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Hussein Cheikh Ali

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# Analyse asymptotique des équations de Hardy-Sobolev dans des espaces singuliers

**Thèse présentée par Hussein CHEIKH ALI**

en vue de l'obtention du grade académique de docteur en Sciences et  
Mathématiques Appliquées  
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Sous la direction du Professeur Denis BONHEURE,  
(Université Libre de Bruxelles) et du Professeur Frédéric ROBERT,  
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*Pour mon Père Youssef*

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**Partie 0: Le cadre**

Le célèbre théorème de Sobolev affirme que,  $n \geq 3$  étant donné, il existe une constante  $C_1(n) > 0$  telle que

$$\|u\|_{L^{2^*}(\mathbb{R}^n)} \leq C_1(n) \|\nabla u\|_{L^2(\mathbb{R}^n)} \text{ pour tout } u \in C_c^\infty(\mathbb{R}^n) \quad (1.1)$$

avec  $2^* := \frac{2n}{n-2}$ . L'exposant  $2^*$  est critique dans le sens suivant: pour tout domaine  $\Omega \subset \mathbb{R}^n$ ,  $H_{1,0}^2(\Omega)$  étant la complétion de  $C_c^\infty(\Omega)$  pour la norme  $u \mapsto \|\nabla u\|_2$ , alors  $H_{1,0}^2(\Omega)$  se plonge continûment dans  $L^q(\Omega)$  pour tout  $1 \leq q \leq 2^*$ , et cette inclusion est compacte si et seulement si  $1 \leq q < 2^*$ . Ce manque de compacité est la principale complexité du problème de Yamabe: Soit  $(M, g)$  une variété Riemannienne lisse et compacte de dimension  $n \geq 3$  sans bord. On note  $Scal_g$  la courbure scalaire. Le problème de Yamabe s'énonce ainsi: Existe-t-il une métrique  $\tilde{g}$  conforme à  $g$  telle que  $Scal_{\tilde{g}}$  soit constante? Cela revient à trouver une solution positive  $u \in C^2(M)$  à l'équation

$$\Delta_g u + c_n Scal_g u = \epsilon u^{2^*-1} \text{ t.q. } \epsilon \in \{-1, 0, 1\}, u > 0 \quad (1.2)$$

où  $\Delta_g = -div_g(\nabla)$  est l'opérateur de Laplace-Beltrami sur  $(M, g)$  et  $c_n := \frac{n-2}{4(n-1)}$ . Ici, l'espace fonctionnel naturel dans lequel travailler est  $H_1^2(M)$ , la complétion de  $C^\infty(M)$  pour la norme  $u \mapsto \|u\|_2 + \|\nabla u\|_2$ . La réponse est positive, et la résolution de ce problème fût une longue histoire. Il implique des conditions locales pour les "grandes" dimensions, et des conditions globales (la positivité de la *masse*) pour la petites dimensions et le cas localement conformément plat. La preuve initiale [112] de Yamabe n'était pas complète et la résolution finale du problème est due à Aubin [2] et à Schoen [100]. La référence classique pour ce problème est l'article de Lee et Parker [82].

Que se passe-t-il si le problème est défini sur un domaine de  $\mathbb{R}^n$ ? Par exemple, peut-on encore résoudre (1.2) et si on l'ajoute une condition au bord de type Dirichlet? C'est l'objet du travail classique de Brezis-Nirenberg [19]. Soit  $\Omega \subset \mathbb{R}^n$ ,  $n \geq 3$ , un domaine ouvert, borné et régulier, et soit  $\alpha$  un nombre réel. Dans l'article de référence [19], Brezis-Nirenberg ont prouvé l'existence d'une



solution  $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$  au problème

$$\begin{cases} \Delta u - \alpha u = u^{2^*-1} & \text{dans } \Omega, \\ u > 0 & \text{dans } \Omega, \\ u = 0 & \text{sur } \partial\Omega, \end{cases} \quad (1.3)$$

avec  $\Delta = -\operatorname{div}(\nabla)$ , dès que  $\alpha > 0$  quand  $n \geq 4$  (et la condition nécessaire  $\alpha < \lambda_1(\Omega)$ ). Le cas  $n = 3$  est plus complexe: l'existence des solutions minimisantes est équivalente à la positivité de la masse de l'opérateur  $\Delta - \alpha$  (voir Druet [36]). L'analyse de l'équation (1.3) (respectivement (1.2)) est étroitement liée à la meilleure constante pour l'inégalité de Sobolev  $H_{1,0}^2(\Omega) \hookrightarrow L^{2^*}(\Omega)$  (respectivement  $H_1^2(M) \hookrightarrow L^{2^*}(M)$ ).

### Partie 0.1: L'inégalités classiques sur $\mathbb{R}^n$ ( $n \geq 3$ )

L'inégalité de Hardy s'écrit comme

$$\frac{(n-2)^2}{4} \int_{\mathbb{R}^n} \frac{u^2}{|x|^2} dx \leq \int_{\mathbb{R}^n} |\nabla u|^2 dx \text{ pour tout } u \in C_c^\infty(\mathbb{R}^n). \quad (1.4)$$

En interpolant les inégalités (1.1) et (1.4), on obtient l'inégalité de Hardy-Sobolev: Pour  $s \in [0, 2]$ , il existe  $C(n, s) > 0$  telle que

$$\left( \int_{\mathbb{R}^n} \frac{|u|^{2^*(s)}}{|x|^s} dx \right)^{\frac{2}{2^*(s)}} \leq C(n, s) \int_{\mathbb{R}^n} |\nabla u|^2 dx \text{ pour tout } u \in C_c^\infty(\mathbb{R}^n), \quad (1.5)$$

où  $2^*(s) := \frac{2(n-s)}{n-2}$  est l'exposant critique de Hardy-Sobolev. On observe que, avec  $s = 0$  on récupère l'inégalité de Sobolev (1.1), et avec  $s = 2$  on récupère l'inégalité de Hardy (1.4). Si  $\gamma < \frac{(n-2)^2}{4}$ , il suit de (1.4) qu'il existe  $C(n, \gamma) > 1$  tel que

$$\frac{1}{C(n, \gamma)} \int_{\mathbb{R}^n} |\nabla u|^2 dx \leq \int_{\mathbb{R}^n} \left( |\nabla u|^2 - \gamma \frac{u^2}{|x|^2} \right) dx \leq C(n, \gamma) \int_{\mathbb{R}^n} |\nabla u|^2 dx$$

pour tout  $u \in C_c^\infty(\mathbb{R}^n)$ . Ainsi, pour  $\gamma < \frac{(n-2)^2}{4}$ , il existe  $C(n, \gamma, s) > 0$  tel que

$$\left( \int_{\mathbb{R}^n} \frac{|u|^{2^*(s)}}{|x|^s} dx \right)^{\frac{2}{2^*(s)}} \leq C(n, \gamma, s) \int_{\mathbb{R}^n} \left( |\nabla u|^2 - \gamma \frac{u^2}{|x|^2} \right) dx \quad (1.6)$$

pour tout  $u \in C_c^\infty(\mathbb{R}^n)$ . L'inégalité de Hardy-Sobolev est un cas particulier de la célèbre famille des inégalités fonctionnelles obtenues par Caffarelli-Kohn-Nirenberg [25] mais a apparemment été découvert aussi par V.P. Il'in, voir [73].

On limite maintenant cette inégalité à un domaine. Soit un domaine  $\Omega$  dans  $\mathbb{R}^n$ ,  $n \geq 3$ , et, pour  $0 < s < 2$ ,  $\gamma \in \mathbb{R}$  et  $a \in L^\infty(\Omega)$ , on définit

$$\mu_{\gamma,s,a}(\Omega) := \inf \{ J_{\gamma,s,a}^\Omega(u) / u \in H_{1,0}^2(\Omega) \setminus \{0\} \},$$

où

$$J_{\gamma,s,a}^\Omega(u) := \frac{\int_\Omega \left( |\nabla u|^2 - \left( \frac{\gamma}{|x|^2} + a(x) \right) u^2 \right) dx}{\left( \int_\Omega \frac{|u|^{2^*(s)}}{|x|^s} dx \right)^{\frac{2}{2^*(s)}}} \text{ pour tout } u \in H_{1,0}^2(\Omega) \setminus \{0\}.$$

Il découle de (1.6) que, pour  $s \in (0, 2)$  et  $\gamma < \frac{(n-2)^2}{4}$ , il existe  $K > 0$  tel que

$$\left( \int_\Omega \frac{|u|^{2^*(s)}}{|x|^s} dx \right)^{\frac{2}{2^*(s)}} \leq K \int_\Omega \left( |\nabla u|^2 - \gamma \frac{u^2}{|x|^2} \right) dx \quad (CKN)$$

pour tout  $u \in C_c^\infty(\Omega)$ . De façon équivalente,  $\mu_{\gamma,s,0}(\Omega) > 0$  lorsque  $\gamma < \frac{(n-2)^2}{4}$ .

Dans cette thèse, nous nous concentrons principalement sur les points critiques de  $J_{\gamma,s,a}^\Omega$ . Soit  $(\overline{M}, g)$  une variété Riemannienne de bord  $\partial M$  et d'intérieur de  $M$ . Soit  $x_0 \in \overline{M}$  un point fixé et soit  $d_g$  la distance Riemannienne sur  $M$ . On fixe  $a, h \in L^\infty(M)$  et  $q \in (1, 2^* - 1)$ . Nous considérons des solutions faibles pour

$$\begin{cases} \Delta_g u - \left( a(x) + \frac{\gamma}{d_g(x,x_0)^2} \right) u = \frac{u^{2^*(s)-1}}{d_g(x,x_0)^s} + h(x)u^q & \text{dans } M; \\ u > 0 & \text{dans } M; \\ u = 0 & \text{sur } \partial M \text{ si } \partial M \neq \emptyset \end{cases} \quad (1.7)$$

## Partie 0.2 Le programme de travail

Cette thèse est divisée en trois parties:

- Partie 1:** On analyse l'existence de solutions pour (1.7) sur un domaine non régulier dans  $\mathbb{R}^n$  avec la singularité au bord en 0 modelé sur le cône. Les solutions sont atteintes en tant que minimiseurs de  $J_{\gamma,s,0}^\Omega$  quand  $h \equiv 0$ , et sont de type Mountain-Pass quand  $h \not\equiv 0$ .
- Partie 2:** Sur une variété Riemannienne  $M$  sans bord et avec  $\gamma = 0$  et  $h \equiv 0$ , nous effectuons une analyse de "blow-up" de solutions de (1.7) de type minimisante. Ceci fournit des informations sur la valeur de la seconde meilleure constante dans l'inégalité fonctionnelle Riemannienne correspondante.
- Partie 3:** Nous étudions la version Paneitz d'ordre 4 de (1.7) avec  $\gamma = s = 0$  sur un domaine borné et régulier dans  $\mathbb{R}^n$ ,  $n \geq 5$ .

## Partie 1: Équations Hardy-Sobolev sur un domaine singulier

Soit  $\Omega$  un domaine borné dans  $\mathbb{R}^n$ ,  $n \geq 3$ , on fixe  $\gamma \in \mathbb{R}$  et  $a \in L^\infty(\Omega)$ . On s'intéresse à l'existence de solutions faibles  $u \in H_{1,0}^2(\Omega)$ ,  $u \not\equiv 0$ , pour

$$\begin{cases} \Delta u - \left( a(x) + \frac{\gamma}{|x|^2} \right) u = \frac{u^{2^*(s)-1}}{|x|^s} & \text{dans } \Omega, \\ u > 0 & \text{p.p. dans } \Omega, \\ u = 0 & \text{sur } \partial\Omega. \end{cases} \quad (HS)$$

L'équation (HS) est l'équation d'Euler-Lagrange associée à  $J_{\gamma,s,a}^\Omega$ . Donc, s'il existe des extrémales positives pour  $\mu_{\gamma,s,a}(\Omega)$  positive, ce sont des solutions pour (HS) à l'homothétie près.

**Le cas  $0 \in \Omega$ :** Le problème prend un sens lorsque  $\gamma < (n-2)^2/4$ , la constante de Hardy classique. Il n'y a pas d'extrémale pour  $\mu_{\gamma,s,0}(\Omega)$  (voir [60]). Il existe une littérature importante sur cette question. Par exemple, nous référons à Ruiz-Willem [99], Smets [102] et la revue [60] par Ghoussoub-Robert.

**Le cas  $0 \in \partial\Omega$ .** L'existence des extrémales pour  $\mu_{\gamma,s,0}(\Omega)$  a été étudiée par Egnell [46] où  $\Omega$  est un cône en 0. Lorsque le domaine est régulier, cette question a été posée par Ghoussoub-Kang [55] et étudiée par Chern-Lin [32] et Ghoussoub-Robert [61]. C'était également intéressant car la courbure moyenne en 0 joue un rôle important. Dans cette thèse (voir [27, 28]), on considère un domaine non lisse modelé sur des cônes réguliers (on les désigne comme des "singularités modèles"). On montre comment la géométrie du cône du modèle influence la valeur de la constante de Hardy sur  $\Omega$ .

À partir de maintenant, on suppose que  $0 \in \partial\Omega$ , et pour simplifier, on définit

$$\mathbb{R}^{k+,n-k} := \mathbb{R}_+^k \times \mathbb{R}^{n-k} \text{ pour tout } k \in \{1, \dots, n\},$$

avec  $\mathbb{R}_+^k := \{x_1, \dots, x_k > 0\}$ . Dans les papiers Cheikh Ali [27],[28], on définit des domaines qui sont modelés sur des cônes:

**Définition 1.1.** On fixe  $1 \leq k \leq n$ . Soit  $\Omega$  un domaine dans  $\mathbb{R}^n$ . On dit que  $x_0 \in \partial\Omega$  a une singularité de type  $(k, n-k)$  s'ils existe  $U, V$  deux ouverts dans  $\mathbb{R}^n$  tel que  $0 \in U$ ,  $x_0 \in V$  et il existe une difféomorphisme  $\phi \in C^\infty(U, V)$  tels que  $\phi(0) = x_0$  et

$$\phi(U \cap \mathbb{R}^{k+,n-k}) = \phi(U) \cap \Omega \text{ et } \phi(U \cap \partial\mathbb{R}^{k+,n-k}) = \phi(U) \cap \partial\Omega,$$

avec l'hypothèse supplémentaire que la différentielle en 0  $d\phi_0$  est une isométrie.

**La motivation pour considérer l'équation (HS)** découle du problème de l'existence des extrémales pour les inégalités de Caffarelli-Kohn-Nirenberg noté par (CKN). Nous adressons les questions suivantes:

**(Q1)** Pour quelles valeurs de  $\gamma \in \mathbb{R}$ , existe-t-il  $K > 0$  telle que l'inégalité  $(CKN)$  est valable pour tous les  $u \in H_{1,0}^2(\Omega)$ ? En d'autres termes, quand avons-nous  $\mu_{\gamma,s,0}(\Omega) > 0$ ?

**(Q2)** La meilleure constante est-elle atteinte? En d'autres termes, est-ce que  $\mu_{\gamma,s,0}(\Omega)$  est atteint par certains  $u \in H_{1,0}^2(\Omega)$ ,  $u \neq 0$  ?

On note que la meilleure constante de Hardy sur  $\mathbb{R}^{k+,n-k}$  est explicite. Comme il est noté dans Ghoussoub-Moradifam [57], nous avons que

$$\gamma_H(\mathbb{R}^{k+,n-k}) = \frac{(n + 2k - 2)^2}{4} \text{ pour tout } k \in \{1, \dots, n\}.$$

La réponse à la première question **(Q1)** dépend de la constante de Hardy. Nous définissons

$$\gamma_H(\Omega) := \mu_{0,2,0}(\Omega) = \inf \left\{ \frac{\int_{\Omega} |\nabla u|^2 dx}{\int_{\Omega} \frac{u^2}{|x|^2} dx}; u \in H_{1,0}^2(\Omega) \setminus \{0\} \right\}. \quad (1.8)$$

Par conséquence, en interpolant l'inégalité de Hardy (1.8) et l'inégalité de Sobolev ((CKN) avec  $\gamma = s = 0$ ), on obtient que

$$\gamma < \gamma_H(\Omega) \Rightarrow \mu_{\gamma,s,0}(\Omega) > 0.$$

On considère la deuxième question **(Q2)**, c'est l'existence des extrémales pour  $\mu_{\gamma,s,0}(\Omega)$ . Le résultat suivant est central pour la suite. La preuve est standard comme la preuve d'Aubin de la conjecture de Yamabe en grandes dimensions [2] où il a noté que la compacité des séquences minimisantes a lieu si l'infimum est strictement inférieur à l'énergie de "Bubble". Dans notre cas ci-dessous, cela se traduit par  $\mu_{\gamma,s,a}(\Omega) < \mu_{\gamma,s,0}(\mathbb{R}^{k+,n-k})$ .

**Théorème 1.1** (Cheikh-Ali [27], [28], voir Chapitres 3, 4). *On suppose  $\Omega \subset \mathbb{R}^n$  un domaine borné tel que  $0 \in \partial\Omega$  a une singularité de type  $(k, n - k)$ . On suppose que  $\gamma < \gamma_H(\mathbb{R}^{k+,n-k})$ ,  $0 \leq s \leq 2$ , et  $\mu_{\gamma,s,a}(\Omega) < \mu_{\gamma,s,0}(\mathbb{R}^{k+,n-k})$ . Alors il existe des extrémales pour  $\mu_{\gamma,s,a}(\Omega)$ . En particulier, il existe un minimiseur  $u$  dans  $H_{1,0}^2(\Omega) \setminus \{0\}$  qui est une solution positive à l'équation*

$$\begin{cases} \Delta u - \left( \frac{\gamma}{|x|^2} + a(x) \right) u = \mu_{\gamma,s,a}(\Omega) \frac{u^{2^*(s)-1}}{|x|^s} & \text{dans } \Omega, \\ u > 0 & \text{dans } \Omega, \\ u = 0 & \text{sur } \partial\Omega. \end{cases}$$

Une condition de type  $\mu_{\gamma,s,a}(\Omega) < \mu_{\gamma,s,0}(\mathbb{R}^{k+,n-k})$  est très classique dans le problème de la meilleure constante, voir Aubin [2], Brezis-Nirenberg [19]. A partir de là, on considère la question suivante:

**(Q3)** Pour lequel  $\gamma < \gamma_H(\mathbb{R}^{k_+,n-k})$  avons-nous  $\mu_{\gamma,s,a}(\Omega) < \mu_{\gamma,s,0}(\mathbb{R}^{k_+,n-k})$ ?

Avant de répondre à cette question, pour  $k \in \{1, \dots, n\}$ , on doit d'abord envisager des solutions modèles de

$$\begin{cases} \Delta u - \frac{\gamma}{|x|^2}u = 0 & \text{dans } \mathbb{R}^{k_+,n-k}; \\ u > 0 & \text{dans } \mathbb{R}^{k_+,n-k}; \\ u = 0 & \text{sur } \partial\mathbb{R}^{k_+,n-k}. \end{cases} \quad (1.9)$$

Soit un nombre réel  $\alpha \in \mathbb{R}$  et on fixe  $\gamma < \gamma_H(\mathbb{R}^{k_+,n-k})$ . Alors

$$U_\alpha \text{ est une solution pour (1.9)} \Leftrightarrow \alpha \in \{\alpha_-, \alpha_+\},$$

où

$$U_\alpha := |x|^{-\alpha-k} \prod_{i=1}^k x_i \text{ et } \alpha_\pm = \alpha_\pm(\gamma, n, k) := \frac{n-2}{2} \pm \sqrt{\gamma_H(\mathbb{R}^{k_+,n-k}) - \gamma}. \quad (1.10)$$

Les fonctions  $U_{\alpha_-}, U_{\alpha_+}$  sont des prototypes de solutions de (1.9). On note que

$$\alpha_- < \frac{n-2}{2} < \alpha_+ \text{ et } U_{\alpha_-} \in H_{1,0,loc}^2(\mathbb{R}^{k_+,n-k}).$$

Le profile  $U_{\alpha_-}$  est également le modèle de comportement des solutions variationnelles à  $(HS)$ . En effet, il découle du résultat de régularité de Felli-Ferrero [50] que, via la carte, une solution variationnelle à  $(HS)$  est comporte comme  $U_{\alpha_-}$  autour la singularité 0. On définit maintenant la dimension critique  $n_{\gamma,k} := \sqrt{4\gamma+1} + 2 - 2k$  (non-entier, voire négative):

$$\left\{ \begin{array}{l} \gamma \text{ est petite (**grande dimension**) : } \gamma \leq \gamma_H(\mathbb{R}^{k_+,n-k}) - \frac{1}{4} \text{ i.e. } n \geq n_{\gamma,k} \\ \gamma \text{ est grande (**petite dimension**) : } \gamma > \gamma_H(\mathbb{R}^{k_+,n-k}) - \frac{1}{4} \text{ i.e. } n < n_{\gamma,k} \end{array} \right\}$$

Ici la stratégie pour obtenir  $\mu_{\gamma,s,a}(\Omega) < \mu_{\gamma,s,0}(\mathbb{R}^{k_+,n-k})$  est la suivante:

1. On prends une extrémale  $V$  positive pour  $\mu_{\gamma,s,0}(\mathbb{R}^{k_+,n-k})$ .
2. On concentre la fonction  $V$  en 0 avec  $\epsilon > 0$  et on revient sur  $\Omega$  par la carte  $\phi$  (voir la Définition 1.1).
3. On calcule et on obtient des intégrales qui, sous de bonnes hypothèses sur la dimension, doivent converger pour obtenir la **courbure généralisée**  $\mathbf{GH}_{\gamma,s}(\Omega)$ .

4. Dans les autres cas, des arguments globaux sont nécessaires et on doit introduire la masse  $m_{\gamma,0}(\Omega)$

D'abord, on a besoin de l'existence des extrémales de  $\mu_{\gamma,s,0}(\mathbb{R}^{k+,n-k})$  pour construire la fonction test. Par Ghoussoub-Robert [60] (voir Section 5), on a le résultat suivant:

**Proposition 1.1.** *On fixe  $\gamma < \gamma_H(\mathbb{R}^{k+,n-k})$ ,  $s \in [0, 2)$  où  $n \geq 3$ , alors*

*Si  $\{s > 0\}$  où  $\{s = 0, \gamma > 0 \text{ et } n \geq 4\}$  alors  $\mu_{\gamma,s,0}(\mathbb{R}^{k+,n-k})$  est atteinte.*

Cette partie est divisée en trois sous-sections:

**Partie 1.1:** On introduit le problème  $(HS)$  dans Cheikh-Ali [27] (voir Chapitre 3). Sous une hypothèse géométrique locale, à savoir que la courbure moyenne généralisée est négative (voir 1.11), on démontre l'existence des extrémales pour l'inégalité de Hardy-Sobolev pertinente pour les grandes dimensions.

**Partie 1.2:** Ensuite, on reprend dans Cheikh-Ali [28] (voir Chapitre 4) la question de la petite dimension qui a été laissée ouverte. On introduit la "masse", qui est une quantité globale dont la positivité garantit l'existence des extrémales de petites dimensions.

**Partie 1.3:** On prouve l'existence de solutions de l'équation initiale avec une perturbation via le lemme du col.

### Partie 1.1: Grandes dimensions et courbure généralisée

**Lorsque  $\gamma$  est petite (dimension grande):** Dans ce cas, on montre comment la géométrie locale induite par le cône autour de la singularité influence la valeur de  $\mu_{\gamma,s,a}(\Omega)$  pour répondre à (Q3). On aura besoin de deux choses importantes:

**I- Définition de la courbure généralisée:** On introduit une nouvelle notion géométrique en la singularité conique qui généralise la "courbure moyenne": cela permet d'obtenir des extrémales pour  $(HS)$ . Pour cela, on écrit le domaine non régulier  $\Omega$  comme l'intersection de domaines réguliers autour de 0: il existe  $\Omega_1, \dots, \Omega_k \subset \mathbb{R}^n$  des domaines réguliers et  $\delta > 0$  tel que

$$\Omega \cap B_\delta(0) = \left( \bigcap_{i=1}^k \Omega_i \right) \cap B_\delta(0).$$

Les  $\Omega_i$  sont localement uniques à permutation près. On fixe  $\Sigma := \bigcap_{i=1}^k \partial\Omega_i$  où  $k \in \{1, \dots, n\}$ . Le vecteur  $\vec{H}_0^\Sigma$  désigne le vecteur de la courbure moyenne en 0 de la sous-variété de la sous-variété orientée  $\Sigma$ . Pour tout  $m = 1, \dots, k$ ,  $II_0^{\partial\Omega^m}$

désigne la second forme fondamentale en 0 de la sous-variété orientée  $\partial\Omega_m$ . La **courbure moyenne généralisée** de  $\Omega$  est définie par:

$$GH_{\gamma,s}(\Omega) := c_{\gamma,s}^1 \sum_{m=1}^k \langle \vec{H}_0^\Sigma, \vec{\nu}_m \rangle + c_{\gamma,s}^2 \sum_{i,m=1, i \neq m}^k II_0^{\partial\Omega_m}(\vec{\nu}_i, \vec{\nu}_i) \quad (1.11)$$

$$+ c_{\gamma,s}^3 \sum_{p,q,m=1, |\{p,q,m\}|=3}^k II_0^{\partial\Omega_m}(\vec{\nu}_p, \vec{\nu}_q)$$

où pour tout  $m = 1, \dots, k$ ,  $\vec{\nu}_m$  est le vecteur normal extérieure en 0 de  $\partial\Omega_m$  et  $c_{\gamma,s}^1, c_{\gamma,s}^2, c_{\gamma,s}^3$  sont des constantes explicites et positives. On se réfère à [27] (voir Chapitre 3) pour plus de détails sur cette courbure.

**II- Symétrie des extrémales pour  $\mu_{\gamma,s,0}(\mathbb{R}^{k+,n-k})$ .** On présente la symétrie des extrémales pour  $\mu_{\gamma,s,0}(\mathbb{R}^{k+,n-k})$ . Le type de symétrie ci-dessous a été prouvé dans plusieurs contextes depuis la contribution pionnière de Caffarelli-Gidas-Spruck [24] (voir Chern-Lin [32] et Ghoussoub-Robert [61] pour les références de type Hardy). Pour  $\gamma < \gamma_H(\mathbb{R}^{k+,n-k})$ ,  $s \in [0, 2)$ , on considère le problème  $(HS)$  sur  $\mathbb{R}^{k+,n-k}$ :

$$\begin{cases} \Delta V - \frac{\gamma}{|x|^2} V = \frac{V^{2^*(s)-1}}{|x|^s} & \text{dans } \mathbb{R}^{k+,n-k}, \\ V \geq 0 & \text{dans } \mathbb{R}^{k+,n-k}, \\ V = 0 & \text{sur } \partial\mathbb{R}^{k+,n-k}. \end{cases} \quad (1.12)$$

On a le Théorème suivant:

**Théorème 1.2** (Cheikh-Ali [27], voir Chapitre 3). *Pour  $\gamma \geq 0$  et si  $V$  est une solution de l'équation (1.12) dans  $C^2(\mathbb{R}^{k+,n-k}) \cap C(\mathbb{R}^{k+,n-k} \setminus \{0\})$  avec  $k \in \{1, \dots, n\}$ , alors  $V \circ \sigma = V$  pour tout isométrie de  $\mathbb{R}^n$  tel que  $\sigma(\mathbb{R}^{k+,n-k}) = \mathbb{R}^{k+,n-k}$ . En particulier:*

• *Il existe  $w \in C^\infty(]0, \infty[^k \times \mathbb{R}^{n-k})$  tel que pour tout  $x_1, \dots, x_k > 0$  et  $x' \in \mathbb{R}^{n-k}$ , on obtient que*

$$V(x_1, \dots, x_k, x') = w(x_1, \dots, x_k, |x'|).$$

•  *$V$  est une fonction symétrique de  $k$  variables: pour tout permutation  $s$  de l'ensemble des indices  $\{1, \dots, k\}$ , on a*

$$V(x_1, \dots, x_k, x_{k+1}, \dots, x_n) = V(x_{s(1)}, \dots, x_{s(k)}, x_{k+1}, \dots, x_n).$$

Au départ, notre intention était de suivre la preuve de Chen-Lin [32]. Cependant, le bord singulier nous a empêché d'utiliser le principe classique de comparaison forte. Nous produisons finalement une preuve robuste en utilisant la méthode de Berestycki-Nirenberg [20] qui n'exigeait pas que les bords soient lisses.

On est alors en position de suivre la stratégie qu'on a présentée ci-dessus. Soit  $V > 0$  une extrémale pour  $\mu_{\gamma,s,0}(\mathbb{R}^{k_+,n-k})$  (quand elle existe). Pour  $\epsilon > 0$ , on définit la fonction test

$$V_\epsilon(x) := \left( \eta \epsilon^{-\frac{n-2}{2}} V(\epsilon^{-1} \cdot) \right) \circ \phi^{-1}(x) \quad (1.13)$$

où la carte  $\phi$  est comme dans la Définition 1.1 et  $\eta$  est une fonction de troncature adaptée. En raison de la régularité de Felli-Ferrero [50], on obtient un comportement précis de  $V$  à l'infini, et donc une asymptotique précise pour  $V_\epsilon$  pour  $\epsilon \rightarrow 0$ . On obtient le résultat suivant:

**Proposition 1.2** (Cheikh Ali [27], voir Chapitre 3). *Soit  $0 \leq \gamma < \gamma_H(\mathbb{R}^{k_+,n-k})$ , et on suppose qu'il existe des extrémales pour  $\mu_{\gamma,s,0}(\mathbb{R}^{k_+,n-k})$ . Alors il existe des constantes positives  $c_{\gamma,s}^\beta$  où  $\beta = 1, \dots, 3$  et pour tout  $m = 1, \dots, k$  telles que:*

1. *Pour  $\gamma < \gamma_H(\mathbb{R}^{k_+,n-k}) - \frac{1}{4}$  (c'est-à-dire  $n > n_{\gamma,k}$ ), on a que*

$$J_{\gamma,s,0}^\Omega(V_\epsilon) = \mu_{\gamma,s,0}(\mathbb{R}^{k_+,n-k}) (1 + GH_{\gamma,s}(\Omega)\epsilon + o(\epsilon)).$$

2. *Pour  $\gamma = \gamma_H(\mathbb{R}^{k_+,n-k}) - \frac{1}{4}$  (c'est-à-dire  $n = n_{\gamma,k}$ ), on a que*

$$J_{\gamma,s,0}^\Omega(V_\epsilon) = \mu_{\gamma,s,0}(\mathbb{R}^{k_+,n-k}) \left( 1 + GH_{\gamma,s}(\Omega)\epsilon \ln \left( \frac{1}{\epsilon} \right) + o \left( \epsilon \ln \left( \frac{1}{\epsilon} \right) \right) \right).$$

avec  $GH_{\gamma,s}(\Omega)$  comme dans (1.11).

Ces expressions ne dépendent que de la géométrie locale du domaine. Ceci est possible car, dans les développements asymptotiques, on observe un phénomène de localisation en grandes dimensions. Un tel phénomène a déjà été observé dans le contexte géométrique du problème de Yamabe (voir Aubin [2]) et dans les EDP euclidiennes non linéaires (voir Brezis-Nirenberg [19]). Cette localisation est possible grâce au choix de la grande dimension  $n \geq n_{\gamma,k}$ : elle correspond au cas  $n \geq 4$  pour le problème de Brezis-Nirenberg.

## Partie 1.2: Phénomène de petites dimensions et existence des extrémales

**Lorsque  $\gamma$  est grand (petites dimensions).** Lorsque  $n < n_{\gamma,k}$ , il est connu depuis le travail pionnier de Schoen [100] que des arguments globaux sont requis et qu'on a besoin d'une notion de masse.

**Définition 1.2 (Masse).** *Soit  $\Omega$  un domaine borné dans  $\mathbb{R}^n$ ,  $n \geq 3$ , tel que  $0 \in \partial\Omega$  a une singularité de type  $(k, n - k)$  avec  $k \in \{1, \dots, n\}$ . On fixe  $\gamma < \gamma_H(\Omega)$*



et  $a \in C^{0,\theta}(\Omega)$  ( $\theta \in (0, 1)$ ). On dit qu'un opérateur coercif  $\Delta - (\gamma|x|^{-2} + a)$  a une masse s'il existe  $G \in C^2(\Omega) \cap H_{1,0,loc}^2(\Omega)$  tel que

$$\begin{cases} \Delta G - \left(\frac{\gamma}{|x|^2} + a(x)\right) G = 0 & \text{dans } \Omega, \\ G > 0 & \text{dans } \Omega, \\ G = 0 & \text{sur } \partial\Omega \setminus \{0\}, \end{cases}$$

et s'il existe  $c \in \mathbb{R}$  tel que

$$G(x) = \prod_{i=1}^k d(x, \partial\Omega_i) (|x|^{-\alpha_+ - k} + c|x|^{-\alpha_- - k} + o(|x|^{-\alpha_- - k})) \text{ lorsque } x \rightarrow 0,$$

avec  $\alpha_{\pm}$  est définie dans (1.10). On définit  $m_{\gamma,a}(\Omega) := c$  comme la masse du bord de l'opérateur  $\Delta - (\gamma|x|^{-2} + a)$ .

On note que la fonction  $G$  est unique, de sorte que la définition de la masse ait un sens. Dans [28], on donne plusieurs situations pour lesquelles la masse est définie. Pour  $\gamma_H(\mathbb{R}^{k_+,n-k_-}) - \frac{1}{4} < \gamma < \gamma_H(\mathbb{R}^{k_+,n-k_-})$ , c'est  $n < n_{\gamma}$ , on construit le profil global:

$$W_{\epsilon}(x) := V_{\epsilon}(x) + \epsilon^{\frac{\alpha_+ - \alpha_-}{2}} \Theta(x), \quad (1.14)$$

où  $V_{\epsilon}$  est défini dans (1.13), et  $\Theta \in H_{1,0}^2(\Omega)$  est telle que

$$\Theta(x) = m_{\gamma,a}(\Omega) \prod_{i=1}^k d(x, \partial\Omega_i) |x|^{-\alpha_- - k} + o\left(\prod_{i=1}^k d(x, \partial\Omega_i) |x|^{-\alpha_- - k}\right),$$

lorsque  $x \rightarrow 0$ .

**Proposition 1.3** (Cheikh Ali [28], voir Chapitre 4). Soit  $\Omega$  un domaine borné dans  $\mathbb{R}^n$ ,  $n \geq 3$ , tel que  $0 \in \partial\Omega$  a une singularité de type  $(k, n - k)$  avec  $k \in \{1, \dots, n\}$ . On fixe  $0 \leq s < 2$ ,  $\gamma < \gamma_H(\Omega)$  et  $a \in C^{0,\theta}(\Omega)$  ( $\theta \in (0, 1)$ ). On suppose qu'il existe des extrémales pour  $\mu_{\gamma,s,0}(\mathbb{R}^{k_+,n-k_-})$ . On suppose que

$$\gamma > \gamma_H(\mathbb{R}^{k_+,n-k_-}) - \frac{1}{4} \text{ (c'est-à-dire } n < n_{\gamma,k}),$$

et que l'opérateur  $\Delta - (\gamma|x|^{-2} + a(x))$  est coercif avec la masse  $m_{\gamma,a}(\Omega)$ . On prend  $(W_{\epsilon})_{\epsilon} \in H_{1,0}^2(\Omega)$  comme dans (1.14). Alors, il existe une constante explicite  $\zeta_{\gamma,s}^0 > 0$  telle que

$$J_{\gamma,s,a}^{\Omega}(W_{\epsilon}) = \mu_{\gamma,s,0}(\mathbb{R}^{k_+,n-k_-}) \left(1 - \zeta_{\gamma,s}^0 m_{\gamma,a}(\Omega) \epsilon^{\alpha_+ - \alpha_-} + o(\epsilon^{\alpha_+ - \alpha_-})\right),$$

lorsque  $\epsilon \rightarrow 0$ .

En regroupant les Propositions 1.1, 1.2 et 1.3, on obtient la réponse suivante aux questions (Q2) et (Q3):

**Théorème 1.3** (Cheikh-Ali [27, 28], voir Chapitres 3, 4). *Soit  $\Omega$  un domaine borné dans  $\mathbb{R}^n$ ,  $n \geq 3$  tel que  $0 \in \partial\Omega$  a une singularité de type  $(k, n - k)$  avec  $k \in \{1, \dots, n\}$ . On fixe  $0 \leq s < 2$  et  $0 \leq \gamma < \gamma_H(\Omega)$ . En outre, on suppose que soit  $\{s > 0\}$ , soit  $\{s = 0, \gamma > 0 \text{ et } n \geq 4\}$ . On suppose que:*

- $GH_{\gamma,s}(\Omega) < 0$  si  $n \geq n_{\gamma,k}$ ,
- La masse  $m_{\gamma,0}(\Omega) > 0$  existe et est positive si  $n < n_{\gamma,k}$ .

*Alors, il existe des extrémales pour  $\mu_{\gamma,s,0}(\Omega)$ . En plus, les extrémales sont des solutions faibles strictement positives sur  $\Omega$  pour l'équation (HS) avec  $a \equiv 0$ .*

Le cas restant  $s = 0, \gamma > 0$  et  $n = 3$  est un peu différent. En effet, dans ce cas, on ne sait pas s'il y a des extrémales ou non pour  $\mu_{\gamma,s,0}(\mathbb{R}^{k+,n-k})$ . Si non, on introduit la masse dans un esprit plus classique. Cette situation est largement développée dans [27, 28].

### Partie 1.3: Une équation de Hardy-Sobolev perturbée

On discute brièvement de l'équation perturbée. On prend  $a, h \in L^\infty(\Omega)$  et  $1 < q < 2^* - 1 = \frac{n+2}{n-2}$  qui sont des paramètres supplémentaires. On s'intéresse à l'existence de solutions  $u \in C^2(\Omega) \cap H_{1,0}^2(\Omega)$  pour l'équation perturbée

$$\begin{cases} \Delta u - \left( a(x) + \frac{\gamma}{|x|^2} \right) u = \frac{u^{2^*(s)-1}}{|x|^s} + h(x)u^{q-1} & \text{dans } \Omega, \\ u > 0 & \text{a.e. dans } \Omega, \\ u = 0 & \text{sur } \partial\Omega. \end{cases} \quad (PHS)$$

Ces solutions sont des points critiques pour la fonction  $E_q : H_{1,0}^2(\Omega) \rightarrow \mathbb{R}$ :

$$E_q(u) := \frac{1}{2} \int_{\Omega} (|\nabla u|^2 + au^2) dx - \frac{1}{2^*(s)} \int_{\Omega} \frac{u_+^{2^*(s)}}{|x|^s} dx - \frac{1}{q+1} \int_{\Omega} hu_+^{q+1} dx$$

pour tout  $u \in H_{1,0}^2(\Omega)$ . Notre outil principal est le Mountain-Pass Lemma (Lemme du col) d'Ambrosetti-Rabinowitz [5] pour produire des points critiques de  $E_q$ .

**Théorème 1.4** (Cheikh-Ali [28], voir Chapitre 4). *Soit  $\Omega$  un domaine borné dans  $\mathbb{R}^n$ ,  $n \geq 3$ , tel que  $0 \in \partial\Omega$  a une singularité de type  $(k, n - k)$  avec  $k \in \{1, \dots, n\}$ . On fixe  $\gamma < \gamma_H(\mathbb{R}^{k+,n-k})$ ,  $a \in C^{0,\theta}(\Omega)$  tel que  $\Delta - (\gamma|x|^{-2} + a(x))$*

est coercif, et  $h \in C^{0,\theta}(\Omega)$  tel que  $h \geq 0$  et soit  $0 \leq s < 2$  et  $1 < q < 2^* - 1$ . On suppose qu'il existe  $u_0 \in H_{1,0}^2(\Omega)$ ,  $u_0 \neq 0$ , tel que

$$\sup_{t \geq 0} E_q(tu_0) < \frac{2-s}{2(n-s)} \mu_{\gamma,s,0}(\mathbb{R}^{k+,n-k})^{\frac{n-s}{2-s}},$$

alors l'équation (PHS) admet une solution non nulle dans  $H_{1,0}^2(\Omega)$  de type Mountain-Pass.

Du coup, trouver des solutions à (PHS) se réduit à la question:

$$\text{(Q4) Quand avons-nous } \sup_{t \geq 0} E_q(tu_0) < \frac{2-s}{2(n-s)} \mu_{\gamma,s,0}(\mathbb{R}^{k+,n-k})^{\frac{n-s}{2-s}}?$$

On répond sur (Q4) en prenant pour  $u_0$  soit  $V_\epsilon$  (voir (1.13)) quand  $n \geq n_{\gamma,k}$ , soit  $W_\epsilon$  (voir (1.14)) lorsque  $n < n_{\gamma,k}$ . On choisit de ne présenter que le cas  $s > 0$ : le cas  $s = 0$  est détaillé dans [28] (voir Chapitre 4):

**Théorème 1.5** (Cheikh-Ali [28], voir Chapitre 4). Soit  $\Omega$  un domaine borné dans  $\mathbb{R}^n$ ,  $n \geq 3$ , tel que  $0 \in \partial\Omega$  a une singularité de type  $(k, n-k)$  avec  $k \in \{1, \dots, n\}$ . Soit  $a, h \in C^{0,\theta}(\Omega)$  ( $\theta \in (0, 1)$ ) tel que  $\Delta - (\gamma|x|^{-2} + a)$  est coercif et  $h \geq 0$ . On considère  $0 < s < 2$  et  $0 \leq \gamma < \gamma_H(\mathbb{R}^{k+,n-k})$ . On fixe  $q \in (1, 2^* - 1)$ . Alors, il existe une solution de type Mountain-Pass  $u \in H_{1,0}^2(\Omega)$  positive pour l'équation Hardy-Schrödinger perturbée (PHS) sous l'une des conditions suivantes:

- $n > n_{\gamma,k}$  et

$$\begin{cases} GH_{\gamma,s}(\Omega) < 0 & \text{si } q+1 < \frac{2n-2}{n-2}, \\ c_1 GH_{\gamma,s}(\Omega) - c_2 h(0) < 0 & \text{si } q+1 = \frac{2n-2}{n-2}, \\ h(0) > 0 & \text{si } q+1 > \frac{2n-2}{n-2}, \end{cases}$$

- $n = n_{\gamma,k}$  et

$$\begin{cases} GH_{\gamma,s}(\Omega) < 0 & \text{si } q+1 \leq \frac{2n-2}{n-2}, \\ h(0) > 0 & \text{si } q+1 > \frac{2n-2}{n-2}, \end{cases}$$

- $n < n_{\gamma,k}$  et

$$\begin{cases} m_{\gamma,a}(\Omega) > 0 & \text{si } q+1 < \frac{2n-2(\alpha_+-\alpha_-)}{n-2}, \\ c_3 m_{\gamma,a}(\Omega) + c_2 h(0) > 0 & \text{si } q+1 = \frac{2n-2(\alpha_+-\alpha_-)}{n-2}, \\ h(0) > 0 & \text{si } q+1 > \frac{2n-2(\alpha_+-\alpha_-)}{n-2}, \end{cases}$$

où  $c_1, c_2, c_3 > 0$  sont des constantes explicites (voir Chapitre 4).

Ce résultat montre l'impact de la non-linéarité sous-critique sur l'existence de solutions. Lorsque la non-linéarité sous-critique est presque linéaire, seule la géométrie de  $\Omega$  commande l'existence. Inversement, lorsqu'elle est proche de la critique, la non-linéarité sous-critique commande l'existence, quelle que soit la géométrie.

## Partie 2: Asymptotiques pour les équations elliptiques de Hardy-Sobolev sur les variétés et les meilleures constantes

Soit  $(M, g)$  une variété Riemannienne compacte de dimension  $n \geq 3$  avec  $\partial M = \emptyset$ . On fixe  $x_0 \in M$  et  $s \in [0, 2)$ . On traite maintenant des équations comme (1.7) avec  $\gamma = 0$  et  $h \equiv 0$ .

Dans la Partie 1, nous nous sommes surtout intéressés aux extrémales des inégalités de Hardy-Sobolev par rapport à la meilleure constante de plongement continu. Dans cette partie, on traite également de l'existence/non-existence des extrémales, mais en nous concentrant sur la seconde meilleure constante associée. En interpolant les inégalités de Sobolev et de Hardy, on obtient l'inégalité de Hardy-Sobolev qui s'écrit sous la forme suivante: il existe  $A, B > 0$  tel que

$$\left( \int_M \frac{|u|^{2^*(s)}}{d_g(x, x_0)^s} dv_g \right)^{\frac{2}{2^*(s)}} \leq A \int_M |\nabla u|_g^2 dv_g + B \int_M u^2 dv_g \quad (1.15)$$

pour tout  $u \in H_1^2(M)$ . Lorsque  $s = 0$ , c'est l'inégalité de Sobolev classique. Des discussions approfondies sur les valeurs optimales de  $A$  et  $B$  pour  $s = 0$  ci-dessus figurent dans la monographie Druet-Hebey [40]. Il a été prouvé par Hebey-Vaugon [70] (le cas classique  $s = 0$ ) et par Jaber [75] ( $s \in (0, 2)$ ) que

$$\mu_{0,s,0}(\mathbb{R}^n)^{-1} = \inf \{ A > 0 \text{ tel que } \exists B > 0 \text{ vérifiant (1.15), } \forall u \in H_1^2(M) \},$$

et l'infimum est atteint avec

$$\mu_{0,s,0}(\mathbb{R}^n) = \inf \left\{ \frac{\int_{\mathbb{R}^n} |\nabla u|^2 dX}{\left( \int_{\mathbb{R}^n} \frac{|u|^{2^*(s)}}{|X|^s} dX \right)^{\frac{2}{2^*(s)}}}, u \in C_c^\infty(\mathbb{R}^n) \right\}$$

qui est la meilleure constante de Hardy-Sobolev (voir Lieb [83] Théorème 4.3 pour la valeur exacte). En plus, il existe  $B > 0$  tel que

$$\left( \int_M \frac{|u|^{2^*(s)}}{d_g(x, x_0)^s} dv_g \right)^{\frac{2}{2^*(s)}} \leq \mu_{0,s,0}(\mathbb{R}^n)^{-1} \left( \int_M |\nabla u|_g^2 dv_g + B \int_M u^2 dv_g \right) \quad (1.16)$$

pour tout  $u \in H_1^2(M)$ . En saturant cette inégalité par rapport à  $B$ , on définit la seconde meilleure constante comme

$$B_s(g) := \inf\{B > 0 \text{ verifiant (1.16) pour tout } u \in H_1^2(M)\},$$

pour obtenir l'inégalité optimale

$$\left( \int_M \frac{|u|^{2^*(s)}}{d_g(x, x_0)^s} dv_g \right)^{\frac{2}{2^*(s)}} \leq \mu_{0,s,0}(\mathbb{R}^n)^{-1} \left( \int_M |\nabla u|_g^2 dv_g + B_s(g) \int_M u^2 dv_g \right) \quad (1.17)$$

pour tout  $u \in H_1^2(M)$ . On dit que  $u_0 \in H_1^2(M)$  est une extrémale pour (1.17) si  $u_0 \not\equiv 0$  et que l'égalité dans (1.17) est valable pour  $u = u_0$ . En plus de l'existence des extrémales, nous nous intéressons à la valeur de la seconde meilleure constante. Lorsque  $s = 0$ , la question a été étudiée par Druet et al. :

**Théorème 1.6** (Les cas  $s = 0$ , [34, 39]). *Soit  $(M, g)$  une variété Riemannienne compacte de dimension  $n \geq 3$ . On suppose que  $s = 0$  et qu'il n'y a pas d'extrémale pour (1.17). Alors*

- $B_0(g) = \frac{n-2}{4(n-1)} \max_M \text{Scal}_g$  si  $n \geq 4$ ;
- La masse de  $\Delta_g + B_0(g)$  s'annule si  $n = 3$ .

La masse sera définie dans la Définition 1.3. On établit le résultat correspondant pour le cas singulier  $s \in (0, 2)$ :

**Théorème 1.7** (Le cas  $s > 0$ , Cheikh-Ali [29], voir Chapitre 5). *Soit  $(M, g)$  une variété Riemannienne de dimension  $n \geq 3$ . On fixe  $x_0 \in M$  et  $s \in (0, 2)$ . On suppose qu'il n'existe pas d'extrémale pour (1.17). Alors*

- $B_s(g) = \frac{(6-s)(n-2)}{12(2n-2-s)} \text{Scal}_g(x_0)$  si  $n \geq 5$ ;
- La masse  $\Delta_g + B_s(g)$  s'annule si  $n = 3$ .

Le cas  $n = 4$  est en cours.

Notre preuve repose sur l'analyse des équations elliptiques critiques dans l'esprit de Druet-Hebey-Robert [41]. Soit  $(a_\alpha)_{\alpha \in \mathbb{N}} \in C^1(M)$  tel que

$$\lim_{\alpha \rightarrow +\infty} a_\alpha = a_\infty \text{ dans } C^1(M). \quad (1.18)$$

On considère que  $(\lambda_\alpha)_\alpha \in (0, +\infty)$  tel que

$$\lim_{\alpha \rightarrow +\infty} \lambda_\alpha = \mu_{0,s,0}(\mathbb{R}^n).$$

On prend une suite de solutions faibles  $(u_\alpha)_\alpha \in H_1^2(M)$  pour

$$\begin{cases} \Delta_g u_\alpha + a_\alpha u_\alpha = \lambda_\alpha \frac{u_\alpha^{2^*(s)-1}}{d_g(x, x_0)^s} & \text{dans } M, \\ u_\alpha \geq 0 & \text{p.p. dans } M. \end{cases} \quad (1.19)$$

On suppose que

$$\|u_\alpha\|_{2^*(s),s} = \left( \int_M \frac{|u_\alpha|^{2^*(s)}}{d_g(x, x_0)^s} dv_g \right)^{\frac{1}{2^*(s)}} = 1,$$

et que

$$u_\alpha \rightharpoonup 0 \text{ faiblement dans } H_1^2(M) \text{ lorsque } \alpha \rightarrow +\infty. \quad (1.20)$$

Nos principaux résultats sont deux descriptions des asymptotiques de  $(u_\alpha)$ . Notons que la régularité et le principe du maximum donnent  $u_\alpha \in C^0(M)$  et  $u_\alpha > 0$ . Ensuite, nous obtenons contrôle ponctuel fort:

**Théorème 1.8.** [Cheikh-Ali [29], voir Chapitre 5] Soit  $M$  une variété Riemannienne compacte de dimension  $n \geq 3$ . On fixe  $x_0 \in M$  et  $s \in (0, 2)$ . Soit  $(a_\alpha)_{\alpha \in \mathbb{N}} \in C^1(M)$  et  $a_\infty \in C^1(M)$  tel que (1.18) vérifiant et  $\Delta_g + a_\infty$  est coercif dans  $M$ . On prend  $(\lambda_\alpha)_\alpha \in \mathbb{R}$  et  $(u_\alpha) \in H_1^2(M)$  tel que (1.18) à (1.20) ont lieu pour tout  $\alpha \in \mathbb{N}$ . Alors, il existe  $C > 0$  tel que,

$$u_\alpha(x) \leq C \frac{\mu_\alpha^{\frac{n-2}{2}}}{\mu_\alpha^{n-2} + d_g(x, x_0)^{n-2}} \text{ pour tout } x \in M, \quad (1.21)$$

où

$$\mu_\alpha := \left( \max_M u_\alpha \right)^{-\frac{2}{n-2}} \quad (1.22)$$

converge vers 0 lorsque  $\alpha \rightarrow +\infty$ .

**Théorème 1.9.** [Cheikh-Ali [29], voir Chapitre 5] Soit  $M$  une variété Riemannienne compacte de dimension  $n \geq 3$ . On fixe  $x_0 \in M$  et  $s \in (0, 2)$ . Soit  $(a_\alpha)_{\alpha \in \mathbb{N}} \in C^1(M)$  et  $a_\infty \in C^1(M)$  tels que  $\Delta_g + a_\infty$  est coercif dans  $M$ . On prend  $(\lambda_\alpha)_\alpha \in \mathbb{R}$  et  $(u_\alpha) \in H_1^2(M)$  tel que (1.18) à (1.20) ont lieu pour tout  $\alpha \in \mathbb{N}$ . Alors,

1. Si  $n \geq 5$ , alors  $a_\infty(x_0) = c_{n,s} \text{Scal}_g(x_0)$ .
2. Si  $n = 3$ , alors  $m_{a_\infty}(x_0) = 0$ .

où  $m_{a_\infty}(x_0)$  est la masse de l'opérateur  $\Delta_g + a_\infty$  (voir la Définition 1.3) et

$$c_{n,s} := \frac{(6-s)(n-2)}{12(2n-2-s)}. \quad (1.23)$$

Le cas  $n = 4$  est en cours.

### Partie 2.1: A propos de la preuve du Théorème 1.8.

Nous établissons des estimations ponctuelles pour des suites arbitraires de solutions de (1.19). Le Théorème 1.8 affirme que le contrôle ponctuel est identique au contrôle du problème (1.19). Avec ce contrôle ponctuel optimal, on peut obtenir plus d'informations sur la localisation du point de "blowup"  $x_0$  et la paramètre de "blowup"  $(\mu_\alpha)_{\alpha \in \mathbb{N}}$ . La preuve du Théorème 1.8 passe par la preuve en deux étapes ci-dessous:

**Étape 1.1.** *On démontre qu'il existe  $\epsilon_0 > 0$  tel que pour tout  $\epsilon \in (0, \epsilon_0)$ , il existe  $C_\epsilon > 0$  tel que*

$$u_\alpha(x) \leq C_\epsilon \frac{\mu_\alpha^{\frac{n-2}{2}-\epsilon}}{d_g(x, x_0)^{n-2-\epsilon}} \text{ pour tout } x \in M \setminus \{x_0\}.$$

**Étape 1.2.** *On démontre qu'il existe  $C > 0$  tel que*

$$d_g(x, x_0)^{n-2} u_\alpha(x_0) u_\alpha(x) \leq C \text{ pour tout } x \in M. \quad (1.24)$$

Pour obtenir les dernière étapes, on s'inspire de Ghoussoub-Robert [62] et Robert [98] (pour plus de détails, (voir Chapitre 5)). Finalement, on utilise (1.24) et la définition de  $\mu_\alpha$  (voir (1.22)), on obtient le résultat attendu.

### Partie 2.2: A propos de la preuve du Théorème 1.9.

Grâce aux estimations dans (1.21), on peut prouver le Théorème 1.9 où  $n \geq 3$ . Lorsque  $n = 3$ , la masse est définie ainsi:

**Définition 1.3.** *[La masse] Soit  $(M, g)$  une variété Riemannienne compacte de dimension  $n = 3$ , et soit  $h \in C^0(M)$  tel que  $\Delta_g + h$  est coercif. Soit  $G_{x_0}$  la fonction Green de  $\Delta_g + h$  en  $x_0$ . Soit  $\eta \in C^\infty(M)$  tel que  $\eta = 1$  autour de  $x_0$ . Alors il existe  $\beta_{x_0} \in H_1^2(M)$  tel que*

$$G_{x_0} = \frac{1}{4\pi} \eta d_g(\cdot, x_0)^{-1} + \beta_{x_0} \text{ dans } M \setminus \{x_0\}.$$

*On a que  $\beta_{x_0} \in H_2^p(M) \cap C^{0,\theta}(M) \cap C^{2,\gamma}(M \setminus \{x_0\})$  pour tout  $p \in (\frac{3}{2}, 3)$  et  $\theta, \gamma \in (0, 1)$ . On définit la masse en  $x_0$  comme  $m_h(x_0) := \beta_{x_0}(x_0)$ , qui est indépendante du choix de  $\eta$ .*

La principale difficulté de notre analyse est due à la non-existence de l'identité de Pohozaev dans le contexte Riemannien. En effet, on doit trouver une carte convenable qui envoie localement  $M$  vers  $\mathbb{R}^n$ . Ici, on s'inspire de Ghoussoub-Robert [62], on prend une suite de solutions faibles  $(u_\alpha)_\alpha \in H_1^2(M)$  pour (1.19).

On fait le changement de variable avec la carte exponentielle  $\exp_{x_0}$  centrée en  $x_0$ , et on définit la fonction suivante:

$$\hat{u}_\alpha(X) := u_\alpha(\exp_{x_0}(X)) \text{ pour } X \in B_\delta(0) \subset \mathbb{R}^n.$$

On note que  $\hat{u}_\alpha$  vérifie localement l'équation (1.19) sur  $\mathbb{R}^n$ . On injecte maintenant  $\hat{u}_\alpha$  dans l'identité classique de Pohozaev sur  $\mathbb{R}^n$  (voir par exemple [62]). On calcule, et on obtient deux parties. Avec l'estimation précise du Théorème 1.8, nous sommes en mesure d'obtenir le comportement asymptotique précis de ces termes. D'ici, la courbure scalaire  $Scal_g(x_0)$  ( $n \geq 4$ ) et la masse  $m_{a_\infty}(x_0)$  ( $n = 3$ ) apparaissent. Finalement, on fait une comparaison et on obtient le résultat de ce Théorème. Le cas  $n = 4$  est en cours. En effet, la majeure partie de l'analyse a été faite dans le Chapitre 5, et la preuve des asymptotique se ramène maintenant à l'obtention de (5.108) (voir Chapitre 5).

### Partie 3: Existence d'une solution non constante à une équation du quatrième ordre avec un exposant critique (En collaboration avec D.Bonheure et R.Nascimento)

Dans cette partie, on considère un domaine borné et régulier  $\Omega \subset \mathbb{R}^n$  ( $n \geq 5$ ). On prend  $\alpha > 0$ , on étudie la multiplicité des solutions de

$$\begin{cases} \Delta^2 u + \Delta u + \alpha u = |u|^{\frac{8}{n-4}} u, & \text{dans } \Omega, \\ \partial_\nu u = \partial_\nu(\Delta u) = 0, & \text{sur } \partial\Omega. \end{cases} \quad (P_\nu)$$

Il y a au moins trois solutions: les solutions constantes  $u \equiv 0$  et  $\pm \alpha^{\frac{n-4}{8}}$ .

(Q5) Y-a-t'il des solutions non constantes à  $(P_\nu)$ ?

Le problème  $(P_\nu)$  est une généralisation du problème de type Brezis-Nirenberg

$$\begin{cases} \Delta u + \alpha u = |u|^{\frac{4}{n-2}} u & \text{dans } \Omega, \\ \partial_\nu u = 0 & \text{sur } \partial\Omega. \end{cases} \quad (1.25)$$

En dépit du fait que le domaine est régulier et qu'il n'existe pas de singularité de type Hardy (contrairement à  $(HS)$ ), la difficulté du problème et la méthodologie d'examen de l'existence de solutions ressemblent beaucoup à celles développées dans la première partie. Dans la suite, on définit

$$H_{2,\nu}^2(\Omega) := \{u \in H_2^2(\Omega) : \partial_\nu u = 0 \text{ sur } \partial\Omega\}.$$



On dit que  $u \in H_{2,\nu}^2(\Omega)$  est une solution faible à  $(P_\nu)$  si

$$\int_{\Omega} (\langle \Delta u, \Delta v \rangle + \langle \nabla u, \nabla v \rangle + uv) dx = \int_{\Omega} u^{2^{**}-1} v dx \text{ pour tout } v \in H_{2,\nu}^2(\Omega),$$

où  $2^{**} := \frac{2n}{n-4}$ . On étudie les solutions faibles de  $(P_\nu)$  comme les minima de la fonctionnelle

$$u \mapsto J(u) = \int_{\Omega} (|\Delta u|^2 + |\nabla u|^2 + \alpha |u|^2) dx,$$

sur

$$M_{\Omega} := \left\{ u \in H_{2,\nu}^2(\Omega) : \int_{\Omega} |u|^{\frac{2n}{n-4}} dx = 1 \right\}.$$

On définit,

$$\Sigma_{\nu}(\Omega) := \inf \{ J(u) \mid u \in M_{\Omega} \}.$$

Là encore, la principale difficulté tient au fait que  $2^{**}$  est critique du point de vue de l'injection de Sobolev. L'injection  $H_{2,\nu}^2(\Omega) \hookrightarrow L^p(\Omega)$  est compacte ssi  $1 \leq p < 2^{**}$ . Avant d'aller plus loin, on établit quelques notations et rappels sur des résultats connus. On note  $D_2^2(\mathbb{R}^n)$  le complété de  $C_c^\infty(\mathbb{R}^n)$  pour la norme  $u \mapsto \|\Delta u\|_2$ . La meilleure constante pour le prolongement de  $D_2^2(\mathbb{R}^n)$  dans  $L^{\frac{2N}{N-4}}(\mathbb{R}^n)$  est caractérisée par

$$S(n) := \inf_{u \in D_2^2(\mathbb{R}^n)} \left\{ \int_{\mathbb{R}^n} |\Delta u|^2 dx : \int_{\mathbb{R}^n} |u|^{\frac{2N}{N-4}} dx = 1 \right\}.$$

Lorsque la non-linéarité dans (1.25) est sous-critique (à savoir lorsque l'exposant  $4/(n-2)$  est remplacé par  $q-2$  par  $2 < q < 2^*$ ). Lin, Ni et Tagaki [80] ont prouvé que la seule solution positive à (1.25), pour  $\alpha > 0$  assez petit, est la solution constante non nulle. Dans le cas critique, Lin et Ni [81] ont soulevé ce résultat de rigidité comme une conjecture.

LA CONJECTURE DE LIN-NI'S: *Pour  $\alpha$  assez petit, (1.25) n'admet que  $\alpha^{(n-2)/4}$  comme une solution positive.*

Cette conjecture a été résolue par Adimurthi-Yadava [8] dans le cas radial. Dans le cas général, la situation est maintenant parfaitement comprise et résolue: nous nous référons à Druet-Robert-Wei [44] pour les références et pour la résolution lorsque  $n = 3$  et  $n \geq 7$  avec une énergie bornée.

Dans le cas sous-critique, il résulte d'un argument d'indice de Morse que la rigidité est cassée pour les grand  $\alpha$ . Dans le cas critique, inspiré par Brezis et Nirenberg, Wang [107], Adimurthi et Mancini [6] ont prouvé que (1.25) admet une solution positive non constante pour chaque  $\alpha > \bar{\alpha} > 0$ .

On revient vers l'équation initiale  $(P_\nu)$ . Notre premier résultat dans cette direction est l'existence d'une solution non constante:

**Théorème 1.10.** [Bonheure-Cheikh Ali-Nascimento [14], voir Chapitre 6] On suppose  $\Omega$  est un domaine ouvert, borné et régulier dans  $\mathbb{R}^n$ . Il existe  $\bar{\alpha} = \bar{\alpha}(n, |\Omega|) > 0$  tel que pour  $\alpha > \bar{\alpha}$ , toute solution d'énergie minimale de l'équation  $(P_\nu)$  est non constante.

Le résultat suivant est de type Lin-Ni:

**Théorème 1.11.** [Bonheure-Cheikh Ali-Nascimento [14], voir Chapitre 6] On suppose que  $\Omega$  est un domaine ouvert, borné et régulier dans  $\mathbb{R}^n$ . Alors, il existe  $\underline{\alpha} = \underline{\alpha}(n, |\Omega|) > 0$  tel que pour  $0 < \alpha < \underline{\alpha}$ , la seule solution d'énergie minimale de l'équation  $(P_\nu)$  est la solution constante  $\alpha^{\frac{n-4}{8}}$ .

### Partie 3.1: A propos de la preuve du Théorème 1.10.

Comme dans la Partie 1, la meilleure constante dans l'inégalité de Sobolev  $H_{2,\nu}^2(\Omega) \hookrightarrow L^{2^{**}}(\Omega)$  va jouer un grand rôle sur l'existence de solution non constante  $(P_\nu)$ . D'abord, on montre que la meilleure constante est l'inégalité de Sobolev est la meilleure constante pour l'espace modèle  $\mathbb{R}_+^n$ :

**Lemme 1.1** (Bonheure-Cheikh Ali-Nascimento [14], voir Chapitre 6). On suppose que  $\Omega$  est un domaine ouvert, borné et régulier dans  $\mathbb{R}^n$  ( $n \geq 5$ ). Alors, pour tout  $\varepsilon > 0$ , il existe  $B(\varepsilon) > 0$  tel que pour tout  $u \in H_{2,\nu}^2(\Omega)$ ,

$$\|u\|_{L^{\frac{2n}{n-4}}(\Omega)}^2 \leq \left( \frac{2^{4/n}}{S(n)} + \varepsilon \right) \|\Delta u\|_{L^2(\Omega)}^2 + B(\varepsilon) \|u\|_{H^1(\Omega)}^2.$$

En plus,  $\Sigma_\nu(\mathbb{R}_+^n) = S(n)/2^{4/n}$  et cet infimum n'est pas atteint.

Un résultat similaire a été démontré pour les domaines singuliers dans la Partie 1 (voir Théorème 1.1 ci-dessus et Chapitre 3). Le résultat d'existence suivant est dans l'esprit d'Aubin [2] et le Théorème 1.1 de la Partie 1:

**Lemme 1.2.** [Bonheure-Cheikh Ali-Nascimento [14], voir Chapitre 6] On suppose que  $\Omega$  un domaine ouvert, borné et régulier dans  $\mathbb{R}^n$  ( $n \geq 5$ ). Si  $\Sigma_\nu(\Omega) < \Sigma_\nu(\mathbb{R}_+^n)$ , alors  $\Sigma_\nu(\Omega)$  est atteint.

Il reste à estimer  $J$  en des fonctions test pertinentes:

1. Pour  $n \geq 5$ , les minimiseurs pour  $S(n)$  sont donnés par la famille à un paramètre

$$x \mapsto u_\varepsilon(x) := \gamma_n \frac{\varepsilon^{\frac{n-4}{2}}}{(\varepsilon^2 + |x|^2)^{\frac{n-4}{2}}}; \gamma_n := [(n-4)(n-2)n(n+2)]^{\frac{n-4}{8}}.$$

2. On fixe  $p_0 \in \partial\Omega$  et on définit la fonction test

$$\psi_\varepsilon(x) := (\eta u_\varepsilon(|\cdot|)) \circ \Phi^{-1}(x) \in H_\nu^2(\Omega).$$

avec  $\Phi$  une carte convenable en  $p_0 \in \partial\Omega$  pour avoir  $\psi_\varepsilon \in H_{2,\nu}^2(\Omega)$  (pour plus de détails (voir le Chapitre 6)),  $\eta$  est une fonction de troncature radiale.

3. On estime  $J(\psi_\varepsilon)$  et on trouve une expression qui dépend de la courbure moyenne  $H(p_0)$  pour tout  $n \geq 5$ , et il existe  $C_n > 0$  tel que

$$J(\psi_\varepsilon) = \begin{cases} \frac{S(n)}{2^{4/n}} - C_n 2^{1-4/n} S(n)^{1-n/4} H(p_0) \varepsilon + o(\varepsilon) & \text{si } n \geq 6, \\ \frac{S(n)}{2^{4/5}} - 2^{14/5} \pi^2 \sqrt[4]{\frac{105}{S(n)}} H(p_0) \varepsilon \log \frac{1}{\varepsilon} + O(\varepsilon) & \text{si } n = 5. \end{cases}$$

À partir de là, la positivité de la courbure moyenne à un certain point de  $\partial\Omega$  et l'expression de  $J(\psi_\varepsilon)$  donnent la condition suffisante  $\Sigma_\nu(\Omega) < \Sigma_\nu(\mathbb{R}_+^n)$ . Enfin, avec le Lemme 1.2, on obtient le Théorème 1.10.

### Partie 3.2: A propos de la preuve du Théorème 1.11

Cette preuve est dans l'esprit de Ni-Takagi [91]. En ce qui concerne le problème original de Lin-Ni, un contrôle point par point est décisif pour obtenir l'unicité de  $\alpha \rightarrow 0$ . On prouve un tel contrôle pour minimiser des solutions minimisantes à  $(P_\nu)$  lorsque  $\alpha > 0$  est suffisamment petit:

**Lemme 1.3** (Bonheure-Cheikh Ali-Nascimento [14], voir Chapitre 6). *On suppose que  $u \in M_\Omega$  atteint  $\Sigma_\nu(\Omega)$  et  $\alpha \leq 1/4$ . Alors  $u > 0$ . Si on choisit  $v$  comme le multiple de  $u$  qui résout*

$$\begin{cases} \Delta^2 v + \Delta v + \alpha v = |v|^{\frac{8}{N-4}} v, & \text{dans } \Omega, \\ \partial_\nu v = \partial_\nu(\Delta v) = 0, & \text{sur } \partial\Omega, \end{cases}$$

alors il existe  $C_0 > 0$  dépend seulement de  $\Omega$  tel que  $\|v\|_\infty \leq C_0$ .

Avec un tel contrôle, on peut prouver que la norme  $L^\infty$  – des solutions minimisantes de  $(P_\nu)$  va uniformément 0 lorsque  $\alpha \rightarrow 0$ . Il est simple d'obtenir que ces solutions sont constantes en utilisant une inégalité de Poincaré. Ceci donne le Théorème 1.11.

## Introduction (English version)

### Part 0: The framework

The celebrated Sobolev theorem asserts that, given  $n \geq 3$ , there exists a constant  $C_1(n) > 0$  such that

$$\|u\|_{L^{2^*}(\mathbb{R}^n)} \leq C_1(n) \|\nabla u\|_{L^2(\mathbb{R}^n)} \text{ for all } u \in C_c^\infty(\mathbb{R}^n) \quad (2.1)$$

where  $2^* := \frac{2n}{n-2}$ . The exponent  $2^*$  is critical in the following sense. For any domain  $\Omega \subset \mathbb{R}^n$ , let  $H_{1,0}^2(\Omega)$  be the completion of  $C_c^\infty(\Omega)$  for the norm  $u \mapsto \|\nabla u\|_2$ . Then  $H_{1,0}^2(\Omega)$  is continuously embedded in  $L^q(\Omega)$  for all  $1 \leq q \leq 2^*$ , and the embedding is compact if  $1 \leq q < 2^*$  where  $\Omega$  is any bounded domain. This lack of compactness is the main complexity of the Yamabe problem. Given a smooth compact Riemannian manifold  $(M, g)$  of dimension  $n \geq 3$  without boundary, we denote the scalar curvature as  $Scal_g$ . The Yamabe problem states as follows: Is there a metric  $\tilde{g}$  conformal to  $g$  such that  $Scal_{\tilde{g}}$  is constant? It amounts to finding a positive solution  $u \in C^2(M)$  to

$$\Delta_g u + c_n Scal_g u = \epsilon u^{2^*-1} \text{ s.t. } \epsilon \in \{-1, 0, 1\}, u > 0 \quad (2.2)$$

where  $\Delta_g = -div_g(\nabla)$  is the Laplace-Beltrami operator on  $(M, g)$  and  $c_n := \frac{n-2}{4(n-1)}$ . Here, the natural functional space in which to work is  $H_1^2(M)$ , the completion of  $C^\infty(M)$  for the norm  $u \mapsto \|u\|_2 + \|\nabla u\|_2$ . The answer is positive, and the resolution of this problem is a long story. It involves local conditions for "large" dimensions, and a global condition (positivity of the so-called *mass*) for "small" dimensions and the locally-conformally flat case. Let us just mention that Yamabe's initial proof [112] was not complete and that the final resolution of the problem is due to Aubin [2] and Schoen [100]. The classical reference for this problem is the survey of Lee and Parker [82].

What happens if the problem is set on a domain of  $\mathbb{R}^n$ ? Can one still solve (2.2) with the addition of Dirichlet boundary conditions for instance? This is the object of the classical work [19] by Brezis-Nirenberg. Let  $\Omega \subset \mathbb{R}^n$ ,  $n \geq 3$ , be an open bounded domain with a smooth boundary, and  $\alpha$  be a real number. In the celebrated paper [19], Brezis-Nirenberg have proved the existence of a solution

$u \in C^2(\Omega) \cap C^0(\overline{\Omega})$  to the problem

$$\begin{cases} \Delta u - \alpha u = u^{2^*-1} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (2.3)$$

as soon as  $\alpha > 0$  when  $n \geq 4$  (and the necessary condition  $\alpha < \lambda_1(\Omega)$ ). The case  $n = 3$  is more intricate: the existence of minimizing solutions is equivalent to the positivity of the mass of the operator  $\Delta - \alpha$  (see Druet [36]).

The analysis of (2.3) (respectively (2.2)) is closely related to the best constant for the Sobolev embedding  $H_{1,0}^2(\Omega) \hookrightarrow L^{2^*}(\Omega)$  (respectively  $H_1^2(M) \hookrightarrow L^{2^*}(M)$ ).

### Part 0.1: Classical inequalities on $\mathbb{R}^n$ ( $n \geq 3$ ).

The Hardy inequality writes

$$\frac{(n-2)^2}{4} \int_{\mathbb{R}^n} \frac{u^2}{|x|^2} dx \leq \int_{\mathbb{R}^n} |\nabla u|^2 dx \text{ for all } u \in C_c^\infty(\mathbb{R}^n). \quad (2.4)$$

Interpolating the inequalities (2.1) and (2.4), we get the Hardy-Sobolev inequality: for  $s \in [0, 2]$ , there exists  $C(n, s) > 0$  such that

$$\left( \int_{\mathbb{R}^n} \frac{|u|^{2^*(s)}}{|x|^s} dx \right)^{\frac{2}{2^*(s)}} \leq C(n, s) \int_{\mathbb{R}^n} |\nabla u|^2 dx \text{ for all } u \in C_c^\infty(\mathbb{R}^n), \quad (2.5)$$

where  $2^*(s) := \frac{2(n-s)}{n-2}$  is the critical Hardy-Sobolev exponent. We observe that, with  $s = 0$  we recover the Sobolev inequality (2.1), and with  $s = 2$  we recover the Hardy inequality (2.4). When  $\gamma < \frac{(n-2)^2}{4}$ , it follows from (2.4) that there exists  $C(n, \gamma) > 1$  such that

$$\frac{1}{C(n, \gamma)} \int_{\mathbb{R}^n} |\nabla u|^2 dx \leq \int_{\mathbb{R}^n} \left( |\nabla u|^2 - \gamma \frac{u^2}{|x|^2} \right) dx \leq C(n, \gamma) \int_{\mathbb{R}^n} |\nabla u|^2 dx$$

for all  $u \in C_c^\infty(\mathbb{R}^n)$ . Therefore, for any  $\gamma < \frac{(n-2)^2}{4}$ , there exists  $C(n, \gamma, s) > 0$  such that

$$\left( \int_{\mathbb{R}^n} \frac{|u|^{2^*(s)}}{|x|^s} dx \right)^{\frac{2}{2^*(s)}} \leq C(n, \gamma, s) \int_{\mathbb{R}^n} \left( |\nabla u|^2 - \gamma \frac{u^2}{|x|^2} \right) dx \quad (2.6)$$

for all  $u \in C_c^\infty(\mathbb{R}^n)$ . This Hardy-Sobolev inequality is a particular case of the celebrated family of functional inequalities obtained by Caffarelli-Kohn-Nirenberg [25] but was apparently discovered earlier by V.P.II'in, see [73]. Let

us now restrict this inequality to a domain. Let  $\Omega$  be a domain of  $\mathbb{R}^n$ ,  $n \geq 3$ , and, for  $0 < s < 2$ ,  $\gamma \in \mathbb{R}$  and  $a \in L^\infty(\Omega)$ , define

$$\mu_{\gamma,s,a}(\Omega) := \inf \left\{ J_{\gamma,s,a}^\Omega(u) / u \in H_{1,0}^2(\Omega) \setminus \{0\} \right\},$$

where

$$J_{\gamma,s,a}^\Omega(u) := \frac{\int_\Omega \left( |\nabla u|^2 - \left( \frac{\gamma}{|x|^2} + a(x) \right) u^2 \right) dx}{\left( \int_\Omega \frac{|u|^{2^*(s)}}{|x|^s} dx \right)^{\frac{2}{2^*(s)}}}.$$

Note that it follows from (2.6) that for  $s \in (0, 2)$  and  $\gamma < \frac{(n-2)^2}{4}$ , there exists  $K > 0$  such that

$$\left( \int_\Omega \frac{|u|^{2^*(s)}}{|x|^s} dx \right)^{\frac{2}{2^*(s)}} \leq K \int_\Omega \left( |\nabla u|^2 - \gamma \frac{u^2}{|x|^2} \right) dx \quad (CKN)$$

for all  $u \in C_c^\infty(\Omega)$ . In other words,  $\mu_{\gamma,s,0}(\Omega) > 0$  when  $\gamma < \frac{(n-2)^2}{4}$ .

In this thesis, we mainly focus on critical points of  $J_{\gamma,s,a}^\Omega$ . Namely, let  $(\overline{M}, g)$  be a compact Riemannian manifold with boundary  $\partial M$  and interior  $M$ . Let  $x_0 \in \overline{M}$  be a fixed point and let  $d_g$  be the Riemannian distance on  $M$ . We fix  $a, h \in L^\infty(M)$  and  $q \in (1, 2^* - 1)$ . We will consider weak solutions to

$$\begin{cases} \Delta_g u - \left( a(x) + \frac{\gamma}{d_g(x,x_0)^2} \right) u = \frac{u^{2^*(s)-1}}{d_g(x,x_0)^s} + h(x)u^q & \text{in } M; \\ u > 0 & \text{in } M; \\ u = 0 & \text{on } \partial M \text{ if } \neq \emptyset \end{cases} \quad (2.7)$$

## Part 0.2: The work program

Basicaly, this thesis is divided into three parts:

- Part 1:** We analyze the existence of solutions to (2.7) on a non-smooth domain in  $\mathbb{R}^n$  with boundary singularity at 0 modeled on a cone. The solutions are achieved as minimizers of  $J_{\gamma,s,0}^\Omega$  when  $h \equiv 0$ , and as Mountain-Pass critical points when  $h \not\equiv 0$ .
- Part 2:** On a compact Riemannian manifold  $M$  with no boundary and with  $\gamma = 0$  and  $h \equiv 0$ , we perform a blow-up analysis of solutions to (2.7) of minimizing type. This yields informations on the value of the second best constant in the related Riemannian functional inequality.
- Part 3:** We study a fourth order Paneitz version of (2.7) with  $\gamma = s = 0$  on a smooth bounded domain of  $\mathbb{R}^n$ ,  $n \geq 5$ .

## Part 1: Hardy-Sobolev equations on singular domains

Let  $\Omega$  be a bounded domain of  $\mathbb{R}^n$ ,  $n \geq 3$ , and fix  $\gamma \in \mathbb{R}$  and  $a \in L^\infty(\Omega)$ . We investigate the existence of weak solutions  $u \in H_{1,0}^2(\Omega)$ ,  $u \neq 0$ , to

$$\begin{cases} \Delta u - \left( a(x) + \frac{\gamma}{|x|^2} \right) u = \frac{u^{2^*(s)-1}}{|x|^s} & \text{in } \Omega, \\ u > 0 & \text{a.e. in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \quad (HS)$$

Up to dilation, equation (HS) is the Euler-Lagrange equation associated to  $J_{\gamma,s,a}^\Omega$ . So if there are extremals for a positive  $\mu_{\gamma,s,a}(\Omega)$ , they are solutions to (HS).

**The case  $0 \in \Omega$ :** The problem makes sense when  $\gamma < (n-2)^2/4$ , the classical Hardy constant. There are no extremals for  $\mu_{\gamma,s,0}(\Omega)$  (see [60]). There is an important literature on this case. For instance, we refer to Ruiz-Willem [99], Smets [102] and the survey [60] by Ghoussoub-Robert.

**The case  $0 \in \partial\Omega$ .** The existence of extremals for  $\mu_{\gamma,s,0}(\Omega)$  has been studied by Egnell [46] when  $\Omega$  is a cone at 0. When the domain is smooth, the question was initiated by Ghoussoub-Kang [55] and studied by Chern-Lin [32] and Ghoussoub-Robert [61]. It turned out to be also interesting as the mean curvature at 0 gets to play an important role. In this thesis (see [27, 28]), we consider nonsmooth domain modeled on some regular cones (we refer them as "model singularities"). We show how the geometry of the model cone influences the value of the Hardy constant on  $\Omega$ .

From now on, we assume that  $0 \in \Omega$ , and for convenience, we define

$$\mathbb{R}^{k+,n-k} := \mathbb{R}_+^k \times \mathbb{R}^{n-k} \text{ for all } k \in \{1, \dots, n\},$$

with  $\mathbb{R}_+^k := \{x_1, \dots, x_k > 0\}$ . In Cheikh Ali [27],[28], we define domains that are modeled on cones:

**Definition 2.1.** We fix  $1 \leq k \leq n$ . Let  $\Omega$  be a domain of  $\mathbb{R}^n$ . We say that  $x_0 \in \partial\Omega$  is a singularity of type  $(k, n-k)$  if there exist  $U, V$  open subsets of  $\mathbb{R}^n$  such that  $0 \in U$ ,  $x_0 \in V$  and there exists a diffeomorphism  $\phi \in C^\infty(U, V)$  such that  $\phi(0) = x_0$  and

$$\phi(U \cap \mathbb{R}^{k+,n-k}) = \phi(U) \cap \Omega \text{ and } \phi(U \cap \partial\mathbb{R}^{k+,n-k}) = \phi(U) \cap \partial\Omega,$$

with the additional hypothesis that the differential at 0  $d\phi_0$  is an isometry.

**The motivation for considering equation (HS)** arises from the problem of existence of extremals for the Caffarelli-Kohn-Nirenberg (CKN) inequalities. We address the following questions:

**(Q1)** For which values of  $\gamma \in \mathbb{R}$  does  $(CKN)$  hold for some  $K > 0$  and all  $u \in H_{1,0}^2(\Omega)$ ? In other words, when do we have  $\mu_{\gamma,s,0}(\Omega) > 0$ ?

**(Q2)** Is the best constant achieved? In other words, is  $\mu_{\gamma,s,0}(\Omega)$  achieved by some  $u \in H_{1,0}^2(\Omega)$ ,  $u \neq 0$ ?

Note that the best Hardy constant for  $\mathbb{R}^{k_+,n-k}$  is explicit. As it is noted in Ghoussoub-Moradifam [57], we have that

$$\gamma_H(\mathbb{R}^{k_+,n-k}) = \frac{(n + 2k - 2)^2}{4} \text{ for all } k \in \{1, \dots, n\}.$$

The answer to the first question **(Q1)** depends on the Hardy constant. We define

$$\gamma_H(\Omega) := \mu_{0,2,0}(\Omega) = \inf \left\{ \frac{\int_{\Omega} |\nabla u|^2 dx}{\int_{\Omega} \frac{u^2}{|x|^2} dx}; u \in H_{1,0}^2(\Omega) \setminus \{0\} \right\}. \quad (2.8)$$

As a consequence, interpolating the Hardy inequality (2.8) and Sobolev inequality  $((CKN)$  with  $\gamma = s = 0$ ), we get that

$$\gamma < \gamma_H(\Omega) \Rightarrow \mu_{\gamma,s,0}(\Omega) > 0.$$

We now tackle the second question **(Q2)**, that is the existence of extremals for  $\mu_{\gamma,s,0}(\Omega)$ . The following result is central for the sequel. The proof is standard since Aubin's proof of the Yamabe conjecture in high dimensions [2] where he noted that the compactness of minimizing sequences is restored if the infimum is strictly below the energy of "Bubble". In our case below, this translates to  $\mu_{\gamma,s,a}(\Omega) < \mu_{\gamma,s,0}(\mathbb{R}^{k_+,n-k})$ .

