only if, (1) $\Delta_1 \in \Omega$ or Δ_1 is the subargument of some $\Delta_2 \in \Omega$, or (2) Δ_1 results from the substitution with a term t of each free occurrence of x in occurrences of formulas in some $\Delta_2 \in \Omega$, or (3) Δ_1 is of the type

$$\begin{array}{ccc} \Delta_2 & & \Delta_n \\ [\alpha_2] & \dots & [\alpha_n] \\ & \Delta_{n+1} & \end{array}$$

for $\Delta_i \in \Omega$ with conclusion α_i $(i \leq n)$ and $\Delta_{n+1} \in \Omega$ with undischarged assumptions $\{\alpha_2, ..., \alpha_n\}$. An argument Δ is *contained* in a set of arguments Ω if, and only if, there is a sequence of arguments $\Delta_1, ..., \Delta_n$ such that $\Delta_n = \Delta$ and for every $i \leq n$, Δ_i is immediately extracted from $\Omega \cup \{\Delta_1, ..., \Delta_{i-1}\}$. Let us assume as usual that the atomic derivations in a base B are analytically valid arguments, and that there is no analytically valid argument for $\bot$.

Definition 8. Δ is *anayitically valid* (*with respect to* B) if, and only if,

1. Δ is closed $\Rightarrow$ Δ contains a closed analytically valid (with respect to B) argument in canonical form;
2. Δ is open $\Rightarrow$ every Δ^σ is analytically valid (with respect to B), where Δ^σ is a closed example of Δ obtained by first replacing all the occurrences of free variables not bound in Δ with closed (B-)terms, so to get an argument Δ_1, and then all the undischarged assumptions in Δ_1 with closed analitically valid (with respect to B) arguments for such assumptions.

As in the case of the notion of valid argument in Section 2.5.2.1, definition 8 proceeds by simultaneous recursion. Similarly, we will have a *global* definition of analytical validity for inferences and inference rules.

Definition 9. A set of inference rules $\Re$ is *analytically valid* (*with respect to* B) if, and only if, for every application Δ of the type

$$\frac{\Delta_1 \qquad \dots \qquad \Delta_n}{\beta} \; \alpha_1, ..., \alpha_m, x_1, ..., x_s$$

of an element of $\Re$, if, for every $i \leq n$, Δ_i is analytically valid (with respect to B), Δ is analytically valid with respect to B.

However, and this is the central point, unlike the notion of validity of 1973,

> when we now for analytical validity demand that the non canonical closed argument contain a closed, canonical, and analytically valid argument [...], we get a condition whose satisfaction depends on what the major premiss means. (Prawitz 2018a, 13)

Nevertheless, we must admit that the recognizability problem is still present. To possess a closed analytically valid argument means *knowing how* to obtain a closed analytically valid argument in canonical form: in this case, the effectiveness of the method at our disposal depends on the constructive character of the operation of extraction, applied in the exact order on the starting argument. However, the possession is not the same as *knowing that* the closed argument is analytically valid. For this purpose, in fact, it is necessary to complete the extraction. Likewise, to possess an open analytically valid argument means *knowing how* to get closed analytically valid arguments, but not also *knowing that* the open argument is analytically valid. Such an understanding could be achieved only by carrying out all the (generally infinite) substitutions required. However, we cannot assume that an analytically valid argument contains information necessary for the desired recognition; there is no upper limit on the complexity of the extractions, and the domain of analytically valid arguments on which to make the substitutions is infinite and not regimented. Again, we could require a proof of the fact that the argument is analytically valid, which would give rise to a regressive explanation, or ask whether the relation "being an analytically valid argument for" is decidable.

The reasoning is the same for the problem of proofs-as-chains; although the analytical validity of an argument now depends on the type of inferences occurring in it, an example similar to that of Section 2.5.2.1 shows, *mutatis mutandis*, the existence of analytically valid arguments with inferences that instantiate inference rules in sets of inference rules not analytically valid. More generally, the distinction between arguments in a canonical form and arguments in a non-canonical form, and the consequent necessity to define the notion of analytical validity by simultaneous recursion, impedes the possibility of characterizing the analytically valid arguments as chains of analytically valid inferences.

This flaw can also be referred to the proofs of 1977 and 2005: a proof-structure π could be a proof simply by virtue of the way in which the non-primitive procedures occurring in it are defined. Following definitions 8 and 9, we could then solve this problem by introducing a notion of *analytical proof*, and a related notion of *analytically acceptable inference*; in any case, we would meet again with the recognizability and the proofs-as-chains problems.

2.5.3.3 Procedures from proofs to proofs

At the beginning of this chapter, we have pointed out that the notion of valid inference is often defined in terms of consequence. According to the intentions of this work, the question we have asked ourselves is if in this way valid inferences are endowed with epistemic power. Model-theoretic consequence, devoid of relevant modal connotations, is not suitable for this purpose. On the other hand, the corresponding proof-theoretic proposal, involving the concepts of valid argument and proof, seems to be on the right track. However, in order to affirm that this proposal is satisfactory, we have to show that valid arguments and proofs are endowed with necessity of thought, and this can be done, it seems, only if we are able to characterize them as chains of valid inferences. In this sense, two problems prevent the attainment of the objectives: 1) the recognizability problem, and 2) the proofs-as-chains problem. There is also a third difficulty concerning the independence of the validity from inferences; although it can be solved through analytical validity, the resulting approach still suffers from the two previous problems.

Be that as it may, valid arguments and proofs are distinguished by one essential feature. The justification procedures of valid arguments are, as Prawitz observes in *An approach to general proof theory and conjecture of a kind of completeness of intuitionistic logic revisited*,

> defined on argument skeletons rather than on arguments, i.e. skeletons together with justifications, and [their values] consist of just argument skeletons instead of skeletons with justifications. (Prawitz 2014, 275)

The (non-primitive) procedures in the proofs of the years 1977 and 2005, on the contrary, are to be understood as operations *from proofs to proofs*. Although the difference is highlighted by Prawitz in the context of a discussion serving purposes other than those we are interested in, we believe it has a particular meaningfulness when connected to another point which, conversely, is relevant for our subject: how should we understand, in the framework of proof-theoretic semantics, *the carrying out of a closed valid non-canonical argument* or *the carrying out of a non-canonical or categorical proof*?

To carry out a closed valid non-canonical argument means building a certain argument structure by means of a series of inferential steps from premises to conclusion, to which possible justification procedures are associated. Since, on the other hand, justification procedures are defined on argument structures and go to argument structures, the process of reduction to canonical form is to be understood as, so to speak, external to, or separated from, that of construction of the argument structure itself. In fact, if it were not so, there should be some inference J that involves, besides the passage from premises to conclusions, the application of the relative reduction to the argument obtained by carrying out the step. In this way, J would be nothing more than (the application of) a constructive procedure and the application of J to argument structures would give rise to something that is *no longer* an argument structure. Then, no justification procedure associated with some inference applied in arguments containing J can be defined on such arguments, nor can any justification procedure return, on the values on which it is defined, arguments where J occurs. In other words, no reduction will be available in these cases. In this regard, it is useful to point out that, in the article from which the previous quotation comes, Prawitz refers to proofs - or rather, to a variant of such proofs, but this does not change the pith of the speech - as to *interpreted proof-terms*. It follows, it seems, that argument structures are then to be understood as *uninterpreted proof-terms*. Which is, after all, obvious, since the same inference rule can, in the proof-theoretic semantics framework with valid arguments, be associated with *more than one* justification procedure. That could never be the case if valid arguments were interpreted proof-terms, in which each inferential step corresponds to the application of a specific constructive procedure.

Of course, if by carrying out a non-canonical or categorical proof we mean, as implied so far, the construction of the method to which it corresponds, we will be in the same situation as for valid arguments. The process of reduction to canonical form will be external to, or separated from, that of construction of the proof itself. However, if for the valid arguments such a circumstance cannot but occur, in accordance with the way in which the justification procedures are defined, the fact that the (non-primitive) procedures are operations from proofs to proofs seems to suggest a way out. Namely, we could consider the carrying out of a non-canonical or categorical proof as the *computation of the method* to which it corresponds. Since such a method generates a canonical proof, it is the *carrying out as such* of a non-canonical or categorical proof - and not a further reduction of the latter - to give as a result the correspondent canonical form. Before giving an account of the consequences of this step, and for the sake of a greater precision, it is perhaps useful to summarize everything with two examples.

Suppose that an agent P is in possession of a closed canonical valid argument

$$\dfrac{\begin{matrix}\Delta_1 \\ \alpha_1\end{matrix} \qquad \begin{matrix}\Delta_2 \\ \alpha_2\end{matrix}}{\alpha_1 \wedge \alpha_2}\,(\wedge_I)$$

for $\alpha_1 \wedge \alpha_2$, with Δ_1 closed canonical valid argument for α_1. Suppose now that P applies $(\wedge_{E,1})$ to such argument, obtaining

$$\dfrac{\dfrac{\begin{matrix}\Delta_1 \\ \alpha_1\end{matrix} \qquad \begin{matrix}\Delta_2 \\ \alpha_2\end{matrix}}{\alpha_1 \wedge \alpha_2}\,(\wedge_I)}{\alpha_1}\,(\wedge_{E,1})$$

In this situation, P is not yet in possession of a closed valid canonical argument for α_1. Thus, the carrying out of the second argument, that is the application of $(\wedge_{E,1})$ to the first, is not sufficient in this sense. P must still reduce what carried out through a further application of $\wedge$-rid. Suppose instead that P is in possession of a canonical proof $O_{\wedge}(d_1, d_2)$ for $\alpha_1 \wedge \alpha_2$, with d_1 canonical proof for α_1. Suppose then that P applies a non-primitive procedure f (corresponding to an application of $(\wedge_{E,1})$) defined by the equation

$$f(O_{\wedge}(\pi_1, \pi_2)) = \pi_1.$$

Now, if by *application of* f, we simply mean that P builds the non-canonical or categorical proof $f(O_{\wedge}(d_1, d_2))$, we can follow a reasoning similar to the previous one; P does not yet have a canonical proof for α_1, since it has to reduce what has been built by a further application of the defining equation of f. If, instead, by *application of* f, we mean that P *computes* f *on* $O_{\wedge}(d_1, d_2)$, namely, that he/she *computes the value of* $f(O_{\wedge}(d_1, d_2))$, the carrying out of the second proof, i.e. the application of f to the first one, will be enough for P to possess a canonical proof for α_1 - namely, according to the defining equation of f, of d_1. As we said, the idea that the performance of inferences corresponds to the application of constructive procedures, in the sense of computing the value of such procedures on certain inputs, is not feasible in the setup with valid arguments - otherwise, alternatively, a modification of this setup would return *de facto* nothing but the setup with proofs.

From this point of view, the implicational case, certainly more complex, should be similarly understood in the following sense. Suppose that P is in possession of a canonical proof $O_{\rightarrow}(e^{\alpha}_{\beta})$ for $\alpha \rightarrow \beta$ and of a proof - possibly non-canonical or categorical - h for α. Suppose now that P applies a non-primitive procedure g (corresponding to an application of $(\rightarrow_E)$) defined by the equation

$$g(O_{\rightarrow}(\pi^{\alpha}_{\beta}), \pi) = \pi^{\alpha}_{\beta}(\pi).$$

If by *application of* g, we mean that P *computes* g *on* $O_{\rightarrow}(e^{\alpha}_{\beta})$, namely, P *computes the value of* $g(O_{\rightarrow}(e^{\alpha}_{\beta}))$, the carrying out of the second proof, i.e. the application of g to the first, will put P in possession of $e^{\alpha}_{\beta}(h)$. Since there is no guarantee that $e^{\alpha}_{\beta}(h)$ is a canonical proof for β, we could deduce that the carrying out of the proof could not give P a canonical proof for β. However, that, in our opinion, and according to the interpretation we are proposing, would be wrong; to say that the carrying out of the intended proof gives P the possession of e^{α}_{β} means to say that, through such carrying out, P *applies* the constructive procedure e^{α}_{β} to h, and since by application we here mean the computation of a non-primitive procedure on certain values, the *value* of the computation of e^{α}_{β} on h is what the carrying out of the intended proof gives P the possession of. Since $O_{\rightarrow}(e^{\alpha}_{\beta})$ is a canonical proof for $\alpha \rightarrow \beta$, e^{α}_{β} will be a hypothetical proof for β from α, and since h is a proof - possibly non-canonical or categorical - for α, $e^{\alpha}_{\beta}(h)$ will be a constructive procedure to obtain a canonical proof d for β. Hence, the carrying out of the intended proof will give P the possession of such d.

In proposing the suggested reading, the first obvious consequence is that we can no longer consider the carrying out of an inferential act as a simple passage from premises to conclusion - mere construction of a procedure of a certain range starting from certain arguments. We must more strongly affirm that it consists in computing the value of the procedure. Then, as the result of the carrying out of a non-canonical or categorical proof is no longer the possession of a method, but the possession of the result of this method, so the carrying out of an inference in a non-canonical or categorical proofs does not give the possession of a procedure, but the result of this procedure. Acceptable inferences can still be defined in terms of procedures that transform canonical proofs of the premises into canonical proofs of the conclusion. However, it is once again the *inferential act as such* - not a subsequent reduction - to give as a result the canonical proof when carried out on canonical proofs.

Moreover, in close connection with the previous point, it is above all necessary to introduce a clear distinction between *objects* built of some procedures, or to which the procedures themselves amount, and *acts* consisting of the application of these procedures. It is easy to realize that the expressions corresponding to non-canonical or categorical proofs are, according to the line of thought we are proposing here, nothing but descriptions of acts carried out by inferential agents, whereas the canonical forms to which such descriptions reduce are the objects that agents possess after carrying out such acts. Even more significantly, this means the only objects we need are the canonical proofs – and a primitive notion, that is, no further analyzable, of constructive function. In other words, the *objectual* approach to non-canonical or categorical proofs, emphasized in Section 2.5.3.1, is replaced by an *operational* conception of the same.

The adoption of this strategy is of particular interest if we turn again to the problems recognisability and of proofs-as-chains. Starting from the latter, we reaffirm that they partly derived from the distinction, at the object level, between canonical proofs and non-canonical or categorical proofs. Hence, it becomes available a solution that follows and completes the reconsideration of the example on the valid closed canonical argument, or on the canonical proof referred to in Section 2.5.2.1 - though, according to what we have illustrated earlier, referable now only to the case of proofs. Inferential validity can be given in terms of objects that the theory considers to be evidence, so that the acts involving such inferences, or to which the same inferences correspond, have an epistemic value exactly by virtue of their production of objects of this type. In addition, since such objects are always canonical, we can abstract, in defining them, and just as in the BHK clauses, from the intrinsic complexity of any constructive procedures in them involved. The problem of recognizability, instead, seems to have now less urgency - albeit it is still present. Indeed, first of all it will no longer be a question of understanding that the *objects* we *possess* enjoy certain properties, but rather of realizing that the *fulfilment* of certain *acts* allows to possess objects that enjoy specific properties. Since these objects are always in canonical form, as already pointed out in the case of the canonical proofs referred to in Section 2.5.2.2, in order to achieve the required understanding it will be sufficient to recognize that the immediate subproofs of (applications of constructive procedures that represent) introduction inferences are categorical proofs of the premises of the introduction inference. Later, we will able to realize how this is exactly the road Prawitz takes with the theory of grounds.

Chapter 3

Evidence in the BHK framework

3.1 Approaching evidence directly

The idea of proofs as chains of valid inferences, to which one is widely inclined, has, as seen, a famous supporter in Descartes and, more deeply, seems to indicate the best route to the justification of the epistemic power of deductive activity. Consequently, the definition of the notion of inference in the global terms of that of proof, peculiar to proof-theoretic semantics, is already an unnatural inversion of the conceptual order. The purpose of giving an account of the strength of proofs, and that of clarifying the link between the latter with valid inferences, are, in any case, only derivatives with respect to the theme to which the present work is dedicated. The question from which we started, in fact, concerns primarily the capacity some inferences have to justify the conclusions under the hypothesis that the premises are justified, and to offer evidence for the second ones from the first ones. Therefore, it is perhaps plausible to abandon a reductive strategy, and rather approach the problem in a direct way, as a platform for, and not as the result of, further theorizing. After all, as Cozzo affirms, Prawitz with the theory of grounds

> developed a notion of valid inference that is immediately connected with the necessity of thought. (Cozzo 2015, 105)

However immediate, the description of valid inferences cannot disregard the basic question: "what is evidence?". Therefore, we now must turn to it.

In the scope of our study, evidence must obviously be understood in relation to inferential acts; and since, as we shall see, according to Prawitz inferences involve judgments or assertions, depending on whether we consider them from a mental or a linguistic point of view, what we must ask ourselves is, more specifically, "what constitutes evidence for a judgment or for an assertion?". Having evidence for a judgment or for an assertion means to be justified in making that judgment or that assertion; the possession of evidence is a guarantee of correctness in judging a proposition as true, or in asserting a sentence. It is then obvious that such possession does not correspond to the simple identification of an abstract object, however epistemically relevant; in addition there must be a recognition, the awareness of the fact that the object at disposal is such as to justify and guarantee correctness. This property, usually called *epistemic transparency* - to which we will return in a specific Chapter of our discussion - is therefore something that, as recently rightly emphasized by Usberti, an adequate *explicans* of the notion of evidence should enjoy:

> to be warranted in believing that β one must not only have evidence e for β, but also base one's belief on e; and to base one's belief on e, one must know that e is evidence for β; analogously, to be warranted in believing that β follows from α one must not only have evidence e for β from α, but also base one's belief on e; and to base one's belief on e, one must know that e is evidence for β from α. (Usberti 2017, 3)

This is an issue that, *mutatis mutandis*, we have already emphasized in connection with the notion of proof - indeed, proofs are universally considered as the most reliable type of evidence we can aspire to.

However, we should now point out that the proposal that could at this stage rise spontaneously, and that would send us back to proof-theoretic semantics - namely to consider as an *explicans* of the notion of evidence that of valid argument of Section 2.5.2.1, or that of proof of Section 2.5.2.2 - would be clearly erroneous. The problem of recognizability which holds both with the valid arguments of 1973 and with the proofs of the years 1977-2005, indicates exactly that the notions of valid argument and proofs, so conceived, cannot reasonably be considered as epistemically transparent. Nevetheless, we have had the opportunity to see how the recognizability problem is, in a sense, less pressing in the case of the notion of canonical proof. Contrary to what happens in the non-canonical or categorical case, in order to know wheter one is in possession of a canonical proof, it will suffice to know that the immediate substructures are proofs for the premises (in the range of the constructive procedure involved) of the conclusive introduction inference. Further on - in the Section on epistemic transparency - we will wonder about the plausibility of the stronger idea that the notion of canonical proof is actually epistemically transparent. However, whenever the investigation were successful, not even the notion of canonical proof could, because of the proofs-as-chains problem, be understood as an adequate *explicans* of the notion of evidence.

Be that as it may, at the end of the previous chapter, we affirmed that, on the basis of a certain reading, the notion of proof could prove to be satisfactory. The fact that, unlike the justifications for the rules in non-introductory form in valid arguments, the procedures in the proofs are operations from proofs to proofs, allows, on an objectual level, to limit the attention to canonical proofs, and to consider non-canonical proofs from an operational point of view, i.e., as computations of the method to which they amount. Thus, in the BHK clauses – and in particular in Prawitz's reformulation – there might be something good, after all. They might be reconsidered as a basis for a rigorous characterization of the notion of evidence. But before embarking on this path, an observation is necessary.

The original BHK clauses do not set the concept of proof for judgments or assertions; on the contrary, they inductively specify the notion of proof for propositions or sentences. This also applies to the modification that Prawitz anticipates in 1977, and makes explicit in 2005; in the latter case, however, the introduction of primitive recognition procedures seems to indicate that the real question concerns rather proofs for judgments or assertions or, at the very least, the performance of proofs for propositions or sentences in the deductive *practice*. A proof of a proposition or sentence $\alpha \wedge \beta$ can be perfectly understood as a simple pair, without any recognition of the fact that the formation of this pair authorizes to assert $\alpha \wedge \beta$, or to judge it true. The subsequent request becomes relevant only when we pass from the level of abstraction - in which certain mathematical objects are structurally related to certain propositions or sentences as their proofs - to the necessity of identifying conditions of correct judgmentability or assertability. The tension between these two ways of looking at proofs is already found in Heyting (Heyting 1931, 1934), as a distinction between *constructions* for propositions or sentences and *realizations* of such construcions. This point was discussed in detail by Göran Sundholm (Sundholm, 1993, 1998) and, as we will see in Section 3.2.2, generalized by the latter in an important distinction between proofs-as-objects and proofs-as-acts. More significantly for the current discussion, however, we find again this observation also in Prawitz. For example, in a very recent article, the Swedish logician claims that

> Heyting's explanation of his epistemic notion of proof was closely linked to his notion of proposition: a proposition expresses an intention of (finding) a construction that satisfies certain condition, while "a proof of a proposition consists in the realization of the construction required by the proposition". A proof in the

> epistemic sense is accordingly an act, and it clearly makes an assertion warranted; "the assertion of a proposition signifies the realization of the intention [expressed by the proposition]", according to Heyting. [...] a proof in the intuitionistic tradition is either a construction intended by a proposition, or a realization of such a construction. (Prawitz 2018a, 16 - 18, ma si veda anche Prawitz 2012b, 2015, 2017)

We will, obviously, deal again with the BHK proofs, as well as with Heyting's conception. This will happen in the context of our exposition of the theory of grounds, when we will see how the latter, especially in the latest Prawitz's articles concerning it, develops within the framework of an essential comparison with the BHK semantics. In the remainder of this chapter, instead, we will deal with the exposition and analysis of two among the numerous formal semantics inspired by the BHK approach, that come conceptually close, though for different reasons, to the theory of grounds. The aim will be, so to speak, to introduce ourselves to the theory of grounds, and to highlight its answers to the difficulties experienced in similar frameworks.

3.2 Two BHK-inspired theories

In this section, we will briefly discuss the *theory of constructions* by Georg Kreisel and Nicolas Goodman, and the *intuitionistic type theory*, formulated instead by Martin-Löf.

Advanced for the first time by Kreisel in *Foundations of intuitionistic logic* (Kreisel 1962), the theory of constructions is not properly a unitary corpus; Kreisel himself, in fact, has proposed alternative versions (in addition to Kreisel 1962, see for example Kreisel 1965). On the other hand, Goodman, further modifying Kreisel's formulation, gave us approaches that, however connected, differ from each other in details not at all irrelevant (Goodman 1968, 1970, 1973). For the aims of our work, anyway, it will be more than enough to turn to a recent article by Walter Dean and Hidenori Kurokawa, *Kreisel's theory of constructions, the Kreisel-Goodman paradox, and the second clause* (Dean & Kurokawa 2016), where the theory of constructions is presented in a form, so to speak, generalized, perfectly sufficient to the intention of

> set down some of the common characteristics of these systems with the [goal] of explaining how Kreisel and Goodman proposed to use the language of the Theory of Constructions to formalize

> Kreisel's reformulation of the BHK clauses. (Dean & Kurokawa 2016, 33 - 34)

There are also many examples of Martin-Löf's intuitionistic type theory. However, our discussion will be confined mainly to *Intuitionistic type theory* (Martin-Löf 1984) - with the exception of references to subsequent works or to a minor bibliography.

3.2.1 The Kreisel-Goodman theory of constructions

In Section 2.5.2.2 we have seen how the original BHK clauses have problems related to the cases of $\rightarrow$ and $\forall$. The simple possession of a constructive procedure that generates a certain result when applied to arguments in its domain, in fact, may not be equivalent to the awareness that the procedure enjoys this property. On the other hand, this awareness seems necessary so that the procedure can rightfully be considered a proof. A possible way out, then, consists in demanding that a proof of an implication or of a universal quantification implies, in addition, a proof of the fact that the procedure has the desired behavior. This is the strategy adopted by Kreisel, who proposes the following reformulation of the clauses ($\rightarrow_P$) and ($\forall_P$) (with respect to a base B):

($\rightarrow_P^K$) a proof with respect to B of $\alpha \rightarrow \beta$ is an ordered pair $\langle f, \pi \rangle$, where f is a constructive procedure such that, for every π_1 proof with respect to B of α, $f(\pi_1)$ is a proof with respect to B of β, and π is a proof with respect to B of the fact f behaves in this way;

($\forall_P^K$) a proof with respect to B of $\forall x \alpha(x)$ is an ordered pair $\langle f, \pi \rangle$, where f is a constructive procedure such that, for every individual k on B, $f(k)$ is a proof with respect to B of $\alpha(k)$, and π is a proof with respect to B of the fact f behaves in this way.

Again in Section 2.5.2.2, we have affirmed that an alternative solution could instead turn to the question of whether the relationship "being a proof of" can be said to be decidable. However, it should be noted, as suggested by Dean and Kurokawa (Dean & Kurokawa 2016, but see also Díez 2000 and Sundholm 1983), that also Kreisel's proposal is, at least to some degree, close to the inquiry on decidability. The fundamental point, though, is that Kreisel starts from the idea that the relation "being a proof of" is definitely decidable, as witnessed by a well-known passage from *Foundations of intuitionistic logic*:

> the *sense* of a mathematical assertion denoted by a linguistic object A is intuitionistically determined (or understood) if we have laid down what constructions constitute a *proof* of A, i.e., if we have a construction r_A such that, for any construction c, $r_A(c) = \top$ if c is a proof of A and $r_A(c) = \bot$ if c is not a proof of A; the logical particles in this explanation are interpred truth functionally, since we are adopting the basic intuitionistic idealization that we can recognize a proof when we see one, and so r_A is decidable. (Kreisel 1962, 201 - 202)

And from this perspective, the adoption of the clauses ($\rightarrow_P^K$) and ($\forall_P^K$) rather than the simple ($\rightarrow_P$) and ($\forall_P$), is justified by the necessity to respect this assumption in light of the problematic interaction, in the BHK clauses, between what Dean and Kurokawa (Dean & Kurokawa 2016) call the problem of *impredicativity* (*de facto* identical to the one referred to earlier when quoting Gentzen 1935-1936 in Section 2.5.3), and, precisely, a derivative problem of decidability.

The most explicit, and in our case relevant, formulation of the problem of impredicativity is traced back to the article by Gödel *An interpretation of the intuitionistic propositional calculus*:

> Heyting's axioms [...] violate the principle [...] that the word "any" can be applied only to those totalities for which we have a finite procedure for generating all their elements. [...] Totalities whose elements cannot be generated by a well-defined procedure are in some sense vague and indefinite as to their borders. And this objection applies particularly to the totality of intuitionistic proofs because of the vagueness of the notion of constructivity. (Gödel 1933, 53)

According to Dean and Kurokawa, Gödel's words identify three aspects that, though being in the original intentions of the author aimed at discussing the distinction between intuitionism and Hilbertian finitism (see Prawitz 1981, Abrusci 1985 and Cellucci 2007), ended up influencing significantly the BHK debate:

> 1) a crucial difference between finitism and intuitionism is that, unlike finitists, intuitionists do not reject the meaningfulness of unrestricted quantification over a potentially infinite domain; 2) the class of constructive proofs form such a totality; but 3) it is not possible to see this class as inductively generated in virtue of the occurrence of the universal quantifier over proofs. (Dean & Kurokawa 2016, 30)

The exact connection between point 1) and points 2) and 3) emerges when one questions on what condition the relation "being a proof of" must satisfy, so that, as put forward by Gödel, intuitionism can claim on finitism the advantage of offering a unitary characterization of, according to the Hilbertian terminology, "real" mathematics and "ideal" mathematics (see again Prawitz 1981, Abrusci 1985 and Cellucci 2007). By referring to *The intended interpretation of intuitionistic logic* by Scott Weinstein (Weinstein 1983), Dean and Kurokawa maintain that the aim is achieved only if the relation is decidable. Quoting from Weinstein,

> it is precisely by admitting as meaningful the notion of a decidable property holding for arbitrary mathematical constructions that intuitionists achieve an interpretation of those sentences which are from Hilbert's point of view devoid of intuitive content. And, for intuitionists, to admit this notion as meaningful is to claim that statements asserting that decidable properties of mathematical constructions hold universally have tolerably clear proof conditions. [...] The intuitionists identify the truth of a mathematical statement, A, with our possession of a construction, c, which is a proof of the statement A. This latter statement, that the construction c is a proof of A, involves no logical operations and is moreover the application [of] a decidable property to a given mathematical construction. (Weinstein 1983, 268)

However, although acceptable, Weinstein's position runs counter to the aforementioned problem of decidability, which emerges even when one assumes, like Kreisel does, that "being a proof of" is a decidable relation:

> if we assume that the proof relation itself is decidable, then the clauses [$(\rightarrow_P)$ and $(\forall_P)$] are all analogous in form to Π^0_1 statements in the language of arithmetic - i.e. they begin with an unrestricted universal quantifier over proofs applied to a decidable matrix. As such statements are not in general decidable in the technical sense of computability theory, it seems that there is reason to worry that they do not satisfy Weinstein's criteria of having "tolerably clear proof conditions". (Dean & Kurokawa 2016, 31 - 32)

It is therefore the impredicative use of the universal quantifier in the clauses $(\rightarrow_P)$ and $(\forall_P)$ that in the end makes the relation "being a proof of" non-decidable in the BHK framework, and since, according to Kreisel, this relation must on the contrary be decidable, he adds the second clause precisely to avoid such charges and thereby also to provide a characterization of the proof relation which could plausibly be regarded as decidable. (Dean & Kurokawa 2016, 32)

Therefore, the theory of constructions is conceived as a formal apparatus in which to interpret the BHK clauses - in the modified version with $(\to_P^K)$ and $(\forall_P^K)$ – in order to offer a rigorous foundation of it. In other words, it aims to define a predicate $\Pi(c, A)$ – to be read as "the construction c proves the formula A" – in a way similar to what Tarski does with the truth predicate. We will therefore have constructions (some of which translate formulas of a reference language) and properties of constructions, both of them mathematical entities governed by particular axioms. It is relevant, in terms of what we will say later, to emphasize that, exactly in order to translate $(\to_P^K)$ and $(\forall_P^K)$, constructions are also allowed for formulas of the type $\Pi(c, A)$, with consequent infinite chains

$$\Pi(c_1, A), \Pi(c_2, \Pi(c_1, A)), ..., \Pi(c_n, \Pi(c_{n-1}, ..., \Pi(c_1, A)...)), ...$$

As pointed out by Sundholm, the general intent of the theory of constructions is therefore more *reductive* than, as for the BHK clauses, explicative:

[Kreisel] wished to set up an abstract theory of proofs (constructions) such that the logical constants could be defined and their properties derived within the logic-free theory of constructions. (Sundholm 1983, 154 - 155)

In the version of Dean and Kurokawa (Dean & Kurokawa 2016), the theory of constructions appears as an equational calculus $\mathtt{C}$ for untyped terms, which includes a symbol for the absurd $\bot$ and one for the validity $\top$, with the usual rules of weakening and *Cut* from Gentzen's sequent calculus (Gentzen 1934-1935), with a symbol of equality $\equiv$ and, more specifically, with functional symbols D, for pair formation, D_i $(i = 1, 2)$ for projection on a pair, λ, for λ-abstraction - ruled by standard equations (see, for example, Hindley, Lercher & Seldin 1975) – and τ, which is definitely the most important operator. According to Kreisel and Goodman's intention, $\tau\ t_1 t_2 \equiv \top$ should be read as "t_2 is a proof of the fact that, for every x, $t_1 x \equiv \top$"; namely, if t_1 is intended as a representation of a formula of the reference language, $\tau\ t_1 t_2 \equiv \top$ means that "t_2 is a proof of the universal closure of the formula of the reference language represented by t_1". Obviously, if t_1 is closed, such should be the formula of the reference language that it represents, and the universal closure will correspond to the formula itself. Therefore, $\tau\ t_1 t_2 \equiv \top$ in this case simply amounts to " t_2 is a proof of formula of the reference language represented by t_1". The rules for τ are:

$$\frac{}{\tau\ t_1t_2 \equiv \top \vdash t_1x \equiv \top}\ (\text{Refl})$$

$$\frac{\Gamma, \tau\ t_1t_2 \equiv \bot \vdash t_3 \equiv t_4 \qquad \Gamma, \tau\ t_1t_2 \equiv \top \vdash t_3 \equiv t_4}{\Gamma \vdash t_3 \equiv t_4}\ (\text{Dec})$$

(Refl) is a (rather) natural principle of reflection, according to which given a proof (of the universal closure) of a formula A of the reference language, (the universal closure of) A is true. In line with the view of Kreisel according to which we recognize a proof when we see one, and in order to guarantee the desired decidability of the predicate $\Pi(c, A)$, the rule (Dec) instead assures the bivalence, and therefore the decidability, of τ:

> a term R can be understood as expressing a binary relation just in case for all pairs of terms s, t, if Rst is defined, then $Rst \equiv \top$ or $Rst \equiv \bot$ may be derived in the theory. The decidability of such a relation R may then be expressed by stating that Rst is defined for all pairs of terms s, t, - i.e. that R is *bivalent*. (Dean & Kurokawa 2016, 36)

The rewriting of the clauses requires at this point only a simple final readjustment. In order to maintain the reductive, or preferably foundative character, of the theory of constructions with respect to the BHK, we need to ensure that the formalization of the latter takes place in a logic-free context; in other words, we must avoid reusing at the meta-linguistic level, as the BHK clauses do, the same logical constants we intend to illustrate at the level of language object. Kreisel and Goodman's answer is based on the decidability of τ, and on the consequent decidability of the predicate $\Pi(c, A)$:

> 1) it is intuitionistically admissible to apply classical propositional logic to decidable statements; 2) if the truth values $\top$ and $\bot$ are taken as abbreviating particular λ-term [in general, $\lambda xy.x$ for $\top$ and $\lambda xy.y$ for $\bot$], it is possible to define bivalent λ-terms $\cap_k$, $\cup_k$, and $\supset_k$ which mimic the classical truth functional connectives $\wedge$, $\vee$, $\rightarrow$ applied to binary terms with k free variables [in the case of $\rightarrow$ in just one variable, we can put $xz \supset_1 yz$ as $\lambda xyz.xzy(\lambda w.\top)z$]; 3) the application of these terms to terms of the form $\Pi(c, A)$ will always yield a term which is defined as long as it can be assured that $\Pi(c, A)$ is itself defined so that it is bivalent. (Dean & Kurokawa 2016, 37 - 38)

For α and β with k free variables indicated by the expression $\vec{x}$, we can then set:

($\wedge_{\mathtt{C}}$) $\Pi(c, A \wedge B) \stackrel{def}{=} \lambda\vec{x}.(\Pi(D_1c, A) \cap_k \Pi(D_2c, B))$

($\vee_{\mathtt{C}}$) $\Pi(c, A \vee B) \stackrel{def}{=} \lambda\vec{x}.(\Pi(D_1c, A) \cup_k \Pi(D_2c, B))$

($\rightarrow_{\mathtt{C}}$) $\Pi(c, A \rightarrow B) \stackrel{def}{=} \lambda\vec{x}.\tau\langle\lambda y.(\Pi(y, A) \supset_k \Pi((D_1c)y, B)), D_2c\rangle$ [brackets and commas are introduced in order to make it easier the comprehension]

($\forall_{\mathtt{C}}$) $\Pi(c, \forall zA(z)) \stackrel{def}{=} \lambda\vec{x}.\tau\langle\lambda y.(\Pi((D_1c)y, A[z/y])), D_2c\rangle$ [brackets and commas are introduced in order to make it easier the comprehension]

($\exists_{\mathtt{C}}$) $\Pi(c, \exists yA(y)) \stackrel{def}{=} \lambda\vec{x}.\Pi(D_2c, A[D_1c/y])$

where $\Pi(c, A \rightarrow B)$ will be valid only if c is a pair $D(t_1, t_2)$ such that t_2 shows that, for every construction y proving A, the construction t_1y obtained by applying t_1 in y proves B. Likewise, $\Pi(c, \forall zA(z))$ will be valid only when c is a pair $D(t_1, t_2)$ such that t_2 proves that, for every term y, the construction t_1y resulting from the application of t_1 in y proves $A[z/y]$. Therefore, ($\rightarrow^K_P$) and ($\forall^K_P$) are faithfully formalized in $\mathtt{C}$.

As already mentioned in Section 2.5.2.2, Prawitz explicitly refuses a solution *à la* Kreisel, stating that the adoption of the second clause is responsible for the onset of regressive explanations. This is strictly linked to the possibility, above taken into account, of having in the theory of constructions infinite chains for iterations of $\Pi(c_1, A)$ on formulas A which are in turn of the type $\Pi(c_2, B)$. However, it must be said that the reasons that lead Prawitz to require that proofs of implications and universal quantifications are something more than mere constructive procedure,s are in many respects different from those which instead lead Kreisel to the adoption of the second clause; while according to Prawitz, already in 1977, it is essential to capture the epistemic power of a proof (since the simple possession of a constructive procedure is not in this sense sufficient, if not accompanied by the additional awareness of the fact that the procedure does have a relevant behavior), for Kreisel instead the question is based on the foundation of the BHK clauses, in the belief that the relation "being a proof of" is decidable, and in the light of the already discussed problems of impredicativity and of the derivative problem of decidability.

The situation therefore seems to be not easily resolvable. On the one hand, Prawitz would appear to pose the following dilemma: a description of implicational proofs as pairs $\langle t_1, t_2\rangle$ with t_2 such that $\Pi(t_2, \forall c(\Pi(c, A) \supset_k \Pi(t_1c, B)))$ would only move the question of the *justificatory capacity* of t_1 - the constructive procedure that from a proof of A yields one of B - to that of t_2 - where it should be noted that, if t_2 is *again* a constructive procedure, the simple possession of t_2 is not sufficient to have a proof of $\forall c(\Pi(c, A) \supset_k \Pi(t_1 c, B))$, so that it is required a third proof t_3 such that $\Pi(t_3, \forall c(\Pi(t_2 c, (\Pi(c, A) \supset_k \Pi(t_1 c, B)))))$, which must itself be a constructive procedure, and so forth. Thus, after rejecting the idea of the second clause, Prawitz wonders if it *exists* any plausible sense in which constructive procedures can be considered as decidable, or, more weakly, recognizable, relative to the property of producing proofs of a certain type when applied to arguments of a certain type. On the other hand, Kreisel, when *founding* the BHK, recovering from them a decidability that he considers evident, regards the operator τ, on which the clauses ($\rightarrow_{\mathtt{C}}$) and ($\forall_{\mathtt{C}}$) are based, as decidable by assumption, so that the quantifier $\forall$ occurring in $\Pi(t_2, \forall c(\Pi(c, A) \supset_k \Pi(t_1 c, B)))$ should be read in *truth-functional* terms; t_2, therefore, should not be understood as a new constructive procedure, so that regression is blocked.

Maybe on account of this discrepancy, Dean and Kurokawa argue that one could reasonably

> wonder on this basis if grasping the second clause interpretation of a formula ever requires that we grasp such an infinite sequence of conditions. (Dean & Kurokawa 2016, 46)

In any case, in a note of *Explaining deductive inference*, Prawitz seems to focus a more radical critique to Kreisel's proposal, which is also valid even when Dean and Kurokawa's objection is accepted:

> Kreisel proposes that a proof of $A \rightarrow B$ or $\forall x A(x)$ is a pair whose second member is a proof of the fact that the first member is a construction that [produces either a proof of B when applied to a proof of A, or a proof of $A(t)$ when applied to a term t]. Thus, he presupposes that we already know what a proof is; it is thought that the second proof establishes a decidable sentence and that a reduction has therefore taken place. (Prawitz 2015, 87)

In other words, it seems to us that Prawitz, in the passage just cited, criticizes - or, at least, critically emphasizes - two points; the decidability by assumption of the operator τ, and the occurrence of the notion of proof in clauses that should fix this notion itself. So, first of all, even assuming that the relation "being a proof of" is decidable, or, in a weaker sense, recognizable (and, as we saw in Section 2.5.2.2, Prawitz is skeptical about this hypothesis, at least in a certain interpretation), this fact should be achieved

- as a result, and not as a starting point - in a framework inspired by, and which formalizes, the BHK clauses, if necessary by imposing conditions and further restrictions on the definitions proposed. Secondly, Prawitz seems to attribute to a formalization of the BHK the explanatory task (as was the task of the BHK themselves) of defining the notion of proof. But in order to do this, we obviously cannot adopt Kreisel's foundationalist perspective, and in particular we cannot assume that we already know what a proof is.

3.2.2 Martin-Löf's intuitionistic type theory

Intuitionistic type theory, developed by Martin-Löf since the 1970s (see Sundholm 2012, Dybjer & Palmgren 2016), is one of the most studied and fruitful research areas in the field of constructive logic and mathematics and, more generally, one of the most innovative approaches in the investigation of the foundations of mathematics. In fact, it is impossible here to give an exhaustive picture of the vast theoretical, conceptual, formal and methodological apparatus to which it amounts, of its manifold philosophical implications and, not least, of its innumerable branches towards more or less related systematizations. Likewise, it would not be appropriate here, since it rather requires a separate treatment, to review the various changes that Martin-Löf has made over the years to his system - not to mention the ones proposed by others - or to illustrate the different philosophical positions that, implied in or inferred from this system, he has gradually articulated. We will therefore limit ourselves to an inevitably partial treatment - even more partial than that reserved for the Kreisel-Goodman theory of constructions - recalling, when useful for the purposes of this work, Prawitz's observations on the work of his Swedish colleague.

The starting point - or at least one of the possible starting points - is the attribution of a central role to the already introduced distinction between propositions on the one hand, and judgments on the other. The interplay between the two concepts, while remaining central throughout the development of Martin-Löf's thought, takes across time on different characteristics, analyzed in different directions, and not always mutually compatible. In any case, in *Intuitionistic type theory*, the text we have decided to refer to, we are told that

> what we combine by means of the logical operations [...] and hold to be true are propositions. When we hold a proposition to be true, we make a judgement:

$$\underbrace{\underbrace{A}_{proposition} \quad \text{is true}}_{judgement}$$

> In particular, the premisses and conclusion of a logical inference are judgements. (Martin-Löf 1984, 3)

From this point of view, the formulas of *intuitionistic type theory* are conceived as representing judgments; in line with what has been said at the end of the previous quotation, moreover, the (applications of) the rules of calculus involve only formulas for judgments, so it is formulas of this type they intend to demonstrate. The fundamental forms of judgment are:

- α is a set (written α `set`)
- α and β are equal sets (written $\alpha = \beta$)
- a is an element of the set α (written $a \in \alpha$)
- a and b are equal elements of the set α (written $a = b \in \alpha$)

Each of these forms receives, in addition, an explanation aimed at fixing the nature of the defined object, the epistemic conditions for asserting of the intended judgment or, finally, the meaning of the latter; for example, in the case of the α `set` judgment, we will have:

> what is a set? What is it that we must know in order to have the right to judge something to be a set? What does a judgement of the form "α is a set" mean? [...] At first sight, we could assume that a set is defined by prescribing how its elements are formed. This we do when we say that the set of natural numbers $\mathbb{N}$ is defined by giving the rules:
>
> $$0 \in \mathbb{N} \qquad \frac{a \in \mathbb{N}}{s(a) \in \mathbb{N}}$$
>
> by which its elements are constructed. However, the weakness of this definition is clear: 10^{10}, for instance, though not obtainable with the given rules, is clearly an element of $\mathbb{N}$, since we know that we can bring it to the form $s(a)$ for some $a \in \mathbb{N}$. We thus have to distinguish the elements which have a form by which we can directly see that they are the result of one of the rules, and call

them canonical, from all other elements, which we will call noncanonical. But then, to be able to define when two noncanonical elements are equal, we must also prescribe how equal canonical elements are formed. So a set α is defined by prescribing how a canonical element of α is formed as well as how two equal canonical elements of α are formed. [...] For example, to the rules for $\mathbb{N}$ above, we must add

$$0 = 0 \in \mathbb{N} \qquad \text{and} \qquad \frac{a = b \in \mathbb{N}}{s(a) = s(b) \in \mathbb{N}}$$

(Martin-Löf 1984, 7 - 8)

Likewise, the meaning of a judgment of the form $\alpha = \beta$ is fixed by requiring that it is valid only when, first, α and β are extensionably equal and, secondly, the equality between the elements of one is preserved in the other. Finally, there are the judgments on the elements of a set, and on the equality between them. A judgment of the form $a \in \alpha$ means that a is a method that, when performed, generates as a result a canonical element of α; thus, two elements a and b of α are equal if, and only if, they generate the same canonical element of α - and this explains the meaning of $a = b \in \alpha$. The equality between sets, and that between elements of the same set, are subject to the usual properties of reflexivity, symmetry and transitivity, fixed through appropriate inference rules. To these are added two more that, so to speak, link the equality between sets and that between elements of the same set: if $a \in \alpha$ (respectively, $a = b \in \alpha$) and $\alpha = \beta$, then $a \in \beta$ (respectively, $a = b \in \beta$).

One of the distinctive traits - if not the main distinguishing trait - of intuitionistic type theory is the generalization of the above forms of judgment to what Martin-Löf calls *hypothetical judgments*, that is to say, judgments dependent on assumptions. Confining ourselves to the case of a single variable, under the common and implicit assumption α `set`, and indicating with $[x \in \alpha]$ the assumption (discharged) "for $x \in \alpha$", we will then have that

- $\beta(x)$ `set` $[x \in \alpha]$
- $\beta(x) = \delta(x)$ $[x \in \alpha]$
- $b(x) \in \beta(x)$ $[x \in \alpha]$
- $b(x) = d(x) \in \beta(x)$ $[x \in \alpha]$

The first form of hypothetical judgment expresses $\beta(x)$ as a set family over α; thus, $\beta(a)$ is a set for every $a \in \alpha$ and $\beta(a)$ and $\beta(x)$ are equal sets for $a = c \in \alpha$. The second one instead means that $\beta(x)$ and $\delta(x)$ are equal families of sets on α; therefore, $\beta(a) = \delta(a)$ for every $a \in \alpha$. As regards the elements, the third typology of hypothetical judgment means that $b(x)$ is a function from α to $\beta(x)$ such that, for every $a \in \alpha$, $b(a) \in \beta(a)$; $b(x)$ is therefore a function the value of which depends on the choice of the argument $a \in \alpha$ on which it is computed. Finally, the fourth case indicates the equality between two functions from α to $\beta(x)$; for each $a \in \alpha$, $b(a) = d(a) \in \beta(a)$. This elucidation is accompanied by different substitution rules - discharging the assumption $[x \in \alpha]$ – that make explicit the already discussed semantic content of the forms of judgment to which they are referred from time to time.

After presenting the meaning of judgments and hypothetical judgments, Martin-Löf deals with the meaning of propositions and sentences, adhering to the intuitionist idea according to which

> a proposition is defined by laying down what counts as a proof of the proposition. (Martin-Löf 1984, 11)

Therefore, he adopts the BHK clauses for propositions and sentences, exposing them in the following form - as usual, with reference to a base B, under the assumption that there are no proofs for $\bot$, and with λ operator of λ-abstraction:

($\wedge^M_P$) a proof over B of $\alpha \wedge \beta$ is (a, b) for a proof over B of α and b proof over B of β

($\vee^M_P$) a proof over B of $\alpha \vee \beta$ is $i(a)$ for a proof over B of α, or $j(b)$ for b proof over B of β

($\rightarrow^M_P$) a proof over B of $\alpha \rightarrow \beta$ is $\lambda x.b(x)$ where $b(x)$ is such that, for every a proof over B of α, $b(a)$ is a proof over B of β

($\forall^M_P$) a proof over B of $\forall x \alpha(x)$ is $\lambda x.b(x)$ where $b(x)$ is such that, for every B-term a, $b(a)$ is a proof over B of $\alpha(a)$

($\exists^M_P$) a proof over B of $\exists x \alpha(x)$ is (a, b) for a B-term and b proof over B of $\alpha(a)$

Martin-Löf's position on these clauses is similar to that adopted by Prawitz in *Meaning and proofs: on the conflict between classical and intuitionistic logic* (Prawitz1977) - and here described in Section 2.5.2.2: they define the notion of canonical proof, but

> an arbitrary proof, in analogy with an arbitrary element of a set, is a method of producing a proof of canonical form. (Martin-Löf 1984, 13)

And precisely in this regard, the analogy between proofs of propositions and arbitrary elements of sets plays a central role in intuitionistic type theory. The latter, in fact, is inspired by the so-called *formulas-as-types conception* of the Curry-Howard isomorphism, which we will discuss more in detail in the next chapter; without going into details, however, we can now see how

> if we take seriously the idea that a proposition is defined by laying down how its canonical proofs are formed [...] and accept that a set is defined by prescribing how its canonical elements are formed, then it is clear that it would only lead to an unnecessary duplication to keep the notions of proposition and set (and the associated notions of proof of a proposition and element of a set) apart. Instead, we simply identify them, that is, treat them as one and the same notion. (Martin-Löf 1984, 13)

Thus, in Martin-Löf's system the formulas of a logical language (although, as we shall see, it is not really possible to speak of a logical language of "reference", since the system integrates syntactic rules for the generation of such formulas) are a special case of a set and, more precisely, they are sets of proofs with canonical elements fixed in such a way as to respect the inductive definition of the BHK clauses. Thus, when α is a formula, a judgment for example of the type $a \in \alpha$ tells us that a is a proof of α, that is to say a method for obtaining a canonical proof of α; similarly, a judgment, for example, of the type $b(x) \in \beta(x)[x \in \alpha]$ with $\beta(x)$ open formula in the free variable x and α as intended domain, tells us that $b(x)$ is a function from α to $\beta(x)$, such that for every element a of α, $b(a)$ is a proof of $\beta(\alpha)$, whereas with $\beta(x) = \beta$ not depending on x and α formula, indicates that $b(x)$ is a function from α to β such that for each proof a of α, $b(a)$ is a proof of β. More generally, if we say that the formulas are *types* of proofs, and we think of the latter as elements typed on their own set, we can also easily conclude that the introduction of hypothetical judgments is equivalent to the introduction of *dependent types* and, correspondingly, of proofs as *elements of dependent type* in families of sets.

What has been said so far clarifies the meaning of the fundamental forms of judgement and hypothetical judgment, corresponding to the formulas of intuitionistic type theory, and of the propositions corresponding to some of the types occurring in such formulas. What we should now do is, at least, to exemplify some of the rules of the system, and to do so, it is first of all worth remembering that there are basically four types of them: of formation, of introduction, of elimination and, finally, of equality. According to Martin-Löf, and with a significant reference to what we have explained in Section 2.5,

> the formation rule says that we can form a certain set (proposition) from certain other sets (propositions) or families of sets (propositional functions). The introduction rules say what are the canonical elements (and equal canonical elements) of the set, thus giving its meaning. The elimination rule shows how we may define functions on the set defined by the introduction rules. The equality rules relate the introduction and elimination rules by showing how a function defined by means of the elimination rule operates on the canonical elements of the set which are generated by the introduction rules. In the interpretation of sets as propositions, the formation rules are used to form proposition, introduction and elimination rules are like those of Gentzen, and the equality rules correspond to the reduction rules of Prawitz. (Martin-Löf 1984, 24)

In any case, it would already be excessive compared to the purposes of this work to expose even only all the rules for a first-order logical language; therefore, in what follows, we will be content to show exclusively those related to the intuitionistically problematic connectives $\rightarrow$ and $\forall$. In them, the λ-abstraction operator corresponds to an analogous operator Π for the formation of sets of functions from and to sets/propositions/types, called by Martin-Lööf *cartesian products of families of sets*. We will also follow Martin-Löf in calling `Ap` the so-called *application* operation for (terms reducing to) λ-terms - that is, if $c = \lambda x.b(x)$ for some $b(x)$, $\texttt{Ap}(c, a) = b(a)$ for every a of an appropriate type. In this way, we can distinguish the application of `Ap` to c and a from the application to a of the function $b(x)$ in the rank of λ - where c reduces to $\lambda x.b(x)$. Of the rules that follow, we will provide a quick elucidation, providing more details only when strictly necessary (the further details are obviously in Martin-Löf 1984, but see also, among others, Usberti 1995 and Dybjer & Palmgren 2016).

Π-formation

$$\frac{\alpha\ \texttt{set} \qquad \begin{matrix}[x \in \alpha] \\ \beta(x)\ \texttt{set}\end{matrix}}{(\Pi x \in \alpha)\beta(x)\ \texttt{set}} \qquad \frac{\alpha = \gamma \qquad \begin{matrix}[x \in \alpha] \\ \beta(x) = \delta(x)\end{matrix}}{(\Pi x \in \alpha)\beta(x) = (\Pi x \in \gamma)\delta(x)}$$

Π-introduction

$$\frac{\begin{matrix}[x \in \alpha] \\ b(x) \in \beta(x)\end{matrix}}{\lambda x.b(x) \in (\Pi x \in \alpha)\beta(x)} \qquad \frac{\begin{matrix}[x \in \alpha] \\ b(x) = d(x) \in \beta(x)\end{matrix}}{\lambda x.b(x) = \lambda x.d(x) \in (\Pi x \in \alpha)\beta(x)}$$

Π-elimination

$$\frac{c \in (\Pi x \in \alpha)\beta(x) \qquad a \in \alpha}{\mathtt{Ap}(c,a) \in \beta(a)} \qquad \frac{c = d \in (\Pi x \in \alpha)\beta(x) \qquad a = g \in \alpha}{\mathtt{Ap}(c,a) = \mathtt{Ap}(c,g) \in \beta(a)}$$

Π-equality

$$\frac{a \in \alpha \qquad \begin{matrix}[x \in \alpha] \\ b(x) \in \beta(x)\end{matrix}}{\mathtt{Ap}(\lambda x.b(x), a) = b(a) \in \beta(a)} \qquad \frac{c \in (\Pi x \in \alpha)\beta(x)}{c = \lambda x.\mathtt{Ap}(c,x) \in (\Pi x \in \alpha)\beta(x)}$$

The rules of intuitionistic type theory can be used to obtain, as a special case, those of a system of natural deduction *à la* Gentzen. This point of view, which is of fundamental significance for the philosophical framework suggested by Martin-Löf, and more specifically in the context of the issues on which the present work is focused, is obtained by passing from a (possibly hypothetical) judgment of the type $a \in \alpha$ to one of the (hypothetical) form α `true`. If we think of α as a proposition, and if we understand propositions as sets/types of proofs a, the underlying idea is at first obvious; α will be true in the case where there is a proof a. Then, by defining $\forall x\beta(x)$ as $(\Pi x \in \alpha)\beta(x)$ and $\alpha \to \beta$ as $(\Pi x \in \alpha)\beta$ - where in the latter case x does not occur free in β - and confining ourselves, when appropriate, to sets/types considered specifically as propositions, we can rewrite the rules of Π-introduction and Π-elimination respectively in the following ways:

$$\frac{\begin{matrix}[x \in \alpha] \\ \beta(x)\ \mathtt{true}\end{matrix}}{\forall x\beta(x)\ \mathtt{true}} \qquad \frac{\forall x\beta(x)\ \mathtt{true} \qquad a \in \alpha}{\beta(a)\ \mathtt{true}}$$

$$\frac{\begin{matrix}[\alpha\ \mathtt{true}] \\ \beta\ \mathtt{true}\end{matrix}}{\alpha \to \beta\ \mathtt{true}} \qquad \frac{\alpha \to \beta\ \mathtt{true} \qquad \alpha\ \mathtt{true}}{\beta\ \mathtt{true}}$$

with the usual restriction on x in what, now, is the equivalent of $(\forall_I)$ in intuitionistic type theory.

The distinction between $a \in \alpha$ type judgments and α `true` judgments makes it possible to illuminate two aspects of crucial importance, and closely related to each other, in the framework under examination: the distinction between two different, and differently relevant from an epistemic point of view, notions of proof, and the status of the notion of truth. The related discussion, which will conclude this section, will be carried out by referring to Martin-Löf's writings which followed *Intuitionistic type theory*, to the seminal Sundholm's article, *Proofs as acts and proofs as object: some questions for Dag Prawitz* (Sundholm 1998), and finally to a fundamental and due - given the aims of our investigation - reference to Prawitz's interpretation in *Truth and proof in intuitionism* (Prawitz 2012b).

Starting from the first of the aforementioned points, intuitionistic type theory includes, on close inspection, proofs of two different genres: first, those that occur in the BHK clauses and in judgments such as $a \in \alpha$, $a = b \in \alpha$, $b(x) \in \beta(x)[x \in \alpha]$ or $b(x) = d(x) \in \beta(x)[x \in \alpha]$, similar to elements of sets/types (possibly dependent) that are, more specifically, *propositions* (possibly under assumptions); on the other hand, there are also proofs built by exploiting the rules of the theory itself, aimed at establishing (possibly depending on assumptions) *judgments* (possibly dependent) of one of the four fundamental forms. Now, placing ourselves in an intuitive and very general perspective, a proof should be something epistemic, endowed with a significant impact on the whole of our knowledge; can we say that this also applies to the framework we are examining? Furthermore, with reference to the previous question, what is the difference between proofs of propositions and proofs of judgments? If we espouse the reading proposed by Prawitz in *Truth and proof in intuitionism* (Prawitz 2012b), we can certainly say that the position of Martin-Löf on such an issue, and particularly on the character of the proofs for propositions, has not always been univocal.

In *On the meanin of the logical constants and the justifications of the logical laws* (Martin-Löf 1985), Martin-Löf extensively discusses the notion of judgment; although it is not possible here to go into the details of a complex and profound treatment, it will still be sufficient to note how the Swedish logician attributes an epistemic weight to the activity of judging, by explicitly linking it to that of knowing. For example, he claims that

> when understood as an act of judging, a judgement is nothing but an act of knowing, and, when understood as that which is judged, it is [...] an object of knowledge. [...] Thus, first of all, we have an ambiguity of the term judgement between the act of

> judging and that which is judged. (Martin-Löf 1985, 20)

This ambiguity is resolved later through the distinction between *judgment* and *evident judgment*:

> if you let G be the proposition that every even number is the sum of two prime numbers, and then look at
>
> $$G \text{ is true,}$$
>
> is it a judgement, or is it not a judgement? Clearly, in one sense, it is, and, in another sense, it is not. It is not a judgement in the sense that it is not known, that is, that it has not been proved, or grasped. But, in another sense, it is a judgement, namely, in the sense that G is true makes perfectly good sense, because G is a proposition which we all understand, and, presumably, we understand what it means for a proposition to be true. [...] it seems better, when there is a need of making the distinction between [...] a judgement before and after it has been proved, or become known, to speak of a judgement and an evident judgement, respectively. (Martin-Löf 1985, 22 - 23)

Thus, the *act* of judging - which, according to what we have just seen, is, so to speak, an act of knowledge - allows, so to speak, to acquire the *object* of the judgment - which, as we have just seen, is the object of knowledge - and, in this way, to make it evident. Again, according to Martin-Löf's words (where, to stick to the previous quotation, we replace "is a proposition" with "is true"),

$$\underbrace{\text{I know } \underbrace{\underbrace{A}_{\text{expression}} \quad \underbrace{[\text{is true}]}_{\text{form of judgement}}}_{\text{judgement}}}_{\text{evident judgement}}$$

> Here is involved, first, an expression A, which should be a complete expression. Second, we have the form ... [is true], which is the form of judgement. Composing these two, we arrive at A [is true], which is a judgement in the first sense. And then, third, we have the act in which I grasp this judgement, and through which it becomes evident. These two together, that is, the judgement and my act of grasping it, become the evident judgement. (Martin-Löf 1985, 27 - 28)

But then, a proof for a judgment must be such as to allow exactly the accomplishment of an act of the type just described; and since this act will be, more specifically, an act of knowledge, proofs for judgments are, almost by definition, endowed with epistemic power. With regard to this circumstance, Martin-Löf seems to have never changed his mind - except of course for some terminological differences, and some variations in the declination of the relation between proofs for propositions or sentences and proofs for judgments, topics that in any case it is not appropriate here to investigate further (see also Martin-Löf 1985, and also Martin-Löf 1994, 1998, and Prawitz 2012b). In the case of proofs for propositions, this do not seem to be the case; in *Intuitionistic type theory* and *On the meaning of the logical constants and the justifications of the logical laws*, as well as in a variety of other articles,

> there is yet no idea about proofs of propositions lacking of epistemic significance. In the absence of any such indication, it must be taken for granted that the notion of proof should be taken in its usual epistemic sense, and thus [...] a proposition is determined by how it is established as true. (Prawitz 2012b, 53)

On the contrary, starting from *Truth and Knowability: on the principles C and K of Michael Dummett* (Martin-Löf 1998),

> the two crucial new ideas are to separate concepts that are epistemic from those that are non-epistemic and to distinguish between two radically different senses of proofs, of which one is epistemic and one is non-epistemic. [...] the epistemic notion appears when we speak of proofs of judgements [...] while the non-epistemic notion appears when we speak of proofs of propositions. (Prawitz 2012b, 58)

It is important at this point to emphasize that, in addressing the issues under examination, Prawitz explicitly refers to a known distinction, introduced and articulated by Sundholm in the famous *Proofs as acts and proofs as objects: some questions for Dag Prawitz* (Sundholm 1998, see also Sundholm 1983, 1993). As stated above, judgments are linked to an *activity* consisting in catching a certain object of knowledge; if this is true, proofs for judgments will ultimately have to take the shape of acts - *proofs-acts*, in the established terminology. On the other hand, proofs for propositions, especially when understood as abstract mathematical entities, and devoid of a real epistemic connotation, will be pure and simple objectual constructions - *proof-objects*, in the established terminology. Proof-objects and proof-acts are, of course, closely linked:

> the proofs occurring in the meaning explanations are constructions [...], whereas what is required for an assertion that a certain proposition is true is that a construction [...] has been carried out. (Sundholm 1998, 187)

> The proof-object is the construction-object, that is, the object of the construction-act that forms part of the proof-act. (Sundholm 1998, 193)

This notwithstanding, even Sundholm, like Martin-Löf, insists that while a proof-act is epistemically relevant,

> a proof-object is a mathematical object like any other, say, a function in a Banach space, or a complex contour-integral, whence, from an epistemological point of view, it is no more forcing than such objects. (Sundholm 1998, 194)

Herein we will see how Prawitz is in a sense critical towards this strong position on the non-epistemic character of proof-objects. However, for the time being, we can use what has been illustrated so far to come back to the notion of truth.

Shortly above, we have argued that, underlying the transition from judgments of the type $a \in \alpha$ to judgments of the α `true` form, there is the idea according to which, understood as a proposition, α will be true only when a proof of it exists. However, this is only a first approximation, and it is clear that the connection between truth and proofs can be further developed along very different paths. More specifically, according to Prawitz, in the writings prior to *Truth and Knowability: on the principles C and K of Michael Dummett*,

> the *truth* of a proposition [...] is equated with the *verifiability* of the proposition; in other words, a proposition α is true if and only if α can be verified. (Prawitz 2012b, 56)

On the other hand, the change of paradigm concerning the essentially non-epistemic status of proof-objects induces a corresponding modification of the understanding of truth: the discourse in terms of verifiability is replaced by a much more abstract perspective, so that

> the *truth* of a proposition α is defined as the existence of a canonical proof-object of α. [...] asked what the proof-objects are after their epistemic connections have been severed, Martin-Löf and Sundholm often answer that they are just *truth-makers*. (Prawitz 2012b, 59)

In any event, there is something that both readings share, and that runs through all the phases of Martin-Löf's thinking. As will be noted, α `true` is not one of the fundamental forms of judgment, and therefore is not a formula of intuitionistic type theory; the suppression of the proof-object a in $a \in \alpha$, thus, causes α `true` to be

> an *incomplete* judgment, or a *shortened* way of saying that a certain object is a construction for α; therefore it should not be understood as a judgment of a new species. [...] if we were to say that α `true` has the same meaning as $\exists x(x \in \alpha)$ we would assume that to the judgment the logical constants can be applied to $x \in \alpha$, which is [...] illegal. (Usberti 1995, 80)

In the article *Analytic and synthetic judgments in type theory* (Martin-Löf 1994), Martin-Löf elaborates on the question, calling *analytical* the judgments of the form $a \in \alpha$, and *synthetic* those of the form α `true`, the latter translated into `proof`(α) `exists`:

> if a judgement of [the first form] is evident at all, then it is evident solely by virtue of the meanings of the terms that occur in it. [...] The synthetic form of judgement is precisely the existential form of judgement that I have just introduced. [...] we clearly have to go beyond what is contained in the judgement itself, namely, to the thing that exists, in order to make an existential judgement evident. (Martin-Löf 1994, 93 - 94)

The distinction between analytical and synthetic judgments in intuitionistic type theory has particular relevance with respect to some issues that we will discuss later in this survey. In fact, Martin-Löf has linked with it significant reflections on the phenomenon of incompleteness, on the one hand, and, on the other, on the question of the decidability of the relation "being a proof-object for (or of the type)":

> if you have [an analytic judgement], then it can be checked, or decided, whether or not that judgement is derivable by means of the formal rules, and the algorithm for doing that is what the computer scientists call the type checking algorithm [...] if you have [an analytic judgement] which contains certain constants, expressing concepts, then no other laws are needed to derive it than the laws which concern precisely those concepts. (Martin-Löf 1994, 97)

In its analytical, and therefore totally explicit, form, intuitionistic type theory - or better, any formal system structured according to the guidelines given in this Section - is thus complete and decidable. Completeness has a number of consequences, some of which, in particular, connected in Prawitz's semantics to the concepts of harmony and conservativity - though it is not appropriate to dwell on them here (for details, see Sundholm 1998). Instead, decidability refers to what we have said about the recognizability of non-canonical arguments as valid - in Section 2.5.2.1 - and of effective procedures as non-canonical proofs, or proofs for implications and universal quantifications - in Section 2.5.2.2. We are therefore in the presence of a point of contact, but also of a possible divergence, between our previous discussion on the way in which this question is organized in Prawitz, and the framework suggested by Martin-Löf. Therefore, in order to end our discussion on intuitionistic type theory, and to engage in the discussion of the actual theory of grounds, it seems important to report what Prawitz says on this point:

> there are two views concerning whether we have the right to assert that a proposition α is true when we have come into possession of a proof-object of α. One view is that it remains to prove that the object in question is a proof-object of α. [...] Another view is that from the way in which a proof-object of α has been constructed, it should be evident that it is a proof-object of α. [...] It is enough to know the meaning of the terms involved and to be aware of how the proof-object has been constructed to know that the construction is a proof-object of the proposition in question. [...] On this second view, a proof-object of a proposition α amounts to what is commonly called a *ground* for asserting that α is true, that is, what a speaker must be in possession of in order to be correct or right in making the assertion. (Prawitz 2012b, 65 - 66)

Part II

Prawitz's theory of grounds

Chapter 4

Inferences, grounds and validity

4.1 From inferences to proofs, via grounds

As anticipated in Chapter 1, and reiterated in Chapter 3, Prawitz's theory of grounds intends to answer the question: how and why have some inferences the epistemic power to confer evidence to the conclusion starting from justified premises? Previously, after noting the relative inadequacy of model-theory and of proof-theoretic semantics, we have suggested the necessity of an approach that formalized the notion of evidence independently from more basic concepts such as truth, valid argument or proof. From this perspective, we have also said that a promising framework seems to be suggested by the BHK clauses, then discussing two theories that are inspired by them in various ways.

In the Kreisel-Goodman theory of constructions, the notion of evidence descends from that of BHK proof, and is formally understood in terms of constructions for formulas of a logical language; in the case of proofs for implications and universal quantifications, a decidability assumption validates Kreisel's second clause, thus solving the problem of recognizability which, as we have seen in Chapter 2, afflicts the proof-theoretic semantics. Also in Martin-Löf's intuitionistic type theory we have a sort of constructions *à la* Kreisel-Goodman; alongside the demonstrations for judgments, to be understood as proof-acts, there are the proofs (used to fix the meaning) of propositions or sentences, which are proof-objects instead. In the interpretation of Prawitz, the latest Martin-Löf and Sundholm agree in denying an epistemic content to these entities, considering them as simple truth-makers. Nevertheless, the fact that intuitionistic type theory is complete and, more significantly, decidable, allows every time to determine whether a certain proof-object is or is not a proof-object for a certain proposition or for a certain sentence. If, on the one hand, Prawitz considers unsatisfactory Kreisel's second clause, he seems from the other to consider the peculiar recognizability of Martin-Löf's proof-objects as a valid reason to look at the latter as - and the use of the term is not casual – grounds for the correctness of judgments or assertions.

In this chapter, we will start our discussion of the theory of grounds; in practice, it will keep us busy until the end of this work. Before that, however, it seems to us necessary to indicate the articles to which Prawitz has entrusted the development of his new approach, and that will therefore constitute our sources. The theory of the grounds is announced for the first time in an article of 2006 entitled *Validity of inferences* (now Prawitz 2013), further developed in *Inference and knowledge* (Prawitz 2009) and *The epistemic significance of valid inference* (Prawitz 2012a), and more systematized in the long and dense *Explaining deductive inference* (Prawitz 2015). Many others articles, although not primarily devoted to it, are nonetheless linked to it in a substantial way: from the aforementioned *Truth and proof in intuitionism* (Prawitz 2012b) to *Truth as an epistemic notion* (Prawitz 2012c), from *An approach to general proof-theory and a conjecture of a kind of completeness of intuitionistic ogic revisited* (Prawitz 2014) to *On the relation between Heyting's and Gentzen's approaches to meaning* (Prawitz 2016a), ending with the latest *The fundamental problem of general proof theory* (Prawitz 2016b) and *The seeming interdependence between the concepts of valid inference and proof* (Prawitz 2017). Our guiding article will be *Explaining deductive inference.* Obviously, however, we will occasionally refer to other writings, as much to complete, as to integrate some points of our scrutiny.

4.1.1 Inferences in the theory of grounds

At the beginning of this survey, while introducing in Section 1.4 the "fundamental task" that the theory of grounds must fulfill, it has been said that a rigorous discussion of the notion of ground presupposes, or at very least accompanies, a clarification of the notion of valid inference - and the additional delineation of an adequate relationship between agents and inferences. The search for a suitable definition of validity, however, has led us to an analysis of the notion of evidence and, therefore, to a preliminary discussion on the concept of ground. This should not be surprising; as we have said in Section 2.5.2, and as we will say again later, in the context we are moving the most appropriate strategy is inspired by the Wittgensteinian idea (Wittgenstein 1953) of meaning as use - and, more precisely, by the related interpretation of Dummett (Dummett 1978c, 1993b). Therefore, in order to approach the description of valid inferences, it is fundamental to understand what is the evidence they confer; however, the reverse is also true, in the sense that the nature of such an evidence will depend significantly on how it is obtained. If the acquisition of evidence occurs by means of valid inferences, then, it is also and above all necessary to ascertain what is meant by inference; in short, the preliminary question on grounds is, in Prawitz's theory, inseparable from the equally fundamental question "what is an inference?".

In Section 1.1 it has been argued that the nature of inferences cannot, nor it should, be uniquely described. Following Cozzo (Cozzo 2015), we have put forward the argument that what inferences are depends on the context. In particular, it is possible to identify seven factors (summarized in table 4.1).

Factors	**Examples of possible options**
Nature of premises and conclusion	Data
	Truth-bearers
	Acts
Agent of inference	Machine
	Impersonal biological entity
	Person
Relation between S and premises and conclusion (Γ, α)	Data (Γ, α) are stored in S
	S is in the neural state (Γ, α)
	S is in the representational state (Γ, α)
	S performs the act (Γ, α)
Relation between premises and conclusion	Pair
	Causal link
	Abstract relation
	Conscious and voluntary act of the agent
Stability	Aleatory
	Refutable
	Conclusive
Character	Private
	Publicly manifestable
Context	Premises, conclusion, relations, agent
	+ other inferences and reasonings
	+ information and knowledge
	+ co-agents and their activity

Table 4.1: `The seven factors of` (Cozzo 2015)

On the other hand, in Section 1.2 we observed that an analysis of the epistemic power of valid inferences requires, for each of the seven abovementioned factors, a rather stringent selection. Nevertheless, although in some cases there seems to be one possible option, in others there remains a sufficient choice; also in this perspective, therefore, the nature of inferences remains undetermined. Thus, what we have to ask is what notion of inference Prawitz refers to in approaching the basic question of his theory of grounds. And it is indeed the same Swedish logician who provides us with the relevant indications; in *Explaining deductive inference*, when introducing his investigation, he points out that

> philosophers and logicians distinguish among other things between inductive, abductive and deductive inferences. Philosophers and psychologist make a quite different distinction between what they call intuitive and reflective inference. (Prawitz 2015, 66)

The distinction between inductive, abductive or deductive inferences concerns exclusively the factor related to the stability of the relationship between premises and conclusion. As anticipated in Section 1.1, the deductive inferences are those of maximum force, establishing the conclusion in a not refutable way – under the assumption that the premises are justified; this does not happen, instead, with the inductive (see, for example, Hawthorne 2018) and abductive (see, for example, Douven 2017) inferences, in which the conclusion, even assuming that the premises are correct, might be questioned by future occurrences. The distinction between intuitive and reflective inferences, on the other hand, does not concern an individual factor, but a multiplicity of factors, of which the most important seem to be those of the relationship between agent and premises-conclusion, on the one hand, and between premises and conclusion as such on the other:

> to make a reflective inference is to be aware of passing to a conclusion from a number of premisses that are explicitly taken to support the conclusion. Most inferences that we make are not reflective but intuitive, that is, we are not aware of making them. (Prawitz 2015, 66)

Therefore, according to Prawitz, in reflective inferences the relationship between agent S and premises and conclusion (Γ, α) consists at least – as we shall see, it is only a first approximation – in the passage that S performs in an aware and voluntary way from Γ to α. However, awareness and voluntariness are here inextricably linked to an additional, fundamental circumstance:

believing to have found in Γ a valid basis for α (where the expression "basis for" is deliberately generic, since it can be specified only after having clarified the nature of premises and conclusion), S performs the inferential act for the evident purpose of substantiating, making known or using the link he/she has identified. The relation between Γ and α, therefore, arises from the fact that S, by inferring, explicitly assumes the responsibility of a support that the elements of Γ have on α.

However, it is worth emphasizing that the description of reflective inferences as conscious acts seems to be compatible also with other readings of the relationship between agent and premises-conclusion, or between premises and conclusion in themselves. It could indeed be required that the agent be aware to possess certain data, or to be in some neural or representational state; similarly, it could be that the agent makes the inference by virtue of the awareness of causal links or abstract relationships between premises and conclusion. If this is true, it should also be noted that a certain understanding of the above relationships is compatible *only* with reflective inferences, since, unlike the others, it seems to be meaningless when referred to the intuitive ones. Both being in a representational state and performing an act in which some premises are taken to support a conclusion imply a conscious and voluntary participation; on the contrary, in order to possess certain data, or to occupy a neural state, awareness and voluntariness are not necessarily required, as they are not in acts triggered by unconscious or automatic causal links. However, there are no doubt two elements that seem to unite all the possible variations to which reflective inferences give rise: first,

> reflective inferences are presumably confined to humans; (Prawitz 2015, 67)

secondly, the character of awareness has as a result that, although they can be performed at a primarily mental level, reflective inferences are publicly – for instance, linguistically – expressible and communicable in the context of fulfillment.

It will not have escaped, at this point, that much of what has been said with regard to reflective inferences is consistent with what, in Section 1.2, we have argued in relation to those endowed with an epistemically compelling force. In fact, the idea that an inference can definitely establish its own conclusion – under the assumption that its premises are justified – does not necessarily require that the act through which this occurs has a reflective nature – think of the examples of Chrysippus' dog or of the *modus tollendo ponens* discussed in Section 1.2. However, if we concentrate on epistemically compelling deductive inferences, it is safe to assume that they are also reflective – even though the opposite, obviously, may not be the case. It would certainly be interesting, but alien to the purposes of our survey, to research the precise links that induction, abduction and deduction have with intuitiveness and reflectivity; here, we can only limit ourselves to emphasize that Prawitz, with the theory of grounds, takes exclusively into account reflective deductive inferences, with the following remark:

> the logical relevance of the distinction between intuitive and reflective inferences may be doubted, but the distinction turns out to be relevant also for logic. (Prawitz 2015, 67)

Finally, it remains to understand Prawitz's outlook of the nature of premises and conclusion and of the context of an inference. And the best way to do it is, very simply, to recall a passage of our author – in which, among other things, we find again clearly summarized what has been said so far about reflective inferences to which the theory of grounds is addressed:

> a reflective inference contains at least a number of assertions or judgements made in the belief that one of them, the conclusion, say β, is supported by the other ones, the premisses, say $\alpha_1, ..., \alpha_n$. An inference in the course of an argument or a proof is not an assertion or judgement to the effect that β *follows* from $\alpha_1, ..., \alpha_n$, but is first of all a transition from some assertions (or judgements) to another one. In other words, it is a compound act that contains the $n + 1$ assertions $\alpha_1, ..., \alpha_n$ and β, and in addition, the claim that the latter is supported by the former, a claim commonly indicated by words like "then", "hence", or "therefore". (Prawitz 2015, 67)

In the inferences which the theory of grounds deals with premises and conclusion should therefore be understood as judgments or assertions; Prawitz is careful in specifying that an inference is not, in itself, the further judgment or assertion according to which the conclusion follows, or is supported by, the premises. On the contrary, it just takes the form of a mere act of passage – although, let us repeat, we are here at a level of first approximation from the first to the second. But what are, more specifically, judgments and assertions?

Following (a certain interpretation of) Frege (Frege 1879), a judgment can be understood as the mental act by which it is claimed that a certain proposition is true; and in the same way that a sentence can be considered as the linguistic equivalent of a proposition, an assertion can be considered as the linguistic equivalent of the judgment, i.e., the linguistic act through which it is claimed (mostly implicitly) that a certain sentence is true. Obviously, depending on how premises and conclusion are intended, namely as judgments or assertions,

> the transition that takes place when we make an inference is either a mental act or a linguistic act (a speech act). [...] I shall call such an act an *inferential transition.* (Prawitz 2015, 68)

Regarding premises, conclusion, and the inference itself, however, Prawitz does not seem to consider the distinction between the mental and the linguistic plan to be really substantial – as we shall see, it will become important in a more specific context of the theory of grounds; in this area, the two levels are almost exactly parallel, so that

> it does not matter for what I am interested in here whether we speak of judgements or assertions in this context and hence whether we take an inferential transition to be mental or linguistic. (Prawitz 2015, 68)

From Frege (Frege 1879) Prawitz also draws the symbol of judgment or assertion $\vdash$ but extending its use so as to account for what he calls judgments and assertions open or under assumptions - where the mental or linguistic act of assumption is reduced to the particular case of judgment or assertion below an assumption identical to the dependent proposition or sentence:

> unlike Frege, I want to account for the fact that an inference may occur in a context where assumptions have been made, and a qualification of what has been said is therefore needed. It should be understood that the assertions in an inferential transition need not be categorical, but may be made under a number of assumptions. [...] I write $\alpha_1, ..., \alpha_n \vdash \beta$ to indicate that the sentence β is asserted under the assumptions $\alpha_1, ..., \alpha_n$. To make an assumption is a speech act of its own, and one may allow an inference to start with premisses that are assumptions, not assertions. It is convenient however not to have to reckon with this additional category of premisses, but to cover it by the case that arises when α is asserted under itself as assumption, that is, an assertion of the form $\alpha \vdash \alpha$. [...] Furthermore, we have to take into account that a sentence that is asserted by a premiss or conclusion need not be closed but may be open. For instance, to take a classical example, we may assume that $\sqrt{2}$ equals a rational number n/m and infer that then 2 equals n^2/m^2. (Prawitz 2015, 68)

At the end of our reasoning on premises and conclusion, it should finally be noted that, although the latter are here similar to those of intuitionistic type theory, where they are understood as judgments or assertions as well, however they differ in what is judged or asserted:

> Like Frege, and unlike Martin-Löf, I do not take an expression of the form "it is true that ...", where the dots stand for a declarative sentence, to be the form of a judgement. To assert a sentence of the form "... is true", where the dots stand for the name of a sentence α, is to make a semantic ascent, as I see it, and is thus not the same as to assert α. (Prawitz 2015, 68)

> Martin-Löf type theory contains rules for how to prove [assertions or judgements of the form $a \in \alpha$]. There are also assertions of propositions in the type theory, but they have the form [α `true`] and are inferred from judgements of the form $a \in \alpha$ [...]. Thus, the assertions in the type theory are, as I see it, on a meta-level as compared to the object level to which the assertions that I am discussing belong. (Prawitz 2015, 97)

Moving towards the context, it turns out to be appropriate to remember what previously has been repeated time and again: the description of the inferential acts as transitions is, in Prawitz's view, only approximate at this stage of the analysis. Therefore the refinement will show how the activity in question actually includes something more – something essential – than premises, conclusions, relationships and agent. First of all, our author observes that

> when we characterize an inference as an act, we may do this on different levels of abstractions. If we pay attention to the agent who performs an inference and the occasion at which it is performed, we are considering an *individual* inference act. By abstracting from the agent and the occasion [...] we get a *generic* inference act. We may further abstract from particular premisses and conclusion of a generic inference and consider only how the logical forms of the sentences involved are related [...]. I shall call what we then get an *inference form.* (Prawitz 2015, 69 - 70)

A distinction analogous to the tripartition into individual inferences, generic inferences and inferential forms may also relate to transitions; more in particular, a *generic* inferential transition may be indicated by an *inferential figure*

$$\frac{\Gamma_1 \vdash \alpha_1 \quad \dots \quad \Gamma_n \vdash \alpha_n}{\Delta \vdash \gamma} \, x_1, ..., x_s$$

with Γ_i $(i \leq n)$ and Δ sets (possibly empty) of propositions or sentences (or possibly open formulas) for $\Delta \subseteq \bigcup_{i \leq n} \Gamma_i$, α_i $(i \leq n)$ and γ propositions or sentences (possibly open formula) for x_h $(h \leq s)$ bound by the transition. The transition contained in an inferential form can instead be indicated by an *inference scheme* of the same type, where, however, the elements in Γ_i $(i \leq n)$ and Δ, as well as α_i $(i \leq n)$ and β, are, for example, meta-variable parameters. Now, Prawitz emphasizes at this point how

> we usually take for granted that everything logically relevant about inference acts can be dealt with when an inference is identified with a set of premisses and a conclusion, in other words, with what individuates a generic inferential transition or the form of such a transition. We can then direct our attention solely to inference figures or schemata and disregards the acts that they represent. (Prawitz 2015, 70)

In the framework described so far, however, this move is not harmless, having on the contrary at least two significant consequences; in fact, the exclusive focus on inferential figures and schemes, leaving out transitions, and therefore *a fortiori* the acts they describe, implies that

> [1] we can see premisses and conclusions as sentences instead of assertions; the Fregean assertion sign may be taken just as a punctuation sign that separates a sentence in an argument from the hypotheses on which it depends. [...] [2] the distinction between intuitive and reflective inferences is of no interest, nor is there any room for it. (Prawitz 2015, 70)

However, as we have already had occasion to mention, Prawitz believes that the distinction between intuitive and reflective inferences is logically relevant. In fact, questioning the power of epistemic compulsion of inferences makes sense only when the latter are intended as reflective, and it can be assumed that the required analysis passes through a deepening and a clarification of the awareness and voluntariness involved in certain deductive activities. In addition, the idea that certain inferences are capable of providing justification seems to adapt only to an understanding of premises and conclusion as judgments or assertions; the acts of this kind are those that can be said correct or incorrect, and therefore authorized or not by reasoning. In other words, limiting oneself to inferential figures and schemes means to endorse the point of view according to which premises and conclusion are abstract entities, between which an equally abstract tie occurs. In Chapter 2, we have already seen how some proposals of this type – validity of inferences in terms of (logical) consequence between premises and conclusion – do not account, in a rather satisfactory way, for the presence of an epistemic bond. Soon, we will see that this line of thought seems, according to a certain reading, totally impracticable with regard to the basic question of the theory of grounds.

However, the risks of a reductionist reading on the nature of inferences seem already entailed in a description of the same as mere transitions. If all that is involved in an inferential act is the passage – however conscious and voluntary – from premises to conclusion, what else can count but an abstract relationship between the former and the latter? If all that we do by making an inference is to move from judgments to judgments, and if the whole thing can satisfactorily be exhausted in the linguistic practice of inserting a "therefore" – or other equivalent expressions – between assertions and assertions, what can we appeal to in order to explain the epistemic constraint if not to a link between the content of judgments and assertions, and to the possibility that it exerts in any, perhaps mysterious, way its force on us? In fact, Prawitz wonders

> does an inference contain something more than an inferential transition, and if so, what? [...] On the one hand, when we make inferences in the course of a proof, all that we announce is normally a number of inferential transitions. On the other hand, there seems to be something more that goes on at an inference, some kind of mental operation that makes us believe, correctly or incorrectly, that we have a good reason to make the assertion that appears as conclusion. (Prawitz 2015, 68 - 69)

This is the focal point of the theory of grounds, namely, the idea of something more than the simple transition, the idea of a mental operation on the basis of which we feel authorized, more or less licitly, to judge as true the conclusion, or to assert it. But before getting to the heart of the question, it is necessary to clear the field of some positions related to what is really at stake when we infer. In fact, through this further reflection on the nature of inferences, we will be able to go back to the notion of evidence.

For the moment, at the end of this chapter, we briefly summarize the ground-theoretic notion of inference by showing how the seven factors of Cozzo (Cozzo 2015) are articulated in Prawitz's framework. This is done in table 4.2 below, by a direct and quick reference to all the results we have obtained so far.

Factors	Typology
Nature of premises and conclusion	Acts (judgments or assertions)
Agent of inference	Person
Relation between agent S and premises and conclusion (Γ, α)	S performs the transition (Γ, α) in a conscious and voluntary way
Relation between premises and conclusion	S assumes responsibility for a support of Γ on α
Stability	Conclusive
Character	Publicly manifestable
Context	Premises, conclusion, assumptions and bound variables, relations, agent, mental operation

Table 4.2: `Inferences in the theory of grounds`

4.1.2 Evidence as the aim of reflective inferences

What has been said so far on the notion of evidence, as well as on the nature of inferences to which the theory of grounds is dedicated, provides a sufficiently clear framework for a renewed approach to valid inferences. This, both in the sense of allowing us a critical evaluation, with an essentially negative outcome, of a certain understanding of inferential validity, and with a specific reference to a deeper and more reasoned articulation of the "fundamental task" referred to in Section 1.4. On the other hand, in *seeking* an adequate description of valid inferences we cannot simply *take for granted* approaches that the logico-mathematical tradition has strengthened. As seen in Chapter 2, in fact, many proposals of respectable paternity and wide dissemination prove to be unsatisfactory. It is thus plausible that a philosophically "honest" treatment requires instead a *comparison* between certain formalizations and *pre-formal* desiderata, in order to consider their appropriateness, with the additional eventuality that, after ascertaining the inadequacy of the existing options, there is need of a new formalization. The Kreisel-Goodman theory of constructions and Martin-Löf's intuitionistic type theory in Chapter 3 – and of course, seen as a constructive semantics, Prawitz's proof-theory in Chapter 2 – are, in this regard, some of the possible examples, although perhaps more related to the problem of the epistemic power of deductive activity as a whole than to that of the epistemic power of the inferences which the deductive activity consists of, and that is also the object of our investigation.

Therefore, it will not come as a surprise that, when understanding the question of valid inferences or, equivalently, when undertaking the "fundamental task", Prawitz starts from what could be considered as a sort of "phenomenological analysis", essentially non-formal, of inferential activity – obviously, with a specific focus on correct inferential activity. A key role in this perspective is played by the three points that, in the previous section, we have seen being obscured by an understanding of inferences as simple premises-conclusions pairs, reducible to the graphic representation of generic transitions as inferential figures, and of inferential forms as inferential schemes: namely, the idea that an inference is an act, that this act is intuitive or reflective, and that premises and conclusion are for their part acts, that is, judgments or assertions. It is only when an inference is understood as an act, in fact, that the distinction between intuitive and reflective inferences makes any sense; the acts can be unconscious, involuntary and automatic or, on the contrary, conscious and deliberate. Whereupon, it is only because of this distinction that acquires importance the observation that conscious and voluntary acts, unlike the others, can be accomplished with a *purpose*; and actually, this seems to be the case with reflective inferences, in which – limiting ourselves to the first approximation of inferential transitions – the transition from premises to conclusion is, as it were, oriented to the latter, involving the "aiming" at a support that the former guarantee, or seem to guarantee, for the latter. In the words of Prawitz,

> we must say something about the point of inferences, why we are interested in making and studying them, and now the inferences must be seen as acts. [...] However, it is the point of reflective inferences that I want to discuss especially, and in this context we can speak of aims. The personal aims of subjects who make inferences may of course differ, but we can speak of an aim that should be present in order that an inference is to count as reflective. As already said, it is an ingredient in what it is to make such an inference that the conclusion is held, correctly or incorrectly, to be supported by the premisses. In view of this, reflective inferences must be understood as aiming at getting support for the conclusion. (Prawitz 2015, 71)

But what is more precisely this purpose? What is meant by the support that the premises secure, or seem to secure, for the conclusion? Well, if an inference involves judgments or assertions, the support will have to be of an *epistemic* type, in the specific sense that the purpose the agent aims to when moving from premises to conclusion is to judge or assert *correctly* the proposition or sentence involved in the latter by virtue of the (possibly hypothetical) *correctness* of the judgments or assertions to which the former amount. In other words, the support for a judgment or assertion cannot but be a *guarantee* of correctness, and the purpose of the inferential transition is exactly to achieve this guarantee (and, in the second instance, to demonstrate or to communicate that it has been obtained). In particular, Prawitz points out how this idea can be declined in different, and, one would say, equivalent ways:

> we may say that the function of inferences in general is to arrive at new beliefs with a sufficient degree of veracity. [...] We may say that the primary aim is to get a good *reason* for the assertion that occurs as conclusion. Since the term reason also stands for cause or motive, another and better way to express the same point is to say that the aim is to get adequate *grounds* for assertions or sufficient *evidence* for the truth of asserted sentences. Since assertions are evaluated among other things with respect to the grounds or evidence the speakers have for making them, we may also say that the aim of reflective inferences is to make assertions *justified* or *warranted.* (Prawitz 2015, 71)

At the beginning of Chapter 2 it has been said that validity of inferences is usually explained by requiring that the conclusion be a (logical) consequence of the premises; in turn, the explanatory standard of the relation of (logical) consequence provides for recourse to a more or less specific notion of truth, and this either by following the general and pre-formal intuition according to which the truth of the premises necessarily implies the truth of the conclusion, or, with reference to the more precise proposals of Bolzano, Tarski and model-theory, by requiring the preservation of a formally understood notion of truth under substitution and interpretation, respectively. However, it is not difficult to guess how, in relation to an explanation *à la* Prawitz of the purpose of inferences, this line of thought turns out to be unsatisfactory; to determine the validity of a *passage* from *judgments to judgments* or from *assertions to assertions* through an *abstract relationship* between *propositions or statements*, in turn based on an equally abstract notion such as that of truth, obscures the essentially *epistemic* nature of the purpose of the passage, and its result in case of legitimacy. In this regard, it is useful to note, however, that what has just been said is independent of which kind of truth is chosen; also an epistemic explanation of the notion, which contrasts the realistic one of Bolzano, Tarski and model-theory, does not seem befitting, since, more in general,

> the aim of a reflective inference cannot be described just in terms of truth. The aim has not been attained when the sentences that is asserted by the conclusion just happens to be true. (Prawitz 2015, 72)

At the end of the passage just mentioned, Prawitz highlights the obvious observation that, in order to have reasons, grounds, evidence, justification or guarantee for judgments or assertions, naturally it is not enough that the propositions judged or the sentences asserted are *de facto* true. We could be totally unaware of this circumstance, and in that case we would obviously not be authorized (nor, probably, willing) to judge as true or to assert (possibly under appropriate assumptions) the inferred propositions or sentences. Besides that, in the first part of the same quotation, the Swedish logician asserts in a stronger, and perhaps less obvious way that the explanation in terms of simple truth already has the effect of blocking an adequate elucidation of the purpose itself of reflective inferences; in fact, if we limited ourselves to saying that by inferring we aim to pass from propositions or sentences (possibly assumed as) true to true propositions or sentences (possibly under assumptions), we would omit the essential trait of this activity, namely taking responsibility for the *epistemic support* that the truth of the premises provides to the truth of the conclusion, of the fact that the conclusion is true *by virtue* of the truth of the premises – here we can note that, when Prawitz speaks in terms of truth, he always uses the word "evidence" for this truth. It is exactly (the identification of) this dependence our reason, ground, evidence, justification or guarantee for (possibly illicit) conclusive judgments or assertions; in other words,

> we expect an inference to afford us with knowledge in a Platonian sense, which is again to say that it should give us not only a true belief, but also a ground for the belief. (Prawitz 2015, 72)

In any case, the failure of an explanation of the purpose of inferences in terms of simple truth does not necessarily imply the inadequacy of a description of inferential validity in terms of truth-preservation (under substitution or interpretation) with respect to the "fundamental task" of the theory of grounds – and indeed, the idea counts some supporters (among the most recent, see Brîncuş 2015). As a matter of fact, we did not talk about this last possibility, and Prawitz himself points out that

> although this does not seem likely in view of the fact that the definition of validity refers to truth and not to any epistemic notion concerned with how truths are established, this should not be excluded off hand. (Prawitz 2015, 74)

However, according to what we have shown in Section 2.3, there is reason to believe that the situation will not be so different from that observed about the epistemic understanding of the modality involved in the relation of (logical) consequence. In fact, what we have shown there is that the Bolzano, Tarski and model-theoretic approaches do not capture the notion of necessity of thought (Prawitz 2005), which, as observed by Cozzo (Cozzo 2015), informs the relation of (logical) consequence of strong inferential traits; it is therefore presumable that the evaluation of these proposals with respect to the "fundamental task" is doomed to fail. Moreover, the same Cozzo proposes a reformulation of the "task" which emphasizes in particular the connection that we are highlighting:

> from the fact that an inference J is valid it should be possible to derive that J is endowed with necessity of thought. Let us say that "the fundamental task" is to devise an analysis of deductive validity which satisfies this condition. (Cozzo 2015, 106)

From the insufficiency of a characterization of the inferential purpose in terms of simple truth, we have thus passed to the question: can we account for the epistemic power of valid inferences by defining them as those in which the conclusion is a (logical) consequence of the premises in the sense of Bolzano, Tarski and model-theory? Obviously, the answer will be positive only if the adoption of the proposed definition takes us on the right direction with respect to the "fundamental task" – that is to say, it allows us to fulfill it through the identification of an adequate relation between agents and inferences, the last step that would remain to be made at that point. But then, we need to turn to a second question: can we go beyond the simple *observation* that this strategy *seems* to fail in light of its incapacity, observed in Section 2.3, to capture an epistemic reading of the relation of (logical) consequence? Well, there seems to be no other possibility here than to confront *directly* with the notion of valid inference, i.e. without passing through (logical) consequence. And as significantly will be seen, this manner of proceeding also involves as, so to speak, an indirect outcome, further and useful elucidations on the most appropriate type of relation to be required between agents and inferences.

For the purpose of a better understanding of this phase of the analysis (in which we will essentially use Prawitz 2009, 2012a and 2013), it is appropriate at this stage to explicitly repeat that "fundamental task" which, from the beginning of this chapter, we have so widely called into question – without, unlike now, a real need for precision: given the two premises

a) I is a valid inference from the set of premises Γ to the conclusion α;

b) A is in possession of a ground for each of the premises in Γ,

which additional condition should be added to a) and b) for the purpose of getting

c) A has a ground for α?

From this point of view, the first and most obvious observation concerns the actual need to accompany a) and b) with an additional premise c); in fact, in conjunction with the derivation of d), it is not sufficient that I is valid, and that the agent is in possession of grounds for each of the premises in Γ. Obviously, it applies here what has already been said regarding the insufficiency of the simple truth of the conclusion with respect to the possibility that the corresponding inference achieves the purpose; on several occasions Prawitz insists on this point, arguing for example that

> given a valid argument or a valid inference from a judgement α to a judgement β, it may be possible for an agent who is already in possession of a ground for α to use this inference to get a ground for β, too. But the agent is not ensured a ground for β, just because of the inference from α to β being valid and the agent being in possession of a ground for α. The agent may simply be ignorant of the existence of this valid inference, in which case its mere existence does not make her justified in making the judgement β; (Prawitz 2009, 183)

> it should be clear that the mere existence of a valid argument with conclusion α and a set Γ of premisses that express already received knowledge does not provide us with the knowledge expressed by α: we may be unaware of the existence of this valid argument and hence may be unable to use it to infer α from Γ. (Prawitz 2012a, 888)

In *Validity of inferences*, the situation is clarified by a reference to Andrew Wiles; the latter, as is well known, proved in 1994 the so-called "Fermat's last theorem", but he previously had to withdraw a first, wrong, attempt to prove it. Therefore,

> we may grant that Andrew Wiles was justified in holding true all the facts from which he started when proving Fermat's last theorem and that the one step inference from these starting points to Fermat's last theorem is valid. But these two assumptions are clearly not enough to make Wiles or anybody else justified in holding Fermat's last theorem true; they were, we may assume, satisfied long before Wiles gave his proof. [...] Wiles would have been justified in asserting Fermat's last theorem as soon as he gave his first incomplete proof, contrarily to the fact that in reality he soon afterwards had to withdraw it because of the discovered gap. (Prawitz 2013, 186 - 187)

Of course, it is one thing to point out the need for a third premise, but another to give one in an adequate form. It is plausible, however, to expect that the choice will be influenced by what is meant by valid inference. Just above, we have contemplated the idea of defining as valid those inferences the conclusion of which is a (logical) consequence of the premises in the sense of Bolzano, Tarki and model-theory. Therefore, the question is which, adopting this framework, the desired condition c) might be. At first sight, we may require that A is *aware* of the validity of I, i.e.

c_1) A knows that $\Gamma \models \alpha$

where, of course, Γ should be read as a set of formulas and no longer as a set of judgments or assertions, and α as a formula and no longer as a judgment or assertion – which implies obvious and consequential amendments, that we shall leave out here, to the formulation of the remaining premises of the "fundamental task". Whether this framework is satisfactory or not is a question that Prawitz, again in *Validity of inferences*, takes explicitly into account. However, at the same time he performs the additional – and essential, for what we are concerned with – observation according to which a), b) and c) – the latter in whatever declination, therefore *a fortiori* in the form c_1) – should be understood not only as merely sufficient, but also and above all necessary conditions of d). More specifically, given the implication

> 1) *if a person knows the inference J from a sentence α to a sentence β to be valid, and is justified in holding α true, then she is also justified in holding β true* [...] we should ask whether the antecedent of this implication describes a way to acquire new knowledge; otherwise, the fact that the implication holds is of little interest. [...] 1) was formulated as a possible response to

> the problem that arose when noting that the mere validity of an inference whose premisses are known to be true does not justify a person in asserting the conclusion, it then being suggested that it is only when a person sees the validity of the inference that she is so justified. Identifying "seeing the validity of an inference" with "knowing the inference to be valid", such knowledge was suggested as a necessary condition for an inference to justify a belief, which we may formulate as follows: 2) *it is only when a person knows an inference to be valid and its premisses to be true that the inference justifies her in holding the conclusion true.* (Prawitz 2013, 187 - 188)

However, Prawitz points out that the joint action of 1) and 2) generates regressive explanations similar to those already described by Bolzano (Bolzano 1837) and Lewis Carroll (Carroll 1895), thus making the overall approach decidedly unsatisfactory. As a matter of fact, completing a "fundamental task" of which the third hypothesis has the form c_1), means to try to

> establish the truth of the implication 1). Assume that a person, call her A, knows a sentence α to be true and let J be an inference from α to a sentence β. Assume further not only that J is valid but that A knows this, as required in 1), if she is to use the inference J to justify her in believing that β. Why should A now be justified in holding β true? Suppose that we argue that, given A's knowledge of the validity of J, A knows that if α is true then β is true [...] and that therefore A may just apply modus ponens and conclude that β is true. But to be an argument showing that A is justified in holding β true, it must also be assumed because of 2) that A knows modus ponens to be a valid inference, (Prawitz 2013, 190)

which, as is evident, requires the addition to the starting assumptions of the further hypothesis that A knows that *modus ponens* is a valid inference, thus resulting in a regress. Strictly speaking, be as it is, the regress does not depend in this case on the fact in itself that the validity of the inferences is understood in the terms of the notion of (logical) consequence *à la* Bolzano, Tarski and model-theory; this is clearly emphasized by Prawitz, who writes as follows:

> the failure to support 1) does not depend on the assumed fact that the Bolzano-Tarski notion of logical consequence lacks a genuine

> modal ingredient, because the same regress seems to arise if we take the validity of the inference to mean that the conclusion is a necessary consequence of the premisses [...]. What has been shown so far is only that a regress arises when we combine the Bolzano-Tarski notion of validity with the idea expresses in 2). (Prawitz 2013, 191 - 192)

However, it is easy to understand how the fact that the approach of Bolzano, Tarski and model-theory lacks whatsoever modal traits, plays a decisive role in the discourse we are carrying out. In fact, even ascribing to c) a minimum force with respect to that of the relation of explicit knowledge postulated by c_1), or in other words requiring generically and simply that

c_2) *A* acknowledges that *I* is valid

is the right characterization of c), it seems inevitable to fall back into invalidating regressive forms:

> when "valid" means that the implication "if α then β" is true for all variations of the content of the non-logical terms of α and β, then this is what the person recognizes, and what seems relevant here is just that she recognizes the truth of this implication (without any variation of the content). From this we want to conclude that she is justified in holding β true. [...] we may assume that she recognizes the validity of modus ponens [...]. But there we are again with this kind of argument that leads to a regress of the Bolzano-Carroll kind. (Prawitz 2013, 195)

Of course, the examples discussed so far do not exhaust the range of subjects that a supporter of the definition of inferential validity as (logical) consequence in the framework of Bolzano, Tarski and model-theory could eventually devise; however, the approach in question seems unsatisfactory on principle and this because of a problem that undermines its very foundations. As argued in Section 2.1, the notion of logical consequence is a special case of a more general notion of deductive consequence; the modal character of the former is transmitted to the latter, since the difference between the two concerns only the invariance of the relation by substitution or interpretation of the non-logical terms. Furthermore, in Section 2.3 we have seen how an analogous distinction seems also to concern the notion of inferential validity. Here, and in the light of the clarifications so far acquired on the concept of inference, we are able to explore this last point:

(ID) an inference I with set of premises Γ and conclusion α is *deductively valid* if, and only if, an agent A who is justified for all the premises in Γ is, through I, epistemically compelled or justified to accept α – where it should be noted that the expression "through I" is deliberately inaccurate, being in fact the object of the current phase of investigation;

(IL) an inference I with set of premises Γ and conclusion α is *logically valid* if, and only if, I is deductively valid solely on the basis of the logical form of the elements in Γ and of α.

As usual, the idea expressed in (IL) can be translated - although not a little problematically - either using Bolzano's substitutional approach (Bolzano 1837) or, if we accept Tarski's criticism with respect to what Etchemendy (Etchemendy 1990) called *persistence principle*, introducing interpretations from the language into (one or more) appropriate sets – thus providing for the possible extension that model-theory inherits from Tarski's original formulation:

(IL_*) an inference I with set of premises Γ and conclusion α is *logically valid* if, and only if, for every substitution Σ, the inference I^Σ with a set of premises Γ^Σ and conclusion α^Σ is deductively valid; an inference I with set of premises Γ and conclusion α is *logically valid* if, and only if, for every interpretation $\mathtt{A}_D$, the inference $\mathtt{A}_D(I)$ with a set of premises $\mathtt{A}_D(\Gamma)$ and conclusion $\mathtt{A}_D(\alpha)$ is deductively valid.

However, (IL_*) is based on a prior notion of deductively valid inference, and it is *precisely* this notion that Prawitz intends to define adequately. On the other hand, as argued again in Section 2.3, both Bolzano and Tarski and model-theory do not seem, with their characterizations, to go further, or better, deeper than the relation of logical consequence, therefore not being able to deal with the notion of deductively valid inference, to which (IL) refers; and even when we wanted to search in them a description of deductive validity, we have already seen that what results is a trivial collapse of the relation of deductive consequence on the material implication, and of the corresponding inference on the circumstance that either one of the premises is false or the conclusion is true. Therefore, Prawitz significantly remarks how

> to say that the conclusion of an inference is a logical consequence of its premisses in the sense of Bolzano-Tarski amounts just to saying two things: 1) that the inference preserves truth (which only means here that if the premisses are true then so is the

> conclusion) and 2) that all inferences of the same logical form do the same. Now, although the first property is certainly *relevant* for the question whether the inference has the power to justify a belief in the conclusion (being a necessary condition for that), it is obviously not sufficient for the inference to have this power. The second property seems not even relevant to this question. (Prawitz 2013, 185)

Once the feasibility of an approach based on the notion of (logical) consequence has been discarded - being the latter understood as in Bolzano, Tarski and model-theory - it remains the other option explored in Chapter 2: to describe as valid those inferences for which there exists a valid argument or a proof – in the sense of Prawitz's proof-theoretic semantics – from the premises to the conclusion, and to weigh the adequacy of this proposal in connection with the "fundamental task". It would indeed seem promising that, as observed in sections 2.5.2.1 and 2.5.2.2, the notions of valid argument and proof allow as a matter of principle to distinguish between deductive validity *strictu sensu* and, more generally, logical validity. In addition, they should be understood as having epistemic connotations that make the respective inferences capable of forcing, or justifying. However, we can from the beginning point out that even in this case c) cannot consist in the requirement that the agent has an explicit knowledge of inferential validity. In other words, when added to a) and b), the condition

c_3) A knows that there is a valid argument or proof from Γ to α

blocks any possibility of explanation, just like c_1). In fact, it seems to be there a wider problem already with the relation of explicit knowledge in itself, the excessive force of which is by itself able to induce regresses:

> there are other objections, not connected with any particular view on the validity of inferences, against the requirement expressed in 2) that one has to know the validity of an inference if one is to be justified by the inference in holding the conclusion true. Such a requirement seems directly to give rise to a circle or a new regress, perhaps a more straightforward one that the regress noted by Bolzano ando Carroll. It appears as soon as we assume that the knowledge of the validity of the inference has to be explicit and ask how we know the validity. If the validity is not immediately evident, it must come about by a demonstration, whose inferences must again be known to be valid according to the requirement

> expressed in 2). These inferences must either be of the same kind as the inferences whose validity we try to demonstrate or be of another kind, and so we get either into a circle or regress. It hardly seems reasonable to think that this circle or regress can be avoided by saying that, for sufficiently many inferences, their validity is immediately evident. (Prawitz 2013, 192)

Therefore, whatever the selected definition of inferential validity, we are forced to abandon the idea that the knowledge of the validity of an inference may serve the purpose of obtaining epistemic constraint or justification through the latter; in conclusion,

> we need to find a relation R between a person P and an inference J in terms of which we can state a condition that satisfies the following demands. On one hand, it is to be substantial enough so that [...] it implies that the person is justified in holding true the conclusion of the inference [...] on the other hand, it is not to be so strong that [...] it cannot be satisfied when taken as a necessary condition for an inference to justify a belief. (Prawitz 2013, 197)

But what can this relationship be? In answering, it becomes essential to remember what we have often written about inferences: against a frequent reductive vision that looks at them as simple sets of premises and conclusion, inferences are to be understood as oriented acts, namely, at their lowest, as transitions between judgments or assertions serving the purpose of getting reason, ground, evidence, justification or guarantee by virtue of a support provided by knowledge (assumed as) acquired.

