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THESE

pour obtenir le grade de docteur

Discipline : Mathématiques

présentée par

Habib Fourti

Intitulée :

***Sur quelques problèmes elliptiques
non linéaires***

Soutenue le 19 décembre 2016 devant le jury composé de :

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À
mon père, ma mère,
mon frère et ma soeur,
mon épouse Warda,
mes enfants : Anes et Iyes.
Affectueusement.

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Notations générales

Symbole	Signification
Ω	Ouvert de \mathbb{R}^N
$\partial\Omega$	Frontière topologique de Ω
dx	Mesure de Lebesgue sur Ω
$d\sigma$	Mesure de surface sur $\partial\Omega$
ν	Normale unitaire extérieure à Ω
$\Pi = \mathbb{R}_+^N$	Demi espace
Δu	Laplacien de u
∇u	Gradient de u
$supp(u)$	Support de la fonction u
$\ \cdot\ _X$	Norme dans l'espace X
$d(x, A)$	Distance entre x et l'ensemble A
B_R	Boule de \mathbb{R}^N de rayon R centrée à l'origine
$B_R(x_0)$	Boule de \mathbb{R}^N de rayon R centrée en x_0
$\mathcal{C}(\Omega)$	Ensemble des fonctions continues dans Ω
$\mathcal{C}^k(\Omega)$	Ensemble des fonctions de classe k dans Ω
$\mathcal{C}^\infty(\Omega)$	Ensemble des fonctions indéfiniment différentiables dans Ω
$\mathcal{C}_c^\infty(\Omega) = \mathcal{D}(\Omega)$	Ensemble des fonctions \mathcal{C}^∞ à support compact dans Ω
$L^p(\Omega)$	Ensemble des fonctions mesurables sur Ω et $\int_\Omega u ^p dx < \infty$
$L^\infty(\Omega)$	Ensemble des fonctions mesurables et bornées sur Ω
$W^{m,p}(\Omega), W_0^{m,p}(\Omega)$	Espaces de Sobolev
$H^m(\Omega)$	L'espace de Sobolev $W^{m,2}(\Omega)$
$H_0^1(\Omega)$	L'espace de Sobolev $W_0^{1,2}(\Omega)$
$H^{-1}(\Omega)$	Le dual de $H_0^1(\Omega)$

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Introduction

Cette thèse a pour objet l'étude de quelques équations aux dérivées partielles non linéaires elliptiques de type Dirichlet ou Neumann. Ce sujet trouve son origine dans des problèmes de géométrie différentielle (problème de la courbure scalaire et problème de Yamabe) ou des problèmes de physique (les équations de Yang-Mills).

Ces équations ont une structure variationnelle, c'est-à-dire que leurs solutions correspondent aux points critiques d'une certaine fonctionnelle I , i.e. solutions de l'équation

$$\nabla I(u) = 0.$$

Ce document comporte trois chapitres. Le premier est entièrement consacré à l'exposé des définitions et résultats nécessaires pour la suite de ce travail. Dans ce chapitre, on présente des résultats de base sur les espaces de Sobolev, puis on aborde l'étude de quelques propriétés concernant des problèmes elliptiques non linéaires et on termine par des résultats classiques sur la théorie de Morse.

Dans le deuxième chapitre, nous traitons un problème de type courbure scalaire prescrite sur les domaines bornés réguliers de \mathbb{R}^3 . Plusieurs résultats ont été prouvés pour $N \geq 4$ et l'objectif étant d'étudier la dimension 3. On montre un résultat d'existence de type Bahri-Coron [4]. Notre méthode consiste à étudier un problème approché sous critique. A l'aide d'une analyse de "blow up" et quelques idées de O. Rey, nous analysons le comportement asymptotique des solutions de ce problème. Enfin, on conclut avec la théorie de Morse.