**Theorem 2.1** (Cheikh-Ali [27], [28], see Chapters 3, 4). *Assume  $\Omega \subset \mathbb{R}^n$  is a bounded domain such that  $0 \in \partial\Omega$  is a singularity of type  $(k, n-k)$ . Assume that  $\gamma < \gamma_H(\mathbb{R}^{k_+,n-k})$ ,  $0 \leq s \leq 2$ , and  $\mu_{\gamma,s,a}(\Omega) < \mu_{\gamma,s,0}(\mathbb{R}^{k_+,n-k})$ . Then there are extremals for  $\mu_{\gamma,s,a}(\Omega)$ . In particular, there exists a minimizer  $u$  in  $H_{1,0}^2(\Omega) \setminus \{0\}$  that is a positive solution to the equation*

$$\begin{cases} \Delta u - \left( \frac{\gamma}{|x|^2} + a(x) \right) u = \mu_{\gamma,s,a}(\Omega) \frac{u^{2^*(s)-1}}{|x|^s} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

The kind of condition  $\mu_{\gamma,s,a}(\Omega) < \mu_{\gamma,s,0}(\mathbb{R}^{k_+,n-k})$  is very classical in best constant problems, see Aubin [2], Brezis-Nirenberg [19]. From here, we consider the following question:

**(Q3)** For which  $\gamma < \gamma_H(\mathbb{R}^{k_+,n-k})$  do we have  $\mu_{\gamma,s,a}(\Omega) < \mu_{\gamma,s,0}(\mathbb{R}^{k_+,n-k})$ ?



Before answering this question, for  $k \in \{1, \dots, n\}$  we need first to consider model solutions to

$$\begin{cases} \Delta u - \frac{\gamma}{|x|^2} u = 0 & \text{in } \mathbb{R}^{k+, n-k}; \\ u > 0 & \text{in } \mathbb{R}^{k+, n-k}; \\ u = 0 & \text{on } \partial \mathbb{R}^{k+, n-k}. \end{cases} \quad (2.9)$$

Let  $\alpha \in \mathbb{R}$  be a real number and fix  $\gamma < \gamma_H(\mathbb{R}^{k+, n-k})$ . Then

$$U_\alpha \text{ is a solution to (2.9)} \Leftrightarrow \alpha \in \{\alpha_-, \alpha_+\},$$

where

$$U_\alpha := |x|^{-\alpha-k} \prod_{i=1}^k x_i \text{ and } \alpha_\pm = \alpha_\pm(\gamma, n, k) := \frac{n-2}{2} \pm \sqrt{\gamma_H(\mathbb{R}^{k+, n-k}) - \gamma}. \quad (2.10)$$

The functions  $U_{\alpha_-}, U_{\alpha_+}$  are prototypes of solution to (2.9) vanishing on  $\partial \mathbb{R}^{k+, n-k}$ . Note that

$$\alpha_- < \frac{n-2}{2} < \alpha_+ \text{ and } U_{\alpha_-} \in H_{1,0,loc}^2(\mathbb{R}^{k+, n-k}).$$

The profile  $U_{\alpha_-}$  is also the model of behavior for variational solutions to  $(HS)$ . Indeed, it follows from the regularity result of Felli-Ferrero [50] that, via a chart, a variational solution to  $(HS)$  is behaving like  $U_{\alpha_-}$  around the singularity 0. We now define the critical dimension  $n_{\gamma,k} := \sqrt{4\gamma + 1} + 2 - 2k$  (possibly non-integer and negative):

$$\left\{ \begin{array}{l} \gamma \text{ is small (**large dimension**) : } \gamma \leq \gamma_H(\mathbb{R}^{k+, n-k}) - \frac{1}{4} \text{ that is } n \geq n_{\gamma,k} \\ \gamma \text{ is large (**small dimension**) : } \gamma > \gamma_H(\mathbb{R}^{k+, n-k}) - \frac{1}{4} \text{ that is } n < n_{\gamma,k} \end{array} \right\}$$

Here is the strategy to get  $\mu_{\gamma,s,a}(\Omega) < \mu_{\gamma,s,0}(\mathbb{R}^{k+, n-k})$ :

1. We take a non-negative extremal  $V$  for  $\mu_{\gamma,s,0}(\mathbb{R}^{k+, n-k})$ .
2. We concentrate the function  $V$  in 0 with  $\epsilon > 0$  and bring it back on  $\Omega$  by the chart  $\phi$  (see Definition 2.1).
3. We calculate and obtain an integral which, under the right assumptions on the dimension, must converge to get the **generalized curvature**  $GH_{\gamma,s}(\Omega)$ .
4. In the other cases, global arguments are needed and we need to introduce the mass  $m_{\gamma,0}(\Omega)$ .

First, we need the existence of extremals for  $\mu_{\gamma,s,0}(\mathbb{R}^{k+,n-k})$  to build the test function. By Ghoussoub-Robert [60] (see Section 5), we have the following result:

**Proposition 2.1.** *Fix  $\gamma < \gamma_H(\mathbb{R}^{k+,n-k})$  and  $s \in [0, 2)$  with  $n \geq 3$ , then*

*If  $\{s > 0\}$  or  $\{s = 0, \gamma > 0 \text{ and } n \geq 4\}$  then  $\mu_{\gamma,s,0}(\mathbb{R}^{k+,n-k})$  is attained .*

This part is divided into three subsections:

- Part 1.1:** We introduce the problem  $(HS)$  in Cheikh-Ali [27] (see Chapter 3). Under a local geometric hypothesis, namely that the generalized mean curvature is negative (see (2.11) below), we prove the existence of extremals for the relevant Hardy-Sobolev inequality for large dimensions.
- Part 1.2:** Next, we tackle in Cheikh-Ali [27] (see Chapter 4) the question of small dimensions that was left open. We introduce a "mass", that is a global quantity, the positivity of which ensures the existence of extremals in small dimensions.
- Part 1.3:** As a byproduct, we prove the existence of solutions to a perturbation of the initial equation via the Mountain Pass Lemma.

### Part 1.1: Large dimensions and generalized curvature

**When  $\gamma$  is small (large dimensions):** In this case, we show how the local geometry induced by the cone around the singularity influences the value of Hardy-Sobolev constant to answer the (Q3), we will need two important things:

**I- Definition of the generalized curvature:** We introduce a new geometric object at the conical singularity that generalizes the "mean curvature": this allows to get extremals for  $(HS)$ . For this, we write the nonsmooth domain  $\Omega$  as the intersection of smooth domains around 0: there exists  $\Omega_1, \dots, \Omega_k \subset \mathbb{R}^n$  smooth bounded domains and  $\delta > 0$  such that

$$\Omega \cap B_\delta(0) = \left( \bigcap_{i=1}^k \Omega_i \right) \cap B_\delta(0).$$

The  $\Omega_i$ 's are locally unique up to permutation. We set  $\Sigma := \bigcap_{i=1}^k \partial\Omega_i$  where  $k \in \{1, \dots, n\}$ . The vector  $\vec{H}_0^\Sigma$  denotes the mean-curvature vector at 0 of the  $(n-k)$ -submanifold  $\Sigma$ . For any  $m = 1, \dots, k$ ,  $II_0^{\partial\Omega_m}$  denotes the second fundamental form at 0 of the oriented  $(n-1)$ -submanifold  $\partial\Omega_m$ . The **generalized**

**mean curvature** of  $\Omega$  is defined by:

$$GH_{\gamma,s}(\Omega) := c_{\gamma,s}^1 \sum_{m=1}^k \langle \vec{H}_0^\Sigma, \vec{\nu}_m \rangle + c_{\gamma,s}^2 \sum_{i,m=1, i \neq m}^k II_0^{\partial\Omega_m}(\vec{\nu}_i, \vec{\nu}_i) \quad (2.11)$$

$$+ c_{\gamma,s}^3 \sum_{p,q,m=1, |\{p,q,m\}|=3}^k II_0^{\partial\Omega_m}(\vec{\nu}_p, \vec{\nu}_q)$$

where for any  $m = 1, \dots, k$ ,  $\vec{\nu}_m$  is the outward normal vector at 0 of  $\partial\Omega_m$  and  $c_{\gamma,s}^1, c_{\gamma,s}^2, c_{\gamma,s}^3$  are positive explicit constants. We refer to [27] (see Chapter 3) for details on this curvature.

**II- Symmetry of the extremals for  $\mu_{\gamma,s,0}(\mathbb{R}^{k+,n-k})$ .** We present the symmetry of the extremals for  $\mu_{\gamma,s,0}(\mathbb{R}^{k+,n-k})$ . The type of symmetry below has been proved in several context since the pioneer contribution of Caffarelli-Gidas-Spruck [24] (see Chern-Lin [32] and Ghoussoub-Robert [61] for Hardy-type references). For  $\gamma < \gamma_H(\mathbb{R}^{k+,n-k})$ ,  $s \in [0, 2)$ , we consider the problem  $(HS)$  on  $\mathbb{R}^{k+,n-k}$ :

$$\begin{cases} \Delta V - \frac{\gamma}{|x|^2} V = \frac{V^{2^*(s)-1}}{|x|^s} & \text{in } \mathbb{R}^{k+,n-k}, \\ V \geq 0 & \text{in } \mathbb{R}^{k+,n-k}, \\ V = 0 & \text{on } \partial\mathbb{R}^{k+,n-k}. \end{cases} \quad (2.12)$$

We have the following theorem:

**Theorem 2.2** (Cheikh Ali [27], see Chapter 3). *For  $\gamma \geq 0$  and if  $V$  is a solution of the equation (2.12) in  $C^2(\mathbb{R}^{k+,n-k}) \cap C(\overline{\mathbb{R}^{k+,n-k}} \setminus \{0\})$  for some  $k \in \{1, \dots, n\}$ , then  $V \circ \sigma = V$  for all isometries of  $\mathbb{R}^n$  such that  $\sigma(\mathbb{R}^{k+,n-k}) = \mathbb{R}^{k+,n-k}$ . In particular:*

- *There exists  $w \in C^\infty(]0, \infty[^k \times \mathbb{R}^{n-k})$  such that for all  $x_1, \dots, x_k > 0$  and for any  $x' \in \mathbb{R}^{n-k}$ , we get that*

$$V(x_1, \dots, x_k, x') = w(x_1, \dots, x_k, |x'|).$$

- *$V$  is a symmetric function of  $k$  variables: for all permutation  $s$  of the set of indices  $\{1, \dots, k\}$ , we have*

$$V(x_1, \dots, x_k, x_{k+1}, \dots, x_n) = V(x_{s(1)}, \dots, x_{s(k)}, x_{k+1}, \dots, x_n).$$

Initially, our intention was to follow the proof of Chen-Lin [32]. However, the singular boundary prevented us from using the classical strong comparison principle. We finally produce a robust proof by using the method of Berestycki-Nirenberg [20] which did not require smoothness of the boundary.

We are then in position to follow the strategy we introduced above. Let  $V > 0$  be an extremal for  $\mu_{\gamma,s,0}(\mathbb{R}^{k+,n-k})$  (when it exists). For  $\epsilon > 0$ , define the test function

$$V_\epsilon(x) := \left( \eta \epsilon^{-\frac{n-2}{2}} V(\epsilon^{-1} \cdot) \right) \circ \phi^{-1}(x) \quad (2.13)$$

where the chart  $\phi$  is as in Definition 2.1 and  $\eta$  is an adapted cutoff function. Due to the regularity by Felli-Ferrero [50], we get a precise behavior of  $V$  at infinity, and therefore precise asymptotics for  $V_\epsilon$  for  $\epsilon \rightarrow 0$ . We get the following:

**Proposition 2.2** (Cheikh Ali [27], see Chapter 3). *Let  $0 \leq \gamma < \gamma_H(\mathbb{R}^{k+,n-k})$ , and assume that there are extremals for  $\mu_{\gamma,s,0}(\mathbb{R}^{k+,n-k})$ . Then there exists positive constants  $c_{\gamma,s}^\beta$  where  $\beta = 1, \dots, 3$  and for all  $m = 1, \dots, k$  such that:*

1. *For  $\gamma < \gamma_H(\mathbb{R}^{k+,n-k}) - \frac{1}{4}$  (that is  $n > n_{\gamma,k}$ ), we have that*

$$J_{\gamma,s,0}^\Omega(V_\epsilon) = \mu_{\gamma,s,0}(\mathbb{R}^{k+,n-k}) (1 + GH_{\gamma,s}(\Omega)\epsilon + o(\epsilon)).$$

2. *For  $\gamma = \gamma_H(\mathbb{R}^{k+,n-k}) - \frac{1}{4}$  (that is  $n = n_{\gamma,k}$ ), we have that*

$$J_{\gamma,s,0}^\Omega(V_\epsilon) = \mu_{\gamma,s,0}(\mathbb{R}^{k+,n-k}) \left( 1 + GH_{\gamma,s}(\Omega)\epsilon \ln \left( \frac{1}{\epsilon} \right) + o \left( \epsilon \ln \left( \frac{1}{\epsilon} \right) \right) \right).$$

with  $GH_{\gamma,s}(\Omega)$  as in (2.11).

These expressions depends only on the local geometry of the domain. This is possible since, in the expansions, we observe a phenomenon of localization in large dimensions. Such a phenomenon has already been observed in the geometric context of the Yamabe problem (see Aubin [2]) and in nonlinear euclidean PDEs (see Brezis-Nirenberg [19]). This localization is possible due to the choice of the large dimension  $n \geq n_{\gamma,k}$ ; corresponding to the case  $n \geq 4$  for the Brezis-Nirenberg problem.

### Part 1.2: Small dimension phenomenon and existence of extremals

**When  $\gamma$  is large (small dimension).** When  $n < n_{\gamma,k}$ , it is known since the pioneering work of Schoen [100] that global arguments are required and that we need a notion of mass.

**Definition 2.2 (Mass).** *Let  $\Omega$  be a bounded domain in  $\mathbb{R}^n$ ,  $n \geq 3$ , such that  $0 \in \partial\Omega$  is a singularity of type  $(k, n - k)$  for some  $k \in \{1, \dots, n\}$ . We fix*

$\gamma < \gamma_H(\Omega)$  and  $a \in C^{0,\theta}(\Omega)$  ( $\theta \in (0, 1)$ ). We say that a coercive operator  $\Delta - (\gamma|x|^{-2} + a)$  has a mass if there exists  $G \in C^2(\Omega) \cap H_{1,0,loc}^2(\Omega)$  such that

$$\begin{cases} \Delta G - \left(\frac{\gamma}{|x|^2} + a(x)\right) G = 0 & \text{in } \Omega, \\ G > 0 & \text{in } \Omega, \\ G = 0 & \text{on } \partial\Omega \setminus \{0\}, \end{cases}$$

and there exists  $c \in \mathbb{R}$  such that

$$G(x) = \prod_{i=1}^k d(x, \partial\Omega_i) (|x|^{-\alpha_+ - k} + c|x|^{-\alpha_- - k} + o(|x|^{-\alpha_- - k})) \text{ as } x \rightarrow 0,$$

where  $\alpha_{\pm}$  is defined in (2.10). We define  $m_{\gamma,a}(\Omega) := c$  as the boundary mass of the operator  $\Delta - (\gamma|x|^{-2} + a)$ .

We remark that the function  $G$  is unique, so that the definition of the mass makes sense. In [28], we give several situations for which the mass is defined. For  $\gamma_H(\mathbb{R}^{k_+,n-k}) - \frac{1}{4} < \gamma < \gamma_H(\mathbb{R}^{k_+,n-k})$ , that is  $n < n_{\gamma}$ , we construct global profiles:

$$W_{\epsilon}(x) := V_{\epsilon}(x) + \epsilon^{\frac{\alpha_+ - \alpha_-}{2}} \Theta(x), \quad (2.14)$$

where  $V_{\epsilon}$  is defined in (2.13), and  $\Theta \in H_{1,0}^2(\Omega)$  is such that

$$\Theta(x) = m_{\gamma,a}(\Omega) \prod_{i=1}^k d(x, \partial\Omega_i) |x|^{-\alpha_- - k} + o\left(\prod_{i=1}^k d(x, \partial\Omega_i) |x|^{-\alpha_- - k}\right) \text{ as } x \rightarrow 0.$$

**Proposition 2.3** (Cheikh Ali [28], see Chapter 4). *Let  $\Omega$  be a bounded domain in  $\mathbb{R}^n$ ,  $n \geq 3$ , such that  $0 \in \partial\Omega$  is a singularity of type  $(k, n - k)$  for some  $k \in \{1, \dots, n\}$ . We fix  $0 \leq s < 2$ ,  $\gamma < \gamma_H(\Omega)$  and  $a \in C^{0,\theta}(\Omega)$  ( $\theta \in (0, 1)$ ). Assume that there are extremals for  $\mu_{\gamma,s,0}(\mathbb{R}^{k_+,n-k})$ . We assume that*

$$\gamma > \gamma_H(\mathbb{R}^{k_+,n-k}) - \frac{1}{4} \text{ (that is } n < n_{\gamma,k}),$$

and that the operator  $\Delta - (\gamma|x|^{-2} + a(x))$  is coercive with a mass  $m_{\gamma,a}(\Omega)$ . We let  $(W_{\epsilon})_{\epsilon} \in H_{1,0}^2(\Omega)$  be as in (2.14). Then, there exists an explicit constant  $\zeta_{\gamma,s}^0 > 0$  such that

$$J_{\gamma,s,a}^{\Omega}(W_{\epsilon}) = \mu_{\gamma,s,0}(\mathbb{R}^{k_+,n-k}) \left(1 - \zeta_{\gamma,s}^0 m_{\gamma,a}(\Omega) \epsilon^{\alpha_+ - \alpha_-} + o(\epsilon^{\alpha_+ - \alpha_-})\right),$$

as  $\epsilon \rightarrow 0$ .

Putting together Propositions 2.1, 2.2 and 2.3, we get the following answer to (Q2) and (Q3):

**Theorem 2.3** (Cheikh-Ali [27, 28], see Chapters 3,4). *Let  $\Omega$  be a bounded domain in  $\mathbb{R}^n$ ,  $n \geq 3$  such that  $0 \in \partial\Omega$  is a singularity of type  $(k, n - k)$  for some  $k \in \{1, \dots, n\}$ . We fix  $0 \leq s < 2$  and  $0 \leq \gamma < \gamma_H(\Omega)$ . In addition, suppose that either  $\{s > 0\}$  or  $\{s = 0, \gamma > 0 \text{ and } n \geq 4\}$ . We assume that:*

- $GH_{\gamma,s}(\Omega) < 0$  if  $n \geq n_{\gamma,k}$ ,
- the mass exists and  $m_{\gamma,0}(\Omega) > 0$  if  $n < n_{\gamma,k}$ .

*Then there are extremals for  $\mu_{\gamma,s,0}(\Omega)$ . Moreover, up to dilation, the extremals are positive weak solutions to (HS) for  $a \equiv 0$ .*

The remaining case  $s = 0, \gamma > 0$  and  $n = 3$  is a bit different. Indeed, in this case, we do not know whether there are extremals or not for  $\mu_{\gamma,s,0}(\mathbb{R}^{k+,n-k})$ . If not, we introduce a mass in a more classical spirit. This situation is developed extensively in [27, 28].

### Part 1.3: A perturbed Hardy-Sobolev equation

We briefly discuss the perturbed equation. We let  $a, h \in L^\infty(\Omega)$  and  $1 < q < 2^* - 1 = \frac{n+2}{n-2}$  be additional parameters. We investigate the existence of solutions  $u \in C^2(\Omega) \cap H_{1,0}^2(\Omega)$  to the perturbed equation

$$\begin{cases} \Delta u - \left(a(x) + \frac{\gamma}{|x|^2}\right) u = \frac{u^{2^*(s)-1}}{|x|^s} + h(x)u^{q-1} & \text{in } \Omega, \\ u > 0 & \text{a.e. in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \quad (PHS)$$

Such solutions are critical points for the functional  $E_q : H_{1,0}^2(\Omega) \rightarrow \mathbb{R}$ :

$$E_q(u) := \frac{1}{2} \int_{\Omega} (|\nabla u|^2 + au^2) dx - \frac{1}{2^*(s)} \int_{\Omega} \frac{u_+^{2^*(s)}}{|x|^s} dx - \frac{1}{q+1} \int_{\Omega} hu_+^{q+1} dx$$

for all  $u \in H_{1,0}^2(\Omega)$ . Our main tool is the Mountain-Pass Lemma by Ambrosetti-Rabinowitz [5] to produce critical points of  $E_q$ .

**Theorem 2.4** (Cheikh-Ali [28], see Chapter 4). *Let  $\Omega$  be a bounded domain in  $\mathbb{R}^n$ ,  $n \geq 3$ , such that  $0 \in \partial\Omega$  is a singularity of type  $(k, n - k)$  for some  $k \in \{1, \dots, n\}$ . We fix  $\gamma < \gamma_H(\mathbb{R}^{k+,n-k})$ ,  $a \in C^{0,\theta}(\Omega)$  such that  $\Delta - (\gamma|x|^{-2} + a(x))$  is coercive, and  $h \in C^{0,\theta}(\Omega)$  such that  $h \geq 0$  and let  $0 \leq s < 2$  and  $1 < q < 2^* - 1$ . Assume that there exists  $u_0 \in H_{1,0}^2(\Omega)$ ,  $u_0 \not\equiv 0$ , such that*

$$\sup_{t \geq 0} E_q(tu_0) < \frac{2-s}{2(n-s)} \mu_{\gamma,s,0}(\mathbb{R}^{k+,n-k})^{\frac{n-s}{2-s}},$$

then equation (PHS) has a non-vanishing solution in  $H_{1,0}^2(\Omega)$  of Mountain-Pass type.

Therefore, finding solutions to (PHS) reduces to the question:

**(Q4)** When do we have that  $\sup_{t \geq 0} E_q(tu_0) < \frac{2-s}{2(n-s)} \mu_{\gamma,s,0}(\mathbb{R}^{k+,n-k})^{\frac{n-s}{2-s}}$ ?

We answer **(Q4)** by taking for  $u_0$  either  $V_\epsilon$  (see (2.13)) when  $n \geq n_{\gamma,k}$ , or  $W_\epsilon$  (see (2.14)) when  $n < n_{\gamma,k}$ . We choose to present only the case  $s > 0$ : the case  $s = 0$  is detailed in [28] (see Chapter 4):

**Theorem 2.5** (Cheikh-Ali [28], see Chapter 4). *Let  $\Omega$  be a bounded domain in  $\mathbb{R}^n$ ,  $n \geq 3$ , such that  $0 \in \partial\Omega$  is a singularity of type  $(k, n-k)$  for some  $k \in \{1, \dots, n\}$ . Let  $a, h \in C^{0,\theta}(\Omega)$  ( $\theta \in (0, 1)$ ) be such that  $\Delta - (\gamma|x|^{-2} + a)$  is coercive and  $h \geq 0$ . Consider  $0 < s < 2$  and  $0 \leq \gamma < \gamma_H(\mathbb{R}^{k+,n-k})$ . We fix  $q \in (1, 2^* - 1)$ . Then, there exists a positive Mountain-Pass solution  $u \in H_{1,0}^2(\Omega)$  to the perturbed Hardy-Schrödinger equation (PHS) under one of the following conditions:*

•  $n > n_{\gamma,k}$  and

$$\begin{cases} GH_{\gamma,s}(\Omega) < 0 & \text{if } q+1 < \frac{2n-2}{n-2}, \\ c_1 GH_{\gamma,s}(\Omega) - c_2 h(0) < 0 & \text{if } q+1 = \frac{2n-2}{n-2}, \\ h(0) > 0 & \text{if } q+1 > \frac{2n-2}{n-2}, \end{cases}$$

•  $n = n_{\gamma,k}$  and

$$\begin{cases} GH_{\gamma,s}(\Omega) < 0 & \text{if } q+1 \leq \frac{2n-2}{n-2}, \\ h(0) > 0 & \text{if } q+1 > \frac{2n-2}{n-2}, \end{cases}$$

•  $n < n_{\gamma,k}$  and

$$\begin{cases} m_{\gamma,a}(\Omega) > 0 & \text{if } q+1 < \frac{2n-2(\alpha_+-\alpha_-)}{n-2}, \\ c_3 m_{\gamma,a}(\Omega) + c_2 h(0) > 0 & \text{if } q+1 = \frac{2n-2(\alpha_+-\alpha_-)}{n-2}, \\ h(0) > 0 & \text{if } q+1 > \frac{2n-2(\alpha_+-\alpha_-)}{n-2}, \end{cases}$$

where  $c_1, c_2, c_3 > 0$  are explicit constants (see Chapter 4).

This result shows how the subcritical nonlinearity has an impact on the existence of solutions. When the subcritical nonlinearity is close to being linear, only the geometry of  $\Omega$  commands the existence. Conversely, when it is close to being critical, the subcritical nonlinearity commands the existence, whatever the geometry is.

## Part 2: Asymptotics for elliptic Hardy-Sobolev equations on manifolds and Best Constants

Let  $(M, g)$  be a compact Riemannian manifold of dimension  $n \geq 3$  with  $\partial M = \emptyset$ . We fix  $x_0 \in M$  and  $s \in [0, 2)$ . We are now dealing with equations like (2.7) with  $\gamma = 0$  and  $h \equiv 0$ . In Part 1, we were mostly interested in extremals for Hardy-Sobolev inequalities in relation with the best constant of the embedding. In the present setting, we are also dealing with the existence/non-existence of extremals, but with a focus on the associated second best constant.

Interpolating the Sobolev and Hardy inequalities, we get the Hardy-Sobolev inequality that writes as follows: there exists  $A, B > 0$  such that

$$\left( \int_M \frac{|u|^{2^*(s)}}{d_g(x, x_0)^s} dv_g \right)^{\frac{2}{2^*(s)}} \leq A \int_M |\nabla u|_g^2 dv_g + B \int_M u^2 dv_g \quad (2.15)$$

for all  $u \in H_1^2(M)$ . When  $s = 0$ , this is the classical Sobolev inequality. Extensive discussions on the optimal values of  $A$  and  $B$  above are in the monograph Druet-Hebey [40]. It was proved by Hebey-Vaugon [70] (the classical case  $s = 0$ ) and by Jaber [75] ( $s \in (0, 2)$ ) that

$$\mu_{0,s,0}(\mathbb{R}^n)^{-1} = \inf \{ A > 0 \text{ such that } \exists B > 0 \text{ s.t. (2.15) holds for all } u \in H_1^2(M) \},$$

and the the infimum is achieved, where

$$\mu_{0,s,0}(\mathbb{R}^n) = \inf \left\{ \frac{\int_{\mathbb{R}^n} |\nabla u|^2 dX}{\left( \int_{\mathbb{R}^n} \frac{|u|^{2^*(s)}}{|X|^s} dX \right)^{\frac{2}{2^*(s)}}}, u \in C_c^\infty(\mathbb{R}^n) \right\}$$

is the best constant in the Hardy-Sobolev inequality (see Lieb [83] Theorem 4.3 for the exact value). Therefore, there exists  $B > 0$  such that

$$\left( \int_M \frac{|u|^{2^*(s)}}{d_g(x, x_0)^s} dv_g \right)^{\frac{2}{2^*(s)}} \leq \mu_{0,s,0}(\mathbb{R}^n)^{-1} \left( \int_M |\nabla u|_g^2 dv_g + B \int_M u^2 dv_g \right) \quad (2.16)$$

for all  $u \in H_1^2(M)$ . Saturating this inequality with respect to  $B$ , we define the second best constant as

$$B_s(g) := \inf \{ B > 0 \text{ such that (2.16) holds for all } u \in H_1^2(M) \},$$

to get the optimal inequality

$$\left( \int_M \frac{|u|^{2^*(s)}}{d_g(x, x_0)^s} dv_g \right)^{\frac{2}{2^*(s)}} \leq \mu_{0,s,0}(\mathbb{R}^n)^{-1} \left( \int_M |\nabla u|_g^2 dv_g + B_s(g) \int_M u^2 dv_g \right) \quad (2.17)$$



for all  $u \in H_1^2(M)$ . We say that  $u_0 \in H_1^2(M)$  is an extremal for (2.17) if  $u_0 \not\equiv 0$  and equality in (2.17) holds for  $u = u_0$ . In addition to the existence of extremals, we are interested in the value of the second best constant. When  $s = 0$ , the issue has been studied by Druet and al.:

**Theorem 2.6** (The case  $s = 0$ , Druet and al.[34, 39]). *Let  $(M, g)$  be a compact Riemannian manifold of dimension  $n \geq 3$ . Assume that  $s = 0$  and that there is no extremal for (2.17). Then*

- $B_0(g) = \frac{n-2}{4(n-1)} \max_M \text{Scal}_g$  if  $n \geq 4$ ;
- The mass of  $\Delta_g + B_0(g)$  vanishes if  $n = 3$ .

The mass will be defined in Definition 2.3. We establish the corresponding result for the singular case  $s \in (0, 2)$ :

**Theorem 2.7** (The case  $s > 0$ , Cheikh-Ali [29], see Chapter 5). *Let  $(M, g)$  be a compact Riemannian manifold of dimension  $n \geq 3$ . We fix  $x_0 \in M$  and  $s \in (0, 2)$ . We assume that there is no extremal for (2.17). Then*

- $B_s(g) = \frac{(6-s)(n-2)}{12(2n-2-s)} \text{Scal}_g(x_0)$  if  $n \geq 5$ ;
- The mass of  $\Delta_g + B_s(g)$  vanishes if  $n = 3$ .

The case  $n = 4$  is still under investigations.

Our proof relies on the blow-up analysis of critical elliptic equations in the spirit of Druet-Hebey-Robert [41]. Let  $(a_\alpha)_{\alpha \in \mathbb{N}} \in C^1(M)$  be such that

$$\lim_{\alpha \rightarrow +\infty} a_\alpha = a_\infty \text{ in } C^1(M). \quad (2.18)$$

We consider  $(\lambda_\alpha)_\alpha \in (0, +\infty)$  such that

$$\lim_{\alpha \rightarrow +\infty} \lambda_\alpha = \mu_{0,s,0}(\mathbb{R}^n).$$

We let  $(u_\alpha)_\alpha \in H_1^2(M)$  be a sequence of weak solutions to

$$\begin{cases} \Delta_g u_\alpha + a_\alpha u_\alpha = \lambda_\alpha \frac{u_\alpha^{2^*(s)-1}}{d_g(x, x_0)^s} & \text{in } M, \\ u_\alpha \geq 0 & \text{a.e. in } M. \end{cases} \quad (2.19)$$

We assume that

$$\|u_\alpha\|_{2^*(s),s} = \left( \int_M \frac{|u_\alpha|^{2^*(s)}}{d_g(x, x_0)^s} dv_g \right)^{\frac{1}{2^*(s)}} = 1,$$

and that

$$u_\alpha \rightharpoonup 0 \text{ as } \alpha \rightarrow +\infty \text{ weakly in } H_1^2(M). \quad (2.20)$$

Our main results are two descriptions of the asymptotics of  $(u_\alpha)$ . Note that regularity and the maximum principle yield  $u_\alpha \in C^0(M)$  and  $u_\alpha > 0$ . Next, we obtain strong pointwise control:

**Theorem 2.8.** [Cheikh-Ali [29], see Chapter 5] *Let  $M$  be a compact Riemannian manifold of dimension  $n \geq 3$ . We fix  $x_0 \in M$  and  $s \in (0, 2)$ . Let  $(a_\alpha)_{\alpha \in \mathbb{N}} \in C^1(M)$  and  $a_\infty \in C^1(M)$  be such that (2.18) holds and  $\Delta_g + a_\infty$  is coercive in  $M$ . We let  $(\lambda_\alpha)_\alpha \in \mathbb{R}$  and  $(u_\alpha) \in H_1^2(M)$  be such that (2.18) to (2.20) hold for all  $\alpha \in \mathbb{N}$ . Then, there exists  $C > 0$  such that,*

$$u_\alpha(x) \leq C \frac{\mu_\alpha^{\frac{n-2}{2}}}{\mu_\alpha^{n-2} + d_g(x, x_0)^{n-2}} \text{ for all } x \in M, \quad (2.21)$$

where

$$\mu_\alpha := \left( \max_M u_\alpha \right)^{-\frac{2}{n-2}} \quad (2.22)$$

converge to 0 as  $\alpha \rightarrow +\infty$ .

**Theorem 2.9.** [Cheikh-Ali [29], see Chapter 5] *Let  $M$  be a compact Riemannian manifold of dimension  $n \geq 3$ . We fix  $x_0 \in M$  and  $s \in (0, 2)$ . Let  $(a_\alpha)_{\alpha \in \mathbb{N}} \in C^1(M)$  and  $a_\infty \in C^1(M)$  be such that (2.18) holds and  $\Delta_g + a_\infty$  is coercive in  $M$ . We let  $(\lambda_\alpha)_\alpha \in \mathbb{R}$  and  $(u_\alpha)_\alpha \in H_1^2(M)$  be such that (2.18) to (2.20) hold for all  $\alpha \in \mathbb{N}$ . Then,*

1. If  $n \geq 5$ , then  $a_\infty(x_0) = c_{n,s} \text{Scal}_g(x_0)$ .
2. If  $n = 3$ , then  $m_{a_\infty}(x_0) = 0$ .

where  $m_{a_\infty}(x_0)$  is the mass of the operator  $\Delta_g + a_\infty$  (see Definition 2.3) and

$$c_{n,s} := \frac{(6-s)(n-2)}{12(2n-2-s)}. \quad (2.23)$$

The case  $n = 4$  is in progress.

### Part 2.1: About the proof of Theorem 2.8.

We establish sharp pointwise estimates for arbitrary sequences of solutions of (2.19). With this optimal pointwise control, we are able to obtain more informations on the localization of the blowup point  $x_0$  and the blowup parameter  $(\mu_\alpha)_{\alpha \in \mathbb{N}}$ . The proof of Theorem 2.8 goes through the proof of two steps below:

**Step 2.1.** We claim that there exists  $\epsilon_0 > 0$  such that for any  $\epsilon \in (0, \epsilon_0)$ , there exists  $C_\epsilon > 0$  such that

$$u_\alpha(x) \leq C_\epsilon \frac{\mu_\alpha^{\frac{n-2}{2}-\epsilon}}{d_g(x, x_0)^{n-2-\epsilon}} \text{ for all } x \in M \setminus \{x_0\}.$$

**Step 2.2.** We claim that there exists  $C > 0$  such that

$$d_g(x, x_0)^{n-2} u_\alpha(x_\alpha) u_\alpha(x) \leq C \text{ for all } x \in M. \quad (2.24)$$

To get the last step, we take inspiration in Ghoussoub-Robert [62] and Robert [98] (for more details, (see Chapter 5)). Finally, Using (2.24) and the definition of  $\mu_\alpha$  (see (2.22)), we get the expect result .

### Part 2.2: About the proof of Theorem 2.9.

Thanks to estimates in (2.21), we can prove the Theorem 2.9 when  $n \geq 3$ . When  $n = 3$ , the mass is defined as follows:

**Definition 2.3.** [The mass] Let  $(M, g)$  be a compact Riemannian manifold of dimension  $n = 3$ , and let  $h \in C^0(M)$  be such that  $\Delta_g + h$  is coercive. Let  $G_{x_0}$  be the Green's function of  $\Delta_g + h$  at  $x_0$ . Let  $\eta \in C^\infty(M)$  such that  $\eta = 1$  around  $x_0$ . Then there exists  $\beta_{x_0} \in H_1^2(M)$  such that

$$G_{x_0} = \frac{1}{4\pi} \eta d_g(\cdot, x_0)^{-1} + \beta_{x_0} \text{ in } M \setminus \{x_0\}.$$

We have that  $\beta_{x_0} \in H_2^p(M) \cap C^{0,\theta}(M) \cap C^{2,\gamma}(M \setminus \{x_0\})$  for all  $p \in (\frac{3}{2}, 3)$  and  $\theta, \gamma \in (0, 1)$ . We define the mass at  $x_0$  as  $m_h(x_0) := \beta_{x_0}(x_0)$ , which is independent of the choice of  $\eta$ .

The main difficulty in our analysis is due to non existence of the Pohozaev identity in the Riemannian context. Indeed, we must get a suitable chart that maps locally  $M$  to  $\mathbb{R}^n$ . From here, inspired by Ghoussoub-Robert [62], we take  $(u_\alpha)_\alpha \in H_1^2(M)$  is a sequence of weak solutions to (2.19). We make a change of variable with the exponential chart  $\exp_{x_0}$  centered at  $x_0$ , and we define the following function

$$\hat{u}_\alpha(X) := u_\alpha(\exp_{x_0}(X)) \text{ for } X \in B_\delta(0) \subset \mathbb{R}^n.$$

We remark that  $\hat{u}_\alpha$  also satisfies an equation like (2.19) locally on  $\mathbb{R}^n$ . Now, we inject  $\hat{u}_\alpha$  in the Pohozaev identity classical on  $\mathbb{R}^n$  (see for instance [62]). We calculate, and we get two parts. With the sharp estimate of Theorem 2.8, we are in position to get the precise asymptotic behavior of these terms. From here, the

Scalar curvature  $Scal_g(x_0)$  ( $n \geq 4$ ) and the mass  $m_{a_\infty}(x_0)$  ( $n = 3$ ) will appear. Finally, we make a comparison and get the result of this Theorem. The case  $n = 4$  is in progress. Indeed, most of the analysis has been made in Chapter 5, and the asymptotics are obtained provided Assumption (5.108) (see Chapter 5).

### Part 3: Existence of a nonconstant solution to a fourth-order equation with critical exponent (In collaboration with D.Bonheure and R.Nascimento)

In this part, we consider a domain  $\Omega \subset \mathbb{R}^n$  ( $n \geq 5$ ) that is bounded and smooth. Given  $\alpha > 0$ , we investigate the multiplicity of solutions to

$$\begin{cases} \Delta^2 u + \Delta u + \alpha u = |u|^{\frac{8}{n-4}} u, & \text{in } \Omega, \\ \partial_\nu u = \partial_\nu(\Delta u) = 0, & \text{on } \partial\Omega. \end{cases} \quad (P_\nu)$$

There are at least three solutions: the constant solutions  $u \equiv 0$  and  $\pm \alpha^{\frac{n-4}{8}}$ .

(Q5) Are there nonconstant solutions to  $(P_\nu)$ ?

Problem  $(P_\nu)$  is a generalization of the Brezis-Nirenberg-type problem

$$\begin{cases} \Delta u + \alpha u = |u|^{\frac{4}{n-2}} u & \text{in } \Omega, \\ \partial_\nu u = 0 & \text{on } \partial\Omega. \end{cases} \quad (2.25)$$

Despite the domain is smooth and there is no Hardy-type singularity (unlike in  $(HS)$ ), the difficulty of the problem and the methodology to investigate the existence of solutions are pretty much like what was developed in Part 1. In the following, we define

$$H_{2,\nu}^2(\Omega) := \{u \in H_2^2(\Omega) : \partial_\nu u = 0 \text{ on } \partial\Omega\}.$$

We say that  $u \in H_{2,\nu}^2(\Omega)$  is a weak solution to  $(P_\nu)$  if

$$\int_{\Omega} (\langle \Delta u, \Delta v \rangle + \langle \nabla u, \nabla v \rangle + uv) dx = \int_{\Omega} u^{2^{**}-1} v dx \text{ for all } v \in H_{2,\nu}^2(\Omega),$$

where  $2^{**} := \frac{2n}{n-4}$ . We investigate weak solutions to  $(P_\nu)$  as minima of the functional

$$u \mapsto J(u) = \int_{\Omega} (|\Delta u|^2 + |\nabla u|^2 + \alpha|u|^2) dx,$$

on

$$M_\Omega := \left\{ u \in H_{2,\nu}^2(\Omega) : \int_\Omega |u|^{\frac{2n}{n-4}} dx = 1 \right\}.$$

We define,

$$\Sigma_\nu(\Omega) := \inf \{ J(u) \mid u \in M_\Omega \}.$$

Again, the main difficulty here is due to the fact that  $2^{**}$  is critical from the viewpoint of the Sobolev embeddings. The embedding  $H_{2,\nu}^2(\Omega)$  is compact in  $L^p(\Omega)$  iff  $1 \leq p < 2^{**}$ . Before proceeding any further, we establish some notations and recall some known results. Denote by  $D_2^2(\mathbb{R}^n)$  the completion of  $C_c^\infty(\mathbb{R}^n)$  for the norm  $u \mapsto \|\Delta u\|_2$ . The best constant for the embedding of  $D_2^2(\mathbb{R}^n)$  into  $L^{\frac{2N}{N-4}}(\mathbb{R}^n)$  is characterized by

$$S(n) := \inf_{u \in D_2^2(\mathbb{R}^n)} \left\{ \int_{\mathbb{R}^n} |\Delta u|^2 dx : \int_{\mathbb{R}^n} |u|^{\frac{2N}{N-4}} dx = 1 \right\}.$$

When the nonlinearity in (2.25) is subcritical (namely when the exponent  $4/(n-2)$  is replaced by  $q-2$  with  $2 < q < 2^*$ ). Lin, Ni and Tagaki [80] have proved that the only positive solution to (2.25), for small  $\alpha > 0$ , is the nonzero constant solution.

In the critical case, Lin and Ni [81] raised this rigidity result as a conjecture. LIN-NI'S CONJECTURE: *For  $\alpha$  small enough, (2.25) admits only  $\alpha^{(n-2)/4}$  as a positive solution.*

This conjecture has been solved by Adimurthi-Yadava [8] in the radial case. In the general case, the situation is now fully understood and solved: we refer to Druet-Robert-Wei [44] for references and for the resolution when  $n = 3$  and  $n \geq 7$  with bounded energy.

In the subcritical case, it follows from a Morse index argument that the rigidity is broken for large  $\alpha$ . In the critical case, inspired by Brezis and Nirenberg, Wang [107], and Adimurthi and Mancini [6] proved that (2.25) admits a non-constant positive solution for every  $\alpha > \bar{\alpha} > 0$ .

We go back to the initial equation  $(P_\nu)$ . Our first result in this direction is the existence of a non-constant solution:

**Theorem 2.10.** *[Bonheure-Cheikh Ali-Nascimento [14], see Chapter 6]] Assume  $\Omega$  is an open bounded subset of  $\mathbb{R}^n$  with smooth boundary. There exists  $\bar{\alpha} = \bar{\alpha}(n, |\Omega|) > 0$  such that for  $\alpha > \bar{\alpha}$ , any least energy solution of Equation  $(P_\nu)$  is nonconstant.*

The following result is of Lin-Ni type:

**Theorem 2.11.** [Bonheure-Cheikh Ali-Nascimento [14], see Chapter 6] Assume  $\Omega$  is an open bounded subset of  $\mathbb{R}^n$  with smooth boundary. Then, there exists  $\underline{\alpha} = \underline{\alpha}(n, |\Omega|) > 0$  such that for  $0 < \alpha < \underline{\alpha}$ , the only least energy solution of Equation  $(P_\nu)$  is the constant solution  $\alpha^{\frac{n-4}{8}}$ .

### Part 3.1: About the proof of Theorem 2.10.

As in the Part 1, the best constants in the Sobolev inequality  $H_{2,\nu}^2(\Omega) \hookrightarrow L^{2^{**}}(\Omega)$  will play a big role on the existence non constant solution of  $(P_\nu)$ . We first prove that the best constant is the Sobolev inequality is the best constant for the model space  $\mathbb{R}_+^n$ :

**Lemma 2.1** (Bonheure-Cheikh Ali-Nascimento [14], see Chapter 6). Assume that  $\Omega$  is a smooth open bounded subset of  $\mathbb{R}^n$  with smooth boundary and  $n \geq 5$ . Then, for every  $\varepsilon > 0$ , there exists  $B(\varepsilon) > 0$  such that for all  $u \in H_{2,\nu}^2(\Omega)$ ,

$$\|u\|_{L^{\frac{2n}{n-4}}(\Omega)}^2 \leq \left( \frac{2^{4/n}}{S(n)} + \varepsilon \right) \|\Delta u\|_{L^2(\Omega)}^2 + B(\varepsilon) \|u\|_{H^1(\Omega)}^2.$$

Moreover,  $\Sigma_\nu(\mathbb{R}_+^n) = S(n)/2^{4/n}$  and the infimum is not achieved.

A similar result was proved for singular domains in Part 1 (see Theorem 2.1 above and Chapter 3). The following existence result is in the spirit of Aubin [2] and Theorem 2.1 of Part 1:

**Lemma 2.2.** [Bonheure-Cheikh Ali-Nascimento [14], see Chapter 6] Assume that  $\Omega$  is an open bounded subset of  $\mathbb{R}^n$  with smooth boundary and  $n \geq 5$ . If  $\Sigma_\nu(\Omega) < \Sigma_\nu(\mathbb{R}_+^n)$ , then the infimum in  $\Sigma_\nu(\Omega)$  is achieved.

It remains to estimate  $J$  at relevant test functions:

1. For  $n \geq 5$ , minimizers for  $S(n)$  are given by the one-parameter family

$$x \mapsto u_\varepsilon(x) := \gamma_n \frac{\varepsilon^{\frac{n-4}{2}}}{(\varepsilon^2 + |x|^2)^{\frac{n-4}{2}}}; \gamma_n := [(n-4)(n-2)n(n+2)]^{\frac{n-4}{8}}.$$

2. We fix  $p_0 \in \partial\Omega$  and we define the test function

$$\psi_\varepsilon(x) := (\eta u_\varepsilon(|\cdot|)) \circ \Phi^{-1}(x) \in H_\nu^2(\Omega).$$

where  $\Phi$  is the suitable chart at  $p_0 \in \partial\Omega$  to get  $\psi_\varepsilon \in H_{2,\nu}^2(\Omega)$  (for more details (see Chapter 6)),  $\eta$  is a radial cut-off function.

3. We estimate  $J(\psi_\epsilon)$  and we derive an expression that depends on the mean curvature  $H(p_0)$  for all  $n \geq 5$ , and there exists  $C_n > 0$  such that

$$J(\psi_\epsilon) = \begin{cases} \frac{S(n)}{2^{4/n}} - C_n 2^{1-4/n} S(n)^{1-n/4} H(p_0) \epsilon + o(\epsilon) & \text{if } n \geq 6, \\ \frac{S(n)}{2^{4/5}} - 2^{14/5} \pi^2 \sqrt[4]{\frac{105}{S(n)}} H(p_0) \epsilon \log \frac{1}{\epsilon} + O(\epsilon) & \text{if } n = 5. \end{cases}$$

From here, the positivity of the mean curvature at some point of  $\partial\Omega$  and the expansion of  $J(\psi_\epsilon)$  yield the necessary condition  $\Sigma_\nu(\Omega) < \Sigma_\nu(\mathbb{R}_+^n)$ . Finally, with the Lemma 2.2 we get Theorem 2.10.

### Part 3.2: About the proof of Theorem 2.11

This proof is in the spirit of Ni-Takagi [91]. As for the Lin-Ni original problem, a pointwise control is decisive to get the uniqueness for  $\alpha \rightarrow 0$ . We prove such a control for minimizing solutions to  $(P_\nu)$  when  $\alpha > 0$  is small enough:

**Lemma 2.3** (Bonheure-Cheikh Ali-Nascimento [14], see Chapter 6). *Assume that  $u \in M_\Omega$  achieves  $\Sigma_\nu(\Omega)$  and  $\alpha \leq 1/4$ . Then  $u > 0$ . If we select  $v$  as the multiple of  $u$  that solves*

$$\begin{cases} \Delta^2 v + \Delta v + \alpha v = |v|^{\frac{8}{N-4}} v, & \text{in } \Omega, \\ \partial_\nu v = \partial_\nu(\Delta v) = 0, & \text{on } \partial\Omega, \end{cases}$$

*then there exists  $C_0 > 0$  depending only on  $\Omega$  such that  $\|v\|_\infty \leq C_0$ .*

With such a control, we can prove that the  $L^\infty$ -norm of minimizing solutions to  $(P_\nu)$  is going to uniformly 0 as  $\alpha \rightarrow 0$ . It is standard to get that these solutions are constant by using a Poincaré inequality. This yields Theorem 2.11.



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## Travaux de l'Auteur:

1. H. Cheikh Ali, Hardy–Sobolev inequalities with singularities on non smooth boundary: Hardy constant and extremals. Part I: Influence of local geometry, *Nonlinear Anal.* 182 (2019), 316–349.
2. H. Cheikh Ali, Hardy-Sobolev inequalities with singularities on non smooth boundary: Hardy constant and extremals. Part 2: small dimensions and the global mass (2019). Submitted.
3. D. Bonheure, H. Cheikh Ali and R. Nascimento, A Paneitz-Branson type equation with Neumann boundary conditions (2019). Accepted (*Advances in Calculus of Variations*).
4. H. Cheikh Ali, The best constant for the Hardy-Sobolev inequality on manifolds (2019).





# **Part I**



## Hardy-Sobolev inequalities with singularities on non smooth boundary: Hardy constant and extremals. Part I: Influence of local geometry

### Abstract

Let  $\Omega$  be a domain of  $\mathbb{R}^n$ ,  $n \geq 3$ . The classical Caffarelli-Kohn-Nirenberg inequality rewrites as the following inequality: for any  $s \in [0, 2]$  and any  $\gamma < \frac{(n-2)^2}{4}$ , there exists a constant  $K(\Omega, \gamma, s) > 0$  such that

$$\left( \int_{\Omega} \frac{|u|^{2^*(s)}}{|x|^s} dx \right)^{\frac{2}{2^*(s)}} \leq K(\Omega, \gamma, s) \int_{\Omega} \left( |\nabla u|^2 - \gamma \frac{u^2}{|x|^2} \right) dx, \quad (HS)$$

for all  $u \in D^{1,2}(\Omega)$  (the completion of  $C_c^\infty(\Omega)$  for the relevant norm). When  $0 \in \Omega$  is an interior point, the range  $(-\infty, \frac{(n-2)^2}{4})$  for  $\gamma$  cannot be improved: moreover, the optimal constant  $K(\Omega, \gamma, s)$  is independent of  $\Omega$  and there is no extremal for  $(HS)$ . But when  $0 \in \partial\Omega$ , the situation turns out to be drastically different since the geometry of the domain impacts:

- the range of  $\gamma$ 's for which  $(HS)$  holds;
- the value of the optimal constant  $K(\Omega, \gamma, s)$ ;
- the existence of extremals for  $(HS)$ .

When  $\Omega$  is smooth, the problem was tackled by Ghoussoub-Robert [61] where the role of the mean curvature was central. In the present paper, we consider a nonsmooth domain with a singularity at 0 modeled on a cone. We show how the local geometry induced by the cone around the singularity influences the value of the Hardy constant on  $\Omega$ . When  $\gamma$  is small, we introduce a new geometric object at the conical singularity that generalizes the "mean curvature": this allows to get extremals for  $(HS)$ .

The case of larger values for  $\gamma$  will be dealt in the forthcoming paper [28]. As an intermediate result, we prove the symmetry of some solutions to singular pdes that has an interest on its own.

### 3.1 Introduction

Let  $\Omega$  be a domain of  $\mathbb{R}^n$ ,  $n \geq 3$ ,  $s \in [0, 2)$  and let us consider the following problem:

$$\begin{cases} -\Delta u - \frac{\gamma}{|x|^2} u = \frac{u^{2^*(s)-1}}{|x|^s} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (3.1)$$

where  $\gamma \in \mathbb{R}$ ,  $2^*(s) := \frac{2(n-s)}{n-2}$  is the critical Hardy-Sobolev exponent and  $\Delta$  is the Euclidean Laplacian that is  $\Delta = \operatorname{div}(\nabla)$ . This equation makes sense for  $u \in D^{1,2}(\Omega)$ , that is the completion of  $C_c^\infty(\Omega)$  with respect to the norm  $u \mapsto \|\nabla u\|_2$ . The motivation for considering equation (3.1) arises from the problem of existence of extremals for the Caffarelli-Kohn-Nirenberg (CKN) inequalities [25]. The Caffarelli-Kohn-Nirenberg inequalities are equivalent to the Hardy-Sobolev inequality (see [61]): for any  $\gamma < \frac{(n-2)^2}{4}$  and  $s \in [0, 2]$ , there exists  $K > 0$  such that

$$\left( \int_{\Omega} \frac{|u|^{2^*(s)}}{|x|^s} dx \right)^{\frac{2}{2^*(s)}} \leq K \int_{\Omega} \left( |\nabla u|^2 - \gamma \frac{u^2}{|x|^2} \right) dx, \quad (3.2)$$

for all  $u \in D^{1,2}(\Omega)$ . More generally, for any  $0 \leq s \leq 2$  and any  $\gamma \in \mathbb{R}$ , we define

$$J_{\gamma,s}^{\Omega}(u) := \frac{\int_{\Omega} |\nabla u|^2 dx - \gamma \int_{\Omega} \frac{u^2}{|x|^2} dx}{\left( \int_{\Omega} \frac{|u|^{2^*(s)}}{|x|^s} dx \right)^{\frac{2}{2^*(s)}}},$$

for  $u \in D^{1,2}(\Omega) \setminus \{0\}$ , and we define

$$\mu_{\gamma,s}(\Omega) = \inf_{u \in D^{1,2}(\Omega) \setminus \{0\}} J_{\gamma,s}^{\Omega}(u). \quad (3.3)$$

If  $u \in D^{1,2}(\Omega) \setminus \{0\}$  achieves the infimum  $\mu_{\gamma,s}(\Omega)$ , and if  $\mu_{\gamma,s}(\Omega) > 0$ , then, up to a constant,  $u$  is a solution to (3.1). We address the following questions:

**(Q1)** For which values of  $\gamma \in \mathbb{R}$  does (3.2) hold for some  $K > 0$  and all  $u \in D^{1,2}(\Omega)$ ? In other words, when do we have  $\mu_{\gamma,s}(\Omega) > 0$ ?

**(Q2)** Is the best constant achieved? In other words, is  $\mu_{\gamma,s}(\Omega)$  achieved by some  $u \in D^{1,2}(\Omega)$ ,  $u \not\equiv 0$ ?

The answer to the first question **(Q1)** depends on the Hardy constant. Define

$$\gamma_H(\Omega) := \mu_{0,2}(\Omega) = \inf \left\{ \frac{\int_{\Omega} |\nabla u|^2 dx}{\int_{\Omega} \frac{u^2}{|x|^2} dx}; u \in C_c^\infty(\Omega) \setminus \{0\} \right\}. \quad (3.4)$$

The classical Hardy inequality reads  $\gamma_H(\mathbb{R}^n) = \frac{(n-2)^2}{4}$  and therefore, we have that  $\gamma_H(\Omega) \geq \frac{(n-2)^2}{4}$ . As a consequence, interpolating the Hardy inequality (3.4) and Sobolev inequality ((3.2) with  $\gamma = s = 0$ ), we get that

$$\gamma < \gamma_H(\Omega) \Rightarrow \mu_{\gamma,s}(\Omega) > 0.$$

When  $0 \in \Omega$  is an interior point, it is classical that  $\gamma_H(\Omega) = \gamma_H(\mathbb{R}^n) = \frac{(n-2)^2}{4}$ . We consider the case  $0 \in \partial\Omega$ . The study of this type of nonlinear singular problems when  $0 \in \partial\Omega$  was initiated by Ghoussoub-Kang [55] and studied by Chern-Lin [32] and Ghoussoub-Robert [61] when  $\Omega$  is a smooth domain.

In this work, we tackle the more intricate case of a non smooth domain. We restrict ourselves to domains modeled locally on  $\mathbb{R}_+^k \times \mathbb{R}^{n-k}$  for all  $k \in \{1, \dots, n\}$ . We define the *model cone* at  $P \in \bar{\Omega}$  as

$$C_P(\Omega) := \left\{ \lim_{t \rightarrow 0} \frac{1}{t} \overrightarrow{PM_t} / t \mapsto M_t \text{ is a curve of } \Omega \text{ and the limit exists} \right\}.$$

When  $\Omega$  is smooth,  $C_{x_0}(\Omega) = \mathbb{R}^n$  if  $x_0 \in \Omega$ . Still in the smooth case,  $C_{x_0}(\Omega)$  is a half-space bounded by the tangent space at  $x_0$  if  $x_0 \in \partial\Omega$ . Moreover, when  $x_0 \in \partial\Omega$ , then  $\partial C_{x_0}(\Omega)$  is exactly the tangent space at  $x_0$ .

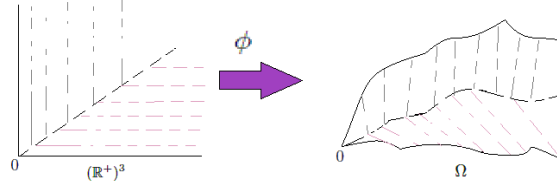
**Definition 3.1.1.** We fix  $1 \leq k \leq n$ . Let  $\Omega$  be a domain of  $\mathbb{R}^n$ . We say that  $x_0 \in \partial\Omega$  is a *singularity of type  $(k, n-k)$*  if there exist  $U, V$  open subsets of  $\mathbb{R}^n$  such that  $0 \in U$ ,  $0 \in V$  and there exists  $\phi \in C^\infty(U, V)$  a diffeomorphism such that  $\phi(0) = x_0$  and

$$\phi(U \cap (\mathbb{R}_+^k \times \mathbb{R}^{n-k})) = \phi(U) \cap \Omega \text{ and } \phi(U \cap \partial(\mathbb{R}_+^k \times \mathbb{R}^{n-k})) = \phi(U) \cap \partial\Omega,$$

with the additional hypothesis that the differential at 0  $d\phi_0$  is an isometry.

As one checks, we have that  $C_0(\Omega) = d\phi_0(\mathbb{R}_+^k \times \mathbb{R}^{n-k})$ , and then  $C_0(\Omega)$  is isometric to  $\mathbb{R}_+^k \times \mathbb{R}^{n-k}$ . In the sequel, we write for convenience

$$\mathbb{R}^{k+,n-k} := \mathbb{R}_+^k \times \mathbb{R}^{n-k} \text{ for all } k \in \{1, \dots, n\}.$$

Figure 3.1: Case:  $k = 3, n - k = 0$ .

For example: When  $\Omega$  is smooth, boundary points are all of type  $(1, n - 1)$ . A general conical singularity is as in Figure 1. We assume that  $0$  is a singularity of type  $(k, n - k)$ . We write the cone as  $C_0(\Omega) = \{r\sigma; r > 0, \sigma \in D\}$  having  $0$  as a vertex included in  $\mathbb{R}^n$ , where  $D$  is the trace of the cone on the sphere  $S^{n-1}$ . More generally, given  $D \subset S^n$ , define the cone  $C := \{r\sigma; r > 0, \sigma \in D\}$ . Then we have that

- If  $D$  is the sphere  $S^n$ , then  $C = \mathbb{R}^n \setminus \{0\}$ .
- If  $D$  is the half-sphere  $S_+^n$ , then  $C$  is the half-space  $\mathbb{R}^{1+, n-1} := \mathbb{R}_+^n$ .
- If  $D = S^{n-1} \cap \mathbb{R}^{k+, n-k}$ , then  $C = \mathbb{R}^{k+, n-k}$  for all  $k \in \{1, \dots, n\}$ .

For such cones, see Ghoussoub-Moradifam [57], the Hardy constant is

$$\gamma_H(C) = \frac{(n-2)^2}{4} + \lambda_1(D),$$

such that  $\lambda_1(D)$  is the first eigenvalue of the Laplacian on  $D \subset S^{n-1}$  with Dirichlet boundary condition. In particular,  $\gamma_H(\mathbb{R}^{k+, n-k}) = \frac{(n+2k-2)^2}{4}$  where  $\lambda_1(D) = k(n+k-2)$  for all  $k \in \{1, \dots, n\}$ . The model cone is the relevant object to consider to understand the Hardy constant of  $\Omega$ :

**Proposition 3.1.1.** *Let  $\Omega$  be a bounded domain of  $\mathbb{R}^n$ . We assume that  $0 \in \partial\Omega$  is a singularity of type  $(k, n - k)$  for some  $k \in \{1, \dots, n\}$ . Then  $\gamma_H$  satisfies the following properties:*

- $\frac{(n-2)^2}{4} < \gamma_H(\Omega) \leq \gamma_H(C_0(\Omega))$ .
- $\gamma_H(\Omega) = \gamma_H(C_0(\Omega))$  for every  $\Omega$  such that  $0 \in \partial\Omega$  and  $\Omega \subset C_0(\Omega)$ .
- If  $\gamma_H(\Omega) < \gamma_H(C_0(\Omega))$ , then it is attained in  $D^{1,2}(\Omega)$ .
- For every  $\epsilon > 0$ , there exists  $\mathbb{R}^{k+, n-k} \subsetneq \Omega_\epsilon \subsetneq \mathbb{R}^n$  with a boundary singularity at  $0$  of type  $(k, n - k)$  such that  $\gamma_H(\mathbb{R}^{k+, n-k}) - \epsilon \leq \gamma_H(\Omega_\epsilon) \leq \gamma_H(\mathbb{R}^{k+, n-k})$ .

The study of the Hardy constant for itself is reminiscent in the literature. Without being exhaustive, we refer to Fall [48], Fall-Musina [49] and the references therein.

We now tackle the second question **(Q2)**, that is the existence of extremals for (3.3). In this framework, the following result is classical:

**Theorem 3.1.1.** *Let  $\Omega \subset \mathbb{R}^n$  be a bounded domain such that  $0 \in \partial\Omega$  is a singularity of type  $(k, n - k)$ . Assume that  $\gamma < \gamma_H(\mathbb{R}^{k+, n-k})$ ,  $0 \leq s \leq 2$ , and  $\mu_{\gamma, s}(\Omega) < \mu_{\gamma, s}(\mathbb{R}^{k+, n-k})$ . Then there are extremals for  $\mu_{\gamma, s}(\Omega)$ . In particular, there exists a minimizer  $u$  in  $D^{1,2}(\Omega) \setminus \{0\}$  that is a positive solution to the equation*

$$(E) \begin{cases} -\Delta u - \frac{\gamma}{|x|^2} u = \mu_{\gamma, s}(\Omega) \frac{u^{2^*(s)-1}}{|x|^s} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

In other words, being below a critical threshold given by the model cone yields existence of extremals. Such a result is reminiscent in the functional inequalities of elliptic type since the work of Trudinger [105] and Aubin [3] on the Yamabe problem. Related results for Hardy-Sobolev equations are in Bartsch-Peng-Zhang [23] and Pucci-Servadei [95].

We now give sufficient conditions to get the existence condition. As for the Yamabe problem, we need to introduce some test-functions cooked up from a model space: here, it is the model cone. In the smooth case, that is  $k = 1$ , the test-functions yield a condition on the mean curvature to recover existence. In our non-smooth context, we must tackle two additional difficulties:

- The mean curvature is not defined, and we must define another geometric quantity.
- The extremals for the model space  $\mathbb{R}^{k+, n-k}$  are not smooth, and the proof of the symmetry in [61] does not extend to our context.

We are able to recover symmetry via a version of the moving-plane method developed by Berestycki and Nirenberg [19]. Concerning the lack of mean curvature, we introduce a new geometric object.

**Definition 3.1.2.** *Let  $\Omega \subset \mathbb{R}^n$  be a domain such that  $0 \in \partial\Omega$  is a singularity of type  $(k, n - k)$ . We define*

$$\Omega_i := \phi(U \cap \{x_i > 0\}) \text{ for all } i = 1, \dots, k, \quad (3.5)$$

where  $(\phi, U)$  is a chart as in the Definition 3.1.1. We have that:



1. For all  $i = 1, \dots, k$ ,  $\Omega_i$  is smooth around  $0 \in \partial\Omega_i$ .
2. Up to permutation, the  $\Omega_i$ 's are locally independent of the chart  $\phi$ .
3. The  $\Omega_i$ 's define locally  $\Omega$ : there exists  $\delta > 0$  such that

$$\Omega \cap B_\delta(0) = \bigcap_{i=1}^k \Omega_i \cap B_\delta(0).$$

For example:

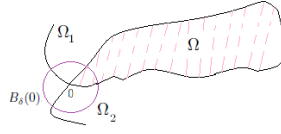


Figure 3.2: Case  $k = 2$ ,  $n - k = 0$ .

**Definition 3.1.3.** Let  $S$  be a submanifold of  $\mathbb{R}^n$ . We let  $II_{x_0}^S$  be the second fundamental form at  $x_0$  of  $S$ , that is

$$\begin{cases} II_{x_0}^S : T_{x_0}S \times T_{x_0}S \times (T_{x_0}S)^\perp & \rightarrow & \mathbb{R} \\ & (X, Y, \eta) & \mapsto & II^S(X, Y, \eta) = \langle \overline{\nabla}_X Y - \nabla_X Y, \eta \rangle_{x_0}. \end{cases}$$

The mean curvature vector at  $x_0 \in S$  is the vector  $\vec{H}_{x_0}^S \in (T_{x_0}S)^\perp$  such that for all  $\eta \in (T_{x_0}S)^\perp$ , we have that

$$\langle \vec{H}_{x_0}^S, \eta \rangle_{x_0} = \text{Trace} \left( (X, Y) \mapsto II_{x_0}^S(X, Y, \eta) \right).$$

For  $k \in \{1, \dots, n\}$  and  $m = 1, \dots, k$ , we define  $\vec{\nu}_m : \partial\Omega_m \rightarrow \mathbb{R}^n$  is the outer unit normal vector of the locally oriented  $\Omega_m$  around 0 where  $\Omega_m$  as in (3.5) (see Definition 3.1.2): this definition makes sense locally around 0. In particular, we have  $\vec{\nu}_m(0) := (0, \dots, 0, -1, 0, \dots, 0)$  when  $d\phi_0 = Id$ . We are in position to get an existence result for small values of  $\gamma$ :

**Theorem 3.1.2.** Let  $\Omega$  be a bounded domain in  $\mathbb{R}^n$  ( $n \geq 3$ ) such that  $0 \in \partial\Omega$  is a singularity of type  $(k, n - k)$  for some  $k \in \{1, \dots, n\}$ . We fix  $0 \leq s < 2$  and  $0 \leq \gamma < \gamma_H(\Omega)$ . Assume that either  $s > 0$ , or that  $\{s = 0, n \geq 4 \text{ and } \gamma > 0\}$ . We assume that

$$0 \leq \gamma \leq \gamma_H(\mathbb{R}^{k_+, n-k}) - \frac{1}{4}.$$

Then there are extremals for  $\mu_{\gamma, s}(\Omega)$  if

$$GH_{\gamma, s}(\Omega) < 0 \tag{3.6}$$

where, for  $\Sigma := \cap_{i=1}^k \partial\Omega_i$ ,  $GH_{\gamma,s}(\Omega)$  is the **generalized mean curvature**

$$\begin{aligned} GH_{\gamma,s}(\Omega) := & c_{\gamma,s}^1 \sum_{m=1}^k \langle \vec{H}_0^\Sigma, \vec{\nu}_m \rangle + c_{\gamma,s}^2 \sum_{i,m=1, i \neq m}^k II_0^{\partial\Omega_m}(\vec{\nu}_i, \vec{\nu}_i) \\ & + c_{\gamma,s}^3 \sum_{p,q,m=1, |\{p,q,m\}|=3}^k II_0^{\partial\Omega_m}(\vec{\nu}_p, \vec{\nu}_q) \end{aligned} \quad (3.7)$$

and  $c_{\gamma,s}^1, c_{\gamma,s}^2, c_{\gamma,s}^3$  are positive explicit constants. By convention, each of the sums above is zero when empty.

The first term in  $GH_{\gamma,s}(\Omega)$  shows the influence of the mean curvature of  $\Sigma = \cap_{i=1}^k \partial\Omega_i$  at 0. The second and third sums outline the influence of the positions of the  $\Omega_m$ 's relatively to each other: these two terms do not appear in the smooth case, that is  $k = 1$ .

When  $k = 1$ , condition (3.6) reads  $\langle \vec{H}_0^{\partial\Omega}, \vec{\nu}_{\partial\Omega} \rangle < 0$ . We then recover the condition of Ghoussoub-Robert [61]. Our condition is *local*: only the local geometry of the boundary at 0 is relevant here. In the paper [28], we deal with the case  $\gamma > \gamma_H(\mathbb{R}^{k+,n-k}) - \frac{1}{4}$ : the test-functions then are different, and the existence condition is global.

For the sake of completeness, we now deal with the remaining cases, still for  $\gamma \leq \gamma_H(\mathbb{R}^{k+,n-k}) - \frac{1}{4}$ .

**Theorem 3.1.3.** *Let  $\Omega$  be a bounded domain in  $\mathbb{R}^n$  ( $n \geq 3$ ) with a singularity of type  $(k, n - k)$  at 0 for some  $k \in \{1, \dots, n\}$ . Then*

1. *If  $\gamma \leq 0$ , then  $\mu_{\gamma,0}(\Omega) = \mu_{0,0}(\mathbb{R}^n)$ , and there is no extremal.*
2. *If  $n = 3$ ,  $0 < \gamma \leq \gamma_H(\mathbb{R}^{k+,n-k}) - \frac{1}{4}$  and there are extremals for  $\mu_{\gamma,0}(\mathbb{R}_+^k \times \mathbb{R}^{3-k})$ , then there are extremals for  $\mu_{\gamma,0}(\Omega)$  if  $GH_{\gamma,0}(\Omega) < 0$ .*
3. *If  $n = 3$ ,  $0 < \gamma$  and there are no extremals for  $\mu_{\gamma,0}(\mathbb{R}^{k+,3-k})$ , then there are extremals for  $\mu_{\gamma,0}(\Omega)$  if  $R_\gamma(x_0) > 0$  for some  $x_0 \in \Omega$ .*

The proof of Theorem 3.1.3 is similar to what was performed in Ghoussoub-Robert [61], and we will only sketch it in Section 3.6, where the interior mass  $R_\gamma(x_0)$  will be defined in Proposition 3.6.1.

Our results are summarized in these tables:

In this paper, some regularity issues will be used very often. Our main tool will be the article [50] by Felli and Ferrero. We also refer to the historical reference Gmira-Véron [65] and to the monograph [33] by Cirstea. As an intermediate step in our analysis, we will prove a symmetry result for the extremals

Hardy Condition	Dimension	Geometric Condition	Extremal
$0 \leq \gamma \leq \gamma_H(\mathbb{R}^{k+,n-k}) - \frac{1}{4}$	$n \geq 3$	$GH_{\gamma,s}(\Omega) < 0$	Yes

Table 3.1: Case  $s > 0$ .

Hardy Condition	Dim.	Geometric Condition	Ext.
$0 < \gamma \leq \gamma_H(\mathbb{R}^{k+,n-k}) - \frac{1}{4}$	$n = 3$	$GH_{\gamma,0}(\Omega) < 0$ and $R_\gamma(x_0) > 0$	Yes
—	$n \geq 4$	$GH_{\gamma,0}(\Omega) < 0$	Yes
$\gamma \leq 0$	$n \geq 3$		No

Table 3.2: Case  $s = 0$ .

of  $\mu_{\gamma,s}(\mathbb{R}^{k+,n-k})$ : with the use of the moving-plane method (see Berestycki-Nirenberg [19]), we will obtain that the symmetries of the domain transfer to the extremals. This will be the object of Theorem 3.4.1.

### 3.2 The best Hardy constant and Hardy Sobolev Inequality

This section is devoted to the analysis of the Hardy constant  $\gamma_H(\Omega)$  and the proof of Proposition 3.1.1:

*Proof of (i) of Proposition 3.1.1:* By definition,  $\frac{(n-2)^2}{4} = \gamma_H(\mathbb{R}^n) \leq \gamma_H(\Omega)$ . We assume by contradiction that  $\gamma_H(\Omega) = \gamma_H(\mathbb{R}^n)$ . We that have  $\mu_{\gamma,2}(\Omega) = \gamma_H(\Omega) - \gamma = \frac{(n-2)^2}{4} - \gamma < \mu_{\gamma,2}(\mathbb{R}^{k+,n-k}) = \frac{(n+2k-2)^2}{4} - \gamma$ . Theorem 3.1.1 yields  $\mu_{\gamma,2}(\Omega)$  is achieved by some  $u_0 \in D^{1,2}(\Omega) \setminus \{0\}$ . Since  $u_0 \in D^{1,2}(\Omega) \subset D^{1,2}(\mathbb{R}^n)$ , we get that  $\gamma_H(\mathbb{R}^n)$  is achieved in  $D^{1,2}(\mathbb{R}^n)$ . Replacing  $u_0$  by  $|u_0|$ , we assume that  $u_0 \geq 0$  on  $\mathbb{R}^n$ . The Euler-Lagrange equation and the maximum principle yield  $u_0 > 0$  on  $\mathbb{R}^n$ , contradicting  $u_0 = 0$  on  $\partial\Omega$ . Therefore  $\frac{(n-2)^2}{4} < \gamma_H(\Omega)$ .

For the other inequality, since  $\Omega$  is a singularity of type  $(k, n-k)$  at 0, we choose a chart  $(U, \phi)$  as in Definition 3.1.1. Without loss of generality, we assume that  $d\phi_0 = Id$  and that  $C_0(\Omega) = \mathbb{R}^{k+,n-k}$ . Let  $\eta \in C_c^\infty(U)$  such that  $\eta(x) = 1$  for  $x \in B_\delta(0)$ , for some  $\delta > 0$  small enough, and consider  $(\alpha_\epsilon)_{\epsilon>0} \in ]0, +\infty[$  such that  $\alpha_\epsilon = o(\epsilon)$  as  $\epsilon \rightarrow 0$ . We define

$$\rho(x) := |x|^{-k-\frac{n-2}{2}} \prod_{i=1}^k x_i \text{ for all } x \in \mathbb{R}^{k+,n-k}.$$

Note that  $\rho \notin D^{1,2}(\mathbb{R}^{k_+,n-k})$ . We fix  $\beta > 1$  and define

$$\rho_\epsilon(x) = \begin{cases} \frac{|x|^\beta}{\epsilon} \rho(x) & \text{if } |x| < \epsilon \\ \rho(x) & \text{if } \epsilon < |x| < \frac{1}{\epsilon} \\ |\epsilon \cdot x|^{-\beta} \rho(x) & \text{if } |x| > \frac{1}{\epsilon}. \end{cases}$$

Note that  $\rho_\epsilon \in D^{1,2}(\mathbb{R}^{k_+,n-k})$ . For  $\epsilon > 0$ , we define

$$u_\epsilon(y) = \eta(\phi^{-1}(y)) \alpha_\epsilon^{\frac{2-n}{2}} \rho_\epsilon(\alpha_\epsilon^{-1} \phi^{-1}(y)) \text{ for any } y \in \phi(U) \cap \Omega, y = \phi(x)$$

and 0 elsewhere. Immediate computations yield

$$\begin{aligned} \int_{\mathbb{R}^n \setminus B_{\epsilon^{-1}}(0)} \frac{\rho_\epsilon^2}{|x|^2} dx &= O(1) \quad \text{and} \quad \int_{B_\epsilon(0)} \frac{\rho_\epsilon^2}{|x|^2} dx = O(1). \\ \int_{\mathbb{R}^n \setminus B_{\epsilon^{-1}}(0)} |\nabla \rho_\epsilon|^2 dx &= O(1) \quad \text{and} \quad \int_{B_\epsilon(0)} |\nabla \rho_\epsilon|^2 dx = O(1) \end{aligned} \quad (3.8)$$

when  $\epsilon \rightarrow 0$ . Since  $d\phi_0 = Id$ , we have

$$\begin{aligned} \int_{\Omega} \frac{|u_\epsilon(y)|^2}{|y|^2} dy &= \int_{\mathbb{R}^{k_+,n-k} \cap U} \frac{|u_\epsilon(\phi(x))|^2}{|\phi(x)|^2} |Jac(\phi(x))| dx \\ &= \int_{B_\delta(0) \cap \mathbb{R}^{k_+,n-k}} \frac{|u_\epsilon(\phi(x))|^2}{|x|^2} |1 + O(|x|)| dx + O(1) \end{aligned} \quad (3.9)$$

Writing  $B_\delta(0) = (B_\delta(0) \setminus B_{\epsilon^{-1}\alpha_\epsilon}(0)) \cup (B_{\epsilon^{-1}\alpha_\epsilon}(0) \setminus B_{\epsilon\alpha_\epsilon}(0)) \cup (B_{\epsilon\alpha_\epsilon}(0))$ , (3.8) yields

$$\int_{(B_\delta(0) \setminus B_{\epsilon^{-1}\alpha_\epsilon}(0)) \cap \mathbb{R}^{k_+,n-k}} \frac{|u_\epsilon(\phi(x))|^2}{|x|^2} dx = O(1) \quad (3.10)$$

$$\int_{B_{\epsilon\alpha_\epsilon}(0) \cap \mathbb{R}^{k_+,n-k}} \frac{|u_\epsilon(\phi(x))|^2}{|x|^2} dx = O(1). \quad (3.11)$$

And,

$$\int_{(B_{\epsilon^{-1}\alpha_\epsilon}(0) \setminus B_{\epsilon\alpha_\epsilon}(0)) \cap \mathbb{R}^{k_+,n-k}} \frac{|u_\epsilon(\phi(x))|^2}{|x|^2} (1 + O(|x|)) dx = W_{D,2} \ln \frac{1}{\epsilon^2} + O(1), \quad (3.12)$$

where  $W_{D,2} := 2 \int_D |\prod_{i=1}^k x_i|^2 d\sigma$  with  $D = S^{n-1} \cap \mathbb{R}^{k_+,n-k}$  for all  $k \in \{1, \dots, n\}$ . We combine (3.9), (3.10) and (3.12)

$$\int_{\Omega} \frac{|u_\epsilon(y)|^2}{|y|^2} dy = W_{D,2} \ln \frac{1}{\epsilon^2} + O(1) \text{ as } \epsilon \rightarrow 0. \quad (3.13)$$

Similar arguments yield

$$\int_{\Omega} |\nabla u_{\epsilon}(y)|^2 dy = W_{D,2} \ln \frac{1}{\epsilon^2} \gamma_H(\mathbb{R}^{k_+,n-k}) + O(1) \text{ as } \epsilon \rightarrow 0. \quad (3.14)$$

By the equations (3.13), (3.14), we get that

$$\frac{\int_{\Omega} |\nabla u_{\epsilon}(y)|^2 dy}{\int_{\Omega} \frac{|u_{\epsilon}(y)|^2}{|y|^2} dy} = \gamma_H(\mathbb{R}^{k_+,n-k}) + o(1) \text{ as } \epsilon \rightarrow 0,$$

and by the definition of  $\gamma_H$ , we get that  $\gamma_H(\Omega) \leq \gamma_H(\mathbb{R}^{k_+,n-k})$ . This proves (i).

*Proof of (ii):* If  $\Omega \subset C_0(\Omega)$ , then the definition yields  $\gamma_H(\Omega) \geq \gamma_H(C_0(\Omega))$ . The reverse inequality is by (i), which yields (ii).

*Proof of (iii):* Is a particular case of Theorem 3.1.1 below when  $s = 2$ .

*Proof of (iv):* By Ghoussoub-Robert [61] we have the following lemma:

**Lemma 3.2.1.** *Let  $(\phi_t)_{t \in \mathbb{N}} \in C^1(\mathbb{R}^n, \mathbb{R}^n)$  be such that,*

$$\lim_{t \rightarrow +\infty} (\|\phi_t - Id_{\mathbb{R}^n}\|_{\infty} + \|\nabla(\phi_t - Id_{\mathbb{R}^n})\|_{\infty}) = 0 \text{ and } \phi_t(0) = 0.$$

*Let  $D \subset \mathbb{R}^n$  be a domain such that  $0 \in \partial D$  (not necessarily bounded nor regular), and set  $D_t := \phi_t(D)$ ,  $\forall t \in \mathbb{N}$ . Then  $0 \in \partial D_t$ , and  $\lim_{t \rightarrow +\infty} \gamma_H(D_t) = \gamma_H(D)$ .*

Let  $\phi \in C^{\infty}(\mathbb{R}^{n-k})$  be such that  $0 \leq \phi \leq 1$ ,  $\phi(0) = 0$  et  $\phi(x'') = 1$  for all  $x'' \in \mathbb{R}^{n-k}$  such that  $|x''| \geq 1$ . For  $t \geq 0$ , define  $\phi_t(x', x'') := (x_1 - t\phi(x''), \dots, x_k - t\phi(x''), x'')$  for all  $(x', x'') \in \mathbb{R}^k \times \mathbb{R}^{n-k}$ . Set  $\widetilde{\Omega}_t := \phi_t(\mathbb{R}^{k_+,n-k})$ . Lemma 3.2.1 yields

$$\lim_{t \rightarrow 0} \gamma_H(\widetilde{\Omega}_t) = \gamma_H(\mathbb{R}^{k_+,n-k}) = \frac{(n+2k-2)^2}{4}.$$

Since  $\phi \geq 0$  and  $\phi(x'') = 1$  for  $x'' \in \mathbb{R}^{n-k}$ ,  $|x''| \geq 1$ , we have that  $\mathbb{R}^{k_+,n-k} \subsetneq \widetilde{\Omega}_t$ . To finish the proof of (iv), we take  $\Omega_{\epsilon} := \widetilde{\Omega}_t$  with  $\epsilon > 0$ ,  $t > 0$  small enough.

**Proposition 3.2.1.** *Let  $\gamma < \gamma_H(\mathbb{R}^{k_+,n-k})$  for all  $k \in \{1, \dots, n\}$  and  $s \in [0, 2]$ . Then, for all  $\epsilon > 0$  there exists  $c_{\epsilon} > 0$  such that for all  $u \in D^{1,2}(\Omega)$ ,*

$$\left[ \int_{\Omega} \frac{|u|^{2^*(s)}}{|x|^s} dx \right]^{\frac{2}{2^*(s)}} \leq (\mu_{\gamma,s}(\mathbb{R}^{k_+,n-k})^{-1} + \epsilon) \int_{\Omega} \left( |\nabla u|^2 - \frac{\gamma}{|x|^2} u^2 \right) dx + c_{\epsilon} \int_{\Omega} u^2 dx. \quad (3.15)$$

*Proof of Proposition 3.2.1:* We choose a chart  $(U, \phi)$  as in Definition 3.1.1. Without loss of generality, we assume that  $d\phi_0 = Id$  and then  $C = C_0(\Omega) = \mathbb{R}^{k+, n-k}$ . Choose  $u \in C_c^\infty(\phi(B_\delta(0)) \cap \Omega)$  and define  $v := u \circ \phi$  for all  $v \in C_c^1(B_\delta(0) \cap C)$ . Define the metric  $g := \phi^{-1*} Eucl$ , where  $Eucl$  is the Euclidean metric. We have that  $|\phi(x)| = |x|(1 + O(|x|))$  and  $|\phi^* Eucl - Eucl|(x) \leq c|x|$  for all  $x \in \mathbb{R}^n$  small enough for some  $c > 0$ .