From an intuitive point of view, it is the inferential act *as such*, without, as it were, further additions, to force or give justification. If it is true that correct deductive practices exert an epistemic force on us, the correct inferential acts of which they are ultimately composed must be the means through which this force is carried on. By *performing* these acts, starting from a certain (assumed as) acquired knowledge, we are eventually induced to accept or take for granted the conclusion. Therefore, Prawitz's natural and plausible proposal is that the desired condition c) simply have the form

c_4) A performs I.

However, we must be careful. What does it mean to perform an inference? In Section 4.1.2 we already warned against the risks of a reductionist reading, according to which performing an inference means only to pass from the judgments expressed by the premises to the judgments expressed by the conclusion or, at a linguistic level, to perform a complex act consisting in asserting the premises, in saying "therefore" or other equivalent expressions, and finally in asserting the conclusion. On that occasion, it was said that such a reconstruction seems to force us to abstract relations, such as that of (logical) consequence, existing between premises and conclusion. To the discussion so far conducted on the insufficiency of this framework, we can now add Prawitz's observations:

> a person announces an inference in the way described, say as a step in a proof, but is not able to defend the inference when it is challenged. Such cases occur actually, and the person may then have to withdraw the inference, although no counter example may have been given. If it later turns out that the inference is in fact valid, perhaps by a long and complicated argument, the person will still not be considered to have had a ground for the conclusion at the time when she asserted it; (Prawitz 2009, 186 - 187)

> Fermat had of course some arguments in mind; his problem was only that the margin where he announced his theorem was too small. Suppose that we found these arguments in some other notes by Fermat and that they could now be shown to be valid. One would still suspect him to be unjustified, if the validity was shown by advanced mathematics that Fermat had no access to: that the argument were stated and happened to be valid do not change the matter. (Prawitz 2012a, 891)

It is therefore necessary to make the inferential acts something more than mere transitions. Actually, this seems to be independent of, and indeed to be a necessary condition for, a definition of the notion of valid inference that allows us to fulfill the "fundamental task". Just as with the notions of evidence and inference, an indissoluble bond unites, in the theory of grounds, the most appropriate description of what to perform an inference means and the adequate definition of valid inference. The main suggestion regarding this problem comes, not surprisingly, from the same Prawitz:

> having arrived at a contradiction under an assumption α, we conclude that the assumption is false, saying "hence, by reduction not-α". [...] the assertion of the conclusion is accompanied by an indication of some kind of operation that is taken to justify it. This is consonant with an intuitive understanding of an inference

> as consisting of something more than just a conclusion and some premisses. Although the conclusion and the premisses may be all that we make explicit, there is also some kind of operation involved thanks to which we see that the conclusion is true given that the premisses are. [...] My suggestion is that in analysing the validity of inference, we should make this operation explicit, and regard an inference as an act by which we acquire a justification or ground for the conclusion by somehow operating on the already available grounds for the premisses. (Prawitz 2013, 199)

At the conclusion of this section, it is perhaps appropriate to make a final point. Suppose we define the validity of inferences as existence of valid arguments or proofs from premises to conclusion, and attribute to the condition c) the form c_4), where "making an inference" is understood in merely transitional terms - the only one description that, without prejudice to the discussion in Section 2.5.3.3, seems appropriate to this framework. Then, the impossibility of fulfilling in this way the "fundamental task" becomes even more evident if we recall the three problems that, in Section 2.5.3, we have seen afflicting proof-theoretic semantics – the recognizability problem, the proofs-as-chains problem, and the problem of independence of validity from inferences of which arguments and proofs are made up. Let I be a valid inference (represented by a constructive procedure f), and suppose that agent A is in possession of evidence for (the types in the domain of f representatives of) the premises of I; in the approach under examination, such an evidence will amount to valid arguments or proofs in canonical form. Applying to this evidence (the procedure f which represents) I, A obtains, respectively, a valid argument Δ or a proof π. Well:

Recognizability if Δ is a non-canonical argument or π a categorical or non-canonical proof (according to the terminology we prefer), on account of the problem of recognizability we cannot assume that being in possession of Δ or π, i.e. knowing how to get a valid argument or a canonical proof, means also to know that Δ or π reduce to such valid argument or canonical proof. Thus we cannot say that A is in possession, albeit indirectly, of evidence for the related conclusion[1];

[1]As already mentioned in Section 2.5.3.3 in the case of proofs, the constructive procedures representing inferences are defined on proofs and give proofs as output; this allows to redefine the inferential act as a computation of a procedure of the intended type. When the constructive procedure stands for a valid inference, it produces canonical proofs if applied to canonical proofs; this instead means that what we actually need at the level of objects is only the notion of canonical proof, and the problem of recognizability, less

Proofs-as-chains on the other hand, if we really want to maintain that the mere possession of Δ or π is sufficient to compel or to justify, we need, as analogously stated in Section 2.5.3, to be able to look at Δ or π as a chains of inferences that satisfy – what we aim to prove for (f representative of) I – the property requested by the "fundamental task". But it is not clear how this can happen, since, on account of the distinction between canonical and non-canonical cases, Δ or π might contain inferences of the same type as (f representative of) I with an identical or greater complexity[2];

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urgent in the canonical case, can thus be referred in the full sense only to the acts performed in the deductive process. The general setup is now, in a sense, less problematic – while still remaining problematic – since, although the act accomplished might not recognizably be such as to produce evidence, we are in possession of the result of this act. The distinction between objects and acts makes it possible to distinguish between what the agent has and what the agent does, and the agent him/herself would occupy mental states reified by always canonical objects that the theory treats as evidence. A similar strategy does not seem possible in the case of valid arguments; here, the fact that justifications are defined on argument structures and give argument structures as output, implies that the reduction - what *de facto* forces or justifies - is external to the construction of the argument, and therefore to the individual inferential steps which this construction amounts to. Nonetheless, one might argue, when we are in possession of a valid non-canonical argument, we are factually in possession of a method to obtain a valid canonical argument – in other terms, a valid non-canonical argument denotes a valid canonical argument (see, for example, Tranchini 2014b). Then, when he/she is in possession of a non-canonical valid argument, the inferential agent occupies a mental state reified by an object that is the result of the method which the valid non-canonical argument amounts to, namely its denotation. However, this objection is problematic; an argument is a linguistic structure, and as such it is already in itself an object. If making a valid non-introductory inference is reductionistically understood as the mere passing from premises to conclusion, by applying the inference the agent comes to occupy a mental state which will be, as an object, the correspondent valid non-canonical argument, and not the result of the method which it amounts to, or its denotation. Otherwise, we need more strongly – and against the hypotheses of the issue we are here carrying out - conceive the performing of a valid inference in a non-introductory form as the application of the related justification. Even assuming the plausibility of such a move – which is doubtful, since justifications are defined on argument structures and give argument structures as output – however it is clear that the resulting framework is *de facto* identical to that of proofs; to make an inference means to apply a certain constructive procedure, and it also becomes necessary to distinguish between objects always canonical that the theory treats as evidence – the valid canonical arguments – and the acts that produce such objects – the valid non-canonical arguments.

[2] In a reductionist view of inferences as mere passages, the proofs-as-chains problem seems to block the way even to an explanation that sets validity of inferences in terms of existence of valid arguments or proofs from premises to conclusion and that attributes to c) the form c_2). In fact, the identification of the validity of (f representative of) I either can take place by performing I (as a construction of the functional given by the constructive procedure f which represents I applied to appropriate values), or can be external to such performance. In the first case, we obtain a picture similar to that suggested for proofs in Section 2.5.3.3 – and, in a purely hypothetical way, for the valid arguments

Validity as independent from inferences if Δ is a non-canonical argument or π a categorical or non-canonical proof (depending on the preferred terminology), this may not depend at all on (f representative of) I, but simply on the justification associated with I (or, in the other case, on the way f is defined). However, Δ and π can have an epistemic weight only by virtue of the fact that they reduce to what the theory treats as evidence – a valid canonical argument or a canonical proof, respectively. In this regard, it is the process of reducing Δ (that involves the reduction associated with I) or the computation of π (which involves an application of the defining equation of f) that proves that Δ and π reduce in the manner described. Therefore, strictly speaking, it is not the fulfillment as such of I (as a construction of the functional given by the constructive procedure f which represents I applied to appropriate values) to give the constraint or the justification required.

4.1.3 Prawitz's notion of ground

At the beginning of the present investigation, and in the discussion so far conducted, we have noted how the basic issue of the theory of grounds connects to that on the nature and role of proofs. The latter are composed of deductively valid inferences, with which they share the property of compelling epistemically to accept the conclusion when the underlying assumptions are accepted. In fact, being justified in judging a proposition as true, or in asserting a sentence, is usually led back to the possession of a proof for that proposition or for that sentence.

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of note 1 above: inferences cannot be simple transitions from premises to conclusion, since they consist rather in the computation of constructive procedures of a certain type; the objects of the theory are always canonical, and the distinction between canonical and non-canonical cases concerns the acts by which these objects are obtained. In the second case, what the agent sees by performing I (as construction of the function given by the constructive procedure f which represents I applied to appropriate values) is that, whenever there are valid closed arguments (possibly not canonical) or proofs (possibly non-canonical or categorical, depending on the terminology we prefer) for the premises, through (f representative of) I we get a valid (possibly non-canonical) argument or a proof (possibly categorical or non-canonical, depending on the terminology we prefer) for the conclusion; which means that the agent recognizes that Δ or π reduce to a valid canonical argument Δ^* or to a canonical proof π^* (possibly identical to the first one). In order that this is sufficient for the agent to be forced or justified towards the conclusion, however, we must be able to look at Δ or π as a chain of inferences that satisfy – what we aim to prove for (f representative of) I – the property required by the "fundamental task". However, as soon as Δ^* or π^* end by introducing $\rightarrow$, the corresponding immediate subargument or substructure could, on account of the distinction between canonical and non-canonical cases, contain inferences of the same type as (f representative of) I with an identical or greater complexity.

The "fundamental task" requires a definition of the notion of deductively valid inference that depends on that of evidence. In this perspective, the fact that proofs are intended as what we have when we are epistemically compelled or justified in judging or in asserting, could lead to define evidence in terms of valid arguments and proofs of proof-theoretic semantics. At the end of Chapter 2, however, we have seen how this way is not practicable; the problems met by proof-theoretic semantics reveal the need for a definition of the notion of deductively valid inference, and therefore of evidence, independent of the notions of valid argument and proof, in such a way that, rather, the latter are definable in terms of the former. This is also consistent with the usual way in which proof themselves are characterized; by defining deductively valid inferences in terms of valid arguments and proofs, proof-theoretic semantics reverses the Cartesian idea of proofs as chains.

However, it is precisely the close link that proofs have both with evidence, and with deductively valid inferences, to make the relation we are talking about problematic. The levelling of the notion of evidence on that of proof may not be after all wrong; what we have when we are epistemically successful should perhaps be distinguished from the activity through which this success is achieved. According to this line of thinking, there are two different notions of proof, or at least two different points of view, summarized in the distinction, which, as we have already pointed out, was introduced by Martin-Löf (Martin-Löf 1984) and developed by Sundholm (Sundholm 1998, but see also Sundholm 1983, 1993), between proof-objects and proof-acts. The two perspectives are interwoven: through the deductively valid inferences of which they consist, proof-acts allow to obtain proof-objects. But then, although it has to be independent, the notion of evidence cannot be totally detached from that of proof and therefore, again, from that of deductively inference.

The two knots highlighted here – the independence of the notions of deductively valid inference and of evidence from that of proof, and the shared articulation of these three notions - offer as many, interconnected, points of access to a characterization of the notion of ground.

4.1.3.1 Evidence states and primitive operations

Although the term "ground" often occurs in the writings of Prawitz dating back to the period of proof-theoretic semantics, the notion associated with it receives, in the theory of grounds, a radically different declination, as well as a more specific scope:

> in some previous works [...] I have identified a ground for a judgement with a proof of the judgement, or I have spoken of grounds

> for sentences and have taken them to be valid arguments. I prefer not to use that terminology now, because I want to take proofs to be built up by inferences, and I do not want to say that an inference constitutes a ground for its conclusion - the question is instead how an inference can deliver a ground for the conclusion. (Prawitz 2009, 191 - 192).

In this new meaning, the concept of ground appears for the first time in the 2006 article *Validity of inferences* – published in 2013:

> To have a ground for a sentence α, as I use the term, will always mean to be justified in holding α true and to know that α. (Prawitz 2013, 175)

> I here use the term *ground for a sentence* to denote what a person needs to be in possession of in order to be justified in holding the sentence true. (Prawitz 2013, 193 - 194)

However clear, this characterization is rather lapidarian. It keeps at least two fundamental questions open: what kind of objects are grounds, and what does it mean to possess them? Well, a first investigation can be found in the immediately following article, *Inference and knowledge*, where Prawitz points out significantly how rational judgments and sincere assertions must be understood

> to be made on good grounds. It is not that an assertion is usually accompanied by the statement of a ground for it; in other words, the speaker often keep her ground for herself. But if the assertion is challenged, the speaker is expected to be able to state a ground for it. To have a ground is thus to be in a state of mind that can manifest itself verbally. (Prawitz 2009, 190 - 191)

The content of the above quoted passage contains two very important pieces of information. First, to possess grounds means occupying a mental state in which the subject has justification for judgments or assertions; secondly, of the available grounds there is often no trace in the linguistic practice, with which the acquisition is associated - although possession could be made explicit. Obviously, these are strictly connected circumstances; for the sake of a greater clarity, however, we will deal with them separately. The reason for the division is that, in our opinion, an analysis of the first aspect will allow to point out suggestions related to the nature of grounds, and of their possession, as well as in this sense some problematic knots; the latter will then be further clarified through the discussion of the second point.

The idea that the possession of grounds can be expressed in terms of mental states of justification is again found in *The epistemic significance of valid inference*, but it is in *Explaining deductive inference* that the setting is offered and investigated in detail, so as to make it clear, and allow us to set out its main implications. In fact - resuming anyway from the already cited *The epistemic significance of valid inference* - Prawitz introduces here the discussion of the concept of ground, and consequently the theory that he articulates around it, with the following words:

> one finds something to be evident by performing a mental act [...]. After having made such an act, one is in an epistemic state, a state of mind, where the truth of a certain sentence is evident, or as I have usually put it, one is in possession of evidence for a certain assertion. (Prawitz 2015, 88)

This last step highlights an aspect of fundamental importance. It introduces a link between *what*, according to the perspective in question, one is in possession when one is in a mental state of justification, and *who* is in this state: the former can be obtained by the latter by performing some kind of operations. It then becomes possible to articulate the analysis, both in the sense of investigating the nature of the grounds through those operations of which they are the outcome and, in the other direction, to show more precisely in what sense the notion of ground should be linked to the states in which we have evidence for judgments or assertions:

> rather than trying to analyse phenomenologically the states of mind where we experience evidence - let us call them evidence states - we have to say what evidence states are possible and what operations are possible for transforming one evidence state to another. [...] To state principles like this, it is convenient to think of evidence states as states where the subject is in possession of certain objects. I shall call these objects *grounds* [...]. I am so to say reifying evidence and am replacing evidence states with states where the subject is in possession of grounds. (Prawitz 2015, 88 - 89)

As one can easily guess, there is a parallelism between the operations the fulfillment of which allows to have grounds and the inferences which Prawitz intends to deal with: by virtue of the "fundamental task" frequently referred to, in fact, they too are, or at least they should in the end be such that, by fulfilling them, one then finds oneself in a state of evidence. Moreover, this similarity will be one of the key points of the theory of grounds; without anticipating what we will say more broadly thereafter, the observation nevertheless allows us to touch a critical point. In the same way to what is illustrated in Section 2.5.2 with respect to the notions of valid argument and proof, also in *Explaining deductive inference* - as indeed in almost all the other articles on the theory of grounds - Prawitz draws attention to the possible circularity resulting from the interplay of two circumstances:

> for logically compound sentences there seems to be no alternative to saying that evidence comes from inference. On the other hand, since not any inference gives evidence, one cannot account for evidence by referring to inferences without saying which inferences one has in mind. (Prawitz 2015, 77)

Again as in Section 2.5.2, the solution to this impasse comes from the adoption of an approach that recalls the Wittgensteinian motto (Wittgenstein 1953) of meaning as use in the anti-holistic articulation of Dummett's interpretation (Dummett 1978c, 1993b):

> it is in the nature of the meaning of some types of sentences that evidence for them can only be explained in terms of certain kinds of inferences. The legitimacy of these inferences is then a datum that has to be accepted as somehow constitutive for the meaning of these sentences. [...] But not all cases of accepted inferences can reasonably be seen as constitutive of the meaning of the involved sentences [...]. For them, it remains to explain why they are legitimate. (Prawitz 2015, 77 - 78)

Now, these words refer exclusively to inferences, although, by virtue of the link with the operations involved in the concept of ground, we can expect that the idea that meaning should be expressed in terms of certain types of inferences is transformed, or can be transformed, into the idea according to which meaning should be expressed in terms of certain types of operations and, therefore, of grounds. In fact, the reference to the problem of meaning is often explicit, and immediately involves the introduction of operations for the production of grounds:

> we may want to say that anyone who knows the meaning of the sentence $t = t$ has access to a state in which she has evidence for asserting it. Similarly, we may want to say that anyone who

> knows the meaning of conjunction and is in a state in which she has evidence for asserting a sentence α as well as for asserting a sentence β, can put herself in a state in which she has evidence for asserting $\alpha \wedge \beta$. [...] We can make the latter even more articulate by saying that there is an operation $\wedge I$ that applied to grounds g_1 and g_2 for asserting α and β, respectively, produces a ground $\wedge I(g_1, g_2)$ for asserting $\alpha \wedge \beta$. (Prawitz 2015, 88 - 89)

Aiming to explain the epistemic power of deductively valid inferences, and indeed precisely for the purpose of doing so, the theory of grounds therefore provides also a characterization of the meaning of propositions or sentences in terms of what counts as ground for them, calling into question primitive and, as such, meaning-constitutive operations. Although Prawitz often proposed this idea, his clearest and most direct formulation can be found in *The epistemic significance of valid inference*, where we are required a

> meaning theory that explains the meaning of our sentences directly in terms of what constitutes grounds for the corresponding assertions [...]. To have a name for how the grounds for asserting $\alpha \wedge \beta$ is formed, let us call it *conjunction grounding*, for short $\wedge I$, a primitive operation that we introduce in this connection. [...] a ground for the assertion of a numerical identity would be obtained by making a certain calculation, and outside of mathematics, a ground for asserting an observation sentence would be got by making an adequate observation. (Prawitz 2012a, 893)

This determination of meaning obviously has a broad combination of consequences; as rightly pointed out by Cozzo, in fact, an operation as $\wedge I$ is also

> the core of a semantic and ontological explanation. It serves to explain the content of the judgement. Its content is that a proposition $\alpha \wedge \beta$ is true. So to explain the content of the judgement is to explain what that proposition is. This is done by Prawitz in terms of the operation of conjunction-grounding: it is "a proposition such that a ground for judging it to be true is formed by bringing together two grounds for affirming the two propositions α and β". (Cozzo 2015, 110; la citazione interna è da Prawitz 2009, 194)

It must however be pointed out that, in relation to empirical, or more generally atomic, propositions or sentences, Prawitz does not go much beyond what we have already mentioned here - a ground for an empirical proposition or sentence is obtained by making an appropriate observation, while a ground for a numerical identity is, for example, obtained by performing a certain computation. This, one could argue, is testimony of the fact that the interest of the Swedish logician is primarily aimed at a determination of the notion of ground, and therefore, as we shall see, of inferential validity, for first-order languages, so that the determination of the meaning of the logical constants, after all one of the classic problems of contemporary mathematical logic and philosophy of logic, becomes a priority. However, we cannot fail to emphasize how the notion of ground on the empirical, and more generally atomic, case seems fraught with conceptual difficulties. We will not enter such a complex issue - the interested reader may refer to the bibliography dedicated to it (see, for example, Brîncuş 2015 and Usberti 2015).

Since the first articles dedicated to the theory of grounds – *Validity of inferences* (Prawitz 2013), *Inference and knowledge* (Prawitz 2009) and *The epistemic significance of valid inference* (Prawitz 2012a) – the notion of ground for propositions or sentences of different logical form is fixed by inductive clauses in which the transition from grounds from propositions or sentences of lower logical complexity to grounds for propositions or sentences of greater logical complexity occurs by virtue of primitive operations, which are thus constitutive of the meaning itself. However, in *Explaining deductive inference* this setting is still present, although it is significantly enriched by some important specifications:

> the grounds will be seen as abstract entities. As such we get to know them via descriptions. To form a ground for an assertion is thus to form a term that denotes the ground, and it is in this way that one comes in possession of the ground. Simultaneously with saying what grounds and operations on grounds there are, I shall therefore indicate a language in which the grounds can be denoted. [...] the grounds that come out of this enterprise will be like intuitionistic constructions and the language in which they will be described will be like the extended lambda calculus [che noi descriveremo più sotto]. The type structure will however be made more fine-grained by using sentences as types following Howard (1980), so that the question whether a term denotes a ground for an assertion of a sentence α coincides with the question of the type of the term. (Prawitz 2015, 89)

First of all, Prawitz takes up the basic idea of the *formulas-as-types conception*, referred to in Section 3.2.2: grounds are typed abstract objects or, in other words, a ground for $\vdash \alpha$ is an abstract object of type α. In addition, Prawitz now suggests the idea - apparently absent in the initial articles - to develop a *formal language of grounds*, of which the terms, also typed, *denote* grounds. The general lines along which such formal language is to be developed is determined by assuming in the first place to dispose of

> a first order language and with that individual terms and what counts as grounds for asserting atomic sentences. (Prawitz 2015, 90)

This first-order language, which we will call *background language*, is what the language of grounds, as it were, speaks of. Therefore, in the language of grounds we will have, in addition, of course, to the individual variables $x, y, ...$, ground-variables $\xi^{\alpha}, \xi^{\beta}, ...$ intended to range over grounds for α, β, and primitive operations related to each of the logical constants of the background language, indicated respectively with $\wedge I$, that we have already introduced above, $\vee I_1$, $\vee I_2$, $\rightarrow I$, $\forall I$, $\exists I$. Among the atomic formulas of the background language, Prawitz usually adds an atomic constant for the absurd $\bot$, and considers $\neg\alpha$ as an abbreviation of $\alpha \rightarrow \bot$.

If the typing of ground-variables is obvious - a ground-variable ξ^{α} obviously having type α - and, we would add, almost explicated by the notation, Prawitz is keen to specify that also

> the ground constants [...] are given with their types. The primitive operations are also to be understood as coming with types, although this may be left implicit. This is harmless, except for $\vee I_1$, $\vee I_2$ and $\exists I$ (Prawitz 2015, 90)

in which case, as will be easily understood in a moment, it becomes essential to know what is the result produced by the operation when applied to terms of the appropriate type. The typing of complex terms will therefore be determined in the following way:

($\wedge_{\tau}$) if T has type α and U has type β, $\wedge I(T, U)$ has type $\alpha \wedge \beta$

($\vee_{\tau}$) if T has type α_i, $\vee I_i[\alpha_i \rhd \alpha_1 \vee \alpha_2](T)$ has type $\alpha_1 \vee \alpha_2$ $(i = 1, 2)$

($\rightarrow_{\tau}$) if T has type β, $\rightarrow I\xi^{\alpha}(T)$ has type $\alpha \rightarrow \beta$

($\forall_{\tau}$) if T has type α, $\forall Ix(T)$ has type $\forall x\alpha$

($\exists_{\tau}$) if T has type $\alpha(t/x)$, $\exists I[\alpha(t/x) \rhd \exists x\alpha(x)](T)$ has type $\exists x\alpha(x)$

where it should be noted that, if in ($\vee_\tau$) the type of $\vee I_i$ were not specified, we could not know the type of the term that $\vee I_i$ produces, since we could not know the second element of the corresponding disjunction; similarly, if in ($\exists_\tau$) the type of $\exists I$ were not specified, we could not know the type of the term that $\exists I$ produces, since we could not know either what term of the starting formula is existentially quantified, or which individual variable is actually used for the existential quantification. Moreover, it should be noted that the primitive operation $\to I$ is applied by having it followed by a ground-variable, which indicates that this operation binds that ground-variable in the immediate subterm - and also allows the determination of the resulting type; in the same way, the primitive operation $\forall I$ is applied by having it followed by an individual variable, which indicates that this operation binds that individual variable in the immediate subterm - and also allows the determination of the resulting type. In the latter case, we have a binding similar to the one in force in the rule of introduction of universal quantification in a Gentzen system; x must not occur free in δ for ξ^δ free in the immediate subterm.

The determination of the typing provides a useful first indication of the terms that, within the formal language under examination, will denote the grounds. However, we can do more, namely

> we may also say directly what grounds there are for different sentence forms. (Prawitz 2015, 91)

The indication of the grounds for propositions or sentences of different logical form is entrusted, as mentioned above, to inductive clauses that involve primitive operations $\wedge I$, $\vee I_1$, $\vee I_2$, $\to I$, $\forall I$, $\exists I$. Established that the individual ground constants available in the language of grounds denote grounds for the atomic formulas that constitute their type - which, as discussed below, is equivalent to relativizing on atomic bases the notion of denotation, and consequently, as we will see, the notion of inferential validity - and that there are no grounds for $\bot$, in the case of $\wedge$, $\vee$ and $\exists$, given α, β, $\exists x\alpha(x)$ closed, and t closed, the clauses will be

- if T denotes a ground for $\vdash \alpha$ and U denotes a ground for $\vdash \beta$, $\wedge I(T, U)$ denotes a ground for $\vdash \alpha \wedge \beta$
- if T denotes a ground for $\vdash \alpha_i$, $\vee I_i[\alpha_i \rhd \alpha_1 \vee \alpha_2](T)$ denotes a ground for $\vdash \alpha_1 \vee \alpha_2$ $(i = 1, 2)$
- if T denotes a ground for $\vdash \alpha(t/x)$, $\exists I[\alpha(t/x) \rhd \exists x\alpha(x)](T)$ denotes a ground for $\vdash \exists x\alpha(x)$

Prawitz introduces this first set of clauses separately from those for $\rightarrow$ and $\forall$, and this because, in the latter case, it is necessary to generalize the discussion on the operations on grounds, going beyond the primitive operations introduced so far.

4.1.3.2 Operations on grounds and open languages

When, in Section 2.5.2.2, we presented the BHK semantics for the first-order logical constants, we also observed how the clauses related to $\rightarrow$ and $\forall$ require a primitive notion of constructive procedure; a BHK proof of $\alpha \rightarrow \beta$ is a constructive procedure that transforms BHK proofs of α in BHK proofs of β, while a BHK proof of $\forall x \alpha(x)$ is a constructive procedure that, for each individual k, produces a BHK proof of $\alpha(k)$. In the theory of grounds the discourse is very similar: a ground for $\vdash \alpha \rightarrow \beta$ will involve a constructive procedure that turns grounds for $\vdash \alpha$ into grounds for $\vdash \beta$, whereas a ground for $\vdash \forall x \alpha(x)$ will involve a constructive procedure which, given an individual k, produces a ground for $\vdash \alpha(k)$. However, how can we understand now the constructive procedures which the theory of grounds refers to?

The only operations which we have dealt with so far are primitive operations for the specification of meaning. When applied to grounds of the appropriate type, primitive operations produce, by definition, grounds of an appropriate type. In this regard, however, Prawitz observes that

> this simple way of getting grounds [...] does not go very far, of course, and in general, one has to define new operations to this end. (Prawitz 2015, 92)

The content of the above quote can be illustrated with a simple example. Given grounds g_1 for $\vdash \alpha_1$ and g_2 for $\vdash \alpha_2$, we can apply to them the primitive operation $\wedge I$ obtaining, by virtue of the clause that fixes what counts as ground for a conjunction, a ground $\wedge I(g_1, g_2)$ for $\vdash \alpha_1 \wedge \alpha_2$. Conversely, given a ground $\wedge I(g_1, g_2)$ for $\vdash \alpha_1 \wedge \alpha_2$, with g_1 ground for α_1 and, more specifically, g_2 ground for $\vdash \alpha_2$, it seems reasonable to affirm that we can apply to $\wedge I(g_1, g_2)$ a "projection" operation, which selects g_2 and thereby returns a ground for $\vdash \alpha_2$. This "projection" is undoubtedly constructive, but it is equally clear that it cannot be any of the primitive operations; primitive operations, in fact, only authorize passages from grounds for propositions or sentences of lower logical complexity to grounds for propositions or sentences of greater logical complexity.

However, given the possibility of contemplating non-primitive operations, it is obvious that the production of grounds by means of them will have to agree with the way the notion of ground is determined. More in detail, a non-primitive operation must be fixed by equations that establish its behavior with regard to the already acquired primitive operations - and therefore, not unlike it happened in proof-theoretic semantics, in compliance with the principle of harmony inspired by the inversion principle (Prawitz 2006). Therefore, with this aim in view, Prawitz states that

> for any closed sentence $\alpha_1 \wedge \alpha_2$ we can define two operations, which I call $\wedge_{E,1}$ and $\wedge_{E,2}$, both of which are to have as domain grounds for $\alpha_1 \wedge \alpha_2$. The intention is that the operation $\wedge_{E,i}$ is always to produce grounds for α_i $(i = 1, 2)$ when applied to grounds for $\alpha_1 \wedge \alpha_2$ [...]. This is attained by letting the two operations be defined by the equations
>
> $$\wedge_{E,1}(\wedge I(g_1, g_2)) = g_1 \text{ and } \wedge_{E,2}(\wedge I(g_1, g_2)) = g_2.$$
>
> The fact the the operation $\wedge_{E,i}$ produces a ground for α_i when applied to a ground g for $\alpha_1 \wedge \alpha_2$ [...] *depends* on what $\wedge$ means, and has to be established by an argument. (Prawitz 2015, 92)

The argument to which Prawitz refers is rather easy - our reconstruction will temporarily disregard certain identity conditions on the primitive operations: given a ground g for $\vdash \alpha_1 \wedge \alpha_2$, by virtue of the clause fixing what counts as ground for a conjunction, g must be of the form $\wedge I(g_1, g_2)$, with g_i ground for $\vdash \alpha_i$; by virtue of the equations that define $\wedge_{E,i}$ i, we will then have $\wedge_{E,i}(g) = \wedge_{E,i}(\wedge I(g_1, g_2)) = g_i$; therefore, for every g ground for $\vdash \alpha_1 \wedge \alpha_2$, $\wedge_{E,i}(g)$ produces a ground for $\vdash \alpha_i$ $(i = 1, 2)$. Other examples of non-primitive operations, provided by Prawitz himself, are: a binary operation $\to E$ for the elimination of implication that produces grounds for $\vdash \beta$ when applied to grounds for $\vdash \alpha \to \beta$ and for $\vdash \alpha$, fixed by the equation

$$\to E(\to I\xi^{\alpha}(T(\xi^{\alpha})), U) = T(U);$$

a binary operation **Barb** for the syllogism in *Barbara*, that produces grounds for $\vdash \forall x(P(x) \to R(x))$ when applied to grounds for $\vdash \forall x(P(x) \to Q(x))$ and for $\vdash \forall x(Q(x) \to R(x))$, fixed by the equation

$$\mathbf{Barb}(\forall Ix(T), \forall Ix(U)) = \forall Ix(\to I\xi^{P(x)}(\to E(U, \to E(T, \xi^{P(x)}))));$$

a binary operation **Mtp** for disjunctive syllogism, that produces grounds for $\vdash \beta$ when applied to grounds for $\vdash \alpha \vee \beta$ and for $\vdash \neg\alpha$, fixed by the equation

$$\mathbf{Mtp}(\vee I_2(g_1), g_2) = g_1.$$

Obviously, in order to give a precise definition of the notion of grounds for implications or universal quantifications, it is necessary to pass from examples of specific operations (be they primitive or non-primitive) to a discussion of a more general nature. In this perspective, a first significant observation concerns the distinction between grounds, understood as reifications of mental states in which we are in possession of a conclusive justification for categorical judgments or assertions, and operations on grounds; the latter, producing grounds of the appropriate type when applied to grounds of appropriate type or, as we will see below, to individuals, can be understood as grounds for hypothetical-open judgments or assertions. It then becomes important to ask oneself how an operation can be identified, and here we can with Prawitz affirm that

> an operation is given by stating the types of its domain and range and, for each argument in the domain, the value it produces for that argument. (Prawitz 2015, 92)

Thus, for example, the primitive operation $\wedge I$ has for domain grounds for $\vdash \alpha$ and grounds for $\vdash \beta$ and as a codomain grounds for $\vdash \alpha \wedge \beta$, which can be also expressed by saying that $\wedge I$ has *operational type*

$$\alpha, \beta \rhd \alpha \wedge \beta.$$

Given a ground g_1 for $\vdash \alpha$ and a ground g_2 for $\vdash \beta$, $\wedge I(g_1, g_2)$ is a ground for $\vdash \alpha \wedge \beta$; of course, being a primitive operation, the production of grounds takes place, as it were, automatically, so that there is no need – nor would it be possible – to further specify the behavior of such operation. The non-primitive operation $\wedge_{E,i}$ will have operational type

$$\alpha_1 \wedge \alpha_2 \rhd \alpha_i$$

($i = 1, 2$). This is guaranteed by the defining equations for $\wedge_{E,i}$ illustrated above. Again by virtue of the corresponding defining equations, we can say that the non-primitive operations $\rightarrow E$, **Barb** and **Mtp** have, respectively, operational types

$$\alpha \rightarrow \beta, \alpha \rhd \beta$$

$$\forall x(P(x) \rightarrow Q(x)), \forall x(Q(x) \rightarrow R(x)) \rhd \forall x(P(x) \rightarrow R(x))$$

$$\alpha \vee \beta, \neg\alpha \rhd \beta$$

As we can easily see, the operational types associated with operations on grounds and that specify their domain and codomain are the same ones that, in the language of grounds, accompany - although generally left implicit by Prawitz - the corresponding operational symbols. Thus, the (primitive) *operation* $\vee I_i$ will have operational type

$$\alpha_i \rhd \alpha_1 \vee \alpha_2$$

and such will be the type of the *operational symbol* $\vee I_i$ $(i = 1, 2)$; in the same way, the (primitive) *operation* $\exists I$ will have operational type

$$\alpha(t/x) \rhd \exists x \alpha(x)$$

and such will be the type of the *operational symbol* $\exists I$. We will resume later on in more detail the discussion on operational types; what we have so far said is sufficient to understand the rest of this section.

The specification of operations on grounds of different nature is by Prawitz conducted by degrees. First of all, the Swedish logician says that an operation is of operational type

> $\alpha_1, ..., \alpha_n \rhd \beta$, where $\alpha_1, ..., \alpha_n, \beta$ are closed sentences, if it is an n-ary effective operation that is defined whenever its ith argument place is filled with a ground of type α_i and then always produces a ground of type β. Such an operation is a ground for the hypothetical assertion $\alpha_1, ..., \alpha_n \vdash \beta$. (Prawitz 2015, 92)

The limitation to closed formulas in the passage above-mentioned is only a workaround aimed at facilitating the understanding, so much so that shortly afterwards Prawitz states that a ground for $\alpha(x_1, ..., x_m)$

> is an effective m-ary operation defined for individual terms that always produces a ground for asserting $\alpha(t_1/x_1, ..., t_m/x_m)$ when applied to $t_1, ..., t_m$. (Prawitz 2015, 92)

As an obvious generalization, we will have that, if an operational type

$$\alpha_1, ..., \alpha_n \rhd \beta$$

involves both open and closed formulas, an operation of this operational type will be

> an n-ary effective operation from operations to operations [...]. It is again a ground for $\alpha_1, ..., \alpha_n \vdash \beta$. (Prawitz 2015, 92)

The last and most complex case discussed by Prawitz is that of an operation of operational type

$$(\Gamma_1 \rhd \alpha_1, ..., \Gamma_n \rhd \alpha_n) \rhd (\Delta \rhd \beta)$$

where Γ_i, Δ are sets of (open or closed) formulas, with $\Delta \subseteq \bigcup_{i \leq n} \Gamma_i$, and that will be

> an n-ary effective operation from operations of the type indicated before the main arrow to operations of the type indicated after the main arrow. (Prawitz 2015, 93)

As a whole, therefore, the theory of grounds uses, like BHK semantics, a primitive notion of constructive procedure; what Prawitz significantly adds, however, is a specification of the different operational types to which these procedures are traceable, so as to identify some of them, different, and of different complexity. However, it is important to remember how, unlike what happens in *Explaining deductive inference* (Prawitz 2015), in *Inference and knowledge* (Prawitz 2009) and in *The epistemic signifance of valid inference* (Prawitz 2012a) Prawitz does not refer to constructive procedures but, in a more Fregean sense (Frege 1891, 2001, but see also Kenny 2003, Tranchini 2018), to *unsaturated* grounds - and, at the same time, *saturated* grounds - for which it is intended a substitutional approach of ground-variables with saturated grounds, and of individual variables with closed terms:

> an unsaturated ground is like a function and is given with a number of open argument places that have to be filled in or saturated by closed grounds so as to become a closed ground. Something is a ground for an assertion β under the assumptions $\alpha_1, ..., \alpha_n$ if and only if it is an n-ary unsaturated ground that becomes a closed ground for β when saturated by closed grounds for $\alpha_1, ..., \alpha_n$. [...] We must [...] consider unsaturated grounds that are unsaturated not only with respect to grounds but also with respect to individuals that can appear as arguments in propositional functions. (Prawitz 2009, 193)

> a *ground for an assertion β under assumptions* $\alpha_1, ..., \alpha_n$ is to be an unsaturated ground which when saturated by grounds for the assertions $\alpha_1, ..., \alpha_n$ becomes a ground for the assertion β. For terms that stand for grounds there is a corresponding distinction between *closed* and *open* terms representing saturated

> and unsaturated grounds [...]. A ground for [...] an open assertion, say $\alpha(x_1, ..., x_n)$, is again an unsaturated ground, say $f(x_1, ..., x_n)$, such that for individuals $a_1, ..., a_n$ in the domain in question, $f(a_1, ..., a_n)$ is a ground for the assertion $\alpha(t_1, ..., t_n)$, where t_i denotes a_i. (Prawitz 2012a, 893 - 894)

Once clarified the nature of the operations involved in the theory of grounds, it becomes particularly easy to provide the clauses that fix what counts as ground for formulas having $\rightarrow$ and $\forall$ as main logical constant. Given α, β and $\forall x\alpha(x)$ closed,

- if T denotes an operation on grounds of operational type $\alpha \rhd \beta$, or equivalently a ground for $\alpha \vdash \beta$, $\rightarrow I\xi^{\alpha}(T)$ denotes a ground for $\vdash \alpha \rightarrow \beta$
- if T denotes an operation on grounds of operational type $\alpha(x)$, or equivalently a ground for $\vdash \alpha(x)$, $\forall Ix(T)$ denotes a ground for $\vdash \forall x\alpha(x)$

As we can easily see, the clauses for $\rightarrow$ and $\forall$ place no limitation to the operations that can be denoted by T in $\rightarrow I\xi^{\alpha}(T)$ or $\forall Ix(T)$; such operations can be primitive and non-primitive - or, more frequently, result from the composition of primitive and non-primitive operations - as well as responding to operational types of whatever complexity. This leads, quite naturally, to two questions. Why should not we limit ourselves only to primitive operations? And also, and regardless of how the previous question is answered, is it possible that the class of operations which the theory of grounds refers to is captured by a closed language of grounds? (Where by closed we mean a language equipped with a finite, or finitely axiomatizable, number of operational symbols for operations from which all the others can be generated, or, better, in terms of which all the others can be defined).

As regards the first question, the need to take into account, together with the primitive operations, also the non-primitive operations is noticeable, we could say, *de facto* and *de iure*. Anticipating what we will say very soon, we understand the primitive operations as those that are applied in inferential passages in introductory form, and the non-primitive operations as those that are applied in non-introductory inferential passages. Now, it is a commonly experienced fact that inferences in non-introductory form are often performed in deductive practice. If we limited ourselves only to primitive operations, therefore, we could not account for this aspect of deductive practice *as it is*. On the other hand, a theory that wants to be able to answer the question about the power of epistemic compulsion of deductively valid inferences, cannot omit to distinguish, in the words of Dummett, between

> the *conditions* for the utterance and the *consequences* of it; (Dummett 1975, 11)

both these poles illuminate different, essential and yet complementary roles that judgments and assertions play in every correct reasoning. However, the primitive operations, when applied in inferential passages in introductory form, capture only the conditions under which judgments and assertions can be reasonably accomplished, whereas, instead, they are the non-primitive ones, when applied in non-introductory inferential passages, to capture the correct consequences of judgments and assertions. If we limited ourselves only to primitive operations, therefore, we could not account for this aspect of deductive practice *as it has to be*.

As for the second question, it is Prawitz himself who suggests a negative answer, when he says that

> the language of grounds is to be understood as open in the sense that symbols for defined operations of a specific type can always be added (Prawitz 2015, 92)

– however, it should be noted that, in the passage under examination, the Swedish logician takes into account only the introduction of new non-primitive operations, defined in the terms of the primitive ones; but nothing prevents, and indeed, as we will soon understand, Prawitz himself seems to suggest the possibility of adding new primitive operations related to new logical constants. The fact that a formal language of grounds should be understood as open in the specified sense, obviously depends on what we have just said regarding the need to take into account operations both primitive and non-primitive; as a matter of principle, the class of valid inferences in non-introductory form - which, as we have said above, and anticipating what we will say shortly, is to be understood as that where non-primitive operations are applied - can be generated by no finitely axiomatizable formal system (for logically valid inferences instead it is different, because, if we accept a reformulation in the theory of grounds of Prawitz's conjecture, which we will discuss in Chapter 7, they may be derivable in first-order intuitionistic logic). Next to this, however, there is another reason by virtue of which a formal language of grounds cannot but be open - in this case, to the addition of primitive operations; according to Gödel's incompleteness theorems, in fact, no closed language of grounds, at least as powerful as a formal system for intuititionistic first-order arithmetic, can express, through its terms, all the possible grounds on the background reference language. Although inserted in a different context, which we will deal with below, this observation is carried out by Prawitz:

> we know because of Gödel's incompleteness result that already for first order arithmetical assertions there is no closed language of grounds in which all grounds for them can be defined; for any such intuitively acceptable closed language of grounds, we can find an assertion and a ground for it that we find intuitively acceptable but that cannot be expressed within that language. (Prawitz 2015, 98)

With these observations, we conclude our discussion of the notion of ground as it appears in Prawitz's writings. To them, we will add only a discussion about what it means to be in *possession of grounds*, a point that will be of crucial importance in some of the issues we will face later.

4.1.3.3 Possession of grounds

Grounds are abstract entities. Since they are what we have when we are in a mental state of justification for judgments or assertions, it seems furthermore reasonable to expect that they can have an *epistemic* character. Last but not least, as a result of the fulfillment of certain acts, they can be *constructed* by appropriate agents. Hence, on the basis on this information, the question we now want to ask ourselves is: in what does the possession of a ground consist?

Following an interpretative line outlined by Cozzo (Cozzo 2015), and in part also suggested by Usberti (Usberti 2015), grounds are nothing more than objects that the philosopher introduces to *describe*, reifying them, the mental states of justification; expressing oneself in terms of being in possession of grounds is only a formally convenient way of *representing* the occupation of the mental states of justification. In particular, Cozzo draws our attention to the need to distinguish

> the three levels of the picture outlined by Prawitz: a mental level, a linguistic level and a mathematical level. [...] to the mental level belong acts of judgements and other mental epistemic acts like observation, calculation, reasoning. [...] The genuine active experience of making an inference also belongs to this level: it is the conscious act of moving mentally from judgements-premises to a judgement-conclusion in such a way that the agent cannot help but recognize that this movement leads the mind from correct premises to a correct conclusion. [...] to the linguistic level belong acts of assertions, and linguistic practices of argumentation in support of assertions [...] to the mathematical level belong

> mathematical representations of the mental level in terms of grounds and operations on grounds. This is the level of abstract entities. The logician resorts to abstract entities in order to construct a theory that makes the mental phenomenon of deduction intelligible. (Cozzo 2015, 108)

This point of view has the very immediate effect of clarifying what we already have pointed out in Section 4.1.3.1: of the possession of grounds, through which the acquisition of evidence is manifested, there is generally no clarification in the linguistic practice. The agent keeps, according to Prawitz's words, the ground for him/herself, since the objects that the theory introduces are something of which he/she is not necessarily aware. An evidence of this would be, in a sense, the fact that

> when an assertion is justified by way of an inference, it is common to indicate this by simply stating the inference [...] and the premisses of the inferences are then often called the ground for the assertion. This way of speaking may be acceptable in an everyday context, but it conceals the problem that we are dealing with, which is probably one reason why the problem has been so neglected. (Prawitz 2009, 191)

As further support, and referring to what was said in Section 4.1.2, we recall that the simple announcing an inferential transition prevents the fulfillment of the "fundamental task". To this end, Prawitz warns, it would not do to simply indicate either the premises, or the fact that they are (possibly) true, or the grounds that (possibly) justify the corresponding judgment or assertion. About the insufficiency of the simple reference to the truth we have already argued again in Section 4.1.2; for the rest, the Swedish logician argues significantly that

> the premisses are judgements or assertions affirming propositions, and the fact that one has judged or asserted them as true cannot constitute a ground for the conclusion, nor can the truth of the propositions affirmed constitute such a ground. [...] It is rather the fact, if it is a fact, that the agent has *grounds* for the premisses that is relevant for her having a ground for the assertion made in the conclusion. But the grounds for the premisses are grounds for *them*, not for the conclusion. (Prawitz 2009, 191)

In conclusion, it seems fair to say that, according to Prawitz's intention, to speak of grounds and of the possession of grounds is not a

> realistic description of the mental act, but is suggested as a theoretical reconstruction of what goes on when we pass from one evidence state to another. (Prawitz 2015, 89)

On the other hand, the Swedish logician often expresses himself in terms of having constructed, and of being mentally in possession of something on the basis of which to have justification. It would be precisely for this circumstance that the possession of grounds may be made explicit - and, in addition, that the grounds may be "named". For example:

> to be in possession of a ground [...] means basically to have made a certain construction in the mind of which the agent is aware, and which she can manifest by naming the construction. (Prawitz 2009, 199)

Put it like that, it becomes possible, and in our opinion fruitfully, to connect the notion of grounds, and in particular the notion of possession of grounds, to the BHK tradition. To do this, we cannot fail to take into account a very recent article, entitled *The seeming interdependence between the concepts of valid inference and proof*, in which Prawitz asserts that

> in the case of having a ground for the assertion of an arithmetical identity because of having made the relevant calculation, one may have recorded the steps of the calculation, and one has then a ground for the assertion also in the concrete sense of a protocol of the justifying act open for inspection. The process by which one finds a construction g for a proposition α can be recorded too. The resulting protocol can be seen as a description of not only the process but also of the obtained object g. (Prawitz 2018b, 5 - 6)

The idea of possession is accompanied, here, by another important suggestion. The system of acts through which a ground is acquired - namely, according to what said in the previous section, the set of operations performed having such ground as outcome - can be recorded; the resulting protocol can be seen, not only as an encoding of that process, but also as a description of the ground itself. Now, these observations are inserted by Prawitz in the context of a discussion and a dialogue with the intuitionistic tradition, and, in particular, as it appears in Heyting's writings. As already anticipated in Section 3.1, Heyting's notion of proof is linked to his way of understanding propositions and assertions. The former express the intention to find a construction, the latter the realization of the intention expressed by the asserted proposition. In this framework, a proof of a proposition would consist in the process of realization of the construction. Also in Section 3.1, we have argued that the use made by Heyting of the term "proof" fluctuates essentially between construction and realization of this construction; in the article we took in account, Prawitz says that

> the term "construction" occurs frequently in Heyting's writing, and [...] one should distinguish between different senses of that word, among them "process of construction" and "object obtained as the result of a process of construction". Heyting uses the term in at least both these two senses, but I think that it is usually clear which one is intended. When he explains what a proof is it seems clear that he has in mind a construction in the sense of a process: to realize the intention expressed by a proposition α is to perform a construction act, which may proceed in steps [...] and which results in the construction of a mathematical object, namely the construction intended by α (Prawitz 2018b, 4; la citazione interna è da Sundholm 1983, 164)

Therefore, *The seeming interdependence between the concepts of valid inference and proof* - as indeed some other works of recent years (see, for example, Prawitz 2012b, 2018a) - looks with renewed interest at the BHK semantics; it suggests, more specifically, that the notion of ground is, in a sense, inspired by it, or, more weakly, that the two approaches look rather and relevantly similar, and that they should be connected to each other in some way - as indeed already suggested by Tranchini (Tranchini 2014a). Prawitz concentrates more directly on the possible circularity in an attempt to define valid inferences and proofs, so as to capture their epistemic power – something that we have already anticipated in the introduction to this section:

> a definition of proof in terms of valid inference requires that the latter concept be explained in terms of evidence or related notions such as ground, justification, or knowledge; since a proof is understood in terms of such epistemic concepts, a valid inference must also be related to these concepts if it is to serve in an explanation of what a proof is. [...] However, if we first explain the concept of proof saying that a proof is a chain of valid inferences, and then explain the validity of an inference by referring among other things to proofs [...] we are moving in a plain circle. (Prawitz 2018b, 2)

Beyond the difficulties inherent in a definition of valid inferences in terms of proofs - about which we have spoken at length in Section 2.5.3 - Prawitz seems to believe that such an approach is already misleading in itself, on account of the inversion of the natural cartesian idea of proofs as chains of valid inferences:

> there may be in principle two ways of breaking up the apparent interdependency between the concepts of proof and valid inference, thereby making explanatory progess with respect to the two concepts: either explaining the validity of inferences without referring to proofs or explaining the concept of proof without referring to the validity of inferences. The second alternative is in my view to put the natural conceptual order upside down. So the first alternative seems to me preferable. (Prawitz 2018b, 2)

From this point of view, Heyting's proposal would be of such importance as to induce the Swedish logician to argue that

> neither Heyting nor his intuitionistic successors explain the concept of proof that I am discussing here but that his notion of construction may be developed so that it gives us what we are looking for, namely a concept of justification or ground not based on the concept of proof. (Prawitz 2018b, 4)

There is clearly a distance between Heyting and Prawitz. For Heyting, both the notion of construction, and that of proof as realization of the intended construction, are essentially non-inferential, which is in clear disagreement with the objective Prawitz aims at with his theory of grounds. Yet, this distance *itself* is revealed, after all, as a winning feature. In the light of the perceived need for a notion of deductively valid inference that, on the one hand, is fixed in terms of evidence and, on the other, allows to define the notion of proof, as well as by virtue of the obvious circularity that would result from the understanding of evidence in terms of proofs, Heyting's constructions offer an apt way of access to the notion of evidence:

> his view does not lead to a general account of the concept of proof, but it offers an explanation of what it is to justify the assertion of a proposition without using the concept of proof. (Prawitz 2018b, 5)

Obviously, the underlying assumption is that the possession of a construction *à la* Heyting corresponds to the possession of evidence for α, namely, to be in a state of justification to judge the proposition α as true, or, equivalently, to assert the sentence α; however, Prawitz himself explicitly affirms that it is not at all unquestionable that it is like this. The analysis of this point, that we will conduct further in the course of our discussion, will involve a formulation of the recognizability problem already raised, in Section 2.5, for valid arguments and proofs in the framework of proof-theoretic semantics.

The constructions about which Heyting speaks, later converged into the clauses of BHK semantics, have been understood in various ways; most of the time, however, as terms of a typed λ-calculus or, with slight modifications, of Kreisel-Goodman theory of constructions or of Martin-Löf's intuitionistic type theory. On the other hand, the Curry-Howard isomorphism seems to substantiate what Prawitz says on the protocols of demonstrative acts that, in addition, denote the result of such acts. It goes without saying, however, that the idea that the possession of grounds is equivalent to the actual possession of a construction does not necessarily imply that what one is in possession of is the term of a typed λ-calculus, of Kreisel-Goodman theory of constructions, or of Martin-Löf's intuitionistic type theory. The vast majority of agents engaged in proof activities, including "professional" mathematicians, is not acquainted with these theoretical setups. Nevertheless, a ground for a proposition or sentence α is an object that results from the reification of an act through which we enter a state of evidence to judge the proposition α true, or to assert the sentence α, an act that is also constitutive of the meaning of α. Proof agents know the meaning of the propositions and of the sentences used, and are aware of the acts performed in deductive practice; as a consequence of that, they can make explicit the grounds as we have understood them above, in a certain form and, possibly, within a more or less detailed theoretical framework. This is what Prawitz says about his way of writing the operation underlying the acquisition of grounds through mathematical induction - indicated with `Ind`:

> an ordinary mathematician would of course not formulate a ground for the conclusion of an induction inference explicitly in the form of `Ind`. But if she reflects on the inference, she may very well think that she can form a ground for the conclusion from the grounds for the premisses, namely by taking the ground for the induction base and successively applying the ground for the induction step as many times as needed. `Ind` simply represents such a mental operation. (Prawitz 2012a, 897)

Reflecting on what has been done, the agent can then realize that he/she has accomplished certain operations, and possibly describe them.[3]

4.1.4 Inference acts and validity

As repeatedly pointed out in the course of this investigation, the fulfilling of the "fundamental task" - set out in Section 1.3, and repeated in Section 4.1.2 – presupposes two steps: identification of an adequate relation between agents and inferences and, obviously, an appropriate definition of the notion of valid inference. Again in Section 4.1.2, we then remarked how Prawitz, after having detected the insufficiency of different options, maintains that the most appropriate relationship between an agent A and an inference I is "A performs I", where it remains to be determined what is meant by "performing an inference".

To perform an inference cannot simply mean announcing it, i.e., to establish its premises, to say "therefore" or other equivalent expressions, and to state their conclusion. This reductionist view, according to which inferences are mere transitions from judgments or assertions to other judgments or other assertions, exposes itself to a clear criticism even when the inferences in question are really deductively valid:

> an argument that happens to be valid but where, without further arguments, the conclusion cannot be seen to follow from the premisses is not considered to constitute a piece of correct reasoning; it is rather viewed as a gap in reasoning until it has been supplemented. (Prawitz 2012a, 890)

Inspired by the inferences in which operations such as discharge of assumptions or binding of variables are performed, Prawitz then proposes to conceive inferential acts as something that involves more than mere linguistic acts, or transitions:

> to infer a conclusion α form a set Γ of premisses may be experienced not just as making the assertion α giving a set Γ of

[3] We should probably anticipate that `Ind` is not, for Prawitz, a primitive operation; apparently, this could contradict what we have argued about the grounds as a reification of acts constitutive of the meaning. Indeed, a term constructed through `Ind` denotes a ground to the extent that it can be transformed into, or rather reduced to, a term which begins with a primitive operation, and which denotes a ground only by virtue of the meaning of the proposition or sentence considered. Thus, grounds must be distinguished from the terms that describe them; they are denoted directly by a certain class of terms, which those that, instead, indirectly denote must reduce to. The description of a ground, which will be more specifically the description of the act performed to obtain it, will therefore be subject to the distinction between direct and indirect denotation.

> premisses as one's reason but rather as *seeing* that the proposition asserted by α is true given that the propositions asserted by the premisses of Γ are true. (Prawitz 2012a, 890)

Thus the idea is that of generalizing the completion of operations to all the inferential acts, and this because

> to characterize correct reasoning we may need to give substance to this metaphorical use of seeing. (Prawitz 2012a, 890)

Seeing is often regarded as something passive; but in a passage already mentioned by us, Cozzo (Cozzo 2015) points out that to see that a certain conclusion follows from certain premises is an *active experience* of necessity of thought, experienced simultaneously with the performance of the valid inferential act. In making a valid inference, the agent *does* something; through this act, he/she has the opportunity to consciously experience the epistemic support that the premisses guarantee for the conclusion. The discussion we have conducted in Section 4.1.3 now allows us to specify more precisely *what* the agent does, and under what conditions the inferential act thus understood can be said *valid*.

Since *Validity of inferences*, and then gradually in all the articles on the theory of grounds, Prawitz has proposed a new conception of inferential act, intended as an application of one of the above described operations on grounds. In particular, in *Explaining deductive inference* he says that to perform an inference means

> in addition to making an inferential transition, to apply an operation to the grounds that one considers oneself to have for the premisses with the intention to get thereby a ground for the conclusion. (Prawitz 2015, 94)

This general characterization is accompanied by further specifications, since, as mentioned in Section 4.1.1, an act of inference can be understood at different levels of abstraction:

> I take an *individual* or *generic inference* to be individuated by what individuates an individual or generic inferential transition from premisses $\alpha_1, ..., \alpha_n$ to a conclusion β, and, in addition, by alleged grounds $g_1, ..., g_n$ for the premisses and an operation ϕ. Conforming to the usual terminology according to which an inference may be unsuccessful, no requirement is put on the alleged grounds and the operation; in other words, $g_1, ..., g_n$ may be any

kind of entities, and ϕ may be any kind of operation. [...] Similarly, an *inference form* is individuated by the form of an inference transition and an operation. (Prawitz 2015, 94)[4]

Individual inferential acts had been described by Prawitz at the outset, as including specific premisses and conclusions, plus an agent and a space-time situation in which the agent makes the inference. The generic inferential acts were instead an abstraction from the agent and from the occasion, whereas inferential forms added a further abstraction from specific premisses and conclusions, taking into account only parameters for the latter. It is therefore natural to take into account a further level, of which we find trace in *Validity of inferences*:

> finally, we may also abstract from the operation left in an inference form, and may then speak about an *inference figure* or *schema*. (Prawitz 2013, 198)

The understanding of inferential acts, at the different levels of abstraction, as applications of operations on the grounds that we believe to have for the

[4]The fact that the grounds in an individual or generic inferential act are only alleged, and that there is no restriction on such alleged grounds or on the operation involved in the act, is an important change that Prawitz makes in *Explaining deductive inference*, compared to the framework proposed in the previous articles. It is essentially due to four objections raised by Cozzo to the original setting, in which Prawitz required that an inferential act involved grounds for the premises – hence, actual grounds – and left open the possibility that the operations involved in these acts were always such as to produce grounds from grounds: 1) only valid inferential acts are inferences, while "it seems reasonable to say that our activity includes acts of inference that can be valid or invalid" (Cozzo 2015, 113); 2) inferential acts for which the agent has no grounds for the premises are not inferential acts, while "it seems reasonable to say that our deductive activity includes inferences (valid or invalid) with mistaken premises" (Cozzo 2015, 114); 3) the theory of grounds would not be able to distinguish invalid inferential acts from valid inferential acts with unjustified premisses, while "it seems reasonable to say that an inference can be valid even if its premises are mistaken" (Cozzo 2015, 114); 4) Prawitz would not be able to account for the power of epistemic compulsion experienced in valid inferential acts with unjustified premises, while "it seems reasonable to say that the experience of necessity of thought also characterizes the transition from mistaken premises" (Cozzo 2015, 114). In light of the new definition, however, Prawitz argues that "an individual or generic inference can err in two ways: the alleged grounds for the premisses may not be such grounds, or the operation may not produce a ground for the conclusion" (Prawitz 2015, 95). However, Usberti argues that Cozzo's fourth objection remains to be settled, since "an assertion based on an entity of any kind may be true or false, but it is difficult to see how he can be rational at all" (Usberti 2017, 11). As a solution to the problems he has identified, Cozzo introduces the interesting notion of *ground-candidate*, "a mathematical representation of the results of epistemic acts underlying mistaken premises"; a ground-candidate "can be a genuine ground or a pseudo-ground" (Cozzo 2015, 114).

premises, suggests at this point a natural definition of the notion of valid inference:

> an individual or generic inference [...] is (*deductively*) *valid*, if $g_1, ..., g_n$ are grounds for $\alpha_1, ..., \alpha_n$ and ϕ is an operation such that $\phi(\alpha_1, ..., \alpha_n)$ is a ground for the conclusion β. [...] [An inference form] is defined as *valid* if the operation when applied to grounds for the premisses produces a ground for the conclusion. [...] for each instance of the form, the operation is of a specific type and is denoted by the corresponding instance of the term. An *inference schema* is defined as valid when there is an operation such that the inference schema together with that operation is a valid inference form. (Prawitz 2015, 94 - 95)

Analogously, focusing on the graphic representations of figures and inferential schemes, given a figure

$$\frac{\Gamma_1 \vdash \alpha_1 \qquad \dots \qquad \Gamma_n \vdash \alpha_n}{\Delta \vdash \beta}$$

the operation it involves – which, if we want, can be indicated in the figure itself – must be, in order that the represented transition be valid,

> an operation of type $(\Gamma_1 \rhd \alpha_1, ..., \Gamma_n \rhd \alpha_n) \rhd (\Delta \rhd \beta)$. [...] the ground for the assertion β under the assumptions Δ, produced when the operation ϕ is applied to grounds for the premisses, can then be denoted by a ground term. (Prawitz 2015, 94 - 95)

We have thus outlined the two remaining factors for the fulfillment of the repeatedly mentioned "fundamental task". Given an individual or generic inferential act I, the relation between an agent A and I will be "A performs I". By virtue of the reconstruction that Prawitz offers of the notion of individual or generic inferential act, I will involve the alleged grounds $g_1, ..., g_n$ for the premisses, and an operation ϕ applicable to $g_1, ..., g_n$. Therefore, the fact that A performs I means for A to apply ϕ to $g_1, ..., g_n$, which can be indicated by $\phi(g_1, ..., g_n)$. If we make the further hypothesis that I is a valid individual or generic inferential act, according to the definition of validity just given $g_1, ..., g_n$ are grounds for the premisses of I, and ϕ is an operation that, when applied to the grounds for the premisses of I, produces a ground for the conclusion of I. Hence, performing I, that is computing, as it were, $\phi(g_1, ..., g_n)$, A obtains the ground searched for. The "fundamental task" seems to be fulfilled, but at this point it becomes important to conduct an observation.

As stated in Chapter 1 of this work, the "fundamental task" is introduced by Prawitz in order to offer a clear conceptual grid in which to articulate in a rigorous and precise manner the underlying issue that the theory of grounds intends to address and resolve: to explain how deductively valid inferences are able to force epistemically to accept the conclusion, given an epistemic acceptance of the premises, namely how such inferences can justify the conclusion provided that the premises are justified. The "fundamental task", on the other hand, is formulated in terms of (possession of) grounds, and thus presupposes that to possess grounds means to have an epistemic compulsion, to be in a state of justification. The notion of ground is then further specified according to the lines indicated in Section 4.1.3; grounds are abstract objects expressed in terms of primitive operations, essential to the determination of meaning, and are obtained either performing such operations, or performing non-primitive operations defined through equations by virtue of which grounds for the elements in the codomain are generated from grounds for the elements in the domain. We can now ask ourselves: is it fair to say that, in the terms of *this* notion of (possession of) grounds, the "fundamental task" is such that its fulfillment actually satisfies the basic question of the theory of grounds? If a ground is to be intended as a description or a reification of a mental state of justification, can we say that the grounds *so as described by Prawitz* adequately respect this basic intuition? Does having a ground, *conceived by Prawitz* as having applied one of the operations described above, actually means to be in possession of evidence for any judgment or any assertion? In connection with these questions, it is important to point out how, in *Explaining deductive inference*, the "fundamental task" is explained by Prawitz not in the terms of the notion of ground, but in the terms of the notion of evidence:

> the task to show that a condition c on generic inferences is sufficient for legitimacy can be spelled out as the task of establishing for any generic inference I and subject A that from the three facts (1) the inference I satisfies the condition c, (2) the subject A has evidence for the premisses of I, (3) A performs I, it can be derived that (4) A gets evidence for the conclusion of I. (Prawitz 2015, 74)

We will return to this question already in Section 4.1.5.2, and then later in the course of our investigation; for the moment, wishing to conclude, it is relevant the observation according to which

> an inference that is valid is so in virtue of the meaning of the sentences involved in the inference. (Prawitz 2015, 95)

The fact that the inferences of the theory of grounds are valid by virtue of their meaning is immediate, for example, in the case of Gentzen's introductions; in the case of other inferences, like Gentzen's eliminations, the guarantee will come from the fact that the equations that define the non-primitive operations applied, produce grounds from grounds - and the notion of ground is fixed in terms of primitive operations constitutive of meaning. The limitation to grounds and operations for first-order logical constants is not necessary, but it becomes essential when we intend to pass from a notion of deductive validity to a notion of logical validity:

> the notion of validity defined above may be called deductive validity to differentiate it from [...] a narrower notion of logical validity, which is now easily defined by using the same strategy as used by Bolzano and Tarski; inferences on various level of abstractions are *logically valid* if they are deductively valid and remain deductively valid for all variations of the meaning of the non-logical vocabulary. (Prawitz 2015, 95)

4.1.5 Advancements and open issues

In sections 4.1.1, 4.1.2, 4.1.3 and 4.1.4 we have provided an account of the theory of grounds as it appears in Prawitz's articles up to *Explaining deductive inference* (Prawitz 2015) - with the integration of *The seeming interdependence between the concepts of valid inference and proof* (Prawitz 2018b). In this section, we intend to deal with the relation between the theory of grounds and the problems that, in Section 2.5.3, we have seen concern proof-theoretic semantics: the proofs-as-chains problem, the recognizability problem and, although much more concisely, the problem of validity as independent from inferences. As we will see, the theory of grounds allows some philosophical and substantial advancements compared to Prawitz's old semantics but, on admission of the Swedish logician himself, it still has certain shortcomings.

4.1.5.1 Proofs-as-chains

If proofs are intended as chains of valid inferences, to explain the power of epistemic compulsion of valid inferences also means to explain the power of epistemic compulsion of proofs. On the other hand, as seen in Section 4.1.3.3, Prawitz warns against the possible circularity in which such an explanation is at risk of becoming bogged, essentially on account of the interdependence between the notions of valid inference and proof. Proof-theoretic semantics, for example, suffered exactly of this defect; the notion of valid inference was defined in it in the terms of the notions of valid argument and proof, so that it proved impossible to appeal to the idea that valid arguments and proofs were chains of valid inferences. In the theory of grounds, however, the notion of valid inference is defined in terms of the notion of ground, and the notion of ground does not in turn presuppose the notion of proof. It is therefore significant that *Explaining deductive inference* ends with a definition of the notion of proof:

> an (individual or generic) proof may be now defined as a chain of valid (individual or generic) inferences. [...] a proof of an assertion does not constitute a ground for the assertion, but produces such a ground. (Prawitz 2015, 93)

The theory of grounds therefore offers two substantial advancements. The first, that descends immediately from what we have just said, is that it offers a notion of valid inference independent of the notions of valid argument and proof, in such a way that on the contrary the former serves as a basis for the characterization of the latter. The second, instead, emerges as soon as we turn to the final part of the previous quote. Here, Prawitz specifies that a proof *is not* a ground, but *produces* a ground; it is not what the justification of a judgment or an assertion is *based on*, but *through* what the evidence for a judgment or an assertion is *obtained*. In a terminology already introduced, the proof-objects of the theory of the grounds are the grounds themselves; proofs, on the other hand, are not proof-objects, but proof-acts. This is relevant to what we called in Section 2.5.3 the proofs-as-chains problem.

The first advancement we have pointed out refers to an observation conducted again in Section 2.5.3. If the power of epistemic compulsion of valid arguments and proofs must be explained by saying that they are made of only valid inferences, and if the latter are explained in terms of valid arguments and proofs, we have an impasse already with, for example, a valid closed canonical argument

$$\dfrac{\begin{array}{c}\Delta_1\\ \alpha\end{array} \qquad \begin{array}{c}\Delta_2\\ \beta\end{array}}{\alpha \wedge \beta}\,(\wedge_I)$$

or a canonical proof $O_\wedge(\pi_1, \pi_2)$; it would obviously be circular to say that this latter inference is epistemically compelling since, as valid, it produces the same valid closed canonical argument or the same canonical proof of which we want to explain the ability to compel epistemically. Obviously, we could argue that this is a badly-posed example, and that the impasse is only apparent; the valid closed canonical argument or the canonical proof are epistemically compelling *when understood as acts*, and this because the last inference is precisely an act that produces the canonical closed valid argument or the canonical proof which, *understood this time as objects*, constitute evidence for the conclusion. But this obviously requires that the theory clearly distinguishes between proof-objects and proof-acts.

In proof-theoretic semantics valid arguments and proofs actually play a double-role: objects that, as formalization of the notion of evidence, serve to determine the meaning, and acts that, as they are aimed at achieving evidence, are required to be made of only valid inferences. However, again in Section 2.5.3.1, we have said that the inability of proof-theoretic semantics to offer an adequate conception of proofs as chains was ultimately attributable precisely to this twofold status. Moreover, even if the notion of valid inference is not based on those of valid arguments or proof, but still on a notion that singles out structures then required to be chains of valid inferences, the distinction between inferences in introductory form and inferences in non-introductory form could induce to circular explanations; we cannot, for example, characterize as valid an implication introduction by requiring that its immediate subchain is a chain of valid inferences since, on account of the distinction between canonical and non-canonical cases, we cannot exclude that the immediate subchain contains inferences of the same type as that we are defining as valid (see again Gentzen 1934 - 1935, Negri & von Plato 2015, Usberti 2015).

In this regard, and following an interpretative line outlined by Usberti (Usberti 2015), it was suggested a way out inspired by the BHK clauses. Leaving aside for the moment the epistemic transparency that could be ascribed - as Usberti does - to the proofs identified by such clauses, it had been pointed out how the fact that they proceeded by induction allowed to abstract from the intrinsic complexity of the functions involved in the problematic clauses for $\rightarrow$ and $\forall$ and, therefore, from the distinction between inferences in introductory form and inferences in non-introductory form, between canonical and non-canonical cases. This had indicated that the proofs-as-chains problem could be worked around by one of the following two strategies. Either the notion of valid inference is defined relative to always canonical objects, fixed by simple induction and by abstracting from the distinction between introductory/canonical and non-introductory/non-canonical cases, a distinction which can then be referred exclusively to the acts in which inferences occur, to be characterized instead as chains of valid inferences. Or, alternatively, the distinction between introductory/canonical and non-introductory/non-canonical cases can be preserved relative to the notion used in the definition of valid inferences, but in such a way that this notion define objects distinct from the acts in which the inferences occur, acts that can then be characterized, once again, as chains of valid inferences. In our opinion, with the theory of grounds Prawitz pursues the first option.[5]

Valid inferences are defined in terms of a notion of ground which, exactly as for the BHK proofs, proceeds by simple induction on the logical form. In the case of the constants $\rightarrow$ and $\forall$, a notion of constructive procedure is assumed as primitive, so as to disregard the complexity inherent in the procedures themselves. The clauses that establish what counts as ground use only meaning-constitutive operations, and the resulting grounds are, as

[5]In *The concepts of proof and ground* (Prawitz 2019), Prawitz seems instead to be pursuing the second of the above strategies. The notion of valid inference is not defined there in terms of the notion of ground, but through a notion of *grounding tree*, for which the Swedish logician validates a distinction between inferences in introductory form and non-introductory form, between canonical and non-canonical cases - through a definition by simultaneous recursion, very similar to that of analytical validity for arguments (Prawitz 2018a) examined in Section 2.5.3.2. Valid arguments and proofs should be understood, one would say, as chains of valid inferences. Significantly, Prawitz argues that "a proof is built up by inference linked to each other, which they supply successively conclusions with grounds, finally resulting in the proof delivering a ground to its final conclusion. [...] the proof has to be taken as an act" (Prawitz 2019, 2) and that "identifying possible grounds for a proposition with grounding trees for, we can understand the performance of an inference and has as immediate the grounds that the agent takes herself to have for the premisses. I shall say that this tree *results* from the inference or that the inference *gives rise* to this tree" (Prawitz 2019, 22). Grounding trees, though not devoid of an obvious epistemic relevance, are however characterized in "ontological" terms; for example, "a grounding tree for proposition explains why the proposition is true. A canonical grounding tree for a compound proposition exhibits the immediate reason why is true by it having immediate subtrees for constituents of which by being true make true. Admittedly, these subtrees may be non-canonical, and if so, they do not on their turn exhibit the immediate ontological grounds why their last propositions are true" (Prawitz 2019, 19 - 20). It is important to note that Prawitz's notion of grounding tree recalls the *ontological grounding trees* that Sundholm, in *A garden of grounding trees* (Sundholm 2011), proposes as distinct from the notion of *epistemic grounding tree*; Sundholm's ontological grounding trees are defined starting from the BHK proofs, while epistemic grounding trees are closer to the derivations of judgments in Martin-Löf's intuitionistic type theory (Martin-Löf 1984). The difference between Prawitz's grounding trees and Sundholm's ontological grounding trees concerns the way to treat open grounding trees. According to Prawitz, an open grounding tree for α from β should be characterized substitutionally, requiring it to give a closed grounding tree for α from each closed grounding tree for β; the construction of the overall grounding tree, therefore, stops, so to speak, on β. According to Sundholm, the construction of the overall grounding tree continues, starting from nodes that instantiate the open ontological grounding tree on each of the possible closed ontological grounding trees for β, which could obviously return structures of infinite width, but finite in "height", i.e., well founded.

it were, always canonical. Referring back to Usberti's words,

> we can see the reasons why Prawitz makes now reference to two different notions, assigning them different roles: to the notion of ground the role of the key notion of the theory of meaning, to the notion of deduction the role of (linguistic presentation of) proof [...]. The crucial difference between the two notions is that grounds for α are defined by recursion on the complexity of α, while a deduction of α is defined by induction on the number of its steps. So, in the case of grounds for $\alpha \to \beta$, it is legitimate to abstract from the intrinsic complexity of the function that transforms each ground for α into a ground for β. (Usberti 2015, 418)

If the objects of the theory of grounds are always canonical, the acts in which (valid) inferences occur are not; valid arguments and proofs, in fact, may involve inferences of any kind, introductory or not, i.e. possibly corresponding to the application of operations that are not meaning-constitutive. The terms of the languages of grounds that, in *Explaining deductive inference*, Prawitz suggests to develop, must denote grounds; but, as the Swedish logician claims in *The seeming interdependence between the concepts of valid inference and proof*, they can be understood also as protocols - we could say encodings - of the process by which the denoted ground can be obtained. Not surprisingly, therefore, Prawitz distinguishes between canonical and non-canonical *terms*, saying that

> a closed ground term whose first symbol is one of the primitive operations is said to be in *canonical form* - the form used to specify the grounds there are for different assertions (Prawitz 2015, 93)

- it goes without saying that if a closed term has as its outermost symbol a non-primitive operation, it is to be considered in non-canonical form. However, this distinction has no parallel at the level of the objects of the theory, namely at the level of grounds.

The relation between grounds and proofs in Prawitz's theory of grounds is analogous to that between constructions/proofs and realizations of such constructions/proofs in Heyting's approach. However, it retraces also what in Martin-Löf's intuitionistic type theory exists between proofs for propositions, on the one hand, and proofs for judgments on the other. The notion of ground for judgments or assertions, in the terms of which the meaning of propositions or sentences is explained, corresponds to the notion of proof for propositions in Martin-Löf, to the point that Prawitz points out as

> the grounds will thereby be among the objects that one comes across within intuitionistic type theory developed by Martin-Löf. (Prawitz 2015, 89)

As we have said in Section 3.2.2, the fact that in Martin-Löf's type theory the relation "to be a proof-object for (or of type)" was decidable, induced Prawitz to look at Martin-Löf's proof-objects as at good candidates for the notion of ground for a judgment or an assertion; this naturally is linked to the recognizability problem, with which we dealt in relation to proof-theoretic semantics and which, with reference to the theory of grounds, we anticipated at the end of the previous Section and will face more broadly in the next one. In this respect, however, a proof-object cannot be considered as an abstract truth-maker, what Prawitz (Prawitz 2012b) criticizes in the last Martin-Löf and in Sundholm; on the contrary, it must have an epistemic import, which is relevant for the justification of judgments or assertions. According to Prawitz, indeed,

> what is of interest here is whether what is defined as a ground for the assertion of a sentence α is not only a truth-maker of α but is really a ground the possession of which makes one justified in asserting α. (Prawitz 2015, 89)

The proofs of the theory of grounds would correspond instead to the proofs of judgments in Martin-Löf - with the reserve, already highlighted in Section 4.1.1, that for Prawitz the form of a judgment is not to be understood as "a is a proof-object for α" or, if we pass from the analytic to the synthetic form, as "α is true", since expressions of this type should be placed on a meta-level.

However, without taking account of the similarities, among Prawitz's, Heyting's and Martin-Löf's frameworks there are also profound differences. The most immediate one concerns undoubtedly the treatment of the distinction between canonical and non-canonical cases, that we have seen to be attributed by Prawitz not to the objects of the theory of grounds, but to the acts through which these objects are produced, and to the terms that codify such acts - proofs, terms for grounds; while in Heyting's writings there seems to be no trace of this theme, it is clearly present in Martin-Löf, where one oscillates from an exclusive attribution of the distinction between canonical cases and not to proofs of propositions, namely to objects - and not therefore to proofs for judgments, which we have instead seen to be acts (Martin -Löf 1984) - and an introduction of the distinction also at the level of proofs for judgments, with a simultaneous explanation of this distinction in terms of that analogous at the level of proofs for propositions (Martin-Löf 1985, Prawitz 2012b).

But the most original trait that the theory of grounds presents with respect to similar approaches is, we think, the *explicit* and *programmatical* introduction of a deep connection between objects and acts, between grounds on the one hand, and terms/proofs on the other. A proof, i.e., a process that allows us to be in possession of a ground, can be described by a term of an appropriate formal language, which in turn will *denote* the ground obtained; the term, which encodes in the formal language the corresponding proof, is a *name* of the denoted ground, i.e. of the ground which could be obtained whenever the proof were carried out. The name provides, if we may say so, a set of *computation instructions*; computing the term means performing the proof, thereby getting the ground to which it leads. A very strong bond is therefore established between the objectual aspect and the operational aspect of the theory, a bond that on the other hand is more implicit in the respective proposals of Heyting and Martin-Löf. In Heyting, the individual steps performed in the realization process of the proof/construction show, as it were, what the construction realized is, in correspondence with the operations of which it is composed (Heyting 1931, 1932, Prawitz 2012b); in Martin-Löf, likewise, the proof of a judgment consists of passages that correspond to the operations involved in the proof of the proposition and in the logical constants of the proposition itself - the latter intended as a type - to the point to allow a type-checking that makes judgments of the type "$a \in \alpha$" decidable (Martin-Löf 1984, 1994, Prawitz 2012b, Sundholm 1983, 1998, 2011). However, it lacks any idea that a proof-act can be described by a term denoting the proof-object of which it is name, and which can be computed in order to obtain actually such a proof-object.

Just above, we have said that this idea is on the contrary introduced by Prawitz in an explicit and programmatical way; explicit because it is openly suggested, as an essential feature of the development, both philosophical and formal, of the theory of grounds, and programmatical because, above all in the latest writings, Prawitz seems to conceive the theory of grounds as including a language of terms for which we can define as much as possible a rigorous and precise notion of denotation with respect to a "universe" of grounds and operations on grounds. This has, in our opinion, the immediate consequence of substantiating the criticism that Prawitz makes to the idea of the latest Martin-Löf and Sundholm according to which proof-objects would be lacking epistemic import (Martin-Löf 1998, Sundholm 1998):

> even if proofs are primarily acts, there are objects that relate in various ways to these acts and are also called proofs or, at least,

> are considered to have epistemic import. Furthermore, proof acts can be noted down, and what we then have are linguistic objects that can be seen as records of the acts. [...] Such a record of an act [...] can be seen as an instruction for how to perform a proof act. It has clearly epistemic import, and we may call it a *representation of a proof* (*act*) when there is need to distinguish it form the act itself. (Prawitz 2012b, 63 - 64)

However, it seems appropriate to point out that the above passage is inserted in the context of a discussion of the positions of Martin-Löf, as inscribed in the intuitionistic tradition of BHK semantics; in this view, we cannot ignore the fact that, with respect to his objection, Prawitz notes in a limiting way how

> we cannot argue that proofs of the BHK-interpretation or proof-objects in general correspond to proofs generated by a fixed set-up of inference rules. (Prawitz 2012b, 64)

4.1.5.2 A recognizability problem

The solution that the theory of grounds offers of the "fundamental task" presupposes, as stated in Section 4.1.4, a question of adequacy of the notion of ground with respect to a pre-formal notion of evidence - and hence, the notion of possession of grounds with respect to a pre-formal notion of possession of evidence. In the concluding remarks of *Explaining deductive inference*, the question is explicitly raised by Prawitz himself:

> does a ground as now defined really amount to evidence? When the assertion is a categorical one, the ground is a truth-maker of the asserted sentence; since the meanings of the sentences are given by laying down what counts as grounds for asserting them, the truth of a sentence does not amount to anything more than the existence of such a ground. Nevertheless one may have doubts about whether to be in possession of a truth-maker of a sentence as understood here really amounts to being justified in asserting the sentence. (Prawitz 2015, 96)

It is important to note that the Swedish logician does not refer here to a notion of (possession of) ground understood as, so to speak, synonym of (possession of) evidence; the questions concern rather the grounds "as now defined" and "as understood here". What one wonders is whether the grounds as described in the theory of grounds appropriately capture the pre-formal notion of ground which the theory itself should comply with.

What could be problematic here? The answer to the question is: the fact that inferential acts are intended as applications of primitive or non-primitive operations, hence as corresponding to terms which, however always denoting canonical objects, can be in canonical or non-canonical form. More specifically, Prawitz argues that

> when a subject performs a valid inference and applies an operation ϕ to what she holds to be grounds for the premisses, she forms a term T that in fact denotes a ground for the drawn conclusion α, but it is not guaranteed in general that she knows that T denotes a ground for α. (Prawitz 2015, 96)

As you may remember, in proof-theoretic semantics there was a problem of recognizability related to valid closed non-canonical arguments, and non-canonical or categorical proofs - depending on the preferred terminology, that of 1977 or that of 2005: to have a valid closed non-canonical argument or a non-canonical or categorical proof means *knowing how* to get a valid closed canonical argument or a canonical proof for the same conclusion, but not also *knowing that* the non-canonical argument or the non-canonical or categorical proof are indeed such as to produce the expected result. A similar difficulty arises in connection with valid open arguments, and hypothetical or general proofs, for which infinite substitutions from non-regimentable domains are required. Well, in the theory of grounds we have something similar; the difference is that, as the non-canonical cases employ non-primitive operations fixed by equations of a certain type, the problem can be reformulated in terms of recognizing that the equations offer a good definition with respect to the operational type understood, so that the operation actually produces grounds of the type indicated in the codomain starting from grounds of the type indicated in the domain. In the words of Prawitz,

> since the meanings of closed atomic sentences are given by what counts as grounds for asserting them, [the agent] should thus know that T denotes a ground for asserting an atomic sentence α when this is how the meaning of α is given. Such knowledge is preserved by introduction inferences, given again that the meanings of the involved sentences are known: the term T obtained by an introduction is in normal form, that is, it has the form $\phi(U)$ of $\phi(U, V)$, where ϕ is a primitive operation and the term U or the terms U and V denote grounds for the premisses - knowing that these terms do so, the agent also knows that T denotes a ground for the conclusion [...]. However, when ϕ is a defined operation,

> the subject needs to reason from how ϕ is defined in order to see that T denotes a ground for the conclusion. If T is a closed term, she can in fact carry out the operations that T is built up of and bring T to normal form in this way, but she may not know this fact. Furthermore, when T is an open term, it denotes a ground for an open assertion or an assertion under assumption, and it is first after appropriate substitutions for the free variables that one of the previous two cases arises. (Prawitz 2015, 97)

Despite having made a valid inference, therefore, the agent may not recognize it. And in the light of the way in which the notion of valid inferential act is reconstructed by Prawitz, the agent might therefore not recognize that he/she made use of an operation that produces grounds for the conclusion when applied to grounds for the premisses; if he/she actually has grounds for the premisses, the agent could thus not recognize that he/she has a ground for the conclusion. Compared to the "fundamental task" that the theory of grounds had aimed to accomplish, this could be problematic to the extent that

> one may ask if to make a valid inference really gives the evidence that one should expect. (Prawitz 2015, 97)

Before turning to the way in which Prawitz deals with the problem of recognizability in the framework of the theory of grounds, and to the answers that he contextually puts forward, it is perhaps appropriate to explain in more detail why this problem exists. A first suggestion obviously comes from what has already been said with reference to valid arguments and proofs in the context of proof-theoretic semantics: the computation of a closed term to its canonical form could be *de facto* impossible for an agent with limited resources of time and memory and, in the case of an open term, the process of substitution of individual variables and ground-variables with, respectively, individuals and grounds from non-regimentable expansions of a given language of grounds could result impossible even if no limitation is put on agent's time and memory. In *Explaining deductive inference* Prawitz advances another argument, in the light of which the recognition of a term as denoting a ground would be said hardly plausible - an argument already anticipated in part for the proofs of proof-theoretic semantics (Prawitz 1977). This is an observation which refers back to Gödel's incompleteness theorems, already mentioned in Section 4.1.3.2 to endorse the character of openness of a language of grounds.

As we will understand when we will talk about the Curry-Howard isomorphism, a closed language of grounds could be understood as a "translation" of a formal system and, in this regard, Prawitz notes that

> it is of course an essential feature of a formal system that it is decidable whether something is a proof in that system. For a closed language of grounds where the term formation is restricted by specifying what operations may be used, it may similarly be decidable whether an expression in the language denotes a ground. (Prawitz 2015, 98)

On the other hand, as we have said, the phenomenon of incompleteness implies that the languages of grounds must enjoy a character of openness to the introduction of new operations. Already at the level of first-order arithmetic, a closed language of grounding Λ would be extensionally inadequate, since it does not manage to express all the possible grounds for judgments or assertions that involve the formulas of the background language; it is therefore necessary to take into account a (infinite) class of expansions Λ^+ of Λ and, although the terms of Λ can be recognizable as denoting grounds, we do not have any guarantee that this assumption of recognition is preserved in all the Λ^+.

However, it is worth noting that while we are talking in terms of recognizability, in *Explaining deductive inference* Prawitz expresses hinself first in terms of luminosity of having evidence for a certain judgment or for a certain assertion and, then, in terms of decidability:

> it may be argued that the condition for having evidence for an assertion is luminous [...]. The crucial question is therefore if it is decidable for an arbitrary definition of an operation, which we may contemplate to add to a given closed language of grounds, whether it always produces a ground of a particular type when applied to grounds in its domain. This is what must hold if we are to say that the property of being a ground is decidable, and it seems to me that we must be sceptical of such an idea, and therefore also of the idea that the condition for something to be a proof or to constitute evidence is luminous. (Prawitz 2015, 97 - 98)

Referring to a precise concept of contemporary mathematical logic, the notion of decidability is certainly more precise, and therefore more stringent than those of luminosity or recognizability. Therefore, in what sense does Prawitz believe that undecidability implies non-luminosity? And why does he pass from a discourse of recognizability (as in Prawitz 1973 or in Prawitz 1977) to one of decidability? That the issue under consideration is to be formulated in the strict terms of the notion of decidability is a thesis put forward by some authors (see, for example, Pagin 1998); nevertheless, one could ask whether there can be any plausible way of understanding the expressions "luminous" or "recognizable", by virtue of which, although not strictly decidable, the fact that an arbitrary term denotes a ground be, more weakly, and perhaps more vaguely, luminous or recognizable. To this gap between decidability and luminosity or recognizability, is addressed for example the following criticism by Usberti:

> suppose that the correct answer to the crucial question is negative, namely that there is an operation O on grounds represented in the formal theory of grounds T by a term K such that if $t_1, ..., t_n$ are terms denoting grounds for $\alpha_1, ..., \alpha_n$ respectively, then $K(t_1, ..., t_n)$ denotes a ground for β, but neither the sentence of T translating "$K(t_1, ..., t_n)$ is a ground for β", nor the sentence translating "$K(t_1, ..., t_n)$ is not a ground for β", is a theorem of T. There is no reason to conclude that it is not *intuitively* evident that $K(t_1, ..., t_n)$ is a ground for β, and therefore that intuitive evidence is not epistemically transparent. [...] Concededly, formal provability is *intended* to catch intuitive evidence, but sometimes it does not succeed, as just Gödel's theorem shows; when this happens, we don't infer that intuitive evidence is different from what it appears to be (for instance, that Gödel's sentence is not intuitively true/evident), but that formal provability is incomplete. (Usberti 2017, 4)

As can be seen, Usberti refers to notions of epistemic transparency and of intuitive evidence that somehow resemble, respectively, the notions of luminosity or recognizability which we are dealing with, on the one hand, and the notion of evidence in the pre-formal sense we have used this far, on the other.[6]

[6] As mentioned several times, Gödel's theorems are used by Prawitz in support of the character of openness of the languages of grounds, and of the impossibility of ensuring that the recognizability/decidability of the fact that a term denotes a ground, which may also be plausibly assumed for the terms of a closed language, is preserved in the expansions of the given language. In our opinion, however, incompleteness is also central for replying to a possible objection to Prawitz's argument: recognizability/decidability is ensured by the fact that, for each of the closed languages, constituting expansions of the given language of grounds, we can, for each of its terms, recognize/decide that such term denotes a

For the recognizability problem related to proof-theoretic semantics, we saw that a possible solution, similar to that adopted by Kreisel (Kreisel 1962) for the BHK clauses, consisted in requiring that a valid argument or a proof for α, were not only a valid argument Δ or a proof π according to Prawitz's definitions, but also an additional valid argument Δ^* or proof π^* for "Δ is a valid argument for α or "π is a proof for α". Similarly, in the case of the recognizability problem of the theory of grounds, the idea would be to require that a valid inference is not only describable by a term T that actually denotes a ground for $\vdash \alpha$, but also by a further term T^* denoting a ground for "T denotes a ground for $\vdash \alpha$". However, Prawitz does not consider this way viable, and in particular argues that to perform an inference

> is not to assert that the inference is valid. Nor is it to make an assertion about the grounds that one has found for the conclusion of the inference. One may of course reflect over the inference that one has made, and, if successful, one may be able to demonstrate that a ground for the conclusion has been obtained and that the inference is valid. But a requirement to the effect that one must have performed such a successful reflection in order that one's conclusion is to be held to be justified would be vulnerable to [...] vicious regresses. (Prawitz 2015, 97)

Prawitz therefore reaffirms his reconstruction of the inferential act, with obvious additional implications concerning the acquired definition of validity for such an act: when we infer, all we do is to make judgments/assertions-premisses, for which we believe to be in possession of grounds, and to pass to a judgment/assertion-conclusion applying an operation that, when the grounds we thought we had for the judgments/assertions-premises are actually such, and when the inferential act is valid, actually produces a ground for the judgment/assertion-conclusion. This activity, however complex, does

ground. In order that this objection achieves its objective, it is necessary to assume that the expansions of a given language of grounds are effectively generable: the algorithm suggested could in this case use a sort of diagonal procedure to cover the terms of all the possible expansions, setting at each step that the term met denotes a ground. But if the starting language of grounds is a language of grounds for first-order arithmetic, the effective generability of all its expansions would be equivalent to the recursive enumerability of all the "truths" of first-order arithmetic - it is sufficient to enumerate the types of the terms in each of these languages; that, by virtue of Gödel's incompleteness theorems, is impossible. An argument similar to that just proposed is also found in (Usberti 1995, 2015). It should also be noted that the feared objection has also another weak point: it makes sense only given for granted that each ground can be expressed by some term in some language of grounds, which is by no means obvious.

not involve and should not be intended to involve a further judgment or assertion which establishes that the performed passage actually produces the expected result. If we adopted this point of view, in fact, a valid inference should be able to produce, as it were, not one, but two grounds; one for the judgment/assertion-conclusion, and the other for the judgment or assertion "the inference is valid", i.e. "the term which this inference corresponds to denotes a ground for the conclusion". Indeed, this second judgment-assertion is not, according to Prawitz, equivalent to the judgment/assertion-conclusion of the inference. And this could in turn, in a quite obvious way, generate regressive explanations, since the second judgment or assertion involves a proposition or sentence that

> is on a meta-level as compared to the former one [...]. To be justified in asserting it, it is of course not sufficient only to produce a truth-maker of α. One must also have a ground for the assertion that what is produced is a truth-maker of α, which has to be delivered by a proof on the meta-level of an assertion of the form ".... is a ground for asserting ...". This proof will in turn depend on its inferences giving evidence for their conclusions. To avoid an infinite regress it seems again to be essential that there are inferences that give evidence for their conclusions without it necessarily being known that they give such evidence. (Prawitz 2015, 97)

Prawitz's reasoning is obviously based on a particular conception of judgments or assertions; to judge that α is true does not also mean to judge that it is true that α is true and, analogously, to assert α does not also mean to assert that α is true - remember what Prawitz says about Martin-Löf's reconstruction of judgments and assertions. With specific reference to assertions, the centrality of this point is in our opinion specified at best in *The seeming interdependence of the concepts of valid inference and proof*:

> to assert that α is true is from a constructive point of view to assert that there is a construction for α, which of course requires that one can specify a construction a for α. To assert a proposition α, I take to be the same as uttering in assertive mood a sentence that expresses α. It certainly requires the speaker to have a construction for α since, as always, she needs a ground for the assertion. But it does not require, one may argue, a proof that so and so is a construction for α, because that there is a construction of α is not what she asserts; it is not a part of the content of the proposition α. (Prawitz 2018b, 9 - 10)

Finally, we consider it appropriate to point out that, although the theory of grounds still suffers from a recognizability problem, this seems to have now less urgency. We can appeal to some of Prawitz's statements, according to which it could already be satisfying to fulfill the "fundamental task" in the manner indicated above, making it

> a conceptual truth that a person who performs a valid inference is aware of making an operation that produces what she takes to be a ground for the conclusion. (Prawitz 2015, 98)

The recognizability problem does not concern the grounds as such, as abstract objects liable to some formalization; the problem concerns the grounds as a clarification of the notion of evidence and, therefore and above all, our being in possession of grounds as being in possession of evidence. To have a ground means having constructed a term that denotes that ground, or better, to have performed acts that can be described as terms that denote the ground to which these acts lead. Now, two key points come into play here in the theory of grounds; first of all, the idea that an inferential act is the application of an operation on the grounds that we think we have for the premises in order to obtain a ground for the conclusion; and, secondly, the clear distinction between always canonical objects, that the theory treats as evidence, and acts that allow to come into possession of these objects. What an agent possesses after having made a valid inference is a *ground*, an always canonical *object* of evidence, not a term; the *term*, which certainly denotes the ground obtained, has however only the role of describing the *act* performed, and can be canonical or not depending on whether the last inference made by the agent is in an introductory form or not. When the inferential act is valid, the agent has grounds for the premisses and, by applying to these grounds the operation associated with the inference, he/she gets - as, as it were, result of a computation - a ground for the conclusion; this activity can be codified by a term, which hypostatizes the agent's activity - and which provides, so to speak, the instructions for the computation. Therefore, a valid inference *objectively* gives the agent the possession of something that the theory treats as evidence.

As we have often said in the course of our discussion, such a solution does not seem possible in the case of the notion of valid argument of proof-theoretic semantics. In Section 2.5.3.3 we have suggested that this impossibility comes from the fact that the justification procedures associated with valid arguments go from argument structures to argument structures, and not from valid arguments to valid arguments. Again in Section 2.5.3.3, we have also suggested that, as regards the notion of proof of proof-theoretic semantics, a solution analogous to that adopted in the theory of grounds, seemed instead possible, but only provided that the fulfillment of an inference is understood as the computation of a procedure, and the canonical objects of evidence are always distinguished from the acts for the production of such objects; and this because the actual procedures involved in the proofs go from proofs to proofs. In passing, let us note that also the operations on grounds, denoted by the operational symbols in terms of languages of grounds, are to be understood as operations *from grounds to grounds*.

4.1.5.3 Independent validity

Alongside the proofs-as-chains and recognizability problems, in Section 2.5.3.2 we had drown attention to another difficulty which, with reference to the notion of valid argument, the proof-theoretic semantics takes into account: namely, the fact that a valid argument is valid might not depend at all on the inferences involved, but only on the justification procedures associated with those in non-introductory form. Again in that section, we then saw how, with *The fundamental problem of general proof theory* (Prawitz 2018a), Prawitz proposes a solution in terms of a relation of containment among argument structures, and a definition of analytic validity for arguments. The question can also refer to proofs: the fact that a certain proof-structure is actually a proof may not derive from the inferences of which the structure is composed, but only from the equations that define the procedures associated with inferences in non-introductory form. Here too, we can define a containment relation among proof-structures, and introduce an idea of analyticity. In this Section we therefore propose a quick examination of the articulation of the problem of validity as independent from inferences in the theory of grounds.

In order to deal with this issue, it is again necessary to refer to one of the most important and innovative basic ideas of the theory of grounds - already mentioned in the previous section: the reconstruction of the notion of inferential act, i.e. the proposition according to which to make an inference means to apply an operation to the grounds that we believe to have for the premises, in order to obtain a ground for the conclusion. This explanatory strategy, in our opinion, permits to answer in a twofold way, positively in one sense, negatively in the other, the question whether it is fair to say, in the theory of grounds, that the fact that an argument structure is valid, or that a proof-structure is a proof, may result independent from the inferences involved, and instead depend only on the way non-introductory inferences are justified.

Let us take into account the following example. Let c_2 be a computation of $2 + 2 = 4$, and let f be an operation of operational type

$$1 + 1 = 2 \rhd 2 + 2 = 4$$

fixed by an equation that makes it a sort of "pointer" on c_2, for example, for every g ground for $\vdash 1 + 1 = 2$,

$$f(g) = c_2$$

and let c_1 finally be a computation of $1 + 1 = 2$. The argument structure (which in the following we will indicate with Δ)

$$\frac{\begin{array}{c} c_1 \\ 1 + 1 = 2 \end{array}}{2 + 2 = 4}\, R$$

in which R is associated with the operation on grounds f, or the proof-structure (which in the following will be indicated with π) $f(c_1)$, are respectively a valid argument (closed and non-canonical) and a proof (closed and non-canonical or categorical); the inference from the premise $1 + 1 = 2$ to the conclusion $2 + 2 = 4$ is indeed valid - which is immediate in the case of the proof-structure, and it becomes so when the inference R in the argument structure is understood as associated with f - and we can assume that the inferences of which the computation c_1 is made of are equally valid, so that we are dealing with concatenations of valid inferences. Well, in a sense, the validity of Δ, or the fact that π is a proof, do not depend properly on R or f, or on the other inferences involved in c_1, but only on the circumstance that f, directly involved in π, and associated with R in Δ, is fixed by an equation such that

$$f(c_1) = c_2$$

and thus produces a ground c_2 for the conclusion when applied to the ground c_1 for the premise. Although c_1 is undoubtedly a ground for the premise, the computation c_2 may be not at all contained, in the sense to which Prawitz aims in *The fundamental problem of general proof theory* (Prawitz 2018a), in the computation c_1. What really matters is, on closer inspection, the existence of c_2, and the fact that it occurs in a certain way in the definition of f; not, therefore, the relation between premise and its ground, and the conclusion in the inference to which f corresponds. The situation is even clearer when, dropping the reference to c_1, we take into account the argument structure (which we will indicate in the following with Δ^*)

$$\frac{1 + 1 = 2}{2 + 2 = 4}\, R$$

in which again R is associated with f, or the proof-structure (which we will indicate in the following with π^*) $f(\xi^{1+1=2})$. Δ^* and π^* are, again and respectively, a valid argument (open) and a proof (hypothetical), but this depends simply on the fact that, whenever we have a ground for $\vdash 1+1=2$, and *whatever* this ground is, calculated on it f produces the ground c_2 for the conclusion.

To remedy this situation, an obvious solution could go in the same direction as the notion of analytical validity for arguments adopted by Prawitz in *The fundamental problem of general proof theory* (Prawitz 2018a) - and to the analogue that can be defined for proofs; we could place restrictions on the operations that can be included in the theory of grounds, in order to exclude operations like f, and therefore validate inferences to which operations of this type and argument and proof-structures involving them can be associated. A central step was the definition of a containment relation among argument structures - and possibly proof-structures - on the basis of which to say that an argument is analytically valid if an analytically valid one in canonical form can be extracted from it; the extraction is limited to the argument structure - and possibly to the proof-structure - on which it is from time to time defined, and therefore it cannot produce as output anything that is not already contained in this structure. Thus, the limitation on operations on grounds could be such as to make the operations available exclusively extraction operations in the sense just indicated; in other words, we could request that for every instance

$$f(g_1, ..., g_n) = g$$

of every defining equation for an n-ary operation f admissible in the theory of grounds, g is contained in $g_1, ..., g_n$, in some precise sense of containment. The suggestion, it seems to us, is equivalent to requiring that the operations authorized in the theory of grounds are always and only compositions of a well delimited set of primitive operations of the type of those introduced by Prawitz in *Constructive semantics* (Prawitz 1971a) - an article that we will discuss instead in Section 4.1.6.2.

Therefore, if on the one hand the problem of validity as independent from inferences seems also to apply to the theory of grounds, on the other hand, as said before, the reconstruction that Prawitz offers of the inferential act gives us reason to maintain that this problem can now be considered less urgent. The absence of link among the inferences of which argument or proof-structures are composed, and the fact that these structures correspond, respectively, to valid arguments or proofs, has in fact a particularly unfortunate consequence: in the case of non-introductory inferences, it is not the inferential act as such to produce evidence for the conclusion, but a reduction procedure through which the inference itself is justified, or an equation which sets the behaviour of the associated operation.

In the theory of grounds, however, we have the idea that, by *performing the inference*, the agent *applies the associated operation*. Obviously, as in the example above, the inference may be valid not by virtue of an actual relation between premisses and their grounds, and conclusion, but solely because the operation produces a ground for the conclusion without, as it were, actually acting on the grounds to which it is applied. Nevertheless, the fact that the performance of an inferential act is now understood to be the application of the corresponding operation, authorizes us to say that it is the inferential act *as such* that produces evidence for the conclusion, providing the ground that, although disconnected from the available grounds, justifies epistemically in judging as true the conclusion, or makes its assertion correct. In other words, completing the last inference in the argument structure

$$\dfrac{\begin{array}{c} c_1 \\ 1+1=2 \end{array}}{2+2=4}\, R$$

or of the proof-structure $f(c_1)$, the agent applies the operation f to c_1, and since f is defined in such a way that, for every ground g for $\vdash 1+1=2$,

$$f(g) = c_2$$

it is thanks to this inference, and not thanks to a reduction procedure of the argument structure, or to a computation of the proof-structure, that the agent takes possession of c_2. It is as if we were saying that passing from the premise $1+1=2$ to the conclusion $2+2=4$ means for the agent finding the computation c_2, and using it to justify the passage. The operation f, i.e. the accomplished inference, *consists* exactly in this.

In conclusion, it seems that making inferences the application of operations on grounds, counterbalances the undesirable effects of an excessive liberality in the definitions of the admissible operations in the theory of grounds. We emphasize in passing that, as stated in Section 2.5.3.3, such a solution seems impossible in the case of proof-theoretic semantics focused on the notion of valid argument, appearing instead available when the basic notion is that of proof. This, again as at the end of the previous section, because the justification procedures are, in the approach with valid arguments, defined from argument structures to argument structures, and not from valid arguments to valid arguments; in the case of proofs, on the contrary, effective procedures go from proofs to proofs, just like the operations on grounds go from grounds to grounds. For what has just been said, and for what we have seen so far as a whole, Tranchini would therefore be right when he maintains that

> the conception developed in the 1977 article anticipates Prawitz's latest works [...] where the approach to the study of proofs starting from the notion of validity of an argument is abandoned, in favour of a more direct approach in terms of the notion of *ground*. (Tranchini 2014a, 507)

4.2 Building on Prawitz's ideas

This Section has a dual purpose. First, after recapitulating the results at which Prawitz's ground-theoretical proposal allows to arrive, and the problems from which it still suffers, we will lay the foundations for the investigation to carry out in the third part of this work; we will indicate the points that, in our opinion, can be further developed, as well as the reasons and advantages that such a proposition offers for a better understanding of the critical knots. In the second part, we will focus on a quick examination of some formal tools, we owe to Prawitz himself or to other authors, which will facilitate and will make clearer our proposal for formalization in Chapters 5, 6 and 7.

4.2.1 Summing up

The theory of grounds undoubtedly has an intrinsic interest; it offers a philosophical and formal framework in which to develop, in an innovative way, an analysis of the notion of deductive validity and, more particularly, to define the notions of valid inference and proof in order to explain their epistemic power - or at the very least to put us on the way of a satisfactory explanation - and their mutual conceptual link. Prawitz has addressed some of the oldest problems of logic since the birth of this discipline, and only that would suffice to justify why this proposal should be taken into account and, possibly, expanded. In addition, the entire theoretical apparatus is based on a notion of ground - and operation on grounds - that refers to some of the most important traditions in contemporary mathematical logic, offering original contributions and original points of view: from intuitionism to constructivism, from Kreisel-Goodman theory of construction to Martin-Löf's intuitionistic type theory, ending with Dummett's criticisms of a realistic approach to the explanation of meaning, and to the corresponding idea of a verificationist semantics.

Of course, we could say that the attention to the problem of epistemic power of valid inferences and proofs, and the way in which the former bind conceptually to the latter, is *from the very beginning* the focus of Prawitz's semantic investigations; and the Swedish logician could not help dialoguing *all along* with intuitionism, constructivism and verificationism. But this very circumstance makes the theory of grounds important for a second reason. Proof-theoretic semantics has structural and philosophical limitations against which the new theory allows - in a definitive way in some cases and in a partial one in others - a doubtless progress.

The proofs-as-chains problem, for example, can, in our opinion, be considered solved. A central role is played here by a twofold circumstance: first of all, the clear distinction between objects of evidence and acts for the production of such objects – proof-objects and proof-acts – and, secondly, the fact that objects are in a form always canonical, and specified through simple induction on the complexity of the formulas, whereas the distinction between canonical and non-canonical cases is referred to the acts - and to the terms that describe them. There still remain the problems of recognizability and of validity as independent from inferences. In the first case, the phenomenon can be traced back just to that distinction between canonical and non-canonical forms which, although limited to (descriptions of) acts for the production of objects of evidence, could precisely for this reason invalidate the possibility of recognizing that an inference is valid - and the problem arises to the extent that *such recognition* is required as a necessary condition for the possession of evidence. That the validity of deductive structures can be independent from the inferences involved, and solely dependent on the way the operations applied are defined, derives instead from the fact that, by its current state of development, the theory of grounds does not impose any restriction on the complexity of the operations on grounds and of their definitions - apart of course that of being, or returning, total constructive operations. However, we have also been able to argue that the last two problems are now in a sense less urgent; thanks to the way the notion of inferential act is reconstructed, they are the valid inferences as such that justify the conclusion, and not the subsequent reduction procedures or computation on the structures in which the inferences occur. Which is immediate with regards to the problem of validity as independent of inferences, and is accompanied, in the case of the recognizability problem, by the further and already mentioned feature that what inferences produce starting from evidence for the premisses are always canonical objects of evidence for the conclusion. Essential for both these solutions, finally, it is the circumstance that - unlike the justification procedures for the inferences in non-introductory form in proof-theoretic semantics based on the notion of valid argument - the operations on grounds are defined from grounds to grounds.