Dans le troisième chapitre, on se propose de donner une relation entre l'indice de Morse et la norme L^∞ des solutions de certaines équations aux dérivées partielles elliptiques sur des ouverts bornés de \mathbb{R}^N , $N \geq 2$, présentant une condition de non linéarité au bord. L'analyse de "blow up" de ces solutions ramène le problème initial

à une équation limite sur un demi espace ce qui nous a conduit à considérer les solutions d'un certain problème de type Liouville. Ces résultats sont similaires à ceux de Bahri-Lions [6].

La suite de cette introduction contient une description des principaux résultats de cette thèse.

0.1 Problème de type courbure scalaire en dimension 3

Ce travail a pour objectif l'étude du problème elliptique non linéaire suivant :

$$(P_0) \quad \begin{cases} -\Delta u = K(x)u^p \text{ dans } \Omega, \\ u > 0 \text{ dans } \Omega, \\ u = 0 \text{ sur } \partial\Omega, \end{cases}$$

où Ω est un ouvert borné régulier de \mathbb{R}^N , $N \geq 3$, K est une fonction strictement positive sur $\bar{\Omega}$ et $p+1 = \frac{2N}{N-2}$. L'exposant p est critique du point de vue des injections de Sobolev, dans le sens où l'injection $H_0^1(\Omega)$ dans $L^{p+1}(\Omega)$ est continue mais pas compacte. Il en résulte que la fonctionnelle associée à (P_0) ne vérifie pas la condition de Palais Smale.

L'intérêt de ce problème réside dans sa ressemblance au problème de la courbure scalaire. Etant donné une variété Riemannienne compacte sans bord (M, g) , le problème de la courbure scalaire consiste à trouver des conditions suffisantes pour qu'une fonction $K : M \rightarrow \mathbb{R}$ donnée soit la courbure scalaire d'une métrique $g' \in [g]$, la classe conforme de g . Ce problème se réduit à résoudre l'équation aux dérivées partielles non linéaire suivante :

$$(E) \quad \begin{cases} -4\frac{n-1}{n-2}\Delta_g u + R_g u = K(x)u^p \text{ dans } M, \\ u > 0 \text{ dans } M. \end{cases}$$

Le problème (E) a fait l'objet de nombreux travaux. Trois approches principales peuvent être citées : la première par minimisation, utilisée par T. Aubin et R. Schoen, la deuxième par approximation sous critique qui correspond à modifier la puissance p par $p - \varepsilon$ où $\varepsilon > 0$. Celle ci est connue aussi sous le nom "analyse de blow up". Elle est introduite par R. Schoen et développée plus tard par Y.Y. Li. Le cas sous-critique (avec une puissance $p - \varepsilon$) admet toujours des solutions (u_ε) grâce à la compacité de l'injection $H_0^1(\Omega) \hookrightarrow L^q(\Omega)$ pour $q < 2N/(N - 2)$. Lorsqu'on fait tendre ε vers

0, il est possible que les solutions qu'on gagne disparaissent. L'analyse de "blow up" consiste à comprendre la forme de ces solutions et comment et à quelle vitesse elles disparaissent. La dernière méthode "points critiques à l'infini" est introduite par A. Bahri [3]. Elle a pu résoudre les cas difficiles laissés par les autres méthodes. Elle consiste à étudier le cas critique directement et comprendre le défaut de compacité. On rappelle certains résultats antérieurs à ce propos. Concernant les variétés de dimension 3 non conformément équivalentes à la sphère, un résultat optimal est obtenu par Escobar-Schoen (1986), à savoir : (E) possède une solution si et seulement si $\max_{M^3} K > 0$. Dans le cas de la sphère, on a une obstruction topologique donnée par Kazdan-Warner (1975). Cette condition donne des exemples de fonctions K sur S^n telles que le problème n'a aucune solution. Ainsi, on devrait chercher des conditions suffisantes sur K pour que le problème admette une solution.