**Step 1:** fix  $\epsilon > 0$ , we first claim that there exists  $\delta > 0$  such that for all  $u \in C_c^1(\phi(B_\delta(0)) \cap \Omega)$ ,

$$\left[ \int_{\Omega} \frac{|u|^{2^*(s)}}{|x|^s} dx \right]^{\frac{2}{2^*(s)}} \leq (\mu_{\gamma,s}(C)^{-1} + \epsilon) \int_{\Omega} \left( |\nabla u|^2 - \frac{\gamma}{|x|^2} u^2 \right) dx. \quad (3.16)$$

*Proof of (3.16):* We have that

$$\begin{aligned} \left[ \int_{\Omega} \frac{|u|^{2^*(s)}}{|x|^s} dx \right]^{\frac{2}{2^*(s)}} &= \left[ \int_{B_\delta(0) \cap C} \frac{|u \circ \phi(x)|^{2^*(s)} |Jac(\phi(x))|}{|\phi(x)|^s} dx \right]^{\frac{2}{2^*(s)}} \\ &\leq (1 + c\delta) \left[ \int_{B_\delta(0) \cap C} \frac{|v|^{2^*(s)}}{|x|^s} dx \right]^{\frac{2}{2^*(s)}} \\ &\leq (1 + c\delta) \mu_{\gamma,s}(C)^{-1} \int_{B_\delta(0) \cap C} \left( |\nabla v|^2 - \frac{\gamma}{|x|^2} v^2 \right) dx \\ &\leq (1 + c\delta) \mu_{\gamma,s}(C)^{-1} \int \left( |\nabla u|_g^2 - \frac{\gamma}{|\phi^{-1}(x)|^2} u^2 \right) |Jac \phi^{-1}(x)| dx \\ &\leq (1 + c_1\delta) \mu_{\gamma,s}(C)^{-1} \int \left( |\nabla u|^2 - \frac{\gamma}{|x|^2} u^2 \right) dx + c_2\delta \int \left( |\nabla u|^2 + \frac{u^2}{|x|^2} \right) dx \end{aligned}$$

where the last three integrals are taken on  $\phi(B_\delta(0)) \cap \Omega$ . This give us

$$\begin{aligned} \left[ \int_{\Omega} \frac{|u|^{2^*(s)}}{|x|^s} dx \right]^{\frac{2}{2^*(s)}} &\leq (1 + c_1\delta) \mu_{\gamma,s}(C)^{-1} \int_{\Omega} \left( |\nabla u|^2 - \frac{\gamma}{|x|^2} u^2 \right) dx \\ &\quad + c_2\delta \int_{\Omega} \left( |\nabla u|^2 + \frac{u^2}{|x|^2} \right) dx. \end{aligned}$$

For all  $v \in C_c^1(\phi(B_\delta(0)) \cap \Omega)$ , we get that

$$\int_{\Omega} \frac{u^2}{|x|^2} dx = \int_{B_\delta(0) \cap C} \frac{v^2}{|x|^2} |1 + O(|x|)| dx \leq (1 + c_1\delta) \int_C \frac{v^2}{|x|^2} dx, \quad (3.17)$$

and,

$$\int_{\Omega} |\nabla u|^2 dx = \int_{B_\delta(0) \cap C} |\nabla v|_{\phi^* Eucl}^2 |1 + O(|x|)| dx \geq (1 - c_2\delta) \int_C |\nabla v|^2 dx, \quad (3.18)$$

where  $c_1, c_2 > 0$  are independent of  $\delta$  and  $v$ . Since  $\gamma < \gamma_H(\mathbb{R}^{k+, n-k})$ , there exists  $c_0 > 0$  for  $\delta$  small enough,

$$c_0^{-1} \int_{\Omega} |\nabla u|^2 dx \leq \int_{\Omega} \left( |\nabla u|^2 - \gamma \frac{u^2}{|x|^2} \right) dx \leq c_0 \int_{\Omega} |\nabla u|^2 dx. \quad (3.19)$$

With (3.17), (3.18) and (3.19), we get (3.16) for  $\delta > 0$  small enough. This ends Step 1.  $\square$

**Step 2:** We prove (3.15) for all  $u \in D^{1,2}(\Omega)$ .

Let  $\eta \in C_c^\infty(\mathbb{R}^n)$  and  $\sqrt{\eta}, \sqrt{1-\eta} \in C^2(\mathbb{R}^n)$  be such that  $\eta(x) = 1$  if  $x \in B_{\delta/2}(0)$  and  $\eta(x) = 0$  if  $x \notin B_\delta(0)$ . We define  $\|w\|_{p,|x|^{-s}} = \left[ \int_{\Omega} \frac{|w|^p}{|x|^s} dx \right]^{\frac{1}{p}}$ . We set  $p = 2^*(s)/2$ . Hölder's inequality yield

$$\begin{aligned} \|u^2\|_{p,|x|^{-s}} &= \|\eta u^2 + (1-\eta)u^2\|_{p,|x|^{-s}} \\ &\leq \|\eta u^2\|_{p,|x|^{-s}} + \|(1-\eta)u^2\|_{p,|x|^{-s}} \\ &\leq \|\sqrt{\eta}u\|_{2^*(s),|x|^{-s}}^2 + \|\sqrt{1-\eta}u\|_{2^*(s),|x|^{-s}}^2, \end{aligned}$$

for all  $u \in C_c^\infty(\Omega)$ . Since  $\sqrt{\eta}u \in C_c^2(B_{\delta_\epsilon} \cap C)$ , we use (3.16) and integrate by parts

$$\begin{aligned} \left[ \int_{\Omega} \frac{|u|^{2^*(s)}}{|x|^s} dx \right]^{\frac{2}{2^*(s)}} &\leq (\mu_{\gamma,s}(C)^{-1} + \epsilon) \int_{\Omega} \left( |\nabla \sqrt{\eta}u|^2 - \frac{\gamma}{|x|^2} \eta u^2 \right) dx \\ &\quad + \|\sqrt{1-\eta}u\|_{2^*(s),|x|^{-s}}^2 \\ &\leq (\mu_{\gamma,s}(C)^{-1} + \epsilon) \int_{\Omega} \eta \left( |\nabla u|^2 - \frac{\gamma}{|x|^2} u^2 \right) dx \\ &\quad + \|\sqrt{1-\eta}u\|_{2^*(s),|x|^{-s}}^2 + c \int_{\Omega} u^2 dx, \end{aligned} \quad (3.20)$$

where  $c > 0$  depends of  $\epsilon > 0$ .

**Case 1:**  $s = 0$ . We claim that

$$\mu_{\gamma,0}(\Omega) \leq \mu_{0,0}(\mathbb{R}^n) \quad (3.21)$$

We prove the claim. Fix  $x_0 \in \Omega$ ,  $x_0 \neq 0$ , and take  $\eta \in C_c^\infty(\Omega)$  such that  $\eta(x) = 1$  around of  $x_0$ . For  $x \in \Omega$  and  $\epsilon > 0$ , we define  $u_\epsilon(x) := \eta(x) \left( \frac{\epsilon}{\epsilon^2 + |x-x_0|^2} \right)^{\frac{n-2}{2}}$  for all  $x \in \Omega$ . Classical computations in the spirit of Aubin [2] yield

$$\lim_{\epsilon \rightarrow 0} \frac{\int_{\Omega} |\nabla u_\epsilon|^2 dx}{\left( \int_{\Omega} u_\epsilon^{2^*} dx \right)^{\frac{2}{2^*}}} = \mu_{0,0}(\mathbb{R}^n)$$

and  $\lim_{\epsilon \rightarrow 0} \int_{\Omega} \frac{u_{\epsilon}^2}{|x|^2} dx = 0$ . This yields (3.21), and the claim is proved.

The Sobolev inequality yields  $\|f\|_{2n/(n-2)}^2 \leq \mu_{0,0}(\mathbb{R}^n)^{-1} \|\nabla f\|_2^2$  for all  $f \in D^{1,2}(\Omega) \subset D^{1,2}(\mathbb{R}^n)$ . We combine these inequalities to get

$$\begin{aligned} \|\sqrt{1-\eta}u\|_{2^*(s),|x|^{-s}}^2 &\leq \mu_{0,0}(\mathbb{R}^n)^{-1} \int_{\Omega} |\nabla(\sqrt{1-\eta}u)|^2 dx \\ &\leq (\mu_{\gamma,s}(C)^{-1} + \epsilon) \int_{\Omega} (1-\eta)|\nabla u|^2 dx \\ &\quad + c \int_{\Omega} u^2 dx. \end{aligned} \quad (3.22)$$

We use the equations (3.20) and (3.22)

$$\begin{aligned} \left[ \int_{\Omega} \frac{|u|^{2^*(s)}}{|x|^s} dx \right]^{\frac{2}{2^*(s)}} &\leq (\mu_{\gamma,s}(C)^{-1} + \epsilon) \int_{\Omega} \eta \left( |\nabla u|^2 - \frac{\gamma}{|x|^2} u^2 \right) dx \\ &\quad + (\mu_{\gamma,s}(C)^{-1} + \epsilon) \int_{\Omega} (1-\eta)|\nabla u|^2 dx + c \int_{\Omega} u^2 dx \\ &\leq (\mu_{\gamma,s}(C)^{-1} + \epsilon) \int_{\Omega} |\nabla u|^2 dx - (\mu_{\gamma,s}(C)^{-1} + \epsilon) \gamma \int_{\Omega \setminus B_{\delta_{\frac{\epsilon}{2}}}(0)} \frac{\eta}{|x|^2} u^2 dx \\ &\quad - (\mu_{\gamma,s}(C)^{-1} + \epsilon) \gamma \int_{B_{\delta_{\frac{\epsilon}{2}}}(0)} \frac{\eta}{|x|^2} u^2 dx + c \int_{\Omega} u^2 dx \\ &\leq (\mu_{\gamma,s}(C)^{-1} + \epsilon) \int_{\Omega} |\nabla u|^2 dx - (\mu_{\gamma,s}(C)^{-1} + \epsilon) \gamma \int_{B_{\delta_{\frac{\epsilon}{2}}}(0)} \frac{u^2}{|x|^2} dx + c \int_{\Omega} u^2 dx \\ &\leq (\mu_{\gamma,s}(C)^{-1} + \epsilon) \int_{\Omega} \left( |\nabla u|^2 - \frac{\gamma}{|x|^2} u^2 \right) dx + c \int_{\Omega} u^2 dx. \end{aligned}$$

**Case 2:**  $0 < s < 2$ . We have that  $2 < 2^*(s) < 2^*$ , let  $\nu > 0$  and by interpolation inequality there exists  $c_{\nu} > 0$ , such that

$$\begin{aligned} \|\sqrt{1-\eta}u\|_{2^*(s),|x|^{-s}}^2 &\leq C \left( \nu \|\sqrt{1-\eta}u\|_{2^*}^2 + c_{\nu} \|\sqrt{1-\eta}u\|_2^2 \right) \\ &\leq C \left( \nu \mu_{0,0}(\mathbb{R}^n)^{-1} \|\nabla(\sqrt{1-\eta}u)\|_2^2 + c_{\nu} \|\sqrt{1-\eta}u\|_2^2 \right). \end{aligned}$$

We choose  $\nu$  such that  $\nu \mu_{0,0}(\mathbb{R}^n)^{-1} \leq \mu_{\gamma,s}^{-1}(C) + \epsilon$ , we get

$$\|\sqrt{1-\eta}u\|_{2^*(s),|x|^{-s}}^2 \leq (\mu_{\gamma,s}^{-1}(C) + \epsilon) \|\nabla(\sqrt{1-\eta}u)\|_2^2 + c_{\nu} \|\sqrt{1-\eta}u\|_2^2. \quad (3.23)$$



By (3.20) and (3.23)

$$\begin{aligned} \left[ \int_{\Omega} \frac{|u|^{2^*(s)}}{|x|^s} dx \right]^{\frac{2}{2^*(s)}} &\leq (\mu_{\gamma,s}(C)^{-1} + \epsilon) \int_{\Omega} \eta \left( |\nabla u|^2 dx - \frac{\gamma}{|x|^2} u^2 \right) dx \\ &+ (\mu_{\gamma,s}^{-1}(C) + \epsilon) \|\nabla(\sqrt{1-\eta}u)\|_2^2 + c_{\nu} \|\sqrt{1-\eta}u\|_2^2 + c \int_{\Omega} u^2 dx \\ &\leq (\mu_{\gamma,s}(C)^{-1} + \epsilon) \int_{\Omega} \left( |\nabla u|^2 - \frac{\gamma}{|x|^2} u^2 \right) dx + c \int_{\Omega} u^2 dx. \end{aligned}$$

**Cas 3:**  $s = 2$ . We have  $2^*(s) = 2$

$$\|\sqrt{1-\eta}u\|_{2^*(s),|x|^{-s}}^2 = \int_{\Omega \setminus B_{\delta/2}(0)} \frac{1-\eta}{|x|^2} u^2 dx \leq c_{\delta} \int_{\Omega} u^2 dx, \quad (3.24)$$

by the equations (3.20) and (3.24) we get the result.  $\square$

**Proposition 3.2.2.** *Let  $\Omega$  be a bounded domain such that  $0 \in \partial\Omega$ .*

- (i) *If  $\gamma < \gamma_H(\mathbb{R}^{k+,n-k})$ , then  $\mu_{\gamma,s}(\Omega) > -\infty$ .*
- (ii) *If  $\gamma > \gamma_H(\mathbb{R}^{k+,n-k})$ , then  $\mu_{\gamma,s}(\Omega) = -\infty$ .*
- (iii) *If  $\gamma < \gamma_H(\Omega)$ , then  $\mu_{\gamma,s}(\Omega) > 0$ .*
- (iv) *If  $\gamma_H(\Omega) < \gamma < \gamma_H(\mathbb{R}^{k+,n-k})$ , then  $0 > \mu_{\gamma,s}(\Omega) > -\infty$ .*
- (v) *If  $\gamma = \gamma_H(\Omega) < \gamma_H(\mathbb{R}^{k+,n-k})$ , then  $\mu_{\gamma,s}(\Omega) = 0$ .*

*Proof of Proposition 3.2.2: Proof of (i):* Let  $\gamma < \gamma_H(\mathbb{R}^{k+,n-k})$  and  $\epsilon > 0$  such that  $(1+\epsilon)\gamma \leq \gamma_H(\mathbb{R}^{k+,n-k})$ . By Proposition 3.2.1 there exist  $c_{\epsilon} > 0$  for any  $u \in D^{1,2}(\Omega)$  such that

$$\gamma_H(\mathbb{R}^{k+,n-k}) \int_{\Omega} \frac{u^2}{|x|^2} dx \leq (1+\epsilon) \int_{\Omega} |\nabla u|^2 dx + c_{\epsilon} \int_{\Omega} u^2 dx.$$

Since  $2^*(s) > 2$  and  $\Omega$  is bounded, Hölder inequality yields  $c_1 > 0$  such that

$$\int_{\Omega} u^2 dx \leq c_1 \left( \int_{\Omega} \frac{u^{2^*(s)}}{|x|^s} dx \right)^{\frac{2}{2^*(s)}}. \quad (3.25)$$

If  $\gamma \geq 0$  and since  $(1 - \gamma\gamma_H(\mathbb{R}^{k+,n-k})^{-1}(1+\epsilon)) \geq 0$ , by (3.25), we get

$$J_{\gamma,s}^{\Omega}(u) = \frac{\int_{\Omega} |\nabla u|^2 dx - \gamma \int_{\Omega} \frac{u^2}{|x|^2} dx}{\left( \int_{\Omega} \frac{u^{2^*(s)}}{|x|^s} dx \right)^{\frac{2}{2^*(s)}}} \geq \frac{-c_2 c_{\epsilon} \gamma}{\gamma_H(\mathbb{R}^{k+,n-k})},$$

then for any  $u \in D^{1,2}(\Omega)$  we have  $\mu_{\gamma,s}(\Omega) > -\infty$ . If  $\gamma < 0$ , then  $\mu_{\gamma,s}(\Omega) \geq \mu_{0,s}(\Omega) > 0$  by Hardy-Sobolev inequality.

*Proof of (ii):* We take  $(u_\epsilon)_{\epsilon>0}$  as in the proof of Proposition 3.1.1 -(i). We get

$$J_{\gamma,s}^\Omega(u_\epsilon) = \left( (\gamma_H(\mathbb{R}^{k_+,n-k}) - \gamma) \frac{W_{D,2}}{W_{D,2}^{\frac{2}{2^*(s)}}} + O(1) \right) \left( \ln\left(\frac{1}{\epsilon^2}\right) \right)^{\frac{2-s}{n-s}},$$

As  $s < 2$  and  $\gamma > \gamma_H(\mathbb{R}^{k_+,n-k})$ , then  $\lim_{\epsilon \rightarrow 0} J_{\gamma,s}^\Omega(u_\epsilon) = -\infty$ , and  $\mu_{\gamma,s}(\Omega) = -\infty$ .

*Proof of (iii):* We fix  $\gamma < \gamma_H(\Omega)$ . For any  $u \in D^{1,2}(\Omega) \setminus \{0\}$ , we have that

$$J_{\gamma,s}^\Omega(u) = \frac{\int_\Omega |\nabla u|^2 dx - \gamma \int_\Omega \frac{u^2}{|x|^2} dx}{\left( \int_\Omega \frac{u^{2^*(s)}}{|x|^s} dx \right)^{\frac{2}{2^*(s)}}} \geq \left( 1 - \frac{\gamma}{\gamma_H(\Omega)} \right) \mu_{0,s}(\Omega),$$

and then  $\mu_{\gamma,s}(\Omega) > 0$ .

*Proof of (iv):* We assume that  $\gamma_H(\Omega) < \gamma < \gamma_H(\mathbb{R}^{k_+,n-k})$ , it follows from Proposition 3.1.1-(iii) that  $\gamma_H(\Omega)$  is attained by some  $u_0$ . We get that  $\mu_{\gamma,s}(\Omega) \leq J_{\gamma,s}^\Omega(u_0) < 0$ .

*Proof of (v):* We now assume that  $\gamma_H(\Omega) = \gamma < \gamma_H(\mathbb{R}^{k_+,n-k})$ . Then  $\mu_{\gamma,s}(\Omega) \geq 0$ . Here again, Proposition 3.1.1 yields an extremal  $u_0 \in D^{1,2}(\Omega)$  for  $\gamma_H(\Omega)$ . We get  $J_{\gamma,s}^\Omega(u_0) = 0$ , and then  $\mu_{\gamma,s}(\Omega) = 0$ .

**Sketch of the proof of Theorem 3.1.1.** The proof is very classical and follows the proof of Proposition 6.2 in [60]. We only sketch it to outline the specific tools we use here. Let  $(u_k)_{k \in \mathbb{N}} \in D^{1,2}(\Omega) \setminus \{0\}$  be a minimizing sequence  $\mu_{\gamma,s}(\Omega)$  such that  $\|u_k\|_{2^*(s),|x|^{-s}}^2 = 1$ . Using Proposition 3.2.1, we get that  $(u_k)_{k \in \mathbb{N}}$  is bounded in  $D^{1,2}(\Omega)$ . As a consequence, up to the extraction of a subsequence, there exists  $u \in D^{1,2}(\Omega)$  such that  $u_k \rightarrow u$  weakly in  $D^{1,2}(\Omega)$  and strongly in  $L^2(\Omega)$  as  $k \rightarrow +\infty$ . We write  $\theta_k := u_k - u$ , so that  $\theta_k \rightarrow 0$  weakly in  $D^{1,2}(\Omega)$  and strongly in  $L^2(\Omega)$  as  $k \rightarrow +\infty$ . We apply the definition (3.3) of  $\mu_{\gamma,s}(\Omega)$  to  $u$  and Proposition 3.2.1 to  $\theta_k$  for  $\epsilon_0 > 0$  small enough. It is then standard to get that  $\theta_k \rightarrow 0$  strongly in  $D^{1,2}(\Omega)$ , and then  $u \not\equiv 0$  is a minimizer for  $\mu_{\gamma,s}(\Omega)$ . As mentioned above, we refer to the proof of Proposition 6.2 in [60] for the method.

### 3.3 Regularity and approximate solutions

We say that  $u \in D_{loc,0}^{1,2}(\Omega)$  if there exists  $\eta \in C_c^\infty(\mathbb{R}^n)$  such that  $\eta \equiv 1$  around 0 and  $\eta u \in D^{1,2}(\Omega)$ . We define  $U_\alpha(x) := |x|^{-\alpha-k} \prod_{i=1}^k x_i$ . As one checks

$$-\Delta U_\alpha - \frac{\gamma}{|x|^2} U_\alpha = 0 \text{ in } \mathbb{R}^{k_+,n-k} \Leftrightarrow \alpha \in \{\alpha_-, \alpha_+\}$$

where

$$\alpha_{\pm} = \frac{n-2}{2} \pm \sqrt{\gamma_H(\mathbb{R}^{k+,n-k}) - \gamma}.$$

Note that  $U_{\alpha_-} \in D^{1,2}(\mathbb{R}^{k+,n-k})_{loc,0}$ . It is the model for more general equations:

**Theorem 3.3.1** (Felli-Ferrero). *(Optimal regularity) Let  $\Omega$  be a domain of  $\mathbb{R}^n$  with a boundary singularity of type  $(k, n-k)$  at 0. We fix  $\gamma < \gamma_H(\mathbb{R}^{k+,n-k})$ . We let  $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$  such that*

$$|f(x, v)| \leq c|v| \left( 1 + \frac{|v|^{2^*(s)-2}}{|x|^s} \right) \text{ for all } x \in \Omega, v \in \mathbb{R}.$$

Let  $u \in D^{1,2}(\mathbb{R}^{k+,n-k})_{loc,0}$ ,  $u > 0$  be a weak solution to

$$-\Delta u - \frac{\gamma + O(|x|^\tau)}{|x|^2} u = f(x, u) \text{ in } D^{1,2}(\Omega)_{loc,0}$$

for some  $\tau > 0$ . Then there exists  $K > 0$  such that

$$\lambda^{\alpha_-} u(\lambda x) \rightarrow K \left( \prod_{i=1}^k x_i \right) |x|^{-\alpha_- - k} \text{ in } B_1(0) \cap \mathbb{R}^{k+,n-k}, \quad (3.26)$$

uniformly in  $C^1$  as  $\lambda \rightarrow 0$ .

This result is essentially in Felli-Ferrero [50]. Applying Theorem 1.1 of Felli-Ferrero [50] to  $u \in D^{1,2}(\Omega)$ , and since  $u > 0$ , we get that

$$\lambda^{\frac{n-2}{2} - \sqrt{\frac{(n-2)^2}{4} + \mu}} u(\lambda x) \rightarrow |x|^{-\frac{n-2}{2} - \sqrt{\frac{(n-2)^2}{4} + \mu}} \psi \left( \frac{x}{|x|} \right) \text{ as } \lambda \rightarrow 0^+,$$

where  $\mu$  is an eigenvalue of  $L_\gamma := -\Delta_{\mathbb{S}^{n-1}} - \gamma$  on  $\mathbb{S}^{n-1} \cap \mathbb{R}^{k+,n-k}$  with Dirichlet boundary condition and  $\psi : \mathbb{S}^{n-1} \rightarrow \mathbb{R}$  is a nontrivial associated eigenfunction. Since  $u > 0$ , then  $\psi \geq 0$ , and then  $\psi > 0$ , so  $\mu = k(n+k-2) - \gamma$  is the first eigenfunction and there exists  $K > 0$  such that  $\psi(x) = K \prod_{i=1}^k x_i$ . This yields (3.26).

**Lemma 3.3.1.** *Assume the  $u \in D^{1,2}(\mathbb{R}^{k+,n-k})_{loc,0}$  is a weak solution of*

$$\begin{cases} -\Delta u - \frac{\gamma + O(|x|^\tau)}{|x|^2} u = 0 & \text{in } D^{1,2}(\mathbb{R}^{k+,n-k})_{loc,0}, \\ u = 0 & \text{on } B_{2\delta} \cap \partial\mathbb{R}^{k+,n-k}, \end{cases} \quad (3.27)$$

for some  $\tau > 0$ . Assume there exists  $c > 0$  such that

$$|u(x)| \leq c|x|^{-\alpha} \text{ for } x \rightarrow 0, x \in \mathbb{R}^{k+,n-k}. \quad (3.28)$$

1. Then, there exists  $c_1 > 0$  such that

$$|\nabla u(x)| \leq c_1 |x|^{-\alpha-1} \text{ as } x \rightarrow 0, x \in \mathbb{R}^{k+,n-k}.$$

2. If  $\lim_{x \rightarrow 0} |x|^\alpha u(x) = 0$ , then  $\lim_{x \rightarrow 0} |x|^{\alpha+1} |\nabla u(x)| = 0$ .

*Proof.* For any  $X \in \mathbb{R}^{k+,n-k}$ , let  $(X_j)_j \in \mathbb{R}^{k+,n-k}$  be such that  $\lim X_j = 0$  as  $j \rightarrow +\infty$ . Take  $r_j = |X_j|$  and  $\theta_j := \frac{X_j}{|X_j|}$ , we have  $\lim_{j \rightarrow +\infty} r_j = 0$ . Define

$$\tilde{u}_j(X) := r_j^\alpha u(r_j X) \text{ for all } j, X \in (B_R(0) \cap \mathbb{R}^{k+,n-k}) \setminus \{0\}.$$

Since  $u$  is a solution of the equation (3.27), we get

$$\begin{cases} -\Delta \tilde{u}_j - \frac{\gamma+o(1)}{|X|^2} \tilde{u}_j = 0 & \text{in } B_R(0) \cap \mathbb{R}^{k+,n-k}, \\ \tilde{u}_j = 0 & \text{in } B_R(0) \cap \partial \mathbb{R}^{k+,n-k}. \end{cases}$$

Here,  $o(1) \rightarrow 0$  in  $C_{loc}^0(\overline{\mathbb{R}^{k+,n-k}} \setminus \{0\})$ . Since  $\lim_{j \rightarrow +\infty} X_j = 0$  and by (3.28), we get that  $|\tilde{u}_j(X)| \leq c |X|^{-\alpha}$  for all  $X \in B_R(0) \cap \mathbb{R}^{k+,n-k}$  and all  $j \in \mathbb{N}$ . It follows from elliptic theory, that there exists  $\tilde{u} \in C^2(\overline{\mathbb{R}^{k+,n-k}} \setminus \{0\})$  such that  $\tilde{u}_j \rightarrow \tilde{u}$  in  $C_{loc}^1(\overline{\mathbb{R}^{k+,n-k}} \setminus \{0\})$ . Take  $\theta := \lim_{j \rightarrow +\infty} \theta_j$  with  $|\theta| = 1$ , we have that

$$\lim_{j \rightarrow +\infty} |x_j|^{\alpha+1} \partial_m u(x_j) = \partial_m \tilde{u}(\theta) \text{ for all } m = 1, \dots, n. \quad (3.29)$$

We assume that there exists  $(x_j)_j \in \mathbb{R}^{k+,n-k}$  such that  $x_j \rightarrow 0$  and

$$|x_j|^{\alpha+1} |\nabla u(x_j)| \rightarrow +\infty \text{ as } j \rightarrow +\infty.$$

Take  $\theta_j = \frac{x_j}{|x_j|}$  and we have  $\lim_{j \rightarrow +\infty} |\nabla \tilde{u}_j(\theta_j)| = +\infty$  contradiction with (3.29). The case when  $\lim_{x \rightarrow 0} |x|^\alpha u(x) = 0$  goes similarly.  $\square$

### 3.4 Symmetry of the extremals for $\mu_{\gamma,s}(\mathbb{R}^{k+,n-k})$

In this section we present the symmetry of the extremals for  $\mu_{\gamma,s}(\mathbb{R}^{k+,n-k})$ . We take inspiration in the proof of the symmetry carried out by Ghoussoub-Robert [61] in half space  $\{x_1 > 0\}$ . For  $\gamma < \gamma_H(\mathbb{R}^{k+,n-k})$ ,  $s \in [0, 2)$ , we consider the problem:

$$\begin{cases} -\Delta u - \frac{\gamma}{|x|^2} u = \frac{u^{2^*(s)-1}}{|x|^s} & \text{in } \mathbb{R}^{k+,n-k}, \\ u \geq 0 & \text{in } \mathbb{R}^{k+,n-k}, \\ u = 0 & \text{on } \partial \mathbb{R}^{k+,n-k}. \end{cases} \quad (3.30)$$

**Theorem 3.4.1.** For  $\gamma \geq 0$  and if  $u$  is solution of the equation (3.30) in  $C^2(\mathbb{R}^{k+,n-k}) \cap C(\overline{\mathbb{R}^{k+,n-k}} \setminus \{0\})$  for all  $k \in \{1, \dots, n\}$ , then  $u \circ \sigma = u$  for all isometries of  $\mathbb{R}^n$  such that  $\sigma(\mathbb{R}^{k+,n-k}) = \mathbb{R}^{k+,n-k}$ . In particular:

- There exists  $w \in C^\infty([0, \infty[^k \times \mathbb{R}^{n-k})$  such that for all  $x_1, \dots, x_k > 0$  and for any  $x' \in \mathbb{R}^{n-k}$ , we get that

$$u(x_1, \dots, x_k, x') = w(x_1, \dots, x_k, |x'|).$$

- $u$  is a symmetric function of  $k$  variables: for all permutation  $s$  of the set of indices  $\{1, \dots, k\}$ , we have

$$u(x_1, \dots, x_k, x_{k+1}, \dots, x_n) = u(x_{s(1)}, \dots, x_{s(k)}, x_{k+1}, \dots, x_n).$$

*Proof of Theorem 3.4.1:* We prove the Theorem. We proceed as in Berestycki-Nirenberg [19] (see Ghoussoub-Robert[59] and Fraenkel[52]). We write for convenience  $p := 2^*(s) - 1$ . We define  $F := B_{\frac{1}{2}}(\frac{1}{2}\vec{e}_1) \cap \mathbb{R}^{k+, n-k}$  and

$$v(x) := |x|^{2-n} u\left(-\vec{e}_1 + \frac{x}{|x|^2}\right) \text{ for all } x \in \overline{F} \setminus \{0\},$$

with  $v(0) = 0$  and  $\vec{e}_1 := (1, 0, \dots, 0)$ . Clearly, this is well defined. We have  $\partial F = F_1 \cup F_2$  where

$$F_1 := \partial B_{\frac{1}{2}}(\frac{1}{2}\vec{e}_1) \cap \mathbb{R}^{k+, n-k} \text{ and } F_2 := \cup_{j=2}^k \left( B_{\frac{1}{2}}(\frac{1}{2}\vec{e}_1) \cap \{x_j = 0\} \right).$$

If  $x \in F_1$ , then  $|x|^2 = x_1$ , we have  $v(x) = 0$  or if  $x \in F_2$ , then  $v(x) = 0$ . Consequently,  $v(x) = 0$  for all  $x \in \partial F \setminus \{0\}$ . We have that  $\vec{e}_1 \in \partial F$ . Since  $|x - |x|^2 \vec{e}_1| = |x||x - \vec{e}_1|$ , we have that

$$-\Delta v = \frac{\gamma}{|x|^2 |x - \vec{e}_1|^2} v + \frac{v^p}{|x|^s |x - \vec{e}_1|^s} \text{ in } F. \quad (3.31)$$

It follows from the assumptions on  $u$  that  $v \in C^2(F) \cap C(\overline{F} \setminus \{0, \vec{e}_1\})$ .

We claim that:

$$v(x'', -x_n) = v(x'', x_n) \text{ for all } x \in F, \quad (3.32)$$

where  $x'' := (x_1, \dots, x_{n-1})$ . Theorem 3.4.1 will be mostly a consequence of this claim.

*Proof of (3.32).* For  $\lambda \in \mathbb{R}$  we define

$$\begin{aligned} T_\lambda &:= \{x \in \mathbb{R}^n; x_n = \lambda\} & ; & \quad x_\lambda := (x'', 2\lambda - x_n). \\ Z(\lambda) &:= \{x \in F; x_n < \lambda\} & ; & \quad Y(\lambda) := \{x \in \mathbb{R}^n; x_\lambda \in Z(\lambda)\}. \end{aligned}$$

Let  $-a := \inf_{x \in F} x_n$ , so that  $Z(\lambda)$  is empty if and only if  $\lambda \leq -a$ . Since

$$|x_\lambda|^2 - |x|^2 = 4\lambda(\lambda - x_n), \quad (3.33)$$

we obtain that  $Y(\lambda) \subset F$  if  $\lambda \leq 0$ . We adapt the moving-plane method. Take  $-a < \lambda < 0$  and define

$$g_\lambda(x) := v(x_\lambda) - v(x) \text{ for all } x \in Z(\lambda).$$

We claim that

$$v(x_\lambda) > v(x) \text{ for } \lambda \in (-a, 0) \text{ and } x \in Z(\lambda). \quad (3.34)$$

We prove the claim (3.34). Since,  $\lambda < 0$ , (3.33) yields  $\{x \in Z(\lambda) \Rightarrow x_\lambda \in F\}$ . Since

$$|x_\lambda - |x_\lambda|^2 \vec{e}_1|^2 - |x - |x|^2 \vec{e}_1|^2 = (|x_\lambda|^2 - |x|^2) [1 + |x_\lambda|^2 + |x|^2 - 2x_1],$$

for all  $x \in \mathbb{R}^n$ ,  $\lambda < 0$  and by (3.33), we obtain that

$$|x_\lambda - |x_\lambda|^2 \vec{e}_1|^2 - |x - |x|^2 \vec{e}_1|^2 < 0 \text{ in } Z(\lambda). \quad (3.35)$$

We define

$$c_\lambda(x) := \begin{cases} \frac{v(x_\lambda)^p - v(x)^p}{v(x_\lambda) - v(x)} & \text{if } v(x_\lambda) \neq v(x). \\ pv^{p-1}(x) & \text{if } v(x_\lambda) = v(x). \end{cases}$$

The equation (3.31) of  $v$ ,  $\gamma \geq 0$  and (3.35) yield

$$\begin{aligned} -\Delta g_\lambda &= \gamma \left[ \frac{v(x)}{|x - |x|^2 \vec{e}_1|^2} - \frac{v(x_\lambda)}{|x_\lambda - |x_\lambda|^2 \vec{e}_1|^2} \right] \\ &\quad + \left[ \frac{v(x)^p}{|x - |x|^2 \vec{e}_1|^s} - \frac{v(x_\lambda)^p}{|x_\lambda - |x_\lambda|^2 \vec{e}_1|^s} \right] \\ &< -\gamma \frac{g_\lambda}{|x - |x|^2 \vec{e}_1|^2} - c_\lambda(x) \frac{g_\lambda}{|x - |x|^2 \vec{e}_1|^s}, \end{aligned}$$

then,

$$-\Delta g_\lambda + d_\lambda g_\lambda < 0 \text{ in } Z(\lambda), \quad (3.36)$$

where  $d_\lambda(x) := \gamma|x - |x|^2 \vec{e}_1|^{-2} + c_\lambda(x)|x - |x|^2 \vec{e}_1|^{-s}$ . We have  $Z(\lambda) = F \cap \{x \in \mathbb{R}^n, x_n < \lambda\}$ , this gives that  $\partial Z(\lambda) \subset \partial F \cup T_\lambda$ . Therefore,

$$g_\lambda(x) \geq 0 \text{ if } x \in \partial Z(\lambda), \quad (3.37)$$

with the strict inequality when  $x \in \partial Z(\lambda) \setminus T_\lambda$  and  $x_\lambda \in F$  and with equality when  $x \in \partial Z(\lambda) \cap T_\lambda$ . Again,  $g_\lambda(x) = 0$  if  $x, x_\lambda$  in  $\partial F \setminus T_\lambda$ .

**Step 3.4.1.** We prove (3.34) for  $\lambda + a > 0$  close to 0.

*Proof of Step 3.4.1:* Since  $x \in Z(\lambda)$ , we have  $x \in F$  and  $x_n < \lambda$ . But  $\lambda < 0$  thus  $x \notin \{0, \vec{e}_1\}$ . On the other hand, we have  $0 < |x| < 1$  and

$$|d_\lambda(x)| \leq \frac{\gamma}{|x|^2||x| - 1|^2} + \left| \frac{c_\lambda(x)}{|x|^s||x| - 1|^s} \right|. \quad (3.38)$$

But  $v \in C(\overline{F} \setminus \{0, \vec{e}_1\})$ , then is a  $c_0 > 0$  such that  $0 \leq v(x) \leq c_0 \sup \overline{F} \setminus \{0, \vec{e}_1\}$ . The definition of  $c_\lambda(x)$  and (3.38), then there exists  $c_1 > 0$  such that  $|d_\lambda(x)| \leq c_1$  for all  $x \in Z(\lambda)$  and  $\lambda < 0$ . Next,  $g_\lambda$  verifies (3.36). For any  $\delta > 0$ , if  $\lambda \in (-a, 0)$  is close to  $-a$ , then  $|Z(\lambda)| \leq \delta$ . It follows from Theorem 3.4.2 that for  $\lambda$  close to  $-a$ , we have

$$g_\lambda(x) \geq 0 \text{ for } x \in \overline{Z(\lambda)}.$$

We now prove (3.34) for  $x \in Z(\lambda)$ . Here again, for any  $\delta > 0$ , then  $|Z(\lambda)| \leq \delta$  for  $\lambda \in (-a, 0)$  close to  $-a$ . Moreover,  $Z(\lambda)$  is bounded and  $g_\lambda$  verifies (3.36). The Maximum principle (Theorem 3.4.2 below) yields  $g_\lambda > 0$  in  $Z(\lambda)$  or  $g_\lambda \equiv 0$ . We assume by contradiction that  $g_\lambda \equiv 0$ . We fix  $x \in \partial F \cap \{x \in \mathbb{R}^n, x_n < \lambda\}$  such that  $v(x) = 0$ . The definition of  $g_\lambda$  yields  $v(x_\lambda) = 0$  and in addition  $x_\lambda \in \partial F$ . Equation (3.33) ( $4\lambda(\lambda - x_n) = 0$ ) yields  $\lambda = 0$ : contradiction with  $-a < \lambda < 0$ . This yields (3.34) and Step 3.4.1 is proved.

We let  $(-a, \beta)$  be the largest open interval in  $(-\infty, 0)$  such that

$$g_\lambda > 0 \text{ in } Z(\lambda) \text{ for all } \lambda \in (-a, \beta).$$

**Step 3.4.2.** We claim that  $\beta = 0$ .

*Proof of Step 3.4.2:* We assume  $\beta < 0$  and we argue by contradiction. Since  $g_\lambda(x)$  for all  $x \in Z(\lambda)$  and all  $\lambda \in (-a, \beta)$ , letting  $\lambda \rightarrow \beta$ , we get that  $g_\beta \geq 0$  for  $x \in Z(\beta)$ . As in the proof of Step 3.4.1, the case  $g_\beta \equiv 0$  is discarded and the maximum principle yields  $g_\beta(x) > 0$  for all  $x \in Z(\beta)$ .

We fix  $\delta > 0$  that will be precised later. We let  $D \subset Z(\beta)$  be a smooth domain such that  $|Z(\beta) \setminus D| < \frac{\delta}{2}$ . Thus  $g_\beta(x) > 0$  when  $x \in \overline{D}$ . For  $0 < \epsilon \leq \epsilon_0$ , we define  $G_\epsilon := Z(\beta + \epsilon) \setminus D$ . We let  $\epsilon_0 > 0$  small enough such that, for any  $\epsilon \in (0, \epsilon_0)$ , we have that  $|G_\epsilon| < \delta$ ,  $\beta + \epsilon < 0$ , and  $g_{\beta+\epsilon} > 0$  in  $D$ . Equation (3.36) yields,

$$-\Delta g_{\beta+\epsilon} + d_{\beta+\epsilon} g_{\beta+\epsilon} < 0 \text{ in } G_\epsilon.$$

With (3.37) and  $g_\beta > 0$  in  $D$ , we get that  $g_{\beta+\epsilon} \geq 0$  on  $\partial G_\epsilon$ . Then, up to taking  $\delta > 0$  small enough, by Theorem 3.4.2 below, we get  $g_{\beta+\epsilon} \geq 0$  for  $x \in \overline{G_\epsilon}$ . As above, the strong maximum principle yields  $g_{\beta+\epsilon} > 0$  for  $x \in G_\epsilon$ . Consequently,

$g_{\beta+\epsilon} > 0$  in  $Z(\beta + \epsilon)$ . This contradicts the maximality of  $\beta$ . Then  $\beta = 0$  and  $g_\lambda(x) > 0$  for  $\lambda \in (-a, 0)$  and  $x \in Z(\lambda)$ . This proves (3.34).

Letting  $\lambda \rightarrow 0$  in (3.34), we get that  $v(x'', -x_n) \geq v(x'', x_n)$  for all  $x \in F$  such that  $x_n \leq 0$ . By symmetry, we get the reverse inequality. This proves (3.32).

**Proof of the first part of Theorem 3.4.1:** Permuting  $x_n$  and any  $x_j$ ,  $j \in \{k+1, \dots, n\}$ , it follows from (3.32) that  $v$  is symmetric with respect the hyperplane  $\{x_j = 0\}$ . Coming back to the definition of  $u$ , we get the desired symmetry.

**Proof of the second part of Theorem 3.4.1.** As above, this will be a consequence of a claim. We claim that

$$u(x_1, x_2, x') = u(x_2, x_1, x') \text{ in } \mathbb{R}^{k+,n-k}. \quad (3.39)$$

*Proof of (3.39).* We define  $E'_{+k} := \{x \in \mathbb{R}^{k+,n-k} ; x_1 - x_2 > 0\} := D'_1 \cap D'_2 \cap (\cap_{i=1}^k D'_i)$  where

$$D'_1 := \{x_1 + x_2 > 0\}, D'_2 := \{x_1 - x_2 > 0\} \text{ et } D'_i := \{x_i > 0\}.$$

We consider the isometry  $\sigma(x) := (\frac{x_1+x_2}{\sqrt{2}}, \frac{x_1-x_2}{\sqrt{2}}, x')$  for  $x := (x_1, x_2, x') \in \mathbb{R} \times \mathbb{R} \times (\mathbb{R}_+^{k-2} \times \mathbb{R}^{n-k})$ . We have that  $\sigma(E'_{+k}) = \mathbb{R}^{k+,n-k}$ . We define  $v(x) := u \circ \sigma(x)$  for all  $x \in E'_{+k}$ . Equation (3.30) of  $u$ , the isometry  $\sigma$  and the definition of  $v$  yield

$$-\Delta v - \frac{\gamma}{|x|^2} v = \frac{v^p}{|x|^s} \text{ in } E'_{+k}. \quad (3.40)$$

For any  $x \in \mathbb{R}^n \setminus \{0\}$ , we define the inversion  $i(x) = -\vec{e}_1 + \frac{x}{|x|^2}$ . We note that:  $i^{-1}(D'_i) = D'_i$ , and then

$$\begin{aligned} x \in i^{-1}(D'_1) &\Leftrightarrow x \in B_{\frac{1}{\sqrt{2}}} \left( \frac{1}{2}(\vec{e}_1 + \vec{e}_2) \right); \\ x \in i^{-1}(D'_2) &\Leftrightarrow x \in B_{\frac{1}{\sqrt{2}}} \left( \frac{1}{2}(\vec{e}_1 - \vec{e}_2) \right). \end{aligned}$$

We define  $\hat{v}(x) := |x|^{2-n} v(i(x))$  for all  $x \in H := i^{-1}(E'_{+k})$ , where  $v(0) = 0$  and  $0, \vec{e}_1 \in \partial H$ . Since  $v$  verifies (3.40) and by the definition of  $\hat{v}$ , we obtain that

$$-\Delta \hat{v} = \frac{\gamma}{|x|^2 |x - \vec{e}_1|^2} \hat{v} + \frac{\hat{v}^p}{|x|^s |x - \vec{e}_1|^s}.$$

We denote that  $\hat{v} \in C^2(H) \cap C(\overline{H} \setminus \{0, \vec{e}_1\})$ . Arguing as in the proof of (3.32), we get that  $\hat{v}(x_1, x_2, x') = \hat{v}(x_1, -x_2, x')$  for all  $x \in H$ . Coming back to  $v$ , and then  $u$ , we get (3.39). As noted above, this yields the second part of Theorem 3.4.1.



**Theorem 3.4.2** (Maximum Principle for small domains). *Let  $\Omega \subset \mathbb{R}^n$  be open domain and  $a \in L^\infty(\Omega)$  such that  $\|a\|_\infty \leq M$ . Then there exists  $\delta(M, n) > 0$  such that we have the following: if  $|\Omega| < \delta$  and  $u \in H^1(\Omega)$  satisfies  $-\Delta u + au \geq 0$  weakly and  $u \geq 0$  on  $\partial\Omega$ , then  $u \geq 0$  in  $\Omega$ .*

*Proof.* This result is cited in Berestycki-Nirenberg [19] and Fraenkel [52]. We give a short independent proof. Since  $-\Delta u + au \geq 0$  weakly, we have that

$$\int_{\Omega} (\langle \nabla u, \nabla \varphi \rangle + au\varphi) dx \geq 0 \text{ for all } \varphi \in H_0^1(\Omega), \varphi \geq 0.$$

We take  $\varphi := u_- := \max\{0, -u\} \in H_0^1(\Omega)$ . Since  $\nabla u_- = -1_{u < 0} \nabla u$  a.e, we get

$$\int_{\Omega} (|\nabla u_-|^2 + au_-^2) dx \leq 0.$$

Since  $u_-^2 \in L^{\frac{2^*}{2}}(\Omega)$ , Hölder's inequality yields

$$\int_{\Omega} |\nabla u_-|^2 dx \leq \|a\|_\infty \text{mes}(\Omega)^{\frac{2}{n}} \|u_-\|_{2^*}^2 \leq \|a\|_\infty \delta^{\frac{2}{n}} \|u_-\|_{2^*}^2. \quad (3.41)$$

On the other hand, it follows from Sobolev's inequality that  $\mu_{0,0}(\mathbb{R}^n) \|u_-\|_{2^*}^2 \leq \|\nabla u_-\|_2^2$ . With (3.41) and  $\delta := [\mu_{0,0}(\mathbb{R}^n)^{-1} \|a\|_\infty 2]^{-\frac{n}{2}}$ , we obtain  $\|u_-\|_{2^*}^2 = 0$ . Therefore  $u \geq 0$  in  $\Omega$ .  $\square$

### 3.5 Existence of extremals: the case of small values of $\gamma$

We estimate the functional  $J_{\gamma,s}^\Omega$  at some natural test-functions. We let  $W \in D^{1,2}(\mathbb{R}^{k_+,n-k})$  be a positive extremal for  $\mu_{\gamma,s}(\mathbb{R}^{k_+,n-k})$ . In other words,

$$J_{\gamma,s}^{\mathbb{R}^{k_+,n-k}}(W) = \frac{\int_{\mathbb{R}^{k_+,n-k}} (|\nabla W|^2 - \frac{\gamma}{|x|^2} W^2) dx}{\left( \int_{\mathbb{R}^{k_+,n-k}} \frac{|W|^{2^*(s)}}{|x|^s} dx \right)^{\frac{2}{2^*(s)}}} = \mu_{\gamma,s}(\mathbb{R}^{k_+,n-k}).$$

Therefore, there exists  $\xi > 0$  such that

$$\begin{cases} -\Delta W - \frac{\gamma}{|x|^2} W = \xi \frac{W^{2^*(s)-1}}{|x|^s} & \text{in } \mathbb{R}^{k_+,n-k}, \\ W > 0 & \text{in } \mathbb{R}^{k_+,n-k}, \\ W = 0 & \text{on } \partial\mathbb{R}^{k_+,n-k}. \end{cases} \quad (3.42)$$

They exist under the assumption that  $s > 0$  or  $\{s = 0, \gamma > 0 \text{ and } n \geq 4\}$  (see Ghoussoub-Robert [61]). By Theorem 3.3.1, there exists  $c > 0$  such that

$$W(x) \leq c|x|^{-\alpha_-} \text{ as } x \rightarrow 0. \quad (3.43)$$

It follows from Lemma 3.3.1, that there exists  $c > 0$  such that

$$|\nabla W(x)| \leq c|x|^{-1-\alpha_-} \text{ as } x \rightarrow 0. \quad (3.44)$$

Define now the Kelvin transform  $\overline{W}(x) := |x|^{2-n}W(\frac{x}{|x|^2})$ , since  $W$  satisfies (3.42), then  $\overline{W}$  also satisfies (3.42). By (3.43), (3.44) and the definition of  $\overline{W}$  we get,

$$W(x) \leq c|x|^{-\alpha_+} \text{ and } |\nabla W(x)| \leq c|x|^{-1-\alpha_+} \text{ as } |x| \rightarrow +\infty. \quad (3.45)$$

For  $r > 0$ , we define  $\tilde{B}_r := (-r, r)^k \times B_r^{n-k}(0)$ , where  $B_r^{n-k}(0)$  is the ball of center 0 and radius  $r$  in  $\mathbb{R}^{n-k}$ . We take the chart  $(\phi, U)$  of Definition 3.1.1 so that

$$\phi(\tilde{B}_{3\delta} \cap \mathbb{R}^{k_+, n-k}) = \phi(\tilde{B}_{3\delta}) \cap \Omega \text{ and } \phi(\tilde{B}_{3\delta} \cap \partial\mathbb{R}^{k_+, n-k}) = \phi(\tilde{B}_{3\delta}) \cap \partial\Omega,$$

where  $\delta > 0$ . We write the chart  $\phi = (\phi^1, \phi^2, \dots, \phi^n)$  and the pull-back metric  $g_{ij}(x) := (\phi_{Eucl}^*(x))_{ij} = (\partial_i\phi(x), \partial_j\phi(x))$  for all  $i, j = 1, \dots, n$ . The Taylor formula of  $g_{ij}(x)$  around 0 writes

$$g_{ij}(x) = \delta_{ij} + H_{ij} + O(|x|^2) \text{ with } H_{ij} := \sum_{l=1}^n [\partial_{il}\phi^j(0) + \partial_{jl}\phi^i(0)]x_l. \quad (3.46)$$

As  $x \rightarrow 0$ , the inverse metric  $g^{-1} = (g^{ij})$  expands as  $g^{-1} = Id_n - (H_{ij})_{1 \leq i, j \leq n} + O(|x|^2)$ , and the volume element is

$$|Jac(\phi)(x)| = 1 + \sum_{i,j=1}^n \partial_{ji}\phi^j(0)x_i + O(|x|^2), \quad (3.47)$$

as  $x \rightarrow 0$ . For any  $\epsilon > 0$ , we define

$$W_\epsilon(x) := \left( \eta \epsilon^{-\frac{n-2}{2}} W(\epsilon^{-1}\cdot) \right) \circ \phi^{-1}(x) \text{ for all } x \in \phi(\tilde{B}_{3\delta}) \cap \Omega \text{ and } 0 \text{ elsewhere,} \quad (3.48)$$

where  $\eta \in C_c^\infty(\mathbb{R}^n)$  is such that  $\eta(x) = 1$  for  $x \in \tilde{B}_\delta(0)$  and  $\eta(x) = 0$  for  $x \notin \tilde{B}_{2\delta}(0)$ . Theorem 3.1.2 will be the consequence of the following estimates:

**Proposition 3.5.1.** *Let  $0 \leq \gamma < \gamma_H(\mathbb{R}^{k_+, n-k}) = \frac{(n+2k-2)^2}{4}$ , and assume that there are extremals for  $\mu_{\gamma,s}(\mathbb{R}^{k_+, n-k})$ . Then there exists  $c_{\gamma,s}^\beta$  positives constants where  $\beta = 1, \dots, 3$  and for all  $k \in \{1, \dots, n\}$  and  $m = 1, \dots, k$  such that:*

1. For  $\gamma < \gamma_H(\mathbb{R}^{k_+, n-k}) - \frac{1}{4}$ , we have that

$$J_{\gamma,s}^\Omega(W_\epsilon) = \mu_{\gamma,s}(\mathbb{R}^{k_+, n-k}) (1 + GH_{\gamma,s}(\Omega)\epsilon + o(\epsilon)). \quad (3.49)$$

2. For  $\gamma = \gamma_H(\mathbb{R}^{k+,n-k}) - \frac{1}{4}$ , we have that

$$J_{\gamma,s}^\Omega(W_\epsilon) = \mu_{\gamma,s}(\mathbb{R}^{k+,n-k}) \left( 1 + GH_{\gamma,s}(\Omega) \epsilon \ln \left( \frac{1}{\epsilon} \right) + o \left( \epsilon \ln \left( \frac{1}{\epsilon} \right) \right) \right). \quad (3.50)$$

With  $GH_{\gamma,s}(\Omega)$  as in (3.7).

*Proof.* Take  $\tilde{B}_{\delta,+k} := \tilde{B}_\delta \cap \mathbb{R}^{k+,n-k}$ . For any family  $(a_\epsilon)_{\epsilon>0} \in \mathbb{R}$ , we define

$$\Theta_\gamma(a_\epsilon) := \begin{cases} o(a_\epsilon) & \text{if } \gamma < \gamma_H(\mathbb{R}^{k+,n-k}) - \frac{1}{4}, \\ O(a_\epsilon) & \text{if } \gamma = \gamma_H(\mathbb{R}^{k+,n-k}) - \frac{1}{4}. \end{cases} \quad \text{as } \epsilon \rightarrow 0.$$

In order to get lighter computations, we take the following conventions: the integral symbol  $\int$  means  $\int_{\tilde{B}_{\epsilon^{-1}\delta,+k}}$ , and  $A_\epsilon^\alpha := \tilde{B}_{\epsilon^{-1}\delta} \cap \{x_\alpha = 0\}$ .

**Step 3.5.1.** We claim that

$$\begin{aligned} \int_\Omega |\nabla W_\epsilon|^2 dx &= \int_{\mathbb{R}^{k+,n-k}} |\nabla W|^2 dx + \epsilon \sum_{1 \leq i \leq k; j \geq 1} \partial_{ji} \phi^j(0) \int |\nabla W|^2 x_i dx \\ &\quad - 2\epsilon \sum_{m=1}^k (A_{1,m} + A_{2,m} + \partial_{mm} \phi^m(0) \int \partial_m W \partial_m W x_m dx \\ &\quad + \sum_{i \geq 1; i \neq m} [\partial_{mi} \phi^i(0) \int \partial_i W \partial_m W x_i dx + \partial_{im} \phi^i(0) \int \partial_i W \partial_i W x_m dx]) \\ &\quad + \sum_{p=1; p \neq m}^k \sum_{q=p+1; q \neq m}^k [\partial_{qp} \phi^m(0) \int \partial_m W \partial_q W x_p dx \\ &\quad + \partial_{pq} \phi^m(0) \int \partial_m W \partial_p W x_q dx] + \Theta_\gamma(\epsilon) \text{ as } \epsilon \rightarrow 0 \end{aligned}$$

where or  $m = 1, \dots, k$ , we define  $x_{0,m} := (x_1, \dots, 0^m, \dots, x_k, x_{k+1}, \dots, x_n)$  and

$$\begin{aligned} A_{1,m} &:= \sum_{i=1; i \neq m}^k \partial_{ii} \phi^m(0) \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} \partial_m W x_i \partial_i W dx. \\ A_{2,m} &:= \sum_{i=k+1}^n \partial_{ii} \phi^m(0) \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} \partial_m W x_i \partial_i W dx. \\ B_{1,m} &:= \sum_{i \geq 1; i \neq m}^k \partial_{ii} \phi^m(0) \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} \frac{|W|^{2^*(\sigma)}}{|x|^\sigma} \frac{x_m}{|x|^2} x_i^2 dx. \\ B_{2,m} &:= \sum_{i=k+1}^n \partial_{ii} \phi^m(0) \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} \frac{|W|^{2^*(\sigma)}}{|x|^\sigma} \frac{x_m}{|x|^2} x_i^2 dx. \end{aligned}$$

Note that

$$\left\{ \begin{array}{l} \alpha_+ - \alpha_- > 1 \Leftrightarrow \gamma < \gamma_H(\mathbb{R}^{k+,n-k}) - \frac{1}{4} \\ \alpha_+ - \alpha_- = 1 \Leftrightarrow \gamma = \gamma_H(\mathbb{R}^{k+,n-k}) - \frac{1}{4} \end{array} \right\} \quad (3.51)$$

*Proof of Step 3.5.1:* By (3.44) and (3.45), there exists  $c_1 > 0$  such that

$$|\nabla W_\epsilon(x)| \leq c_1 \epsilon^{\alpha_+ - \frac{n-2}{2}} |x|^{-1-\alpha_+} \text{ for any } x \in \Omega. \quad (3.52)$$

Therefore,

$$\begin{aligned} & \int_{\phi((\tilde{B}_{3\delta} \setminus \tilde{B}_\delta) \cap \mathbb{R}^{k+,n-k})} |\nabla W_\epsilon|^2 dx \\ & \leq c_1^2 \epsilon^{2\alpha_+ - n + 2} \int_{\phi((\tilde{B}_{3\delta} \setminus \tilde{B}_\delta) \cap \mathbb{R}^{k+,n-k})} |x|^{-2-2\alpha_+} dx \end{aligned}$$

since  $2\alpha_+ - n + 2 = \alpha_+ - \alpha_-$ , we get that

$$\int_{\phi((\tilde{B}_{3\delta} \setminus \tilde{B}_\delta) \cap \mathbb{R}^{k+,n-k})} |\nabla W_\epsilon|^2 dx = \Theta_\gamma(\epsilon) \text{ as } \epsilon \rightarrow 0.$$

Then,

$$\int_{\Omega} |\nabla W_\epsilon|^2 dx = \int_{\tilde{B}_{\delta,+k}} |\nabla(W_\epsilon \circ \phi)|_{\phi^* Eucl}^2 |Jac(\phi)| dx + \Theta_\gamma(\epsilon) \text{ as } \epsilon \rightarrow 0.$$

It follows from (3.46) and for any  $\theta \in (0, 1]$  that

$$\begin{aligned} \int_{\Omega} |\nabla W_\epsilon|^2 dx &= \int |\nabla(W_\epsilon \circ \phi)|_{Eucl}^2 |Jac(\phi)| dx \\ &\quad - \sum_{i,j \geq 1} \int H_{ij} \partial_i(W_\epsilon \circ \phi) \partial_j(W_\epsilon \circ \phi) |Jac(\phi)| dx \\ &\quad + O\left( \int_{\tilde{B}_{\delta,+k}} |x|^{1+\theta} |\nabla(W_\epsilon \circ \phi)|^2 dx \right) + \Theta_\gamma(\epsilon) \text{ as } \epsilon \rightarrow 0. \end{aligned}$$

Using (3.46), we get  $\sum_{i,j \geq 1} H_{ij} = 2 \sum_{i,j,l \geq 1} \partial_{il} \phi^j(0) x_l$ , and then

$$\begin{aligned} \int_{\Omega} |\nabla W_\epsilon|^2 dx &= \int_{\tilde{B}_{\delta,+k}} |\nabla(W_\epsilon \circ \phi)|_{Eucl}^2 |Jac(\phi)| dx \\ &\quad - 2 \sum_{i,j,l \geq 1} \partial_{ij} \phi^l(0) \int_{\tilde{B}_{\delta,+k}} \partial_l(W_\epsilon \circ \phi) \partial_i(W_\epsilon \circ \phi) x_j |Jac(\phi)| dx \quad (3.53) \end{aligned}$$

$$+ O\left( \int_{\tilde{B}_{\delta,+k}} |x|^{1+\theta} |\nabla(W_\epsilon \circ \phi)|^2 dx \right) + \Theta_\gamma(\epsilon) \quad (3.54)$$

as  $\epsilon \rightarrow 0$ . The two equations (3.47), (3.48) and the change of variable  $x := \epsilon y$  yield as  $\epsilon \rightarrow 0$ ,

$$\begin{aligned} & \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} |\nabla(W_\epsilon \circ \phi)|_{Eucd}^2 |Jac(\phi)| dx = \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} |\nabla W|^2 dx \\ & + \epsilon \sum_{1 \leq i \leq k; j \geq 1} \partial_{ji} \phi^j(0) \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} |\nabla W|^2 x_i dx \\ & + \epsilon \sum_{k+1 \leq i \leq n; j \geq 1} \partial_{ji} \phi^j(0) \int |\nabla W|^2 x_i dx + O\left(\int |x|^2 |\nabla(W_\epsilon \circ \phi)|^2 dx\right) \end{aligned} \quad (3.55)$$

and

$$\begin{aligned} & \int_{\tilde{B}_{\delta,+k}} \partial_l(W_\epsilon \circ \phi) \partial_i(W_\epsilon \circ \phi) x_j |Jac(\phi)| dx \\ & = \epsilon \int \partial_l W \partial_i W x_j dx + O\left(\int |x|^2 |\nabla(W_\epsilon \circ \phi)|^2 dx\right). \end{aligned} \quad (3.56)$$

Plugging together (3.54), (3.55), (3.56) yields

$$\begin{aligned} & \int_{\Omega} |\nabla W_\epsilon|^2 dx = \int |\nabla W|^2 dx + \epsilon \sum_{1 \leq i \leq k; j \geq 1} \partial_{ji} \phi^j(0) \int |\nabla W|^2 x_i dx \\ & + \epsilon \sum_{k+1 \leq i \leq n; j \geq 1} \partial_{ji} \phi^j(0) \int |\nabla W|^2 x_i dx - 2\epsilon \sum_{i,j,l \geq 1} \partial_{ij} \phi^l(0) \int \partial_l W \partial_i W x_j dx \\ & + O\left(\int |x|^{1+\theta} |\nabla(W_\epsilon \circ \phi)|^2 dx\right) + \Theta_\gamma(\epsilon) \text{ as } \epsilon \rightarrow 0. \end{aligned}$$

- If  $\gamma = \gamma_H(\mathbb{R}^{k+,n-k}) - \frac{1}{4}$ , we choose  $\theta \in (0, 1)$ .
- If  $\gamma < \gamma_H(\mathbb{R}^{k+,n-k}) - \frac{1}{4}$ , we choose  $0 < \theta < \alpha_+ - \alpha_- - 1$  (see (3.51)).

Therefore, it follows from (3.52) that we have as  $\epsilon \rightarrow 0$  that,

$$\int_{\tilde{B}_{\delta,+k}} |x|^{1+\theta} |\nabla(W_\epsilon \circ \phi)|^2 dx = \Theta_\gamma(\epsilon). \quad (3.57)$$

Since  $\gamma \geq 0$ , we use the symmetry of  $W$  (see Theorem 3.4.1). For  $i \geq k+1$ ,  $W$  and  $\mathbb{R}^{k+,n-k}$  are invariant by  $x \rightarrow (x_1, \dots, -x_i, \dots, x_n)$ , then a change of variables yields

$$\int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} |\nabla W|^2 x_i dx = - \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} |\nabla W|^2 x_i dx = 0. \quad (3.58)$$

This equality and (3.57) yield

$$\begin{aligned} \int_{\Omega} |\nabla W_{\epsilon}|^2 dx &= \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} |\nabla W|^2 dx + \epsilon \sum_{1 \leq i \leq k; j \geq 1} \partial_{ji} \phi^j(0) \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} |\nabla W|^2 x_i dx \\ &\quad - 2\epsilon \sum_{i,j,l \geq 1} \partial_{ij} \phi^l(0) \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} \partial_l W \partial_i W x_j dx + \Theta_{\gamma}(\epsilon) \text{ as } \epsilon \rightarrow 0. \end{aligned} \quad (3.59)$$

The inequation (3.45) and  $-2 - 2\alpha_+ + n = -(\alpha_+ - \alpha_-)$  yields,

$$\left| \int_{\mathbb{R}^{k_+,n-k} \setminus \tilde{B}_{\epsilon^{-1}\delta,+k}} |\nabla W|^2 dx \right| \leq c^2 \int_{\mathbb{R}^{k_+,n-k} \setminus \tilde{B}_{\delta,+k}} |x|^{-2-2\alpha_+} dx \leq c_1 \epsilon^{\alpha_+ - \alpha_-},$$

therefore,

$$\int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} |\nabla W|^2 dx = \int_{\mathbb{R}^{k_+,n-k}} |\nabla W|^2 dx + \Theta_{\gamma}(\epsilon) \text{ as } \epsilon \rightarrow 0. \quad (3.60)$$

Using again the symmetry of  $W$  as in (3.58), we get

$$\begin{aligned} &\sum_{i,j,l \geq 1} \partial_{ij} \phi^l(0) \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} \partial_l W \partial_i W x_j dx \\ &= \sum_{m=1}^k \left( A_{1,m} + A_{2,m} + \partial_{mm} \phi^m(0) \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} \partial_m W \partial_m W x_m dx \right. \\ &\quad \left. + \sum_{i \geq 1; i \neq m} \left[ \partial_{mi} \phi^i(0) \int \partial_i W \partial_m W x_i dx + \partial_{im} \phi^i(0) \int \partial_i W \partial_i W x_m dx \right] \right. \\ &\quad \left. + \sum_{p=1; p \neq m}^k \sum_{q=p+1; q \neq m}^k \left[ \partial_{qp} \phi^m(0) \int \partial_m W \partial_q W x_p dx \right. \right. \\ &\quad \left. \left. + \partial_{pq} \phi^m(0) \int \partial_m W \partial_p W x_q dx \right] \right). \end{aligned}$$

Combining (3.59), (3.60) and the last equation, we get Step 1.  $\square$

**Step 3.5.2.** We fix  $\sigma \in [0, 2]$ . We claim that

$$\begin{aligned}
& \int_{\Omega} \frac{|W_{\epsilon}|^{2^*(\sigma)}}{|x|^{\sigma}} dx \\
&= \int_{\mathbb{R}^{k_+, n-k}} \frac{|W|^{2^*(\sigma)}}{|x|^{\sigma}} dx + \epsilon \sum_{1 \leq i \leq k; j \geq 1} \partial_{ji} \phi^j(0) \int_{\tilde{B}_{\epsilon^{-1}\delta, +k}} \frac{|W|^{2^*(\sigma)}}{|x|^{\sigma}} x_i dx \\
&\quad - \epsilon \frac{\sigma}{2} \sum_{m=1}^k \left( B_{1,m} + B_{2,m} + \partial_{mm} \phi^m(0) \int_{\tilde{B}_{\epsilon^{-1}\delta, +k}} \frac{|W|^{2^*(\sigma)}}{|x|^{\sigma}} \frac{x_m}{|x|^2} x_m^2 dx \right. \\
&\quad + 2 \sum_{i \geq 1; i \neq m} \partial_{mi} \phi^i(0) \int_{\tilde{B}_{\epsilon^{-1}\delta, +k}} \frac{|W|^{2^*(\sigma)}}{|x|^{\sigma}} \frac{x_i}{|x|^2} x_m x_i dx \\
&\quad \left. + 2 \sum_{p=1; p \neq m}^k \sum_{q=p+1; q \neq m}^k \partial_{qp} \phi^m(0) \int_{\tilde{B}_{\epsilon^{-1}\delta, +k}} \frac{|W|^{2^*(\sigma)}}{|x|^{\sigma}} \frac{x_m}{|x|^2} x_q x_p dx \right) + \Theta_{\gamma}(\epsilon).
\end{aligned}$$

*Proof of Step 3.5.2:* Equations (3.44) and (3.45) yield

$$|W_{\epsilon}(x)| \leq c\epsilon^{\alpha_+ - \frac{n-2}{2}} |x|^{-\alpha_+} \text{ for all } \epsilon > 0 \text{ and } x \in \Omega, \quad (3.61)$$

this implies,

$$\left| \int_{\phi(\tilde{B}_{3\delta} \setminus \tilde{B}_{\delta}) \cap \Omega} \frac{|W_{\epsilon}|^{2^*(\sigma)}}{|x|^{\sigma}} dx \right| \leq c^{2^*(\sigma)} \epsilon^{2^*(\sigma)(\alpha_+ - \frac{n-2}{2})} \int_{\phi(\tilde{B}_{3\delta} \setminus \tilde{B}_{\delta}) \cap \Omega} |x|^{-\alpha_+ 2^*(\sigma) - \sigma} dx$$

and then, since  $2^*(\sigma) \geq 2$  and  $\alpha_+ + \alpha_- = n - 2$ , we get that

$$\int_{\phi(\tilde{B}_{3\delta} \setminus \tilde{B}_{\delta}) \cap \Omega} \frac{|W_{\epsilon}|^{2^*(\sigma)}}{|x|^{\sigma}} dx = \Theta_{\gamma}(\epsilon).$$

Therefore,

$$\int_{\Omega} \frac{|W_{\epsilon}|^{2^*(\sigma)}}{|x|^{\sigma}} dx = \int_{\tilde{B}_{\delta, +k}} \frac{|W_{\epsilon} \circ \phi|^{2^*(\sigma)}}{|\phi(x)|^{\sigma}} |Jac(\phi)| dx + \Theta_{\gamma}(\epsilon) \text{ as } \epsilon \rightarrow 0. \quad (3.62)$$

We choose  $\theta \in (0, 1)$  as follows.

- If  $\gamma < \gamma_H(\mathbb{R}^{k_+, n-k}) - \frac{1}{4}$  or  $\{\gamma = \gamma_H(\mathbb{R}^{k_+, n-k}) - \frac{1}{4} \text{ and } \sigma < 2\}$  we choose  $\theta \in (0, (\alpha_+ - \alpha_-) \frac{2^*(\sigma)}{2} - 1) \cap (0, 1)$ .
- If  $\gamma = \gamma_H(\mathbb{R}^{k_+, n-k}) - \frac{1}{4}$  and  $\sigma = 2$ , we choose  $0 < \theta < 1$ .

This choice makes sense due to (3.51). Since  $d\phi_0 = Id$ , a Taylor expansion yields

$$|\phi(x)|^{-\sigma} = |x|^{-\sigma} \left[ 1 - \frac{\sigma}{2|x|^2} \sum_{i,j,l \geq 1} \partial_{ij} \phi^l(0) x_l x_i x_j + O(|x|^{1+\theta}) \right] \text{ as } \epsilon \rightarrow 0. \quad (3.63)$$

Inequality (3.61) yields,

$$\int_{\tilde{B}_{\delta,+k}} \frac{|W_\epsilon \circ \phi|^{2^*(\sigma)} |x|^{1+\theta}}{|\phi(x)|^\sigma} dx = \Theta_\gamma(\epsilon).$$

The estimates (3.62), (3.63) and the last equation get,

$$\begin{aligned} \int_{\Omega} \frac{|W_\epsilon|^{2^*(\sigma)}}{|x|^\sigma} dx &= \int_{\tilde{B}_{\delta,+k}} \frac{|W_\epsilon \circ \phi|^{2^*(\sigma)}}{|x|^\sigma} |Jac(\phi)| dx \\ &- \frac{\sigma}{2} \sum_{i,j,l \geq 1} \partial_{ij} \phi^l(0) \int_{\tilde{B}_{\delta,+k}} \frac{|W_\epsilon \circ \phi|^{2^*(\sigma)}}{|x|^\sigma} \frac{x_l}{|x|^2} x_i x_j |Jac(\phi)| dx + \Theta_\gamma(\epsilon) \end{aligned} \quad (3.64)$$

In view of (3.47), (3.48) and the change of variable  $x := \epsilon y$  yield as  $\epsilon \rightarrow 0$ ,

$$\begin{aligned} &\int_{\tilde{B}_{\delta,+k}} \frac{|W_\epsilon|^{2^*(\sigma)}}{|x|^\sigma} |Jac(\phi)| dx \quad (3.65) \\ &= \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} \frac{|W|^{2^*(\sigma)}}{|x|^\sigma} dx + \epsilon \sum_{1 \leq i \leq k; j \geq 1} \partial_{ji} \phi^j(0) \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} \frac{|W|^{2^*(\sigma)}}{|x|^\sigma} x_i dx \\ &+ \epsilon \sum_{k+1 \leq i \leq n; j \geq 1} \partial_{ji} \phi^j(0) \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} \frac{|W|^{2^*(\sigma)}}{|x|^\sigma} x_i dx + \Theta_\gamma(\epsilon). \end{aligned}$$

And,

$$\int_{\tilde{B}_{\delta,+k}} \frac{|W_\epsilon|^{2^*(\sigma)}}{|x|^\sigma} \frac{x_l}{|x|^2} x_i x_j |Jac(\phi)| dx = \epsilon \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} \frac{|W|^{2^*(\sigma)}}{|x|^\sigma} \frac{x_l}{|x|^2} x_i x_j dx + \Theta_\gamma(\epsilon). \quad (3.66)$$

Plugging together (3.64), (3.65), (3.66) yields,

$$\begin{aligned} \int_{\Omega} \frac{|W_\epsilon|^{2^*(\sigma)}}{|x|^\sigma} dx &= \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} \frac{|W|^{2^*(\sigma)}}{|x|^\sigma} dx + \epsilon \sum_{1 \leq i \leq k; j \geq 1} \partial_{ji} \phi^j(0) \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} \frac{|W|^{2^*(\sigma)}}{|x|^\sigma} x_i dx \\ &+ \epsilon \sum_{k+1 \leq i \leq n; j \geq 1} \partial_{ji} \phi^j(0) \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} \frac{|W|^{2^*(\sigma)}}{|x|^\sigma} x_i dx \\ &- \epsilon \frac{\sigma}{2} \sum_{i,j,l \geq 1} \partial_{ij} \phi^l(0) \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} \frac{|W|^{2^*(\sigma)}}{|x|^\sigma} \frac{x_l}{|x|^2} x_i x_j dx + \Theta_\gamma(\epsilon). \end{aligned}$$



By equation (3.45), we have

$$\int_{\mathbb{R}^{k_+, n-k} \setminus \tilde{B}_{\epsilon^{-1}\delta, +k}} \frac{|W|^{2^*(\sigma)}}{|x|^\sigma} dx = \Theta_\gamma(\epsilon).$$

Since  $\gamma \geq 0$ , using the symmetry of  $W$  as in (3.58) and the last equation,

$$\begin{aligned} \int_{\Omega} \frac{|W_\epsilon|^{2^*(\sigma)}}{|x|^\sigma} dx &= \int_{\mathbb{R}^{k_+, n-k}} \frac{|W|^{2^*(\sigma)}}{|x|^\sigma} dx \\ &+ \epsilon \sum_{1 \leq i \leq k; j \geq 1} \partial_{ji} \phi^j(0) \int_{\tilde{B}_{\epsilon^{-1}\delta, +k}} \frac{|W|^{2^*(\sigma)}}{|x|^\sigma} x_i dx \\ &- \epsilon \frac{\sigma}{2} \sum_{i, j, l \geq 1} \partial_{ij} \phi^l(0) \int_{\tilde{B}_{\epsilon^{-1}\delta, +k}} \frac{|W|^{2^*(\sigma)}}{|x|^\sigma} \frac{x_l}{|x|^2} x_i x_j dx + \Theta_\gamma(\epsilon). \end{aligned} \quad (3.67)$$

We use again the symmetry of  $W$ ,

$$\begin{aligned} &\sum_{i, j, l \geq 1} \partial_{ij} \phi^l(0) \int_{\tilde{B}_{\epsilon^{-1}\delta, +k}} \frac{|W|^{2^*(\sigma)}}{|x|^\sigma} \frac{x_l}{|x|^2} x_i x_j dx = \sum_{m=1}^k (B_{1,m} + B_{2,m} \\ &+ \partial_{mm} \phi^m(0) \int_{\tilde{B}_{\epsilon^{-1}\delta, +k}} \frac{|W|^{2^*(\sigma)}}{|x|^\sigma} \frac{x_m}{|x|^2} x_m^2 dx \\ &+ \sum_{i \geq 1; i \neq m} \left[ \partial_{mi} \phi^i(0) \int_{\tilde{B}_{\epsilon^{-1}\delta, +k}} \frac{|W|^{2^*(\sigma)}}{|x|^\sigma} \frac{x_i}{|x|^2} x_m x_i dx \right. \\ &\left. + \partial_{im} \phi^i(0) \int_{\tilde{B}_{\epsilon^{-1}\delta, +k}} \frac{|W|^{2^*(\sigma)}}{|x|^\sigma} \frac{x_i}{|x|^2} x_i x_m dx \right] \\ &+ \sum_{p=1; p \neq m}^k \sum_{q=p+1; q \neq m}^k \left[ \partial_{qp} \phi^m(0) \int_{\tilde{B}_{\epsilon^{-1}\delta, +k}} \frac{|W|^{2^*(\sigma)}}{|x|^\sigma} \frac{x_m}{|x|^2} x_q x_p dx \right. \\ &\left. + \partial_{pq} \phi^m(0) \int_{\tilde{B}_{\epsilon^{-1}\delta, +k}} \frac{|W|^{2^*(\sigma)}}{|x|^\sigma} \frac{x_m}{|x|^2} x_p x_q dx \right] \Big). \end{aligned}$$

Replace the last equation in (3.67), we get Step 2.  $\square$

**Step 3.5.3.** We now prove (3.49) and (3.50). We fix  $m \in \{1, \dots, k\}$ . For:  $i = 1, \dots, n$ ;  $l = k+1, \dots, n$ ;  $p = 1, \dots, k$  and  $q = p+1, \dots, k$  such that  $i, p, q \neq m$ ,

we define

$$\begin{aligned}
M_{p,m} &:= \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} \partial_m W x_p \partial_p W \, dx \quad \text{and} \quad M_{l,m} := \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} \partial_m W x_l \partial_l W \, dx. \\
K_{i,m} &:= \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} \partial_i W \partial_m W x_i \, dx \quad \text{and} \quad J_{i,m} := \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} \partial_i W \partial_i W x_m \, dx. \\
L_{m,p,q} &:= \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} \partial_m W \partial_p W x_q \, dx \quad \text{and} \quad N_{m,p,q} := \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} \partial_m W \partial_q W x_p \, dx. \\
I_m &:= \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} \partial_m W \partial_m W x_m \, dx.
\end{aligned}$$

**Lemma 3.5.1.** *Here  $\xi > 0$  and  $s \in [0, 2]$ , we have as  $\epsilon \rightarrow 0$  that:*

$$\begin{aligned}
2I_m &= \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} \frac{x_m^2}{|x|^2} x_m \left( \xi \frac{s}{2^*(s)} \frac{W^{2^*(s)}}{|x|^s} + \gamma \frac{W^2}{|x|^2} \right) dx \\
&\quad + \xi \left( 1 - \frac{2}{2^*(s)} \right) \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} x_m \frac{W^{2^*(s)}}{|x|^s} dx + \Theta_\gamma(1). \\
2M_{p,m} &= \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} \frac{x_p^2}{|x|^2} x_m \left( \xi \frac{s}{2^*(s)} \frac{|W|^{2^*(s)}}{|x|^s} + \gamma \frac{W^2}{|x|^2} \right) dx \\
&\quad - \int_{\tilde{B}_{\epsilon^{-1}\delta} \cap \{x_m=0\}} \frac{x_p^2 |\partial_m W|^2}{2} d\sigma + \Theta_\gamma(1). \\
2M_{l,m} &= \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} \frac{x_l^2}{|x|^2} x_m \left( \xi \frac{s}{2^*(s)} \frac{|W|^{2^*(s)}}{|x|^s} + \gamma \frac{W^2}{|x|^2} \right) dx \\
&\quad - \int_{\tilde{B}_{\epsilon^{-1}\delta} \cap \{x_m=0\}} \frac{x_l^2 |\partial_m W|^2}{2} d\sigma + \Theta_\gamma(1). \\
K_{i,m} + J_{i,m} &= \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} \frac{x_i^2}{|x|^2} x_m \left( \xi \frac{s}{2^*(s)} \frac{W^{2^*(s)}}{|x|^s} + \gamma \frac{W^2}{|x|^2} \right) dx \\
&\quad + \left( \frac{1}{2} - \frac{1}{2^*(s)} \right) \xi \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} x_m \frac{W^{2^*(s)}}{|x|^s} dx + \Theta_\gamma(1). \\
L_{m,p,q} + N_{m,p,q} &= \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} \frac{x_q x_p}{|x|^2} x_m \left( \xi \frac{s}{2^*(s)} \frac{W^{2^*(s)}}{|x|^s} + \gamma \frac{W^2}{|x|^2} \right) dx \\
&\quad - \frac{1}{2} \int_{\tilde{B}_{\epsilon^{-1}\delta} \cap \{x_m=0\}} x_q x_p (\partial_m W)^2 dx + \Theta_\gamma(1).
\end{aligned}$$

*Proof of Lemma 3.5.1.* We first state two preliminary remarks. First

$$\int_{\partial\tilde{B}_{\epsilon^{-1}\delta}\cap\mathbb{R}^{k_+,n-k}} (W^2 + |x|W|\nabla W| + |x|^2|\nabla W|^2) dx = \Theta_\gamma(1). \quad (3.68)$$

Another remark we will use often is that

$$\partial_i W(x) = 0 \text{ if } x_j = 0, j \neq i, j \leq k \quad (3.69)$$

We want to calculate the value of

$$I_m = \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} \partial_m W \partial_m W x_m dx = \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} (\partial_m W)^2 \partial_m \left( \frac{x_m^2}{2} \right) dx.$$

For any domain  $D$ , we define  $\nu$  as the outer normal vector at a boundary point of  $D$  when this makes sense. For any  $j = 1, \dots, n$ ,  $\nu_j$  denote the  $j$ th coordinate. In the sequel, the normal vector will be defined except on lower dimensional portions of the boundary and the computations will be valid. On  $\{x_\alpha = 0\} = \partial\{x_\alpha > 0\}$ , the outer normal vector is  $(0, \dots, -1, \dots, 0) = (\nu_{\alpha,i})_{i=1,\dots,n}$  where  $\nu_{i,j} := -\delta_{ij}$  for  $i = 1, \dots, k$  and  $j \geq 1$ . Since  $W(x_{0,m}) = 0$ , (3.68) and integrations by parts yield

$$\begin{aligned} I_m &= - \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} x_m^2 \partial_m W \partial_{mm} W dx + \int_{\partial(\tilde{B}_{\epsilon^{-1}\delta,+k})} \frac{x_m^2 (\partial_m W)^2}{2} \nu_m dx \\ &= - \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} x_m^2 \partial_m W [\Delta W - \sum_{i \geq 1; i \neq m} \partial_{ii} W] dx \\ &\quad + O \left( \int_{\partial\tilde{B}_{\epsilon^{-1}\delta}\cap\mathbb{R}^{k_+,n-k}} |x|^2 |\nabla W|^2 dx \right) \\ &= \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} x_m^2 \partial_m W (-\Delta W) dx \\ &\quad + \sum_{i \geq 1; i \neq m} \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} x_m^2 \partial_m W \partial_{ii} W dx + \Theta_\gamma(1) \\ &= \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} x_m^2 \partial_m W (-\Delta W) dx - \sum_{i \geq 1; i \neq m} \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} x_m^2 \partial_{im} W \partial_i W dx \\ &\quad + \sum_{i \geq 1; i \neq m} \int_{\partial(\tilde{B}_{\epsilon^{-1}\delta,+k})} x_m^2 \partial_m W \partial_i W \nu_i dx + \Theta_\gamma(1) \\ &= \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} x_m^2 \partial_m W (-\Delta W) dx - \sum_{i \geq 1; i \neq m} \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} x_m^2 \partial_m \left( \frac{(\partial_i W)^2}{2} \right) dx \\ &\quad + \sum_{i \geq 1; i \neq m} \sum_{\alpha=1}^k \int_{B_{\epsilon^{-1}\delta} \cap \{x_\alpha=0\}} x_m^2 \partial_m W \partial_i W \nu_{\alpha,i} d\sigma + \Theta_\gamma(1). \end{aligned}$$

Using again the integrations by parts and (3.68), we get

$$\begin{aligned}
I_m &= \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} x_m^2 \partial_m W (-\Delta W) dx \\
&\quad - \sum_{i \geq 1; i \neq m} \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} x_m^2 \partial_m \left( \frac{(\partial_i W)^2}{2} \right) dx + \Theta_\gamma(1) \\
&= \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} x_m^2 \partial_m W (-\Delta W) dx + \sum_{i \geq 1; i \neq m} \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} x_m (\partial_i W)^2 dx \\
&\quad - \sum_{i \geq 1; i \neq m} \sum_{\alpha=1}^k \int_{\tilde{B}_{\epsilon^{-1}\delta} \cap \{x_\alpha=0\}} x_m^2 \frac{(\partial_i W)^2}{2} \nu_{\alpha,m} dx + \Theta_\gamma(1) \\
&= \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} x_m^2 \partial_m W (-\Delta W) dx \\
&\quad + \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} x_m (|\nabla W|^2 - (\partial_m W)^2) dx + \Theta_\gamma(1) \\
&= \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} x_m^2 \partial_m W (-\Delta W) dx + \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} x_m |\nabla W|^2 dx \\
&\quad - \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} x_m (\partial_m W)^2 dx + \Theta_\gamma(1) \\
&= \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} x_m^2 \partial_m W (-\Delta W) dx + \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} x_m |\nabla W|^2 dx - I_m + \Theta_\gamma(1).
\end{aligned}$$

With equation (3.42), we then get

$$\begin{aligned}
2I_m &= \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} x_m^2 \partial_m W \left( \xi \frac{W^{2^*(s)-1}}{|x|^s} + \gamma \frac{W}{|x|^2} \right) dx \\
&\quad + \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} x_m^2 |\nabla W|^2 dx + \Theta_\gamma(1).
\end{aligned} \tag{3.70}$$