4.2.2 For a development of the theory of grounds

In addressing the recognizability problem, Prawitz discarded as a bearer of regressions, the idea that an inference should produce not only a ground for its conclusion, but also a ground for its same validity. The only way out, the Swedish logician suggested, would be that to authorize only inferences that produce evidence for the conclusion, without it being necessary to recognize this circumstance. But a solution of this type is equivalent to placing restrictions on the inferences that the theory of grounds validates and, for the way in which the notion of inferential act is conceived, on the operations on grounds associated with such acts; since the critical operations are the non-primitive ones, moreover, the proposal is equivalent to set a limit on the possible equations for the definition of operations on grounds. This idea, which in *Explaining deductive inference* is only feared, and which, as we have seen, also concerns the problem of validity as independent from inferences in terms of limitation of operations on grounds to extraction operations (Prawitz 2018a, 2019), is more substantiated in *The seeming interdependence between the concepts of valid inference and proof*:

> consider the case when ϕ is [...] defined as a unary operation from constructions for a proposition α by saying that $\phi(g)$ is the construction denoted by the term exhibited by a certain Turing machine when it stops after having been started with a term that denotes g, and assume that the output of the Turing machine is always a term denoting a construction for the proposition β when the input is a term denoting a construction for α. Then, in fact, ϕ satisfies the conditions imposed on operations, but since [...] it may not be obvious from its definition that it does so, nobody would regard $\phi(T)$, where T is known to be a term denoting a ground for asserting α, as a ground for asserting β before it was proved that the Turing machine always behaves as assumed. A restriction on the defined operations used in forming constructions for propositions is consequently required if the finding of a construction for a proposition is to be considered a ground for the assertion of the propositions. (Prawitz 2018b, 10)

The passage is interesting for at least two reasons. First of all, Prawitz takes into account a *specific* way of understanding operations on grounds. An operation on grounds of operational type

$$\alpha \rhd \beta$$

is defined in terms of Turing machines: for each term T that denotes a ground g, $\phi(g)$ is the construction denoted by the output of a Turing machine on input T, so that, if g is a ground for $\vdash \alpha$, $\phi(g)$ is a ground for $\vdash \beta$. On account of the *halting problem* (Turing 1936), an approach of this type is obviously unsatisfactory, and indeed reveals the need for restrictions on definitions and on the defined operations. However, there is another remarkable point, of a more general and, as it were, methodological relevance. By proposing a specific way to characterize definitions and defined operations, proving then their inadequacy and finally suggesting the identification of appropriate restrictions, Prawitz seems to indicate the main road along which to develop further his theory of grounds: a general, formally rigorous theory of the total constructive operations which, starting from the grounds for the elements in the domain of the intended operational type, produce grounds for the element in the codomain of the intended operational type. As proof of this, Prawitz reaffirms that

> the concept of proof should be defined in such a way that it becomes decidable whether something is a proof, but how this is to be achieved is seldom indicated, except of course in the case of formal proofs. The terms that denote constructions are supposed to be typed, and whether an expression has a type is decidable, but the rules for typing already assume that the demands put on the defined operations are fulfilled. What is needed in order to make it decidable whether something is a proof within the conceptual framework discussed here is a method to decide whether a proposed definition of an operation fulfills the demand that it is put on operations on constructions, (Prawitz 2018b, 10)

and concludes by emphasizing the need - which manifests itself at this point as priority for a possible, appropriate solution to the recognizability problem - of

> a general criterion for when a proposed definition of an operation on grounds guarantees that what is defined is really a total operation of the kind demanded here. (Prawitz 2018, 10)

A precise indication of the restrictions that make the theory of grounds able to respect the epistemic presuppositions on which it is based, cannot fail to go through a general theory of operations on grounds, and of the equations that fix their behaviour. It would then become possible to indicate rigorously which operations and defining equations can be in principle taken into account, and therefore to have a clearer idea of the problems of recognizability and validity as independent from inferences. To outline punctually the class of operations and equations involved would allow in other words to give a more precise content to questions such as: what are the operations we are to recognize as capable of producing a ground of a certain type starting from grounds of a certain type, and what are the equations we are to be able to recognize as defining operations with a certain operational type? What operations, when associated with appropriate inferences, make them valid not by virtue of a bond between premises and conclusion, but only by virtue of the defining equations? On such a general framework we could then, as a further refinement of the analysis, introduce appropriate restrictions; namely, to identify, within a wider class of operations on grounds and of the related defining equations, a subclass that respects certain criteria or parameters, and that employs basic formal resources more satisfying, more restricted and, therefore, more easily manageable - for example, by requiring that the operations on grounds should be defined by means of equations in which only the elements of a well delimited set of default operations occur, not differently from the way it occurs, for example, with the recursive functions in Gödel's approach (Van Dalen 1994).

In the next part of this work - chapters 5, 6 and 7 - we put forward a first development proposal along the lines just indicated. Our goal will be a formalization of the theory of grounds that serves as a general scheme for a possible, complete systematization of the theory itself; though not penetrating specifically into the possible restrictions to be adopted, we will provide a general context in which such restrictions can then be inserted, as well as definitions and related results in order to frame and dealing appropriately with operations on grounds and equations. On the latter we will not impose any limit, except that of, respectively, being or returning total constructive operations, but we will indicate the way in which these objects, even in the unconstraint way with which we introduce them, must, in our opinion interact with some notions on which, given Prawitz's indications, we have considered appropriate to focus our attention.

The first, obvious passage to complete is to provide a general definition of the notion of operational type; furthermore, we will identify some classes of operational types and describe precisely the behaviour of arbitrary operations on grounds associable with the types of one or the other class. Each of these operations will be understood as defined by an appropriate group of equations that set their behaviour; the equations are by assumption such as to identify total constructive operations from (operations on) grounds to (operations on) grounds. This seems in allegiance with what Prawitz himself maintains; as we have already shown, the Swedish logician points out how an operation on grounds is set by establishing the type of domain and codomain, and an equation that, according to the operational type thus obtained, shows constructively what values the operation produces in the codomain when computed on the values in the domain. But that is not all.

In *Explaining deductive inference*, Prawitz suggests in fact to develop languages of grounds consisting of terms that denote grounds and operations on grounds; he also indicates quite accurately and with clear examples, how the alphabet of a language of grounds should be made - ground-variables, individual constants, operational symbols for primitive and non-primitive operations - and how, starting from this language, the various terms can be constructed. All the elements of the alphabet, and the terms themselves, are to be understood as typed starting from formulas of a background language, and as related to a certain set of derivations in an atomic system of reference. In light of this, it will therefore prove necessary first to define rigorously the notion of base for a language of grounds - the bases provide, so to speak, the necessary material for typing and for the elementary deductive apparatus. Once this is done, we can introduce a general notion of language of grounds; each of the languages identified by our definition will consist of a typed alphabet of the kind indicated by Prawitz, and of a set of typed terms constructed from the elements of this alphabet. A guiding intuition in this phase of the analysis will be the following: in order to be able to be defined as such, a language of grounds will necessarily have to contain, among the elements of its alphabet, at least all the individual constants corresponding to all the closed atomic derivations in the atomic system of reference, and at least all the operational symbols corresponding to the primitive operations associated with the different logical constants, as well as, obviously, all the ground variables. Although arbitrary, the choice is motivated by two main reasons.

The first, of a philosophical nature, is that a language of grounds must speak, precisely, of grounds, and since the latter are defined in terms of primitive operations, it would be strange - although possible - that a language which speaks of grounds does not contain operational symbols corresponding to the operations required to define the grounds themselves. As for the second reason, it is instead more technical, and is linked to the formal systems which, in relation to each of the languages of grounds we define, we will develop in Chapter 6. In such systems it must be possible to express the idea that a term of which the outermost operational symbol corresponds to a non-primitive operation is "equal" to a term, of the same type as the starting one, of which the outermost operational symbol corresponds instead to a primitive operation - and, in the case of closed terms, equality is to be actually provable. Now, if the language on which the system acts lacked the operational symbols corresponding to the primitive operations, it would be impossible to capture the idea of equality - and, obviously, make proofs available on the closed terms.

Since the terms of a language of grounds must denote grounds and operations on grounds, among our objectives there will also be that of defining rigorously the notion of denotation, understood - drawing inspiration from the interpretative functions used in model-theory - as a function which appropriately associates the terms of the language of grounding with grounds or operations on grounds. However, for its part, the concept of denotation must be articulated with respect to another essential feature which, as we have observed, Prawitz refers to languages of grounds: a language of grounds must be open to the introduction of new operational symbols. But how to capture this character of openness?

Our proposal will move here in the direction of a definition of the notion of expansion of a language of grounds; the general idea is that, given a language of grounds with a certain set of individual constants and operational symbols, its expansion is obtained by adding to it either new individual constants, which denote new derivations in a larger atomic system (possibly in a broader background language), or new operational symbols, which denote new operations on grounds. To the general concept of expansion of a language of grounds we will apply then some distinctions that identify features that we consider relevant; the expansion may be primitive or non-primitive, depending on whether there is or not the addition of new individual constants (i.e., an actual change of the atomic system of reference - possibly, of the background language - and consequently of the base) or new primitive operational symbols, or conservative or non-conservative, depending on whether the new linguistic resources of the expansion allow or not to express grounds that were not expressible in the starting language (i.e., depending on whether some of the terms of the expansion denote or not grounds that were not denoted by any of the terms of the starting language).

Languages of grounds, denotation and expansions - with all the notions and the related results - will be the subject of Chapter 5. The formal languages considered here, however, will have a fundamental limitation: they contain only terms whereas, when generally speaking of a formal language, it comes natural to think of formulas that express properties which the terms can enjoy or not. Nevertheless, what are, in the case we are interested in here, these properties? Already at the end of Chapter 5, and then more widely in Chapter 6, we will identify two of them: the fact that a term denotes or not a ground, and the fact that two terms are equal to each other or not. Now, if in a sense it can be clear from now, albeit only in an approximate way, what means the fact that a term denotes a ground, this idea of denotation must be developed if we want to get a clearer idea of what we mean when we say that two terms may or may not be the same, namely, of what we mean, in the framework that we will be proposing, by equality among the terms of a language of grounds.

A term will denote a ground to the extent that the operational symbols of which this term consists are combined in such a way that the operations on grounds to which they correspond allow a computation, the result of which is a ground - it goes without saying that since grounds are defined starting from primitive operations applied to objects that denote grounds for judgments or assertions of the appropriate type, the computation must have as output an object constructed by applying a primitive operation to objects that denote grounds for judgments or assertions of an appropriate type. To say that two terms are equal, then, means saying that two terms denote the same ground - i.e., that the computation resulting from both produces as a result the same object constructed by applying a primitive operation to objects that denote grounds for judgments or assertions of the appropriate type. The enrichment of the grounding language via binary predicates "T denotes a ground for $\vdash \alpha$" and "T and U are equal" - together with the addition of appropriate first-order logical constants - will allow us to have a language in which to express (circumstances relating to) the important properties identified above.

Obviously, it is one thing to have a language in which certain (circumstances relating to the) properties in question are expressed, another is to have a formal tool that allows to establish whether these properties hold or not. Some of the formulas of the enriched language of grounds will hold and others will not. But how to distinguish the ones from the others? A way to answer to this question is obviously to go back to the notion of denotation and, extending this notion from the terms to the formulas, to focus on the truth of the latter; a formula that expresses that a certain term denotes a ground is true if, and only if, the denotation of the term is actually a ground, while a formula expressing that two terms are equal is true if, and only if, the denotation of the two terms is actually the same - and so on for the other constants, depending on whether we want to use a classical semantics or a constructive semantics. The road we will follow in Chapter 6, however, will rather consist in providing a formal system in which some formulas are derivable from an adequate set of inference rules - possibly under a certain number of assumptions.

The introduction of formal systems of grounding, namely the possibility to prove certain properties of denotation or equality of terms for grounds, in our opinion, has a double interest. First of all, that of making explicit, rigorous and mechanically provable the implications of the background assumptions of the theory of grounds. Once it is established that the latter is to be formalized by resorting to formal languages of which the terms denote grounds, in the sense of being combinations of operational symbols, such that a computation of operations on grounds to which these symbols correspond gives as output a canonical object denoting a ground according to certain clauses, it might be interesting to prove precisely and not ambiguously which computation is required to actually verify whether a term denotes a ground, which terms have the same denotation, and which general laws apply to (classes of) terms having a certain form, or operational symbols fixed by certain defining equations. Moreover, since a formal system must be specified starting from a well-defined set of rules, the development of formal systems for the various languages of grounds will allow to identify clearly which deductive principles are underlying the design of the theory of grounds. We will realize, for example, how the philosophical interests that enliven Prawitz's discourse - and, above all, the importance that the Swedish logician attributes to the recognizability problem - seem to make use of indispensable rules corresponding to Dummett's so-called *fundamental assumption* (Dummett 1991); in addition, it will be possible to indicate clearly which parts two systems of grounding for two different languages of grounds share, and where instead they differ.

From a general point of view, however, the introduction of formal systems of grounding has a relevance similar to that of the definition of formal languages of grounds with associated denotation functions: to have a different, and possibly fruitful, point of view on the problems of recognizability and of validity as independent of inferences. But there is something else; both languages of grounds, and systems of grounding will allow a deeper understanding of issues more closely related to the general semantic properties of the theory of grounds, first of all the question of completeness which, as is well known, is the subject of a famous conjecture elaborated by Prawitz in *Towards a foundation of a general proof theory* (Prawitz 1973) - although with a reference to proof-theoretic semantics focused on the notion of valid argument - and proposed by him, in a different form, in *An approach to general proof theory and a conjecture of a kind of completeness revisited* (Prawitz 2014). We will deal with the whole of these issues in Chapter 7.

According to what we have said so far, it seems to us that a formalization of the theory of grounds has also an *intrinsic* interest. To better understand the problems that this theorys poses, we believe we need to know *what* it is, or *how* it can be further refined; but to understand in detail what the theory of grounds is, and how it can be formally systematized, is also *in and of itself* important, if only for the originality and progress that, in reference to certain essential problems of logic, and as we hope to have already amply proved, Prawitz is able to obtain with this new approach.

4.3 Towards a formal approach to grounds

A full understanding of the formal apparatus which, starting from the next section, we will attribute to the theory of grounds, requires in our opinion the introduction of some technical tools, as well as a quick exposition of the formalizations that Prawitz has developed over the years, and that result similar or conceptually linked to the ground-theoretic approach. As for the first point, we will refer to the so-called Curry-Howard isomorphism - for the exposure of which we will use, in addition to *The notion formula-as-types of construction* by William Alvin Howard (Howard 1980), *Proofs and types* by Jean-Yves Girard, Paul Taylor and Yves Lafont (Girard, Taylor & Lafont 1993) and, for some details, *Lectures on the Curry-Howard isomorphism*, by Morten Heine Sørensen and Pawel Urzyczy (Sørensen & Urzyczyn 2006); Prawitz's aforementioned proposals are instead contained in *Constructive semantics* (Prawitz 1970) and *On the relationship between the Heyting and Gentzen approaches meaning* (Prawitz 2016).

4.3.1 The Curry-Howard isomorphism

Described for the first time in *The formula-as-type notion of construction* (Howard 1980) with reference to the only constants $\wedge$, $\rightarrow$ and $\forall$, and subsequently also extended to the constants $\vee$, $\exists$ and $\bot$, the Curry-Howard isomorphism - named after its creators, Haskell Curry and Howard - identifies a deep connection between Gentzen's system of natural deduction for first-order intuitionistic logic `IL` - referred to in Section 2.5.1 - and what we will call typed first-order λ-calculus. The interest of the relation is substantially twofold: first, the isomorphism defines a bi-univocal correspondence between derivations of `IL` and terms of the typed λ-calculus, offering an operational interpretation of derivations; to such a bijection, which alone of course is not naturally sufficient to guarantee isomorphism, an invariance of normalization, moudulo the bijection, is added, together with the derived properties - which, again in Section 2.5.1, we have seen proven by Prawitz (Prawitz 2006) for `IL`.

In Section 3.2.2 we have said that, among the theories influenced by the Curry-Howard isomorphism, there is Martin-Löf' intuitionistic type theory. In that section, we have also argued that one of the basic moves consists in the adoption of the *formulas-as-types conception*. Therefore, referring here the same inspiration to the specific of the derivations in IL,

> the rules of natural deduction then appear as a special way of constructing functions: a deduction of β on the hypotheses $\alpha_1, ..., \alpha_n$ can be seen as a function $t[x_1, ..., x_n]$ which associates to elements $a_i \in \alpha_i$ a result $t[a_1, ..., a_n] \in \beta$. (Girard, Taylor & Lafont 1993, 11)

In order to substantiate the idea of translating IL derivations into functional expressions, it is necessary to provide a formal language in the which functional expressions can be expressed rigorously. This language will be referred to a first-order logic language L, and its types will be the elements of the set $\mathtt{FORM}_L \cup \{0\}$ with 0 constant type for terms of L. The language will have individual typed variables, and functional symbols D, for pair-formation, D_i $(i = 1, 2)$, for pair-projections, $\mathtt{inj}_i^\beta$ $(i = 1, 2)$, for formation of a term of type $\alpha \vee \beta$ $(i = 1)$ or $\beta \vee \alpha$ $(i = 2)$ from a term of type α, $\mathtt{case}\ [\alpha, \beta]\ t_1\ \mathtt{of}\ t_2$, for proof by cases on a term of type $\alpha \vee \beta$, λ, for λ abstraction on (possibly typed) individual variables, application of a λ-abstraction, and $\mathtt{let}\ [y_j, \alpha(y_j)]\ t_1\ \mathtt{in}\ t_2$, for proof on a specific t_1 on a term t_2 of type $\exists x \alpha(x)$.

The analogous of the maximal formulas in the derivations of IL is, in the typed λ-calculus, the concept of *redex*. A redex is a term of such calculus of one of the following forms:

- $D_i(D(t_1, t_2))$ $(i = 1, 2)$
- $\mathtt{case}\ [\alpha, \beta]\ \mathtt{inj}_i^\beta(t_1)\ \mathtt{of}\ t_2\ \mathtt{or}\ t_3$ $(i = 1, 2)$
- $(\lambda \xi^\alpha . t_1) t_2$
- $(\lambda x.t)k$
- $\mathtt{let}\ [x, \alpha(x)]\ D(k, t_1)\ \mathtt{in}\ t_2$

t is said in *normal form* if, and only if, it does not contain redexes. Some equations fix the computation of terms in non-normal form:

$$D_i(D(t_1, t_2)) = t_i\ (i = 1, 2)$$

$$\mathtt{case}\ [\alpha, \beta]\ \mathtt{inj}_1^\beta(t_1)\ \mathtt{of}\ t_2\ \mathtt{or}\ t_3 = t_2[t_1/\xi^\alpha]$$

$$\mathtt{case}\ [\alpha, \beta]\ \mathtt{inj}_2^\beta(t_1)\ \mathtt{of}\ t_2\ \mathtt{or}\ t_3 = t_3[t_1/\xi^\alpha]$$

$$(\lambda \xi^\alpha . t_1) t_2 = t_1[t_2/\xi^\alpha]$$

$$(\lambda x.t)k = t[k/x]$$

$$\texttt{let } [x, \alpha(x)]\ D(k, t_1) \texttt{ in } t_2 = t_2[t_1/\xi^{\alpha(x)}]$$

As can be seen, these equations reflect in the typed λ-calculus Prawitz's reductions for IL. And similarly to what happened there, it is possible to ensure that the result of the computation of a redex of type α with a set of free typed variables Γ is still a term of type α with a set of free typed variables $\Gamma^* \subseteq \Gamma$; it is sufficient to adopt an analogue of Convention 3, of Section 2.5.1, after having introduced a distinction between free and bound variables, and having established that x in $\lambda x.t$ and let $[x, \alpha(x)]$ t_1 in t_2 is the *proper variable* of, respectively, λ-abstraction and let/in.

Convention 10. In every t (1) free and bound variables are all distinct - property (FB) - and (2) proper and non-proper variables are all distinct, and each proper variable is used in at most one application of λ-abstraction or let/in - property (PN).

Strictly speaking, the typed calculus we have outlined is a functional equivalent of ML; in order to be able to complete it to IL, we must find the way to express $(\bot)$. Therefore we add $\bot_\alpha$ ($\alpha \in \texttt{FORM}_L$) to the operational symbols, in such a way that, for each t of type $\bot$, $\bot_\alpha(t)$ has type α. With this modification, we could also have redexes of the type

- $D_i(\bot_{\alpha_1 \wedge \alpha_2}(t))$ $(i = 1, 2)$
- case $[\alpha, \beta]$ $\bot_{\alpha \vee \beta}(t_1)$ of t_2 or t_3
- $\bot_{\alpha \to \beta}(t_1)t_2$
- $\bot_{\forall x \alpha(x)}(t)k$
- let $[x, \alpha(x)]$ $\bot_{\exists x \alpha(x)}(t_1)$ in t_2

about which, however, we avoid worrying by resorting to an analogue of the theorem quoted, regarding the corresponding case of the derivations of IL, again in Section 2.5.1: for every t_1 of type α with set of free typed variables Γ, there is t_2 of type α with set of free typed variables $\Gamma^* \subseteq \Gamma$ such that, for every application of $\bot_\beta$ in t_2, $\beta \in \texttt{ATOM}_L$. At this point, we can enunciate a theorem of normal form.

Theorem 11. For every t_1 of type α with set of free typed variables Γ, there is t_2 in normal form of type α with set of free typed variables $\Gamma^* \subseteq \Gamma$.

More precisely, and similarly to what was done in Section 2.5.2.1 for the arguments in the more general framework of Prawitz's proof-theoretic semantics, we can talk about *normalizability.* t_1 immediately reduces to t_2 if, and only if, t_2 can be obtained by eliminating a redex from t_1 through the above mentioned equations. $t_1, ..., t_n$ is said a *reduction sequence* if, and only if, for every $i \leq n$, t_i immediately reduces to t_{i+1}. t_1 *reduces* to t_2 if, and only if, there is a reduction sequence that begins with t_1 and ends with t_2. t_1 is called *normalizable* if, and only if, there is t_2 in normal form such that t_1 reduces to t_2. Hence, Theorem 11 can be reformulated saying that each t is normalizable.

The frequent analogies so far referred to between IL derivations and terms of the typed λ-calculus are not random: the Curry-Howard isomorphism, in fact, allows to obtain the abovementioned properties of the terms - including normalization - as derived properties of a bijection with corresponding derivations of IL, and vice versa. The bijection can be defined by induction on the complexity of $\Delta \in \texttt{IL}$:

$$\alpha \quad \stackrel{\iota}{\Longrightarrow} \quad \xi^{\alpha}$$

$$\dfrac{\begin{matrix}\Delta_1 & \Delta_2 \\ \alpha & \beta\end{matrix}}{\alpha \wedge \beta}\,(\wedge_I) \quad \stackrel{\iota}{\Longrightarrow} \quad D(\iota(\Delta_1), \iota(\Delta_2))$$

$$\dfrac{\begin{matrix}\Delta \\ \alpha_1 \wedge \alpha_2\end{matrix}}{\alpha_i}\,(\wedge_{E,i}),\ i = 1, 2 \quad \stackrel{\iota}{\Longrightarrow} \quad D_i(\iota(\Delta)),\ i = 1, 2$$

$$\dfrac{\begin{matrix}\Delta \\ \alpha_i\end{matrix}}{\alpha_1 \vee \alpha_2}\,(\vee_I),\ i = 1, 2 \quad \stackrel{\iota}{\Longrightarrow} \quad \texttt{inj}_i^{\alpha_j}(\iota(\Delta)),\ i, j = 1, 2,\ i \neq j$$

$$\dfrac{\begin{matrix} & [\alpha] & [\beta] \\ \Delta_1 & \Delta_2 & \Delta_3 \\ \alpha \vee \beta & \gamma & \gamma\end{matrix}}{\gamma}\,(\vee_E) \quad \stackrel{\iota}{\Longrightarrow} \quad \texttt{case}\ [\alpha, \beta]\ \iota(\Delta_1)\ \texttt{of}\ \iota(\Delta_2)\ \texttt{or}\ \iota(\Delta_3)$$

$$\dfrac{\begin{matrix}[\alpha] \\ \Delta \\ \beta\end{matrix}}{\alpha \to \beta}\,(\to_I) \quad \stackrel{\iota}{\Longrightarrow} \quad \lambda\xi^{\alpha}.\iota(\Delta)$$

$$\dfrac{\begin{matrix}\Delta_1 \\ \alpha \to \beta\end{matrix} \quad \begin{matrix}\Delta_2 \\ \alpha\end{matrix}}{\beta}\,(\to_E) \quad \overset{\iota}{\Longrightarrow} \quad \iota(\Delta_1)\iota(\Delta_2)$$

$$\dfrac{\begin{matrix}\Delta(x) \\ \alpha(x)\end{matrix}}{\forall y\alpha(y/x)}\,(\forall_I) \quad \overset{\iota}{\Longrightarrow} \quad \lambda y.\iota(\Delta(y/x))$$

$$\dfrac{\begin{matrix}\Delta \\ \forall x\alpha(x)\end{matrix}}{\alpha(k)}\,(\forall_E) \quad \overset{\iota}{\Longrightarrow} \quad \iota(\Delta)k$$

$$\dfrac{\begin{matrix}\Delta \\ \alpha(k/x)\end{matrix}}{\exists x\alpha(x)}\,(\exists_I) \quad \overset{\iota}{\Longrightarrow} \quad D(k,\iota(\Delta))$$

$$\dfrac{\begin{matrix}\Delta_1 \\ \exists y\alpha(y/x)\end{matrix} \quad \begin{matrix}[\alpha(x)] \\ \Delta_2(x) \\ \beta\end{matrix}}{\beta}\,(\exists_E) \quad \overset{\iota}{\Longrightarrow} \quad \texttt{let } [x,\alpha(x)]\ \iota(\Delta_1)\ \texttt{in } \iota(\Delta_2(x))$$

$$\dfrac{\begin{matrix}\Delta \\ \bot\end{matrix}}{\alpha}\,(\bot) \quad \overset{\iota}{\Longrightarrow} \quad \bot_\alpha(\iota(\Delta))$$

The untyped variables in terms of the typed λ-calculus should be understood as indexed, and doing the same thing (as is usual, after all) for the assumption in the derivations of `IL`, it is easy to prove that ι is injective and surjective and, therefore, that it is a bijection. Isomorphism is guaranteed by the fact that Δ_1 immediately reduces to Δ_2 if, and only if, $\iota(\Delta_1)$ reduces immediately to $\iota(\Delta_2)$ – though it has to be said that, beyond normalization, isomorphism also relates to other properties, which we will omit here. So,

> the process of reducing a derivation of `IL` to its normal form can be seen as the process of computing a term. (Usberti 1995, 52)

The basic idea of the Curry-Howard isomorphism is, as you can easily notice, essential for the theory of grounds. In a sense, we can say that the latter illuminates an issue that may arise at this point spontaneously. Isomorphism holds between a formal system and a *closed* λ-calculus; on the other hand, we have seen that the theory of grounds - as already proof-theoretic semantics - sets no limit on the type of usable inferences in the argument structures, thus returning a sort of formal system *open* to the addition of ever new rules. The constructive character of each of these open languages, thus provides a valid semantic basis with respect to the epistemic interests to which Prawitz is oriented.

4.3.2 Constructions and translations

The article *Constructive semantics*, in which Prawitz anticipates – in a somewhat surprising way – the argument that he will then resume with the theory of the grounds only many years later, dates back to 1970.

Constructive semantics revolves around two notions. The first, that of *typed construction object*, determines a set having among its elements both the formulas of a first-order logical language L – on a base B defined as in Section 2.5.1, with relative Post system $\mathtt{S}$ and a set $\mathtt{DER_S}$ of atomic derivations - and those called by Prawitz *constructions* for these formulas, in the terms of which to fix the meaning of the logical constants. The *construction terms*, instead, belong to a formal language in itself, and denote the above constructions under a specific interpretation of the symbols occurring in them. However, the initial definition is that of type: the types will be 0 for terms and formulas of L, and derivations of $\mathtt{DER_S}$, and $\langle \tau_1, \tau_2 \rangle$ or $\tau_1(\tau_2)$, with types already acquired τ_1 and τ_2, for, respectively, pairs and functions from objects of type τ_1 to objects of type τ_2. Terms, formulas, atomic derivations, pairs and constructive procedures determine a set of typed construction objects; the latter is first of all useful for establishing a typing of the formulas of L - with the aim of establishing a correspondence between each formula of L and each construction (term) for this formula.

- $\alpha : 0$ for $\alpha \in \mathtt{ATOM}_L$
- $\alpha_1 : \tau_1, \alpha_2 : \tau_2 \Rightarrow \alpha_1 \wedge \alpha_2 : \langle \tau_1, \tau_2 \rangle$
- $\alpha_1 : \tau_1, \alpha_2 : \tau_2 \Rightarrow \alpha_1 \vee \alpha_2 : \langle \langle \tau_1, \tau_2 \rangle, 0 \rangle$
- $\alpha_1 : \tau_1, \alpha_2 : \tau_2 \Rightarrow \alpha_1 \rightarrow \alpha_2 : \tau_1(\tau_2)$
- $\alpha : \tau \Rightarrow \exists x \alpha(x) : \langle 0, \tau \rangle$
- $\alpha : \tau \Rightarrow \forall x \alpha(x) : 0(\tau)$

Moreover, it becomes possible to establish inductively the notion ω *is a construction for* α *on* B, indicated with $C(\omega, \alpha, B)$:

- $\alpha \in \mathtt{ATOM}_L \Rightarrow \omega \in \mathtt{DER_S}$
- α is of the form $\alpha_1 \wedge \alpha_2 \Rightarrow \omega$ is of the form $\langle \omega_1, \omega_2 \rangle$ with $C(\omega_1, \alpha_1, B)$ and $C(\omega_2, \alpha_2, B)$
- α is of the form $\alpha_1 \vee \alpha_2 \Rightarrow \omega$ is of the form $\langle \langle \omega_1, \omega_2 \rangle, \alpha_i \rangle$ with $i, j = 1, 2$, $i \neq j$, $C(\omega_i, \alpha_i, B)$, and ω_j arbitrary object of type α_j
- α is of the form $\alpha_1 \rightarrow \alpha_2 \Rightarrow \omega$ is an object of type of α such that, for every ω_1 such that $C(\omega_1, \alpha_1, B)$, $C(\omega(\omega_1), \alpha_2, B)$ [Prawitz takes into account here also extensions B^+ of B, but we will leave out this detail to comply with the subsequent developments of proof-theoretic semantics (Prawitz 1973)]
- α is of the form $\exists x \alpha(x) \Rightarrow \omega$ is of the form $\langle t, \omega_1 \rangle$ with $t \in \mathtt{TERM}_L$ and $C(\omega_1, \alpha(t), B)$
- α is of the form $\forall x \alpha(x) \Rightarrow \omega$ is an object of type of α such that, for every $t \in \mathtt{TERM}_L$, $C(\omega(t), \alpha(t), B)$

Given the previous definitions, it is easy to verify that $C(\omega, \alpha, B)$ if, and only if, ω and α are objects of the same type. We will say then that α closed is *constructively true on B* if, and only if, it exists ω such that $C(\omega, \alpha, B)$; the same will apply for α open if, and only if, it exists ω such that $C(\omega, CL(\alpha), B)$. Therefore, generally speaking, α is *constructively valid* if, and only if, α is constructively true on every B. We will say that α is intuitionistically true on B if, and only if, α is constructively true on B for B consistent; then α will be *intuitionistically valid* if, and only if, α is intuitionistically true on each consistent B.

The construction terms are in this framework introduced by Prawitz in order to prove that ML and IL - as in Definition 7 - are correct in terms of construtive and intuitionistic validity. To do this, we need a formal language with typed individual variables and functional symbols D, for pair formation, D_i $(i = 1, 2)$, for projection on pair, λ, for λ-abstraction on typed individual variables, application of a λ-abstraction, $\mathtt{c}$, for a sort of "choice function" such that $\mathtt{c}(t_1, t_2, t_3, t_4) = t_1$ if $t_3 = t_4$ and $\mathtt{c}(t_1, t_2, t_3, t_4) = t_2$ if $t_3 \neq t_4$, $\mathtt{S}$, for a substitution function such that $\mathtt{S}(t_1, ..., t_n, u_1, ..., u_n, z) = z[u_1, ..., u_n / t_1, ..., t_n]$. With *value of a term t* we intend the term t^* such that $t = t^*$; it is easy to prove that, for each t, there exists a unique t^*, and furthermore that t and t^* always have the same type. Therefore, given $\alpha_1, ..., \alpha_n, \beta$ formulas of L with $FV(\alpha_1) \cup ... \cup FV(\alpha_n) \cup FV(\beta) = \{y_1, ..., y_m\}$, we will say that a term t is (*intuitionistically*) *appropriate for* $\alpha_1, ..., \alpha_n \vdash \beta$ if, and only if, (1) the free variables of t are a subset of $\{x_1^0, ..., x_m^0, x^{\tau_1}, ..., x^{\tau_n}\}$, with $\alpha_i : \tau_i$

($i \leq n$) and moreover (2) for every (consistent) base B on L, for every $t_1, ..., t_n, u_1, ..., u_m$ objects on B, called z the term obtained from t by replacing x^{τ_i} with t_i and x_i^0 with u_i, $C(t_i, \mathtt{S}(y_1, ..., y_m, u_1, ..., u_m, \alpha_i), B)$ ($i \leq n$) implies $C(z, \mathtt{S}(y_1, ..., y_m, u_1, ..., u_m, \beta), B)$. Thus, if $n = 0$, t is intuitionistically appropriate for $\alpha_1, ..., \alpha_n \vdash \beta$ if, and only if, for every (consistent) base B, $C(t, CL(\beta), B)$. We can now prove that, if $\alpha_1, ..., \alpha_n \vdash_{\mathtt{ML[IL]}} \beta$, there is a term of the just described formal language which is (intuitionistically) appropriate for $\alpha_1, ..., \alpha_n \vdash \beta$. As an easy corollary, if $\vdash_{\mathtt{ML[IL]}} \alpha$, then α is, respectively, constructively or intuitionisticall valid: it is indeed ensured of a term t appropriate for $\vdash \alpha$ - and hence such that $C(t, CL(\alpha), B)$ for every (consistent) B. In conclusion, it seems to us important to make two observations.

The first concerns the relation between the program elaborated by Prawitz in *Constructive semantics* and that we find in the articles on proof-theoretic semantics. Undoubtedly, the above terms of the formal language can be extended either, generalizing the Curry-Howards isomorphism, to encodings of a certain class of valid arguments - in particular, valid arguments the valid inferences (and inference rules) of which in non-introductory form are justified by resorting exclusively to selection or composition of, and to substitution in, subarguments - or to particular rewritings of a certain class of proofs - obtained with limitations similar to the previous ones; it is also true that constructions can be viewed as encodings of valid arguments, or rewriting of proofs, in canonical form - despite the above restrictions. However, in *Constructive semantics* Prawitz clearly distinguishes the two notions: a construction is a typed object, as it were, always in canonical form, whereas a term is a typed expression of a specific formal language that can be in canonical or non-canonical form. There is nothing like that in proof-theoretic semantics and, from this point of view, *Constructive semantics* is particularly interesting with respect to the proofs-as-chains and recognizability problems, and with respect to their possible solution. However, it should also be noted that in *Constructive semantics* it is missing, or at least it is not explicitly formulated, any mention of the problem of (valid) inferences; it becomes therefore significant to emphasize how the theory of grounds resumes, with particular reference to this problem, many of the ideas of *Constructive semantics*.

The last point concerns the formal language that we have defined. It is designed for the definition of terms suitable to denote constructions, and this in view of a proof of correctness of `ML` and `IL` in terms of validity, respectively, constructive or intuitionistic. To do this, a closed language, in a certain sense equivalent to the typed λ-calculus from the previous section, is sufficient. What we have already observed on that occasion is therefore valid; the general character of the theory of grounds requires to take into account open languages. With respect to this issue, of particular significance is an article in which Prawitz elaborates a translation of valid arguments in BHK proofs, and vice versa.

As we have seen, Prawitz's proof-theoretic semantics referred to in Section 2.5 has, among others, two important sources of inspiration: on the one hand, BHK semantics and, on the other hand, Gentzen's suggestion that the introduction rules establish, or more weakly mirror, the meaning of logic constants - whereas the elimination ones are univocal functions, and as such justifiable, of the corresponding introductions. This dual matrix corresponds to the parallel plans on which, in Prawitz's framework, the notion of proof can be investigated: as an abstract object by virtue of which propositions or sentences are correctly judged as true or asserted, or as a linguistic structure, essentially inferential, dedicated to establishing the correctness of judgments or assertions.

We have also said that Prawitz, in a sense, enriches the picture of the BHK clauses, introducing the same distinction for proofs between canonical and non-canonical cases which, in a natural way, applies in the case of valid arguments; this, for its part, could give hope that the two approaches are not only parallel, but even equi-extensional. More in detail, a non-canonical proof could be made correspond to a valid non-canonical argument, understanding the constructive functions that occur in the former as translations of the rules that occur in the latter; therefore, to the inferences - or inference rules - in a non-introductory form there will correspond applications of constructive procedures - or constructive procedures - the behavior of which is fixed by equations that translate into operational terms the reductions of the starting rules.

This issue is dealt with in the article *On the relation between Heyting's and Gentzen's approaches to meaning* (Prawitz 2016), in which Prawitz observes significantly how

> Gentzen was concerned with what justifies inferences and thereby with what makes something a valid form of reasoning. These concerns were absent from Heyting's explanations of mathematical propositions and assertions. The constructions that Heyting refers to in his meaning explanations [...] are mathematical objects [...]. They are not proofs built up from inferences. (Prawitz 2016, 5 - 6)

This conceptual difference, however, may not be so profound as to prevent the translation of one approach into the other, and vice versa. According to Prawitz himself, moreover, the differences

> do not rule out the possibility that the existence of such proofs nevertheless comes materially to the same. For instance, a BHK-proof of an implication $\alpha \rightarrow \beta$ is defined as an operation that takes a BHK-proof of α into one of β, and a closed Gentzen proof of $\alpha \rightarrow \beta$ affords similarly a construction that takes a Gentzen proof of α into one of β [...]. Such similarities may make one expect that one can construct a BHK-proof given a Gentzen proof and vice versa. (Prawitz 2016, 6)

Prawitz then proposes to define a mapping from closed valid arguments to BHK proofs and, vice versa, a mapping from BHK proofs to closed valid arguments. The intention is that this mapping is, more strongly, a constructive bijection: given a BHK proof, it is possible to construct a closed valid argument that corresponds to it and, vice versa, from a closed valid argument we can extract an appropriate BHK proof. It should be pointed out, however, that Prawitz's translations here are referred to a notion of a closed valid argument that is significantly different from that illustrated in Section 2.5.2.1; in addition, they also concern a notion of "constructive validity", obtained by understanding intuitionistically the existential involved in the definition of a closed valid argument, and a notion of "strong validity" that we have omitted in this work, but that Prawitz had already introduced, although in a different form, in the repeatedly mentioned *Towards a foundation of a general proof theory* (Prawitz 1973). Without going into detail, we limit ourselves here to summarize the results achieved by the Swedish logician: (1) the translation from closed valid arguments to BHK proofs only works when "valid" means "strongly valid" or "valid in a constructive sense"; (2) the translation from BHK proofs to closed valid arguments only works when "valid" does not mean "strongly valid". In conclusion, Prawitz is unable to achieve a full equi-extensionality, showing on the contrary how the latter depends in an essential way on the type of validity attributed to the argumental structures.

Part III

A formal theory of grounding

Chapter 5

Languages of grounding

5.1 General overview

In this Chapter we identify a class of *languages of grounding*, namely formal languages the terms of which express *grounds* or *operations on grounds*. The link between terms, on the one hand, and grounds or operations on grounds that they express, on the other, is specified through a notion of *denotation*, defined as a function from languages of grounding to grounds or operations on grounds. Since some limitative results show that, for each language of grounding at least as powerful as Heyting first-order arithmetic, there are grounds or operations on grounds denoted by no term, no language of grounding, with the relative denotation function, can be called "definitive". It is therefore necessary to introduce a notion of *expansion* of language of grounding.

The project of this part of the analysis seems to be in line with the suggestions of Prawitz; in his articles on the theory of grounds, and chiefly in *Explaining deductive inference* (Prawitz 2015), it seems that the Swedish logician moves on the double level of a "universe" of grounds and operations on grounds, and of terms that describe the elements of this "universe". The overall focus of these writings is however specifically and mainly philosophical, so that the more strictly "mathematical" aspects, by acting, so to speak, in the background, remain in an embryonic state. Undoubtedly Prawitz provides some clear and paradigmatic examples, but this seems to happen more in the perspective of a substantiation of the general framework in which the reflection is articulated, than with the purpose of a systematic development of the formal context as such. Add to this the fact that the indications available in this sense leave open many and different options on how to continue the discussion, and on how eventually to expand it. The definitions that we will gradually propose will be guided by some background intuitions, and often accompanied by further definitions and related results that specify – in our opinion significantly – the global framework. However, before going into the formal discussion, and in order to provide the reader a better understanding of the reasons and goals that have informed our choices, we consider it appropriate to provide an overview in which to summarize and anticipate the subsequent reasoning.

5.1.1 From grounds to terms, through denotation

When we talk about grounds, we should keep in mind two basic ideas - reasserted by Prawitz several times during his investigation. The first illuminates and substantiates the understanding of the grounds as *abstract* objects. A ground is what we must be in possession of when we judge or assert correctly; it is the reification of a mental state of justification for judgments or assertions. The second concerns the understanding of the grounds as *epistemic* objects. A ground is obtained by performing certain operations; when we deal with grounds, in order to say what they are and which they are, we cannot exempt us from talking about the operations by which we pass from a mental state of justification to another. This suggests that, strictly speaking, the grounds are not the only objects of the correspondent theory; other, and equally important objects are the operations on grounds. But grounds and operations on grounds are notions inseparably related to each other.

5.1.1.1 Grounds, operations, judgements/assertions

A ground justifies a *certain* judgment or a *certain* assertion; it is the reification of a mental state related to a *well-determined* judgment, or to a *well-determined* assertion. In the same way, an operation on grounds allows us to pass from grounds that justify *certain* judgments or *certain* assertions to a *certain* judgment or to a *certain* assertion; the passage takes place from a mental state of justification for a *well-determined* group of judgments or assertions, to a *well-determined* judgment or to a *well-determined* assertion. It therefore makes sense to talk about grounds *for* a specific judgment or a specific assertion, and about an operation on grounds *for* a specific judgment or a specific assertion *on the base of* other specific judgments or other specific assertions. The link that is so established between grounds and operations on grounds, on the one hand, and judgments or assertions on the other, allows us to set more precisely the different types of grounds and operations on grounds, and to clarify the link between the former and the latter. Obviously, in order to make the investigation precise, well-founded and not circular, we need a starting point from which it can start.

From the perspective of what Schroeder-Heister (Schroeder-Heister 2008, 2012), and with him Kosta Došen (Došen 2015), have called *dogma of the primacy of the categorical over the hypothetical*, the initial notion is that of ground for a categorical judgment or assertion. Categorical judgments and assertions involve propositions or sentences, which we will understand here as expressed by closed formulas. In this case, the grounds will be fixed by the clauses we have illustrated in Section 4.1.3.2 – and which will be resumed below; in particular, on par with the proposition or sentence involved in the judgment or assertion for which the ground is a ground, it will be a closed object – the primitive operation for the main logical constant of the proposition or sentence of reference is applied to objects, of which the linguistic expression is devoid of individual variables or free ground variables, or binds all the individual variables or ground variables free within the arguments to which it is applied. Obviously, the fact that to the closed character of a categorical judgment or assertion corresponds the closed character of its ground depends on the fact that the justification of a categorical judgment or assertion cannot be based on the reference to *arbitrary* individuals or on *assumptions*; if the ground must act as a justification, it will not have to require individual variables or free ground-variables.

After having acquired the notion of ground for a categorical judgment or assertion, we can introduce a first, elementary class of operation on grounds, which we will call of *first level*. While the grounds for categorical judgments or assertions are closed, a first level operation on grounds is an object that, more generally, can apply to individuals in an appropriate domain, or to grounds. In describing first-level operations, we will proceed gradually. We will start from those that only apply to individuals, then we will pass to those that only apply to grounds, and finally, as a combination of these first two cases, we will discuss those that apply both to individuals and to grounds. On the other hand, the values produced by these operations will always be grounds.

In what follows, we will understand general judgements and assertions as involving open formulas, and hence as having the form $\vdash \alpha(x_1, ..., x_n)$. A ground for $\vdash \alpha(x_1, ..., x_n)$ is to be understood as a function f that associates individuals from a reference domain D to grounds for $\vdash \alpha(k_1, ..., k_n)$, with $k_i \in D$ $(i \leq n)$. If we indicate with $\mathtt{Gr}^{\alpha(k_1,...,k_n)}$ the class of the grounds for $\vdash \alpha(k_1, ..., k_n)$, we will thus have

$$f \colon D^n \to \bigcup_{k_1,...,k_n \in D} \mathtt{Gr}^{\alpha(k_1,...,k_n)}$$

and

$$f(k_1, ..., k_n) = g \Leftrightarrow g \in \mathtt{Gr}^{\alpha(k_1,...,k_n)}$$

The function will therefore have a linguistic expression with individual variables

$$f(x_1, ..., x_n)$$

such that, for every $k_i \in D$ $(i \leq n)$,

$$f(k_1, ..., k_n) \in \mathtt{Gr}^{\alpha(k_1,...,k_n)}$$

i.e., is a ground for $\vdash \alpha(k_1, ..., k_n)$. In this case, we will speak of an operation of operational type

$$\alpha(x_1, ..., x_n).$$

For general judgments or assertions, we have correctness if what has been judged or affirmed is valid, depending on no assumption, for arbitrary individuals corresponding to the free individual variables involved in the propositions or sentences of reference; the grounds for general judgments or assertions can be understood as functions of which the linguistic expressions involve exclusively these individual free variables and which, when applied to individuals in the reference domain, produce grounds for the categorical judgments or assertions obtained by performing the same replacement in the propositions or sentences of reference. Obviously, the fact that to the openness of a general judgment or assertion, in the terms of the occurrence of only individual free variables, corresponds the openness of its ground in terms of the occurrence of only individual free variables, depends on the fact that the justification of a general judgment or assertion is based on the reference to only arbitrary individuals. Therefore, if the ground must act as a justification, its linguistic expression will have to involve only free individual variables.

Then, we have hypothetical judgments or assertions, involving a proposition or sentence depending on other propositions or sentences, and hence having the form $\alpha_1, ..., \alpha_n \vdash \beta$, for $\alpha_1, ..., \alpha_n, \beta$ closed. A ground for $\alpha_1, ..., \alpha_n \vdash \beta$ is to be understood as a function f that associates grounds for $\vdash \alpha_i$ $(i \leq n)$ to grounds for $\vdash \beta$. If we indicate with $\mathtt{Gr}^{\alpha_i}$, $\mathtt{Gr}^{\beta}$ the class of the grounds for $\vdash \alpha_i$ $(i \leq n)$ and $\vdash \beta$, respectively, we will thus have

$$f \colon \mathtt{Gr}^{\alpha_1} \times \cdots \times \mathtt{Gr}^{\alpha_n} \to \mathtt{Gr}^{\beta}.$$

The function will therefore have a linguistic expression with typed variables - that we have called, and will call in the sequel, ground-variables -

$$f(\xi^{\alpha_1}, ..., \xi^{\alpha_n})$$

such that, for every $g_i \in \mathtt{Gr}^{\alpha_i}$ $(i \leq n)$,

$$f(g_1, ..., g_n) \in \mathtt{Gr}^{\beta}$$

i.e., is a ground for $\vdash \beta$. In this case, we will speak of an operation on grounds of operational type

$$\alpha_1, ...\alpha_n \rhd \beta.$$

We have here correctness if what has been judged or affirmed is valid, without the reference to arbitrary individuals, depending on a certain number of closed assumptions; the grounds for hypothetical judgments or assertions can be understood as functions of which the linguistic expressions involve exclusively free ground-variables, each of an appropriate closed type, and which, when appropriately applied to grounds, produce grounds for the dependent categorical judgment or assertion. Obviously the fact that to the openness of a hypothetical judgment or assertion, in the terms of the dependence of a certain proposition or sentence on other propositions or sentences, corresponds the openness of their grounds in terms of the occurrence of only free ground variables, each of the appropriate closed type, depends on the fact that the justification of a hypothetical judgment or assertion refers to closed assumptions. Therefore, if the ground must act as a justification, it will have to involve only free ground-variables, each of the appropriate closed type.

By combining hypothetical judgments or assertions with general judgments or assertions, we obtain open-hypothetical judgments or assertions. They involve a possibly open formula depending on other possibly open formulas, and thus have the form $\alpha_1, ..., \alpha_m \vdash \beta$, for $\alpha_1, ..., \alpha_m, \beta$ possibly open. A ground for $\alpha_1, ..., \alpha_m \vdash \beta$ is to be understood as a function f that operates in the following way. Let D be our reference domain, and let $\{x_1, ..., x_n\}$ be the set of the individual variables occurring free in $\alpha_1, ..., \alpha_m, \beta$. Given a n-tuple $\langle k_1, ..., k_n \rangle$ of elements of D, and given a m-tuple $\langle g_1, ..., g_m \rangle$ of grounds such that g_i is a ground for $\vdash \alpha_i[k_1, ..., k_n/x_1, ..., x_n]$ $(i \leq m)$, f associates to the pair $\langle \langle k_1, ..., k_n \rangle, \langle g_1, ..., g_m \rangle \rangle$ a ground for $\vdash \beta[k_1, ..., k_n/x_1, ..., x_n]$. So, let us indicate with $\mathtt{Gr}^{\alpha_i[k_1,...,k_n/x_1,...,x_n]}$ the class of the grounds for $\vdash \alpha_i[k_1, ..., k_n/x_1, ..., x_n]$ $(i \leq m)$, and given

$$D^n \times \bigcup_{k_1,...,k_n \in D} (\mathtt{Gr}^{\alpha_1[k_1,...,k_n/x_1,...,x_n]} \times \cdots \times \mathtt{Gr}^{\alpha_m[k_1,...,k_n/x_1,...,x_n]})$$

let us consider its subclass

$$B = \{\langle\langle k_1, ..., k_n\rangle, \langle g_1, ..., g_m\rangle\rangle \mid g_i \in \mathtt{Gr}^{\alpha_i[k_1,...,k_n/x_1,...,x_n]}, i \leq m\}.$$

We will therefore have

$$f\colon B \to \bigcup_{k_1,...,k_n \in D} \mathtt{Gr}^{\beta[k_1,...,k_n/x_1,...,x_n]}$$

and

$$f(\langle\langle k_1, ..., k_n\rangle, \langle g_1, ..., g_m\rangle\rangle) = g \Leftrightarrow g \in \mathtt{Gr}^{\beta[k_1,...,k_n/x_1,...,x_n]}.$$

Observe that this case would have been sufficient to describe all the first-level operations. Starting from it, indeed, we can obtain the other two by putting

$$\{x_1, ..., x_n\} = \emptyset \text{ or } \{\xi^{\alpha_1}, ..., \xi^{\alpha_m}\} = \emptyset.$$

The function will therefore have a linguistic expression with individual and ground-variables

$$f(x_1, ..., x_n, \xi^{\alpha_1}, ..., \xi^{\alpha_m})$$

such that, for every $k_i \in D$ $(i \leq m)$,

$$f(k_1, ..., k_n, \xi^{\alpha_1[k_1,...,k_n/x_1,...,x_n]}, ..., \xi^{\alpha_m[k_1,...,k_n/x_1,...,x_n]})$$

is an operation of operational type

$$\alpha_1[k_1, ..., k_n/x_1, ..., x_n], ..., \alpha_m[k_1, ..., k_n/x_1, ..., x_n] \rhd \beta[k_1, ..., k_n/x_1, ..., x_n]$$

that is, for every g_j ground for $\vdash \alpha_j[k_1, ..., k_n/x_1, ..., x_n]$ $(j \leq m)$,

$$f(k_1, ..., k_n, g_1, ..., g_m)$$

is a ground for $\vdash \beta[k_1, ..., k_n/x_1, ..., x_n]$. In this case, we will speak of an operation of operational type

$$\alpha_1, ..., \alpha_m \rhd \beta.$$

We have here correctness if what has been judged or affirmed is valid for arbitrary individuals, corresponding to the free individual variables in the propositions or sentences of reference, and dependent on a certain number of possibly open assumptions. The grounds for hypothetical-general judgments or assertions can be understood as functions of which the linguistic expressions involve the free individual variables of the propositions or sentences of reference, and free ground-variables of the appropriate type, and which, when appropriately applied to grounds (possibly on formulas in which the free individual variables have been replaced with names of individuals of the reference domain), produce a ground for the dependent (categorical or general) judgment or assertion (with possible replacements of free individual variables with individuals from the reference domain). Obviously the fact that to the openness of a hypothetical-general judgment or assertion, in terms of the dependence on a certain formula possibly open on other formulas possibly open, corresponds the openness of their grounds in terms of the occurrence of individual variables and free ground-variables of the appropriate type, depends on the fact that the justification of a hypothetical-general judgment or assertion is based on the reference to arbitrary individuals, and involves possibly open assumptions. If the ground must act as a justification, it will have to involve individual variables and free ground-variables of the appropriate type.

Starting from the notion of operation on grounds of the first level, we can outline a more complex class of operations on grounds, which we will call of *second level*. In addition to individuals and grounds, these objects can also apply to first-level operations on grounds.

Here, we have to take into account hypothetical, or general-hypothetical judgments or assertions, where at least one of the assumptions, and possibly also the depending judgment or assertion, is again a hypothetical, or general-hypothetical judgment or assertion. The general form will hence be $\tau_1, ..., \tau_m \vdash \tau_{m+1}$, where τ_i $(i \leq m+1)$ is either a formula, or a hypothetical, or general-hypothetical judgment or assertion (we have the restriction that, if τ_{m+1} is a hypothetical, or general-hypothetical judgment or assertion, the set of its assumptions is contained in the union set of the assumptions of all the τ_i, for $i \leq m$). A ground for $\tau_1, ..., \tau_m \vdash \tau_{m+1}$ is to be understood as a function f that operates in the following way. Let D be our reference domain, and let $\{x_1, ..., x_n\}$ be the set of the individual variables occurring free in the *codomains* of the $\tau_1, ..., \tau_m, \tau_{m+1}$. Given a n-tuple $\langle k_1, ..., k_n \rangle$ of elements of D, and given a m-tuple $\langle g_1, ..., g_m \rangle$ of grounds or operations on grounds such that g_i is a ground or an operation on grounds of operational type $\vdash \tau_i[k_1, ..., k_n/x_1, ..., x_n]$ $(i \leq m)$, f associates to the pair $\langle \langle k_1, ..., k_n \rangle, \langle g_1, ..., g_m \rangle \rangle$ a ground or operation on grounds of operational type $\tau_{m+1}[k_1, ..., k_n/x_1, ..., x_n]$. Thus, if we indicate with $\mathtt{Gr}^{\tau_i}$ the class of the grounds or of the operations on grounds of operational type τ_i $(i \leq m)$, we proceed in the same way as above, by singling out a corresponding subclass B of

$$D^n \times \bigcup_{k_1,...,k_n \in D} (\mathtt{Gr}^{\tau_1[k_1,...,k_n/x_1,...,x_n]} \times \cdots \times \mathtt{Gr}^{\tau_m[k_1,...,k_n/x_1,...,x_n]}).$$

The fact that only the individual variables in the codomains are mentioned depends on the fact that, in operations of this kind, the arguments may be operations on grounds with any domain *whatsoever* - hence, the variables of the domains may not be actually there. In conclusion, we will have a linguistic expression

$$f(x_1, ..., x_n, \xi^{\tau_1}, ..., \xi^{\tau_m})$$

and we will speak of an operation of type

$$\tau_1, ..., \tau_m \rhd \tau_{m+1}.$$

It applies what we have said about the operations on grounds of first level related to individual variables and ground-variables, although we have to bear in mind that, in this case, some assumptions could in turn be hypothetical-general judgements or assertions.

Some operations on grounds may bind individual variables, in which case the linguistic expression of the operation will not involve the individual variables that it binds involved in the judgment or assertion of which it is ground, and the dependent judgment or assertion will not contain any of the bound individual variables, or it will contain them as individual variables universally quantified. An operation on grounds that binds individual variables will be subject to a restriction similar to that on the introduction rule for $\forall$ or of the elimination rulw for $\exists$ in Gentzen's natural deduction systems. More generally, operations on grounds are subject to restrictions that follow those on the rules for the quantifiers in Gentzen's natural deduction systems, related to the proper variables (introduction of $\forall$ and elimination of $\exists$) and to the being free of a term for a variable in a formula (introduction of $\exists$ and elimination of $\forall$). Some operations on grounds may also bind ground-variables, in which case the judgment or dependent assertion will involve a judgment or assertion that is no longer dependent on the assumptions corresponding to the type of the bound ground-variables.

In our description of the various kinds of operations, we have so far only mentioned their domain and codomain. However, the operations that we will deal with, are not to be understood as functions in a standard set-theoretic sense, i.e., as laws that associate, without further specifications, elements of the domain to elements of the codomain. A ground-theoretic operation is also determined by another parameter, namely, by equations that defines it, by showing how the arguments in the domain are "computed" by the function, and hence how the value of the function on those arguments can be obtained, and which this value is. In addition, the "computation" instruction exhibited by the equation must be such that the function it defines is constructive. Through the equation, it must therefore be possible to effectively "compute" the function on every argument of the domain, to get thereby the corresponding argument in the codomain.

The notion of second-level operation on grounds requires that of first-level operation on grounds, and the latter in turn requires the notion of ground for categorical judgments or assertions. Therefore, we might think that, in giving the clauses that fix the notion of ground for categorical judgments or assertions, there is no need to resort, when proceeding inductively, to notions other than that we intend to fix. However, as we have seen in Section 4.1.3.2, this is not the case, since the definition of ground for categorical judgments or assertions of the form $\alpha \to \beta$ or $\forall x \alpha(x)$ requires the notion of first-level operation on ground as ground for, respectively, hypothetical judgments or assertions of the form $\alpha \vdash \beta$, and general judgments or assertions of the form $\alpha(x)$. Now, if it is really true that the notion of first-level operation on grounds is conceptually posterior to that of ground for categorical judgments or assertion, it is also true that the second cannot be defined without resorting to the first. However, the overall definition is not circular, and this is because the clauses proceed via simultaneous recursion - taking advantage of the fact that primitive operations concern the passage from judgments or assertions which involve formulas of lower logical complexity to judgments or assertions that involve formulas of greater logical complexity. Therefore, in order to know what a ground is for $\alpha \to \beta$ we need to know what a ground is for $\alpha \vdash \beta$, and to know this we need to know what is a ground for $\vdash \alpha$ and what is a ground $\vdash \beta$; in the same way, in order to know what a ground is for $\forall x \alpha(x)$, we need to know what a ground is for $\alpha(x)$, and to know this we need to know what a ground is for $\vdash \alpha(k)$, for k individual in the reference domain.

5.1.1.2 Denotation of terms

The languages of grounding that we will be developing will consist of terms to be understood as "names" of some of the "inhabitant" of our "universe". The terms are typed on a first-order logical language, related to a base, and starting from a similar typing of the elements of the alphabet. When defining the denotation, we will follow the following general scheme: a closed term of type α denotes a ground for the categorical judgment or assertion $\vdash \alpha$; an open term with only free individual variables $x_1, ..., x_n$ of type $\alpha(x_1, ..., x_n)$ denotes a ground for the general judgment or assertion $\alpha(x_1, ..., x_n)$, namely an operation

$$f(x_1, ..., x_n)$$

of operational type

$$\alpha(x_1, ..., x_n);$$

an open term with only ground-variables $\xi^{\alpha_1}, ..., \xi^{\alpha_n}$ of type β denotes a ground for the hypothetical judgment or assertion $\alpha_1, ..., \alpha_n \vdash \beta$, namely an operation on grounds

$$f(\xi^{\alpha_1}, ..., \xi^{\alpha_n})$$

of operational type

$$\alpha_1, ..., \alpha_n \rhd \beta;$$

an open term with free individual variables $x_1, ..., x_n$ and free ground-variables $\xi^{\alpha_1}, ..., \xi^{\alpha_m}$ of type β denotes a ground for the general-hypothetical judgment or assertion $\alpha_1, ..., \alpha_m \vdash \beta$, namely an operation on grounds

$$f(x_1, ..., x_n, \xi^{\alpha_1}, ..., \xi^{\alpha_m})$$

of operational type

$$\alpha_1, ..., \alpha_m \rhd \beta.$$

The second-level operation on grounds are denoted by no term, since the languages of grounding do not include variables typed on first-level operations on grounds (see Prawitz 2015).

5.1.1.3 A summary scheme

To sum up and clarify what we have been saying so far, we propose a summary scheme, showing how the objects of our "universe" are associated, on the one hand, to the judgments or assertions for which they are grounds, and on the other, to the terms of the languages of grounding. The schema will be accompanied by some additional remarks, that will serve to introduce us to the following sections.

Act	**Object**	**Term**
Categorical $\vdash \alpha$	Closed ground g	Closed of type α
General $\vdash \alpha(x_1, ..., x_n)$	Operation on grounds $f(x_1, ..., x_n)$ of operational type $\alpha(x_1, ..., x_n)$	Open of type $\alpha(x_1, ..., x_n)$ with free variables $x_1, ..., x_n$

Hypothetical $\alpha_1, ..., \alpha_n \vdash \beta$	Operation on grounds $f(\xi^{\alpha_1}, ..., \xi^{\alpha_n})$ of operational type $\alpha_1, ..., \alpha_n \rhd \beta$	Open of type β with free variables $\xi^{\alpha_1}, ..., \xi^{\alpha_n}$
General-hypothetical $\alpha_1, ..., \alpha_m \vdash \beta$	Operation on grounds $f(x_1, ..., x_n, \xi^{\alpha_1}, ..., \xi^{\alpha_m})$ of operational type $\alpha_1, ..., \alpha_m \rhd \beta$	Open of type β with free variables $x_1, ..., x_n$ and $\xi^{\alpha_1}, ..., \xi^{\alpha_m}$
General hypothetical $\tau_1, ..., \tau_m \vdash \tau_{m+1}$ for some τ_i general-hypothetical ($i \leq m$, possibly also $i = m + 1$)	Operation on grounds $f(x_1, ..., x_n, \xi^{\tau_1}, ..., \xi^{\tau_m})$ of operational type $\tau_1, ..., \tau_m \rhd \tau_{m+1}$	None

Table 5.1: `Acts, objects and denoting terms`

It is now worthwhile making three observations. First of all, we anticipate that the impossibility to express second-level operations will be overcome by attributing denotation to the operational symbols for the construction of terms. Indeed, denotation of terms is defined inductively, on the basis of a denotation function for the elements of the alphabet of the language of grounding. In the alphabet, in turn, we will have operational symbols typed on operational types for *first-level* operations, depending on the operations that such symbols are meant to express. If the symbol binds ground-variables on the index i, the denotation function for the alphabet will assign to it a *second-level* operation. The i-th assumption of the operational type of this operation is a hypothetical or general-hypothetical judgment or assertion with as assumption the assumption typing the bound ground-variable, and with as depending judgment or assertion the formula with index i in the operational type of the symbol. In other words, if the type of the operational symbol has β as i-th assumption, and if the symbol binds the variable ξ^α on the index i, the denotation of this symbol will be a second-level operation on grounds, the operational type of which has as i-th assumption the hypothetical or general-hypothetical judgment or assertion $\alpha \vdash \beta$. Alternatively, we could have let the languages of grounding contain variables typed on first-level operations on grounds. We shall actually pursue this strategy, but only starting from Chapter 6, with reference to the formal systems that we will develop therein. For the moment, we prefer to leave it aside, in order to avoid an excessive degree of complexity in languages of grounding and in the definition of denotation.

The second observation concerns the general framework that we are in process of outlining. Our "universe" is "inhabited" by: saturated grounds, for categorical judgments or assertions; unsaturated grounds, that is, functions defined on individual or grounds, for general, hypothetical, or general-hypothetical judgements or assertions of first level, and functions defined on individual, grounds and first-level operations on grounds for general, or general-hypothetical judgements or assertions of second level. To this dichotomy there corresponds, at the linguistic level, the one between closed terms, with no free individual or ground-variables, and open terms, where we instead have free occurrences of individual or ground-variables. The relevant notions for the unsaturated cases are defined by essentially resorting to the saturated cases. That a first-level operation has a certain operational type is explained by requiring that the operation gives the right saturated value when applied to saturated arguments, and that a second-level operation has a certain operational type is explained by requiring that the operation gives the right unsaturated value of lower level when applied to unsaturated arguments of lower level. As we will see when defining the denotation of the elements of the alphabet and of the terms of the languages of grounding, this line of thought applies also to the linguistic level. Given an open term, its denotation will be established by referring to the saturated *denotata* of its closed instances.

Thus, our point of view is strongly fregean (see in particular 1884, 1891, 2001). Frege's objects are here our saturated grounds, whereas Frege's functions/concepts are our unsaturated grounds, that is, our operations from individuals, grounds and operations, to grounds and operations. On the other hand, it seems to us that this standpoint is in line with the one that Prawitz himself adopts, above all in his first ground-theoretic papers (Prawitz 2009, 2012a, 2013). As we said in Chapter 4, the expressions "saturated" and "unsaturated", as well as the idea of grounds and their expressions being closed/complete or open/incomplete, is present also in the earliest ground-theoretic writings of the Swedish logician (the same conclusion seems to be upheld by Tranchini 2014a, 2018).

This remark has, it seems to us, an important and deep consequence, concerning the issue about the identity of the "inhabitants" of our universe, and of the terms of the languages of grounding that denote these "inhabitants", as well as the issue about the structure and the behavior of the equations that define the operations on grounds. The saturated grounds can be said to be identical when it is possible to show compositionally that their subparts are such. Instead, the operations can be said to be identical when they have the same domain, and they produce the same saturated values when applied to the same saturated arguments or, at the second level, when this property holds for the unsaturated values generated out of equal unsaturated values. Similarly, two closed terms are identical when they reduce or expand one to another, whereas two open terms are identical when this property is preserved for all their closed instances. If we now focus on defining equations, the latter will not in general fix an operation by showing to which operation, or combination of operations, the first is equal. Rather, the equations show which values the operations produce when they are applied to saturated values. Or better, an equation of the first kind will be admissible only under previous recognition of the fact that *definiendum* and *definiens* produce the same saturated values on the same saturated arguments or, at the second level, the same unsaturated values of lower level on the same unsaturated arguments of lower level. Hence, the approach we will adopt relatively to the identity of the operations and of the terms that denote them, will be, not intensional, but extensional - to put it with Frege, our approach will not single out a link between the properties of the operations and of their computation instructions, but it will require to pass through their courses-of-values.

The circumstance that identity between operations, as well as identity between terms denoting such operations, cannot be estabilished, or used in definitions, without passing through the saturated/closed case, reminds of a distinction between *open derivations* and *open valid arguments*, in the respective frameworks of normalization theory and proof-theoretic semantics - the former being a theory of the structural properties of the derivations of a formal system, obtained by means of reduction procedures, whereas the latter is an investigation about the semantic properties of given argument structures, obtained by means of appropriate justifications for non-introductory rules. The *normalizability* of an open derivation is a property of the derivation as such, and it can be obtained by applying *syntactic* operations to the derivation itself, without referring to its closed instances. Instead, the *validity* of an open argument structure requires to look at the closed instances of the structure, and to perform on them *adequate* operations, that respect semantic *desiderata* concerning the whole class of valid arguments. Thanks to what Schroeder-Heister (Schroeder-Heister 2006) calls the *fundamental corollary* of normalization theory, the two notions are of course linked, and we could even say that the second is a kind of "semantization" of the first. Though, they are also conceptually distinct, and refers to two levels philosophically as well as formally different. Both in this chapter, and in the next one, we will face again with questions concerning identity. The similarity between how this notion is framed within the theory of grounds, and the distinction between open derivations and open valid arguments, will then be sharper - in particular, when discussing *identity axioms* occurring in the formal systems developed in Chapter 6.

We conclude with the third observation, which also brings us back to valid arguments in proof-theoretic semantics. Throughout the investigation, for the purpose of giving a more concrete idea of what the ground-theoretic operations are, and of how they behave according to their operational type, we will appeal, although in a purely illustrative way, to proof-theoretic valid arguments. In particular, we will adopt the following schema - see Section 2.5.2.1: a closed valid argument can be understood as representing (a term that denotes) a saturated ground; an open valid argument or a valid inference rule can be understood as representing (in the case of first-level operations, a term that denotes) an unsaturated ground, that is, a first- or second-level operation on grounds. It is however important to keep in mind that, while the idea that valid arguments can be looked at as grounds or operations on grounds is more or less unproblematic, it may not hold that every ground or operation on grounds corresponds to a valid argument (see Prawitz 2016 for limitative results going in this direction).

5.1.1.4 Proper and improper grounds

With reference to the last observation of the previous section, let us consider the following argument structure:

$$\dfrac{\dfrac{\dfrac{\overset{1}{[\alpha(x)]}}{\alpha(x) \rightarrow \alpha(x)}\,(\rightarrow_I),\,1}{(\beta(y) \rightarrow \beta(y)) \rightarrow (\alpha(x) \rightarrow \alpha(x))}\,(\rightarrow_I) \qquad \dfrac{\overset{2}{[\beta(y)]}}{\beta(y) \rightarrow \beta(y)}\,(\rightarrow_I),\,2}{\alpha(x) \rightarrow \alpha(x)}\,(\rightarrow_E)$$

It is an open valid argument for $\alpha(x) \rightarrow \alpha(x)$, that involves the free individual variables x, y. According to the schema above, we could therefore make it correspond to an operation on grounds

$$f(x, y)$$

of operational type

$$\alpha(x) \rightarrow \alpha(x)$$

thereby having more individual variables than those actually occurring in the operational type. The operation will then be denoted by a term with free individual variables x, y, and hence with more free individual variables than those occurring in the operational type of the operation denoted by the term. Of course, the observation applies equally well also to operations defined on grounds or on operations on grounds, that is, operations whose linguistic expression involves ground-variables or variables for operations - as well as to terms with free ground-variables denoting first-level operations defined on grounds.

The point at issue, thus, is that the schema above was too "simplified", as it implies symmetries that do not necessarily hold between operational types and arities of the respective operations. In general, an operation on grounds may be such that its linguistic expression has to involve *more* free individual variables than those occurring in its operational type; a term denoting this operation may contain *more* free individual variables than those occurring in the corresponding operational type. In spite of this difficulty, however, throughout our investigation we will ensure that an operational type for a first-level operation on grounds involves a set of free individual variables that is *contained* in the set of the free individual variables occurring in the expression of an operation on grounds having that operational type - apart from bindings. In other words, if x occurs free in an operational type τ, then the expression of any operation on grounds ϕ having type τ involves x, and x occurs free in all terms (if any) denoting ϕ - apart from bindings. If the set of the individual variables occurring free in τ *coincides* with those of the expression of ϕ, we will say that ϕ is a *proper* ground for the judgment or assertion corresponding to τ.

The setup we have adopted can be justified as follows: the open valid argument for $\alpha(x) \rightarrow \alpha(x)$ exemplified above *is not* in normal form and, through Prawitz's reduction for $\rightarrow$ described in Section 2.5.1, it can be reduced to

$$\dfrac{\overset{1}{[\alpha(x)]}}{\alpha(x) \rightarrow \alpha(x)} \ (\rightarrow_I),\ 1$$

which can in turn be understood as an operation on grounds

$$f(x)$$

of operational type

$$\alpha(x) \rightarrow \alpha(x)$$

the expression of which involves all and only the individual variables occurring free in its type. Likewise, the term denoting the operation that can be associated to the open valid argument can be reduced to a term denoting the operation that can be associated to the reduced open valid argument, and that contains only x as free individual variable. If we generalize this observation, it seems plausible to assume that operations on grounds relative to more individual variables than those occurring in their operational type, and corresponding terms (if any), contain some "detours", and can be reduced to operations on grounds and terms that require all and only the individual variables occurring free in the corresponding operational type. Therefore, the operations

$$f(x, y) \text{ and } f(x)$$

above, have both operational type

$$\alpha(x) \rightarrow \alpha(x)$$

and hence both constitute a ground for the general judgment or assertion $\vdash \alpha(x) \rightarrow \alpha(x)$; but while the second, according to our terminology, is a proper ground, the first one will be called *improper* ground.

5.1.1.5 Total constructive functions

We have said that operations on grounds are functions. However, here we have to reiterate more specifically what many times anticipated throughout Chapter 4; the functions we are going to take into account are, more precisely, *total constructive*. But what do we mean by this expression? In particular, is it possible to refer our explanation to a more basic *definition* of the notion of total constructive function?

To say that an operation on grounds is a total function, means that the operation is a function that converges on all the values in the definition domain. Let us give just two examples: an operation on grounds

$$f(x_1, ..., x_n)$$

of operational type

$$\alpha(x_1, ..., x_m)$$

$(m \leq n)$ is a total function such that, *for every* sequence $k_1, ..., k_n$ of individuals in the reference domain,

$$f(k_1, ..., k_n)$$

is a ground for $\vdash \alpha(k_1, ..., k_m)$: likewise, an operation on grounds

$$f(\xi^{\alpha_1}, ..., \xi^{\alpha_n})$$

of operational type

$$\xi^{\alpha_1}, ..., \xi^{\alpha_n} \rhd \beta$$

is a total function such that, *for every* g_i ground for $\vdash \alpha_i$ $(i \leq n)$,

$$f(g_1, ..., g_n)$$

is a ground for $\vdash \beta$. The reasoning proceeds analogous for first-level operations on grounds of other kinds, as well as, of course, for second-level operations on grounds. The case of totality is therefore quite easily manageable. How are things, on the other hand, with the expression "constructive"?

From a general point of view, it would appear that, in the approach adopted by Prawitz, constructiveness is to be understood as a sort of *effective computability*. An operation on grounds is a constructive function in the sense that, whenever applied to specific values in the definition domain, it is actually possible to compute it so as to obtain the output to which those specific values correspond. After all, it is no coincidence if, both in Prawitz, and in frameworks philosophically akin to the theory of grounds, to the expression "constructive function" is often preferred that of "effective procedure" - and in the present work, we do have often used these expressions as synonyms; an effective procedure is a set of computation instructions such that, by using those instructions, the computation can actually be carried out.

Contemporary mathematical logic offers a series of devices that translate, in formally rigorous terms, the concept of *effectively computable function*, although often exclusively in the context of an investigation into the foundations of mathematics: recursive functions developed by Gödel (Gödel 1931), Peter (Peter 1931) and Stephen Cole Kleene (Kleene 1936) and, within the framework of foundational investigations, Turing machines (Turing 1936 - 1937) and, finally, λ-calculus, which we have already dealt with, and introduced by Alonzo Church (Church 1932). The three theories just mentioned are provably equivalent, in the sense that they declare as effectively computable the same functions; on this base, the so-called *Church-Turing thesis* (Copeland 2017) affirms that a function is effectively computable if, and only if, it is recursive, or equivalently if it is implementable in a Turing machine, or equivalently if it can be translated into a λ-term. When linking a pre-formal or intuitive concept, such as that of effectively computable function, to mathematically precise notions, such as those of recursive function, Turing machine or λ-term, it is often said that the Church-Turing thesis is not subject to a proof or final confutation. However, we might be tempted to exploit it here, in order to define more precisely the functions related to operations on grounds.

We have said that the idea that an operation on grounds is a constructive function can be equated with the idea that an operation on grounds is an effectively computable function. Therefore, we could be led to understand operations on grounds as, more specifically, recursive functions, or Turing machines, or λ-terms; even if the latter are mainly related to a study of the foundations of mathematics, the ideas underlying their definitions could be adapted to the theory of grounds, so to have a well-defined notion of operation on grounds, a well-describable class of operations on grounds. However, this strategy appears problematic, on account of a circularity pointed out by Peter in *Rekursivitat und Konstruktivitat* (Peter 1959).

Although Peter primarily refers to the BHK semantics, from one hand, and to recursive functions, on the other, the central point of her argument is that the definition of recursive function, and therefore definitions of equivalent notions, such as those of a Turing machine or λ-term, cannot be employed in a constructive determination of the meaning of the logical constants without falling into vicious circles. The definitions, in fact, involve an existential quantifier - in the case of recursive functions, as a condition of existence for a scheme of equations that produces expected values, and as the existence condition for the zeros of the μ-operator. Now, this existential quantifier cannot obviously be read in non-constructive terms, since the characterization of the meaning of the logical constants would result in that case inadequate; but if the existential must be read in necessarily constructive terms, the characterization would clearly be circular.

As part of the exposition of BHK semantics in *Meaning and proofs: on the conflict between classical and intuitionistic logic* (Prawitz 1977), Prawitz refers explicitly to Peter's argument (quoting Peter 1959); the Swedish logician concludes that the notion of constructive function employed in the BHK clauses for $\rightarrow$ and $\forall$ must be assumed to be primitive and not further analyzable. But also the clauses that fix what counts as ground for categorical judgments or assertions are intended as a constructive determination of the meaning of logical constants, and therefore also for them it applies the argument according to which it would be circular to specify the operations on grounds, involved in the clauses related to $\rightarrow$ and $\forall$ and intended as constructive functions, in terms of recursive functions, Turing machines and λ-terms. In light of this, in the course of our investigation we will assume therefore as primitive the notion of constructive function. We will limit ourselves just to say that an operation on grounds must be constructive in the sense of being actually computable, on the basis of the consideration made above.

Be that as it may, the character of computability of operations on grounds will be guaranteed by conceiving the latter not as set-theoretic objects - associations from (operations on) grounds that constitute their domain to (operations on) grounds that constitute their codomain - but, in addition, as defined by groups of equations that render them *transformation methods*; the equations have to be such as to actually provide a way to transform the grounds or the operations on grounds in the definition domain into grounds or operations on grounds into the codomain. In this sense, the operations on grounds can be conceived as a generalization of the reduction procedures used by Prawitz in the proof of the normalization theorems (Prawitz 2006) - illustrated in Section 2.5.1 - or of the equations for the elimination of redexes in typed λ-calculus - illustrated in Section 4.3.1.

5.1.2 Languages and expansions

The discourse conducted so far, concerning grounds and operations on grounds, and the way in which the terms of our languages of grounding denote such objects, constitutes the main road along which we intend the development of the languages of grounding as such. We will proceed following the points corresponding to the three sections below.

5.1.2.1 Atomic bases

Grounds and operations on grounds concern well-defined judgments or well-defined assertions, and therefore specific formulas. A ground or an operation on grounds will always be *for* a certain judgment or assertion, which can be expressed also by saying that grounds and operations on grounds have *type* certain (combinations of) formulas. Consequently, also the terms that denote grounds and operations on grounds will be typed, and this starting from a more elementary typing of the elements of the alphabet of the language of grounding to which the terms belong. We call the language to which types belong a *background language*. Here, a background language will always be a first-order logical language.

As is well-known, a first-order logical language comprises first of all a well-specified alphabet, starting from which we define recursively a set of terms and a set of formulas. Beyond the usual logical constants of first-order $\wedge$, $\vee$, $\rightarrow$, $\forall$ and $\exists$, and an atomic constant $\bot$ for the absurd, the alphabets are characterized by a certain *set of individual constants*, a certain *set of relational symbols* and a certain *set of functional symbols* – of which the first and the third can be empty. For such sets it is possible to define a formal atomic system, which we will understand later as a Post-system of the type of that for first-order arithmetic described in Section 2.5.1. The set of individual constants, the set of relational symbols, the set of functional symbols, and, finally the Post-system related to them, constitute altogether an atomic base for a language of grounding on the first-order logical language in question. The atomic bases play a twofold role.

The first is to provide, through the atomic system, a set of individual constants for the language of grounding; to each derivation Δ in the atomic system, it will correspond, in the language of grounding, one and only one individual constant δ , "canonical name" of Δ. The individual constants are some of the terms with minimal complexity – therefore act as a base for the construction of more complex terms – and, in case they do not contain individual or free ground-variable variables, are the only ones that denote grounds for the formulas - obviously atomic - on which they are typed. Furthermore, as a matter of principle, through the atomic bases, it should be possible to *determine the intended meaning* of individual constants, relational symbols and functional symbols, and consequently, of the terms and formulas of the background language. Obviously this is essential when we want to know what the grounds or the operations on grounds denoted by the elements of the alphabet and by the terms of the language of grounding are grounds or operations on grounds for. But, more precisely, how does this determination of meaning happen?

Once again, the atomic system is central. In fact, by providing rules that involve individual constants, relational symbols and functional symbols, it sets the behavior of these symbols and, depending on the role they consequently play in the atomic derivations, permits to consider them as well-defined objects. An example will help to understand better this point. Let us take into account the following rules of a Post-system for first-order arithmetic:

$$\frac{0 \doteq s(t)}{\bot}\,(s_1) \qquad \frac{}{t + 0 \doteq t}\,(+_1) \qquad \frac{}{t \cdot 0 \doteq 0}\,(\cdot_1)$$

Assuming we know the meaning of the functional symbols s – successor function - $+$ - addition - and $\cdot$ - multiplication - they tell us that the individual constant 0 of a first-order arithmetic language behaves in such a way that: 0 is the successor of no term, the result of the addition of a term to 0 is the term itself, and finally the result of the product of each term for 0 is 0. The rules allow us to consider the individual constant 0 as the zero of natural numbers.

It is however clear that a determination of the meaning of this type requires that the rules of the atomic system enjoy some structural properties. The meaning of a symbol will be fixed by a certain set of rules, and each of these rules must plausibly concern this symbol. This remark, apparently simple, actually raises numerous and difficult questions. What is meant by "concern"? Under what conditions and in what way a group of rules concerning a symbol determine its meaning? Is it possible to say that some rules concern a symbol in a more direct way than others, so as to contribute more significantly to the determination of the meaning of the symbol? Will a rule concerning a certain symbol be able to concern others, and if so – as seems to be the case, observing that the rules for 0 set out above also concern the symbols s, $+$ and $\cdot$ - which must be the relationships between the rules concerning the symbol and those that concern the other symbols involved? An answer even if only partially exhaustive to questions like these would need a discussion to itself, and in fact there is a whole theory in literature designed to clarify, resolve and articulate on such issues; it is the theory of meaning based on the notion of an *immediate argumental role* developed by Cozzo first in *Teoria del significato and filosofia della logica* (Cozzo 1994b) and then, with significant differences, in *Meaning and argument. A theory of meaning centered on immediate argumental role* (Cozzo 1994a). Therefore, for reasons of space, in the sequel of our discussion we will pass over the problems just outlined, assuming as primitive the thorny notion of rule for a symbol; it will be tacitly understood as proposed by Cozzo (especially in Cozzo 1994a).

5.1.2.2 Expanding a core language

A language of grounding involves terms built on an alphabet containing - in addition to ground-variables and individual constants - a limited group of operational symbols designed to denote operations on grounds, of which the operational type has a well-defined structure. On the other hand, by explicit indication of Prawitz (Prawitz 2015), a language of grounding must be conceived as open, where "open" means that new operational symbols, for new operations on grounds, can be indefinitely added to it. As we have already argued, this request has a double motivation.

The first derives from the fact that the operations on grounds are associated with inferential acts. Obviously, inferential activity is not limited to a set of finitely axiomatizable rules; if one of the aspects of deductive practice - the conditions for judging or asserting correctly - can be fully described in terms of a well specifiable number of *introduction* rules, the other pole - the correct consequences of judgments or assertions - concerns rules in a *non-introductory* form, and is on the contrary totally open. Therefore, there will be no finitely axiomatizable set operations on grounds - and operational symbols expressing them - able to capture the deductive practice as a whole. An obvious objection could be: a finitely axiomatizable set of operations on grounds - and of symbols expressing them - could be *complete*, in the sense of allowing the definition of *all* the possible operations on grounds on a given domain. In that case, we could satisfy us with a closed language of grounding. Well, even if it is true that some languages of grounding could be complete with respect to certain domains, from Gödel's incompleteness theorems we know that this can never hold for any domain. Therefore, once again, if the target of the theory of grounds must be the explanation of the deduction in its entirety, no closed language of grounding can be said satisfactory.

To account for the character of openness, we will adopt the following strategy: first we will define a general notion of language of grounding, and then a notion of expansion of a language of grounding, so as to identify a class of languages of grounding. Although each element of the class is a closed language of grounding, the whole class can in a sense be intended as an open language of grounding. In doing so, we will keep to two basic ideas.

First of all, the general notion of language of grounding will be based on a special type of language of grounding, which we will call *core language*. Given an atomic base on a background language, the alphabet of a core language on this base will contain only three groups of elements: (1) typed ground-variables, (2) individual constants and (3) operational symbols which correspond to all and only the primitive operations on grounds, namely to all and only the operations on grounds involved in the clauses that fix what counts as ground for categorical judgments or assertions. From this alphabet, we can then recursively specify the set of terms of the core language; having done that, our definition of language of grounding will be such that each language of grounding on a certain atomic base is nothing but an expansion of a core language on the same base. Having obtained the general notion of grounding language, we will finally define the notion of expansion of a grounding language in general. Ultimately, therefore, the conceptual scheme is

$$\begin{gathered}\text{core language} \Rightarrow \text{expansion of core language}\\ = \text{language of grounding in general} \Rightarrow\\ \text{expansion of a language of grounding in general.}\end{gathered}$$

Obviously, the prefigured explicative order is not inevitable. In other words, from a purely formal point of view, it is not at all necessary that each language of grounding contains a core language as its sublanguage; it is in principle possible to provide definitions such that the operational symbols of a language of grounding do not correspond to *any* of the primitive operations on grounds. But what would be the point of a language of grounding of this sort? Which semantic and philosophical importance could it claim within the general project of Prawitz's theory of grounds?

A language of grounding must be a formal apparatus so as to speak, in a precise and rigorous way, of grounds and of the ways by which they can be obtained; the terms of a language of grounding, in particular, must *denote* or *name* grounds through a structure that *describes* or *codifies* an epistemic procedure by which the denoted ground can be obtained. Using a metaphor related to Computer Science, the terms of a language of grounding form a set of programs, the execution of which on certain inputs produces as output an object to which is semantically ascribed a central epistemic role. Hence, a language of groudning that does not contain even a single operational symbol corresponding to the primitive operations on grounds, would be a language devoid of (part of) the expressive resources required for the determination of the central semantic objects. Naturally, this gap could be filled by the denotation function; being an association of terms of a language of grounding to grounds and operations on grounds, it ensures in any case an epistemic content to the language of grounding. However, if the denoted objects always depend on a limited number of primitive operations, why could we not require that each language of grounding possesses names for all these operations? Therefore, in conclusion, although the request that every language of grounding contains a core language is not formally unavoidable, we will adopt it because it is formally *convenient* and philosophically *plausible* - or at least more plausible than an approach in which it is not adopted.

The second guiding idea for the discussion of languages of grounding and their expansions concerns the conditions that are required to be respected so that a certain formal apparatus counts as a language of grounding - possible expansion of a smaller language of grounding. In this case, our principle will be simply the following: in a language of grounding, each operational symbol F must be associated with an operational type

$$\alpha_1, ..., \alpha_n \rhd \beta$$

and bind individual and ground- variables such that, called Γ_i the set of the ground-variables bound by F on index i, there is an operation on grounds f of operational type

$$(\Gamma_1 \rhd \alpha_1, ..., \Gamma_n \rhd \alpha_n) \rhd \beta$$

such that x is bound by F on index j if, and only if, x is bound by f on index j, and where $\Gamma_i \rhd \alpha_i$ is simply α_i if $\Gamma_i = \emptyset$ $(i, j \leq n)$.

The obvious consequence of this request is that it becomes possible to prove that all terms of a language of grounding denote a ground or an operation on grounds; it will be enough that each operational symbol of the language of grounding denote one of the corresponding operations on grounds that we had required as existing at the time of the introduction of the symbol. But what is the plausibility of this circumstance?

A possible answer is simply to say that a language of grounding is, in fact, a language of grounding, and it would no longer be such if some of its terms did not denote. As mentioned above, the interest of a language of grounding, in the general framework proposed by the theory of grounds, is to have a formal tool for denoting grounds and for the description of the ways in which they can be obtained.

On the other hand, not without theoretical insight would be languages of grounding in which some operational symbols cannot denote any operation on grounds - think of a symbol with operational type

$$\alpha \to \alpha \rhd \bot$$

or of the operational types

$$\alpha_i \rhd \alpha_1 \wedge \alpha_2 \text{ and } \alpha_1 \vee \alpha_2 \rhd \alpha_i \ (i = 1, 2)$$

or finally an operational symbol associated with the operational type

$$\alpha(x) \rhd \forall x \alpha(x)$$

and that *does not* bind the individual variable x - and where, as a consequence, there are terms that do not denote either grounds or operations on grounds. In languages like these, once properly defined a denotation function, it would become possible to study under what conditions a term does not denote, and under what conditions it denotes even though involving non-denoting subterms. Indeed, the very definition of the notion of denotation would take a significant specific weight: it must in fact take into account the general intuition according to which, in the framework to which the theory of grounds is inspired, a term does not denote when its operational symbols are so combined that the computation of the operations to which these symbols correspond does not end in a canonical form, namely, it does not yield an object constructed by applying primitive operations to denoting objects. But if the type of an operational symbol is not inhabited by any operation on grounds, the operation to which the symbol corresponds must be set by a "defective" equation; consequently, if the computation of the combination does not end in a canonical form, it diverges into an infinite chain

$$f_1(x) = f_2(y) = ... = f_n(z) = ...$$

or, more specifically, into a *loop*

$$f_1(x) = f_2(y) = ... = f_n(z) = f_1(x) = f_2(y) = ...$$

This opens the way to many, in our opinion interesting, questions. To mention just a few: which operational types admit "non-defective" equations for the respective operations on grounds? According to what criteria and how can we determine if an equation for the definition of an operation on grounds is "defective" or not? Is it possible that an equation is "defective", if it uses in its *definiens* only operations of which the equations, on the contrary, are not? If an operational type is not inhabited by any operation, is it always possible to associate a contextual equation to it, however "defective", in which *definiendum* and *definiens* have the same type, and the *definiendum* does not involve more variables than the *definiens*, or cannot we not to resort to explicit definitions, which involve other operations the equations of which are possibly "defective"? More in general, what does it mean that an equation is "defective"? And how should it be made a reference "universe" of a language of grounding that contains non-denoting operational symbols and terms? In addition, an approach that authorizes the introduction of non-denoting expressions might be fruitfully linked to some more or less recent research sectors in the contemporary mathematical logic. Think, as a mere partial example, of Dana Scott's computation theory which, as denotational semantics for programming languages, considers interpretations on partially ordered domains, so as to account for divergent programs, or whose denotation is only approximate (Scott 1970, Cardone 2017); or of the line of investigation that Neil Tennant inaugurated with the proof-theoretical treatment of paradoxes, by putting the focus on the links between derivations and paradoxical rules, on the one hand, and normalization procedures on the other, and by attributing a pivotal role to looping reductions (Tennant 1982, 1995, 2016, although the suggestion can be already found in Prawitz 2006; Tennant's approach and conclusions have been expanded or developed by several scholars - among them, for example, Tranchini 2014b, 2018 - and often criticized or denounced as partial by many others - see, for example, Petrolo & Pistone 2018). However, in the course of our work, we will not deal, as announced, with such an approach.

Before moving on to the next section, we would like to make a final remark. So far we have only talked about terms of a language of grounding but, if a language of grounding must be actually configured as a language, it seems natural to require that it involves also formulas, and therefore predicates for the formation of atomic formulas and logical constants. In this chapter, we will consider only the terms of a language of grouding; the complete version of these languages, with an explanation of the properties their formulas express, will be discussed in Chapter 6.

5.1.2.3 Primitiveness and conservativity

In light of the definitions we will provide, there will essentially be two ways to expand a language of grounding on a base B_1:

- by adding *new individual constants* for atomic derivations in a *proper expansion* B_2 of B_1, or
- by adding *new operational symbols* for *non-primitive* operations on ground.

Obviously, the two ways can be combined, in the sense that an expansion of a grounding language may contain *both* new individual constants *and* new operational symbols.

As regards the first expansion mode, we note first of all that it consists, *de facto*, in an expansion of the atomic base. Since an atomic base is nothing more than a quadruple relative to a first-order language, a proper expansion of an atomic base will be an atomic base in which one of the elements is a superset of the corresponding element of the unexpanded base. And since an expansion of the first type of a grounding language on a base B_1 contains new individual constants for atomic derivations in a proper expansion B_2 of B_1, this means *at a minimum* that the Post-system of the base B_2 contains the Post-system of the base B_1 as its own subsystem. If the other elements of B_2 are unchanged compared to the corresponding elements of B_1, the Post-system of B_2 does nothing but add new rules for the elements of the same alphabet on which already the Post-system of B_1 acted; then, the language of grounding and its expansion will have the same background language. However, it is not obviously necessary that this is the case. B_2, in fact, could contain new individual constants, new relational symbols or new functional symbols compared to those available in B_1, and its Post-system could provide rules for the new symbols - in addition to the already contemplated case of new rules for old symbols; in this case, the background language of the expansion will be an expansion of the background language of the unexpanded grounding language.

As we have said, an atomic base is intended to fix the meaning of some of the elements of the background language alphabet. A change of base therefore implies a change of meaning - and its extension in the case of new linguistic or deductive resources. Hence, when we expand a language of grounding according to the first of the above mentioned modes, we add new *primitive elements* with respect to the denotation of grounds and operation on grounds; an individual constant without free individual or ground-variable variables, in fact, will denote by definition a ground for the (atomic) formula on which it is typed. Accordingly, we will call *primitive* an expansion thus obtained.

We are induced at this point to think, in a completely correct way, that another type of primitive expansion could consist in adding to a language of grounding operational symbols for primitive operations on grounds different from those related to $\wedge$, $\vee$, $\rightarrow$, $\forall$ and $\exists$, namely, operational symbols for operations involved in the definition of the concept of ground for categorical judgments or assertions of formulas the main logical constant of which is not any of the constants $\wedge$, $\vee$, $\rightarrow$, $\forall$ and $\exists$. It is not difficult to realize, however, that contemplating such a type of expansion requires changing the set of logical constants of the background language. In this work, though, apart from a quick example, we will not deal with expansions of this kind; in other words, our analysis will be limited to first-order logic, whence the aforementioned restriction that a background language is always only a first-order logical language. At the base of our choice, there are essentially two reasons.

First of all, it does not seem to hold any real interest the addition to $\wedge$, $\vee$, $\rightarrow$, $\forall$ and $\exists$ of non-modal logical constants that do not involve a passage to orders higher than the first - think of the constant $\leftrightarrow$, or of Sheffer's constant (Sheffer 1913) $\mid$ and of its dual $\downarrow$; Prawitz himself, in fact, proved that the first-order logical constants are functionally complete (Prawitz 1979; see also Schroeder-Heister 1984). On the other hand, we know that the addition of second-order quantifiers leads to many difficulties in the semantic frameworks of constructivist, or more generically verificationist, inspiration; the breach of the Fregean principle of compositionality goes together with reliable phenomena of unpredicativity that require treatments *ad hoc*, or substantial changes that avoid the emergence of paradoxes (see, among others, Pistone 2015). The problem already concerns a formulation of the clauses for determining what counts as ground for categorical judgments or assertions of formulas quantified at the second order, and therefore the primitive operations on grounds for second-order quantifiers; actually, it is reasonable to expect that we find here the same phenomenon we have with the rules of introduction of these quantifiers. We can borrow here Cozzo's words, who notes how

> the introduction rule for the second-order existential quantifier has the form:

$$\frac{\alpha(T)}{\exists X\alpha(X)}$$

> where T is a predicative (second-order) term that may contain the formula $\exists X\alpha(X)$ as a part [...] the premise $\alpha(T)$ could be much more complex than the conclusion $\exists X\alpha(X)$ and the rule violates the molecularity requirement. (Cozzo 1994b, 113)

Likewise, the clause fixing what counts as ground for $\exists X\alpha(X)$ should appeal to a primitive operation, that we can indicate with $\exists^2 I$, saying that $\exists^2 I(g)$ is a ground for $\vdash \exists X\alpha(X)$ if, and only if, g is a ground for $\alpha(T)$; but T could contain $\exists X\alpha(X)$ as its subformula, and this could make the expected inductive nature of the clauses problematic.[1]

If an expansion obtained by adding individual constants is primitive, an expansion obtained by adding operational symbols for non-primitive operations on grounds can obviously be called *non-primitive*. A non-primitive operation on grounds, in fact, will have to be fixed by one or more equations by virtue of which, whenever applied to grounds for the types of its domain, the operation returns a ground for the type of its codomain; but the notion of ground is fixed in terms of primitive operations, so a non-primitive operation is "harmonic" with respect to the determination of the meaning that these primitive operations provide. In the same way, the terms of an expansion obtained by adding operational symbols for non-primitive operations on grounds will not contain new primitive concepts compared to the unexpanded language of grounding, but they will denote on the basis of a notion of ground - and therefore of a determination of the meaning - already available before the expansion. When expanding a language of grounding in a non-primitive way, we just have to comply with the request mentioned in Section 5.1.2.2: the added operational symbol must be of an operational type, and a binding of individual variables and ground-variables, such that there is a corresponding operation on grounds, of the appropriate operational type.

Together with the distinction between primitive and non-primitive expansions, we would introduce also another classification: conservative and

[1] The restriction to the first order does not disregard the great interest of an approach that includes expansions of languages of grounding with operational symbols for primitive operations related to logical constants of higher order than the first. This is a point that we will leave out for mere reasons of space, and that could be fruitfully developed later. In a sense, our discussion will be equivalent to a formal framework for the analysis of *first-order inferential validity*; hence, to authorize expansions with operational symbols for primitive operations on grounds would be equivalent to a formal framework for the analysis of *inferential validity of n-th order*, namely of *inferential validity as such.*

non-conservative expansions. The basic idea is the following. Let Λ_1 be a grounding language on a base B on L, and let

$$\mathtt{Gr}_B^{\Lambda_1} = \{g \mid g \text{ is a ground over } B \text{ denoted by some term of } \Lambda_1\}$$

- i.e. grounds on B denoted by some term of Λ_1. Let Λ_2 be now an expansion of Λ_1, and let

$$\mathtt{Gr}_L^{\Lambda_2} = \{g \mid g \text{ is a ground with type in } L \text{ denoted by some term of } \Lambda_2\}$$

- i.e. grounds of type in L denoted by some term of Λ_2. Since Λ_2 is an expansion of Λ_1, we will have that $\mathtt{Gr}_B^{\Lambda_1} \subseteq \mathtt{Gr}_L^{\Lambda_2}$. If the inclusion is strict, that is in the case of $\mathtt{Gr}_B^{\Lambda_1} \subset \mathtt{Gr}_L^{\Lambda_2}$, we will say that Λ_2 is a *non-conservative* expansion of Λ_1; when the two sets are equal, namely in the case of $\mathtt{Gr}_B^{\Lambda_1} = \mathtt{Gr}_L^{\Lambda_2}$, we will say that Λ_2 is a *conservative* expansion of Λ_1.

In other words, an expansion is non-conservative if, and only if, it contains terms that denote grounds based on the unexpanded language of grounding that were not denoted by any term in the unexpanded language of grounding; an expansion is non-conservative if, and only if, it allows us to express new grounds on the same base as the language of grounding that it enhances. The expansion, on the contrary, is conservative if, and only if, all the grounds that it allows to express on the old base were already expressible in the unexpanded language of grounding; if all the objects that its terms "name", were already "nameable" in the old language of grounding. Obviously, the notion is relativized to a fixed denotation function.

It should be noted that the kind of conservativeness, which we are dealing with here, is significantly different from the usually used notion of *conservativity of formal systems*. Given a formal system Σ on a language L, and given an expansion Σ^+ of Σ, we say that Σ^+ is conservative on Σ if, and only if, for each finite set Γ of formulas of L, for each formula α of L, if $\Gamma \vdash_{\Sigma^+} \alpha$, then $\Gamma \vdash_{\Sigma} \alpha$. In terms of languages of grounding, this would amount to requiring that, given a language of grounding Λ_1 on a base B, and given an expansion Λ_2 of Λ_1, Λ_2 is conservative on Λ_1 if, and only if, for each $\alpha_1, ..., \alpha_n \vdash \beta$ on B, if there exists a term of Λ_2 denoting a ground g_1 on B for $\alpha_1, ..., \alpha_n \vdash \beta$, then there exists a term of Λ_1 denoting a ground g_2 on B for $\alpha_1, ..., \alpha_n \vdash \beta$. This type of conservativity, however, does not clearly imply that $g_1 = g_2$; the kind of conservativity we outlined, however, requires exactly that the class of grounds on B denoted by the terms of Λ_2 is identical to the class of grounds on B denoted by the terms of Λ_1. The second type of conservativeness clearly implies the first, but the opposite does not always apply.

In other words, it is not a conservativity of *provability*, but a conservativity, so to speak, of *denotation*. If, as a mere way of example, we undertsand a term as a proof, and a ground as the canonical form to which this proof reduces - if the term is closed - or as the method that this proof denotes - if the term is open - the conservativity proposed here corresponds to the idea that a conservative expansion of a language of grounding adds names for proof-methods that were already available in the unexpanded language, but does not involve a substantial increase in the deductive power of the language itself. On the contrary, in a non-conservative expansion, we dispose of new proof-methods, for judgments or assertions not provable in the unexpanded language of grounding, and possibly for judgments or assertions already provable in the unexpanded language of grounding.

Finally, in the course of our discussion we will show how the distinction between primitive and non-primitive expansions does not coincide at all with the distinction between conservative and non-conservative expansions. We will provide in practice examples of expansions: (1) primitive and conservative; (2) primitive and non-conservative; (3) non-primitive and conservative; (4) non-primitive and non-conservative. Point 1 will be proved through an expansion of a language of grounding corresponding to a Gentzen's natural deduction system for first-order intuitionist logic obtained by adding an operational symbol for a primitive operation on grounds for the logical constant $\leftrightarrow$. Point 2 will be proved through an expansion of a language of grounding on a base for first-order arithmetic with corresponding Post-system, obtained by adding to the base a reflection principle - and 1 and 2 will be the only examples in which we will take into account expansions of bases by adding new logical constants. Point 3 will be proved through an expansion of a language of grounding corresponding to a Gentzen's natural deduction system for first-order intuitionist logic obtained by adding an operational symbol for a non-primitive operation on grounds for disjunctive syllogism. Finally, Point 4 will be proved through an expansion of a language of grounding corresponding to a Gentzen's natural deduction system for first-order intuitionistic logic with only introduction rules, towards a Gentzen's natural deduction system for first-order intuitionistic logic with also elimination rules.

5.2 A class of languages

Let us now begin the delineation of our class of grounding language. We will proceed in the following order: first, we will define a background language and a notion of base on a background language; then we will discuss the notion of operational type, different notions of operations on grounds of different operational types, and finally we will provide the clauses which fix the notion of ground for categorical judgments or assertions for formulas of different logical kinds; we will then go to the definitions of core language of grounding, of language of grounding in general and of expansion of a language of grounding; the next step will be the definition of the notion of denotation, and the proof of some related results; we will conclude with the distinction between primitive and non-primitive, conservative and non-conservative expansions. Throughout our discussion, we shall of course provide examples that make operational the definitions at the same time offered.

5.2.1 Background language and bases

The *background languages* we are going to take into account are always *first-order logical languages* of the kind already used in previous points of our investigation. The setting is similar to that of *Logic and structure* by Dirk Van Dalen (Van Dalen 1994), although it leaves out the reference to the notion of similarity type.

Definition 12. A *first-order logical language* L is characterised by an alphabet, and by sets of terms and formulas defined recursively starting from that alphabet. The *alphabet* $\mathtt{AL}_L$ consists of:

- individual variables x_i $(i \in \mathbb{N})$
- relational symbols P^n_j $(n, j \in \mathbb{N})$
- functional symbols ϕ^m_h $(m, h \in \mathbb{N})$
- individual contants $\|c_i\|$ $(i \in \mathbb{N})$
- logical constants $\wedge$, $\vee$, $\rightarrow$, $\forall$, $\exists$, $\bot$
- brackets and commas as auxiliary symbols

The set $\mathtt{TERM}_L$ of the *terms of* L is the smallest set X such that

- $x_i \in X$ $(i \in \mathbb{N})$
- $\|c_i\| \in X$ $(i \in I)$
- $t_1, ..., t_m \in X \Rightarrow \phi^m_h(t_1, ..., t_m) \in X$

The set $\mathtt{FORM}_L$ of the *formulas of* L is the smallest set X such that

- $\bot \in X$
- $t_1, ..., t_n \in \mathtt{TERM}_L \Rightarrow P^n_j(t_1, ..., t_n) \in X$

[These first two clauses define the set of the *atomic* formulas of L, indicated with $\mathtt{ATOM}_L$]

- $\alpha, \beta \in X \Rightarrow \alpha \star \beta \in X$ (where $\star$ is one of the symbols $\wedge, \vee, \rightarrow$, which we will indicate more briefly with $\star = \wedge, \vee, \rightarrow$)
- $\alpha \in X \Rightarrow \star\, x_i\, \alpha \in X$ ($\star = \forall, \exists$, $i \in \mathbb{N}$)

As regards negation, it is defined by putting

$$\neg\alpha \stackrel{def}{=} \alpha \rightarrow \bot$$

so it is not a primitive symbol of the language.
When this does not generate ambiguity, we will omit indices and subscripts to lighten up the notation.

On first-order logical languages, we can introduce technical, although very important, definitions. We will provide them without further explanation, as they are quite standard notions.

Definition 13. The set $FV(t)$ of the *free variables* of t is defined inductively as follows:

- $FV(x) = \{x\}$
- $FV(\|c\|) = \emptyset$
- $FV(\phi(t_1, ..., t_n)) = FV(t_1) \cup ... \cup FV(t_n)$

The set $BV(t)$ of the *bound variables* of t is $\emptyset$. t is *closed* if, and only if, $FV(t) = \emptyset$.

Definition 14. The set $FV(\alpha)$ of the *free variables* of α is defined inductively as follows:

- $FV(\bot) = \emptyset$
- $FV(P(t_1, ..., t_n)) = FV(t_1) \cup ... \cup FV(t_n)$
- $FV(\alpha \star \beta) = FV(\alpha) \cup FV(\beta)$ ($\star = \wedge, \vee, \rightarrow$)
- $FV(\star\ x\ \alpha) = FV(\alpha) - \{x\}$ ($\star = \forall, \exists$)

The set $BV(\alpha)$ of the *bound variables* of α is defined inductively as follows:

- $BV(\bot) = BV(P(t_1, ..., t_n)) = \emptyset$
- $BV(\alpha \star \beta) = BV(\alpha) \cup BV(\beta)$ $(\star = \wedge, \vee, \rightarrow)$
- $BV(\star\ x\ \alpha) = BV(\alpha) \cup \{x\}$ $(\star = \forall, \exists)$

α is *closed* if, and only if, $FV(\alpha) = \emptyset$.