En 1991, Bahri et Coron [4] ont pu résoudre le cas de la sphère S^3 en utilisant des arguments topologiques. Sous certaines conditions sur K , ils ont montré que le problème (E) admette une solution si

$$1 \neq \sum_{y: \nabla K(y)=0, -\Delta K(y)>0} (-1)^{3-ind(K;y)}$$

où $ind(K; y)$ est l'indice de Morse de K au point critique y . Plus tard, Ben Ayed, Chen, Chtioui et Hammami [8] ont traité le cas des variétés de dimension 4 et ils ont pu établir un résultat de type Bahri-Coron. Ensuite, Ben Ayed et Hammami [9] ont mis en oeuvre un résultat similaire pour le cas des ouverts bornés réguliers de \mathbb{R}^4 .

Plusieurs résultats ont été prouvés pour $N \geq 4$. En fait, l'objet de cette partie est de traiter le cas des ouverts bornés réguliers de \mathbb{R}^3 .

0.1.1 Résultat d'existence

Avant d'énoncer le premier théorème, on commence par définir quelques notions qu'on utilisera par la suite. On désigne par G et H respectivement la fonction de Green du Laplacien avec la condition de Dirichlet au bord sur Ω , et sa partie régulière, i.e.

$$\begin{cases} G(x, y) = \frac{1}{|x - y|} - H(x, y) & (x, y) \in \Omega \times \Omega \\ \Delta_x H = 0 \text{ dans } \Omega \times \Omega, & G(\cdot, y) = 0 \text{ sur } \partial\Omega \ \forall y \in \Omega. \end{cases}$$

Pour $k \in \mathbb{N}^*$ et $x = (x_1, \dots, x_k) \in \Omega^k$ où $x_i \neq x_j$ pour $i \neq j$, on pose $M(x) = (m_{ij})_{1 \leq i, j \leq k}$ la matrice définie par

$$m_{ii} = \frac{H(x_i, x_i)}{\sqrt{K(x_i)}}; \quad m_{ij} = -\frac{G(x_i, x_j)}{(K(x_i)K(x_j))^{\frac{1}{4}}} \quad (0.1.1)$$

et on désigne par $\rho(x)$ la plus petite valeur propre de $M(x)$.

On admettra les hypothèses suivantes :

(H₁) K est une fonction strictement positive de classe \mathcal{C}^3 sur $\bar{\Omega} \subset \mathbb{R}^3$ admettant seulement des points critiques non dégénérés y_1, \dots, y_m .

(H₂) Pour chaque $x \in \partial\Omega$, $\frac{\partial K(x)}{\partial \nu} < 0$ où ν est la normale extérieure à $\partial\Omega$.

(H₃) Pour tout s -uplet $\tau_s = (i_1, \dots, i_s) \in (1, \dots, m)^s$ avec $i_j \neq i_q$ pour $j \neq q$, $M(\tau_s) = M(y_{i_1}, \dots, y_{i_s})$ est non dégénérée.

On a un résultat de type Bahri-Coron.

Théorème 0.1.1. *Supposons que les hypothèses (H₁), (H₂) et (H₃) sont satisfaites. Si*

$$1 \neq \sum_{s=1}^m \sum_{\tau_s=(i_1, \dots, i_s)/M(\tau_s)>0} (-1)^{4s-1-\sum_{j=1}^s k_{i_j}},$$

où $k_{i_j} = \text{ind}(K, y_{i_j})$ et $m = \#\{y/\nabla K(y) = 0\}$, alors (P_0) possède au moins une solution.

Pour prouver ce résultat, on va suivre la méthode d'approximation qui consiste à bien comprendre les propriétés des solutions du problème approché de (P_0) .