Integrating by parts, using that  $W$  vanishes on  $\partial\mathbb{R}^{k_+,n-k}$ , we get that

$$\begin{aligned}
&\int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} x_m^2 \partial_m W \frac{W^{2^*(\sigma)-1}}{|x|^\sigma} dx = \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} x_m^2 |x|^{-\sigma} \partial_m \left( \frac{W^{2^*(\sigma)}}{2^*(\sigma)} \right) dx \\
&= - \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} \partial_m (x_m^2 |x|^{-\sigma}) \frac{W^{2^*(\sigma)}}{2^*(\sigma)} dx + \int_{\partial(\tilde{B}_{\epsilon^{-1}\delta,+k})} x_m^2 |x|^{-\sigma} \frac{W^{2^*(\sigma)}}{2^*(\sigma)} \nu_m dx \\
&= - \frac{2}{2^*(\sigma)} \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} x_m \frac{W^{2^*(\sigma)}}{|x|^\sigma} dx \\
&\quad + \frac{\sigma}{2^*(\sigma)} \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} x_m^2 x_m \frac{W^{2^*(\sigma)}}{|x|^{\sigma+2}} dx + \Theta_\gamma(1)
\end{aligned} \tag{3.71}$$

as  $\epsilon \rightarrow 0$ . We claim that

$$\int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} x_m |\nabla W|^2 dx = \gamma \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} x_m \frac{W^2}{|x|^2} dx + \xi \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} x_m \frac{W^{2^*(s)}}{|x|^s} dx + \Theta_\gamma(1). \quad (3.72)$$

*Proof of (3.72).* We multiply equation (3.42) by  $x_m W$  and integrate by parts to get

$$\begin{aligned} \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} x_m |\nabla W|^2 dx &= - \int \nabla(x_m) W \nabla W dx + \int_{\partial} x_m W \partial_\nu W dx \\ &\quad + \gamma \int x_m \frac{W^2}{|x|^2} dx + \xi \int x_m \frac{W^{2^*(s)}}{|x|^s} dx \\ &= - \int \nabla(x_m) \nabla \left( \frac{W^2}{2} \right) dx + \int_{\partial} x_m W \partial_\nu W dx \\ &\quad + \gamma \int x_m \frac{W^2}{|x|^2} dx + \xi \int x_m \frac{W^{2^*(s)}}{|x|^s} dx \\ &= - \int_{\partial} \frac{W^2}{2} \partial_\nu x_m dx + \int_{\partial} x_m W \partial_\nu W dx \quad (3.73) \\ &\quad + \gamma \int x_m \frac{W^2}{|x|^2} dx + \xi \int x_m \frac{W^{2^*(s)}}{|x|^s} dx \end{aligned}$$

where all integrals are taken on  $\tilde{B}_{\epsilon^{-1}\delta,+k}$  or  $\partial\tilde{B}_{\epsilon^{-1}\delta,+k}$ . Since  $W$  vanishes on  $\partial\mathbb{R}^{k_+,n-k}$  and by (3.68), we have

$$\int_{\partial\tilde{B}_{\epsilon^{-1}\delta,+k}} x_m W \partial_\nu W dx = \int_{\partial\tilde{B}_{\epsilon^{-1}\delta} \cap \mathbb{R}^{k_+,n-k}} x_m W \partial_\nu W dx = \Theta_\gamma(1). \quad (3.74)$$

And,

$$\int_{\partial\tilde{B}_{\epsilon^{-1}\delta,+k}} \frac{W^2}{2} \partial_\nu x_m dx = \int_{\partial\tilde{B}_{\epsilon^{-1}\delta} \cap \mathbb{R}^{k_+,n-k}} \frac{W^2}{2} \partial_\nu x_m dx = \Theta_\gamma(1). \quad (3.75)$$

Then (3.73), (3.74) and (3.75) yields (3.72).  $\square$

Combining (3.70), (3.71) and (3.72), we obtain

$$\begin{aligned}
2I_m &= \xi \left[ -\frac{2}{2^*(s)} \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} x_m \frac{W^{2^*(s)}}{|x|^s} dx + \frac{s}{2^*(s)} \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} x_m^2 x_m \frac{W^{2^*(s)}}{|x|^{s+2}} dx \right] \\
&\quad + \gamma \left[ -\int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} x_m \frac{W^2}{|x|^2} dx + \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} x_m^2 x_m \frac{W^2}{|x|^{2+2}} dx \right] \\
&\quad + \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} x_m |\nabla W|^2 dx + \Theta_\gamma(1) \\
&= \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} \frac{x_m^2}{|x|^2} x_m \left[ \xi \frac{s}{2^*(s)} \frac{W^{2^*(s)}}{|x|^s} + \gamma \frac{W^2}{|x|^2} \right] dx \\
&\quad - \xi \frac{2}{2^*(s)} \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} x_m \frac{W^{2^*(s)}}{|x|^s} dx - \gamma \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} x_m \frac{W^2}{|x|^2} dx \\
&\quad + \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} x_m |\nabla W|^2 dx + \Theta_\gamma(1)
\end{aligned}$$

And then

$$\begin{aligned}
2I_m &= \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} \frac{x_m^2}{|x|^2} x_m \left[ \xi \frac{s}{2^*(s)} \frac{W^{2^*(s)}}{|x|^s} + \gamma \frac{W^2}{|x|^2} \right] dx \quad (3.76) \\
&\quad - \xi \frac{2}{2^*(s)} \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} x_m \frac{W^{2^*(s)}}{|x|^s} dx - \gamma \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} x_m \frac{W^2}{|x|^2} dx \\
&\quad + \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} x_m \left[ \xi \frac{W^{2^*(s)}}{|x|^s} + \gamma \frac{W^2}{|x|^2} \right] dx + \Theta_\gamma(1),
\end{aligned}$$

by the last equality, we obtain the value of  $I_m$ . We now fix  $m, p \in \{1, \dots, k\}$  such that  $p \neq m$ . Integrating by parts, we get that

$$\begin{aligned}
M_{p,m} &= \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} \partial_m W x_p \partial_p W dx = \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} \partial_m W \partial_p \left( \frac{x_p^2}{2} \right) \partial_p W dx \\
&= - \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} \frac{x_p^2}{2} \partial_p (\partial_m W \partial_p W) dx + \int_{\partial(\tilde{B}_{\epsilon^{-1}\delta,+k})} \partial_m W \frac{x_p^2}{2} \partial_p W \nu_p dx,
\end{aligned}$$

with  $\nu_{i,j} := -\delta_{ij}$  for  $i = 1, \dots, k$  and  $j \geq 1$ , since  $W(x_{0,m}) = 0$ , we have that

$$\begin{aligned}
M_{p,m} &= - \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} \frac{x_p^2}{2} \partial_p (\partial_m W \partial_p W) dx \\
&+ \int_{\tilde{B}_{\epsilon^{-1}\delta} \cap \partial \mathbb{R}^{k+,n-k}} \partial_m W \frac{x_p^2}{2} \partial_p W \nu_p d\sigma + O \left( \int_{\mathbb{R}^{k+,n-k} \cap \partial \tilde{B}_{\epsilon^{-1}\delta}} |x|^2 |\nabla W|^2 d\sigma \right) \\
&= - \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} \frac{x_p^2}{2} \partial_p (\partial_m W \partial_p W) dx + \sum_{\alpha=1}^k \int_{\tilde{B}_{\epsilon^{-1}\delta} \cap \{x_\alpha=0\}} \partial_m W \frac{x_p^2}{2} \partial_p W \nu_{\alpha,p} d\sigma \\
&+ O \left( \int_{\mathbb{R}^{k+,n-k} \cap \partial \tilde{B}_{\epsilon^{-1}\delta}} |x|^2 |\nabla W|^2 d\sigma \right) \\
&= - \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} \frac{x_p^2}{2} [\partial_{mp} W \partial_p W + \partial_m W \partial_{pp} W] dx + \Theta_\gamma(1) \\
&= \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} \frac{x_p^2}{2} \partial_m W [-\Delta W] dx - \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} \frac{x_p^2}{2} \partial_m \left( \frac{|\partial_p W|^2}{2} \right) dx \\
&+ \sum_{j \geq 1; j \neq p} \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} \frac{x_p^2}{2} \partial_m W \partial_{jj} W dx + \Theta_\gamma(1) \\
&= \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} \frac{x_p^2}{2} \partial_m W [-\Delta W] dx - \int_{\partial(\tilde{B}_{\epsilon^{-1}\delta,+k})} \frac{x_p^2}{4} |\partial_p W|^2 \nu_m d\sigma \\
&+ \sum_{j \geq 1; j \neq p} \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} \frac{x_p^2}{2} \partial_m W \partial_{jj} W dx + \Theta_\gamma(1) \\
&= \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} \frac{x_p^2}{2} \partial_m W [-\Delta W] dx - \sum_{\alpha=1}^k \int_{\tilde{B}_{\epsilon^{-1}\delta} \cap \{x_\alpha=0\}} \frac{x_p^2}{4} |\partial_p W|^2 \nu_{\alpha,m} d\sigma \\
&+ \sum_{j \geq 1; j \neq p} \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} \frac{x_p^2}{2} \partial_m W \partial_{jj} W dx + \Theta_\gamma(1)
\end{aligned}$$

And then

$$\begin{aligned}
M_{p,m} &= \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} \frac{x_p^2}{2} \partial_m W [-\Delta W] dx - \sum_{j \geq 1; j \neq p} \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} \frac{x_p^2}{2} \partial_{jm} W \partial_j W dx \\
&+ \sum_{j \geq 1; j \neq p} \sum_{\alpha=1}^k \int_{\tilde{B}_{\epsilon^{-1}\delta} \cap \{x_\alpha=0\}} \frac{x_p^2}{2} \partial_m W \partial_j W \nu_{\alpha,j} d\sigma + \Theta_\gamma(1).
\end{aligned}$$

So we have

$$\begin{aligned}
M_{p,m} &= \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} \frac{x_p^2}{2} \partial_m W [-\Delta W] dx - \sum_{j \geq 1; j \neq p} \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} \partial_m \left( \frac{x_p^2}{4} |\partial_j W|^2 \right) dx \\
&+ \int_{\tilde{B}_{\epsilon^{-1}\delta} \cap \{x_m=0\}} \frac{x_p^2}{2} |\partial_m W|^2 \nu_{m,m} d\sigma + \Theta_\gamma(1) \\
&= \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} \frac{x_p^2}{2} \partial_m W [-\Delta W] dx \\
&- \sum_{j \geq 1; j \neq p} \sum_{\alpha=1}^k \int_{\tilde{B}_{\epsilon^{-1}\delta} \cap \{x_\alpha=0\}} \frac{x_p^2}{4} |\partial_j W|^2 \nu_{\alpha,m} d\sigma \\
&+ \int_{\tilde{B}_{\epsilon^{-1}\delta} \cap \{x_m=0\}} \frac{x_p^2}{2} |\partial_m W|^2 \nu_{m,m} d\sigma + \Theta_\gamma(1) \\
&= \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} \frac{x_p^2}{2} \partial_m W [-\Delta W] dx - \int_{\tilde{B}_{\epsilon^{-1}\delta} \cap \{x_m=0\}} \frac{x_p^2}{4} |\partial_m W|^2 d\sigma \\
&+ \Theta_\gamma(1).
\end{aligned}$$

Moreover, using (3.42), we have that

$$\begin{aligned}
M_{p,m} &= \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} \frac{x_p^2}{2} \partial_m W \left( \frac{\gamma}{|x|^2} W + \xi \frac{W^{2^*(s)-1}}{|x|^s} \right) dx \\
&- \int_{\tilde{B}_{\epsilon^{-1}\delta} \cap \{x_m=0\}} \frac{x_p^2 |\partial_m W|^2}{4} d\sigma + \Theta_\gamma(1).
\end{aligned}$$

Using again that  $W$  vanishes on  $\partial\mathbb{R}^{k+,n-k}$ , we get that

$$\begin{aligned}
&\int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} x_p^2 \partial_m W \frac{W^{2^*(\sigma)-1}}{|x|^\sigma} dx = \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} x_p^2 |x|^{-\sigma} \partial_m \left( \frac{W^{2^*(\sigma)}}{2^*(\sigma)} \right) dx \\
&= \frac{\sigma}{2^*(\sigma)} \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} \frac{x_p^2 x_m}{|x|^{\sigma+2}} W^{2^*(\sigma)} dx + O \left( \int_{\partial\tilde{B}_{\epsilon^{-1}\delta} \cap \mathbb{R}^{k+,n-k}} |x|^{2-\sigma} W^{2^*(\sigma)} d\sigma \right) \\
&= \frac{\sigma}{2^*(\sigma)} \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} \frac{x_p^2 x_m}{|x|^{\sigma+2}} W^{2^*(\sigma)} dx + \Theta_\gamma(1) \text{ as } \epsilon \rightarrow 0.
\end{aligned}$$



Moreover,

$$\begin{aligned} M_{p,m} &= \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} \frac{x_p^2 x_m}{2|x|^2} \left( \xi \frac{s}{2^*(s)} \frac{W^{2^*(s)}}{|x|^s} + \gamma \frac{W^2}{|x|^2} \right) dx \\ &\quad - \int_{\tilde{B}_{\epsilon^{-1}\delta} \cap \{x_m=0\}} \frac{x_p^2 |\partial_m W|^2}{4} d\sigma + \Theta_\gamma(1). \end{aligned}$$

The proof is similiar for  $M_{l,m}$  for all  $l \geq k+1$ . Fix  $m \in \{1, \dots, k\}$  and  $i \geq 1$  such that  $i \neq m$ , we have that

$$K_{i,m} : = \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} \partial_i W \partial_m W x_i dx = \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} \partial_i W \partial_m W x_i \partial_m x_m dx.$$

Integrating by parts again and using (3.69), we get

$$\begin{aligned} K_{i,m} &= - \int x_i x_m \partial_i W \partial_{mm} W dx - \int x_i x_m \partial_m W \partial_{mi} W dx \\ &\quad + \sum_{\alpha=1}^k \int_{A_\epsilon^\alpha} x_i x_m \partial_m W \partial_i W \nu_{\alpha,m} dx + \Theta_\gamma(1) \\ &= \int x_i x_m \partial_i W (-\Delta W) dx + \sum_{j \geq 1; j \neq m} \int x_i x_m \partial_i W \partial_{jj} W dx \\ &\quad - \frac{1}{2} \int x_i x_m \partial_i (\partial_m W)^2 dx + \Theta_\gamma(1) \\ &= \int x_i x_m \partial_i W (-\Delta W) dx - \int x_m \partial_i W \partial_i W dx \\ &\quad - \sum_{j \geq 1; j \neq m} \int x_i x_m \partial_{ij} W \partial_j W dx + \frac{1}{2} \int x_m (\partial_m W)^2 dx \\ &\quad - \frac{1}{2} \sum_{\alpha=1}^k \int_{A_\epsilon^\alpha} x_i x_m (\partial_m W)^2 \nu_{\alpha,i} dx + \Theta_\gamma(1) \\ &= \int x_i x_m \partial_i W (-\Delta W) dx - J_{i,m} - \frac{1}{2} \sum_{j \geq 1; j \neq m} \int x_i x_m \partial_i (\partial_j W)^2 dx \\ &\quad + \frac{1}{2} \int x_m (\partial_m W)^2 dx + \Theta_\gamma(1) \end{aligned}$$

$$\begin{aligned}
&= \int x_i x_m \partial_i W (-\Delta W) dx - J_{i,m} + \frac{1}{2} \sum_{j \geq 1; j \neq m} \int x_m (\partial_j W)^2 dx \\
&\quad - \frac{1}{2} \sum_{j \geq 1; j \neq m} \sum_{\alpha=1}^k \int_{A_\epsilon^\alpha} x_i x_m (\partial_j W)^2 \nu_{\alpha,i} dx + \frac{1}{2} \int x_m (\partial_m W)^2 dx + \Theta_\gamma(1) \\
&= \int x_i x_m \partial_i W (-\Delta W) dx - J_{i,m} + \frac{1}{2} \sum_{j \geq 1; j \neq m} \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} x_m (\partial_j W)^2 dx \\
&\quad + \frac{1}{2} \int x_m (\partial_m W)^2 dx + \Theta_\gamma(1) \\
&= \int x_i x_m \partial_i W (-\Delta W) dx - J_{i,m} + \frac{1}{2} \int x_m |\nabla W|^2 dx + \Theta_\gamma(1),
\end{aligned}$$

since  $W$  is a solution to (3.42), then there exists  $\xi > 0$  such that

$$\begin{aligned}
K_{i,m} + J_{i,m} &= \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} x_i x_m \partial_i W \left( \xi \frac{W^{2^*(s)-1}}{|x|^s} + \gamma \frac{W}{|x|^2} \right) dx \\
&\quad + \frac{1}{2} \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} x_m |\nabla W|^2 dx + \Theta_\gamma(1).
\end{aligned}$$

Since  $W$  vanishes on  $\partial\mathbb{R}^{k+,n-k}$ , we get

$$\begin{aligned}
&\int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} x_i x_m \partial_i W \frac{W^{2^*(\sigma)-1}}{|x|^\sigma} dx = \frac{1}{2^*(\sigma)} \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} \frac{x_i x_m}{|x|^\sigma} \partial_i (W^{2^*(\sigma)}) dx \\
&= -\frac{1}{2^*(\sigma)} \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} x_m \frac{W^{2^*(\sigma)}}{|x|^\sigma} dx + \frac{\sigma}{2^*(\sigma)} \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} x_i^2 x_m \frac{W^{2^*(\sigma)}}{|x|^{\sigma+2}} dx + \Theta_\gamma(1).
\end{aligned}$$

Then with (3.72)

$$\begin{aligned}
K_{i,m} + J_{i,m} &= \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} \frac{x_i^2}{|x|^2} x_m \left[ \xi \frac{s}{2^*(s)} \frac{W^{2^*(s)}}{|x|^s} + \gamma \frac{W^2}{|x|^2} \right] dx \\
&\quad + \left( \frac{1}{2} - \frac{1}{2^*(s)} \right) \xi \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} x_m \frac{W^{2^*(s)}}{|x|^s} dx + \Theta_\gamma(1).
\end{aligned}$$

Fix  $m \in \{1, \dots, k\}$ ,  $p \in \{1, \dots, k\}$  and  $q \in \{p+1, \dots, k\}$  such that  $p, q \neq m$ . We get

$$L_{m,p,q} := \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} \partial_m W \partial_p W x_q dx = \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} \partial_m W \partial_p W x_q \partial_p x_p dx.$$

Using again the integrations by parts, (3.68) and (3.69), we get

$$\begin{aligned}
L_{m,p,q} &= - \int x_q x_p \partial_m W \partial_{pp} W \, dx - \int x_p x_q \partial_p W \partial_{mp} W \, dx \\
&+ \sum_{\alpha=1}^k \int_{A_\epsilon^\alpha} x_q x_p \partial_m W \partial_p W \nu_{\alpha,p} \, dx + \Theta_\gamma(1) \\
&= \int x_q x_p \partial_m W (-\Delta W) \, dx + \sum_{j \geq 1; j \neq p} \int x_q x_p \partial_m W \partial_{jj} W \, dx \\
&- \frac{1}{2} \int x_q x_p \partial_m (\partial_p W)^2 \, dx + \Theta_\gamma(1) \\
&= \int x_q x_p \partial_m W (-\Delta W) \, dx - \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} x_p \partial_m W \partial_q W \, dx \\
&- \sum_{j \geq 1; j \neq p} \int x_p x_q \partial_{jm} W \partial_j W \, dx + \sum_{j \geq 1; j \neq p} \sum_{\alpha=1}^k \int_{A_\epsilon^\alpha} x_q x_p \partial_m W \partial_j W \nu_{\alpha,j} \, dx \\
&- \frac{1}{2} \sum_{\alpha=1}^k \int_{A_\epsilon^\alpha} x_q x_p (\partial_p W)^2 \nu_{\alpha,m} \, dx + \Theta_\gamma(1) \\
&= \int x_q x_p \partial_m W (-\Delta W) \, dx - N_{m,p,q} - \frac{1}{2} \sum_{j \geq 1; j \neq p} \int x_p x_q \partial_m (\partial_j W)^2 \, dx \\
&+ \int_{A_\epsilon^m} x_q x_p (\partial_m W)^2 \nu_{m,m} \, dx + \Theta_\gamma(1) \\
&= \int x_q x_p \partial_m W (-\Delta W) \, dx - N_{m,p,q} - \frac{1}{2} \sum_{j \geq 1; j \neq p} \sum_{\alpha=1}^k \int_{A_\epsilon^\alpha} x_q x_p (\partial_j W)^2 \nu_{\alpha,m} \, dx \\
&+ \int_{A_\epsilon^m} x_q x_p (\partial_m W)^2 \nu_{m,m} \, dx + \Theta_\gamma(1) \\
&= \int x_q x_p \partial_m W (-\Delta W) \, dx - N_{m,p,q} + \frac{1}{2} \int_{A_\epsilon^m} x_q x_p (\partial_m W)^2 \nu_{m,m} \, dx + \Theta_\gamma(1),
\end{aligned}$$

with  $A_\epsilon^\alpha := \tilde{B}_{\epsilon^{-1}\delta} \cap \{x_\alpha = 0\}$ , other integrals being taken on  $\tilde{B}_{\epsilon^{-1}\delta,+k}$ . With (3.42), we then get

$$\begin{aligned}
L_{m,p,q} + N_{m,p,q} &= \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} x_q x_p \partial_m W \left( \xi \frac{W^{2^*(s)-1}}{|x|^s} + \gamma \frac{W}{|x|^2} \right) \, dx \\
&+ \frac{1}{2} \int_{\tilde{B}_{\epsilon^{-1}\delta} \cap \{x_m=0\}} x_q x_p (\partial_m W)^2 \nu_{m,m} \, dx + \Theta_\gamma(1).
\end{aligned}$$

Integrating by parts, using that  $W$  vanishes on  $\partial\mathbb{R}^{k_+,n-k_-}$ , for  $\sigma \in [0, 2]$ , we get that

$$\begin{aligned} \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} x_q x_p \partial_m W \frac{W^{2^*(\sigma)-1}}{|x|^\sigma} dx &= \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} x_q x_p |x|^{-\sigma} \partial_m \left( \frac{W^{2^*(\sigma)}}{2^*(\sigma)} \right) dx \\ &= \frac{\sigma}{2^*(\sigma)} \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} x_q x_p x_m \frac{W^{2^*(\sigma)}}{|x|^{\sigma+2}} dx + \Theta_\gamma(1) \text{ as } \epsilon \rightarrow 0. \end{aligned}$$

And then

$$\begin{aligned} L_{m,p,q} + N_{m,p,q} &= \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} \frac{x_q x_p x_m}{|x|^2} \left( \xi \frac{s}{2^*(s)} \frac{W^{2^*(s)}}{|x|^s} + \gamma \frac{W^2}{|x|^2} \right) dx \\ &\quad - \frac{1}{2} \int_{\tilde{B}_{\epsilon^{-1}\delta} \cap \{x_m=0\}} x_q x_p (\partial_m W)^2 dx + \Theta_\gamma(1). \end{aligned}$$

This ends the proof of Lemma 3.5.1. □

We define (all integrals are taken on  $\tilde{B}_{\epsilon^{-1}\delta,+k}$ )

$$\begin{aligned} A_\epsilon &:= \sum_{1 \leq i \leq k; j \geq 1} \partial_{ji} \phi^j(0) \left( \int |\nabla W|^2 x_i dx - \gamma \int \frac{|W|^2}{|x|^2} x_i dx \right) \\ &\quad - 2 \sum_{m=1}^k \sum_{i=1; i \neq m}^k \partial_{ii} \phi^m(0) \left( \int \partial_m W x_i \partial_i W dx + \gamma \int \frac{|W|^2}{|x|^2} \frac{x_m}{|x|^2} x_i^2 dx \right) \\ &\quad - 2 \sum_{m=1}^k \sum_{i=k+1}^n \partial_{ii} \phi^m(0) \left( \int \partial_m W x_i \partial_i W dx + \gamma \int \frac{|W|^2}{|x|^2} \frac{x_m}{|x|^2} x_i^2 dx \right) \\ &\quad - 2 \sum_{m=1}^k \partial_{mm} \phi^m(0) \left( \int \partial_m W \partial_m W x_m dx + \gamma \int \frac{|W|^2}{|x|^2} \frac{x_m}{|x|^2} x_m x_m dx \right) \\ &\quad - 2 \sum_{m=1}^k \sum_{i \geq 1; i \neq m} \partial_{mi} \phi^i(0) \left( \int \partial_i W \partial_m W x_i dx + \gamma \int \frac{|W|^2}{|x|^2} \frac{x_i}{|x|^2} x_m x_i dx \right) \\ &\quad - 2 \sum_{m=1}^k \sum_{i \geq 1; i \neq m} \partial_{im} \phi^i(0) \left( \int \partial_i W \partial_i W x_m dx + \gamma \int \frac{|W|^2}{|x|^2} \frac{x_i}{|x|^2} x_i x_m dx \right) \\ &\quad + \sum_{m=1}^k \sum_{p=1; p \neq m}^k \sum_{q=p+1; q \neq m}^k \partial_{qp} \phi^m(0) \left( -2 \int \partial_m W \partial_q W x_p dx + \gamma \int \frac{|W|^2}{|x|^2} \frac{x_m}{|x|^2} x_q x_p dx \right) \\ &\quad + \sum_{m=1}^k \sum_{p=1; p \neq m}^k \sum_{q=p+1; q \neq m}^k \partial_{qp} \phi^m(0) \left( -2 \int \partial_m W \partial_p W x_q dx + \gamma \int \frac{|W|^2}{|x|^2} \frac{x_m}{|x|^2} x_p x_q dx \right) \end{aligned}$$

and

$$\begin{aligned}
B_\epsilon &:= \sum_{1 \leq i \leq k; j \geq 1} \partial_{ji} \phi^j(0) \int \frac{|W|^{2^*(s)}}{|x|^s} x_i dx \\
&- \frac{s}{2} \sum_{m=1}^k \sum_{i=1; i \neq m}^k \partial_{ii} \phi^m(0) \int \frac{|W|^{2^*(s)}}{|x|^s} \frac{x_m}{|x|^2} x_i^2 dx \\
&- \frac{s}{2} \sum_{m=1}^k \sum_{i=k+1}^n \partial_{ii} \phi^m(0) \int \frac{|W|^{2^*(s)}}{|x|^s} \frac{x_m}{|x|^2} x_i^2 dx \\
&- \frac{s}{2} \sum_{m=1}^k \partial_{mm} \phi^m(0) \int \frac{|W|^{2^*(s)}}{|x|^s} \frac{x_m}{|x|^2} x_m^2 dx \\
&- s \sum_{m=1}^k \sum_{i \geq 1; i \neq m} \partial_{mi} \phi^i(0) \int \frac{|W|^{2^*(s)}}{|x|^s} \frac{x_i^2}{|x|^2} x_m dx \\
&- s \sum_{m=1}^k \sum_{p=1; p \neq m}^k \sum_{q=p+1; q \neq m}^k \partial_{qp} \phi^m(0) \int \frac{|W|^{2^*(s)}}{|x|^s} \frac{x_m}{|x|^2} x_q x_p dx
\end{aligned}$$

Steps 3.5.1 and 3.5.2 yield

$$\begin{aligned}
\int_{\Omega} \left( |\nabla W_\epsilon|^2 - \gamma \frac{|W_\epsilon|^2}{|x|^2} \right) dx &= \int_{\mathbb{R}^{k_+, n-k}} \left( |\nabla W|^2 - \gamma \frac{|W|^2}{|x|^2} \right) dx \\
&\quad + A_\epsilon \epsilon + \Theta_\gamma(\epsilon), \\
\int_{\Omega} \frac{W_\epsilon^{2^*(s)}}{|x|^s} dx &= \int_{\mathbb{R}^{k_+, n-k}} \frac{W^{2^*(s)}}{|x|^s} dx + \epsilon B_\epsilon + \Theta_\gamma(\epsilon)
\end{aligned}$$

It follows from (3.42) that

$$\int_{\mathbb{R}^{k_+, n-k}} \left( |\nabla W|^2 - \gamma \frac{W^2}{|x|^2} \right) dx = \xi \int_{\mathbb{R}^{k_+, n-k}} \frac{W^{2^*(s)}}{|x|^s} dx.$$

Since  $W$  is an extremal for the Euclidean inequality, we have that

$$\frac{\int_{\mathbb{R}^{k_+, n-k}} \left( |\nabla W|^2 - \frac{\gamma}{|x|^2} W^2 \right) dx}{\left( \int_{\mathbb{R}^{k_+, n-k}} \frac{W^{2^*(s)}}{|x|^s} dx \right)^{\frac{2}{2^*(s)}}} = \mu_{\gamma, s}(\mathbb{R}^{k_+, n-k}).$$

Note that, for  $\gamma \leq \gamma_H(\mathbb{R}^{k_+, n-k}) - \frac{1}{4}$ , we have that  $\lim_{\epsilon \rightarrow 0} A_\epsilon \epsilon = \lim_{\epsilon \rightarrow 0} B_\epsilon \epsilon = 0$ . Therefore, the above estimates yield

$$\begin{aligned}
J_{\gamma, s}^\Omega(W_\epsilon) &= \mu_{\gamma, s}(\mathbb{R}^{k_+, n-k}) \left( 1 + \frac{1}{\xi \int_{\mathbb{R}^{k_+, n-k}} \frac{W^{2^*(s)}}{|x|^s} dx} \left( A_\epsilon - \frac{2\xi}{2^*(s)} B_\epsilon \right) \right) \epsilon \\
&\quad + \Theta_\gamma(\epsilon).
\end{aligned}$$

In the following formula, all the integrals are on  $\tilde{B}_{\epsilon^{-1}\delta,+k}$  and  $F(x) := \gamma \frac{W^2}{|x|^2} + \xi \frac{W^{2^*(s)}}{|x|^s}$ . Using the notations of Step 3.5.3 and Lemma 3.5.1, we get

$$\begin{aligned}
A_\epsilon - \frac{2\xi}{2^*(s)} B_\epsilon &= \sum_{i=1}^k \sum_j \partial_{ji} \phi^j(0) \xi \left(1 - \frac{2}{2^*(s)}\right) \int \frac{|W|^{2^*(s)}}{|x|^s} x_i dx \\
&+ \sum_{m=1}^k \sum_{i=1, i \neq m}^k \partial_{ii} \phi^m(0) \left(-2M_{im} + \int \frac{x_i^2}{|x|^2} x_m F(x) dx\right) \\
&+ \sum_{m=1}^k \sum_{i=k+1}^n \partial_{ii} \phi^m(0) \left(-2M_{im} + \int \frac{x_i^2}{|x|^2} x_m F(x) dx\right) \\
&+ \sum_{m=1}^k \partial_{mm} \phi^m(0) \left(-2I_m + \int \frac{x_m^2}{|x|^2} x_m F(x) dx\right) \\
&+ \sum_{m=1}^k \sum_{i \geq 1; i \neq m} \partial_{mi} \phi^i(0) \left(-2K_{im} + \int \frac{x_i^2}{|x|^2} x_m F(x) dx\right) \\
&+ \sum_{m=1}^k \sum_{i \geq 1; i \neq m} \partial_{mi} \phi^i(0) \left(-2J_{im} + \int \frac{x_i^2}{|x|^2} x_m F(x) dx\right) \\
&+ \sum_{m=1}^k \sum_{p=1; p \neq m}^k \sum_{q=p+1; q \neq m}^k \partial_{qp} \phi^m(0) \left(-2N_{m,p,q} + \int \frac{x_p x_q}{|x|^2} x_m F(x) dx\right) \\
&+ \sum_{m=1}^k \sum_{p=1; p \neq m}^k \sum_{q=p+1; q \neq m}^k \partial_{qp} \phi^m(0) \left(-2L_{m,p,q} + \int \frac{x_p x_q}{|x|^2} x_m F(x) dx\right) \\
&= \sum_{i=1}^k \sum_j \partial_{ji} \phi^j(0) \xi \left(1 - \frac{2}{2^*(s)}\right) \int \frac{|W|^{2^*(s)}}{|x|^s} x_i dx \\
&+ \frac{1}{2} \sum_{m=1}^k \sum_{i=1, i \neq m}^k \partial_{ii} \phi^m(0) \int_{\tilde{B}_{\epsilon^{-1}\delta} \cap \{x_m=0\}} x_i^2 |\partial_m W|^2 d\sigma \\
&+ \frac{1}{2} \sum_{m=1}^k \sum_{i=k+1}^n \partial_{ii} \phi^m(0) \int_{\tilde{B}_{\epsilon^{-1}\delta} \cap \{x_m=0\}} x_i^2 |\partial_m W|^2 d\sigma \\
&- \xi \left(1 - \frac{2}{2^*(s)}\right) \partial_{mm} \phi^m(0) \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} x_m \frac{W^{2^*(s)}}{|x|^s} dx \\
&- \xi \left(1 - \frac{2}{2^*(s)}\right) \sum_{i \geq 1; i \neq m} \partial_{im} \phi^i(0) \int_{\tilde{B}_{\epsilon^{-1}\delta,+k}} x_m \frac{W^{2^*(s)}}{|x|^s} dx
\end{aligned}$$

$$+ \sum_{m=1}^k \sum_{p=1; p \neq m}^k \sum_{q=p+1; q \neq m}^k \partial_{qp} \phi^m(0) \int_{\tilde{B}_{\epsilon-1\delta} \cap \{x_m=0\}} x_q x_p |\partial_m W|^2 d\sigma.$$

Then,

$$\begin{aligned} A_\epsilon - \frac{2\xi}{2^*(s)} B_\epsilon &= \frac{1}{2} \sum_{m=1}^k \sum_{i=1, i \neq m}^k \partial_{ii} \phi^m(0) \int_{\tilde{B}_{\epsilon-1\delta} \cap \{x_m=0\}} x_i^2 |\partial_m W|^2 d\sigma \\ &+ \frac{1}{2} \sum_{m=1}^k \sum_{i=k+1}^n \partial_{ii} \phi^m(0) \int_{\tilde{B}_{\epsilon-1\delta} \cap \{x_m=0\}} x_i^2 |\partial_m W|^2 d\sigma \\ &+ \sum_{m=1}^k \sum_{p=1; p \neq m}^k \sum_{q=p+1; q \neq m}^k \partial_{qp} \phi^m(0) \int_{\tilde{B}_{\epsilon-1\delta} \cap \{x_m=0\}} x_q x_p |\partial_m W|^2 d\sigma. \end{aligned}$$

With the symmetries of  $W$  (see Theorem 3.4.1), there exists  $\alpha_\epsilon, \beta_\epsilon, \tau_\epsilon > 0$  such that

$$\begin{aligned} \int_{\tilde{B}_{\epsilon-1\delta} \cap \{x_m=0\}} x_i^2 |\partial_m W|^2 d\sigma &= \alpha_\epsilon && \text{if } i = 1, \dots, k, i \neq m \\ \int_{\tilde{B}_{\epsilon-1\delta} \cap \{x_m=0\}} x_i^2 |\partial_m W|^2 d\sigma &= \beta_\epsilon && \text{if } i = k+1, \dots, n \\ \int_{\tilde{B}_{\epsilon-1\delta} \cap \{x_m=0\}} x_q x_p |\partial_m W|^2 d\sigma &= \tau_\epsilon && \text{if } p, q, m \in \{1, \dots, k\} \text{ are distinct} \end{aligned}$$

Then, we get that

$$\begin{aligned} A_\epsilon - \frac{2\xi}{2^*(s)} B_\epsilon &= \frac{\alpha_\epsilon}{2} \sum_{m=1}^k \sum_{i=1, i \neq m}^k \partial_{ii} \phi^m(0) \\ &+ \frac{\beta_\epsilon}{2} \sum_{m=1}^k \sum_{i=k+1}^n \partial_{ii} \phi^m(0) + \tau_\epsilon \sum_{m=1}^k \sum_{p=1; p \neq m}^k \sum_{q=p+1; q \neq m}^k \partial_{qp} \phi^m(0). \end{aligned}$$

We distinguish two cases:

**Case 1:**  $\gamma < \gamma_H(\mathbb{R}^{k+, n-k}) - \frac{1}{4}$ , that is  $\alpha_+ - \alpha_- > 1$ . It follows from the pointwise control (3.44) that  $x \mapsto |x|^2 |\nabla W|^2 \in L^1(\mathbb{R}^{k+, n-k} \cap \{x_m = 0\})$ , therefore

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \alpha_\epsilon &= 2c_{\gamma, s}^2 := \int_{\mathbb{R}^{k+, n-k} \cap \{x_m=0\}} x_i^2 |\partial_m W|^2 d\sigma > 0 && \text{if } i = 1, \dots, k, i \neq m \\ \lim_{\epsilon \rightarrow 0} \beta_\epsilon &= 2c_{\gamma, s}^1 := \int_{\mathbb{R}^{k+, n-k} \cap \{x_m=0\}} x_i^2 |\partial_m W|^2 d\sigma > 0 && \text{if } i = k+1, \dots, n \\ \lim_{\epsilon \rightarrow 0} \tau_\epsilon &= c_{\gamma, s}^3 := \int_{\mathbb{R}^{k+, n-k} \cap \{x_m=0\}} x_q x_p |\partial_m W|^2 d\sigma > 0 && \text{if } p, q, m \in \{1, \dots, k\} \text{ dist.} \end{aligned}$$

Consequently,

$$\begin{aligned} A_\epsilon - \frac{2\xi}{2^*(s)} B_\epsilon &= c_{\gamma, s}^2 \sum_{m=1}^k \sum_{i=1, i \neq m}^k \partial_{ii} \phi^m(0) \\ &+ c_{\gamma, s}^1 \sum_{m=1}^k \sum_{i=k+1}^n \partial_{ii} \phi^m(0) + c_{\gamma, s}^3 \sum_{m=1}^k \sum_{p=1; p \neq m}^k \sum_{q=p+1; q \neq m}^k \partial_{qp} \phi^m(0) + o(1) \end{aligned}$$

**Case 2:**  $\gamma = \gamma_H(\mathbb{R}^{k+,n-k}) - \frac{1}{4}$ , that is  $\alpha_+ - \alpha_- = 1$ . It follows from (3.26) that

$$\lim_{\lambda \rightarrow 0} \lambda^{\alpha_-} |x|^{\alpha_- + k} \partial_m W(\lambda x) = K \left( \prod_{j=1, j \neq m}^k x_j - (\alpha_- + k) \frac{p(x)x_m}{|x|^2} \right),$$

where  $p(x) := \prod_{j=1}^k x_j$ . As in the proof of (3.45), a Kelvin transform yields

$$\lim_{\lambda \rightarrow +\infty} \lambda^{\alpha_+} |x|^{\alpha_+ + k} \partial_m W(\lambda x) = K \prod_{j=1, j \neq m}^k x_j \text{ on } \{x_m = 0\}. \quad (3.77)$$

We claim that

$$\int_{\tilde{B}_{\epsilon^{-1}\delta} \cap \{x_m=0\}} x_i^2 |\partial_m W|^2 dx = 2c_{\gamma,s}^2 \ln \left( \frac{1}{\epsilon} \right) + o \left( \ln \left( \frac{1}{\epsilon} \right) \right) \text{ as } \epsilon \rightarrow 0, \quad (3.78)$$

where,

$$c_{\gamma,s}^2 := \frac{K^2}{2} \int_{S^{n-2} \cap (\{x_m=0\} \cap \mathbb{R}^{k+,n-k})} \sigma_i^2 \left( \prod_{j=1, j \neq m}^k \sigma_j \right)^2 d\sigma$$

is independent of  $i \in \{1, \dots, k\}, i \neq m$ . We prove the claim. Since  $n-2-2\alpha_+ = -1$ , we have

$$\begin{aligned} \int_{\tilde{B}_{\epsilon^{-1}\delta} \cap \{x_m=0\}} x_i^2 |\partial_m W|^2 dx &= \int_{(\tilde{B}_{\epsilon^{-1}\delta} \setminus \tilde{B}_1) \cap \{x_m=0\}} x_i^2 |\partial_m W|^2 dx + O(1) \\ &= \int_1^{\epsilon^{-1}\delta} \frac{f(r)}{r} dr + O(1), \end{aligned} \quad (3.79)$$

where

$$f(r) := \int_{S^{n-2} \cap (\{x_m=0\} \cap \mathbb{R}^{k+,n-k})} r^{2\alpha_+} \sigma_i^2 |\partial_{\sigma,m} W(r\sigma)|^2 d\sigma.$$

It follows from the uniform convergence in (3.77) that  $\lim_{r \rightarrow +\infty} f(r) = 2c_{\gamma,s}^2$ . Then (3.77) and (3.79) yield (3.78) and then the claim.

Similarly, there exists explicit constants  $c_{\gamma,s}^2, c_{\gamma,s}^3 > 0$  such that

$$\begin{aligned} \int_{\tilde{B}_{\epsilon^{-1}\delta} \cap \{x_m=0\}} x_i^2 |\partial_m W|^2 d\sigma &= 2c_{\gamma,s}^1 \ln \left( \frac{1}{\epsilon} \right) + o \left( \ln \left( \frac{1}{\epsilon} \right) \right); \\ \int_{\tilde{B}_{\epsilon^{-1}\delta} \cap \{x_m=0\}} x_q x_p |\partial_m W|^2 d\sigma &= c_{\gamma,s}^3 \ln \left( \frac{1}{\epsilon} \right) + o \left( \ln \left( \frac{1}{\epsilon} \right) \right) \end{aligned}$$



for  $i \geq k+1$  and  $p, q, m \in \{1, \dots, k\}$  all distinct. Therefore

$$\begin{aligned} A_\epsilon - \frac{2\xi}{2^*(s)} B_\epsilon &= \left( c_{\gamma,s}^2 \sum_{m=1}^k \sum_{i=1, i \neq m}^k \partial_{ii} \phi^m(0) + c_{\gamma,s}^1 \sum_{m=1}^k \sum_{i=k+1}^n \partial_{ii} \phi^m(0) \right. \\ &\quad \left. + c_{\gamma,s}^3 \sum_{m=1}^k \sum_{p=1; p \neq m}^k \sum_{q=p+1; q \neq m}^k \partial_{qp} \phi^m(0) \right) \ln \left( \frac{1}{\epsilon} \right) + o \left( \ln \left( \frac{1}{\epsilon} \right) \right) \end{aligned}$$

We are left with writing the expressions of Cases 1 and 2 intrinsically. We refer to Definition 3.1.3. For any  $1 \leq i_1, i_2 \leq n$  such that  $i_1, i_2 \neq m$ , we have

$$\begin{aligned} \partial_{i_1 i_2} \phi^m(0) &= -\langle \vec{\nu}_m(0), \partial_{i_1 i_2} \phi(0) \rangle = \langle \partial_{i_1}(\vec{\nu}_m \circ \phi)(0), \partial_{i_2} \phi(0) \rangle \\ &= II_0^{\partial \Omega_m}(\partial_{i_1} \phi, \partial_{i_2} \phi) := II_{i_1 i_2}^m. \end{aligned}$$

For  $p \neq m$ , we have  $\vec{\nu}_p \in (T_0 \partial \Omega_m)^\perp$  and

$$\sum_{p,q,m=1, |\{p,q,m\}|=3}^k II_0^{\partial \Omega_m}(\vec{\nu}_p, \vec{\nu}_q) = \sum_{p=1; p \neq m}^k \sum_{q=p+1; q \neq m}^k \partial_{pq} \phi^m(0).$$

Define  $\Sigma := \cap_{j=1}^k \partial \Omega_j$ . We have that

$$\sum_{m=1}^k \langle \vec{H}_0^\Sigma, \vec{\nu}_m \rangle = \sum_{m=1}^k \sum_{i=k+1}^n \partial_{ii} \phi^m(0),$$

and,

$$\sum_{m=1}^k \sum_{i=1, i \neq m}^k \partial_{ii} \phi^m(0) = \sum_{i,m=1, i \neq m}^k II_0^{\partial \Omega_m}(\vec{\nu}_i, \vec{\nu}_i).$$

□

Theorem 3.1.2 is a straightforward application of Theorem 3.1.1 and Proposition 3.5.1.

### 3.6 Proof of Theorem 3.1.3

**Point (1):** we assume that  $s = 0$  and  $\gamma \leq 0$ . It follows from the definition that  $\mu_{\gamma,0}(\Omega) \geq \mu_{0,0}(\mathbb{R}^n)$ . With the reverse inequality (3.21), we get that  $\mu_{\gamma,0}(\Omega) = \mu_{0,0}(\mathbb{R}^n)$ . If there was an extremal for  $\mu_{\gamma,0}(\Omega)$ , it would also be an extremal for  $\mu_{0,0}(\mathbb{R}^n)$ , with no compact support, contradicting the boundedness of  $\Omega$ . This proves (1) of Theorem 3.1.3.

**Point (2):** Point (2) of Theorem 3.1.3 is a straightforward application of Theorem 3.1.1 and Proposition 3.5.1.

**Point (3):** We assume that  $n = 3$ ,  $s = 0$ ,  $\gamma > 0$  and there is no extremal for  $\mu_{\gamma,0}(\mathbb{R}_+^k \times \mathbb{R}^{3-k})$ . In this situation, see Proposition 1.3 of [61], we have that  $\mu_{\gamma,0}(\mathbb{R}_+^k \times \mathbb{R}^{3-k}) = \mu_{0,0}(\mathbb{R}^3)$ . The following proposition is as in [61]:

**Proposition 3.6.1.** *Let  $\Omega \subset \mathbb{R}^3$  be an open domain such that  $0 \in \partial\Omega$ . Fix  $x_0 \in \Omega$ . If  $\gamma \in (0, \gamma_H(\Omega))$ , then the equation*

$$\begin{cases} -\Delta G - \frac{\gamma}{|x|^2} G = 0; G > 0 & \text{in } \Omega \setminus \{x_0\} \\ G = 0 & \text{on } \partial\Omega \setminus \{0\} \end{cases}$$

has a solution  $G \in C^2(\Omega \setminus \{x_0\}) \cap D_1^2(\Omega \setminus \{x_0\})_{loc,0}$ , that is unique up to multiplication by a constant. Moreover, for any  $x_0 \in \Omega$ , there exists a unique  $R_\gamma(x_0) \in \mathbb{R}$  independent of the choice of  $G$  and  $c_G > 0$  such that

$$G(x) = c_G \left( \frac{1}{|x - x_0|} + R_\gamma(x_0) \right) + o(1) \text{ as } x \rightarrow x_0.$$

The proof is similar to the proof of Proposition 10.1 in [61]. Cooking-up some test-functions  $(u_\epsilon)_{\epsilon>0}$  as in Lemma 10.2 of [61], we get that  $\mu_{\gamma,0}(\Omega) \leq J_{\gamma,s}^\Omega(u_\epsilon) < \mu_{0,0}(\mathbb{R}^3) = \mu_{\gamma,0}(\mathbb{R}^3)$  when  $R_\gamma(x_0) > 0$  for some  $x_0 \in \Omega$ . Point (3) of Theorem 3.1.3 is then a consequence of Theorem 3.1.1.



## Hardy-Sobolev inequalities with singularities on non smooth boundary. Part 2: Influence of the global geometry in small dimensions.

### Abstract

We consider Hardy-Sobolev nonlinear equations on domains with singularities. We introduced this problem in Cheikh-Ali [27]. Under a local geometric hypothesis, namely that the generalized mean curvature is negative (see (4.6) below), we proved the existence of extremals for the relevant Hardy-Sobolev inequality for large dimensions. In the present paper, we tackle the question of small dimensions that was left open. We introduce a "mass", that is a global quantity, the positivity of which ensures the existence of extremals in small dimensions. As a byproduct, we prove the existence of solutions to a perturbation of the initial equation via the Mountain Pass Lemma.

### 4.1 Introduction

Let  $\Omega$  be a bounded domain of  $\mathbb{R}^n$ ,  $n \geq 3$ . We fix  $s \in [0, 2]$  and  $\gamma \in \mathbb{R}$ . It follows from the classical Caffarelli-Kohn-Nirenberg inequalities [25] that if  $\gamma < \frac{(n-2)^2}{4}$ , there exists  $K > 0$  such that

$$\left( \int_{\Omega} \frac{|u|^{2^*(s)}}{|x|^s} dx \right)^{\frac{2}{2^*(s)}} \leq K \int_{\Omega} \left( |\nabla u|^2 - \gamma \frac{u^2}{|x|^2} \right) dx, \quad (4.1)$$

for all  $u \in D^{1,2}(\Omega)$ , where  $2^*(s) := \frac{2(n-s)}{n-2}$  and  $D^{1,2}(\Omega)$  is the completion of  $C_c^\infty(\Omega)$  with respect to the norm  $u \mapsto \|\nabla u\|_2$ . We define the Hardy constant by

$$\gamma_H(\Omega) := \inf \left\{ \frac{\int_\Omega |\nabla u|^2 dx}{\int_\Omega \frac{u^2}{|x|^2} dx}; u \in D^{1,2}(\Omega) \setminus \{0\} \right\} > 0.$$

The classical Hardy inequality reads  $\gamma_H(\mathbb{R}^n) = \frac{(n-2)^2}{4}$  and therefore, we have that  $\gamma_H(\Omega) \geq \frac{(n-2)^2}{4}$ . We refer to [27] for discussions and properties of the Hardy constant. As one checks, for any  $\gamma < \gamma_H(\Omega)$ , there exists  $K = K(\Omega, \gamma, s) > 0$  such that (4.1) holds for all  $u \in D^{1,2}(\Omega)$ . For  $a \in L^\infty(\Omega)$ , we define

$$\mu_{\gamma,s,a}(\Omega) = \inf_{u \in D^{1,2}(\Omega) \setminus \{0\}} J_{\gamma,s,a}^\Omega(u),$$

where

$$J_{\gamma,s,a}^\Omega(u) := \frac{\int_\Omega \left( |\nabla u|^2 - \left( \frac{\gamma}{|x|^2} + a(x) \right) u^2 \right) dx}{\left( \int_\Omega \frac{|u|^{2^*(s)}}{|x|^s} dx \right)^{\frac{2}{2^*(s)}}},$$

so that

$$\mu_{\gamma,s,a}(\Omega) \left( \int_\Omega \frac{|u|^{2^*(s)}}{|x|^s} dx \right)^{\frac{2}{2^*(s)}} \leq \int_\Omega \left( |\nabla u|^2 - \left( \frac{\gamma}{|x|^2} + a(x) \right) u^2 \right) dx, \quad (4.2)$$

for all  $u \in D^{1,2}(\Omega)$ . As in [27], we address the question of the existence of extremals for (4.2), more precisely

**Q:** Is there  $u \in D^{1,2}(\Omega) \setminus \{0\}$  equality holds in (4.2)?

When  $0 \in \Omega$ , there are no extremals for  $\mu_{\gamma,s,0}(\Omega)$  (see [60]). From now on, we assume that  $0 \in \partial\Omega$ . When  $\Omega$  is a smooth domain, criteria for existence are in Ghoussoub-Robert [61]: in particular, there is a dichotomy between large dimension (where the criterion is local) and the small dimensions (where the criterion is global). In [27], we studied the case of domains that are modeled on cones:

**Definition 4.1.1.** We fix  $1 \leq k \leq n$ . Let  $\Omega$  be a domain of  $\mathbb{R}^n$ . We say that  $x_0 \in \partial\Omega$  is a singularity of type  $(k, n-k)$  if there exist  $U, V$  open subsets of  $\mathbb{R}^n$  such that  $0 \in U$ ,  $x_0 \in V$  and there exists  $\phi \in C^\infty(U, V)$  a diffeomorphism such that  $\phi(0) = x_0$  and

$$\phi(U \cap (\mathbb{R}_+^k \times \mathbb{R}^{n-k})) = \phi(U) \cap \Omega \text{ and } \phi(U \cap \partial(\mathbb{R}_+^k \times \mathbb{R}^{n-k})) = \phi(U) \cap \partial\Omega,$$

with the additional hypothesis that the differential at 0  $d\phi_0$  is an isometry.

In the sequel, we write  $\mathbb{R}^{k_+,n-k} := \mathbb{R}_+^k \times \mathbb{R}^{n-k}$ . We have that (see [27])

$$\gamma_H(\mathbb{R}^{k_+,n-k}) = \frac{(n-2+2k)^2}{4}.$$

We have proved the following:

**Theorem 4.1.1** (Cheikh-Ali [27]). *Let  $\Omega$  be a bounded domain in  $\mathbb{R}^n$ ,  $n \geq 3$ , such that  $0 \in \partial\Omega$  is a singularity of type  $(k, n-k)$  for some  $k \in \{1, \dots, n\}$ . We fix  $0 \leq s < 2$  and  $0 \leq \gamma < \gamma_H(\Omega)$ . Assume that either  $s > 0$ , or that  $\{s = 0, n \geq 4 \text{ and } \gamma > 0\}$ . We assume that*

$$\gamma \leq \gamma_H(\mathbb{R}^{k_+,n-k}) - \frac{1}{4} \text{ that is } n \geq n_{\gamma,k} := \sqrt{4\gamma+1} + 2 - 2k. \quad (4.3)$$

Then there are extremals for  $\mu_{\gamma,s,0}(\Omega)$  if

$$GH_{\gamma,s}(\Omega) < 0$$

where  $GH_{\gamma,s}(\Omega)$  is the **generalized mean curvature** defined below in (4.6).

This result is for large dimension  $n \geq n_{\gamma,k}$  (see (4.3)). In the present article, we tackle the case of the remaining small dimensions. The argument based on local geometry performed for the proof of Theorem 4.1.1 is not working here. Here, the global geometry has an impact: in order to obtain extremals, we must introduce a "mass" in the spirit of Schoen [101] and Schoen-Yau [101]. Concerning low dimension phenomena, we refer to the pioneer work of Brezis-Nirenberg [19], Jannelli [77] and the more recent reference Ghoussoub-Robert [60] for further discussions. Our main theorem is the following:

**Theorem 4.1.2.** *Let  $\Omega$  be a bounded domain in  $\mathbb{R}^n$ ,  $n \geq 3$ , such that  $0 \in \partial\Omega$  is a singularity of type  $(k, n-k)$  for some  $k \in \{1, \dots, n\}$ . We fix  $0 \leq s < 2$ ,  $\gamma < \gamma_H(\Omega)$  and  $a \in C^{0,\theta}(\Omega)$  ( $\theta \in (0, 1)$ ). Assume that either  $s > 0$ , or that  $\{s = 0, n \geq 4 \text{ and } \gamma > 0\}$ . We assume that*

$$\gamma > \gamma_H(\mathbb{R}^{k_+,n-k}) - \frac{1}{4} \text{ that is } n < n_{\gamma,k}.$$

We assume that the operator  $-\Delta - (\gamma|x|^{-2} + a(x))$  is coercive and has a mass  $m_{\gamma,a}(\Omega)$  (see Definition 4.2.2), and that  $m_{\gamma,a}(\Omega) > 0$ . Then there are extremals for  $\mu_{\gamma,s,a}(\Omega)$ . In particular, there exists  $u \in C^{2,\theta}(\Omega) \cap D^{1,2}(\Omega)$  such that

$$\begin{cases} -\Delta u - \left(\frac{\gamma}{|x|^2} + a(x)\right) u = \frac{u^{2^*(s)-1}}{|x|^s} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \quad (4.4)$$

In the second part of this paper, we consider the perturbative Hardy-Schrödinger equation. Given  $a, h \in C^{0,\theta}(\Omega)$  for some  $\theta \in (0, 1)$  and  $q \in (1, 2^* - 1)$ , we investigate the existence of solutions  $u \in C^2(\Omega) \cap D^{1,2}(\Omega)$  to

$$\begin{cases} -\Delta u - \left(\frac{\gamma}{|x|^2} + a(x)\right) u = \frac{u^{2^*(s)-1}}{|x|^s} + h(x)u^q & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \quad (4.5)$$

We refer to Brezis-Nirenberg [19] ( $\gamma = 0$  and  $s = 0$  on a smooth domain  $\Omega$ ), Ghoussoub-Yuan [66] ( $\gamma = 0$ ,  $s > 0$  and  $0 \in \Omega$ ), Ghoussoub-Kang [55] and Jaber [75] ( $\gamma = 0$ ,  $s > 0$  and  $0 \in \partial\Omega$ ). In the Riemannian context with no boundary, still for  $\gamma = 0$ , we refer to Djadli [34] when  $s = 0$ , and to Jaber [76] for  $s > 0$  and  $\gamma = 0$ .

The case  $a, h \equiv 0$  was tackled in [27] for  $n \geq n_{\gamma,k}$  for nonsmooth domains. We prove the following:

**Theorem 4.1.3.** *Let  $\Omega$  be a bounded domain in  $\mathbb{R}^n$ ,  $n \geq 3$ , such that  $0 \in \partial\Omega$  is a singularity of type  $(k, n - k)$  for some  $k \in \{1, \dots, n\}$ . Let  $a, h \in C^{0,\theta}(\Omega)$  ( $\theta \in (0, 1)$ ) such that  $-\Delta - (\gamma|x|^{-2} + a)$  is coercive and  $h \geq 0$ . Consider  $s \in [0, 2)$  and  $\gamma < \gamma_H(\mathbb{R}^{k+,n-k})$ . Assume that either  $s > 0$ , or that  $\{s = 0, n \geq 4 \text{ and } \gamma > 0\}$ . We fix  $q \in (1, 2^* - 1)$ . Then, there exists a positive mountain pass solution  $u \in D^{1,2}(\Omega)$  to the perturbative Hardy-Schrödinger equation (4.5) under one of the following conditions:*

- $0 \leq \gamma < \gamma_H(\mathbb{R}^{k+,n-k}) - \frac{1}{4}$ , and

$$\begin{cases} GH_{\gamma,s}(\Omega) < 0 & \text{if } q + 1 < \frac{2n-2}{n-2}, \\ c_1 GH_{\gamma,s}(\Omega) - c_2 h(0) < 0 & \text{if } q + 1 = \frac{2n-2}{n-2}, \\ h(0) > 0 & \text{if } q + 1 > \frac{2n-2}{n-2}, \end{cases}$$

- $0 \leq \gamma = \gamma_H(\mathbb{R}^{k+,n-k}) - \frac{1}{4}$ , and

$$\begin{cases} GH_{\gamma,s}(\Omega) < 0 & \text{if } q + 1 \leq \frac{2n-2}{n-2}, \\ h(0) > 0 & \text{if } q + 1 > \frac{2n-2}{n-2}, \end{cases}$$

- $\gamma > \gamma_H(\mathbb{R}^{k+,n-k}) - \frac{1}{4}$ , and

$$\begin{cases} m_{\gamma,a}(\Omega) > 0 & \text{if } q + 1 < \frac{2n-2(\alpha_+-\alpha_-)}{n-2}, \\ c_3 m_{\gamma,a}(\Omega) + c_2 h(0) > 0 & \text{if } q + 1 = \frac{2n-2(\alpha_+-\alpha_-)}{n-2}, \\ h(0) > 0 & \text{if } q + 1 > \frac{2n-2(\alpha_+-\alpha_-)}{n-2}, \end{cases}$$

where  $\alpha_+ - \alpha_- = 2\sqrt{\gamma_H(\mathbb{R}^{k+,n-k})} - \gamma$  (see (4.7) below),  $c_1, c_2, c_3 > 0$  are defined in (4.69) and  $m_{\gamma,a}(\Omega)$  is the mass of  $\Omega$  at 0.

This result shows how the subcritical nonlinearity has an impact on the existence of solutions. When the subcritical nonlinearity is close to being linear, only the geometry of  $\Omega$  commands the existence. Conversely, when it is close to being critical, the subcritical nonlinearity commands the existence, whatever the geometry is.

**Notation:** In the sequel,  $C$  denotes a positive constant. Its value might change from a page to another, and even from one line to another.

## 4.2 Definition of the generalized curvature and the mass

### Generalized curvature.

#### Definition 4.2.1.

$$\Omega_i := \phi(U \cap \{x_i > 0\}) \text{ for all } i = 1, \dots, k,$$

where  $(\phi, U)$  is a chart as in Definition 4.1.1. We have that:

- For all  $i = 1, \dots, k$ ,  $\Omega_i$  is smooth around  $0 \in \partial\Omega_i$ .
- Up to permutation, the  $\Omega_i$ 's are locally independent of the chart  $\phi$ .
- The  $\Omega_i$ 's define locally  $\Omega$ : there exists  $\delta > 0$  such that

$$\Omega \cap B_\delta(0) = \bigcap_{i=1}^k \Omega_i \cap B_\delta(0).$$

We set  $\Sigma := \bigcap_{i=1}^k \partial\Omega_i$  where  $k \in \{1, \dots, n\}$ . The vector  $\vec{H}_0^\Sigma$  denotes the mean-curvature vector at 0 of the  $(n - k)$ -submanifold  $\Sigma$ . For any  $m = 1, \dots, k$ ,  $II_0^{\partial\Omega_m}$  denotes the second fundamental form at 0 of the oriented  $(n - 1)$ -submanifold  $\partial\Omega_m$ . The **generalized mean curvature** of  $\Omega$  is defined by:

$$\begin{aligned} GH_{\gamma,s}(\Omega) &:= c_{\gamma,s}^1 \sum_{m=1}^k \langle \vec{H}_0^\Sigma, \vec{\nu}_m \rangle + c_{\gamma,s}^2 \sum_{i,m=1, i \neq m}^k II_0^{\partial\Omega_m}(\vec{\nu}_i, \vec{\nu}_i) \\ &\quad + c_{\gamma,s}^3 \sum_{p,q,m=1, |\{p,q,m\}|=3}^k II_0^{\partial\Omega_m}(\vec{\nu}_p, \vec{\nu}_q) \end{aligned} \quad (4.6)$$



where for any  $m = 1, \dots, k$ ,  $\vec{\nu}_m$  is the outward normal vector at 0 of  $\partial\Omega_m$  and  $c_{\gamma,s}^1, c_{\gamma,s}^2, c_{\gamma,s}^3$  are positive explicit constants. We refer to [27] for details on this curvature.

**The mass.** Let  $\alpha \in \mathbb{R}$  be a real number and fix  $\gamma < \gamma_H(\mathbb{R}^{k+,n-k})$ . Then

$$\left(-\Delta - \frac{\gamma}{|x|^2}\right) S_\alpha = 0 \Leftrightarrow \alpha \in \{\alpha_-, \alpha_+\},$$

where:

$$S_\alpha := |x|^{-\alpha-k} \prod_{i=1}^k x_i \text{ and } \alpha_\pm = \alpha_\pm(\gamma, n, k) := \frac{n-2}{2} \pm \sqrt{\gamma_H(\mathbb{R}^{k+,n-k}) - \gamma}. \quad (4.7)$$

The functions  $S_{\alpha_-}, S_{\alpha_+}$  are prototypes of solution to (4.4) vanishing on  $\partial\mathbb{R}^{k+,n-k}$ .

**Definition 4.2.2.** Let  $\Omega$  be a bounded domain in  $\mathbb{R}^n$ ,  $n \geq 3$ , such that  $0 \in \partial\Omega$  is a singularity of type  $(k, n-k)$  for some  $k \in \{1, \dots, n\}$ . We fix  $\gamma < \gamma_H(\Omega)$  and  $a \in C^{0,\theta}(\Omega)$  ( $\theta \in (0, 1)$ ). We say that a coercive operator  $-\Delta - (\gamma|x|^{-2} + a)$  has a mass if there exists  $G \in C^2(\Omega) \cap D_{loc,0}^{1,2}(\Omega)$  such that

$$\begin{cases} -\Delta G - \left(\frac{\gamma}{|x|^2} + a(x)\right) G = 0 & \text{in } \Omega, \\ G > 0 & \text{in } \Omega, \\ G = 0 & \text{on } \partial\Omega \setminus \{0\}, \end{cases} \quad (4.8)$$

and there exists  $c \in \mathbb{R}$  such that

$$G(x) = \prod_{i=1}^k d(x, \partial\Omega_i) (|x|^{-\alpha_+-k} + c|x|^{-\alpha_--k} + o(|x|^{-\alpha_--k})) \text{ as } x \rightarrow 0. \quad (4.9)$$

Then the function  $G$  is unique, and we define  $m_{\gamma,a}(\Omega) := c$  as the boundary mass of the operator  $-\Delta - (\gamma|x|^{-2} + a)$ .

Examples of domains with positive or negative mass are in Section 4.5 below.

### 4.3 Some background results

We start with the following classical result:

**Theorem 4.3.1.** [See Cheikh-Ali [27]] Let  $\Omega$  be a bounded domain in  $\mathbb{R}^n$ ,  $n \geq 3$ , such that  $0 \in \partial\Omega$  is a singularity of type  $(k, n-k)$  for some  $k \in \{1, \dots, n\}$ . Assume that  $\gamma < \gamma_H(\mathbb{R}^{k+,n-k})$ ,  $0 \leq s \leq 2$ , et  $\mu_{\gamma,s,a}(\Omega) < \mu_{\gamma,s,0}(\mathbb{R}^{k+,n-k})$ . Then there are extremals for  $\mu_{\gamma,s,a}(\Omega)$ .

Indeed, Theorem 4.3.1 was proved in [27] when  $a \equiv 0$ . The proof extends to the general case with no effort. Recall now an optimal regularity theorem.

**Theorem 4.3.2.** [See Felli-Ferrero [50] and [27]] *Let  $\Omega$  be a bounded domain in  $\mathbb{R}^n$ ,  $n \geq 3$ , such that  $0 \in \partial\Omega$  is a singularity of type  $(k, n - k)$  for some  $k \in \{1, \dots, n\}$ . We fix  $\gamma < \gamma_H(\mathbb{R}^{k+, n-k})$ . Let  $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$  be a Caratheodory function such that*

$$|f(x, v)| \leq C|v| \left( 1 + \frac{|v|^{2^*(s)-2}}{|x|^s} \right) \text{ for all } x \in \Omega, v \in \mathbb{R}. \quad (4.10)$$

Let  $u \in D^{1,2}(\Omega)_{loc,0}$ , be a weak solution to

$$-\Delta u - \frac{\gamma + O(|x|^\tau)}{|x|^2} u = f(x, u) \text{ in } D^{1,2}(\Omega)_{loc,0}$$

for some  $\tau > 0$ . Then there exists  $K \in \mathbb{R}$  such that

$$\lambda^{\alpha-} u(\lambda\phi(x)) \rightarrow K|x|^{-\alpha-} \frac{\prod_{i=1}^k x_i}{|x|^k} \text{ in } B_1(0) \cap \mathbb{R}^{k+, n-k},$$

uniformly in  $C^1$  as  $\lambda \rightarrow 0$ , where  $\phi$  is a chart as in Definition 4.1.1.

In section 4.4, we will need the following lemma:

**Lemma 4.3.1.** [See Cheikh-Ali [27]] *Assume the  $u \in D^{1,2}(\mathbb{R}^{k+, n-k})_{loc,0}$  is a weak solution of*

$$\begin{cases} -\Delta u - \frac{\gamma + O(|x|^\tau)}{|x|^2} u = 0 & \text{in } D^{1,2}(\mathbb{R}^{k+, n-k})_{loc,0}, \\ u = 0 & \text{on } B_{2\delta}(0) \cap \partial\mathbb{R}^{k+, n-k}, \end{cases}$$

for some  $\tau > 0$  and  $\alpha \in \{\alpha_-, \alpha_+\}$ . Assume there exists  $c > 0$  such that

$$|u(x)| \leq c|x|^{-\alpha} \text{ for } x \rightarrow 0, x \in \mathbb{R}^{k+, n-k}.$$

- Then, there exists  $c_1 > 0$  such that

$$|\nabla u(x)| \leq c_1|x|^{-\alpha-1} \text{ as } x \rightarrow 0, x \in \mathbb{R}^{k+, n-k}.$$

- If  $\lim_{x \rightarrow 0} |x|^\alpha u(x) = 0$ , then  $\lim_{x \rightarrow 0} |x|^{\alpha+1} |\nabla u(x)| = 0$ .

#### 4.4 Test-functions estimates for the mass: proof of Theorem 4.1.2

Let  $U \in D^{1,2}(\mathbb{R}^{k_+,n-k})$  be a positive extremal for  $\mu_{\gamma,s,0}(\mathbb{R}^{k_+,n-k})$ . Then

$$J_{\gamma,s,0}^{\mathbb{R}^{k_+,n-k}}(U) = \frac{\int_{\mathbb{R}^{k_+,n-k}} (|\nabla U|^2 - \gamma|x|^{-2}U^2) dx}{\left(\int_{\mathbb{R}^{k_+,n-k}} |x|^{-s}|U|^{2^*(s)} dx\right)^{\frac{2}{2^*(s)}}} = \mu_{\gamma,s,0}(\mathbb{R}^{k_+,n-k}).$$

Therefore, there exists  $\xi > 0$  such that

$$\begin{cases} -\Delta U - \gamma|x|^{-2}U = \xi|x|^{-s}U^{2^*(s)-1} & \text{in } \mathbb{R}^{k_+,n-k}, \\ U > 0 & \text{in } \mathbb{R}^{k_+,n-k}, \\ U = 0 & \text{on } \partial\mathbb{R}^{k_+,n-k}. \end{cases} \quad (4.11)$$

For  $r > 0$ , we define

$$B_r := B_r(0) \text{ and } B_{r,+} := B_r(0) \cap \mathbb{R}^{k_+,n-k}. \quad (4.12)$$

Therefore, with  $\delta > 0$  small, the chart  $\phi$  of Definition 4.1.1 yields

$$\phi(B_{3\delta} \cap \mathbb{R}^{k_+,n-k}) = \phi(B_{3\delta}) \cap \Omega \text{ and } \phi(B_{3\delta} \cap \partial\mathbb{R}^{k_+,n-k}) = \phi(B_{3\delta}) \cap \partial\Omega.$$

We fix  $\eta \in C_c^\infty(\mathbb{R}^n)$  such that

$$\eta(x) = \begin{cases} 1 & \text{for } x \in B_\delta, \\ 0 & \text{for } x \notin B_{2\delta}. \end{cases} \quad (4.13)$$

Define also for convenience,

$$p(x) := \prod_{i=1}^k d(x, \partial\Omega_i) \text{ for all } x \in \Omega \text{ and } v(x) := \prod_{i=1}^k x_i \text{ for all } x \in \mathbb{R}^{k_+,n-k}. \quad (4.14)$$

Equation (4.9) allows us to define  $\Theta \in \Omega \rightarrow \mathbb{R}$  such that

$$G(x) = (\eta v|x|^{-\alpha_+ - k}) \circ \phi^{-1}(x) + \Theta(x) \text{ for any } x \in \Omega,$$

where  $\phi$  as in Definition 4.1.1. We then get that  $\Theta \in D^{1,2}(\Omega)$  and

$$\Theta(x) = m_{\gamma,a}(\Omega)p(x)|x|^{-\alpha_- - k} + o(p(x)|x|^{-\alpha_- - k}) \text{ as } x \rightarrow 0. \quad (4.15)$$

Note that that

$$\left\{ \gamma > \gamma_H(\mathbb{R}_+^k \times \mathbb{R}^{n-k}) - \frac{1}{4} \right\} \Leftrightarrow \{\alpha_+ - \alpha_- < 1\} \Leftrightarrow \{n < n_{\gamma,k}\}. \quad (4.16)$$

Since  $U$  satisfies (4.11), Theorem 4.3.2 yields  $K_1 > 0$  such that

$$\lim_{\lambda \rightarrow 0^+} \lambda^{\alpha_-} U(\lambda x) = K_1 v(x) |x|^{-\alpha_- - k} \text{ in } B_1(0) \cap \mathbb{R}^{k_+, n-k}. \quad (4.17)$$

The regularity applied to the Kelvin transform  $x \mapsto \bar{U}(x) := |x|^{2-n} U(\frac{x}{|x|^2})$  yields

$$\lim_{\lambda \rightarrow +\infty} \lambda^{\alpha_+} U(\lambda x) = K_2 v(x) |x|^{-\alpha_+ - k} \text{ in } B_1(0) \cap \mathbb{R}^{k_+, n-k}, \quad (4.18)$$

for some  $K_2 > 0$ . Up to multiplying  $U$  by a positive constant, we assume that  $K_2 = 1$ . Equation (4.17), the Kelvin transform and Lemma 4.3.1 yield

$$|U(x)| \leq C|x|^{-\alpha_+} \text{ and } |\nabla U(x)| \leq C|x|^{-1-\alpha_+} \text{ for any } x \in \mathbb{R}^{k_+, n-k}. \quad (4.19)$$

For  $\epsilon > 0$ , we define

$$U_\epsilon(x) := \epsilon^{-\frac{n-2}{2}} U(\epsilon^{-1}x) \text{ for all } x \in \mathbb{R}^{k_+, n-k} \quad (4.20)$$

and

$$u_\epsilon(x) := (\eta U_\epsilon) \circ \phi^{-1}(x) \text{ for } x \in \Omega \text{ and } \tilde{u}_\epsilon := u_\epsilon + \epsilon^{\frac{\alpha_+ - \alpha_-}{2}} \Theta. \quad (4.21)$$

The main result of this paper is the following:

**Proposition 4.4.1.** *Let  $\Omega$  be a bounded domain in  $\mathbb{R}^n$ ,  $n \geq 3$  such that  $0 \in \partial\Omega$  is a singularity of type  $(k, n-k)$  for some  $k \in \{1, \dots, n\}$ . We fix  $0 \leq s < 2$ ,  $\gamma < \gamma_H(\Omega)$  and  $a \in C^{0,\theta}(\Omega)$  ( $\theta \in (0, 1)$ ). Assume that there are extremals for  $\mu_{\gamma,s,0}(\mathbb{R}^{k_+, n-k})$ . We assume that*

$$\gamma > \gamma_H(\mathbb{R}^{k_+, n-k}) - \frac{1}{4} \text{ that is } n < n_{\gamma,k},$$

and that the operator  $-\Delta - (\gamma|x|^{-2} + a(x))$  is coercive with a mass  $m_{\gamma,a}(\Omega)$ . We let  $(\tilde{u}_\epsilon)_\epsilon \in D^{1,2}(\Omega)$  as in (4.21). Then

$$J_{\gamma,s,a}^\Omega(\tilde{u}_\epsilon) = \mu_{\gamma,s,0}(\mathbb{R}^{k_+, n-k}) (1 - \zeta_{\gamma,s}^0 m_{\gamma,a}(\Omega) \epsilon^{\alpha_+ - \alpha_-} + o(\epsilon^{\alpha_+ - \alpha_-})) \text{ as } \epsilon \rightarrow 0,$$

where

$$\zeta_{\gamma,s}^0 := (\alpha_+ - \alpha_-) C_{k,n} \left( \xi \int_{\mathbb{R}^{k_+, n-k}} \frac{U^{2^*(s)}}{|x|^s} dx \right)^{-1} > 0, \quad (4.22)$$

where  $C_{k,n}$  is defined in (4.24).

As one checks, Theorem 4.1.2 is a direct consequence of the combination of Proposition 4.4.1 and Theorem 4.3.1.

This section is devoted to the proof of Proposition 4.4.1.

*Proof of Proposition 4.4.1:* It follows from the uniform convergence in  $C^1$  of the equation (4.18), the definitions of  $\tilde{u}_\epsilon$  and  $G$ , we denote that

$$\lim_{\epsilon \rightarrow 0} \frac{\tilde{u}_\epsilon}{\epsilon^{\frac{\alpha_+ - \alpha_-}{2}}} = G \text{ dans } C_{loc}^1(\Omega) \cap D_{loc,0}^{1,2}(\Omega). \quad (4.23)$$

Define the constant

$$C_{k,n} := \int_{S^{n-1} \cap \mathbb{R}^{k+,n-k}} \left( \prod_{i=1}^k x_i \right)^2 d\sigma. \quad (4.24)$$

In the sequel,  $\vartheta_\rho^\epsilon$  will denote a quantity such

$$\lim_{\rho \rightarrow 0} \lim_{\epsilon \rightarrow 0} \vartheta_\rho^\epsilon = 0.$$

For convenience, we define

$$N_{\gamma,a}(w) := |\nabla w|^2 - (\gamma|x|^{-2} + a)w^2.$$

**Step 4.4.1.** For any  $\rho > 0$ , we claim that

$$\int_{\Omega \setminus \phi(B_{\rho,+})} N_{\gamma,a}(\tilde{u}_\epsilon) dx = \epsilon^{\alpha_+ - \alpha_-} (\alpha_+ C_{k,n} \rho^{n-2\alpha_+ - 2} + m_{\gamma,a}(\Omega)(n-2)C_{k,n} + \vartheta_\rho^\epsilon),$$

as  $\epsilon \rightarrow 0$  with the constant  $C_{k,n}$  is defined in (4.24).

*Proof of Step 4.4.1:* From the equation (4.23), we observe that

$$\lim_{\epsilon \rightarrow 0} \epsilon^{-(\alpha_+ - \alpha_-)} \int_{\Omega \setminus \phi(B_{\rho,+})} N_{\gamma,a}(\tilde{u}_\epsilon) dx = \int_{\Omega \setminus \phi(B_{\rho,+})} N_{\gamma,a}(G) dx.$$

Since  $G$  satisfies (4.8) and vanishes on  $\partial\Omega \setminus \{0\}$ , integrations by parts yield

$$\begin{aligned} \int_{\Omega \setminus \phi(B_{\rho,+})} N_{\gamma,a}(G) dx &= \int_{\Omega \setminus \phi(B_{\rho,+})} (-\Delta G - (\gamma|x|^2 + a(x))G) dx \\ &\quad - \int_{\phi(\partial(B_{\rho,+}))} G \partial_\nu G d\sigma \\ &= - \int_{(\partial B_\rho(0)) \cap \mathbb{R}^{k+,n-k}} (G \circ \phi) \partial_{\phi_* \nu} (G \circ \phi) d(\phi^* \sigma) \end{aligned} \quad (4.25)$$

where  $\nu(x)$  is the outer normal vector of  $B_\rho(0)$  at  $x \in \partial B_\rho(0)$ . We will now find the value of  $(G \circ \phi)\partial_{\phi_*\nu}(G \circ \phi)$ . The definitions of  $v$  and  $G$  yields,

$$(G \circ \phi)(x) = v(x)|x|^{-\alpha_+ - k} + m_{\gamma,a}(\Omega)v(x)|x|^{-\alpha_- - k} + o(v(x)|x|^{-\alpha_- - k}) \text{ as } x \rightarrow 0. \quad (4.26)$$

From  $\Theta$  and the uniform convergence in  $C^1$  of  $G$ , we have for all  $l = 1, \dots, n$  that

$$\partial_l(\Theta \circ \phi) = \partial_l(m_{\gamma,a}(\Omega)v|x|^{-\alpha_- - k}) + o(|x|^{-\alpha_- - 1}) \text{ as } x \rightarrow 0. \quad (4.27)$$

Moreover, the definition of  $G$  yields,

$$\begin{aligned} \partial_l(G \circ \phi) &= \partial_l v (|x|^{-\alpha_+ - k} + m_{\gamma,a}(\Omega)|x|^{-\alpha_- - k}) \\ &\quad - x_l v ((\alpha_+ + k)|x|^{-\alpha_+ - k - 2} + (\alpha_- + k)m_{\gamma,a}(\Omega)|x|^{-\alpha_- - k - 2}) + o(|x|^{-\alpha_- - 1}). \end{aligned}$$

In view of,

$$\phi_*\nu(x) = \frac{x}{|x|} + O(|x|) \text{ as } x \rightarrow 0 \text{ and } \alpha_+ < \alpha_- + 1,$$

we obtain as  $x \rightarrow 0$  that,

$$\partial_{\phi_*\nu}(G \circ \phi) = -v(\alpha_+|x|^{-\alpha_+ - k - 1} + m_{\gamma,a}(\Omega)\alpha_-|x|^{-\alpha_- - k - 1}) + o(|x|^{-\alpha_- - 1}). \quad (4.28)$$

We combine the equations (4.26), (4.28) and since  $\alpha_+ + \alpha_- = n - 2$ ,  $-2\alpha_- - 1 > 1 - n$ ,  $\alpha_+ - \alpha_- < 1$ , we get

$$\begin{aligned} &-(G \circ \phi)\partial_{\phi_*\nu}(G \circ \phi) \\ &= v^2(\alpha_+|x|^{-2\alpha_+ - 2k - 1} + m_{\gamma,a}(\Omega)(n - 2)|x|^{-n + 1 - 2k}) + o(|x|^{1 - n}). \end{aligned}$$

Moreover, using again the definition of  $v$ ,

$$\begin{aligned} &-\int_{\partial B_{\rho,+}} (G \circ \phi)\partial_{\phi_*\nu}(G \circ \phi)d(\phi^*\sigma) \\ &= \alpha_+ C_{k,n}\rho^{n - 2\alpha_+ - 2} + m_{\gamma,a}(\Omega)(n - 2)C_{k,n} + \vartheta_\rho, \end{aligned}$$

where  $\lim_{\rho \rightarrow 0} \vartheta_\rho = 0$  and  $C_{k,n}$  is defined in (4.24). Plugging the last equation in (4.25) yields Step 4.4.1.  $\square$

**Step 4.4.2.** We claim that, as  $\epsilon \rightarrow 0$ ,

$$\begin{aligned} &\int_{\Omega} N_{\gamma,a}(\tilde{u}_\epsilon) dx \\ &= \xi \int_{\mathbb{R}^{k_+, n-k}} |x|^{-s} U^{2^*(s)} dx + m_{\gamma,a}(\Omega)(n - 2)C_{k,n}\epsilon^{\alpha_+ - \alpha_-} + o(\epsilon^{\alpha_+ - \alpha_-}). \end{aligned}$$

*Proof of Step 4.4.2:* With the definition (4.21) of  $\tilde{u}_\epsilon$ , for any  $x \in \mathbb{R}^{k_+, n-k_-}$ , we get

$$\tilde{u}_\epsilon \circ \phi(x) = U_\epsilon(x) + \epsilon^{\frac{\alpha_+ - \alpha_-}{2}} \Theta \circ \phi(x) \text{ for all } x \in B_{\delta, +}. \quad (4.29)$$

Fix  $\rho \in ]0, \delta[$  that we will eventually let go to 0. We define

$$I_{\epsilon, \rho} := \int_{\phi(B_{\rho, +})} (|\nabla \tilde{u}_\epsilon|^2 - (\gamma|x|^{-2} + a) \tilde{u}_\epsilon^2) dx.$$

Let  $\phi^* Eucl$  be the pullback of the Euclidean metric. With (4.29), we get

$$\begin{aligned} I_{\epsilon, \rho} &= \int_{B_{\rho, +}} \left( |\nabla(\tilde{u}_\epsilon \circ \phi)|_{\phi^* Eucl}^2 - \left( \frac{\gamma}{|\phi(x)|^2} + a \circ \phi \right) (\tilde{u}_\epsilon \circ \phi)^2 \right) |Jac(\phi)| dx \\ &= \int_{B_{\rho, +}} \left( |\nabla U_\epsilon|_{\phi^* Eucl}^2 - \left( \frac{\gamma}{|\phi(x)|^2} + a \circ \phi \right) U_\epsilon^2 \right) |Jac(\phi)| dx \\ &\quad + 2\epsilon^{\frac{\alpha_+ - \alpha_-}{2}} \int_{B_{\rho, +}} \left( \langle \nabla U_\epsilon, \nabla(\Theta \circ \phi) \rangle_{\phi^* Eucl} \right. \\ &\quad \left. - \left( \frac{\gamma}{|\phi(x)|^2} + a \circ \phi \right) (\Theta \circ \phi) U_\epsilon \right) |Jac(\phi)| dx \\ &\quad + \epsilon^{\alpha_+ - \alpha_-} \int_{B_{\rho, +}} \left( |\nabla(\Theta \circ \phi)|_{\phi^* Eucl}^2 \right. \\ &\quad \left. - \left( \frac{\gamma}{|\phi(x)|^2} + a \circ \phi \right) (\Theta \circ \phi)^2 \right) |Jac(\phi)| dx. \end{aligned}$$

Since  $d\phi_0 = Id_{\mathbb{R}^n}$ ,  $\phi^* Eucl = Eucl + O(|x|)$ . Since  $\Theta \in D^{1,2}(\Omega)$ , we get that

$$\begin{aligned} I_{\epsilon, \rho} &= \int_{B_{\rho, +}} \left( |\nabla U_\epsilon|_{Eucl}^2 - \left( \frac{\gamma}{|x|^2} + a \circ \phi \right) U_\epsilon^2 \right) dx \\ &\quad + O \left( \int_{B_{\rho, +}} |x| (|\nabla U_\epsilon|_{Eucl}^2 + |x|^{-2} U_\epsilon^2) dx \right) \\ &\quad + 2\epsilon^{\frac{\alpha_+ - \alpha_-}{2}} \int_{B_{\rho, +}} \left( \langle \nabla U_\epsilon, \nabla(\Theta \circ \phi) \rangle_{Eucl} - \left( \frac{\gamma}{|x|^2} + a \circ \phi \right) (\Theta \circ \phi) U_\epsilon \right) dx \\ &\quad + O \left( \epsilon^{\frac{\alpha_+ - \alpha_-}{2}} \int_{B_{\rho, +}} |x| (|\nabla U_\epsilon| \cdot |\nabla(\Theta \circ \phi)| + |x|^{-2} (\Theta \circ \phi) U_\epsilon) dx \right) \\ &\quad + \epsilon^{\alpha_+ - \alpha_-} \vartheta_\rho^\epsilon \end{aligned}$$

as  $\epsilon \rightarrow 0$ . The explicit expression (4.20) of  $U_\epsilon$ , (4.19) and  $n > 2\alpha_+$  yield

$$\begin{aligned} \int_{B_{\rho, +}} U_\epsilon^2 dx &= O \left( \epsilon^{\alpha_+ - \alpha_-} \int_0^\rho r^{n-2\alpha_+ - 1} dr \right) \\ &= \epsilon^{\alpha_+ - \alpha_-} \vartheta_\epsilon^\rho. \end{aligned} \quad (4.30)$$

The definition of  $\Theta$  and  $\alpha_+ + \alpha_- = n - 2$  give

$$\begin{aligned} \int_{B_{\rho,+}} a \circ \phi(\Theta \circ \phi) U_\epsilon dx &= O\left(\epsilon^{\alpha_+ - \alpha_-} \int_0^\rho r dr\right) \\ &= \epsilon^{\alpha_+ - \alpha_-} \vartheta_\rho^\epsilon. \end{aligned} \quad (4.31)$$

We combine the equations (4.17), (4.19), (4.27), (4.30) and (4.31),

$$\begin{aligned} I_{\epsilon,\rho} &= \int_{B_{\rho,+}} \left( |\nabla U_\epsilon|_{Eucl}^2 - \frac{\gamma}{|x|^2} U_\epsilon^2 \right) dx \\ &\quad + 2\epsilon^{\frac{\alpha_+ - \alpha_-}{2}} \int_{B_{\rho,+}} \left( \langle \nabla U_\epsilon, \nabla(\Theta \circ \phi) \rangle_{Eucl} - \frac{\gamma}{|x|^2} (\Theta \circ \phi) U_\epsilon \right) dx \\ &\quad + \epsilon^{\alpha_+ - \alpha_-} \vartheta_\rho^\epsilon \text{ as } \epsilon \rightarrow 0. \end{aligned}$$

Using again the integrations by parts, since  $U_\epsilon$  and  $\Theta \circ \phi$  vanish on  $\partial\mathbb{R}^{k_+,n-k} \setminus \{0\}$ , we have as  $\epsilon \rightarrow 0$  that

$$\begin{aligned} I_{\epsilon,\rho} &= \int_{B_{\rho,+}} U_\epsilon (-\Delta U_\epsilon - \gamma|x|^{-2}U_\epsilon) dx + \int_{\mathbb{R}^{k_+,n-k} \cap \partial B_\rho(0)} U_\epsilon \partial_\nu U_\epsilon d\sigma \\ &\quad + 2\epsilon^{\frac{\alpha_+ - \alpha_-}{2}} \left( \int_{B_{\rho,+}} (\Theta \circ \phi) (-\Delta U_\epsilon - \gamma|x|^{-2}U_\epsilon) dx \right. \\ &\quad \left. + \int_{\mathbb{R}^{k_+,n-k} \cap \partial B_\rho(0)} (\Theta \circ \phi) \partial_\nu U_\epsilon d\sigma \right) + \epsilon^{\alpha_+ - \alpha_-} \vartheta_\rho^\epsilon. \end{aligned} \quad (4.32)$$

We claim as  $\epsilon \rightarrow 0$  that

$$\int_{\mathbb{R}^{k_+,n-k} \cap \partial B_\rho(0)} (\Theta \circ \phi) \partial_\nu U_\epsilon d\sigma = -\alpha_+ \epsilon^{\frac{\alpha_+ - \alpha_-}{2}} m_{\gamma,a}(\Omega) C_{k,n} + o(\epsilon^{\frac{\alpha_+ - \alpha_-}{2}}), \quad (4.33)$$

and

$$\int_{\mathbb{R}^{k_+,n-k} \cap \partial B_\rho(0)} U_\epsilon \partial_\nu U_\epsilon d\sigma = -\alpha_+ C_{k,n} \epsilon^{\alpha_+ - \alpha_-} \rho^{n-2\alpha_+ - 2} + o(\epsilon^{\alpha_+ - \alpha_-} \rho^{n-2-2\alpha_+}). \quad (4.34)$$

We prove the claim. It follows from the uniform convergence in  $C^1$  of the equation (4.18), we have for all  $l = 1, \dots, n$

$$\lim_{\lambda \rightarrow +\infty} \lambda^{\alpha_+} \partial_l U(\lambda x) = |x|^{-\alpha_+ - k} \left( \delta_{l \leq k} \prod_{j=1; j \neq l}^k x_j - (\alpha_+ + k) \frac{v(x) x_l}{|x|^2} \right), \quad (4.35)$$

where  $v$  is defined in (4.14). The definition of  $U_\epsilon$  and (4.19) yield

$$\partial_l U_\epsilon = \epsilon^{\frac{\alpha_+ - \alpha_-}{2}} \left( |x|^{-\alpha_+ - k} \left( \delta_{l \leq k} \prod_{j=1; j \neq l}^k x_j - (\alpha_+ + k) \frac{x_l}{|x|^2} v \right) + o(|x|^{-\alpha_+ - 1}) \right).$$



Since  $\nu(x) = |x|^{-1}x$  is the outer normal vector of  $B_\rho(0)$ , we then get

$$\partial_\nu U_\epsilon = \epsilon^{\frac{\alpha_+ - \alpha_-}{2}} \left( -\alpha_+ \nu |x|^{-\alpha_+ - k - 1} + o(|x|^{-\alpha_+ - 1}) \right), \quad (4.36)$$

as  $\epsilon \rightarrow 0$  uniformly on compact subsets of  $\subset \overline{\mathbb{R}^{k_+, n-k}} \setminus \{0\}$ . From  $\Theta$  and  $\alpha_+ + \alpha_- = n - 2$ , and (4.16), we obtain as  $\epsilon \rightarrow 0$  that

$$(\Theta \circ \phi) \partial_\nu U_\epsilon = \epsilon^{\frac{\alpha_+ - \alpha_-}{2}} \left( -\alpha_+ m_{\gamma, a}(\Omega) \epsilon^{\frac{\alpha_+ - \alpha_-}{2}} \nu^2 |x|^{-n+1-2k} + o(|x|^{1-n}) \right).$$

Therefore, we get (4.33). The definition of  $U_\epsilon$  and the equations (4.17) and (4.36) yield

$$U_\epsilon \partial_\nu U_\epsilon = -\alpha_+ \epsilon^{\alpha_+ - \alpha_-} \nu^2 |x|^{-2\alpha_+ - 2k - 1} + o(\epsilon^{\alpha_+ - \alpha_-} |x|^{-2\alpha_+ - 1}),$$

as  $\epsilon \rightarrow 0$  uniformly locally in  $\subset \overline{\mathbb{R}^{k_+, n-k}} \setminus \{0\}$ . This yields (4.34) and proves the claim.