With $FV(\Gamma)$ we indicate the set

$$\{FV(\gamma) \mid \gamma \in \Gamma\}$$

and in what follows we will always assume that $FV(\alpha) \cap BV(\alpha) = \emptyset$.

Definition 15. A *substitution* of x with t in s is a function $\mathtt{TERM}_L \rightarrow \mathtt{TERM}_L$ defined inductively as follows:

- $y[t/x] = \begin{cases} y & \text{if } y \neq x \\ t & \text{if } y = x \end{cases}$
- $\|c\|[t/x] = \|c\|$
- $\phi(t_1, ..., t_n)[t/x] = \phi(t_1[t/x], ..., t_n[t/x])$

Definition 16. A *substitution* of x with t in α is a function $\mathtt{FORM}_L \rightarrow \mathtt{FORM}_L$ defined inductively as follows:

- $\bot[t/x] = \bot$
- $P(t_1, ..., t_n)[t/x] = P(t_1[t/x], ..., t_n[t/x])$
- $(\alpha \star \beta)[t/x] = \alpha[t/x] \star \beta[t/x]$ $(\star = \wedge, \vee, \rightarrow)$
- $(\star\ y\ \alpha)[t/x] = \begin{cases} \star\ y\ \alpha[t/x] & \text{if } y \neq x \\ \star\ y\ \alpha & \text{if } y = x \end{cases} \qquad (\star = \forall, \exists)$

Definition 17. A *substitution* of β with γ in α, for $\beta \in \mathtt{ATOM}_L$, is a function $\mathtt{FORM}_L \rightarrow \mathtt{FORM}_L$ defined inductively as follows:

- $\alpha \in \mathtt{ATOM}_L \Rightarrow \alpha[\gamma/\beta] = \begin{cases} \alpha & \text{if } \alpha \neq \beta \\ \gamma & \text{if } \alpha = \beta \end{cases}$
- $(\alpha \star \delta)[\gamma/\beta] = \alpha[\gamma/\beta] \star \delta[\gamma/\beta]$ $(\star = \wedge, \vee, \rightarrow)$
- $(\star\ x\ \alpha)[\gamma/\beta] = \star\ x\ \alpha[\gamma/\beta]$ $(\star = \forall, \exists)$

All the functions described above can be generalized to simultaneous substitutions for an arbitrary number of variables or atomic formulas. In this regard, the notation:

$$(...((\diamond[*_1/\circ_1])[*_2/\circ_2])...[*_n/\circ_n]) \stackrel{def}{=} \diamond[*_1, ..., *_n/\circ_1, ..., \circ_n]$$

will indicate a simultaneous substitution of n variables or atomic formulas – $\diamond$ is a term or formula, $\circ$ a variable or an atomic formula, $*$ a term or a formula, respectively.

Definition 18. t is *free for x in α* if, and only if,

- $\alpha \in \mathtt{ATOM}_L$
- $\alpha = \beta \star \gamma$ and t is free for x in β and γ $(\star = \wedge, \vee, \rightarrow)$
- $\alpha = \star\ y\ \beta$, $y = x$ or $y \neq x$, $y \notin FV(t)$ and t is free for x in β

Definition 19. γ is *free for β in α*, for $\beta \in \mathtt{ATOM}_L$ if, and only if,

- $\alpha \in \mathtt{ATOM}_L$
- $\alpha = \delta \star \epsilon$ and γ is free for β in δ and ϵ $(\star = \wedge, \vee, \rightarrow)$
- $\alpha = \star\ y\ \delta$, $y \notin FV(\gamma)$ and γ is free for β in δ $(\star = \forall, \exists)$

The definitions of free term for variable in formula, and of free formula for formula in formula, are intended to isolate the substitutions that have the following properties: (1) given a term t and a formula α, no variable of t becomes bound in $\alpha[t/x]$ and (2) given two formulas α and γ, no free variable of γ becomes bound in $\alpha[\gamma/\beta]$. As it is with substitutions, also the two previous definitions can be generalized to the case of an arbitrary number of variables or formulas. Hereafter, we will assume that all the substitutions satisfy the aforementioned properties (1) and (2).

Definition 20. Given a first-order logical language L_1, an *expansion* of L_1 is a first-order logical language L_2 such that $\mathtt{AL}_{L_1} \subseteq \mathtt{AL}_{L_2}$.

Turning now to the notion of base, it requires a preliminary notion of atomic system, and a preliminary clarification of the relationship between an atomic system and the language on which it acts. The atomic systems that we will take into account are always Post-systems, namely pairs $\langle L, \Re \rangle$, with L first-order logical language and $\Re$ finitely axiomatizable set of rules

$$\frac{\alpha_1 \quad \dots \quad \alpha_n}{\beta}$$

relating to individual constants, relational symbols or functional symbols of L and such that:

- $n \geq 0$ (if $n = 0$ the rules is an axiom);
- for every $i \leq n$, $\alpha_i \in \texttt{ATOM}_L$ and $\alpha_i \neq \bot$;
- $\beta \in \texttt{ATOM}_L$ and, if $x \in FV(\beta)$, there is $i \leq n$ such that $x \in FV(\alpha_i)$.

Contrary to what has been done previously, we will leave here open the possibility that the rules of a Post-system bind variables or assumptions. Well, the basic idea, which we have already mentioned, is that the meaning of the elements of the alphabet of a background language is fixed by the rules of a Post-system. By way of example, the Post-system for first-order arithmetic, set out in Section 2.5.1, consists of rules by virtue of which: the constant 0 can be interpreted as the zero of the natural numbers; the symbol s can be interpreted as the successor function on natural numbers; the symbol $+$ can be interpreted as addition on natural numbers; the symbol $\cdot$ can be interpreted as multiplication on natural numbers; the symbol $\doteq$ can be interpreted as the relation of equality on natural numbers. In this perspective, we have to understand the following definition.

Definition 21. An atomic system $\texttt{S} = \langle L_1, \Re \rangle$ is an *atomic system for* L_2 if, and only if, L_2 is an expansion of L_1. $\texttt{S}$ *totally interprets* L_2 if, and only if, $L_1 = L_2$. Otherwise, $\texttt{S}$ *partially interprets* L_2.

If $\texttt{S}$ is a language for L – whether it provides a total interpretation or the interpretation is only partial - its rules are intended to determine the meaning of the elements of the alphabet of L; we can also say that they are *rules for such symbols.* As already mentioned, we will imply the notion of "rule for a symbol", wanting with it to refer to the circumstance that a rule contains a symbol so as to allow its interpretation. A detailed development of the notion of "rules for a symbol" is found in Cozzo's theory of the meaning (Cozzo 1994a, 1994b). As a limit case, we have the system $\langle \emptyset, \emptyset \rangle$, to which we can give the name of empty system. Whatever L, from definition 21 it follows that the empty system is a system for L; on it, though, L remains totally uninterpreted.

Definition 22. Given an atomic system $\texttt{S}_1 = \langle L_1, \Re_1 \rangle$, an *expansion* of $\texttt{S}_1$ is an atomic system $\texttt{S}_2 = \langle L_2, \Re_2 \rangle$ with L_2 expansion of L_1.

Definition 23. Given a first-order logical language L, we indicate: with $\mathtt{R}$ the set of the relational symbols of L; with $\mathtt{F}$ the set of the functional symbols of L; with $\mathtt{C}$ the set of the individual constants of L. Let finally $\mathtt{S}$ be an atomic system for L. We will say that the quadruple

$$\langle \mathtt{R}, \mathtt{F}, \mathtt{C}, \mathtt{S} \rangle$$

is an *atomic base* on L. We will cal L *background language* of the base.

For the sake of greater simplicity, we will exclude bases of which the atomic systems interpret partially the background language; in other words, the latter must be either not interpreted, or totally interpreted - in the first case, we will sometimes talk about *logical base*.

Convention 24. Given a base B on L, if the atomic system $\mathtt{S}$ of B is not empty, that is, if B is not a logical base, $\mathtt{S}$ totally interprets L.

Definition 25. Let B be a base on L_1 with atomic system $\mathtt{S}_1$. We will say that a base on L_2, for L_2 expansion of L_1, with atomic system $\mathtt{S}_2$ expansion of $\mathtt{S}_1$, is an *expansion* of B.

By convention 24, in the case of a real expansion of $\mathtt{R}$, $\mathtt{F}$, or $\mathtt{C}$, and if the atomic system $\mathtt{S}$ of B is not empty, also $\mathtt{S}$ is actually expanded.

5.2.2 Operational types, operations and clauses

In this section, we introduce the notion of operation on grounds and, subsequently, the definition of the notion of ground for categorical judgments or assertions. To make this possible, we need a preliminary notion of operational type.

5.2.2.1 Operational types

An operation on grounds, as already mentioned, is a total constructive function with a certain domain and a certain codomain; domain and codomain are nothing but classes of individuals or (operations on) grounds for certain judgments or for certain assertions on a certain background language. When applied to an element of the domain, then, the operation on grounds produces a ground for the judgment or assertion that constitutes the codomain. An operation on grounds will therefore be identified by two parameters; domain/codomain, and an equation indicating how the operation transforms each element of the domain into an element of the codomain. As for the first parameter, we can therefore talk of an *operational type*, since it fixes the type of inputs and outputs of the operation.

The notion of operational type will mainly have the purpose of classifying the possible operations on grounds according to the lines just indicated, but it will be used by us also for another purpose. When we introduce the languages of grounding, in fact, all the elements of the alphabet, and all the terms built starting from them will have to have a certain type. Now, for the ground-variables the individual constants and all terms, a type is nothing but a formula of the background language; but the alphabet of a language of grounding also includes operational symbols, the type of which will be instead, more specifically, an operational type. The reason for this is simple: an operational symbol allows the creation of a term of a certain type starting from terms of another type, and the operational type assigned to it has exactly the purpose of making this circumstance explicit.

Obviously, a close link exists between the operations on grounds and the operational symbols of a language of grounding: the latter must be understood as denoting the former. A denotation function will therefore ensure that, if an operational symbol has a certain operational type, its *denotatum* is an operation on grounds having the same operational type. Therefore, expressing ourselves in a concise but effective way, the "syntactic" typing and the "semantic" typing coincide. It must be said, however, that not always an operational symbol having a certain operational type denotes an operation on grounds having the same operational type: if an operational symbol F binds the assumption α on an index i, and if the i-th entry on the domain of the operational type of F is β, F must denote an operation on grounds the operational type of which has as i-th entry of the domain operations on grounds from α to β. The difference between the operational types of the operational symbols, and those of the operations on grounds, depends on the fact that the languages of grounding have no variables for operations on grounds; in any case, this difference can be remedied through an appropriate definition of the denotation function.

The operational types will therefore be understood by us as strings of formulas on a background language. Each string will be divided into two by a main sign $\rhd$: what is *on the left* of the main sign $\rhd$ will constitute the domain of an operation on grounds having that operational type, while what is *on the right* of this $\rhd$ will be understood as the codomain of the operation on grounds having that operational type.

Definition 26. Let L be a first-order logical language. An *operational type* on L is $\alpha \in \mathtt{FORM}_L$, or an expression

$$\tau_1, ..., \tau_n \rhd \tau_{n+1}$$

such that:

- $n \geq 1$ and
- τ_i $(i \leq n)$ is
  - $\Gamma_i \rhd \alpha_i$ with $\Gamma_i \subset \texttt{FORM}_L$, $\Gamma_i \neq \emptyset$ and $\alpha_i \in \texttt{FORM}_L$ or
  - α_i with $\alpha_i \in \texttt{FORM}_L$

  and τ_{n+1} is
  - $\Delta \rhd \beta$ con $\Delta \subseteq \bigcup_{i \leq n} \Gamma_i$, $\Delta \neq \emptyset$ and $\beta \in \texttt{FORM}_L$ or
  - β con $\beta \in \texttt{FORM}_L$

We will say that $\tau_1, ..., \tau_n$ is the *domain* of the operational type, and that τ_i $(i \leq n)$ is an *entry* of the domain. We will say that τ_{n+1} is the *codomain.*

In other words: if the operational type has a non-empty domain, the i-th entry of the domain is

$$\Gamma_i \rhd \alpha_i$$

when Γ_i is not empty, otherwise it is simply α_i, and instead the codomain will have the form

$$\Delta \rhd \beta$$

for Δ non-empty subset of the union of all the Γ_i, otherwise simply β; if instead the operational type has an empty domain, it is simply a formula. As generic examples, the following are all operational types:

- $\alpha(x_1, ..., x_n)$
- $(\alpha_1(x_1, ..., x_n) \rhd \beta_1(y_1, ..., y_m), \alpha_2 \rhd \beta_2) \rhd \delta(z_1, ..., z_p)$
- $(\alpha_1(x_1, ..., x_n) \rhd \beta_1(y_1, ..., y_m), \alpha_2 \rhd \beta_2) \rhd (\alpha_2 \rhd \delta(z_1, ..., z_p))$

or again, with reference to a language for first-order arithmetic, the following are all operational types:

- $\forall x(0 \geq x)$
- $\forall x(x \leq y \wedge y \leq z), 9^4 = s(s(0)) \rhd \forall w(w = s(s(s(\sqrt{64}))))$
- $(\exists x(x \leq y), 0 \geq 2^5, 0 = s(s(0)) \rhd 0 = 1) \rhd (0 = s(s(x)) \rhd \forall y(y \neq 7!))$

5.2.2.2 First- and second-order operations

We can now proceed to the notion of *operation on grounds of operational type* τ. In order to characterize it, we must bear in in mind two important points. First of all, operations on grounds are always *operations on a base*, namely, operations related to a certain background language, and to a certain atomic system for this language. In light of what said in Section 5.2.1, this does not cause problems; we already possess the necessary theoretical concepts.

Secondly, in order that the class of operations on grounds to be well founded, we need its definition to be based on a default notion, starting from which to proceed recursively. Well, this notion is that of ground for a categorical judgment or assertion. A ground for a categorical judgment or assertion is a *closed* object, whereas an operation on grounds is an *open* object, applicable to individuals and grounds, and which produces closed objects of the appropriate type when applied to closed objects of the appropriate type. However, although this is the explanatory order, we will reverse it; and this because, although the notion of ground for a categorical judgment or assertion conceptually comes before that of operation on grounds, it requires it. This circumstance, for its part, does not imply circularity, since the notion of ground for categorical judgments or assertions is fixed by simultaneous recursion, so that the notion of operation on grounds is assumed on a level of lower complexity. Therefore, in conclusion, when in the following we meet the notion of ground for categorical judgments or assertions, we shall need to refer to the clauses of Section 5.2.2.4.

As we have already anticipated, it is convenient to distinguish between *first-level* operations, the domain of which has no entries of the type

$$\Gamma \rhd \alpha$$

and *second-level* operations, the domain of which has at least one entry of that type. In order to define the second-level operations it is necessary to have defined those of first-level, from which it is therefore appropriate to begin. In characterizing first-level operations, however, we will proceed by steps, providing examples of increasingly complex operations, and passing only at a later stage to the general description.

The simplest of first-level operations on grounds are those with an empty domain. Given a base B on L, a B-operation on grounds of operational type

$$\alpha(x_1, ..., x_n)$$

with $n \geq 0$, is a total constructive function

$$f(x_1, ..., x_{n+m})$$

with $m \geq 1$ if $n = 0$, and $m \geq 0$ otherwise, such that, for every sequence $k_1, ..., k_{n+m}$ of individual in domain of B,

$$f(k_1, ..., k_{n+m})$$

is a ground on B for $\vdash \alpha(k_1, ..., k_n)$. Let us take an example. Let B be a base on a language for first-order arithmetic with Post-system as indicated in Section 2.5.1, and let us consider the following reasoning:

1. let $0 < x$
2. let $x < s(y) + z$
3. by transitivity, $0 < s(y) + z$

therefore, discharging assumptions 1 and 2,

4. se $0 < x$ and $x < s(y) + z$, $0 < s(y) + z$.

It is easy to realize that, if we replace x, y, z with the positive integers whatsoever n, m, p, we have something that justifies the categorical judgment or assertion "if $0 < n$ and $n < s(m) + p$, then $0 < s(m) + p$". The proposed reasoning can hence be understood as the expression on three individual variables of a total constructive function that, given as input three positive integers, returns a ground related to an open formula the free individual variables of which are replaced by the chosen numbers.

Before continuing, we will introduce a wording that will conveniently facilitate many of our subsequent definitions. A B-operation on grounds of operational type

$$\alpha(x_1, ..., x_n)$$

will also be called ground on B for $\vdash \alpha(x_1, ..., x_n)$. It should be noted that this way of speaking is in a sense ambiguous, because the expression of the operation could involve more individual variables than the free ones occurring in its operational type; for example, if α is closed it is expected that a ground for $\vdash \alpha$ is an equally closed object. To resolve that, we can then say that a ground for $\vdash \alpha$ is a *proper ground* if, and only if, it is an object the expression of which involves all and only the elements of $FV(\alpha)$, and that, otherwise, it is an *improper ground*.

A second very simple type of first-level operation is that in which the domain is non-empty, but all the formulas involved in the operational type are closed. Given a base B, a B-operation on grounds of operational type

$$\alpha_1, ..., \alpha_n \rhd \beta$$

with $\alpha_1, ..., \alpha_n, \beta$ closed and $n \geq 1$, is a total constructive function

$$f(\xi^{\alpha_1}, ..., \xi^{\alpha_n})$$

such that, for every g_i ground on B for $\vdash \alpha_i$ $(i \leq n)$

$$f(g_1, ..., g_n)$$

is a ground on B for $\vdash \beta$. Let us take an example. Let B be again a base on a language for first-order arithmetic, with Post-system as indicated in Section 2.5.1, and let us consider the following reasoning:

1. let us make the hypothesis $s(s(0)) = s(0) + s(0)$
2. let us make the hypothesis $s(0) = s(0) \cdot s(0)$

hence

3. $s(s(0)) = s(0) + s(0)$ and $s(0) = s(0) \cdot s(0)$.

It is easy to realize that, if we put before the hypotheses 1 and 2 some computations that justify these hypotheses, we obtain something that justifies the categorical judgment or assertion "$s(s(0)) = s(0) + s(0)$ and $s(0) = s(0) \cdot s(0)$". The proposed reasoning can therefore be understood as the expression on two ground-variables of a total constructive function that, given as input two computations, returns a ground related to a closed formula depending, in the reasoning, on the conclusions of the computation.

We have presented the cases of first-level operations on grounds, the domain of which is empty, or the domain of which is non-empty but all the formulas involved in the operational type are closed. Let us move on to the more general case, namely the one in which the domain of the operations on grounds is non-empty, and some of the formulas involved in the operational type are open. Unlike with what we have done so far, let us start from the examples here, and provide at a later stage the general characterization.

As a first example, let us take into account the following instance of $(\wedge_I)$ - all the free individual variables are explicitly indicated -

$$\frac{\alpha(x) \qquad \beta(y)}{\alpha(x) \wedge \beta(y)} (\wedge_I)$$

and, first of all, replace the individual variables x, y with closed terms t_1, t_2, so as to obtain the instance

$$\frac{\alpha(t_1/x) \qquad \beta(t_2/y)}{\alpha(t_1/x) \wedge \beta(t_2/y)} \ (\wedge_I)$$

and then, finally, let us put before the assumptions $\alpha(t_1/x)$ and $\beta(t_2/y)$ two closed valid arguments Δ_1, Δ_2, for each of them respectively - we assume that there are -

$$\frac{\begin{array}{cc} \Delta_1 & \Delta_2 \\ \alpha(t_1/x) & \beta(t_2/y) \end{array}}{\alpha(t_1/x) \wedge \beta(t_2/y)} \ (\wedge_I)$$

so as to obtain a closed valid argument for $\alpha(t_1/x) \wedge \beta(t_2/y)$. If we accept to take, in a purely illustrative way, a closed valid argument having a certain conclusion as a ground for that conclusion, we can say that the instance of $(\wedge_I)$ from which we started expresses a total constructive function which, taken as input two closed terms, and then two grounds for the assumptions obtained by replacing with the closed terms the individual free variables in the original assumptions, returns a ground for the conclusion obtained by replacing with the closed terms the individual free variables in the original conclusion. In other words, we can look at the instance of $(\wedge_I)$ as the expression of an operation on ground of operational type

$$\alpha(x), \beta(y) \rhd \alpha(x) \wedge \beta(y).$$

The operations on grounds that we are taking into account here can bind variables. The paradigmatic case is that of the rule $(\forall_I)$, of which we take into account first the form - all the free individual variables are explicitly indicated -

$$\frac{\alpha(x)}{\forall x \alpha(x)} \ (\forall_I)$$

where, if the rule is correctly applied, x is the proper variable bound by the rule itself. As we know, a correct application

$$\frac{\begin{array}{c} \Delta(x) \\ \alpha(x) \end{array}}{\forall x \alpha(x)} \ (\forall_I)$$

of the rule, requires that x does not occur free in any of the non-discharged assumptions of $\Delta(x)$, which will happen in particular if $\Delta(x)$ is an argument without non-discharged assumptions; therefore, if we accept to understand a valid argument for $\alpha(x)$ without non-discharged assumptions as the expression of an operation on ground of operational type

$$\alpha(x)$$

and a valid closed argument for $\forall x \alpha(x)$ as a ground for $\vdash \forall x \alpha(x)$, we can say that the rule from which we started expresses a total constructive function that, taken as input an operation on grounds of operational type

$$\alpha(x)$$

returns a ground for $\vdash \forall x \alpha(x)$, binding x. Same thing is when the rule $(\forall_I)$ is considered in the form - all the free individual variables are explicitly indicated -

$$\frac{\alpha(x, y)}{\forall x \alpha(x, y)} \, (\forall_I)$$

where, if the rule is correctly applied, x is the proper variable bound by the rule itself. Here, too, a correct application of the rule

$$\frac{\begin{array}{c}\Delta(x, y) \\ \alpha(x, y)\end{array}}{\forall x \alpha(x, y)} \, (\forall_I)$$

requires that x does not occur free in any of the non-discharged assumptions of $\Delta(x, y)$, which will happen, for example, when $\Delta(x, y)$ is an argument without non-discharged assumptions; therefore, if we replace y - which occurs free - with whatever closed term t, and if we accept to take a valid argument for $\alpha(x, t)$ without non-discharged assumptions as the expression of an operation on grounds of operational type

$$\alpha(x, t)$$

as well as a closed valid argument for $\forall x \alpha(x, t)$ as a ground for $\vdash \forall x \alpha(x, t)$, we can say that the rule from which we started expresses a total constructive function which, taken as input a closed term t replacing the free individual variable y, and taken as input an operation on grounds of operational type

$$\alpha(x, t)$$

returns a ground for $\vdash \forall x \alpha(x, t)$, binding x. In other words, the two last examples allow us to talk about operations on grounds with respective operational types

$$\alpha(x) \rhd \forall x \alpha(x) \text{ and } \alpha(x, y) \rhd \forall x \alpha(x, y)$$

which, by binding the individual variable x, do not involve x - namely x is not to occur free in their linguistic expression.

At this point, we consider it appropriate to make an observation. As we have noticed, while in the example

$$\frac{\alpha(x) \qquad \beta(y)}{\alpha(x) \wedge \beta(y)} (\wedge_I)$$

we had been able to say that the instance of $(\forall_I)$ can be understood as an operation on grounds of operational type

$$\alpha(x), \beta(y) \rhd \alpha(x) \wedge \beta(y)$$

in the case of examples

$$\frac{\alpha(x)}{\forall x \alpha(x)} (\forall_I) \qquad \frac{\alpha(x, y)}{\forall x \alpha(x, y)} (\forall_I)$$

we cannot say that the instances of $(\forall_I)$ can be understood as operations on grounds of operational type

$$\alpha(x) \rhd \forall x \alpha(x) \text{ and } \alpha(x, y) \rhd \forall x \alpha(x, y),$$

and this is simply because the argument structures considered are not instances of $(\forall_I)$, since they violate the restriction on this rule; what we have said, in fact, is only that those particular *formulations* of $(\forall_I)$ can be seen as operations of the intended type. Well, among the correct applications of $(\forall_I)$, obviously there are not only those mentioned, relating to arguments for $\forall x \alpha(x)$ and $\forall x \alpha(x, y)$ without non-discharged assumptions; there are also those in which $(\forall_I)$ is applied to arguments for $\forall x \alpha(x)$ and $\forall x \alpha(x, y)$, of which the non-discharged assumptions Γ_1 and Γ_2 do not contain x free. Should we not have taken into account also the latter case? Certainly we should, but the problem is that valid arguments of the form

$$\begin{array}{cc} \Gamma_1 & \Gamma_2 \\ \vdots & \vdots \\ \alpha(x) & \alpha(x, y) \end{array}$$

should be understood as operations on grounds of the operational type

$$\Gamma_1 \rhd \alpha(x) \text{ and } \Gamma_2 \rhd \alpha(x, y)$$

therefore, they could be operations on grounds of the exact kind of those we are defining. The definition would thus be circular. Thus, we limit ourselves to define the first-level operations on grounds, with a non-empty domain and of an operational type containing open formulas, based on operations on grounds with an empty domain; we will deal with the most general case when we discuss the composition of operations on grounds.

Before moving on to a general characterization, however, we need a few notations. We indicate with $\underline{x}_i$ and $\underline{y}$ *sequences of distinct individual variables*. The variables of the first sequence, namely $\underline{x}_i$, have an *index* in correspondence with an operational type where $\underline{x}_i$ is intended to occur; an element of $\underline{x}_i$ will therefore have the form x_i, where i is the index of the entry of the operational type where $\underline{x}_i$ occurs. Given sequences $\underline{x}_1, ..., \underline{x}_{n+1}\underline{x}$, we will say that $\underline{x}$ is a *subsequence* of $\underline{x}_1, ..., \underline{x}_{n+1}$ when each element of $\underline{x}$ has the form x_i, for some x_i element of $\underline{x}_i$ $(i \leq n+1)$. The expression $\underline{x}_1 \ \dots \ \underline{x}_{n+1} - \underline{x}$ indicates the (possibly empty) sequence obtained from $\underline{x}_1 \ \dots \ \underline{x}_{n+1}$ by *deleting* all the elements of $\underline{x}$. With $(\underline{x}_1 \ \dots \ \underline{x}_{n+1} - \underline{x}) \ \underline{y}$ we indicate the sequence obtained by letting the elements of $\underline{y}$ follow those of $(\underline{x}_1 \ \dots \ \underline{x}_{n+1} - \underline{x})$. We indicate with $\underline{s}$ a *sequence of individuals* in a reference domain. With $\mathfrak{L}$ we indicate the *length of a sequence*. Given a sequence $\underline{v}$, and given $\mathfrak{L} \ \underline{v} \leq \mathfrak{L} \ \underline{s}$, with $\underline{s}/\underline{v}$ we indicate the result of replacing, in the sequence $\underline{v}$, the j-th element of $\underline{v}$ with the j-th element of $\underline{s}$ $(j \leq \mathfrak{L} \ \underline{v})$.

This laid down, let B be as usual a base on L. A B-operation on grounds of operational type

$$\alpha_1(\underline{x}_1), ..., \alpha_n(\underline{x}_n) \rhd \beta(\underline{x}_{n+1})$$

the binds the elements of a subsequence $\underline{x}$ of $\underline{x}_1 \ ... \ \underline{x}_n$, is a total constructive function

$$f((\underline{x}_1 \ \dots \ \underline{x}_{n+1} - \underline{x}) \ \underline{y}, \xi^{\alpha_1(\underline{x}_1)}, ..., \xi^{\alpha_n(\underline{x}_n)})$$

where $\underline{y}$ (possibly empty) contains new variables, and to it it applies a restriction that we will see very soon. But first, let us conclude the general characterization. The function is total constructive in the sense that, for every $\underline{s}$ in the domain of B with $\mathfrak{L} \ \underline{s} = \mathfrak{L} \ [(\underline{x}_1 \ \dots \ \underline{x}_{n+1} - \underline{x}) \ \underline{y}]$, for every g_i ground on B (proper or improper) for $\vdash \alpha_i(\underline{s}/\underline{x}_i)$ $(i \leq n)$,

$$f(\underline{s}/[(\underline{x}_1 \ \dots \ \underline{x}_{n+1} - \underline{x}) \ \underline{y}], g_1, ..., g_n)$$

is a ground on B (proper or improper) for $\vdash \beta(\underline{s}/\underline{x}_{n+1})$. When the operation binds x_i, we say that the binding occurs on the i-th entry; the same variable can also occur in the sequence on an entry of index $j \neq i$, without binding on j.[2]

The linguistic expression of the operation therefore involves variables that are not bound by it; in particular, what interests us, an operation on grounds of operational type

$$\alpha(x) \rhd \beta$$

with β closed and that binds x, must be a total constructive function

$$f(\xi^{\alpha(x)})$$

such that, for every (proper or improper) ground $h(x, z_1, ..., z_m)$ $(m \geq 0)$ for $\vdash \alpha(x)$,

$$f(h(x, z_1, ..., z_m))$$

is a (proper or improper) ground for $\vdash \beta$.

We now come to the restriction on $\underline{y}$. It can be non-empty only in the case when there are entries of the operational type such that the operation on grounds at issue *does not* bind individual variables on the index of such entries; in other words, $\underline{y}$ is empty if the function bounds variables on every $i \leq n$.

Why do we require not to have fresh variables if there are no indexes where there is no binding? Although this actually depends on restriction (2) introduced in the next section, we can explain the point already here. In order to do this, we resort again, in a purely illustrative way, and as we have done so far, to valid arguments. The expression of an operation on grounds of operational type

$$\alpha(x) \rhd \beta$$

with β closed and that *does not* bind variables can, according to our definition, involve individual variables other than x. It can be understood as an open valid argument of the form

[2]Recall that the elements of $\underline{x}_1 \ \dots \ \underline{x}_{n+1} - \underline{x}$ are indexed; obviously, if the same variable occurs with more indexes, it counts as a *single* variable when involved in the operation. Moreover, we could have considered the case of "vacuous" quantification, that is, when the operation binds also variables other than those occurring indexed in $\underline{x}$. This would have required the binding, not of a subsequence $\underline{x}$ of $\underline{x}_1 \ \dots \ \underline{x}_{n+1}$, but of any sequence $\underline{z}$ with variables whose indexes range from 1 to $n+1$. In this case, we do not need anymore the restriction on $\underline{y}$ told about below. Finally, it is of course understood that the replacement of the elements of $\underline{s}$ takes place on the unbound variables.

$$\begin{array}{c}\alpha(x)\\ \Delta(x, y)\\ \beta\end{array}$$

where the fact that y is different from x means that $\Delta(x, y)$ involves formulas in the free variable y which do not appear either among the undischarged assumptions, or in the conclusion. However, the restriction on y excludes this possibility in the case of an operation of the same operational type, which however *binds* x. The point is that such an operation cannot be seen as an argument in the assumption $\alpha(x)$, since there must be the restriction according to which the bound variables do not occur free in the undischarged assumptions. Rather, the operation should be considered here as a rule

$$\frac{\alpha(x)}{\beta}$$

which binds x and is subject to the aforementioned restriction. Therefore, we assume that the only variables that can occur there are those of the entry of the operational type, and of the corresponding codomain. It should be noted that the restriction on the variables bound in the undischarged assumptions has not been explicitly introduced by us yet - we will do it in the next section, when we will talk about the composition of the operations on grounds; what we are here limited ourselves to requiring is that the operation produces a ground for β when applied to grounds for $\alpha(x)$, and a ground for $\alpha(x)$ can be understood as an open argument for $\Delta(x)$ without undischarged assumptions.

As previously done, let us emphasize finally that a B-operation on grounds of operational type

$$\alpha_1(\underline{x}_1), ..., \alpha_n(\underline{x}_n) \rhd \beta(\underline{x}_{n+1})$$

will also be called a ground on B for $\alpha_1(\underline{x}_1), ..., \alpha_n(\underline{x}_n) \vdash \beta(\underline{x}_{n+1})$; to avoid ambiguity, it is a *proper* ground if, and only if, the additional sequence $\underline{y}$ is empty, or, otherwise, an *improper* ground.

We now move on to the second-level operations on grounds, namely those of which the operational type has entries

$$\Gamma \rhd \alpha.$$

Here, too, we will start with some examples, providing only later the general characterization. We will use again as an example a representation in terms of argument structures. Given first a structure of the type

$$\dfrac{\begin{array}{cc}\Gamma_1 & \Gamma_2 \\ \vdots & \vdots \\ \alpha_1 & \alpha_2\end{array}}{\alpha \wedge \alpha_2}\,(\wedge_I)$$

it is clear that, if we replace the dots with two open valid arguments

$$\begin{array}{c}\Gamma_i \\ \Delta_i \\ \alpha_i\end{array}$$

with $i = 1, 2$, we obtain the open valid argument

$$\dfrac{\begin{array}{cc}\Gamma_1 & \Gamma_2 \\ \Delta_1 & \Delta_2 \\ \alpha_1 & \alpha_2\end{array}}{\alpha_1 \wedge \alpha_2}\,(\wedge_I)$$

and it is therefore clear that the starting argument structure corresponds to a total constructive function which, taken as input grounds for $\Gamma_i \vdash \alpha_i$ $(i = 1, 2)$, gives a ground for $\Gamma_1, \Gamma_2 \vdash \alpha_1 \wedge \alpha_2$. The operational type is

$$(\Gamma_1 \rhd \alpha_1, \Gamma_2 \rhd \alpha_2) \rhd (\Gamma_1, \Gamma_2 \rhd \alpha_1 \wedge \alpha_2).$$

Second-level operations on grounds can bind ground-variables, thus recalling the rules that bind assumptions. The paradigmatic case is

$$\dfrac{\begin{array}{c}[\alpha] \\ \vdots \\ \beta\end{array}}{\alpha \rightarrow \beta}\,(\rightarrow_I)$$

understandable as a total constructional function which, taken as input a ground for $\alpha \vdash \beta$, returns a ground for $\vdash \alpha \rightarrow \beta$, binding the ground-variable which corresponds to the assumption α. The operational type is therefore

$$(\alpha \rhd \beta) \rhd \alpha \rightarrow \beta.$$

Obviously, the binding of the ground-variables can be accompanied by that of individual variables. The example is, in this case, the rule

$$\dfrac{\begin{array}{cc} & [\alpha(x)] \\ & \vdots \\ \exists x \alpha(x) & \beta\end{array}}{\beta}\,(\exists_E)$$

understandable, given the restrictions on x, to a total construction function which, taken as input grounds for $\vdash \exists x \alpha(x)$ and for $\alpha(x) \rhd \beta$, returns a ground for $\vdash \beta$, binding the ground-variable corresponding to the assumption $\alpha(x)$ and, on the index of $\alpha(x)$, the variable x. The operational type is

$$\exists x \alpha(x), (\alpha(x) \rhd \beta) \rhd \beta.$$

We consider it appropriate to make an observation at this point. When equated with an operation on grounds, the rule $(\rightarrow_I)$ could be defined on grounds for $\Delta \vdash \beta$ whatever Δ and return grounds on these values for $\Delta - \{\alpha\} \vdash \alpha \rightarrow \beta$. It will still be sufficient to define the operation only relative to grounds for $\alpha \vdash \beta$, and then recover the more general case when we discuss the composition of operations on grounds. However, in intuitive terms, let us assume that an operation

$$f(\xi^{\alpha \rhd \beta})$$

has operational type

$$(\alpha \rhd \beta) \rhd \alpha \rightarrow \beta$$

and let

$$h(\xi^{\gamma_1}, ..., \xi^{\gamma_n}, \xi^{\alpha})$$

be a ground for $\gamma_1, ..., \gamma_n, \alpha \vdash \beta$. For every g_i ground for $\vdash \gamma_i$ $(i \leq n)$,

$$h(g_1, ..., g_n, \xi^{\alpha})$$

is then a ground for $\alpha \vdash \beta$, so that

$$f(h(g_1, ..., g_n, \xi^{\alpha}))$$

is a ground for $\vdash \alpha \rightarrow \beta$. Hence, the composition

$$f(h(\xi^{\gamma_1}, ..., \xi^{\gamma_n}, \xi^{\alpha}))$$

is a ground for $\gamma_1, ..., \gamma_n \vdash \alpha \rightarrow \beta$. As limiting case, we will assume that, for every g ground for $\vdash \beta$,

$$f(g)$$

is a ground for $\vdash \alpha \rightarrow \beta$, so that, given a ground

$$h(\xi^{\gamma_1}, ..., \xi^{\gamma_n})$$

for $\gamma_1, ..., \gamma_n \vdash \beta$, for every g_i ground per γ_i ($i \leq n$),

$$f(h(g_1, ..., g_n))$$

is a ground for $\vdash \alpha \rightarrow \beta$, and hence the composition

$$f(h(\xi^{\gamma_1}, ..., \xi^{\gamma_n}))$$

is again a ground for $\gamma_1, ..., \gamma_n \vdash \alpha \rightarrow \beta$. This corresponds to an application

$$\dfrac{\begin{matrix}\gamma_1 & \ldots & \gamma_n \\ & \Delta & \\ & \beta & \end{matrix}}{\alpha \rightarrow \beta}\ (\rightarrow_I)$$

where the assumption α is "vacuously" discharged. A similar discourse can be conducted for $(\exists_E)$. Therefore, the limitation does not depend on circularity problems - as for $(\forall_I)$ before - but on specific purposes of exposition convenience.

Before the general characterization, we need some notation. We indicate with $\underline{\xi}$ a *sequence of distinct ground-variables*. The ground-variables in $\underline{\xi}$ have an *index*, in correspondence with an operational type where the types of the ground-variables of $\underline{\xi}$ are intended to occur; an element of $\underline{\xi}$ will hence have the form ξ^{α}_i, where i is the index of the entry of the operational type where α occurs. As regards the sequences of individuals and of individual variables, to the previous notations the following must be added. Given an operational type

$$\tau_1, ..., \tau_n \rhd \tau_{n+1}$$

we indicate with $\underline{x}^{\uparrow}_i$ the sequence of the individual varibales occurring altogether in the domain of τ_i, and with $\underline{x}^{\downarrow}_i$ the sequence of the individual variables occurring in the codomain of τ_i. Obviously, if τ_i has empty domain, $\underline{x}^{\uparrow}_i$ is the empty sequence ($i \leq n+1$). Therefore, the indexing is of type $x^{\uparrow}_i$ or $x^{\downarrow}_i$. With this established, given a base B on L, a B-operation on grounds of operational type

$$\tau_1, ..., \tau_n \rhd \tau_{n+1}$$

that binds the elements of a subsequence $\underline{x}$ of $\underline{x}^{\uparrow}_1\ \underline{x}^{\downarrow}_1\ ...\ \underline{x}^{\uparrow}_{n+1}\ \underline{x}^{\downarrow}_{n+1}$, and of a sequence $\underline{\xi}$ such that ξ^{α}_i is in $\underline{\xi}$ if (but not only if, that is, we may have vacuous binding) α is in the domain of some τ_i but not in that of τ_{n+1}, is a total constructive function

$$f((\underline{x}_1^{\downarrow} \text{ ... } \underline{x}_{n+1}^{\downarrow} - \underline{x})\ \underline{y}, \xi^{\tau_1}, ..., \xi^{\tau_n})$$

such that, for every $\underline{s}$ with $\mathfrak{L}\ \underline{s} = \mathfrak{L}\ [(\underline{x}_1^{\downarrow} \text{ ... } \underline{x}_{n+1}^{\downarrow} - \underline{x})\ \underline{y}]$, for every g_i ground on B (proper or improper) for $\tau_i[\underline{s}/(\underline{x}_1^{\downarrow} \text{ ... } \underline{x}_{n+1}^{\downarrow} - \underline{x})]$ $(i \leq n)$

$$f((\underline{x}_1^{\downarrow} \text{ ... } \underline{x}_{n+1}^{\downarrow} - \underline{x})\ \underline{y}, g_1, ..., g_n)$$

is a ground on B (proper or improper) for $\tau_{n+1}[\underline{s}/(\underline{x}_1^{\downarrow} \text{ ... } \underline{x}_{n+1}^{\downarrow} - \underline{x})]$ (or $\vdash \tau_{n+1}[\underline{s}/(\underline{x}_1^{\downarrow} \text{ ... } \underline{x}_{n+1}^{\downarrow} - \underline{x})]$).[3] If the operation binds ξ_i^{α}, we will say that the binding occurs on the i-th entry; the same ground-variable may occur in the sequence on an entry of index $j \neq i$, without being bound on j. Then, called α_i the codomain of τ_i, for τ_i with non-empty domain, and called β the codomain of τ_{n+1}, or simply the codomain of the entire operational type if τ_{n+1} has empty domain, we admit, finally, the limiting case that allows the "vacuous" discharge: for every g_i (proper or improper) ground on B for $\vdash \alpha_i[\underline{s}/(\underline{x}_1^{\downarrow} \text{ ... } \underline{x}_{n+1}^{\downarrow} - \underline{x})]$ $(i \leq n)$,

$$f(\underline{s}/(\underline{x}_1^{\downarrow} \text{ ... } \underline{x}_{n+1}^{\downarrow} - \underline{x})\ \underline{y}, g_1, ..., g_n)$$

is a (proper or improper) ground on B for $\vdash \beta[\underline{s}/(\underline{x}_1^{\downarrow} \text{ ... } \underline{x}_{n+1}^{\downarrow} - \underline{x})]$. There are clearly also "mixed cases", where some g_i are (proper or improper) grounds on B for τ_i, and other are (proper or improper) grounds on B only for the codomain of τ_i $(i \leq n)$; therefore, in the codomain of τ_{n+1} will appear only those domains taken into account in the application.

The linguistic expression of the operation involves individual variables that are not bound by it; it should also be noted that it also involves ground-variables that could be typed on an operational type for a first-level operation. Obviously, on the additional sequence y we make the same request as on first-level operations, adding the same restriction on the indexes where the ground-variables occur in the operations. Finally, it should be noted that the operation acts only on the individual variables that occur in the codomains of the various τ_i, and which are not bound $(i \leq n+1)$. And all this is because the function, once applied to specific values, might not be defined on the domain variables, simply because these specific values, as we will see when we deal with the composition of the operations on grounds, may have an operational type with *any* domain.

We conclude this Section by pointing out that a B-operation on grounds of operational type

$$\tau_1, ..., \tau_n \rhd \tau_{n+1}$$

[3]See note 2 in the present chapter.

will be also called a ground on B for $\tau_1, ..., \tau_n \vdash \tau_{n+1}$. It will be a *proper* ground when the additional sequence $\underline{y}$ is empty, otherwise, an *improper* ground.

5.2.2.3 Plugging operations and restrictions

We now introduce the notion of *composition of operations on grounds*. The term "composition" is used here by analogy with that used in standard set-theory: given

$$f\colon B \to C \text{ and } h\colon A \to B$$

we can consider the composite function of f and h

$$f \circ h\colon A \to C$$

by putting

$$f \circ h(x) = f(h(x)).$$

Exactly in the same way, given n operations on grounds

$$h_1(\underline{x}_1, \xi^{\tau_1^1}, ..., \xi^{\tau_{m_1}^1}), ..., h_n(\underline{x}_n, \xi^{\tau_1^n}, ..., \xi^{\tau_{m_n}^n})$$

each of operational type

$$\tau_1^i, ..., \tau_{m_i}^i \rhd \tau_{m_i+1}^i$$

$(i \leq n)$ where $\tau_{m_i+1}^i$ is α_i if $\tau_{m_i+1}^i$ has empty domain, and it has codomain α_i otherwise, and given an operation on grounds

$$f(\underline{x}, \xi^{\tau_1}, ..., \xi^{\tau_n})$$

of operational type

$$\tau_1, ..., \tau_n \rhd \tau_{n+1}$$

where τ_i is α_i if τ_i has empty domain, and has codomain α_i otherwise $(i \leq n)$, we take as defined the composite of f and $h_1 \dots h_n$

$$f \circ (h_1 \dots h_n)$$

of operational type

$$\sigma_1, ..., \sigma_n \rhd \sigma_{n+1}$$

by putting

$$f \circ (h_1 \dots h_n) = f(\underline{x}, h_1(\underline{x}_1, \xi^{\tau_1^1}, ..., \xi^{\tau_{m_1}^1}), ..., h_n(\underline{x}_n, \xi^{\tau_1^n}, ..., \xi^{\tau_{m_n}^n})).$$

As regards the operational type of the composite, it is determined as follows:

- each σ_i $(i \leq n)$ is the set of all the $\tau_{m_j}^i$ $(j \leq m_i)$, *minus* the assumptions corresponding to the type of the ground-variables bound by f on index i;
- as for σ_{n+1}, its codomain is the union of all the domains of the various τ_j^i with non-empty domain *minus* the assumptions corresponding to the type of the ground-variables bound by h_i on index j, and *minus* the assumptions corresponding to the type of the ground-variables bound by f on index i $(i \leq n,\ j \leq m_i)$, while the codomain of σ_{n+1} will be that of τ_{n+1}, or simply τ_{n+1} if the latter has empty domain.

In order to better understand the functioning of the composition, we will give four practical examples. We will rely again on a representation in terms of valid arguments; moreover, we will disregard the involvement of free individual variables. As a first case, let us suppose we have two valid arguments

$$\begin{array}{cc} \alpha & \gamma \\ \Delta_1 & \Delta_2 \\ \beta & \alpha \end{array}$$

to be understood as operations on grounds

$$f(\xi^\alpha) \text{ and } h(\xi^\gamma)$$

of respective operational types

$$\alpha \rhd \beta \text{ and } \gamma \rhd \alpha;$$

then, the composite of f and h can be understood as the open valid argument

$$\begin{array}{c} \gamma \\ \Delta_2 \\ \alpha \\ \Delta_1 \\ \beta \end{array}$$

obtained by "composing" Δ_1 and Δ_2, and therefore understandable as an operation on grounds

$$f(h(\xi^{\gamma}))$$

of operational type

$$\gamma \rhd \beta.$$

Let us now suppose that we have a valid argument

$$\begin{array}{c} \alpha \\ \Delta \\ \gamma \end{array}$$

to be understood again as an operation on grounds

$$f(\xi^{\alpha})$$

of operational type

$$\alpha \rhd \beta$$

and let us suppose we have a rule

$$\cfrac{\begin{array}{cc} \gamma_1 & [\gamma_2] \\ & \vdots \\ & \gamma_3 \end{array}}{\alpha}$$

to be understood as an operation on grounds

$$h(\xi^{\gamma_1, \gamma_2 \rhd \gamma_3})$$

of operational type

$$(\gamma_1, \gamma_2 \rhd \gamma_3) \rhd (\gamma_1 \rhd \alpha);$$

binding the ground-variable of type γ_2; then, the composite of f and h can be understood as the structure

$$\begin{array}{c} \cfrac{\begin{array}{cc} \gamma_1 & [\gamma_2] \\ & \vdots \\ & \gamma_3 \end{array}}{\alpha} \\ \Delta \\ \beta \end{array}$$

obtained by "composing" Δ with the rule we had supposed to have, and therefore understandable as an operation on grounds

$$f(h(\xi^{\gamma_1, \gamma_2 \rhd \gamma_3}))$$

of operational type

$$(\gamma_1, \gamma_2 \rhd \gamma_3) \rhd (\gamma_1 \rhd \beta).$$

Moving on to the third example, let us consider the usual rule

$$\dfrac{\begin{matrix}[\alpha] \\ \vdots \\ \beta\end{matrix}}{\alpha \rightarrow \beta}\ (\rightarrow_I)$$

to be understood as an operation on grounds

$$f(\xi^{\alpha \rhd \beta})$$

of operational type

$$(\alpha \rhd \beta) \rhd \alpha \rightarrow \beta$$

that binds the ground-variable of type α, and let us suppose we have a valid argument

$$\begin{matrix}\gamma \quad \alpha \\ \Delta \\ \beta\end{matrix}$$

to be understood as an operation on grounds

$$h(\xi^{\gamma}, \xi^{\alpha})$$

of operational type

$$\gamma, \alpha \rhd \beta;$$

then, the composite of f and h can be understood as the valid argument

$$\dfrac{\begin{matrix}\gamma \quad [\alpha] \\ \Delta \\ \beta\end{matrix}}{\alpha \rightarrow \beta}\ (\rightarrow_I)$$

understandable as an operation on grounds

$$f(h(\xi^{\gamma}, \xi^{\alpha}))$$

of operational type

$$\gamma \rhd \alpha \to \beta.$$

Finally, let us suppose to have a rule

$$\dfrac{\begin{matrix} \gamma_1 & & [\gamma_2] \\ & \vdots & \\ & \alpha & \end{matrix}}{\beta}$$

to be understood as an operation on grounds

$$f(\xi^{\gamma_1, \gamma_2 \rhd \alpha})$$

of operational type

$$(\gamma_1, \gamma_2 \rhd \alpha) \rhd (\gamma_1 \rhd \beta)$$

that binds the ground-variable of type γ_2, and let us suppose to have the rule

$$\dfrac{\begin{matrix} \delta_1 & [\delta_2] & \gamma_2 \\ & \vdots & \\ & \delta_3 & \end{matrix}}{\alpha}$$

to be understood as an operation on grounds

$$h(\xi^{\delta_1, \delta_2, \gamma_2 \rhd \delta_3})$$

of operational type

$$(\delta_1, \delta_2, \gamma_2 \rhd \delta_3) \rhd (\delta_1, \gamma_2 \rhd \alpha)$$

that binds the ground-variable of type δ_2; the composite will be the structure

$$\dfrac{\dfrac{\begin{matrix} \delta_1 & [\delta_2] & [\gamma_2] \\ & \vdots & \\ & \delta_3 & \end{matrix}}{\alpha}\, \delta_2}{\beta}\, \gamma_2$$

understandable as an operation on grounds

$$f(h(\xi^{\delta_1, \delta_2, \gamma_2 \rhd \delta_3}))$$

of operational type

$$(\delta_1, \delta_2, \gamma_2 \rhd \delta_3) \rhd (\delta_1 \rhd \beta).$$

With regard to the examples discussed so far, we would like to make two observations. First of all, we observe that the examples cover respectively the following four cases:

- a first-level operation f is applied to a first-level operation h, where the codomain of h corresponds to *one of the entries of the domain* of f;
- a first-level operation f is applied to a second-level operation h, where the (codomain of the) codomain of h corresponds to *one of the entries of the domain* of f;
- a second-level operation f is applied to a first-level operation h, where the codomain of h corresponds to the *(codomain of) one of the entries of the domain* of f;
- a second-level operation f is applied to a second-level operation h, where the (codomain of the) codomain of h corresponds to the *(codomain of) one of the entries of the domain* of f.

In all cases, the operational type of the composite corresponds to the operational type of an operation on grounds of level ≤ 2; these will be the only types of composition that we will take into account in the following.

The second observation related to the examples concerns the individual variables. Although, in order not to complicate our discussion, we have done without mention them, their treatment will be subject to two important restrictions, which we will indicate below as restrictions (1) and (2).

After this preliminary discussion, we can turn to a general characterization, on the basis of which to state a result. Let B be a base on L, and let

$$f(\underline{x}, \xi^{\tau_1}, ..., \xi^{\tau_n})$$

be a B-operation on grounds of operational type

$$\tau_1, ..., \tau_n \rhd \tau_{n+1}$$

where τ_i is α_i if τ_i has empty domain, and it has α_i as codomain otherwise. Then, for every $i \leq n$, for every

$$h_i(\underline{x}_i, \xi^{\tau^i_1}, ..., \xi^{\tau^i_{m_i}})$$

(proper or improper) ground on B for $\tau^i_1, ..., \tau^i_{m_i} \vdash \tau^i_{m_i+1}$ ($m_i \geq 0$) where $\tau^i_{m_i+1}$ is α_i if $\tau^i_{m_i+1}$ has empty domain, and it has α_i as codomain otherwise, and which moreover respects the restrictions (1) and (2) below, we call

$$f(\underline{x}, \ \dots \ h_i(\underline{x}_i, \xi^{\tau^i_1}, ..., \xi^{\tau^i_{m_i}}) \ \dots)$$

a *composite* operation of f and $\dots \ h_i \ \dots$ - where the dots refers to the indexes from 1 to n on which the composition takes place. We will also say that

$$h_i(\underline{x}_i, \xi^{\tau^i_1}, ..., \xi^{\tau^i_{m_i}})$$

is plugged into f on index i.[4]

[4]Observe that the individual variables involved in h_i - in compliance with the restrictions told about below - are here to be understood as all having an index i, corresponding to the index on which h_i is plugged. We left out these details here so as not to burden the notation. Additionally, it should be observed that the composite of f and h_i is to be understood as involving, relatively to index i, all the variables involved in f on i, and all those involved in h_i; nonetheless, if f binds x on index i, and h_i involves x, then the composite is *not* to be understood as involving x. By illustratively resorting to λ-abstraction, when we compose f with h_i, we will have

$$\dots \lambda x_i \dots (\dots h_i(\dots x_i \dots) \dots).$$

Finally, we take the opportunity to observe that also the ground-variables involved in f, and that f binds, should have indexes - the same way as indexes are attributed to the assumptions in a derivation in natural deduction; we can adopt the convention that the index of a ξ^α involved in f is the index of the entry of the operational type where α occurs, and that the index of a bound ground-variable is the index where it is bound. When we plug h_i, then, we have to require that the ground-variables ξ^α involved in h_i are re-indexed, to avoid conflicts with indexes of ground-variables ξ^α involved in f; this can be done by requiring that the ground-variables of h_i have the additional index i. In this way, given

$$h_i(\dots \xi^\alpha_j \dots),$$

after the plugging we will have

$$\dots \lambda \xi^\alpha_i \dots (\dots \xi^\alpha_j, h_i(\dots \xi^\alpha_{j,i} \dots) \dots);$$

f is meant to bind *all* the $\xi^\alpha_{j,i}$ involved in h_i.

Proposition 27. Let B be a base on L, and let

$$f(\underline{x}, \xi^{\tau_1}, ..., \xi^{\tau_n})$$

be a B-operation on grounds of operational type

$$\tau_1, ..., \tau_n \rhd \tau_{n+1}$$

and, for $i \leq n$, let

$$h_i(\underline{x}_i, \xi^{\tau^i_1}, ..., \xi^{\tau^i_{m_i}})$$

be a (proper or improper) ground on B for $\tau^i_1, ..., \tau^i_{m_i} \vdash \tau^i_{m_i+1}$ plugged into f on index i. Then, the composite of f and $\dots\ h_i\ \dots$ is a (proper or improper) ground on B for

$$\dots\ \{\tau^i_1, ..., \tau^i_{m_i}\} - \nu_i\ \dots\ \vdash (\dots\ ((\{\sigma^i_s, ..., \sigma^i_t\} - \{\nu^i_s, ..., \nu^i_t\}) - \nu_i)\ \dots\ \rhd \beta)$$

where

- ν_i is the set of the type of the ground-variables bound by f on index i;
- $\{\nu^i_s, ..., \nu^i_t\}$ is the set of the ground-variables bound by h_i on indexes $s, ..., t$ ($s, ..., t \leq m_i$);
- $\{\sigma^i_s, ..., \sigma^i_t\}$ is the set of the domains of the entries of the operational type of h_i with non-empty domain;
- β is τ_{n+1} if τ_{n+1} has empty domain, and it is the codomain of τ_{n+1} otherwise.

Moreover, given a sequence $\underline{s}$ of individuals on B which respects the restrictions (1) and (2) below,

$$f(\underline{s}/\underline{x},\ \dots\ h_i(\underline{s}/\underline{x}_i, \xi^{\tau^i_1[\underline{s}/\underline{x}_i]}, ..., \xi^{\tau^i_{m_i}[\underline{s}/\underline{x}_i]})\ \dots\ )$$

- where the substitution holds for some or all of the free variables in the operational type - is a (proper or improper) ground on B for the judgment or assertion above, where the free variables are replaced by $\underline{s}$.

As regards the proof of the proposition, we refer to the examples given in the preliminary discussion. Instead, we turn now to the restrictions on the individual variables, mentioned when defining the notion of composition of two (or more) operations on grounds, and in the statement of the proposition. The restrictions are to be understood as a generalization of those on the rules of introduction and elimination of the quantifiers in a Gentzen's natural deduction system for first-order logic. The first generalizes that on

$$\frac{\forall x\alpha(x)}{\alpha(t/x)}\ (\forall_E) \qquad \frac{\alpha(t/x)}{\exists x\alpha(x)}\ (\exists_I)$$

where t is required to be free for x in $\alpha(x)$.

Restriction (1): given an operation on grounds of operational type

$$\tau_1, ..., \tau_n \rhd \tau_{n+1}$$

if, for some $i \leq n$, τ_i has empty domain and is of the form

$$\ldots\ \forall x\ \ldots\ \alpha_i(x, t/y)$$

or it has non-empty domain and its codomain has such form, and if τ_{n+1} has empty domain and is of the form

$$\ldots\ \exists y\ \ldots\ \beta(u/x, y)$$

or it has non-empty domain and its codomain has such form, then

- t is free for y in $\beta(u/x, y)$ and
- u is free for x in $\alpha(x, t/y)$.

As regards the second restriction, it is a generalization of that on

$$\frac{\alpha(x)}{\forall y\alpha(y/x)}\ (\forall_I) \qquad \frac{\exists x\alpha(y/x) \qquad \begin{matrix}[\alpha(x)] \\ \vdots \\ \beta\end{matrix}}{\beta}\ (\exists_E)$$

where, as we know, for x it must hold that it does not occur free neither (on the branch where it is bound) in non-discharged assumptions on which the premise (on that branch) depends and which are different from assumptions bound by the rule, nor in the conclusion of the rule.

Restriction (2): given an operation on grounds f of operational type

$$\tau_1, ..., \tau_n \rhd \tau_{n+1}$$

that binds x on index i, a (proper or improper) ground h for $\tau^i_1, ..., \tau^i_{m_i} \vdash \tau^i_{m_i+1}$ plugged into f on index i must be such that, for every $j \leq m_i$

- if τ^i_j has non-empty domain Γ, then, for every $\beta \in \Gamma$, x occurs free in β if, and only if, h binds a ground-variable of type β on index j or f binds a ground-variable of type β on index i.

Moreover, x does not occur free in τ_{n+1} if τ_{n+1} has empty domain, or in the codomain of τ_{n+1} if τ_{n+1} has non-empty domain.

5.2.2.4 Ground-clauses and identity of operations

Once the discussion on operations on grounds has been completed, we can now introduce the clauses that establish the notion of ground for categorical judgments or assertions. Since judgments and assertions must be categorical, the mentioned formulas and terms will always be closed, or will be open on condition that the individual variables involved in them are bound in the formula having the logical form which the clause deals with. Therefore, given a base B on L, and given canonical names δ for the derivations in the atomic system of B, we will have as follows.

(At_G) for every δ closed with conclusion α, δ is a ground on B for $\vdash \alpha$

($\wedge_G$) g_1 is a ground on B for $\vdash \alpha$ and g_2 is a ground on B for $\vdash \beta \Leftrightarrow \wedge I(g_1, g_2)$ is a ground on B for $\vdash \alpha \wedge \beta$

($\vee_G$) g is a ground on B for $\vdash \alpha_i \Leftrightarrow \vee I\langle \alpha_i \rhd \alpha_1 \vee \alpha_2 \rangle(g)$ is a ground on B for $\vdash \alpha_1 \vee \alpha_2$ $(i = 1, 2)$

($\rightarrow_G$) $f(\xi^\alpha)$ is a ground on B for $\alpha \vdash \beta \Leftrightarrow \rightarrow I\xi^\alpha(f(\xi^\alpha))$ is a ground on B for $\vdash \alpha \rightarrow \beta$

($\forall_G$) $f(x)$ is a ground on B for $\vdash \alpha(x) \Leftrightarrow \forall Ix(f(x))$ is a ground on B for $\vdash \forall x\alpha(x)$

($\exists_G$) g is a ground on B for $\vdash \alpha(t) \Leftrightarrow \exists I\langle \alpha(t) \rhd \exists x\alpha(x) \rangle(g)$ is a ground on B for $\vdash \exists x\alpha(x)$

($\bot_G$) there is no ground on B for $\vdash \bot$

$\wedge I$, $\vee I$, $\rightarrow I$, $\forall I$ and $\exists I$ are to be understood as (the only) *primitive* operations. Primitivenes depends on the fact that these operations are *meaning-constitutive*; in other words, to know the meaning of a formula α with k as main logical constant, we need to know that there is an operation kI that yields a ground for $\vdash \alpha$ when applied to the grounds for the immediate subformulas of α.The primitive operation will hence have by default the following respective operational types:

($\wedge$) $\alpha, \beta \rhd \alpha \wedge \beta$

($\vee$) $\alpha_i \rhd \alpha_1 \vee \alpha_2$ $(i = 1, 2)$

($\rightarrow$) $(\alpha \rhd \beta) \rhd \alpha \rightarrow \beta$

($\forall$) $\alpha(x) \rhd \forall x\alpha(x)$

$(\exists)$ $\alpha(t) \rhd \exists x \alpha(x)$

Furthermore, it should be noted that $\rightarrow I$ binds a ground-variable of type α, while $\forall I$ binds the individual variable x. Finally, by virtue of the clause $(\bot_G)$, we can take as defined on every base an operation

$$\bot_\alpha(\xi^\bot)$$

with operational type

$$\bot \rhd \alpha;$$

indeed, if there is no ground on any base for $\vdash \bot$, the antecedent of the definition of operation on grounds having that operational type is vacuously satisfied, so that the conditions required by the definition are vacuously satisfied too. In other words, introducing the clause $(\bot_G)$ is tantamount to require that the atomic bases are always consistent.

Finally, we emphasize that the primitive operations undergo some identity conditions. These conditions have to be explicitly stated since the operations, because of their primitiveness, cannot be further analyzed.

- $g_i = h_i$ $(i = 1, 2) \Leftrightarrow \wedge I(g_1, g_2) = \wedge I(h_1, h_2)$
- $g = h \Leftrightarrow \vee I(g) = \vee I(h)$
- $f_1(\xi^\alpha) = f_2(\xi^\alpha) \Leftrightarrow \rightarrow I\xi^\alpha(f_1(\xi^\alpha)) = \rightarrow I\xi^\alpha(f_2(\xi^\alpha))$
- $f_1(x) = f_2(x) \Leftrightarrow \forall Ix(f_1(x)) = \forall Ix(f_2(x))$
- $g = h \Leftrightarrow \exists I(g) = \exists I(h)$
- $g = h \Leftrightarrow \bot_\alpha(g) = \bot_\alpha(h)$

The identity conditions for categorical judgments or assertions of implications and universal quantifications show that it is also necessary to introduce identity conditions for operations on grounds. From this point of view, here and in the sequel, we will adopt an *extensional* approach, according to which, given to operations on grounds f_1 and f_2, $f_1 = f_2$ if, and only if, (1) f_1 and f_2 have the same domain and (2) f_1 and f_2 produce the same saturated values when applied to the same saturated arguments. This definition can be taken to proceed by simultaneous recursion, based on the identity conditions for grounds for categorical judgments or assertions. With regard to this point, we would like to make two observations.

The first is that another possible way of dealing with identity of operations on grounds - and hence, of grounds for categorical judgments or assertions - would be that of understanding this notion *intensionally*. As we will see in detail in Section 5.2.4.5, operations on grounds are determined, not only by their operational type, but also by an equation that define them, in the sense of indicating *how* they produce values in the codomain when applied to arguments in the domain, and *which* these values are. Two operations on grounds can be therefore said to be intensionally identical when they are, so to say, defined by the same equation and, if they are composite, when they are composed of the same operations, applied in the same order. It follows that the identity we have defined in an extensional way is weaker than intensional identity, so that it would perhaps be more correct to speak of *equivalence*, and to deserve the expression "identity" to the intensional standpoint - and, if we adopt this terminology, it will hold that identity implies equivalence, but not viceversa.

Moreover, even if we will not deal with the intensional version, and limit ourselves only to the quick mention of the previous observation, and to another reference in Section 5.2.4.4, there is a point of our investigation where we will pursue a more intensional line of thought. This will happen during our discussion of the notions of universal grounds and operations on grounds, in Section 5.2.4.6.[5]

5.2.3 Core languages and expansions

Grounds and operations on grounds are some of the objects denoted by the terms of the languages of grounding that we will be defining; these are the "inhabitans" of our "universe", and the terms of the languages of grounding, by referring to this "semantics universe", will denote some of them.

[5]The choice of the extensional approach has also other consequences. The first is that the requirement that two identical operations have identical domains blocks the possibility of declaring as identical two operations that, if we understand them as open valid arguments or valid rules, reduce one to another - because in reduction the domain of one of them may be only contained in the domain of the other. The second is that we could have identical operations, that would however correspond to open valid arguments or valid rules in normal form, and that hence do not reduce one to another. If these circumstances are unpleasant, besides the notions of extensional identity and intensional identity, we may introduce a third, that we may call *reductive identity* - or *reductive equivalence* if we have called equivalence the extensional identity - and indicate with $\equiv$. Then, $f_1 \equiv f_2$ if, and only if, the domain of f_1 is contained in that of f_2 or viceversa, and the operations produce the same saturated values when applied to the same saturated values corresponding to the intersection of the domains. The extensional identity/equivalence implies the reductive one, but not viceversa.

5.2.3.1 Core language over an atomic base

The notion from which we will start is that of core language.

Definition 28. Let B be a base on a first-order logical language L. A *core language* $\mathtt{G}$ *on* B is determined by an alphabet and by a set of terms. The *alphabet* is the following - we indicate with $\mathtt{DER_S}$ the set of the derivations of the atomic system $\mathtt{S}$ of B -

- individual constants δ_i [δ_i is (a name for) the i-th element of $\mathtt{DER_S}$]
- ground-variables ξ_i^α ($\alpha \in \mathtt{FORM}_L$, $i \in \mathbb{N}$)
- operational symbols F of operational type $\langle \ddagger \rangle$:
  - $\wedge_I\langle \alpha, \beta \rhd \alpha \wedge \beta \rangle$ ($\alpha, \beta \in \mathtt{FORM}_L$)
  - $\vee_I\langle \alpha_i \rhd \alpha_1 \vee \alpha_2 \rangle$ ($i = 1, 2$, $\alpha_1, \alpha_2 \in \mathtt{FORM}_L$)
  - $\rightarrow_I \langle \beta \rhd \alpha \rightarrow \beta \rangle$ ($\alpha, \beta \in \mathtt{FORM}_L$)
  - $\forall_I\langle \alpha(x_i) \rhd \forall y_j \alpha(y_j/x_i) \rangle$ ($i, j \in \mathbb{N}$, $\alpha \in \mathtt{FORM}_L$)
  - $\exists_I\langle \alpha(t/x_i) \rhd \exists x_i \alpha(x_i) \rangle$ ($t \in \mathtt{TERM}_L$, $\alpha \in \mathtt{FORM}_L$, $i \in \mathbb{N}$)
  - $\bot_\alpha\langle \bot \rhd \alpha \rangle$ ($\alpha \in \mathtt{FORM}_L$)
- Parentheses and comma as auxiliary symbols

In order not to excessively burden the notation, and whenever this does not give rise to ambiguity, we will omit the explicit mention of the typing of the operational symbols, as well as indexes and subscripts. The notation $T : \alpha \in X$ indicates that T is an element of X of type α. The set $\mathtt{TERM_G}$ of the *terms* of $\mathtt{G}$ is the least set X such that

- $\delta : \alpha \in X$ (α conclusion of δ)
- $\xi^\alpha : \alpha \in X$
- $T : \alpha, U : \beta \in X \Rightarrow \wedge I(T, U) : \alpha \wedge \beta \in X$
- $T : \alpha_i \Rightarrow \vee I\langle \alpha_i \rhd \alpha_1 \vee \alpha_2 \rangle : \alpha_1 \vee \alpha_2 \in X$ ($i = 1, 2$)
- $T : \beta \in X \Rightarrow \rightarrow I\xi^\alpha(T) : \alpha \rightarrow \beta \in X$ ($\alpha \in \mathtt{FORM}_L$)
- $T : \alpha(x) \in X \Rightarrow \forall I y(T) : \forall y \alpha(y/x) \in X$
- $T : \alpha(t/x) \in X \Rightarrow \exists I\langle \alpha(t/x) \rhd \exists x \alpha(x) \rangle(T) : \exists x \alpha(x) \in X$

- $T : \bot \Rightarrow \bot_\alpha(T) : \alpha \in X$

In $\forall Iy(T)$, it must not happen that $y \in FV(\alpha)$ where ξ^α occurs free in T; y is called the *proper variable* of the term. In $\exists I(T)$, t must be free for x in $\alpha(x)$.

Definition 28 must be accompanied by a series of technical, but important definitions. We provide them without further explanations, only by pointing out that the notions they identify must be introduced, in a similar way, in all the languages of grounding.

Definition 29. The set $S(T)$ of the *subterms* of T is inductively defined as follows:

- $S(\delta) = \{\delta\}$
- $S(\xi^\alpha) = \{\xi^\alpha\}$
- $S(\wedge I(T, U)) = S(T) \cup S(U) \cup \{\wedge I(T, U)\}$
- $S(\vee I(T)) = S(T) \cup \{\vee I(T)\}$
- $S(\rightarrow I\xi^\alpha(T)) = S(T) \cup \{\rightarrow I\xi^\alpha(T)\}$
- $S(\forall Ix(T))) = S(T) \cup \{\forall Ix(T)\}$
- $S(\exists I(T)) = S(T) \cup \{\exists I(T)\}$
- $S(\bot_\alpha(T)) = S(T) \cup \{\bot_\alpha(T)\}$

Definition 30. The set $FV^I(T)$ of the *free individual variables* of T is defined inductively as follows:

- $FV^I(\delta) = \{x_1, ..., x_n\}$, for $x_1, ..., x_n$ occurring free in δ
- $FV^I(\xi^\alpha) = FV(\alpha)$
- $FV^I(\wedge I(T, U)) = FV^I(T) \cup FV^I(U)$
- $FV^I(\vee I\langle \alpha_i \rhd \alpha_1 \vee \alpha_2 \rangle(T)) = FV^I(T) \cup FV(\alpha_j)$ $(i, j = 1, 2,\ i \neq j)$
- $FV^I(\rightarrow I\xi^\alpha(T)) = FV^I(T) \cup FV(\alpha)$
- $FV^I(\forall Ix(T)) = FV^I(T) - \{x\}$
- $FV^I(\exists I(T)) = FV^I(T)$

- $FV^I(\bot_\alpha(T)) = FV^I(T)$

The set $BV^I(T)$ of the *bound individual variables* of T is defined inductively as follows:

- $BV^I(\delta) = \{x_1, ..., x_n\}$, for $x_1, ..., x_n$ occurring bound in δ
- $BV^I(\xi^\alpha) = BV(\alpha)$
- $BV^I(\wedge I(T, U)) = BV^I(T) \cup BV^I(U)$
- $BV^I(\vee I\langle\alpha_i \rhd \alpha_1 \vee \alpha_2\rangle(T)) = BV^I(T) \cup BV(\alpha_j)$ $(i, j = 1, 2,\ i \neq j)$
- $BV^I(\rightarrow I\xi^\alpha(T)) = BV^I(T)$
- $BV^I(\forall Ix(T)) = BV^I(T) \cup \{x\}$
- $BV^I(\exists I(T)) = BV^I(T)$
- $BV^I(\bot_\alpha(T)) = BV^I(T)$

Definition 31. The set $FV^T(U)$ of the *free ground-variables* of U is defined inductively as follows:

- $FV^T(\delta) = \{\xi^{\alpha_1}, ..., \xi^{\alpha_n}\}$ $(\alpha_1, ..., \alpha_n$ undischarged assumptions of $\delta)$
- $FV^T(\xi^\alpha) = \{\xi^\alpha\}$
- $FV^T(\wedge I(U, Z)) = FV^T(U) \cup FV^T(Z)$
- $FV^T(\vee_I(U)) = FV^T(U)$
- $FV^T(\rightarrow I\xi^\alpha(U)) = FV^T(U) - \{\xi^\alpha\}$
- $FV^T(\forall Ix(U)) = FV^T(U)$
- $FV^T(\exists I(U)) = FV^T(U)$
- $FV^T(\bot_\alpha(U)) = FV^T(U)$

The set $BV^T(U)$ of the *bound ground-variables* of U is defined inductively as follows:

- $BV^T(\delta) = \{\xi^{\alpha_1}, ..., \xi^{\alpha_n}\}$ $(\alpha_1, ..., \alpha_n$ discharged assumptions of $\delta)$
- $BV^T(\xi^\alpha) = \emptyset$

- $BV^T(\wedge I(U, Z)) = BV^T(U) \cup BV^T(Z)$
- $BV^T(\vee I(U)) = BV^T(U)$
- $BV^T(\rightarrow I\xi^\alpha(U)) = BV^T(U) \cup \{\xi^\alpha\}$
- $BV^T(\forall Ix(U)) = BV^T(U)$
- $BV^T(\exists I(U)) = BV^T(U)$
- $BV^T(\bot_\alpha(U)) = BV^T(U)$

U is *closed* if, and only if, $FV^I(U) = FV^T(U) = \emptyset$.

For the sake of greater simplicity, we finally adopt a rather standard convention concerning free and bound variables, and proper variables; a similar convention has already been mentioned with regard to the derivations in the Gentzen's natural deduction, as well as to typed λ-calculus. As in the notions of the previous definitions, also this convention is intended to apply to all languages of grounding.

Convention 32. In every T (1) $FV^I(T) \cap BV^I(T) = \emptyset$ - property (FB) - and (2) proper and non-proper variables are all distinct from each other, and each proper variables occurs in at most one $\forall Ix(U)$ - property (PN).

5.2.3.2 Languages of grounding and expansions

The notion of core language serves as a basis for defining the notion of language of grounding in general, and for the expansion of a language of grounding in general. As pointed out several times, we will ensure that every language of grounding on a given base contains a core language on that base as its proper or improper sublanguage; in other words, the expression "language of grounding on a base B" has the same meaning as "(proper or improper) expansion of a core language on B". It is worth repeating that this strategy is not formally unavoidable, but we will pursue it as philosophically plausible in the general framework of the theory of grounds – as well as having the effect of facilitating the demonstration of some results.

Before moving on to the actual definitions, however, we must make some preliminary remarks. To begin with, a core language on B can be expanded in two ways - possibly combined:

1. by adding new individual constants that name derivations in an expansion of the atomic system of B, or

2. by adding new operational symbols.

However, point 2 requires a more in-depth discussion. Firstly, we specify that, as was already in the case of the core language of definition 28, an operational symbol F will be, in the alphabet of a language of grounding, accompanied by an operational type $\langle \ddagger \rangle$ on L; the construction of the terms of the language of grounding, therefore, will proceed by adding to a certain number of appropriate terms, not simply F, but more particularly $F\langle \ddagger \rangle$ - although, except for ambiguities, the operational type will be left implicit. Thus, when we add operational symbols, the latter are to be understood as typed; on the other hand it is clear that an operational type, associated to an operational symbol F, can be any string of formulas of L - in spite of the existence of an appropriate operation on grounds having that operational type. The latter fact, for its part, could mean that, if the quantity of the operational symbols added is countable but not finite, and if the operational types associated with the new operational symbols constitute a non-decidable set, the grounding language itself ends up being not "axiomatizable". And we clearly want to exclude such possibility.

For this reason, in the definitions that follow we will use the expression "operational type instance of a decidable scheme of operational types". This, in practice, means that point 2 above, is divided into the following subpoints:

2_1 the operational symbols added are *finite* in number, or

2_2 the operational symbols added are of a countable but non-finite quantity, and classifiable in a finite number of groups since the operational type of each falls into a finite number of structural descriptions of operational types – for example, the type matches the structural description $\alpha_1 \wedge \alpha_2 \rhd \alpha_i$ $(i = 1, 2)$.

Therefore, in conclusion, using in our opinion a fitting metaphor, the languages of grounding that we will be defining can be equated with *finitely axiomatizable* or, more weakly, *axiomatizable* systems. In other words, the notion of scheme of operational types scheme is analogous to that of scheme of axioms.

As a further premise, we want to point out that the addition of operational symbols will be subject to the following restrictions. First of all, the operational type associated with the operational symbol must be such that there is a *corresponding operation on grounds of the appropriate operational type*. The expression "appropriate" must be specified in the following sense. Each operational symbol will have an operational type for a *first-level* operation on grounds; if, however, the symbol is intended to bind ground-variables, the required appropriate operation will be of the *second level*. In particular, if the symbol has β as the i-th entry of the type, and binds a ground-variable of type on α index i, the i-th entry of the type of the operation will have a non-empty domain, containing α, and codomain β - that is, it will be of the form

$$\Gamma \rhd \beta$$

for $\alpha \in \Gamma$. We can now move on to the actual definitions.

Definition 33. Let $\mathtt{G}_1$ be a core language on a base B_1 on a language L_1. We will say that a *language of grounding* is a core language $\mathtt{G}_2$ on a base B_2 on a language L_2, with B_2 expansion of B_1, or a language Λ specified by

- an *alphabet* $\mathtt{Al}$ such that, called $\mathtt{Al}_{\mathtt{G}_2}$ the alphabet of a $\mathtt{G}_2$ as above,

$$\mathtt{Al}_{\mathtt{G}_2} \subseteq \mathtt{Al} \text{ and } \mathtt{Al} - \mathtt{Al}_{\mathtt{G}_2} = \{F_1, ..., F_n\}$$

  where, for every $i \leq n$, F_i can bind a sequence of individual variables $\underline{x}$ or a sequence of ground-variables $\underline{\xi}$, and to it we attribute operational types

$$\alpha_1(\underline{x}_1), ..., \alpha_n(\underline{x}_m) \rhd \beta(\underline{x}_{m+1})$$

  instances of a decidable schemes of operational types on L_2 such that there is a B_2-operation on grounds of which the expression f has the following properties:

  1. f involves all and only the individual variables of the sequence $\underline{x}_1, ..., \underline{x}_m, \underline{x}_{m+1} - \underline{x}$
  2. the operational type of f has a codomain $\beta(\underline{x}_{m+1})$ and, for every $j \leq m$, the entry of index j has a non-empty domain and codomain $\alpha_j(\underline{x}_j)$ if F_i binds ground-variables on j, and the domain of the entry of index j is the set of the types of the ground-variables that F_i binds on j, otherwise the j-th entry is $\alpha_j(\underline{x}_j)$;
  3. F_i binds x and ξ^α on j and k respectively if, and only if, f binds x and ξ^α on j and k respectively $(i, j \leq m)$;

- a set of *terms* $\mathtt{TERM}_\Lambda$ specified, for every $F \in \mathtt{Al} - \mathtt{Al}_{\mathtt{G}_2}$, by a clause that, in compliance with the operational types $\langle \ddagger \rangle$ associated to F, has the form

$$U_i : \alpha_i(\underline{x}_i) \in X \ (i \leq m) \Rightarrow F\langle \ddagger \rangle \ \underline{x} \ \underline{\xi} \ (U_1, ..., U_m) : \beta(\underline{x}_{m+1}) \in X;$$

and that also respects an analogue of the restrictions (1) and (2) on terms free for individual variables in formulas and on proper variables.

Given a language of grounding Λ, we agree to call *primitive* the operational symbols $\wedge I$, $\vee I$, $\rightarrow I$, $\forall I$ and $\exists I$, *non-primitive* all the others. We then have the following definition.

Definition 34. Let Λ_1 be a language of grounding on a base B_1 with set of non-primitive operational symbols $\mathbb{F}_1$. An *expansion* Λ_2 of Λ_1 is a language of grounding on a base B_2, with B_2 expansion of B_1, and of which the set $\mathbb{F}_2$ of the non-primitive operational symbols is such that $\mathbb{F}_1 \subseteq \mathbb{F}_2$.

Before concluding the section, we will conduct some observations, which descend immediately from the definitions just introduced, or integrate them in a way fruitful for the continuation of our investigation.

First of all, note that each language of grounding is trivially an expansion of itself; furthermore, if Λ_2 is an expansion of Λ_1, and Λ_3 is an expansion of Λ_2, Λ_3 is an expansion of Λ_1. The expansion relation on languages of grounding is therefore reflexive and transitive. Each language of grounding on B is also trivially an expansion of a core language on B and, if the atomic system of B is non-empty, it is also trivially an expansion of a core language on a logical base involving the same individual constants, and the same relational and functional symbols of B. Obviously, there may be cases of non-trivial core sublanguages; for example, a grounding language on first-order arithmetic – having as atomic system the Post-system $\{\texttt{EQ}\} \cup \{\texttt{SAM}\}$ of Section 2.5.1 - is also an expansion: of the core language on the base

$$\langle \{\doteq\}, \emptyset, \emptyset, \{\texttt{EQ}\} \rangle;$$

of the core language on the base

$$\langle \{\doteq\}, \{s\}, \{0\}, \{\texttt{EQ} \cup \{(s_1), (s_2)\}\} \rangle;$$

of the core language on the base

$$\langle \{\doteq\}, \{s, +\}, \{0\}, \{\texttt{EQ} \cup \{(s_1), (s_2), (+_1), (+_2)\}\} \rangle.$$

On the other hand, other combinations based on first-order arithmetic are not possible, because they violate either definition 21, by providing an atomic system with rules on a language that cannot be constructed starting from the other elements of the base itself, or convention 24, according to which the atomic system of a base must totally interpret the background language.

Another important observation concerns individual variables and free ground variables in terms of a language of grounding. As anticipated in the discussion of the core languages, we will assume as specified on each Λ the following notions - and, on their basis, we will assume that on each Λ applies the convention 32:

- set of the subterms of a term;
- set of the free and bound individual variables of a term;
- set of the free and bound ground-variables of a term.

Well, these notions have to be specified inductively, as in definitions 29, 30, 31, in such a way that, for every $F\ \underline{x}\ \underline{\xi}\ (\dots\ U\ \dots) : \alpha \in \texttt{TERM}_\Lambda$, the two following circumstances hold - it is easy to verify that they hold for a core language:

(a) $FV^I(F\ \underline{x}\ \underline{\xi}\ (\dots\ U_i\ \dots))$ is equal to the union of (1) $FV(\alpha)$ and (2) of the free individual variables of U_i other than those in $\underline{x}$ bound on index i;

(b) $FV^T(F\ \underline{x}\ \underline{\xi}\ (\dots\ U_i\ \dots))$ is equal to the union of the free ground-variables of U_i other than those in $\underline{\xi}$ bound on index i.

Obviously, $i \leq n$, where n is the ariety of F. Therefore, each term T of the type α in Λ contains: among its free individual variables, all and only those of α, plus those of its immediate subterms not bound by the outermost operational symbol of T on those subterms; among its free ground variables, all and only those of its immediate subterms not bound by the outermost operational symbol of T on those subterms.

5.2.3.3 Gentzen-language and Heyting arithmetics

To complete our discussion on languages of grounding and their expansion, we give two examples. The first language we present will be called "Gentzen-language"; as we will show by exhibiting a sort of Curry-Howard isomorphism, it constitutes a functional translation of first-order intuitionistic logic in a Gentzen's natural deduction system. The second language, on the other hand, will be called "language of grounding for first-order Heyting arithmetic"; it can still be understood as the functional translation of its equivalent in Gentzen's natural deduction.

Gentzen-language
Let L be a first-oder logical language, B a base on L with atomic system $\mathtt{S}$, and $\mathtt{G}$ a core language on B. Let us consider an expansion $\mathtt{Gen}$ of $\mathtt{G}$.

Definition 35. The language of $\mathtt{Gen}$ is specified starting from an alphabet which contains that of $\mathtt{G}$ plus

- operational symbols F of operational type $\langle\ddagger\rangle$:
  - $\wedge_{E,i}\langle\alpha_1 \wedge \alpha_2 \rhd \alpha_i\rangle$ ($\alpha_i \in \mathtt{FORM}_L$, $i = 1, 2$)
  - $\vee E\langle\alpha \vee \beta, \delta, \delta \rhd \delta\rangle$ ($\alpha, \beta, \delta \in \mathtt{FORM}_L$)
  - $\rightarrow E\langle\alpha \rightarrow \beta, \alpha \rhd \beta\rangle$ ($\alpha, \beta \in \mathtt{FORM}_L$)
  - $\forall E\langle\forall x_i\alpha(x_i) \rhd \alpha(t/x_i)\rangle$ ($\alpha \in \mathtt{FORM}_L, t \in \mathtt{TERM}_L, i \in \mathbb{N}$)
  - $\exists E\langle\exists x_i\alpha(x_i), \beta \rhd \beta\rangle$ ($\alpha, \beta \in \mathtt{FORM}_L$, $i \in \mathbb{N}$)

In order not to excessively burden the notation, and whenever this does not create ambiguity, we will omit the explicit mention of the typing of the operational symbols, as well as indexes and subscripts. The notation $T : \alpha \in X$ indicates that T is an element of X of type α. The set $\mathtt{TERM}_{\mathtt{Gen}}$ of the *terms* of $\mathtt{Gen}$ is the smallest set X such that

- $\mathtt{TERM}_{\mathtt{G}} \subset X$
- $T : \alpha_1 \wedge \alpha_2 \in X \Rightarrow \wedge_{E,i}(T) : \alpha_i \in X$
- $T : \alpha \vee \beta, U_1 : \delta, U_2 : \delta \in X \Rightarrow \vee E\ \xi^\alpha\ \xi^\beta(T, U_1, U_2) : \delta \in X$
- $T : \alpha \rightarrow \beta, U : \alpha \in X \Rightarrow\ \rightarrow E(T, U) : \beta \in X$
- $T : \forall x\alpha(x) \in X \Rightarrow \forall E\langle\forall x\alpha(x) \rhd \alpha(t/x)\rangle(T) : \alpha(t) \in X$
- $T : \exists x\alpha(x), U : \beta \in X \Rightarrow \exists E\ x\ \xi^{\alpha(x)}(T, U) : \beta \in X$

In $\forall E(T)$, t must be free for x in $\alpha(x)$. In $\exists E\ x\ \xi^{\alpha(x)}(T, U)$, it must hold that $x \notin FV(\delta)$, for $\xi^\delta \in FV^T(U)$ with $\delta \neq \alpha(x)$, and $x \notin FV(\beta)$; x is called *proper variable* of the term.

By way of example, we explicitly define for $\mathtt{Gen}$ the notions we have said are to be defined for each grounding language.

Definition 36. The set $S(T)$ of the *subterms* of T is like in definition 30 if $T \in \mathtt{G}$ and, in the other cases,

- $S(\wedge_{E,i}(T)) = S(T) \cup \{\wedge_{E,i}(T)\}$
- $S(\vee E\ \xi^{\alpha}\ \xi^{\beta}(T, U_1, U_2)) = S(T) \cup S(U_1) \cup S(U_2) \cup \{\vee E\ \xi^{\alpha}\ \xi^{\beta}(T, U_1, U_2)\}$
- $S(\rightarrow E(T, U)) = S(T) \cup S(U) \cup \{\rightarrow E(T, U)\}$
- $S(\forall E(T)) = S(T) \cup \{\forall E(T)\}$
- $S(\exists E\ x\ \xi^{\alpha(x)}(T, U)) = S(T) \cup S(U) \cup \{\exists E\ x\ \xi^{\alpha(x)}(T, U)\}$

Definition 37. The set $FV^I(T)$ of the *free individual variables* of T is like in definition 31 and, in the other cases,

- $FV^I(\wedge_{E,i}(T)) = FV^I(T)$
- $FV^I(\vee E\ \xi^{\alpha}\ \xi^{\beta}(T, U_1, U_2)) = FV^I(T) \cup FV^I(U_1) \cup FV^I(U_2)$
- $FV^I(\rightarrow E(T, U)) = FV^I(T) \cup FV^I(U)$
- $FV^I(\forall E\langle \forall x \alpha(x) \rhd \alpha(t/x)\rangle(T)) = FV^I(T) \cup FV(\alpha(t/x))$
- $FV^I(\exists E\ x\ \xi^{\alpha(x)}(T, U)) = FV^I(T) \cup (FV^I(U) - \{x\})$

The set $BV^I(T)$ of the *bound individual variables* of T is like in definition 31 and, in the other cases,

- $BV^I(\wedge_{E,i}(T)) = BV^I(T)$
- $BV^I(\vee E\ \xi^{\alpha}\ \xi^{\beta}(T, U_1, U_2)) = BV^I(T) \cup BV^I(U_1) \cup BV^I(U_2)$
- $BV^I(\rightarrow E(T, U)) = BV^I(T) \cup BV^I(U)$
- $BV^I(\forall E(T)) = BV^I(T)$
- $BV^I(\exists E\ x\ \xi^{\alpha(x)}(T, U)) = BV^I(T) \cup (BV^I(U) \cup \{x\})$

Definition 38. The set $FV^T(U)$ of the *free ground-variables* of U is like in definition 32 if $U \in \mathtt{G}$ and, in the other cases,

- $FV^T(\wedge_{E,i}(U)) = FV^T(U)$
- $FV^T(\vee E\ \xi^{\alpha}\ \xi^{\beta}(U, Z_1, Z_2)) =$
  $= FV^T(U) \cup (FV^T(Z_1) - \{\xi^{\alpha}\}) \cup (FV^T(Z_2) - \{\xi^{\beta}\})$
- $FV^T(\rightarrow E(U, Z)) = FV^T(U) \cup FV^T(Z)$

- $FV^T(\forall E(U)) = FV^T(U)$
- $FV^T(\exists E\ x\ \xi^{\alpha(x)}(U,Z)) = FV^T(U) \cup (FV^T(Z) - \{\xi^{\alpha(x)}\})$

The set $BV^T(U)$ of the *bound ground-variables* of U is like in definition 32 if $U \in \mathtt{G}$ and, in the other cases,

- $BV^T(\wedge_{E,i}(U)) = BV^T(U)$
- $BV^T(\vee E\ \xi^\alpha\ \xi^\beta(U,Z_1,Z_2)) =$
  $= BV^T(U) \cup (BV^T(Z_1) \cup \{\xi^\alpha\}) \cup (BV^T(Z_2) \cup \{\xi^\beta\})$
- $BV^T(\to E(U,Z)) = BV^T(U) \cup BV^T(Z)$
- $BV^T(\forall E(U)) = BV^T(U)$
- $BV^T(\exists E\ x\ \xi^{\alpha(x)}(U,Z)) = BV^T(U) \cup (BV^T(Z) \cup \{\xi^{\alpha(x)}\})$

Furthermore we assume that convention 32 applies. Let us establish an isomorphism similar to the Curry-Howard one, by giving first the following function:

$$i\text{-th derivation } \mathtt{S} \quad \stackrel{\iota}{\Longrightarrow} \quad \delta_i$$

$$\alpha \quad \stackrel{\iota}{\Longrightarrow} \quad \xi^\alpha$$

$$\frac{\begin{array}{cc}\Delta_1 & \Delta_2 \\ \alpha & \beta\end{array}}{\alpha \wedge \beta}\,(\wedge_I) \quad \stackrel{\iota}{\Longrightarrow} \quad \wedge I(\iota(\Delta_1), \iota(\Delta_2))$$

$$\frac{\begin{array}{c}\Delta \\ \alpha_1 \wedge \alpha_2\end{array}}{\alpha_i}\,(\wedge_{E,i}),\ i = 1,2 \quad \stackrel{\iota}{\Longrightarrow} \quad \wedge_{E,i}(\iota(\Delta)),\ i = 1,2$$

$$\frac{\begin{array}{c}\Delta \\ \alpha_i\end{array}}{\alpha_1 \vee \alpha_2}\,(\vee_I),\ i = 1,2 \quad \stackrel{\iota}{\Longrightarrow} \quad \vee I\langle \alpha_i \rhd \alpha_1 \vee \alpha_2\rangle(\iota(\Delta))$$

$$\frac{\begin{array}{ccc} & [\alpha] & [\beta] \\ \Delta_1 & \Delta_2 & \Delta_3 \\ \alpha \vee \beta & \gamma & \gamma\end{array}}{\gamma}\,(\vee_E) \quad \stackrel{\iota}{\Longrightarrow} \quad \vee E\ \xi^\alpha\ \xi^\beta(\iota(\Delta_1), \iota(\Delta_2), \iota(\Delta_3))$$

$$\cfrac{\begin{matrix}[\alpha]\\ \Delta\\ \beta\end{matrix}}{\alpha \to \beta}\,(\to_I) \quad \stackrel{\iota}{\Longrightarrow} \quad \to I\xi^{\alpha}(\iota(\Delta))$$

$$\cfrac{\begin{matrix}\Delta_1\\ \alpha \to \beta\end{matrix} \qquad \begin{matrix}\Delta_2\\ \alpha\end{matrix}}{\beta}\,(\to_E) \quad \stackrel{\iota}{\Longrightarrow} \quad \to E(\iota(\Delta_1), \iota(\Delta_2))$$

$$\cfrac{\begin{matrix}\Delta(x)\\ \alpha(x)\end{matrix}}{\forall y\alpha(y/x)}\,(\forall_I) \quad \stackrel{\iota}{\Longrightarrow} \quad \forall Ix(\iota(\Delta(x)))$$

$$\cfrac{\begin{matrix}\Delta\\ \forall x\alpha(x)\end{matrix}}{\alpha(t)}\,(\forall_E) \quad \stackrel{\iota}{\Longrightarrow} \quad \forall E(\iota(\Delta))$$

$$\cfrac{\begin{matrix}\Delta\\ \alpha(t/x)\end{matrix}}{\exists x\alpha(x)}\,(\exists_I) \quad \stackrel{\iota}{\Longrightarrow} \quad \exists I(\iota(\Delta))$$

$$\cfrac{\begin{matrix}\Delta_1\\ \exists y\alpha(y/x)\end{matrix} \qquad \begin{matrix}[\alpha(x)]\\ \Delta_2(x)\\ \beta\end{matrix}}{\beta}\,(\exists_E) \quad \stackrel{\iota}{\Longrightarrow} \quad \exists E\ x\ \xi^{\alpha(x)}(\iota(\Delta_1), \iota(\Delta_2(x))$$

$$\cfrac{\begin{matrix}\Delta\\ \bot\end{matrix}}{\alpha}\,(\bot) \quad \stackrel{\iota}{\Longrightarrow} \quad \bot_{\alpha}(\iota(\Delta))$$

It can be proved that ι is a bijective function, so the following holds:

Proposition 39. For every derivation in IL of the kind

$$\begin{matrix}\alpha_1 & \dots & \alpha_n\\ & \Delta(x_1, ..., x_m) & \\ & \beta & \end{matrix}$$

there is $U : \beta \in \texttt{TERM}_{\texttt{Gen}}$ with $FV^I(U) = \{x_1, ..., x_m\}$ and $FV^T(U) = \{\xi^{\alpha_1}, ..., \xi^{\alpha_n}\}$, and viceversa.

Moreover, as we will see more in detail in the next Section and in the next chapter, it is possible to associate to the non-primitive operational symbols of Gen some equations that establish the behavior of (the operations on grounds denoted by) such symbols, in all similar to the equations for the elimination of the redexes in the typed λ-calculus. This allows us to introduce, for the terms of Gen, a notion of reduction that the bijection ι preserves isomorphically, with respect to the corresponding reduction relation for the derivations of IL. Therefore, ι is an isomorphism with respect to this reducibility relation, and therefore with respect to the relation of reduction to normal form.

Heyting arithmetic

Let L be a first-order logical language with individual constant 0, relational constant $\doteq$, and functional constants s, $+$, $\cdot$. Let then B be a base on L with atomic system the Post-system S for first-order arithmetic of Section 2.5.1, and let Gen be a Gentzen-language on B. Let us consider an expansion GHA of Gen.

Definition 40. The language of GHA is specified starting from an alphabet that contains that of Gen plus

- operational symbols F of operational type $\langle \ddagger \rangle$:
    - $\mathtt{Ind}\langle \alpha(0), \alpha(s(x)) \rhd \alpha(t)\rangle$ $(t \in \mathtt{TERM}_L)$

The notation $T : \alpha \in X$ indicates that T is an element of X of type α. The set $\mathtt{TERM}_{\mathtt{GHA}}$ of the *terms* of GHA is the smallest set X such that:

- $\mathtt{TERM}_{\mathtt{Gen}} \subset X$
- $T : \alpha(0), U : \alpha(s(x)) \in X \Rightarrow \mathtt{Ind}\ x\ \xi^{\alpha(x)}(T, U) : \alpha(t) \in X$

In $\mathtt{Ind}\ x\ \xi^{\alpha(x)}(T, U)$, it must non be $x \in FV(\beta)$, for $\xi^{\beta} \in FV(U)$ with $\beta \neq \alpha(x)$, and it must not be $x \in FV(\alpha(t))$.

Here too we have an isomorphism, by adding to the previous bijection ι the clause

$$\dfrac{\begin{matrix} \\ \Delta_1 \\ \alpha(0) \end{matrix} \qquad \begin{matrix} [\alpha(x)] \\ \Delta_2(x) \\ \alpha(s(x)/x) \end{matrix}}{\alpha(t)}\, IND \quad \stackrel{\iota}{\Longrightarrow} \quad \mathtt{Ind}\ x\ \xi^{\alpha(x)}(\iota(\Delta_1), \iota(\Delta_2(x))).$$

Proposition 41. For every derivation in HA of the kind

$$\frac{\alpha_1 \quad \dots \quad \alpha_n}{\beta}\,\Delta(x_1, ..., x_m)$$

there is $U : \beta \in \texttt{TERM}_{\texttt{GHA}}$ with $FV^I(U) = \{x_1, ..., x_m\}$ and $FV^T(U) = \{\xi^{\alpha_1}, ..., \xi^{\alpha_n}\}$, and viceversa.

Speaking of GHA, there is also another important observation to make. According to Gödel's first incompleteness theorem, we know that there is a closed formula $G \in \texttt{FORM}_L$ such that $\nvdash_{\texttt{HA}} G$ and $\nvdash_{\texttt{HA}} \neg G$. On the basis of the bijection ι, the following proposition will therefore apply:

Proposition 42. There is $G \in \texttt{FORM}_L$ closed such that there is neither $T : G \in \texttt{TERM}_{\texttt{GHA}}$, nor $T : \neg G \in \texttt{TERM}_{\texttt{GHA}}$, for T closed.

On the other hand, it is often said that, however undecidable in HA, G is a true formula on the intended model $\mathbb{N}$. Thus, if by truth of G we mean the existence of a ground for $\vdash G$, and if we assume that the terms of a language of grounding denote ground for judgments or assertions involving formulas of the background language, this will also mean that there is a ground for $\vdash G$ inexpressible in GHA. This observation therefore provides us a decisive reason to consider a language of grounding as open to the introduction of new linguistic resources: in order to express grounds on languages at least as rich as those of first-order arithmetic, we should be able to expand any language of grounding on these languages - namely, there is no language that allows us to express all the grounds. As we will see more in detail later, an expansion that allows to express grounds previously inexpressible, will be considered as a non-conservative expansion of the source language.

5.2.4 Denotation

We now provide a precise definition of denotation. As a preliminary remark before the entire discussion, it should be recalled that the underlying intuition is that a base with a non-empty atomic system B authorizes one single interpretation $\texttt{int}_B$ of the elements of the alphabet, and therefore of the terms, of the background language. For example, on the Post-system in Section 2.5.1,

$$\texttt{int}_B(0) = 0 \in \mathbb{N}$$

$$\texttt{int}_B(\doteq) = \{(n, m) \mid n = m, n, m \in \mathbb{N}\} \subseteq \mathbb{N}^2$$

$$\texttt{int}_B(s) = s \colon \mathbb{N} \rightarrow \mathbb{N} \text{ (successor function)}$$

$$\mathtt{int}_B(+) = +\colon \mathbb{N}^2 \to \mathbb{N} \text{ (addition)}$$

$$\mathtt{int}_B(\cdot) = \cdot\colon \mathbb{N}^2 \to \mathbb{N} \text{ (multiplication)}$$

If B is logical, instead, namely, if its atomic system is empty, the idea is that the background language remains uninterpreted. In that case, we can understand $\mathtt{int}_B$ as the identity function on $\mathtt{TERM}_L$, i.e.

$$\mathtt{int}_B(t) = t, \text{ for every } t \in \mathtt{TERM}_L.$$

5.2.4.1 Denotation for alphabet and terms

As in the case of model-theory, the definition of the notion of denotation proceeds in two steps: fixed the denotation of the elements of the alphabet, on its base we establish the denotation of the terms of the language by induction on the complexity.

Definition 43. Let Λ be a language of grounding on a base B with atomic system $\mathtt{S}$. The *denotation* of the alphabet of Λ is determined by a function den^* defined as follows:

- $den^*(\delta_i) = i$-th derivation of $\mathtt{S}$
- $den^*(\xi_i^{\alpha(x_1,...,x_n)}) = \mathtt{Id}(\xi_i^{\alpha(x_1,...,x_n)})$ $(n \geq 0)$, where $\mathtt{Id}$ is the identity function on the operational type

$$\alpha(x_1, ..., x_n) \rhd \alpha(x_1, ..., x_n)$$

- $den^*(F) = F$, if F a primitive operational symbol
- if instead F is a non-primitive operational symbol of operational type

$$\alpha_1(\underline{x}_1), ..., \alpha_m(\underline{x}_n) \rhd \beta(\underline{x}_{n+1})$$

  then $den^*(F)$ is one of the B-operations on grounds assumed as existing relatively to the operational type of F as indicated in definition 33.

Definition 44. Let Λ be a language of grounding on a base B with atomic system $\mathtt{S}$, and let den^* be a denotation of the elements of the alphabet of Λ. The *denotation* of the terms of Λ *associated with* den^* is determined by the function den defined as follows:

- $den(\delta) = den^*(\delta)$ and $den(\xi^\alpha) = den^*(\xi^\alpha)$
- $den(F\ \underline{x}\ \underline{\xi}\ (T_1, ..., T_n)) = den^*(F)(den(T_1), ..., den(T_n))$ $(n \geq 0)$

With regard to the last clause of the definition just given, it should be noted that it can be considered well posed if, and only if

$$den(F\ \underline{x}\ \underline{\xi}\ (T_1, ..., T_n))$$

is always well-defined. Now, since

$$den(F\ \underline{x}\ \underline{\xi}\ (T_1, ..., T_n)) = den^*(F)(den(T_1), ..., den(T_n))$$

and since, by virtue of definition 43, $den^*(F)$ is a B-operation on grounds of a certain operational type, what must be actually ensured is that $den(T_i)$ $(i \leq n)$ is a (proper or improper) ground on B for an appropriate judgment or assertion, which meets the conditions required for the B-operation on grounds $den^*(F)$. A way to do it is to prove, by induction on the complexity of the terms of Λ, that all terms denote (proper or improper) grounds on B for appropriate judgments or assertions. And this is the content of a denotation theorem set forth in the following section.

5.2.4.2 Denotation theorem

We should state beforehand that, although conducted only on some of the many possible subcases, the proof of the theorem is very long. Indeed, for an understanding of what follows it is sufficient to know only the claim of the theorem, and that it is based essentially on the fact that each operational symbol of any language of grounding must, as according definition 33, be associated with operational types such that there is a corresponding operation on grounds of the appropriate operational type.

Theorem 45. Let Λ be a language of grounding on a base B, let den^* be a denotation function of the elements of the alphabet of Λ, and let den be the denotation function of the terms of Λ associated with den^*. Let then $U : \beta \in \mathtt{TERM}_\Lambda$ with

$$FV^I(U) = \{x_1, ..., x_n\} \text{ and } FV^T(U) = \{\xi^{\alpha_1}, ...\xi^{\alpha_m}\}\ (n, m \geq 0).$$

Then, $den(U)$ is a (proper or improper) ground on B for $\alpha_1, ..., \alpha_m \vdash \beta$, in all and only the individual variables $x_1, ..., x_n$.

Proof. We prove the theorem by induction on the complexity of $U : \beta \in \mathtt{TERM}_\Lambda$.

- for U constant or ground-variable, the result holds trivially in virtue of definition 44;
- let us assume the result is proved for every term less complex than U. For clarification purposes, we distinguish two cases:

  (Case 1) U is closed. By virtue of observation (a) in Section 5.2.3.2, it must hold that

  $$FV(\beta) \subseteq FV^I(U),$$

  and hence, since we have assumed

  $$FV^I(U) = \emptyset,$$

  it must hold

  $$FV(\beta) = \emptyset.$$

  We have therefore to prove that U is a *proper* ground on B for $\vdash \beta$, with β closed. For the sake of simplicity, and without loss of generality, we assume that U has the form

  $$F\ x\ \xi^{\gamma}(Z)$$

  with $Z : \alpha \in \texttt{TERM}_{\Lambda}$. Let us observe first of all that F must have operational type

  $$\alpha \rhd \beta.$$

  Moreover, by virtue of observation (a) in Section 5.2.3.2, it must hold that

  $$FV(\alpha) \subseteq FV^I(Z)$$

  and that

  $$FV^I(Z) \subseteq \{x\}.$$

Hence, $den^*(F)$ must be a B-operation on gorunds

$$f(\xi^{\gamma \rhd \alpha})$$

of operational type

$$(\gamma \rhd \alpha) \rhd \beta$$

binding x and ξ^γ, for closed or open α. As for Z, after what already established with regard to $FV^I(Z)$, we observe that, in accordance with observation (b) again in Section 5.2.3.2, it must apply that

$$FV^T(Z) \subseteq \{\xi^\gamma\}$$

and, again by observations (a) and (b) in Section 5.2.3.2, it will hold that

$$FV^T(Z) \neq \emptyset \Rightarrow \text{for every } y \neq x \in FV(\gamma),\, y \in BV^I(Z).$$

To lighten the matter, this last circumstance will be reduced to the assumption according to which

$$FV(\gamma) \subseteq FV^I(Z)$$

although it should be kept in mind that this is only an imprecise, although useful, simplification. We therefore have a series of subcases to take into account. The first is that in which Z is itself closed; by induction hypothesis, $den(Z)$ is a ground g on B for $\vdash \alpha$, with α closed. But then,

$$den(F\ x\ \xi^\gamma(Z)) = den^*(F)(den(Z)) = f(g)$$

is a ground on B of the required type. The other subcase is that where Z is open, which in turn has the three subcases of the disjunction

$$FV^I(Z) = \{x\} \text{ or } FV^T(Z) = \{\xi^\gamma\}.$$

Let us consider only two of them. Let us first of all suppose that

$$FV^I(Z) = \{x\} \text{ and } FV^T(Z) = \emptyset.$$

By induction hypothesis, $den(Z)$ is a B-operation on grounds

$$h(x)$$

of operational type

$$\alpha$$

and is hence a ground on B for $\vdash \alpha$, proper if α is open, improper otherwise. But then,

$$den(F\ x\ \xi^\gamma(Z)) = den^*(F)(den(Z)) = f(h(x))$$

is a ground on B of the required kind. Let us finally consider the case

$$FV^I(Z) = \{x\} \text{ and } FV^T(Z) = \{\xi^\gamma\}.$$

By induction hypothesis, $den(Z)$ is a B-operation on grounds

$$h(x, \xi^\gamma)$$

of operational type

$$\gamma \rhd \alpha$$

and is hence a ground on B for $\gamma \vdash \alpha$, proper if γ or α are open, improper otherwise. But then,

$$den(F\ x\ \xi^\gamma(Z)) = den^*(F)(den(Z)) = f(h(x, \xi^\gamma))$$

is a ground on B of the required kind;

(Case 2) U is open with

$FV^I(U) = \{x_1, ..., x_n\}$ and $FV^T(U) = \{\xi^{\alpha_1}, ..., \xi^{\alpha_m}\}$ ($n > 0$ or $m > 0$).