0.1.2 Approximation du problème et analyse de blow up

Pour tout $\varepsilon > 0$, on pose le problème (P_ε) suivant :

$$(P_\varepsilon) \quad \begin{cases} -\Delta u = Ku^{p-\varepsilon} \text{ dans } \Omega, \\ u > 0 \text{ dans } \Omega, \\ u = 0 \text{ sur } \partial\Omega, \end{cases}$$

où Ω est un ouvert borné régulier de \mathbb{R}^N , $N \geq 3$ et $p+1 = \frac{2N}{N-2}$. Dans ce qui suit, on s'intéresse plus particulièrement au comportement asymptotique par rapport à ε des solutions du problème (P_ε) quand $\varepsilon > 0$ tend vers 0. On peut aborder le problème (P_0) via la recherche des points critiques non triviaux de la fonctionnelle définie sur $H_0^1(\Omega)$ par :

$$I_\varepsilon(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 - \frac{1}{p-\varepsilon} \int_\Omega K |u|^{p-\varepsilon}.$$

Les solutions de (P_ε) correspondraient aux points critiques positifs de I_ε et inversement.

Le problème (P_ε) a été traité dans plusieurs travaux dont on cite l'article de Y.Y. Li [20] qui étudie le cas de la sphère S^N pour $N \geq 4$. Concernant le cas des domaines bornés et réguliers de \mathbb{R}^N , $N \geq 4$, Bahri-Li-Rey [7] ont traité le même problème (P_ε) dans le cas où la fonction K est identiquement constante égale à 1, connu sous le nom de problème de Yamabe. Plus tard, O. Rey [26] a pu étendre le même résultat pour la dimension $N = 3$, qui pose certaines difficultés par rapport aux dimensions supérieures.

Dans la suite, on prendra $N = 3$ ce qui entraîne que $p = 5$.

Soient $\lambda \in \mathbb{R}_+^*$ et $a \in \mathbb{R}^3$, nous désignons par $\delta_{a,\lambda}$ la fonction définie sur \mathbb{R}^3 par

$$\delta_{a,\lambda}(x) = \sqrt[4]{3} \frac{\lambda^{\frac{1}{2}}}{(1 + \lambda^2 |x - a|^2)^{\frac{1}{2}}}.$$

Ces fonctions sont les seules solutions de l'équation :

$$-\Delta u = u^5 \text{ dans } \mathbb{R}^3, \quad u > 0, \quad u \in L^6(\mathbb{R}^3) \quad \text{et} \quad \nabla u \in (L^2(\mathbb{R}^3))^3.$$

Soit $P\delta_{a,\lambda}$ la projection de $\delta_{a,\lambda}$ sur $H_0^1(\Omega)$; c'est-à-dire

$$\Delta P\delta_{a,\lambda} = \Delta \delta_{a,\lambda} \text{ sur } \Omega, \quad P\delta_{a,\lambda} = 0 \text{ sur } \partial\Omega.$$

On définit aussi

$$\begin{aligned} F_x &: (0, +\infty)^k \longrightarrow \mathbb{R} \\ \Lambda &= (\Lambda_1, \dots, \Lambda_k) \longrightarrow \frac{1}{2} \Lambda M(x)^t \Lambda - \sum_{i=1}^k \frac{\ln \Lambda_i}{\sqrt{K(x_i)}}. \end{aligned}$$

Etant supposé que la famille des solutions (u_ε) est bornée dans $H_0^1(\Omega)$, l'analyse de "blow up" entraîne que [22] [28] :

$$u_\varepsilon = u_0 + \sum_{i=1}^k \frac{1}{\sqrt[4]{K(a_i^\varepsilon)}} P\delta_{a_i^\varepsilon, \lambda_i^\varepsilon} + v_\varepsilon.$$

où u_0 est une solution de (P_0) ou identiquement nulle et $\|v_\varepsilon\|_{H_0^1} \rightarrow 0$. De plus on a :