We combine equations (4.32), (4.33) and (4.34) to get

$$\begin{aligned} I_{\epsilon, \rho} &= \int_{B_{\rho, +}} U_\epsilon (-\Delta U_\epsilon - \gamma |x|^{-2} U_\epsilon) dx - \alpha_+ C_{k, n} \epsilon^{\alpha_+ - \alpha_-} \rho^{n-2\alpha_+ - 2} \\ &+ 2\epsilon^{\frac{\alpha_+ - \alpha_-}{2}} \int_{B_{\rho, +}} (\Theta \circ \phi) (-\Delta U_\epsilon - \gamma |x|^{-2} U_\epsilon) dx \\ &- 2\alpha_+ \epsilon^{\alpha_+ - \alpha_-} m_{\gamma, a}(\Omega) C_{k, n} + \epsilon^{\alpha_+ - \alpha_-} \vartheta_\rho^\epsilon. \end{aligned}$$

Since  $U$  satisfies the equation (4.11) and by the definition (4.20) of  $U_\epsilon$ , we have

$$-\Delta U_\epsilon - \gamma |x|^{-2} U_\epsilon = \xi |x|^{-s} U_\epsilon^{2^*(s)-1}.$$

Therefore, we get as  $\epsilon \rightarrow 0$  that

$$\begin{aligned} I_{\epsilon, \rho} &= \xi \int_{B_{\rho, +}} \frac{U_\epsilon^{2^*(s)}}{|x|^s} dx - \alpha_+ C_{k, n} \epsilon^{\alpha_+ - \alpha_-} \rho^{n-2\alpha_+ - 2} \\ &+ 2\epsilon^{\frac{\alpha_+ - \alpha_-}{2}} \xi \int_{B_{\rho, +}} (\Theta \circ \phi) \frac{U_\epsilon^{2^*(s)-1}}{|x|^s} dx \\ &- 2\alpha_+ \epsilon^{\alpha_+ - \alpha_-} m_{\gamma, a}(\Omega) C_{k, n} + \epsilon^{\alpha_+ - \alpha_-} \vartheta_\rho^\epsilon. \end{aligned} \quad (4.37)$$

The definition (4.20) of  $U_\epsilon$  and (4.19) yield

$$\left| \xi \int_{\mathbb{R}^{k_+, n-k} \setminus (B_{\rho, +})} \frac{U_\epsilon^{2^*(s)}}{|x|^s} dx \right| \leq C \epsilon^{\frac{2^*(s)}{2}(\alpha_+ - \alpha_-)}.$$

Therefore, with  $2^*(s) > 2$ , we get

$$\xi \int_{B_{\rho,+}} \frac{U_\epsilon^{2^*(s)}}{|x|^s} dx = \xi \int_{\mathbb{R}^{k_+,n-k}} \frac{U^{2^*(s)}}{|x|^s} dx + o(\epsilon^{\alpha_+ - \alpha_-}) \text{ as } \epsilon \rightarrow 0. \quad (4.38)$$

The definition (4.14), (4.20) and the control (4.19) yield

$$\begin{aligned} \int_{\mathbb{R}^{k_+,n-k} \setminus (B_{\rho,+})} v \frac{U_\epsilon^{2^*(s)-1}}{|x|^{s+\alpha_-+k}} dx &= O\left(\epsilon^{\frac{\alpha_+ - \alpha_-}{2}} \int_{\epsilon^{-1}\rho}^{+\infty} r^{(1-\frac{2^*(s)}{2})(\alpha_+ - \alpha_-) - 1} dr\right) \\ &= \epsilon^{\frac{2^*(s)-1}{2}(\alpha_+ - \alpha_-)} \vartheta_\rho^\epsilon. \end{aligned}$$

Therefore, with the definition of  $\Theta$  we get as  $\epsilon \rightarrow 0$  that

$$\begin{aligned} \xi \int_{B_{\rho,+}} \frac{U_\epsilon^{2^*(s)-1}}{|x|^s} \Theta \circ \phi dx &= \xi m_{\gamma,a}(\Omega) \int_{B_{\rho,+}} v \frac{U_\epsilon^{2^*(s)-1}}{|x|^{s+\alpha_-+k}} dx \\ &+ o\left(\int_{B_{\rho,+}} v \frac{U_\epsilon^{2^*(s)-1}}{|x|^{s+\alpha_-+k}} dx\right) \\ &= \epsilon^{\frac{\alpha_+ - \alpha_-}{2}} \left(m_{\gamma,a}(\Omega) \xi \int_{\mathbb{R}^{k_+,n-k}} v \frac{U^{2^*(s)-1}}{|x|^{s+\alpha_-+k}} dx + \vartheta_\epsilon^\rho\right). \end{aligned} \quad (4.39)$$

Since  $(-\Delta - \gamma|x|^{-2})(v|x|^{-\alpha_- - k}) = 0$ , using integrations by parts and since  $U$  vanishes on  $\partial\Omega \setminus \{0\}$ , we obtain that

$$\begin{aligned} \xi \int_{\mathbb{R}^{k_+,n-k}} v \frac{U^{2^*(s)-1}}{|x|^{s+\alpha_-+k}} dx &= \lim_{R \rightarrow +\infty} \int_{B_{R,+}} v|x|^{-\alpha_- - k} (-\Delta U - \gamma|x|^{-2}U) dx \\ &= \lim_{R \rightarrow +\infty} \left[ \int_{B_{R,+}} U (-\Delta - \gamma|x|^{-2})(v|x|^{-\alpha_- - k}) dx \right. \\ &\quad \left. - \int_{\mathbb{R}^{k_+,n-k} \cap \partial B_R} \partial_\nu U v|x|^{-\alpha_- - k} d\sigma \right]. \end{aligned} \quad (4.40)$$

Arguing as for (4.36), it follows from (4.35), that, as  $R \rightarrow +\infty$

$$\partial_\nu U = -\alpha_+ v|x|^{-\alpha_+ - k - 1} + o(|x|^{-\alpha_+ - 1}) \text{ uniformly for } x \in \partial B_R(0) \cap \mathbb{R}^{k_+,n-k}.$$

Moreover, since  $\alpha_+ + \alpha_- = n - 2$  we get

$$\partial_\nu U v|x|^{-\alpha_- - k} = -\alpha_+ v^2|x|^{-(n+2k-1)} + o(|x|^{1-n}).$$

The last equation yields,

$$\lim_{R \rightarrow +\infty} \int_{\mathbb{R}^{k_+,n-k} \cap \partial B_R(0)} \partial_\nu U v|x|^{-\alpha_- - k} d\sigma = -\alpha_+ C_{k,n}.$$

Then, by (4.40)

$$\xi \int_{\mathbb{R}^{k_+, n-k}} v \frac{U^{2^*(s)-1}}{|x|^{s+\alpha_++k}} dx = \alpha_+ C_{k,n}. \quad (4.41)$$

Combining (4.39) and (4.41), we get

$$\xi \int_{B_{\rho,+}} \frac{U_\epsilon^{2^*(s)-1}}{|x|^s} \Theta \circ \phi dx = \epsilon^{\frac{\alpha_+-\alpha_-}{2}} (\alpha_+ m_{\gamma,a}(\Omega) C_{k,n} + \vartheta_\epsilon^\rho) \text{ as } \epsilon \rightarrow 0. \quad (4.42)$$

Next, the equations (4.37), (4.38) and (4.42) yields,

$$I_{\epsilon,\rho} = \xi \int_{\mathbb{R}^{k_+, n-k}} \frac{U_\epsilon^{2^*(s)}}{|x|^s} dx - \alpha_+ C_{k,n} \epsilon^{\alpha_+-\alpha_-} \rho^{n-2\alpha_+-2} + o(\epsilon^{\alpha_+-\alpha_-}).$$

In the other hand, using Step 4.4.1 the definition of  $I_{\epsilon,\rho}$  and the last equation, we get Step 4.4.2  $\square$

**Step 4.4.3.** We claim as  $\epsilon \rightarrow 0$  that,

$$\int_{\Omega} \frac{\tilde{u}_\epsilon^{2^*(s)}}{|x|^s} dx = \int_{\mathbb{R}^{k_+, n-k}} \frac{U_\epsilon^{2^*(s)}}{|x|^s} dx + 2^*(s) \alpha_+ m_{\gamma,a}(\Omega) \xi^{-1} C_{k,n} \epsilon^{\alpha_+-\alpha_-} + o(\epsilon^{\alpha_+-\alpha_-}).$$

*Proof of Step 4.4.3:* We fix  $\rho > 0$ . The definitions of  $\tilde{u}_\epsilon$  and  $\Theta$ , and  $2^*(s) > 2$  yield

$$\int_{\phi(B_{2\delta,+} \setminus B_{\delta,+})} \frac{\tilde{u}_\epsilon^{2^*(s)}}{|x|^s} dx = o(\epsilon^{\alpha_+-\alpha_-}), \quad (4.43)$$

with the definition (4.12). Equations (4.15), (4.17), (4.29) and (4.43) yield

$$\int_{\Omega} \frac{\tilde{u}_\epsilon^{2^*(s)}}{|x|^s} dx = \int_{B_{\delta,+}} \frac{|U_\epsilon + \epsilon^{\frac{\alpha_+-\alpha_-}{2}} (\Theta \circ \phi)|^{2^*(s)}}{|x|^s} dx + o(\epsilon^{\alpha_+-\alpha_-}),$$

as  $\epsilon \rightarrow 0$ .

$$\begin{aligned} \int_{\Omega} \frac{\tilde{u}_\epsilon^{2^*(s)}}{|x|^s} dx &= \int_{B_{\delta,+}} \left( \frac{U_\epsilon^{2^*(s)}}{|x|^s} + 2^*(s) \epsilon^{\frac{\alpha_+-\alpha_-}{2}} \frac{U_\epsilon^{2^*(s)-1}}{|x|^s} (\Theta \circ \phi) \right) dx \\ &+ \int_{B_{\delta,+}} O \left( \epsilon^{\alpha_+-\alpha_-} \frac{U_\epsilon^{2^*(s)-2}}{|x|^s} \Theta \circ \phi^2 + \epsilon^{\frac{2^*(s)}{2}(\alpha_+-\alpha_-)} |\Theta \circ \phi|^{2^*(s)} \right) dx \\ &+ o(\epsilon^{\alpha_+-\alpha_-}). \end{aligned}$$

It follows from the definitions of  $\Theta$  and  $U_\epsilon$ ,

$$\begin{aligned} \int_{B_{\delta,+}} \frac{U_\epsilon^{2^*(s)-2}}{|x|^s} (\epsilon^{\frac{\alpha_+ - \alpha_-}{2}} (\Theta \circ \phi))^2 dx &= O \left( \epsilon^{2(\alpha_+ - \alpha_-)} \int_0^{\epsilon^{-1}\rho} r^{\frac{2^*(s)}{2}(\alpha_+ - \alpha_-) - 1} dr \right) \\ &= \epsilon^{\alpha_+ - \alpha_-} \vartheta_\epsilon^\rho. \end{aligned} \quad (4.44)$$

And,

$$\begin{aligned} \int_{B_{\delta,+}} (\epsilon^{\frac{\alpha_+ - \alpha_-}{2}} \Theta \circ \phi)^{2^*(s)} |x|^{-s} dx &= O \left( \epsilon^{(\alpha_+ - \alpha_-) \frac{2^*(s)}{2}} \int_0^\rho r^{\frac{2^*(s)}{2}(\alpha_+ - \alpha_-) - 1} dr \right) \\ &= \epsilon^{\alpha_+ - \alpha_-} \vartheta_\epsilon^\rho. \end{aligned} \quad (4.45)$$

The equations (4.44) et (4.45) yield as  $\epsilon \rightarrow 0$  that,

$$\int_\Omega \frac{\tilde{u}_\epsilon^{2^*(s)}}{|x|^s} dx = \int_{B_{\delta,+}} \left( \frac{U_\epsilon^{2^*(s)}}{|x|^s} + 2^*(s) \epsilon^{\frac{\alpha_+ - \alpha_-}{2}} \frac{U_\epsilon^{2^*(s)-1}}{|x|^s} (\Theta \circ \phi) \right) dx + \epsilon^{\alpha_+ - \alpha_-} \vartheta_\epsilon^\rho. \quad (4.46)$$

Therefore, for all  $\xi > 0$  the equations (4.38), (4.42) and (4.46) yield the result.  $\square$

**Step 4.4.4.** We are now in position to prove Proposition 4.4.1.

*Proof of Step 4.4.4:* By Step 4.4.3, we have that

$$\begin{aligned} \left( \int_\Omega \frac{\tilde{u}_\epsilon^{2^*(s)}}{|x|^s} dx \right)^{\frac{2}{2^*(s)}} &= \left( \int_{\mathbb{R}^{k_+, n-k_+}} \frac{U_\epsilon^{2^*(s)}}{|x|^s} dx \right)^{\frac{2}{2^*(s)}} \\ &+ 2\alpha_+ m_{\gamma,a}(\Omega) \xi^{-1} C_{k,n} \epsilon^{\alpha_+ - \alpha_-} \left( \int_{\mathbb{R}^{k_+, n-k_+}} \frac{U_\epsilon^{2^*(s)}}{|x|^s} dx \right)^{\frac{2}{2^*(s)} - 1} + o(\epsilon^{\alpha_+ - \alpha_-}). \end{aligned} \quad (4.47)$$

We go back to the definition of  $J_{\gamma,a,s}^\Omega$ , Step 4.4.3, equation (4.47) and since  $U$  satisfies (4.11), we get as  $\epsilon \rightarrow 0$  that

$$J_{\gamma,s,a}^\Omega(\tilde{u}_\epsilon) = J_{\gamma,s,0}^{\mathbb{R}^{k_+, n-k_+}}(U) \left( 1 - m_{\gamma,a}(\Omega) \zeta_{\gamma,s}^0 \epsilon^{\alpha_+ - \alpha_-} + o(\epsilon^{\alpha_+ - \alpha_-}) \right),$$

where  $\zeta_{\gamma,s}^0$  is defined in (4.22). This ends the proof of Proposition 4.4.1.  $\square$

Combining Proposition 4.4.1 and Theorem 4.3.1 yields Theorem 4.1.2.

## 4.5 Examples of mass

In this section, we discuss the existence and the sign of the mass. An example of existence of mass is as follows:

**Proposition 4.5.1.** *Let  $\Omega$  be a bounded domain in  $\mathbb{R}^n$ ,  $n \geq 3$  such that  $0 \in \partial\Omega$  is a singularity of type  $(k, n - k)$  for some  $k \in \{1, \dots, n\}$ . We assume that  $\gamma > \gamma_H(\mathbb{R}^{k+, n-k}) - 1/4$  and that*

$$\Omega \cap B_\delta(0) = \mathbb{R}^{k+, n-k} \cap B_\delta(0) \text{ for some } \delta > 0. \quad (4.48)$$

We assume that  $\gamma_H(\mathbb{R}^{k+, n-k}) - \frac{1}{4} < \gamma < \gamma_H(\Omega)$ , that  $a \in C^{0,\theta}(\Omega)$  vanishes around 0 and that  $-\Delta - (\gamma|x|^{-2} + a(x))$  is coercive. Then the mass is defined.

*Proof of Proposition 4.5.1.* We fix  $\eta$  as in (4.13). For  $a \in C^{0,\theta}(\Omega)$  that vanishes around 0, define on  $\Omega$  the function

$$g := \left( -\Delta - \frac{\gamma}{|x|^2} - a(x) \right) (\eta S_{\alpha_+}),$$

where  $S_{\alpha_+}$  is defined in (4.7) such that  $-\Delta S_{\alpha_+} - \gamma|x|^{-2}S_{\alpha_+} = 0$  on  $\mathbb{R}^{k+, n-k}$ . Note that this definition makes sense when the support of  $\eta$  is small enough due to (4.48) and  $a$  vanishes around 0. In particular  $g(x) = 0$  around 0. Therefore, we have  $g \in L^{\frac{2n}{n+2}}(\Omega) = (L^{2^*}(\Omega))' \subset (D^{1,2}(\Omega))'$ . Since the operator  $-\Delta - (\gamma|x|^{-2} + a)$  is coercive, there exists  $w \in D^{1,2}(\Omega)$  such that

$$\begin{cases} \left( -\Delta - \frac{\gamma}{|x|^2} - a(x) \right) w = g & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega. \end{cases}$$

Since  $g$  vanishes around 0, Theorem 4.3.2 and the change of variable  $y = \lambda x$  that there exists  $K \in \mathbb{R}$  such that

$$w(x) = K \frac{v(x)}{|x|^{\alpha_- + k}} + o\left(\frac{v(x)}{|x|^{\alpha_- + k}}\right) \text{ as } x \rightarrow 0,$$

where  $v$  is as in (4.14). For all  $x \in \bar{\Omega} \setminus \{0\}$ , we define the function  $G_0 := \eta S_{\alpha_+} - w$ . The definition of  $w$  yields

$$\begin{cases} \left( -\Delta - \frac{\gamma}{|x|^2} - a(x) \right) G_0 = 0 & \text{in } \Omega, \\ G_0 = 0 & \text{on } \partial\Omega \setminus \{0\}. \end{cases}$$

For  $\delta_0 > 0$  small enough, the definitions of  $S_{\alpha_+}$ ,  $w$  and  $\alpha_- < \alpha_+$  yield

$$G_0(x) = v(x)|x|^{-\alpha_+ - k} (1 + o(1)) \text{ in } \mathbb{R}^{k+, n-k} \cap B_{\delta_0},$$

#### 4.6. Proof of Theorem 4.1.3: functional background for the perturbed equation

with  $o(1) \rightarrow 0$  as  $x \rightarrow 0$ . Therefore,  $G_0 > 0$  in  $\mathbb{R}^{k+,n-k} \cap B_{\delta_0}$ . Then coercivity and the comparison principle yield  $G_0 > 0$  in  $\Omega$ . Moreover, we have that

$$G_0(x) = v(x) (|x|^{-\alpha+k} - K|x|^{-\alpha-k} + o(|x|^{-\alpha-k})),$$

as  $x \rightarrow 0$ . Then the mass at 0 of  $-\Delta - (\gamma|x|^{-2} + a(x))$  is defined and  $m_{\gamma,a}(\Omega) = -K$ . This proves Proposition 4.5.1.

We now discuss briefly examples of negative and positive mass. Here, the reference is Section 9 of Ghoussoub-Robert [61]. We still assume (4.48) and that  $\gamma > \gamma_H(\mathbb{R}^{k+,n-k}) - 1/4$ , so that the mass  $m_{\gamma,0}(\Omega)$  is defined. When  $\Omega \subset \mathbb{R}^{k+,n-k}$ , due to the comparison principle, we get that  $G_0 < S_{\alpha+}$ , and  $m_{\gamma,0}(\Omega) < 0$ . Arguing as in [61], we are able to define the mass of a domain  $\tilde{\Omega} \supset \mathbb{R}^{k+,n-k}$ , for which  $m_{\gamma,0}(\tilde{\Omega}) > 0$ : then, defining  $\tilde{\Omega}_R := \tilde{\Omega} \cap B_R(0)$ , we get that  $\lim_{R \rightarrow +\infty} m_{\gamma,0}(\tilde{\Omega}_R) = m_{\gamma,0}(\tilde{\Omega}) > 0$ . So for  $R > 0$  large, we get examples of bounded domains with a singularity of type  $(k, n - k)$  at 0 and with positive mass.

#### 4.6 Proof of Theorem 4.1.3: functional background for the perturbed equation

In this section, we proceed as in Jaber [76]. A Palais-Smale sequence for  $G : E \rightarrow \mathbb{R}$  is a sequence  $(u_m)_{m \in \mathbb{N}} \in E$  such that there exists  $\beta \in \mathbb{R}$  such that

$$G(u_m) \rightarrow \beta \text{ for all } m \in \mathbb{N} \text{ and } G'(u_m) \rightarrow 0 \text{ in } E',$$

as  $m \rightarrow +\infty$ . Here, we say that the Palais-Smale sequence is at level  $\beta$ . The main tool is the Mountain Pass Lemma of Ambrosetti-Rabinowitz [5]:

**Theorem 4.6.1** (Mountain-Pass Lemma [5]). *Let  $G \in C^1(E, \mathbb{R})$  where  $(E, \|\cdot\|)$  is a Banach space. We assume that  $G(0) = 0$  and that*

- *There exists  $\lambda, r > 0$  such that  $G(u) \geq \lambda$  for all  $u \in E$  such that  $\|u\| = r$ ,*
- *There exists  $u_0$  in  $E$  such that  $\limsup_{t \rightarrow +\infty} G(tu_0) < 0$ .*

*We consider  $t_0 > 0$  large such that  $\|t_0 u_0\| > r$  and  $G(t_0 u_0) < 0$ , and*

$$\beta = \inf_{c \in \Gamma} \sup G(c(t)),$$

*where*

$$\Gamma = \{c : [0, 1] \rightarrow E \text{ s.t. } c(0) = 0, c(1) = t_0 u_0\}.$$

*Then, there exists a Palais-Smale sequence at level  $\beta$  for  $G$ . Moreover, we have that  $\beta \leq \sup_{t \geq 0} G(tu_0)$ .*

Weak solutions of (4.5) are to the nonzero critical points of the functional

$$E_q(u) := \frac{1}{2} \int_{\Omega} \left( |\nabla u|^2 - \left( \frac{\gamma}{|x|^2} + a \right) u^2 \right) dx - \int_{\Omega} \frac{u_+^{2^*(s)}}{2^*(s)|x|^s} dx - \int_{\Omega} \frac{hu_+^{q+1}}{q+1} dx,$$

for any  $u \in D^{1,2}(\Omega)$  and where  $u_+ = \max\{u, 0\}$ . In the sequel, we assume that the operator  $-\Delta - \left( \frac{\gamma}{|x|^2} + a(x) \right)$  is coercive that there exists  $c_0 > 0$  such that

$$\int_{\Omega} \left( |\nabla w|^2 - \left( \frac{\gamma}{|x|^2} + a \right) w^2 \right) dx \geq c_0 \int_{\Omega} |\nabla w|^2 dx \text{ for all } w \in D^{1,2}(\Omega). \quad (4.49)$$

**Proposition 4.6.1.** Fix  $u_0 \in D^{1,2}(\Omega)$  such that  $u_0 \geq 0$ ,  $u_0 \not\equiv 0$ . Then there exists a sequence  $(u_m)_{m \in \mathbb{N}} \in D^{1,2}(\Omega)$  that is a Palais-Smale sequence for  $E_q$  at level  $\beta$  such that  $0 < \beta \leq \sup_{t \geq 0} E_q(tu_0)$ .

*Proof of Proposition 4.6.1:* Clearly  $E_q \in C^1(D^{1,2}(\Omega))$ . Note that  $E_q(0) = 0$ . It follows from (4.49) and the Sobolev and Hardy-Sobolev embeddings that there exist  $c_0, c_1, c_2 > 0$  such that

$$E_q(u) \geq c_0 \|u\|^2 - c_1 \|u\|^{2^*(s)} - c_2 \|u\|^{q+1} \text{ for all } u \in D^{1,2}(\Omega). \quad (4.50)$$

Define  $f(r) = r^2 [c_0 - c_1 r^{2^*(s)-2} - c_2 r^{q-1}] := r^2 g(r)$  and since  $2^*(s), q+1 > 2$  we have  $g(r) \rightarrow c_0$  as  $r \rightarrow 0$ . Then there exists  $r_0 > 0$  such that  $r < r_0$ , we have  $g(r) > \frac{c_0}{2}$ . Therefore, for all  $u \in D^{1,2}(\Omega)$  such that  $\|u\| = \frac{r_0}{2}$  and by (4.50), we have  $E_q(u) \geq \frac{c_0 r_0^2}{8} := \lambda$ . We fix  $u_0 \in D^{1,2}(\Omega)$ ,  $u_0 \not\equiv 0$ . We have that

$$\begin{aligned} E_q(tu_0) &= \frac{t^2}{2} \int_{\Omega} \left( |\nabla u_0|^2 - \left( \frac{\gamma}{|x|^2} + a \right) u_0^2 \right) dx \\ &\quad - \frac{t^{2^*(s)}}{2^*(s)} \int_{\Omega} \frac{|u_0|^{2^*(s)}}{|x|^s} dx - \frac{t^{q+1}}{q+1} \int_{\Omega} h|u_0|^{q+1} dx \\ &:= \frac{t^2}{2} R_1 - \frac{t^{2^*(s)}}{2^*(s)} R_2 - \frac{t^{q+1}}{q+1} R_3 \leq t^{2^*(s)} \left( \frac{t^{2-2^*(s)}}{2} R_1 - R_2 \right), \end{aligned}$$

where  $R_1, R_2 > 0$  and  $R_3 \geq 0$ . Since  $2^*(s) > 2$ , we have  $E_q(tu_0) \rightarrow -\infty$  as  $t \rightarrow +\infty$ . Then  $\limsup_{t \rightarrow +\infty} E_q(tu_0) < 0$ . We consider  $t_0 > 0$  large such that  $\|t_0 u_0\| > r$  and  $E_q(t_0 u_0) < 0$ . For  $t \in [0, 1]$ , we have  $E_q(c(t)) \geq \lambda$  and then there exists

$$\beta := \inf_{c \in \Gamma} \sup E_q(c(t)) \geq \lambda > 0.$$

Proposition 4.6.1 then follows from Theorem 4.6.1.  $\square$

#### 4.6. Proof of Theorem 4.1.3: functional background for the perturbed equation

**Proposition 4.6.2.** *Let  $\Omega$  be a bounded domain in  $\mathbb{R}^n$ ,  $n \geq 3$  such that  $0 \in \partial\Omega$  is a singularity of type  $(k, n-k)$  for some  $k \in \{1, \dots, n\}$ . We fix  $a, h \in C^{0,\theta}(\Omega)$ ,  $\theta \in (0, 1)$ . We assume that  $h \geq 0$  and that (4.49) holds. We fix  $\gamma < \gamma_H(\mathbb{R}^{k+,n-k})$  and  $\beta \in \mathbb{R}$  such that*

$$\beta < \frac{2-s}{2(n-s)} \mu_{\gamma,s,0}(\mathbb{R}^{k+,n-k})^{\frac{n-s}{2-s}}. \quad (4.51)$$

*Then, for any Palais-Smale sequence  $(u_m)_{m \in \mathbb{N}} \in D^{1,2}(\Omega)$  for  $E_q$  at level  $\beta$ , there exists  $u \in D^{1,2}(\Omega)$  such that  $E_q(u) = \beta$  and we have  $u_m$  converges strongly in  $D^{1,2}(\Omega)$  up to a subsequence. Moreover, we have  $E'_q(u) = 0$ .*

*Proof of Proposition 4.6.2:* Take  $(u_m)_{m \in \mathbb{N}} \in D^{1,2}(\Omega)$  a Palais-Smale sequence for  $E_q$  such that

$$E_q(u_m) \rightarrow \beta \text{ and } E'_q(u_m) \rightarrow 0 \text{ in } D^{1,2}(\Omega)'.$$

**Step 4.6.1.** *We claim that  $u_m$  is bounded in  $D^{1,2}(\Omega)$ .*

*Proof of Step 4.6.1:* The coercivity (4.21) and the definition of  $E_q$  yield

$$\|u_m\|^2 \leq 2c_0^{-1} \left( E_q(u_m) + \frac{1}{2^*(s)} \int_{\Omega} (u_m)_+^{2^*(s)} dx + \frac{1}{q+1} \int_{\Omega} h(u_m)_+^{q+1} dx \right). \quad (4.52)$$

Since  $E'_q(u_m) \rightarrow 0$  in  $D^{1,2}(\Omega)'$ , we observe that

$$\begin{aligned} \int_{\Omega} \left( |\nabla u_m|^2 - \left( \frac{\gamma}{|x|^2} + a \right) u_m^2 \right) dx &= \int_{\Omega} \frac{(u_m)_+^{2^*(s)}}{|x|^s} dx \\ &\quad + \int_{\Omega} h(u_m)_+^{q+1} dx + o(\|u_m\|). \end{aligned}$$

The definition of the energy  $E_q$  and the last equation yield

$$\begin{aligned} 2E_q(u_m) &= \left( 1 - \frac{2}{2^*(s)} \right) \int_{\Omega} \frac{(u_m)_+^{2^*(s)}}{|x|^s} dx \\ &\quad + \left( 1 - \frac{2}{q+1} \right) \int_{\Omega} h(u_m)_+^{q+1} dx + o(\|u_m\|). \end{aligned} \quad (4.53)$$

Moreover, since  $E_q(u_m) \rightarrow \beta$  as  $m \rightarrow +\infty$ ,  $h \geq 0$  and  $q+1 > 2$ , we obtain that

$$\begin{aligned} \left( 1 - \frac{2}{2^*(s)} \right) \int_{\Omega} \frac{(u_m)_+^{2^*(s)}}{|x|^s} dx &= 2E_q(u_m) \\ &\quad - \left( 1 - \frac{2}{q+1} \right) \int_{\Omega} h(u_m)_+^{q+1} dx + o(\|u_m\|) \\ &\leq 2\beta + o(\|u_m\|), \end{aligned}$$



therefore,

$$\left(1 - \frac{2}{2^*(s)}\right) \int_{\Omega} \frac{(u_m)_+^{2^*(s)}}{|x|^s} dx = O(1) + o(\|u_m\|). \quad (4.54)$$

Similar and but  $2^*(s) > 2$ , we have

$$\left(1 - \frac{2}{q+1}\right) \int_{\Omega} h(u_m)_+^{q+1} dx = O(1) + o(\|u_m\|). \quad (4.55)$$

(4.52) and (3.71) give

$$\|u_m\|^2 \leq c_0^{-1} \left( \int_{\Omega} \frac{(u_m)_+^{2^*(s)}}{|x|^s} dx + \int_{\Omega} h(u_m)_+^{q+1} dx \right) + o(\|u_m\|). \quad (4.56)$$

The equations (4.54), (4.55) and (4.56) yields,

$$\|u_m\|^2 = O(1) + o(\|u_m\|),$$

as  $m \rightarrow +\infty$ . This proves Step 4.6.1.  $\square$

Therefore, up to a subsequence, there exists  $u \in D^{1,2}(\Omega)$  such that

$$\begin{cases} u_m \rightharpoonup u & \text{weakly in } D^{1,2}(\Omega), \\ u_m \rightarrow u & \text{strongly in } L^p(\Omega) \text{ for all } 1 < p < 2^*. \end{cases} \quad (4.57)$$

Moreover, we have  $E'_q(u) = 0$ .

**Step 4.6.2.** We claim that, as  $m \rightarrow +\infty$

$$\int_{\Omega} \left( |\nabla(u_m - u)|^2 - \gamma \frac{(u_m - u)^2}{|x|^2} \right) dx = \int_{\Omega} \frac{(u_m - u)_+^{2^*(s)}}{|x|^s} dx + o(1), \quad (4.58)$$

and,

$$\frac{2-s}{2(n-s)} \int_{\Omega} \left( |\nabla(u_m - u)|^2 - \gamma \frac{(u_m - u)^2}{|x|^2} \right) dx \leq \beta + o(1). \quad (4.59)$$

*Proof of Step 4.6.2:* We denote that

$$\begin{aligned} \langle E'_q(u_m), \varphi \rangle &= \int_{\Omega} \left( (\nabla u_m, \nabla \varphi) - \left( \frac{\gamma}{|x|^2} + a \right) u_m \varphi \right) dx \\ &\quad - \int_{\Omega} \frac{(u_m)_+^{2^*(s)-1}}{|x|^s} \varphi dx - \int_{\Omega} h(u_m)_+^q \varphi dx, \end{aligned}$$

#### 4.6. Proof of Theorem 4.1.3: functional background for the perturbed equation

for all  $\varphi \in D^{1,2}(\Omega)$ . We observe that

$$\begin{aligned}
 o(1) &= \langle E'_q(u_m) - E'_q(u), u_m - u \rangle \\
 &= \int_{\Omega} \left( |\nabla(u_m - u)|^2 - \left( \frac{\gamma}{|x|^2} + a \right) (u_m - u)^2 \right) dx \quad (4.60) \\
 &\quad - \int_{\Omega} \left( (u_m)_+^{2^*(s)-1} - u_+^{2^*(s)-1} \right) \frac{(u_m - u)}{|x|^s} dx \\
 &\quad - \int_{\Omega} h((u_m)_+^q - u_+^q)(u_m - u) dx.
 \end{aligned}$$

Since  $u_m \rightharpoonup u$  weakly in  $D^{1,2}(\Omega)$ , integration theory yields

$$\lim_{m \rightarrow +\infty} \int_{\Omega} \frac{(u_m)_+^{2^*(s)-1}}{|x|^s} u dx = \int_{\Omega} \frac{u_+^{2^*(s)}}{|x|^s} dx = \lim_{m \rightarrow +\infty} \int_{\Omega} \frac{u_+^{2^*(s)-1}}{|x|^s} u_m dx. \quad (4.61)$$

The equation (4.57) yields,

$$\int_{\Omega} h(u_m - u) ((u_m)_+^q - u_+^q) dx = \int_{\Omega} h(u_m - u)^{q+1} dx + o(1) = o(1), \quad (4.62)$$

as  $m \rightarrow +\infty$ . Combining (4.60), (4.61) and (4.62), we get as  $m \rightarrow +\infty$  that

$$\begin{aligned}
 &\int_{\Omega} \left( |\nabla(u_m - u)|^2 - \left( \frac{\gamma}{|x|^2} + a \right) (u_m - u)^2 \right) dx \\
 &= \int_{\Omega} \left( (u_m)_+^{2^*(s)} - u_+^{2^*(s)} \right) \frac{dx}{|x|^s} + o(1). \quad (4.63)
 \end{aligned}$$

Since  $2^*(s) > 1$ , we get that

$$\left| (u_m)_+^{2^*(s)} - u_+^{2^*(s)} - (u_m - u)_+^{2^*(s)} \right| \leq C (|u_m - u|^{2^*(s)-1} |u| + |u|^{2^*(s)-1} |u_m - u|),$$

for some  $C > 0$  independent of  $m$ . Therefore, but (4.57) we have

$$\int_{\Omega} \left( (u_m)_+^{2^*(s)} - (u_m - u)_+^{2^*(s)} \right) \frac{dx}{|x|^s} = \int_{\Omega} \frac{u_+^{2^*(s)}}{|x|^s} dx + o(1). \quad (4.64)$$

Since  $u_m \rightarrow u$  strongly in  $L^2(\Omega)$  as  $m \rightarrow +\infty$  and by (4.63), (4.64), we obtain (4.58). With (4.57) we have that

$$\begin{aligned}
 E_q(u_m) - E_q(u) &= \frac{1}{2} \int_{\Omega} \left( |\nabla(u_m - u)|^2 - \gamma \frac{(u_m - u)^2}{|x|^2} \right) dx \\
 &\quad - \frac{1}{2^*(s)} \int_{\Omega} \left( (u_m)_+^{2^*(s)} - u_+^{2^*(s)} \right) \frac{dx}{|x|^s} + o(1).
 \end{aligned}$$

With (4.58), we get

$$E_q(u_m) - E_q(u) = \left( \frac{1}{2} - \frac{1}{2^*(s)} \right) \int_{\Omega} \left( |\nabla(u_m - u)|^2 - \gamma \frac{(u_m - u)^2}{|x|^2} \right) dx + o(1).$$

Since  $u$  is a solution of (4.5) then  $E_q(u) \geq 0$ , and  $E_q(u_m) \rightarrow \beta$  as  $m \rightarrow +\infty$ . We then get (4.59). This proves Step 4.6.2.  $\square$

**Step 4.6.3.** We claim that

$$\lim_{m \rightarrow +\infty} u_m = u \text{ in } D^{1,2}(\Omega). \quad (4.65)$$

*Proof of Step 4.6.3:* Let  $\gamma < \gamma_H(\mathbb{R}^{k_+, n-k})$  for all  $k \in \{1, \dots, n\}$  and by the Proposition 2.1 in Cheikh-Ali [27], then for all  $\epsilon > 0$  there exists  $c_\epsilon > 0$  such that for all  $v \in D^{1,2}(\Omega)$ ,

$$\begin{aligned} \left( \int_{\Omega} \frac{|v|^{2^*(s)}}{|x|^s} dx \right)^{\frac{2}{2^*(s)}} &\leq (\mu_{\gamma, s, 0}(\mathbb{R}^{k_+, n-k})^{-1} + \epsilon) \int_{\Omega} \left( |\nabla v|^2 - \frac{\gamma}{|x|^2} v^2 \right) dx \\ &\quad + c_\epsilon \int_{\Omega} v^2 dx. \end{aligned}$$

Take  $\theta_m = u_m - u$ . Since  $u_m$  converges to  $u$  in  $L^2(\Omega)$  taking  $v = \theta_m$  yields

$$\begin{aligned} \left( \int_{\Omega} \frac{(\theta_m)_+^{2^*(s)}}{|x|^s} dx \right)^{\frac{2}{2^*(s)}} &\leq (\mu_{\gamma, s, 0}(\mathbb{R}^{k_+, n-k})^{-1} + \epsilon) \int_{\Omega} \left( |\nabla \theta_m|^2 - \frac{\gamma}{|x|^2} \theta_m^2 \right) dx \\ &\quad + o(1). \end{aligned} \quad (4.66)$$

We write  $N(\theta_m) := \int_{\Omega} \left( |\nabla \theta_m|^2 - \frac{\gamma}{|x|^2} \theta_m^2 \right) dx$ . By (4.58) and (4.66), we get that

$$N(\theta_m)^{\frac{2}{2^*(s)}} \left( 1 - (\mu_{\gamma, s, 0}(\mathbb{R}^{k_+, n-k})^{-1} + \epsilon) N(\theta_m)^{1 - \frac{2}{2^*(s)}} \right) \leq o(1).$$

With (4.59) and the last inequation, we get that, as  $\epsilon \rightarrow 0$ ,

$$N(\theta_m)^{\frac{2}{2^*(s)}} \left( 1 - (\mu_{\gamma, s, 0}(\mathbb{R}^{k_+, n-k})^{-1} + \epsilon) \left( \frac{2(n-s)\beta}{2-s} \right)^{\frac{2^*(s)-2}{2^*(s)}} + o(1) \right) \leq o(1). \quad (4.67)$$

With the assumption (4.51) and (4.67), taking  $\epsilon > 0$  small enough, we get that  $N(\theta_m) \rightarrow 0$  as  $m \rightarrow +\infty$  and by coercivity, we obtain (4.65).  $\square$

With Step 4.6.3 and since  $E_q(u_m) \rightarrow \beta$  as  $m \rightarrow +\infty$ , we get that  $E_q(u) = \beta$ . This ends the proof of Proposition 4.6.2.  $\square$

**Theorem 4.6.2.** *Let  $\Omega$  be a bounded domain in  $\mathbb{R}^n$ ,  $n \geq 3$ , such that  $0 \in \partial\Omega$  is a singularity of type  $(k, n-k)$  for some  $k \in \{1, \dots, n\}$ . We fix  $\gamma < \gamma_H(\mathbb{R}^{k+, n-k})$ . We fix  $a \in C^{0,\theta}(\Omega)$  such that  $-\Delta - (\gamma|x|^{-2} + a(x))$  is coercive, and  $h \in C^{0,\theta}(\Omega)$  such that  $h \geq 0$ . We fix  $0 \leq s < 2$  and  $1 < q < 2^* - 1$ . We assume that there exists  $u_0 \in D^{1,2}(\Omega)$ ,  $u_0 \not\equiv 0$ , such that*

$$\sup_{t \geq 0} E_q(tu_0) < \frac{2-s}{2(n-s)} \mu_{\gamma,s,0}(\mathbb{R}^{k+, n-k})^{\frac{n-s}{2-s}}. \quad (4.68)$$

Then equation (4.5) has a non-vanishing solution in  $D^{1,2}(\Omega)$ .

*Proof of Theorem 4.6.2:* By Proposition 4.6.1, there exists a Palais-Smale sequence  $(u_m)_{m \in \mathbb{N}} \in D^{1,2}(\Omega)$  for  $E_q$  at level  $\beta > 0$  such that  $\beta \leq \sup_{t \geq 0} E_q(tu_0)$ . It then follows from Proposition 4.6.2 that, up to a subsequence,  $(u_m)$  converges strongly to  $u$  in  $D^{1,2}(\Omega)$ . Then  $E_q(u) = \beta > 0$ , so  $u \not\equiv 0$ , and  $E'_q(u) = 0$ . Coercivity and  $E'_q(u)[u_-] = 0$  yield  $u \geq 0$ . Regularity theory and Hopf's principle yield  $u \in C^{2,\theta}(\Omega)$  and  $u > 0$ . Then  $u$  is a solution of (4.5). This proves Theorem 4.6.2.  $\square$

## 4.7 Proof of Theorem 4.1.3: Test-Functions estimates

The main result of this section is the following:

**Proposition 4.7.1.** *For  $\gamma < \gamma_H(\mathbb{R}^{k+, n-k})$  and fix  $0 \leq s < 2$ . We assume that there are extremals for  $\mu_{\gamma,s,0}(\mathbb{R}^{k+, n-k})$ , we let  $U$  as in (4.11) be such an extremal. We let  $(u_\epsilon)_\epsilon$  and  $(\tilde{u}_\epsilon)_\epsilon$  as in (4.21). Then,*

(a) *For  $0 \leq \gamma < \gamma_H(\mathbb{R}^{k+, n-k}) - \frac{1}{4}$ , we have*

$$\sup_{t \geq 0} E_q(tu_\epsilon) := \beta_0 + \begin{cases} c_1 GH_{\gamma,s}(\Omega) \epsilon + o(\epsilon) & \text{if } q+1 < \frac{2n-2}{n-2}, \\ (c_1 GH_{\gamma,s}(\Omega) - c_2 h(0)) \epsilon + o(\epsilon) & \text{if } q+1 = \frac{2n-2}{n-2}, \\ -c_2 h(0) \epsilon^{n - \frac{(q+1)(n-2)}{2}} + o(\epsilon^{n - \frac{(q+1)(n-2)}{2}}) & \text{if } q+1 > \frac{2n-2}{n-2}. \end{cases}$$

(b) *For  $0 \leq \gamma = \gamma_H(\mathbb{R}^{k+, n-k}) - \frac{1}{4}$ , we have*

$$\sup_{t \geq 0} E_q(tu_\epsilon) := \beta_0 + \begin{cases} c_1 GH_{\gamma,s}(\Omega) \epsilon \ln\left(\frac{1}{\epsilon}\right) + o\left(\epsilon \ln\left(\frac{1}{\epsilon}\right)\right) & \text{if } q+1 \leq \frac{2n-2}{n-2}, \\ -c_2 h(0) \epsilon^{n - \frac{(q+1)(n-2)}{2}} + o\left(\epsilon^{n - \frac{(q+1)(n-2)}{2}}\right) & \text{if } q+1 > \frac{2n-2}{n-2}. \end{cases}$$

(c) *For  $\gamma > \gamma_H(\mathbb{R}^{k+, n-k}) - \frac{1}{4}$ , we have*

$$\sup_{t \geq 0} E_q(t\tilde{u}_\epsilon) := \beta_0 + \begin{cases} -c_3 m_\gamma(\Omega) \epsilon^{\alpha_+ - \alpha_-} + o(\epsilon^{\alpha_+ - \alpha_-}) & \text{if } q+1 < q_{\alpha_\pm}, \\ -(c_3 m_\gamma(\Omega) + c_2 h(0)) \epsilon^{\alpha_+ - \alpha_-} + o(\epsilon^{\alpha_+ - \alpha_-}) & \text{if } q+1 = q_{\alpha_\pm}, \\ -c_2 h(0) \epsilon^{n - \frac{(q+1)(n-2)}{2}} + o\left(\epsilon^{n - \frac{(q+1)(n-2)}{2}}\right) & \text{if } q+1 > q_{\alpha_\pm}, \end{cases}$$

where  $\beta_0 = \frac{2-s}{2(n-s)}\mu_{\gamma,s,0}(\mathbb{R}^{k_+,n-k})^{\frac{n-s}{2-s}}$  and  $q_{\alpha_{\pm}} = \frac{2n-2(\alpha_+-\alpha_-)}{n-2}$ ,

$$\begin{cases} c_1 = \frac{\mu_{\gamma,s,0}(\mathbb{R}^{k_+,n-k})^{\frac{2^*(s)}{2^*(s)-2}}}{2} \left( \xi \int_{\mathbb{R}^{k_+,n-k}} \frac{U^{2^*(s)}}{|x|^s} dx \right)^{-1}, \\ c_2 = \frac{\xi^{\frac{q+1}{2^*(s)-2}}}{q+1} \int_{\mathbb{R}^{k_+,n-k}} U^{q+1} dx, \\ c_3 = \mu_{\gamma,s,0}(\mathbb{R}^{k_+,n-k})^{\frac{2^*(s)}{2^*(s)-2}} \frac{\alpha_+-\alpha_-}{2} \frac{\int_{S^{n-1} \cap \mathbb{R}^{k_+,n-k}} \left( \prod_{i=1}^k x_i \right)^2 d\sigma}{\xi \int_{\mathbb{R}^{k_+,n-k}} \frac{U^{2^*(s)}}{|x|^s} dx}. \end{cases} \quad (4.69)$$

Theorem 4.6.2 and Proposition 4.7.1 yield Theorem 4.1.3.

*Proof of Proposition 4.7.1:* We define the test-function sequence  $(Z_\epsilon)_{\epsilon>0}$  by

$$Z_\epsilon(x) := \begin{cases} u_\epsilon & \text{if } \gamma \leq \gamma_H(\mathbb{R}^{k_+,n-k}) - \frac{1}{4}, \\ \tilde{u}_\epsilon & \text{if } \gamma > \gamma_H(\mathbb{R}^{k_+,n-k}) - \frac{1}{4}, \end{cases}$$

where  $u_\epsilon$  and  $\tilde{u}_\epsilon$  are as in the definition (4.21). We have:

$$E_q(tZ_\epsilon) = \frac{t^2}{2}R_\epsilon - \frac{t^{2^*(s)}}{2^*(s)}B_\epsilon - \frac{t^{q+1}}{q+1}C_{h,\epsilon},$$

when  $\epsilon \rightarrow 0$  where:

$$\begin{aligned} R_\epsilon &:= \int_{\Omega} \left( |\nabla Z_\epsilon|^2 - \left( \frac{\gamma}{|x|^2} + a(x) \right) Z_\epsilon^2 \right) dx \\ B_\epsilon &:= \int_{\Omega} \frac{Z_\epsilon^{2^*(s)}}{|x|^s} dx \text{ and } C_{h,\epsilon} := \int_{\Omega} h Z_\epsilon^{q+1} dx. \end{aligned}$$

**Step 4.7.1.** We fix  $f \in C^{0,\theta}(\Omega)$ ,  $\theta \in (0, 1)$ , and  $p \in [1, 2^*)$ . We claim that

$$\int_{\Omega} f |Z_\epsilon|^{p+1} dx = \begin{cases} f(0)\epsilon^{n-\frac{n-2}{2}(p+1)} \int_{\mathbb{R}^{k_+,n-k}} U^{p+1} dx + o\left(\epsilon^{n-\frac{n-2}{2}(p+1)}\right) & \text{if } n < p_+, \\ O\left(\epsilon^{\frac{p+1}{2}(\alpha_+-\alpha_-)} \ln\left(\frac{1}{\epsilon}\right)\right) & \text{if } n = p_+, \\ O\left(\epsilon^{\frac{p+1}{2}(\alpha_+-\alpha_-)}\right) & \text{if } n > p_+, \end{cases}$$

where  $p_+ = (p+1)\alpha_+$ . Moreover, we have

$$\int_{\Omega} f |Z_\epsilon|^{p+1} dx \rightarrow 0 \text{ as } \epsilon \rightarrow 0. \quad (4.70)$$

*Proof of Step 4.7.1:* Note that it follows from (4.19) that

$$0 < U_\epsilon(x) \leq C\epsilon^{\frac{\alpha_+-\alpha_-}{2}}|x|^{-\alpha_+} \text{ for all } x \in \mathbb{R}^{k_+,n-k} \text{ and } \epsilon > 0. \quad (4.71)$$

We first prove Step 4.7.1 for  $u_\epsilon$ , postponing the case of  $\tilde{u}_\epsilon$ , and then  $Z_\epsilon$ , to the end of the proof. We distinguish three cases:

*Case 1:* We assume that  $n > (p+1)\alpha_+$ . It follows from (4.71) that

$$\left| \int_{\Omega} f |u_\epsilon|^{p+1} dx \right| \leq C \epsilon^{\frac{p+1}{2}(\alpha_+ - \alpha_-)} \int_{\Omega} |x|^{-(p+1)\alpha_+} dx \leq C \epsilon^{\frac{p+1}{2}(\alpha_+ - \alpha_-)}$$

as  $\epsilon \rightarrow 0$ . This proves Step 4.7.1 for  $u_\epsilon$  when  $n > (p+1)\alpha_+$ .

*Case 2:* We assume that  $n = (p+1)\alpha_+$ . With (4.71), we get that

$$\begin{aligned} \left| \int_{\Omega} f |u_\epsilon|^{p+1} dx \right| &\leq C \epsilon^{n - \frac{n-2}{2}(p+1)} + C \int_{B_{\delta,+}} |u_\epsilon|^{p+1} dx \\ &\leq C \epsilon^{n - \frac{n-2}{2}(p+1)} + C \epsilon^{n - \frac{n-2}{2}(p+1)} \int_{B_{\delta\epsilon^{-1},+}} U^{p+1} dx \\ &\leq C \epsilon^{n - \frac{n-2}{2}(p+1)} + C \epsilon^{n - \frac{n-2}{2}(p+1)} \int_1^{\epsilon^{-1}\delta} r^{-1} dr \\ &\leq C \epsilon^{\frac{p+1}{2}(\alpha_+ - \alpha_-)} \ln \left( \frac{1}{\epsilon} \right) \end{aligned}$$

*Case 3:* We assume that  $n < (p+1)\alpha_+$ . It follows from (4.71) that

$$\int_{\Omega \setminus \phi(B_{\delta,+})} f |u_\epsilon|^{p+1} dx = O \left( \epsilon^{\frac{p+1}{2}(\alpha_+ - \alpha_-)} \right) \text{ as } \epsilon \rightarrow 0.$$

Independently, since  $f \in C^{0,\theta}(\Omega)$ , we have that

$$\begin{aligned} \int_{\phi(B_{\delta,+})} f |u_\epsilon|^{p+1} dx &= \int_{B_{\delta,+}} f \circ \phi \cdot U_\epsilon^{p+1} |\text{Jac } \phi| dx \\ &= \epsilon^{n - \frac{n-2}{2}(p+1)} f(0) \int_{B_{\delta\epsilon^{-1},+}} U^{p+1} dx + O \left( \int_{B_{\delta,+}} |x|^\theta |U_\epsilon|^{p+1} dx \right) \end{aligned} \quad (4.72)$$

Since  $n < (p+1)\alpha_+$ , it follows from (4.19) that  $U \in L^{p+1}(\mathbb{R}^{k_+,n-k})$  and that

$$\begin{aligned} \int_{B_{\delta\epsilon^{-1},+}} U^{p+1} dx &= \int_{\mathbb{R}^{k_+,n-k}} U^{p+1} dx + O \left( \int_{\mathbb{R}^{k_+,n-k} \setminus B_{\delta\epsilon^{-1},+}} U^{p+1} dx \right) \\ &= \int_{\mathbb{R}^{k_+,n-k}} U^{p+1} dx + O \left( \int_{\epsilon^{-1}\delta}^{\infty} r^{n-(p+1)\alpha_+-1} dr \right) \\ &= \int_{\mathbb{R}^{k_+,n-k}} U^{p+1} dx + O \left( \epsilon^{(p+1)\alpha_+-n} \right) \end{aligned} \quad (4.73)$$

We claim that

$$\int_{B_{\delta,+}} |x|^\theta |U_\epsilon|^{p+1} dx = o\left(\epsilon^{n-\frac{n-2}{2}(p+1)}\right) \text{ as } \epsilon \rightarrow 0. \quad (4.74)$$

Indeed, when  $\theta+n > (p+1)\alpha_+$ , we argue as in Case 1. When  $\theta+n = (p+1)\alpha_+$ , we argue as in Case 2. When  $\theta+n < (p+1)\alpha_+$ , we make a change of variable  $y = \epsilon^{-1}x$  and we argue as in (4.73). This yields (4.74). Putting (4.73) and (4.74) in (4.72) yields Step 4.7.1 for  $u_\epsilon$  in Case 2.

We now prove Step 4.7.1. When  $\gamma \leq \gamma_H(\mathbb{R}^{k_+,n-k}) - \frac{1}{4}$ ,  $Z_\epsilon = u_\epsilon$ , and we are done. When  $\gamma > \gamma_H(\mathbb{R}^{k_+,n-k}) - \frac{1}{4}$ ,  $Z_\epsilon = \tilde{u}_\epsilon$ . With the definition (4.21), we get that

$$\begin{aligned} \int_{\Omega} f |\tilde{u}_\epsilon|^{p+1} dx &= \int_{\Omega} f \left| u_\epsilon + \epsilon^{\frac{\alpha_+ - \alpha_-}{2}} \Theta \right|^{p+1} dx \\ &= \int_{\Omega} f |u_\epsilon|^{p+1} dx + O\left(\epsilon^{\frac{\alpha_+ - \alpha_-}{2}} \int_{\Omega} |u_\epsilon|^p |\Theta| dx\right) \\ &\quad + O\left(\epsilon^{\frac{p+1}{2}(\alpha_+ - \alpha_-)} \int_{\Omega} |\Theta|^{p+1} dx\right) \end{aligned} \quad (4.75)$$

Since  $\Theta \in D^{1,2}(\Omega)$  and  $p+1 < 2^*$ , we get that  $\Theta \in L^{p+1}(\Omega)$ . It follows from (4.15) that  $|\Theta(x)| \leq C|x|^{-\alpha_-}$  for all  $x \in \Omega$ . Arguing as in Cases 1, 2, 3 above, we get that the second term in the right-hand-side of (4.75) is dominated by  $\int_{\Omega} |u_\epsilon|^{p+1} dx$ . Then Step 4.7.1 for  $\gamma > \gamma_H(\mathbb{R}^{k_+,n-k}) - 1/4$  follows.  $\square$

By Cheikh-Ali [27] and Step 4.7.1 for the case  $\gamma \leq \gamma_H(\mathbb{R}^{k_+,n-k}) - 1/4$  and Steps 4.4.2 and 4.4.3 for the case  $\gamma > \gamma_H(\mathbb{R}^{k_+,n-k}) - 1/4$ , we get that, as  $\epsilon \rightarrow 0$ ,

$$R_\epsilon \rightarrow R_0 := \xi \int_{\mathbb{R}^{k_+,n-k}} \frac{U^{2^*(s)}}{|x|^s} dx \text{ and } B_\epsilon \rightarrow B_0 := \int_{\mathbb{R}^{k_+,n-k}} \frac{U^{2^*(s)}}{|x|^s} dx. \quad (4.76)$$

**Step 4.7.2.** We claim that for all  $\epsilon > 0$ , then there exists a unique  $t_\epsilon$  such that

$$\sup_{t \geq 0} E_q(tZ_\epsilon) = E_q(t_\epsilon Z_\epsilon). \quad (4.77)$$

Moreover,  $t_\epsilon$  verifies

$$t_\epsilon = S_\epsilon [1 - C_0 C_{h,\epsilon} + o(C_{h,\epsilon})], \quad (4.78)$$

where  $S_\epsilon := (R_\epsilon B_\epsilon^{-1})^{\frac{1}{2^*(s)-2}}$ ,  $C_0 > 0$  and  $t_\epsilon \rightarrow t_0$  as  $\epsilon \rightarrow 0$ .

*Proof of Step 4.7.2:* We have that  $\partial_t E_q(tZ_\epsilon) = 0$  iff  $t = 0$  or  $g_\epsilon(t) = R_\epsilon$  where  $g_\epsilon(t) := B_\epsilon t^{2^*(s)-2} + C_{h,\epsilon} t^{q-1}$ . Since  $B_\epsilon, C_{h,\epsilon} \geq 0$  and  $g_\epsilon$  is a strictly increasing map i.e  $g_\epsilon(t) - R_\epsilon$  also, and since  $R_\epsilon > 0$  we have  $g_\epsilon(0) - R_\epsilon < 0$  then, there exists  $t_\epsilon > 0$  unique verifying  $g_\epsilon(t_\epsilon) = R_\epsilon$  such that (4.77) holds. Since  $g_\epsilon(t_\epsilon) = R_\epsilon$ , we get

$$t_\epsilon \leq S_\epsilon := (R_\epsilon B_\epsilon^{-1})^{\frac{1}{2^*(s)-2}}.$$

We are using (4.76), (4.70) and (4.11) to get that  $S_\epsilon \rightarrow (R_0 B_0^{-1})^{\frac{1}{2^*(s)-2}} = \xi^{\frac{1}{2^*(s)-2}}$  as  $\epsilon \rightarrow 0$ . Therefore,  $t_\epsilon$  is bounded and there exists  $t_0$  such that  $t_\epsilon \rightarrow t_0$  up to extraction. Since  $g_\epsilon(t_\epsilon) = R_\epsilon$  and  $C_{h,\epsilon} \rightarrow 0$  as  $\epsilon \rightarrow 0$ , we obtain that

$$\begin{aligned} t_\epsilon &= [R_\epsilon B_\epsilon^{-1} - C_{h,\epsilon} B_\epsilon^{-1} t_\epsilon^{q-1}]^{\frac{1}{2^*(s)-2}} \\ &= S_\epsilon [1 - C_{h,\epsilon} R_\epsilon^{-1} t_\epsilon^{q-1}]^{\frac{1}{2^*(s)-2}} = S_\epsilon [1 - C_0 C_{h,\epsilon} + o(C_{h,\epsilon})], \end{aligned}$$

where  $C_0 := \frac{R_0^{-1} t_0^{q-1}}{2^*(s)-2}$  and  $t_0 = \xi^{\frac{1}{2^*(s)-2}}$ . This yields (4.78) and Step 4.7.2.  $\square$

**Step 4.7.3.** We claim that

$$E_q(t_\epsilon Z_\epsilon) = \frac{2-s}{2(n-s)} (J_{\gamma,s,a}^\Omega(Z_\epsilon))^{\frac{2^*(s)}{2^*(s)-2}} - \frac{\xi^{\frac{q+1}{2^*(s)-2}}}{q+1} C_{h,\epsilon} + o(C_{h,\epsilon}).$$

*Proof of Step 4.7.3:* The expression (4.78) of Step 4.7.2 and (4.70) yield

$$\begin{aligned} E_q(t_\epsilon Z_\epsilon) &= \frac{t_\epsilon^2}{2} R_\epsilon - \frac{t_\epsilon^{2^*(s)}}{2^*(s)} B_\epsilon - \frac{t_\epsilon^{q+1}}{q+1} C_{h,\epsilon} \\ &= \frac{S_\epsilon^2 [1 - C_0 C_{h,\epsilon} + o(C_{h,\epsilon})]^2}{2} R_\epsilon - \frac{S_\epsilon^{2^*(s)} [1 - C_0 C_{h,\epsilon} + o(C_{h,\epsilon})]^{2^*(s)}}{2^*(s)} B_\epsilon \\ &\quad - \frac{S_\epsilon^{q+1} [1 - C_0 C_{h,\epsilon} + o(C_{h,\epsilon})]^{q+1}}{q+1} C_{h,\epsilon} \\ &= \frac{S_\epsilon^2 [1 - 2C_0 C_{h,\epsilon} + o(C_{h,\epsilon})]}{2} R_\epsilon - \frac{S_\epsilon^{2^*(s)} [1 - C_0 2^*(s) C_{h,\epsilon} + o(C_{h,\epsilon})]}{2^*(s)} B_\epsilon \\ &\quad - \frac{S_\epsilon^{q+1} [1 - (q+1)C_0 C_{h,\epsilon} + o(C_{h,\epsilon})]}{q+1} C_{h,\epsilon}, \end{aligned}$$

then,

$$\begin{aligned} E_q(t_\epsilon Z_\epsilon) &= \frac{S_\epsilon^2}{2} R_\epsilon - \frac{S_\epsilon^{2^*(s)}}{2^*(s)} B_\epsilon - \frac{S_\epsilon^{q+1}}{q+1} C_{h,\epsilon} \\ &\quad - C_0 C_{h,\epsilon} [S_\epsilon^2 R_\epsilon - S_\epsilon^{2^*(s)} B_\epsilon - S_\epsilon^{q+1} C_{h,\epsilon}] + o(C_{h,\epsilon}). \end{aligned}$$



Since  $S_\epsilon := (R_\epsilon B_\epsilon^{-1})^{\frac{1}{2^*(s)-2}}$  and  $C_{h,\epsilon} \rightarrow 0$  as  $\epsilon \rightarrow 0$ , this yields Step 4.7.3.  $\square$

**Proof of Proposition 4.7.1 when  $0 \leq \gamma \leq \gamma_H(\mathbb{R}^{k_+,n-k}) - \frac{1}{4}$ .** In this case, we recall that  $Z_\epsilon(x) = u_\epsilon(x)$ . Note that

$$\left\{ \gamma < (=) \gamma_H(\mathbb{R}^{k_+,n-k}) - \frac{1}{4} \right\} \Leftrightarrow \{ \alpha_+ - \alpha_- > (=) 1 \}.$$

It was proved in Proposition 5.1 in Cheikh-Ali [27] that

- For  $\gamma < \gamma_H(\mathbb{R}^{k_+,n-k}) - \frac{1}{4}$ , we have that

$$J_{\gamma,s,0}^\Omega(u_\epsilon) = \mu_{\gamma,s,0}(\mathbb{R}^{k_+,n-k}) (1 + \kappa GH_{\gamma,s}(\Omega)\epsilon + o(\epsilon)). \quad (4.79)$$

- For  $\gamma = \gamma_H(\mathbb{R}^{k_+,n-k}) - \frac{1}{4}$ , we have that

$$J_{\gamma,s,0}^\Omega(u_\epsilon) = \mu_{\gamma,s,0}(\mathbb{R}^{k_+,n-k}) \left( 1 + \kappa GH_{\gamma,s}(\Omega)\epsilon \ln\left(\frac{1}{\epsilon}\right) + o\left(\epsilon \ln\left(\frac{1}{\epsilon}\right)\right) \right), \quad (4.80)$$

where  $\kappa := \left( \xi \int_{\mathbb{R}^{k_+,n-k}} \frac{U^{2^*(s)}}{|x|^s} dx \right)^{-1}$  and  $GH_{\gamma,s}(\Omega)$  is defined in (4.6). It follows from Step 4.7.1 that  $\int_\Omega u_\epsilon^2 dx = o(\epsilon)$  if  $\alpha_+ - \alpha_- > 1$ , and  $O(\epsilon)$  if  $\alpha_+ - \alpha_- = 1$ . Therefore (4.79) and (4.80) hold unchanged with the potential  $a$ .

*Case 1:* We assume that  $n < (q+1)\alpha_+$ . It follows from Step 4.7.1 that

$$C_{h,\epsilon} = \int_\Omega h|u_\epsilon|^{q+1} dx = h(0)\epsilon^{n-\frac{n-2}{2}(q+1)} \int_{\mathbb{R}^{k_+,n-k}} U^{q+1} dx + o\left(\epsilon^{n-\frac{n-2}{2}(q+1)}\right)$$

as  $\epsilon \rightarrow 0$ . Then, when  $n < (q+1)\alpha_+$ , we get Case (a) of Proposition 4.7.1 follows by combining Step 4.7.3, (4.79), (4.80), the estimate of  $C_{h,\epsilon}$  and studying the relative positions of  $n - \frac{n-2}{2}(q+1)$  and 1.

*Case 2:* We assume that  $n \geq (q+1)\alpha_+$ . Since  $\alpha_+ - \alpha_- \geq 1$  and  $q > 1$ , we then get that

$$n - \frac{n-2}{2}(q+1) - 1 = (n - (q+1)\alpha_+) + \frac{q+1}{2}(\alpha_+ - \alpha_-) - 1 > 0.$$

Then, for  $n \geq (q+1)\alpha_+$ , Cases (a) and (b) of Proposition 4.7.1 follows by the same arguments as in Case 1.

This proves Cases (a) and (b) of Proposition 4.7.1.  $\square$

**Proof of Proposition 4.7.1 when  $\gamma > \gamma_H(\mathbb{R}^{k_+,n-k}) - \frac{1}{4}$ .** Proposition 4.4.1 yields

$$J_{\gamma,s,a}^\Omega(\tilde{u}_\epsilon) = \mu_{\gamma,s,0}(\mathbb{R}^{k_+,n-k}) (1 - \zeta_{\gamma,s}^0 m_{\gamma,a}(\Omega)\epsilon^{\alpha_+ - \alpha_-} + o(\epsilon^{\alpha_+ - \alpha_-})) \quad (4.81)$$

as  $\epsilon \rightarrow 0$ . Here, we compare  $n - \frac{n-2}{2}(q+1)$  and  $\alpha_+ - \alpha_-$ . Note that

$$n - \frac{n-2}{2}(q+1) - (\alpha_+ - \alpha_-) = n - (q+1)\alpha_+ + \frac{q-1}{2}(\alpha_+ - \alpha_-).$$

Therefore, since  $q > 1$ , when  $n \geq (q+1)\alpha_+$ , we have that  $n - \frac{n-2}{2}(q+1) > \alpha_+ - \alpha_-$ . As for the case  $\gamma \leq \gamma_H(\mathbb{R}^{k_+, n-k}) - \frac{1}{4}$ , we get Case (b) of Proposition 4.7.1 by studying the relative positions of  $n - \frac{n-2}{2}(q+1)$  and  $\alpha_+ - \alpha_-$  and using Step 4.7.1 and (4.81).

This proves Case (c) of Proposition 4.7.1.  $\square$

All these cases prove Proposition 4.7.1. As already mentioned, Theorem 4.6.2 and Proposition 4.7.1 yield Theorem 4.1.3.



## **Part II**



## The second best constant for the Hardy-Sobolev inequality on manifolds

### 5.1 Introduction

Let  $(M, g)$  be a compact Riemannian manifold of dimension  $n \geq 3$  with  $\partial M = \emptyset$ . We fix  $x_0 \in M$  and  $s \in [0, 2)$ . Interpolating the Sobolev and Hardy inequalities, we get the Hardy-Sobolev inequality that writes as follows: there exists  $A, B > 0$  such that

$$\left( \int_M \frac{|u|^{2^*(s)}}{d_g(x, x_0)^s} dv_g \right)^{\frac{2}{2^*(s)}} \leq A \int_M |\nabla u|_g^2 dv_g + B \int_M u^2 dv_g \quad (5.1)$$

for all  $u \in H_1^2(M)$ , where  $2^*(s) := \frac{2(n-s)}{n-2}$ ,  $dv_g$  is the Riemannian element of volume and  $H_1^2(M)$  is the completion of  $C_c^\infty(M)$  for the norm  $u \mapsto \|u\|_2 + \|\nabla u\|_2$ . When  $s = 0$ , this is the classical Sobolev inequality. Extensive discussions on the optimal values of  $A$  and  $B$  above are in the monograph Druet-Hebey [40]. It was proved by Hebey-Vaugon [70] (the classical case  $s = 0$ ) and by Jaber [75] ( $s \in (0, 2)$ ) that

$$\mu_s(\mathbb{R}^n)^{-1} = \inf \{ A > 0 \text{ such that } \exists B > 0 \text{ such that (5.1) holds for all } u \in H_1^2(M) \},$$

and that the infimum is achieved, where

$$\mu_s(\mathbb{R}^n) = \inf \left\{ \frac{\int_{\mathbb{R}^n} |\nabla u|^2 dX}{\left( \int_{\mathbb{R}^n} \frac{|u|^{2^*(s)}}{|X|^s} dX \right)^{\frac{2}{2^*(s)}}}, u \in C_c^\infty(\mathbb{R}^n) \right\}$$

is the best constant in the Hardy-Sobolev inequality (see Lieb [83] Theorem 4.3 for the value). Therefore, there exists  $B > 0$  such that

$$\left( \int_M \frac{|u|^{2^*(s)}}{d_g(x, x_0)^s} dv_g \right)^{\frac{2}{2^*(s)}} \leq \mu_s(\mathbb{R}^n)^{-1} \left( \int_M |\nabla u|_g^2 dv_g + B \int_M u^2 dv_g \right) \quad (5.2)$$

for all  $u \in H_1^2(M)$ . Saturating this inequality with respect to  $B$ , we define the second best constant as

$$B_s(g) := \inf\{B > 0 \text{ such that (5.2) holds for all } u \in H_1^2(M)\},$$

to get the optimal inequality

$$\left( \int_M \frac{|u|^{2^*(s)}}{d_g(x, x_0)^s} dv_g \right)^{\frac{2}{2^*(s)}} \leq \mu_s(\mathbb{R}^n)^{-1} \left( \int_M |\nabla u|_g^2 dv_g + B_s(g) \int_M u^2 dv_g \right) \quad (5.3)$$

for all  $u \in H_1^2(M)$ . In this paper, we are interested in the value of the second best constant. We say that  $u_0 \in H_1^2(M)$  is an extremal for (5.3) if  $u_0 \neq 0$  and equality in (5.3) holds for  $u = u_0$ . When  $s = 0$ , the issue has been studied by Druet and al.:

**Theorem 5.1.1** (The case  $s = 0$ , [34, 39]). *Let  $(M, g)$  be a compact Riemannian manifold of dimension  $n \geq 3$ . Assume that  $s = 0$  and that there is no extremal for (5.3). Then*

- $B_0(g) = \frac{n-2}{4(n-1)} \max_M \text{Scal}_g$  if  $n \geq 4$ ;
- The mass of  $\Delta_g + B_0(g)$  vanishes if  $n = 3$ .

The mass will be defined in Proposition-Definition 1. We establish the same result for the singular case  $s \in (0, 2)$ :

**Theorem 5.1.2** (The case  $s > 0$ ). *Let  $(M, g)$  be a compact Riemannian manifold of dimension  $n \geq 3$ . We fix  $x_0 \in M$  and  $s \in (0, 2)$ . We assume that there is no extremal for (5.3). Then*

- $B_s(g) = \frac{(6-s)(n-2)}{12(2n-2-s)} \text{Scal}_g(x_0)$  if  $n \geq 5$ ;
- The mass of  $\Delta_g + B_s(g)$  vanishes if  $n = 3$ .

The case  $n = 4$  is still under investigations.

Our proof relies on the blow-up analysis of critical elliptic equations in the spirit of Druet-Hebey-Robert [41]. Let  $(a_\alpha)_{\alpha \in \mathbb{N}} \in C^1(M)$  such that

$$\lim_{\alpha \rightarrow +\infty} a_\alpha = a_\infty \text{ in } C^1(M). \quad (5.4)$$

We assume uniform coercivity, that is there exists  $c_0 > 0$  such that

$$\int_M (|\nabla w|_g^2 + a_\alpha w^2) dv_g \geq c_0 \int_M w^2 dv_g \text{ for all } w \in H_1^2(M). \quad (5.5)$$

Note that this equivalent to the coercivity of  $\Delta_g + a_\infty$ . We consider  $(\lambda_\alpha)_\alpha \in (0, +\infty)$  such that

$$\lim_{\alpha \rightarrow +\infty} \lambda_\alpha = \mu_s(\mathbb{R}^n). \quad (5.6)$$

We let  $(u_\alpha)_\alpha \in H_1^2(M)$  is a sequence of weak solutions to

$$\begin{cases} \Delta_g u_\alpha + a_\alpha u_\alpha = \lambda_\alpha \frac{u_\alpha^{2^*(s)-1}}{d_g(x, x_0)^s} & \text{in } M, \\ u_\alpha \geq 0 & \text{a.e. in } M, \end{cases} \quad (5.7)$$

where  $\Delta_g := -\text{div}_g(\nabla)$  is the Laplace-Beltrami operator. We assume that

$$\|u_\alpha\|_{2^*(s), s} = \left( \int_M \frac{|u_\alpha|^{2^*(s)}}{d_g(x, x_0)^s} dv_g \right)^{\frac{1}{2^*(s)}} = 1, \quad (5.8)$$

and that

$$u_\alpha \rightharpoonup 0 \text{ as } \alpha \rightarrow +\infty \text{ weakly in } H_1^2(M). \quad (5.9)$$

It follows from the regularity and the maximum principle of Jaber [74] that  $u_\alpha \in C^{0, \beta_1}(M) \cap C_{loc}^{2, \beta_2}(M \setminus \{x_0\})$ ,  $\beta_1 \in (0, \min(1, 2 - s))$ ,  $\beta_2 \in (0, 1)$  and  $u_\alpha > 0$ . Therefore, since  $M$  is compact and  $u_\alpha \in C^0(M)$ , then there exists  $x_\alpha \in M$  and  $\mu_\alpha > 0$  such that

$$\mu_\alpha := \left( \max_M u_\alpha \right)^{-\frac{2}{n-2}} = (u_\alpha(x_\alpha))^{-\frac{2}{n-2}}. \quad (5.10)$$

We prove two descriptions of the asymptotics of  $(u_\alpha)$ :

**Theorem 5.1.3.** *Let  $M$  be a compact Riemannian manifold of dimension  $n \geq 3$ . We fix  $x_0 \in M$  and  $s \in (0, 2)$ . Let  $(a_\alpha)_{\alpha \in \mathbb{N}} \in C^1(M)$  and  $a_\infty \in C^1(M)$  be such that (5.4) holds and  $\Delta_g + a_\infty$  is coercive in  $M$ . We let  $(\lambda_\alpha)_\alpha \in \mathbb{R}$  and  $(u_\alpha)_\alpha \in H_1^2(M)$  be such that (5.4) to (5.10) hold for all  $\alpha \in \mathbb{N}$ . Then, there exists  $C > 0$  such that,*

$$u_\alpha(x) \leq C \frac{\mu_\alpha^{\frac{n-2}{2}}}{\mu_\alpha^{n-2} + d_g(x, x_0)^{n-2}} \text{ for all } x \in M, \quad (5.11)$$

where  $\mu_\alpha \rightarrow 0$  as  $\alpha \rightarrow +\infty$  is as in (5.10), that is  $\mu_\alpha^{-\frac{n-2}{2}} = \max_M u_\alpha$ .



**Theorem 5.1.4.** *Let  $M$  be a compact Riemannian manifold of dimension  $n \geq 3$ . We fix  $x_0 \in M$  and  $s \in (0, 2)$ . Let  $(a_\alpha)_{\alpha \in \mathbb{N}} \in C^1(M)$  and  $a_\infty \in C^1(M)$  be such that (5.4) holds and  $\Delta_g + a_\infty$  is coercive in  $M$ . We let  $(\lambda_\alpha)_\alpha \in \mathbb{R}$  and  $(u_\alpha)_\alpha \in H_1^2(M)$  be such that (5.4) to (5.10) hold for all  $\alpha \in \mathbb{N}$ . Then,*

1. *If  $n \geq 5$ , then  $a_\infty(x_0) = c_{n,s} \text{Scal}_g(x_0)$ .*
2. *If  $n = 3$ , then  $m_{a_\infty}(x_0) = 0$ .*

where  $m_{a_\infty}(x_0)$  is the mass of the operator  $\Delta_g + a_\infty$  (see Proposition-Definition 1) and

$$c_{n,s} := \frac{(6-s)(n-2)}{12(2n-2-s)}. \quad (5.12)$$

The case  $n = 4$  is in progress. The mass is defined as follows:

**Proposition-Definition 1.** *[The mass] Let  $(M, g)$  be a compact Riemannian manifold of dimension  $n = 3$ , and let  $h \in C^0(M)$  be such that  $\Delta_g + h$  is coercive. Let  $G_{x_0}$  be the Green's function of  $\Delta_g + h$  at  $x_0$ . Let  $\eta \in C^\infty(M)$  such that  $\eta = 1$  around  $x_0$ . Then there exists  $\beta_{x_0} \in H_1^2(M)$  such that*

$$G_{x_0} = \frac{1}{4\pi} \eta d_g(\cdot, x_0)^{-1} + \beta_{x_0} \text{ in } M \setminus \{x_0\}. \quad (5.13)$$

We have that  $\beta_{x_0} \in H_2^p(M) \cap C^{0,\theta}(M) \cap C^{2,\gamma}(M \setminus \{x_0\})$  for all  $p \in (\frac{3}{2}, 3)$  and  $\theta, \gamma \in (0, 1)$ . We define the mass at  $x_0$  as  $m_h(x_0) := \beta_{x_0}(x_0)$ , which is independent of the choice of  $\eta$ .

Theorem 5.1.4 yields a necessary condition for the existence of solutions to (5.7) that blow-up with minimal energy. Conversely, in a work in progress [29], we show that this is a necessary condition by constructing an example via the finite-dimensional reduction in the spirit of Micheletti-Pistoia-Vétois [88].