By observations (a) and (b) of Section 5.2.3.2, it must hold that

$$\bigcup_{i \leq m} FV(\alpha_i) \cup FV(\beta) \subseteq FV^I(U).$$

We must therefore prove that $den(U)$ is a B-operation on grounds

$$f(x_1, ..., x_n, \xi^{\alpha_1}, ..., \xi^{\alpha_m})$$

of operational type

$$\alpha_1, ..., \alpha_m \rhd \beta$$

which will be a ground on B for $\alpha_1, ..., \alpha_m \vdash \beta$, proper if

$$\bigcup_{i \leq m} FV(\alpha_i) \cup FV(\beta) = FV^I(U)$$

and improper if

$$\bigcup_{i \leq m} FV(\alpha_i) \cup FV(\beta) \subset FV^I(U).$$

For the sake of simplicity, and without loss of generality, let us assume that U has the form

$$F\ x_1\ \xi^{\gamma_1}(Z)$$

with $Z : \alpha \in \texttt{TERM}_\Lambda$ and

$$FV^I(F\ x_1\ \xi^{\gamma_1}(Z)) = \{x_2\} \text{ and } FV^T(F\ x_1\ \xi^{\gamma_1}(Z)) = \{\xi^{\gamma_2}\}.$$

Let us observe first of all that F must have operational type

$$\alpha \rhd \beta.$$

Moreover, by virtue of observation (a) in Section 5.2.3.2, and of restriction (2), it must hold that

$$FV(\beta) \subseteq \{x_2\}$$

and furthermore, again by virtue of observation (a) in Section 5.2.3.2, it must hold that

$$FV(\alpha) \subseteq FV^I(Z)$$

and that

$$FV^I(Z) \subseteq \{x_1, x_2\}.$$

Hence, $den^*(F)$ can be: either a B-operation on grounds

$$f(\xi^{\gamma_1 \rhd \alpha})$$

of operational type

$$(\gamma_1 \rhd \alpha) \rhd \beta$$

if

$$FV(\alpha) \cup FV(\beta) \subseteq \{x_1\};$$

or a B-operation on grounds

$$f(x_2, \xi^{\gamma_1 \rhd \alpha})$$

of operational type

$$(\gamma_1 \rhd \alpha) \rhd \beta$$

if

$$x_2 \in FV(\alpha) \cup FV(\beta).$$

In all cases, $den^*(F)$ binds x_1 and ξ^{γ_1}. As for Z, after what already established about $FV^I(Z)$, we observe that, by observation (b) in Section 2.5.3.2, it must hold that

$$FV^T(Z) \subseteq \{\xi^{\gamma_1}, \xi^{\gamma_2}\}.$$

Hence, again by observations (a) and (b) in Section 2.5.3.2, we will have

$$FV^T(Z) \neq \emptyset \Rightarrow \text{for every } y \neq x_j \in FV(\gamma_i)\ (i,j=1,2),\ y \in BV^I(Z).$$

To lighten the matter, this latter circumstance will be reduced to the assumption according to which

$$FV(\gamma_i)\ (i=1,2) \subseteq FV^I(Z)$$

although it should be kept in mind that this is only an imprecise, although useful, simplification. Observe anyway that, by observation (b) in Section 2.5.3.2, it cannot be

$$FV^T(Z) = \emptyset$$

since

$$\xi^{\gamma_2} \in FV^T(F\ x\ \xi(Z));$$

hence

$$\xi^{\gamma_2} \in FV^T(Z)$$

and

$$FV(\gamma_2) \subseteq FV^I(Z).$$

We thus have a series of cases to take into account. First of all, the one in which $FV^I(Z) = \emptyset$. Here, we have to take into account the two subcases of the disjunction

$$FV^T(Z) = \{\xi^{\gamma_2}\} \text{ or } FV^T(Z) = \{\xi^{\gamma_1}, \xi^{\gamma_2}\}.$$

Consider only the second, and suppose that

$$FV^T(Z) = \{\xi^{\gamma_1}, \xi^{\gamma_2}\}.$$

By induction hypothesis, $den(Z)$ is a B-operation on grounds

$$h(\xi^{\gamma_1}, \xi^{\gamma_2})$$

of operational type

$$\gamma_1, \gamma_2 \rhd \alpha$$

with γ_1, γ_2 and α closed, and is hence a proper ground on B for $\gamma_1, \gamma_2 \vdash \alpha$. Since

$$x_2 \in FV^I(F\ x\ \xi(Z)),$$

it must hold that

$$FV(\beta) = \{x_2\},$$

and hence $den^*(F)$ will be a B-operation on grounds

$$f(x_2, \xi^{\gamma_1 \rhd \alpha})$$

of operational type

$$(\gamma_1 \rhd \alpha) \rhd \beta.$$

But then

$$den(F\ x\ \xi(Z)) = den^*(F)(den(Z)) = f(x_2, h(\xi^{\gamma_1, \gamma_2}))$$

is a ground on B of the required kind. Let us move on to the case in which $FV^I(Z) \neq \emptyset$. We have the three subcases of the disjunction

$$FV^I(Z) = \{x_1\} \text{ or } FV^I(Z) = \{x_2\} \text{ or } FV^I(Z) = \{x_1, x_2\}.$$

each of which must be articulated with respect to the two subcases of the disjunction

$$FV^T(Z) = \{\xi^{\gamma_2}\} \text{ or } FV^T(Z) = \{\xi^{\gamma_1}, \xi^{\gamma_2}\}.$$

We consider only some of them. Let us suppose first of all that

$$FV^I(Z) = \{x_1\} \text{ and } FV^T(Z) = \{\xi^{\gamma_2}\}.$$

Here, by restriction (2), we must have

$$x_1 \notin FV(\gamma_2).$$

Since

$$x_2 \in FV^I(F\ x\ \xi^{\gamma_1}(Z)),$$

we have only two possibilities. The first one is

$$FV(\gamma_2) = \emptyset,\ FV(\alpha) = \emptyset \text{ and } FV(\beta) = \{x_2\},$$

in which case, by induction hypothesis, $den(Z)$ is a B-operation on grounds

$$h(x_1, \xi^{\gamma_2})$$

of operational type

$$\gamma_2 \rhd \alpha,$$

namely, an improper ground on B for $\gamma_2 \vdash \alpha$, and $den^*(F)$ is a B-operation on grounds

$$f(x_2, \xi^{\gamma_1 \rhd \alpha})$$

of operational type

$$(\gamma_1 \rhd \alpha) \rhd \beta;$$

but then,

$$den(F\ x\ \xi(Z)) = den^* F(den(Z)) = f(x_2, h(x_1, \xi^{\gamma_2}))$$

is a ground on B of the required kind. The second possibility is

$$FV(\xi^{\gamma_2}) = \emptyset,\ FV(\alpha) = \{x_1\} \text{ and } FV(\beta) = \{x_2\},$$

in which case, by induction hypothesis, $den(Z)$ is a B-operation on grounds

$$h(x_1, \xi^{\gamma_2})$$

of operational type

$$\gamma_2 \rhd \alpha,$$

namely, a proper ground on B for $\gamma_2 \vdash \alpha$, and $den^*(F)$ is a B-operation on grounds

$$f(x_2, \xi^{\gamma_1 \rhd \alpha})$$

of operational type

$$(\gamma_1 \rhd \alpha) \rhd \beta;$$

but then,

$$den(F\ x\ \xi(Z)) = den^*F(den(Z)) = f(x_2, h(x_1, \xi^{\gamma_2}))$$

is a ground on B of the required kind. Let us now consider the case

$$FV^I(Z) = \{x_2\} \text{ and } FV^T(Z) = \{\xi^{\gamma_1}, \xi^{\gamma_2}\}.$$

Here we have four groups of possibilities. A first group of possibilities is summarized in

$$FV(\gamma_1) \cup FV(\gamma_2) \cup FV(\alpha) = \{x_2\} \text{ and } FV(\beta) = \emptyset,$$

in which case, by induction hypothesis, $den(Z)$ is a B-operation on grounds

$$h(x_2, \xi^{\gamma_1}, \xi^{\gamma_2})$$

of operational type

$$\gamma_1, \gamma_2 \rhd \alpha,$$

namely, a proper ground on B for $\gamma_1, \gamma_2 \vdash \alpha$, and $den^*(F)$ is a B-operation on grounds

$$f(\xi^{\gamma_1 \rhd \beta})$$

of operational type

$$(\gamma_1 \rhd \alpha) \rhd \beta;$$

but then,

$$den(F\ x\ \xi(Z)) = den^*(F)(den(Z)) = f(h(x_2, \xi^{\gamma_1}, \xi^{\gamma_2}))$$

is a ground on B of the required kind. A second group of possibilities is instead summarized in

$$FV(\gamma_1) \cup FV(\gamma_2) \cup FV(\alpha) = \{x_2\} \text{ and } FV(\beta) = \{x_2\},$$

in which case, by induction hypothesis, $den(Z)$ is B-operation on grounds

$$h(x_2, \xi^{\gamma_1}, \xi^{\gamma_2})$$

of operational type

$$\gamma_1, \gamma_2 \rhd \alpha,$$

namely, a proper ground on B for $\gamma_1, \gamma_2 \vdash \alpha$, and $den^*(F)$ is a B-operation on grounds

$$f(x_2, \xi^{\gamma_1 \rhd \beta})$$

of operational type

$$(\gamma_1 \rhd \alpha) \rhd \beta;$$

but then,

$$den(F\ x\ \xi(Z)) = den^*(F)(den(Z)) = f(x_2, h(x_2, \xi^{\gamma_1}, \xi^{\gamma_2}))$$

is a ground on B of the required kind. The third possibility is expressed by the conjunction

$$FV(\gamma_1) \cup FV(\gamma_2) \cup FV(\alpha) = \emptyset \text{ and } FV(\beta) = \emptyset,$$

in which case, by induction hypothesis, $den(Z)$ is a B-operation on grounds

$$h(x_2, \xi^{\gamma_1}, \xi^{\gamma_2})$$

of operational type

$$\gamma_1, \gamma_2 \rhd \alpha,$$

namely, an improper ground on B for $\gamma_1, \gamma_2 \vdash \alpha$, and $den^*(F)$ is a B-operation on grounds

$$f(\xi^{\gamma_1 \rhd \beta})$$

of operational type

$$(\gamma_1 \rhd \alpha) \rhd \beta;$$

but then,

$$den(F\ x\ \xi(Z)) = den^*(F)(den(Z)) = f(h(x_2, \xi^{\gamma_1}, \xi^{\gamma_2}))$$

is a ground on B of the required kind. A fourth possibility is expressed by the conjunction

$$FV(\gamma_1) \cup FV(\gamma_2) \cup FV(\alpha) = \emptyset \text{ and } FV(\beta) = \{x_2\},$$

in which case, by induction hypothesis, $den(Z)$ is a B-operation on grounds

$$h(x_2, \xi^{\gamma_1}, \xi^{\gamma_2})$$

of operational type

$$\gamma_1, \gamma_2 \rhd \alpha,$$

namely, an improper ground on B for $\gamma_1, \gamma_2 \vdash \alpha$, and $den^*(F)$ is a B-operation on grounds

$$f(x_2, \xi^{\gamma_1 \rhd \beta})$$

of operational type

$$(\gamma_1 \rhd \alpha) \rhd \beta;$$

but then,

$$den(F\ x\ \xi(Z)) = den^*(F)(den(Z)) = f(x_2, h(x_2, \xi^{\gamma_1}, \xi^{\gamma_2}))$$

is a ground on B of the required kind. Let us finally consider the case

$$FV^I(Z) = \{x_1, x_2\} \text{ and } FV^T(Z) = \{\xi^{\gamma_1}, \xi^{\gamma_2}\}.$$

Again by restriction (2), we will have that

$$x_1 \notin FV(\gamma_2).$$

We have again four groups of possibilities. The first one is summarized in

$$FV(\gamma_1) \cup FV(\gamma_2) \cup FV(\alpha) \subseteq \{x_i\}\ (i = 1, 2) \text{ and } FV(\beta) = \emptyset,$$

in which case, by induction hypothesis, $den(Z)$ is a B-operation on grounds

$$h(x_1, x_2, \xi^{\gamma_1}, \xi^{\gamma_2})$$

of operational type

$$\gamma_1, \gamma_2 \rhd \alpha,$$

namely, an improper ground on B for $\gamma_1, \gamma_2 \vdash \alpha$, and $den^*(F)$ is a B-operation on grounds

$$f(\xi^{\gamma_1 \rhd \beta})$$

of operational type

$$(\gamma_1 \rhd \alpha) \rhd \beta;$$

but then,

$$den(F\ x\ \xi(Z)) = den^*(F)(den(Z)) = f(h(x_1, x_2, \xi^{\gamma_1}, \xi^{\gamma_2}))$$

is a ground on B of the required kind. A second group of possibilities is summarized in

$$FV(\gamma_1) \cup FV(\gamma_2) \cup FV(\alpha) \subseteq \{x_i\}\ (i = 1, 2) \text{ and } FV(\beta) = \{x_2\},$$

in which case, by induction hypothesis, $den(Z)$ is a B-operation on grounds

$$h(x_1, x_2, \xi^{\gamma_1}, \xi^{\gamma_2})$$

of operational type

$$\gamma_1, \gamma_2 \rhd \alpha,$$

namely, an improper ground on B for $\gamma_1, \gamma_2 \vdash \alpha$, and $den^*(F)$ is a B-operation on grounds

$$f(x_2, \xi^{\gamma_1 \rhd \beta})$$

of operational type

$$(\gamma_1 \rhd \alpha) \rhd \beta;$$

but then,

$$den(F\ x\ \xi(Z)) = den^*(F)(den(Z)) = f(x_2, h(x_1, x_2, \xi^{\gamma_1}, \xi^{\gamma_2}))$$

is a ground on B of the required kind. A third possibility is expressed by the conjunction

$$FV(\gamma_1) \cup FV(\gamma_2) \cup FV(\alpha) = \{x_1, x_2\} \text{ and } FV(\beta) = \emptyset,$$

in which case, by induction hypothesis, $den(Z)$ is a B-operation on grounds

$$h(x_1, x_2, \xi^{\gamma_1}, \xi^{\gamma_2})$$

of operational type

$$\gamma_1, \gamma_2 \rhd \alpha,$$

namely, a proper ground on B for $\gamma_1, \gamma_2 \vdash \alpha$, and $den^*(F)$ is a B-operation on grounds

$$f(\xi^{\gamma_1 \rhd \beta})$$

of operational type

$$(\gamma_1 \rhd \alpha) \rhd \beta;$$

but then,

$$den(F\ x\ \xi(Z)) = den^*(F)(den(Z)) = f(h(x_1, x_2, \xi^{\gamma_1}, \xi^{\gamma_2}))$$

is a ground on B of the required kind. A fourth possibility is expressed by the conjunction

$$FV(\gamma_1) \cup FV(\gamma_2) \cup FV(\alpha) = \{x_1, x_2\} \text{ and } FV(\beta) = \{x_2\},$$

in which case, by induction hypothesis, $den(Z)$ is a B-operation on grounds

$$h(x_1, x_2, \xi^{\gamma_1}, \xi^{\gamma_2})$$

of operational type

$$\gamma_2 \rhd \alpha,$$

namely, a proper ground on B for $\gamma_1, \gamma_2 \vdash \alpha$, and $den^*(F)$ is a B-operation on grounds

$$f(x_2, \xi^{\gamma_1 \rhd \beta})$$

of operational type

$$(\gamma_1 \rhd \alpha) \rhd \beta;$$

but then,

$$den(F\ x\ \xi(Z)) = den^*(F)(den(Z)) = f(x_2, h(x_1, x_2, \xi^{\gamma_1}, \xi^{\gamma_2}))$$

is a ground on B of the required kind.

On more complex terms the reasoning is analogous. □

5.2.4.3 Closure under canonical form

The denotation theorem guarantees a sort of "correctness" for language of grounding, with respect to the grounds set on a reference base by the clauses (At_G) - ($\bot_G$), as well as with respect to the operations on grounds on this base. Indeed, it shows that each term of each language of grounding denotes either a ground for a categorical judgment or assertion, or a first-level operation on grounds. The propitious circumstance depends on the fact that, in languages of grounding, we have authorized only operational symbols with operational types and bindings such that there are corresponding operations on grounds on the reference base.

In this section, we want to conduct an observation that is linked to the denotation theorem, and that will allow us to identify a first property on the basis of which languages of grounding can be classified. Before we begin, however, we need a definition.

Definition 46. Let Λ be a language of grounding, and let T be a term of Λ. We will say that T is in *canonical form* if, and only if, it is an individual constant of Λ, or a ground-variable, or the outermost operational symbol of T is $\wedge I$ or $\vee I$ or $\rightarrow I$ or $\forall I$ or $\exists I$.

Well, given a grounding language Λ, and given $T : \alpha \in \texttt{TERM}_\Lambda$ closed, according to the denotation theorem we know that, for each denotation function den^* of the elements of the alphabet of Λ, and called den the denotation function of the terms of Λ associated with den^*, $den(T)$ is a proper ground for $\vdash \alpha$ with α closed, namely a ground as required by the clause (k_G), where k is the main logical constant of α. Now, we know that grounds of this kind are formed by applying primitive operations to appropriate objects, and we know also that these primitive operations correspond to operational symbols present in *every* grounding language. All these observations, then, could lead us to conclude that it must exist $U : \alpha \in \texttt{TERM}_\Lambda$ closed and in *canonical form* such that $den(T) = den(U)$. In other words, we could be led to believe that every proper ground for a categorical judgment or assertion denoted by some term of Λ has, *in the same* Λ, a canonical name. If it is easy to realize that this applies when α is atomic, an example will show however, that this circumstance does not apply in general.

Given an atomic base B on a language L, let us take into account two classes of operations

$$h_i(\xi^{\alpha_1 \wedge \alpha_2})\ (i = 1, 2)$$

of respective operational types

$$\alpha_1 \wedge \alpha_2 \rhd \alpha_i \ (i = 1, 2),$$

with $\alpha_1, \alpha_2 \in \texttt{FORM}_L$ closed, fixed by the following, respective equations: for every $\wedge I(g_1, g_2)$ proper ground on B for $\vdash \alpha_1 \wedge \alpha_2$,

$$h_i(\wedge I(g_1, g_2)) = g_i.$$

These operations are, in particular, B-operations on grounds; indeed, since $\wedge I(g_1, g_2)$ is a proper ground on B for $\vdash \alpha_1 \wedge \alpha_2$, g_i is a proper ground on B for $\vdash \alpha_i$ $(i = 1, 2)$. Whence the conclusion that

$$\rightarrow I\xi^{\alpha_1 \wedge \alpha_2}(h_i(\xi^{\alpha_1 \wedge \alpha_2}))$$

is a proper ground on B for $\vdash \alpha_1 \wedge \alpha_2 \rightarrow \alpha_i$. Let us then take into account a class of operations

$$f(\xi^{\alpha_i \rightarrow \alpha_1 \vee \alpha_2}) \ (i = 1, 2)$$

of operational type

$$\alpha_i \rightarrow \alpha_1 \vee \alpha_2 \rhd \alpha_1 \wedge \alpha_2 \rightarrow \alpha_i$$

with $\alpha_1, \alpha_2 \in \texttt{FORM}_L$ closed, fixed by the following equations: for every g proper ground on B for $\vdash \alpha_i \rightarrow \alpha_1 \vee \alpha_2$,

$$f(g) = \rightarrow I\xi^{\alpha_1 \wedge \alpha_2}(h_i(\xi^{\alpha_1 \wedge \alpha_2})).$$

These operations are, for what has been just stated about the term on the right in the last equation, B-operations on grounds.

Given now a core language on B, let us expand it to a language of grounding Λ by adding to it a class of operational symbols

$$F\langle \alpha_i \rightarrow \alpha_1 \vee \alpha_2 \rhd \alpha_1 \wedge \alpha_2 \rightarrow \alpha_i \rangle \ (i = 1, 2)$$

with $\alpha_1, \alpha_2 \in \texttt{FORM}_L$ closed, and let us put

$$den^*(F\langle \alpha_i \rightarrow \alpha_1 \vee \alpha_2 \rhd \alpha_1 \wedge \alpha_2 \rightarrow \alpha_i \rangle) = f(\xi^{\alpha_i \rightarrow \alpha_1 \vee \alpha_2}).$$

Now, in Λ we surely have a closed term that denotes a proper ground g on B for $\vdash \alpha_i \rightarrow \alpha_1 \vee \alpha_2$, that is

$$\rightarrow I\xi^{\alpha_i}(\vee I\langle \alpha_i \rhd \alpha_1 \vee \alpha_2 \rangle(\xi^{\alpha_i})).$$

Consequently, the term of Λ

$$F(\rightarrow I\xi^{\alpha_i}(\vee I\langle \alpha_i \rhd \alpha_1 \vee \alpha_2\rangle(\xi^{\alpha_i})))$$

denotes a proper ground on B for$\vdash \alpha_1 \wedge \alpha_2 \rightarrow \alpha_i$, as shown by the following computation:

$$\begin{gathered} den(F(\rightarrow I\xi^{\alpha_i}(\vee I\langle \alpha_i \rhd \alpha_1 \vee \alpha_2\rangle(\xi^{\alpha_i})))) = \\ den^*(F)(den(\rightarrow I\xi^{\alpha_i}(\vee I\langle \alpha_i \rhd \alpha_1 \vee \alpha_2\rangle(\xi^{\alpha_i})))) = \\ f(g) = \rightarrow I\xi^{\alpha_1 \wedge \alpha_2}(h_i(\xi^{\alpha_1 \wedge \alpha_2})). \end{gathered}$$

The question is now if it exists a term $U : \alpha_1 \wedge \alpha_2 \rightarrow \alpha_i \in \texttt{TERM}_\Lambda$ closed and in canonical form, such that

$$den(F(\rightarrow I\xi^{\alpha_i}(\vee I\langle \alpha_i \rhd \alpha_1 \vee \alpha_2\rangle(\xi^{\alpha_i})))) = den(U)$$

that is, such that

$$den(U) = \rightarrow I\xi^{\alpha_1 \wedge \alpha_2}(h_i(\xi^{\alpha_1 \wedge \alpha_2})).$$

Now, if such a term existed, since it should have the form

$$\rightarrow I\xi^{\alpha_1 \wedge \alpha_2}(Z)$$

with $Z : \alpha_i \in \texttt{TERM}_\Lambda$, we would have that

$$\begin{gathered} den(U) = den(\rightarrow I\xi^{\alpha_1 \wedge \alpha_2}(Z)) = den^*(\rightarrow I)(den(Z)) = \\ \rightarrow I\xi^{\alpha_1 \wedge \alpha_2}(den(Z)) = \rightarrow I\xi^{\alpha_1 \wedge \alpha_2}(h_i(\xi^{\alpha_1 \wedge \alpha_2})) \end{gathered}$$

and hence we should have that

$$den(Z) = h_i(\xi^{\alpha_1 \wedge \alpha_2}).$$

And this, by the denotation theorem, will occur when

$$FV^T(Z) = \{\xi^{\alpha_1 \wedge \alpha_2}\}.$$

Well, note that Λ can be equated to a natural deduction system *à la* Gentzen, say Σ, which consists of only introduction rules, plus the pair of rules

$$\frac{\alpha_i \rightarrow \alpha_1 \vee \alpha_2}{\alpha_1 \wedge \alpha_2 \rightarrow \alpha_i}\, F$$

with respect to which we can define a translation ι similar to that for the Curry-Howard isomorphism. On introduction rules, ι acts as that defined for `Gen` in Section 5.2.4.5, while on F it behaves as follows:

$$\frac{\begin{matrix}\Delta \\ \alpha_i \to \alpha_1 \vee \alpha_2\end{matrix}}{\alpha_1 \wedge \alpha_2 \to \alpha_i}\, F \quad \stackrel{\iota}{\Longrightarrow} \quad F\langle \alpha_i \to \alpha_1 \vee \alpha_2 \rhd \alpha_1 \wedge \alpha_2 \to \alpha_i \rangle(\iota(\Delta))$$

Therefore, a Z of the required kind can exists if, and only if, $\alpha_1 \wedge \alpha_2 \vdash_\Sigma \alpha_i$, which, as can be easily seen, is not possible. Since the property in question is not trivial, we highlight it with a definition.

Definition 47. Let Λ be a language of grounding, let den^* be a denotation function for the elements of the alphabet of Λ, and let den be the denotation function for the terms of Λ associated with den^*. Λ is *closed under canonical form with respect to den* if, and only if, for every $T : \alpha \in \texttt{TERM}_\Lambda$ closed, there is $U : \alpha \in \texttt{TERM}_\Lambda$ closed and in canonical form such that $den(T) = den(U)$.[6]

5.2.4.4 Variant and invariant denotation

At this point, we are able to illustrate an aspect - in our opinion interesting - concerning the interaction between the languages of grounding and our "universe" of grounds and operations on grounds. We will prove, in fact, that every language of grounding on an atomic base can be expanded to a language of grounding on the same base, closed under canonical form with respect to an expansion of a denotation function defined on the original language. The proof includes several subcases, one of which requires a strategy that we could define as *duplication* of operational symbols. After the proof, however, we will conduct a brief discussion on this strategy, to see whether and how much it is plausible, and to show how it suggests the possibility of a denotational approach different from the one presented here. Finally, we will show how the alternative denotational approach allows for an easier proof of the same result as above.

Be as it is, however, we note that the proof, in each of the two forms we will present here, is based on a convention, also in two forms, depending on the denotational approach chosen. We said that the denotation theorem guarantees a sort of "correctness" of the languages of grounding; well, the convention consists in taking on also the "completeness". We will not dwell here in the analysis of the major or minor plausibility of this assumption, we will limit ourselves to fix it, firstly, in the following form.

[6]A good example of language of grounding closed under canonical form is the Gentzen-language of Section 5.2.3.3, when the denotation function is meant to associate to the non-primitive operational symbols an analogue of Prawitz's reduction for Gentzen's elimination rules. A pivotal role is played by what Schroeder-Heister (Schroeder-Heister) calls the *fundamental corollary* of Prawitz's normalization theory, according to which, if $\vdash_{\texttt{IL}} \alpha$, there is in IL a closed derivation of α ending with an introduction rule; something similar, via Curry-Howard, applies in the case of typed λ-calculus.

Convention 48. Let B be an atomic base. For every (proper or improper) ground g on B for $\alpha_1, ..., \alpha_n \vdash \beta$, there is a language of grounding Λ on B such that, for some denotation function den^* for the elements of the alphabet of Λ, and called den the denotation function for the terms of Λ associated with den^*, there is $U \in \mathtt{TERM}_\Lambda$ such that $den(U) = g$.

Now we will prove the theorem, in its first form.

Theorem 49. Let be B be an atomic base, let Λ be a language of grounding on B, let den_0^* be a denotation function for the elements of the alphabet of Λ, and let den_0 be the denotation function for the terms of Λ associated with den_0^*. Then, for every $T : \alpha \in \mathtt{TERM}_\Lambda$ closed, there is an expansion Λ^+ on B of Λ such that, for some denotation function for the elements of the alphabet of Λ^+, and called den the denotation function for the terms of Λ^+ associated with den^*, there is $U \in \mathtt{TERM}_{\Lambda^+}$ in canonical form such that $den_0(T) = den(T) = den(U)$.

Proof. We reason by cases on the logical form of α. As regards the logically complex cases, we take into account only the one with $\wedge$.

Let α be atomic. In this case, $den_0(T)$ is a closed derivation Δ for α in the atomic system of B. In Λ we will have a name δ of Δ. We can hence put $\Lambda^+ = \Lambda$, $den^* = den_0^*$ and, therefore, $den = den_0$. Indeed

$$den_0(T) = \Delta = den_0^*(\delta) = den_0(\delta).$$

Let α be of the form $\alpha_1 \wedge \alpha_2$. In this case, $den_\Lambda(T)$ is $\wedge I(g_1, g_2)$, with g_i proper ground on$B$ for $\vdash \alpha_i$ $(i = 1, 2)$. By convention 48, there exist

- a language of grounding Λ_1 on B such that, for some denotation function den_1^* for the elements of the alphabet of Λ_1, and called den_1 the denotation function for the terms of Λ_1 associated to den_1^*, there is $U_1 \in \mathtt{TERM}_{\Lambda_1}$ such that $den_1(U_1) = g_1$;
- a language of grounding Λ_2 on B such that, for some denotation function den_2^* for the elements of the alphabet of Λ_2, and called den_2 the denotation function for the terms of Λ_2 associated with den_2^*, there is $U_2 \in \mathtt{TERM}_{\Lambda_2}$ such that $den_2(U_2) = g_2$.

Now, called $\mathtt{Al}_\Lambda$, $\mathtt{Al}_{\Lambda_1}$ and $\mathtt{Al}_{\Lambda_2}$ the respective alphabets of Λ, Λ_1 and Λ_2, if we have

$$\mathtt{Al}_\Lambda \cap \mathtt{Al}_{\Lambda_1} \cap \mathtt{Al}_{\Lambda_2} = \emptyset,$$

let us consider the language of grounding Λ^+ of which the alphabet is

$$\mathtt{Al}_\Lambda \cup \mathtt{Al}_{\Lambda_1} \cup \mathtt{Al}_{\Lambda_2}$$

and let us define the denotation function den^* for the elements of the alphabet of Λ^+ such that

$$\begin{cases} den^*(x) = den^*_0(x) & \text{for } x \in \mathtt{Al}_\Lambda \\ den^*(x) = den^*_1(x) & \text{for } x \in \mathtt{Al}_{\Lambda_1} \\ den^*(x) = den^*_2(x) & \text{for } x \in \mathtt{Al}_{\Lambda_2} \end{cases}$$

which will be associated to a denotation function den of the terms of Λ^+ such that

$$den(T) = den_0(T) \text{ and } den(U_1) = den_1(U_1) \text{ and } den(U_2) = den_2(U_2).$$

But then, from the following computation

$$\begin{gathered} den(T) = den_0(T) = \wedge I(g_1, g_2) = \\ \wedge I(den_1(U_1), den_2(U_2)) = \\ \wedge I(den(U_1), den(U_2)) = den(\wedge I(U_1, U_2)). \end{gathered}$$

we conclude that

$$den_0(T) = den(T) = den(\wedge I(U_1, U_2)).$$

Let us suppose then that it holds

$$\mathtt{Al}_\Lambda \cap \mathtt{Al}_{\Lambda_1} \cap \mathtt{Al}_{\Lambda_2} \neq \emptyset$$

and, called $\mathbb{F}_T$, $\mathbb{F}_{U_1}$ and $\mathbb{F}_{U_2}$ the sets of the non-primitive operational symbols respectively occurring in T, U_1 and U_2, let us suppose that it holds

$$\mathbb{F}_T \cap \mathbb{F}_{U_1} \cap \mathbb{F}_{U_2} = \emptyset.$$

Also in this case, let us consider the language of grounding Λ^+ of which the alphabet is

$$\mathtt{Al}_\Lambda \cup \mathtt{Al}_{\Lambda_1} \cup \mathtt{Al}_{\Lambda_2}$$

and let us define a denotation function den^* for the elements of the alphabet of Λ^+ such that

$$\begin{cases} den^*\restriction_{\mathbb{F}_T}(x) = den^*_0(x) \\ den^*\restriction_{\mathbb{F}_{U_1}}(x) = den^*_1(x) \\ den^*\restriction_{\mathbb{F}_{U_2}}(x) = den^*_2(x) \end{cases}$$

which will be again associated with a denotation function *den* of the terms of Λ^+ such that, again,

$$den(T) = den_0(T) \text{ and } den(U_1) = den_1(U_1) \text{ and } den(U_2) = den_2(U_2).$$

But then, again from the computation above, we conclude that

$$den_0(T) = den(T) = den(\wedge I(U_1, U_2)).$$

Let us finally suppose that it holds

$$\mathbb{F}_T \cap \mathbb{F}_{U_1} \cap \mathbb{F}_{U_2} \neq \emptyset.$$

If den_0^*, den_1^* and den_2^* are such that

$$den_0(T) = den_i(T) \text{ and } den_i(U_i) = den_j(U_i) \ (i = 1, 2,\ j = 0, 1, 2,\ i \neq j)$$

let us consider again the language of grounding Λ^+ with alphabet

$$\texttt{Al}_\Lambda \cup \texttt{Al}_{\Lambda_1} \cup \texttt{Al}_{\Lambda_2}$$

and let us define a denotation function for the elements of the alphabet of Λ^+

$$den^* = den_i$$

where is i is indifferently $0, 1$ or 2. The denotation function *den* associated with den^* will be such that

$$den(T) = den_0(T) \text{ and } den(U_1) = den_1(U_1) \text{ and } den(U_2) = den_2(U_2).$$

But then, from the computation above, we conclude that

$$den_0(T) = den(T) = den(\wedge I(U_1, U_2)).$$

If instead we have

$$den_0(T) \neq den_i(T) \text{ or } den_i(U_i) \neq den_j(U_i) \ (i = 1, 2,\ j = 0, 1, 2,\ i \neq j)$$

let us consider the language of grounding Λ^+ with alphabet

$$\texttt{Al}_\Lambda \cup \texttt{Al}_{\Lambda_1} \cup \texttt{Al}_{\Lambda_2}$$

and moreover, for every operational symbol

$$F\langle \ddagger \rangle \in \mathbb{F}_T \cap \mathbb{F}_{U_1} \cap \mathbb{F}_{U_2},$$

two fresh operational symbols $F^*\langle \ddagger \rangle$, $F^{**}\langle \ddagger \rangle$ (duplication). Then we put

$$\mathbb{F}^* = \{F^*\langle\ddagger\rangle \mid F\langle\ddagger\rangle \in \mathbb{F}_T \cap \mathbb{F}_{U_1} \cap \mathbb{F}_{U_2}\}$$
$$\mathbb{F}^{**} = \{F^{**}\langle\ddagger\rangle \mid F\langle\ddagger\rangle \in \mathbb{F}_T \cap \mathbb{F}_{U_1} \cap \mathbb{F}_{U_2}\}$$

and we define den^* in such a way that

$$\begin{cases} den^* \restriction_{\mathbb{F}^*}(x) = den_1^*(x) \\ den^* \restriction_{\mathbb{F}^{**}}(x) = den_2^*(x) \end{cases}$$

Let now:

- U_1^* be the term of Λ^+ obtained from U_1 by replacing every

  $$F\langle\ddagger\rangle \in \mathbb{F}_T \cap \mathbb{F}_{U_1} \cap \mathbb{F}_{U_2}$$

  with $F^*\langle\ddagger\rangle$;

- U_2^{**} be the term of Λ^+ obtained from U_2 by replacing every

  $$F\langle\ddagger\rangle \in \mathbb{F}_T \cap \mathbb{F}_{U_1} \cap \mathbb{F}_{U_2}$$

  with $F^{**}\langle\ddagger\rangle$.

The denotation function den of the terms of Λ^+ associated with den^* will be hence such that

$$den(T) = den_0(T) \text{ and } den(U_1^*) = den_1(U_1) \text{ and } den(U_2^{**}) = den_2(U_2).$$

But then, the following computation

$$den(T) = den_0(T) = \wedge I(g_1, g_2) =$$
$$\wedge I(den_1(U_1), den_2(U_2)) =$$
$$\wedge I(den(U_1^*), den(U_2^{**})) = den(\wedge I(U_1^*, U_2^{**}))$$

shows that

$$den_0(T) = den(T) = den(\wedge I(U_1^*, U_2^{**})).$$

The result is hence proven. □

The crucial step of the proof is what we have called of duplication. It consists of adding, for each operational symbol which is part of at least two of the terms T, U_1 and U_2, an operational symbol that has the same operational type, but is indicated with a different "label", and not attributed to any of the operational symbols already present. The passage proves to be necessary insofar as, unlike the other cases, if at least two among T, U_1 and U_2 share some operational symbol, it may not be possible to define on Λ^+ a den such that it holds

$$den(T) = den_0(T) \text{ and } den(U_1) = den_1(U_1) \text{ and } den(U_2) = den_2(U_2)$$

and this because it could hold

$$den_0(T) \neq den_i(T) \text{ or } den_i(U_i) \neq den_j(U_i) \ (i = 1, 2, \ j = 0, 1, 2, \ i \neq j).$$

One might obviously ask if duplication is a plausible, and therefore viable strategy. And doubts arise due to the fact that a language of grounding can be understood as a formal system, as exemplified from the bijective translations of the Gentzen-language and the language of grounding for Heyting's first-order arithmetic. If this is true, in fact, a language of grounding in which there are duplicate symbols, would be like a formal system with rules where premises, conclusions, discharge of assumptions and binding of variables are equal, and where what changes is only the name of the rule. For example, by leaving out the discharge of assumptions and the binding of variables, we could have

$$\frac{\alpha_1 \quad \dots \quad \alpha_n}{\beta}\, F \qquad \frac{\alpha_1 \quad \dots \quad \alpha_n}{\beta}\, F^*$$

Now, if we adopt the point of view that an inference rule is nothing but the set of its instances, we easily conclude that F and F^* are the *same* rule. From this point of view, therefore, we would say that the duplication is a senseless strategy.

However, this objection could perhaps be answered as follows. Obviously, a language of grounding can be understood as a formal system; on the other hand, though, a language of grounding is really such only when on it a denotation has been defined, namely a function that allows us to understand the purely syntactic symbols of which it consists of as actual operations on grounds. A language of grounding is such when the terms it contains are names of grounds; and the terms will be names of grounds not when understood as mere *proof-terms* but, more strongly, when a specific denotation function has made them *interpreted proof-terms* (see Prawitz 2014). Thus, if a language of grounding is comparable to a formal system, a *language of grounding* on which it has been defined a denotation is comparable to a formal system where each rule is identified, not only through premises, conclusion, discharge of assumptions and binding of variables, but also by a reduction procedure associated to it. Then, despite having the same set of instances, the rules F and F^* are different if associated with different reduction procedures.

Operations on grounds are to be intended as identified by a double parameter. The first, which we have already discussed extensively, concerns their operational type; the operational type states, so to speak, the domain and the codomain of the operation, or rather the nature of the inputs on which the operation can be computed, and the nature of the output that the operation produces when applied to appropriate inputs. The second parameter is given instead by a defining equation that, associated with the operation, specifies it indicating how the operation acts on its inputs, and what output it generates on each of these inputs; it is precisely this defining equation that, once we have established inputs and outputs, really identifies the operation, providing us with the computation instructions of which it consists.

Two operations can be distinguished *extensionally*, when they produce different values on the same arguments, or *intensionally*, when they are defined by different equations. Obviously, if two operations are extensionally different, they will also be intensionally different; however, as we will see more widely later, the vice-versa does not apply. Be as it is, and although investigations related to identity issues similar to those we have mentioned here are at the center of intense research in some areas of contemporary mathematical logic (especially in the field of Martin-Löf's intuitionist type theory, see Martin-Löf 1975a, 1975b; for a recent worh, see Klev 2018), we want here to confine ourselves to remarking how, to the same operational type, they can match several operations on grounds. A very simple example is the following. Let us take into account the following two classes of grounds for $\vdash \alpha \to (\alpha \to \alpha)$, with α closed:

$$g_1 = \to I\xi_2^{\alpha}(\to I\xi_1^{\alpha}(\mathtt{Id}(\xi^{\alpha_1})) \text{ and } g_2 = \to I\xi_1^{\alpha}(\to I\xi_2^{\alpha}(\mathtt{Id}(\xi^{\alpha_1})).$$

Well, for any closed β, to the scheme of operational types

$$\beta \rhd \alpha \to (\alpha \to \alpha)$$

we can associate two distinct classes of operations on grounds

$$f_i(\xi^{\beta}) \ (i = 1, 2)$$

defined by respective equations

$$f_i(\xi^{\beta}) = g_i.$$

f_1 and f_2 produce different values on the same arguments, and this is evidenced by the fact that they are fixed by different equations.

Now, when we have defined the languages of grounding, we have authorized only operational symbols F, of which the operational types and bindings were such that there were corresponding operations on grounds f with adequate operational types and bindings. And when we have defined the denotation den^* of the elements of the alphabet of a language of grounding, we have required that $den^*(F)$ were one of the appropriate operations on grounds that it was supposed to exist. Since to the same operational type there could be associated several operations on grounds, on the same language of grounding it could therefore be possible to define several den^*s. Such an approach could be defined as a *variant* denotation, since it considers a language of grounding as a non-interpreted syntactic apparatus, therefore subject to different interpretations.

The variant denotation approach is very similar to the one adopted in model-theory, where the same logical language is interpreted on multiple set structures, and where therefore the same theory can have different models. The idea that a formal language should be understood as non-interpreted, however, has not always been dominant in logic, and it was not, for example, in Frege or in Russell & Whitehead (Frege 1879, 1884, 1893-1903; Russell & Whitehead 1962; for a recent reconstruction of this point, see Sundholm 2018). Therefore, it might make sense here to take into account an alternative approach, which could be defined as *invariant* denotation.

In an invariant denotation approach, an operational symbol F added to a language of grounding Λ it is not a syntactic symbol that can be later interpreted on one of the possible operations on grounds that are supposed to exist; on the contrary, it is a *name* for a *specific* operation f, belonging to the class of operations on grounds with the operational type and the bindings attributed to F. To conform with this basic idea, therefore, when we define a denotation function den^* for the elements of the alphabet of Λ, we no longer have to require that $den^*(F)$ is *one of the possible* operations on grounds befitting to the operational type and to the bindings of F, but to establish more strongly $den^*(F) = f$. Then it is obvious that den^* is the only denotation function that can be defined for the elements of the alphabet of Λ, and therefore equally unique will be the denotation function *den* of the terms of Λ associated with den^*; in other words, we can no longer speak of *a* denotation function for Λ, but of *the* denotation function of Λ or, if you prefer, of the denotation function *associated with* Λ. We can indicate this by writing den^*_Λ for the denotation function of the elements of the alphabet associated with Λ, and den_Λ for the denotation function of the terms of Λ associated with den^*_Λ.

Now, in our opinion it is interesting to note how the adoption of an invariant denotation approach gives us a much easier proof of a result analogous to that established by theorem 49. The first thing to do is to reformulate convention 48 on the invariant denotation, which will be done as follows.

Convention 50. Let B be an atomic base. Then, for every (proper or improper) ground g on B for $\alpha_1, ..., \alpha_n \vdash \beta$, there is a language of grounding Λ on B such that, for some $U \in \texttt{TERM}_\Lambda$, $den_\Lambda(U) = g$.

Theorem 51. Let B be an atomic base, and let Λ be a language of grounding on B. Then, for every $T : \alpha \in \texttt{TERM}_\Lambda$ closed, there is an expansion Λ^+ on B of Λ such that there is $U \in \texttt{TERM}_{\Lambda^+}$ in canonical form, such that $den_{\Lambda^+}(T) = den_{\Lambda^+}(U)$.

Proof. We reason by cases on the logical form of α. As regards the logically complex cases, also in this case we limit ourselves to the one with $\wedge$. Since we are adopting the invariant denotation approach, for every language of grounding $\mathfrak{G}$ on B such that $T \in \texttt{TERM}_\mathfrak{G}$, it must hold that $den_\Lambda(T) = den_\mathfrak{G}(T)$.

Let α be atomic. In this case, $den_\Lambda(T)$ is a closed derivation Δ for α in the atomic system of B. In Λ we will have a name δ of Δ, and we can hence put $\Lambda^+ = \Lambda$. Indeed

$$den_\Lambda(T) = \Delta = den^*_\Lambda(\delta) = den_\Lambda(\delta).$$

Let α be of the form $\alpha_1 \wedge \alpha_2$. In this case, $den_\Lambda(T)$ is $\wedge I(g_1, g_2)$, with g_i proper ground on B for $\vdash \alpha_i$ $(i = 1, 2)$. By convention 50, there are

- a language of grounding Λ_1 on B such that, for some $U_1 \in \texttt{TERM}_{\Lambda_1}$, $den_{\Lambda_1}(U) = g_1$;
- a language of grounding Λ_2 on B such that, for some $U_2 \in \texttt{TERM}_{\Lambda_2}$, $den_{\Lambda_2}(U_2) = g_2$.

Called $\texttt{Al}_\Lambda$, $\texttt{Al}_{\Lambda_1}$ and $\texttt{Al}_{\Lambda_2}$ the respective alphabets of Λ, Λ_1 and Λ_2, let us consider the language of grounding Λ^+ of which the alphabet is

$$\texttt{Al}_\Lambda \cup \texttt{Al}_{\Lambda_1} \cup \texttt{Al}_{\Lambda_2}.$$

For T we have already noted that the adoption of the invariant denotation approach implies that $den_\Lambda(T) = den_{\Lambda^+}(T)$; this will also hold for U_i, that is, $den_{\Lambda_i}(U_i) = den_{\Lambda^+}(U_i)$ $(i = 1, 2)$. Therefore, we have

$$\begin{gathered} den_{\Lambda^+}(T) = den_\Lambda(T) = \wedge I(g_1, g_2) = \\ \wedge I(den_{\Lambda_1}(U_1), den_{\Lambda_2}(U_2)) = \\ \wedge I(den_{\Lambda^+}(U_1), den_{\Lambda^+}(U_2)) = den_{\Lambda^+}(\wedge I(U_1, U_2)) \end{gathered}$$

and the result is thus proven. □

5.2.4.5 Defining equations and empty functions

In the previous section, we made a first mention of the second parameter which identifies an operation on grounds: a defining equation that shows *how*, by acting on the arguments in its definition domain, the operation produces values in its codomain, and *which* the produced values are. But what do these equations look like, more precisely?

In order to answer the question in detail, we need a theory of the defining equations of the operations on grounds. Well, whatever the lines along which we intend to develop this theory, it is plausible to expect that there will not be a single one. A request that must surely be respected by all is that the equations give back total constructive functions: functions converging on each element in the definition domain, namely that on each value of the indicated domain, produce a value in the indicated codomain, and that, based on the instructions from the equations, are also computable in an effective sense. But, this set out, many and different are the possible options. We can require that the equations respect those only two criteria, which would mean having a very large class of operations, but hardly describable. Or, if we want to have a more manageable class of operations, we can add new bounds, for example by requesting that the operations do not produce values that are not already contained, in some sense, in the values to which they are applied.

The aim of this work is *not* a general theory of defining equations for operations on grounds. Although we will go a little further in the problem in Chapter 7, we want to give here only some *examples*; in this section, therefore, we will show which equations can be given for operations on grounds that correspond to the elimination rules in a system of natural deduction for first-order intuitionist logic, and the induction rule in Heyting's first-order arithmetic. These operations are associated, through denotation, to the non-primitive operational symbols of the Gentzen-language and of the language of grounding for Heyting's first-order arithmetic, both defined in Section 5.2.3.3. Before starting, we need to make the obvious observation that operations on grounds denoted by the operational symbols $\wedge I$, $\vee I$, $\rightarrow I$, $\forall I$, $\exists I$ do not need, as primitive, any further specification.

First of all, let us discuss operational types with only closed formulas. As regards the operational symbol $\wedge_{E,i}\langle \alpha_1 \wedge \alpha_2 \rhd \alpha_i \rangle$, we can request that

$$den^*(\wedge_{E,i}\langle \alpha_1 \wedge \alpha_2 \rhd \alpha_i \rangle)$$

$(i = 1, 2)$ is the operation

$$f(\xi^{\alpha_1 \wedge \alpha_2})$$

of operational type

$$\alpha_1 \wedge \alpha_2 \rhd \alpha_i$$

such that, for every $\wedge I(g_1, g_2)$ proper ground for $\vdash \alpha_1 \wedge \alpha_2$,

$$f(\wedge I(g_1, g_2)) = g_i.$$

As regards the operational symbol $\vee E\langle \alpha_1 \vee \alpha_2, \beta, \beta \rangle$, we can request that

$$den^*(\vee E\langle \alpha_1 \vee \alpha_2, \beta, \beta \rangle)$$

is the operation

$$f(\xi^{\alpha_1 \vee \alpha_2}, \xi^{\alpha_1 \rhd \beta}, \xi^{\alpha_2 \rhd \beta})$$

of operational type

$$\alpha_1 \vee \alpha_2, (\alpha_1 \rhd \beta), (\alpha_2 \rhd \beta) \rhd \beta$$

that binds the ground-variable ξ^{α_1} on the second entry, and the ground-variable ξ^{α_2} on the third, such that, for every $\vee I\langle \alpha_i \rhd \alpha_1 \vee \alpha_2 \rangle(g)$ proper ground for $\vdash \alpha_1 \vee \alpha_2$, for every $h_i(\xi^{\alpha_i})$ proper ground for $\alpha_i \vdash \beta$ $(i = 1, 2)$

$$f(\vee I\langle \alpha_i \rhd \alpha_1 \vee \alpha_2 \rangle(g), h_1(\xi^{\alpha_1}), h_2(\xi^{\alpha_2})) = h_i(\vee I(g)).$$

As regards the operational symbol $\rightarrow E\langle \alpha \rightarrow \beta, \alpha \rhd \beta \rangle$, we can request that

$$den^*(\rightarrow E\langle \alpha \rightarrow \beta, \alpha \rhd \beta \rangle)$$

is the operation

$$f(\xi^{\alpha \rightarrow \beta}, \xi^{\alpha})$$

of operational type

$$\alpha \rightarrow \beta, \alpha \rhd \beta$$

such that, for every $\rightarrow I\xi^{\alpha}(h(\xi^{\alpha}))$ proper ground for $\vdash \alpha \rightarrow \beta$, for every g proper ground for $\vdash \alpha$,

$$f(\rightarrow I\xi^{\alpha}(h(\xi^{\alpha})), g) = h(g).$$

As regards the operational symbol $\exists E\langle \exists x\alpha(x), \beta \rhd \beta \rangle$, we can request that

$$den^*(\exists E\langle \exists x\alpha(x), \beta \rhd \beta\rangle)$$

is the operation

$$f(\xi^{\exists x\alpha(x)}, \xi^{\alpha(x)\rhd\beta})$$

of operational type

$$\exists x\alpha(x), (\alpha(x) \rhd \beta) \rhd \beta$$

that binds the individual variable x and the ground variable $\xi^{\alpha(x)}$ on the second entry, such that, for every $\exists I(g)$ proper ground for $\vdash \exists x\alpha(x)$, with g proper ground for $\vdash \alpha(t)$, for every $h(x, \xi^{\alpha(x)})$ proper ground for $\alpha(x) \vdash \beta$,

$$f(\exists I(g), h(x, \xi^{\alpha(x)})) = h(t, g).$$

Finally, as regards the operational symbol $\forall E\langle \forall x\alpha(x) \rhd \alpha(t/x)\rangle$, we can require that

$$den^*(\forall E\langle \forall x\alpha(x) \rhd \alpha(t)\rangle)$$

is the operation

$$f(\xi^{\forall x\alpha(x)})$$

of operational type

$$\forall x\alpha(x) \rhd \alpha(t)$$

such that, for every $\forall Ix(h(x))$ proper ground for $\vdash \forall x\alpha(x)$,

$$f(\forall Ix(h(x))) = h(t).$$

Of all the operations thus defined, we can show that they are, more in particular, and given any atomic base B, B-operations on grounds of the indicated operational type. We will show only the cases related to $\vee E$ and a $\exists E$. Starting with $\vee E$, let us call $f_\vee$ the operation associated with it, defined by the equation indicated above. Let us then suppose that G is a proper ground on B for $\vdash \alpha_1 \vee \alpha_2$. By virtue of the clause $(\vee_G)$, G must be of the form

$$\vee I\langle \alpha_i \rhd \alpha_1 \vee \alpha_2\rangle(G_i)$$

with G_i proper ground on B for $\vdash \alpha_i$ $(i = 1, 2)$. By virtue of the restrictions on identity relative to primitive operations, G_i is also unique. Let us suppose now that $h_i(\xi^{\alpha_i})$ is a proper ground on B for $\alpha_i \vdash \beta$ $(i = 1, 2)$. Then, for every g_i proper ground on B for $\vdash \alpha_i$, $h_i(g_i)$ is a proper ground on B for $\vdash \beta$. But then, $h_i(G_i)$ is a proper ground on B for $\vdash \beta$, and since it holds

$$f_\vee(\vee I\langle \alpha_i \rhd \alpha_1 \vee \alpha_2\rangle(G_i), h_1(\xi^{\alpha_1}), h_2(\xi^{\alpha_2})) = h_i(G_i)$$

the result is proven. Turning now to $\exists E$, let us call $f_\exists$ the operation associated with it, defined by the equation indicated above. Let us suppose then that G is a proper ground on B for $\vdash \exists x \alpha(x)$. By virtue of the clause $(\exists_G)$, G must be of the form

$$\exists I(G_1)$$

with G_1 proper ground on B for $\vdash \alpha(t)$, for some term t on the background language of B. By virtue of the restrictions on identity relative to primitive operations, G_1 is also unique. Let us suppose now that $h(x, \xi^{\alpha(x)})$ is a proper ground on B for $\alpha(x) \vdash \beta$. Then, for every term u on the background language of B, for every g proper ground on B for $\vdash \alpha(u)$, $h(u, g)$ is a proper ground on B for $\vdash \beta$. But then, $h(t, G_1)$ is a proper ground on B for $\vdash \beta$, and since it holds

$$f_\exists(\exists E(G_1), h(x, \xi^{\alpha(x)})) = h(t, G_1)$$

the result is proven. The remaining cases are similar.

The language of grounding for Heyting's first-order arithmetic contains all the operational symbols of the Gentzen-language, plus the operational symbol $\texttt{Ind}\langle \alpha(0), \alpha(s(x)) \rhd \alpha(t)\rangle$. In this case, we can request that

$$den^*(\texttt{Ind}\langle \alpha(0), \alpha(s(x)) \rhd \alpha(t)\rangle)$$

is the operation

$$f(\xi^{\alpha(0)}, \xi^{\alpha(x) \rhd \alpha(s(x))})$$

of operational type

$$\alpha(0), (\alpha(x) \rhd \alpha(s(x))) \rhd \alpha(t)$$

that binds the individual variable x and the ground-variable $\xi^{\alpha(x)}$ on the second entry, such that, for every proper ground g for $\vdash \alpha(0)$, for every $h(x, \xi^{\alpha(x)})$ proper ground for $\alpha(x) \vdash \alpha(s(x))$,

$$f(g, h(x, \xi^{\alpha(x)})) =$$
$$\begin{cases} g & \text{if } t = 0 \\ h(den^*(\texttt{Ind}\langle \alpha(0), \alpha(s(x)) \rhd \alpha(t-1) \rangle)(g, h(x, \xi^{\alpha(x)}))) & \text{if } t > 0 \end{cases}$$

Let us prove that the operation defined in this way is actually a B-operation on grounds of the indicated operational type, where B is a base for a logical language for first-order arithmetic with atomic system the Post-system referred to in Section 2.5.1. Let us call $f_{\texttt{Ind}}$ the operation associated with Ind, defined by the equation indicated above. Let us suppose that G is a proper ground on B for $\vdash \alpha(0)$, and that $h(x, \xi^{\alpha(x)})$ is a proper ground on B for $\alpha(x) \vdash \alpha(s(x))$. The latter circumstance implies that, for every term u on the background language of B, for every proper ground g on B for $\vdash \alpha(u)$, $h(u, g)$ is a proper ground on B for $\vdash \alpha(s(u))$. Let us distinguish two cases: if $t = 0$,

$$f_{\texttt{Ind}}(G, h(x, \xi^{\alpha(x)})) = G$$

and we are done; if $t > 0$, by (meta)inductive hypothesis, we know that

$$den^*(\texttt{Ind}\langle \alpha(0), \alpha(s(x)) \rhd \alpha(t-1) \rangle)(G, h(x, \xi^{\alpha(x)}))$$

is a proper ground G_1 on B for $\vdash \alpha(t-1)$, so that $h(t-1, G_1)$ will be a proper ground on B for $\vdash \alpha(t)$.

As for the operational types with open formulas, the discourse is substantially similar. The corresponding B-operations on grounds are defined by equations that imply substitutions with closed terms of the individual variables involved in the expression of the operation, and of grounds for the entries of the operational type domain obtained after an analogous substitution in the starting operational type.[7] Thus, for example, as regards the operational symbol

[7] The passage through the closed case is stricly necessary when, in defining the operation, we cannot help but specify that one of the argument to which the operation acts is constructed by applying a primitive operation on grounds for formulas of a lower complexity. The fact that a ground has such shape, in fact, is guaranteed only in the case of grounds for judgments or assertions that involve closed formuale. In this sense, operations on operational types with open formulas are defined in terms of an analogue of those which Schroeder-Heister (Schroeder-Heister 1984) calls *variants* of derivations with free variables; we could say that the operation is defined by its *closed* variants. In some cases, however, the passage is superfluous. This happens, for example, when the operation is defined by resorting, in the *definiens*, to other operations for which equations on the closed variants are already known - in such circumstances, indeed, the reference to a specific form of the arguments of the domain is not required even in the closed case. Another case where the substitutions of the free variables is redundant is that of an operation of which the

$$\exists E\langle \exists x\alpha(x,\underline{y}), \beta(\underline{z}) \rhd \beta(\underline{z})\rangle$$

it will be understood as the operation

$$f(\underline{y}, \underline{z}, \xi^{\exists x\alpha(x,\underline{y})}, \xi^{\alpha(x,\underline{y})\rhd\beta(\underline{z})})$$

of operational type

$$\exists x\alpha(x,\underline{z}), (\alpha(x,\underline{y}) \rhd \beta(\underline{z})) \rhd \beta(\underline{z})$$

such that, for sequence of closed terms $\underline{t}$, with $\mathfrak{L}\ \underline{t} \geq \mathfrak{L}\ \underline{y} + \mathfrak{L}\ \underline{z}$, for every $\exists I(g)$ proper ground for $\vdash \exists x\alpha(x, \underline{t}/\underline{y})$, with g proper ground for $\vdash \alpha(u, \underline{t}/\underline{y})$, for every $h(x, \xi^{\alpha(x,\underline{t}/\underline{y})})$ proper ground for $\alpha(x, \underline{t}/\underline{y}) \vdash \beta(\underline{t}/\underline{z})$,

$$f(\exists I(g), h(x, \xi^{\alpha(x,\underline{t}/\underline{y})})) = h(u, g).$$

It remains to be discussed the case of the operational symbol $\perp_\alpha$. Clearly, we need to ensure that $den^*(\perp_\alpha)\langle \perp \rhd \alpha\rangle$ corresponds to an operation on grounds $\perp_\alpha$ of operational type

$$\perp \rhd \alpha.$$

But what will be, more precisely, the behavior of this operation? The answer comes from the clause $(\perp_G)$, which states that, whatever the atomic base B, there are no grounds on B for $\vdash \perp$. The B-operation on grounds $\perp_\alpha$ can then be set as the *empty function* - the function having an empty domain. By virtue of the clause $(\perp_G)$, this total effective function does exist - and indeed it is the only one on the intended operational type - since the condition for a total effective function to be a B-operation on grounds is vacuously satisfied. Note that, in a sense, the clause $(\perp_G)$ states the meaning of the constant $\perp$, so that to the B-operation $\perp_\alpha$ we should not associate any definition; but even when such definition were required, it will be the *empty definition* – the definition with 0 equations; the role of a definition must in fact be that of fixing the behavior of $\perp_\alpha$ on the grounds for $\vdash \perp$, and this condition is, again, vacuously satisfied by the empty definition.

What has been said about $\perp_\alpha$ provides a very important observation. Given an atomic base B, and given an operational type on the language of B

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operational type has only non-empty domain entries of the type $\Gamma(\underline{x}) \rhd \alpha(\underline{y})$; whether the individual variables are replaced or not by closed terms, a ground for $\Gamma(\underline{x}) \vdash \alpha(\underline{y})$ or for $\Gamma(\underline{t}/\underline{x}) \vdash \alpha(\underline{t}/\underline{y})$ may not at all be constructed by applying a primitive operation. As for the latter case, however, we could require that the equation is given relatively to the *value* of the operation of type $\Gamma(\underline{t}/\underline{x}) \rhd \alpha(\underline{t}/\underline{y})$ *applied* to grounds for the elements of $\Gamma(\underline{t}/\underline{x})$, in such a way to reason on grounds for $\vdash \alpha(\underline{t}/\underline{x})$.

$$\tau_1, ..., \tau_n \rhd \tau_{n+1},$$

if there exists τ_i $(i \leq n)$ such that there are no grounds for τ_i on B, a B-operation on grounds of this operational type exists trivially and is unique; it is the empty function. Then, a non-primitive operational symbol that corresponds to an operation on grounds of the operational type just indicated, is *automatically* authorized in a language of grounding on B; such as in the case of $\bot_\alpha$, the denotation of the operational symbol in question will be the empty function associated with the intended operational type, and the definition of the B-operation on grounds denoted will be the empty definition again. It is interesting to note that, in case there is no change of base, the denotation can remain stable throughout the expansions. In the case of a change of base, from B to B^+, instead, it might be necessary to associate to the operational symbol a B^+-operation that is no longer the empty function, and of which the definition is therefore no longer the empty definition. And what is more, it could also happen that an operational type of which the only B-operation on grounds is the empty function, ceases to have B^+-operations on grounds; then, an operational symbol automatically authorized in a language of grounding on B, would no longer be authorized in a language of grounding on B^+, so that none of these languages on B^+ can be an expansion of the language of B. Some examples will help to better understand these points.

Let B be a logical base on a language L for first-order arithmetic and, for every $\alpha \in \texttt{ATOM}_L$ closed, let us consider the operational type

$$\alpha \rhd \neg\exists x(0 \doteq s(x)).$$

Clearly, there is no ground on B neither for $\vdash \alpha$, whatever is α, nor for $\vdash \neg\exists x(0 \doteq s(x))$. Hence, B-operations on grounds having such operational types exist, and they are unique: they are the empty functions - let us say $f^\alpha_\emptyset$. We can therefore build a language of grounding Λ on B that has a non-primitive operational symbol F of operational type

$$\alpha \rhd \neg\exists x(0 \doteq s(x))$$

for every $\alpha \in \texttt{ATOM}_L$ closed, and require that

$$den^*(F\langle \alpha \rhd \neg\exists x(0 \doteq s(x))\rangle) = f^\alpha_\emptyset.$$

Let us now consider an expansion B^+ of B that has as atomic system a Post-system $\texttt{S}$ for first-order arithmatic as in Section 2.5.1. The first thing that can be detected is that, now, there is a ground on B^+ for $\vdash \neg\exists x(0 \doteq s(x))$, namely

$$\to I\xi^{\exists x(0 \doteq s(x))}(\exists E\ x\ \xi^{0 \doteq s(x)}(\xi^{\exists x(0 \doteq s(x))}, \delta))$$

where δ is the derivation in $\mathtt{S}$

$$\frac{0 \doteq s(x)}{\bot}\ (s_1)$$

Analogously, there will be grounds on B^+ for some $\vdash \alpha$, precisely if, and only if, $\vdash_{\mathtt{S}} \alpha$. Let now Λ^+ be a language of grounding on B^+, expanding Λ. For the α such that $\nvdash_{\mathtt{S}} \alpha$, we can continue to associate the operational symbols F of Λ of operational type

$$\alpha \rhd \neg \exists x(0 \doteq s(x))$$

with the empty function $f^{\alpha}_{\emptyset}$; however, it is clear that, as soon as $\vdash_{\mathtt{S}} \alpha$, there will no longer be any empty function $f^{\alpha}_{\emptyset}$ of the intended operational type, so that the suggested denotation can no longer work. In such cases, then, we can ensure that

$$den^*(F\langle \alpha \rhd \neg \exists x(0 \doteq s(x))\rangle)$$

is the operation

$$f(\xi^{\alpha})$$

such that, for every g ground on B^+ for $\vdash \alpha$,

$$f(g) = \to I\xi^{\exists x(0 \doteq s(x))}(\exists E\ x\ \xi^{0 \doteq s(x)}(\xi^{\exists x(0 \doteq s(x))}, \delta)).$$

As can be seen, hence, we can actually pass from Λ to Λ^+, but a denotation function defined on Λ cannot remain invariant on Λ^+.

Let again B be a logical base on a language L for first-order arithmetic as in Section 2.5.1 and, for every $\alpha \in \mathtt{ATOM}_L$ closed, let us consider the operational type

$$\alpha \rhd 0 \doteq s(0).$$

Clearly, there is no ground on B neither for $\vdash \alpha$, whatever is α, nor for $\vdash 0 \doteq s(0)$. Hence, B-operations on grounds having such operational types exist, and they are also unique: they are the empty functions - say $f^{\alpha}_{\emptyset}$. We can therefore build a language of grounding Λ_1 on B, having a non-primitive operational symbol F of operational type

$$\alpha \rhd 0 \doteq s(0)$$

for every $\alpha \in \texttt{ATOM}_L$ closed, and require that

$$den^*(F\langle \alpha \rhd 0 \doteq s(0) \rangle) = f^{\alpha}_{\emptyset}.$$

Let us consider an expansion B^+ of B that has as atomic system a Post-system $\texttt{S}$ for first-order arithmetic as in Section 2.5.1. Obviously, there is still no ground on B^+ for $\vdash 0 \doteq s(0)$, since $\nvdash_{\texttt{S}} 0 \doteq s(0)$. There will instead be grounds on B^+ for some $\vdash \alpha$, precisely if, and only if, $\vdash_{\texttt{S}} \alpha$. Let now Λ_2 be a language of grounding on B^+. For the α such that $\nvdash_{\texttt{S}} \alpha$, we can continue to associate the operational symbols F of Λ_1 of operational type

$$\alpha \rhd 0 \doteq s(0)$$

with the empty function $f^{\alpha}_{\emptyset}$. Though, when $\vdash_{\texttt{S}} \alpha$, not only there will be no empty function $f^{\alpha}_{\emptyset}$ of the intended operational type, so that the suggested denotation can no longer work, but in addition, and because of the lack of grounds on B^+ for $\vdash 0 \doteq s(0)$, there can be no B^+-operation on grounds of the intended operational type, and hence, some of the operational symbols F considered above are not admissible. In this case, we cannot expand Λ_1 toward Λ_2.

5.2.4.6 Universal grounds, operations and terms

So far, the notions of ground and operation on grounds have been understood as relating to atomic bases; in fact, we have always spoken of grounds and operations on *specific* atomic bases. Something similar can be told regarding the notions of language of grounding and denotation; languages of grounding are always constructed with reference to *given* atomic bases (possibly empty), so that the denotation of the elements of the alphabet of a language of grounding and, consequently, that of the terms, will be related to grounds and operations on grounds on that base. It is natural at this point to ask whether it is possible to generalize the notions mentioned for all the possible bases; if a base determines an interpretation of individual, relational and functional constants of the background language an invariance on all the possible bases will also be an invariance on every possible interpretation of the non-logical terms.

In the previous Section we have provided an example of interpretation of non-primitive operational symbols of a Gentzen-language. As you probably noticed, in developing the discourse we have not made any reference to the atomic base on which the Gentzen-language under examination was intended to act; likewise, when showing the adequacy of the definitions of the operations associated, we have taken into account arbitrary atomic bases.

This means that, what was said then, applies whatever the atomic base of reference, namely that the procedure can be generalized, exactly in the same terms and with the same formal choices, for all the possible atomic bases. This is not the case, instead, of the language of grounding for first-order arithmetic; here, in fact, it has proved necessary to make explicit that the reference atomic base contains a language for first-order arithmetic and, above all, a Post-system for this theory. And this because the proof of the adequacy of the definition of the operation associated to the operational symbol `Ind` cannot disregard the meaning of the symbol of successor, and, consequently, the fact that we are operating on a structure on which a principle of (meta)induction applies. So, while the definitions chosen for the denotation of the non-primitive operational symbols of the Gentzen-language work on any atomic base, the definition chosen for the denotation of `Ind` could fail, with respect to the intended operational type, on some atomic bases. In light of this illustrative observation, we now introduce the following characterization.

Given a base B_1 on L and a ground g on B_1, we will say that g is a *universal ground* if, and only if, for every L^+ expansion of L, for every base B_2 on L^+, g is a ground on B_2. In the case of terms of language of grounding, the notion of universality is fixed in an analogous way.

Definition 52. Let Λ be a language of grounding, and let *den** be a denotation function for the elements of the alphabet of Λ. We will say that $T \in \mathtt{TERM}_\Lambda$ is *universal with respect to den** if, and only if, for every element x of the alphabet of Λ in T, $den^*(x)$ is a universal ground.

It is easy to realize that definition 52 implies that, if T is universal with respect to *den**, then for every L^+ expansion of the background language of T, for every B on L^+, it is always possible to define a language of grounding on B of which the set of terms contains T.

The universality of a term of a language of grounding is hence relative to a denotation function for the elements of the alphabet of the language. One may prefer a relativization to a denotation function for terms, by requiring that the values of such function on universal terms are universal grounds. However, if we assume a plausible convention, definition 52 is sufficient for such a result to obtain.

Convention 53. Let B be a base on L, and let

$$f(\underline{x}, \xi^{\tau_1}, ..., \xi^{\tau_n})$$

be a B-operation on grounds that is more in particular a universal ground. For every sequence $\underline{s}$ of individuals on the domain of B, for every composite operation obtained by plugging a ground g_i on B for $\tau_i[\underline{s}/\underline{x}]$ on index i, if g_i is a universal ground, then

$$f(\underline{s}/\underline{x}, \dots\ g_i\ \dots)$$

is a universal ground.

Let us then prove the announced result.

Proposition 54. Let Λ be a language of grounding, den^* be a denotation function for the elements of the alphabet of Λ, den be the denotation function for the terms of Λ associated with den^* and $T \in \texttt{TERM}_\Lambda$ be universal with respect to den^*. Then, $den(T)$ is a universal ground.

Proof. By induction on the complexity of T. The case of ground-variables is obvious, since the identity function for the given operational type is cleary a universal ground. If T is a constant, then it cannot be universal with respect to den^*, since for example a constant will not count as a ground on a logical base. If T is

$$F\ \underline{x}\ \underline{\xi}\ (U_1, ..., U_n)$$

then, by corollary 57 below, every U_i is universal with respect to den^* and hence, by induction hypothesis, $den(U_i)$ is a universal ground. By hypothesis on the universality of T with respect to den^*, moreover, by definition 52 we know that $den^*(F)$ is a universal ground. Therefore, since

$$den(F\ \underline{x}\ \underline{\xi}\ (U_1, ..., U_n)) = den^*(F)(den(U_1), ..., den(U_n)),$$

the result follows immediately from convention 53. □

Observe that we also have an analogue of proposition 54 if, instead of requiring that T is universal with respect to den^*, we require that every element of the alphabet occurring in T is a ground on a base B on an expansion of the background language of T. In this case, convention 53 is to be restricted to admissibility over bases and, by applying it, we obtain that $den(T)$ is a ground on B.

It may be interesting to note that the inverse of proposition 54 does not apply. Let us show this with two counter-examples. First of all, given a *non-logical* base B_1, and an appropriate language of grounding Λ on B_1,

$$\wedge_{E,1}(\wedge I(\rightarrow I\xi^\alpha(\xi^\alpha), \delta))$$

with δ individual constant in the atomic system of B_1, is such that, with respect to the den^* where $den^*(\wedge_{E,1})$ is as indicated in Section 5.2.4.5,

$$den(\wedge_{E,1}(\wedge I(\rightarrow I\xi^{\alpha}(\xi^{\alpha}), \delta))$$

is a universal ground - where *den* is the denotation function for terms of Λ associated with den^{*}. Indeed,

$$den(\wedge_{E,1}(\wedge I(\rightarrow I\xi^{\alpha}(\xi^{\alpha}), \delta))) = \rightarrow I\xi^{\alpha}(\mathtt{Id}(\xi^{\alpha})).$$

But the term is not universal with respect to den^{*}. The constant δ involved in it will not denote a ground on a *logical* base B_2.

Given a *logical* base B_1 on a language for first-order arithmetic as in Section 2.5.1, and an appropriate language of grounding Λ_1 on B_1 that has in its alphabet a non-primitive operational symbol F of operational type τ

$$s(s(0)) \doteq s(0) + s(0) \rhd \neg\exists x(0 \doteq s(x))$$

the term

$$\wedge_{E,1}(\wedge I(\rightarrow I\xi^{\alpha}(\xi^{\alpha}), \rightarrow I\xi^{\tau}(F(\xi^{\tau})))$$

is such that, with respect to the den_1^{*} where $den_1^{*}(\wedge_{E,1})$ is as indicated in Section 5.2.4.5, and $den_1^{*}(F)$ is the empty function on the intended operational type,

$$den_1(\wedge_{E,1}(\wedge I(\rightarrow I\xi^{\alpha}(\xi^{\alpha}), \rightarrow I\xi^{\tau}(F(\xi^{\tau}))))$$

is a universal ground - where den_1 is the denotation function for terms of Λ_1 associated with den_1^{*}. Indeed, we have here exactly the same situation as in the previous example,

$$den_1(\wedge_{E,1}(\wedge I(\rightarrow I\xi^{\alpha}(\xi^{\alpha}), \rightarrow I\xi^{\tau}(F(\xi^{\tau}))))) = \rightarrow I\xi^{\alpha}(\mathtt{Id}(\xi^{\alpha})).$$

But the term is not universal with respect to den_1^{*}. Given a base B_2 on the same background language, with Post-system $\mathtt{S}$ for first-order arithmetic as in Section 2.5.1, the empty function on the intended operational type is not admissible on B_2, since $\vdash_{\mathtt{S}} s(s(0)) \doteq s(0) + s(0)$.

Since the fact of denoting grounds or operations on grounds is a necessary, but not sufficient condition for a term to be universal, one may wonder if there is a necessary and sufficient condition that, in terms of denotation, expresses the universality of a term. The following theorem is meant to answer this question.

Theorem 55. Let B_1 be a base on L with atomic system $\mathtt{S}$, Λ be a language of grounding on B_1, and den^* be a denotation function for the elements of the alphabet of Λ. $T \in \mathtt{TERM}_\Lambda$ is universal with respect to den^* if, and only if,

a) $S(T) \cap \mathtt{DER}_\mathtt{S} = \emptyset$ - namely, T does not involve individual constants - and

b) for every non-primitive operational symbol F of Λ occurring in T, $den^*(F)$ is a universal ground.

Proof. ($\Longrightarrow$) Let T be universal with respect to den^*. Let us prove a). Definition 52 implies that, given a logical base B_2 on L, every element x of the alphabet of Λ occurring in T denotes a ground on B_2. But if it was $S(T) \cap \mathtt{DER}_\mathtt{S} \neq \emptyset$, this could not happen since, being B_2 logical, an individual constant occurring in T could not denote a ground on B_2. Hence, $S(T) \cap \mathtt{DER}_\mathtt{S} = \emptyset$. The other point, b), is trivial. Again by definition 52, indeed, we know that for every element x of the alphabet of Λ, $den^*(x)$ is a universal ground. But then, for every non-primitive operational symbol F of T, $den^*(F)$ is a universal ground, and the result is therefore proven.
($\Longleftarrow$) Let now $S(T) \cap \mathtt{DER}_\mathtt{S} = \emptyset$ and, for every non-primitive operational symbol F of Λ_1 occurring in T, let $den^*(F)$ be a universal ground. This means that T only contains ground-variables, primitive operational symbols, and non-primitive operational symbols. And it is clear that the images of ground-variables and primitive operational symbols through den^* are universal grounds. The result is therefore proven. □

Corollary 56. Let Λ be a language of grounding on a base B, and den^* be a denotation function for elements of the alphabet of Λ. $T \in \mathtt{TERM}_\Lambda$ is universal with respect to den^* if, and only if, for every $U \in S(T)$, U is universal with respect to den^*.

Proof. Immediate from theorem 55. □

To conclude this section, let us make a remark. To say that a ground on a base is universal means that, for all the operations on grounds involved in it, the equations that define these operations on that base, define operations on grounds of the same operational type on whatever base. It is, in a way, a kind of intensional approach; the operations on ground are equal on all bases in the sense of being defined always by the same equations. An analogous extensional approach does not seem to make sense; an operation on grounds may in fact have different domains and codomains on different bases, even if defined by the same equations - think of an operation on grounds for the elimination of conjunction from $p_1 \wedge p_2$ to p_i $(i = 1, 2)$ defined in a standard way, on a logical base, and then on a base having p_i as axiom in the atomic system. The notion of universality is the only point of our investigation where we adopt an intensional standpoint since, as said, we will understand in general equality of functions in an extensional way.

5.2.5 Primitiveness and conservativity

In introducing and discussing languages of grounding, we have so far proposed a single classification parameter; a grounding language can be or cannot be closed under canonical form - with respect to a denotation function of the elements of the alphabet. This feature is not very stringent because, as we have proven, any language of grounding can be extended to one that is closed in a canonical form with respect to an expansion of a given denotation. In this last section, we propose on the contrary to discuss and exemplify two deeper classification criteria of languages of grounding; a language can be the expansion of another, in a primitive/non-primitive, and conservative/non-conservative sense. Starting with primitiveness, it is fixed by the following, simple definition.

Definition 57. Let Λ_1 be a language of grounding on an atomic base B_1, and let Λ_2 be an expansion of Λ_1 on an atomic base B_2. We will then say that Λ_2 is a *primitive* expansion of Λ_1 if, and only if, B_2 is a proper expansion of B_1.

Thus, a language of grounding expands another one primitively, when it is relative to an atomic base that is a proper expansion of the original one. Instead, the expansion will be non-primitive when the only new linguistic resources are non-primitive operational symbols. In this regard, it seems appropriate to make two observations.

First of all, why do we call "primitive" an expansion that acts on an atomic base that contains strictly the starting one? In order to answer this question, we must bear in mind convention 24, according to which the system of an atomic base must totally interpret the background language. According to the way we have set up our discussion, it follows that, if an atomic base of an incoming language is the proper expansion of the atomic base of a starting language, the incoming Post-system is the proper expansion of the starting Post-system. In fact, given a base B_1

$$\langle \mathtt{R}, \mathtt{F}, \mathtt{C}, \mathtt{S} \rangle,$$

suppose that a proper expansion B_2 of its *does not* expand properly $\mathtt{S}$; if B_2 is a proper expansion, and if $\mathtt{S}$ remains unchanged, B_2 will have to involve proper super-sets of at least one of the components of the background language of B_1, namely, it will have to be of the type

$$\langle \mathtt{R}^+, \mathtt{F}^+, \mathtt{C}^+, \mathtt{S} \rangle,$$

with $\mathtt{R} \subset \mathtt{R}^+$ or $\mathtt{F} \subset \mathtt{F}^+$ or $\mathtt{C} \subset \mathtt{C}^+$. But then, the background language of B_2 is a proper expansion of the background language of B_1. And since $\mathtt{S}$ is an atomic system for the background language of B_1, from definition 21 it follows that the background language of B_2 is the proper expansion of the language of $\mathtt{S}$. But $\mathtt{S}$ is also the atomic system of B_2, so the background language of B_2 is only partially interpreted by $\mathtt{S}$. Therefore, a language of grounding on B_2 would violate convention 24.

That being said, it is easier to understand in what sense a primitive expansion of a language of grounding is, actually, primitive. The expansion is relative to an atomic base that, besides acting on a background language possibly enriched, it involves a more powerful atomic system; hence, there will be *new* atomic derivations, and therefore, in the expansion, *new* individual constants which, by definition, constitute grounds for judgments or assertions involving atomic formulas. These constants are primitive elements in so far as they can be neither reduced to elements of the starting language of grounding, nor defined in such terms; they are, so to speak, new axioms, which cannot be further justified, and which instead contribute to the determination of meaning. This is not the case with the non-primitive operational symbols, which, as stated, must be defined in terms of equations which fix their behavior in a harmonious way with respect to the determination of meaning established by the clauses (At_G) - $(\exists_G)$.

At this point, in a correct and natural way, it could be argued that our definition of primitive expansion is too restricted. Indeed, if the primitiveness depends on the addition of elements non-reducible, not further justifiable, undefinable, which contribute to the determination of meaning, why do we not take into account also expansions, the primitiveness of which depends on the addition of *new primitive operational symbols*, related to new *logical constants*? This would require taking into account, in addition to the set of relational and functional symbols, and of the individual constants that constitute the atomic base, also the set $\Im$ of the logical constants of the background language, i.e.

$$\langle \mathtt{R}, \mathtt{F}, \mathtt{C}, \mathtt{S} \rangle \text{ and } \Im.$$

Therefore, a proper expansion of a language of grounding on such a base would be a language of grounding on an atomic base and, in addition, on a new set of logical constants

$$\langle \mathtt{R}^+, \mathtt{F}^+, \mathtt{C}^+, \mathtt{S}^+ \rangle \text{ and } \Im^+.$$

with $\mathtt{S} \subset \mathtt{S}^+$, and $\mathtt{R} \subset \mathtt{R}^+$ or $\mathtt{F} \subset \mathtt{F}^+$ or $\mathtt{C} \subset \mathtt{C}^+$ or $\Im \subset \Im^+$. Now, such a strategy is undoubtedly feasible, correct, and probably full of interesting consequences; however, we will not deal with it below - except for an example at the end of this section. The reasons why we leave it out, in addition to those obvious of space and time, are essentially two. If the new primitive operational symbols are related to logical non-modal constants that do not determine the passage to orders higher than the first, the primitive expansions are certainly possible, but not particularly significant. Indeed, it has been shown (Prawitz 1979, Schroeder-Heister 1984) that the usual intuitionistic first-order logical constants of first order - those of which we have dealt with so far - are functionally complete with respect to all the possible first-order intuitionistic logical constants, and therefore sufficient to express them. On the other hand, if the new primitive operational symbols are relative to logical constants of orders higher than the first - for example, second-order quantifiers - an approach such as that of the theory of grounds, inserted in the verificationist tradition, could encounter problems related to Dummett's molecularity requirement (see Cozzo 1994b), or to phenomena of loss of compositionality or paradoxicality (see Pistone 2015).

Now to the notion of conservativeness. As seen, Gödel's incompleteness implies that not all the grounds for judgments or assertions on a language for first-order arithmetic can be expressed in a language of grounding for Heyting's first-order arithmetic. Expansions in which these grounds on the contrary can be constructed, can then be considered as non-conservative expansions. The next definition is nothing more than a generalization of this idea.

Definition 58. Let Λ_1 be a language of grounding on a background language L, let Λ_2 be an expansion of Λ_1 on an atomic base B on an expansion L^+ of L, let den^* be a denotation function for the elements of the alphabet of Λ_2, and let den be the denotation function for the terms of Λ_2 associated with den^*. Λ_2 is *conservative with respect to* den^* *on* Λ_1 if, and only if, for every $T : \alpha \in \mathtt{TERM}_{\Lambda_2}$ with $\alpha \in \mathtt{FORM}_L$, there is $U \in \mathtt{TERM}_{\Lambda_1}$ such that $den(T) = den(U)$.

An expansion is therefore conservative if all the objects related to formulas or operational types of the starting background language, denoted by some of its terms, were already denoted by some term in the not expanded language. We now want to make three observations.

First of all, the notion of conservativeness introduced here is different from that used for theories - given a logical language L, a formal system Σ on L, and an expansion Σ^+ of Σ, we say that Σ^+ is conservative on Σ if, and only if, for each finite $\Gamma \subset \texttt{FORM}_L$, for every $\alpha \in \texttt{FORM}_L$, if $\Gamma \vdash_{\Sigma^+} \alpha$, then $\Gamma \vdash_{\Sigma} \alpha$. Indeed, it is not simply required that all that is provable in the expansion, related to formulas of the original language, is provable also in the unexpanded language; in other words, it is not required that, for each term $T : \alpha \in \texttt{TERM}_{\Lambda_2}$ with an empty set of ground-variables, and for $\alpha \in \texttt{FORM}_L$, there is $U : \alpha \in \texttt{TERM}_{\Lambda_1}$ with an empty set of ground-variables. This is not conservativeness of provability, but conservativeness of denotation; all the deductive means - grounds and operations on grounds - relative to a fixed atomic base, and denoted in a conservative expansion - conceivable, via Curry-Howard's isomorphism, as a formal system - are already available in the unexpanded grounding language – also conceivable, via Curry-Howard isomorphism, as a formal system. It is easy to prove that the first type of conservativeness, the standard one, is implied by the second; the opposite, however, might not apply.

As a second point, note that, as in the case of universality, if we adopt also here an invariant denotation approach, we do not need to relativize the notion of conservativeness to a specific denotation function for the elements of the alphabet.

Finally, given Λ_2 conservative with respect to a certain den^* - to which it is associated a certain den - on Λ_1, and said B_1 and B_2 the bases of, respectively, Λ_1 and Λ_2, it applies what follows. Each ground on B_2 for judgments or assertions relative to formulas of the background language of Λ_2, denoted by some term of Λ_2 via den, must be, more specifically, a ground on B_1; in the same way, each B_2-operation on grounds of which the operational type involves only formulas of the background language of Λ_1, denoted by some term of Λ_2 via den, must, more specifically, be equal to a B_1-operation on grounds. If $B_1 = B_2$, the observation is trivial. If vice versa B_2 is a proper expansion of B_1, the conservativeness of Λ_2 on Λ_2 implies that: B_2 does not "add" any new ground, nor operations on grounds extensionally different, compared to grounds and operations on grounds already available in B_1 for the language which B_1 refers to or; there are grounds on B_2 or B_2-operations on grounds which do not correspond to any term of Λ_2.

To conclude our discussion, we implement our definitions, by providing examples that show how the concepts of primitiveness and conservativeness are not extensionally equivalent.

Example 1 (Non-primitive non-conservative expansion)
The expansion `Gen` of a core language `G` on whatever atomic base B for a first-order language - namely, a Gentzen language as displayed in Section 5.2.3.3 - is obviously non-primitive; however, it is non-conservative with respect to the denotation function den^* of Section 5.2.4.5. For example, the term

$$\to I\xi^{\alpha}(\to I\xi^{\alpha\to\beta}(\to E(\xi^{\alpha\to\beta}, \xi^{\alpha}))$$

is clearly a ground on B for $\vdash \alpha \to ((\alpha \to \beta) \to \beta)$ not denoted by any term of `G`. More in particular, the term

$$\to E(\xi^{\alpha\to\beta}, \xi^{\alpha})$$

denotes a B-operation on grounds not denoted by any term of `G`; in fact, β is not derivable from $\alpha \to \beta$ and α in a formal system that contains only Gentzen's introduction rules.

Examples 2 (Non-primitive conservative expansion)
Given `Gen` on whatever atomic base B, let us take into account the following expansion $\texttt{Gen}^+$.

Definition 59. The language $\texttt{Gen}^+$ is specified by an alphabet which contains that of `Gen` plus

- operational symbols F of operational type $\langle \ddagger \rangle$:
  - $F\langle \alpha \vee \beta, \neg\alpha \rhd \beta \rangle$

The set $\texttt{TERM}_{\texttt{Gen}^+}$ of the *terms of* $\texttt{Gen}^+$ is the smallest set X such that

- $\texttt{TERM}_{\texttt{Gen}} \subset X$
- $T : \alpha \vee \beta, U : \neg\alpha \in X \Rightarrow F(T, U) : \beta \in X$.

Then, with regard to the atomic base B, let us consider a denotation function den^* for the elements of the alphabet of $\texttt{Gen}^+$ which, on non-primitive operational symbols of `Gen`, behaves as indicated in Section 5.2.4.5, and that, for α and β closed, is such that

$$den^*(F\langle \alpha \vee \beta, \neg\alpha \rhd \beta \rangle)$$

is the operation

$$f(\xi^{\alpha\vee\beta}, \xi^{\neg\alpha})$$

such that, for every proper ground $\vee I(g)$ on B for $\vdash \alpha \vee \beta$, for every proper ground u on B for $\vdash \neg\alpha$,

$$f(\vee I(g), u) = g.$$

The latter is a total constructive function which, from every proper ground on B for $\vdash \alpha \vee \beta$, from every proper ground on B for $\vdash \neg\alpha$, produces a proper ground on B for $\vdash \beta$, and is hence a B-operation on grounds of operational type

$$\alpha \vee \beta, \neg\alpha \rhd \beta;$$

to prove it: given $\vee I(G_1)$ proper ground on B for $\vdash \alpha \vee \beta$, by virtue of the clauses and of the identity conditions on $\vee I$, G_1 is a proper ground on B for $\vdash \alpha$ or a proper ground on B for $\vdash \beta$. Likewise, given a proper ground G_2 on B for $\vdash \neg\alpha$, G_2 is of the form $\rightarrow I\xi^{\alpha}(h(\xi^{\alpha}))$ and, by virtue of the clauses and of the identity conditions on $\rightarrow I$, $h(\xi^{\alpha})$ is a proper ground on B for $\alpha \vdash \bot$. If now G_1 was a proper ground on B for $\vdash \alpha$, $h(G_1)$ would be a proper ground on B for $\vdash \bot$, and this is impossible by virtue of the clause on $\bot$; it follows that G_1 must be a proper ground on B for $\vdash \beta$. The result now follows immediately from an application of the defining equation, namely

$$f(\vee I(G_1), G_2) = G_1.$$

$\texttt{Gen}^+$ is clearly a non-primitive expansion of $\texttt{Gen}$; additionally, it is conservative with respect to den^* on $\texttt{Gen}$. To make this clear, let us show that the non-primitive operational symbol $F\langle \alpha \vee \beta, \neg\alpha \rhd \beta \rangle$ can be "rewritten" in the language $\texttt{Gen}$ by leaving unchanged the denotation function den for the terms of $\texttt{Gen}^+$ associated with den^*; more in particular, let us exhibit an open term $T \in \texttt{TERM}_{\texttt{Gen}}$ such that

$$den^*(F\langle \alpha \vee \beta, \neg\alpha \rhd \beta \rangle) = den(F(\xi^{\alpha \vee \beta}, \xi^{\neg\alpha})) = den(T)$$

- observe that the first identity holds in general, since F does not bind ground-variables. However, we want beforehand to observe that this strategy, strictly speaking, does not consist in showing that all the grounds on B and all the B-operations on grounds expressible in $\texttt{Gen}^+$ via den can already be expressed in $\texttt{Gen}$ via den; the suggested procedure is actually more reminiscent of the one by which it is shown that an inference rule in a certain formal system is derivable in a subsystem of the latter. In light of a result that we will soon demonstrate, however, what is achieved actually implies the conservativeness of $\texttt{Gen}^+$ with respect to den^* on $\texttt{Gen}$. Well, let us take into account the following term of $\texttt{Gen}$:

$$\vee E\ \xi^{\alpha}\ \xi^{\beta}(\xi^{\alpha\vee\beta}, \bot_{\beta}(\rightarrow E(\xi^{\alpha}, \xi^{\neg\alpha})), \xi^{\beta})$$

that by the denotation theorem - but also in the light of a simple computation - denotes via *den* a B-operation on grounds

$$f_2(\xi^{\alpha\vee\beta}, \xi^{\neg\alpha})$$

of operational type

$$\alpha \vee \beta, \neg\alpha \rhd \beta.$$

Let us now show the extensional identity

$$f_1(\xi^{\alpha\vee\beta}, \xi^{\neg\alpha}) = f_2(\xi^{\alpha\vee\beta}, \xi^{\neg\alpha})$$

that is, knowing that, for α and β closed, for every $\vee I(g)$ proper ground on B for $\vdash \alpha \vee \beta$, where g is a proper ground on B for $\vdash \beta$, for every u proper ground on B for $\vdash \neg\alpha$, we have

$$den(F(\xi^{\alpha\vee\beta}, \xi^{\neg\alpha}))(\vee I(g), u) = den^*(F\langle\alpha \vee \beta, \neg\alpha \rhd \beta\rangle)(\vee I(g), u) = \\ f_1(\vee I(g), u) = g$$

we also have

$$den(\vee E\ \xi^{\alpha}\ \xi^{\beta}(\xi^{\alpha\vee\beta}, \bot_{\beta}(\rightarrow E(\xi^{\alpha}, \xi^{\neg\alpha})), \xi^{\beta}))(\vee I(g), u) = g.$$

Taking into account the definition of the denotation for terms, the behavior of the operations on grounds, and finally the equations that define the operations associated by den^* with $\vee E$, $\bot_{\beta}$ and $\rightarrow E$ shown in Section 5.2.4.5, we will have the following computation - where we leave out some obvious passages:

$$den(\vee E\ \xi^{\alpha}\ \xi^{\beta}(\xi^{\alpha\vee\beta}, \bot_{\beta}(\rightarrow E(\xi^{\alpha}, \xi^{\neg\alpha})), \xi^{\beta}))(\vee I(g), u) = \\ den^*(\vee E)(den^*(\xi^{\alpha\vee\beta}), den^*(\bot_{\beta})(den^*(\rightarrow E)(\xi^{\alpha}, u))), den^*(\xi^{\beta}))(\vee I(g), u) = \\ den^*(\vee E)(\vee I(g), den^*(\bot_{\beta})(den^*(\rightarrow E)(\xi^{\alpha}, u))), den^*(\xi^{\beta})) = \\ den^*(\vee E)(\vee I(g), den^*(\bot_{\beta})(den^*(\rightarrow E)(\xi^{\alpha}, u))), \mathtt{Id}(\xi^{\beta})) = \mathtt{Id}(g) = g.$$

As a more general result, we prove the following theorem.

Theorem 60. Let Λ_1 be a language of grounding on an atomic base B, Λ_2 be a non-primitive expansion of Λ_1, den^* be a denotation function for elements of the alphabet of Λ_2, *den* the denotation function for terms of Λ_2 associated with den^*. Let us moreover suppose that, for every non-primitive operational symbol F of Λ_2, there is a composition

$$h = f_1 \circ \cdots \circ f_n$$

(for some $i \in \mathbb{N}$) of B-operations on grounds such that

- for every $i \leq n$, there is an operational symbol F_i of Λ_1 such that $f_i = den^*(F_i)$
- $den^*(F) = h$.

Then, Λ_2 is conservative with respect to den^* on Λ_1.

Proof. By induction on the complexity of $T \in \mathtt{TERM}_{\Lambda_2}$:

- the atomic case is trivial by non-primitivity of Λ_2 with respect to Λ_1;
- without loss of generality, let us consider the term

$$F\ \xi^\alpha\ x\ (U).$$

By denotation theorem, $den(U)$ is a ground on B for some appropriate judgment or assertion. By induction hypothesis, moreover, there is $Z \in \mathtt{TERM}_{\Lambda_1}$ of the same type as U (not necessarily distinct from U) such that $den(U) = den(Z)$. If F is an operational symbol of Λ_1, the required term will be

$$F\ \xi^\alpha\ x\ (Z)$$

which, as easily seen, has the same type as the starting term. Indeed - possibly under substitution $(*/\star)$ of free individual variables with individuals, and of free ground-variables with grounds on B for the closed formulas thereby obtained - we will have the computation

$$den(F\ \xi^\alpha\ x\ (U))(*/\star) = den^*(F)(den(U)(*/\star)) = \\ den^*(F)(den(Z)(*/\star)) = den(F\ \xi^\alpha\ x\ (Z))(*/\star)$$

whence, for the arbitrariness of the eventual substitution $(*/\star)$, we conclude that

$$den(F\ \xi^\alpha\ x\ (U)) = den(F\ \xi^\alpha\ x\ (Z)).$$

If instead F is a primitive operational symbol, or if it is not a non-primitive operational symbol of Λ_1, by the hypotheses of the theorem we know that there exists a B-operation on grounds h resulting from the composition of B-operations on grounds $f_1, ..., f_n$ (for some $n \in \mathbb{N}$) for which it holds that, for every $i \leq n$, $f_i = den^*(F_i)$, with F_i operational symbol of Λ_1, and moreover $den^*(F) = h$. Let us apply - which is permitted - the composite h to $den(Z)$; by replacing each f_i with the corresponding F_i, we will have a term Λ_1 with Z as a subterm, which we indicate with

$$F_1, ..., F_n(Z).$$

Well - possibly under substitution $(*/\star)$ of free individual variables with individuals, and of free ground-variables with grounds on B for the closed formulas thereby obtained, and under appropriate substitutions of $den(U)$ and $den(Z)$ for variables in the composition of $f_1, ..., f_n$ - we have the computation

$$den(F\ \xi^\alpha\ x\ (U))(*/\star) = den^*(F)(den(U)(*/\star)) = h(den(U)(*/\star)) =$$
$$f_1 \circ \cdots \circ f_n(den(U)(*/\star)) = f_1 \circ \cdots \circ f_n(den(Z)(*/\star)) =$$
$$den^*(F_1), ..., den^*(F_n)(den(Z)(*/\star)) = den(F_1, ..., F_n(Z))(*/\star)$$

whence, for the arbitrariness of the eventual substitution $(*/\star)$, we again conclude that

$$den(F\ \xi^\alpha\ x\ (U)) = den(F_1, ..., F_n(Z)).$$

The result is hence proven.[8] □

[8] Theorem 60 authorizes an observation in our opinion interesting. Suppose that, to each language of grounding Λ on an atomic base B on a background language L, it is always possible to associate, via Curry-Howard isomorphism, a formal system Σ_Λ on L which includes the atomic system of B. We then indicate the standard conservativeness among theories, though extending it to the case of admissibility of rules under substitution, as *conservativeness on provability*, and indicate the conservativeness we set in definition 58 as a *conservativeness on denotation*. The following result applies. Given a language of grounding Λ_1 on an atomic base B, and given a non-primitive expansion Λ_2 of Λ_2, there exists a denotation function den^* of the elements of the alphabet of Λ_2 such that Λ_2 is a conservative expansion on the denotation of Λ_1 with respect to den^* if, and only if, Σ_{Λ_2} is a conservative expansion on the provability of Σ_{Λ_1}. The left-right direction of the equivalence is a rather immediate consequence of theorem 60; it is sufficient to choose as den^* the function that associates to each non-primitive operational symbol F of Λ_2

Note that if a non-primitive operational symbol F of Λ_2 does not bind either individual or ground-variables, we can, as in the case of disjunctive syllogism in $\texttt{Gen}^+$, simply require that Λ_1 contains an open term, of which the free individual variables are all and only those that occur in the operational type of F, and of which the free ground-variables are all and only those the type of which occur in the operational type of F. However, as soon as F binds at least on individual or ground-variable, this strategy can no longer work, as operations on grounds include operations on grounds in the domain of their operational type that are not denoted by any term, since languages of grounding do not have ground-variables of type $\Gamma \rhd \alpha$ (see Prawitz 2015, 93). Note also that variables of type $\Gamma \rhd \alpha$ have been used by us at the level of B-operations on grounds, and not at the level of terms. Incidentally, therefore, an obvious way to remedy these difficulties, which we will not deal with here, and which we will only deal partially in the next chapter, would be to enrich languages of grounding with ground-variables $\xi^{\Gamma \rhd \alpha}$, meant to range over operations on grounds of appropriate operational type.

We conclude with two quick examples of primitive non-conservative and of primitive conservative expansion, respectively. Abandoning for a while the limitation to the first-order background languages adopted so far, let us consider the possibility, mentioned above, of taking into account not just the bases, but

$$\langle \texttt{R}, \texttt{F}, \texttt{C}, \texttt{S} \rangle \text{ and } \Im$$

where $\Im$ is the set of the logical constants of the background language.

Example 3 (Primitive non-conservative expansion)
Given an atomic base B on a language L with a set of logical constants $\Im$ for first-order arithmetic with Post-system as in the Section 2.5.1, it is known that there exists an expansion B^+ of B on an expansion L^+ of L in which, for an appropriate $\Im^+$ we can express a reflection principle allowing to prove Gödel's formula G.

a coding of the derivation of the operational type of F in Σ_{Λ_1}. The left-right direction is obvious in the case of closed terms, and in the case of open terms that correspond to the denotation of primitive operational symbols that do not bind individual or ground-variables. In the case instead of non-primitive operational symbols that bind individual or ground-variables, we know that each of their applications, namely that each instance of the corresponding rule on specific terms or specific derivations, corresponds to a derivation in Σ_{Λ_1}, which guarantees the admissibility of the rule under substitutions in Σ_{Λ_1}. A true derivability of rules of this type of Σ_{Λ_2} in Σ_{Λ_1} might not be generally available.

Given the language of grounding GHA on B for Heyting's first-order arithmetic, let us build a primitive expansion $\texttt{GHA}^+$ on B^+ with terms representing the reflection principle; for an appropriate denotation function den^* for the elements of the alphabet of $\texttt{GHA}^+$, and an appropriate denotation function den for the terms of $\texttt{GHA}^+$ associated with den^*, $\texttt{GHA}^+$ is non-conservative, since there is a closed term $T : G$, such that $den(T)$ is a ground on B^+ for $\vdash G$ - and such a term was, by virtue of Gödel's incompleteness term, absent in GHA.

Example 4 (Primitive conservative expansion)
Let Gen be a Gentzen-language as in Section 5.2.3.3 on whatever atomic base B on a first-order logical language L with set of individual constants

$$\Im = \{\wedge, \vee, \rightarrow, \forall, \exists\}.$$

Let us consider a primitive expansion on an atomic base B^+ on an expansion L^+ of L which is identical to L except for the set of the logical constants, the latter being

$$\Im^+ = \Im \cup \{\leftrightarrow\}.$$

Definition 61. The languagr $\texttt{Gen}^+$ is specified by an alphabet which contains that of Gen plus

- operational symbols F of operational type $\langle \ddagger \rangle$:
  - $\leftrightarrow I\langle \alpha_1, \alpha_2 \rhd \alpha_1 \leftrightarrow \alpha_2 \rangle$
  - $\leftrightarrow E\langle \alpha_1 \leftrightarrow \alpha_2, \alpha_i \rhd \alpha_j \rangle$ $(i, j = 1, 2,\ i \neq j)$

The set $\texttt{TERM}_{\texttt{Gen}^+}$ of the *terms* of $\texttt{Gen}^+$ is the smallest set X such that

- $\texttt{TERM}_{\texttt{Gen}} \subset X$
- $T : \alpha_1, U : \alpha_2 \in X \Rightarrow \leftrightarrow I\ \xi^{\alpha_1}\ \xi^{\alpha_2}\ (T, U) : \alpha_1 \leftrightarrow \alpha_2 \in X$
- $T : \alpha_1 \leftrightarrow \alpha_2, U : \alpha_i \in X \Rightarrow \leftrightarrow E(T, U) : \alpha_j \in X$.

The primitive operational symbol $\leftrightarrow I$ is used in a clause that fixes what counts as a ground on whatever atomic base B for categorical judgment or assertions $\vdash \alpha_1 \leftrightarrow \alpha_2$, that is

($\leftrightarrow_G$) $f_1(\xi^{\alpha_1})$ is a ground on B for $\alpha_1 \vdash \alpha_2$ and $f_2(\xi^{\alpha_2})$ is a ground on B for $\alpha_2 \vdash \alpha_1 \Leftrightarrow \leftrightarrow I\ \xi^{\alpha_1}\ \xi^{\alpha_2}\ (f_1(\xi^{\alpha_1}), h(\xi^{\alpha_2}))$ is a ground on B for $\vdash \alpha_1 \leftrightarrow \alpha_2$.

to which we must add the identity condition

$$f_i(\xi^{\alpha_i}) = h_i(\xi^{\alpha_i})\ (i = 1, 2)$$
$$\Leftrightarrow \leftrightarrow I\ \xi^{\alpha_1}\ \xi^{\alpha_2}\ (f_1(\xi^{\alpha_1}), f_2(\xi^{\alpha_2})) = \leftrightarrow I\ \xi^{\alpha_1}\ \xi^{\alpha_2}\ (h_1(\xi^{\alpha_1}), h_2(\xi^{\alpha_2})).$$

Thus, the primitive operational symbol $\leftrightarrow I$ can be matched by default a B-operation on grounds of operational type

$$(\alpha_1 \rhd \alpha_2), (\alpha_2 \rhd \alpha_1) \rhd \alpha_1 \leftrightarrow \alpha_2.$$

Let us now consider a denotation function for the elements of the alphabet of `Gen`$^+$ which, on the non-primitive operational symbols of `Gen` behaves as indicated in Section 5.2.3.3, and such that, instead, for α_1, α_2 closed,

$$den^*(\leftrightarrow E\langle \alpha_1 \leftrightarrow \alpha_2, \alpha_i \rhd \alpha_j \rangle)$$

is the operation

$$f(\xi^{\alpha_1 \leftrightarrow \alpha_2}, \xi^{\alpha_i})$$

of operational type

$$\alpha_1 \leftrightarrow \alpha_2, \alpha_i \rhd \alpha_j$$

such that, for every $\leftrightarrow I\ \xi^{\alpha_1}\ \xi^{\alpha_2}\ (f_1(\xi^{\alpha_1}), f_2(\xi^{\alpha_2}))$ proper ground for $\vdash \alpha_1 \leftrightarrow \alpha_2$, for every g proper ground for $\vdash \alpha_i$,

$$f(\leftrightarrow I\ \xi^{\alpha_1}\ \xi^{\alpha_2}\ (f_1(\xi^{\alpha_1}), f_2(\xi^{\alpha_2})), g) = f_i(g)$$

The operation is an operation on grounds of the indicated operational type. Indeed, let us suppose that G_1 is a proper ground on B for $\vdash \alpha_1 \leftrightarrow \alpha_2$. By virtue of the clause ($\leftrightarrow_G$), G_1 is unique and is of the form

$$\leftrightarrow I\ \xi^{\alpha_1}\ \xi^{\alpha_2}\ (f_1(\xi^{\alpha_1}), f_2(\xi^{\alpha_2}))$$

with $f_i(\xi^{\alpha_i})$ proper ground on B for $\alpha_i \vdash \alpha_j$. Then, for every proper ground g on B for $\alpha_i \vdash \alpha_j$, $f_i(g)$ is a proper ground on B for $\vdash \alpha_j$. If we now suppose that G_2 is a proper ground on B for $\vdash \alpha_i$, $f_i(G_2)$ is a proper ground on B for $\vdash \alpha_j$, so that the computation

$$f(\leftrightarrow I\ \xi^{\alpha_1}\ \xi^{\alpha_2}\ (f_1(\xi^{\alpha_1}), f_2(\xi^{\alpha_2}), G_2) = f_i(G_2)$$

shows what we required.