- $a_i^\varepsilon \in \Omega, a_i^\varepsilon \rightarrow \bar{a}_i \in \bar{\Omega}, \forall i,$
- $\varepsilon \log \lambda_i^\varepsilon \rightarrow 0, \lambda_i^\varepsilon d_i \rightarrow +\infty, \forall i$ (où $d_i = d(a_i, \partial\Omega)$) et
- $\varepsilon_{i,j} = \left(\frac{\lambda_i^\varepsilon}{\lambda_j^\varepsilon} + \frac{\lambda_j^\varepsilon}{\lambda_i^\varepsilon} + \lambda_i^\varepsilon \lambda_j^\varepsilon |a_i^\varepsilon - a_j^\varepsilon|^2 \right)^{-\frac{1}{2}} \rightarrow 0, \forall i, j, i \neq j.$

Pour le problème de Yamabe, R. Schoen [27] prouve qu'on a $k = 0$ ou $u_0 \equiv 0$.

Dans notre cas, pour prouver Théorème 0.1.1, en raisonnant par l'absurde, on suppose désormais que le problème (P_0) n'a pas de solution et donc $u_0 \equiv 0$ dans le développement précédent. Plus précisément, le résultat suivant présente une caractérisation des variables a_i^ε et λ_i^ε .

Théorème 0.1.2. *On suppose que les hypothèses (H_1) , (H_2) et (H_3) sont satisfaites et que (P_0) n'admet pas de solution. Soit A une constante positive assez grande. On suppose que (u_ε) est une suite de solutions de (P_ε) qui satisfait $I_\varepsilon(u_\varepsilon) < A$. Alors, elle doit exploser en k points $\bar{a}_1, \dots, \bar{a}_k$ de $\bar{\Omega}$ avec $k \leq \sharp\{y/\nabla K(y) = 0\}$. Ces solutions devraient s'écrire sous la forme :*

$$u_\varepsilon = \sum_{i=1}^k \frac{1}{\sqrt[4]{K(a_i^\varepsilon)}} P\delta_{a_i^\varepsilon, \lambda_i^\varepsilon} + v_\varepsilon, \text{ avec } \|v_\varepsilon\|_{H_0^1} \rightarrow 0.$$

De plus, on a

1. $(a_1^\varepsilon, \dots, a_k^\varepsilon) \in \Omega_{d_0}^k$, avec $d_0 = d_0(\Omega) > 0$ et $\Omega_{d_0} = \{x \in \Omega / d(x, \partial\Omega) > d_0\}$
2. $|a_i^\varepsilon - a_j^\varepsilon| \geq d'_0 \quad \forall i \neq j$, avec $d'_0 > 0$.
3. $\forall i \neq j, c \leq \frac{\lambda_i^\varepsilon}{\lambda_j^\varepsilon} \leq C$ pour certaines constantes positives c et C .
4. $\forall i \in \{1, \dots, k\}$, $a_i^\varepsilon \rightarrow y_i$ où y_i est un point critique de K tel que $\rho(y_1, \dots, y_k) > 0$.
5. $\varepsilon^{-\frac{1}{2}}\Lambda \rightarrow c\bar{\Lambda}$ où $\bar{\Lambda}$ est un point critique de F_y où $y = (y_1, \dots, y_k)$ ainsi défini dans 4, $\Lambda := (\frac{1}{\sqrt{\lambda_1^\varepsilon}}, \dots, \frac{1}{\sqrt{\lambda_k^\varepsilon}})$ et c est une certaine constante strictement positive.

Remarque 0.1.1. *Les propriétés 1, 2 et 3 sont prises comme hypothèses dans les travaux [7] et [26] concernant le problème de Yamabe (résultats prouvés par Han et R. Schoen pour le problème de Yamabe).*

Notons que, dans ces travaux [7] et [26], la fonction $K \equiv 1$. Par contre, dans notre cas, la fonction K est non identiquement constante. On signale que, pour $K(x) = |x|^\gamma$ et $\Omega = B(0, 1)$, Cao-Peng [11] ont montré que le point maximum de la fonction minimisante de l'énergie associée au problème converge vers le bord de Ω lorsque ε tend vers 0. Ce résultat impose une justification de la première propriété dans notre cas. Pour éliminer ce genre d'exemple, on a imposé à la fonction K l'hypothèse (H_2) à savoir $\partial K / \partial \nu < 0$ qui va jouer un rôle important pour pousser les points de concentration à l'intérieur du domaine. C'est la difficulté essentielle de notre travail. Dans ce genre de problème, la partie v_ε ne joue pas un rôle essentiel. On commence par donner son estimation.