The role of the scalar curvature in blow-up analysis has been outlined since the reference paper [38] of Druet for  $s = 0$ . In the singular Hardy-Sobolev case ( $s \in (0, 2)$ ), the critical threshold  $c_{n,s} \text{Scal}_g(x_0)$  was first observed by Jaber [74] who proved that there is a solution  $u \in H_1^2(M) \cap C^0(M)$  to

$$\Delta_g u + hu = \frac{u^{2^*(s)-1}}{d_g(x, x_0)^s}; \quad u > 0 \text{ in } M.$$

as soon as  $h(x_0) < c_{n,s} \text{Scal}_g(x_0)$  where  $h \in C^0(M)$  and  $\Delta_g + h$  is coercive. More recently, it was proved by Chen [31] that for any potential  $h \in C^1(M)$

such that  $\Delta_g + h$  is coercive, then there is a blowing-up family of solutions  $(u_\epsilon)_{\epsilon>0}$  to

$$\Delta_g u_\epsilon + h u_\epsilon = \frac{u_\epsilon^{2^*(s)-1-\epsilon}}{d_g(x, x_0)^s}; \quad u_\epsilon > 0 \text{ in } M.$$

when  $h(x_0) > c_{n,s} \text{Scal}_g(x_0)$ .

## 5.2 Preliminary blow-up analysis

We let  $(a_\alpha)_\alpha \in C^1(M)$ ,  $a_\infty \in C^1(M)$ ,  $(\lambda_\alpha)_\alpha \in \mathbb{R}$  be such that (5.4)-(5.10) hold.

**Lemma 5.2.1.** *We claim that*

$$\lim_{\alpha \rightarrow +\infty} u_\alpha = 0 \text{ in } C_{loc}^0(M \setminus \{x_0\}).$$

*Proof of Lemma 5.2.1:* We take  $y \in M \setminus \{x_0\}$ ,  $r_y = \frac{1}{3}d_g(y, x_0)$ . Since  $u_\alpha$  verifies the equation (5.6), we have

$$\Delta_g u_\alpha = H_\alpha u_\alpha \text{ in } B_{2r_y}(y),$$

where the function

$$H_\alpha(x) := a_\alpha + \lambda_\alpha \frac{u_\alpha^{2^*(s)-2}}{d_g(x, x_0)^s}.$$

Since  $a_\alpha \rightarrow a_\infty$  in  $C^1$ , for any  $r \in (\frac{n}{2}, \frac{n}{2-s})$ , then there exists  $c_0 > 0$  independent of  $\alpha$  such that

$$\int_{B_{2r_y}(y)} H_\alpha^r dv_g \leq c_0.$$

Using the Theorem 8.11 in Gilbarg-Trudinger [63], that there exists  $C_{n,s,y,c_0} > 0$  independent of  $\alpha$  such that

$$\max_{B_{r_y}(y)} u_\alpha \leq C_{n,s,y,c_0} \|u_\alpha\|_{L^2(B_{2r_y}(y))}.$$

Therefore, by the convergence in (5.9), we get

$$\|u_\alpha\|_{L^\infty(B_{r_y}(y))} \rightarrow 0 \text{ as } \alpha \rightarrow +\infty.$$

A covering argument yields Lemma 5.2.1. □

**Lemma 5.2.2.** *We claim that*

$$\sup_{x \in M} u_\alpha(x) = +\infty \text{ as } \alpha \rightarrow +\infty. \quad (5.14)$$

*Proof of Lemma 5.2.2:* If (5.14) does not hold, then there exists  $C > 0$  such that

$$u_\alpha \leq C \text{ for all } x \in M.$$

The convergence (5.9) and Lebesgue's Convergence Theorem yield

$$\lim_{\alpha \rightarrow +\infty} \|u_\alpha\|_{2^*(s),s} = 0,$$

contradiction (5.8). This proves Lemma 5.2.2.  $\square$

It follows from Lemmas 5.2.1 and 5.2.2 that

$$x_\alpha \rightarrow x_0 \text{ as } \alpha \rightarrow +\infty. \quad (5.15)$$

We divide the proof of Theorem 5.1.3 in several steps:

**Step 5.2.1.** *We claim that*

$$d_g(x_\alpha, x_0) = o(\mu_\alpha) \text{ as } \alpha \rightarrow +\infty.$$

*Proof of Step 5.2.1:* Since  $x_\alpha \rightarrow x_0$  as  $\alpha \rightarrow +\infty$ , taking  $z_\alpha = x_\alpha$  in Theorem 5.7.1, we get that  $d_g(x_\alpha, x_0) = O(\mu_\alpha)$  as  $\alpha \rightarrow +\infty$ . We define the rescaled metric  $\bar{g}_\alpha(x) := (\exp_{x_\alpha}^* g)(\mu_\alpha X)$  in  $B_{\delta_0^{-1}\mu_\alpha}(0)$  and

$$\bar{u}_\alpha(X) := \mu_\alpha^{\frac{n-2}{2}} u_\alpha(\exp_{x_\alpha}(\mu_\alpha X)) \text{ for all } X \in B_{\delta_0^{-1}\mu_\alpha}(0) \subset \mathbb{R}^n.$$

Here,  $\exp_{x_\alpha} : B_{\delta_0}(0) \rightarrow B_{\delta_0}(x_0) \subset M$  is the exponential map at  $x_\alpha$ . It follows from Theorem 5.7.1 that

$$\bar{u}_\alpha \rightarrow \tilde{u} \text{ in } C_c^0(\mathbb{R}^n) \text{ as } \alpha \rightarrow +\infty,$$

where  $\tilde{u}$  is as in Theorem 5.7.1. Since  $\bar{u}_\alpha(0) = 1 = \max \bar{u}_\alpha$ , we get

$$\tilde{u}(0) = \lim_{\alpha \rightarrow +\infty} \bar{u}_\alpha(0) = 1.$$

On the other hand, we have  $\|\bar{u}_\alpha\|_\infty = 1$  thus 0 is a maximum of  $\tilde{u}$ . We define  $X_{0,\alpha} := \mu_\alpha^{-1} \exp_{x_\alpha}^{-1}(x_0)$  such that  $X_0 := \lim_{\alpha \rightarrow +\infty} X_{0,\alpha}$ . Using the explicit form of  $\tilde{u}$  in Theorem 5.7.1 that  $\tilde{u}(X) \leq \tilde{u}(X_0)$  for all  $X \in \mathbb{R}^n$ . This yields  $X_0 = 0$ . We have that

$$d_g(x_\alpha, x_0) = \mu_\alpha d_{\bar{g}_\alpha}(X_{0,\alpha}, 0) = \mu_\alpha |X_{0,\alpha}| = o(\mu_\alpha).$$

This yields Step 5.2.1.  $\square$

We now define the metric

$$\tilde{g}_\alpha(x) := (\exp_{x_0}^* g)(\mu_\alpha X) \text{ in } B_{\delta_0^{-1}\mu_\alpha}(0), \quad (5.16)$$

and the rescaled function

$$\tilde{u}_\alpha(X) := \mu_\alpha^{\frac{n-2}{2}} u_\alpha(\exp_{x_0}(\mu_\alpha X)) \text{ for all } X \in B_{\delta_0\mu_\alpha^{-1}}(0) \subset \mathbb{R}^n. \quad (5.17)$$

Equation (5.7) rewrites

$$\Delta_{\tilde{g}_\alpha} \tilde{u}_\alpha + \tilde{a}_\alpha \tilde{u}_\alpha = \lambda_\alpha \frac{\tilde{u}_\alpha^{2^*(s)-1}}{|X|^s} \text{ in } B_{\delta_0\mu_\alpha^{-1}}(0), \quad (5.18)$$

where  $\tilde{a}_\alpha(X) := \mu_\alpha^2 a_\alpha(\exp_{x_0}(\mu_\alpha X)) \rightarrow 0$  in  $C_{loc}^1(\mathbb{R}^n)$  as  $\alpha \rightarrow +\infty$ .

**Step 5.2.2.** *We claim that,*

$$\lim_{\alpha \rightarrow +\infty} \tilde{u}_\alpha = \tilde{u}, \quad (5.19)$$

in  $C_{loc}^2(\mathbb{R}^n \setminus \{0\})$  and uniformly in  $C_{loc}^{0,\beta}(\mathbb{R}^n)$ , for all  $\beta \in (0, \min\{1, 2-s\})$ .  
Where

$$\tilde{u}(X) = \left( \frac{K^{2-s}}{K^{2-s} + |X|^{2-s}} \right)^{\frac{n-2}{2-s}} \text{ for all } X \in \mathbb{R}^n,$$

with

$$K^{2-s} = (n-2)(n-s)\mu_s(\mathbb{R}^n)^{-1}. \quad (5.20)$$

In particular,  $\tilde{u}$  verifies

$$\Delta_{Eucl} \tilde{u} = \mu_s(\mathbb{R}^n) \frac{\tilde{u}^{2^*(s)-1}}{|X|^s} \text{ in } \mathbb{R}^n \text{ and } \int_{\mathbb{R}^n} \frac{\tilde{u}^{2^*(s)}}{|X|^s} dX = 1, \quad (5.21)$$

where  $Eucl$  is the Euclidean metric of  $\mathbb{R}^n$ . Moreover,

$$\lim_{R \rightarrow +\infty} \lim_{\alpha \rightarrow +\infty} \int_{M \setminus B_{R\mu_\alpha}(x_0)} \frac{u_\alpha^{2^*(s)}}{d_g(x, x_0)^s} dv_g = 0. \quad (5.22)$$

*Proof of Step 5.2.2:* Using Step 5.2.1 and applying again Theorem 5.7.1 with  $z_\alpha = x_0$ , we get the convergence of  $\tilde{u}_\alpha$  (see (5.17)). Now, we want to proof (5.22), taking the change of variable  $X = \mu_\alpha^{-1} \exp_{x_0}^{-1}(x)$ , applying Lebesgue's Convergence Theorem and from the uniform convergence in  $C_{loc}^{0,\beta}(\mathbb{R}^n)$ , for all  $\beta \in (0, \min 1, 2-s)$  of the equation (5.19) and using (5.143), we get

$$\begin{aligned} \lim_{R \rightarrow +\infty} \lim_{\alpha \rightarrow +\infty} \int_{B_{R\mu_\alpha}(x_0)} \frac{u_\alpha^{2^*(s)}}{d_g(x, x_0)^s} dv_g &= \lim_{R \rightarrow +\infty} \lim_{\alpha \rightarrow +\infty} \int_{B_R(0)} \frac{\tilde{u}_\alpha^{2^*(s)}}{|X|^s} dv_{\tilde{g}_\alpha} \\ &= \lim_{R \rightarrow +\infty} \int_{B_R(0)} \frac{\tilde{u}^{2^*(s)}}{|X|^s} dX \\ &= \int_{\mathbb{R}^n} \frac{\tilde{u}^{2^*(s)}}{|X|^s} dX = 1. \end{aligned} \quad (5.23)$$

It follows from  $\|u_\alpha\|_{2^*(s),s}^{2^*(s)} = 1$  and (5.23), we get (5.22). This ends Step 5.2.2.  $\square$

**Step 5.2.3.** We claim that for any  $R > 0$ ,

$$\tilde{u}_\alpha \rightarrow \tilde{u} \text{ in } H_1^2(B_R(0)) \text{ as } \alpha \rightarrow +\infty. \quad (5.24)$$

*Proof of Step 5.2.3:* We rewrite (5.18) as

$$\Delta_{\tilde{g}_\alpha} \tilde{u}_\alpha = f_\alpha := \lambda_\alpha \frac{\tilde{u}_\alpha^{2^*(s)-1}}{|X|^s} - \tilde{a}_\alpha \tilde{u}_\alpha.$$

It follows from (5.19) that  $f_\alpha(X) \rightarrow f(X) = \mu_{s,0}(\mathbb{R}^n) \frac{\tilde{u}^{2^*(s)-1}(X)}{|X|^s} \in C_{loc}^{0,\beta}(\mathbb{R}^n \setminus \{0\})$ . For any  $R > 0$ , we have

$$\|f_\alpha\|_{L^p(B_{2R}(0))} \leq \| |X|^{-s} \|_{L^p(B_{2R}(0))} \|\tilde{u}_\alpha\|_{L^\infty(B_{2R}(0))}. \quad (5.25)$$

It follows from (5.19) that  $(\tilde{u}_\alpha)_\alpha$  is bounded in  $L_{loc}^\infty$ . Since  $X \rightarrow |X|^{-s} \in L_{loc}^p(\mathbb{R}^n)$  for  $1 < p < \frac{n}{s}$ , then for such  $p$ , we have that  $(f_\alpha)_\alpha$  is bounded in  $L^p(B_{2R}(0))$ . By standard elliptic theory (see for instance [63]), we get that

$$\|\tilde{u}_\alpha\|_{H_2^p(B_0(R))} \leq C (\|f_\alpha\|_{L^p(B_{2R}(0))} + \|\tilde{u}_\alpha\|_{L^p(B_{2R}(0))}).$$

We define  $p^*$  such that  $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}$ . If  $p^* \leq 0$ ,  $H_1^p(B_R(0))$  is compactly embedded in  $L^2(B_R(0))$ . Now, if  $p^* > 0$ , we have  $H_1^p(B_R(0))$  is compactly embedded in  $L^q(B_R(0))$  for  $1 \leq q < p^*$  and  $L^{2^*}(B_R(0)) \hookrightarrow L^2(B_R(0))$  iff  $2 \leq p^* \iff p \geq \frac{2n}{n+2}$ . Since,  $s \in (0, 2)$  then there exists  $p > 1$  such that  $p \in (\frac{2n}{n+2}, \frac{n}{s})$  and then  $(\tilde{u}_\alpha)$  is bounded in  $H_2^p(B_0(R)) \hookrightarrow H_1^2(B_0(R))$ . Since the embedding is compact, up to extraction, we get (5.24) and ends Step 5.2.3.  $\square$

**Step 5.2.4.** We claim that there exists  $C > 0$  such that

$$d_g(x, x_0)^{\frac{n-2}{2}} u_\alpha(x) \leq C \text{ for all } x \in M \text{ and } \alpha > 0. \quad (5.26)$$

*Proof of Step 5.2.4:* We follow the arguments of Jaber [75] (see also Druet [34] and Hebey [67]). We argue by contradiction and assume that there exists  $(y_\alpha)_\alpha \in M$  such that

$$\sup_{x \in M} d_g(x, x_0)^{\frac{n-2}{2}} u_\alpha(x) = d_g(y_\alpha, x_0)^{\frac{n-2}{2}} u_\alpha(y_\alpha) \rightarrow +\infty \text{ as } \alpha \rightarrow +\infty. \quad (5.27)$$

Since  $M$  is compact, with (5.27) we obtain that  $\lim_{\alpha \rightarrow +\infty} u_\alpha(y_\alpha) = +\infty$ . Thanks again to Lemma 5.2.1, we obtain that, up to a subsequence,

$$\lim_{\alpha \rightarrow +\infty} y_\alpha = x_0. \quad (5.28)$$

For  $\alpha > 0$ , given  $\nu_\alpha := u_\alpha(y_\alpha)^{-\frac{2}{n-2}}$ , and moreover

$$\nu_\alpha \rightarrow 0 \text{ as } \alpha \rightarrow +\infty. \quad (5.29)$$

We adopt the following notation:  $\theta_R$  will denote any quantity such that there exists  $\theta : \mathbb{R} \rightarrow \mathbb{R}$  such that

$$\lim_{R \rightarrow +\infty} \theta_R = 0.$$

We claim that

$$\int_{B_{\nu_\alpha}(y_\alpha)} \frac{u_\alpha^{2^*(s)}}{d_g(x, x_0)^s} dv_g = o(1) \text{ as } \alpha \rightarrow \infty. \quad (5.30)$$

*Proof of (5.30):* For any  $R > 0$ ,

$$\int_{B_\delta(x_0) \setminus B_{R\mu_\alpha}(x_0)} \frac{u_\alpha^{2^*(s)}}{d_g(x, x_0)^s} dv_g \leq \int_{M \setminus B_{R\mu_\alpha}(x_0)} \frac{u_\alpha^{2^*(s)}}{d_g(x, x_0)^s} dv_g.$$

Therefore, with the equation (5.22) in the Step 5.2.2, we get

$$\int_{B_\delta(x_0) \setminus B_{R\mu_\alpha}(x_0)} \frac{u_\alpha^{2^*(s)}}{d_g(x, x_0)^s} dv_g = \theta_R + o(1). \quad (5.31)$$

On the other hand, equations (5.28) and (5.29) yield  $B_{\nu_\alpha}(y_\alpha) \setminus B_\delta(x_0) = \emptyset$ , and

$$\begin{aligned} \int_{B_{\nu_\alpha}(y_\alpha)} \frac{u_\alpha^{2^*(s)}}{d_g(x, x_0)^s} dv_g &= \int_{B_{\nu_\alpha}(y_\alpha) \cap B_\delta(x_0)} \frac{u_\alpha^{2^*(s)}}{d_g(x, x_0)^s} dv_g \\ &= \int_{B_{\nu_\alpha}(y_\alpha) \cap (B_\delta(x_0) \setminus B_{R\mu_\alpha}(x_0))} \frac{u_\alpha^{2^*(s)}}{d_g(x, x_0)^s} dv_g \\ &\quad + \int_{B_{\nu_\alpha}(y_\alpha) \cap B_{R\mu_\alpha}(x_0)} \frac{u_\alpha^{2^*(s)}}{d_g(x, x_0)^s} dv_g \\ &\leq \theta_R + \int_{B_{\nu_\alpha}(y_\alpha) \cap B_{R\mu_\alpha}(x_0)} \frac{u_\alpha^{2^*(s)}}{d_g(x, x_0)^s} dv_g + o(1), \end{aligned} \quad (5.32)$$

we get the last inequality with (5.31). We distinguish two cases:

**Case 1:** If  $B_{\nu_\alpha}(y_\alpha) \cap B_{R\mu_\alpha}(x_0) = \emptyset$ , we obtain the result (5.30).

**Case 2:** If  $B_{\nu_\alpha}(y_\alpha) \cap B_{R\mu_\alpha}(x_0) \neq \emptyset$ . Then,

$$d_g(y_\alpha, x_0) \leq \nu_\alpha + R\mu_\alpha. \quad (5.33)$$

It follows from the definition of  $\nu_\alpha$  and (5.27), we get

$$\lim_{\alpha \rightarrow +\infty} \frac{\nu_\alpha}{d_g(y_\alpha, x_0)} = 0. \quad (5.34)$$

Combining the equations (5.33) and (5.34)

$$d_g(y_\alpha, x_0) = O(\mu_\alpha) \text{ and } \nu_\alpha = o(\mu_\alpha) \text{ as } \alpha \rightarrow +\infty. \quad (5.35)$$

We now consider an exponential chart  $(\Omega_0, \exp_{x_0}^{-1})$  centered at  $x_0$  such that  $\exp_{x_0}^{-1}(\Omega_0) = B_{r_0}(0)$ ,  $r_0 \in (0, i_g(M))$ . We take  $\tilde{Y}_\alpha = \mu_\alpha^{-1} \exp_{x_0}^{-1}(y_\alpha)$ . By compactness arguments, there exists  $c > 1$  such that for all  $X, Y \in \mathbb{R}^n$ ,

$$\mu_\alpha |X|, \mu_\alpha |Y| < r_0,$$

and,

$$\frac{1}{c} |X - Y| \leq d_{\tilde{g}_\alpha}(X, Y) \leq c |X - Y|.$$

Therefore, we have:

$$\mu_\alpha^{-1} \exp_{x_0}^{-1}(B_{\nu_\alpha}(y_\alpha)) \subset B_{c \frac{\nu_\alpha}{\mu_\alpha}}(\tilde{Y}_\alpha). \quad (5.36)$$

And by equation (5.35),

$$\tilde{Y}_\alpha = O(d_{\tilde{g}_\alpha}(\tilde{Y}_\alpha, 0)) = O(\mu_\alpha^{-1} d_g(y_\alpha, x_0)) = O(1). \quad (5.37)$$

By equations (5.34), (5.35) and the change of variables  $X = \mu_\alpha^{-1} \exp_{x_0}^{-1}(x)$  yields,

$$\begin{aligned} \int_{B_{\nu_\alpha}(y_\alpha) \cap B_{R\mu_\alpha}(x_0)} \frac{u_\alpha^{2^*(s)}}{d_g(x, x_0)^s} dv_g &\leq \int_{\mu_\alpha^{-1} \exp_{x_0}^{-1}(B_{\nu_\alpha}(y_\alpha))} \frac{\tilde{u}_\alpha^{2^*(s)}}{d_{\tilde{g}_\alpha}(X, 0)^s} dv_{\tilde{g}_\alpha} \\ &\leq \int_{B_{c \frac{\nu_\alpha}{\mu_\alpha}}(\tilde{Y}_\alpha)} \frac{\tilde{u}_\alpha^{2^*(s)}}{d_{\tilde{g}_\alpha}(X, 0)^s} dv_{\tilde{g}_\alpha} \end{aligned}$$

It follows from the equation  $\nu_\alpha = o(\mu_\alpha)$  and Lebesgue's Convergence Theorem,

$$\lim_{\alpha \rightarrow +\infty} \int_{B_{\nu_\alpha}(y_\alpha) \cap B_{R\mu_\alpha}(x_0)} \frac{u_\alpha^{2^*(s)}}{d_g(x, x_0)^s} dv_g = 0.$$

The latest equation and (5.32) yield (5.30). This proves the claim.  $\square$

We take now a family  $(\Omega_\alpha, \exp_{y_\alpha}^{-1})_{\alpha>0}$  of exponential charts centered at  $y_\alpha$ . Set  $r_0 \in (0, i_g(M))$ , we define

$$\hat{u}_\alpha(X) = \nu_\alpha^{\frac{n-2}{2}} u_\alpha(\exp_{y_\alpha}(\nu_\alpha X)) \text{ on } B_{r_0\nu_\alpha^{-1}}(0) \subset \mathbb{R}^n,$$

and the metric,

$$\hat{g}_\alpha(X) = \exp_{y_\alpha}^* g(\nu_\alpha X) \text{ on } \mathbb{R}^n.$$

Since  $u_\alpha$  verifies the equation (5.7), we get  $\hat{u}_\alpha$  verifies also weakly

$$\Delta_{\hat{g}_\alpha} \hat{u}_\alpha + \hat{a}_\alpha \hat{u}_\alpha = \lambda_\alpha \frac{\hat{u}_\alpha^{2^*(s)-1}}{d_{\hat{g}_\alpha}(X, X_{0,\alpha})^s} \text{ in } \mathbb{R}^n,$$

where  $\hat{a}_\alpha(X) := \nu_\alpha^2 a_\alpha(\exp_{y_\alpha}(\nu_\alpha X)) \rightarrow 0$  as  $\alpha \rightarrow +\infty$  and  $X_{0,\alpha} = \mu_\alpha^{-1} \exp_{y_\alpha}^{-1}(x_0)$ .

We claim that

$$\hat{u}_\alpha \rightarrow \hat{u} \neq 0 \text{ in } C_{loc}^0(\mathbb{R}^n) \text{ as } \alpha \rightarrow +\infty. \quad (5.38)$$

We prove (5.38). Using the definition of  $\hat{u}_\alpha$  and the equation (5.27), we get

$$\hat{u}_\alpha(X) \leq \left( \frac{d_g(x_0, y_\alpha)}{d_g(\exp_{y_\alpha}(\nu_\alpha X), x_0)} \right)^{\frac{n-2}{2}} \text{ for all } X \in B_{r_0\nu_\alpha^{-1}}(0). \quad (5.39)$$

On the other hand, using the triangular inequality and for any  $X \in B_R(0)$ , we obtain that

$$\begin{aligned} d_g(\exp_{y_\alpha}(\nu_\alpha X), x_0) &\geq d_g(x_0, y_\alpha) - d_g(\exp_{y_\alpha}(\nu_\alpha X), y_\alpha) \\ &= d_g(x_0, y_\alpha) - \nu_\alpha |X| \\ &\geq d_g(x_0, y_\alpha) - \nu_\alpha R. \end{aligned}$$

Therefore, it follows from the equation (5.39), we have for all  $X \in B_R(0)$  that,

$$\hat{u}_\alpha(X) \leq \left( \frac{1}{1 - \frac{\nu_\alpha R}{d_g(x_0, y_\alpha)}} \right)^{\frac{n-2}{2}}.$$

Moreover, with (5.33), we obtain for all  $X \in B_R(0)$ , that  $\hat{u}_\alpha(X) \leq 1 + o(1)$  in  $C^0(B_R(0))$ . Using again the definition  $\nu_\alpha$ , we have  $\hat{u}_\alpha(0) = 1$  for all  $\alpha > 0$ . Elliptic Theory yields  $\hat{u}_\alpha \rightarrow \hat{u}$  in  $C_{loc}^0(\mathbb{R}^n)$  and we have also that  $\hat{u}(0) = \lim_{\alpha \rightarrow +\infty} \hat{u}_\alpha(0) = 1$ . This yields (5.38) and the claim is proved.



Using Lebesgue's Convergence Theorem, the definition of  $\hat{u}_\alpha$  and (5.30), take  $X = \nu_\alpha^{-1} \exp_{y_\alpha}^{-1}(x)$ , we obtain that

$$\begin{aligned} \int_{B_1(0)} \frac{\hat{u}^{2^*(s)}}{|X|^s} dX &= \lim_{\alpha \rightarrow +\infty} \int_{B_1(0)} \frac{\hat{u}_\alpha^{2^*(s)}}{d_{\hat{g}_\alpha}(X, X_{0,\alpha})^s} dv_{\hat{g}_\alpha} \\ &= \lim_{\alpha \rightarrow +\infty} \int_{B_{\nu_\alpha}(y_\alpha)} \frac{u_\alpha^{2^*(s)}}{d_g(x, x_0)^s} dv_g \\ &= 0. \end{aligned}$$

with  $\theta_R \rightarrow 0$  as  $R \rightarrow +\infty$ . Which yields  $\hat{u} \equiv 0$  in  $B_1(0)$ , contradicting with  $\hat{u} \in C^0(B_1(0))$  and  $\hat{u}(0) = 1$ . This completes the proof of Step 5.2.4.  $\square$

**Step 5.2.5.** *We claim that*

$$\lim_{R \rightarrow +\infty} \lim_{\alpha \rightarrow +\infty} \sup_{x \in M \setminus B_{R\mu_\alpha}(x_0)} d_g(x, x_0)^{\frac{n-2}{2}} u_\alpha(x) = 0. \quad (5.40)$$

*Proof of Step 5.2.5:* The proof is a refinement of Step 5.2.4. We omit it and we refer to [39] and Chapter 4 in Druet-Hebey-Robert [41] where the case  $s = 0$  is dealt with.  $\square$

### 5.3 Refined blowup analysis: proof of Theorem 5.1.3

We let  $(u_\alpha)_\alpha \in H_1^2(M)$ ,  $(a_\alpha)_\alpha \in C^1(M)$ ,  $a_\infty \in C^1(M)$ ,  $(\lambda_\alpha)_\alpha \in \mathbb{R}$  be such that (5.4)-(5.10) hold. The next Step towards the proof of Theorem 5.1.3 is the following:

**Step 5.3.1.** *We claim that there exists  $\epsilon_0 > 0$  such that for any  $\epsilon \in (0, \epsilon_0)$ , there exists  $C_\epsilon > 0$  such that*

$$u_\alpha(x) \leq C_\epsilon \frac{\mu_\alpha^{\frac{n-2}{2}-\epsilon}}{d_g(x, x_0)^{n-2-\epsilon}} \text{ for all } x \in M \setminus \{x_0\}. \quad (5.41)$$

*Proof of Step 5.3.1:* Let  $G$  be the Green function on  $M$  at  $x_0$  of  $\Delta_g + (a_\infty - \xi)$  where  $\xi > 0$ . Up to taking  $\xi$  small enough, the operator is coercive and the Green's function is defined on  $M \setminus \{x_0\}$ . In others words,  $G$  satisfies

$$\Delta_g G + (a_\infty - \xi)G = 0 \text{ in } M \setminus \{x_0\}. \quad (5.42)$$

Estimates of the Green's function (see Robert[98]) yield for  $\delta_0 > 0$  small the existence of  $C_i > 0$  for  $i = 1, 2, 3$  such that

$$C_2 d_g(x, x_0)^{2-n} \leq G(x_0, x) \leq C_1 d_g(x, x_0)^{2-n}, \quad (5.43)$$

and,

$$|\nabla G(x_0, x)|_g \geq C_3 d_g(x, x_0)^{1-n}, \quad (5.44)$$

for all  $\alpha \in \mathbb{N}$  and all  $x \in B_{\delta_0}(x_0) \setminus \{x_0\}$ . Define the operator

$$M_{g,\alpha} := \Delta_g + a_\alpha - \lambda_\alpha \frac{u_\alpha^{2^*(s)-2}}{d_g(x, x_0)^s}.$$

*Step 5.3.1.1:* We claim that there exists  $\nu_0 \in (0, 1)$  and  $R_0 > 0$  such that for any  $\nu \in (0, \nu_0)$  and  $R > R_0$ , we have that

$$M_{g,\alpha} G^{1-\nu} > 0 \text{ for all } x \in M \setminus B_{R\mu_\alpha}(x_0). \quad (5.45)$$

*Proof of Step 5.3.1.1:* With (5.42), we get that

$$\frac{M_{g,\alpha} G^{1-\nu}}{G^{1-\nu}}(x) = a_\alpha - a_\infty + \nu(a_\infty - \xi) + \xi + \nu(1-\nu) \left| \frac{\nabla G}{G} \right|_g^2 - \lambda_\alpha \frac{u_\alpha^{2^*(s)-2}}{d_g(x, x_0)^s},$$

for all  $x \in M \setminus \{x_0\}$ . Using again (5.4), there exists  $\alpha_0$  for all  $\alpha > \alpha_0$  such that

$$a_\alpha(x) - a_\infty(x) \geq -\frac{\xi}{2} \text{ for all } x \in M. \quad (5.46)$$

Take now  $\nu_0 \in (0, 1)$  and we let  $\nu \in (0, \nu_0)$ , we get that

$$\frac{M_{g,\alpha} G^{1-\nu}}{G^{1-\nu}}(x) \geq \frac{\xi}{4} + \nu(1-\nu) \left| \frac{\nabla G}{G} \right|_g^2 - \lambda_\alpha \frac{u_\alpha^{2^*(s)-2}}{d_g(x, x_0)^s}. \quad (5.47)$$

Fix  $\rho > 0$ , it follows from the result of the Step 5.2.5 that there exists  $R_0 > 0$  such that for any  $R > R_0$  and for  $\alpha > 0$  large enough, we get that

$$d_g(x, x_0)^{\frac{n-2}{2}} u_\alpha(x) \leq \rho \text{ for } x \in M \setminus B_{R\mu_\alpha}(x_0). \quad (5.48)$$

We let  $\nu \in (0, \nu_0)$  and  $R > R_0$ . We first let  $x \in M$  such that  $d_g(x, x_0) \geq \delta_0$ , then from Corollary 5.2.1

$$\lim_{\alpha \rightarrow +\infty} u_\alpha(x) = 0 \text{ in } M \setminus B_{\delta_0}(x_0). \quad (5.49)$$

With (5.47), we obtain that

$$\frac{M_{g,\alpha} G^{1-\nu}}{G^{1-\nu}}(x) \geq \frac{\xi}{4} - 2\mu_s(\mathbb{R}^n) \frac{u_\alpha^{2^*(s)-2}}{\delta_0^s},$$

and  $\alpha \in \mathbb{N}$ . The Step 5.3.2 yields (5.45) when  $d_g(x, x_0) \geq \delta_0$ .

We now take  $x \in B_{\delta_0}(x_0) \setminus B_{R\mu_\alpha}(x_0)$ . Using again (5.47), (5.48), (5.43) and (5.44), we get that

$$\frac{M_{g,\alpha}G^{1-\nu}}{G^{1-\nu}}(x) \geq \frac{1}{d_g(x, x_0)^2} \left( \nu(1-\nu) \left( \frac{C_3}{C_1} \right)^2 - 2\mu_s(\mathbb{R}^n)\rho^{2^*(s)-2} \right).$$

Up to taking  $\rho > 0$  small enough, we obtain (5.45) when  $x \in B_{\delta_0}(x_0) \setminus B_{R\mu_\alpha}(x_0)$ . This ends Step 5.3.1.1.  $\square$

*Step 5.3.1.2:* We claim that there exists  $C_R > 0$  such that

$$u_\alpha(x) \leq C_R \mu_\alpha^{\frac{n-2}{2}-\nu(n-2)} G(x)^{1-\nu} \text{ for any } x \in \partial B_{R\mu_\alpha}(x_0) \text{ and } \alpha \in \mathbb{N}.$$

*Proof of Step 5.3.1.2:* It follows from (5.17), (5.19) and (5.43) then

$$\begin{aligned} u_\alpha(x) &\leq C \mu_\alpha^{-\frac{n-2}{2}} \\ &= C \mu_\alpha^{-\frac{n-2}{2}} d_g(x, x_0)^{-(2-n)(1-\nu)} d_g(x, x_0)^{(2-n)(1-\nu)} \\ &\leq C C_2^{\nu-1} \mu_\alpha^{-\frac{n-2}{2}} d_g(x, x_0)^{(n-2)(1-\nu)} G(x)^{1-\nu} \\ &\leq C C_2^{\nu-1} R^{(n-2)(1-\nu)} \mu_\alpha^{\frac{n-2}{2}-\nu(n-2)} G(x)^{1-\nu}. \end{aligned}$$

This ends Step 5.3.1.2.  $\square$

*Step 5.3.1.3:* We claim that

$$u_\alpha(x) \leq C_R \mu_\alpha^{\frac{n-2}{2}-\nu(n-2)} G(x)^{1-\nu} \text{ for any } x \in M \setminus B_{R\mu_\alpha}(x_0).$$

*Proof of Step 5.3.1.3:* Define the function

$$v_\alpha := C_R \mu_\alpha^{\frac{n-2}{2}-\nu(n-2)} G(x)^{1-\nu} - u_\alpha.$$

Since  $u_\alpha$  verifies (5.7) and by (5.45), we observe that

$$\begin{aligned} M_{g,\alpha}v_\alpha &= C_R \mu_\alpha^{\frac{n-2}{2}-\nu(n-2)} M_{g,\alpha}G^{1-\nu} - M_{g,\alpha}u_\alpha \\ &= C_R \mu_\alpha^{\frac{n-2}{2}-\nu(n-2)} M_{g,\alpha}G^{1-\nu} > 0 \text{ in } M \setminus B_{R\mu_\alpha}(x_0). \end{aligned}$$

Then Step 5.3.1.3 follows from this inequality, Step 5.3.1.2 and the comparison principle (See Berestycki–Nirenberg–Varadhan [21]). This ends Step 5.3.1.3.  $\square$

We are in position to finish the proof of Step 5.3.1. It follows from, Step 5.3.1.3 that

$$u_\alpha(x) \leq C'_R \frac{\mu_\alpha^{\frac{n-2}{2}-\nu(n-2)}}{d_g(x, x_0)^{(n-2)(1-\nu)}} \text{ for all } x \in M \setminus B_{R\mu_\alpha}(x_0). \quad (5.50)$$

On the other hand, in (5.10), for  $x \in B_{R\mu_\alpha}(x_0) \setminus \{x_0\}$  and  $\nu \in (0, \nu_0)$

$$\begin{aligned} u_\alpha(x) &\leq \mu_\alpha^{-\frac{n-2}{2}} \leq \mu_\alpha^{\frac{n-2}{2}-\nu(n-2)} \mu_\alpha^{(\nu-1)(n-2)} \\ &\leq R^{(1-\nu)(n-2)} \frac{\mu_\alpha^{\frac{n-2}{2}-\nu(n-2)}}{d_g(x, x_0)^{(1-\nu)(n-2)}} \text{ for all } x \in B_{R\mu_\alpha}(x_0). \end{aligned} \quad (5.51)$$

Up to taking  $C'_R$  larger and  $\epsilon = (n-2)\nu$ , by (5.50) that the inequalities (5.41). This ends Step 5.3.1.  $\square$

**Step 5.3.2.** We claim that there exists  $C > 0$  such that

$$d_g(x, x_0)^{n-2} u_\alpha(x_\alpha) u_\alpha(x) \leq C \text{ for all } x \in M. \quad (5.52)$$

*Proof of Step 5.3.2:* We let  $(y_\alpha)_\alpha \in M$  be such that

$$\sup_{x \in M} d_g(x, x_0)^{n-2} u_\alpha(x_\alpha) u_\alpha(x) = d_g(y_\alpha, x_0)^{n-2} u_\alpha(x_\alpha) u_\alpha(y_\alpha).$$

The claim is equivalent to proving that for any  $y_\alpha$ , we have that

$$d_g(y_\alpha, x_0)^{n-2} u_\alpha(x_\alpha) u_\alpha(y_\alpha) = O(1) \text{ as } \alpha \rightarrow +\infty.$$

We distinguish two cases:

**Case 1:** We assume that  $d_g(y_\alpha, x_0) = O(\mu_\alpha)$  as  $\alpha \rightarrow +\infty$ . Therefore, it follows from the definition of  $\mu_\alpha$  that

$$d_g(y_\alpha, x_0)^{n-2} u_\alpha(x_\alpha) u_\alpha(y_\alpha) \leq C \mu_\alpha^{n-2} u_\alpha^2(x_\alpha) \leq C.$$

This yields (5.52).

**Case 2:** We assume that

$$\lim_{\alpha \rightarrow +\infty} \frac{d_g(y_\alpha, x_0)}{\mu_\alpha} = +\infty. \quad (5.53)$$

Let now  $G_\alpha$  the Green's function of  $\Delta_g + a_\alpha$  in  $M$ . Green's representation formula and standard estimates on the Green's function (see (5.43)-(5.39) and Robert [98]), then there exists  $C > 0$  such that

$$\begin{aligned} u_\alpha(y_\alpha) &= \int_M G_\alpha(y_\alpha, x) \lambda_\alpha \frac{u_\alpha^{2^*(s)-1}(x)}{d_g(x, x_0)^s} dv_g \\ &\leq C \int_M d_g(x, y_\alpha)^{2-n} \lambda_\alpha \frac{u_\alpha^{2^*(s)-1}(x)}{d_g(x, x_0)^s} dv_g. \end{aligned} \quad (5.54)$$

We fix  $R > 0$  and we write  $M := \cup_{i=1}^4 \Omega_{i,\alpha}$  where

$$\begin{aligned} \Omega_{1,\alpha} &:= B_{R\mu_\alpha}(x_0), \\ \Omega_{2,\alpha} &:= \left\{ R\mu_\alpha < d_g(x, x_0) < \frac{d_g(y_\alpha, x_0)}{2} \right\}, \\ \Omega_{3,\alpha} &:= \left\{ \frac{d_g(y_\alpha, x_0)}{2} < d_g(x, x_0) < 2d_g(y_\alpha, x_0) \right\}, \\ \Omega_{4,\alpha} &:= \{d_g(x, x_0) \geq 2d_g(y_\alpha, x_0)\} \cap M. \end{aligned}$$

*Step 5.3.2.1:* We first deal with  $\Omega_{1,\alpha}$ .

Using (5.53), we fix  $C_0 > R$ . For  $\alpha$  large, we have that

$$d_g(y_\alpha, x_0) \geq C_0 \mu_\alpha \geq \frac{C_0}{R} d_g(x, x_0) \text{ for all } x \in \Omega_{1,\alpha}.$$

Then since  $C_0 > R > 1$ , we get

$$\begin{aligned} d_g(x, y_\alpha) &\geq d_g(y_\alpha, x_0) - d_g(x, x_0) \\ &\geq \left(1 - \frac{R}{C_0}\right) d_g(y_\alpha, x_0). \end{aligned}$$

We use the result of Step 5.2.4, (5.53) and take  $x = \exp_{x_0}(\mu_\alpha X)$ , then for  $R > 1$  there exists  $C > 0$  such that

$$\begin{aligned} &\left| \int_{\Omega_{1,\alpha}} d_g(x, y_\alpha)^{2-n} \frac{u_\alpha^{2^*(s)-1}(x)}{d_g(x, x_0)^s} dv_g \right| \\ &\leq C d_g(y_\alpha, x_0)^{2-n} \int_{\Omega_{1,\alpha}} \frac{u_\alpha^{2^*(s)-1}(x)}{d_g(x, x_0)^s} dv_g \\ &\leq C \mu_\alpha^{\frac{n-2}{2}} d_g(y_\alpha, x_0)^{2-n} \int_{B_R(0)} \frac{\tilde{u}_\alpha^{2^*(s)-1}(X)}{|X|^s} dv_{\tilde{g}_\alpha}, \end{aligned} \quad (5.55)$$

where  $\tilde{u}_\alpha, \tilde{g}_\alpha$  are defined in (5.17), (5.16). Since  $\tilde{u}_\alpha \leq 1$ , by applying Lebesgue's Convergence Theorem and thanks to Step 5.2.2, we get that

$$\lim_{\alpha \rightarrow +\infty} \int_{B_R(0)} \frac{\tilde{u}_\alpha^{2^*(s)-1}(X)}{|X|^s} dv_{\tilde{g}_\alpha} = \int_{B_R(0)} \frac{\tilde{u}^{2^*(s)-1}}{|X|^s} dX. \quad (5.56)$$

Combining (5.55) and (5.56) yields

$$\left| \int_{\Omega_{1,\alpha}} d_g(x, y_\alpha)^{2-n} \frac{u_\alpha^{2^*(s)-1}(x)}{d_g(x, x_0)^s} dv_g \right| \leq C \mu_\alpha^{\frac{n-2}{2}} d_g(y_\alpha, x_0)^{2-n}. \quad (5.57)$$

*Step 5.3.2.2:* We deal with  $\Omega_{2,\alpha}$ .

Noting that  $d_g(x, y_\alpha) \geq d_g(y_\alpha, x_0) - d_g(x, x_0) \geq (1 - \frac{1}{2}) d_g(y_\alpha, x_0)$  for all  $x \in \Omega_{2,\alpha}$ , we argue as in Step 5.3.2.1 by using (5.41) with  $\epsilon > 0$  small to get

$$\begin{aligned} & \left| \int_{\Omega_{2,\alpha}} d_g(x, y_\alpha)^{2-n} \frac{u_\alpha^{2^*(s)-1}(x)}{d_g(x, x_0)^s} dv_g \right| \\ & \leq C d_g(y_\alpha, x_0)^{2-n} \int_{\Omega_{2,\alpha}} \frac{u_\alpha^{2^*(s)-1}(x)}{d_g(x, x_0)^s} dv_g \\ & \leq C \mu_\alpha^{\frac{(n-2)-\epsilon}{2}(2^*(s)-1)} d_g(y_\alpha, x_0)^{2-n} \int_{\Omega_{2,\alpha}} d_g(x, x_0)^{-s-(n-2-\epsilon)(2^*(s)-1)} dv_g \\ & \leq C \mu_\alpha^{\frac{(n-2)-\epsilon}{2}(2^*(s)-1)} d_g(y_\alpha, x_0)^{2-n} \int_{M \setminus B_{R\mu_\alpha}(x_0)} d_g(x, x_0)^{-s-(n-2-\epsilon)(2^*(s)-1)} dv_g \end{aligned}$$

Taking  $X = \exp_{x_0}^{-1}(x)$  and  $\hat{g} = \exp_{x_0}^* g$  on  $\mathbb{R}^n$ , we get

$$\begin{aligned} & \left| \int_{\Omega_{2,\alpha}} d_g(x, y_\alpha)^{2-n} \frac{u_\alpha^{2^*(s)-1}(x)}{d_g(x, x_0)^s} dv_g \right| \\ & \leq C \mu_\alpha^{\frac{(n-2)-\epsilon}{2}(2^*(s)-1)} d_g(y_\alpha, x_0)^{2-n} \int_{\mathbb{R}^n \setminus B_{R\mu_\alpha}(0)} |X|^{-s-(n-2-\epsilon)(2^*(s)-1)} dv_{\hat{g}} \\ & \leq C \mu_\alpha^{\frac{(n-2)-\epsilon}{2}(2^*(s)-1)} d_g(y_\alpha, x_0)^{2-n} \int_{\mathbb{R}^n \setminus B_{R\mu_\alpha}(0)} |X|^{-s-(n-2-\epsilon)(2^*(s)-1)} dX \\ & \leq C \mu_\alpha^{\frac{n-2}{2}} d_g(y_\alpha, x_0)^{2-n} \int_{\mathbb{R}^n \setminus B_R(0)} |X|^{-s-(n-2-\epsilon)(2^*(s)-1)} dX \\ & \leq C \mu_\alpha^{\frac{n-2}{2}} d_g(y_\alpha, x_0)^{2-n} \int_R^{+\infty} r^{s-2+\epsilon(2^*(s)-1)-1} dr, \end{aligned}$$

Take  $\epsilon$  small and we have that

$$\left| \int_{\Omega_{2,\alpha}} d_g(x, y_\alpha)^{2-n} \frac{u_\alpha^{2^*(s)-1}(x)}{d_g(x, x_0)^s} dv_g \right| \leq C_R \mu_\alpha^{\frac{n-2}{2}} d_g(y_\alpha, x_0)^{2-n}, \quad (5.58)$$

as  $\alpha \rightarrow +\infty$ , where  $C_R \rightarrow 0$  as  $R \rightarrow +\infty$ .

*Step 5.3.2.3:* We deal with  $\Omega_{3,\alpha}$ . For  $\epsilon > 0$  small in the control (5.41), we get

$$\begin{aligned} & \left| \int_{\Omega_{3,\alpha}} d_g(x, y_\alpha)^{2-n} \frac{u_\alpha^{2^*(s)-1}(x)}{d_g(x, x_0)^s} dv_g \right| \\ & \leq C \mu_\alpha^{\frac{(n-2)-\epsilon}{2}(2^*(s)-1)} d_g(y_\alpha, x_0)^{-s-(n-2-\epsilon)(2^*(s)-1)} \int_{\Omega_{3,\alpha}} d_g(x, y_\alpha)^{2-n} dv_g. \end{aligned}$$

Taking  $x = \exp_{x_0}(X)$  and  $y_\alpha = \exp_{x_0}(Y_\alpha)$ , we get that

$$\begin{aligned}
 & \left| \int_{\Omega_{3,\alpha}} d_g(x, y_\alpha)^{2-n} \frac{u_\alpha^{2^*(s)-1}(x)}{d_g(x, x_0)^s} dv_g \right| \\
 & \leq C \mu_\alpha^{(\frac{n-2}{2}-\epsilon)(2^*(s)-1)} d_g(y_\alpha, x_0)^{-s-(n-2-\epsilon)(2^*(s)-1)} \int_{\frac{1}{2}|Y_\alpha| < |X| < 2|Y_\alpha|} |X - Y_\alpha|^{2-n} dv_{\hat{g}} \\
 & \leq C \mu_\alpha^{(\frac{n-2}{2}-\epsilon)(2^*(s)-1)} d_g(y_\alpha, x_0)^{-s-(n-2-\epsilon)(2^*(s)-1)} \int_{\frac{1}{2}|Y_\alpha| < |X| < 2|Y_\alpha|} |X - Y_\alpha|^{2-n} dX \\
 & \leq C \mu_\alpha^{(\frac{n-2}{2}-\epsilon)(2^*(s)-1)} d_g(y_\alpha, x_0)^{-s-(n-2-\epsilon)(2^*(s)-1)} |Y_\alpha|^2 \int_{\frac{1}{2}|X| < |X| < 2|X|} \left| X - \frac{Y_\alpha}{|Y_\alpha|} \right|^{2-n} dX \\
 & \leq C \mu_\alpha^{(\frac{n-2}{2}-\epsilon)(2^*(s)-1)} d_g(y_\alpha, x_0)^{-s-(n-2-\epsilon)(2^*(s)-1)} d_g(y_\alpha, x_0)^2 \int_{|X| < 2} \left| X - \frac{Y_\alpha}{|Y_\alpha|} \right|^{2-n} dX \\
 & \leq C \mu_\alpha^{(\frac{n-2}{2}-\epsilon)(2^*(s)-1)} d_g(y_\alpha, x_0)^{2-n-\frac{n-2}{2}(2^*(s)-1)+\epsilon(2^*(s)-1)} \int_{|X| < 3} |X|^{2-n} dX \\
 & \leq C \mu_\alpha^{\frac{n-2}{2}} d_g(y_\alpha, x_0)^{2-n} \left( \frac{\mu_\alpha}{d_g(y_\alpha, x_0)} \right)^{(\frac{n-2}{2})(2^*(s)-2)-\epsilon(2^*(s)-1)}.
 \end{aligned}$$

Just take  $\epsilon > 0$  small, hence  $(\frac{n-2}{2})(2^*(s)-2) - \epsilon(2^*(s)-1) > 0$  and we obtain that,

$$\begin{aligned}
 & \left| \int_{\Omega_{3,\alpha}} d_g(x, y_\alpha)^{2-n} \frac{u_\alpha^{2^*(s)-1}(x)}{d_g(x, x_0)^s} dv_g \right| \\
 & \leq C \mu_\alpha^{\frac{n-2}{2}} d_g(y_\alpha, x_0)^{2-n} \left( \frac{\mu_\alpha}{d_g(y_\alpha, x_0)} \right)^{(\frac{n-2}{2})(2^*(s)-2)-\epsilon(2^*(s)-1)} \quad (5.59)
 \end{aligned}$$

**Step 5.3.2.4:** We deal with  $\Omega_{4,\alpha}$ . For  $x \in \Omega_{4,\alpha}$ , we have that

$$\begin{aligned}
 d_g(x, y_\alpha) & \geq d_g(x, x_0) - d_g(y_\alpha, x_0) \\
 & \geq \left( 1 - \frac{1}{2} \right) d_g(x, x_0) = \frac{1}{2} d_g(x, x_0).
 \end{aligned}$$

Taking  $X = \exp_{x_0}^{-1}(x)$  and  $Y_\alpha = \exp_{x_0}^{-1}(y_\alpha)$ , and we obtain that

$$\begin{aligned}
 & \left| \int_{\Omega_{4,\alpha}} d_g(x, y_\alpha)^{2-n} \frac{u_\alpha^{2^*(s)-1}(x)}{d_g(x, x_0)^s} dv_g \right| \\
 & \leq C \mu_\alpha^{(\frac{n-2}{2}-\epsilon)(2^*(s)-1)} \int_{\Omega_{4,\alpha}} d_g(x, x_0)^{2-n-s-(n-2-\epsilon)(2^*(s)-1)} dv_g \\
 & \leq C \mu_\alpha^{(\frac{n-2}{2}-\epsilon)(2^*(s)-1)} \int_{B_\delta(0) \setminus B_{2|Y_\alpha|}(0)} |X|^{2-n-s-(n-2-\epsilon)(2^*(s)-1)} dv_{\hat{g}} \\
 & \leq C \mu_\alpha^{(\frac{n-2}{2}-\epsilon)(2^*(s)-1)} \int_{B_\delta(0) \setminus B_{2|Y_\alpha|}(0)} |X|^{2-n-s-(n-2-\epsilon)(2^*(s)-1)} dX \\
 & \leq C \mu_\alpha^{(\frac{n-2}{2}-\epsilon)(2^*(s)-1)} \int_{2|Y_\alpha|}^{+\infty} r^{-n+s+\epsilon(2^*(s)-1)-1} dr.
 \end{aligned}$$

Take  $\epsilon$  small and we get

$$\begin{aligned} & \left| \int_{\Omega_{4,\alpha}} d_g(x, y_\alpha)^{2-n} \frac{u_\alpha^{2^*(s)-1}(x)}{d_g(x, x_0)^s} dv_g \right| \\ & \leq C \mu_\alpha^{\frac{n-2}{2}} d_g(y_\alpha, x_0)^{2-n} \left( \frac{\mu_\alpha}{d_g(x_0, y_\alpha)} \right)^{(\frac{n-2}{2})(2^*(s)-2) - \epsilon(2^*(s)-1)}. \end{aligned} \quad (5.60)$$

Plugging the equations (5.57)-(5.60) in (5.54), we get (5.52). This ends Step 5.3.2.  $\square$

**Step 5.3.3.** We claim that there exists  $C > 0$ , such that

$$u_\alpha(x) \leq C \frac{\mu_\alpha^{\frac{n-2}{2}}}{\mu_\alpha^{n-2} + d_g(x, x_0)^{n-2}} \text{ for all } x \in M. \quad (5.61)$$

*Proof of Step 5.3.3:* Using (5.83) and the definition of  $\mu_\alpha$  (see (5.10)), we have

$$\begin{aligned} (\mu_\alpha^{n-2} + d_g(x, x_0)^{n-2}) \mu_\alpha^{-\frac{n-2}{2}} u_\alpha(x) & \leq \mu_\alpha^{\frac{n-2}{2}} u_\alpha(x) + C \\ & \leq \mu_\alpha^{\frac{n-2}{2}} u_\alpha(x_\alpha) + C \\ & \leq 1 + C. \end{aligned}$$

This proves Theorem 5.1.3 and ends Step 5.3.3.  $\square$

As a first remark, it follows from the definition (5.17) and the pointwise control (5.11) of Theorem 5.1.3 that

$$\tilde{u}_\alpha(X) \leq \frac{C}{(1 + |X|^2)^{\frac{n-2}{2}}} \text{ in } B_{\mu_\alpha^{-1}\delta_0}(0). \quad (5.62)$$

**Proposition 5.3.1.** We claim that

$$\lim_{\alpha \rightarrow +\infty} \frac{u_\alpha}{\mu_\alpha^{\frac{n-2}{2}}} = d_n G_{x_0} \text{ in } C_{loc}^2(M \setminus \{x_0\}), \quad (5.63)$$

where,

$$d_n := \mu_s(\mathbb{R}^n) \int_{\mathbb{R}^n} \frac{\tilde{u}^{2^*(s)-1}}{|X|^s} dX, \quad (5.64)$$

and  $G_{x_0}$  is the Green's function for  $\Delta_g + a_\infty$  on  $M$  at  $x_0$ .



*Proof of Proposition 5.3.1:* We define  $v_\alpha := \mu_\alpha^{-\frac{n-2}{2}} u_\alpha$ . Equation (5.7) rewrites

$$\begin{cases} \Delta_g v_\alpha + a_\alpha v_\alpha = \lambda_\alpha \mu_\alpha^{\frac{n-2}{2}(2^*(s)-2)} \frac{v_\alpha^{2^*(s)-1}}{d_g(x, x_0)^s} & \text{in } M \setminus \{x_0\}, \\ v_\alpha \geq 0 & \text{a.e. in } M \setminus \{x_0\}. \end{cases} \quad (5.65)$$

We fix  $y \in M$  such that  $y \neq x_0$ . We choose  $\delta' \in (0, \delta)$  such that  $d_g(y, x_0) > \delta'$ . Let  $G_\alpha$  be the Green's function of  $\Delta_g + a_\alpha$ . Green's representation formula yields,

$$\begin{aligned} v_\alpha(y) &= \mu_\alpha^{-\frac{n-2}{2}} \lambda_\alpha \int_M G_\alpha(y, x) \frac{u_\alpha^{2^*(s)-1}}{d_g(x, x_0)^s} dv_g \\ &= \mu_\alpha^{-\frac{n-2}{2}} \lambda_\alpha \left( \int_{B_{\delta'}(x_0)} G_\alpha(y, x) \frac{u_\alpha^{2^*(s)-1}}{d_g(x, x_0)^s} dv_g \right. \\ &\quad \left. + \int_{M \setminus B_{\delta'}(x_0)} G_\alpha(y, x) \frac{u_\alpha^{2^*(s)-1}}{d_g(x, x_0)^s} dv_g \right). \end{aligned}$$

On the other hand, since  $d_g(x, y) \geq \frac{\delta'}{2}$  in the second integral, using the estimation of  $G_\alpha$  (see (5.43)) and the Theorem 5.1.3, we get

$$\begin{aligned} &\int_{M \setminus B_{\delta'}(x_0)} G_\alpha(y, x) \frac{u_\alpha^{2^*(s)-1}}{d_g(x, x_0)^s} dv_g \\ &\leq C \frac{\mu_\alpha^{\frac{n-2}{2}(2^*(s)-1)}}{\delta'^{s+(n-2)(2^*(s)-1)}} \int_{M \setminus B_{\delta'}(x_0)} d_g(x, y)^{2-n} dv_g \\ &\leq C_{\delta'} \mu_\alpha^{\frac{n-2}{2}(2^*(s)-1)} Vol_g(M), \end{aligned}$$

Since  $M$  is compact and taking  $x = \exp_{x_0}(\mu_\alpha X)$ , we obtain as  $\alpha \rightarrow +\infty$  that

$$\begin{aligned} v_\alpha(y) &= \mu_\alpha^{-\frac{n-2}{2}} \lambda_\alpha \int_{B_{\delta'}(x_0)} G_\alpha(y, x) \frac{u_\alpha^{2^*(s)-1}}{d_g(x, x_0)^s} dv_g + O(\mu_\alpha^{2-s}) \\ &= \lambda_\alpha \int_{B_{\delta' \mu_\alpha^{-1}}(0)} G_\alpha(y, \exp_{x_0}(\mu_\alpha X)) \frac{\tilde{u}_\alpha^{2^*(s)-1}}{|X|^s} dv_{\tilde{g}_\alpha} + O(\mu_\alpha^{2-s}) \end{aligned}$$

Thanks again to Step 5.2.2, (5.6), the pointwise control (5.62) and Lebesgue's Convergence Theorem, we get

$$\lim_{\alpha \rightarrow +\infty} v_\alpha(y) = d_n G(y, x_0), \quad (5.66)$$

where  $d_n := \mu_s(\mathbb{R}^n) \int_{\mathbb{R}^n} \frac{\tilde{u}^{2^*(s)-1}}{|X|^s} dX$ . The definition of  $v_\alpha$  and the result of the Step 4, Step 7 yields

$$v_\alpha(x) \leq c d_g(x, x_0)^{2-n} \text{ for all } x \in M \text{ and } \alpha \in \mathbb{N}.$$

Then,  $v_\alpha$  is bounded in  $L_{loc}^\infty(M \setminus \{x_0\})$ . Then (5.65), (5.66) and elliptic theory the limit (5.66) in  $C_{loc}^2(M \setminus \{x_0\})$ . This proves Proposition 5.3.1.  $\square$

## 5.4 Direct consequences of Theorem 5.1.3

**Proposition 5.4.1.** *Let  $(u_\alpha)_\alpha$  be as in Theorem 5.1.3. Let  $(y_\alpha)_\alpha \in M$  be such that  $y_\alpha \rightarrow y_0$  as  $\alpha \rightarrow +\infty$ . Then*

$$\lim_{\alpha \rightarrow +\infty} \left( \frac{\mu_\alpha^{2-s} + \frac{d_g(y_\alpha, x_0)^{2-s}}{K^{2-s}}}{\mu_\alpha^{\frac{2-s}{2}}} \right)^{\frac{n-2}{2-s}} u_\alpha(y_\alpha) = \begin{cases} 1 & \text{if } y_0 = x_0, \\ d_n \left( \frac{d_g(y_0, x_0)}{K} \right)^{n-2} G(y_0, x_0) & \text{if } y_0 \neq x_0. \end{cases} \quad (5.67)$$

where,

$$K^{2-s} = (n-2)(n-s)\mu_s(\mathbb{R}^n)^{-1} \text{ and } d_n := \mu_s(\mathbb{R}^n) \int_{\mathbb{R}^n} \frac{\tilde{u}^{2^*(s)-1}}{|X|^s} dX, \quad (5.68)$$

and  $G_{x_0}$  is the Green's function for  $\Delta_g + a_\infty$  on  $M$  at  $x_0$ .

As a consequence, we get that

**Corollary 5.4.1.** *Let  $(u_\alpha)_\alpha$  be as in Theorem 5.1.3. Then there exists  $C > 1$  such that*

$$\frac{1}{C} \frac{\mu_\alpha^{\frac{n-2}{2}}}{\left( \mu_\alpha^{2-s} + \frac{d_g(x, x_0)^{2-s}}{K^{2-s}} \right)^{\frac{n-2}{2-s}}} \leq u_\alpha(x) \leq C \frac{\mu_\alpha^{\frac{n-2}{2}}}{\left( \mu_\alpha^{2-s} + \frac{d_g(x, x_0)^{2-s}}{K^{2-s}} \right)^{\frac{n-2}{2-s}}}. \quad (5.69)$$

*Proof of Proposition 5.4.1:* We recall  $\tilde{u}_\alpha(X) := \mu_\alpha^{\frac{n-2}{2}} u_\alpha(\exp_{x_0}(\mu_\alpha X))$  and satisfies (5.18). It follows from (5.19) in Step 5.2.2 that  $\lim_{\alpha \rightarrow +\infty} \tilde{u}_\alpha = \tilde{u}$  in  $C_{loc}^2(\mathbb{R}^n \setminus \{0\}) \cap C_{loc}^0(\mathbb{R}^n)$ , where  $\tilde{u}$  is as in Step 5.2.2 and satisfies (5.21). We fix  $\delta' > 0$ . Let  $G_\alpha$  be the Green's function of  $\Delta_g + a_\alpha$ . We fix  $\delta' > 0$ . As in the proof of Proposition 5.3.1, we have as  $\alpha \rightarrow +\infty$  that

$$u_\alpha(y_\alpha) = \lambda_\alpha \int_{B_{\delta'}(x_0)} G_\alpha(y_\alpha, x) \frac{u_\alpha^{2^*(s)-1}}{d_g(x, x_0)^s} dv_g + o\left(\mu_\alpha^{\frac{n-2}{2}}\right). \quad (5.70)$$

**Case 1:** We first assume that  $\lim_{\alpha \rightarrow +\infty} y_\alpha = y_0 \neq x_0$ . The result is a direct consequence of (5.63).

**Case 2:** We assume that  $\lim_{\alpha \rightarrow +\infty} y_\alpha = x_0$ .

**Case 2.1:** We assume that there exists  $L \in \mathbb{R}$  such that

$$\frac{d_g(y_\alpha, x_0)}{\mu_\alpha} \rightarrow L \in \mathbb{R} \text{ as } \alpha \rightarrow +\infty. \quad (5.71)$$

We let  $Y_\alpha \in B_{\delta\mu_\alpha^{-1}}(0)$  be such that  $y_\alpha = \exp_{x_0}(\mu_\alpha Y_\alpha)$ . It follows from (5.71) that

$$|Y_\alpha| \rightarrow L \text{ as } \alpha \rightarrow +\infty. \quad (5.72)$$

We have that

$$d_g(y_\alpha, x_0)^{n-2} \mu_\alpha^{-\frac{n-2}{2}} u_\alpha(y_\alpha) = \left( \frac{d_g(y_\alpha, x_0)}{\mu_\alpha} \right)^{n-2} \tilde{u}_\alpha(Y_\alpha).$$

It then follows from the convergence (5.19), (5.71) and (5.72) that

$$\lim_{\alpha \rightarrow +\infty} d_g(y_\alpha, x_0)^{n-2} \mu_\alpha^{-\frac{n-2}{2}} u_\alpha(y_\alpha) = \left( \frac{L^{2-s}}{1 + \frac{L^{2-s}}{K^{2-s}}} \right)^{\frac{n-2}{2-s}}. \quad (5.73)$$

**Case 2.2:** We assume that

$$y_\alpha \rightarrow x_0 \text{ and } \frac{d_g(y_\alpha, x_0)}{\mu_\alpha} \rightarrow +\infty \text{ as } \alpha \rightarrow +\infty. \quad (5.74)$$

Coming back to (5.70), we have as  $\alpha \rightarrow +\infty$  that,

$$\begin{aligned} & d_g(y_\alpha, x_0)^{n-2} \mu_\alpha^{-\frac{n-2}{2}} u_\alpha(y_\alpha) \\ &= d_g(y_\alpha, x_0)^{n-2} \mu_\alpha^{-\frac{n-2}{2}} \lambda_\alpha \left( \int_{D_{1,\alpha}} G_\alpha(y_\alpha, x) \frac{u_\alpha^{2^*(s)-1}}{d_g(x, x_0)^s} dv_g \right. \\ & \quad \left. + \int_{D_{2,\alpha}} G_\alpha(y_\alpha, x) \frac{u_\alpha^{2^*(s)-1}}{d_g(x, x_0)^s} dv_g \right) + O\left(\mu_\alpha^{\frac{n-2}{2}(2^*(s)-2)}\right), \end{aligned}$$

with,

$$D_{1,\alpha} := \left\{ x \in B_\delta(x_0); d_g(y_\alpha, x) \geq \frac{1}{2} d_g(y_\alpha, x_0) \right\} \text{ and } D_{2,\alpha} := B_\delta(x_0) \setminus D_{1,\alpha}.$$

With a change of variable, we get

$$\mu_\alpha^{-\frac{n-2}{2}} \int_{D_{1,\alpha}} G_\alpha(y_\alpha, x) \frac{u_\alpha^{2^*(s)-1}}{d_g(x, x_0)^s} dv_g = \int_{D'_{1,\alpha}} G_\alpha(y_\alpha, \exp_{x_0}(\mu_\alpha X)) \frac{\tilde{u}_\alpha^{2^*(s)-1}}{|X|^s} dv_{\tilde{g}_\alpha}, \quad (5.75)$$

where  $D'_{1,\alpha} = \mu_\alpha^{-1} \exp_{x_0}(D_{1,\alpha})$ . For  $R > 0$ , we take  $X \in B_R(0)$  and  $z_\alpha := \exp_{x_0}(\mu_\alpha X)$ , by (5.74) we have that

$$\frac{d_g(y_\alpha, z_\alpha)}{\mu_\alpha} \rightarrow +\infty \text{ as } \alpha \rightarrow +\infty. \quad (5.76)$$

Writing,

$$d_g(y_\alpha, z_\alpha) - d_g(z_\alpha, x_0) \leq d_g(y_\alpha, x_0) \leq d_g(y_\alpha, z_\alpha) + d_g(z_\alpha, x_0),$$

and nothing that  $d_g(z_\alpha, x_0) = \mu_\alpha |X|$ , we obtain that

$$1 - |X| \frac{\mu_\alpha}{d_g(y_\alpha, z_\alpha)} \leq \frac{d_g(y_\alpha, x_0)}{d_g(y_\alpha, z_\alpha)} \leq 1 + |X| \frac{\mu_\alpha}{d_g(y_\alpha, z_\alpha)},$$

therefore, with (5.76), we get

$$\lim_{\alpha \rightarrow +\infty} \frac{d_g(y_\alpha, x_0)}{d_g(y_\alpha, z_\alpha)} = 1. \quad (5.77)$$

Therefore for all  $R > 0$ , we have that  $B_R(0) \subset D'_{1,\alpha}$  for  $\alpha \gg 1$ . Moreover, since  $d_g(y_\alpha, z_\alpha) \rightarrow 0$  as  $\alpha \rightarrow +\infty$  and by Proposition 12 in Robert [98], we have

$$\begin{aligned} \lim_{\alpha \rightarrow +\infty} d_g(y_\alpha, x_0)^{n-2} G_\alpha(y_\alpha, z_\alpha) &= \left( \frac{d_g(y_\alpha, x_0)}{d_g(y_\alpha, z_\alpha)} \right)^{n-2} d_g(y_\alpha, z_\alpha)^{n-2} G_\alpha(y_\alpha, z_\alpha) \\ &= \frac{1}{(n-2)\omega_{n-1}}, \end{aligned}$$

where  $\omega_{n-1}$  is the volume of the unit  $(n-1)$ -sphere. It then follows from (5.75), (5.43) the pointwise control (5.62) and Lebesgue's Convergence Theorem that

$$\begin{aligned} \lim_{\alpha \rightarrow +\infty} d_g(y_\alpha, x_0)^{n-2} \mu_\alpha^{-\frac{n-2}{2}} \lambda_\alpha \int_{D_{1,\alpha}} G_\alpha(y_\alpha, x) \frac{u_\alpha^{2^*(s)-1}}{d_g(x, x_0)^s} dv_g \\ = \mu_s(\mathbb{R}^n) \frac{1}{(n-2)\omega_{n-1}} \int_{\mathbb{R}^n} \frac{\tilde{u}^{2^*(s)-1}}{|X|^s} dX \end{aligned} \quad (5.78)$$

With a change of variable, we have that

$$\begin{aligned}
 \int_{\mathbb{R}^n} \frac{\tilde{u}^{2^*(s)-1}}{|X|^s} dX &= \int_{\mathbb{R}^n} |X|^{-s} \left(1 + \frac{|X|^{2-s}}{K^{2-s}}\right)^{-\frac{n-2}{2-s}(2^*(s)-1)} dX \\
 &= K^{n-s} \int_{\mathbb{R}^n} |X|^{-s} (1 + |X|^{2-s})^{-\frac{n-2}{2-s}(2^*(s)-1)} dX \\
 &= K^{n-s} \omega_{n-1} \int_0^{+\infty} \frac{r^{n-s-1}}{(1+r^{2-s})^{\frac{n-2}{2-s}(2^*(s)-1)}} dr \\
 &= K^{n-s} \frac{\omega_{n-1}}{2-s} \int_0^{+\infty} \frac{t^{\frac{n-2}{2-s}}}{(1+t)^{\frac{n-2}{2-s}(2^*(s)-1)}} dt \\
 &= K^{n-s} \frac{\omega_{n-1}}{2-s} \int_0^{+\infty} \frac{t^{\frac{n-s}{2-s}-1}}{(1+t)^{\frac{n-2s+2}{2-s}}} dt \\
 &= K^{n-s} \frac{\omega_{n-1}}{2-s} \frac{\Gamma(\frac{n-s}{2-s})\Gamma(1)}{\Gamma(\frac{n-s}{2-s}+1)} = K^{n-s} \frac{\omega_{n-1}}{n-s}.
 \end{aligned}$$

Since  $\mu_s(\mathbb{R}^n) = K^{s-2}(n-2)(n-s)$  and by (5.78), we get

$$\lim_{\alpha \rightarrow +\infty} d_g(y_\alpha, x_0)^{n-2} \mu_\alpha^{-\frac{n-2}{2}} \lambda_\alpha \int_{D_{1,\alpha}} G_\alpha(y_\alpha, x) \frac{u_\alpha^{2^*(s)-1}}{d_g(x, x_0)^s} dv_g = K^{n-2}. \quad (5.79)$$

Note that  $d_g(x, x_0) \geq \frac{1}{2}d_g(y_\alpha, x_0)$  for all  $x \in D_{2,\alpha}$ . Then, it follows from (5.43) and (5.11) that

$$\begin{aligned}
 &\mu_\alpha^{-\frac{n-2}{2}} d_g(y_\alpha, x_0)^{n-2} \int_{D_{2,\alpha}} G_\alpha(y_\alpha, x) \frac{u_\alpha^{2^*(s)-1}}{d_g(x, x_0)^s} dv_g \\
 &\leq C \left(\frac{\mu_\alpha}{d_g(y_\alpha, x_0)}\right)^{2-s} \frac{1}{d_g(y_\alpha, x_0)^2} \int_{D_{2,\alpha}} d_g(y_\alpha, x)^{2-n} dv_g \\
 &\leq C \left(\frac{\mu_\alpha}{d_g(y_\alpha, x_0)}\right)^{2-s} = o(1). \quad (5.80)
 \end{aligned}$$

Combining (5.79), (5.80) and, we get that

$$\lim_{\alpha \rightarrow +\infty} d_g(y_\alpha, x_0)^{n-2} \mu_\alpha^{-\frac{n-2}{2}} u_\alpha(y_\alpha) = K^{n-2}. \quad (5.81)$$

Proposition 5.4.1 is a direct consequence of (5.63), (5.73) and (5.81).  $\square$

*Proof of Corollary 5.4.1:* We define

$$v_\alpha(x) := \left(\frac{\mu_\alpha^{2-s} + \frac{d_g(x, x_0)^{2-s}}{K^{2-s}}}{\mu_\alpha^{\frac{2-s}{2}}}\right)^{\frac{n-2}{2-s}} u_\alpha(x) \quad (5.82)$$

for all  $x \in M$  and  $\alpha \in \mathbb{N}$ . We let  $(y_\alpha)_\alpha \in M$  be such that  $v_\alpha(y_\alpha) = \min_{x \in M} v_\alpha(x)$  for all  $\alpha \in \mathbb{N}$ . Since  $G > 0$ , it follows from Proposition 5.4.1 that there exists  $c_0 > 0$  such that  $v_\alpha(y_\alpha) \geq c_0$  for all  $\alpha \in \mathbb{N}$ . This yields the lower bound of Corollary 5.4.1. The upper bound is (5.11). This proves Corollary 5.4.1.  $\square$

**Proposition 5.4.2.** *For all  $R > 0$ , we claim that there exists  $C > 0$  such that*

$$|\nabla u_\alpha(x)|_g \leq C \frac{\mu_\alpha^{\frac{n-2}{2}}}{(d_g(x, x_0)^2 + \mu_\alpha^2)^{\frac{n-1}{2}}} \text{ for all } x \in M \setminus B_{R\mu_\alpha}(x_0), \quad (5.83)$$

as  $\alpha \rightarrow +\infty$ .

*Proof of Proposition 5.4.2:* Let  $(y_\alpha)_\alpha \in M$  be such that

$$\begin{aligned} & \sup_{x \in M} (d_g(x, x_0)^{n-1} + \mu_\alpha^{n-1}) u_\alpha(x_\alpha) |\nabla u_\alpha(x)|_g \\ &= (d_g(y_\alpha, x_0)^{n-1} + \mu_\alpha^{n-1}) u_\alpha(x_\alpha) |\nabla u_\alpha(y_\alpha)|. \end{aligned}$$

The claim is equivalent to proving that for any  $y_\alpha$ , we have that

$$(d_g(y_\alpha, x_0)^{n-1} + \mu_\alpha^{n-1}) u_\alpha(x_\alpha) |\nabla u_\alpha(y_\alpha)|_g = O(1) \text{ as } \alpha \rightarrow +\infty.$$

We let  $G_\alpha$  be the Green's function of  $\Delta_g + a_\alpha$  in  $M$ . Green's representation formula and the estimates (5.44) yield  $C > 0$  such that

$$\begin{aligned} |\nabla u_\alpha(y_\alpha)| &\leq \int_M |\nabla G_\alpha(y_\alpha, x)|_g \lambda_\alpha \frac{u_\alpha^{2^*(s)-1}(x)}{d_g(x, x_0)^s} dv_g(x) \\ &\leq C \int_M d_g(x, y_\alpha)^{1-n} \frac{u_\alpha^{2^*(s)-1}(x)}{d_g(x, x_0)^s} dv_g(x). \end{aligned} \quad (5.84)$$

More generally, we prove that for any sequence  $(y_\alpha)_\alpha \in M$  such that  $d_g(y_\alpha, x_0) \geq R\mu_\alpha$  for some  $R > 0$ , then there exist  $C > 0$  such that

$$|\nabla u_\alpha(y_\alpha)| \leq C \frac{\mu_\alpha^{\frac{n-2}{2}}}{\mu_\alpha^{n-1} + d_g(y_\alpha, x_0)^{n-1}} \text{ as } \alpha \rightarrow +\infty. \quad (5.85)$$

We prove (5.85). We fix  $r_0 \in (0, R_0)$ . We write  $M := \cup_{i=1}^4 \Omega_{i,\alpha}$  where

$$\begin{aligned} \Omega_{1,\alpha} &:= B_{R_0\mu_\alpha}(x_0), \\ \Omega_{2,\alpha} &:= \left\{ R_0\mu_\alpha < d_g(x, x_0) < \frac{d_g(y_\alpha, x_0)}{2} \right\}, \\ \Omega_{3,\alpha} &:= \left\{ \frac{d_g(y_\alpha, x_0)}{2} < d_g(x, x_0) < 2d_g(y_\alpha, x_0) \right\}, \\ \Omega_{4,\alpha} &:= \{d_g(x, x_0) \geq 2d_g(y_\alpha, x_0)\}. \end{aligned}$$

We argue as in Step 5.3.2 to prove (5.85).

**Case 1:** The domain  $\Omega_{1,\alpha}$ . As one checks, for all  $x \in \Omega_{1,\alpha}$ , we have that

$$d_g(y_\alpha, x_0) \geq R\mu_\alpha \geq \frac{R}{R_0}d_g(x, x_0),$$

then for  $R_0 \in (0, R)$ , we have that

$$d_g(x, y_\alpha) \geq d_g(y_\alpha, x_0) - d_g(x, x_0) \geq \left(1 - \frac{R_0}{R}\right) d_g(y_\alpha, x_0).$$

With the change of variables  $x = \exp_{x_0}(\mu_\alpha X)$ , we get that

$$\begin{aligned} & \left| \int_{\Omega_{1,\alpha}} d_g(x, y_\alpha)^{1-n} \frac{u_\alpha^{2^*(s)-1}(x)}{d_g(x, x_0)^s} dv_g \right| \\ & \leq C d_g(y_\alpha, x_0)^{1-n} \int_{\Omega_{1,\alpha}} \frac{u_\alpha^{2^*(s)-1}(x)}{d_g(x, x_0)^s} dv_g \\ & \leq C \mu_\alpha^{\frac{n-2}{2}} d_g(y_\alpha, x_0)^{1-n} \int_{B_{R_0}(0)} \frac{\tilde{u}_\alpha^{2^*(s)-1}(X)}{|X|^s} dv_{\tilde{g}_\alpha}, \end{aligned} \quad (5.86)$$

where  $\tilde{u}_\alpha, \tilde{g}_\alpha$  are defined in (5.17), (5.16). By applying Lebesgue's Dominated Convergence Theorem and thanks to Step 5.2.2, we get that

$$\lim_{R \rightarrow +\infty} \lim_{\alpha \rightarrow +\infty} \int_{B_{R_0}(0)} \frac{\tilde{u}_\alpha^{2^*(s)-1}(X)}{|X|^s} dv_{\tilde{g}_\alpha} = \int_{\mathbb{R}^n} \frac{\tilde{u}^{2^*(s)-1}}{|X|^s} dX. \quad (5.87)$$

Combining (5.86) and (5.87) yields

$$\left| \int_{\Omega_{1,\alpha}} d_g(x, y_\alpha)^{1-n} \frac{u_\alpha^{2^*(s)-1}(x)}{d_g(x, x_0)^s} dv_g \right| \leq C \mu_\alpha^{\frac{n-2}{2}} d_g(y_\alpha, x_0)^{1-n}. \quad (5.88)$$

**Case 2:** We now consider the domain  $\Omega_{2,\alpha}$ . As one checks, for all  $x \in \Omega_{2,\alpha}$ , we have that  $d_g(x, y_\alpha) \geq \frac{1}{2}d_g(y_\alpha, x_0)$ . With the upper bound (5.11), we then get that

$$\begin{aligned} & \left| \int_{\Omega_{2,\alpha}} d_g(x, y_\alpha)^{1-n} \frac{u_\alpha^{2^*(s)-1}(x)}{d_g(x, x_0)^s} dv_g \right| \\ & \leq C \mu_\alpha^{\frac{n-2s+2}{2}} d_g(y_\alpha, x_0)^{1-n} \int_{\Omega_{2,\alpha}} \frac{1}{d_g(x, x_0)^{s+(n-2)(2^*(s)-1)}} dv_g, \end{aligned}$$

Taking  $X = \exp_{x_0}^{-1}(x)$  and  $\hat{g} = \exp_{x_0}^* g$  on  $\mathbb{R}^n$ , we get

$$\begin{aligned} & \left| \int_{\Omega_{2,\alpha}} d_g(x, y_\alpha)^{1-n} \frac{u_\alpha^{2^*(s)-1}(x)}{d_g(x, x_0)^s} dv_g \right| \\ & \leq C \mu_\alpha^{\frac{n-2s+2}{2}} d_g(y_\alpha, x_0)^{1-n} \int_{\mathbb{R}^n \setminus B_{R_0 \mu_\alpha}(0)} |X|^{s-2-n} dv_{\hat{g}} \\ & \leq C \mu_\alpha^{\frac{n-2}{2}} d_g(y_\alpha, x_0)^{1-n} \int_{\mathbb{R}^n \setminus B_{R_0}(0)} |X|^{s-2-n} dX \\ & \leq C \mu_\alpha^{\frac{n-2}{2}} d_g(y_\alpha, x_0)^{1-n} \int_{R_0}^{+\infty} r^{s-2-1} dr, \end{aligned}$$

since  $s \in (0, 2)$  we have that

$$\left| \int_{\Omega_{2,\alpha}} d_g(x, y_\alpha)^{1-n} \frac{u_\alpha^{2^*(s)-1}(x)}{d_g(x, x_0)^s} dv_g \right| \leq C \mu_\alpha^{\frac{n-2}{2}} d_g(y_\alpha, x_0)^{1-n}, \quad (5.89)$$

as  $\alpha \rightarrow +\infty$ .