Now, $\texttt{Gen}^+$ can be understood as a Gentzen's natural deduction system $\texttt{IL}^+$ for first-order intuitionistic logic, to which we have added the introduction rule

$$\dfrac{\begin{matrix}[\alpha_1] & & [\alpha_2] \\ \vdots & & \vdots \\ \alpha_2 & & \alpha_1\end{matrix}}{\alpha_1 \leftrightarrow \alpha_2}\ (\leftrightarrow_I)$$

and the elimination rules

$$\dfrac{\alpha_1 \leftrightarrow \alpha_2 \qquad \alpha_i}{\alpha_j}\ (\leftrightarrow_E),\ i, j = 1, 2,\ i \neq j$$

The bijection to be taken into account works, on the rules of first-order intuitionistic logic, as indicated in Section 5.2.3.3, and on the additional rules it works as follows:

$$\dfrac{\begin{matrix}[\alpha_1] & & [\alpha_2] \\ \Delta_1 & & \Delta_2 \\ \alpha_2 & & \alpha_1\end{matrix}}{\alpha_1 \leftrightarrow \alpha_2}\ (\leftrightarrow_I) \quad \overset{\iota}{\Longrightarrow} \quad \leftrightarrow I\ \xi^{\alpha_1}\ \xi^{\alpha_2}\ (\iota(\Delta_1), \iota(\Delta_2))$$

$$\dfrac{\begin{matrix}\Delta_1 & & \Delta_2 \\ \alpha_1 \leftrightarrow \alpha_2 & & \alpha_i\end{matrix}}{\alpha_j}\ (\leftrightarrow_E) \quad \overset{\iota}{\Longrightarrow} \quad \leftrightarrow E(\iota(\Delta_1), \iota(\Delta_2))$$

The denotation function den^* for the elements of the alphabet of $\texttt{Gen}^+$ we choose, instead, associates Prawitz's standard reductions to the standard elimination rules, and to the rules $(\leftrightarrow_E)$ it associates the reduction

$$\dfrac{\dfrac{\begin{matrix}[\alpha_1] & & [\alpha_2] \\ \Delta_1 & & \Delta_2 \\ \alpha_2 & & \alpha_1\end{matrix}}{\alpha_1 \leftrightarrow \alpha_2}\ (\leftrightarrow_I) \qquad \begin{matrix}\Delta_3 \\ \alpha_i\end{matrix}}{\alpha_j}\ (\leftrightarrow_E) \qquad \leftrightarrow_j\text{-rid} \qquad \begin{matrix}\Delta_3 \\ [\alpha_i] \\ \Delta_i \\ \alpha_j\end{matrix}$$

In the light of the reductions thus defined, it becomes possible to prove on $\texttt{IL}^+$ a normalization theorem, with a subformula principle by virtue of which every derivation in $\texttt{DER}_{\texttt{IL}^+}$ such that

$$\alpha_1, ..., \alpha_n \vdash_{\texttt{IL}^+} \beta$$

with $\alpha_1, ..., \alpha_n, \beta \in \mathtt{FORM}_L$ can be transformed - by means of reductions and/or expansion - into a derivation in $\mathtt{DER}_{\mathtt{IL}}$ such that

$$\alpha_1, ..., \alpha_n \vdash_{\mathtt{IL}} \beta.$$

Through the bijection we mentioned above, and by defining appropriate reduction or expansion relations among terms of $\mathtt{Gen}^+$, we will hence have that every $T : \beta \in \mathtt{TERM}_{\mathtt{Gen}^+}$ with

$$FV^I(T) = \{x_1, ..., x_n\} \text{ and } FV^T(T) = \{\xi^{\alpha_1}, ..., \xi^{\alpha_m}\}$$

can be transformed into a term $U : \beta \in \mathtt{TERM}_{\mathtt{Gen}}$ with

$$FV^I(U) = \{x_1, ..., x_n\} \text{ and } FV^T(U) = \{\xi^{\alpha_1}, ..., \xi^{\alpha_m}\}$$

and since appropriate reductions and expansions permit to preserve denotation, we will have that $den(T) = den(U)$. Observe that $den(T)$ is a ground on B^+ which, in the light of what has been shown, is more in particular a ground on B. Therefore, although primitive, $\mathtt{Gen}^+$ is a conservative expansion of $\mathtt{Gen}$.

The concepts of primitiveness and conservativeness, therefore, even if overlapping, are not equivalent, nor is any of the two contained in the other. Primitiveness concerns the presence or absence of new linguistic resources that modify or expand the meaning of the starting background language, while conservativeness concerns the possibility of describing new deductive means that were inexpressible in the starting language of grounding. If the addition of new linguistic resources leaves the relative deductive means unchanged from the meaning of the starting background language, we will not have an increase in the deductive power of the starting grounding language; similarly, if the increase in the deductive power of the starting language of grounding does not affect the meaning of the starting background language, the meaning remains unchanged.

Chapter 6

Systems of grounding

6.1 General overview

This Chapter is given up to the proposal and development of a class of *formal systems of grounding*; each of the elements of the class is intended as related to one of the languages of grounding identified in the previous chapter. The aim of the systems of grounding is to provide rules that allow us to deduce relevant properties of the terms of languages of grounding. The properties will be essentially two: the fact that a term denotes a ground for a certain judgment or a certain assertion, and the fact that two terms denote the same ground, namely the same object closed when both of them are closed, or extensionally equal operations when they are open.

Since the class of languages to which the systems refer is infinite, also the class of the systems of grounding is infinite. However, each system has a standard structure, i.e. the same kinds of rules involved. Firstly, there are rules for the predicate which indicates that a certain term is a ground for a certain judgment or a certain assertion; then, rules for an identity predicate; and finally rules for the usual constants of first-order intuitionist logic. The systems differ by only a restricted subgroup of some, but not all, of the aforementioned groups of rules - exactly how the various grounding languages differ only by the set of individual constants, or of non-primitive operational symbols that they involve.

When introducing the class of the systems of grounding, we will start from a specific example. In other words, we will develop a detailed system of grounding for a Gentzen-language as indicated in Section 5.2.3.3, although enriched with new expressive resources, and will show how it enjoys some important properties. Based on this example, we will then generalize our argument, and will therefore substantiate the request that the properties enjoyed by the Gentzen-language apply in each of the elements of the class.

As we did in the previous chapter, however, we want to conduct first some preliminary observations, aimed at facilitating the reading and understanding of what we will say later, as well as at indicating our aims, and the reasons which have motivated our formal choices.

6.1.1 Denotation and identity

As you may have noticed, the languages of grounding we have been presenting in the previous sections consist, strictly speaking, only of terms. It is natural to wonder, then, whether it is possible to enrich them with formulas and, if so, what such formulas should express. Obviously, the formulas will have to indicate, by means of suitable predicates, properties of the terms, so that the previous question is reduced to which the most relevant properties of the terms in question are.

A first property is indicated by the denotation theorem: all the terms of a language of grounding denote grounds. The general scheme provided is that, if the term is closed, it denotes a ground for a categorical judgment or assertion, as fixed by the clauses (At_G) - ($\exists_G$), while, if the term is open, it denotes a ground for a general, hypothetical, or general-hypothetical judgment or assertion, namely an operation on grounds that produces grounds for categorical judgments or assertions, when applied to appropriate arguments - grounds for categorical judgements or assertions, possibly after substitution of individual variables. In the case of closed terms, this property will be indicated with

$$Gr(T, \beta)$$

for closed T and β. in other words, in developing a system of grounding, we will enrich the language of grounding to which it refers with a binary predicate $Gr(..., ---)$.

Given this enrichment of the expressive power of a language of grounding, and turning from the closed case to the open one, we suppose first of all that T contains a sequence $\underline{x}$ of free individual variables, and does not contain free ground-variables. Then

$$Gr(T(\underline{x}), \beta)$$

indicates that $T(\underline{x})$ is an operation on grounds of operational type

$$\beta$$

for $FV(\beta) \subseteq FV^I(T(\underline{x}))$, that is, for every appropriate sequence $\underline{t}$ of closed terms, the closed term $T(\underline{t}/\underline{x})$ is a ground for $\vdash \beta(\underline{t})$, that is

$$Gr(T(\underline{t}/\underline{x}), \beta(\underline{t})).$$

If instead T contains free ground-variables $\xi^{\alpha_1}, ..., \xi^{\alpha_n}$, and it does not contain free individual variables,

$$Gr(T(\xi^{\alpha_1}, ..., \xi^{\alpha_n}), \beta)$$

for closed $\alpha_1, ..., \alpha_n, \beta$, indicates that $T(\xi^{\alpha_1}, ..., \xi^{\alpha_n})$ is an operation on grounds of operational type

$$\alpha_1, ..., \alpha_n \rhd \beta,$$

that, for every closed term $U_i : \alpha_i$ $(i \leq n)$ in expansions of the language of grounding to which T belongs, the closed term $T(U_1, ..., U_n)$ is a ground for $\vdash \beta$, that is

$$Gr(T(U_1, ..., U_n), \beta).$$

Finally, if T contains a sequence of free individual variables $x_1, ..., x_n$, and of free ground-variables $\xi^{\alpha_1}, ..., \xi^{\alpha_m}$,

$$Gr(T(x_1, ..., x_n, \xi^{\alpha_1}, ..., \xi^{\alpha_m}), \beta)$$

indicates as before that T is an operation on grounds of operational type

$$\alpha_1, ..., \alpha_m \rhd \beta$$

for

$$FV(\alpha_1) \cup ... \cup FV(\alpha_m) \cup FV(\beta) \subseteq \{x_1, ..., x_n\},$$

that is, for every appropriate sequence of closed terms $t_1, ..., t_n$, for every closed term $U_i : \alpha_i(t_1, ..., t_n)$ $(i \leq m)$ in expansions of the language of grounding to which T belongs, the closed term $T(t_1, ..., t_n/x_1, ..., x_n, U_1, ..., U_m)$ is a ground for $\vdash \beta(t_1, ..., t_n)$, that is

$$Gr(T(t_1, ..., t_n/x_1, ..., x_n, U_1, ..., U_m), \beta(t_1, ..., t_n)).$$

With regard to what has been said so far, we would like to make at this point two clarifications. The first is the following: in the case of a term $T(\underline{x}, \xi^{\alpha_1}, ..., \xi^{\alpha_n})$ denoting an operation on grounds of operational type

$$\alpha_1, ..., \alpha_n \rhd \beta,$$

which, as said, is expressed by the formula

$$Gr(T(\underline{x}, \xi^{\alpha_1}, ..., \xi^{\alpha_n}), \beta),$$

we have noted that this means requesting that, after appropriate substitutions of ground-variables, in turn possibly following the substitution of individual variables, the term in question denotes a ground for a categorical judgment or assertion, which is expressed by the formula

$$Gr(T(\underline{t}/\underline{x}, U_1, ..., U_n), \beta).$$

Of course, the legitimacy of this statement depends on the fact that the terms used to replace the ground-variables - possibily after substitution of the individual variables - are such as to denote grounds for categorical judgments or assertions involving the type of the ground-variable substituted - possibly after replacing individual variables. If the type is $\alpha_i(\underline{t})$, and if the term denoting a ground for $\vdash \alpha_i(\underline{t})$ is U_i $(i \leq n)$, then, the fact that the starting term denotes a ground after the indicated substitutions depends on

$$Gr(U_i, \alpha_i(\underline{t})),$$

a dependence that can be expressed with

$$\begin{array}{c} \ldots Gr(U_i, \alpha_i(\underline{t})) \ldots \\ \vdots \\ Gr(T(\underline{t}/\underline{x}, \ldots U_i \ldots), \beta) \end{array}$$

However, since this must apply to *every possible* U_i that enjoys this property, we can say that the fact that the starting term denotes an operation on grounds of a given operational type, depends on the *assumption*

$$Gr(\xi^{\alpha_i}, \alpha_i)$$

where ξ^{α_i} is the ground-variable occurring in the starting term, a dependence that can be expressed with

$$\begin{array}{c} \ldots Gr(\xi^{\alpha_i}, \alpha_i) \ldots \\ \vdots \\ Gr(T(\underline{x}, \ldots \xi^{\alpha}_i \ldots), \beta) \end{array}$$

In developing the systems of grounding, we will follow the intuition contained in this observation. More precisely; since a term T without ground-variables must denote a ground for a categorical or general judgment or assertion, that is an operation on grounds of which the domain is empty, we will ensure that the fact that this term denotes that ground is provable as theorem of the system, that is

$$\vdash Gr(T(\underline{x}), \alpha);$$

since instead a term T in which free ground-variables occur $\xi^{\alpha_1}, ..., \xi^{\alpha_n}$ denotes a ground for a hypothetical, or general-hypothetical judgement or assertion, that is an operation on grounds with a non-empty domain and co-domain β, we will ensure that the fact that this term denotes that ground is provable in dependence on assumptions which require that each of the free ground-variables denotes a ground for a judgment or assertion involving the type of the ground-variable, i.e.

$$Gr(\xi^{\alpha_1}, \alpha_1), ..., Gr(\xi^{\alpha_n}, \alpha_n) \vdash Gr(T(\underline{x}, \xi^{\alpha_1}, ..., \xi^{\alpha_n}), \beta).$$

As a second clarification, we premise that in enriching the languages of grounding with the binary predicate $Gr(..., ---)$, we will impose the following restriction: called Λ the language of grounding to which T belongs, $Gr(T, \alpha)$ is a formula of the enriched grounding language if, and only if, $T : \alpha \in \texttt{TERM}_\Lambda$ - namely, T is a term of type α in Λ. This restriction has a rather "practical" reason, and responds also to a criterion of formal convenience. In the light of the close connection between the syntactic typing of the terms of a language of grounding and the formulas involved in the judgment or assertion, which that term denotes a ground for, it would make no sense to authorize, for example, a formula such as $Gr(T, \beta)$ if T has type α instead. Indeed, from the denotation theorem we know that T must be a ground for α, or an operation on grounds having as codomain α. If, on the contrary, we wanted to be liberal, we would have some unnecessary complications. First of all, we should authorize formulas in which, in fact, there is no connection between the syntactic typing and the semantic value of a term, formulas that, being consequently "wrong" from the beginning, should not play any role in the theory of grounds. As a result, since a system of grounding, while having to comply with certain basic semantic assumptions, is configured as a mere deductive apparatus, it would become possible to demonstrate results like

$$\Gamma_1 \vdash Gr(T, \beta).$$

These theorems determine a "deductive over-generation"; the system proves what we expect to show, that is

$$\Gamma_2 \vdash Gr(T, \alpha),$$

and in addition other results that, in semantic terms, are incorrect. The whole system would be incorrect with respect to the reference semantics, but in a very peculiar sense - that is why we talked about "deductive over-generation", and not about incorrectness *strictu sensu.* For each of the derivations that lead to incorrect results, in fact, it is possible to find a substitution of formulas of the background language that returns the derivation of a correct result; and this substitution of formulas is exactly what is obtained by introducing the aforementioned restriction, i.e. by ensuring that in each formula of the derivation constructed with the predicate $Gr(..., ---)$, the term shown on the left has type the formula shown on the right.

We come now to the second property of the terms of a language of grounding, which we intend to deal with. It is indicated by the theorems which prove that each language of grounding can be expanded to a closed one in canonical form - with respect to a certain denotation function. If a closed term T denotes a ground, it will denote the same ground of a canonical closed term U. As already seen, T and U do not necessarily belong to the same language; but this is guaranteed, for an appropriate choice of the denotation function, if the language is closed in canonical form. Within these languages, therefore, we can pass from an identity of the denotation of T and U, to an identity between T and U expressible in the enriched grounding language, namely

$$T \approx U.$$

To this end, we will therefore introduce into languages of grounding a binary predicate $... \approx ---$, that, being intended as an identity predicate, will enjoy the usual properties of reflexivity, symmetry, transitivity, and substitution of identicals. For this reason, although neither T nor U are canonical, the identity between them will apply if both of them reduce to the same canonical closed term Z.

If this is what holds for the identity between closed terms, what will happen in the case of open terms? We will adopt in this case, as already done in the previous chapter, a criterion of *extensional identity.* If T and U contain free individual or ground-variables, each of them denotes a ground for a general, or general-hypothetical judgment or assertion, namely an operation on grounds having a certain operational type. They are identical if, and only if, they denote the same ground, which in this case means to say that they denote the same operation on grounds; and the latter circumstance will apply if, and only if, the operations they denote return the same values on the same arguments. In other words, assumed that the free individual variables constitute a sequence $\underline{x}$, and the ground-variables a sequence $\underline{\xi}$, we will have that

$$T(\underline{x}, \underline{\xi}) \approx U(\underline{x}, \underline{\xi})$$

if, and only if, for all the appropriate sequences of closed individual terms $\underline{t}$, and of terms for closed grounds $\underline{Z}$,

$$T(\underline{t}/\underline{x}, \underline{Z}/\underline{\xi}) \approx U(\underline{t}/\underline{x}, \underline{Z}/\underline{\xi}).$$

Also in relation to the discussion on the identity predicate so far conducted, we consider it appropriate to make two clarifications. The first concerns a point which we will discuss in more detail in the next section, and which we anticipate here only briefly. As we know, operations on grounds denoted by non-primitive operational symbols of a language of grounding are to be intended as defined by certain equations; when developing systems of grounding, we will take up this idea, even if internalizing, so to speak, the defining equations in the system itself. In other words, instead of *mapping* the non-primitive operational symbols on operations defined by certain equations, through denotation functions that connect the language of grounding to our "universe" of grounds and operations on grounds, we will provide identity *axioms* that directly concern non-primitive operational symbols, namely equations that can be used in the derivations of the system to eliminate the non-primitive operational symbols, transforming some terms constructed with such symbols in others that the system considers as identical to them. Now, even if the axioms faithfully reflect the equations that, in the denotational approach, define the operation denoted by the non-primitive operational symbol in question, the two ways of proceeding differ for an essential point: while an operation on grounds is defined by an equation that shows how the operation behaves on closed *objects*, the axiom that, in the system, fixes the behavior of a non-primitive operational symbol, provides a *syntactic* method of transformation, connected only with the type of its arguments, regardless of the presence or absence of free variables in these arguments. By way of example, let us take into account the non-primitive operational symbol $\wedge_{E,i}\langle \alpha_1 \wedge \alpha_2 \rhd \alpha_i \rangle$; according to the standard interpretation, that symbol is associated to an operation on grounds

$$f(\xi^{\alpha_1 \wedge \alpha_2})$$

of operational type

$$\alpha_1 \wedge \alpha_2 \rhd \alpha_i$$

fixed by requiring that, for every $\wedge I(g_1, g_2)$ *ground for* $\vdash \alpha_1 \wedge \alpha_2$, with g_i ground for $\vdash \alpha_i$,

$$f(\wedge I(g_1, g_2)) = g_i.$$

The equivalent in a system of grounding will be an axiom for the predicate ... $\approx - - -$ of the type

$$\frac{}{\wedge_{E,i}(\wedge I(T_1, T_2)) \approx T_i} R_\wedge$$

where T_i is whatever term of type α_i. Now let α_2 be a closed atomic formula, and let δ be (the name of) a closed derivation of α_2 in an atomic system that occurs in the base B of a Gentzen-language on which we have defined a system of grounding. From a denotational point of view, the term

$$\wedge_{E,2}(\wedge I(\xi^{\alpha_1}, \delta))$$

denotes a B-operation on grounds

$$f(\xi^{\alpha_1})$$

of operational type

$$\alpha_1 \rhd \alpha_2;$$

indeed

$$den(\wedge_{E,2}(\wedge I(\xi^{\alpha_1}, \delta)) = den^*(\wedge_{E,2})(\wedge I(den^*(\xi^{\alpha_1}), den^*(\delta)) = f(\wedge I(\mathtt{Id}, \delta))$$

and, for every g ground on B for $\vdash \alpha_1$,

$$f(\wedge I(\mathtt{Id}, \delta))(g) = f(\wedge I(\mathtt{Id}(g), \delta)) = f(\wedge I(g, \delta)) = \delta.$$

Obviously, the operation will give δ on whatever ground g for α_1, but in the approach in question, we cannot conclude that the operation is equal to δ, namely, that the denotation of the starting term is the same as for δ. An operation on grounds, being open, is an object different from a ground for a categorical judgment or assertion, which is instead a closed object. In the system, on the other hand, it will be possible to prove that the starting term and δ are identical, and therefore that this term is a ground for α_2; in fact, assuming it plausible to have an axiom

$$\frac{}{Gr(\delta, \alpha_2)}\ \texttt{C}$$

we will have, via an equally plausible principle of preservation of the denotation $\approx_P$,

$$\dfrac{\dfrac{}{\wedge_{E,2}(\wedge I(\xi^{\alpha_1}, \delta)) \approx \delta}\ R_\wedge \qquad \dfrac{}{Gr(\delta, \alpha_2)}\ \texttt{C}}{Gr(\wedge_{E,2}(\wedge I(\xi^{\alpha_1}, \delta)), \alpha_2)}\ \approx_P$$

The central point is, therefore, that the identity fixed by the predicate $... \approx ---$ is an equivalence relation among terms that reduce or expand from/to each other via syntactic transformations that generalize the reduction or expansion procedures in Prawitz's theory of normalization, or the β–reduction or η-expansion in the theory of λ-conversion. Despite this asymmetry, however, in a grounding system it will be possible to demonstrate identities that faithfully reflect identity in the denotational approach; in other words, it will apply as a result that if $den(T) = den(U)$, then, once internalized with appropriate axioms the equations relative to *den* in a system of grounding for the language to which T and U belong, we will have that

$$\vdash T \approx U.$$

The second clarification is similar to the second clarification already carried out for the predicate $Gr(..., ---)$. Namely, related to the predicate $... \approx ---$, we will adopt the following restriction: called Λ the language of grounding to which T and U belong, $T \approx U$ is a formula of the enriched language of grounding if, and only if, $T, U : \alpha \in \texttt{TERM}_\Lambda$ - namely, T and U have identical type α in Λ. Here too, the restriction is aimed at avoiding the incorrectness of a grounding system which is, in the peculiar sense indicated above, a "deductive super-generation".

Finally, at the conclusion of the discussion conducted in this section, concerning the predicates $Gr(..., ---)$ and $... \approx ---$, and in order to connect with what we will say in the next section, we point out the following. As we have said, the fact that a term T in which ground-variables $\xi^{\alpha_1}, ..., \xi^{\alpha_n}$ occur, denotes an operation on grounds of operational type

$$\alpha_1, ..., \alpha_n \rhd \beta$$

can be represented as

$$\begin{array}{c} Gr(\xi^{\alpha_i}, \alpha_i) \\ \vdots \\ Gr(T(\underline{x}, \ldots \xi^{\alpha_i} \ldots), \beta) \end{array}$$

giving as result in a system of grounding

$$Gr(\xi^{\alpha_1}, \alpha_1), ..., Gr(\xi^{\alpha_n}, \alpha_n) \vdash Gr(T(\underline{x}, \xi^{\alpha_1}, ..., \xi^{\alpha_n}), \beta).$$

By introducing an appropriate universal quantifier – indicated with $\forall^G$ so as to distinguish it from the quantifier $\forall$ in action in the background language, and indicating with $\wedge^G$ and $\rightarrow^G$ conjunction and implication of the system of grounding so as to distinguish them from those in action in the background language - the circumstance can therefore also be expressed as

$$\vdash \forall^G \underline{x}\ \forall^G \xi^{\alpha_1} ... \forall^G \xi^{\alpha_n} (\bigwedge\nolimits_{i \leq n}^{G} Gr(\xi^{\alpha_i}, \alpha_i) \rightarrow^G Gr(T(\underline{x}, \xi^{\alpha_1}, ..., \xi^{\alpha_n}), \beta)).$$

Analogously, the identity

$$T(\underline{x}, \underline{\xi}) \approx U(\underline{x}, \underline{\xi})$$

can be expressed, without explicit reference to substitutions, with

$$\forall^G \underline{x}\ \forall^G \underline{\xi} (T(\underline{x}, \underline{\xi}) \approx U(\underline{x}, \underline{\xi})).$$

Such a use of quantifiers will, as we shall see, be central to the proof of two important types of results: the well-definition of a non-primitive operational symbol with respect to an operational type intended, and the rewritability, *salva denotatione*, of a non-primitive operational symbol of an expansion of a language of grounding into the original grounding language.

6.1.2 Aims and outcomes of a deductive approach

What is the point of introducing formal systems on languages of grounding enriched with the predicates $Gr(..., ---)$ and $... \approx ---$? With languages of grounding of this type it will be possible to express properties which, as mentioned in the previous section, concern the denotation and the denotational identity of terms. Formal systems, then, go beyond the simple expressiveness, enabling us to prove such properties.

The theoretical apparatus presented in the previous chapter, in the terms of denotation functions from languages of grounding to a "universe" of grounds and operations on grounds, actually allows, in itself, to prove that a term denotes a ground, which this ground is, and if and when two terms share the same denotation. The advantage that an approach in terms of formal system offers is to allow the achievement of these results through a clear and well-defined set of rules, and this in turn has a triple, useful implication.

Obviously, the first is to make the provability of the intended properties *precise* and, so to speak, *mechanical*; once given the rules of the system, assuming that they are accepted as adequate, there is no need to reflect on the type of denotation chosen for the language of grounding under examination, and of transforming the terms in the objects they denote, being on the contrary sufficient, remaining in the language itself, to apply those rules in the right order so as to achieve the desired result. If this is a merit that the approach in deductive terms has in common with any investigation based on formal systems, the second, more important, consequence, in part already anticipated, is proper to the discourse that is being conducted here: the proof of the properties can be obtained within the language of grounding, of which the terms we want to prove those properties belong. Apart from the asymmetry identified in the previous section, concerning the denotational identity and the provability of this identity in a system, an immediate effect of this point is to make the analysis easier and clearer; for example, the computation of one or more closed terms to the canonical form, aimed at a possible proof of their identity, no longer requires the application of the denotation function, since it occurs directly within the system, as direct computation of the terms themselves.

In order to illustrate the third point, which is also the central one, we must instead conduct a preliminary discussion, and illustrate how the systems of grounding, we are dealing with in this chapter, *are made*. All the systems share the following groups of rules:

- rules for the predicate $Gr(..., ---)$;
- rules for the predicate $... \approx ---$;
- introduction and elimination rules of a Gentzen natural deduction system for first-order intuitionistic logic.

As for the rules for the predicate $Gr(..., ---)$, they are further divided into two subgroups:

- rules of type introduction;
- rules of type elimination.

Now, the rules of type introduction are nothing more than a translation, in deductive terms, of the clauses (At_G) - ($\exists_G$). For example, for each closed individual constant δ of the language of grounding to which the system refers, and that "names" a closed derivation of an atomic formula α in the atomic base of reference, we will have an axiom

$$\frac{}{Gr(\delta, \alpha)}\ \mathtt{C}$$

while, in the case of for example $\wedge$, we will have the rule

$$\frac{Gr(T, \alpha) \qquad Gr(U, \beta)}{Gr(\wedge I(T, U), \alpha \wedge \beta)} \wedge I$$

The rules of type introduction can also be seen as introduction rules of the appropriate primitive operational symbol.

On the other hand, the rules of type elimination express, relative to the main logical constant of the type, Dummett's *fundamental assumption* (Dummett 1991): if T denotes a ground for a formula with main logical constant k, then it must be possible to reduce T to a canonical term which starts with kI that denots a ground for the same formula. These rules will take the form of generalized elimination rules, to which the usual discharges and restrictions apply: for example, in the case of $\wedge$, we will have

$$\frac{Gr(T, \alpha \wedge \beta) \qquad \begin{matrix} [T \approx \wedge I(\xi^\alpha, \xi^\beta)] \quad [Gr(\xi^\alpha, \alpha)] \quad [Gr(\xi^\beta, \beta)] \\ \vdots \\ A \end{matrix}}{A} D_\wedge$$

The rule discharges the assumptions $T \approx \wedge I(\xi^\alpha, \xi^\beta)$, $Gr(\xi^\alpha, \alpha)$, and $Gr(\xi^\beta, \beta)$, and it can be applied if, in the derivation of A as a minor premise, ξ^α does not occur free in A, nor in assumptions on which A depends other than $T \approx \wedge I(\xi^\alpha, \xi^\beta)$ and $Gr(\xi^\alpha, \alpha)$, and ξ^β does not occur free in A, nor in assumptions on which A depends other than $T \approx \wedge I(\xi^\alpha, \xi^\beta)$ and $Gr(\xi^\beta, \beta)$ - we can additionally require that ξ^α and ξ^β are fresh variables, not previously used in the derivation of $Gr(T, \alpha \wedge \beta)$. Observe that this rule is equivalent - via usual rules for first-order intutionistic logic, and using as before a conjunction $\wedge^G$, and an existential quantifier, that we indicate with $\exists^G$ to distinguish it from that acting in the background language - to that, perhaps more perspicuous in form,

$$\frac{Gr(T, \alpha \wedge \beta)}{\exists^G \xi^\alpha \exists^G \xi^\beta (T \approx \wedge I(\xi^\alpha, \xi^\beta) \wedge^G (Gr(\xi^\alpha, \alpha) \wedge^G Gr(\xi^\beta, \beta)))}$$

In the margins of what has been said about the rules of introduction and elimination of type, it may be useful to specify that the enriched languages of grounding, other than containing the predicates $Gr(..., ---)$ and $... \approx ---$, will also have two new linguistic resources, resulting from "technical" reasons. In the alphabet of the enriched language of grounding, we will first have a variable for terms of type $\alpha(x)$ defined on individual variables x, intended to represent operations on grounds of operational type

$$\alpha(x)$$

or operations on grounds with codomain $\alpha(x)$ for x not free in any of the entries of the domain. These variables will be represented with the notation $\mathtt{h}^{\alpha(x)}(x)$, where x can be substituted by any term t in the background language. The reason for the introduction of these variables is related to the possibility to express, in a system of grounding, the rule of type elimination for terms T of type $\forall x\alpha(x)$; in one of the assumptions on which the minor premise of the rule depends, in fact, we will find the dischargeable formula

$$T \approx \forall Ix(\mathtt{h}^{\alpha(x)}(x))$$

which, in this specific case, indicates that T is reducible to a term in canonical form of the same type, and which is a ground for the formula of which it is type - in fact, the other dischargeable assumption will be

$$\forall^G x(Gr(\mathtt{h}^{\alpha(x)}(x), \alpha(x)))$$

- that is to say that T is identical to a term in canonical form the immediate subterm of which is an operation on grounds of operational type

$$\alpha(x)$$

or an operation on grounds with codomain $\alpha(x)$ for x not free in any of the entries of the domain.

We will also have a variable for terms of type β defined on ground-variables of arbitrary type α, intended to represent operations on grounds with codomain β with α arbitrary in the domain. These variables will be represented with the notation $\mathtt{f}^{\beta}(\xi^{\alpha})$, for whatever α in the set of formulas of the background language, where ξ^{α} can be substituted by any term of type α in the language of grounding. As for the requirements that determine this second addition, we limit ourselves only to point out that, without it, the system would demonstrate a series of incorrect results, above all in connection with operational symbols that bind ground-variables. In order to understand this point, it is perhaps sufficient to reflect on the formulation of the clause that sets what counts as ground for formulas with $\rightarrow$ as main logical constant, which, with the variables for operations on grounds with domain β, can be written - and, as we will see, proven in the system - in the form

$$Gr(\rightarrow I\xi^{\alpha}(\mathtt{f}^{\beta}(\xi^{\alpha})), \alpha \rightarrow \beta) \Leftrightarrow \forall^G \xi^{\alpha}(Gr(\xi^{\alpha}, \alpha) \rightarrow^G Gr(\mathtt{f}^{\beta}(\xi^{\alpha}), \beta))$$

while, without these variables, it seems that we can only ask for a problematic empty quantification

$$Gr(\rightarrow I\xi^{\alpha}(\xi^{\beta}), \alpha \rightarrow \beta) \Leftrightarrow \forall^{G}\xi^{\alpha}(Gr(\xi^{\alpha}, \alpha) \rightarrow^{G} Gr(\xi^{\beta}, \beta)).$$

However, on the variables for operations on grounds with codomain β we will not define any quantification - which we will do instead for the standard ground-variables; not because this cannot be done, even with interesting consequences and developments, but simply because such a move is not strictly necessary with respect to the purposes we set ourselves here.

Now we come to the rules for the predicate $... \approx ---$. They are further divided into four subgroups:

- rules that determine an equivalence relation that preserves the denotation;
- substitution rules with primitive operational symbols;
- equations for non-primitive operational symbols;
- substitution rules with non-primitive operational symbols;

The first of the four subgroups includes the usual rules of reflexivity, symmetry and transitivity. To these we add a rule that states the *preservation of the denotation* among identical terms, which we have already anticipated, of which the form is

$$\frac{T \approx U \qquad Gr(U, \alpha)}{Gr(T, \alpha)} \approx_{P}$$

The substitution rules with primitive operational symbols express the substitution principle of identicals in relation to the primitive operational symbols of a language of grounding. These rules are bidirectional, allowing to pass from identity between terms of minor complexity to identity between terms of greater complexity, and vice versa. In the case of $\wedge$, for example, we will have

$$\frac{T_1 \approx U_1 \qquad T_2 \approx U_2}{\wedge I(T_1, T_2) \approx \wedge I(U_1, U_2)} \approx^{\wedge}_{1} \qquad \frac{\wedge I(T_1, T_2) \approx \wedge I(U_1, U_2)}{T_i \approx U_i} \approx^{\wedge}_{2i}$$

Analogously, the substitution rules with non-primitive operational symbols express the substitution principle of identicals on non-primitive operational symbols of a language of grounding. In this case, however, the rules are unidirectional, that is they go only from the identity of less complex terms to that of more complex terms; for example, again in the case of $\wedge$, we will have

$$\frac{T \approx U}{\wedge_{E,i}(T) \approx \wedge_{E,i}(U)} \approx^{\wedge}_{3i}$$

– obviously for T and U of appropriate type. The reason for the unidirectionality is obvious. Some non-primitive operational symbols, such as $\wedge_{E,i}$, are defined in such a way that, when applied to certain terms, they reduce these terms to others, "deleting" some subterms in the application arguments; while identity can be guaranteed in the case of terms which result at the end of the reduction, it may not apply to those "deleted", and therefore on the arguments of application themselves.

Finally, as regards the equations for non-primitive operational symbols, we have already talked about them to some extent in the previous section. They are a sort of "internalization" of the defining equations of the operations on grounds which, in the denotational approach, non-primitives operational symbols are meant to denote. While a denotation function on a language of grounding connects the syntactic components of a language of grounding to a "universe" of grounds and operations on grounds, within a formal system we will have a series of axioms that regulate the deductive behavior of the syntax; the defining equations which, via denotation, determine which function is represented by that non-primitive operational symbol, become, in the formal system, equations that tell us how, in well-defined contexts, the non-primitive operational symbols can be eliminated, and how the terms in which they occur are transformed into others provably identical. Using again $\wedge$ as a case study, for the non-primitive operational symbol $\wedge_{E,i}$, we will have, as already said, the axiom

$$\frac{}{\wedge_{E,i}(\wedge I(T_1, T_2)) \approx T_i} R_{\wedge}$$

for T_1 and T_2 whatever terms of the language of grounding. Obviously, so as the same operational symbol can be interpreted on several operations on grounds – unless an *invariant* denotation approach is adopted – in the same way there could be different types of equations that regulate the deductive behaviour of that symbol in a system. We have said that operations on grounds, as, in fact, operations *on grounds*, are to be understood as functions defined on closed objects; on the other hand, the equations of a formal system are rather configured as methods of syntactic transformation, and as such they concern only the type of the term to which the symbol in question applies, regardless of the fact that this term is closed or open. Now this has consequences on the functioning of the identity predicate, discussed in the previous section, and also on the type of restrictions we must explicitly request to an equation, so as it can be included among the axioms of the calculus. More specifically, since the operations on grounds are to be intended as functions, we can assume that the equations that define them are such to respect by default conditions such as identity of domain and codomain between *definiendum* and *definiens*, and linearity of the operation with respect to substitution – namely, that $(f(g))[*/\star] = f(g[*/\star])$; on the other hand, since the equations of a system fix the deductive behavior of syntactic symbols, we must request explicitly that *definiendum* and *definiens* have the same type, that the *definiens* is not defined on more variables than they occur in the *definiendum*, and that all the instances of the equation make it possible to move from the substitution on the whole term to the substitution on its arguments.

The last set of rules for a system of grounding includes the usual rules of a Gentzen's natural deduction system for standard constants in first-order intuitionistic logic. The language of these rules is obviously different from the background language on which the language of grounding is intended to act; to the enriched language of grounding it will in fact be added first-order logical constants k, which, in order to distinguish them from the corresponding constants of the background language, will be indicated, as mentioned above, with k^G – therefore, in addition to the already mentioned constants $\wedge^G$, $\rightarrow^G$, $\forall^G$ and $\exists^G$, we will have also $\vee^G$ and $\bot^G$. The propositional constants link formulas in the usual way, while the quantifiers act on two different types of variables; individual variables and ground-variables - where it has to be noted that, while the ground-variables explicitly occur in a term, the individual variables occur in the term when they occur in the type of its ground variables.

With regard to the logical part of systems of grounding, we wish to emphasize just one point. If thanks to the predicates $Gr(..., ---)$ and $... \approx ---$ we can demonstrate denotation properties of individual terms, or identities between/among two or more terms, the fact that with logical constants it is possible to pass from atomic formulas to logically complex formulas allows instead to prove general properties of entire classes of terms, or general properties of components of the alphabet of the enriched language of grounding. In particular, it becomes possible to prove general relations of denotation and identity, which in turn allows us to derive, mainly, theorems that represent the conditions for denoting grounds with a certain structure – namely, an analogue of the clauses (At_G) - $(\exists_G)$ – and theorems that guarantee a valid definition of non-primitive operational symbols. This observation refers directly to what we have just left open, namely the third advantage of the deductive approach we are proposing here.

The identification of groups of rules that are adequate with respect to the underlying aims of the theory of grounds, and satisfactory with respect to the proof of fundamental properties that in this theory are expected to hold, has, in our opinion, above all the advantage of *clearly explicitating* what the deductive means are to which the approach proposed by Prawitz must appeal. Once found a minimum set of principles, the approach itself can be *extended*, in a precise and schematic way, and according to the objectives we aim at. A paradigmatic example of what we mean to say here are, we think, the rules of type elimination.

In the theory of grounds it is essential to exploit, as an ingredient in the proof of the general properties, the principle according to which closed non-canonical terms, as denoting grounds, have the same denotation of, or more generally can be reduced to, closed canonical terms. Let us take into account, for example, the following words of Prawitz:

> the fact that the operation $\wedge_{E,i}$ always produces a ground for α_1 when applied to a ground g for $\alpha_1 \wedge \alpha_2$ is not an expression of what $\wedge$ means [...]. Instead it *depends* on what $\wedge$ means, and has to be established by an argument: Firstly, in view of [the] clause [...] specifying the grounds for $\alpha_1 \wedge \alpha_2$ exhaustively, g must have the form $\wedge I(g_1, g_2)$ [...]. Secondly, because of the identity condition, g_1 and g_2 are unique. Hence, according to the equations that define the operations, $\wedge_{E,i}(\wedge I(g_1, g_2)) = g_i$, where g_i is a ground for asserting α_i. (Prawitz 2015, 92)

Prawitz intends to prove that the non-primitive operation (denoted by the non-primitive operational symbol) $\wedge_{E,i}$ is well defined; to do it, in addition to the structural properties of primitive operations such as the identity conditions to which they are subject, the Swedish logician make use unavoidably of the principle according to which a ground for $\vdash \alpha_1 \wedge \alpha_2$ has a certain form – and therefore, in the way in which we have set the discourse here, that a term that denotes such ground has the same denotation of, or more generally is reducible to a canonical term.

On the other hand, to prove that non-primitive operations (and, as a cosenquence, that non-primitive operational symbols that denote them) are well-defined is one of the key points of the theory of grounds, so that Prawitz, as we have seen, comes to see as crucial the recognizability/decidability problem that we have widely discussed above. The circumstance depends, in our opinion and primarily, on the aforementioned "fundamental task" that the theory of the grounds intends to fulfill, namely to explain how and why deductively valid inferences can force us epistemically; in doing so, certainly we cannot avoid making sure that the operations on grounds the theory associates to the inferential passages are, in the concrete examples, well-posed, and guaranteeing that this circumstance is actually ascertainable - otherwise, no inferential agent would be willing to use that inference or, even if it used it, it could never become aware of the correctness of its deductive acts, which seems contrary to the idea that the agent could be epistemically forced by such acts. As mentioned above, this is a point which profoundly distinguishes Prawitz's theory of grounds from similar theories, such as the Kreisel-Goodman theory of constructions and Martin-Löf's intuitionistic type theory, both illustrated in Chapter 3; to borrow from Prawitz – with words already quoted above -

> it is often said that the concept of proof should be defined in such a way that it becomes decidable whether something is a proof, but how this is to be achieved is seldom indicated, except of course in the case of formal proofs. The terms that denote constructions are supposed to be typed, and whether an expression has a type is decidable, but the rules for typing already assume that the demands put on the defined operations are fulfilled. (Prawitz 2018b, 10)

As a result, it seemed necessary to explicitly adopt Dummett's fundamental assumption (Dummett 1991) among the rules of the class of formal systems for the theory of grounds. As maybe you will notice, and in a not surprising way, our systems share many aspects with Martin-Löf's intuitionistic type theory; while a difference is precisely the absence, in the latter, of rules for the fundamental assumption, although it should be noted that Martin-Löf uses it, *outside the system*, to make it evident the validity of some of its rules (Martin-Löf 1984).

6.1.3 Three kinds of theorems

In proposing systems of grounding, we will focus mainly on three types of results that these systems allow to obtain. The three typologies can in turn be grouped into two groups: theorems *of* the system, and meta-theorems, that is, theorems *about* the system.

Among the theorems of the system, there first appear the analogues of the clauses (At_G) - ($\exists_G$). In other words, we will show how the translations of these clauses in enriched languages of grounding are derivable in systems of grounding on these languages. The clause for $\wedge$ will become, for example the formula

$$Gr(\xi^{\alpha}, \alpha) \wedge^{G} Gr(\xi^{\beta}, \beta) \Leftrightarrow Gr(\wedge I(\xi^{\alpha}, \xi^{\beta}), \alpha \wedge \beta)$$

derivable from the empty set of assumptions – and therefore universally quantifiable.

Along with these results, we will have next the derivability of the well-definition of non-primitive operational symbols, with respect to the operational type of the operation on grounds they are intended to represent. Again in the case of $\wedge$, the formula

$$\forall^G \xi^{\alpha_1 \wedge \alpha_2}(Gr(\xi^{\alpha_1 \wedge \alpha_2}, \alpha_1 \wedge \alpha_2) \rightarrow^G Gr(\wedge_{E,i}(\xi^{\alpha_1 \wedge \alpha_2}), \alpha_i))$$

is a theorem of a system for an enriched Gentzen-language of grounding. It says that, for each ground for $\vdash \alpha_1 \wedge \alpha_2$, the application of $\wedge_{E,i}$ to such ground produces a ground for $\vdash \alpha_i$. The derivation, which makes an essential appeal both to the defining equation of $\wedge_{E,i}$ and to Dummett's fundamental assumption, follows through and through that of Prawitz, mentioned in the previous section.

These two types of theorems are essential to obtain the result of the other typology, the theorem about systems. The meta-theorem is a deductive analogue of the denotation theorem and, as already anticipated, consists in ensuring that, given

$$FV^I(U) = \{x_1, ..., x_n\} \text{ and } FV^T(U) = \{\xi^{\alpha_1}, ..., \xi^{\alpha_m}\}$$

with $U : \beta$, we have

$$Gr(\xi^{\alpha_1}, \alpha_1), ..., Gr(\xi^{\alpha_m}, \alpha_m) \vdash Gr(U(x_1, ..., x_n, \xi^{\alpha_1}, ..., \xi^{\alpha_m}), \beta).$$

In other words, in systems of grounding it is possible to prove that all terms denote, in dependence on the assumption that the free ground-variables denote grounds for the respective types. This also means, put another way, that a term denotes a ground for a categorical or general judgment or assertion, if it does not contain free occurrences of ground-variables, and a ground for a hypothetical or general-hypothetical judgment or assertion, if it contains them. The meta-theorem just mentioned is therefore a sort of correctness result for systems of grounding, compared to the denotational approach developed in the previous Chapter - let us remember that we had interpreted the denotation theorem itself as a kind of correctness result of the languages of grounding with respect to our "universe" of grounds and operations on grounds.

Finally, a last type of theorems internal to the systems concerns the rewritability of non-primitive operational symbols of a given language of grounding in terms of operational symbols of its sublanguage of grounding. In this case, we will provide as an example the rewriting of the disjunctive syllogism in the Gentzen-language of Section 5.2.3.3 – appropriately enriched - according to the lines indicated in example 2 of Section 5.2.5.

Obviously, since the class of languages of grounding identified in the previous Chapter is infinite, also the class of systems of grounding on such languages will be infinite. Which is why we cannot actually show for each of the systems in our class that they prove the three types of results mentioned above. The actual illustration will be carried out inside of the only system of grounding that we will exemplify, which acts on a Gentzen-language of grounding - as defined in Section 5.2.3.3, and appropriately enriched. Otherwise, we will limit ourselves just to indicate *how* and *where* systems of grounding differ from each other. In this regard, it is well to remember that languages of grounding may differ with respect to the bases on which they operate, and in this case they involve different individual constants, or with respect to the non-primitive operational symbols they are endowed with – or both. It follows that systems of grounding can distinguish from each other for the axioms concerning different individual constants and also, and mainly, for the rules concerning identity in connection with non-primitives operational symbols – namely the equations for such symbols and the contextual substitution of identical.

Now, it turns out that the proof of the analogue of the clauses (At_G) - ($\exists_G$) involves neither defining equations for non-primitive symbols, nor substitution of identicals with these symbols. In other words, the derivations we will provide in the Gentzen-language system, can be transposed without any variation in whatever system of grounding. As for the meta-theorem, instead, the following applies: if the system derives the well-definitions of non-primitive operational symbols, then the denotation of each term is derivable in the system - along the lines indicated above. The whole thing is reduced therefore to the derivation of the well-definition of non-primitive operational symbols, and the possibility of the derivation will obviously depend on the equations associated with these symbols. As we have said, the equations must respect some specific parameters, but there is obviously the general parameter according to which the equation must allow to consider the symbol as an operation on grounds of an appropriate operational type; if this is guaranteed, the derivation of the well-definition can be assumed in a non-problematic way to all the systems of the class.[1]

[1]There would be another point to discuss, but an in-depth discussion of it would lead too far. This is the possibility of deriving, in systems of grounding, formulas of which the main logical sign is the negation, namely formulas of the type $A \rightarrow^G \bot^G$. Since, as mentioned, in introducing the predicate $Gr(..., ---)$, we require that it be applicable to a term T and to a formula α of the background language if, and only if, T actually has type α in the language of grounding of the system, and since of each term it is possible to prove – depending or not on assumptions – that it denotes a ground for that formula, or an operation on grounds having that formula as codomain, then no formula of the

6.2 Deduction over the Gentzen-language

As previously said, we focus first on a system of grounding for a Gentzen-language as presented in Section 5.2.3.3, enriched with the expressive resources we discussed in the introductory overview.

6.2.1 An enriched Gentzen-language

We refer from now on to the Gentzen-language `Gen` of Section 5.2.3.3. We provide first of all some definitions that complete those already introduced, again in Section 5.2.3.3, of a subterm of a term, of free and bound individual variable in a term, of free and bound ground-variable in a term. However, first of all and in order to simplify the discussion, regardless of the definability of notions related to atomic bases (notions that can in any case be easily specified), we adopt the following, important convention.

Convention 62. In the following, we will assume that the set of individ-

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type $Gr(T, \alpha) \rightarrow^G \bot^G$ will be derivable in the system. Actually, the derivability of such formulas would be possible only in one of the two following circumstances: (1) T does not have type α or (2) T has type α, but does not denote. Now, as for (1), as we have seen, the violation of syntactic typing has unpleasant consequences; in any case, even if we wanted to renounce the restriction, the derivation should be based on axioms that tell us that a term cannot denote a ground or an operation on grounds if it does not have type the formula for which it denotes a ground, or which appears in its codomain. As for (2), instead, it should be somehow possible to derive in the system that there is no succession of equations that "reduce" the term to a canonical form U for each immediate subargument Z of which it holds that $Gr(Z, \beta)$, for some appropriate β; if, on the contrary, the term is open, we should prove that there is a substitution that returns an instance of the term for which the previous circumstance holds. Such a derivation should be based on rules that reason on "reductions", rather than on terms; we should verify wheter these rules stand out necessarily on a higher order than that of the systems of grounding we are here considering, or it is somehow possible to "internalize" them, placing them on the same level of rules for denotation and identity. In any case, to follow this road would require the introduction of non-denoting terms, and therefore the authorization of languages of grounding of the type discussed in Section 5.1.2.2. Regarding the predicate $... \approx ---$, it applies more or less what has already been said for the predicate $Gr(..., ---)$: if T and U have a different type, the derivation of $T \approx U \rightarrow^G \bot^G$ requires what shown at point (1) above, while if T and U have the same type, but denote different grounds, or different operations on grounds, or one of the two does not denote, it could be indispensable to reason about the "reductions" of T and U, as in (2) above, showing that there is none that goes from one to the other. In any case, some negative formulas are actually derivable, what we will achieve by discussing the derivability of the clause $(\bot_G)$; in general, however, the derivation depends essentially on the use of the logical part, so that the formulas will always be logically complex. Note that what we have said here is similar to a discourse already carried out, in the introduction to the Chapter 5, on the possibility of having non-denoting terms in languages of grounding.

ual constants of Gen is limited to individual constants without free ground-variables.

Definition 63. A *substitution* of x with t in T is a function $\texttt{TERM}_{\texttt{Gen}} \rightarrow \texttt{TERM}_{\texttt{Gen}}$ inductively defined as follows

- $\delta^\alpha[t/x]$ is (the name for) the derivation in the atomic system of the reference base obtained by replacing t with x in δ^α
- $\xi^\alpha[t/x] = \xi^{\alpha[t/x]}$
- $\wedge_I(T, U)[t/x] = \wedge_I(T[t/x], U[t/x])$
- $\wedge_{E,i}(T)[t/x] = \wedge_{E,i}(T[t/x])$
- $\vee_I(T)[t/x] = \vee_I(T[t/x])$
- $\vee_E\ \xi^\alpha\ \xi^\beta\ (T, U_1, U_2)[t/x] = \vee_E\ \xi^{\alpha[t/x]}\ \xi^{\beta[t/x]}\ (T[t/x], U_1[t/x], U_2[t/x])$
- $\rightarrow_I \xi^\alpha(T)[t/x] = \rightarrow I\xi^{\alpha[t/x]}(T[t/x])$
- $\rightarrow E(T, U)[t/x] = \rightarrow E(T[t/x], U[t/x])$
- $\forall Iy(T)[t/x] \ = \begin{cases} \forall Iy(T) \\ \quad \text{if } y = x \\ \forall y(T[t/x]) \\ \quad \text{if } y \neq x \text{ and } y \notin FV(t) \\ \forall Iz((T[z/y])[t/x]) \\ \quad \text{if } y \neq x \text{ and } y \in FV(t),\ z \notin FV(t), FV^I(T) \end{cases}$
- $\forall E(T)[t/x] \ = \begin{cases} \forall E(T) \\ \quad \text{if } T : \forall y\alpha(y), \forall E(T) : \alpha(s/y), x \in FV(s) \text{ and} \\ \quad t \text{ not free for } y \text{ in } \alpha(y) \\ \forall E(T[t/x]) \\ \quad \text{otherwise} \end{cases}$
- $\exists I(T)[t/x] \ = \begin{cases} \exists I(T) \\ \quad \text{if } T : \alpha(s/y), \exists I(T) : \exists y\alpha(y), x \in FV(s) \text{ and} \\ \quad t \text{ not free for } y \text{ in } \alpha(y) \\ \exists I(T[t/x]) \\ \quad \text{otherwise} \end{cases}$

- $\exists E\ y\ \xi^{\alpha(y)}(T,U)[t/x] = \begin{cases} \exists E\ y\ \xi^{\alpha(y)}(T[t/x],U) \\ \quad \text{if } y = x \\ \exists\ y\ \xi^{\alpha(y)[t/x]}(T[t/x],U[t/x]) \\ \quad \text{if } y \neq x \text{ and } y \notin FV(t) \\ \exists E\ z\ \xi^{\alpha(z/y)}(T[t/x],(U[z/y])(t/x])) \\ \quad \text{if } y \neq x \text{ and } y \in FV(t),\ z \notin FV(t), FV^{I}(T) \end{cases}$

Recall that, for the substitution of x with t in α, defined on the notion of first-order logical language, we have assumed to consider only substitutions such that t is free for x in α, in the way that this notion was defined on that occasion.

Definition 64. A *substitution* of ξ^{α} with $T:\alpha$ in U is a function $\mathtt{TERM}_{\mathtt{Gen}} \rightarrow \mathtt{TERM}_{\mathtt{Gen}}$ inductively defined as follows

- $\delta^{\alpha}[T/\xi^{\alpha}] = \delta^{\alpha}$ - recall the convention on individual constants
- $\xi^{\beta}[T/\xi^{\alpha}] = \begin{cases} T & \text{if } \alpha = \beta \\ \xi^{\beta} & \text{if } \alpha \neq \beta \end{cases}$
- $\wedge I(U,Z)[T/\xi^{\alpha}] = \wedge I(U[T/\xi^{\alpha}], Z[T/\xi^{\alpha}])$
- $\wedge_{E,i}(U)[T/\xi^{\alpha}] = \wedge_{E,i}(U[T/\xi^{\alpha}])$
- $\vee I(U)[T/\xi^{\alpha}] = \vee I(U[T/\xi^{\alpha}])$
- $\vee E\ \xi^{\beta}\ \xi^{\gamma}(U,Z_1,Z_2)[T/\xi^{\alpha}] = \begin{cases} \vee E\ \xi^{\beta}\ \xi^{\gamma}(U[T/\xi^{\alpha}],Z_1,Z_2) \\ \quad \text{if } \alpha = \beta = \gamma \\ \vee E\ \xi^{\beta}\ \xi^{\gamma}(U[T/\xi^{\alpha}],Z_1[T/\xi^{\alpha}],Z_2) \\ \quad \text{if } \alpha \neq \beta, \alpha = \gamma \\ \vee E\ \xi^{\beta}\ \xi^{\gamma}(U[T/\xi^{\alpha}],Z_1,Z_2[T/\xi^{\alpha}]) \\ \quad \text{if } \alpha = \beta, \alpha \neq \gamma \\ \vee E\ \xi^{\beta}\ \xi^{\gamma}(U[T/\xi^{\alpha}],Z_1[T/\xi^{\alpha}],Z_2[T/\xi^{\alpha}]) \\ \quad \text{if } \alpha \neq \beta \neq \gamma \end{cases}$
- $\rightarrow I\xi^{\beta}(U)[T/\xi^{\alpha}] = \begin{cases} \rightarrow I\xi^{\beta}(U) & \text{if } \alpha = \beta \\ \rightarrow I\xi^{\beta}(U[T/\xi^{\alpha}]) & \text{if } \alpha \neq \beta \end{cases}$
- $\rightarrow E(U,Z)[T/\xi^{\alpha}] = \rightarrow E(U[T/\xi^{\alpha}], Z[T/\xi^{\alpha}])$

- $\forall Ix(U)[T/\xi^\alpha] = \forall Ix(U[T/\xi^\alpha])$
- $\forall E(U)[T/\xi^\alpha] = \forall E(U[T/\xi^\alpha])$
- $\exists I(U)[T/\xi^\alpha] = \exists I(U[T/\xi^\alpha])$
- $\exists E\ x\ \xi^{\beta(x)}(U,Z)[T/\xi^\alpha] = \begin{cases} \exists E\ x\ \xi^{\beta(x)}(U[T/\xi^\alpha], Z) \\ \quad \text{if } \alpha = \beta(x) \\ \exists E\ x\ \xi^{\beta(x)}(U[T/\xi^\alpha], Z[T/\xi^\alpha]) \\ \quad \text{if } \alpha \neq \beta(x) \end{cases}$

Observe that, in some cases, when replacing ξ^α with $T : \alpha$ in terms of type $\vee E\ \xi^\beta\ \xi^\gamma(U, Z_1, Z_2)$, $\rightarrow I\xi^\beta(T)$ or $\exists E\ x\ \xi^{\beta(x)}(U, Z)$, it could happen that, respectively, ξ^β or ξ^γ, ξ^β and $\xi^{\beta(x)}$ are free in T. This is not problematic: the variables in the terms of $\texttt{Gen}^+$ should be understood as indexed, and the binding of such variables as referred to specific indexes. Thus, the typed variables occurring free in T remain such after the substitution.

Definition 65. The language $\texttt{Gen}^+$ on an atomic base B on a background language L is specified starting from an alphabet containing that of $\texttt{Gen}$ of B - restricted to individual constants without free ground-variables - plus

- functional variables $\texttt{h}_i^{\alpha(x)}$, $\texttt{f}_j^\alpha$ ($\alpha(x), \alpha \in \texttt{FORM}_L$, $i, j \in \mathbb{N}$)
- binary relational symbols : and $\approx$
- logical constants $\wedge^G$, $\vee^G$, $\rightarrow^G$, $\forall^G$, $\exists^G$ and $\bot^G$

In order not to excessively burden the notation, and whenever this does not create ambiguity, we will omit indices and subscripts. The set $\texttt{TERM}_{\texttt{Gen}^+}$ of the *terms* of $\texttt{Gen}^+$ is the smallest set X such that

- $\texttt{TERM}_{\texttt{Gen}} \subset X$
- $\texttt{h}^{\alpha(x)}(t) \in X$ ($t \in \texttt{TERM}_L$)
- $T : \alpha \in \texttt{TERM}_{\texttt{Gen}} \Rightarrow \texttt{f}^\beta(T) : \beta \in X$

plus recursive clauses for forming complex terms starting from the operational symbols of $\texttt{Gen}$ in all analogous to those of definitions 28 and 35. The set $\texttt{FORM}_{\texttt{Gen}^+}$ of the *formulas* of $\texttt{Gen}^+$ is the smallest set X such that

- $T : \alpha \in \texttt{TERM}_{\texttt{Gen}^+} \Rightarrow Gr(T, \alpha) \in X$

- $T, U : \alpha \in \texttt{TERM}_{\texttt{Gen}^+} \Rightarrow T \approx U \in X$
- $\bot^G \in X$
- $A, B \in X \Rightarrow A \star B \in X$ ($\star = \wedge^G, \vee^G, \to^G$, $\neg A \overset{def}{=} A \to^G \bot^G$)
- $A \in X \Rightarrow \star\, \varepsilon\, A \in X$ ($\star = \forall^G, \exists^G$, $\varepsilon = x_i, \xi^\alpha$ for $i \in \mathbb{N}$ and $\alpha \in \texttt{FORM}_L$)

On the language $\texttt{Gen}^+$ just specified, it is now necessary re-define all the notions we have already defined for $\texttt{Gen}$.

Definition 66. The set $S(T)$ of the *subterms* of T is like in definition 36 if $T \in \texttt{TERM}_{\texttt{Gen}}$ and, in the other cases,

$$S(\texttt{h}^{\alpha(x)}(t)) = \{\texttt{h}^{\alpha(x)}(t)\} \text{ and } S(\texttt{f}^\alpha(T)) = S(T) \cup \{\texttt{f}^\alpha(T)\}$$

Definition 67. The set $FV^I(T)$ of the *free individual variables* of T is like in definition 37 if $T \in \texttt{TERM}_{\texttt{Gen}}$ and, in the other cases,

$$FV^I(\texttt{h}^{\alpha(x)}(t)) = FV(t) \text{ and } FV^I(\texttt{f}^\alpha(T)) = FV^I(T)$$

The set $BV^I(T)$ of the *bound individual variables* of T is like in definition 37 if $T \in \texttt{TERM}_{\texttt{Gen}}$ and, in the other cases,

$$BV^I(\texttt{h}^{\alpha(x)}(t)) = BV(t) \text{ and } BV^I(\texttt{f}^\alpha(T)) = BV^I(T)$$

Definition 68. The set $FV^T(U)$ of the *free-ground variables* of U is like in definition 38 if $U \in \texttt{TERM}_{\texttt{Gen}}$ and, in the other cases,

$$FV^T(\texttt{h}^{\alpha(x)}(t)) = \emptyset \text{ and } FV^T(\texttt{f}^\alpha(U)) = FV^T(U)$$

The set $BV^T(U)$ of the *bound ground-variables* of U is like in definition 38 if $U \in \texttt{TERM}_{\texttt{Gen}}$ and, in the other cases,

$$BV^T(\texttt{h}^{\alpha(x)}(t)) = \emptyset \text{ and } BV^T(\texttt{f}^\alpha(U)) = BV^T(U)$$

It should be noted that, in terms of the types $\texttt{h}^{\alpha(x)}$ and $\texttt{f}^\beta(\xi^{\alpha(x)})$, there would be reason to claim that there are actually *two* and *three* free variables, respectively; the first - according to definition 66 - is in both of them the individual variable x, the second of the second term - according to definition 67 - is the ground variable $\xi^{\alpha(x)}$, while the second of the first term and the third of the second one - not identified by any of our previous definitions - are the functional variables, respectively, $\texttt{h}^{\alpha(x)}$ or $\texttt{f}^\beta$. The latter are a sort of second order variables, similar to the variable $\phi(\hat{x})$ of Bertrand Russell and Alfred North Whitehead's *Principia Mathematica* (Russell & Whitehead 1962). However, for the aims of this investigation, we do not consider it necessary to introduce functional variables into the scope of the free variables of a term – neither substitution nor, as anticipated, universal or existential quantification on such variables.

Definition 69. A *substitution* of x with u in T is a function $\mathtt{TERM}_{\mathtt{Gen}^+} \to \mathtt{TERM}_{\mathtt{Gen}^+}$ like in definition 63 if $T \in \mathtt{TERM}_{\mathtt{Gen}}$ and, in the other cases,

$$\mathtt{h}^{\alpha(x)}(t)[u/x] = \mathtt{h}^{\alpha(x)[u/x]}(t[u/x]) \text{ and } \mathtt{f}^{\alpha}(T)[u/x] = \mathtt{f}^{\alpha}(T[u/x])$$

Definition 70. A *substitution* of ξ^α with $T : \alpha \in \mathtt{TERM}_{\mathtt{Gen}}$ in U is a function $\mathtt{TERM}_{\mathtt{Gen}^+} \to \mathtt{TERM}_{\mathtt{Gen}^+}$ like in definition 64 if $U \in \mathtt{TERM}_{\mathtt{Gen}}$ and, in the other cases,

$$\mathtt{h}^{\alpha(x)}(t)[T/\xi^\alpha] = \mathtt{h}^{\alpha(x)}(t) \text{ and } \mathtt{f}^{\alpha}(U)[T/\xi^\alpha] = \mathtt{f}^{\alpha}(U[T/\xi^\alpha])$$

It should be noted that definition 70 binds the substitution of ground-variables to terms of `Gen`. In particular, hence, it prevents substitution of the type $[\mathtt{f}^\alpha(\xi^\alpha)/\xi^\alpha]$, which would give rise to expressions

$$\mathtt{f}^\alpha(\xi^\alpha)[\mathtt{f}^\alpha(\xi^\alpha)/\xi^\alpha] = \mathtt{f}^\alpha(\xi^\alpha[\mathtt{f}^\alpha(\xi^\alpha)/\xi^\alpha]) = \mathtt{f}^\alpha(\mathtt{f}^\alpha(\xi^\alpha))$$

not belonging, according to definition 64, to $\mathtt{TERM}_{\mathtt{Gen}^+}$. Finally, we conclude this Section with some technical definitions for the formulas of $\mathtt{Gen}^+$.

Definition 71. The set $FV^I(A)$ of the *free individual variables* of A is defined inductively as follows -

- $FV^I(Gr(T,\alpha)) = FV^I(T)$ - recall that $FV(\alpha) \subseteq FV^I(T)$
- $FV^I(T \approx U) = FV^I(T) \cup FV^I(U)$
- $FV^I(\bot^G) = \emptyset$
- $FV^I(A \star B) = FV^I(A) \cup FV^I(B)$ $(\star = \wedge^G, \vee^G, \to^G)$
- $FV^I(\star\, \varepsilon\, A) = FV^I(A) - \{\varepsilon\}$ $(\star = \forall^G, \exists^G)$

The set $BV^I(A)$ of the *bound individual variables* of A is defined inductively as follows

- $BV^I(Gr(T,\alpha)) = BV^I(T)$ - observe that $BV(\alpha) \subseteq BV^I(T)$
- $BV^I(T \approx U) = BV^I(T) \cup BV^I(U)$

- $BV^I(\bot^G) = \emptyset$
- $BV^I(A \star B) = BV^I(A) \cup BV^I(B)$ $(\star = \wedge^G, \vee^G, \rightarrow^G)$
- $BV^I(\star\ \varepsilon\ A) = \begin{cases} BV^I(A) \cup \{\varepsilon\} & \text{if } \varepsilon \text{ is of type } x \\ BV^I(A) & \text{otherwise} \end{cases}$ $(\star = \forall^G, \exists^G)$

Definition 72. The set $FV^T(A)$ of the *free ground-variables* of A is defined inductively as follows

- $FV^T(Gr(U, \alpha)) = FV^T(U)$
- $FV^T(U \approx Z) = FV^T(U) \cup FV^T(Z)$
- $FV^T(\bot^G) = \emptyset$
- $FV^T(A \star B) = FV^T(A) \star FV^T(B)$ $(\star = \wedge^G, \vee^G, \rightarrow^G)$
- $FV^T(\star\ \varepsilon\ A) = FV^T(A) - \{\varepsilon\}$ $(\star = \forall^G, \exists^G)$

The set $BV^T(A)$ of the *bound ground-variables* of A is defined inductively as follows

- $BV^T(Gr(U, \alpha)) = BV^T(U)$
- $BV^T(U \approx Z) = BV^T(U) \cup BV^T(Z)$
- $BV^T(\bot^G) = \emptyset$
- $BV^T(A \star B) = BV^T(A) \cup BV^T(B)$ $(\star = \wedge^G, \vee^G, \rightarrow^G)$
- $BV^T(\star\ \varepsilon\ A) = \begin{cases} BV^T(A) \cup \{\varepsilon\} & \text{if } \varepsilon \text{ is of type } \xi^\alpha \\ BV^T(A) & \text{otherwise} \end{cases}$

A is *closed* if, and only if, $FV^I(A) = FV^T(A) = \emptyset$.

Definition 73. A *substitution* of x with t in A is a function $\texttt{FORM}_{\texttt{Gen}^+} \rightarrow \texttt{FORM}_{\texttt{Gen}^+}$ inductively defined as follows

- $(Gr(T, \alpha))[t/x] = Gr(T[t/x], \alpha[t/x])$
- $(T \approx U)[t/x] = (T[t/x] \approx U[t/x])$
- $\bot^G[t/x] = \bot^G$

- $(A \star B)[t/x] = (A[t/x] \star B[t/x])$ $(\star = \wedge^G, \vee^G, \rightarrow^G)$
- $(\star\ \varepsilon\ A)[t/x] = \begin{cases} \star\ \xi^\alpha[t/x]\ A[t/x] & \\ \star\ y\ A[t/x] & \text{if } y \neq x \\ \star\ y\ A & \text{if } y = x \end{cases}$ $(\star = \forall^G, \exists^G)$

Definition 74. A *substitution* of ξ^α with $T : \alpha$ in A is a function $\texttt{FORM}_{\texttt{Gen}^+} \rightarrow \texttt{FORM}_{\texttt{Gen}^+}$ defined inductively as follows

- $(Gr(U, \beta))[T/\xi^\alpha] = Gr(U[T/\xi^\alpha], \beta)$
- $(U \approx Z)[T/\xi^\alpha] = (U[T/\xi^\alpha] \approx Z[T/\xi^\alpha])$
- $\bot^G[T/\xi^\alpha] = \bot^G$
- $(A \star B)[T/\xi^\alpha] = (A[T/\xi^\alpha] \star B[T/\xi^\alpha])$ $(\star = \wedge^G, \vee^G, \rightarrow^G)$
- $(\star\ \varepsilon\ A)[T/\xi^\alpha] = \begin{cases} \star\ \varepsilon\ A[T/\xi^\alpha] & \text{if } \varepsilon \neq \xi^\alpha \\ \star\ \varepsilon\ A & \text{if } \varepsilon = \xi^\alpha \end{cases}$ $(\star = \forall^G, \exists^G)$

As in the case of the substitution functions defined relative to a first-order logical language, also the substitution functions for terms and formulas of $\texttt{Gen}^+$ can be generalized to the case of $n > 1$ concurrent substitutions. In the following two further definitions, $\texttt{ATOM}_{\texttt{Gen}^+}$ is the set of the atomic formulas of $\texttt{FORM}_{\texttt{Gen}^+}$, obtained starting from definition 65.

Definition 75. t is *free* for x in A if, and only if,

- $A \in \texttt{ATOM}_{\texttt{Gen}^+}$
- $A = B \star C$ and t is free for x in B and C $(\star = \wedge^G, \vee^G, \rightarrow^G)$
- $A = \star\ \varepsilon\ B$, $\varepsilon = x$ or $\varepsilon \neq x$, $\varepsilon \notin FV(t)$ and t is free for x in B $(\star = \forall^G, \exists^G)$

Definition 76. $U : \alpha$ is free for ξ^α in A if, and only if,

- $A \in \texttt{ATOM}_{\texttt{Gen}^+}$
- $A = B \star C$ and U is free for ξ^α in B and C $(\star = \wedge^G, \vee^G, \rightarrow^G)$
- $A = \star\ \varepsilon\ B$, $\varepsilon = \xi^\alpha$ or $\varepsilon \neq \xi^\alpha$, $\varepsilon \notin FV^I(U)$, $\varepsilon \notin FV^T(U)$ and U is free for ξ^α in B $(\star = \forall^G, \exists^G)$

At the end of this section, we introduce the following notational convention. Instead of writing $Gr(T, \alpha)$, in order to lighten the notation and the layout of the rules we simply write $T : \alpha$. Therefore, from now on, and except when the context implies something different, $T : \alpha$ stands for "T is a ground for $\vdash \alpha$".

6.2.2 A system for the enriched Gentzen-language

We now introduce a formal system on the Gentzen-extended language of definition 65. As anticipated, we will essentially have rules for the atomic and logical formulas. Among the first, the typing rules will in turn be divided into type introduction and type elimination rules, the identity rules into identity rules as an equivalence relation that preserves the denotation, rules for the replacement of identicals with primitive and non-primitive operational symbols, and equations for non-primitive operational symbols.

6.2.2.1 Typing rules I - typing introductions

The first rules of the system concern the individual constants of $\texttt{Gen}^+$, axiomatically intended as grounds for the atomic formulas of the background language that constitute their type. Thus, forevery $\delta : \alpha \in \texttt{TERM}_{\texttt{Gen}^+}$, bearing in mind the convention we adopted at the beginning of our discussion, namely, to take into account only individual constants without ground-variables, we will have that

$$\frac{}{\delta : \alpha}\ \texttt{C}$$

The next set of rules authorizes the introduction of primitive operational symbols of $\texttt{Gen}^+$, corresponding to the left-right direction of the clauses $(\wedge_G)$ - $(\exists_G)$.

$$\frac{T : \alpha \qquad U : \beta}{\wedge I(T, U) : \alpha \wedge \beta}\ \wedge I \qquad \frac{T : \alpha_i}{\vee I[\alpha_i \rhd \alpha_1 \vee \alpha_2](T) : \alpha_1 \vee \alpha_2}\ \vee I\ (i = 1, 2)$$

$$\frac{\begin{matrix}[\xi^\alpha : \alpha] \\ \vdots \\ T(\xi^\alpha) : \beta\end{matrix}}{\rightarrow I\xi^\alpha(T(\xi^\alpha)) : \alpha \rightarrow \beta} \rightarrow I$$

$$\frac{T(x) : \alpha(x)}{\forall I y T(y/x) : \forall y \alpha(y/x)}\ \forall I \qquad \frac{T : \alpha(t/x)}{\exists I[\alpha(t/x) \rhd \exists x \alpha(x)](T) : \exists x \alpha(x)}\ \exists I$$

We have two restrictions. In $\rightarrow I$, the ground variable ξ^α must not occur free in any undischarged assumption on which the premise $T(\xi^\alpha) : \beta$ depends, other than $\xi^\alpha : \alpha$. In $\forall I$, the individual variable x must not occur free in any undischarged assumption on which the premise $U(x) : \alpha(x)$ depends, nor in β for $\xi^\beta \in FV^T(U(x))$. We will say that ξ^α and x are, respectively, the *proper variables* of $\rightarrow I$ and $\forall I$. As for the primitive operational symbol $\bot_\alpha$, the rule

$$\frac{T : \bot}{\bot_\alpha(T) : \alpha}$$

is derivable in the system we are building, starting – it could be expected – from a rule that expresses the idea that the existence of grounds for $\bot$ generates inconsistency.

6.2.2.2 Typing rules II - Dummett's assumption

We now turn to a second set of rules, corresponding to the direction right-left of the clauses ($\wedge_G$) - ($\exists_G$). These rules will take the form of generalized elimination rules, and will express the idea of the so-called *Dummett's fundamental assumption* (Dummett 1991): if T denotes a ground for a formula with main logical constant k, then it must be possible to reduce T to a canonical term that begins with kI denoting a ground for the same formula. Starting with $\wedge$, we have

$$\dfrac{T : \alpha \wedge \beta \qquad \begin{array}{c}[T \approx \wedge I(\xi^\alpha, \xi^\beta)] \quad [\xi^\alpha : \alpha] \quad [\xi^\beta : \beta] \\ \vdots \\ A\end{array}}{A}\, D_\wedge$$

For the constant $\vee$, we distinguish two cases, depending on whether the form of the major premise involves term beginning with a primitive symbol - labelled $D^c_\vee$ - or not - labelled $D^{nc}_\vee$.

$$\dfrac{\vee I(U) : \alpha_1 \vee \alpha_2 \qquad \begin{array}{c}[\vee I(U) \approx \vee I(\xi^{\alpha_i})] \quad [\xi^{\alpha_i} : \alpha_i] \\ \vdots \\ A\end{array}}{A}\, D^c_\vee$$

$$\dfrac{T : \alpha_1 \vee \alpha_2 \qquad \begin{array}{c}[T \approx \vee I(\xi^{\alpha_1})] \quad [\xi^{\alpha_1} : \alpha_1] \\ \vdots \\ A\end{array} \qquad \begin{array}{c}[T \approx \vee I(\xi^{\alpha_2})] \quad [\xi^{\alpha_2} : \alpha_2] \\ \vdots \\ A\end{array}}{A}\, D^{nc}_\vee$$

The double formulation is somehow justified by the fact that, when a term of the type $\alpha_1 \vee \alpha_2$ is in canonical form, in it occurs a subterm of which, thanks to the typing of $\vee_I$, we explicitly know the type - which, on the other hand, is not guaranteed when the term of the type $\alpha_1 \vee \alpha_2$ is in non-canonical form, as in the case of the ground-variable $\xi^{\alpha_1 \vee \alpha_2}$. Thus, in the case of the canonical form, we have a formulation, as it were, obliged on the assumption of the minor premise of the elimination rule of the type; in the case of the non-canonical form we can choose between two options. But the distinction depends also on the need to respect the syntactic typing of the terms, and therefore on the restriction of the predicate $Gr(..., ---)$; when we derive an analogue of the clause $(\vee_G)$, if T is $\vee I[\alpha_i \rhd \alpha_1 \vee \alpha_2](\xi^{\alpha_i})$ with $i = 1, 2$, we cannot assume $\xi^{\alpha_i} : \alpha_j$ with $j = 1, 2$ and $j \neq i$, since this would not be a formula of the language.

Now we conclude with the elimination rules of type for the remaining constants.

$$\dfrac{T : \alpha \to \beta \qquad \begin{array}{c}[T \approx \to I\xi^{\alpha}(\mathtt{f}^{\beta}(\xi^{\alpha}))] \quad [\forall^G \xi^{\alpha}(\xi^{\alpha} : \alpha \to^G \mathtt{f}^{\beta}(\xi^{\alpha}) : \beta)] \\ \vdots \\ A\end{array}}{A} D_{\to}$$

$$\dfrac{T : \forall x \alpha(x) \qquad \begin{array}{c}[T \approx \forall I x(\mathtt{h}^{\alpha(x)}(x))] \quad [\forall^G x(\mathtt{h}^{\alpha(x)}(x) : \alpha(x))] \\ \vdots \\ A\end{array}}{A} D_{\forall}$$

$$\dfrac{T : \exists x \alpha(x) \qquad \begin{array}{c}[T \approx \exists I(\xi^{\alpha(x)})] \quad [\xi^{\alpha(x)} : \alpha(x)] \\ \vdots \\ A\end{array}}{A} D_{\exists}$$

Some restrictions. In $D_{\wedge}$, ξ^{α} and ξ^{β} must not occur free in any undischarged assumption on which A depends, other than $T \approx \wedge I(\xi^{\alpha}, \xi^{\beta})$, $\xi^{\alpha} : \alpha$ or $\xi^{\beta} : \beta$. In $D^{c}_{\vee}$, ξ^{α_i} must not occur free in any undischarged assumption on which A depends, other than $\vee I(U) \approx \vee I(\xi^{\alpha_i})$ or $\xi^{\alpha_i} : \alpha_i$. In $D^{nc}_{\vee}$, ξ^{α_1} and ξ^{α_2} must not occur free in any undischarged assumption on which A respectively depends on its two occurrences, and respectively other than $T \approx \vee I(\xi^{\alpha_1})$ or $\xi^{\alpha_1} : \alpha_1$ and $T \approx \vee I(\xi^{\alpha_2})$ or $\xi^{\alpha_2} : \alpha_2$. In $D_{\to}$, $\mathtt{f}^{\beta}$ must not occur free in any undischarged assumption on which A depends, other than $T \approx \to I\xi^{\alpha}(\mathtt{f}^{\beta}(\xi^{\alpha}))$ or $\forall^G \xi^{\alpha}(\xi^{\alpha} : \alpha \to^G \mathtt{f}^{\beta}(\xi^{\alpha}) : \beta)$. In $D_{\forall}$, $\mathtt{h}^{\alpha(t)}$, for every $t \in \texttt{TERM}_L$, must not occur free in any undischarged assumption on which A depends and which, for $t = x$, is different from $T \approx \forall Ix(\texttt{h}^{\alpha(x)}(x))$ or $\forall^G x(\texttt{h}^{\alpha(x)}(x) : \alpha(x))$. Finally, in $D_\exists$, x and $\xi^{\alpha(x)}$ must not occur free in any undischarged assumption on which A depends, other than $T \approx \exists I(\xi^{\alpha(x)})$ or $\xi^{\alpha(x)} : \alpha(x)$. The variables on which the restrictions apply are called *proper variables* of the respective rules. Each proper variable - ξ^α and ξ^β for $D_\wedge$; ξ^{α_i} for $D^c_\vee$; ξ^{α_1} and ξ^{α_2} for $D^{nc}_\vee$; $\texttt{f}^\beta$ for $D_\rightarrow$; $\texttt{h}^{\alpha(x)}$ for $D_\forall$; x and $\xi^{\alpha(x)}$ for $D_\exists$ - must not occur free in the conclusion of the corresponding rule.

The last rule of the group we are analyzing concerns the clause $(\bot_G)$, and captures the idea that there are no grounds for $\vdash \bot$.

$$\frac{T : \bot}{\bot^G}\ \bot$$

6.2.2.3 Identity rules

We now deal with the binary predicate ... $\approx - - -$. The first set of rules concerns the obvious properties of reflexivity, symmetry and transitivity.

$$\frac{}{T \approx T}\approx_R \qquad \frac{T \approx U}{U \approx T}\approx_S \qquad \frac{U \approx Z \qquad Z \approx V}{U \approx V}\approx_T$$

To these we add a rule that establishes the preservation of the denotation respect to identity.

$$\frac{T \approx U \qquad U : \alpha}{T : \alpha}\approx_P$$

We then need rules that attribute to primitive operational symbols a functional behaviour.

$$\frac{T_1 \approx U_1 \qquad T_2 \approx U_2}{\wedge I(T_1, T_2) \approx \wedge I(U_1, U_2)}\approx^\wedge_1 \qquad \frac{\wedge I(T_1, T_2) \approx \wedge I(U_1, U_2)}{T_i \approx U_i}\approx^\wedge_{2i},\ i = 1, 2$$

$$\frac{T \approx U}{\vee I(T) \approx \vee I(U)}\approx^\vee_1 \qquad \frac{\vee I(T) \approx \vee I(U)}{T \approx U}\approx^\vee_2,\ T \text{ and } U \text{ have the same type}$$

$$\frac{T \approx U}{\rightarrow I\xi^\alpha(T) \approx \rightarrow I\xi^\alpha(U)}\approx^\rightarrow_1 \qquad \frac{\rightarrow I\xi^\alpha(T) \approx \rightarrow I\xi^\alpha(U)}{T \approx U}\approx^\rightarrow_2$$

$$\frac{T \approx U}{\forall Ix(T) \approx \forall Ix(U)}\approx^\forall_1 \qquad \frac{\forall Ix(T) \approx \forall Ix(U)}{T \approx U}\approx^\forall_2$$

$$\frac{T \approx U}{\exists I(T) \approx \exists I(U)} \approx_1^{\exists} \qquad \frac{\exists I(T) \approx \exists I(U)}{T \approx U} \approx_2^{\exists}$$

$$\frac{T \approx U}{\bot_\alpha(T) \approx \bot_\alpha(U)} \approx_1^{\bot} \qquad \frac{\bot_\alpha(T) \approx \bot_\alpha(U)}{T \approx U} \approx_2^{\bot}$$

We have rules for the identity constituted by (schemes of) equations that set the behaviour of non-primitive operational symbols.

$$\frac{}{\wedge_{E,i}(\wedge I(T_1, T_2)) \approx T_i} R_\wedge,\ i = 1, 2$$

$$\frac{}{\vee E\ \xi^{\alpha_1}\ \xi^{\alpha_2}(\vee I(T_i), U_1(\xi^{\alpha_1}), U_2(\xi^{\alpha_2})) \approx U_i(T_i)} R_\vee,\ i = 1, 2$$

$$\frac{}{\to E(\to I\xi^\alpha(T(\xi^\alpha)), U) \approx T(U)} R_\to$$

$$\frac{}{\forall E[\forall x\alpha(x) \rhd \alpha(t/x)](\forall Ix(T(x)) \approx T(t)} R_\forall$$

$$\frac{}{\exists E\ x\ \xi^{\alpha(x)}(\exists I[\alpha(t/x) \rhd \exists x\alpha(x)](T), U(x, \xi^{\alpha(x)})) \approx U(t, T)} R_\exists$$

Finally, the substitution of identicals with non-primitive operational symbols.

$$\frac{T \approx U}{\wedge_{E,i}(T) \approx \wedge_{E,i}(U)} \approx_{3i}^{\wedge},\ i = 1, 2$$

$$\frac{T \approx U \qquad Z_1(\xi^\alpha) \approx Z_2(\xi^\alpha) \qquad V_1(\xi^\beta) \approx V_2(\xi^\beta)}{\vee E\xi^\alpha\xi^\beta(T, Z_1(\xi^\alpha), V_1(\xi^\beta)) \approx \vee E\xi^\alpha\xi^\beta(U, Z_2(\xi^\alpha), V_2(\xi^\beta))} \approx_3^{\vee}$$

$$\frac{T_1 \approx T_2 \qquad U_1 \approx U_2}{\to E(T_1, U_1) \approx \to E(T_2, U_2)} \approx_3^{\to}$$

$$\frac{T \approx U}{\forall E(T) \approx \forall E(U)} \approx_3^{\forall}$$

$$\frac{T \approx U \qquad Z(\xi^{\alpha(x)}) \approx V(\xi^{\alpha(x)})}{\exists E\ x\ \xi^{\alpha(x)}(T, Z(\xi^{\alpha(x)})) \approx \exists E\ x\ \xi^{\alpha(x)}(U, V(\xi^{\alpha(x)}))} \approx_3^{\exists}$$

However, it is not plausible to introduce also the inverses of each of these last rules. Let us take into account, for example, a rule of the type

$$\frac{\wedge_{E,1}(T) \approx \wedge_{E,1}(U)}{T \approx U}$$

Let now T be the term $\wedge I(\xi^\alpha, \xi^\beta)$ and let U be the term $\wedge I(\xi^\alpha, \xi^\gamma)$; via the appropriate instances of $R_\wedge$

$$\frac{}{\wedge_{E,1}(\wedge I(\xi^\alpha, \xi^\beta)) \approx \xi^\alpha}\, R_\wedge \qquad \frac{}{\wedge_{E,1}(\wedge I(\xi^\alpha, \xi^\gamma)) \approx \xi^\alpha}\, R_\wedge$$

we can easily derive the conclusion $\wedge_{E,1}(\wedge_I(\xi^\alpha, \xi^\beta)) \approx \wedge_{E,1}(\wedge_I(\xi^\alpha, \xi^\gamma))$. However, although surely $\xi^\alpha \approx \xi^\alpha$, if $\beta \neq \gamma$, $\wedge I(\xi^\alpha, \xi^\beta) \approx \wedge I(\xi^\alpha, \xi^\gamma)$ is not well-formed.

6.2.2.4 Logic

Finally, we add the logic. At variance with Gentzen's `IL`, universal and existential quantification rules operate on both individual and ground-variables.

$$\frac{A \quad B}{A \wedge^G B}\,(\wedge^G_I) \qquad \frac{A_1 \wedge^G A_2}{A_i}\,(\wedge^G_{E,i}),\ i = 1,2$$

$$\frac{A_i}{A_1 \vee^G A_2}\,(\vee^G_I),\ i = 1,2 \qquad \frac{A \vee^G B \quad \begin{matrix}[A]\\ \vdots \\ C\end{matrix} \quad \begin{matrix}[B]\\ \vdots \\ C\end{matrix}}{C}\,(\vee^G_E)$$

$$\frac{\begin{matrix}[A]\\ \vdots \\ B\end{matrix}}{A \rightarrow^G B}\,(\rightarrow^G_I) \qquad \frac{A \rightarrow^G B \quad A}{B}\,(\rightarrow^G_E)$$

$$\frac{A(\varepsilon)}{\forall^G \theta A(\theta/\varepsilon)}\,(\forall^G_I) \qquad \frac{\forall^G \varepsilon A(\varepsilon)}{A(\mathfrak{T}/\varepsilon)}\,(\forall^G_E)$$

$$\frac{A(\mathfrak{T}/\varepsilon)}{\exists^G \varepsilon A(\varepsilon)}\,(\exists^G_I) \qquad \frac{\exists^G \theta A(\theta/\varepsilon) \quad \begin{matrix}[A(\varepsilon)]\\ \vdots \\ B\end{matrix}}{B}\,(\exists^G_E)$$

$$\frac{\bot^G}{A}\,(\bot^G)$$

We have the usual restrictions about the rules on quantificators. Moreover, since quantifiers can act on variables of two different types, we have the following restrictions: in $(\forall^G_E)$ and $(\exists^G_I)$, if ε is of type ξ^α, $\mathfrak{T}$ must be of type α. We will call our system GG - that is, Gentzen-grounding.

6.2.2.5 Derivations

We can now define the set of the derivations of our system. Since the definition proceeds by trivial induction, we will provide only few examples.

Definition 77. The set $\mathtt{DER}_{\mathtt{GG}}$ of the *derivations* of GG is the smallest set X defined by standard induction, as in the following examples

- the single node $A \in X$ for every $A \in \mathtt{FORM}_{\mathtt{Gen}^+}$
- $\dfrac{}{\delta^\alpha : \alpha}\,\mathtt{C}$ and $\dfrac{}{T \approx T}\approx_R \in X$, for every $\delta, T \in \mathtt{TERM}_{\mathtt{Gen}^+}$
- every instance of $R_{\wedge,i}$, $R_\vee$, $R_\to$, $R_\forall$ and $R_\exists \in X$
- $\begin{array}{c}\Delta\\ T(x) : \alpha(x)\end{array} \in X$ in compliance with the restriction on $\forall I \Rightarrow$

$$\dfrac{\begin{array}{c}\Delta\\ T(x) : \alpha(x)\end{array}}{\forall I y T(y/x) : \forall y \alpha(y/x)}\,\forall I \qquad \in X$$

- $\begin{array}{c}\Delta_1\\ T : \alpha \vee \beta\end{array}$ and $\begin{array}{cc}T \approx \vee I(\xi^\alpha) & \xi^\alpha : \alpha\\ \multicolumn{2}{c}{\Delta_2}\\ \multicolumn{2}{c}{A}\end{array}$ and $\begin{array}{cc}T \approx \vee I(\xi^\beta) & \xi^\beta : \beta\\ \multicolumn{2}{c}{\Delta_3}\\ \multicolumn{2}{c}{A}\end{array}$ $\in X$,

  in compliance with the restriction on $D^{nc}_\vee \Rightarrow$

$$\dfrac{\begin{array}{c}\Delta_1\\ T : \alpha \vee \beta\end{array} \quad \begin{array}{cc}[T \approx \vee I(\xi^\alpha)] & [\xi^\alpha : \alpha]\\ \multicolumn{2}{c}{\Delta_2}\\ \multicolumn{2}{c}{A}\end{array} \quad \begin{array}{cc}[T \approx \vee I(\xi^\beta)] & [\xi^\beta : \beta]\\ \multicolumn{2}{c}{\Delta_3}\\ \multicolumn{2}{c}{A}\end{array}}{A}\,D^{nc}_\vee$$

  $\in X$

- $\begin{array}{c}\Delta_1\\ T:\alpha\to\beta\end{array}$ and $\begin{array}{cc}T\approx\,\to I\xi^{\alpha}(\mathtt{f}^{\beta}(\xi^{\alpha})) & \forall^{G}\xi^{\alpha}(\xi^{\alpha}:\alpha\to^{G}\mathtt{f}^{\beta}(\xi^{\alpha}):\beta)\\ & \Delta_2\\ & A\end{array}$ $\in X$, in compliance with the restriction on $D_{\to}$ $\Rightarrow$

$$\dfrac{\begin{array}{c}\Delta_1\\ T:\alpha\to\beta\end{array}\quad\begin{array}{cc}[T\approx\,\to I\xi^{\alpha}(\mathtt{f}^{\beta}(\xi^{\alpha}))] & [\forall^{G}\xi^{\alpha}(\xi^{\alpha}:\alpha\to^{G}\mathtt{f}^{\beta}(\xi^{\alpha}):\beta)]\\ & \Delta_2\\ & A\end{array}}{A}\,D_{\to}$$

$\in X$

- $\begin{array}{c}\Delta_1\\ T:\forall x\alpha(x)\end{array}$ and $\begin{array}{cc}T\approx\forall Ix(\xi^{\alpha(x)}) & \xi^{\alpha(x)}:\alpha(x)\\ \Delta_2 & \\ A & \end{array}$ $\in X$, in compliance with the restriction on $D_{\forall}$ $\Rightarrow$

$$\dfrac{\begin{array}{c}\Delta_1\\ T:\forall x\alpha(x)\end{array}\quad\begin{array}{cc}[T\approx\forall Ix(\xi^{\alpha(x)})] & [\xi^{\alpha(x)}:\alpha(x)]\\ \Delta_2 & \\ A & \end{array}}{A}\,D_{\forall}$$

$\in X$

- $\begin{array}{c}\Delta\\ T\approx U\end{array}$ $\in X \Rightarrow$

$$\dfrac{\begin{array}{c}\Delta_1\\ T\approx U\end{array}}{U\approx T}\approx_S \quad \in X$$

- $\begin{array}{c}\Delta_1\\ T\approx U\end{array}$ e $\begin{array}{c}\Delta_2\\ U:\alpha\end{array}$ $\in X \Rightarrow$

$$\dfrac{\begin{array}{c}\Delta_1\\ T\approx U\end{array}\quad\begin{array}{c}\Delta_2\\ U:\alpha\end{array}}{T:\alpha}\approx_P \quad \in X$$

- $\dfrac{\Delta}{T \approx U} \in X \Rightarrow$

$$\dfrac{\dfrac{\Delta}{T \approx U}}{\rightarrow I\xi^{\alpha}(T) \approx \rightarrow I\xi^{\alpha}(U)} \approx_1^{\rightarrow} \quad \in X$$

- $\dfrac{\Delta}{\rightarrow I\xi^{\alpha}(T) \approx \rightarrow I\xi^{\alpha}(U)} \in X \Rightarrow$

$$\dfrac{\dfrac{\Delta}{\rightarrow I\xi^{\alpha}(T) \approx \rightarrow I\xi^{\alpha}(U)}}{T \approx U} \approx_2^{\rightarrow} \quad \in X$$

- $\dfrac{\Delta_1}{T \approx U}$ and $\dfrac{\Delta_2}{Z_1(\xi^{\alpha}) \approx Z_2(\xi^{\alpha})}$ and $\dfrac{\Delta_3}{V_1(\xi^{\beta}) \approx V_2(\xi^{\beta})} \in X \Rightarrow$

$$\dfrac{\dfrac{\Delta_1}{T \approx U} \quad \dfrac{\Delta_2}{Z_1(\xi^{\alpha}) \approx Z_2(\xi^{\alpha})} \quad \dfrac{\Delta_3}{V_1(\xi^{\beta}) \approx V_2(\xi^{\beta})}}{\vee E\xi^{\alpha}\xi^{\beta}(T, Z_1(\xi^{\alpha}), V_1(\xi^{\beta})) \approx \vee E\xi^{\alpha}\xi^{\beta}(U, Z_2(\xi^{\alpha}), V_2(\xi^{\beta}))} \approx_3^{\vee} \quad \in X$$

The other cases are analogous - on the logic the definition mirrors the one for `IL`. A is *derivable* from $\mathfrak{G}$ in `GG`, indicated with $\mathfrak{G} \vdash_{\texttt{GG}} A$, if, and only if, there is $\Delta \in \texttt{DER}_{\texttt{GG}}$ with set of undischarged assumptions $\mathfrak{G}$ and conclusion A.

6.2.3 Some Results

We now show how `GG` permits to prove some theorems, i.e. theorems in the system, and one meta-theorem, i.e. a theorem about the system. The results are, as announced, the following:

- analogue of the clauses (At_G) - $(\bot_G)$;
- well-definition of non-primitive operational symbos;
- meta-theorem establishing that every term is provably denoting, possibly under assumptions;
- rewritability of non-primitive operational symbols.

6.2.3.1 Clauses

GG allows us to derive clauses (At_G) - ($\bot_G$) as theorems. It should be noted how, in light of the type introduction rules this is, from a certain point of view, a foregone result.