Lemme 0.1.2. $\|v_\varepsilon\|_{H_0^1} = O\left(\varepsilon + \sum_{i=1}^k \left(\frac{1}{\lambda_i^\varepsilon d_i} + \frac{|\nabla K(a_i^\varepsilon)|}{\lambda_i^\varepsilon}\right) + \sum_{i \neq j} \varepsilon_{i,j} \log^{\frac{1}{3}}(\varepsilon_{i,j}^{-1})\right)$.

Cette estimation n'est pas suffisante pour justifier que les points de concentration sont loins du bord. En effet, dans certains développements, $O(\|v_\varepsilon\|^2)$ apparait qui est de l'ordre $1/(\lambda^\varepsilon d)^2$. Connaissant que dans ces développements la partie principale est de l'ordre c/λ^ε , ainsi lorsque le point s'approche du bord, il est possible que c/λ^ε ne domine pas $1/(\lambda^\varepsilon d)^2$. Pour contourner cette difficulté, en utilisant la méthode de "blow up" développée par Y.Y. Li dans [20], on prouve d'abord que

Proposition 0.1.3. *Il existe une constante positive c telle que*

$$|a_i^\varepsilon - a_j^\varepsilon| \geq c \max(d_i, d_j), \forall i \neq j.$$

Cette dernière information nous a permis d'adapter les idées d'O. Rey. On signale que dans [26], O. Rey a pris des boules disjointes $B(a_i^\varepsilon, r)$ où r est un rayon constant. Dans notre cas, la proposition précédente nous a permis de prendre des boules disjointes $B(a_i^\varepsilon, cd_i)$. Ainsi on doit faire attention à l'influence de la distance d_i sur nos estimations. Cette démarche consiste à estimer la partie impaire de v qui nous aboutit à une amélioration de l'équation (E_{a_i}) où on fait apparaître des quantités de l'ordre $1/(\lambda^\varepsilon d)^2$ dans le terme principal.

Proposition 0.1.4. *Soit a_i^ε un point proche du bord de Ω . Alors, on a*

$$\begin{aligned} & \frac{\Gamma_3}{(K(a_i^\varepsilon))^{\frac{5}{4}}} \frac{\nabla K(a_i^\varepsilon)}{\lambda_i} (1 + o(1)) - \frac{\Gamma_4}{\sqrt[4]{K(a_i^\varepsilon)} (\lambda_i^\varepsilon)^2} \frac{\partial H}{\partial x}(a_i^\varepsilon, a_i^\varepsilon) \\ & + \sum_{j \neq i} \frac{\Gamma_4}{\sqrt[4]{K(a_j^\varepsilon)} (\lambda_i^\varepsilon)^{\frac{3}{2}} (\lambda_j^\varepsilon)^{\frac{1}{2}}} \frac{\partial G}{\partial x}(a_i^\varepsilon, a_j^\varepsilon) = o\left(\varepsilon^2 + \sum_{j=1}^k \frac{1}{(\lambda_j^\varepsilon d_j)^2}\right). \end{aligned} \quad (0.1.2)$$

En faisant le produit scalaire des termes principaux de (0.1.2) par la normale extérieure au bord ν_i au point a_i , on remarque que $-\partial H / \partial \nu_i(a_i^\varepsilon, a_i^\varepsilon) < 0$ et $\partial G / \partial \nu_i(a_i^\varepsilon, a_j^\varepsilon) < 0$ pour $d_i \leq d_j$. En imposant que $\partial K / \partial \nu < 0$, on aura que tous les termes sont négatifs. Ce qui présente une information très importante dans notre approche. Elle nous permet de dégager une contradiction si on suppose que la plus petite distance d_i tend vers 0. Ainsi, les propriétés 1 et 2 de Théorème 0.1.2 sont justifiées. Concernant le reste des propriétés, elles découlent en suivant les idées introduites dans [7].