**Case 3:** We consider the domain  $\Omega_{3,\alpha}$ . Using (5.11), there exists  $C > 0$  such that

$$\begin{aligned} & \left| \int_{\Omega_{3,\alpha}} d_g(x, y_\alpha)^{1-n} \frac{u_\alpha^{2^*(s)-1}(x)}{d_g(x, x_0)^s} dv_g \right| \\ & \leq C \mu_\alpha^{\frac{n-2}{2}(2^*(s)-1)} d_g(y_\alpha, x_0)^{-s-(n-2)(2^*(s)-1)} \int_{\Omega_{3,\alpha}} d_g(x, y_\alpha)^{1-n} dv_g. \end{aligned}$$

Taking  $x = \exp_{x_0}(X)$  and  $y_\alpha = \exp_{x_0}(Y_\alpha)$ , we get that

$$\begin{aligned} & \left| \int_{\Omega_{3,\alpha}} d_g(x, y_\alpha)^{2-n} \frac{u_\alpha^{2^*(s)-1}(x)}{d_g(x, x_0)^s} dv_g \right| \\ & \leq C \mu_\alpha^{\frac{n-2}{2}(2^*(s)-1)} d_g(y_\alpha, x_0)^{-s-(n-2)(2^*(s)-1)} \int_{\frac{1}{2}|Y_\alpha| < |X| < 2|Y_\alpha|} |X - Y_\alpha|^{1-n} dv_{\hat{g}} \\ & \leq C \mu_\alpha^{\frac{n-2}{2}(2^*(s)-1)} d_g(y_\alpha, x_0)^{-s-(n-2)(2^*(s)-1)} \int_{\frac{1}{2}|Y_\alpha| < |X| < 2|Y_\alpha|} |X - Y_\alpha|^{1-n} dX \\ & \leq C \mu_\alpha^{\frac{n-2}{2}(2^*(s)-1)} d_g(y_\alpha, x_0)^{-s-(n-2)(2^*(s)-1)} |Y_\alpha| \int_{|X| < 3} |X|^{1-n} dX \\ & \leq C \mu_\alpha^{\frac{n-2}{2}} d_g(y_\alpha, x_0)^{1-n} \left( \frac{\mu_\alpha}{d_g(y_\alpha, x_0)} \right)^{\left(\frac{n-2}{2}\right)(2^*(s)-2)} \end{aligned}$$



Since  $\frac{n-2}{2}(2^*(s) - 2) = 2 - s > 0$  we obtain that,

$$\begin{aligned} & \left| \int_{\Omega_{3,\alpha}} d_g(x, y_\alpha)^{2-n} \frac{u_\alpha^{2^*(s)-1}(x)}{d_g(x, x_0)^s} dv_g \right| \\ & \leq C \mu_\alpha^{\frac{n-2}{2}} d_g(y_\alpha, x_0)^{1-n} \left( \frac{\mu_\alpha}{d_g(y_\alpha, x_0)} \right)^{\frac{n-2}{2}(2^*(s)-2)}. \end{aligned} \quad (5.90)$$

**Case 4:** We now consider the domain  $\Omega_{4,\alpha}$ . For any  $x \in \Omega_{4,\alpha}$ , we have that  $d_g(x, y_\alpha) \geq d_g(x, x_0) - d_g(y_\alpha, x_0) \geq \frac{d_g(x, x_0)}{2}$ . By (5.11) and taking  $X = \exp_{x_0}^{-1}(x)$ ,  $Y_\alpha = \exp_{x_0}^{-1}(y_\alpha)$ , then there exists  $C > 0$  such that

$$\begin{aligned} & \left| \int_{\Omega_{4,\alpha}} d_g(x, y_\alpha)^{1-n} \frac{u_\alpha^{2^*(s)-1}(x)}{d_g(x, x_0)^s} dv_g \right| \\ & \leq C \mu_\alpha^{\frac{n-2}{2}(2^*(s)-1)} \int_{\Omega_{4,\alpha}} d_g(x, x_0)^{1-n-s-(n-2)(2^*(s)-1)} dv_g \\ & \leq C \mu_\alpha^{\frac{n-2}{2}(2^*(s)-1)} \int_{B_\delta(0) \setminus B_{2|Y_\alpha|}(0)} |X|^{1-n-s-(n-2)(2^*(s)-1)} dv_{\hat{g}} \\ & \leq C \mu_\alpha^{\frac{n-2}{2}(2^*(s)-1)} \int_{2|Y_\alpha|}^\delta r^{-n+s-2} dr. \end{aligned}$$

Since  $s \in (0, 2)$  we get

$$\begin{aligned} & \left| \int_{\Omega_{4,\alpha}} d_g(x, y_\alpha)^{2-n} \frac{u_\alpha^{2^*(s)-1}(x)}{d_g(x, x_0)^s} dv_g \right| \\ & \leq C \mu_\alpha^{\frac{n-2}{2}} d_g(y_\alpha, x_0)^{2-n} \left( \frac{\mu_\alpha}{d_g(y_\alpha, x_0)} \right)^{\frac{n-2}{2}(2^*(s)-2)}. \end{aligned} \quad (5.91)$$

Plugging together (5.88)-(5.91) yields

$$|\nabla u_\alpha(y_\alpha)| \leq C \frac{\mu_\alpha^{\frac{n-2}{2}}}{d_g(y_\alpha, x_0)^{n-1}}. \quad (5.92)$$

Since  $d_g(y_\alpha, x_0) \geq R\mu_\alpha$ , we then get

$$\begin{aligned} (\mu_\alpha^{n-1} + d_g(y_\alpha, x_0)^{n-1}) \mu_\alpha^{-\frac{n-2}{2}} |\nabla u_\alpha(y_\alpha)| & \leq C \left( \frac{\mu_\alpha^{n-1}}{d_g(y_\alpha, x_0)^{n-1}} + 1 \right) \\ & \leq C (R^{1-n} + 1) \end{aligned}$$

and therefore (5.85). As mentioned at the beginning of the proof, this yields (5.83). This ends the proof of Proposition 5.4.2.  $\square$

## 5.5 Pohozaev identity and proof of Theorem 5.1.4

We let  $(u_\alpha)_\alpha \in H_1^2(M)$ ,  $(\lambda_\alpha)_\alpha \in \mathbb{R}$ ,  $(a_\alpha)_\alpha \in C^1(M)$  and  $a_\infty \in C^1(M)$  be such that (5.4) to (5.10) hold. In the sequel, we fix  $\delta \in (0, \frac{i_g(M)}{2})$  where  $i_g(M) > 0$  is the injectivity radius of  $(M, g)$ . We define the following function,

$$\hat{u}_\alpha(X) := u_\alpha(\exp_{x_0}(X)) \text{ for all } X \in B_\delta(0) \subset \mathbb{R}^n, \quad (5.93)$$

where  $\exp_{x_0} : B_\delta(0) \rightarrow B_\delta(x_0) \subset M$  is the exponential map at  $x_0$ . We define also the metric

$$\hat{g}(X) := (\exp_{x_0}^* g)(X) \text{ on } \mathbb{R}^n.$$

It then follows from (5.7) that

$$\Delta_{\hat{g}} \hat{u}_\alpha + \hat{a}_\alpha \hat{u}_\alpha = \lambda_\alpha \frac{\hat{u}_\alpha^{2^*(s)-1}}{|X|^s} \text{ weakly in } B_{\delta\mu_\alpha^{-1}}(0), \quad (5.94)$$

where  $\hat{a}_\alpha = a_\alpha(\exp_{x_0}(X))$ . The Pohozaev identity writes (see for instance [62])

$$\begin{aligned} & \int_{B_\delta(0)} \left( X^l \partial_l \hat{u}_\alpha + \frac{n-2}{2} \hat{u}_\alpha \right) \left( \Delta_{Eucl} \hat{u}_\alpha - \lambda_\alpha \frac{\hat{u}_\alpha^{2^*(s)-1}}{|X|^s} \right) dX \\ &= \int_{\partial B_\delta(0)} (X, \nu) \left( \frac{|\nabla \hat{u}_\alpha|^2}{2} - \frac{\lambda_\alpha}{2^*(s)} \frac{\hat{u}_\alpha^{2^*(s)}}{|X|^s} \right) - \left( X^l \partial_l \hat{u}_\alpha + \frac{n-2}{2} \hat{u}_\alpha \right) \partial_\nu \hat{u}_\alpha d\sigma. \end{aligned}$$

where  $\nu(X)$  is the outer normal vector of  $B_\delta(0)$  at  $X \in \partial B_\delta(0)$ , that is  $\nu(X) = \frac{X}{|X|}$ . With (5.94), the Pohozaev identity writes

$$C_\alpha + D_\alpha = B_\alpha. \quad (5.95)$$

with

$$B_\alpha := \int_{\partial B_\delta(0)} (X, \nu) \left( \frac{|\nabla \hat{u}_\alpha|^2}{2} - \frac{\lambda_\alpha}{2^*(s)} \frac{\hat{u}_\alpha^{2^*(s)}}{|X|^s} \right) - \left( X^l \partial_l \hat{u}_\alpha + \frac{n-2}{2} \hat{u}_\alpha \right) \partial_\nu \hat{u}_\alpha d\sigma.$$

$$C_\alpha := - \int_{B_\delta(0)} \left( X^l \partial_l \hat{u}_\alpha + \frac{n-2}{2} \hat{u}_\alpha \right) \hat{a}_\alpha \hat{u}_\alpha dX,$$

and,

$$D_\alpha := - \int_{B_\delta(0)} \left( X^l \partial_l \hat{u}_\alpha + \frac{n-2}{2} \hat{u}_\alpha \right) (\Delta_{\hat{g}} \hat{u}_\alpha - \Delta_{Eucl} \hat{u}_\alpha) dX.$$

We are going to estimate these terms separately.

**Step 5.5.1.** *We claim that*

$$\lim_{\alpha \rightarrow +\infty} \frac{B_\alpha}{\mu_\alpha^{n-2}} = d_n^2 \left( \int_{\partial B_\delta(0)} \frac{\delta}{2} |\nabla \hat{G}_{x_0}|^2 - \frac{1}{\delta} \left( \langle X, \nabla \hat{G}_{x_0} \rangle^2 + \frac{n-2}{2} \langle X, \nabla \hat{G}_{x_0} \rangle \hat{G}_{x_0} \right) d\sigma \right), \quad (5.96)$$

as  $\alpha \rightarrow +\infty$ , where  $d_n$  is defined in (5.64), and  $\hat{G}_{x_0}(X) = G(x_0, \exp_{x_0}(X))$ .

*Proof of the claim:* It follows from the definition of  $\hat{u}_\alpha$  that

$$\begin{aligned} \mu_\alpha^{-(n-2)} B_\alpha &= \int_{\partial B_\delta(0)} (X, \nu) \left( \frac{|\mu_\alpha^{-\frac{n-2}{2}} \nabla \hat{u}_\alpha|^2}{2} - \frac{\lambda_\alpha}{2^*(s)} \mu_\alpha^{2-s} \frac{\left( \mu_\alpha^{-\frac{n-2}{2}} \hat{u}_\alpha \right)^{2^*(s)}}{|X|^s} \right) \\ &\quad - \left( X^l \mu_\alpha^{-\frac{n-2}{2}} \partial_l \hat{u}_\alpha + \frac{n-2}{2} \mu_\alpha^{-\frac{n-2}{2}} \hat{u}_\alpha \right) \mu_\alpha^{-\frac{n-2}{2}} \partial_\nu \hat{u}_\alpha d\sigma. \end{aligned}$$

Since  $s < 2$ , the convergence of Proposition 5.3.1 yields (5.96). This proves the claim.  $\square$

In this section, we will extensively use the following consequences of the point-wise estimates (5.11) and (5.83):

$$\hat{u}_\alpha(X) \leq C \left( \frac{\mu_\alpha}{\mu_\alpha^2 + |X|^2} \right)^{\frac{n-2}{2}} \text{ in } B_\delta(0), \quad (5.97)$$

$$|\nabla \hat{u}_\alpha|(X) \leq C \frac{\mu_\alpha^{\frac{n-2}{2}}}{(\mu_\alpha^2 + |X|^2)^{\frac{n-1}{2}}} \text{ in } B_\delta(0) \setminus B_{R\mu_\alpha}(0), \quad (5.98)$$

and

$$\tilde{u}_\alpha(X) \leq C \left( \frac{1}{1 + |X|^2} \right)^{\frac{n-2}{2}} \text{ in } B_{\mu_\alpha^{-1}\delta}(0), \quad (5.99)$$

$$|\nabla \tilde{u}_\alpha|(X) \leq C \frac{1}{(1 + |X|^2)^{\frac{n-1}{2}}} \text{ in } B_{\mu_\alpha^{-1}\delta}(0) \setminus B_R(0). \quad (5.100)$$

**Step 5.5.2.** *We claim that, as  $\alpha \rightarrow +\infty$ ,*

$$C_\alpha = \begin{cases} \mu_\alpha^2 \ln\left(\frac{1}{\mu_\alpha}\right) (\omega_3 K^4 a_\infty(x_0) + o(1)) & \text{if } n = 4, \\ \mu_\alpha^2 (a_\infty(x_0) \int_{\mathbb{R}^n} \tilde{u}^2 dX + o(1)) & \text{if } n \geq 5, \end{cases}$$

where  $K$  is defined in (5.20).

*Proof of the claim:* Using the definition of  $C_\alpha$  and integrating by parts, we get

$$\begin{aligned}
C_\alpha &= - \int_{B_\delta(0)} \left( X^l \hat{a}_\alpha \partial_l \left( \frac{\hat{u}_\alpha^2}{2} \right) + \frac{n-2}{2} \hat{a}_\alpha \hat{u}_\alpha^2 \right) dX \\
&= - \int_{B_\delta(0)} \left( -\frac{n}{2} \hat{a}_\alpha \hat{u}_\alpha^2 - X^l \partial_l \hat{a}_\alpha \frac{\hat{u}_\alpha^2}{2} + \frac{n-2}{2} \hat{a}_\alpha \hat{u}_\alpha^2 \right) dX \\
&\quad - \frac{1}{2} \int_{\partial B_\delta(0)} (X, \nu) \hat{a}_\alpha \hat{u}_\alpha^2 d\sigma, \\
&= \int_{B_\delta(0)} \left( \hat{a}_\alpha + \frac{X^l \partial_l \hat{a}_\alpha}{2} \right) \hat{u}_\alpha^2 dX - \frac{1}{2} \int_{\partial B_\delta(0)} (X, \nu) \hat{a}_\alpha \hat{u}_\alpha^2 d\sigma \quad (5.101)
\end{aligned}$$

With (5.97), we then get that

$$\begin{aligned}
\left| \int_{\partial B_\delta(0)} (X, \nu) \hat{a}_\alpha \hat{u}_\alpha^2 d\sigma \right| &\leq C(\delta) \mu_\alpha^{n-2} \int_{\partial B_\delta(0)} \frac{1}{(\mu_\alpha^{n-2} + |X|^{n-2})^2} d\sigma \\
&\leq C(\delta) \mu_\alpha^{n-2} \int_{\partial B_\delta(0)} \frac{1}{|X|^{2(n-2)}} d\sigma,
\end{aligned}$$

then,

$$\int_{\partial B_\delta(0)} (X, \nu) \hat{a}_\alpha \hat{u}_\alpha^2 d\sigma = O(\mu_\alpha^{n-2}). \quad (5.102)$$

Moreover, with (5.101) we have

$$C_\alpha = \int_{B_\delta(0)} \left( \hat{a}_\alpha + \frac{X^l \partial_l \hat{a}_\alpha}{2} \right) \hat{u}_\alpha^2 dX + O(\mu_\alpha^{n-2}) \text{ as } \alpha \rightarrow +\infty. \quad (5.103)$$

We now define

$$\varphi_\alpha(X) := \hat{a}_\alpha + \frac{X^l \partial_l \hat{a}_\alpha}{2}. \quad (5.104)$$

We distinguish three cases:

**Case 1:** If  $n \geq 5$ , with a change of variable  $X = \mu_\alpha Y$ , we get that

$$\int_{B_\delta(0)} \varphi_\alpha(X) \hat{u}_\alpha^2 dX = \mu_\alpha^2 \int_{B_{\mu_\alpha^{-1}\delta}(0)} \varphi_\alpha(\mu_\alpha X) \tilde{u}_\alpha^2 dX,$$

where  $\tilde{u}_\alpha$  is as in (5.18). Since  $\mu_\alpha \rightarrow 0$  as  $\alpha \rightarrow +\infty$ , and by (5.104), (5.4), we get

$$\lim_{\alpha \rightarrow +\infty} \varphi_\alpha(\mu_\alpha X) = a_\infty(x_0).$$

Since  $n \geq 5$ , we have that  $X \mapsto (1 + |X|^2)^{\frac{n-2}{2}} \in L^2(\mathbb{R}^n)$ . Therefore, with the pointwise control (5.99), Lebesgue's dominated convergence theorem and Step 5.2.2 yield Step 5.5.2 for  $n \geq 5$ .

**Case 2:** If  $n = 4$ , we have that

$$\begin{aligned}
 K^{-4} \int_{B_{\delta\mu_\alpha^{-1}}(0)} \frac{1}{\left(1 + \left(\frac{|X|}{K}\right)^{2-s}\right)^{\frac{4}{2-s}}} dX &= \int_{B_{\delta\mu_\alpha^{-1}}(0)} \frac{1}{(K^{2-s} + |X|^{2-s})^{\frac{4}{2-s}}} dX \\
 &= \omega_3 \int_1^{\delta\mu_\alpha^{-1}} \frac{r^3}{(K^{2-s} + r^{2-s})^{\frac{4}{2-s}}} + O(1) = \omega_3 \int_1^{\delta\mu_\alpha^{-1}} \frac{1}{r} dr + O(1) \\
 &= \omega_3 \ln\left(\frac{\delta}{\mu_\alpha}\right) + O(1). \tag{5.105}
 \end{aligned}$$

Therefore, it follows from Proposition 5.4.1, for any  $\epsilon > 0$ , there exists  $\delta_\epsilon > 0$  such that, up to a subsequence, for any  $\alpha$  and any  $X \in B_{\delta_\epsilon}(0)$ ,

$$\frac{1}{1 + \epsilon} \frac{1}{\left(1 + \left(\frac{|X|}{K}\right)^{2-s}\right)^{\frac{2}{2-s}}} \leq \tilde{u}_\alpha(X) \leq (1 + \epsilon) \frac{1}{\left(1 + \left(\frac{|X|}{K}\right)^{2-s}\right)^{\frac{2}{2-s}}} \tag{5.106}$$

Combining the last equation and (5.105), by letting  $\alpha \rightarrow +\infty$  and then  $\epsilon \rightarrow 0$ , we obtain that

$$\lim_{\alpha \rightarrow +\infty} \frac{1}{\ln\left(\frac{1}{\mu_\alpha}\right)} \int_{B_{\delta\mu_\alpha^{-1}}(0)} \varphi_\alpha(X) \hat{u}_\alpha^2 dX = \omega_3 K^4 a_\infty(x_0). \tag{5.107}$$

This yields Step 5.5.2 for  $n = 4$ . These two cases yield Step 5.5.2.  $\square$

**Assumption 5.5.1** (An assumption in the case  $n = 4$ ). For  $i, j, \beta_1, \beta_2 \geq 1$ , and  $n = 4$ , we **assume** that

$$\frac{\int_{B_{\delta\mu_\alpha^{-1}}(0)} X^{\beta_1} X^{\beta_2} \partial_i \tilde{u}_\alpha \partial_j \tilde{u}_\alpha dX}{\mu_\alpha^2 \ln\left(\frac{1}{\mu_\alpha}\right)} = 4K^4 \int_{S^{n-1}} \sigma^i \sigma^j \sigma^{\beta_1} \sigma^{\beta_2} d\sigma. \tag{5.108}$$

The proof of this limit is in progress. The results for  $n = 4$  below are stated provided this limit holds.

We are left with estimating  $D_\alpha$ . Recall that  $-(\Delta_{\hat{g}} - \Delta_{Eucl}) = (\hat{g}^{ij} - \delta^{ij}) \partial_{ij} - \hat{g}^{ij} \hat{\Gamma}_{ij}^k \partial_k$  and the Christoffel symbols are  $\hat{\Gamma}_{ij}^k := \frac{1}{2} \hat{g}^{kp} (\partial_i \hat{g}_{jp} + \partial_j \hat{g}_{ip} - \partial_p \hat{g}_{ij})$ . Then, we write

$$D_\alpha = D_{1,\alpha} - D_{2,\alpha} + \frac{n-2}{2} D_{3,\alpha} - \frac{n-2}{2} D_{4,\alpha}, \tag{5.109}$$

where

$$\begin{aligned} D_{1,\alpha} &:= \int_{B_\delta(0)} (\hat{g}^{ij} - \delta^{ij}) X^l \partial_l \hat{u}_\alpha \partial_{ij} \hat{u}_\alpha dX, & D_{2,\alpha} &:= \int_{B_\delta(0)} \hat{g}^{ij} X^l \hat{\Gamma}_{ij}^k \partial_l \hat{u}_\alpha \partial_k \hat{u}_\alpha dX, \\ D_{3,\alpha} &:= \int_{B_\delta(0)} (\hat{g}^{ij} - \delta^{ij}) \hat{u}_\alpha \partial_{ij} \hat{u}_\alpha dX, & D_{4,\alpha} &:= \int_{B_\delta(0)} \hat{g}^{ij} \hat{u}_\alpha \hat{\Gamma}_{ij}^k \partial_k \hat{u}_\alpha dX \end{aligned} \quad (5.110)$$

We now estimate the  $D_{i,\alpha}$ 's separately. Note that, since the exponential map is normal at 0, we have that  $\partial_{\beta_1} \hat{g}^{ij}(0) = 0$  for all  $i, j, \beta_1 = 1, \dots, n$ . For  $i, j, k = 1, \dots, n$ , the Taylor formula around 0 writes,

$$\Gamma_{ij}^k(X) = \sum_{m=1}^n X^m \partial_m \Gamma_{ij}^k(0) + O(|X|^2), \quad (5.111)$$

and,

$$\hat{g}^{ij}(X) - \delta^{ij} = \frac{1}{2} \sum_{\beta_1, \beta_2=1}^n X^{\beta_1} X^{\beta_2} \partial_{\beta_1 \beta_2} \hat{g}^{ij}(0) + O(|X|^3). \quad (5.112)$$

**Step 5.5.3.** *We claim that*

$$\int_{B_\delta(0)} |X|^3 |\nabla \hat{u}_\alpha|^2 dX = \begin{cases} o(\mu_\alpha^2) & \text{if } n \geq 5, \\ O(\mu_\alpha^2) & \text{if } n = 4, \\ O(\delta \mu_\alpha) & \text{if } n = 3. \end{cases} \quad (5.113)$$

And,

$$\int_{B_\delta(0)} |X| \hat{u}_\alpha^2 dX = \begin{cases} o(\mu_\alpha^2) & \text{if } n \geq 5, \\ O(\mu_\alpha^2) & \text{if } n = 4, \\ O(\delta \mu_\alpha) & \text{if } n = 3. \end{cases} \quad (5.114)$$

*Proof of the claim:* Estimate (5.114), this is a direct consequence of the upper bound (5.97). We deal with (5.113). We fix  $R > 0$  and we write

$$\begin{aligned} & \int_{B_\delta(0)} |X|^3 |\nabla \hat{u}_\alpha|^2 dX = \int_{B_{R\mu_\alpha}(0)} |X|^3 |\nabla \hat{u}_\alpha|^2 dX \\ & + \int_{B_\delta(0) \setminus B_{R\mu_\alpha}(0)} |X|^3 |\nabla \hat{u}_\alpha|^2 dX \\ & = \mu_\alpha^3 \int_{B_R(0)} |X|^3 |\nabla \tilde{u}_\alpha|^2 dX + \int_{B_\delta(0) \setminus B_{R\mu_\alpha}(0)} |X|^3 |\nabla \hat{u}_\alpha|^2 dX, \end{aligned}$$

where  $\tilde{u}_\alpha$  is as in (5.17). It follows from the strong convergence of (5.24) that  $\int_{B_R(0)} |X|^3 |\nabla \tilde{u}_\alpha|^2 dX = O(1)$  as  $\alpha \rightarrow +\infty$ . As for (5.114), the control of the integral on  $B_\delta(0) \setminus B_{R\mu_\alpha}(0)$  is a direct consequence of (5.100). This yields (5.114). This proves the claim.  $\square$

**Step 5.5.4.** We estimate  $D_{2,\alpha}$  for  $n \geq 4$ .

Since  $\hat{g}^{ij} - \delta^{ij} = O(|X|^2)$  as  $X \rightarrow 0$  and by (5.111), we estimate as  $\alpha \rightarrow +\infty$  that,

$$\begin{aligned}
 D_{2,\alpha} &:= \int_{B_\delta(0)} \hat{g}^{ij} X^l \hat{\Gamma}_{ij}^k \partial_l \hat{u}_\alpha \partial_k \hat{u}_\alpha dX \\
 &= \delta^{ij} \int_{B_\delta(0)} X^l \hat{\Gamma}_{ij}^k \partial_l \hat{u}_\alpha \partial_k \hat{u}_\alpha dX + O\left(\int_{B_\delta(0)} |X|^3 \hat{\Gamma}_{ij}^k \partial_l \hat{u}_\alpha \partial_k \hat{u}_\alpha dX\right) \\
 &= \sum_{m=1}^n \partial_m \hat{\Gamma}_{ii}^k(0) \int_{B_\delta(0)} X^l X^m \partial_l \hat{u}_\alpha \partial_k \hat{u}_\alpha dX \\
 &+ O\left(\int_{B_\delta(0)} |X|^3 |\nabla \hat{u}_\alpha|^2 dX\right). \tag{5.115}
 \end{aligned}$$

A change of variable  $Y = \mu_\alpha^{-1} X$  and the estimates (5.115) and (5.113) yield

$$D_{2,\alpha} = \mu_\alpha^2 \sum_{m=1}^n \partial_m \hat{\Gamma}_{ii}^k(0) \int_{B_{\delta\mu_\alpha^{-1}}(0)} X^l X^m \partial_l \tilde{u}_\alpha \partial_k \tilde{u}_\alpha dX + \begin{cases} o(\mu_\alpha^2) & \text{if } n \geq 5, \\ O(\mu_\alpha^2) & \text{if } n = 4. \end{cases} \tag{5.116}$$

**Case 1:  $n \geq 5$ .** Since  $X \mapsto |X|^2 ((1 + |X|^2)^{(1-n)/2})^2 \in L^1(\mathbb{R}^n)$  for  $n \geq 5$ , it follows from the strong convergence (5.24), the pointwise convergence of Step 5.2.2, the pointwise control (5.100), the Lebesgue dominated convergence theorem and the radial symmetry of  $\tilde{u}$  that

$$\begin{aligned}
 D_{2,\alpha} &= \mu_\alpha^2 \sum_{m=1}^n \partial_m \hat{\Gamma}_{ii}^k(0) \left( \int_{B_R(0)} X^l X^m \partial_l \tilde{u}_\alpha \partial_k \tilde{u}_\alpha dX \right. \\
 &\quad \left. + \int_{B_{\delta\mu_\alpha^{-1}}(0) \setminus B_R(0)} X^l X^m \partial_l \tilde{u}_\alpha \partial_k \tilde{u}_\alpha dX \right) + o(\mu_\alpha^2) \\
 &= \mu_\alpha^2 \sum_{m=1}^n \partial_m \hat{\Gamma}_{ii}^k(0) \left( \int_{B_R(0)} X^l X^m \partial_l \tilde{u} \partial_k \tilde{u} dX \right. \\
 &\quad \left. + \int_{\mathbb{R}^n \setminus B_R(0)} X^l X^m \partial_l \tilde{u} \partial_k \tilde{u} dX \right) + o(\mu_\alpha^2) \\
 &= \mu_\alpha^2 \sum_{m=1}^n \partial_m \hat{\Gamma}_{ii}^k(0) \int_{\mathbb{R}^n} X^l X^m \partial_l \tilde{u} \partial_k \tilde{u} dX + o(\mu_\alpha^2) \\
 &= \mu_\alpha^2 \sum_{m=1}^n \partial_m \hat{\Gamma}_{ii}^k(0) \int_{\mathbb{R}^n} X^m X^k (\tilde{u}')^2 dX + o(\mu_\alpha^2)
 \end{aligned}$$

$$\begin{aligned}
&= \mu_\alpha^2 \sum_{m=1}^n \partial_m \hat{\Gamma}_{ii}^k(0) \int_{\mathbb{R}^n} X^m X^k |\nabla \tilde{u}|^2 dX + o(\mu_\alpha^2) \\
&= \mu_\alpha^2 \sum_{m,k=1}^n \partial_m \hat{\Gamma}_{ii}^k(0) \int_{S^{n-1}} \theta^m \theta^k d\theta \int_0^{+\infty} r^2 |\nabla_r \tilde{u}|^2 dr + o(\mu_\alpha^2).
\end{aligned}$$

With the symmetries of the sphere, we have that  $\int_{S^{n-1}} \theta^m \theta^k d\theta = \delta^{mk} \frac{\omega_{n-1}}{n}$ . Hence,

$$\begin{aligned}
D_{2,\alpha} &= \frac{\mu_\alpha^2}{n} \omega_{n-1} \sum_{k=1}^n \partial_k \hat{\Gamma}_{ii}^k(0) \int_0^{+\infty} r^2 |\nabla_r \tilde{u}|^2 dr + o(\mu_\alpha^2) \\
&= \frac{\mu_\alpha^2}{n} \sum_{k=1}^n \partial_k \hat{\Gamma}_{ii}^k(0) \int_{\mathbb{R}^n} |X|^2 |\nabla \tilde{u}|^2 dX + o(\mu_\alpha^2).
\end{aligned}$$

**Case 2:  $n = 4$ . IF (5.108) holds,** we have

$$\lim_{\alpha \rightarrow +\infty} \frac{\mu_\alpha^{-2}}{\ln\left(\frac{1}{\mu_\alpha}\right)} D_{2,\alpha} = K^4 \omega_3 \partial_k \hat{\Gamma}_{ii}^k(0).$$

**Step 5.5.5.** We estimate  $D_{3,\alpha}$  for  $n \geq 4$ .

We estimate the term  $D_{3,\alpha}$ , thanks of (5.112), (5.113), (5.114) and the integrations by parts give,

$$\begin{aligned}
D_{3,\alpha} &= \int_{B_\delta(0)} (\hat{g}^{ij} - \delta^{ij}) \hat{u}_\alpha \partial_{ij} \hat{u}_\alpha dX \\
&= - \left( \int_{B_\delta(0)} \partial_i \hat{g}^{ij} \hat{u}_\alpha \partial_j \hat{u}_\alpha dX + \int_{B_\delta(0)} (\hat{g}^{ij} - \delta^{ij}) \partial_i \hat{u}_\alpha \partial_j \hat{u}_\alpha dX \right) \\
&+ O \left( \int_{\partial B_\delta(0)} |X|^2 |\nabla \hat{u}_\alpha| \hat{u}_\alpha d\sigma \right) \\
&= -\frac{1}{2} \left( \int_{B_\delta(0)} \partial_i \hat{g}^{ij} \partial_j (\hat{u}_\alpha)^2 dX + \partial_{\beta_1 \beta_2} \hat{g}^{ij}(0) \int_{B_\delta(0)} X^{\beta_1} X^{\beta_2} \partial_i \hat{u}_\alpha \partial_j \hat{u}_\alpha dX \right) \\
&+ O \left( \int_{B_\delta(0)} |X|^3 |\nabla \hat{u}_\alpha|^2 dX \right) + O \left( \int_{\partial B_\delta(0)} |X|^2 |\nabla \hat{u}_\alpha| \hat{u}_\alpha d\sigma \right) \\
&= -\frac{1}{2} \left( - \int_{B_\delta(0)} \partial_{ij} \hat{g}^{ij} \hat{u}_\alpha^2 dX + \partial_{\beta_1 \beta_2} \hat{g}^{ij}(0) \int_{B_\delta(0)} X^{\beta_1} X^{\beta_2} \partial_i \hat{u}_\alpha \partial_j \hat{u}_\alpha dX \right) \\
&+ O \left( \int_{B_\delta(0)} |X|^3 |\nabla \hat{u}_\alpha|^2 dX \right) + O \left( \int_{\partial B_\delta(0)} (|X|^2 |\nabla \hat{u}_\alpha| \hat{u}_\alpha + |X| \hat{u}_\alpha^2) d\sigma \right) \quad (5.117)
\end{aligned}$$



Therefore,

$$\begin{aligned}
 D_{3,\alpha} &= -\frac{1}{2} \left( -\partial_{ij} \hat{g}^{ij}(0) \int_{B_\delta(0)} \hat{u}_\alpha^2 dX \right. \\
 &\quad \left. + \partial_{\beta_1 \beta_2} \hat{g}^{ij}(0) \int_{B_\delta(0)} X^{\beta_1} X^{\beta_2} \partial_i \hat{u}_\alpha \partial_j \hat{u}_\alpha dX \right) \quad (5.118) \\
 &\quad + \begin{cases} o(\mu_\alpha^2) & \text{if } n \geq 5, \\ O(\mu_\alpha^2) & \text{if } n = 4. \end{cases}
 \end{aligned}$$

**Case 1:  $n \geq 5$ .** Here again, since  $X \mapsto |X|^2 ((1 + |X|^2)^{(1-n)/2})^2 \in L^1(\mathbb{R}^n)$  for  $n \geq 5$ , it follows from the strong convergence (5.24), the pointwise convergence of Step 5.2.2, the pointwise control (5.100), the Lebesgue dominated convergence theorem that

$$\begin{aligned}
 D_{3,\alpha} &= \frac{\mu_\alpha^2}{2} \left( \partial_{ij} \hat{g}^{ij}(0) \int_{B_{\delta\mu_\alpha^{-1}}(0)} \tilde{u}_\alpha^2 dX \right. \\
 &\quad \left. - \partial_{\beta_1 \beta_2} \hat{g}^{ij}(0) \int_{B_{\delta\mu_\alpha^{-1}}(0)} X^{\beta_1} X^{\beta_2} \partial_i \tilde{u}_\alpha \partial_j \tilde{u}_\alpha dX \right) + o(\mu_\alpha^2) \\
 &= \frac{\mu_\alpha^2}{2} \left( \partial_{ij} \hat{g}^{ij}(0) \int_{\mathbb{R}^n} \tilde{u}^2 dX \right. \\
 &\quad \left. - \partial_{\beta_1 \beta_2} \hat{g}^{ij}(0) \int_{\mathbb{R}^n} X^{\beta_1} X^{\beta_2} \partial_i \tilde{u} \partial_j \tilde{u} dX \right) + o(\mu_\alpha^2).
 \end{aligned}$$

**Case 2:  $n = 4$ .** It follows from (5.118) and take  $Y = \mu_\alpha^{-1} X$  that,

$$\begin{aligned}
 D_{3,\alpha} &= \frac{\mu_\alpha^2}{2} \left( \partial_{ij} \hat{g}^{ij}(0) \int_{B_{\delta\mu_\alpha^{-1}}(0)} \tilde{u}_\alpha^2 dX \right. \\
 &\quad \left. - \partial_{\beta_1 \beta_2} \hat{g}^{ij}(0) \int_{B_{\delta\mu_\alpha^{-1}}(0)} X^{\beta_1} X^{\beta_2} \partial_i \tilde{u}_\alpha \partial_j \tilde{u}_\alpha dX \right) + O(\mu_\alpha^2)
 \end{aligned}$$

Withing again the equations (5.105), (5.106), we get

$$\lim_{\alpha \rightarrow +\infty} \frac{1}{\ln(\frac{1}{\mu_\alpha})} \int_{B_{\delta\mu_\alpha^{-1}}(0)} \tilde{u}_\alpha^2 dX = \omega_3 K^4.$$

Then, **IF (5.108) holds**, we get that

$$\lim_{\alpha \rightarrow +\infty} \frac{\mu_\alpha^2}{\ln(\frac{1}{\mu_\alpha})} D_{3,\alpha} = \frac{\omega_3}{12} K^4 (4\partial_{ij} \hat{g}^{ij}(0) - \partial_{\beta_1 \beta_1} \hat{g}^{ii}(0)).$$

**Step 5.5.6.** We estimate  $D_{4,\alpha}$  for  $n \geq 4$ .

Using again the integrations by parts, we get

$$D_{4,\alpha} = -\frac{1}{2} \sum_{k=1}^n \int_{B_\delta(0)} \partial_k \hat{\Gamma}_{ii}^k \hat{u}_\alpha^2 dX + \frac{1}{2} \int_{\partial B_\delta(0)} \hat{\Gamma}_{ii}^k \hat{u}_\alpha^2 \vec{\nu}_k dX + O\left(\int_{B_\delta(0)} |X| \hat{u}_\alpha^2 dX\right).$$

With (5.114), we get

$$D_{4,\alpha} = -\frac{1}{2} \sum_{k=1}^n \partial_k \hat{\Gamma}_{ii}^k(0) \int_{B_\delta(0)} \hat{u}_\alpha^2 dX + \begin{cases} o(\mu_\alpha^2) & \text{if } n \geq 5, \\ O(\mu_\alpha^2) & \text{if } n = 4. \end{cases} \quad (5.119)$$

With a change of variable  $Y = \mu_\alpha^{-1} X$ , we obtain that

$$D_{4,\alpha} = -\frac{\mu_\alpha^2}{2} \sum_{k=1}^n \partial_k \hat{\Gamma}_{ii}^k(0) \int_{B_{\delta\mu_\alpha^{-1}}(0)} \tilde{u}_\alpha^2 dX + \begin{cases} o(\mu_\alpha^2) & \text{if } n \geq 5, \\ O(\mu_\alpha^2) & \text{if } n = 4. \end{cases} \quad (5.120)$$

**Case 1:  $n \geq 5$ .** Here  $X \mapsto (1 + |X|^2)^{1-n/2} \in L^2(\mathbb{R}^n)$ . Then with the pointwise convergence of Step 5.2.2 and the pointwise control (5.99), Lebesgue's dominated convergence theorem yields

$$D_{4,\alpha} = -\frac{\mu_\alpha^2}{2} \partial_k \hat{\Gamma}_{ii}^k(0) \int_{\mathbb{R}^n} \tilde{u}_\alpha^2 dX + o(\mu_\alpha^2).$$

**Case 2:  $n = 4$ .** It follows from (5.105) and (5.106) that,

$$\lim_{\alpha \rightarrow +\infty} \frac{1}{\ln\left(\frac{1}{\mu_\alpha}\right)} \int_{B_{\delta\mu_\alpha^{-1}}(0)} \tilde{u}_\alpha^2 dX = \omega_3 K^4. \quad (5.121)$$

Combining (5.120) and (5.121),

$$D_{4,\alpha} = -\frac{\mu_\alpha^2}{2} \ln\left(\frac{1}{\mu_\alpha}\right) K^4 \partial_k \hat{\Gamma}_{ii}^k(0) (1 + o(1)) \text{ as } \alpha \rightarrow +\infty.$$

**Step 5.5.7.** We now deal with  $D_{1,\alpha}$  for  $n \geq 4$ .

We write

$$b^{ijl} = (g^{ij} - \delta^{ij}) X^l \text{ for all } i, j, l = 1, \dots, n.$$

$$\begin{aligned} D_{1,\alpha} &= \int_{B_\delta(0)} b^{ijl} \partial_l \hat{u}_\alpha \partial_{ij} \hat{u}_\alpha dX \\ &= - \int_{B_\delta(0)} \partial_i b^{ijl} \partial_l \hat{u}_\alpha \partial_j \hat{u}_\alpha dX - \int_{B_\delta(0)} b^{ijl} \partial_j \hat{u}_\alpha \partial_{il} \hat{u}_\alpha dX \\ &\quad + \int_{\partial B_\delta(0)} b^{ijl} \partial_j \hat{u}_\alpha \partial_l \hat{u}_\alpha \vec{\nu}_i dX. \end{aligned} \quad (5.122)$$

Using the integrations by parts and since  $b^{ijl} = b^{jil}$ , we get that

$$\begin{aligned}
 D'_{1,\alpha} &:= - \int_{B_\delta(0)} b^{ijl} \partial_j \hat{u}_\alpha \partial_{il} \hat{u}_\alpha dX \\
 &= \int_{B_\delta(0)} b^{ijl} \partial_{lj} \hat{u}_\alpha \partial_i \hat{u}_\alpha dX + \int_{B_\delta(0)} \partial_l (b^{ijl}) \partial_j \hat{u}_\alpha \partial_i \hat{u}_\alpha dX \\
 &\quad - \int_{\partial B_\delta(0)} b^{ijl} \partial_j \hat{u}_\alpha \partial_i \hat{u}_\alpha \vec{\nu}_l dX \\
 &= \int_{B_\delta(0)} b^{jil} \partial_{li} \hat{u}_\alpha \partial_j \hat{u}_\alpha dX + \int_{B_\delta(0)} \partial_l (b^{ijl}) \partial_j \hat{u}_\alpha \partial_i \hat{u}_\alpha dX \\
 &\quad - \int_{\partial B_\delta(0)} b^{ijl} \partial_j \hat{u}_\alpha \partial_i \hat{u}_\alpha \vec{\nu}_l dX \\
 &= \int_{B_\delta(0)} b^{ijl} \partial_j \hat{u}_\alpha \partial_{il} \hat{u}_\alpha dX + \int_{B_\delta(0)} \partial_l (b^{ijl}) \partial_j \hat{u}_\alpha \partial_i \hat{u}_\alpha dX \\
 &\quad - \int_{\partial B_\delta(0)} b^{ijl} \partial_j \hat{u}_\alpha \partial_i \hat{u}_\alpha \vec{\nu}_l dX \\
 &= -D'_{1,\alpha} + \int_{B_\delta(0)} \partial_l (b^{ijl}) \partial_j \hat{u}_\alpha \partial_i \hat{u}_\alpha dX - \int_{\partial B_\delta(0)} b^{ijl} \partial_j \hat{u}_\alpha \partial_i \hat{u}_\alpha \vec{\nu}_l dX,
 \end{aligned}$$

then,

$$2D'_{1,\alpha} = \int_{B_\delta(0)} \partial_l b^{ijl} \partial_j \hat{u}_\alpha \partial_i \hat{u}_\alpha dX - \int_{\partial B_\delta(0)} b^{ijl} \partial_j \hat{u}_\alpha \partial_i \hat{u}_\alpha \vec{\nu}_l dX. \quad (5.123)$$

Combining (5.122) and (5.123), we get

$$\begin{aligned}
 D_{1,\alpha} &= - \int_{B_\delta(0)} \partial_i b^{ijl} \partial_l \hat{u}_\alpha \partial_j \hat{u}_\alpha dX + \frac{1}{2} \int_{B_\delta(0)} \partial_l b^{ijl} \partial_j \hat{u}_\alpha \partial_i \hat{u}_\alpha dX \\
 &\quad + \int_{\partial B_\delta(0)} b^{ijl} \partial_j \hat{u}_\alpha \partial_l \hat{u}_\alpha \vec{\nu}_i dX - \frac{1}{2} \int_{\partial B_\delta(0)} b^{ijl} \partial_j \hat{u}_\alpha \partial_i \hat{u}_\alpha \vec{\nu}_l dX \quad (5.124)
 \end{aligned}$$

With (5.98), we get

$$\int_{\partial B_\delta(0)} b^{ijl} \partial_j \hat{u}_\alpha \partial_l \hat{u}_\alpha \vec{\nu}_i dX = O(\mu_\alpha^{n-2}).$$

Therefore, thanks of (5.112) and (5.113), we obtain that

$$\begin{aligned}
 D_{1,\alpha} &= - \int_{B_\delta(0)} \partial_i b^{ijl} \partial_l \hat{u}_\alpha \partial_j \hat{u}_\alpha dX + \frac{1}{2} \int_{B_\delta(0)} \partial_l b^{ijl} \partial_j \hat{u}_\alpha \partial_i \hat{u}_\alpha dX + O(\mu_\alpha^{n-2}) \\
 &= - \int_{B_\delta(0)} X^l \partial_i \hat{g}^{ij} \partial_l \hat{u}_\alpha \partial_j \hat{u}_\alpha dX - \int_{B_\delta(0)} (\hat{g}^{ij} - \delta^{ij}) \delta^{il} \partial_l \hat{u}_\alpha \partial_j \hat{u}_\alpha dX \\
 &\quad + \frac{1}{2} \int_{B_\delta(0)} X^l \partial_l \hat{g}^{ij} \partial_j \hat{u}_\alpha \partial_i \hat{u}_\alpha dX + \frac{n}{2} \int_{B_\delta(0)} (\hat{g}^{ij} - \delta^{ij}) \partial_j \hat{u}_\alpha \partial_i \hat{u}_\alpha dX + O(\mu_\alpha^{n-2}) \\
 &= -\partial_{i\beta_1} \hat{g}^{ij}(0) \int_{B_\delta(0)} X^{\beta_1} X^l \partial_l \hat{u}_\alpha \partial_j \hat{u}_\alpha dX - \frac{1}{2} \partial_{\beta_1 \beta_2} \hat{g}^{ij}(0) \int_{B_\delta(0)} X^{\beta_1} X^{\beta_2} \delta^{il} \partial_l \hat{u}_\alpha \partial_j \hat{u}_\alpha dX \\
 &\quad + \frac{1}{2} \partial_{l\beta_1} \hat{g}^{ij}(0) \int_{B_\delta(0)} X^{\beta_1} X^l \partial_i \hat{u}_\alpha \partial_j \hat{u}_\alpha dX + \frac{n}{4} \partial_{\beta_1 \beta_2} \hat{g}^{ij}(0) \int_{B_\delta(0)} X^{\beta_1} X^{\beta_2} \partial_i \hat{u}_\alpha \partial_j \hat{u}_\alpha dX \\
 &\quad + O\left(\int_{B_\delta(0)} |X|^3 |\nabla \hat{u}_\alpha|^2 dX\right) + O(\mu_\alpha^{n-2})
 \end{aligned}$$

$$\begin{aligned}
&= -\partial_{i\beta_1}\hat{g}^{ij}(0) \int_{B_\delta(0)} X^{\beta_1} X^l \partial_l \hat{u}_\alpha \partial_j \hat{u}_\alpha dX - \frac{1}{2} \partial_{\beta_1\beta_2}\hat{g}^{ij}(0) \int_{B_\delta(0)} X^{\beta_1} X^{\beta_2} \partial_i \hat{u}_\alpha \partial_j \hat{u}_\alpha dX \\
&\quad + \frac{1}{2} \partial_{l\beta_1}\hat{g}^{ij}(0) \int_{B_\delta(0)} X^{\beta_1} X^l \partial_i \hat{u}_\alpha \partial_j \hat{u}_\alpha dX + \frac{n}{4} \partial_{\beta_1\beta_2}\hat{g}^{ij}(0) \int_{B_\delta(0)} X^{\beta_1} X^{\beta_2} \partial_i \hat{u}_\alpha \partial_j \hat{u}_\alpha dX \\
&\quad + \begin{cases} o(\mu_\alpha^2) & \text{if } n \geq 5, \\ O(\mu_\alpha^2) & \text{if } n = 4. \end{cases}
\end{aligned}$$

With (5.113), we observe that

$$\begin{aligned}
D_{1,\alpha} &= -\partial_{i\beta_1}\hat{g}^{ij}(0) \int_{B_\delta(0)} X^{\beta_1} X^l \partial_l \hat{u}_\alpha \partial_j \hat{u}_\alpha dX - \frac{1}{2} \partial_{\beta_1\beta_2}\hat{g}^{ij}(0) \int_{B_\delta(0)} X^{\beta_1} X^{\beta_2} \partial_i \hat{u}_\alpha \partial_j \hat{u}_\alpha dX \\
&\quad + \frac{1}{2} \partial_{l\beta_1}\hat{g}^{ij}(0) \int_{B_\delta(0)} X^{\beta_1} X^l \partial_i \hat{u}_\alpha \partial_j \hat{u}_\alpha dX \tag{5.125} \\
&\quad + \frac{n}{4} \partial_{\beta_1\beta_2}\hat{g}^{ij}(0) \int_{B_\delta(0)} X^{\beta_1} X^{\beta_2} \partial_i \hat{u}_\alpha \partial_j \hat{u}_\alpha dX + \begin{cases} o(\mu_\alpha^2) & \text{if } n \geq 5, \\ O(\mu_\alpha^2) & \text{if } n = 4. \end{cases}
\end{aligned}$$

With the change of variable  $Y = \mu_\alpha^{-1}X$ , we get

$$\begin{aligned}
D_{1,\alpha} &= \mu_\alpha^2 \left( -\partial_{i\beta_1}\hat{g}^{ij}(0) \int_{B_{\delta\mu_\alpha^{-1}}(0)} X^{\beta_1} X^l \partial_l \tilde{u}_\alpha \partial_j \tilde{u}_\alpha dX \right. \\
&\quad \left. + \frac{n}{4} \partial_{\beta_1\beta_2}\hat{g}^{ij}(0) \int_{B_{\delta\mu_\alpha^{-1}}(0)} X^{\beta_1} X^{\beta_2} \partial_i \tilde{u}_\alpha \partial_j \tilde{u}_\alpha dX \right) \tag{5.126} \\
&\quad + \begin{cases} o(\mu_\alpha^2) & \text{if } n \geq 5, \\ O(\mu_\alpha^2) & \text{if } n = 4. \end{cases}
\end{aligned}$$

**Case 1:  $n \geq 5$ .** We have that  $X \mapsto |X|^2 (1 + |X|^{n-1})^{-2} \in L^1(\mathbb{R}^n)$ . Therefore, the strong convergence (5.24), the pointwise convergence of Step 5.2.2, the pointwise control (5.100) and Lebesgue's Convergence Theorem yield

$$\begin{aligned}
D_{1,\alpha} &= \mu_\alpha^2 \left( -\partial_{i\beta_1}\hat{g}^{ij}(0) \int_{\mathbb{R}^n} X^{\beta_1} X^l \partial_l \tilde{u} \partial_j \tilde{u} dX \right. \\
&\quad \left. + \frac{n}{4} \partial_{\beta_1\beta_2}\hat{g}^{ij}(0) \int_{\mathbb{R}^n} X^{\beta_1} X^{\beta_2} \partial_i \tilde{u} \partial_j \tilde{u} dX \right) + o(\mu_\alpha^2),
\end{aligned}$$

Moreover, since  $\tilde{u}$  is a radially symmetrical, we get

$$\begin{aligned}
 D_{1,\alpha} &= \mu_\alpha^2 \left( -\partial_{i\beta_1} \hat{g}^{ij}(0) \int_{\mathbb{R}^n} X^{\beta_1} X^j (\tilde{u}')^2 dX \right. \\
 &\quad \left. + \frac{n}{4} \partial_{\beta_1\beta_2} \hat{g}^{ij}(0) \int_{\mathbb{R}^n} X^{\beta_1} X^{\beta_2} \partial_i \tilde{u} \partial_j \tilde{u} dX \right) + o(\mu_\alpha^2) \\
 &= \mu_\alpha^2 \left( -\partial_{i\beta_1} \hat{g}^{ij}(0) \int_{S^{n-1}} \theta^{\beta_1} \theta^j d\theta \int_0^{+\infty} r^{n+1} |\nabla_r \tilde{u}|^2 dr \right. \\
 &\quad \left. + \frac{n}{4} \partial_{\beta_1\beta_2} \hat{g}^{ij}(0) \int_{\mathbb{R}^n} X^{\beta_1} X^{\beta_2} \partial_i \tilde{u} \partial_j \tilde{u} dX \right) + o(\mu_\alpha^2) \\
 &= \mu_\alpha^2 \left( -\frac{1}{n} \omega_{n-1} \partial_{\beta_1 i} \hat{g}^{ij}(0) \delta^{\beta_1 j} \int_0^{+\infty} r^{n+1} |\nabla_r \tilde{u}|^2 dr \right. \\
 &\quad \left. + \frac{n}{4} \partial_{\beta_1\beta_2} \hat{g}^{ij}(0) \int_{\mathbb{R}^n} X^{\beta_1} X^{\beta_2} \partial_i \tilde{u} \partial_j \tilde{u} dX \right) + o(\mu_\alpha^2),
 \end{aligned}$$

then,

$$\begin{aligned}
 D_{1,\alpha} &= \mu_\alpha^2 \left( -\frac{1}{n} \partial_{ij} \hat{g}^{ij}(0) \int_{\mathbb{R}^n} |X|^2 |\nabla \tilde{u}|^2 dX \right. \\
 &\quad \left. + \frac{n}{4} \partial_{\beta_1\beta_2} \hat{g}^{ij}(0) \int_{\mathbb{R}^n} X^{\beta_1} X^{\beta_2} \partial_i \tilde{u} \partial_j \tilde{u} dX \right) + o(\mu_\alpha^2).
 \end{aligned}$$

**Case 2:  $n = 4$ . IF (5.108) holds,** we have

$$\lim_{\alpha \rightarrow +\infty} \frac{\mu_\alpha^{-2}}{\ln(\frac{1}{\mu_\alpha})} D_{1,\alpha} = \frac{\omega_3}{6} K^4 (-4 \partial_{ij} \hat{g}^{ij}(0) + \partial_{\beta_1\beta_1} \hat{g}^{ii}(0)).$$

**Step 5.5.8.** We get as  $\alpha \rightarrow +\infty$  that,

$$D_\alpha = \begin{cases} O(\delta\mu_\alpha) & \text{if } n = 3, \\ -\mu_\alpha^2 \ln(\frac{1}{\mu_\alpha}) \frac{1}{6} \text{Scal}_g(x_0) \omega_3 K^4 (1 + o(1)) & \text{if } n = 4 \text{ and (5.108) holds,} \\ -\mu_\alpha^2 c_{n,s} \text{Scal}_g(x_0) \int_{\mathbb{R}^n} \tilde{u}^2 dX + o(\mu_\alpha^2) & \text{if } n \geq 5. \end{cases}$$

where  $c_{n,s}$ ,  $K$  are defined in (5.12), (5.20).

*Proof of Step 5.5.8:* For  $n \geq 5$ , the steps above yield

$$\begin{aligned}
& D_{1,\alpha} + \frac{n-2}{2}D_{3,\alpha} \\
&= \mu_\alpha^2 \left( -\frac{1}{n} \partial_{ij} \hat{g}^{ij}(0) \int_{\mathbb{R}^n} |X|^2 |\nabla \tilde{u}|^2 dX + \frac{n-2}{4} \partial_{ij} \hat{g}^{ij}(0) \int_{\mathbb{R}^n} \tilde{u}^2 dX \right. \\
&\quad \left. + \frac{1}{2} \partial_{\beta_1 \beta_2} \hat{g}^{ij}(0) \int_{\mathbb{R}^n} X^{\beta_1} X^{\beta_2} \partial_i \tilde{u} \partial_j \tilde{u} dX \right) + o(\mu_\alpha^2) \\
&= \mu_\alpha^2 \left( -\frac{1}{n} \partial_{ij} \hat{g}^{ij}(0) \int_{\mathbb{R}^n} |X|^2 |\nabla \tilde{u}|^2 dX + \frac{n-2}{4} \partial_{ij} \hat{g}^{ij}(0) \int_{\mathbb{R}^n} \tilde{u}^2 dX \right. \\
&\quad \left. + \frac{w_{n-1}^{-1}}{2} \partial_{\beta_1 \beta_2} \hat{g}^{ij}(0) \int_{S^{n-1}} \sigma^i \sigma^j \sigma^{\beta_1} \sigma^{\beta_2} d\sigma \int_{\mathbb{R}^n} |X|^2 |\nabla \tilde{u}|^2 dX \right) + o(\mu_\alpha^2).
\end{aligned}$$

It follows from [13] that

$$\int_{S^{n-1}} \sigma^i \sigma^j \sigma^{\beta_1} \sigma^{\beta_2} d\sigma = \frac{1}{n(n+2)} w_{n-1} (\delta^{ij} \delta^{\beta_1 \beta_2} + \delta^{i\beta_1} \delta^{j\beta_2} + \delta^{i\beta_2} \delta^{j\beta_1}).$$

Therefore we get

$$\begin{aligned}
& D_{1,\alpha} + \frac{n-2}{2}D_{3,\alpha} \\
&= \mu_\alpha^2 \left( \frac{1}{n} \left( -\partial_{ij} \hat{g}^{ij}(0) + \frac{1}{2(n+2)} (\partial_{\beta_1 \beta_1} \hat{g}^{ii}(0) + 2\partial_{ij} \hat{g}^{ij}(0)) \right) \int_{\mathbb{R}^n} |X|^2 |\nabla \tilde{u}|^2 dX \right. \\
&\quad \left. + \frac{n-2}{4} \partial_{ij} \hat{g}^{ij}(0) \int_{\mathbb{R}^n} \tilde{u}^2 dX \right) + o(\mu_\alpha^2) \\
&= \mu_\alpha^2 \left( \frac{1}{n} \left( -\frac{n+1}{(n+2)} \partial_{ij} \hat{g}^{ij}(0) + \frac{1}{2(n+2)} \partial_{\beta_1 \beta_1} \hat{g}^{ii}(0) \right) \int_{\mathbb{R}^n} |X|^2 |\nabla \tilde{u}|^2 dX \right. \\
&\quad \left. + \frac{n-2}{4} \partial_{ij} \hat{g}^{ij}(0) \int_{\mathbb{R}^n} \tilde{u}^2 dX \right) + o(\mu_\alpha^2).
\end{aligned}$$

Therefore, using the definition of  $D_\alpha$ , we get

$$\begin{aligned}
D_\alpha &= D_{1,\alpha} - D_{2,\alpha} + \frac{n-2}{2}D_{3,\alpha} - \frac{n-2}{2}D_{4,\alpha} \\
&= \mu_\alpha^2 \left( \frac{1}{n} \left( -\frac{n+1}{(n+2)} \partial_{ij} \hat{g}^{ij}(0) \right. \right. \\
&\quad \left. \left. + \frac{1}{2(n+2)} \partial_{\beta_1 \beta_1} \hat{g}^{ii}(0) - \partial_k \hat{\Gamma}_{ii}^k(0) \right) \int_{\mathbb{R}^n} |X|^2 |\nabla \tilde{u}|^2 dX \right. \\
&\quad \left. + \frac{n-2}{4} \left( \partial_{ij} \hat{g}^{ij}(0) + \partial_k \hat{\Gamma}_{ii}^k(0) \right) \int_{\mathbb{R}^n} \tilde{u}^2 dX \right) + o(\mu_\alpha^2). \quad (5.127)
\end{aligned}$$

Since  $\hat{g}^{ij}\hat{g}_{ij} = Id_n$  and  $\partial_k\hat{g}^{ij}(0) = 0$ , we get

$$\partial_{ij}\hat{g}^{ij}(0) = -\partial_{ij}\hat{g}_{ij}(0) \text{ for } i, j = 1, \dots, n. \quad (5.128)$$

Combining (5.127) and (5.128), we obtain that

$$\begin{aligned} D_\alpha &= \mu_\alpha^2 \left( \frac{1}{n} \left( \frac{n+1}{n+2} \partial_{ij}\hat{g}_{ij}(0) \right. \right. \\ &\quad \left. \left. - \frac{1}{2(n+2)} \partial_{\beta_1\beta_1}\hat{g}_{ii}(0) - \partial_k\hat{\Gamma}_{ii}^k(0) \right) \int_{\mathbb{R}^n} |X|^2 |\nabla\tilde{u}|^2 dX \right. \\ &\quad \left. + \frac{n-2}{4} \left( -\partial_{ij}\hat{g}_{ij}(0) + \partial_k\hat{\Gamma}_{ii}^k(0) \right) \int_{\mathbb{R}^n} \tilde{u}^2 dX \right) + o(\mu_\alpha^2). \end{aligned} \quad (5.129)$$

By Jaber [74], for  $s \in (0, 2)$  we have that

$$\frac{\int_{\mathbb{R}^n} |X|^2 |\nabla\tilde{u}|^2 dX}{\int_{\mathbb{R}^n} \tilde{u}^2 dX} = \frac{n(n-2)(n+2-s)}{2(2n-2-s)}. \quad (5.130)$$

On the other hand, Cartan's expansion of the metric  $g$  in the exponential chart  $(B_\delta(x_0), \exp_{x_0}^{-1})$  yields

$$g_{ij}(x) = \delta_{ij} + \frac{1}{3}R_{ipqj}(x_0)x^p x^q + O(r^3),$$

where  $r := d_g(x, x_0)$ . Since  $g$  is  $C^\infty$ , we have that

$$\begin{aligned} \partial_{\beta_1\beta_2}g_{ij}(x_0) &= \frac{1}{3}(R_{ipqj}(x_0)\delta_{p\beta_2}\delta_{q\beta_1} + R_{ipqj}(x_0)\delta_{p\beta_1}\delta_{q\beta_2}) \\ &= \frac{1}{3}(R_{i\beta_2\beta_1j}(x_0) + R_{i\beta_1\beta_2j}(x_0)). \end{aligned} \quad (5.131)$$

The Bianchi identities and the symmetry yields  $R_{iijj} = 0$  and  $R_{ij\alpha\beta} = -R_{ij\beta\alpha}$ . Since  $R_{ijij} = Scal_g(x_0)$ , we then get that

$$\sum_{i,j=1}^n \partial_{ij}g_{ij}(x_0) = \frac{1}{3}Scal_g(x_0) \text{ and } \sum_{i,\beta_1=1}^n \partial_{\beta_1\beta_1}g_{ii}(x_0) = -\frac{2}{3}Scal_g(x_0). \quad (5.132)$$

Now, using the Christoffel symbols and  $\partial_k g^{ij}(0) = 0$ , we obtain that

$$\begin{aligned} \partial_k\Gamma_{ii}^k(x_0) &= \frac{1}{2}(\partial_{ki}g_{ik} + \partial_{ki}g_{ik} - \partial_{kk}g_{ii})(x_0) \\ &= \frac{1}{6}(R_{iikk} + R_{ikik} - R_{iikk} + R_{ikik} - 2R_{ikki}), \end{aligned}$$

then we have

$$\sum_{i,k=1}^n \partial_k \Gamma_{ij}^k(x_0) = \frac{2}{3} Scal_g(x_0). \quad (5.133)$$

Combining (5.129), (5.130), (5.132) and (5.133),

$$\begin{aligned} D_\alpha &= \mu_\alpha^2 \left( \left( \frac{n+1}{n+2} \partial_{ij} \hat{g}_{ij}(0) - \frac{1}{2(n+2)} \partial_{\beta_1 \beta_1} \hat{g}_{ii}(0) - \partial_k \hat{\Gamma}_{ij}^k(0) \right) \frac{(n-2)(n+2-s)}{2(2n-2-s)} \right. \\ &\quad \left. + \frac{n-2}{4} \left( -\partial_{ij} \hat{g}_{ij}(0) + \partial_k \hat{\Gamma}_{ii}^k(0) \right) \right) \int_{\mathbb{R}^n} \tilde{u}^2 dX + o(\mu_\alpha^2) \\ &= \mu_\alpha^2 \left( \left( \frac{n+1}{3(n+2)} + \frac{1}{3(n+2)} - \frac{2}{3} \right) Scal_g(x_0) \frac{(n-2)(n+2-s)}{2(2n-2-s)} \right. \\ &\quad \left. + \frac{n-2}{4} Scal_g(x_0) \left( -\frac{1}{3} + \frac{2}{3} \right) \right) \int_{\mathbb{R}^n} \tilde{u}^2 dX + o(\mu_\alpha^2) \\ &= \mu_\alpha^2 Scal_g(x_0) \left( -\frac{(n-2)(n+2-s)}{6(2n-2-s)} + \frac{n-2}{12} \right) \int_{\mathbb{R}^n} \tilde{u}^2 dX + o(\mu_\alpha^2) \end{aligned}$$

This ends Step 5.5.8 for  $n \geq 5$ . The analysis is similar when  $n = 4$  if (5.108) holds.  $\square$

**Step 5.5.9.** We prove Theorem 5.1.4 for  $n \geq 5$  and for  $n = 4$  provided that (5.108) holds.

First, using the definitions (5.95) of  $B_\alpha$ ,  $C_\alpha$  and  $D_\alpha$  and thanks of the Step 5.5.2 to 5.5.8, we get

$$C_\alpha + D_\alpha = \begin{cases} \mu_\alpha^2 (a_\infty(x_0) - c(n, s) Scal_g(x_0)) \int_{\mathbb{R}^n} \tilde{u}^2 dX + o(\mu_\alpha^2) & \text{if } n \geq 5, \\ \mu_\alpha^2 \ln\left(\frac{1}{\mu_\alpha}\right) \left( (a_\infty(x_0) - \frac{1}{6} Scal_g(x_0)) \omega_3 K^4 + o(1) \right) & \text{if } n = 4 \text{ and (5.108) holds,} \\ O(\delta \mu_\alpha) & \text{if } n = 3. \end{cases} \quad (5.134)$$

We distinguish three cases:

**Case 1:** If  $n \geq 5$ , (5.96) and (5.134) yield

$$(a_\infty(x_0) - c(n, s) Scal_g(x_0)) \int_{\mathbb{R}^n} \tilde{u}^2 dX = O(\mu_\alpha^{n-4}) = o(1)$$

and then  $a_\infty(x_0) = c(n, s) Scal_g(x_0)$ , with  $c(n, s)$  as in (5.12).

**Case 2:** If  $n = 4$ , the proof is similar.

**Step 5.5.10.** We prove Theorem 5.1.4 when  $n = 3$ .



*Step 5.5.10.1:* We claim that

$$C_\alpha + D_\alpha = O(\delta\mu_\alpha) \text{ as } \alpha \rightarrow +\infty. \quad (5.135)$$

We prove the claim. It follows from (5.101) that

$$C_\alpha = O\left(\int_{B_\delta(0)} \hat{u}_\alpha^2 dx + \int_{\partial B_\delta(0)} |X| \hat{u}_\alpha^2 d\sigma\right)$$

as  $\alpha \rightarrow +\infty$ . The definitions (5.110) of  $D_{i,\alpha}$ ,  $i = 2, 4$  yield

$$D_{2,\alpha} = O\left(\int_{B_\delta(0)} |X|^2 |\nabla \hat{u}_\alpha|^2 dx\right) \text{ and } D_{4,\alpha} = O\left(\int_{B_\delta(0)} |X| \cdot |\nabla \hat{u}_\alpha| \hat{u}_\alpha dx\right).$$

The identity (5.124) yields

$$D_{1,\alpha} = O\left(\int_{B_\delta(0)} |X|^2 |\nabla \hat{u}_\alpha|^2 dx + \int_{\partial B_\delta(0)} |X|^3 |\nabla \hat{u}_\alpha|^2 d\sigma\right).$$

It follows from (5.117) that

$$D_{3,\alpha} = O\left(\int_{B_\delta(0)} (\hat{u}_\alpha^2 + |X|^2 |\nabla \hat{u}_\alpha|^2) dx + \int_{\partial B_\delta(0)} (|X|^3 |\nabla \hat{u}_\alpha|^2 + |X| \hat{u}_\alpha^2) d\sigma\right).$$

Therefore, with (5.109), we get that

$$C_\alpha + D_\alpha = O\left(\int_{B_\delta(0)} (\hat{u}_\alpha^2 + |X|^2 |\nabla \hat{u}_\alpha|^2) dx + \int_{\partial B_\delta(0)} (|X|^3 |\nabla \hat{u}_\alpha|^2 + |X| \hat{u}_\alpha^2) d\sigma\right).$$

It then follows from (5.61) and (5.83) that

$$C_\alpha + D_\alpha = O\left(\mu_\alpha \int_{B_\delta(0)} |X|^{-2} dx + \mu_\alpha \int_{\partial B_\delta(0)} |X|^{-1} d\sigma\right) = O(\delta\mu_\alpha)$$

since  $n = 3$ . This proves (5.135).

*Step 5.5.10.2:* We write the Green's function as in (5.13) with  $\beta_{x_0} \in C^2(M \setminus \{x_0\}) \cap C^{0,\theta}(M)$  where  $\theta \in (0, 1)$ . In particular,

$$\hat{G}_{x_0}(x) := G(x_0, \exp_{x_0}(X)) = \frac{1}{4\pi|X|} + \beta_{x_0}(\exp_{x_0}(X)) \text{ for all } x \in B_\delta(0). \quad (5.136)$$

Combining (5.96) and (5.134), we get that

$$d_3^2 \int_{\partial B_\delta(0)} \delta \left( \frac{|\nabla \hat{G}_{x_0}|^2}{2} + \hat{a}_\infty \frac{\hat{G}_{x_0}^2}{2} \right) - \frac{1}{\delta} \left( \langle X, \nabla \hat{G}_{x_0} \rangle^2 + \frac{1}{2} \langle X, \nabla \hat{G}_{x_0} \rangle \hat{G}_{x_0} \right) d\sigma = O(\delta) \quad (5.137)$$

From (5.136), we denote that:

$$\begin{aligned} |\nabla \hat{G}_{x_0}|^2 &= \frac{1}{16\pi^2\delta^4} + |\nabla \beta_{x_0}|^2 - \frac{1}{2\pi\delta^3} \langle X, \nabla \beta_{x_0} \rangle, \\ \hat{G}_{x_0}^2 &= \frac{1}{16\pi^2\delta^2} + \beta_{x_0}^2 + \frac{1}{2\pi\delta} \langle X, \beta_{x_0} \rangle, \\ \langle X, \nabla \hat{G}_{x_0} \rangle^2 &= \frac{1}{16\pi^2\delta^2} + \langle X, \nabla \beta_{x_0} \rangle^2 - \frac{1}{2\pi\delta} \langle X, \nabla \beta_{x_0} \rangle, \\ \langle X, \nabla \hat{G}_{x_0} \rangle \hat{G}_{x_0} &= -\frac{1}{16\pi^2\delta^2} - \frac{1}{4\pi\delta} \beta_{x_0} + \frac{\langle X, \nabla \beta_{x_0} \rangle}{4\pi\delta} + \langle X, \nabla \beta_{x_0} \rangle \beta_{x_0}. \end{aligned}$$

Let's replace all the terms in (5.137), we get

$$\begin{aligned} & d_n^2 \int_{\partial B_\delta(0)} \delta \left( \frac{|\nabla \beta_{x_0}|^2}{2} + \hat{a}_\infty \frac{\beta_{x_0}^2}{2} \right) - \frac{1}{4\pi\delta^2} \langle X, \nabla \beta_{x_0} \rangle + \hat{a}_\infty \left( \frac{1}{16\pi^2\delta} + \frac{1}{2\pi} \langle X, \beta_{x_0} \rangle \right) \\ & - \frac{1}{\delta} \left( \langle X, \nabla \beta_{x_0} \rangle^2 - \frac{1}{2\pi\delta} \langle X, \nabla \beta_{x_0} \rangle + \frac{1}{2} \left( \frac{-1}{4\pi\delta} \beta_{x_0} + \frac{\langle X, \nabla \beta_{x_0} \rangle}{4\pi\delta} + \langle X, \nabla \beta_{x_0} \rangle \beta_{x_0} \right) \right) d\sigma = O(\delta). \end{aligned}$$

We note that,

$$\lim_{\delta \rightarrow 0} \sup_{X \in \partial B_\delta(0)} \langle X, \nabla \beta_{x_0} \rangle = 0.$$

We multiply the last equation by  $\delta^2$  and passing the limit  $\delta \rightarrow 0$ , we get that  $\beta_{x_0}(x_0) := \beta_{x_0}(\exp_{x_0}(0)) = 0$ , so the mass vanishes at  $x_0$ .  $\square$

## 5.6 Proof of Theorem 5.1.2

We assume that there is no extremal of (5.3), i.e. for all  $u \in H_1^2(M) \setminus \{0\}$ , we have that

$$\|u\|_{2^*(s)}^2 < \mu_s(\mathbb{R}^n)^{-1} \left( \int_M |\nabla u|^2 dv_g + B_s(g) \int_M u^2 dv_g \right). \quad (5.138)$$

We define  $a_\alpha(x) := B_s(g) - \frac{1}{\alpha} > 0$  for all  $x \in M$  and  $\alpha > 0$  large. We define the functional by

$$J_\alpha(u) = \frac{\int_M (|\nabla u|^2 dv_g + a_\alpha u^2) dv_g}{\left( \int_M \frac{u^{2^*(s)}}{d_g(x, x_0)^s} dv_g \right)^{\frac{2}{2^*(s)}}} \text{ for } u \in H_1^2(M) \setminus \{0\}.$$

Therefore, it follows from the definition of  $B_s(g)$ , there exists  $w \in H_1^2(M) \setminus \{0\}$  such that  $J_\alpha(w) < \mu_s(\mathbb{R}^n)$ , and therefore

$$\inf_{u \in N_s(M)} J_\alpha(u) < \mu_s(\mathbb{R}^n), \quad (5.139)$$

where

$$N_s(M) := \{u \in H_1^2(M), \|u\|_{2^*(s)} = 1\}.$$

Set

$$\lambda_\alpha := \inf_{u \in N_s(M)} J_\alpha(u).$$

By the assumption (5.139) leads to the existence of a non negative minimzer  $u_\alpha \in N_s(M)$  for  $\lambda_\alpha$ . The Euler-Lagrange's equation for  $u_\alpha$  is then

$$\Delta_g u_\alpha + a_\alpha u_\alpha = \lambda_\alpha \frac{u_\alpha^{2^*(s)-1}}{d_g(x, x_0)^s}. \quad (5.140)$$

It follows from the regularity and the maximum principle of Jaber [74] that  $u_\alpha \in C^{0, \beta_1}(M) \cap C_{loc}^{2, \beta_2}(M \setminus \{x_0\})$ ,  $\beta_1 \in (0, \min(1, 2 - s))$ ,  $\beta_2 \in (0, 1)$  and  $u_\alpha > 0$ .

**Step 5.6.1.** *We claim that,*

$$u_\alpha \rightharpoonup 0 \text{ weakly in } H_1^2(M) \text{ as } \alpha \rightarrow +\infty.$$

*Proof of Step 5.6.1:* For any  $\alpha > 0$ , we have  $\|u_\alpha\|_{2^*(s)} = 1$  and  $J_\alpha(u_\alpha) = \lambda_\alpha < \mu_s(\mathbb{R}^n)$ , and we get  $(u_\alpha)_{\alpha > 0}$  is bounded in  $H_1^2(M)$ . Then, there exists  $u_0 \in H_1^2(M)$  such that  $u_\alpha \rightharpoonup u_0$  in  $H_1^2(M)$  as  $\alpha \rightarrow +\infty$ . If  $u_0 \not\equiv 0$ , taking the limit in equation (5.140), we get

$$\Delta_g u_0 + B_s(g)u_0 = \lambda \frac{u_0^{2^*(s)-1}}{d_g(x, x_0)^s}. \quad (5.141)$$

By the inequality (5.138) and (5.141), we have

$$\mu_s(\mathbb{R}^n) < \frac{\left(\int_M |\nabla u_0|^2 dv_g + B_s(g) \int_M u_0^2 dv_g\right)}{\|u_0\|_{2^*(s)}^2} = \lambda \left(\int_M \frac{u_0^{2^*(s)}}{d_g(x, x_0)^s}\right)^{1 - \frac{2}{2^*(s)}}$$

Since  $\lambda \leq \mu_s(\mathbb{R}^n)$  and

$$\int_M \frac{u_0^{2^*(s)}}{d_g(x, x_0)^s} dv_g \leq \liminf_{\alpha \rightarrow +\infty} \int_M \frac{u_\alpha^{2^*(s)}}{d_g(x, x_0)^s} dv_g = 1.$$

We get that,  $\lambda = \mu_s(\mathbb{R}^n)$ . Therefore,  $u_0$  is a nonzero extremal function of (5.138) contradiction. Hence  $u_0 \equiv 0$ . □

**Step 5.6.2.** *We claim that,*

$$\lambda_\alpha \rightarrow \mu_s(\mathbb{R}^n) \text{ as } \alpha \rightarrow +\infty.$$

*Proof of Step 5.6.2:* Since for all  $\alpha > 0$ , we have  $0 < \lambda_\alpha < \mu_s(\mathbb{R}^n)$  then, up to a subsequence,  $\lambda_\alpha \rightarrow \lambda \leq \mu_s(\mathbb{R}^n)$  as  $\alpha \rightarrow +\infty$ . We proceed by contradiction and assume that  $\lambda \neq \mu_s(\mathbb{R}^n)$ . Then there exists  $\epsilon_0$  and  $\alpha_0 > 0$  such that for all  $\alpha > \alpha_0$ ,

$$\mu_s(\mathbb{R}^n) > \lambda + \epsilon_0.$$

Thanks of Jaber [74], there exists  $B_1$  such that for all  $\alpha > 0$ , we have

$$\left( \int_M \frac{|u_\alpha|^{2^*(s)}}{d_g(x, x_0)^s} dv_g \right)^{\frac{2}{2^*(s)}} \leq \mu_s(\mathbb{R}^n)^{-1} \int_M |\nabla u_\alpha|_g^2 dv_g + B_1 \int_M u_\alpha^2 dv_g.$$

By the last Step and since the embedding of  $H_1^2(M)$  in  $L^2(M)$  is compact,

$$u_\alpha \rightarrow 0 \text{ in } L^2(M) \text{ as } \alpha \rightarrow +\infty.$$

Therefore,  $\|u_\alpha\|_{2^*(s)} = 1$  and  $J_\alpha(u_\alpha) = \lambda_\alpha$ , we have

$$1 \leq \frac{\lambda_\alpha}{\lambda + \epsilon_0} + o(1).$$

Letting  $\alpha \rightarrow +\infty$  in the last relation, we obtain that  $\frac{\lambda}{\lambda + \epsilon_0} \geq 1$ , a contradiction since  $\lambda \geq 0$  and  $\epsilon_0 > 0$ .  $\square$

We are in position to prove Theorem (5.1.4). Since  $u_\alpha$  above satisfies the hypothesis of Theorem (5.1.4), we have that  $B_s(g) = c_{n,s} \text{Scal}_g(x_0)$  if  $n > 4$  and  $m_{B_s(g)}(x_0) = 0$  if  $n=3$ .  $\square$ .

## 5.7 Appendix

These results and their proofs are closely to the work of Jaber [75]. In the sequel, we fix  $\delta_0 \in (0, i_g(M))$  where  $i_g(M) > 0$  is the injectivity radius of  $(M, g)$ . We fix  $\eta_0 \in C^\infty(B_{\frac{3\delta_0}{4}}(0) \subset \mathbb{R}^n)$  such that  $\eta \equiv 1$  in  $B_{\frac{\delta_0}{2}}(0)$ .

**Theorem 5.7.1.** *We let  $(u_\alpha)_{\alpha>0}$  be as in (5.7). We consider a sequence  $(z_\alpha)_{\alpha>0} \in M$  such that  $\lim_{\alpha \rightarrow +\infty} z_\alpha = x_0$ . We define the function*

$$\tilde{u}_\alpha(X) := \mu_\alpha^{\frac{n-2}{2}} u_\alpha(\exp_{z_\alpha}(\mu_\alpha X)) \text{ for all } X \in B_{\mu_\alpha^{-1}\delta_0}(0) \subset \mathbb{R}^n, \quad (5.142)$$

where  $\exp_{z_\alpha} : B_{\delta_0}(0) \rightarrow B_{\delta_0}(z_\alpha) \subset M$  is the exponential map at  $z_\alpha$ . We assume that

$$d_g(x_\alpha, z_\alpha) = O(\mu_\alpha) \text{ when } \alpha \rightarrow +\infty.$$

Then,

$$d_g(z_\alpha, x_0) = O(\mu_\alpha) \text{ when } \alpha \rightarrow +\infty,$$

and, up to a subsequence,  $\eta_\alpha \tilde{u}_\alpha$  converge to  $\tilde{u}$  weakly in  $D_1^2(\mathbb{R}^n)$  and uniformly in  $C_{loc}^{0,\beta}(\mathbb{R}^n)$ , for all  $\beta \in (0, \min\{1, 2-s\})$ , where  $\eta_\alpha := \eta_0(\mu_\alpha \cdot)$  and

$$\tilde{u}(X) = \left( \frac{(c_0^{2-s}(n-2)(n-s)\mu_s(\mathbb{R}^n)^{-1})^{\frac{1}{2}}}{c_0^{2-s} + |X - X_0|^{2-s}} \right)^{\frac{n-2}{2-s}} \quad \text{for all } X \in \mathbb{R}^n,$$

with  $X_0 \in \mathbb{R}^n$ ,  $c_0 > 0$ . In particular,  $\tilde{u}$  satisfies

$$\Delta_{Eucl} \tilde{u} = \mu_s(\mathbb{R}^n) \frac{\tilde{u}^{2^*(s)-1}}{|X - X_0|^s} \text{ in } \mathbb{R}^n \text{ and } \int_{\mathbb{R}^n} \frac{\tilde{u}^{2^*(s)}}{|X - X_0|^s} dX = 1, \quad (5.143)$$

where  $Eucl$  is the Euclidean metric of  $\mathbb{R}^n$ .

*Proof.* We define the metric  $\bar{g}_\alpha(X) := (\exp_{z_\alpha}^* g)(\mu_\alpha X)$  in  $\mathbb{R}^n$  and we consider the vector  $X_{0,\alpha} = \mu_\alpha^{-1} \exp_{z_\alpha}^{-1}(x_0)$ . Since  $u_\alpha$  verifies the equation (5.7), we get  $\tilde{u}_\alpha$  verifies also weakly

$$\Delta_{\bar{g}_\alpha} \tilde{u}_\alpha + \tilde{a}_\alpha \tilde{u}_\alpha = \lambda_\alpha \frac{\tilde{u}_\alpha^{2^*(s)-1}}{d_{\bar{g}_\alpha}(X, X_{0,\alpha})^s} \text{ in } \mathbb{R}^n,$$

where  $\tilde{a}_\alpha(X) := \mu_\alpha^2 a_\alpha(\exp_{z_\alpha}(\mu_\alpha X)) \rightarrow 0$  as  $\alpha \rightarrow +\infty$ . Next, we follow the same proof of Theorem 2 in Jaber [75] and we get Theorem 5.7.1.  $\square$

## **Part III**



## A Paneitz-Branson type equation with Neumann boundary conditions

### Abstract

We consider the best constant in a critical Sobolev inequality of second order. We show non-rigidity for the optimizers above a certain threshold, namely we prove that the best constant is achieved by a non-constant solution of the associated fourth-order elliptic problem under Neumann boundary conditions. Our arguments rely on asymptotic estimates of the Rayleigh quotient. We also show rigidity below another threshold.

### 6.1 Introduction

It is well-known that the Sobolev inequality

$$S\|u\|_{L^{2_*}(\mathbb{R}^N)} \leq \|Du\|_{L^2(\mathbb{R}^N)}, \quad (6.1)$$

where  $2_* = 2N/(N - 2)$ ,  $N \geq 3$  and  $S = S(N)$  is a positive constant, plays a fundamental role in geometric analysis. A simple scaling argument shows that the exponent  $2_*$  is the only possible one in the inequality. This very same scaling argument implies that the embedding of  $H_0^1(\Omega)$  into  $L^{2_*}(\Omega)$ , where  $\Omega$  is a bounded open set, cannot be compact. This lack of compactness is the genesis of one of the main complexity in the celebrated Yamabe problem [112]. Assume that  $N \geq 3$  and  $(M, g)$  is a compact Riemannian  $N$ -dimensional manifold with scalar curvature  $R_g$ . The Yamabe Problem consists in looking for a metric  $g'$  conformally equivalent to  $g$  such that the scalar curvature  $R_{g'} \equiv 1$ . It happens that this problem amounts to finding a positive solution of

$$-4\frac{N-1}{N-2}\Delta_g u + R_g u = |u|^{\frac{4}{N-2}}u,$$



where  $\Delta_g$  denotes the Laplace-Beltrami operator on  $M$ . Then  $g' = u^{\frac{4}{N-2}}g$  is a conformal metric satisfying  $R_{g'} \equiv 1$ . The first approach by Yamabe was corrected by Trudinger [105]. We refer to [4, 82] for the history of the problem and to Aubin [2, 3], Schoen [100] and Schoen and Yau [101] for the main breakthroughs in its resolution.

Assume now that  $\Omega \subset \mathbb{R}^N$  ( $N \geq 3$ ) is an open bounded domain with a smooth boundary, and  $\alpha$  is a positive real number. In their seminal paper, Brezis and Nirenberg [19] have proved that the existence of a (positive) solution to the problem

$$-\Delta u + \alpha u = |u|^{\frac{4}{N-2}}u \text{ in } \Omega, \quad (6.2)$$

under homogeneous Dirichlet boundary condition, is closely related to the best constant for the Sobolev embedding of  $H_0^1(\Omega)$  into  $L^{\frac{2N}{N-2}}(\Omega)$ . Their arguments are inspired by the work of Aubin [2] on the Yamabe problem. The idea consists in minimizing the Rayleigh quotient

$$Q_\alpha(u) = \frac{\int_\Omega (|\nabla u|^2 + \alpha|u|^2) dx}{\left(\int_\Omega |u|^{\frac{2N}{N-2}} dx\right)^{\frac{N-2}{N}}}, \quad u \in H_0^1(\Omega) \setminus \{0\},$$

and in evaluating  $Q_\alpha$  at test functions of the form  $u(x) = \varphi(x)(\varepsilon + |x|^2)^{-\frac{N-2}{2}}$ , where  $\varphi$  is a cut-off function. The functions  $(\varepsilon + |x|^2)^{-\frac{N-2}{2}}$  play a natural role because they are extremal functions for the Sobolev inequality (6.1), see, for instance [2, 103].

Brezis [12, Section 6.4] suggested to study (6.2) under Neumann boundary condition. It happens that the equation is then related to models in mathematical biology such as the Keller-Segel model [47, 81, 91] for chemotaxis and the shadow system of Gierer and Meinhardt [56, 89].

With Neumann boundary conditions, the equation (6.2) admits the constant solutions  $u \equiv 0$  and  $u \equiv \alpha^{(N-2)/4}$ . When the nonlinearity in (6.2) is subcritical (namely when the exponent  $4/(N-2)$  is replaced by  $q-2$  with  $2 < q < 2_*$ ), Lin, Ni and Tagaki [81] have proved that the only positive solution to (6.2), for small  $\alpha > 0$ , is the nonzero constant solution. As a byproduct, this yields directly the sharp constant  $C(\alpha) = \alpha^{\frac{1}{2}}|\Omega|^{\frac{1}{2} - \frac{1}{q}}$  in the inequality

$$\left(\int_\Omega (|\nabla u|^2 + \alpha|u|^2) dx\right)^{1/2} \geq C(\alpha) \left(\int_\Omega |u|^q dx\right)^{\frac{1}{q}},$$

for  $u \in H^1(\Omega)$ . In the critical case, Lin and Ni [80] raised this rigidity result as a conjecture.