The clause (At_G) is respected by the fact that, for every closed derivation in the atomic system of the reference base for a closed atomic formula α of the reference language, to which it corresponds an individual constant δ,

$$\dfrac{}{\delta : \alpha}\,\mathtt{C}$$

and, therefore, $\vdash_{\mathtt{GG}} \delta : \alpha$. In proving the theorems for the other clauses, we will refer to open formulas that, being demonstrated starting from the empty set of assumptions, can be universally generalized. Starting from clause ($\land_G$), it corresponds to the pair of theorems

$1_\land$ $\vdash_{\mathtt{GG}} (\xi^\alpha : \alpha \land^G \xi^\beta : \beta) \to^G \land I(\xi^\alpha, \xi^\beta) : \alpha \land \beta$

$2_\land$ $\vdash_{\mathtt{GG}} \land I(\xi^\alpha, \xi^\beta) : \alpha \land \beta \to^G (\xi^\alpha : \alpha \land^G \xi^\beta : \beta)$

The derivation of $1_\land$ is the following

$$\dfrac{\dfrac{\dfrac{\overset{1}{[\xi^\alpha : \alpha \land^G \xi^\beta : \beta]}}{\xi^\alpha : \alpha}\,(\land^G_{E,1}) \qquad \dfrac{\overset{1}{[\xi^\alpha : \alpha \land^G \xi^\beta : \beta]}}{\xi^\beta : \beta}\,(\land^G_{E,2})}{\land I(\xi^\alpha, \xi^\beta) : \alpha \land \beta}\,\land I}{(\xi^\alpha : \alpha \land^G \xi^\beta : \beta) \to^G \land I(\xi^\alpha, \xi^\beta) : \alpha \land \beta}\,(\to^G_I),\ 1$$

As regards the derivation of $2_\land$, let Δ be the following derivation

$$\dfrac{\dfrac{\dfrac{\overset{2}{[\land I(\xi^\alpha, \xi^\beta) \approx \land I(\xi^\alpha_1, \xi^\beta_1)]}}{\xi^\alpha \approx \xi^\alpha_1}\,\approx^\land_{21} \qquad \overset{3}{[\xi^\alpha_1 : \alpha]}}{\xi^\alpha : \alpha}\,\approx_P \qquad \dfrac{\dfrac{\overset{2}{[\land I(\xi^\alpha, \xi^\beta) \approx \land I(\xi^\alpha_1, \xi^\beta_1)]}}{\xi^\beta \approx \xi^\beta_1}\,\approx^\land_{22} \qquad \overset{4}{[\xi^\beta_1 : \beta]}}{\xi^\beta : \beta}\,\approx_P}{\xi^\alpha : \alpha \land^G \xi^\beta : \beta}\,(\land^G_I)$$

then, the derivation of $2_\land$ is the following

$$\dfrac{\dfrac{\overset{1}{\land I(\xi^\alpha, \xi^\beta) : \alpha \land \beta} \qquad \Delta}{\xi^\alpha : \alpha \land^G \xi^\beta : \beta}\,D_\land,\ 2, 3, 4}{\land I(\xi^\alpha, \xi^\beta) : \alpha \land \beta \to^G (\xi^\alpha : \alpha \land^G \xi^\beta : \beta)}\,(\to^G_I),\ 1$$

As for the clause $(\vee_G)$, it corresponds to the pair of theorems

$$1_\vee \ \vdash_{\texttt{GG}} (\xi^{\alpha_1} : \alpha_1 \to^G \vee I(\xi^{\alpha_1}) : \alpha_1 \vee \alpha_2) \wedge^G (\xi^{\alpha_2} : \alpha_2 \to^G \vee I(\xi^{\alpha_2}) : \alpha_1 \vee \alpha_2)$$

$$2_\vee \ \vdash_{\texttt{GG}} (\vee I(\xi^{\alpha_1}) : \alpha_1 \vee \alpha_2 \to^G \xi^{\alpha_1} : \alpha_1) \wedge^G (\vee I(\xi^{\alpha_2}) : \alpha_1 \vee \alpha_2 \to^G \xi^{\alpha_2} : \alpha_2)$$

The derivations of each of the conjuncts $1_\vee$ and $2_\vee$ are, respectively, the following $(i = 1, 2)$

$$\dfrac{\dfrac{\overset{1}{[\xi^{\alpha_i} : \alpha_i]}}{\vee I(\xi^{\alpha_i}) : \alpha_1 \vee \alpha_2}\,\vee I}{\xi^{\alpha_i} : \alpha_i \to^G \vee I(\xi^{\alpha_i}) : \alpha_1 \vee \alpha_2}\,(\to^G_I),\ 1$$

$$\dfrac{\dfrac{\overset{1}{[\vee I(\xi^{\alpha_i}) : \alpha_1 \vee \alpha_2]} \qquad \dfrac{\dfrac{\overset{2}{[\vee I(\xi^{\alpha_i}) \approx \vee I(\xi_1^{\alpha_i})]}}{\xi^{\alpha_i} \approx \xi_1^{\alpha_i}}\,\approx_2^\vee \qquad \overset{3}{[\xi_1^{\alpha_i} : \alpha_i]}}{\xi^{\alpha_i} : \alpha_i}\,\approx_P}{\xi^{\alpha_i} : \alpha_i}\,D^c_\vee,\ 2, 3}{\vee I(\xi^{\alpha_i}) : \alpha_1 \vee \alpha_2 \to^G \xi^{\alpha_i} : \alpha_i}\,(\to^G_I),\ 1$$

and it is then sufficient to apply a passage of $(\wedge^G_I)$ to complete the derivation. The clause $(\to_G)$ corresponds to the pair of theorems

$$1_\to \ \vdash_{\texttt{GG}} \forall^G \xi^\alpha(\xi^\alpha : \alpha \to^G \texttt{f}^\beta(\xi^\alpha) : \beta) \to^G \to I\xi^\alpha(\texttt{f}^\beta(\xi^\alpha)) : \alpha \to \beta$$

$$2_\to \ \vdash_{\texttt{GG}} \to I\xi^\alpha(\texttt{f}^\beta(\xi^\alpha)) : \alpha \to \beta \to^G \forall^G \xi^\alpha(\xi^\alpha : \alpha \to^G \texttt{f}^\beta(\xi^\alpha) : \beta)$$

The derivation of $1_\to$ is the following

$$\dfrac{\dfrac{\dfrac{\dfrac{\overset{1}{[\forall^G \xi^\alpha(\xi^\alpha : \alpha \to^G \texttt{f}^\beta(\xi^\alpha) : \beta)]}}{\xi^\alpha : \alpha \to^G \texttt{f}^\beta(\xi^\alpha) : \beta}\,(\forall^G_E) \qquad \overset{2}{[\xi^\alpha : \alpha]}}{\texttt{f}^\beta(\xi^\alpha) : \beta}\,(\to^G_E)}{\to I\xi^\alpha(\texttt{f}^\beta(\xi^\alpha)) : \alpha \to \beta}\,\to I,\ 2}{\forall^G \xi^\alpha(\xi^\alpha : \alpha \to^G \xi^\beta : \beta) \to^G \to I\xi^\alpha(\texttt{f}^\beta(\xi^\alpha)) : \alpha \to \beta}\,(\to^G_I),\ 1$$

As for the derivation of $2_\to$, let Δ be the following derivation

$$
\dfrac{\dfrac{\overset{2}{[\to I\xi^{\alpha}(\mathtt{f}^{\beta}(\xi^{\alpha})) \approx\ \to I\xi^{\alpha}(\mathtt{f}_1^{\beta}(\xi^{\alpha}))]}}{\mathtt{f}^{\beta}(\xi^{\alpha}) \approx \mathtt{f}_1^{\beta}(\xi^{\alpha})}\approx_2^{\to} \qquad \dfrac{\dfrac{\overset{3}{[\forall^G\xi^{\alpha}(\xi^{\alpha} : \alpha \to^G \mathtt{f}_1^{\beta}(\xi^{\alpha}) : \beta)]}}{\xi^{\alpha} : \alpha \to^G \mathtt{f}_1^{\beta}(\xi^{\alpha}) : \beta}(\forall_E^G) \qquad \overset{4}{[\xi^{\alpha} : \alpha]}}{\mathtt{f}_1^{\beta}(\xi^{\alpha}) : \beta}(\to_E^G)}{\dfrac{\dfrac{\mathtt{f}^{\beta}(\xi^{\alpha}) : \beta}{\xi^{\alpha} : \alpha \to^G \mathtt{f}^{\beta}(\xi^{\alpha}) : \beta}(\to_I^G),\ 4}{\forall^G\xi^{\alpha}(\xi^{\alpha} : \alpha \to^G \mathtt{f}^{\beta}(\xi^{\alpha}) : \beta)}(\forall_I^G)}\approx_P
$$

then, the derivation of $2_{\to}$ is the following

$$
\dfrac{\dfrac{\overset{1}{\to I\xi^{\alpha}(\mathtt{f}^{\beta}(\xi^{\alpha})) : \alpha \to \beta} \qquad \Delta}{\forall^G\xi^{\alpha}(\xi^{\alpha} : \alpha \to^G \mathtt{f}^{\beta}(\xi^{\alpha}) : \beta)} D_{\to},\ 2, 3}{\to I\xi^{\alpha}(\mathtt{f}^{\beta}(\xi^{\alpha})) : \alpha \to \beta \to^G \forall^G\xi^{\alpha}(\xi^{\alpha} : \alpha \to^G \mathtt{f}^{\beta}(\xi^{\alpha}) : \beta)}(\to_I^G),\ 1
$$

With regard to the theorems corresponding to the clause $(\to_G)$, it is perhaps appropriate at this point an observation; in GG we did not authorize the quantification on functional variables – which would have acted as a kind of second-order quantification. If we had done so, both the derivation of $1_{\to}$ and that of $2_{\to}$ could have continued binding universally $\mathtt{f}^{\beta}$, so as to obtain the pair of theorems

$1^*_{\to}$ $\vdash_{\mathtt{GG}} \forall^G \mathtt{f}^{\beta}(\forall^G\xi^{\alpha}(\xi^{\alpha} : \alpha \to^G \mathtt{f}^{\beta}(\xi^{\alpha}) : \beta) \to^G\ \to I\xi^{\alpha}(\mathtt{f}^{\beta}(\xi^{\alpha})) : \alpha \to \beta)$

$2^*_{\to}$ $\vdash_{\mathtt{GG}} \forall^G \mathtt{f}^{\beta}(\to I\xi^{\alpha}(\mathtt{f}^{\beta}(\xi^{\alpha})) : \alpha \to \beta \to^G \forall^G\xi^{\alpha}(\xi^{\alpha} : \alpha \to^G \mathtt{f}^{\beta}(\xi^{\alpha}) : \beta))$

The clause $(\forall_G)$ will correspond to the pair of theorems

$1_{\forall}$ $\vdash_{\mathtt{GG}} \forall^G x(\mathtt{h}^{\alpha(x)}(x) : \alpha(x)) \to^G \forall I x(\mathtt{h}^{\alpha(x)}(x)) : \forall x\alpha(x)$

$2_{\forall}$ $\vdash_{\mathtt{GG}} \forall I x(\mathtt{h}^{\alpha(x)}(x)) : \forall x\alpha(x) \to^G \forall^G x(\mathtt{h}^{\alpha(x)}(x) : \alpha(x))$

The derivation of $1_{\forall}$ is the following

$$
\dfrac{\dfrac{\dfrac{\overset{1}{[\forall^G x(\mathtt{h}^{\alpha(x)}(x) : \alpha(x))]}}{\mathtt{h}^{\alpha(x)}(x) : \alpha(x)}(\forall_E^G)}{\forall I x(\mathtt{h}^{\alpha(x)}(x)) : \forall x\alpha(x)}\forall I}{\forall^G x(\mathtt{h}^{\alpha(x)}(x) : \alpha(x)) \to^G \forall I x(\mathtt{h}^{\alpha(x)}(x)) : \forall x\alpha(x)}(\to_I^G),\ 1
$$

As for the derivation of $2_{\forall}$, it is the following

$$
\dfrac{\dfrac{\dfrac{\overset{1}{[\forall Ix(\mathtt{h}^{\alpha(x)}(x)):\forall x\alpha(x)]}\qquad \dfrac{\dfrac{\overset{2}{[\forall Ix(\mathtt{h}^{\alpha(x)}(x))\approx\forall Ix(\mathtt{h}_1^{\alpha(x)}(x))]}}{\mathtt{h}^{\alpha(x)}(x)\approx\mathtt{h}_1^{\alpha(x)}(x)}\approx_2^{\forall}\qquad \overset{3}{[\mathtt{h}_1^{\alpha(x)}(x):\alpha(x)]}}{\mathtt{h}^{\alpha(x)}(x):\alpha(x)}\approx_P}{\mathtt{h}^{\alpha(x)}(x):\alpha(x)}\,D_{\forall},\,2,3}{\forall^G x(\mathtt{h}^{\alpha(x)}(x):\alpha(x))}\,(\forall_I^G)}{\forall Ix(\mathtt{h}^{\alpha(x)}(x)):\forall x\alpha(x)\rightarrow^G\forall^G x(\mathtt{h}^{\alpha(x)}(x):\alpha(x))}\,(\rightarrow_I^G),\,1
$$

What we said concerning the quantification over functional variables in the case of $\rightarrow$, also applies here. By authorizing it, we will have been able to obtain the pair of theorems

$1_{\forall}^{*}\ \vdash_{\mathtt{GG}} \forall^G\mathtt{h}^{\alpha(x)}(\forall^G x(\mathtt{h}^{\alpha(x)}(x):\alpha(x))\rightarrow^G\forall Ix(\mathtt{h}^{\alpha(x)}(x)):\forall x\alpha(x))$

$2_{\forall}^{*}\ \vdash_{\mathtt{GG}} \forall^G\mathtt{h}^{\alpha(x)}(\forall Ix(\mathtt{h}^{\alpha(x)}(x)):\forall x\alpha(x)\rightarrow^G\forall^G x(\mathtt{h}^{\alpha(x)}(x):\alpha(x)))$

The clause $(\exists_G)$ corresponds to the pair of theorem

$1_{\exists}\ \vdash_{\mathtt{GG}} \exists^G x(\xi^{\alpha(x)}:\alpha(x))\rightarrow^G\exists^G x(\exists I(\xi^{\alpha(x)}):\exists x\alpha(x))$

$2_{\exists}\ \vdash_{\mathtt{GG}} \exists^G x(\exists I(\xi^{\alpha(x)}):\exists x\alpha(x))\rightarrow^G\exists^G x(\xi^{\alpha(x)}:\alpha(x))$

The derivation of $1_{\exists}$ is the following

$$
\dfrac{\dfrac{\overset{1}{[\exists^G x(\xi^{\alpha(x)}:\alpha(x))]}\qquad \dfrac{\dfrac{\overset{2}{[\xi^{\alpha(x)}:\alpha(x)]}}{\exists I(\xi^{\alpha(x)}):\exists x\alpha(x)}\,(\exists I)}{\exists^G x(\exists I(\xi^{\alpha(x)}):\exists x\alpha(x))}\,(\exists_I^G)}{\exists^G x(\exists I(\xi^{\alpha(x)}):\exists x\alpha(x))}\,(\exists_E^G),\,2}{\exists^G x(\xi^{\alpha(x)}:\alpha(x))\rightarrow^G\exists^G x(\exists I(\xi^{\alpha(x)}):\exists x\alpha(x))}\,(\rightarrow_I^G),\,1
$$

As for the derivation of $2_{\exists}$, it is the following

$$
\dfrac{\dfrac{\overset{1}{[\exists^G x(\exists I(\xi^{\alpha(x)}):\exists x\alpha(x))]}\qquad \dfrac{\overset{2}{[\exists I(\xi^{\alpha(x)}):\exists x\alpha(x)]}\qquad \dfrac{\dfrac{\dfrac{\dfrac{\overset{3}{[\exists I(\xi^{\alpha(x)})\approx\exists I(\xi_1^{\alpha(x)})]}}{\xi^{\alpha(x)}\approx\xi_1^{\alpha(x)}}\approx_2^{\exists}\qquad \overset{4}{[\xi_1^{\alpha(x)}:\alpha(x)]}}{\xi^{\alpha(x)}:\alpha(x)}\approx_P}{\exists^G x(\xi^{\alpha(x)}:\alpha(x))}\,(\exists_I^G)}{}\,D_{\exists},\,3,4}{\exists^G x(\xi^{\alpha(x)}:\alpha(x))}\,(\exists_E^G),\,2}{\exists^G x(\xi^{\alpha(x)}:\alpha(x))}}{\exists^G x(\exists I(\xi^{\alpha(x)}):\exists x\alpha(x))\rightarrow^G\exists^G x(\xi^{\alpha(x)}:\alpha(x))}\,(\rightarrow_I^G),\,1
$$

Finally, it remains the clause $(\bot_G)$, which will be expressed by the theorem

$$\vdash_{\texttt{GG}} \neg\, \exists^G \xi^\bot(\xi^\bot : \bot)$$

the derivation of which is the following

$$\cfrac{\cfrac{\overset{1}{[\exists^G \xi^\bot(\xi^\bot : \bot)]} \qquad \cfrac{\overset{2}{[\xi^\bot : \bot]}}{\bot^G}\,\bot}{\bot^G}\,(\exists_E^G),\,2}{\neg\, \exists^G \xi^\bot(\xi^\bot : \bot)}\,(\to_I^G),\,1$$

6.2.3.2 Definitions checking

A second group of theorems of GG on which we focus shows how the definitions of the non-primitive operational symbols, established through the axioms $R_\wedge$ - $R_\exists$, are in place with respect to the intended operational types. So, the fact that $\wedge_{E,i}$ is fixed in such a way as to behave as an operation on grounds of operational type

$$\alpha_1 \wedge \alpha_2 \rhd \alpha_i$$

will be expressed by the theorem

$$\vdash_{\texttt{GG}} \forall^G \xi^{\alpha_1\wedge\alpha_2}(\xi^{\alpha_1\wedge\alpha_2} : \alpha_1 \wedge \alpha_2 \to^G \wedge_{E,i}(\xi^{\alpha_1\wedge\alpha_2}) : \alpha_i)$$

for which, called Δ the following derivation

$$\cfrac{\cfrac{\cfrac{\overset{2}{[\xi^{\alpha_1\wedge\alpha_2} \approx \wedge I(\xi^{\alpha_1}, \xi^{\alpha_2})]}}{\wedge_{E,i}(\xi^{\alpha_1\wedge\alpha_2}) \approx \wedge_{E,i}(\wedge I(\xi^{\alpha_1}, \xi^{\alpha_2}))}\,\approx_3^\wedge \qquad \cfrac{}{\wedge_{E,i}(\wedge I(\xi^{\alpha_1}, \xi^{\alpha_2})) \approx \xi^{\alpha_i}}\,R_\wedge}{\wedge_{E,i}(\xi^{\alpha_1\wedge\alpha_2}) \approx \xi^{\alpha_i}}\,\approx_T \qquad \overset{3}{[\xi^{\alpha_i} : \alpha_i]}}{\wedge_{E,i}(\xi^{\alpha_1\wedge\alpha_2}) : \alpha_i}$$

we have the following derivation

$$\cfrac{\cfrac{\cfrac{\overset{1}{[\xi^{\alpha_1\wedge\alpha_2} : \alpha_1 \wedge \alpha_2]} \qquad \Delta}{\wedge_{E,i}(\xi^{\alpha_1\wedge\alpha_2}) : \alpha_i}\,D_\wedge,\,2,3}{\xi^{\alpha_1\wedge\alpha_2} : \alpha_1 \wedge \alpha_2 \to^G \wedge_{E,i}(\xi^{\alpha_1\wedge\alpha_2}) : \alpha_i}\,(\to_I^G),\,1}{\forall^G \xi^{\alpha_1\wedge\alpha_2}(\xi^{\alpha_1\wedge\alpha_2} : \alpha_1 \wedge \alpha_2 \to^G \wedge_{E,i}(\xi^{\alpha_1\wedge\alpha_2}) : \alpha_i)}\,(\forall_I^G)$$

The fact that $\vee_E$ is fixed so as to behave as an operation on grounds of operational type

$$\alpha_1 \vee \alpha_2, (\alpha_1 \rhd \beta), (\alpha_2 \rhd \beta) \rhd \beta$$

will be expressed by the theorem - indicated with $\forall^G \xi^{\alpha_1 \vee \alpha_2}\ Th_\vee$ -

$$\vdash_{\texttt{GG}} \forall^G \xi^{\alpha_1 \vee \alpha_2}(((\xi^{\alpha_1 \vee \alpha_2} : \alpha_1 \vee \alpha_2 \wedge^G \forall^G \xi^{\alpha_1}(\xi^{\alpha_1} : \alpha_1 \to^G \mathtt{f}_1^\beta(\xi^{\alpha_1}) : \beta)) \wedge^G$$

$$\forall^G \xi^{\alpha_2}(\xi^{\alpha_2} : \alpha_2 \to^G \mathtt{f}_2^\beta(\xi^{\alpha_2}) : \beta)) \to^G \vee E \xi^{\alpha_1} \xi^{\alpha_2}(\xi^{\alpha_1 \vee \alpha_2}, \mathtt{f}_1^\beta(\xi^{\alpha_1}), \mathtt{f}_2^\beta(\xi^{\alpha_2})) : \beta)$$

for which, called Δ_1^0 the derivation

$$\dfrac{\dfrac{\overset{1}{[(\xi^{\alpha_1 \vee \alpha_2} : \alpha_1 \vee \alpha_2 \wedge^G \forall^G \xi^{\alpha_1}(\xi^{\alpha_1} : \alpha_1 \to^G \mathtt{f}_1^\beta(\xi^{\alpha_1}) : \beta_1)) \wedge^G \forall^G \xi^{\alpha_2}(\xi^{\alpha_2} : \alpha_2 \to^G \mathtt{f}_2^\beta(\xi^{\alpha_2}) : \beta_2)]}}{\xi^{\alpha_1 \vee \alpha_2} : \alpha_1 \vee \alpha_2 \wedge^G \forall^G \xi^{\alpha_1}(\xi^{\alpha_1} : \alpha_1 \to^G \mathtt{f}_1^\beta(\xi^{\alpha_1}) : \beta)}\,(\wedge^G_{E,1})}{\xi^{\alpha_1 \vee \alpha_2} : \alpha_1 \vee \alpha_2}\,(\wedge^G_{E,1})$$

and called Δ_2^0 the derivation

$$\dfrac{\dfrac{\overset{1}{[(\xi^{\alpha_1 \vee \alpha_2} : \alpha_1 \vee \alpha_2 \wedge^G \forall^G \xi^{\alpha_1}(\xi^{\alpha_1} : \alpha_1 \to^G \mathtt{f}_1^\beta(\xi^{\alpha_1}) : \beta_1)) \wedge^G \forall^G \xi^{\alpha_2}(\xi^{\alpha_2} : \alpha_2 \to^G \mathtt{f}_2^\beta(\xi^{\alpha_2}) : \beta_2)]}}{\xi^{\alpha_1 \vee \alpha_2} : \alpha_1 \vee \alpha_2 \wedge^G \forall^G \xi^{\alpha_1}(\xi^{\alpha_1} : \alpha_1 \to^G \mathtt{f}_1^\beta(\xi^{\alpha_1}) : \beta)}\,(\wedge^G_{E,1})}{\forall^G \xi^{\alpha_1}(\xi^{\alpha_1} : \alpha_1 \to^G \mathtt{f}_1^\beta(\xi^{\alpha_1}) : \beta)}\,(\wedge^G_{E,2})$$

and called Δ_3^0 the derivation

$$\dfrac{\overset{1}{[(\xi^{\alpha_1 \vee \alpha_2} : \alpha_1 \vee \alpha_2 \wedge^G \forall^G \xi^{\alpha_1}(\xi^{\alpha_1} : \alpha_1 \to^G \mathtt{f}_1^\beta(\xi^{\alpha_1}) : \beta)) \wedge^G \forall^G \xi^{\alpha_2}(\xi^{\alpha_2} : \alpha_2 \to^G \mathtt{f}_1^\beta(\xi^{\alpha_1}) : \beta)]}}{\forall^G \xi^{\alpha_2}(\xi^{\alpha_2} : \alpha_2 \to^G \mathtt{f}_1^\beta(\xi^{\alpha_1}) : \beta)}\,(\wedge^G_{E,2})$$

and called $\Delta_1^{1,1}$ the derivation

$$\dfrac{\overset{2}{[\xi^{\alpha_1 \vee \alpha_2} \approx \vee I(\xi^{\alpha_1})]} \qquad \dfrac{}{\mathtt{f}_1^\beta(\xi^{\alpha_1}) \approx \mathtt{f}_1^\beta(\xi^{\alpha_1})}\approx_R \qquad \dfrac{}{\mathtt{f}_2^\beta(\xi^{\alpha_2}) \approx \mathtt{f}_2^\beta(\xi^{\alpha_2})}\approx_R}{\vee E \xi^{\alpha_1} \xi^{\alpha_2}(\xi^{\alpha_1 \vee \alpha_2}, \mathtt{f}_1^\beta(\xi^{\alpha_1}), \mathtt{f}_2^\beta(\xi^{\alpha_2})) \approx \vee E \xi^{\alpha_1} \xi^{\alpha_2}(\vee I(\xi^{\alpha_1}), \mathtt{f}_1^\beta(\xi^{\alpha_1}), \mathtt{f}_2^\beta(\xi^{\alpha_2}))}\approx_3^\vee$$

and called $\Delta_1^{1,2}$ the derivation

$$\dfrac{\Delta_1^{1,1} \qquad \dfrac{}{\vee E \xi^{\alpha_1} \xi^{\alpha_2}(\vee I(\xi^{\alpha_1}), \xi_1^\beta, \xi_2^\beta) \approx \mathtt{f}_1^\beta(\xi^{\alpha_1})}R_\vee}{\vee E \xi^{\alpha_1} \xi^{\alpha_2}(\xi^{\alpha_1 \vee \alpha_2}, \mathtt{f}_1^\beta(\xi^{\alpha_1}), \mathtt{f}_1^\beta(\xi^{\alpha_1})) \approx \mathtt{f}_1^\beta(\xi^{\alpha_1})}\approx_T$$

and called Δ_1^2 the derivation

$$\dfrac{\dfrac{\begin{matrix}\Delta_2^0 \\ \forall \xi^{\alpha_1}(\xi^{\alpha_1} : \alpha_1 \to^G \mathtt{f}_1^\beta(\xi^{\alpha_1}) : \beta)\end{matrix}}{\xi^{\alpha_1} : \alpha_1 \to^G \mathtt{f}_1^\beta(\xi^{\alpha_1}) : \beta}\,(\forall_E^G) \qquad \begin{matrix}3 \\ [\xi^{\alpha_1} : \alpha_1]\end{matrix}}{\mathtt{f}_1^\beta(\xi^{\alpha_1}) : \beta}\,(\to_E^G)$$

and called $\Delta_1^{2,1}$ the derivation

$$\dfrac{\begin{matrix}4 \\ [\xi^{\alpha_1 \vee \alpha_2} \approx \vee I(\xi^{\alpha_2})]\end{matrix} \qquad \dfrac{}{\mathtt{f}_1^\beta(\xi^{\alpha_1}) \approx \mathtt{f}_1^\beta(\xi^{\alpha_1})}\,\approx_R \qquad \dfrac{}{\mathtt{f}_2^\beta(\xi^{\alpha_1}) \approx \mathtt{f}_1^\beta(\xi^{\alpha_2})}\,\approx_R}{\vee E \xi^{\alpha_1}\xi^{\alpha_2}(\xi^{\alpha_1 \vee \alpha_2}, \mathtt{f}_1^\beta(\xi^{\alpha_1}), \mathtt{f}_2^\beta(\xi^{\alpha_2})) \approx \vee E \xi^{\alpha_1}\xi^{\alpha_2}(\vee I(\xi^{\alpha_2}), \mathtt{f}_2^\beta(\xi^{\alpha_2}), \mathtt{f}_2^\beta(\xi^{\alpha_2}))}\,\approx_3^\vee$$

and called $\Delta_1^{2,2}$ the derivation

$$\dfrac{\Delta_2^{2,1} \qquad \dfrac{}{\vee E \xi^{\alpha_1}\xi^{\alpha_2}(\vee I(\xi^{\alpha_2}), \mathtt{f}_1^\beta(\xi^{\alpha_1}), \mathtt{f}_2^\beta(\xi^{\alpha_2})) \approx \mathtt{f}_1^\beta(\xi^{\alpha_1})}\,R_\vee}{\vee E \xi^{\alpha_1}\xi^{\alpha_2}(\xi^{\alpha_1 \vee \alpha_2}, \mathtt{f}_1^\beta(\xi^{\alpha_1}), \mathtt{f}_2^\beta(\xi^{\alpha_2})) \approx \mathtt{f}_2^\beta(\xi^{\alpha_2})}\,\approx_T$$

and called Δ_2^2 the derivation

$$\dfrac{\dfrac{\begin{matrix}\Delta_3^0 \\ \forall \xi^{\alpha_2}(\xi^{\alpha_2} : \alpha_2 \to^G \mathtt{f}_2^\beta(\xi^{\alpha_2}) : \beta)\end{matrix}}{\xi^{\alpha_2} : \alpha_2 \to^G \mathtt{f}_2^\beta(\xi^{\alpha_2}) : \beta}\,(\forall_E^G) \qquad \begin{matrix}5 \\ [\xi^{\alpha_2} : \alpha_2]\end{matrix}}{\mathtt{f}_2^\beta(\xi^{\alpha_2}) : \beta}\,(\to_E^G)$$

the derivation of $\forall^G \xi^{\alpha_1 \vee \alpha_2} Th_\vee$ will be - again, note that authorizing the quantification on the functional variables, it is possible, at the end of the following derivation, to quantify universally on $\mathtt{f}_1^\beta$ and $\mathtt{f}_2^\beta$ -

$$\dfrac{\dfrac{\dfrac{\Delta_1^0 \qquad \dfrac{\Delta_1^{1,2} \qquad \Delta_1^2}{\vee E \xi^{\alpha_1}\xi^{\alpha_2}(\xi^{\alpha_1 \vee \alpha_2}, \mathtt{f}_1^\beta(\xi^{\alpha_1}), \mathtt{f}_2^\beta(\xi^{\alpha_2})) : \beta}\,\approx_P \qquad \dfrac{\Delta_1^{2,2} \qquad \Delta_2^2}{\vee E \xi^{\alpha_1}\xi^{\alpha_2}(\xi^{\alpha_1 \vee \alpha_2}, \mathtt{f}_1^\beta(\xi^{\alpha_1}), \mathtt{f}_1^\beta(\xi^{\alpha_1})) : \beta}\,\approx_P}{\vee E \xi^{\alpha_1}\xi^{\alpha_2}(\xi^{\alpha_1 \vee \alpha_2}, \mathtt{f}_1^\beta(\xi^{\alpha_1}), \mathtt{f}_2^\beta(\xi^{\alpha_2})) : \beta}\,D_\vee^{nc},\ 2/5}{Th_\vee}\,(\to_I^G),\ 1}{\forall^G \xi^{\alpha_1 \vee \alpha_2}\ Th_\vee}\,(\forall_I^G)$$

The fact that $\to E$ corresponds to an operation on grounds of operational type

$$\alpha \to \beta, \alpha \rhd \beta$$

will be expressed by the theorem

$$\vdash_{\texttt{GG}} \forall^G \xi^{\alpha\to\beta} \forall^G \xi^\alpha (\xi^{\alpha\to\beta} : \alpha \to \beta \wedge^G \xi^\alpha : \alpha \to^G \to E(\xi^{\alpha\to\beta}, \xi^\alpha) : \beta)$$

for which, called Δ_1 the following derivation

$$\dfrac{\dfrac{\overset{2}{[\xi^{\alpha\to\beta} \approx \to I\xi^\alpha(\mathtt{f}^\beta(\xi^\alpha))]} \qquad \dfrac{}{\xi^\alpha \approx \xi^\alpha}\approx_R}{\to E(\xi^{\alpha\to\beta},\xi^\alpha) \approx \to E(\to I\xi^\alpha(\mathtt{f}^\beta(\xi^\alpha)),\xi^\alpha)}\approx_3^{\to} \qquad \dfrac{}{\to E(\to I\xi^\alpha(\mathtt{f}^\beta(\xi^\alpha)),\xi^\alpha)\approx \mathtt{f}^\beta(\xi^\alpha)}\, R_\to}{\to E(\xi^{\alpha\to\beta},\xi^\alpha)\approx \mathtt{f}^\beta(\xi^\alpha)}$$

and called Δ_2 the following derivation

$$\dfrac{\dfrac{\overset{3}{[\forall^G\xi^\alpha(\xi^\alpha : \alpha \to^G \mathtt{f}^\beta(\xi^\alpha) : \beta)]}}{\xi^\alpha : \alpha \to^G \mathtt{f}^\beta(\xi^\alpha) : \beta}\,(\forall^G_E) \qquad \dfrac{\overset{1}{[\xi^{\alpha\to\beta} : \alpha \to \beta \wedge^G \xi^\alpha : \alpha]}}{\xi^\alpha : \alpha}\,(\wedge^G_{E,2})}{\mathtt{f}^\beta(\xi^\alpha) : \beta}\,(\to^G_E)$$

we will have the following derivation

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overset{1}{[\xi^{\alpha\to\beta} : \alpha \to \beta \wedge^G \xi^\alpha : \alpha]}}{\xi^{\alpha\to\beta} : \alpha \to \beta}\,(\wedge^G_{E,1}) \qquad \dfrac{\Delta_1 \qquad \Delta_2}{\to E(\xi^{\alpha\to\beta},\xi^\alpha) : \beta}\approx_P}{\to E(\xi^{\alpha\to\beta},\xi^\alpha) : \beta}\, D_\to, 2, 3}{\xi^{\alpha\to\beta} : \alpha \to \beta \wedge^G \xi^\alpha : \alpha \to^G \to E(\xi^{\alpha\to\beta},\xi^\alpha) : \beta}\,(\to^G_I), 1}{\forall^G\xi^\alpha(\xi^{\alpha\to\beta} : \alpha \to \beta \wedge^G \xi^\alpha : \alpha \to^G \to E(\xi^{\alpha\to\beta},\xi^\alpha) : \beta)}\,(\forall^G_I)}{\forall^G\xi^{\alpha\to\beta}\forall^G\xi^\alpha(\xi^{\alpha\to\beta} : \alpha \to \beta \wedge^G \xi^\alpha : \alpha \to^G \to E(\xi^{\alpha\to\beta},\xi^\alpha) : \beta)}\,(\forall^G_I)$$

We want $\forall E$ to be an operation on grounds of operational type

$$\forall x\alpha(x) \rhd \alpha(t)$$

- where, clearly, the term t will be appropriately the one occurring in the type of the operational symbol on which we are operating - and we have indeed the theorem

$$\vdash_{\texttt{GG}} \forall^G \xi^{\forall x\alpha(x)}(\xi^{\forall x\alpha(x)} : \forall x\alpha(x) \to^G \forall E(\xi^{\forall x\alpha(x)}) : \alpha(t))$$

for which, called Δ the following derivation

$$\dfrac{\dfrac{\dfrac{\overset{2}{[\xi^{\forall x\alpha(x)} \approx \forall I x(\mathtt{h}^{\alpha(x)}(x))]}}{\forall E(\xi^{\forall x\alpha(x)}) \approx \forall E(\forall I x(\mathtt{h}^{\alpha(x)}(x)))}\approx_3^{\forall} \qquad \dfrac{}{\forall E(\forall I x(\mathtt{h}^{\alpha(x)}(x))) \approx \mathtt{h}^{\alpha(t)}(t)}R_{\forall}}{\forall E(\xi^{\forall x\alpha(x)}) \approx \mathtt{h}^{\alpha(t)}(t)}\approx_T \qquad \dfrac{\overset{3}{[\forall^G x(\mathtt{h}^{\alpha(x)}(x) : \alpha(x))]}}{\mathtt{h}^{\alpha(t)}(t) : \alpha(t)}(\forall_E^G)}{\forall E(\xi^{\forall x\alpha(x)}) : \alpha(t)}\approx_P$$

we will have the following derivation

$$\dfrac{\dfrac{\dfrac{\overset{1}{[\xi^{\forall x\alpha(x)} : \forall x\alpha(x)]} \qquad \Delta}{\forall E(\xi^{\forall x\alpha(x)}) : \alpha(t)} D_{\forall},\ 2,3}{\xi^{\forall x\alpha(x)} : \forall x\alpha(x) \rightarrow^G \forall E(\xi^{\forall x\alpha(x)}) : \alpha(t)} (\rightarrow_I^G),\ 1}{\forall^G \xi^{\forall x\alpha(x)}(\xi^{\forall x\alpha(x)} : \forall x\alpha(x) \rightarrow^G \forall E(\xi^{\forall x\alpha(x)}) : \alpha(t))} (\forall_I^G)$$

Finally, the last property we have to show on the non-primitive operational symbols of `Gen` is that $\exists E$ is defined so as to coincide with an operation on grounds of operational type

$$\exists x\alpha(x), (\alpha(x) \rhd \beta) \rhd \beta$$

we have the theorem - indicated with $\forall^G \xi^{\exists x\alpha(x)}\ Th_{\exists}$ -

$$\vdash_{\mathtt{GG}} \forall^G \xi^{\exists x\alpha(x)}(\xi^{\exists x\alpha(x)} : \exists x\alpha(x) \wedge^G \forall^G x \forall^G \xi^{\alpha(x)}(\xi^{\alpha(x)} : \alpha(x) \rightarrow^G \mathtt{f}^{\beta}(\xi^{\alpha(x)}) : \beta)$$
$$\rightarrow^G \exists E\ x\ \xi^{\alpha(x)}(\xi^{\exists x\alpha(x)}, \mathtt{f}^{\beta}(\xi^{\alpha(x)})) : \beta)$$

for which, called Δ_0 the derivation

$$\dfrac{\overset{1}{[\xi^{\exists x\alpha(x)} : \exists x\alpha(x) \wedge^G \forall^G x \forall^G \xi^{\alpha(x)}(\xi^{\alpha(x)} : \alpha(x) \rightarrow^G \mathtt{f}^{\beta}(\xi^{\alpha(x)}) : \beta)]}}{\xi^{\exists x\alpha(x)} : \exists x\alpha(x)} (\wedge_{E,1}^G)$$

and called Δ_1^1 the derivation

$$\dfrac{\overset{2}{[\xi^{\exists x\alpha(x)} \approx \exists I(\xi^{\alpha(x)})]} \qquad \dfrac{}{\mathtt{f}^{\beta}(\xi^{\alpha(x)}) \approx \mathtt{f}^{\beta}(\xi^{\alpha(x)})}\approx_R}{\exists E\ x\ \xi^{\alpha(x)}(\xi^{\exists x\alpha(x)}, \mathtt{f}^{\beta}(\xi^{\alpha(x)})) \approx \exists E\ x\ \xi^{\alpha(x)}(\exists I(\xi^{\alpha(x)}), \mathtt{f}^{\beta}(\xi^{\alpha(x)}))}\approx_3^{\exists}$$

and called Δ_1^2 the derivation

$$\dfrac{\Delta_1^1 \qquad \dfrac{}{\exists E\ x\ \xi^{\alpha(x)}(\exists I(\xi^{\alpha(x)}), \mathtt{f}^{\beta}(\xi^{\alpha(x)}))) \approx \mathtt{f}^{\beta}(\xi^{\alpha(x)})} R_{\exists}}{\exists E\ x\ \xi^{\alpha(x)}(\xi^{\exists x\alpha(x)}, \xi^{\beta}) \approx \mathtt{f}^{\beta}(\xi^{\alpha(x)})}$$

and called Δ_2 the derivation

$$\cfrac{\cfrac{\cfrac{\cfrac{\overset{1}{[\xi^{\exists x\alpha(x)} : \exists x\alpha(x) \wedge^G \forall^G x \forall^G \xi^{\alpha(x)}(\xi^{\alpha(x)} : \alpha(x) \to^G \mathtt{f}^\beta(\xi^{\alpha(x)}) : \beta)]}}{\forall^G x \forall^G \xi^{\alpha(x)}(\xi^{\alpha(x)} : \alpha(x) \to^G \mathtt{f}^\beta(\xi^{\alpha(x)}) : \beta)} (\wedge^G_{E,2})}{\forall^G \xi^{\alpha(x)}(\xi^{\alpha(x)} : \alpha(x) \to^G \mathtt{f}^\beta(\xi^{\alpha(x)}) : \beta)} (\forall^G_E)}{\xi^{\alpha(x)} : \alpha(x) \to^G \mathtt{f}^\beta(\xi^{\alpha(x)}) : \beta} (\forall^G_E) \qquad \overset{3}{[\xi^{\alpha(x)} : \alpha(x)]}}{\mathtt{f}^\beta(\xi^{\alpha(x)}) : \beta} (\to^G_E)$$

we will have the derivation - again, note that, by authorizing the quantification on the functional variables, we can, at the end of the following derivation, universally quantify on $\mathtt{f}^\beta$ -

$$\cfrac{\cfrac{\Delta_0 \qquad \cfrac{\Delta_1^2 \qquad \Delta_2}{\exists E\ x\ \xi^{\alpha(x)}(\xi^{\exists x\alpha(x)}, \mathtt{f}^\beta(\xi^{\alpha(x)})) : \beta} \approx_P}{\exists E\ x\ \xi^{\alpha(x)}(\xi^{\exists x\alpha(x)}, \mathtt{f}^\beta(\xi^{\alpha(x)})) : \beta} D_\exists, 2, 3}{\cfrac{Th_\exists}{\forall^G \xi^{\exists x\alpha(x)}\ Th_\exists} (\forall^G_I)} (\to^G_I), 1$$

6.2.3.3 Provability of denotation

We now prove the announced analogue of the denotation theorem.

Theorem 78. Let be $U : \beta \in \mathtt{TERM}_{\mathtt{Gen}}$ with $FV^T(U) = \{\xi^{\alpha_1}, ..., \xi^{\alpha_n}\}$. Then $\xi^{\alpha_1} : \alpha_1, ..., \xi^{\alpha_n} : \alpha_n \vdash_{\mathtt{GG}} U : \beta$.

Proof. By induction on the complexity of U

- for individual constants and ground-variables the result is trivial
- $\wedge I(Z_1, Z_2) : \beta_1 \wedge \beta_2$ with $Z_1 : \alpha_1$, $Z_2 : \alpha_2$ and $FV^T(\wedge I(Z_1, Z_2)) = FV^T(Z_1) \cup FV^T(Z_2) \Rightarrow$ by induction hypothesis, there are $\Delta_1, \Delta_2 \in \mathtt{DER}_{\mathtt{GG}}$ that satisfy the required properties, so that

$$\cfrac{\begin{matrix}\Delta_1 \\ Z_1 : \beta_1\end{matrix} \qquad \begin{matrix}\Delta_2 \\ Z_2 : \beta_2\end{matrix}}{\wedge I(Z_1, Z_2) : \beta_1 \wedge \beta_2} \wedge I$$

  satisfies the required properties
- $\wedge_{E,i}(Z) : \beta_i$ $(i = 1, 2)$ with $Z : \beta_1 \wedge \beta_2$ and $FV^T(\wedge_{E,i}(Z)) = FV^T(Z) \Rightarrow$ by induction hypothesis, there is $\Delta \in \mathtt{DER}_{\mathtt{GG}}$ that satisfies the required properties, so that - exploiting a theorem of the previous Section -

$$\dfrac{\begin{matrix}\Delta \\ Z : \beta_1 \wedge \beta_2\end{matrix} \qquad \dfrac{\dfrac{}{\forall^G \xi^{\beta_1 \wedge \beta_2}(\xi^{\beta_1 \wedge \beta_2} : \beta_1 \wedge \beta_2 \to^G \wedge_{E,i}(\xi^{\beta_1 \wedge \beta_2}) : \beta_i)}}{Z : \beta_1 \wedge \beta_2 \to^G \wedge_{E,i}(Z) : \beta_i}\,(\forall^G_E)}{\wedge_{E,i}(Z) : \beta_i}\,(\to^G_E)$$

satisfies the required properties

- $\vee I(Z) : \alpha_1 \vee \alpha_2$ with $Z : \alpha_i$ $(i = 1, 2)$ and $FV^T(\vee I(Z)) = FV^T(Z) \Rightarrow$ by induction hypothesis, there is $\Delta \in \mathtt{DER_{GG}}$ that satisfies the required properties, so that

$$\dfrac{\begin{matrix}\Delta \\ Z : \alpha_i\end{matrix}}{\vee I(Z) : \alpha_1 \vee \alpha_2}\,\vee I$$

satisfies the required properties

- $\vee E \xi^{\beta_1} \xi^{\beta_2}(Z_1, Z_2, Z_3) : \beta_3$ with $Z_1 : \beta_1 \vee \beta_2$, $Z_2 : \beta_3$, $Z_3 : \beta_3$ and

$$\begin{gathered} FV^T(\vee E \xi^{\beta_1} \xi^{\beta_2}(Z_1, Z_2, Z_3)) = \\ FV^T(Z_1) \cup (FV^T(Z_2) - \{\xi^{\beta_1}\}) \cup (FV^T(Z_3) - \{\xi^{\beta_2}\}) \end{gathered}$$

$\Rightarrow$ by induction hypothesis, there are $\Delta_1, \Delta_2, \Delta_3 \in \mathtt{DER_{GG}}$ that satisfy the required properties, so that, called Δ the following derivation

$$\dfrac{\dfrac{\begin{matrix}\Delta_1 \\ Z_1 : \beta_1 \vee \beta_2\end{matrix} \qquad \dfrac{\dfrac{\begin{matrix}[\xi^{\beta_1} : \beta_1] \\ \Delta_2 \\ Z_2 : \beta_3\end{matrix}}{\xi^{\beta_1} : \beta_1 \to^G Z_2 : \beta_3}\,(\to^G_I)}{\forall^G \xi^{\beta_1}(\xi^{\beta_1} : \beta_1 \to^G Z_2 : \beta_3)}\,(\forall^G_I)}{Z : \beta_1 \vee \beta_2 \wedge^G \forall^G \xi^{\beta_1}(\xi^{\beta_1} : \beta_1 \to^G Z_2 : \beta_3)}\,(\wedge^G_I) \qquad \dfrac{\dfrac{\begin{matrix}[\xi^{\beta_2} : \beta_2] \\ \Delta_3 \\ Z : \beta_3\end{matrix}}{\xi^{\beta_2} : \beta_2 \to^G Z_3 : \beta_3}\,(\to^G_I)}{\forall^G \xi^{\beta_2}(\xi^{\beta_2} : \beta_2 \to^G Z_3 : \beta_3)}\,(\forall^G_I)}{(Z : \beta_1 \vee \beta_2 \wedge^G \forall^G \xi^{\beta_1}(\xi^{\beta_1} : \beta_1 \to^G Z_2 : \beta_3)) \wedge^G \forall^G \xi^{\beta_2}(\xi^{\beta_2} : \beta_2 \to^G Z_3 : \beta_3)}\,(\wedge^G_I)$$

we will have that - exploiting a theorem of the previous Section -

$$\dfrac{\Delta \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\forall^G \xi^{\beta_1 \vee \beta_2} \forall^G \xi_1^{\beta_3} \forall^G \xi_2^{\beta_3}\ Th_\vee}}{\forall^G \xi_1^{\beta_3} \forall^G \xi_2^{\beta_3}\ Th_\vee[Z_1/\xi^{\beta_1 \vee \beta_2}]}\,(\forall^G E)}{\forall^G \xi_2^{\beta_3}\ (Th_\vee[Z_1/\xi^{\beta_1 \vee \beta_2}])[Z_2/\xi_1^{\beta_3}]}\,(\forall^G E)}{((Th_\vee[Z_1/\xi^{\beta_1 \vee \beta_2}])[Z_2/\xi_1^{\beta}])[Z_3/\xi_2^{\beta_3}]}\,(\forall^G E)}{\vee E \xi^{\beta_1} \xi^{\beta_2}(Z_1, Z_2, Z_3) : \beta_3}\,(\to^G_E)$$

satisfies the required properties

- $\to I\xi^{\beta_1}(Z) : \beta_1 \to \beta_2$ with $Z : \beta_2$ and $FV^T(\to I\xi^{\beta_1}(Z)) = FV^T(Z) - \{\xi^{\beta_1}\} \Rightarrow$ by induction hypothesis, there is $\Delta \in \mathtt{DER_{GG}}$ that satisfies the required properties, so that

$$\dfrac{\begin{array}{c}[\xi^{\beta_1} : \beta_1] \\ \Delta \\ Z : \beta_2\end{array}}{\to I\xi^{\beta_1}(Z) : \beta_1 \to \beta_2} \to I$$

that satisfies the required properties

- $\to E(Z_1, Z_2) : \beta_2$ con $Z_1 : \beta_1 \to \beta_2$, $Z_2 : \beta_1$ and $FV^T(\to E(Z_1, Z_2)) = FV^T(Z_1) \cup FV^T(Z_2) \Rightarrow$ by induction hypothesis, there are $\Delta_1, \Delta_2 \in \mathtt{DER_{GG}}$ that satisfies the required properties, so that, called Δ the following derivation

$$\dfrac{\begin{array}{c}\Delta_1 \\ Z_1 : \beta_1 \to \beta_2\end{array} \qquad \begin{array}{c}\Delta_2 \\ Z_2 : \beta_1\end{array}}{Z_1 : \beta_1 \to \beta_2 \wedge^G Z_2 : \beta_1} (\wedge^G_I)$$

we will have that - exploiting a theorem of the previous Section -

$$\dfrac{\Delta \qquad \dfrac{\dfrac{\dfrac{}{\forall^G \xi^{\beta_1 \to \beta_2} \forall^G \xi^{\beta_1} (\xi^{\beta_1 \to \beta_2} : \beta_1 \to \beta_2 \wedge^G \xi^{\beta_1} : \beta_1 \to^G \to E(\xi^{\beta_1 \to \beta_2}, \xi^{\beta_1}) : \beta_2)}}{\forall^G \xi^{\beta_1} (Z_1 : \beta_1 \to \beta_2 \wedge^G \xi^{\beta_1} \to^G \to E(Z_1, \xi^{\beta_1}) : \beta_2)} (\forall^G_E)}{Z_1 : \beta_1 \to \beta_2 \wedge^G Z_2 : \beta_1 \to^G \to E(Z_1, Z_2) : \beta_2} (\forall^G_E)}{\to E(Z_1, Z_2) : \beta_2} (\to^G_E)$$

satisfies the required properties

- $\forall Ix(Z) : \forall x\beta(x)$ with $Z : \beta(x)$ and $FV^T(\forall Ix(Z)) = FV^T(Z) \Rightarrow$ by induction hypothesis, there is $\Delta \in \mathtt{DER_{GG}}$ that satisfies the required properties (recall that, for the hypotheses of the theorem, $\forall Ix(Z) \in \mathtt{TERM_{Gen}}$, therefore x cannot occur free in γ for $\xi^\gamma \in FV^T(Z)$ and hence, again by induction hypothesis, x does not occur free in any undischarged assumption of Δ), so that

$$\dfrac{\begin{array}{c}\Delta \\ Z : \beta(x)\end{array}}{\forall xI(Z) : \forall x\beta(x)} \forall I$$

satisfies the required properties

- $\forall E(Z) : \beta(t)$ with $Z : \forall x\beta(x)$ and $FV^T(\forall E(Z)) = FV^T(Z) \Rightarrow$ by induction hypothesis, there is $\Delta \in \mathtt{DER_{GG}}$ that satisfies the required properties, so that - exploiting a theorem of the previous Section -

$$\dfrac{\begin{matrix}\Delta \\ Z : \forall x\beta(x)\end{matrix} \qquad \dfrac{\overline{\forall^G \xi^{\forall x\beta(x)}(\xi^{\forall x\beta(x)} : \forall x\beta(x) \to^G \forall E(\xi^{\forall x\beta(x)}) : \beta(t))}}{Z : \forall x\beta(x) \to^G \forall E(Z) : \beta(t)}\,(\forall_E^G)}{\forall E(Z) : \beta(t)}\,(\to_E^G)$$

satisfies the required properties

- $\exists I(Z) : \exists x\beta(x)$ with $Z : \beta(t)$ and $FV^T(\exists I(Z)) = FV^T(Z) \Rightarrow$ by induction hypothesis, there is $\Delta \in \mathtt{DER_{GG}}$ that satisfies the required properties, so that

$$\dfrac{\begin{matrix}\Delta \\ Z : \beta(t)\end{matrix}}{\exists I(Z) : \exists x\beta(x)}\,\exists I$$

satisfies the required properties

- $\exists E\ x\ \xi^{\beta_1(x)}(Z_1, Z_2) : \beta_2$ with $Z_1 : \exists x\beta_1(x)$, $Z_2 : \beta_2$ and

$$FV^T(\exists E\ x\ \xi^{\beta_1(x)}(Z_1, Z_2)) = FV^T(Z_1) \cup (FV^T(Z_2) - \{\xi^{\beta_1(x)}\})$$

$\Rightarrow$ by induction hypothesis, there are $\Delta_1, \Delta_2 \in \mathtt{DER_{GG}}$ that satisfy the required properties, so that, called Δ the following derivation

$$\dfrac{\begin{matrix}\Delta_1 \\ Z_1 : \exists x\beta_1(x)\end{matrix} \qquad \dfrac{\dfrac{\dfrac{\begin{matrix}[\xi^{\beta_1(x)} : \beta_1(x)] \\ \Delta_2 \\ Z_2 : \beta_2\end{matrix}}{\xi^{\beta_1(x)} : \beta_1(x) \to^G Z_2 : \beta_2}\,(\to_I^G)}{\forall^G \xi^{\beta_1(x)}(\xi^{\beta_1(x)} : \beta_1(x) \to^G Z_2 : \beta_2)}\,(\forall_I^G)}{\forall^G x \forall^G \xi^{\beta_1(x)}(\xi^{\beta_1(x)} : \beta_1(x) \to^G Z_2 : \beta_2)}\,(\forall_I^G)}{Z_1 : \exists x\beta_1(x) \wedge^G \forall^G x \forall^G \xi^{\beta_1(x)}(\xi^{\beta_1(x)} : \beta_1(x) \to^G Z_2 : \beta_2)}\,(\wedge_I^G)$$

we will have that - exploiting a theorem of the previous Section -

$$\dfrac{\Delta \qquad \dfrac{\dfrac{\dfrac{}{\forall^G \xi^{\exists x \beta_1(x)} \forall^G \xi^{\beta_2}\ Th_\exists}}{\forall^G \xi^{\beta_2}\ Th_\exists[Z_1/\xi^{\exists x \beta_1(x)}]}\,(\forall^G_E)}{(Th_\exists[Z_1/\xi^{\exists x \beta_1(x)}])[Z_2/\xi^{\beta_2}]}\,(\forall^G_E)}{\exists E\ x\ \xi^{\beta_1(x)}(Z_1, Z_2) : \beta_2}\,(\rightarrow^G_E)$$

satisfies the required properties

The theorem is hence proven. □

6.2.3.4 Rewriting operational symbols

By way of example, we show a case of rewritability of an operational symbol of an expansion of `Gen`$^+$ obtained by adding an operational symbol

$$F\langle \alpha \vee \beta, \neg\alpha \rhd \beta\rangle$$

as in definition 59 of example 2 in Section 5.2.5. We expand contextually `GG` by adding to it the following rules for the substitution of identicals on F and for the defining equation that sets the behavior of F:

$$\dfrac{T_1 \approx T_2 \qquad U_1 \approx U_2}{F(T_1, U_1) \approx F(T_2, U_2)} \approx^F \qquad \dfrac{}{F(\vee I\langle \beta \rhd \alpha \vee \beta\rangle(T), U) \approx T}\, R_F$$

In the expansion of `GG` thus obtained, it becomes possible to prove the following theorem – which we indicate with $\forall^G \xi^{\alpha\vee\beta}\ \forall^G \xi^{\neg\alpha}\ Th_F$ -

$$\forall^G \xi^{\alpha\vee\beta}\ \forall^G \xi^{\neg\alpha}\ (\xi^{\alpha\vee\beta} : \alpha \vee \beta \wedge^G \xi^{\neg\alpha} : \neg\alpha \rightarrow^G F(\xi^{\alpha\vee\beta}, \xi^{\neg\alpha}) \approx \\ \vee E \xi^\alpha \xi^\beta(\xi^{\alpha\vee\beta}, \bot_\beta(\rightarrow E(\xi^\alpha, \xi^{\neg\alpha})), \xi^\beta)).$$

In the derivation, we take for granted some of the previous theorems on the valid definition of non-primitive operational symbols. Let first Δ_1^0 be the following derivation

$$\dfrac{\dfrac{\overset{2}{[\xi^{\alpha\vee\beta} : \alpha \vee \beta \wedge^G \xi^{\neg\alpha} : \alpha]}}{\xi^{\neg\alpha} : \neg\alpha}\,(\wedge^G_{E,2}) \qquad \overset{3}{[\xi^\alpha : \alpha]}}{\xi^{\neg\alpha} : \neg\alpha \wedge^G \xi^\alpha : \alpha}\,(\wedge^G_I)$$

and let Δ_1 be the following derivation

$$\dfrac{\dfrac{\dfrac{\Delta_1^0 \qquad \dfrac{\dfrac{\dfrac{}{\forall^G\xi^{\neg\alpha}\forall^G\xi^{\alpha}(\xi^{\neg\alpha}:\neg\alpha \wedge^G \xi^{\alpha}:\alpha \to^G \to E(\xi^{\neg\alpha},\xi^{\alpha}):\bot)}}{\forall^G\xi^{\alpha}(\xi^{\neg\alpha}:\neg\alpha \wedge^G \xi^{\alpha}:\alpha \to^G \to E(\xi^{\neg\alpha},\xi^{\alpha}):\bot)}\,(\forall^G_E)}{\xi^{\neg\alpha}:\neg\alpha \wedge^G \xi^{\alpha}:\alpha \to^G \to E(\xi^{\neg\alpha},\xi^{\alpha}):\bot}\,(\forall^G_E)}{\to E(\xi^{\neg\alpha},\xi^{\alpha}):\bot}\,(\to^G_E)}{\bot^G}\,\bot}{F(\xi^{\alpha\vee\beta},\xi^{\neg\alpha}) \approx \vee E\xi^{\alpha}\xi^{\beta}(\xi^{\alpha\vee\beta},\bot_\beta(\to E(\xi^{\alpha},\xi^{\neg\alpha})),\xi^{\beta})}\,(\bot^G)$$

and let Δ_2^0 be the following derivation

$$\dfrac{\dfrac{\overset{4}{[\xi^{\alpha\vee\beta}\approx\vee I(\xi^\beta)]} \qquad \dfrac{}{\xi^{\neg\alpha}\approx\xi^{\neg\alpha}}\approx_R}{F(\xi^{\alpha\vee\beta},\xi^{\neg\alpha})\approx F(\vee I(\xi^\beta),\xi^{\neg\alpha})}\approx^F \qquad \dfrac{}{F(\vee I(\xi^\beta),\xi^{\neg\alpha})\approx\xi^\beta}R_F}{F(\xi^{\alpha\vee\beta},\xi^{\neg\alpha})\approx\xi^\beta}\approx_T$$

and let $\Delta_2^{1,1}$ be the following derivation

$$\dfrac{\overset{5}{[\xi^{\alpha\vee\beta}\approx\vee I(\xi^\beta)]} \qquad \dfrac{}{\bot_\beta(\to E(\xi^{\neg\alpha},\xi^\alpha))\approx\bot_\beta(\to E(\xi^{\neg\alpha},\xi^\alpha))}\approx_R \qquad \dfrac{}{\xi^\beta\approx\xi^\beta}\approx_R}{\vee E\xi^\alpha\xi^\beta(\xi^{\alpha\vee\beta},\bot_\beta(\to E(\xi^{\neg\alpha},\xi^\alpha)),\xi^\beta)\approx\vee E\xi^\alpha\xi^\beta(\vee I(\xi^\beta),\bot_\beta(\to E(\xi^{\neg\alpha},\xi^\alpha)),\xi^\beta)}\approx_3^\vee$$

and let Δ_2^1 be the following derivation

$$\dfrac{\dfrac{\Delta_2^{1,1}\qquad \dfrac{}{\vee E\xi^\alpha\xi^\beta(\vee I(\xi^\beta),\bot_\beta(\to E(\xi^{\neg\alpha},\xi^\alpha)),\xi^\beta)\approx\xi^\beta}\vee_R}{\vee E\xi^\alpha\xi^\beta(\xi^{\alpha\vee\beta},\bot_\beta(\to E(\xi^{\neg\alpha},\xi^\alpha)),\xi^\beta)\approx\xi^\beta}\approx_T}{\xi^\beta\approx\vee E\xi^\alpha\xi^\beta(\xi^{\alpha\vee\beta},\bot_\beta(\to E(\xi^{\neg\alpha},\xi^\alpha)),\xi^\beta)}\approx_S$$

and let Δ_2 be the following derivation

$$\dfrac{\Delta_2^0 \qquad \Delta_2^1}{F(\xi^{\alpha\vee\beta},\xi^{\neg\alpha})\approx\vee E\xi^\alpha\xi^\beta(\xi^{\alpha\vee\beta},\bot_\beta(\to E(\xi^{\neg\alpha},\xi^\alpha)),\xi^\beta)}\approx_T$$

then the derivation of the theorem is the following

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overset{1}{[\xi^{\alpha\vee\beta}:\alpha\vee\beta\wedge^G\xi^{\neg\alpha}:\neg\alpha]}}{\xi^{\alpha\vee\beta}:\alpha\vee\beta}\,(\wedge^G_{E,1}) \qquad \Delta_1 \qquad \Delta_2}{F(\xi^{\alpha\vee\beta},\xi^{\neg\alpha})\approx\vee E\xi^\alpha\xi^\beta(\xi^{\alpha\vee\beta},\bot_\beta(\to E(\xi^{\neg\alpha},\xi^\alpha)),\xi^\beta)}\,D_\vee,3,4,5}{Th_F}\,(\to^G_I),1,2}{\forall^G\xi^{\neg\alpha}\;Th_F}\,(\forall^G I)}{\forall^G\xi^{\alpha\vee\beta}\;\forall^G\xi^{\neg\alpha}\;Th_F}\,(\forall^G_I)$$

6.2.3.5 A problem

In this Section we emphasize a difficulty inherent in the system of grounding for the enriched Gentzen-language. As we have already anticipated, the inverse of theorem 78 does not apply. In other words, there are terms $U : \beta \in \texttt{TERM}_{\texttt{Gen}}$ such that $\xi^{\alpha_1} : \alpha_1, ..., \xi^{\alpha_n} : \alpha_n \vdash_{\texttt{GG}} U : \beta$, but $\{\xi^{\alpha_1}, ..., \xi^{\alpha_n}\} \subset FV^T(U)$; more specifically, this means that there are terms with occurrences of free ground-variables (which therefore should correspond not to grounds, but to operations on grounds) that the system proves, depending on an empty set of assumptions, to denote grounds for a formula. A very simple example is the following; given (the name for) an atomic derivation δ for an atomic formula α, we will have

$$\dfrac{\dfrac{}{\wedge_{E,1}(\wedge I(\delta, \xi^\beta)) \approx \delta}\, R_\wedge \qquad \dfrac{}{\delta : \alpha}\, \texttt{C}}{\wedge_{E,1}(\wedge I(\delta, \xi^\beta)) : \alpha}\, \approx_P$$

so that $\vdash_{\texttt{GG}} \wedge_{E,1}(\wedge I(\delta, \xi^\beta)) : \alpha$ for $FV^T(\wedge_{E,1}(\wedge I(\delta, \xi^\beta))) = \{\xi^\beta\} \neq \emptyset$. In general, we can say that the phenomenon depends on a difference between the definition of the denotation and the axioms for the elimination of non-primitive symbols of which GG is endowed. In the first case, we have defined the denotation of open terms by referring to *closed instances* of the latter, so that they corresponded to operations defined on *grounds*; in the second, instead, we have introduced identity rules that authorize "syntactic transformations" defined, without restrictions, on both open terms and closed terms - and which, in this sense, correspond to the normalization operations through which Prawitz (Prawitz 2006) proves his correspondent theorems.

6.3 A class of systems

What has been said so far is a good exemplification to get an idea of the elements of the class of systems that can be associated with the various languages of grounding of the previous Chapter - appropriately enriched. And this is not for the structure alone that GG presents, but also and above all for some of the properties it enjoys, which are, or are expected to be, valid in any system of grounding that can be considered appropriate.

6.3.1 Invariant and characteristic rules

GG contains two main groups of rules: the first is assigned the task to determine the behaviour of atomic formulas, while the second concerns the logic of the system. The rules for the atomic case are in turn of two types: some concern constructs involving the binary predicate ... : − − − − namely, given the notational simplification we have adopted, $Gr(..., ---)$ – whereas others concern constructs involving the binary predicate ... ≈ − − −. The discourse does not stop there, since, as it is easy to notice, there are further subdistinctions. As for ... : − − −, we have two classes: introduction of type, and elimination of type based on Dummett's fundamental assumption (Dummett 1991). As for ... ≈ − − −, we have four classes instead: determination of an equivalence relation that preserves the denotation, substitution of identicals on primitive operational symbols (in the two directions), equations that define non-primitive operational symbols, and substitution of identicals on non-primitive operational symbols (in only one direction). To recap:

- Atomic formulas
  - Rules for ... : − − − (namely $Gr(..., - - -)$)
    - Type introduction
    - Type elimination
  - Rules for ... ≈ − − −
    - Equivalence relation that preserves denotation
    - Substitution for primitive operational symbols
    - Definition of non-primitive operational symbols
    - Substitution for non-primitive operational symbols
- Logic

Well, if from GG we now want to generalize to systems of grounding $\Lambda\mathtt{G}$ on arbitrary - and appropriately enriched - languages of grounding Λ, some of the rules for GG will also be rules of $\Lambda\mathtt{G}$ – and we will use for them the expression "invariant rules" – while other rules will be specific to $\Lambda\mathtt{G}$ – and for them we will use the expression "characteristic rules". The different systems of grounding will differ with respect to the characteristic rules, and the intersection of all these systems (on the same base) will be consisting of the invariant rules; in the same way, a system $\Lambda\mathtt{G}_2$ can be considered as an expansion of a system $\Lambda\mathtt{G}_1$ if the set of rules characteristic of $\Lambda\mathtt{G}_1$ is contained in the set of the rules characteristic of $\Lambda\mathtt{G}_2$ (on the same base). More specifically, taking up the previous scheme, below we indicate bold the rules to be considered characteristic of a system of grounding, being all the others to be considered invariants.

- Atomic formulas

– Rules for ... : – – – (namely $Gr(..., - - -)$)
    * Type introduction
    * Type elimination
– Rules for ... $\approx$ – – –
    * Equivalence relation that preserves denotation
    * Substitution for primitive operational symbols
    * **Definition of non-primitive operational symbols**
    * **Substitution for non-primitive operational symbols**

- Logic

We immediately notice that the distinction between invariant and characteristic rules faithfully follows the distinction between core language of grounding and language of grounding in general: beyond the logic and the determination of an identity relation between terms, the invariant rules concern the primitive operational symbols (core languages of grounding), whereas the characteristics rules vary depending on the non-primitive operational symbols of the language of grounding of reference (arbitrary languages of grounding).

6.3.2 General form of characteristic rules

Given a generic - and appropriately enriched - language of grounding Λ, it is at this point rather easy to suggest the general form of a substitution rule on non-primitive operational symbols in an appropriate system of grounding on Λ; if F is a non-primitive operational symbol of Λ with operational type

$$\alpha_1, ..., \alpha_n \rhd \beta$$

that binds a sequence of ground-variables ξ and a sequence of individual variables x, we will have that, for every $T_i, U_i : \alpha_i$ $(i \leq n)$,

$$\frac{T_1 \approx U_1 \qquad \dots \qquad T_n \approx U_n}{F\ \xi\ x\ (T_1, ..., T_n) \approx F\ \xi\ x\ (U_1, ..., U_n)} \approx^F$$

However, it is less obvious what the general form of a scheme of equations should be for the definition of F. The idea is, of course, that the schema fixes F so that the latter can be considered as a total constructive function, of which the operational type has, in correspondence of the operational type and the binding of individual variables and ground-variables intended, the lines indicated by the definition of the notion of denotation in the previous chapter. In this sense, the schemes of equations for the definition of F must plausibly undergo some restrictions, by virtue of which the just mentioned *desideratum* can be considered satisfied; among these restrictions, we can obviously request an analogue of those placed on the reduction procedures for the argument structures, which Prawitz introduces in *Towards a foundation of a general proof theory* (Prawitz 1973), and that we have expounded in Section 2.5.2.1. Thus, called R_F the scheme of equations that defines F, we must have that: (1) each instance of R_F is of the form

$$\frac{}{F\ \xi\ x\ (U_1, ..., U_n) \approx Z}\ R_F$$

for some $U_1, ..., U_n \in \texttt{TERM}_\Lambda$; (2) for every instance of R_F of the type indicated above, $FV^T(F\ \xi\ x\ (U_1, ..., U_n)) \subseteq FV^T(Z)$ - observe how, unlike what was done for the reduction procedures on the argumentative structures discussed in Section 2.5.2.1, there is no need here to request that $F\ \xi\ x\ (U_1, ..., U_n)$ and Z have the same type, since otherwise the conclusion of R_F would not be a formula of Λ; (3) for every instance of R_F of the type indicated above, for every substitution $(*/\circ)$ of ground-variables with terms, there is an instance of R_F of the form

$$\frac{}{(F\ \xi\ x\ (U_1, ..., U_n))(*/\circ) \approx V}\ R_F$$

and, called $\Lambda\texttt{G}$ the system where R_F occurs, it holds that

$$\vdash_{\Lambda\texttt{G}} V \approx Z(*/\circ)$$

- that is, if we consider an application of R_F as an application of the function to which F is inteded to correspond, we require linearity on substitution; with a slight but intuitive abuse of notation, $R_F(F\ \xi\ x\ (U_1, ..., U_n))(*/\circ) = Z(*/\circ) \approx V = R_F(F\ \xi\ x\ (U_1, ..., U_n)(*/\circ))$.

As we can easily see, the derivations for the analogue of the clauses $(\wedge_G)$ - $(\bot_G)$ exposed with respect to GG do not employ any of the characteristic rules of GG; it follows that, in any arbitrary system of grounding $\Lambda\texttt{G}$ for an arbitrary – and appropriately enriched – language of grounding Λ, these derivations are still available. This does not apply, of course, to the derivations that show the adequacy of the definitions for the non-primitive operational symbols of GG; in this case, it is essential the use of schemes equations characteristic of GG, so that an arbitrary system of grounding $\Lambda\texttt{G}$ on an arbitrary – and opportunely enriched – language of grounding Λ will be able to prove such results only if Λ is an expansion of $\texttt{Gen}^+$, and consequently $\Lambda\texttt{G}$ an expansion of GG. However, it seems plausible to assume that any system of grounding, with schemes of equations appropriate to the definition of the non-primitive operational symbols of the language of grounding of reference, is able to prove the adequacy of these schemes; as a particular example, given a language of grounding – appropriately enriched – Λ with an operational symbol F which is associated with an operational type

$$\alpha_1, \alpha_2, \alpha_3 \rhd \beta$$

and that binds an individual variable x on index 2, an individual variable y on index 3, and an assumption δ on index 3, and given a scheme of equations R_F for the definition of F, it is expected that a system of grounding $\Lambda\mathtt{G}$ for Λ proves

$$F\ x\ y\ \xi^{\delta}\ (\xi^{\alpha_1}, \xi^{\alpha_2}, \mathtt{f}^{\alpha_3}(\xi^{\delta})) : \beta$$

from the assumption

$$\xi^{\alpha_1} : \alpha_1 \wedge^G \forall x(\xi^{\alpha_2} : \alpha_2) \wedge^G \forall y\ \forall \xi^{\delta}(\xi^{\delta} : \delta \rightarrow^G \mathtt{f}^{\alpha_3}(\xi^{\delta}) : \alpha_3)$$

in such a way that it is then possible to universally quantify over $\xi^{\alpha_1}, \xi^{\alpha_2}$, and possibly $\mathtt{f}^{\alpha_3}$ - exactly as done for the non-primitive operational symbols of $\mathtt{Gen}^+$. Finally, it seems plausible to assume that for a system of grounding that respects the properties just indicated it is also possible to prove an analogue of theorem 78.

6.3.3 Reductions and permutations

So as for $\mathtt{GG}$, each system of grounding $\Lambda\mathtt{G}$ contains the rules of first-order intuitionistic logic; therefore, the derivations of this system could present detours, with corresponding maximal formulas defined as in Section 2.5.1. In order to eliminate redundancies of this type, we can use the reductions $\wedge$-rid, $\vee$-rid, $\rightarrow$-rid, $\forall$-rid and $\exists$-rid that Prawitz (Prawitz 2006) utilizes for the proof of its normalization theorems, and that we have already presented again in Section 2.5.1 – which is why we will assume them here as acquired.

6.3.3.1 Reductions of maximal points

At variance with a simple intuitionistic first-order logic on a standard first-order logical language, however, $\Lambda\mathtt{G}$ could also contain also detours of another type, in which the maximal points are conclusions of type introduction rules, and major premises of type elimination rules. Modifying the terminology so far adopted for reasons that will appear clear below, we will use in this case the term *cut-type*, reserving the expression *cut-formula* to the standard redundancies on the rules of intuitionistic first-order logic.

So as with the cut-formulas, also with the cut-types it is possible to associate some reductions. We show them below, starting with the easiest cases of $\wedge$, $\vee$ and $\exists$. As for $\wedge$,

$$\dfrac{\dfrac{\begin{matrix}\Delta_1 & \Delta_2\\ T:\alpha & U:\beta\end{matrix}}{\boldsymbol{\wedge I(T,U):\alpha\wedge\beta}}\wedge I \qquad \begin{matrix}[\wedge I(T,U)\approx\wedge I(\xi^\alpha,\xi^\beta)] \quad [\xi^\alpha:\alpha] \quad [\xi^\beta:\beta]\\ \Delta_3(\xi^\alpha,\xi^\beta)\\ A\end{matrix}}{A}D_\wedge$$

it converts through $\mathtt{T}_\wedge$-rid in

$$\begin{matrix}\dfrac{}{\wedge I(T,U)\approx\wedge I(T,U)}\approx_R \quad \begin{matrix}\Delta_1\\ T:\alpha\end{matrix} \quad \begin{matrix}\Delta_2\\ U:\beta\end{matrix}\\ \Delta_3(T/\xi^\alpha,U/\xi^\beta)\\ A\end{matrix}$$

As for $\vee$,

$$\dfrac{\dfrac{\begin{matrix}\Delta_1\\ T:\alpha_i\end{matrix}}{\boldsymbol{\vee I[\alpha_i\rhd\alpha_1\vee\alpha_2](T):\alpha_1\vee\alpha_2}}\vee I \qquad \begin{matrix}[\vee I(T)\approx\vee I(\xi^{\alpha_i})] \quad [\xi^{\alpha_i}:\alpha_i]\\ \Delta_2(\xi^{\alpha_i})\\ A\end{matrix}}{A}D^c_\vee$$

it converts through $\mathtt{T}_\vee$-rid in

$$\begin{matrix}\dfrac{}{\vee I(T)\approx\vee I(T)}\approx_R \quad \begin{matrix}\Delta_1\\ T:\alpha_i\end{matrix}\\ \Delta_2(T/\xi^{\alpha_i})\\ A\end{matrix}$$

As for $\exists$,

$$\dfrac{\dfrac{\begin{matrix}\Delta_1\\ T:\alpha(t/x)\end{matrix}}{\boldsymbol{\exists I[\alpha(t/x)\rhd\exists x\alpha(x)](T):\exists x\alpha(x)}}\exists I \qquad \begin{matrix}[\exists I(T)\approx\exists I(\xi^{\alpha(x)})] \quad [\xi^{\alpha(x)}:\alpha(x)]\\ \Delta_2(x,\xi^{\alpha(x)})\\ A\end{matrix}}{A}D_\exists$$

it converts through $\mathtt{T}_{\exists}$-rid in

$$\dfrac{}{\exists I(T) \approx \exists I(T)} \approx_R \quad \begin{array}{c} \Delta_1 \\ T : \alpha(t/x) \end{array}$$
$$\Delta_2(t/x, T/\xi^{\alpha(x)})$$
$$A$$

We hence conclude with the more comples cases of $\rightarrow$ and $\forall$, requiring an "additional" derivation after conversion - that is, not a mere composition of previously given derivations. As for $\rightarrow$,

$$\dfrac{\dfrac{\begin{array}{c}[\xi^{\alpha} : \alpha] \\ \Delta_1(\xi^{\alpha}) \\ T(\xi^{\alpha}) : \beta\end{array}}{\boldsymbol{\rightarrow I\xi^{\alpha}(T(\xi^{\alpha})) : \alpha \rightarrow \beta}} \rightarrow I \qquad \begin{array}{c}[\rightarrow I\xi^{\alpha}(T(\xi^{\alpha})) \approx \rightarrow I\xi^{\alpha}(\mathtt{f}^{\beta}(\xi^{\alpha}))] \quad [\forall^{G}\xi^{\alpha}(\xi^{\alpha} : \alpha \rightarrow^{G} \mathtt{f}^{\beta}(\xi^{\alpha}) : \beta)] \\ \Delta_2(\mathtt{f}^{\beta}) \\ A\end{array}}{A} D_{\rightarrow}$$

it converts through $\mathtt{T}_{\rightarrow}$-rid in

$$\dfrac{}{\rightarrow I\xi^{\alpha}(T(\xi^{\alpha})) \approx \rightarrow I\xi^{\alpha}(T(\xi^{\alpha}))} \approx_R \quad \dfrac{\dfrac{\begin{array}{c}[\xi^{\alpha} : \alpha] \\ \Delta_1(\xi^{\alpha}) \\ T(\xi^{\alpha}) : \beta\end{array}}{\xi^{\alpha} : \alpha \rightarrow^{G} T(\xi^{\alpha}) : \beta} (\rightarrow^{G}_{I})}{\forall^{G}\xi^{\alpha}(\xi^{\alpha} : \alpha \rightarrow^{G} T(\xi^{\alpha}) : \beta)} (\forall^{G}_{I})$$
$$\Delta_2(T/\mathtt{f}^{\beta})$$
$$A$$

As for $\forall$,

$$\dfrac{\dfrac{\begin{array}{c}\Delta_1(x) \\ T(x) : \alpha(x)\end{array}}{\boldsymbol{\forall Ix(T(x)) : \forall x\alpha(x)}} \forall I \qquad \begin{array}{c}[\forall Ix(T(x)) \approx \forall Ix(\mathtt{h}^{\alpha(x)}(x))] \quad [\forall^{G}x(\mathtt{h}^{\alpha(x)}(x) : \alpha(x))] \\ \Delta_2(\mathtt{h}^{\alpha(x)}) \\ A\end{array}}{A} D_{\forall}$$

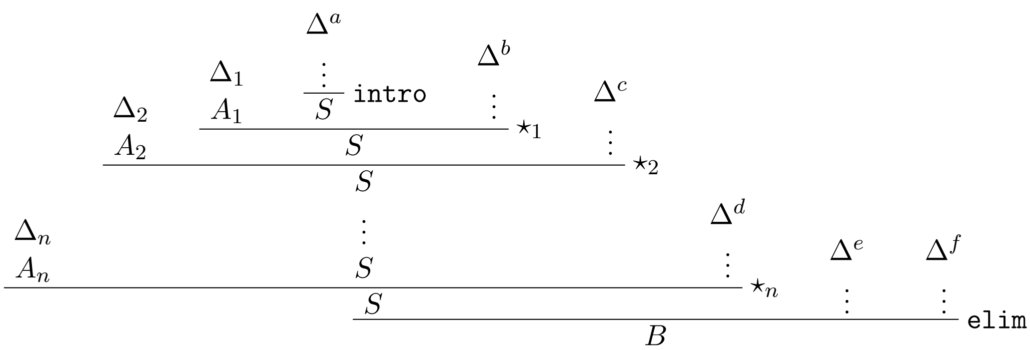
it converts through $\mathtt{T}_{\forall}$-rid in

$$\dfrac{}{\forall Ix(T(x)) \approx \forall Ix(T(x))} \approx_R \quad \dfrac{\begin{array}{c}\Delta_1(x) \\ T(x) : \alpha(x)\end{array}}{\forall^{G}x(T(x) : \alpha(x))} (\forall^{G}_{E})$$
$$\Delta_2(T/\mathtt{h}^{\alpha(x)})$$
$$A$$

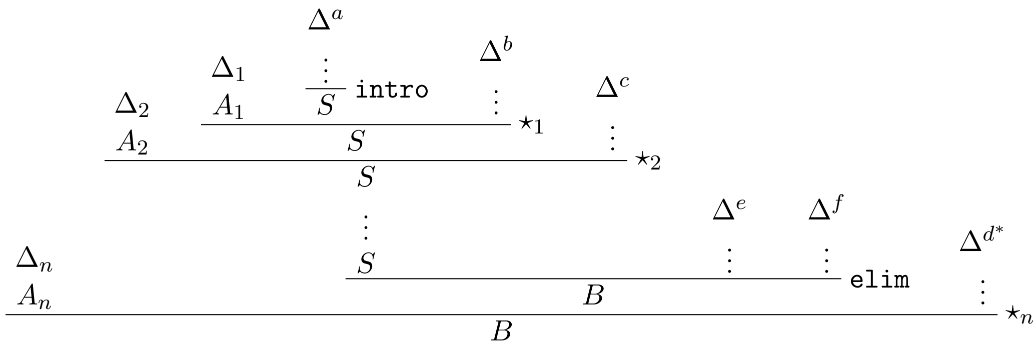
6.3.3.2 Cut-segments and permutations

When proving his normalization theorems, Prawitz not only uses the reductions we have shown in Section 2.5.1, but also the so-called *permutations*. Although we have not needed any of these operations so far, which is why we have been able to avoid mentioning them, they are required also in the systems of grounding we have been developing, so that we have to deal with them here in more detail.

The necessity for permutations depends, in first-order intuitionistic logic, on the presence of the rules $(\vee_E)$ and $(\exists_E)$. It could indeed happen that, during a derivation, there is a concatenation of applications of such rules in which one or both the minor premises of the first application are obtained by introduction, and the conclusion of the latter is a major premise of an elimination - given that, obviously, other minor premises of the intermediate applications could be obtained by introduction. This also applies to our systems of grounding, and also with reference to type elimination rules. In other words, from a very general point of view, we may find ourselves in the following situation:



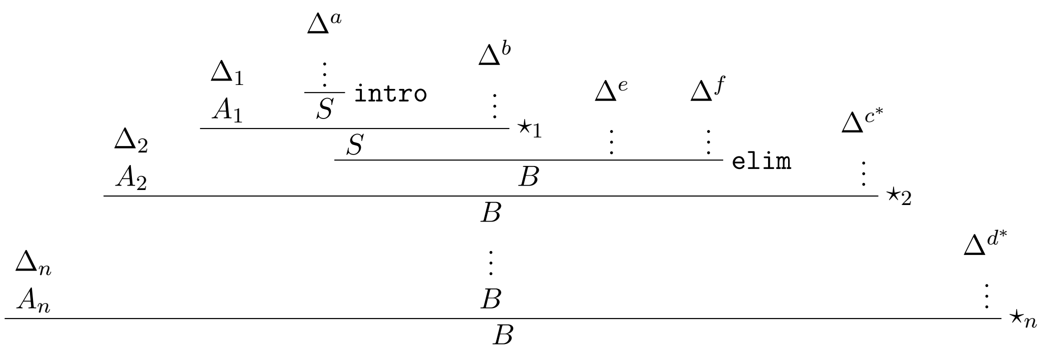
where: Δ^b, Δ^c and Δ^d could in turn end with an introduction; Δ^b, Δ^c, Δ^d, Δ^e and Δ^f could be "empty", that is, the rules to which they correspond could have a number of premises lower than that indicated in our general representation; $\star_i$ $(i \leq n)$ is $(\vee_E^G)$, or $(\exists_E^G)$, or a type elimination rule. It would make sense, in the presence of such cases, to consider S a maximal point of the derivation, even if its occurrence as a conclusion of an introduction could be very far from its occurrence as a major premise of elimination. In order "to shorten" this distance, and apply the required reduction, we can use a series of permutations, which consist in "taking up" the last elimination rule within the derivations of the minor premises of the various $\star_i$ $(i \leq n)$. The procedure is to be applied, in our general representation, n times, and within it the first step would then be



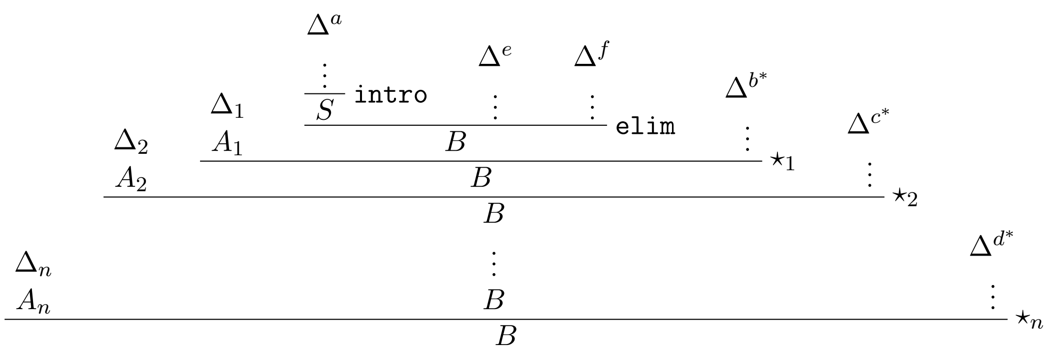
where Δ^{d^*} is obtained applying

$$\dfrac{S \qquad \begin{matrix}\Delta^e \\ \vdots\end{matrix} \qquad \begin{matrix}\Delta^f \\ \vdots\end{matrix}}{B}\ \texttt{elim}$$

to the conclusion of Δ^d. After $n-1$ passages of this kind, we get to



where Δ^{c^*} is obtained from Δ^c as indicated for Δ^{d^*}. The last passage finally returns



where for Δ^{b^*} applies the reasoning made for Δ^{d^*} and Δ^{c^*}. In the derivation obtained, as we can see, the occurrence of S as conclusion of an introduction is the same as that in which S occurs as major premise of elimination. We now provide concrete examples of permutations. Suppose first of all we are in the following situation:

$$\cfrac{\cfrac{\begin{matrix}\Delta_1\\ T:\alpha\wedge\beta\end{matrix} \qquad \begin{matrix}[T\approx\wedge I(\xi^{\alpha},\xi^{\beta})] \quad [\xi^{\alpha}:\alpha] \quad [\xi^{\beta}:\beta]\\ \Delta_2\\ A\end{matrix}}{A}\,D_{\wedge} \qquad \Delta_3 \qquad \Delta_4}{\begin{matrix}B\\ \Delta_5\end{matrix}}$$

then the permutation returns

$$\cfrac{\begin{matrix}\Delta_1\\ T:\alpha\wedge\beta\end{matrix} \qquad \begin{matrix}[T\approx\wedge I(\xi^{\alpha},\xi^{\beta})] \quad [\xi^{\alpha}:\alpha] \quad [\xi^{\beta}:\beta]\\ \cfrac{\begin{matrix}\Delta_2\\ A\end{matrix} \qquad \Delta_3 \qquad \Delta_4}{B}\end{matrix}}{\begin{matrix}B\\ \Delta_5\end{matrix}}\,D_{\wedge}$$

Suppose instead we are in the situation

$$\cfrac{\cfrac{\begin{matrix}\Delta_1\\ T:\alpha_1\vee\alpha_2\end{matrix} \qquad \begin{matrix}[T\approx\vee I(\xi^{\alpha_1})] \quad [\xi^{\alpha_1}:\alpha_1]\\ \Delta_2\\ A\end{matrix} \qquad \begin{matrix}[T\approx\vee I(\xi^{\alpha_2})] \quad [\xi^{\alpha_2}:\alpha_2]\\ \Delta_3\\ A\end{matrix}}{A}\,D_{\vee}^{nc} \qquad \Delta_4 \qquad \Delta_5}{\begin{matrix}B\\ \Delta_6\end{matrix}}$$

then the permutation returns

$$\cfrac{\begin{matrix}\Delta_1\\ T:\alpha_1\vee\alpha_2\end{matrix} \qquad \Delta_7 \qquad \Delta_8}{\begin{matrix}B\\ \Delta_6\end{matrix}}\,D_{\vee}^{nc}$$

where Δ_j $(j=7,8)$ is

$$\begin{matrix}[T\approx\vee I(\xi^{\alpha_i})] \quad [\xi^{\alpha_i}:\alpha_i]\\ \cfrac{\begin{matrix}\Delta_{i+1}\\ A\end{matrix} \qquad \Delta_4 \qquad \Delta_5}{B}\end{matrix}$$

for $i = 1, 2$. Finally, suppose we are in the situation

$$\begin{array}{c}\dfrac{\dfrac{\begin{array}{c}\Delta_1\\ A\vee^G B\end{array}\quad\begin{array}{c}[A]\\ \Delta_2\\ C\end{array}\quad\begin{array}{c}[B]\\ \Delta_3\\ C\end{array}}{C}\,(\vee^G_E)\qquad \Delta_4\qquad \Delta_5}{D}\\ \Delta_6\end{array}$$

then the permutation returns

$$\begin{array}{c}\dfrac{\begin{array}{c}\Delta_1\\ A\vee^G B\end{array}\quad\dfrac{\begin{array}{c}[A]\\ \Delta_2\\ C\end{array}\quad\Delta_4\quad\Delta_5}{D}\quad\dfrac{\begin{array}{c}[B]\\ \Delta_3\\ C\end{array}\quad\Delta_4\quad\Delta_5}{D}}{D}\,(\vee^G_E)\\ \Delta_6\end{array}$$

In conclusion, let us say that the reductions and permutations so far described work on condition that a series of conventions is adopted on individual variables and proper ground-variables, which we will discuss in the next section. Finally, we give a precise definition of what has been proven so far as a general principle.

Definition 79. Let $\Lambda\mathtt{G}$ be a system of grounding for a language of grounding Λ, and let be $\Delta \in \mathtt{DER}_{\Lambda\mathtt{G}}$. We will call *cut-segment* in Δ a sequence σ of n occurrences of a formula S in Δ such that:

- the first occurrene of S is the conclusion of an application of a (type) introduction rule;
- for every $i < n$, the $(i+1)$-th occurrence of S is the conclusion of an application of $(\vee^G_E)$, or of $(\exists^G_E)$, or of a type elimination rule, where the i-th occurrence of S appears as minor premise;
- the n-th occurrence of S is tha major premise of a (type) elimination rule.

Given a cut-segment $\sigma = A_1, ..., A_n$, we will say that σ has *length* n. Moreover, we will find it convenient to use the following expressions: A is *the formula* of σ; A_i is an *occurrence* of σ ($i \leq n$) - clearly, if σ is a cut-segment in a derivation Δ, the occurrences of the formula of σ are occurrences of the formula of σ in Δ. Finally, given two cut-segments σ_1, σ_2, we will say that σ_1 is *disjoint* from σ_2 if, and only if, none of the occurrences of σ_1 is also an occurrence of σ_2 - that is, when the intersection of the occurrences is empty.

Note that the notion of cut-type is a special case of definition 79, for $i = 1$ and σ conclusion of a type introduction rule and major premise of a type elimination rule; similarly, the notion of cut-formula is obtained for $i = 1$ and σ conclusion of an introduction rule and major premise of an elimination rule.

6.3.3.3 Conventions on $(\bot^G)$ and proper variables

As announced, we introduce two useful conventions on the systems of grounding we are dealing with. They will also allow us an easier proof of normalization, the result we will turn to in the next section.

The first convention concerns the applications of the rule $(\bot^G)$, with which we could have "redundant" passages on the elimination rules. For example

$$\dfrac{\dfrac{\begin{array}{c}\Delta\\ \bot^G\end{array}}{A_1 \wedge^G A_2}\,(\bot^G)}{A_i}\,(\wedge^G_{E,i}),\ i = 1, 2$$

Here, it is clear that we could avoid applying $(\wedge^G_{E,i})$, namely that this passage, in each derivation it occurs in, can be without problems transformed into

$$\dfrac{\begin{array}{c}\Delta\\ \bot^G\end{array}}{A_i}\,(\bot^G)$$

An analogous argument applies to all other "logical" elimination rules, so that, as already done for $(\bot)$ in the natural deduction systems in Section 2.5.1, and for the operator $\bot_\alpha$ in the typed λ-calculus referred to in Section 4.3.1, we adopt the following convention.

Convention 80. Given a formal system of grounding $\Lambda\mathtt{G}$ on a language of grounding Λ, for every $\Delta \in \mathtt{DER}_{\Lambda\mathtt{G}}$, for every application of $(\bot^G)$ in Δ, the conclusion of $(\bot^G)$ is an atomic formula of Λ.

The second convention also has an analogue in conventions already adopted for the natural deduction in Section 2.5.1, and for the type λ-calculus in Section 4.3.1. It is aimed at preventing conflicts between proper variables as a result of the application of the reductions.

Convention 81. Given a formal system of grounding $\Lambda\mathtt{G}$ on a language of grounding Λ, for every $\Delta \in \mathtt{DER}_{\Lambda\mathtt{G}}$ (1) free and bound individual and ground-variables are all distinct from one another - property (FB) - and (2) proper and non-proper individual and ground-variables are all distinct from one another, and each proper individual or ground-variable is used at most in one application of a type elimination rule, or of $(\forall^G_E)$ or of $(\exists^G_E)$ - property (PN).

As for the natural deduction and for the typed λ-calculus, convention 81 is based on the possibility of "renaming", without loss of generality, free and bound, proper and non-proper, individual and ground-variables, in derivations that violate the convention itself.

6.3.4 Normalization

Starting from the reductions and permutations discussed in the previous section, it becomes possible to request a normalization result for arbitrary systems of grounding $\Lambda\mathtt{G}$ on arbitrary – and appropriately enriched – languages of grounding Λ having an arbitrary first-order logical language L as background language. In order to understand what we mean here by normalization, and on the basis of what tools the goal can be achieved, we need some preliminary definitions.

6.3.4.1 Measure and degree

As in Prawitz's normalization theorems for Gentzen's natural deduction systems, normalization in systems of grounding is based on the idea of eliminating all the cut-segments possibly occurring in the derivations. And, as in the case of Prawitz's proof, here too the proof of the result will be carried out by proving that a derivation containing cut-segments can be transformed into another that contains cut-segments of lower complexity, so that, repeating this procedure in a finite number of steps, the complexity of cut-segments tends to 0, namely, all the cut-segments at last disappear. The fundamental core is therefore constituted by the notion of complexity of a cut-segment, a precise determination of which requires a method of numerical "measurement" of the formula that occurs in the cut-segment.

However, unlike cuts in natural deduction, which are only of logical type, in systems of grounding we also have cuts related to rules of type introduction and type elimination. This means that our cuts will have to be equipped with a "double measure". The first of such "measures", referring to possible cut-types, will be nothing more than the measure of the most complex of the formulas of the background language that appears as a type in the formula of the language of grounding; the second, instead, will be a standard "measure" of the complexity of the formula of the language of grounding.

In the definitions that follow, we adopt the following notational conventions. With L we denote the background language of the language of grounding Λ; consequently, with $\mathtt{FORM}_L$ and $\mathtt{ATOM}_L$ we indicate respectively the set of the formulas and the set of the atomic formulas of L, while with $\mathtt{FORM}_\Lambda$ and $\mathtt{ATOM}_\Lambda$ we indicate respectively the set of the formulas and the set of the atomic formulas of Λ. Let us assume as defined in a standard way the order relations $\leq$ and $<$ on $\mathbb{N}^2$ – that is the "less than/equal to" and "less than" relations on natural numbers. We will also use the standard function

$$\mathtt{max}\colon \mathbb{N}^2 \to \mathbb{N}$$

such that

$$\mathtt{max}(n, m) = \begin{cases} n & \text{if } m < n \\ m & \text{if } n \leq m \end{cases}$$

Finally, called $\wp(\mathbb{N})$ the power set of $\mathbb{N}$, we put

$$\mathbb{F} = \{S \mid S \in \wp(\mathbb{N}) \text{ and } S \text{ finite}\}$$

and we define the function

$$\mathtt{MAX}\colon \mathbb{F} \to \mathbb{N}$$

such that

$$\mathtt{MAX}(S) = \begin{cases} 0 & \text{if } S = \emptyset \\ n \in S & \text{such that, for every } m \in S, m \leq n \end{cases}$$

Definition 82. The *measure* of $\alpha \in \mathtt{FORM}_L$ is a function $k^1\colon \mathtt{FORM}_L \to \mathbb{N}$ inductively defined as follows:

- $\alpha \in \mathtt{ATOM}_L \Rightarrow k^1(\alpha) = 0$
- $\alpha = \beta \star \gamma \Rightarrow k^1(\alpha) = \mathtt{max}(k^1(\beta), k^1(\gamma)) + 1$ $(\star = \wedge, \vee, \to)$
- $\alpha = \star\ x\ \beta \Rightarrow k^1(\alpha) = k^1(\beta) + 1$ $(\star = \forall, \exists)$

Given $A \in \mathtt{FORM}_\Lambda$, we put

$$T_A = \{k^1(\alpha) \mid U : \alpha \text{ subformula } A\}$$

Definition 83. The *type-measure* of $A \in \mathtt{FORM}_\Lambda$ is a function $\tau \colon \mathtt{FORM}_\Lambda \to \mathbb{N}$ such that

$$\tau(A) = \mathtt{MAX}(T_A).$$

Definition 84. The *logical measure* of $A \in \mathtt{FORM}_\Lambda$ is a function $k^2 \colon \mathtt{FORM}_\Lambda \to \mathbb{N}$ inductively defined as follows:

- $A \in \mathtt{ATOM}_\Lambda \Rightarrow k^2(A) = 0$
- $A = B \star C \to k^2(A) = \mathtt{max}(k^2(B), k^2(C)) + 1$ $(\star = \wedge^G, \vee^G, \to^G)$
- $A = \star\, \varepsilon\, B \Rightarrow k^2(A) = k^2(B) + 1$ $(\star = \forall^G, \exists^G,\ \varepsilon = x, \xi^\alpha)$

Definition 85. The *measure* of $A \in \mathtt{FORM}_\Lambda$ is a function $\mu \colon \mathtt{FORM}_\Lambda \to \mathbb{N}^2$ such that

$$\mu(A) = (\tau(A), k^2(A)).$$

Once the measure has been set, on the formulas of Λ it is possible to define a strict order relation. Since the measure is double, the order will be of an "alphabetic" kind, that is, for every $A, B \in \mathtt{FORM}_\Lambda$,

$$\mu(A) < \mu(B) \Leftrightarrow \begin{cases} \tau(A) < \tau(B) & \text{or} \\ k^2(A) < k^2(B) & \text{for } \tau(A) = \tau(B) \end{cases}$$

It might not be clear why, for the measurement of $A \in \mathtt{FORM}_\Lambda$, we use also $\tau(A)$, and not only, as usually happens, $k^2(A)$. The reason for this procedure is twofold. First of all, the presence of $\tau(A)$ serves to measure cut-types. Suppose we are in the following situation:

$$\dfrac{\dfrac{\begin{matrix}[\xi^\alpha : \alpha] \\ \Delta_1 \\ T(\xi^\alpha) : \beta\end{matrix}}{\to I\xi^\alpha(T(\xi^\alpha)) : \alpha \to \beta} \to I \qquad \begin{matrix}[\to I\xi^\alpha(T(\xi^\alpha)) \approx \to I\xi^\alpha(\mathtt{f}^\beta(\xi^\alpha))] \qquad [\forall^G \xi^\alpha(\xi^\alpha : \alpha \to^G \mathtt{f}^\beta(\xi^\alpha) : \beta)] \\ \Delta_2 \\ A\end{matrix}}{\begin{matrix}A \\ \Delta_3\end{matrix}} D_\to$$

Here, we have the cut-type

$$\to I\xi^\alpha(T(\xi^\alpha)) : \alpha \to \beta.$$

In order to prove normalization, we must be able to attribute it a measure, so as to show that a derivation containing cuts can be transformed into another one that contains less complex cuts, or, simply, fewer cuts of the same complexity. Well, since

$$(\tau(\to I\xi^\alpha(T(\xi^\alpha)):\alpha\to\beta), k^2(\to I\xi^\alpha(T(\xi^\alpha)):\alpha\to\beta) = (k^1(\alpha\to\beta), 0)$$

we will have that

$$\mu(\to I\xi^\alpha(T(\xi^\alpha)):\alpha\to\beta) = (k^1(\alpha\to\beta), 0).$$

However, there is also a second reason. When we apply the reductions, we could have as result derivations containing cut-types or cut-formulas which were not in the starting derivation. Now, since as we have said we want to be able to prove that a derivation containing cuts can be transformed into another one with less complex cuts, or with fewer cuts of the same complexity, we must ensure, in the case of cut-types, that even if new cut-formulas arise following the reduction of the latter, such cut-formulas have a lower measure than the eliminated cut-types – that the measure is lower in the case of the production of new cut-types depends on the fact that the new cut-types will have as type formulas which were subformulas of the starting cut-types. Suppose we are in the following situation:

$$\begin{matrix} \dfrac{\dfrac{\begin{matrix}[\xi^\alpha:\alpha]\\ \Delta_1\\ T(\xi^\alpha):\beta\end{matrix}}{\to I\xi^\alpha(T(\xi^\alpha)):\alpha\to\beta}\to I \qquad \begin{matrix}[\to I\xi^\alpha(T(\xi^\alpha))\approx\to I\xi^\alpha(\mathtt{f}^\beta(\xi^\alpha))]\\ \\ \Delta_2(\mathtt{f}^\beta)\\ A\end{matrix}}{A}D_\to \quad \dfrac{[\forall^G\xi^\alpha(\xi^\alpha:\alpha\to^G\mathtt{f}^\beta(\xi^\alpha):\beta)]}{U:\alpha\to^G\mathtt{f}^\beta(U):\beta}(\forall^G_E) \\ \Delta_3 \end{matrix}$$

By applying $\mathtt{T}_\to$-rid, we obtain

$$\begin{matrix} \dfrac{}{\to I\xi^\alpha(T(\xi^\alpha))\approx\to I\xi^\alpha(T(\xi^\alpha))} \qquad \begin{matrix}\dfrac{\dfrac{\dfrac{\begin{matrix}[\xi^\alpha:\alpha]\\ \Delta_1\\ T(\xi^\alpha):\beta\end{matrix}}{\xi^\alpha:\alpha\to^G T(\xi^\alpha):\beta}(\to^G_I)}{\forall^G\xi^\alpha(\xi^\alpha:\alpha\to^G T(\xi^\alpha):\beta)}(\forall^G_I)}{U:\alpha\to^G T(U):\beta}(\forall^G_E)\\ \Delta_2(T/\mathtt{f}^\beta)\end{matrix} \\ A \\ \Delta_3 \end{matrix}$$

where we have a *new* cut-formula, i.e.

$$\forall^G \xi^\alpha(\xi^\alpha : \alpha \to^G T(\xi^\alpha) : \beta).$$

Let us compare the measure of the eliminated cut-type with the measure of the new cut-formula. As we have seen,

$$\mu(\to I\xi^\alpha(T(\xi^\alpha)) : \alpha \to \beta) = (k^1(\alpha \to \beta), 0).$$

On the other hand,

$$\mu(\forall^G \xi^\alpha(\xi^\alpha : \alpha \to^G T(\xi^\alpha) : \beta)) = (\mathtt{max}(k^1(\alpha), k^1(\beta)), 2).$$

Now, since from definition 82

$$k^1(\alpha \to \beta) = \mathtt{max}(k^1(\alpha), k^1(\beta)) + 1,$$

we will have that

$$\mathtt{max}(k^1(\alpha), k^1(\beta)) < k^1(\alpha \to \beta)$$

and hence, by the alphabetic order defined above,

$$\mu(\forall^G \xi^\alpha(\xi^\alpha : \alpha \to^G T(\xi^\alpha) : \beta)) < \mu(\to I\xi^\alpha(T(\xi^\alpha)) : \alpha \to \beta).$$

As happens in Prawitz's standard normalization theory, also the reductions associated with logical rules could produce new cut-types or new cut-formulas. In these cases, however, the following applies. Let A be a cut-formula, and let B be a new cut-type or cut-formula produced by the reduction on A; since B is a subformula of A, we will have that

$$\tau(B) \leq \tau(A)$$

and

$$k^2(B) < k^2(A).$$

Therefore,

$$\mu(B) < \mu(A).$$

Definition 86. Given Δ, let σ be a cut-segment in Δ the formula of which is A. The *measure* of σ - indicated with $\mu(\sigma)$ - is $\mu(A)$.

To every $\Delta \in \mathtt{DER}_{\Lambda\mathtt{G}}$ we associate the set

$$M_\Delta = \{\mu(\sigma) \mid \sigma \text{ cut-segment of } \Delta\}.$$

Definition 87. The *degree* of Δ is a function $\delta\colon \mathtt{DER}_{\Lambda\mathtt{G}} \to \mathbb{N}^2$ such that

$$\delta(\Delta) = (\mathtt{MAX}(M_\Delta), n)$$

where n is the sum of the lengths of the cut-segments σ of Δ such that $\mu(\sigma) = \mathtt{MAX}(M_\Delta)$.

Also on the derivations of $\Lambda\mathtt{G}$ it is possible to define a strict order relation. It is again of an "alphabetic" kind, that is, for every $\Delta_1, \Delta_2 \in \mathtt{DER}_{\Lambda\mathtt{G}}$ such that $\delta(\Delta_i) = (\mathtt{MAX}(M_{\Delta_i}), n_i)$ $(i = 1, 2)$,

$$\delta(\Delta_1) < \delta(\Delta_2) \Leftrightarrow \begin{cases} \mathtt{MAX}(M_{\Delta_1}) < \mathtt{MAX}(M_{\Delta_2}) & \text{or} \\ n_1 < n_2 & \text{for } \mathtt{MAX}(M_{\Delta_1}) = \mathtt{MAX}(M_{\Delta_2}) \end{cases}$$

Note that $\mathtt{MAX}(M_{\Delta_i})$ is again a pair, for which the alphabetical order defined above applies. So $\mathtt{MAX}(M_{\Delta_1}) < \mathtt{MAX}(M_{\Delta_2})$ if, and only if, in the most complex cut-segments of Δ_1, the most complex formulas α that appear in formulas $T : \alpha$ have a lower complexity than the most complex formulas β that, in the most complex cut-segments of Δ_2, appear in formulas $U : \beta$, or, in the case of equal complexities of formulas α and β just taken into account, if the most complex cut-segments of Δ_1 consist of formulas A having a lower complexity than the formulas B which constitute the most complex cut-segments of Δ_2. In this regard, it should be emphasized that both the order on the formulas of Λ, and that on the derivations of $\Lambda\mathtt{G}$, are in a sense arbitrary, and the choice is justified in light of the demonstrative strategy which, with reference to the reductions set out in Section 6.3.3.1, we will adopt to prove normalization. As we have seen, the idea is that since cut-types are atomic formulas of Λ, what counts for their complexity is not the logical complexity, but that of the formula that types the term; so that, when and if new cuts arise after the elimination of a cut-type, we must ensure that it decreases the complexity of the formulas that type the terms in the new cuts, and not the logical complexity of the cuts as such – it is obvious the impossibility that logical complexity decreases, since a cut-type is an atomic formula. We conclude with the following definition.

Definition 88. Δ is said *in normal form* if, and only if,

$$\delta(\Delta) = ((0, 0), 0).$$

Otherwise Δ is said *in non-normal form.*

6.3.4.2 Normalization theorem

Thanks to the reductions and permutations described in sections 6.3.3.1 and 6.3.3.2, we can introduce a reducibility relation between derivations.

Definition 89. Δ_a *immediately reduces* to Δ_b – indicated with $\Delta_a \succeq \Delta_b$ – if, and only if, $\Delta_a = \Delta_b$, or Δ_b can be obtained from Δ_a by applying one of the reductions for cut-types, or one of the reductions for cut-formulas, or a permutation. Moreover, Δ_a *reduces* to Δ_b – indicated with $\Delta_a \succ \Delta_b$ – if, and only if, there exists a sequence $\Delta_1, ..., \Delta_n$ with $\Delta_1 = \Delta_a$, $\Delta_n = \Delta_b$, and $\Delta_i \succeq \Delta_{i+1}$ for every $i \leq n$.

It is, as we can easily see, an order relation on derivations, that is, a reflexive, asymmetrical and transitive relation. We can transform it into an equivalence relation, either by requiring that two derivations are in relation if one can be reduced to the other or vice versa (see for example Hindley, Lercher & Seldin 1975, with reference to a typed or untyped λ-calculus), or equivalently by placing side by side the reduction and permutation operations with appropriate expansion operations (see for example Francez 2015).

The proof of the normalization theorem is accomplished in three steps. In the first, given a non-normal derivation Δ, we prove that it contains a subderivation Δ^* the conclusion of which is the last occurrence of a cut-segment of maximal measure in Δ, and such that all the cut-segments of Δ^* (if any) have a smaller size than the first element of the degree of Δ. In the second step, we prove that such a Δ^* exists, which enjoys also the property either of not being "side-connected" to any subderivation of Δ, the conclusion of which is an occurrence of a cut-segment of maximum size in Δ, or of not containing cut-segments of maximal measure in Δ. These first two steps allow the identification of a good "starting point" for the normalization procedure. It will indeed be convenient to choose a subderivation of a non-normal derivation Δ that has the characteristics described in the first two steps, in such a way that, by applying reductions or permutations, the degree of Δ can be progressively taken to the minimum size. The strategy is basically the same as that used by Prawitz (Prawitz 2006) for his own normalization theorems. The only difference between our systems of grounding and a Gentzen's natural deduction system for first-order intuitionistic logic consists after all in the fact that, here, we have also type introduction and type elimination rules, and corresponding cut-types. Therefore, we have to make sure simply that if new cut-types or new cut-formulas B arise after the application of an appropriate reduction to a cut-type or to a cut-formula A, then $\mu(B) < \mu(A)$. However, we can easily ensure this – the examples in Section 6.3.4.1 are the least obvious cases, and are intended to prove exactly that. In the following we will use the concepts of *length* of a derivation, and of *subderivation* of a derivation, defined in a standard way.

Proposition 90. If Δ is a non-normal derivation, there is a cut-segment σ in Δ such that $\mu(\sigma) = \mathtt{MAX}(M_\Delta)$, and such that, for every cut-segment σ^* of the subderivation Δ^* of Δ the conclusion of which is the first occurrence of σ, $\mu(\sigma^*) < \mathtt{MAX}(M_\Delta)$.

Proof. By induction on the length of Δ.

If Δ is an assumption, or the application of an axiom, the antecedent of the theorem is false, so that the theorem is trivially true.

Let then Δ be of length $\mathfrak{L}$, and suppose the theorem proven for every derivation of length $\mathfrak{L}^* < \mathfrak{L}$. We must distinguish a series of subcases:

- if Δ is normal the antecedent of the theorem is false, so that the theorem is trivially true;
- if Δ is non-normal, we distinguish two cases:
  - Δ ends with the application J of a (type) elimination rule, and the major premise of J is the last occurrence of a cut-segment σ^1 of Δ, and $\mu(\sigma^1) = \mathtt{MAX}(M_\Delta)$, and finally, for every cut-segment σ^* of the subderivation of Δ the conclusion of which is the first occurrence of σ^1, $\mu(\sigma^*) < \mathtt{MAX}(M_\Delta)$. Then we put $\sigma = \sigma^1$;
  - Δ does not end with the application of a (type) elimination rule, or ends with the application J of a (type) elimination rule but the major premise A of which is not last occurrence of cut-segments of Δ, or A is last occurrence of cut-segments σ^1 of Δ but it does not hold that $\mu(\sigma^1) = \mathtt{MAX}(M_\Delta)$, or $\mu(\sigma^1) = \mathtt{MAX}(M_\Delta)$ but for each σ^1 there are cut-segments σ^2 in the subderivation of Δ the conclusion of which is the first occurrence of σ^1 such that $\mu(\sigma^2) = \mathtt{MAX}(M_\Delta)$. But then, there is a non-normal subderivation Δ^* of Δ of length $\mathfrak{L}^* < \mathfrak{L}$, and such that $\mathtt{MAX}(M_{\Delta^*}) = \mathtt{MAX}(M_\Delta)$. By induction hypothesis, there is a cut-segment σ^* in Δ^* such that $\mu(\sigma^*) = \mathtt{MAX}(M_{\Delta^*})$, and such that, called Δ^{**} the subderivation of Δ^* the conclusion of which is the first occurrence of σ^*, for every cut-segment σ^{**} of Δ^{**}, $\mu(\sigma^{**}) < \mathtt{MAX}(M_{\Delta^*})$. Now, since $\mathtt{MAX}(M_{\Delta^*}) = \mathtt{MAX}(M_\Delta)$, we will have that $\mu(\sigma^*) = \mathtt{MAX}(M_\Delta)$ and, for every cut-segment σ^{**} in Δ^{**}, $\mu(\sigma^{**}) < \mathtt{MAX}(M_\Delta)$. We then put $\sigma = \sigma^*$.

This completes all possible cases, and the proposition is thus proven. □

In what follows, we will call *local maximum point* of Δ a cut-segment of Δ having the properties indicated by proposition 90.

Proposition 91. Given a non-normal derivation Δ, and given a local maximum point σ^1 of Δ the last occurrence of which is major premise of the application J of a (type) elimination rule, let us suppose there is a subderivation Δ^* of Δ with the following properties:

(i) the conclusion of Δ^* is one of the minor premises of J;

(ii) there is a cut-segment σ^* in Δ such that $\mu(\sigma^*) = \mathtt{MAX}(M_\Delta)$, and such that some of its occurrences are in Δ^*.

Then, there is a local maximum point σ^2 of Δ disjoint from σ^1. In addition, let $\sigma^1, \sigma^2, \sigma^3, ..., \sigma^n$ be a sequence of cut-segments in Δ such that σ^1 and σ^2 are as above and, for every $2 < i \leq n$, σ^i is a local maximum point of Δ for which the circumstances (i) and (ii) hold, and σ^i stands with σ^{i-1} in the same relation as the one occurring between σ^2 and σ^1. Then, there is a local maximum point σ^{n+1} of Δ such that, for every σ^i ($i \leq n$), σ^{n+1} is disjoint from σ^i.

Proof. Let Δ, σ^1, Δ^* and σ^* be as in the hypotheses. We distinguish two cases:

1. the formula of σ^* does not occur as conclusion of Δ^*. But then, all the occurrences of σ^* are occurrences of Δ^*. Hence, Δ^* is non-normal. Moreover, since $\mu(\sigma^*) = \mathtt{MAX}(M_\Delta)$, we will have that $\mathtt{MAX}(M_{\Delta^*}) = \mathtt{MAX}(M_\Delta)$. By proposition 90, Δ^* has a local maximum point σ^2 that, since $\mu(\sigma^2) = \mathtt{MAX}(M_{\Delta^*}) = \mathtt{MAX}(M_\Delta)$, is also a local maximum point of Δ. Moreover, it cleary holds that σ^2 is disjoint from σ^1;

2. the formula of σ^* does not occur as conclusion of Δ^*. Again two cases:

   - σ^* is a local maximum point of Δ, and in this case we can put $\sigma^2 = \sigma^*$, since clearly σ^* is disjoint from σ^1;
   - σ^* is not a local maximum point of Δ. Then, called Δ^{**} the subderivation of Δ the conclusion of which is the first occurrence of σ^*, there is a cut-segment σ^{**} in Δ^{**} such that $\mu(\sigma^{**}) = \mathtt{MAX}(M_\Delta)$. But then Δ^{**} is non-normal, and since $\mu(\sigma^{**}) = \mathtt{MAX}(M_\Delta)$, we will have that $\mathtt{MAX}(M_{\Delta^{**}}) = \mathtt{MAX}(M_\Delta)$. Here we can repeat point 1.

Let then $\sigma^1, \sigma^2, \sigma^3, ..., \sigma^n$ be the required sequence of cut-segments in Δ. Since for σ^n (i) and (ii) hold, by applying to σ^n the procedure just concluded for σ^1, we can find a local maximum point σ^{n+1} of Δ disjoint from σ^n. Then, observe that, for every $2 \leq i \leq n+1$, either among the occurrences of σ^i there is one of the minor premises of the application of the (type) elimination rule where the last occurrence of σ^{i-1} is major premise, or σ^i is in a subderivation of Δ having this minor premise as a conclusion. Then, if there is σ^i ($i \leq n-1$) not disjoint from σ^{n+1}, there is a subderivation Δ^a of Δ ending with the application J^a of a (type) elimination rule the major premise of which is the last occurrence of σ^n, and such that, for some subderivation Δ^b of Δ (possibly identical to Δ^a), Δ^b ends with the application J^b of a (type) elimination rule the major premise of which is the conclusion of a subderivation of Δ having Δ^a as subderivation, and one of the minor premises of which is an occurrence of σ^n, or the conclusion of a subderivation of Δ having Δ^a as subderivation. But none of these cases is clearly possible. Therefore, there cannot be σ^i ($i \leq n-1$) not disjoint from σ^{n+1}. □

Proposition 92. If Δ is a non-normal derivation, then there is a local maximum point σ of Δ of which the last occurrence is the major premise of the application J of a (type) elimination rule and, for every subderivation Δ^* of Δ of which the conclusion is one of the minor premises of J, for every cut-segment σ^* of Δ in Δ^*, or that has the conclusion of Δ^* among its occurrences, $\mu(\sigma^*) < \mathtt{MAX}(M_\Delta)$.

Proof. If there were not a σ as required by the proposition, by proposition 91 Δ would contain an infinite number of local maximum points, which is clearly impossible. □

Theorem 93. Let Δ be a non-normal derivation. Then, there is a derivation Δ^* such that $\Delta \succeq \Delta^*$ and $\delta(\Delta^*) < \delta(\Delta)$.

Proof. Let Δ be non-normal, and let be $\delta(\Delta) = (\mathtt{MAX}(M_\Delta), n)$. Let us choose a local maximum point σ of Δ that enjoys the property indicated by proposition 92. If σ has length greater than 1, by applying a permutation on the last occurrence of σ, we obtain a derivation Δ^* with $\delta(\Delta^*) = (\mathtt{MAX}(M_{\Delta^*}), n^*)$ for $\mathtt{MAX}(M_{\Delta^*}) = \mathtt{MAX}(M_\Delta)$ and $n^* < n$, whence $\delta(\Delta^*) < \delta(\Delta)$. If instead σ has length 1, namely σ is a cut-type or a cut-formula, by applying to it a reduction, we obtain a derivation Δ^* for which one of the following circumstances applies:

- Δ^* does not contain cut-segments not already occurring in Δ. In this case: either all the cut-segments of Δ^* have lower measure than the cut-segments of Δ of maximal measure, so that $\mathtt{MAX}(M_{\Delta^*}) < \mathtt{MAX}(M_\Delta)$, and hence $\delta(\Delta^*) < \delta(\Delta)$; or Δ^* contains cut-segments of the same measure of the cut-segments of Δ of maximal measure, so that $\delta(\Delta^*) = (\mathtt{MAX}(M_{\Delta^*}), n^*)$ for $\mathtt{MAX}(M_{\Delta^*}) = \mathtt{MAX}(M_\Delta)$ and $n^* < n$, whence $\delta(\Delta^*) < \delta(\Delta)$;

- Δ^* contains cut-segments which were not already in Δ. In this case, it is easily seen - also thanks to the examples in Section 6.3.4.1 - that the following circumstance holds: if B is the formula of a cut-segmenta produced by the reduction of a cut-type or a cut-formula A, then $\mu(B) < \mu(A)$. Hence, either $\mathtt{MAX}(M_{\Delta^*}) < \mathtt{MAX}(M_\Delta)$, whence $\delta(\Delta^*) < \delta(\Delta)$, or $\delta(\Delta^*) = (\mathtt{MAX}(M_{\Delta^*}), n^*)$ for $\mathtt{MAX}(M_{\Delta^*}) = \mathtt{MAX}(M_\Delta)$ and $n^* < n$, whence again $\delta(\Delta^*) < \delta(\Delta)$.