0.1.3 Construction de solutions du problème approché

Dans cette partie, on justifie que, pour toute configuration des points y_1, \dots, y_s tels que $\rho(y_1, \dots, y_s) > 0$, il existe une solution (u_ε) du problème (P_ε) qui explose en y_1, \dots, y_s . Précisément, on a :

Théorème 0.1.3. *Soient K une fonction positive de classe \mathcal{C}^2 sur $\overline{\Omega}$ et y_1, \dots, y_k des points critiques non dégénérés distincts de K tels que $\rho(y_1, \dots, y_k) > 0$. Alors, il existe $\varepsilon_0 > 0$ tel que pour tout $\varepsilon \in (0, \varepsilon_0]$, (P_ε) admet une solution de la forme*

$$u_\varepsilon = \sum_{i=1}^k \alpha_i^\varepsilon P \delta_{a_i^\varepsilon, \lambda_i^\varepsilon} + v_\varepsilon$$

satisfaisant

$\left| \alpha_i^\varepsilon - \frac{1}{\sqrt[4]{k(a_i^\varepsilon)}} \right| = O(\varepsilon |\ln \varepsilon|)$, $\lambda_i^\varepsilon \varepsilon \rightarrow \bar{c}$, $|a_i^\varepsilon - y_i| = O(\varepsilon |\ln \varepsilon|)$ et $\|v_\varepsilon\|_{H_0^1} \rightarrow 0$ lorsque $\varepsilon \rightarrow 0$.

De plus, on a $|\nabla u_\varepsilon|^2 \rightarrow S_3^{3/2} \sum_{i=1}^k \frac{1}{\sqrt[4]{K(y_i)}} \delta_{y_i}$ dans $\mathcal{D}'(\Omega)$ lorsque $\varepsilon \rightarrow 0$, où δ_{y_i} désigne la masse de Dirac au point y_i .

L'idée de la preuve de ce théorème est introduite pour la première fois dans [7]. Elle consiste à réduire le problème à trouver les a_i^ε , λ_i^ε , α_i^ε et v_ε . La première étape consiste à minimiser par rapport à v_ε . Ensuite, le problème revient à chercher un point critique dans un espace de dimension finie. Cette méthode est devenue classique et la difficulté essentielle est de bien estimer certains développements de la dérivée de la fonctionnelle associée au problème (P_ε) .

Remarque 0.1.5. *Sous les hypothèses de non dégénérescence des points critiques de K et des matrices associées, des idées de Y.Y. Li [21] entraînent l'unicité de la solution u_ε qui explose en y_1, \dots, y_s .*

Une fois Théorèmes 0.1.2 et 0.1.3 sont établis, en utilisant la théorie de Morse [23] et la déformation des ensembles de niveau, on montre Théorème 0.1.1.

0.2 Fonctions harmoniques avec condition de Neumann non linéaire au bord et leurs indices de Morse

L'indice de Morse permet d'avoir des résultats d'existence, de classification et de régularité des solutions d'un certain problème. Il est alors clairement utile d'essayer

de le relier à d'autres propriétés de ces solutions, telle que la norme L^∞ , afin de mieux comprendre les équations elliptiques sur-linéaires avec un comportement sous-critique sur les domaines bornés. Ce sujet de recherche a fait l'objet de plusieurs travaux dont on cite l'article de Bahri-Lions [6]. Ils ont étudié le problème suivant

$$-\Delta u = f(x, u) \text{ dans } \Omega, \quad u = 0 \text{ sur } \partial\Omega. \quad (0.2.1)$$