**LIN-NI'S CONJECTURE:** *For  $\alpha$  small enough, Equation (6.2) under Neumann boundary condition admits only  $\alpha^{(N-2)/4}$  as a positive solution.*

In the subcritical case, it is easily seen from a Morse index argument that the rigidity is broken for large  $\alpha$ . In the critical case, inspired by Brezis and Nirenberg, Wang [107, Theorem 3.1], and Adimurthi and Mancini [6, Theorem 1.1] proved that Equation (6.2) under Neumann boundary conditions admits a non-constant (least energy) positive solution  $u(\alpha)$  for every  $\alpha > \bar{\alpha} > 0$ . These least energy solutions  $u(\alpha)$  have the following concentration property [7, 90]: they are single-peaked in the sense that every  $u(\alpha)$ , for  $\alpha > 0$  sufficiently large, attains its unique maximum at a point  $p(\alpha) \in \partial\Omega$ , and  $p(\alpha) \rightarrow p_0 \in \partial\Omega$  as  $\alpha \rightarrow \infty$ , with  $H(p_0) = \max_{p \in \partial\Omega} H(p)$ , where  $H(p)$  is the mean curvature of  $\Omega$  at  $p \in \partial\Omega$ . Such concentration behaviour was shown in the subcritical case by Ni and Takagi [92, 93].

In the last three decades, lots of progress has been made towards proving or disproving Lin-Ni's conjecture. It is a difficult task to exhaust all the related literature concerning this conjecture and it is not our purpose here. Nevertheless, we give a short overview of the main results regarding this conjecture. In case  $\Omega = B_R(0)$ , and  $u$  is a radial function, the conjecture was studied by Adimurthi and Yadava [8], and Budd, Knaap and Peletier [22]. Namely, they investigated the problem

$$\begin{cases} -\Delta u + \alpha u = u^{\frac{N+2}{N-2}} & \text{in } B_R(0) \\ u \text{ is radial and } u > 0 & \text{in } B_R(0) \\ \partial_r u = 0 & \text{on } \partial B_R(0), \end{cases} \quad (6.3)$$

where  $\partial_r u := \frac{x}{|x|} \cdot \nabla u$ . They have established the following result.

**Theorem A.** *For  $\alpha > 0$  sufficiently small, the following statements hold:*

- (a) *If  $N = 3$  or  $N \geq 7$ , then (6.3) admits only the constant solution.*
- (b) *If  $N \in \{4, 5, 6\}$ , then (6.3) admits a nonconstant solution.*

Theorem A highlights that the validity of Lin-Ni's conjecture depends on the dimension. The proof of Theorem A uses radial symmetry to reduce (6.3) to the ODE

$$-u'' - \frac{N-1}{r}u' + \alpha u = u^{\frac{N+2}{N-2}} \quad \text{in } (0, R),$$

with the boundary condition  $u'(R) = 0$  (the second boundary condition  $u'(0) = 0$  comes from the assumption of radial symmetry of the solution). With regard to Lin-Ni's conjecture in general domains, such an approach cannot be applied. When  $\Omega$  is a convex domain, Zhu [114] proved that the conjecture is true if  $N = 3$ , see also [68, 111]. In case  $\Omega$  is a smooth bounded domain, and the mean curvature of  $\Omega$  is positive along the boundary  $\partial\Omega$ , Druet, Robert and Wei [44] have proved that Lin-Ni's conjecture is true for  $N = 3$  and  $N \geq 7$ , assuming

a bound on the energy of solutions. If  $\Omega$  is any smooth and bounded domain, Rey and Wei [97] have proved that the conjecture is false if  $N = 5$ . The Lin-Ni conjecture is wrong in all dimensions in non-convex domains [108] and in dimension  $N \geq 4$  for convex domains [109]. If we restrict our attention to least energy solutions, Adimurthi and Yadava have proved that the conjecture holds in every dimension [9].

Motivated by the above results, our purpose in this paper is to study an analogue of Equation (6.2) involving a fourth-order elliptic operator. Namely, we assume that  $\Omega \subset \mathbb{R}^N$  ( $N \geq 5$ ) is an open bounded set with smooth boundary, and  $\alpha$  is a positive parameter. We are interested in the following problem

$$\begin{cases} \Delta^2 u - \Delta u + \alpha u = |u|^{\frac{8}{N-4}} u, & \text{in } \Omega, \\ \partial_\nu u = \partial_\nu(\Delta u) = 0, & \text{on } \partial\Omega. \end{cases} \quad (P_\nu)$$

The linear operator  $\Delta^2 - \Delta + \alpha$  is often referred to as a Paneitz-Branson type operator [35] with constant coefficients. If  $(M, g)$  is a compact Riemannian manifold of dimension  $N \geq 5$  and  $Q_g$  is its  $Q$  curvature [11], the prescribed  $Q$  curvature problem consists in finding metric of constant  $Q$  curvature in the conformal class of  $g$ , see for instance [26, 43, 87]. This amounts to finding a positive solution to

$$P_g(u) = |u|^{\frac{8}{N-4}} u, \quad (6.4)$$

where  $P_g$  is the Paneitz operator [94], i.e.,

$$P_g u := \Delta_g^2 u - \operatorname{div}_g(A_g du) + hu,$$

with

$$A_g = \frac{(N-2)^2 + 4}{2(N-1)(N-2)} R_g g - \frac{4}{N-2} \operatorname{Ric}_g, \quad (6.5)$$

where  $R_g$  (resp.  $\operatorname{Ric}_g$ ) stands for the scalar curvature (resp. Ricci curvature), and  $h = \frac{N-4}{2} Q_g$  where  $Q_g$  is the  $Q$  curvature which is defined by

$$Q_g = \frac{1}{2(N-1)} \Delta_g R_g + \frac{N^3 - 4N^2 + 16N - 16}{8(N-1)^2(N-2)^2} R_g^2 - \frac{2}{(N-2)^2} |\operatorname{Ric}_g|_g^2.$$

Equation (6.4) is referred to as the Paneitz-Branson equation. In addition to the above mentioned contributions, we refer to [10, 69, 71, 72, 79, 96, 113] and the references therein for a glance to recent results. If  $(M, g)$  is Einstein ( $\operatorname{Ric}_g = \lambda g$ ,  $\lambda \in \mathbb{R}$ ), then the Paneitz-Branson operator takes the form

$$P_g u = \Delta_g^2 u + b \Delta_g u + cu, \quad (6.6)$$

where  $b = \frac{N^2 - 2N - 4}{2(N - 1)}\lambda$  and  $c = \frac{N(N - 4)(N^2 - 4)}{16(N - 1)^2}\lambda^2$ , see [69]. Observe that in the geometrical context,  $b$  and  $c$  in (6.6) can have the same sign while we take  $b = -1$  and  $\alpha > 0$  in  $(P_\nu)$ . The case of a positive Laplacian interacting with the bi-Laplacian will be considered in a future work.

Again, Equation  $(P_\nu)$  is critical since the  $L^{\frac{2N}{N-4}}$ -norm scales like the  $L^2$ -norm of the Laplacian. The problem admits two constant solutions, namely,  $u_0 = 0$ , and  $u_1 = \alpha^{\frac{N-4}{8}}$ . When the power nonlinearity is subcritical, one can prove in a standard way (adapting for instance [91]) that any positive solution is constant (and nonzero) when  $\alpha$  is small whereas this rigidity breaks down for large  $\alpha$ . Our main concern is to establish a non rigidity result for  $(P_\nu)$  when  $\alpha$  is large. To this end, we establish some Sobolev inequalities of second order with respect to the functional space associated to variational solutions to  $(P_\nu)$ . To the best of our knowledge, there are not many works in the literature dealing with the boundary conditions of  $(P_\nu)$ . For a general overview of this subject we refer to the work of Berchio and Gazzola [15], where the problem of embeddings of second order Sobolev spaces with traces on the boundary has been studied. For more insight on polyharmonic operators, we refer to [53].

Our main result is stated as follows. Let  $H_\nu^2(\Omega) = \{u \in H^2(\Omega) : \partial_\nu u = 0 \text{ on } \partial\Omega\}$ , and

$$\Sigma_\nu(\Omega) := \inf_{u \in M_\Omega} J(u), \tag{6.7}$$

where

$$J(u) = \int_\Omega (|\Delta u|^2 + |\nabla u|^2 + \alpha|u|^2) dx,$$

and

$$M_\Omega = \left\{ u \in H_\nu^2(\Omega) : \int_\Omega |u|^{\frac{2N}{N-4}} dx = 1 \right\}.$$

A solution of  $(P_\nu)$  is said to be of least energy if its  $L^{\frac{2N}{N-4}}$  normalized multiple is an optimizer for (6.7).

**Theorem 6.1.1.** *Assume  $\Omega$  is an open bounded subset of  $\mathbb{R}^N$  with smooth boundary. There exists  $\bar{\alpha} = \bar{\alpha}(N, |\Omega|) > 0$  such that for  $\alpha > \bar{\alpha}$ , any least energy solution of Equation  $(P_\nu)$  is nonconstant.*

It is worth mentioning that when  $(P_\nu)$  is considered in a smooth compact Riemannian manifold, Felli, Hebey and Robert [51] have established that for any  $\Lambda > 0$ , there exists  $\alpha_0 > 0$  such that for  $\alpha \geq \alpha_0$ , the above equation does not have a solution whose energy is smaller than  $\Lambda$ .

Since  $(P_\nu)$  is critical, the existence of a nontrivial solution does not follow directly from standard variational methods. Moreover, the functional setting brings new difficulties in comparison to the second order counterpart (6.2). To overcome the lack of compactness, we follow the arguments introduced in Aubin [2], and developed in Brezis and Nirenberg [19]. However, due to the boundary conditions of  $(P_\nu)$ , we cannot apply the arguments from Adimurthi and Mancini [6], Wang [107, Theorem 3.1] nor those from Berchio and Gazzola [15]. As a way out, our approach consists in making a change of coordinates in such a way that part of the boundary  $\partial\Omega$  will be diffeomorphic to a flat subset of  $\mathbb{R}^N$ . Roughly speaking, the idea is to straighten out the boundary and then to estimate the Rayleigh quotient by choosing suitable test functions adapted to these new coordinates. In this new coordinate system, we establish the following second order Sobolev inequality. The Sobolev constant  $S$  is defined from now on by

$$S = \inf_{u \in \mathcal{D}^{2,2}(\mathbb{R}^N)} \left\{ \int_{\mathbb{R}^N} |\Delta u|^2 dx : \int_{\mathbb{R}^N} |u|^{\frac{2N}{N-4}} dx = 1 \right\}.$$

**Lemma 6.1.1.** *Assume that  $\Omega$  is an open bounded subset of  $\mathbb{R}^N$  with smooth boundary and  $N \geq 5$ . Then, for every  $\varepsilon > 0$ , there exists  $B(\varepsilon) > 0$  such that for all  $u \in H_\nu^2(\Omega)$ ,*

$$\|u\|_{L^{\frac{2N}{N-4}}(\Omega)}^2 \leq \left( \frac{2^{4/N}}{S} + \varepsilon \right) \|\Delta u\|_{L^2(\Omega)}^2 + B(\varepsilon) \|u\|_{H^1(\Omega)}^2. \quad (6.8)$$

Moreover,  $\Sigma_\nu(\mathbb{R}_+^N) = S/2^{4/N}$  and the infimum is not achieved.

This lemma is the key to prove Theorem 6.1.1. We believe this Sobolev inequality has its own interest and can be useful in other situations. As in the second order case, if we focus on least energy solutions, then  $(P_\nu)$  has only the constant solution  $\alpha^{\frac{N-4}{8}}$  when  $\alpha$  is small enough.

**Theorem 6.1.2.** *Assume  $\Omega$  is an open bounded subset of  $\mathbb{R}^N$  with smooth boundary. Then, there exists  $\underline{\alpha} = \underline{\alpha}(N, |\Omega|) > 0$  such that for  $0 < \alpha < \underline{\alpha}$ , the only least energy solution of Equation  $(P_\nu)$  is the constant solution  $\alpha^{\frac{N-4}{8}}$ .*

A natural question arises from Theorem 6.1.1 and Theorem 6.1.2: where does rigidity of the minimizer break down? A tempting conjecture is that the rigidity is lost when the constant solution loses its stability. However this is still open even for the second order equation (6.3), see for instance [16, 42].

The manuscript is organized as follows. In Section 6.2, we settle the functional setting and recall some known facts regarding best constants for embeddings of second order Sobolev spaces. In Section 6.3, we establish a relation between the best constant for the second order Sobolev embedding and that of

the functional space associated to  $(P_\nu)$ . In Section 6.4, by taking into account the smoothness of the boundary  $\partial\Omega$  and the effect of the principal curvatures, we establish some asymptotic estimates, and we give the proof of Theorem 6.1.1. In Section 6.5, we establish some Sobolev inequalities of second order. Section 6.6 contains the proof of the rigidity theorem for small  $\alpha$ . In forthcoming works, we will consider the counterpart of Lin-Ni's conjecture for small  $\alpha$  and study the critical dimensions.

## 6.2 Preliminaries

In this section we settle the functional setting regarding  $(P_\nu)$ , and recall some known facts about the best constants of some second-order Sobolev embeddings.

A classical result in the theory of Sobolev spaces claims that if  $\Omega \not\cong \mathbb{R}^N$  is a smooth domain, then any function in  $H^2(\Omega)$  admits some traces on the boundary  $\partial\Omega$ , see, for instance, [1, Theorem 7.53], or [104, Lemmas 16.1 & 16.2]. In particular, there exists a linear continuous operator

$$\text{Tr} : H^2(\Omega) \rightarrow H^{3/2}(\partial\Omega)$$

such that  $\text{Tr} u = \partial_\nu u|_{\partial\Omega}$  for all  $u \in C^1(\bar{\Omega})$ . In the sense of traces, the kernel of the operator  $\text{Tr}$  gives rise to the following proper subspace of  $H^2(\Omega)$ ,

$$H_\nu^2(\Omega) := \{u \in H^2(\Omega) : \partial_\nu u = 0 \text{ on } \partial\Omega\}.$$

We recall that  $H^2(\Omega)$  is a Hilbert space endowed with the inner product defined through

$$\langle u, v \rangle = \int_{\Omega} (D^2 u D^2 v + Du Dv + uv) dx \quad \text{for all } u, v \in H^2(\Omega).$$

Using regularity theory, see, for instance, [64, Theorem 8.12], or [78, Chapter 1, Section 6, Theorem 4], we infer that

$$(u, u) \mapsto \|u\|_{H_\nu^2(\Omega)} := \left( \int_{\Omega} (|\Delta u|^2 + |\nabla u|^2 + \alpha|u|^2) dx \right)^{1/2}$$

defines an equivalent norm in  $H_\nu^2(\Omega)$  when  $\alpha > 0$ .

Note that by integration by parts,  $H_\nu^2(\Omega)$  is the natural space for (weak) solutions to  $(P_\nu)$ . To obtain nontrivial least energy solutions to  $(P_\nu)$ , we consider the minimization problem (6.7).

Before proceeding any further, we establish some notations and recall some known results. Denote by  $\mathcal{D}^{2,2}(\mathbb{R}^N)$  the closure of the space of smooth compactly supported functions in  $\mathbb{R}^N$  with respect to the norm  $\|D^2 \cdot\|_{L^2(\mathbb{R}^N)}$ . Note

that integration by parts two times together with a density argument show that  $\|D^2\phi\|_{L^2(\mathbb{R}^N)} = \|\Delta\phi\|_{L^2(\mathbb{R}^N)}$  for all  $\phi \in \mathcal{D}^{2,2}(\mathbb{R}^N)$ . It is well-known that the best constant for the embedding of  $\mathcal{D}^{2,2}(\mathbb{R}^N)$  into  $L^{\frac{2N}{N-4}}(\mathbb{R}^N)$  might be characterized by

$$S := \inf_{u \in \mathcal{D}^{2,2}(\mathbb{R}^N)} \left\{ \int_{\mathbb{R}^N} |\Delta u|^2 dx : \int_{\mathbb{R}^N} |u|^{\frac{2N}{N-4}} dx = 1 \right\}. \quad (6.9)$$

We recall that Lieb [83, Section IV], and Lions [86, Theorem I.1] (see also [85]) have proved that there exists a minimizer for (6.9), which is uniquely determined up to translations and dilations. Namely, the minimizer is given by the one-parameter family

$$u_\varepsilon(x) := \gamma_N \frac{\varepsilon^{\frac{N-4}{2}}}{(\varepsilon^2 + |x|^2)^{\frac{N-4}{2}}}, \quad (6.10)$$

where

$$\gamma_N := [(N-4)(N-2)N(N+2)]^{\frac{N-4}{8}}. \quad (6.11)$$

With the above expression, the constant  $S$  can be evaluated explicitly

$$S = \pi^2(N-4)(N-2)N(N+2) \left( \frac{\Gamma(\frac{N}{2})}{\Gamma(N)} \right)^{\frac{4}{N}}.$$

Note that  $u_\varepsilon(x) = \varepsilon^{-\frac{N-4}{2}} u_1(\frac{x}{\varepsilon})$ , and  $u_\varepsilon$  satisfies the equation

$$\Delta^2 u_\varepsilon = u_\varepsilon |u_\varepsilon|^{\frac{8}{N-4}} \quad \text{in } \mathbb{R}^N. \quad (6.12)$$

In fact, all positive solutions of the above equation are given by the  $\varepsilon$ -family (6.10). In regard to this result, see, for instance, [45, Theorem 2.1], [84, Theorem 1.3], and [110, Theorem 1.3].

Now we recall that for any smooth bounded domain  $\Omega \subset \mathbb{R}^N$ ,  $H_0^2(\Omega)$  and  $H^2 \cap H_0^1(\Omega)$  are Hilbert spaces endowed with the equivalent norm defined by

$$u \mapsto \left( \int_{\Omega} |\Delta u|^2 dx \right)^{1/2}, \quad (6.13)$$

see, for instance, [53, Theorem 2.31]. Here we note that  $H^2 \cap H_0^1(\Omega)$  is the space where variational solutions to fourth-order elliptic PDEs are sought when complemented with the so-called homogenous Navier boundary conditions along the boundary,  $u = \Delta u = 0$  on  $\partial\Omega$ , while  $H_0^2(\Omega)$  is the functional space for variational solutions when Dirichlet boundary conditions are considered,  $u = \partial_\nu u = 0$  on  $\partial\Omega$ . Observe that  $H^2 \cap H_0^1(\Omega)$  strictly contains  $H_0^2(\Omega)$ .

The question whether or not the best constants for the embeddings of  $H_0^2(\Omega)$  and  $H^2 \cap H_0^1(\Omega)$  into  $L^{\frac{2N}{N-4}}(\Omega)$  are equal, and independent of the domain, was

investigated by van der Vorst [106, Theorems 1 and 2]. He has shown that for any smooth domain  $\Omega \subset \mathbb{R}^N$ ,

$$\begin{aligned} S &= \inf_{u \in H_0^2(\Omega)} \left\{ \int_{\Omega} |\Delta u|^2 dx : \int_{\Omega} |u|^{\frac{2N}{N-4}} dx = 1 \right\} \\ &= \inf_{u \in H^2 \cap H_0^1(\Omega)} \left\{ \int_{\Omega} |\Delta u|^2 dx : \int_{\Omega} |u|^{\frac{2N}{N-4}} dx = 1 \right\} \end{aligned}$$

and that the infimum is never achieved when  $\Omega$  is bounded. However, a crucial part of the proof is not carried out in full detail. In addition, it is not clear that [106, Lemma A1] can be proved using an extension argument. In regard to this result, we refer to [54, Theorem 1].

In contrast with the above results, we cannot expect to obtain the same conclusions with respect to the space  $H_\nu^2(\Omega)$  since (6.13) is no longer a norm in  $H_\nu^2(\Omega)$ .

### 6.3 A relation between $\Sigma_\nu(\mathbb{R}^N)$ and $S$

In this section we show that  $\Sigma_\nu(\mathbb{R}^N)$  and  $S$  are equal. For convenience, throughout the rest of this paper we denote

$$2^* = \frac{2N}{N-4}.$$

Recall that as a consequence of the density of the space of smooth compactly supported functions in  $\mathbb{R}^N$  with respect to the  $H^2$ -Sobolev norm,

$$H_0^2(\mathbb{R}^N) = H^2 \cap H_0^1(\mathbb{R}^N) = H_\nu^2(\mathbb{R}^N) = H^2(\mathbb{R}^N).$$

Our next result is inspired by [15, Theorem 1(i)].

**Lemma 6.3.1.** *Assume  $N \geq 5$  and let  $S$  be defined as in (6.9). Then, for any  $\alpha > 0$ ,*

$$\Sigma_\nu(\mathbb{R}^N) = S, \quad \text{and the infimum is never achieved.}$$

*Proof.* We begin by noticing that

$$\begin{aligned} S &= \inf_{u \in \mathcal{D}^{2,2}(\mathbb{R}^N)} \left\{ \int_{\mathbb{R}^N} |\Delta u|^2 dx : \int_{\mathbb{R}^N} |u|^{2^*} dx = 1 \right\} \\ &\leq \inf_{u \in H^2(\mathbb{R}^N)} \left\{ \int_{\mathbb{R}^N} |\Delta u|^2 dx : \int_{\mathbb{R}^N} |u|^{2^*} dx = 1 \right\} \\ &\leq \inf_{u \in H^2(\mathbb{R}^N)} \left\{ \int_{\mathbb{R}^N} (|\Delta u|^2 + |\nabla u|^2 + \alpha |u|^2) dx : \int_{\mathbb{R}^N} |u|^{2^*} dx = 1 \right\} \\ &= \Sigma_\nu(\mathbb{R}^N), \end{aligned} \tag{6.14}$$



where in the first inequality we have used the fact that  $H^2(\mathbb{R}^N) \subset \mathcal{D}^{2,2}(\mathbb{R}^N)$ . Now, in order to show the reverse inequality in (6.14) we proceed as follows.

*Step one:* For all  $N \geq 5$ , there holds  $\Sigma_\nu(\mathbb{R}^N) \leq S$ . We construct a suitable minimizing sequence for which  $\Sigma_\nu(\mathbb{R}^N) \leq S$ . For convenience we write  $|x| = r$ . For all  $\varepsilon > 0$ , we consider the function

$$\begin{aligned} \vartheta_\varepsilon(r) &:= u_\varepsilon(r) - u_\varepsilon(1) \\ &= \gamma_N \varepsilon^{\frac{N-4}{2}} \left( \frac{1}{(\varepsilon^2 + r^2)^{\frac{N-4}{2}}} - \frac{1}{(\varepsilon^2 + 1)^{\frac{N-4}{2}}} \right). \end{aligned}$$

Now we set

$$z_\varepsilon(r) = \begin{cases} \vartheta_\varepsilon(r), & \text{if } 0 < r \leq 1/2 \\ w_\varepsilon(r), & \text{if } 1/2 \leq r \leq 1 \\ 0, & \text{if } r \geq 1, \end{cases} \quad (6.15)$$

where  $w_\varepsilon(r) := a(\varepsilon)(r-1)^3 + b(\varepsilon)(r-1)^2$ , with  $a(\varepsilon)$ , and  $b(\varepsilon)$  chosen in such a way that for  $r_0 = 1/2$ ,

$$w_\varepsilon(r_0) = \vartheta_\varepsilon(r_0), \quad \text{and} \quad \partial_r w_\varepsilon(r_0) = \partial_r \vartheta_\varepsilon(r_0).$$

In particular,

$$a(\varepsilon) = O(\varepsilon^{\frac{N-4}{2}}), \quad \text{and} \quad b(\varepsilon) = O(\varepsilon^{\frac{N-4}{2}}). \quad (6.16)$$

In this way, for every  $N \geq 5$ , since  $z_\varepsilon$  is a  $C^1$  gluing,  $w_\varepsilon(1) = 0$ , and  $\partial_r w_\varepsilon(1) = 0$ , we infer that  $z_\varepsilon \in H^2(\mathbb{R}^N)$ .

Next, we seek an upper bound for the functional  $J$  evaluated at  $z_\varepsilon / \|z_\varepsilon\|_{L^{2^*}(\mathbb{R}^N)}$ . Indeed, arguing as in [53, (7.58)],

$$\int_{|x| \leq 1/2} |\Delta u_\varepsilon|^2 dx = S^{N/4} + O(\varepsilon^{N-4}). \quad (6.17)$$

From this together with (6.16),

$$\begin{aligned} \int_{\mathbb{R}^N} |\Delta z_\varepsilon(|x|)|^2 dx &= \int_{|x| \leq 1/2} |\Delta \vartheta_\varepsilon(|x|)|^2 dx + \int_{1/2 \leq |x| \leq 1} |\Delta w_\varepsilon(|x|)|^2 dx \\ &= \int_{|x| \leq 1/2} |\Delta u_\varepsilon(|x|)|^2 dx + o(1) \\ &= S^{N/4} + o(1). \end{aligned} \quad (6.18)$$

Similarly, by (6.16),

$$\int_{\mathbb{R}^N} |z_\varepsilon(|x|)|^{2^*} dx = S^{N/4} + o(1), \quad \text{and} \quad \int_{\mathbb{R}^N} |z_\varepsilon(|x|)|^2 dx = o(1). \quad (6.19)$$

Since  $z_\varepsilon \in H^2(\mathbb{R}^N)$  we use interpolation between  $\|\Delta z_\varepsilon\|_{L^2(\mathbb{R}^N)}$  and  $\|z_\varepsilon\|_{L^2(\mathbb{R}^N)}$  to get an estimate for the  $L^2$ -norm of  $\nabla z_\varepsilon$ . Namely, by (6.18) and (6.19),

$$\int_{\mathbb{R}^N} |\nabla z_\varepsilon(|x|)|^2 dx = o(1). \quad (6.20)$$

Finally, we set  $Z_\varepsilon := z_\varepsilon / \|z_\varepsilon\|_{L^{2^*}(\mathbb{R}^N)}$  so that  $Z_\varepsilon \in M_{\mathbb{R}^N}$ . Therefore, in view of (6.18)-(6.20),

$$\begin{aligned} \Sigma_\nu(\mathbb{R}^N) &\leq J(Z_\varepsilon), \quad \text{for all } \varepsilon > 0, \\ &= \frac{S^{N/4} + o(1)}{(S^{N/4} + o(1))^{\frac{N-4}{N}}} \quad \text{as } \varepsilon \rightarrow 0. \end{aligned}$$

This proves that  $\Sigma_\nu(\mathbb{R}^N) \leq S$ , and hence the first part of the lemma follows.

*Step two:  $\Sigma_\nu(\mathbb{R}^N)$  is never achieved.* Seeking a contradiction, we assume that there exists a function  $u \in M_{\mathbb{R}^N}$  which achieves equality in (6.7). Define for  $\lambda > 0$  the rescaled function  $u_\lambda(x) := \lambda^{\frac{N-4}{2}} u(\lambda x)$  so that  $\|u_\lambda\|_{L^{2^*}(\mathbb{R}^N)}^{2^*} = 1$ . Thus,

$$\begin{aligned} \int_{\mathbb{R}^N} (|\Delta u|^2 + |\nabla u|^2 + \alpha|u|^2) dx &\leq \int_{\mathbb{R}^N} (|\Delta u_\lambda|^2 + |\nabla u_\lambda|^2 + \alpha|u_\lambda|^2) dx \\ &\leq \int_{\mathbb{R}^N} |\Delta u|^2 dx + \frac{1}{\lambda^2} \int_{\mathbb{R}^N} |\nabla u|^2 dx \\ &\quad + \frac{\alpha}{\lambda^4} \int_{\mathbb{R}^N} |u|^2 dx. \end{aligned}$$

By sending  $\lambda$  to infinity in the above inequality we obtain a contradiction.  $\square$

Now we recall that by the Sobolev embedding theorem, there exist positive constants  $A$  and  $B$  such that for any  $u \in H^2(\Omega)$ ,

$$\|u\|_{L^{2^*}(\Omega)}^2 \leq A\|\Delta u\|_{L^2(\Omega)}^2 + B\|u\|_{H^1(\Omega)}^2.$$

The task of finding the best constants in the above inequality has been extensively studied in the last years. In this regard, we refer to [35, 67]. In this direction, we will prove that for every  $\varepsilon > 0$ , there exists  $B(\varepsilon) > 0$  such that for all  $u \in H_\nu^2(\Omega)$ ,

$$\|u\|_{L^{2^*}(\Omega)}^2 \leq \left( \frac{2^{4/N}}{S} + \varepsilon \right) \|\Delta u\|_{L^2(\Omega)}^2 + B(\varepsilon)\|u\|_{H^1(\Omega)}^2. \quad (6.21)$$

Moreover,  $\Sigma_\nu(\mathbb{R}_+^N) = S/2^{4/N}$ , and the infimum is not achieved. This is the content of Lemma 6.1.1 proved in Section 6.5. As a consequence of inequality (6.21), we establish the following result.

**Lemma 6.3.2.** *Assume that  $\Omega$  is an open bounded subset of  $\mathbb{R}^N$  with smooth boundary and  $N \geq 5$ . If  $\Sigma_\nu(\Omega) < S/2^{4/N}$ , then the infimum in (6.7) is achieved.*

*Proof.* Let  $(u_k)_{k \in \mathbb{N}} \subset M_\Omega$  be a minimizing sequence for  $\Sigma_\nu(\Omega)$ . Since  $J$  is the square of a norm in  $H_\nu^2(\Omega)$  we deduce that the sequence  $(u_k)_{k \in \mathbb{N}}$  is bounded in  $H_\nu^2(\Omega)$ . Consequently, up to extracting a subsequence, there exists  $u \in H_\nu^2(\Omega)$  with

$$\begin{cases} u_k \rightharpoonup u & \text{weakly in } H_\nu^2(\Omega) \\ u_k \rightharpoonup u & \text{weakly in } L^{2^*}(\Omega) \\ u_k \rightarrow u & \text{strongly in } H^1(\Omega) \\ u_k(x) \rightarrow u(x) & \text{a.e. in } \Omega. \end{cases} \quad (6.22)$$

*Step one:* There holds  $u \not\equiv 0$ . Seeking a contradiction, we assume that  $u \equiv 0$ . By (6.22),

$$u_k \rightarrow 0 \quad \text{strongly in } H^1(\Omega). \quad (6.23)$$

Recall that  $\|u_k\|_{L^{2^*}(\Omega)} = 1$ . Thus,

$$\begin{aligned} \lim_{k \rightarrow \infty} \int_\Omega |\Delta u_k|^2 dx &\leq \lim_{k \rightarrow \infty} J(u_k) \\ &= \Sigma_\nu(\Omega) \\ &\leq \Sigma_\nu(\Omega) \left( \frac{2^{4/N}}{S} + \varepsilon \right) \int_\Omega |\Delta u_k|^2 dx + o(1) \quad (\text{by (6.8)}) \end{aligned}$$

for every  $\varepsilon > 0$ . Note that  $\Sigma_\nu(\Omega) > 0$ . Hence, as a consequence of the above inequality,

$$1 \leq \Sigma_\nu(\Omega) \left( \frac{2^{4/N}}{S} + \varepsilon \right),$$

which contradicts our assumption  $\Sigma_\nu(\Omega) < S/2^{4/N}$ . Therefore, we conclude that  $u \not\equiv 0$ .

*Step two: Strong convergence in  $L^{2^*}(\Omega)$ .* By Vitali theorem,

$$\begin{aligned} \int_\Omega |u_k|^{2^*} dx - \int_\Omega |u_k - u|^{2^*} dx &= - \int_\Omega \int_0^1 \frac{d}{dt} |u_k - tu|^{2^*} dt dx \\ &= 2^* \int_\Omega \int_0^1 u(u_k - tu) |u_k - tu|^{2^*-2} dt dx \\ &= 2^* \int_\Omega \int_0^1 u(u - tu) |u - tu|^{2^*-2} dt dx + o(1) \\ &= \int_\Omega |u|^{2^*} dx + o(1). \end{aligned}$$

Since  $(u_k)_{k \in \mathbb{N}} \subset M_\Omega$ ,

$$1 - \int_{\Omega} |u_k - u|^{2^*} dx = \int_{\Omega} |u|^{2^*} dx + o(1). \quad (6.24)$$

By weak convergence in  $H_\nu^2(\Omega)$ ,

$$\begin{aligned} J(u_k) &= J(u_k - u) + 2\langle u_k - u, u \rangle_{H_\nu^2(\Omega)} + J(u) \\ &= J(u_k - u) + J(u) + o(1), \end{aligned} \quad (6.25)$$

and by strong convergence in  $H^1(\Omega)$ ,

$$J(u_k) = \int_{\Omega} |\Delta u_k - \Delta u|^2 dx + J(u) + o(1). \quad (6.26)$$

From Step one,

$$\tilde{u} := \frac{u}{\|u\|_{L^{2^*}(\Omega)}} \in M_\Omega$$

and since  $J(\tilde{u}) \geq \Sigma_\nu(\Omega)$ ,

$$J(u) \geq \Sigma_\nu(\Omega) \|u\|_{L^{2^*}(\Omega)}^2. \quad (6.27)$$

Thus,

$$\begin{aligned} \Sigma_\nu(\Omega) &= \int_{\Omega} |\Delta u_k - \Delta u|^2 dx + J(u) + o(1) \quad (\text{by (6.26)}) \\ &\geq \left( \frac{2^{4/N}}{S} + \varepsilon \right)^{-1} \|u_k - u\|_{L^{2^*}(\Omega)}^2 + J(u) + o(1) \quad (\text{by (6.8) and (6.23)}) \\ &\geq \left( \frac{2^{4/N}}{S} + \varepsilon \right)^{-1} \|u_k - u\|_{L^{2^*}(\Omega)}^2 + \Sigma_\nu(\Omega) \|u\|_{L^{2^*}(\Omega)}^2 + o(1) \quad (\text{by (6.27)}) \\ &= \left[ \left( \frac{2^{4/N}}{S} + \varepsilon \right)^{-1} - \Sigma_\nu(\Omega) \right] \|u_k - u\|_{L^{2^*}(\Omega)}^2 \\ &\quad + \Sigma_\nu(\Omega) \left( \|u_k - u\|_{L^{2^*}(\Omega)}^2 + \|u\|_{L^{2^*}(\Omega)}^2 \right) + o(1) \\ &\geq \left[ \left( \frac{2^{4/N}}{S} + \varepsilon \right)^{-1} - \Sigma_\nu(\Omega) \right] \|u_k - u\|_{L^{2^*}(\Omega)}^2 \\ &\quad + \Sigma_\nu(\Omega) \left( \|u_k - u\|_{L^{2^*}(\Omega)}^{2^*} + \|u\|_{L^{2^*}(\Omega)}^{2^*} \right)^{\frac{2}{2^*}} + o(1) \\ &= \left[ \left( \frac{2^{4/N}}{S} + \varepsilon \right)^{-1} - \Sigma_\nu(\Omega) \right] \|u_k - u\|_{L^{2^*}(\Omega)}^2 + \Sigma_\nu(\Omega) + o(1) \quad (\text{by (6.24)}). \end{aligned}$$

Since by assumption we have  $\Sigma_\nu(\Omega) < S/2^{4/N}$ , we deduce that  $u_k \rightarrow u$  strongly in  $L^{2^*}(\Omega)$ , and  $\|u\|_{L^{2^*}(\Omega)} = 1$ .

*Step three: Strong convergence in  $H_\nu^2(\Omega)$ .* By weak lower semi-continuity of  $J$ , and since  $u \in M_\Omega$ ,

$$\Sigma_\nu(\Omega) \leq J(u) \leq \liminf_{k \rightarrow \infty} J(u_k) = \Sigma_\nu(\Omega).$$

Therefore, combining this with (6.25) we conclude that  $u_k \rightarrow u$  strongly in  $H_\nu^2(\Omega)$ .  $\square$

## 6.4 Asymptotic estimates

In this section we take into account the smoothness of the boundary  $\partial\Omega$  and the effect of the principal curvatures at some boundary point.

Before proceeding any further, we settle the geometrical aspect of our problem. Since  $\Omega \subset \mathbb{R}^N$  is a bounded set, there exists a ball of radius  $R_0 > 0$  such that  $\Omega \subset B_{R_0}$ . In view of the smoothness of  $\Omega$ , there exists  $\bar{x} \in \partial\Omega$  such that in a neighborhood of  $\bar{x}$ , we have that  $\Omega$  lies on one side of the tangent plane at  $\bar{x}$ , and the mean curvature with respect to the unit outward normal at  $\bar{x}$  is positive. Due to the invariance of rotations and translations, by a change of variables we may assume that  $\bar{x}$  is the origin, that the tangent hyperplane coincides with  $\{x_N = 0\}$ , and that  $\Omega \subset \mathbb{R}_+^N = \{x = (x', x_N) : x_N > 0\}$ . By the fact that  $\Omega$  is a smooth subset, there are  $R > 0$ , and a smooth function  $\rho : \{x' \in \mathbb{R}^{N-1} : |x'| < R\} \rightarrow \mathbb{R}_+$  such that

$$\begin{cases} \Omega \cap B_R = \{(x', x_N) \in B_R : x_N > \rho(x')\} \\ \partial\Omega \cap B_R = \{(x', x_N) \in B_R : x_N = \rho(x')\}. \end{cases}$$

Since the curvature is positive at the origin, there are real constants  $(\kappa_j)_{j=1}^{N-1}$ , which are called the principal curvatures, that satisfy

$$H_N(0) := \frac{2}{N-1} \sum_{j=1}^{N-1} \kappa_j > 0, \quad \text{and} \quad \rho(x') = \sum_{j=1}^{N-1} \kappa_j x_j^2 + O(|x'|^3) \quad \text{as } |x'| \rightarrow 0. \quad (6.28)$$

Recall that a crucial point in getting compactness in the proof of Lemma 6.3.2 was the assumption  $\Sigma_\nu(\Omega) < S/2^{N/4}$ . In our next result we establish this inequality.

**Lemma 6.4.1.** *Assume that  $\Omega$  is an open bounded subset of  $\mathbb{R}^N$  with smooth boundary, and  $N \geq 5$ . Then, there holds*

$$\Sigma_\nu(\Omega) < \frac{S}{2^{4/N}}.$$

*Proof. Step one: Straightening the boundary.* Note that for any  $x \in \partial\Omega \cap B_R$ , we have that  $x = (x', \rho(x'))$ , where  $\rho$  is defined in (6.28). Consequently, an outward orthogonal vector to the tangent space is given by

$$\nu(x) = \begin{bmatrix} \nabla \rho(x') \\ -1 \end{bmatrix}.$$

For some open subset  $V$  of  $\mathbb{R}^N$ , we define

$$\begin{aligned} \Phi: V \subset \mathbb{R}^N &\rightarrow \mathbb{R}^N \\ (y', y_N) &\mapsto (y', \rho(y')) - y_N \nu(y', \rho(y')). \end{aligned} \quad (6.29)$$

Observe that the Jacobian matrix of  $\Phi$  is given by

$$D\Phi = \begin{bmatrix} 1 - y_N \frac{\partial \nu_1}{\partial y_1} & -y_N \frac{\partial \nu_1}{\partial y_2} & \cdots & -y_N \frac{\partial \nu_1}{\partial y_{N-1}} & -\nu_1 \\ -y_N \frac{\partial \nu_2}{\partial y_1} & 1 - y_N \frac{\partial \nu_2}{\partial y_2} & \cdots & -y_N \frac{\partial \nu_2}{\partial y_{N-1}} & -\nu_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -y_N \frac{\partial \nu_{N-1}}{\partial y_1} & -y_N \frac{\partial \nu_{N-1}}{\partial y_2} & \cdots & 1 - y_N \frac{\partial \nu_{N-1}}{\partial y_{N-1}} & -\nu_{N-1} \\ \nu_1 & \nu_2 & \cdots & \nu_{N-1} & 1 \end{bmatrix}. \quad (6.30)$$

From this, we immediately deduce that for  $(y', y_N) = (0, 0)$ , there holds  $D\Phi(0, 0) = \text{Id}$ , where  $\text{Id}$  is the identity matrix of size  $N$ . By the Inverse Function Theorem there exist  $r_0 > 0$ , and  $U$  an open subset of  $\mathbb{R}^N$  such that  $\Phi: B_{r_0}^+ \rightarrow \Omega \cap U$  is a smooth diffeomorphism, where  $B_{r_0}^+ := B_{r_0} \cap \{y_N > 0\}$ . Now, let  $\eta$  be a  $C^\infty$  radial fixed cut-off function with  $0 \leq \eta \leq 1$ , and

$$\eta(r) = \begin{cases} 1, & \text{if } r \leq r_0/4 \\ 0, & \text{if } r \geq r_0/2. \end{cases}$$

Set

$$\varphi_\varepsilon(y) := \eta(|y|)u_\varepsilon(|y|),$$

where  $u_\varepsilon$  is defined in (6.10). As a consequence, the following function is well-defined

$$\psi_\varepsilon(x) := \varphi_\varepsilon \circ \Phi^{-1}(x).$$

Note that for  $x = (x', \rho(x')) \in \partial\Omega \cap B_R$ ,

$$\begin{aligned} \lim_{t \rightarrow 0} \frac{\psi_\varepsilon(x) - \psi_\varepsilon(x - t\nu(x))}{t} &= \lim_{t \rightarrow 0} \frac{\varphi_\varepsilon(\Phi^{-1}(x)) - \varphi_\varepsilon(\Phi^{-1}(x - t\nu(x)))}{t} \\ &= \lim_{t \rightarrow 0} \frac{\varphi_\varepsilon(y', 0) - \varphi_\varepsilon(y', t)}{t} \\ &= -\partial_{y_N} \varphi_\varepsilon(y', 0) \\ &= 0, \end{aligned}$$

where in the last equality we have used the fact that  $\varphi_\varepsilon$  is a radial function. Therefore  $\psi_\varepsilon$  belongs to  $H_\nu^2(\Omega)$ .

By (6.28) and (6.30),

$$D\Phi(y', y_N) = \text{Id} + A(y', y_N) + O(|(y', y_N)|^2), \quad (6.31)$$

where

$$A(y', y_N) = \begin{bmatrix} -2y_N\kappa_1 & 0 & \dots & 0 & -2\kappa_1y_1 \\ 0 & -2y_N\kappa_2 & \dots & 0 & -2\kappa_2y_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & -2y_N\kappa_{N-1} & -2\kappa_{N-1}y_{N-1} \\ 2\kappa_1y_1 & 2\kappa_2y_2 & \dots & 2\kappa_{N-1}y_{N-1} & 0 \end{bmatrix}.$$

In addition, since  $D\Phi$  is an invertible matrix,

$$D\Phi^{-1}(x) = (D\Phi(y))^{-1},$$

where  $x = \Phi(y)$ . Thus,

$$D\Phi^{-1}(x) = \text{Id} - A(y) + O(|y|^2). \quad (6.32)$$

Henceforth, for convenience we write

$$y = \Phi^{-1}(x), \quad \text{and} \quad y_j = (\Phi^{-1}(x))_j \quad \text{for } j = 1, \dots, N.$$

In view of the above notation, by (6.32) the elements  $(D\Phi^{-1}(x))_{ij}$  of the matrix

$D\Phi^{-1}(x)$  are given by

$$\begin{cases} \frac{\partial y_j}{\partial x_j} = 1 + 2y_N \kappa_j + O(|y|^2), & j \in \{1, \dots, N-1\}, \\ \frac{\partial y_i}{\partial x_N} = 2\kappa_i y_i + O(|y|^2), & i \in \{1, \dots, N-1\}, \\ \frac{\partial y_N}{\partial x_j} = -2\kappa_j y_j + O(|y|^2), & j \in \{1, \dots, N-1\}, \\ \frac{\partial y_i}{\partial x_j} = O(|y|^2), & i \neq j \text{ and } i, j \in \{1, \dots, N-1\}, \\ \frac{\partial y_N}{\partial x_N} = 1 + O(|y|^2). \end{cases} \quad (6.33)$$

Now using the chain rule, for any  $j \in \{1, \dots, N\}$ ,

$$\frac{\partial \psi_\varepsilon(x)}{\partial x_j} = \sum_{l=1}^N \frac{\partial \varphi_\varepsilon(y)}{\partial y_l} \frac{\partial y_l}{\partial x_j}, \quad (6.34)$$

and

$$\frac{\partial^2 \psi_\varepsilon(x)}{\partial x_j^2} = \sum_{k,l=1}^N \frac{\partial^2 \varphi_\varepsilon(y)}{\partial y_k \partial y_l} \left( \frac{\partial y_k}{\partial x_j} \right) \left( \frac{\partial y_l}{\partial x_j} \right) + \sum_{l=1}^N \frac{\partial \varphi_\varepsilon(y)}{\partial y_l} \frac{\partial^2 y_l}{\partial x_j^2}. \quad (6.35)$$

*Step two: Estimate for  $\|\Delta \psi_\varepsilon\|_{L^2(\Omega)}^2$ .* We begin by computing the derivatives of  $u_\varepsilon$ . Notice that for  $l, k \in \{1, \dots, N\}$  fixed,

$$\frac{\partial u_\varepsilon(y)}{\partial y_l} = -\frac{\gamma_N(N-4)\varepsilon^{\frac{N-4}{2}}}{(\varepsilon^2 + |y|^2)^{\frac{N-2}{2}}} y_l,$$

and

$$\frac{\partial^2 u_\varepsilon(y)}{\partial y_k \partial y_l} = \gamma_N(N-4)\varepsilon^{\frac{N-4}{2}} \left( \frac{-\delta_{lk}}{(\varepsilon^2 + |y|^2)^{\frac{N-2}{2}}} + \frac{(N-2)y_k y_l}{(\varepsilon^2 + |y|^2)^{\frac{N}{2}}} \right),$$

where  $\delta_{lk}$  is the Kronecker delta, that is,  $\delta_{lk} = 1$  if  $l = k$ , and  $\delta_{lk} = 0$  otherwise. Since we are interested in an estimate for the  $L^2$ -norm of  $\Delta \psi_\varepsilon$ , it is enough to compute the derivatives of  $\psi_\varepsilon$  when  $\varphi_\varepsilon \equiv u_\varepsilon$ . In this situation, from (6.33) for



$j \in \{1, \dots, N-1\}$ ,

$$\begin{aligned}
\frac{\partial^2 \psi_\varepsilon(x)}{\partial x_j^2} &= \frac{\partial^2 u_\varepsilon(y)}{\partial y_j^2} \left( \frac{\partial y_j}{\partial x_j} \right)^2 + 2 \frac{\partial^2 u_\varepsilon(y)}{\partial y_j \partial y_N} \left( \frac{\partial y_j}{\partial x_j} \right) \left( \frac{\partial y_N}{\partial x_j} \right) \\
&\quad + \frac{\partial u_\varepsilon(y)}{\partial y_N} \frac{\partial^2 y_N}{\partial x_j^2} + O \left( \frac{\varepsilon^{\frac{N-4}{2}} |y|^2}{(\varepsilon^2 + |y|^2)^{\frac{N-2}{2}}} \right) \\
&= \frac{\partial^2 u_\varepsilon(y)}{\partial y_j^2} (1 + 4y_N \kappa_j) - 4\kappa_j y_j \frac{\partial^2 u_\varepsilon(y)}{\partial y_j \partial y_N} \\
&\quad - 2\kappa_j \frac{\partial u_\varepsilon(y)}{\partial y_N} + O \left( \frac{\varepsilon^{\frac{N-4}{2}}}{(\varepsilon^2 + |y|^2)^{\frac{N-4}{2}}} \right) \\
&= \frac{\partial^2 u_\varepsilon(y)}{\partial y_j^2} - \frac{2\gamma_N(N-4)\varepsilon^{\frac{N-4}{2}}}{(\varepsilon^2 + |y|^2)^{\frac{N-2}{2}}} y_N \kappa_j + O \left( \frac{\varepsilon^{\frac{N-4}{2}}}{(\varepsilon^2 + |y|^2)^{\frac{N-4}{2}}} \right).
\end{aligned} \tag{6.36}$$

In case  $j = N$ ,

$$\begin{aligned}
\frac{\partial^2 \psi_\varepsilon(x)}{\partial x_N^2} &= \frac{\partial^2 u_\varepsilon(y)}{\partial y_N^2} + 2 \sum_{j=1}^{N-1} \frac{\partial^2 u_\varepsilon(y)}{\partial y_j \partial y_N} \left( \frac{\partial y_j}{\partial x_N} \right) \left( \frac{\partial y_N}{\partial x_N} \right) + O \left( \frac{\varepsilon^{\frac{N-4}{2}}}{(\varepsilon^2 + |y|^2)^{\frac{N-4}{2}}} \right) \\
&= \frac{\partial^2 u_\varepsilon(y)}{\partial y_N^2} + 4\gamma_N(N-4)(N-2) \sum_{j=1}^{N-1} \frac{\kappa_j y_j^2 y_N \varepsilon^{\frac{N-4}{2}}}{(\varepsilon^2 + |y|^2)^{\frac{N}{2}}} \\
&\quad + O \left( \frac{\varepsilon^{\frac{N-4}{2}}}{(\varepsilon^2 + |y|^2)^{\frac{N-4}{2}}} \right).
\end{aligned} \tag{6.37}$$

Thus, by (6.36) and (6.37),

$$\begin{aligned}
\Delta \psi_\varepsilon(x) &= \Delta u_\varepsilon(y) - \frac{\gamma_N(N-4)(N-1)H_N(0)\varepsilon^{\frac{N-4}{2}}}{(\varepsilon^2 + |y|^2)^{\frac{N-2}{2}}} y_N \\
&\quad + 4\gamma_N(N-4)(N-2) \sum_{j=1}^{N-1} \frac{\kappa_j y_j^2 y_N \varepsilon^{\frac{N-4}{2}}}{(\varepsilon^2 + |y|^2)^{\frac{N}{2}}} + O \left( \frac{\varepsilon^{\frac{N-4}{2}}}{(\varepsilon^2 + |y|^2)^{\frac{N-4}{2}}} \right).
\end{aligned}$$

Recall that

$$\Delta u_\varepsilon(y) = - \frac{\gamma_N(N-4)\varepsilon^{\frac{N-4}{2}}(N\varepsilon^2 + 2|y|^2)}{(\varepsilon^2 + |y|^2)^{\frac{N}{2}}} < 0.$$

Observe that due to the form of the matrix  $D\Phi(y)$ ,

$$\det D\Phi(y) = 1 - (N-1)H_N(0)y_N + O(|y|^2).$$

Henceforth, for convenience we denote

$$d_N = \gamma_N^2(N-4)^2(N-1). \quad (6.38)$$

Thus,

$$\begin{aligned} \int_{\Omega} |\Delta \psi_{\varepsilon}(x)|^2 dx &= \int_{\Omega \cap U} |\Delta \psi_{\varepsilon}(x)|^2 dx + O(\varepsilon^{N-4}) \\ &= \int_{B_{r_0}^+} |\Delta \psi_{\varepsilon}(\Phi(y))|^2 |\det D\Phi(y)| dy + O(\varepsilon^{N-4}) \\ &= I_1 + I_2 + I_3 + I_4 + I_5, \end{aligned}$$

where

$$\begin{aligned} I_1 &= \int_{B_{r_0/2}^+} |\Delta u_{\varepsilon}(y)|^2 dy, \\ I_2 &= -d_N H_N(0) \varepsilon^{N-4} \int_{B_{r_0/2}^+} \frac{(N\varepsilon^2 + 2|y|^2)^2}{(\varepsilon^2 + |y|^2)^N} y_N dy, \\ I_3 &= 2d_N H_N(0) \varepsilon^{N-4} \int_{B_{r_0/2}^+} \frac{(N\varepsilon^2 + 2|y|^2)}{(\varepsilon^2 + |y|^2)^{N-1}} y_N dy, \\ I_4 &= -\frac{8d_N(N-2)}{N-1} \varepsilon^{N-4} \sum_{j=1}^{N-1} \kappa_j \int_{B_{r_0/2}^+} \frac{(N\varepsilon^2 + 2|y|^2)}{(\varepsilon^2 + |y|^2)^N} y_j^2 y_N dy, \end{aligned}$$

and

$$I_5 = \begin{cases} O(\varepsilon^2), & \text{if } N \geq 7 \\ O(\varepsilon^2 \log \frac{1}{\varepsilon}), & \text{if } N = 6 \\ O(\varepsilon), & \text{if } N = 5. \end{cases}$$

In this way, by (6.17),

$$I_1 = \frac{S^{N/4}}{2} + O(\varepsilon^{N-4}),$$

and we estimate  $I_2 + I_3$  as follows,

$$\begin{aligned} I_2 + I_3 &= -d_N H_N(0)(N-2) \varepsilon^{N-2} \int_{B_{r_0/2}^+} \frac{(N\varepsilon^2 + 2|y|^2)}{(\varepsilon^2 + |y|^2)^N} y_N dy \\ &= -d_N H_N(0)(N-2) J_1 \varepsilon + o(\varepsilon), \end{aligned}$$

where

$$J_1 = \int_{\mathbb{R}_+^N} \frac{N + 2|y|^2}{(1 + |y|^2)^N} y_N dy > 0.$$

To estimate  $I_4$  we first note that by symmetry,

$$\int_{B_{r_0/2}^+} \frac{N\varepsilon^2 + 2|y|^2}{(\varepsilon^2 + |y|^2)^N} y_j^2 y_N dy = \frac{1}{N-1} \int_{B_{r_0/2}^+} \frac{(N\varepsilon^2 + 2|y|^2)(|y|^2 - y_N^2)}{(\varepsilon^2 + |y|^2)^N} y_N dy.$$

Thus,

$$\begin{aligned} I_4 &= -\frac{4d_N(N-2)}{N-1} H_N(0) \varepsilon^{N-4} \int_{B_{r_0/2}^+} \frac{(N\varepsilon^2 + 2|y|^2)(|y|^2 - y_N^2)}{(\varepsilon^2 + |y|^2)^N} y_N dy \\ &= -\frac{4d_N(N-2)}{N-1} H_N(0) \varepsilon \int_{B_{r_0/2\varepsilon}^+} \frac{(N + 2|y|^2)(|y|^2 - y_N^2)}{(1 + |y|^2)^N} y_N dy \\ &= -\begin{cases} \frac{4d_N(N-2)}{N-1} H_N(0) J_2 \varepsilon + o(\varepsilon), & \text{if } N \geq 6 \\ 8\pi^2 \sqrt[4]{105} H_5(0) \varepsilon \log \frac{1}{\varepsilon} + O(\varepsilon), & \text{if } N = 5, \end{cases} \end{aligned}$$

where

$$J_2 = \int_{\mathbb{R}_+^N} \frac{(N + 2|y|^2)(|y|^2 - y_N^2)}{(1 + |y|^2)^N} y_N dy > 0.$$

Consequently,

$$\begin{aligned} \int_{\Omega} |\Delta \psi_\varepsilon|^2 dx &= \frac{S^{N/4}}{2} + o(\varepsilon) \tag{6.39} \\ &- \begin{cases} d_N H_N(0) (N-2) \left( J_1 + \frac{4J_2}{N-1} \right) \varepsilon + o(\varepsilon), & \text{if } N \geq 6 \\ 8\pi^2 \sqrt[4]{105} H_5(0) \varepsilon \log \frac{1}{\varepsilon} + O(\varepsilon), & \text{if } N = 5. \end{cases} \end{aligned}$$

*Step three: Estimate for  $\|\psi_\varepsilon\|_{L^{2^*}(\Omega)}^{2^*}$ .* Arguing as in the previous step,

$$\begin{aligned} \int_{\Omega} |\psi_\varepsilon(x)|^{2^*} dx &= \int_{\Omega \cap U} |\psi_\varepsilon(x)|^{2^*} dx + O(\varepsilon^N) \\ &= \int_{B_{r_0}^+} |\psi_\varepsilon(\Phi(y))|^{2^*} |\det D\Phi(y)| dy + O(\varepsilon^N) \\ &= \int_{B_{r_0/2}^+} |u_\varepsilon(y)|^{2^*} (1 - (N-1)H_N(0)y_N + O(|y|^2)) dy + O(\varepsilon^N) \\ &= \frac{S^{N/4}}{2} + O(\varepsilon^2) - \gamma_N^{2^*} (N-1) H_N(0) \varepsilon^N \int_{B_{r_0/2}^+} \frac{y_N}{(\varepsilon^2 + |y|^2)^N} dy \\ &= \frac{S^{N/4}}{2} + O(\varepsilon^2) - \gamma_N^{2^*} (N-1) H_N(0) J_3 \varepsilon, \tag{6.40} \end{aligned}$$

where

$$J_3 = \int_{\mathbb{R}_+^N} \frac{y_N}{(1 + |y|^2)^N} dy > 0.$$

*Step four: Estimate for  $\|\nabla\psi_\varepsilon\|_{L^2(\Omega)}^2$ .* Arguing as previously,

$$\int_{\Omega} |\nabla\psi_\varepsilon(x)|^2 dx \leq \begin{cases} O(\varepsilon^2), & \text{if } N \geq 7 \\ O(\varepsilon^2 \log \frac{1}{\varepsilon}), & \text{if } N = 6 \\ O(\varepsilon), & \text{if } N = 5. \end{cases} \quad (6.41)$$

*Step five: Estimate for  $\|\psi_\varepsilon\|_{L^2(\Omega)}^2$ .* In the same way,

$$\int_{\Omega} |\psi_\varepsilon(x)|^2 dx \leq \begin{cases} O(\varepsilon^4), & \text{if } N \geq 9 \\ O(\varepsilon^4 \log \frac{1}{\varepsilon}), & \text{if } N = 8 \\ O(\varepsilon^{N-4}), & \text{if } N \in \{5, 6, 7\}. \end{cases} \quad (6.42)$$

*Step six: Conclusion.* By (6.39)-(6.42), for  $N \geq 6$ ,

$$\begin{aligned} \Sigma_\nu(\Omega) &\leq \frac{\int_{\Omega} |\Delta\psi_\varepsilon|^2 dx + \int_{\Omega} |\nabla\psi_\varepsilon|^2 dx + \alpha \int_{\Omega} |\psi_\varepsilon|^2 dx}{\left(\int_{\Omega} |\psi_\varepsilon|^{2^*} dx\right)^{\frac{2}{2^*}}} \\ &\leq \frac{S}{2^{4/N}} + o(\varepsilon) \\ &\quad - 2^{1-4/N} S^{1-N/4} H_N(0) \left[ d_N(N-2) \left( J_1 + \frac{4J_2}{N-1} \right) \right. \\ &\quad \left. - \frac{(N-4)(N-1)}{N} \gamma_N^{2^*} J_3 \right] \varepsilon. \end{aligned}$$

Now recall the explicit values of  $\gamma_N$ , and  $d_N$  in (6.11) and (6.38), respectively. In order to show that the term between the brackets is positive, it is enough to guarantee that

$$\beta_N := \frac{1}{N+2} \int_{\mathbb{R}_+^N} \frac{N+2|y|^2}{(1+|y|^2)^N} y_N dy - \int_{\mathbb{R}_+^N} \frac{y_N}{(1+|y|^2)^N} dy \text{ is positive.}$$

Denote by  $\mathbb{S}^{N-1}$  the unit sphere and by  $c(N)$  a positive constant that depends on  $N$ . Then,

$$\begin{aligned}
\beta_N &= \int_{\mathbb{S}^{N-1} \cap \mathbb{R}_+^N} y_N d\sigma \left( \frac{1}{N+2} \int_0^\infty \frac{N+2r^2}{(1+r^2)^N} r^N dr - \int_0^\infty \frac{r^N}{(1+r^2)^N} dr \right) \\
&= c(N) \int_0^\infty \frac{r^2-1}{(1+r^2)^N} r^N dr \\
&= c(N) \left( \int_0^\infty \frac{t^{\frac{N+1}{2}}}{(1+t)^N} dt - \int_0^\infty \frac{t^{\frac{N-1}{2}}}{(1+t)^N} dt \right) \\
&= c(N) \left[ \frac{\Gamma\left(\frac{N+1}{2}+1\right) \Gamma\left(\frac{N-1}{2}-1\right) - \Gamma\left(\frac{N+1}{2}\right) \Gamma\left(\frac{N-1}{2}\right)}{\Gamma(N)} \right] \\
&= c(N) \frac{\Gamma\left(\frac{N-3}{2}\right) \Gamma\left(\frac{N+1}{2}\right)}{\Gamma(N)},
\end{aligned}$$

which yields  $\beta_N > 0$ . Now, going back to the above inequality, and making  $\varepsilon$  sufficiently small we get our result for  $N \geq 6$ .

In case  $N = 5$ ,

$$\begin{aligned}
\Sigma_\nu(\Omega) &\leq \frac{S}{2^{4/N}} + O(\varepsilon) - 2^{14/5} \pi^2 \sqrt[4]{\frac{105}{S}} H_5(0) \varepsilon \log \frac{1}{\varepsilon} \\
&< \frac{S}{2^{4/N}},
\end{aligned}$$

provided  $\varepsilon$  is sufficiently small. This completes the proof.  $\square$

Now we are in position to give the proof of Theorem 6.1.1.

*Proof of Theorem 6.1.1.* By Lemmas 6.3.2 and 6.4.1, there exists a minimizer  $u \in M_\Omega$  for  $\Sigma_\nu(\Omega)$ . Now, we have to rule out  $u$  as the constant solution  $u_1 = \alpha^{\frac{N-4}{8}}$ . To this end, note that

$$\frac{\int_\Omega (|\Delta u_1|^2 + |\nabla u_1|^2 + \alpha|u_1|^2) dx}{\left(\int_\Omega |u_1|^{2^*} dx\right)^{\frac{2}{2^*}}} = \alpha|\Omega|^{4/N},$$

where  $|\Omega|$  stands for the Lebesgue measure of  $\Omega$ . Then, we are done if we have  $\bar{\alpha} > 0$  for which

$$\alpha|\Omega|^{4/N} > \Sigma_\nu(\Omega), \quad \text{for all } \alpha \geq \bar{\alpha}.$$

By Lemma 6.4.1, the above inequality follows by taking  $\bar{\alpha} = S/(2|\Omega|)^{4/N}$ . This completes the proof.  $\square$

## 6.5 A Sobolev inequality of second order

Our aim in this section is to prove Lemma 6.1.1. Our approach consists in providing a sharp inequality in  $H_\nu^2(\mathbb{R}_+^N)$ , and then by a partition of unity argument we establish our result for functions in  $H_\nu^2(\Omega)$ .

*proof of Lemma 6.1.1. Step one:* There holds  $\Sigma_\nu(\mathbb{R}_+^N) = S/2^{4/N}$ , and the infimum is not achieved. Consider the function  $z_\varepsilon$  defined in (6.15). By symmetry, (6.18), (6.19), and (6.20) we infer that  $z_\varepsilon \in H_\nu^2(\mathbb{R}_+^N)$  satisfies

$$\begin{aligned} \int_{\mathbb{R}_+^N} |\Delta z_\varepsilon(|x|)|^2 dx &= \frac{S^{N/4}}{2} + o(1), & \int_{\mathbb{R}_+^N} |\nabla z_\varepsilon(|x|)|^2 dx &= o(1), \\ \int_{\mathbb{R}_+^N} |z_\varepsilon(|x|)|^2 dx &= o(1), & \text{and} & \int_{\mathbb{R}_+^N} |z_\varepsilon(|x|)|^{2^*} dx &= \frac{S^{N/4}}{2} + o(1). \end{aligned}$$

As a consequence,

$$\lim_{\varepsilon \rightarrow 0} \frac{\int_{\mathbb{R}_+^N} (|\Delta z_\varepsilon|^2 + |\nabla z_\varepsilon|^2 + \alpha |z_\varepsilon|^2) dx}{\left( \int_{\mathbb{R}_+^N} |z_\varepsilon|^{2^*} dx \right)^{\frac{2}{2^*}}} = \frac{S}{2^{4/N}},$$

which shows that

$$\Sigma_\nu(\mathbb{R}_+^N) \leq \frac{S}{2^{4/N}}. \quad (6.43)$$

Now we argue by contradiction, that is, assume that there exists  $\phi \in H_\nu^2(\mathbb{R}_+^N)$  such that

$$\frac{\int_{\mathbb{R}_+^N} (|\Delta \phi|^2 + |\nabla \phi|^2 + \alpha |\phi|^2) dx}{\left( \int_{\mathbb{R}_+^N} |\phi|^{2^*} dx \right)^{\frac{2}{2^*}}} \leq \frac{S}{2^{4/N}}. \quad (6.44)$$

Define  $\tilde{\phi}$  as the reflection of  $\phi$  with respect to the  $x_N$ -axis,

$$\tilde{\phi}(x) = \begin{cases} \phi(x', x_N), & \text{if } x_N \geq 0 \\ \phi(x', -x_N), & \text{if } x_N < 0. \end{cases}$$

Since  $\partial_\nu \phi = 0$  along  $\partial \mathbb{R}_+^N$  it is easily seen that  $\tilde{\phi}$  belongs to  $H^2(\mathbb{R}^N)$ . Then, using the symmetry (doubling the integrals),

$$\frac{\int_{\mathbb{R}^N} (|\Delta \tilde{\phi}|^2 + |\nabla \tilde{\phi}|^2 + \alpha |\tilde{\phi}|^2) dx}{\left( \int_{\mathbb{R}^N} |\tilde{\phi}|^{2^*} dx \right)^{\frac{2}{2^*}}} \leq S.$$

However, this is a contradiction with Lemma 6.3.1. Therefore, there exists no  $\phi$  that belongs to  $H_\nu^2(\mathbb{R}_+^N)$  such that (6.44) holds. In other words,

$$\frac{\int_{\mathbb{R}_+^N} (|\Delta\phi|^2 + |\nabla\phi|^2 + \alpha|\phi|^2) dx}{\left(\int_{\mathbb{R}_+^N} |\phi|^{2^*} dx\right)^{\frac{2}{2^*}}} > \frac{S}{2^{4/N}}, \quad \text{for all } \phi \in H_\nu^2(\mathbb{R}_+^N).$$

The above inequality combined with (6.43) implies that  $\Sigma_\nu(\mathbb{R}_+^N) = S/2^{4/N}$ , and the infimum is not achieved.

*Step two: A partition of unity argument.* Since  $\bar{\Omega}$  is a compact set, we can find finitely many points  $x_i \in \bar{\Omega}$ , radii  $r_i > 0$ , with corresponding sets  $\Omega_i = \Omega \cap B_{r_i}(x_i)$  such that

$$\bar{\Omega} \subset \bigcup_{i=1}^n \Omega_i.$$

Up to increasing the number of open sets, we can assume that  $x_i \in \partial\Omega$  whenever  $\Omega_i \cap \partial\Omega \neq \emptyset$ . Now let  $(\tilde{\zeta}_i)_{i=1}^n$  be a smooth partition of unity subordinated to the covering  $(\Omega_i)_{i=1}^n$ . We split the set of indices as

$$\{1, 2, \dots, n\} = \mathcal{I} \cup \mathcal{J},$$

where  $\mathcal{I}$  contains the indices with  $x_i \in \Omega$  while  $\mathcal{J}$  contains the indices with  $x_i \in \partial\Omega$ .

*Case one:*  $\Omega_i \cap \Omega = \emptyset$ . We set

$$\zeta_i = \frac{\tilde{\zeta}_i^5}{\sum_{i=1}^n \tilde{\zeta}_i^5}.$$

By construction,  $(\zeta_i)_{i=1}^n$  is a partition of unity subordinated to the covering  $(\Omega_i)_{i=1}^n$  such that  $\zeta_i^{1/2} \in C^2(\bar{\Omega})$ . We denote by  $c_1$ , and  $c_2$  real positive constants such that  $|\nabla\zeta_i^{1/2}| \leq c_1$ , and  $|\Delta\zeta_i^{1/2}| \leq c_2$ .

Now choose any function  $\phi \in H^2(\mathbb{R}^N)$ . Consequently,  $\zeta_i^{1/2}\phi \in H^2(\mathbb{R}^N)$ ,

and  $\text{supp}(\zeta_i^{1/2}\phi) \subset \Omega_i$ . By Lemma 6.3.1, for  $\varepsilon_0 > 0$ ,

$$\begin{aligned} \sum_{i \in \mathcal{I}} \left( \int_{\Omega_i} |\zeta_i^{1/2}\phi|^{2^*} dx \right)^{\frac{2}{2^*}} &\leq \sum_{i \in \mathcal{I}} \left( \int_{\mathbb{R}^N} |\zeta_i^{1/2}\phi|^{2^*} dx \right)^{\frac{2}{2^*}} \\ &\leq \frac{1}{S} \sum_{i \in \mathcal{I}} \int_{\mathbb{R}^N} |\Delta(\zeta_i^{1/2}\phi)|^2 dx \\ &\leq \frac{2^{4/N}}{S} \sum_{i \in \mathcal{I}} \left[ \int_{\mathbb{R}^N} \left( \zeta_i^{1/2} |\Delta\phi| \right. \right. \\ &\quad \left. \left. + 2|\nabla\zeta_i^{1/2}| |\nabla\phi| + |\phi| |\Delta\zeta_i^{1/2}| \right)^2 dx \right] \\ &\leq \frac{2^{4/N}}{S} \left[ (1 + \varepsilon_0)^2 \|\Delta\phi\|_{L^2(\Omega)}^2 + B(\varepsilon_0) \|\phi\|_{H^1(\Omega)}^2 \right], \end{aligned}$$

where in the last inequality we have used Young inequality two times. Note that, for  $\varepsilon_0 > 0$  sufficiently small,

$$\frac{2^{4/N}}{S} (1 + \varepsilon_0)^2 \leq \frac{2^{4/N}}{S} + \varepsilon \quad \text{for } \varepsilon > 0,$$

so that,

$$\sum_{i \in \mathcal{I}} \left( \int_{\Omega_i} |\zeta_i^{1/2}\phi|^{2^*} dx \right)^{\frac{2}{2^*}} \leq \left( \frac{2^{4/N}}{S} + \varepsilon \right) \|\Delta\phi\|_{L^2(\Omega)}^2 + B(\varepsilon) \|\phi\|_{H^1(\Omega)}^2. \quad (6.45)$$

*Case two:*  $\Omega_i \cap \partial\Omega \neq \emptyset$ . In this case, for every  $i \in \mathcal{J}$  we consider the maps

$$\Phi_i^{-1} : \Omega_i \cap \partial\Omega \rightarrow V_i \subset \mathbb{R}_+^N$$

as defined in (6.29), where  $V_i$  is some open subset. As previously observed, these maps have the property that in this new coordinate system, any  $\phi \in H_\nu^2(\Omega)$  implies that  $(\zeta_i^{1/2}\phi) \circ \Phi_i$  belongs to  $H_\nu^2(\mathbb{R}_+^N)$  for every  $i \in \mathcal{J}$ . In this way, we may assume

$$|\det D\Phi(y)| \leq 1 + \varepsilon_0$$

for  $\varepsilon_0 > 0$  small enough, otherwise we may rearrange our covering in such a way that the sets  $(\Omega)_{i \in \mathcal{J}}$  have smaller sizes. For convenience, we write  $\vartheta_i(y) =$



$(\zeta_i^{1/2}\phi) \circ \Phi(y)$ . By the previous step,

$$\begin{aligned}
\sum_{i \in \mathcal{J}} \left( \int_{\Omega_i \cap \partial\Omega} |(\zeta_i^{1/2}\phi)(x)|^{2^*} dx \right)^{\frac{2}{2^*}} &= \sum_{i \in \mathcal{J}} \left( \int_{V_i} |\zeta_i^{1/2}\phi \circ \Phi(y)|^{2^*} |\det D\Phi(y)| dy \right)^{\frac{2}{2^*}} \\
&\leq (1 + \varepsilon_0)^{\frac{2}{2^*}} \sum_{i \in \mathcal{J}} \left( \int_{\mathbb{R}_+^N} |\vartheta_i(y)|^{2^*} dy \right)^{\frac{2}{2^*}} \\
&\leq \frac{2^{4/N}}{S} (1 + \varepsilon_0)^{\frac{2}{2^*}} \sum_{i \in \mathcal{J}} \left( \int_{\mathbb{R}_+^N} |\Delta\vartheta_i(y)|^2 dy \right. \\
&\quad \left. + \int_{\mathbb{R}_+^N} |\nabla\vartheta_i(y)|^2 dy + \alpha \int_{\mathbb{R}_+^N} |\vartheta_i(y)|^2 dy \right). \tag{6.46}
\end{aligned}$$

Now we recall that by (6.33) we may find  $\varepsilon_1 > 0$  small enough such that

$$\begin{cases} |\Delta\vartheta_i| \leq (1 + \varepsilon_1)|\Delta(\zeta_i^{1/2}\phi)| + \varepsilon_1|\nabla(\zeta_i^{1/2}\phi)| + \varepsilon_1|\zeta_i^{1/2}\phi| \\ |\nabla\vartheta_i| \leq (1 + \varepsilon_1)|\nabla(\zeta_i^{1/2}\phi)| + \varepsilon_1|\zeta_i^{1/2}\phi| \\ |\vartheta_i| \leq (1 + \varepsilon_1)|\zeta_i^{1/2}\phi|. \end{cases} \tag{6.47}$$

In addition,

$$\begin{cases} |\Delta(\zeta_i^{1/2}\phi)| \leq \zeta_i^{1/2}|\Delta\phi| + 2|\nabla\zeta_i^{1/2}||\nabla\phi| + |\phi||\Delta\zeta_i^{1/2}| \\ |\nabla(\zeta_i^{1/2}\phi)| \leq \zeta_i^{1/2}|\nabla\phi| + |\nabla\zeta_i^{1/2}||\phi|. \end{cases} \tag{6.48}$$

Then, by (6.47), and (6.48) together with Young inequality,

$$\sum_{i \in \mathcal{J}} \int_{\mathbb{R}_+^N} |\Delta\vartheta_i|^2 dy = (1 + \varepsilon_1)^2 \int_{\Omega} |\Delta\phi|^2 dy + B(\varepsilon_1)\|\phi\|_{H^1(\Omega)}^2. \tag{6.49}$$

Similarly,

$$\sum_{i \in \mathcal{J}} \int_{\mathbb{R}_+^N} |\nabla\vartheta_i|^2 dy \leq B(\varepsilon_1)\|\phi\|_{H^1(\Omega)}^2, \tag{6.50}$$

and

$$\sum_{i \in \mathcal{J}} \int_{\mathbb{R}_+^N} |\vartheta_i|^2 dy \leq (1 + \varepsilon_1)^2 \int_{\Omega} |\phi|^2 dy. \tag{6.51}$$

As previously, for  $\varepsilon_0, \varepsilon_1 > 0$  sufficiently small,

$$\frac{2^{4/N}}{S} (1 + \varepsilon_0)^{\frac{2}{2^*}} (1 + \varepsilon_1)^2 \leq \frac{2^{4/N}}{S} + \varepsilon \quad \text{for } \varepsilon > 0.$$

Hence, by inserting (6.49)-(6.51) into (6.46),

$$\sum_{i \in \mathcal{J}} \left( \int_{\Omega_i \cap \partial\Omega} |\zeta_i^{1/2} \phi|^{2^*} dx \right)^{\frac{2}{2^*}} \leq \left( \frac{2^{4/N}}{S} + \varepsilon \right) \|\Delta\phi\|_{L^2(\Omega)}^2 + B(\varepsilon) \|\phi\|_{H^1(\Omega)}^2.$$

Therefore, by the above inequality together with (6.45), for any  $\varepsilon > 0$ ,

$$\begin{aligned} \|\phi\|_{L^{2^*}(\Omega)}^2 &= \|\phi^2\|_{L^{2^*/2}(\Omega)} \\ &= \left\| \sum_{i=1}^n \zeta_i \phi^2 \right\|_{L^{2^*/2}(\Omega)} \\ &\leq \sum_{i=1}^n \|\zeta_i \phi^2\|_{L^{2^*/2}(\Omega)} \\ &= \sum_{i=1}^n \|\zeta_i^{1/2} \phi\|_{L^{2^*}(\Omega)}^2 \\ &= \sum_{i \in \mathcal{I}} \left( \int_{\Omega_i} |\zeta_i^{1/2} \phi|^{2^*} dx \right)^{\frac{2}{2^*}} + \sum_{i \in \mathcal{J}} \left( \int_{\Omega_i \cap \partial\Omega} |\zeta_i^{1/2} \phi|^{2^*} dx \right)^{\frac{2}{2^*}} \\ &\leq \left( \frac{2^{4/N}}{S} + \varepsilon \right) \|\Delta\phi\|_{L^2(\Omega)}^2 + B(\varepsilon) \|\phi\|_{H^1(\Omega)}^2. \end{aligned}$$

This completes the proof.  $\square$

## 6.6 Minimizing solutions for small $\alpha$

In this section, we prove the rigidity result for minimizing solutions when  $\alpha \rightarrow 0$ . The proof follows almost directly from the one of [9] for the second order case. We start with a  $L^q$ -bound, for  $1 \leq q \leq \frac{N+4}{N-4}$ , on positive solutions. The proof easily follows by integrating the equation.

**Lemma 6.6.1.** *Any nonnegative solution  $u$  of  $(P_\nu)$  satisfies*

$$\alpha \int_{\Omega} u dx = \int_{\Omega} |u|^{\frac{N+4}{N-4}} dx \leq \alpha^{\frac{N+4}{8}} |\Omega|.$$

The bound is clearly sharp. In the subcritical case, one can use elliptic regularity to bootstrap the corresponding estimate to get a better bound or use Gidas-Spruck blow-up technique [58] to show directly that  $u$  converges uniformly to zero as  $\alpha \rightarrow 0$ , see for instance [16, 91]. In the critical case, we need a further hypothesis to improve the bound as shown by the next lemma.

**Lemma 6.6.2.** *Assume that  $\alpha_k \rightarrow 0$  and the sequence  $(v_k)_k \subset H_\nu^2(\Omega)$  satisfying*

$$\begin{cases} \Delta^2 v_k - \Delta v_k + \alpha_k v_k = |v_k|^{\frac{8}{N-4}} v_k, & \text{in } \Omega, \\ \partial_\nu v_k = \partial_\nu(\Delta v_k) = 0, & \text{on } \partial\Omega, \end{cases} \quad (6.52)$$

*is such that  $v_k \geq 0$  and  $\sup_k \|v_k\|_{L^q(\Omega)} < \infty$  for some  $q > 2^*$ . Then  $\|v_k\|_{L^\infty(\Omega)} \rightarrow 0$ .*

*Proof.* Using elliptic regularity, one shows that a  $L^q$ -bound on  $v_k$ , with  $q = s(N+4)/(N-4)$ , gives a  $W^{4,s}$  estimate and therefore a  $L^{\frac{Ns}{N-4s}}$  bound. This allows to start a bootstrap argument if  $s > 2N/(N+4)$  and to deduce an a priori bound in  $C^{0,\gamma}(\bar{\Omega})$  for some  $0 < \gamma < 1$ . Lemma 6.6.1 then shows (with a simple interpolation argument) that  $v_k$  converges uniformly to 0 as  $k \rightarrow \infty$ .  $\square$

Observe that solutions with a priori finite energy are merely a priori bounded in  $L^{2^*}$  so that Lemma 6.6.3 cannot be used for those solutions. The next lemma shows that minimizing solutions are bounded in  $L^\infty$ .

**Lemma 6.6.3.** *Assume that  $u \in M_\Omega$  achieves  $\Sigma_\nu(\Omega)$  and  $\alpha \leq 1/4$ . Then  $u > 0$ . If we select  $v$  as the multiple of  $u$  that solves*

$$\begin{cases} \Delta^2 v - \Delta v + \alpha v = |v|^{\frac{8}{N-4}} v, & \text{in } \Omega, \\ \partial_\nu v = \partial_\nu(\Delta v) = 0, & \text{on } \partial\Omega, \end{cases}$$

*then*

$$\limsup_{\alpha \rightarrow 0} \|v\|_{L^\infty(\Omega)} < \infty.$$

*Proof.* Observe first that we know from Lemma 6.3.2 and Lemma 6.4.1 that  $\Sigma_\nu(\Omega)$  is indeed achieved. When  $\alpha$  is small enough (or the measure of  $\Omega$  is small enough), this is in fact simpler to show since for  $u_1 = 1$ , we have

$$\frac{\int_\Omega (|\Delta u_1|^2 + |\nabla u_1|^2 + \alpha |u_1|^2) dx}{\left(\int_\Omega |u_1|^{2^*} dx\right)^{\frac{2}{2^*}}} = \alpha |\Omega|^{4/N} < S/2^{4/N}.$$

It is by now standard to show that if  $u$  changes sign, then  $u$  cannot be a minimizer, see e.g., [17, 18]. We sketch the argument for completeness. We can write

$$\Sigma_\nu(\Omega) = \frac{\int_\Omega |-\Delta u + \frac{1}{2}u|^2 dx + (\alpha - \frac{1}{4}) \int_\Omega |u|^2 dx}{\left(\int_\Omega |u|^{2^*} dx\right)^{\frac{2}{2^*}}}.$$

If  $-\Delta(-)u + \frac{1}{2}(-)u \geq 0$  then  $(-)u > 0$  by the strong maximum principle (observe  $u \not\equiv 0$ ). If not, take  $v \in H_\nu^2(\Omega)$  to be the unique solution of

$$-\Delta v + \frac{1}{2}v = |-\Delta u + \frac{1}{2}u|, \quad x \in \Omega.$$

By the strong maximum principle, we infer that  $v(x) > |u(x)|$  in  $\Omega$  so that

$$\begin{aligned} 0 &\leq \int_{\Omega} \left| -\Delta v + \frac{1}{2}v \right|^2 dx + \left(\alpha - \frac{1}{4}\right) \int_{\Omega} |v|^2 dx \\ &< \int_{\Omega} \left| -\Delta u + \frac{1}{2}u \right|^2 dx + \left(\alpha - \frac{1}{4}\right) \int_{\Omega} |u|^2 dx \end{aligned}$$

and

$$\int_{\Omega} |v|^{2^*} dx > \int_{\Omega} |u|^{2^*} dx.$$

This contradicts the fact that  $u$  is a minimizer.

Now, since  $u$  is a minimizer and  $u \in M_{\Omega}$ , we have

$$\int_{\Omega} (|\Delta u|^2 + |\nabla u|^2 + \alpha|u|^2) dx \leq \alpha|\Omega|^{4/N}.$$

Take  $\alpha_k \rightarrow 0$  and denote by  $(u_k)_{k \in \mathbb{N}}$  a sequence of minimizers. Then

$$\int_{\Omega} (|\Delta u_k|^2 + |\nabla u_k|^2) dx \rightarrow 0 \quad \text{as } k \rightarrow \infty$$

and  $(u_k)_{k \in \mathbb{N}}$  is bounded in  $L^2(\Omega)$ . Observe that  $u_k$  solves the equation

$$\begin{cases} \Delta^2 u_k - \Delta u_k + \alpha_k u_k = \mu_k |u_k|^{\frac{8}{N-4}} u_k, & \text{in } \Omega, \\ \partial_{\nu} u_k = \partial_{\nu} (\Delta u_k) = 0, & \text{on } \partial\Omega, \end{cases}$$

where  $\mu_k = \Sigma_{\nu, \alpha_k}(\Omega) \leq \alpha_k |\Omega|^{4/N}$ . Define  $v_k = \mu_k^{\frac{N-4}{8}} u_k$  so that (6.52) holds. Interior estimates show that  $v_k$  is smooth. Clearly  $(v_k)_{k \in \mathbb{N}}$  converges strongly to zero in  $H^2(\Omega)$ . To show that  $\limsup_k \|v_k\|_{L^{\infty}(\Omega)} < \infty$ , one can just borrow the blow-up argument of Gidas and Spruck (arguing therefore by contradiction) used in [9, Lemma 2.1] by taking the blow-up profile

$$w_k(y) := t_k^{\frac{N-4}{2}} v_k(x_k + t_k y),$$

where  $(x_k)_{k \in \mathbb{N}} \subset \bar{\Omega}$  is such that

$$M_k = v_k(x_k) = \|v_k\|_{L^{\infty}(\Omega)}$$

and

$$M_k t_k^{\frac{N-4}{2}} = 1.$$

□

*Proof of Theorem 6.1.2.* Lemma 6.6.3 combined with Lemma 6.6.2 imply any sequence of minimizers uniformly vanish as  $\alpha \rightarrow 0$ . Using Poincaré inequality and the uniform convergence to zero, one then shows that  $u - \frac{1}{|\Omega|} \int_{\Omega} u dx = 0$  when  $\alpha$  is small enough. This is the original argument of Ni and Takagi [91], see also [9, 16]. □



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