This completes all possible cases, and at this point it is enough to observe that $\Delta \succeq \Delta^*$. □

Corollary 94. For every derivation $\Delta \in \mathtt{DER}_{\Lambda\mathtt{G}}$, there is a normal derivation Δ^* such that $\Delta \succ \Delta^*$.

Proof. If Δ is normal, we put $\Delta^* = \Delta$, since clearly $\Delta \succeq \Delta$. Otherwise, by virtue of theorem 93, there is a derivation Δ^1 such that $\Delta \succeq \Delta^1$ and $\delta(\Delta^1) < \delta(\Delta)$. If Δ^1 is normal, by putting $\Delta^* = \Delta^1$ we are done. Otherwise, by virtue of theorem 93, there is a derivation Δ^2 such that $\Delta^1 \succeq \Delta^2$ and $\delta(\Delta^2) < \delta(\Delta^1)$. Again, if Δ^2 is normal, by putting $\Delta^* = \Delta^2$ we are done, since $\Delta \succeq \Delta^1$ and $\Delta^1 \succeq \Delta^2$ imply that $\Delta \succ \Delta^2$. Otherwise, we start again and, after a finite number n of steps, we will find a normal derivation Δ^n such that $\Delta \succeq \Delta^1 \succeq \Delta^2 \succeq ... \succeq \Delta^{n-1} \succeq \Delta^n$, whence $\Delta \succ \Delta^n$. By putting $\Delta^* = \Delta^n$, the result is hence proven. □

At the end of this chapter, we conduct only two quick observations. The first is that corollary 94, which is the result of normalization, is independent of the characteristic rules of the system of grounding to which it refers. Thus, any derivation of whatever system of grounding can be reduced to a normal form. The second observation is the following: generally, a normalization result is accompanied by a series of results implied by it – concerning for example the form of normal derivations, or the subformula property, or the Craig's interpolation theorem (see Prawitz 2006) – so that we might wonder if and to what extent they apply also in our case. Here, however, we will not deal with these issues, both because they are not strictly related to the objectives we have set ourselves, and for mere reasons of space.

Chapter 7

Completeness and recognizability

7.1 About completeness

In the second part of this work, we described the theory of grounds as it has been articulated so far by Prawitz. Among the issues presented there, in this Chapter we will concentrate on two of them. We will deal with the first one in this section, taking as a starting point the notions of validity of inferences and of inference rules, on an atomic base or in a more universally logical sense.

7.1.1 From validity to universal validity

In Chapter 4 we saw that an inference is to be understood as *valid* on B if, and only if, there is a B-operation on grounds associated with it, that is, a B-operation that has as domain the premises, and as (codomain of the) codomain the conclusion of the inference, as well as appropriate discharges of assumptions and bindings of variables; on the other hand, an inference rule is *valid* on B if, and only if, there is a "structural" B-operation on grounds associable to it, where the notion of structurality is to be understood in the sense indicated by following words of Prawitz:

> this operation has no specific type, but is specified by a term with ambiguous types; for each instance of the form, the operation is of a specific type and is denoted by the corresponding instance of that term. (Prawitz 2015, 95)

A *proof* on B is a finite chain of inferences valid on B. In the following, we indicate with

$$\Gamma \models_B \alpha$$

the fact that there is a proof on B having Γ as set of assumptions, α as conclusion, and a set of free individual variables equal to that of the individual variables occurring free in Γ and α. We will also say that α is a *B-consequence* of Γ.

Abstracting from the notion of validity on a base, it is possible to obtain a universal notion of validity. Namely, we could say that an inference and an inference rule are *valid* if, and only if, for every base B, they are valid on B. Actually, this characterization is slightly different from the one we used in Chapter 4 when, quoting the words of Prawitz, we had understood logical validity as persistence of validity under variation of the content of the non-logical symbols. However, the two formulations are equivalent if we take into account what we said in Chapter 5 about the atomic bases, namely, that they determine the meaning of the individual constants, and of the functional and relational symbols of the background language; then, the persistence of validity on all atomic bases coincides with the persistence of validity under variation of the content of the non-logical symbols.

On the other hand, there is perhaps a more important observation to make. The fact that validity means validity on every atomic base B can be understood essentially in two ways:

($\models_1$) for every atomic base B, the inference can be associated to a B-operation on grounds, and the inference rule can be associated to a "structural" B-operation on grounds;

($\models_2$) the inference can be associated to a universal operation on grounds, and the inference rule can be associated to a "structural" universal operation on grounds.

– the notion of universal operation on grounds is here to be understood as defined in Section 5.2.4.6. Obviously, ($\models_2$) implies ($\models_1$). Having to choose between the two suggested formulations, we adopt ($\models_2$).[1] An inference is

[1] In Section 7.1.4, when adapting to our framework a proof by Piecha and Schroeder-Heister (Piecha & Schroeder-Heister 2018), we will instead use ($\models_1$). However, observe that there could be reasons to maintain that also ($\models_1$) implies ($\models_2$), and hence that the two formulations above are equivalent. Indeed, suppose that, for every atomic base B, there is a B-operation on grounds f_B - which, for the sake of simplicity, we suppose to be not defined on individuals - of an appropriate operational type

$$\tau_1, ..., \tau_n \rhd \tau_{n+1}$$

- which, by the assumption on f_B, will have no free individual variables - by virtue of which an inference J can be said to be valid on B. Let us now define an operation f of the same operational type as f_B by requiring that, for every B, for every g_i ground on B for τ_i ($i \leq n$), *valid* if, and only if, there is a universal operation on grounds associated to it, having as domain the premises, and as (codomain of the) codomain the conclusion of the inference, as well as appropriate discharges of assumptions and bindings of variables; an inference rule is *valid* if, and only if, there is a universal "structural" operation on grounds associated to it. The notion of universal "structural" operation on grounds is also to be understood here in the sense indicated by the words of Prawitz quoted above. A *proof* is a finite chain of valid inferences. We denote finally with

$$\Gamma \models_2 \alpha$$

$$f(g_1, ..., g_n) = f_B(g_1, ..., g_n).$$

Thanks to this definition, the operation f could be understood as a universal operation on grounds of the indicated operational type. However, we have to bear in mind that in the definition of f it is *essential* to quantify over all bases; on the other hand, an operation like the one denoted by $\wedge_{E,i}$ in Section 5.2.4.5 is to be understood as universal in the sense that there is one equal on all bases - i.e., the operation can be defined in the same way on all bases. Here, we can maybe reason as follows. Given an atomic base B on a first-order logical language L, for every $\alpha \in \texttt{FORM}_L$ let us consider the set

$$\texttt{Gr}^{\alpha}_{B} = \{g \mid g \text{ is a ground on } B \text{ for } \vdash \alpha\}.$$

Then, called $\mathfrak{B}$ the set of all the atomic bases on L, let us consider the class

$$\texttt{Gr}^{\alpha} = \bigcup_{B \in \mathfrak{B}} \texttt{Gr}^{\alpha}_{B}$$

Let us suppose that, for every B, there is a constructive function

$$f_B \colon \texttt{Gr}^{\alpha}_{B} \to \texttt{Gr}^{\beta}_{B}.$$

Now, an operation on grounds which is universal in the sense we have indicated in Section 5.2.4.5 will be a constructive function

$$f^* \colon \texttt{Gr}^{\alpha} \to \texttt{Gr}^{\beta}$$

such that, for every $g \in \texttt{Gr}^{\alpha}$, the value $f^*(g)$ can be specified by the same defining equation *whatever* the class $\texttt{Gr}^{\alpha}_{B}$ to which g belongs is. Instead, an operation on grounds which is universal in the sense required to pass from ($\models_1$) to ($\models_2$), will be a function

$$f^{**} \colon \texttt{Gr}^{\alpha} \to \texttt{Gr}^{\beta}$$

such that, for every $g \in \texttt{Gr}^{\alpha}$, $f^{**}(g) = f_B(g)$, for $g \in \texttt{Gr}^{\alpha}_{B}$. It seems that, in order to pass from f^{**} to f^*, we need a sort of axiom of choice, which however is known to be valid (and even provable) in certain constructivist setups. A similar argument obviously applies if, from the specific inferences, we turn to inference rules.

the fact that there is a proof having Γ as set of assumptions, α as a conclusion, and a set of free individual variables equal to that of the individual variables occurring free in Γ and α. We will also say that α is a *consequence* of Γ. We conclude this Section with an observation.

Observation 95. $\Gamma \models_B \alpha$ if, and only if, there is a proper ground on B for $\Gamma \vdash \alpha$, that is, a B-operation on grounds of operational type

$$\Gamma \rhd \alpha$$

relative to all and only the individual variables occurring free in Γ and α. Likewise, $\Gamma \models_2 \alpha$ if, and only if, there is a proper universal ground for $\Gamma \vdash \alpha$, that is, a universal operation on grounds of operational type

$$\Gamma \rhd \alpha$$

relative to all and only the individual variables occurring free in Γ and α.

The correctness of the observation can be seen in the following way. According to the definition of $\Gamma \models_B \alpha$, first of all, there is a proof π on B with set of assumptions Γ, conclusion α, and of which the free individual variables are all and only the individual variables occurring overall free in Γ and α. A proof on B is a chain of inferences valid on B. Let $J_1, ..., J_n$ be the inferences valid on B involved in π. For each $i \leq n$, J_i can be associated to a B-operation on grounds f_i. But then, π can be conceived as a composite B-operation on grounds $f_1 \circ \cdots \circ f_n$, and, more in particular, for the assumption on free individual variables of π, as a proper ground on B for $\Gamma \vdash \alpha$. The latter will, by definition, be a B-operation on grounds as indicated in Observation 95. By definition of $\Gamma \models_2 \alpha$, instead, there is a proof π with set of assumptions Γ, conclusion α, and of which the free individual variables are all and only the individual variables occurring overall free in Γ and α. A proof is a chain of valid inference. Let $J_1, ..., J_n$ be the valid inferences involved in π. For each $i \leq n$, J_i can be associated to a universal operation on grounds f_i. But then, π can be conceived as a composite universal operation on grounds $f_1 \circ \cdots \circ f_n$, and, more in particular, for the assumption on free individual variables of π, as a proper universal ground for $\Gamma \vdash \alpha$. The latter will, by definition, be a universal operation on grounds as indicated in Observation 95.

7.1.2 Correctness of first-order intuitionistic logic

On the base of the definitions offered in the previous section, it is easy to establish that all the inference rules of Gentzen's intuitionistic first-order logic, constituting the system IL of Section 2.5.1, are valid in the intended sense. For the introduction rules, there exist clearly universal operations on grounds - the primitive ones involved in the clauses (At_G) - ($\bot_G$) – that, appropriately associated with each instance of the rules, make these instances easily valid; the operations we defined in Section 5.2.4.5, associated to the non-primitive operational symbols of a Gentzen-language, are instead universal operations on grounds which, when appropriately associated to instances of the elimination rules, make them similarly valid. We have then, almost immediately, a result of *correctness* of IL with respect to the notion of logical consequence above defined.

Theorem 96. $\Gamma \vdash_{\mathtt{IL}} \alpha \Rightarrow \Gamma \models_2 \alpha$.

Proof. Let Δ be a derivation of α from Γ in $\mathtt{DER}_{\mathtt{IL}}$. Since all the inference rules of IL are valid, all the instances of such rules occurring in Δ will be valid as well. So, Δ is a finite chain of valid inferences in the intended sense, and hence it is a proof of α from the set of assumptions Γ. □

7.1.3 Accounts of ground-theoretic completeness

In a rather natural way, the question arises at this point whether, in addition to being correct with respect to the ground-theoretical notion of logical consequence, Gentzen's natural deduction for first-order intuitionist logic is also in this sense complete:

$$\Gamma \models_2 \alpha \Rightarrow \Gamma \vdash_{\mathtt{IL}} \alpha.$$

With reference to his proof-theoretic semantics based on the notion of valid argument, illustrated by us in Section 2.5.2.1, Prawitz raises the issue about completeness in terms of a famous conjecture. The formulation we will mention here is from the 1973 *Towards a foundation of a general proof-theory* (Prawitz 1973), although we limit ourselves to the notion of "simple" validity, leaving out strong validity, and we refer to intuitionistic first-order logic, rather than to minimal first-order logic. The notion of derivability of a rule in IL is the standard one (see, for example, von Plato 2014), while the notion of inference rule valid according to proof-theoretic semantics is the one we provided in definition 6.

Conjecture 97 (Prawitz's conjecture)**.** If an inference rule R is valid, then R is derivable in IL.

Since we also have a result of correctness of IL for the proof-theoretic semantics, if conjecture 97 were correct we would have that the set of valid inference rules according to proof-theoretic semantics is identical to the set of rules derivable in IL (for an overview of the surveys related to Prawitz's conjecture, see Piecha 2016; the refutation relative to the proof-theoretic semantics is found in Piecha & Schroeder-Heister 2018).

In the theory of grounds, as we understand it, an inference rule is valid if, and only if, there is a universal "structural" operation on grounds associated to it – where "structural", we reassert it, is to be understood in the sense indicated by the above mentioned words of Prawitz. In the light of this definition, the conjecture can be reformulated as follows.

Conjecture 98 (Ground-theoretic completeness conjecture)**.** Given a decidable scheme of operational types

$$\tau_1, ..., \tau_n \rhd \tau_{n+1},$$

if there is a universal "structural" operation on grounds, each instance of which has operational type an instance of

$$\tau_1, ..., \tau_n \rhd \tau_{n+1},$$

and that binds a sequence of individual variables $\underline{x}$ and a sequence of ground-variables $\underline{\xi}$, then the inference rule

$$\frac{\sigma_1 \qquad \dots \qquad \sigma_n}{\beta}\ \underline{x}, \mathfrak{A}$$

is derivable in IL, where

- for every $i \leq n$, σ_i is τ_i, if τ_i has empty domain, and

  $$\begin{array}{c}\Gamma_i \\ \vdots \\ \alpha_i\end{array}$$

  if τ_i has non-empty domain Γ_i and codomain α_i;
- $\mathfrak{A}$ is a set of assumptions discharged by the rule such that, if the universal "structural" operation on grounds binds ξ^γ on index i, then the rule discharges γ on index i ($i \leq n$);
- β is the codomain of τ_{n+1}.

Proposition 99. If conjecture 98 is correct, $\Gamma \models_2 \alpha \Rightarrow \Gamma \vdash_{\texttt{IL}} \alpha$.

Proof. Let us suppose that $\Gamma \models_2 \alpha$, namely that there is a proof π with set of assumptions Γ, conclusion α, and of which the individual variables are all and only those occurring free Γ and α. π is a chain of valid inferences $J_1, ..., J_n$ and, from observation 95, it can be understood as a composite universal operation on grounds $f_1 \circ \cdots \circ f_n$, where each f_i is associable to an inference J_i $(i \leq n)$. We can now proceed in two alternative ways.
(1) Given

$$\Gamma = \{\beta_1, ..., \beta_n\}$$

and taken any atomic base B, we can define an operation

$$h(\xi^{\beta_1}, ..., \xi^{\beta_n})$$

by requiring that, for every g_i ground on B for $\vdash \beta_i$ $(i \leq n)$,

$$h(g_1, ..., g_n) = f_1 \circ \cdots \circ f_n(g_1, ..., g_n)$$

Since $f_1 \circ \cdots \circ f_n$ is a composite universal operation on grounds, $f_1 \circ \cdots \circ f_n$ is more in particular a universal operation on grounds, and hence also a B-operation on grounds. Therefore

$$h(\xi^{\beta_1}, ..., \xi^{\beta_n})$$

is a B-operation on grounds. On the other hand, due to the arbitrariness of B, it is more in general a universal operation on grounds. According to conjecture 98, then, the rule

$$\frac{\beta_1 \qquad \ldots \qquad \beta_n}{\alpha}$$

is derivable in IL. By definition of derivability, we conclude that $\Gamma \vdash_{\texttt{IL}} \alpha$.
(2) Conjecture 98 implies that each f_i corresponds to a rule, say R_i, derivable in IL. By replacing in π each J_i with the derivation of R_i, we obtain a derivation of α from Γ in IL, i.e. $\Gamma \vdash_{\texttt{IL}} \alpha$. □

Conjecture 98 limits itself to saying the following: if a decidable scheme of operational types S is "inhabited" by some universal "structural" operation on grounds f, then we can derive in IL a rule which has as premise the domain of the decidable schema of operational types, as conclusion the (codomain of the) codomain of this scheme, and which binds individual variables and discharges assumptions in accordance with the bindings of the universal "structural" operation on grounds supposed existing. On the other hand, in the light of the isomorphism, described in Section 5.2.3.3, between IL and the Gentzen-language Gen, conjecture 98 implies that there exists a denotation function den^* for the elements of the alphabet of `Gen` such that there is a composite universal "structural" operation on grounds h that "inhabits" S, of which the instances are of the form

$$den^*(F_1) \circ \cdots \circ den^*(F_n)$$

where, for every $i \leq n$, F_i is an operational symbol of `Gen`. However, it is not in general guaranteed that $f = h$; in other words, it is not certain that, while "inhabiting" the same scheme of operational types, f and h are the same operation. To require this means requiring something stronger, that we could call *full-completeness*.

Conjecture 100 (Full-completeness ground-theoretic conjecture)**.** Let $\texttt{Gen}^+$ be a non-primitive expansion of `Gen`, obtained by adding to `Gen` a non-primitive operational symbol F, and let den_1^* be a denotation function for the elements of the alphabet of $\texttt{Gen}^+$ such that $den_1^*(F)$ is a universal ground. Then, there is a denotation function den_2^* for the elements of the alphabet of `Gen` such that

$$den_1^*(F) = den_2^*(F_1) \circ \cdots \circ den_2^*(F_n)$$

where, for every $i \leq n$, F_i is an operational symbol of `Gen`.

If conjecture 100 is correct, each non-primitive expansion of a Gentzen-language, obtained by adding an operational symbol that has as operational type an instance of a decidable scheme of operational types inhabited by some universal "structural" operation on grounds, is conservative on the Gentzen-language with respect to appropriate denotation functions.

Proposition 101. Let $\texttt{Gen}^+$ be a non-primitive expansion of `Gen`, obtained by adding to `Gen` a non-primitive operational symbol F, and let den_1^* be a denotation function for the elements of the alphabet of $\texttt{Gen}^+$ such that $den_1^*(F)$ is a universal operation on grounds. Then, if conjecture 100 is correct, there is a denotation function den_2^* for the elements of the alphabet of $\texttt{Gen}^+$ such that $\texttt{Gen}^+$ is conservative with respect to den_2^* on `Gen`.

Proof. Let $\texttt{Gen}^+$ and den_1^* be as in the hypotheses. Since we are supposing that conjecture 100 is correct, there will be a denotation function den_3^* for the elements of the alphabet of `Gen` such that

$$den_1^*(F) = den_3^*(F_1) \circ \cdots \circ den_3^*(F_n)$$

where, for every $i \leq n$, F_i is an operational symbol of `Gen`. Hence, let us define the denotation function den_2^* for the elements of the alphabet of $\texttt{Gen}^+$ such that, for every operational symbol x of $\texttt{Gen}^+$,

$$den_2^*(x) = \begin{cases} den_1^*(x) & \text{if } x \in \mathtt{Al}_{\mathtt{Gen}^+} - \mathtt{Al}_{\mathtt{Gen}} \\ den_3^*(x) & \text{if } x \in \mathtt{Al}_{\mathtt{Gen}} \end{cases}$$

On account of the way in which den_2^* has been defined, we will have that

$$den_2^*(F) = den_2^*(F_1) \circ \cdots \circ den_2^*(F_n)$$

where, for every $i \leq n$, F_i is an operational symbol of Gen. The situation in which we find ourselves satisfies, relatively to den_2^* and to an arbitrary atomic base B, the hypotheses of theorem 60 (rewritability of operational symbols implies conservativity). Hence, $\mathtt{Gen}^+$ is conservative with respect to den_2^* over Gen.[2] □

Another consequence of conjecture 100 on which we wish to draw attention, concerns the systems of grounding outlined in Chapter 6. First of all, it is worth repeating that the scheme of equations that, in a system of grounding, fix the deductive behavior of non-primitive operational symbols of the language of grounding of the system, can be understood as "internalizations" of the schemes of equations that set the behaviour of the operations which, via denotation functions, these symbols are intended to represent. Therefore, given an expansion $\mathtt{Gen}^+$ of Gen, obtained by adding a non-primitive operational symbol F, let den_1^* be a denotation function for the elements of the alphabet of $\mathtt{Gen}^+$ such that $den_1^*(F)$ is a universal operation on grounds f. The latter, wil be defined by m equations ($m \geq 1$), possibly involving other operations on grounds f_j^i ($i \in \mathbb{N}$, $j \leq m$) in the following way

[2] Proposition 101 shows that, given the hypotheses, conjecture 100 implies the existence of a denotation function for the elements of the alphabet of $\mathtt{Gen}^+$ with respect to which $\mathtt{Gen}^+$ is conservative over Gen. Again under the same hypotheses, it would be possible to obtain also the inverse implication by enriching the languages of grounding as indicated in Chapter 6, i.e., by introducing variables of the type $\mathtt{h}^{\alpha(x)}$ and $\mathtt{f}^{\beta}$. With this approach, first of all, also the inverse of theorem 60 would hold. Let Λ^+ be a non-primitive expansion of a language of grounding Λ, and let F be a non-primitive operational symbol properly in Λ^+. Let us suppose that F binds an individual variable x on index i, where the i-th entry of the operational type of F is $\alpha_i(x)$, and a ground-variable ξ^γ on index j, where the j-th entry of the operational type of F is α_j. Let finally den^* be a denotation function for the elements of the alphabet of Λ^+, suitably defined also on $\mathtt{h}^{\alpha(x)}$ and $\mathtt{f}^{\alpha}$, and let den be the denotation function for the terms of Λ associated with den^*. Well, $den^*(F) = den^*(F_1) \circ \cdots \circ den^*(F_n)$ - where, for every $i \leq n$, F_i is an operational symbol of Λ - if, and only if, there is $T \in \mathtt{TERM}_\Lambda$ such that $den(F\ \underline{x}\ \underline{\xi}(\ldots, \mathtt{h}^{\alpha_i(x)}, \mathtt{f}^{\alpha_j}(\xi^\gamma), \ldots)) = den(T)$. In other words, in the enriched languages of grounding, it becomes possible to build terms the denotation of which is identical to that of operational symbols binding individual and ground-variables, just as it happens in the non-enriched ones for the operational symbols that do not bind any individual or ground-variables - for example $den^*(\wedge_{E,i}) = den(\wedge_{E,i}(\xi^{\alpha \wedge \beta}))$. Hence, the existence of a denotation function den^* with respect to which $\mathtt{Gen}^+$ is conservative over Gen, implies the rewritability via den^* of the operational symbols of $\mathtt{Gen}^+$ in Gen.

$$\begin{cases} f = f_1^1 \circ \dots \circ f_1^{n_1} & \text{se ...} \\ \vdots & \vdots \\ f = f_m^1 \circ \dots \circ f_m^{n_m} & \text{se ...} \end{cases}$$

In general, it is not guaranteed that, for every f_j^i ($i \leq n_j$, $j \leq m$) in the defining equation of an operational symbol x_1 of $\texttt{Gen}^+$, there is an operational symbol x_2 of $\texttt{Gen}^+$ such that $den_1^*(x_2) = f_j^i$. However, let us suppose that there is an expansion $\texttt{Gen}^{++}$ of $\texttt{Gen}^+$ with non-primitive operational symbols such that it is possible to define a denotation function den_2^* for the elements of the alphabet of $\texttt{Gen}^{++}$ that complies with the condition above. We will say that $\texttt{Gen}^{++}$ is *self-contained* with respect to den_2^* - clearly, $den_1^*(F) = den_2^*(F) = f$.[3]

We now enrich $\texttt{Gen}^{++}$ appropriately, and call this enrichment $\texttt{Gen}_1^{++}$, in such a way that it is possible to define on $\texttt{Gen}_1^{++}$ an appropriate system of grounding Σ which "internalizes" the schemes of equations that fix $den_2^*(x)$, for every non-primitive operational symbol x of $\texttt{Gen}^{++}$. We will say that Σ *totally interprets* $\texttt{Gen}_1^{++}$ with respect to den_2^*. If Σ complies with appropriate provability conditions, it becomes possible to obtain the following proposition.

Proposition 102. Let $\texttt{Gen}^+$ be a non-primitive expansion of $\texttt{Gen}$, obtained by adding to $\texttt{Gen}$ a non-primitive operational symbol F, and let den_1^* be a denotation function for the elements of the alphabet of $\texttt{Gen}^+$ such that $den_1^*(F)$ is a universal operation on grounds. Then, conjecture 100 is equivalent to the following condition. Let

- $\texttt{Gen}^{++}$ be an expansion of $\texttt{Gen}^+$, and den_2^* a denotation function for the elements of the alphabet of $\texttt{Gen}^{++}$ such that $\texttt{Gen}^{++}$ is self-contained with respect to den_2^*, and
- Σ be a system of grounding on an appropriate enrichment $\texttt{Gen}_1^{++}$ of $\texttt{Gen}^{++}$ which totally interprets $\texttt{Gen}_1^{++}$ with respect to den_2^*.

Then, there is a denotation function den_3^* for the elements of the alphabet of $\texttt{Gen}$ with respect to which $\texttt{Gen}$ is self-contained, and such that, given an appropriate enrichment $\texttt{Gen}^*$ of $\texttt{Gen}$, and a system of grounding Σ^* which totally interprets $\texttt{Gen}^*$ with respect to den_3^*, for every $T \in \texttt{TERM}_{\texttt{Gen}_1^{++}}$, there is $U \in \texttt{TERM}_{\texttt{Gen}^*}$ such that

[3] Observe that a self-contained expansion may not be available. For example, the defining equations that can be associated to the various operational symbols may involve an infinite chain: f_1 requires f_2 requires ... requires f_n requires f_{n+1} requires .... Here, we have no hope to find a *finite* appropriate language of grounding, which takes into account all the f_i ($i \in \mathbb{N}$).

$$\vdash_{\Sigma \cup \Sigma^*} T \approx U.$$

Proof. We limit ourselves to general indications. A language of grounding self-contained with respect to a denotation function of the elements of its alphabet, is a language of grounding of which the non-primitive operational symbols denote operations whose definitions involve, if they involve them, only operations that are in their turn denoted, through the same denotation function, by some other non-primitive operational symbol of the same language. On the other hand, a system of grounding that totally interprets this language of grounding with respect to this denotation function, "internalizes" all the definitions of all the operations required in order to "know the meaning" of each non-primitive operational symbol.
($\Longrightarrow$) From conjecture 100 we know that $den^*_2(F)$ will be the same operation as an appropriate combination of operations denoted, via an appropriate denotation function, which we may icall den^*_3, by operational symbols of the Gentzen-language. On the other hand, den^*_2 is "internalized" in Σ, whereas den^*_3 is "internalized" in Σ^*, so that $\Sigma \cup \Sigma^*$ totally interprets both $\mathtt{Gen}_1^{++}$ and $\mathtt{Gen}^*$. It is sufficient at this point to ensure that $\Sigma \cup \Sigma^*$ allows to derive the rewritability of F in terms of operational symbols of $\mathtt{Gen}$, in compliance with the identity between the operation denoted by F via den^*_2 and the combination of operations denoted by operational symbols of $\mathtt{Gen}$ via den^*_3 - for an example, see Section 6.2.3.4; if this obtains, we can replace each occurrence of F with its rewriting in $\mathtt{Gen}$ *salva identitate*. But provability of rewriting is something that, if Σ and Σ^* are appropriate systems of grounding, we can guarantee.
($\Longleftarrow$) This direction is even more straightforward, thanks to the hypotheses made on the systems of grounding. If the systems of grounding "internalize" the defining equations denoted by the operational symbols via den^*_2 and den^*_3, we do obtain the rewritability of the operation denoted by F via den^*_1 - recall that $den^*_1(F) = den^*_2(F)$ - in terms of an appropriate combination of operations denoted by the operational symbols of $\mathtt{Gen}$ via den^*_3. For example, if F binds ξ^γ on index i, and x and ξ^δ on index j, we will have that there is $U \in \mathtt{TERM}_{\mathtt{Gen}^*}$ such that

$$\vdash_{\Sigma \cup \Sigma^*} F \ldots x\ \xi^\gamma\ \xi^\delta \ldots (\ldots \mathtt{f}^{\alpha_i}(\xi^\gamma), \mathtt{f}^{\alpha_j}(\xi^\delta) \ldots) \approx U$$

a result starting from which we can then quantify universally. □

7.1.4 Incompleteness of intuitionistic logic

Conjecture 97, in a certain particular articulation, has recently been proved false by Thomas Piecha and Schroeder-Heister in *Incompleteness of intu-*

itionistic propositional logic with respect to proof-theoretic semantic (Piecha & Schroeder-Heister 2018). Here, we aim to show how the proof by Piecha and Schroeder-Heister can also be applied to the framework of the theory of grounds. First of all, given a first-order logical language L, given $\Gamma \subset \texttt{FORM}_L$ finite, and given an atomic base B on L, with

$$\models_B \Gamma$$

we indicate the circumstance that $\models_B \beta$ for every $\beta \in \Gamma$. We then have the following propositions.

Proposition 103. For every atomic base B, $\Gamma \models_B \alpha \Leftrightarrow (\models_B \Gamma \Rightarrow \models_B \alpha)$.

Proof. ($\Longrightarrow$) For arbitrary B, let us suppose that $\Gamma \models_B \alpha$, namely that there is a proof π on B with set of assumptions Γ, conclusion α, and the individual variables of which are all and only those occurring free in Γ and α. As a chain of inferences valid on B, say $J_1, ..., J_n$, from observation 95 π can be understood as a composite B-operation on grounds $f_1 \circ \cdots \circ f_n$, where each f_i can be associated with an inference J_i ($i \leq n$) and, more in particular, for the restriction on free individual variables, as a proper ground on B for $\Gamma \vdash \alpha$. Let us suppose then $\models_B \Gamma$ and, put

$$\Gamma = \{\beta_1, ..., \beta_n\},$$

let π_i be a proof on B of β_i the individual variables of which are all and only those occurring free in β_i ($i \leq n$). As a chain of inferences valid on B, say $J^i_1, ..., J^i_{m_i}$, from observation 95 π_i can be understood as a composite B-operation on grounds $f^i_1 \circ \cdots \circ f^i_{m_i}$, where each f^i_j can be associated with an inference J^i_j ($j \leq m_i$) and, more in particular, for the restriction on free individual variables, as a proper ground on B for $\vdash \beta_i$. But then

$$f_1 \circ \cdots \circ f_n((f^1_1 \circ \cdots \circ f^1_{m_1}) \ldots (f^n_1 \circ \cdots \circ f^n_{m_n}))$$

returns a proper ground on B for $\vdash \alpha$. Each of the B-operations on grounds involved in such ground, can be associated to an inference, which will hence turn out to be valid on B; moreover, according to way in which the B-operations are combined, the inferences valid on B can be combined so as to obtain a proof on B of α. Hence, $\models_B \alpha$.
($\Longleftarrow$) Put again

$$\Gamma = \{\beta_1, ..., \beta_n\},$$

since, according to observation 95, each proof on B for β_i or α can be understood as a proper ground on B, say g_i for $\vdash \beta_i$ or g for $\vdash \alpha$ ($i \leq n$), it is sufficient to define an operation

$$h(\underline{x}, \xi^{\beta_1}, ..., \xi^{\beta_n})$$

such that, for every g_i proper ground on B for $\vdash \beta_i$ as indicated above, points on a specific g as indicated above, namely

$$h(\underline{x}, g_1, ..., g_n) = g.$$

We then have a B-operation on grounds of operational type

$$\Gamma \rhd \alpha$$

which, when associated to the inference

$$\frac{\beta_1 \qquad \dots \qquad \beta_n}{\alpha}$$

makes it valid on B, so that $\Gamma \models_B \alpha$ - observe that the required operation can be understood simply as the empty function if it is not the case that, for every $i \leq n$, there is a proof on B for β_i. □

Proposition 104. For every atomic base B:

($\rightarrow$) $\models_B \alpha \rightarrow \beta$ if, and only if, $\alpha \models_B \beta$

($\vee$) $\models_B \alpha \vee \beta$ if, and only if, $\models_B \alpha$ or $\models_B \beta$

Proof. The proof is trivial, in the light of observation 95, and of the clauses ($\rightarrow_G$) and ($\vee_G$). □

In order to reconstruct the reasoning of Piecha and Schroeder-Heister, it will be enough to limit ourselves to propositional logic and, as Piecha and Schroeder-Heister do, to rely on some preliminary results, which will be stated, without proof, in the following proposition.

Proposition 105. In `IL` the following circumstances hold:

1. Harrop's rule

   $$\frac{\neg\alpha \rightarrow (\beta_1 \vee \beta_2)}{(\neg\alpha \rightarrow \beta_1) \vee (\neg\alpha \rightarrow \beta_2)}$$

   is not derivable (Harrop 1960);

2. disjunctions can be eliminated from negated formulas, so as to obtain formulas without $\vee$ and inter-derivable, by applying the following results

$$\begin{cases} \neg(\alpha \vee \beta) \vdash_{\texttt{IL}} \neg\alpha \wedge \neg\beta \text{ and } \neg\alpha \wedge \neg\beta \vdash_{\texttt{IL}} \neg(\alpha \vee \beta) \\ \neg(\alpha \wedge \beta) \vdash_{\texttt{IL}} \neg(\neg\neg\alpha \wedge \neg\neg\beta) \text{ and } \neg(\neg\neg\alpha \wedge \neg\neg\beta) \vdash_{\texttt{IL}} \neg(\alpha \wedge \beta) \\ \neg(\alpha \to \beta) \vdash_{\texttt{IL}} \neg\neg\alpha \wedge \neg\beta \text{ and } \neg\neg\alpha \wedge \neg\beta \vdash_{\texttt{IL}} \neg(\alpha \to \beta) \end{cases}$$

At this point, we adapt to the relation $\models_B$ a *generalized disjunction property* - which in the following we will indicate with (GDP$_{\models_B}$):

$$\vee \text{ does not occur in } \Gamma \Rightarrow (\Gamma \models_B \alpha \vee \beta \Rightarrow \Gamma \models_B \alpha \text{ or } \Gamma \models_B \beta).$$

For the last step, and unlike what has been done so far, we do not mean any longer validity in the sense of ($\models_2$), and adopt instead ($\models_1$) – both of them as in Section 7.1.1. So an inference is *valid* if, and only if, for every atomic base B, there is a B-operation on grounds associated with it, i.e. having as domain the premises, and as (codomain of the codomain) the conclusion of the inference, as well as appropriate discarghes of assumptions and bindings of variables; an inference rule is *valid* if, and only if, for every atomic base B, there is a "structural" B-operation on grounds associated with it. The notion of "structural" operation on grounds, must be understood, as usual, in the sense indicated by the words of Prawitz quoted again in Section 7.1.1. We indicate with

$$\Gamma \models_1 \alpha$$

the fact that, for every atomic base B, there is a proof on B having Γ as set of assumptions, α as conclusion, and a set of individual free variables equal to that of the individual variables occurring free in Γ and α. We have the following two results.

Proposition 106. $\Gamma \models_2 \alpha \Rightarrow \Gamma \models_1 \alpha$

Proof. Suppose $\Gamma \models_2 \alpha$, namely, that a proof π exists having Γ as set of assumptions, α as a conclusion, and a set of free individual variables equal to that of the individual variables occurring free in Γ and α. Then, for every atomic base B, π is a proof on B having Γ as set of assumptions, α as a conclusion, and a set of free individual variables equal to that of the individual variables occurring free in Γ and α. Hence, for every atomic base B, there is a proof on B having Γ as set of assumptions, α as a conclusion, and a set of free individual variables equal to that of the individual variables occurring free in Γ and α. Therefore, $\Gamma \models_1 \alpha$. □

Proposition 107. $\Gamma \vdash_{\texttt{IL}} \alpha \Rightarrow \Gamma \models_1 \alpha$

Proof. It follows immediately from the correcteness theorem and from the previous proposition. □

We shall now proceed to the results of Piecha and Schroeder-Heister.

Theorem 108 (Piecha & Schroeder-Heister)**.** If, for every atomic base B, ($\mathrm{GDP}_{\models_B}$) holds, then Harrop's rule is valid.

Proof. Since we are adopting ($\models_1$), what we must prove is that, starting from the hypotheses of the theorem, for every atomic base B, there is a B-operation on grounds of operational type

$$\neg\alpha \to (\beta_1 \vee \beta_2) \rhd (\neg\alpha \to \beta_1) \vee (\neg\alpha \to \beta_2).$$

By virtue of observation 95, this can be done by showing that, starting from the hypotheses of the theorem, for every atomic base B

$$\neg\alpha \to (\beta_1 \vee \beta_2) \models_B (\neg\alpha \to \beta_1) \vee (\neg\alpha \to \beta_2).$$

Let B be an arbitrary base, and let us suppose that

$$\models_B \neg\alpha \to (\beta_1 \vee \beta_2).$$

By proposition 104 we will have that

$$\neg\alpha \models_B \beta_1 \vee \beta_2.$$

By proposition 105, there is a formula α^* without $\vee$ such that

$$\neg\alpha \vdash_{\mathtt{IL}} \alpha^* \text{ and } \alpha^* \vdash_{\mathtt{IL}} \neg\alpha.$$

By proposition 107, we have in particular that

$$\alpha^* \models_1 \neg\alpha$$

and hence, for our specific base B,

$$\alpha^* \models_B \neg\alpha.$$

Again by virtue of observation 95, and of proposition 27 in Section 5.2.2.3 about the composition of B-operations on grounds, we will therefore have that

$$\alpha^* \models_B \beta_1 \vee \beta_2.$$

Since we are assuming ($\mathrm{GDP}_{\models_B}$), we will have

$$\alpha^* \models_B \beta_i \ (i = 1, 2).$$

Again by proposition 107, we have that

$$\neg\alpha \models_B \alpha^*$$

and hence, again by virtue of observation 95, and of proposition 27 in Section 5.2.2.3 about the composition of B-operations on grounds,

$$\neg\alpha \models_B \beta_i.$$

By proposition 104, hence,

$$\models_B \neg\alpha \to \beta_i$$

and, again by proposition 104,

$$\models_B (\neg\alpha \to \beta_1) \vee (\neg\alpha \to \beta_2).$$

In conclusion, we have proven what follows:

$$\models_B \neg\alpha \to (\beta_1 \vee \beta_2) \Rightarrow \models_B (\neg\alpha \to \beta_1) \vee (\neg\alpha \to \beta_2).$$

By proposition 103, we can thus conclude that

$$\neg\alpha \to (\beta_1 \vee \beta_2) \models_B (\neg\alpha \to \beta_1) \vee (\neg\alpha \to \beta_2).$$

The result now ensues immediately from the arbitrariness of B. □

Theorem 109 (Piecha & Schroeder-Heister)**.** (GDP$_{\models_B}$) holds for every atomic base B.

Proof. Let B be an arbitrary base, and let us suppose that $\Gamma \models_B \alpha \vee \beta$. Then, by proposition 103, we have that

$$\models_B \Gamma \Rightarrow \models_B \alpha \vee \beta$$

and, by proposition 104,

$$\models_B \Gamma \Rightarrow (\models_B \alpha \text{ or } \models_B \beta).$$

By adopting classical logic in the meta-language, we have

$$(\models_B \Gamma \Rightarrow \models_B \alpha) \text{ or } (\models_B \Gamma \Rightarrow \models_B \beta)$$

and, by proposition 103,

$$\Gamma \models_B \alpha \text{ or } \Gamma \models_B \beta.$$

The result now ensues immediately from the arbitrariness of B. □

Corollary 110 (Piecha & Schroeder-Heister)**.** Harrop's rule is valid.

Proof. It ensues immediately from theorems 108 and 109, bearing in mind that we are adopting ($\models_1$). □

Corollary 111 (Piecha & Schroeder-Heister)**.** For some Γ and α, $\Gamma \models_1 \alpha$ and $\Gamma \nvdash_{\texttt{IL}} \alpha$.

Proof. Putting

$$\Gamma = \neg\alpha \rightarrow (\beta_1 \vee \beta_2) \text{ and } \alpha = (\neg\alpha \rightarrow \beta_1) \vee (\neg\alpha \rightarrow \beta_2)$$

and again bearing in mind that we are adopting ($\models_1$), from corollary 110 we know that Harrop's rule is valid, and hence that $\Gamma \models_1 \alpha$. On the other hand, from proposition 105, we know that Harrop's rule is not derivable in `IL`, and hence that $\Gamma \nvdash_{\texttt{IL}} \alpha$. □

In conclusion, if we decline the notion of validity along the lines indicated by ($\models_1$), the proof of Piecha and Schroeder-Heister shows that `IL` is incomplete with respect to the theory of grounds. Note that it is not difficult to outline an analogue of conjecture 98 which implies the completeness of `IL` also with respect to ($\models_1$) – just like conjecture 98, as shown by proposition 99, implies the completeness of `IL` with respect to ($\models_2$). The result of Piecha and Schroeder-Heister would therefore imply also the falsity of this possible conjecture. At the conclusion of their work, Piecha and Schroeder-Heister observe significantly that

> the incompleteness of intuitionistic logic with respect to such a semantics therefore raises the question of whether there is an intermediate logic between intuitionistic and classical logic which is complete with respect to it. (Piecha & Schroeder-Heister 2018, 13)

However, the proof of the fundamental theorem 109 resorts in an essential way to the adoption of classical logic in meta-language. However, in this regard, Piecha and Schroeder-Heister point out that, despite the adoption of classical logic in meta-language, their result

> as a negative result, is as devastating for the completeness conjecture as a constructive proof. (Piecha & Schroeder-Heister 2018, 12).

Secondly, it should be noted that the proof of Piecha and Schroeder-Heister applies to ($\models_1$), but it is not clear if it also applies to ($\models_2$). Although ($\models_2$) certainly implies ($\models_1$), the reverse could not hold – but see note 1 in this chapter. In addition, proposition 103 would be said to be false when formulated on ($\models_2$); in particular, it seems there is reason to claim that there are Γ and α such that $\models_2 \Gamma \Rightarrow \models_2 \alpha$, but it does not hold that $\Gamma \models_2 \alpha$. Take $\Gamma = \{p\}$ and $\alpha = q$; obviously, neither $\models_2 p$, nor $\models_2 q$, but we can construct an atomic base that has p as unique axiom, and on which therefore no operation on grounds can exist of operational type

$$p \rhd q.$$[4]

Be that as it may, we can adapt the aforementioned observation of Piecha and Schroeder-Heister to our case. If intuitionist logic is not complete with respect to the theory of grounds, then there could be some intermediate logic for which completeness applies. With regard to this logic, it would remain to verify if it, besides being complete in the sense indicated by conjecture 98, is also full-complete in the sense indicated by conjecture 100. If this were the case, after drawing up a language of grounding that, on a par of `Gen` for intuitionistic logic, constitutes a functional isomorphic translation of this intermediate logic, we will have the results – in our opinion interesting – suggested by propositions 101 and 102.

7.2 Recognizability and equations

In the second part of this chapter, we intend to deal more extensively with the recognizability problem inherited, as seen above, by the theory of grounds, albeit in a new articulation, from proof-theoretic semantics, and that the theory of grounds more generally shares with the clauses for $\rightarrow$ and $\forall$ in BHK semantics. This will be done by following a dual directive.

[4] Piecha and Schroeder-Heister build their proof of incompleteness on the basis of some general semantic principles. Among these there is also the following: $\Gamma \models \alpha \Leftrightarrow$ for every atomic base B, ($\models_B \Gamma \Rightarrow \models_B \alpha$). As said, in our frameworks it holds on the other hand that: for every atomic base B, $\Gamma \models_B \alpha \Leftrightarrow (\models_B \Gamma \Rightarrow \models_B \alpha)$. Therefore, if Piecha and Schroeder-Heister's semantic principle held also in our framework, we would have that: $\Gamma \models \alpha \Leftrightarrow$ for every atomic base B, $\Gamma \models_B \alpha$. Now, this is exactly how the relation ($\models_1$) is characterised, i.e., $\Gamma \models_1 \alpha$ if, and only if, for every atomic base B, there is an operation f such that f is a B-operation on grounds of operational type $\Gamma \rhd \alpha$. But the equivalence seems not to work in the case of ($\models_2$), where we have instead an inversion of the quantifiers: $\Gamma \models_2 \alpha$ if, and only if, there is an operation f such that, for every atomic base B, f is a B-operation on grounds of operational type $\Gamma \rhd \alpha$. It should be noted that the latter seems to be the way in which Prawitz also characterizes his notion of valid inference in *Towards a foundation of a general proof theory* (Prawitz 1973) and in *An approach to general proof theory and to conjecture of a kind of completeness of intuitionistic logic revisited* (Prawitz 2014).

First, we will show how the assumption of recognizability is subject to a double reading, a weak one and a strong one. If you accept some background theses - in particular, the adoption of classical logic in the meta-language - the weak reading, unlike the strong one, would seem plausible. On the other hand, if the basic theses are rejected - in particular, if intuitionistic logic is used also in meta-language – we question the plausibility also of the weak reading.

Secondly, we will see how the recognizability problem involves, when set in the conceptual framework of the theory of grounds, the issue of a general theory of defining equations for non-primitive operations on grounds. We will provide general guidelines for a first classification of these equations, investigating which of these validate, and which, on the contrary, lose the desired recognizability.

7.2.1 Local and global recognizability

In Chapter 2, we have extensively discussed a recognizability problem that Prawitz's proof-theoretic semantics inherits, so to speak, from the BHK clauses. As for the latter, epistemic needs could lead to request that a proof of $\alpha \rightarrow \beta$ is not simply a constructive function which, when applied to any BHK proof of α, turns it into a BHK proof of β, and that a proof of $\forall x \alpha(x)$ is not simply a constructive function which, when applied to whatever individual k in a reference domain, produces a BHK proof of $\alpha(k)$. Since proofs have for us an epistemic relevance only when we are able to recognize that they prove what we intend to prove, we might be right to postulate that the above functions must be accompanied by the recognition of the fact that they have an appropriate behaviour. Likewise, we might require, for the valid arguments and for the proofs of Prawitz's proof-theoretic semantics, that it is recognizable that a valid closed non-canonical argument or a non-canonical/categorical proof reduce, respectively, to a valid closed canonical argument and to a canonical proof, and that a valid open argument or a hypothetical-general proof are such that all their closed instances are valid arguments and proofs.

In the theory of grounds, the problem becomes that of recognizing that a term of an appropriate language of grounding denotes a ground for a certain judgment or a certain assertion. If the term is closed, it must be possible to recognize that it reduces to a canonical form which denotes a ground according to the clauses ($\wedge_G$) - ($\exists_G$) and, if the term is instead open, it must be possible to recognize that it denotes an operation on grounds of the appropriate operational type, namely that each time the free individuals variables are replaced with closed individual terms, and the ground-variables (in the type of which the individual variables have been replaced with individual terms) are replaced with closed terms that denote grounds, the term obtained denotes a ground. The crucial case here is that of the non-primitive operational symbols, since the primitive ones denote operations on grounds involved in the explanation of meaning, and therefore produce grounds from grounds. From the recognizability problem of the denotation of a term we can then pass to that of the recognizability of the fact that an operation on grounds has a certain operational type, and since the operations on grounds are defined by equations aimed at ensuring that the function is constructively convergent on all the values in the intended domain, so as to produce for each of these values a ground in the intended codomain, the problem is reduced to that of the recognizability of the good position of the defining equations. In conclusion, therefore, and very generally, given an arbitrary operation on grounds f of operational type τ defined by an equation ε, what we have to ask ourselves is whether it is possible to recognize that ε fixes f so that f actually has the operational type τ. As we have seen in Chapter 5, it is in many cases possible to prove that specific equations define appropriately specific operations on grounds – it is the case of the equations that fix the denotation of the operational symbols $\wedge_{E,i}$, $\vee E$, $\rightarrow E$, $\forall E$, $\exists E$ and $\mathtt{Ind}$. To raise the recognizability issue means asking oneself whether the recognition guaranteed by proofs of that type is possible for arbitrary operations, fixed by equally arbitrary equations.

Thus formulated, the recognizability problem can in our opinion be understood along two different, equally suitable, *degrees of generality*. In the weaker degree, that we could call *local recognition*, we can ask ourselves whether it is possible to recognize, for every equation ε that defines an operation on grounds f of operational type τ, that ε defines f so that the latter has actually an operational type τ, but without requiring additionally that the recognition is achieved through a method that works also in the case of other equations for other operations. In the stronger degree, that we could call *global recognizability*, we can instead ask ourselves if it is possible to recognize, for every equation ε that defines an operation on grounds f of operational type τ, that ε defines f so that the latter has actually an operational type τ, and this through a universal method that allows to reach homogeneously the same type of recognition for all possible equations for all possible operations. In the context of local recognition, therefore, recognition is always achieved, only through a "case by case" method. Within the framework of global recognizability, on the other hand, the method that guarantees recognizability is unique, and uniformly applicable to all cases. We therefore have two different theses of recognizability, which differ from one another for a different order of the universal and existential quantifiers:

(L) for every equation ε that defines an operation on grounds f of operational type τ, there is a procedure $\wp$ such that $\wp$ allows to recognize that ε defines an operation on grounds f of operational type τ;

(G) there is a procedure $\wp$ such that, for every equation ε that defines an operation on grounds f of operational type τ, $\wp$ allows to recognize that ε defines an operation on grounds f of operational type τ.

Therefore, in asking whether the problem of recognizability can be solved or not, we can ask ourselves, for each of the theses (L) and (G), whether they are plausible or not. Obviously, both (L) and (G) guarantee a recognition that really matches the epistemic needs of the theory of grounds.

At first glance, one would say that (L) and (G) are completely different theses; in particular, (L), unlike (G), would seem plausible. An operation on grounds, in fact, is an epistemic object and, as such, it should always be in principle possible to recognize its properties, included that of having a certain operational type. The equation that defines an operation of this type, therefore, must be such as to attribute it, in a recognizable way, the corresponding operational type. In addition, if we accept classical logic, the negation of (L) would correspond to the existence of an equation ε that defines an operation on grounds f of operational type τ, such that, whatever the procedure $\wp$ is, $\wp$ does not permit to recognize that ε defines an operation on grounds f of operational type τ; namely, we would have an *epistemic object* with *absolutely unknowable* properties, a sort of *contradictio in terminis.* On the other hand, (G) requires the existence of a universal procedure, homogeneous and uniform, which works on all equations. If by "recognizability" we mean "decidability", the existence of such a procedure is clearly impossible. But, also in a less restricted reading, (G) would seem to put forward an excessive claim. Even if we abandoned the idea that of the notion of recognizability we could give an accurate reading in terms of the notion of decidability, and therefore even if we renounced to consider recognition procedures as decision algorithms, it would not make much sense to invoke the existence of a universal, homogeneous and uniform recognition method, unless we are able to give at the very least a generic description of the instructions it contains. On the other hand, it is far from obvious how - and indeed very unlikely that - such a description can ever be found.

In substantiating the plausibility of (L), we referred to the use of classical logic. However, if the logic we intend to use in dealing with the sustainability of (L) is not the classical one, but the intuitionistic one, or more generically the constructivist one, then also the plausibility of (L) is called into question. The problem is that, in this case, it would seem possible to argue that (L) implies (G), and therefore that, if we accept (L), we are consequently forced to accept also (G). The way to this conclusion is reminiscent of that with which the intuitionists justify the validity of the axiom of choice. According to Martin-Löf – whose intuitionistic type theory actually *proves* the axiom of choice:

> the usual argument in intuitionistic mathematics, based on the intuitionistic interpretation of the logical constants, is roughly as follows: to prove $\forall x \exists y C(x,y) \rightarrow \exists f \forall x C(x, f(x))$, assume that we have a proof of the antecedent. This means that we have a method which, applied to an arbitrary x, yields a proof of $\exists y C(x,y)$, that is, a pair consisting of an element y and a proof of $C(x,y)$. Let f be the method which, to an arbitrarily given x, assigns the first component of this pair. Then $C(x, f(x))$ holds for an arbitrary x, and hence so does the consequent. (Martin-Löf 1984, 50)

By adapting this reasoning to our case, we can therefore take into account the implication

$$\forall \varepsilon \ \exists \wp_\varepsilon \ R(\varepsilon, \wp_\varepsilon) \rightarrow \exists \wp \ \forall \varepsilon \ R(\varepsilon, \wp(\varepsilon))$$

where R is the binary predicate "the procedure ... allows the recognition on the equation $---$". If we suppose to have a proof of the antecedent, we will have a constructive function F_1 such that, for every equation ε,

$$F_1(\varepsilon) = (\wp_\varepsilon, \pi_\varepsilon)$$

where $\wp_\varepsilon$ is a recognition procedure, and π_ε is a proof of $R(\varepsilon, \wp_\varepsilon)$. We can now define two operations, F_2 and F_3, such that, for every equation ε,

$$F_2(\varepsilon) = d_1(F_1(\varepsilon)) \text{ and } F_3(\varepsilon) = d_2(F_1(\varepsilon))$$

where d_1 and d_2 are, respectively, left and right projection. By using λ-abstraction, we will then have

$$F_2 = \lambda \varepsilon . d_1(F_1(\varepsilon)) \text{ and } F_3 = \lambda \varepsilon . d_2(F_1(\varepsilon))$$

and therefore the sought proof of the implication will be

$$\lambda F_1.(F_2, F_3).$$

As regards the implication, it is perhaps appropriate to observe that both $\wp_\varepsilon$ and $\wp$ are functions of ε, but in a significantly different way. $\wp_\varepsilon$ is a procedure which, when applied to ε, allows us to recognize, as it were, in a direct way that it defines an operation on grounds having a certain operational type; $\wp$ when applied to ε, it guarantees the same recognition, but through the choice of an appropriate $\wp_\varepsilon$.

7.2.2 Parameters and structure of equations

If, in the framework proposed so far, the recognizability problem is declined so as to understand "recognizable" as a synonym of "decidable", its solution has, as mentioned, certainly a negative nature. The reason for this is simple: since we have no limitation on the class of eligible operations in the theory, nor therefore any restriction on that of the correspondent defining equations, there is reason to expect such classes to be so large to meet all the "algorithmic limitations" imposed by Gödel's theorems.

However, if this is true, it is also true that *exactly* the absence of more precise indications on the operations and on the structure of their defining equations makes it difficult to understand the recognizability problem as a problem of decidability. Decidability is a precise mathematical notion, and to raise a question of decidability seems to make sense only within a context formally well specified; on the other hand, the class of operations of the theory of grounds, as well as the class of equations that define them, as they have been understood so far, have rather blurred, or at least very liberal boundaries. This does not mean, of course, that a weaker reading of the expression "recognizability" puts us on the way to a positive solution. Beyond the difficulties highlighted in the previous section, if "recognizable" does not mean "decidable", then it is not at all clear what the term may indicate, or what is the most appropriate understanding with respect to the purposes desired.

Regardless of how the recognizability problem is intended, a possibility to have more precise answers – even if not necessarily univocal – can therefore come from a restriction of the classes of operations and of their defining equations. In other words, we could indicate a certain number of basic properties that operations must comply with, and a certain number of parameters that the equations that define them must satisfy. Once that is done, having a more rigorous idea of the type of accepted operations and equations, we could expect an equally rigorous formulation of the recognizability problem, and of its possible solution – solution that, obviously, could be positive in some cases, negative in others.

In our opinion, a fruitful proposal in this sense – that, however, we will limit ourselves to suggest in the very great lines - could be inspired, on the one hand, by the reflections that lead Prawitz (Prawitz 2018a) to the notion of analytically valid argument, discussed in Section 2.5.3.2, and on the other hand to *Constructive semantics*, the article of the Swedish logician (Prawitz 1971a) the content of which was illustrated in Section 4.3.2. The notion of analytically valid argument is, as you may remember, based on a notion of containment such that a closed non-canonical argument can be said to be analytically valid if it contains an analytically valid closed canonical one. In order to determine whether containment holds or not, Prawitz authorizes two operations:

1. extraction of a subargument from a given argument;
2. substitution of free individual variables with terms, and of assumptions with closed analytically valid arguments for such assumptions, in a given argument.

In *Constructive semantics*, however, Prawitz aims at a notion of construction, and to a class of terms that denote constructions; this, in order to have a framework with respect to which to prove the correctness of minimal and intuitionistic logic. The terms are constructed using operational symbols with intended interpretation as follows:

a) pair formation;

b) left and right projection on pair;

c) λ-abstraction;

d) application for a λ-abstraction;

e) 4-ary choice function, that selects the first or second element of its arguments depending on where the third and the fourth are or not equal

- there would also be a function replacing individual and typed variables, but here we can leave it aside. The operations of points a) and c) are primitive: λ-abstraction allows the construction of canonical objects for $\alpha \rightarrow \beta$ or $\forall x\alpha(x)$, when applied, respectively, to an operation that transforms constructions of α into constructions of β binding ξ^α, or to an operation that transforms individuals k from a reference domain into constructions of $\alpha(k)$ binding x; pair formation obviously allows the construction of canonical objects for $\alpha \wedge \beta$ – when applied to constructions of α and β – but, if we want, also for $\alpha_1 \vee \alpha_2$ – pair with first element a construction g of α_i $(i = 1, 2)$ and with second element an indication of what disjunct g is a construction – and for $\exists x\alpha(x)$ – pair with first element a construction of $\alpha(t)$ and second element the term t. The operations at points b), d) and e) are instead non-primitive, and correspond quite well to the operations referred to at points 1 and 2; application for the λ-abstraction recalls the operation of point 2, the latter understood as the application of an argument with individual variables and assumptions on specific terms and closed analytically valid arguments; projection on pair is, obviously, an example of extraction from an argument that contains at least two immediate subarguments, while the choice function could allow us to choose which of these two subarguments should be picked up.

With his notion of analytically valid argument, Prawitz seems to suggest that the reduction or justification procedure associated with a rule in non-introductory form, should be such that the argument it provides as output is obtained from the argument in input by applying only the operations 1 and 2; on the other hand, in *Constructive semantics*, Prawitz shows the adequacy of Gentzen's elimination rules by offering a functional interpretation of the latter that contains only the operations a) - e). According to what we have just seen, we can generalize what Prawitz does in *Constructive semantics* and, relying on the similarities highlighted between *Constructive semantics* and the notion of analytically valid argument, request that non-canonical cases are justified by resorting to a well-defined inventory of operations; more specifically, we could request that the defining equation of whatever operation on ground F is such as to express F as a combination of some or all the operations of formation of pair, projection on pair, λ-abstraction, application of λ-abstraction, and function of choice, and of no other operation.[5]

[5] Let us indicate pair formation with D, projection on a pair with D^i $(i = 1, 2)$, and application of λ-abstraction with App, and let us suppose that a ground for $\vdash \alpha_1 \vee \alpha_2$ is of the form (g, i), with g ground for $\vdash \alpha_i$ $(i = 1, 2)$ - o, if you prefer, $((g, i), \alpha_j)$ $(j = 1, 2)$, if you also want to indicate the other disjunct - and that a ground for $\exists x \alpha(x)$ is of the form (g, t), with g ground for $\vdash \alpha(t)$. We can define the operational symbols $\wedge_{E,i}$, $\vee E$, $\rightarrow E$, $\forall E$ and $\exists E$ as follows:

$$\wedge_{E,i}(\xi^{\alpha_1 \wedge \alpha_2}) = D^i(\xi^{\alpha_1 \wedge \alpha_2})$$

$$\vee E\ \xi^{\alpha_1}\ \xi^{\alpha_2}(\xi^{\alpha_1 \vee \alpha_2}, h_1(\xi^{\alpha_1}), h_2(\xi^{\alpha_2})) =$$
$$\texttt{App}(\lambda \xi^{\alpha_{D^2(\xi^{\alpha_1 \vee \alpha_2})}}.h_{D^2(\xi^{\alpha_1 \vee \alpha_2})}(\xi^{\alpha_{D^2(\xi^{\alpha_1 \vee \alpha_2})}}), D^1(\xi^{\alpha_1 \vee \alpha_2}))$$

$$\rightarrow E(\xi^{\alpha \rightarrow \beta}, \xi^{\alpha}) = \texttt{App}(\xi^{\alpha \rightarrow \beta}, \xi^{\alpha})$$

$$\forall E(\xi^{\forall x \alpha(x)}, y) = \texttt{App}(\xi^{\forall x \alpha(x)}, y)$$

$$\exists E\ x\ \xi^{\alpha(x)}(\xi^{\exists x \alpha(x)}, h(x, \xi^{\alpha(x)})) =$$
$$\texttt{App}(\texttt{App}(\lambda x \lambda \xi^{\alpha(x)}.(h(x, \xi^{\alpha(x)}), D^2(\xi^{\exists x \alpha(x)})), D^1(\xi^{\exists x \alpha(x)}))$$

Note that, strictly speaking, we did not make use of the function of choice; the latter serves only in the case of the elimination of the disjunction, and we have preferred a binding of the index of the argument to be chosen to the projection on the second element of the possible ground for $\vdash \alpha_1 \vee \alpha_2$ - we should have written $D^2(D^1(\xi^{\alpha_1 \vee \alpha_2}))$ if the ground for $\vdash \alpha_1 \vee \alpha_2$ is of the form $((g, i), \alpha_j)$.

To go into the technical details of this proposal, or of other similar, would lead us far beyond the scope of this work; therefore, we will be satisfied with the suggestion made, turning now to a last, conclusive observation.

The question we want to ask ourselves is the following: are there general properties of the defining equations that are relevant to occurring or not of recognizability? The raised point is independent of the "width" of the classes of the operations and of the admissible equations, and therefore of the restrictions placed on them, and it depends solely on the fact that, as definitions, the equations can be of many different types, present more or less complex structures, enjoy the most diversified characteristics; which of these types, which of these structures, and which of these characteristics could allow recognizability, and which, on the contrary, make it lose? Obviously, it could be argued that the different typologies, structures and features depend on restrictions placed on the eligible operations and equations; but perhaps we can face the problem from a more general point of view.

The reflection on the relationship between the general form of a defining equation for an operation on grounds, and the problem of recognition, could in fact allow to link the theory of grounds to a sort of *theory of definitions*. The survey on the different typologies of definitions, and on the criteria of their acceptability, is ancient and very general, but it has often received a specific and more in-depth attention (among the various works that can be mentioned in this regard, consider for example Padoa 1901, Carnap 1928, and Suppes 1999; in what they call *revision theory of truth*, and with particular reference to the paradoxes and to truth, also Belnap 1993 and Gupta & Belnap 1993 deal with the theory of definitions, and in this regard we can take into account also Kramer 2016). Here, we intend to propose a first classification of the possible defining equations for operations on grounds into three macro-groups. However, as a preliminary step, we first highlight some points.

We have so far said that an operation on grounds f is fixed by an equation ε, but this is actually inaccurate. Since f is to be understood as defined on a whole class of arguments, which share a certain structure or a certain typing, it would have been more correct to say that f is determined by a scheme of equations Σ, so that each instance of the schema sets the behaviour of f on specific arguments, indicating specific values. But this is not enough either. In general, in fact, to define f it could be necessary to resort not to a single equation scheme, but to a system of equational schemes $\Sigma_1, ..., \Sigma_m$; in a system, we will have also conditional clauses $c_1, ..., c_m$ such that, in the presence of the configuration c_i, the scheme to apply to compute f will be Σ_i ($i \leq m$) – it goes without saying that the set of conditional clauses is intended to cover all the possible cases of application of f, relative to the intended domain. The general form of the equational definition of an n-ary operation can therefore be represented graphically as follows:

$$f(a_1, ..., a_n) = \begin{cases} \Sigma_1 & \text{if } c_1 \\ \vdots & \vdots \\ \Sigma_m & \text{if } c_m \end{cases}$$

Each Σ_i ($i \leq m$) could in turn resort to other operations $f_1^i, ..., f_p^i$, that we will indicate writing Σ_i as

$$f_1^i \circ \cdots \circ f_{p_i}^i(a_1, ..., a_n).$$

In the following, we will find it convenient to call

$$f(a_1, ..., a_n)$$

the *definiendum*, the system of equational schemes the *definiens*, and $a_1, ..., a_n$ the *equational context*. Obviously, we take into account the case of $m = 1$, and also that of $m = 0$ – the latter covers the empty function, for whichsee Section 5.2.4.5.

So let f be an operation on grounds, and let $\mathfrak{S}$ be the system that defines it. We will say that f is *recursively defined* if, and only if, there exists a scheme of equations Σ of $\mathfrak{S}$ in which f appears. An operation on grounds is defined recursively when the definiendum appears in the definiens, and the graphical representation can be understood as having the form

$$f(a_1, ..., a_n) = \begin{cases} \Sigma_1 & \text{if } c_1 \\ \vdots & \vdots \\ f_1^p \circ \cdots \circ f \circ \cdots \circ f_{q_p}^p(a_1, ..., a_n) & \text{if } c_p \\ \vdots & \vdots \\ \Sigma_m & \text{if } c_m \end{cases}$$

Obviously, we are assuming that the definition is well posed, which in particular means that it is not circular, namely it does not create, in specific cases of computation of the value of f, either infinite chains, or *loops*. A definition for recursion will then proceed, in general, by setting the value of f on arguments of a certain complexity, in terms of the value of f on arguments of lower complexity. In fact, we have already come across in the course of our discussion an operation on grounds recursively defined; it is the operation

`Ind`, referred to in Section 5.2.4.5, accounting for the induction principle in Heyting's first-order arithmetic.

As for recursive definitions, it seems very difficult to imagine a positive solution to the problem of recognition, and this also beyond the limitative results that, in contemporary mathematical logic, apply to cases of analogous definitions in arithmetic – just think alone of Turing's *halting problem* (Turing 1936 - 1937). The fact that the definiendum appears in the definiens, makes the definitions for recursion extremely complex, so that the task of establishing that this circumstance does not cause circularity will be complex, if not more. In addition, having to reason about the degrees of the arguments the operation applies on, also requires to reason about the form of such arguments – form on which the complexity will depend, according to the domain of reference – and about the reciprocal relation between arguments of different complexity. Finally, since the behaviour of the operation on arguments of a certain complexity is fixed in terms of the behaviour of the operation on arguments of lower complexity, it becomes essential also to reason about the nature of the domain intended, in order to guarantee that the ordering of the operational behaviour with the varying complexity of the arguments is harmonic with respect to the structure of the domain to which the arguments belong – the definition could be well posed with respect to certain domains, but fail on others.

As an example, let us take into account `Ind`. In order to verify whether the system of equational schemes that defines it – illustrated by us in Section 5.2.4.5 - is well posed, we have to complete a series of passages that are not at all trivial. First of all, to establish that the value of the operation is well defined with respect to a case of minimal complexity. Then, to note that the reduction from a case of complexity δ to the case of complexity $\delta - 1$ works by virtue of the reducibility of the case of complexity $\delta - 1$ to gradually less complex cases, up to the minimum degree, and that this applies whatever the degree. These first steps may to be taken, and play the role that is required from them, only already knowing that relevant application arguments – natural numbers – can be canonically represented in the atomic system as 0, or $s(s(...(s(0))...)$, for a certain quantity of s, on the understanding that the latter represents the successor function; this must serve to establish that the reduction from the application case of complexity δ to the application case of complexity $\delta - 1$, whatever δ, defines the function on all possible relevant arguments – again, the natural numbers; and, moreover, this also means that it is required some understanding of the fact that the intended domain $\mathbb{N}$ is of a nature such to guarantee that the whole of these steps is sufficient to establish that the operation is well defined – namely, that $\mathbb{N}$ has, as it were, an inductive structure, so much so that we, when showing the adequacy of

Ind, had to resort to a meta-induction principle (which is something Prawitz does, 2012a).

Definitions by recursion represent a first macro-group, and a second one is obtained simply by denying that the system that sets up an operation is such as to define the latter recursively – namely, f is defined in a *non-recursive* way if, and only if, in its defining system $\mathfrak{S}$ there is no Σ in which f appears. Non-recursively defined operations can in turn be divided into two classes.

Let f be an operation on grounds, and let $\mathfrak{S}$ be the system that defines it. We will say that f is defined *contextually* if, and only if, there is a scheme of equations of $\mathfrak{S}$ such that each component of the expression of Σ already occurs in the equational context. Contextual definitions are those for the operations on grounds associated with the operational symbols $\wedge_{E,i}$ $(i = 1, 2)$, $\vee E$, $\rightarrow E$, $\forall E$ and $\exists E$, which we discussed again in Section 5.2.4.5, and the definition of the operation f corresponding to the disjunctive syllogism, therefore of operational type

$$\alpha \vee \beta, \neg\alpha \rhd \beta,$$

when it is written as

$$f(\vee I(g_1), g_2) = g_1.$$

Although proving that an operation defined contextually is well posed can be much easier than obtaining an analogous result for a recursively defined operation, even in the case of contextual definitions recognizability is problematic - especially when there are no restrictions on the classes of eligible operations and equations. The problem is that, here too, recognition can only be obtained by reasoning on the equational context, and generally a contextual definition will have to point out that the arguments to which the operation is applied have some specific structure, a structure from which, being at least one of the schemes of equations dependent on the context, descends the good position of the same definition. For example, in the case of the definitions of the operations associated to the operational symbols $\wedge_{E,i}$ $(i = 1, 2)$, $\vee E$, $\rightarrow E$, $\forall E$ and $\exists E$, the recognition of their good position will depend essentially on the form of the grounds for judgments or assertions on propositions or sentences of different logical form. In addition, for a lot of cases it often becomes necessary to reflect on the general structure of the expected operational type, and on the consequences that from this structure we can draw as regards the arguments to which the defined operation is applied - for example, in the case of the operation for the disjunctive syllogism defined above, it should be noted that if g_2 is a ground for $\vdash \neg\alpha$, then g_1

must necessarily be a ground for $\vdash \beta$, because if it were a ground for $\vdash \alpha$ we could, by combining it with g_2, get a ground for $\vdash \bot$. Such reflections usually take place at a meta-level and, as already in the case of the systems of equation schemes, there seems to be no upper bound to their potential complexity.

Before describing the last class of operations and their definitions, we need to make an observation. The recognition of the well position of the operations defined recursively, or in a non-recursive but contextual way, could require, as stated, to reason both on the form of the arguments to which the operations apply, and on the domain to which these arguments belong. This means that the recognition procedure will depend, in a rather specific sense, on the context of application of the operation. In concrete terms, we can take into account the derivations of systems of grounding of Chapter 6, Section 6.2.3.2, which prove the adequacy of the non-primitive operational symbols. In them, it is necessary to make use of the rule that captures Dummett's *fundamental assumption* for the logical constant to which the symbol refers, as well as of the defining equation of the symbol itself; the structure of the derivation will vary as the logical constant considered varies. Now, we could try to identify general properties, independent of the context, shared by all the recognition procedures of this type – and by all the derivations of this type - in order to define a recognition procedure to apply uniformly in all cases. This would mean to substantiate the thesis (G) of Section 7.2.1; on the other hand, though, although the task could in principle be possible for formal closed frameworks, as a system of grounding, the fact that the recognition procedure has to depend in essential points on the context seems a insurmountable obstacle to the attainment of what we have called a global recognition.

Now we come to the last type of operations defined in a non-recursive way. Let f be an operation on grounds, and let $\mathfrak{S}$ be the system that defines it. We will say that f is *reductively* defined if, and only if: (1) in no component of the expression of the equational context occur other operations on grounds and (2) each scheme of the equations Σ of $\mathfrak{S}$ is a composition of a non-null number of operations on grounds (obviously different from f, since we are assuming that the definition is non-recursive). The definition may therefore be represented graphically as

$$f(a_1, ..., a_n) = \begin{cases} f_1^1 \circ \cdots \circ f_{p_1}^1(a_1, ..., a_n) & \text{if } c_1 \\ \vdots & \vdots \\ f_1^m \circ \cdots \circ f_{p_m}^m(a_1, ..., a_n) & \text{if } c_m \end{cases}$$

with $p_i \geq 1$ $(i \leq m)$, and where no a_j $(j \leq n)$ mentions operations on grounds of any sort – in other words, a_j can be understood as a meta-variable for grounds or operations on grounds. The expression "reductive" can be explained by the fact that $\mathfrak{S}$ does nothing but "reduce" the behavior of f to the behaviour (of the combination) of other operations; therefore, we do not take into account in the strict sense the application context of f, but only the application context of other operations in the terms of which f can be defined. An example of reductive definition is that of the operation f corresponding to the disjunctive syllogism, therefore of the operational type

$$\alpha \vee \beta, \neg\alpha \rhd \beta,$$

whenever it is defined by the equation scheme – for convenience, we indicate the operations involved in the definiens with the operational symbols corresponding to them in the usual interpretation of these symbols –

$$f(g_1, g_2) = \vee E\ \xi^{\alpha}\ \xi^{\beta}(g_1, \bot_{\beta}(\rightarrow E(\xi^{\alpha}, g_2)), \xi^{\beta})$$

where, as noted, there is no further specification on the structure of g_1 and g_2 – it is only necessary to suppose that they are grounds for, respectively, $\vdash \alpha \vee \beta$ and $\vdash \neg\alpha$ – and where the *definiens* is a composition of a non-null number of operations on grounds other than f.

Reductive definitions are a kind of "rewriting" of the operation to define through a finite number of other operations – reasoning by analogy, we could say that they correspond to the derivation of a non-primitive rule in a formal system, the rules of which have already been justified with appropriate reduction procedures. Hence, in this case we can indicate a criterion that, when respected, guarantees recognizability: if, of all the operations involved in the definition of f, we know that they are well posed, we can conclude that also f is well posed – or better, we just have to verify that the combination of operations in the defining system defining of f respects the typing, but this can be done through a simple *type-checking* algorithm. Obviously, we may not know that the operations used in the definition of f they are well posed. However, we have to bear in mind that when we handle operations on grounds, we can do it within a language or a system of grounding so as described in chapters 5 and 6; and since these languages and systems are obtained through expansions of less rich languages and systems, we could guarantee – inductively, so to speak – the recognizability on languages and systems in which the definitions (of the denotation) of non-primitive symbols are reductively given, by appealing only (to the operations denoted by) the operational symbols of the languages and systems prior to expansion.

Conclusion

The theory of grounds undoubtedly shows strong comparisons with the whole *corpus* of Prawitz's semantic investigations. The questions he aims to answer are, after all, the same that had already inspired proof-theoretic semantics: what are correct inferences and reasoning, and why are they able to convey knowledge, justification, and epistemic constraint? However, radically different, we could say almost diametrically opposite, is the perspective from which these questions are understood and addressed.

The notions of deductively correct inference and reasoning are closely related, and the treatment of one cannot but lead to the characterization of the other. That is all the more true in the context of an explanation of the epistemic power they both enjoy; the reasoning works thanks to the "goodness" of the inferences that compose it, and the latter are acceptable if they can be satisfactorily used to construct a convincing reasoning. But then, where can we start the analysis? Shall we start from reasoning, or from inferences?

Proof-theoretic semantics prefers the notions of valid argument and proof, and reconstructs the notion of inferential validity in the global terms of the structures in which these inferences occur. The theory of grounds, instead, reverses the order and, by adopting a local point of view, uses valid inferences as its bases, defining then valid arguments and proofs as concatenations. To use a slogan, we pass from validity as *transmission* of justification, to validity as *production* of justification. The change of direction, apparently simple, requires a series of measures, and is fraught with interesting consequences.

If we have decided not to describe the validity of inferences in terms of the structures in which they occur, on what basis then do we define it? Prawitz's answer, it seems to us, cannot avoid traveling along two paths, at this point quite natural. First of all, to distinguish among the acts in which the inferences appear, and this by virtue of what these acts have an epistemic relevance for; the act leads to a state of justification reified by Prawitz in terms of the notion of ground, and therefore considered as an object. On the other hand, in order the bind to work, we must make sure that the act is *per se* able to generate the object. Therefore, the inference will no longer be described like a mere transition to a certain conclusion from certain premises; otherwise, what it produces would no longer be distinguishable from (a description of) the act constructed by performing it, and furthermore, in non-canonical cases, it would be necessary to go through a further justification, a reduction "external" to the inferential process. Here is the theoretical location of the idea of understanding inferences as applications of operations on grounds, and the grounds as objects specifiable by simple induction, obtained by applying only primitive, meaning-constitutive operations. In addition, operations range from grounds to grounds and not, as for the reductions of valid arguments in proof-theoretic semantics, from non-interpreted structures to non-interpreted structures.

The characterization of inferential validity on the basis of objects defined inductively starting from primitive operations, and the consequent shift of the distinction between canonical/non-canonical at the level only (of description) of the acts, leads to a first, fundamental result: a non-circular and satisfactory explanation of the reciprocal link between inferences and correct reasoning. In order to clarify the epistemic force, the theory of grounds seems to be far more convincing than any path offered by proof-theoretic semantics; not like in the latter the set of definitions is not formally correct, but it is obvious that formal correctness is a condition only necessary, not sufficient, to answer a question in the last analysis philosophical.

The last obstacle to overcome towards a successful explanation of deductive compulsion is then that problem of recognizability, shared by the theory of grounds with proof-theoretic semantics. The fact that the inferential operations go from grounds to grounds, and the fact that the latter result from the application of primitive operations, suggests a minimum progress, although a progress limited to cases of inferences from premises devoid of free individual variables, not dependent on assumptions, and for which we already have grounds. In all the other cases the problem remains, both at the semantic level, as recognition of the good position of equations that define operations on grounds, and at the syntactic level, as recognition of the valid denotation of operational symbols or terms of languages of grounding, and the adequacy of the identity axioms that regulate their behaviour in systems of grounding.

There are various ways to deal with, or at least investigate, the issue. The most immediate one perhaps originates from a reflection on the notion of recognizability. What kind of recognition should we request? Is it decidability, or any pragmatically describable operation? Is it to be independent of the context, so as to be performed individually by the inferential agent, or it has to go through a co-operation with other co-agents, in a shared and maybe dialogic epistemic context? Does it have a conclusive character, or does it lead to a temporary acceptance, in principle revisable, of the inference we have recognized as valid?

Given the magnitude of the theoretical breadth taken on by Prawitz, if we want to pursue the "hard line", for example by requiring recognizability to be in all respects decidability, there is no hope of succeeding. Some cracks could be opened, also for a more precise formulation of the question as such, through strong restrictions on the class of operations and defining equations admitted, aimed at making its boundaries less extensive and blurred. The adoption of the "soft line" seems instead to leave more possibilities for maneuver, although it exposes itself to the risk of making the concept of recognizability inaccurate. In this case, however, many interesting developments are looming, as well as connections with more or less recent lines of research. A contextual approach, such as that suggested by Cozzo (Cozzo 2015, 2017), could enlarge towards dialectical and dialogical perspectives, as an integral part of the contemporary argumentation theory (see for example Cantù & Testa 2006). A further alternative is Usberti's internalist perspective (Usberti 2015, 2017 and, for a systematic and independent discussion, 2019); here, the recognizability problem is linked to the delicate question of the grounding of empirical judgments or assertionss, as well as to the characterization in terms of cognitive states, equipped by default of checking algorithms that guarantee the recognition and accessibility of relevant epistemic properties.

From a more formal point of view, the question of recognizability leads to a general theory of admissible defining equations in the theory of grounds. Structure, typology and general form of these equations, and consequently properties of the operations they define, could constitute stand-alone subjects of studies, regardless of the restrictions that can be made for a more precisely defined framework. However, the approach taken in chapters 5 and 6 also suggests a link with another line of research.

Languages and systems of grounding we presented constitute, as can be easily seen, a hierarchy. Climbing this hierarchy means adding to the given languages of grounding new individual constants, which represent new derivations in a new atomic base, and new non-primitive operational symbols, or adding to the given systems of grounding new axioms for new individual constants, and new axioms of identity for new non-primitive operational symbols. Each language of grounding can be intended as a translation, through an appropriate extension of the Curry-Howard isomorphism, of a (finitely) axiomatizable (first-order) system, so that for each of these languages we could assume of having decision algorithms that verify the denotation of the terms, and for each of the formal systems of grounding related to these languages we could assume the recursive enumerability of derivations. From this point of view, it would certainly have importance, on account of the recognizability problem, the following question: is it possible to find a method to recursively generate all the possible expansions of a given language of grounding, with all the possible defining equations for the operations denoted by the operational symbols of these languages? And if so, to what extent?

Certainly, it applies for this question what has already been said about the declination of the recognizability problem in the most stringent terms of the notion of decidability; unless the class of equations and operations they define is specified according to stricter parameters, it makes little sense to wonder about the existence of a recursive method of generation. Moreover we must bear in mind the limitative results that come into play as soon as Prawitz's project is intended with the generality to which it aspires; among the bases of the languages of grounding we have also to take into account those having an expressive power greater than or equal to the expressive power of a base for Heyting's first-order arithmetic, so that Gödel's theorems imply the impossibility of recursive generation, and the contemporary internal decidability of each of the languages generated - pain the recursive enumerability of the truths of first-oder arithmetic.

Once these considerations have been made, however, the study of specific classes of expansions, of their properties, and of their obtainability starting from elements already known, may have a certain interest, and give interesting results. We limit ourselves to two examples: the class of the *non-primitive expansions* of a Gentzen-language (on a certain base), which would correspond to the class of the expansions of first-order intuitionistic logic (on a certain base) obtained by adding constructively valid non-introductory rules; the class of the *primitive expansions* of a language of grounding for Heyting's first-order arithmetic, obtained by adding ground-theoretical equivalents of the reflection principle.

As for the first example, it seems to limit the range to the extensions really relevant for the survey about completeness. After proving the falsity of Prawitz's conjecture (Piecha & Schroeder-Heister 2018), Piecha and Schroeder-Heister envisage that completeness with respect to proof-theoretic semantics - and therefore to the theory of grounds - could apply for some intermediate logic. On the other hand, we should verify whether proof-theoretic semantics - and therefore the theory of grounds – can be understood as semantics of rules, not derivable, but admissible in intuitionistic first-order logic (for the notion of admissibility, see for example von Plato 2014). However, it should be borne in mind that, at least in a non-substitutional articulation of the notion of admissibility, there are rules admissible in `IL`, but not valid either in proof-theoretic semantics, or in the theory of the grounds – whatever the order of the quantifiers in the definition of validity. If we limit ourselves to propositional intuitionist logic, for example, the rule

$$\frac{p}{\alpha}$$

with p propositional variable and α whatsoever, results admissible but not valid. Found a complete system, and verified that – given possible restrictions on the notion of admissibility – Prawitz's semantics captures the intuitionistically admissible rules, we could study the issue of the recursive generability of the expansions under examination in connection with that of the recursive enumerability of the derivations of rules, or of the proofs of admissibility in such systems.

As for the second example, a study of the expansions contemplated in it could perhaps be fruitfully linked to Solomon Feferman's research (see for example Feferman 1958, 1988, 1991, 2013, 2016) on the generation of formal arithmetic systems through transfinite iteration of the reflection principle or the truth-operator.

Finally, beyond what has been said so far, and from a much more general point of view, it could have some interest to make reference to the overall picture of the theory of grounds, to the way in which it is linked to the basic ideas of the previous proof-theoretic semantics, to the formalization of its main notions, and to the recognizability problem that, in various facets, arises, assumptions, positions and tools in the research field of *absolute provability*. Taken into account the vastness of the sector, and the amount of literature produced in it, at this point we can only limit ourselves to a quick indication of those cues we believe to be more conceptually related to the type of our argumentation.

In a recent article entitled *Informal and absolute provability: some remarks from a gödelian perspective* (Crocco 2018), Gabriella Crocco compares the notion of *informal rigour*, as everybody knows defined by Kreisel (Kreisel 1960, 1967, 1987), with that of *absolute proof*, in the sense defined by John Myhill (Myhill 1960), on the one hand, and by Gödel (see mainly Gödel 1946, 1972; see also Wang 1996) on the other. It turns out that, despite the terminological differences, and barring the indubitable conceptual differences, the positions of Kreisel and Myhill are similar, whereas that of Gödel seems at the very least broader, if not opposite. Kreisel's proposal proceeds initially *ex negativo*, by contrast to the analysis of formal rigour specified by Turing (Turing 1936 - 1937):

> formal rigour does *not* apply to the discovery or choice of formal rules nor of notions; neither of basic notions such as *set* in so-called classical mathematics, nor of technical notions such as

> *group* or *tensor product* [...]. The "old fashioned" idea is that one obtains rules and definitions by analysing intuitive notions and putting down their properties. [...] What the old fashioned idea assumes is quite simply that the intuitive notions are *significant*, be it in the external world or in thought. (Kreisel 1967, 138)

Kreisel then resumes, albeit in a different perspective, a strategy adopted years before, aimed at characterizing the *informal concept of proof* by means of iterated extensions of finitary fragments of arithmetic (Kreisel 1960) – reminiscent of what Turing did (Turing 1939), and later developed by Feferman (Feferman 1958, 1988, 1991). The applications reviewed – such as higher-order axiomatizations and independence results in set theory, logical consequence and relative notion of completeness, non-standard models – make it clear that

> informal rigour precedes formalisation, since it concerns the discovery and understanding of intuitive notions, and succeeds it, since it tries to adjust the analysis of the intuitive notions to the formal properties and relations expressed in the formal system representing them, in the dialectical process of discovery of concepts, definitions and rules [...] the notion of informal rigour, or provability, that Kreisel analyses, has to do with the traditional question of whether an axiomatic characterisation of an informal concept is correct, and how to decide an undecided question, going beyond the formalisation with the help of informal principles. (Crocco 2018, 3)

Myhill's notion of absolute proof is inspired by Gödel's incompleteness theorems, pointing out how they show, for each formal system sufficiently powerful, the existence of correct inferences not accessible to that system – where "correct" cannot be understood either as a syntactic, namely relative to a formal system, or as a semantic notion, which Myhill interprets in the sense of preservation of truth:

> Gödel's argument establishes that *there exists for any correct formal system containing the arithmetic of natural numbers, correct inferences which cannot be carried out in that system* - where "correct inferences" is a notion neither of syntax nor of semantics. (Myhill 1960, 463)

On this basis, Myhill professes an ideal of adherence to effective mathematical practice, which in turn recalls an ideal of proof with respect to which the various historically produced formalisms would take the form of approximations from time to time more refined. In any case, even here, this corresponds to the search for axioms, definitions and rules for axiomatic (formal or not formal) theories. (Crocco 2018, 4)

What seems common to both Kreisel and Myhill is that the respective notions of informal rigour and absolute proof are relative to more or less generic axiomatizations, with respect to notions that, in a sense, inspire, guide and lead to such formal approaches. Between the formal and informal level a sort of dialectical relationship is established informally, so that

> neither Kreisel's notion of informal rigour nor Myhill's notion of absolute provability are conceived in opposition to the search for axiomatic systems. Both consider this search as essential for the analysis of logical dependencies among mathematical propositions and among the concepts they contain [...] both Kreisel and Myhill address the question of the possibility for us to have access to the limit of such approximations, and more specifically of the possibility of a uniform and general method to reach this limit, i.e. to achieve a sort of general and informal method of discovery and analysis. (Crocco 2018, 5)

However, the very fact of asking this question seems to envisage the possibility of a further generalization:

> in this respect the important question would be: is there a general notion of provability [...] independent of the subject matter of arithmetic, analysis or set theory? (Crocco 2018, 5)

If Kreisel and Myhill – the latter with reservations – answer negatively, the question leads almost immediately to Gödel's perspective, who first distinguishes two meanings of the expression "formal" and, accordingly, "informal".

The first, to be found in *On an hitherto unutilized extension of the finitary standpoint* (Gödel 1958), is linked to a discussion of the concept of abstract notion, so as used by Paul Bernays in *Sur le platonisme dans les mathématiques* (Bernays 1953). Paraphrasing Bernays, Gödel recalls that, since the consistency proof for a first-order system for arithmetic cannot be based on proof-means weaker than the system itself, it is necessary to go beyond the finitary point of view - *à la* Hilbert (Hilbert 1985) – and hence beyond its relying on principles the acceptability of which is linked to mere combinatorial properties and is therefore intuitive in nature. We need to turn to abstract notions based on – in the words of Gödel – mental constructs, and insights of such constructs, that capture properties of the signs involved, not combinatorial, but linked to meaning. Hence, when referring to Gentzen's consistency proof (Gentzen 1936) based on the ω-rule, Gödel argues that

> we cannot acquire such knowledge *intuitively* by passing stepwise from smaller to large ordinals; we can only gain knowledge abstractly by means of notions of higher type. This is achieved by means of the abstract notion of "accessibility", which is defined by our being able to give an informally understood proof that a certain kind of inference is valid (Gödel 1958, 243)

In passing, as for the translation of "informally" from the German "inhaltlich", and its correspondents in Gentzen, the reader can refer to Došen & Adžić's *Gödel on deduction* (Došen & Adžić 2017b). This kind of informality comes close Kreisel, since

> what is formal [...] is symbolic, concrete and intuitive (in the Hilbertian sense), for it concerns signs and combinations of signs. On the contrary, what is informal [...] is abstract, non spatio temporal, i.e. it cannot be apprehended by our five senses, but only through insights concerning the meaning of the signs. (Crocco 2018, 7)

In any case, to this use of the expression "formal", which Crocco indicates with formal$_1$, Gödel prefers another one, of which there is trace in the fragment 8.6.1 of Wang's text, *A logical journey. From Gödel to philosophy*:

> the general concepts of logic occur in every subject. A formal science applies to every concept and every object. (Wang 1996, 274)

Therefore, unlike formal$_1$, what Crocco indicates with formal$_2$

> explicitly means "universal applicability", in the largest possible sense, i.e. not merely "universal applicability to objects" (that is, applicability to objects irrespective of what has been called the semantic domain of a concept, i.e. the specific domain of objects relative to which it is significant), but also "universal applicability to concepts", without any restriction of type. [...] the general concepts of logic ([...] conjunction, negation, existence, generality, but also the concept of object, the concept of concept and the relation of application of a concept to its arguments) occur in

> every domain, and that is the reason why logic is a formal$_2$ discipline. Logical concepts are genuine concepts, i.e. concepts with their own meaning, and certainly not syncategorematic notions of formal$_1$ theories. (Crocco 2018, 7 - 8)

Another dyad corresponds to the dyad formal$_1$/formal$_2$, concerning, this time strictly speaking, the Gödelian notion of absolute proof. Crocco distinguishes between a *weak* type, and a *strict* type, and emphasizes how

> "absolute" means "non-relative to any particular formal system or formalized language", but there is also more than that. [...] Gödel considered Turing's definition "a miracle" (Gödel 1946, 150). It is a miracle because, being independent of any language and formal systems, it cannot be subject to any diagonalisation. It is also a miracle because it is strictly independent of any domain of things, contrary to the equivalent notions of recursive function, defined on natural numbers, and of Church's calculability, defined on lambda terms. Since such an analysis of mechanical computability is possible, it may also exist for absolute provability. (Crocco 2018, 9)

Both these readings occur in *Remarks before the Princeton bicentennial conference on problems in mathematics* (but see also Gödel 1972), first with a reference to the possibility of generating a hierarchy of formal systems by transfinite reiteration of certain operations on a given formal system, and then with abstraction from formal contexts of sorts:

> there cannot exist any formalism which could embrace all these steps; but this does not exclude that all the steps (or at least all of them which give something new in the domain of propositions in which you are interested) could be described and collected together in some non constructive way [...] the concepts arrived at or envisaged were not absolute in the strictest sense, but only with respect to a certain system of things, namely the sets as conceived in axiomatic set-theory; i.e. although there exist proofs and definitions not falling under these concepts, these definitions and proofs give, or are to give, nothing new within the domain of sets and of propositions expressible in term of "sets", "$\in$" and the logical constants. (Gödel 1946, 151 - 153)

Thus, we have a certain parallelism: the notion of in-formal$_1$ corresponds to weak absoluteness, but Gödel extends the conception of Kreisel and of Myhill, contemplating a notion of strict absoluteness, which corresponds instead to that of formal$_2$. In the first case, we have independence from formal languages and systems, in the second also independence from specific domains or typing of concepts. To borrow again from Crocco,

> the weak sense includes Kreisel's notion of informal proofs [...] and Myhill's notion of absolute provability. They both presuppose a sort of reflection principle, and a kind of completion of the sequences of iterated steps. The strict sense [...] presupposes independence of any domain of applicability and is therefore fully formal$_2$. [...] The use of absolute concepts (in the weak sense) provides a sort of epistemological fixed-point, which implies the stability and the completability of knowledge in the domain. Nevertheless [...] if mechanical computability is strictly absolute, the possibility of strictly absolute [...] provability seems conceivable. (Crocco 2018, 10 - 11)

On the basis of the theoretical framework just offered, we may wonder whether Prawitz's provability can be linked to the aforementioned research on absolute provability and, if so, whether this connection goes in the direction of informal rigour in Kreisel and of absolute provability in Myhill, or rather towards Gödel's formal$_2$ or strict absolute provability. The notions of valid argument, proof and ground, which in various ways occur in Prawitz's more or less recent investigations, are intended to capture a semantic idea of provability, that is, a general concept that does not coincide with that of derivation in a formal system – although inspired by results in normalization theory for specific formal systems. Obviously, a detailed and exhaustive answer would require a separate work; we will limit ourselves therefore only to indicate some points, as possible perspectives for any future research.

Kreisel's and Myhill's informal/absolute provability has with the formal one a dialectical relationship, going beyond it and, at the same time, preserving its results and its structural properties. The identification of new rules and of new definitions, which allows the transition to more and more powerful formal systems, is carried out through a process that, from time to time, depends on acquired results, limitative and non-limitative. Consequently, the process is rigorous but informal, in the sense that it goes beyond what can be achieved in a given system starting from the conformation of the system itself. It is therefore a sort of generalization, with respect to which Kreisel and Myhill suggest an ideal method of approximation.

Also the semantic approach to provability proposed by Prawitz qualifies as general, in the sense of not being reducible to some specific formal system, and this to the extent that factual circumstances or limitative results testify to the existence of deductive methods acceptable but inaccessible to closed groups of rules and axioms. The deductive practice ordinarily considered acceptable does not seem to be limited to any system, fixed or *a priori* determinable; nor, if we turn to systems powerful at least as much as first-order arithmetic, the existence of such a system is possible, as proven by the undecidable sentences of Gödel's theorems. This, in the first place, influences the nature of the notions of valid argument, proof and ground as such, a point for which we can report the following quote from Prawitz:

> the presence of $\to$ and $\forall$ have the effect that in general the conditions for asserting a sentence cannot be exhausted by any formal system; or better: although the general form of the condition for something to be a canonical proof of a sentence $\alpha \to \beta$ is formally stated [in the BHK clauses] in which only the subsentences α and β are mentioned, there is no formal system generating all the procedures that transform canonical proofs of α to canonical proofs of β [...]. Consequently, while the operations of forming canonical proofs run parallel to the introduction rules of Gentzen's system of natural deduction, it is clear that the rules for asserting a sentence do not amount to inference rules of any formal system. (Prawitz 1977, 29)

However, the implications also concern the formal framework in which valid arguments, proofs and grounds should be considered. In the case of the theory of grounds, in particular, a language of grounding cannot be closed, requiring on the contrary, indefinite additions of ever new operational symbols for ever new operations on grounds. And here again, Prawitz refers to incompleteness:

> we know because of Gödel's incompleteness result that already for first order arithmetical assertions there is no closed language of grounds in which all grounds for them can be defined; for any such intuitively acceptable closed language of grounds, we can find an assertion and a ground for it that we find intuitively acceptable but that cannot be expressed within that language. (Prawitz 2015, 98)

The intuitive acceptability of a ground, not expressible in a closed language of grounding requires to expand this language – where it would be interesting to investigate, also reconnecting to the points risen just above, if the expansion process of languages of grounding can be understood in the sense of Kreisel's informal rigour. The perspectives of Kreisel, Myhill and Prawitz therefore seem to have some points in common.

Does the same applies to a comparison with Gödel's strict absoluteness, the formal$_2$? It is not easy to answer this question, if only because, as Crocco points out,

> there is no way to characterize this strict notion of absolute provability except through a "negative" analogy with Turing's notion of mechanical computability. As the latter is strictly absolute, we cannot exclude that the former is also able to be defined in a strict absolute way. (Crocco 2018, 12)

On the other hand, Prawitz focuses on structural aspects and semantically relevant properties of proofs, on an increasingly *general* theory completely distinct from its Hilbertian *reductive* analogue (Prawitz 1973, Abrusci 1985, Cellucci 1978, 2007) – which, moreover, seems to occur also in similar approaches, such as in Martin-Löf's intuitionistic type theory (1984), or Girard's work in Linear Logic, Geometry of Interaction and Ludics (Girard 1987, 1989, 1990, 2001; see also Girard, Taylor & Lafont 1993). In this regard, if Crocco notes that

> proof theory is certainly not part of the formalist paradigm, exactly because in the tradition of Gentzen's work it tends toward a natural analysis of first-order logic with intensional tools. The steps in which a proof can be analysed, the search for its "geometrical" structure through cut elimination, are all tools for explaining why a proof is epistemically compelling and how inferences have the epistemic power of justifying mathematical propositions and judgments, (Crocco 2018, 14)

Gödel also often went in the direction of a general proof-theory, as evidenced by some of his notes (for which see Došen & Adžić 2016, 2017a, 2017b).

In any case, a first problem encountered when linking Prawitz's investigations to Gödel's absolute provability in the strict sense arises from the different perspective that one and the other adopt about the relation between inferences and reasoning. This is very evident in the case of the theory of grounds, where the validity of inferences is defined locally, and has a priority over the notion of proof; for Gödel, on the other hand, an inference is

> something that, attached to a proof, gives as a result a proof, where a proof [...] is not "a sequence of expressions satisfying

certain formal conditions, but a sequence of thoughts convincing a sound mind". (Crocco & Piccolomini d'Aragona 2018, 4; la citazione interna è da Gödel 1995, 341)

The reasons that lead Gödel to this approach, however, also explain the differences from proof-theoretic semantics. Certainly, in the latter, inferential validity is defined globally, in terms of the notion of proof, but the analysis is based on intended compositional properties, focusing on an aspect – canonicity – which is after all local to the extent that it attributes a privileged role to inference rules or inferences of a specific type – introduction of logical constants. Gödel's point of view appears instead much broader, global almost in the sense of holistic, or at least not strictly compositional. The aforementioned analogy with Turing goes in fact far beyond the results of an analysis of computability in terms of Turing machines:

> Gödel thinks that if we could attain for the concept of (human) provability, an analysis similar to Turing's (and therefore strictly absolute), we should be able to analyse what "provability in every domain" means. Therefore, we should be able to give a universal uniform method for solving every problem in all possible domains. [...] There are two general characteristics of a Turing machine in its calculating process: determinism [...] and finiteness [...]. However, *systematic methods* for actualising the development of our understanding of the abstract terms implied in a proof can, for humans, (contrary to computers), converge towards an infinity of distinguishable states of mind. (Crocco 2018, 10)

Therefore, it would appear that the question of the systematic method for the understanding of *abstract terms* leads Gödel, among other things, to identify in *non-locality* the essential character of strict absolute provability:

> non-locality seems to be the core of Gödel's argument. It implies that the acquisition of evidence for human subjects cannot be reduced to elementary steps given once and for all. An inference can become evident when we take into account the global - i.e. non-local - features of what has been proved on the basis of previous evidence. (Crocco & Piccolomini d'Aragona 2018, 4)

A thorough analysis of the distinction between Gödel's non-locality and Turing's determinism can be found in Webb's (Webb 1990) introduction to *Some remarks on the undecidability results* (Gödel 1972a). According to Webb - as well as to Feferman, as Webb himself reports - Turing's determinism is connected with Gödel's acceptance of Turing machines as adequate for defining the notion of mechanical procedure - together with some diagonalization issues. On the other hand, Webb also remarks that Turing's finiteness condition on mental states concerns memory, and that therefore Gödel's idea might be that the convergence to infinity of the number and precision of abstract terms implies that the number of *distinguishable* states of mind also converges to infinity. This sheds light upon Gödel's *own* view, as distinct from Turing's one, on abstract terms and states. Abstract terms, no matter how complex, can be understood more and more precisely, so that we can enter more and more complex states of mind. For Turing, the lack of states of unlimited complexity can be compensated - as in the case of Turing Universal Machine - by enriching the list of symbols writable on the tape.

The fact that strict absoluteness is independent of language and of the typing of concepts is another feature that, we believe, puts a distance between Prawitz and Gödel. The notions of valid argument and proof in proof-theoretic semantics, or of ground in the related theory, are always defined on, or obtained by abstraction from, bases that fix the meaning of individual constants, functional symbols and relational symbols of a background language; and even when the base is empty, purely logical, as it were, the fact remains – maximally evident in the theory of grounds – that semantic objects are typed on terms and formulas of the background language. On the contrary, Gödel's proposal seems to go rather in the direction of a purely structural treatment of proofs, an identification of principles and global properties in deductive methods on whatever domain and whatever language. Given Gödel's interest in Turing's work on mechanical computability, this could materialize in the delineation of the "movements" which, in as a general manner as possible, can be performed during a demonstrative act.

In this regard, as well as at the end of our work, we remark that in recent times Girard (Girard 2001) has introduced Ludics, a sort of logic of the *dynamics* of the proofs. The key notion here is that of *interaction* between a *proponent* and an *opponent*, and deductive activity is explained through that, more basilar, of "proof-game" or "proof-search"; from a technical point of view, the interaction corresponds to cut-elimination on para-proofs with (linearly) orthogonal conclusions, of which the normal form is intended to correspond to a locally "winning" proof – which will be a proof if "winning" in all possible interactions. Significantly, Ludics is totally untyped, and the types – namely the formulas of the logical language - are reconstructed internally, on the basis of some properties attributable to certain sets of para-proofs. It is not a case, after all, that Girard uses for his theory the slogan "from the rules of logic, to the logic of rules".

It is not possible here to go into the details of Ludics, but the question concerning the relation between Prawitz and Gödel could, in the light of the connections just highlighted, lead to that, in our opinion equally interesting, of an embedding of the theory of grounds, and of its basic ideas, into Girard's framework (for a first, purely indicative suggestion in this direction, see Catta & Piccolomini d'Aragona 2019). The differences concerning the typing – and further questions, such as, for example, the fact that Ludics is designed primarily for a second-order linear logic system, and the fact that it comes closer to *bidirectionalism* rather than to verificationism - could be bypassed by bearing in mind some recent works. To name only a few: Marie-Renée Fleury and Myriam Quatrini deal, among other things, with definability of first-order quantifiers in Ludics (Fleury & Quatrini 2004, Quatrini 2014), Eugenia Sironi (Sironi 2014) investigates the translatability into Ludics of Martin-Löf's (dependent) types, at the atomic level, or at the first order, some articles of Schroeder-Heister offer good hints to insert Ludics bidirectionalism in the field of investigation of generalized elimination rules (Schroeder-Heister 2009, 2012).

Therefore, the research perspectives are plentiful, even in concluding the discourse so far carried out, taking into account the approaches that can be drawn for further developments and insights.

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Résumé

Dans la récente théorie des grounds, Prawitz développe ses investigations sémantiques dans la direction d'une analyse de l'origine et de la nature du pouvoir que les inférences valides exercent sur des agents engagés dans l'activité déductive ; à savoir, le pouvoir d'obliger épistémiquement à accepter les conclusions, si l'on en a accepté les prémisses. Un ground, *grosso modo*, est ce dont on est en possession lorsqu'on est justifié à affirmer un certain énoncé. Les grounds peuvent être construits en accomplissant des opérations qui permettent le passage d'un état de justification à un autre. Un acte d'inférence consiste à l'application d'une opération des grounds pour les prémisses aux grounds pour la conclusion. La théorie des grounds présente des avancements indubitables par rapport à la précédente approche de Prawitz, la *proof-theoretic semantics*. En particulier, la théorie des grounds offre une définition du concept d'inférence valide en vertu de laquelle il devient possible de faire dépendre la contrainte épistémique des démonstrations de celle des inférences valides dont ces démonstrations se composent. Mais théorie des grounds et *proof-theoretic semantics* partagent un problème ; dans l'une comme dans l'autre, inférences valides et démonstrations pourraient être telles qu'il est impossible, pour des agents qui les utilisent, de reconnaître le fait qu'elles justifient leur conclusion. Dans ce travail, nous allons développer le cadre formel de la proposition de Prawitz en introduisant un « univers » de grounds et opérations sur grounds, ainsi que des langages formels de grounding dont les termes dénotent grounds ou opérations sur grounds. Nous proposerons aussi des systèmes de grounding, à l'aide desquels démontrer des propriétés significatives des termes des langages de grounding. Finalement, nous nous occuperons de deux questions concernant langages et systèmes. Tout d'abord, celle de la complétude de la logique intuitionniste par rapport à la théorie des grounds ; en second lieu, nous poursuivrons une analyse du problème de reconnaissabilité déjà évoqué.

Riassunto

Nella recente teoria dei grounds, Prawitz sviluppa le sue indagini semantiche nella direzione di un'analisi dell'origine e della natura di quella speciale forza che le inferenze valide esercitano su agenti impegnati nell'attività deduttiva: la forza di costringere epistemicamente ad accettare le conclusioni, se se ne sono accettate le premesse o le ipotesi. Un ground è, *grosso modo*, ciò di cui si è in possesso quando si è giustificati nell'asserire un certo enunciato. I grounds possono essere costruiti compiendo operazioni che consentano il passaggio da uno stato di giustificazione all'altro. Un atto inferenziale consiste nell'applicazione di un'operazione dai grounds per le premesse ai grounds per la conclusione. La teoria dei grounds presenta indubbi avanzamenti rispetto al precedente approccio di Prawitz, la *proof-theoretic semantics*. In particolare, la teoria dei grounds offre una definizione della nozione di inferenza valida in virtù della quale diventa possibile far dipendere la costrizione epistemica esercitata delle dimostrazioni da quella esercitata dalle inferenze valide di cui le dimostrazioni si compongono. Ma teoria dei grounds e *proof-theoretic semantics* condividono un problema; nell'una come nell'altra, inferenze valide e dimostrazioni potrebbero essere tali da risultare impossibile, ad agenti che ne facciano uso nella concreta pratica deduttiva, un riconoscimento del fatto che esse giustificano la loro conclusione. In questo lavoro, il quadro formale della proposta di Prawitz sarà da noi articolato introducendo un "universo" di grounds ed operazioni su grounds e, poi, linguaggi formali di grounding i cui termini denotano grounds od operazioni su grounds. Accanto ai linguaggi di grounding, proporremo anche sistemi di grounding, in cui dimostrare proprietà rilevanti dei termini dei linguaggi di grounding. Ci occuperemo infine di due questioni relative a linguaggi e sistemi. Innanzitutto, la questione della completezza della logica intuizionista rispetto alla teoria dei grounds; in secondo luogo, perseguiremo una disamina del succitato problema di riconoscibilità.