Sous certaines conditions sur f , ils ont pu démontré qu'une suite de solutions $(u_n)_n$ de (0.2.1) est bornée dans $L^\infty(\bar{\Omega})$ si et seulement si $(i(u_n))_n$ est bornée. En utilisant un argument de "blow up" classique, ils se sont ramenés à traiter les systèmes suivants

$$\begin{cases} -\Delta u = |u|^{q-1}u & \text{dans } \mathbb{R}^N \\ u \in \mathcal{C}_b^2(\mathbb{R}^N), \quad i(u) < \infty \end{cases}$$

et

$$\begin{cases} -\Delta u = |u|^{q-1}u & \text{dans } \mathbb{R}_+^N, \quad u = 0 \text{ sur } \partial\mathbb{R}_+^N \\ u \in \mathcal{C}_b^2(\mathbb{R}_+^N), \quad i(u) < \infty \end{cases}$$

où $i(u)$ est l'indice de Morse de u , \mathcal{C}_b^2 représente l'ensemble des fonctions bornées de classe \mathcal{C}^2 , $1 < q < \frac{N+2}{N-2}$ lorsque $N \geq 3$ et $q \in (1, \infty)$ lorsque $N = 2$. Ils ont prouvé que les problèmes précédents possèdent uniquement la solution triviale $u \equiv 0$. Leur démonstration utilise un argument spectral combiné avec une version de l'identité bien connue de Pohozaev [24]. Ce théorème de type Liouville étend le résultat de non existence des solutions strictement positives dans [14] à celles changeant de signe d'indice de Morse fini. Plus tard, Harrabi et al [17, 18] généralisent le travail de Bahri et Lions pour d'autres non-linéarités. Récemment, Harrabi et al [16] ont étudié le problème (0.2.1) avec la condition de Neumann au lieu de Dirichlet.

0.2.1 Estimation L^∞ a priori des solutions d'une équation elliptique non linéaire ayant un indice de Morse fini

Dans cette section, on s'intéresse à la relation entre la norme L^∞ et l'indice de Morse des solutions du problème suivant :

$$(P) \quad \begin{cases} \Delta u = 0 & \text{dans } \Omega, \\ \frac{\partial u}{\partial \nu} = f(x, u) & \text{sur } \partial\Omega, \end{cases}$$

où Ω est un ouvert borné régulier et $f(x, t)$ est continue sur $\partial\Omega \times \mathbb{R}$, différentiable par rapport à t et $\frac{\partial f}{\partial t}$ est continue sur $\partial\Omega \times \mathbb{R}$. On traite le cas sur-linéaire :

$$\lim_{|t| \rightarrow +\infty} f(x, t)t^{-1} = +\infty, \quad \text{uniformément sur } \partial\Omega \quad (0.2.2)$$

à croissance sous critique

$$|f(x, t)| \leq a(1 + |t|^p), \quad a > 0, \quad (0.2.3)$$

où p satisfait

$$1 < p < N/(N - 2) \text{ si } N \geq 3 \text{ et } p \in (1, \infty) \text{ si } N = 2. \quad (0.2.4)$$

Ces conditions entraînent qu'on ait dans le cas sur-linéaire et sous critique. Par conséquent, en utilisant la compacité de l'injection $H^1(\Omega) \hookrightarrow L^{p+1}(\partial\Omega)$, des méthodes variationnelles justifient l'existence de solutions pour le problème (P) .

Dans la suite, on suppose que f vérifie

$$\lim_{|t| \rightarrow +\infty} f(x, t)t^{-1}|t|^{-(p-1)} = b \quad \text{uniformément sur } \partial\Omega, \quad \text{où } b > 0 \text{ et} \quad (0.2.5)$$

p satisfait (0.2.4). Notons que la condition (0.2.5) implique (0.2.2) et (0.2.3).

Définition 0.2.1. L'indice de Morse d'une solution du problème (P) , noté $i(u)$, est la dimension de l'espace de négativité de sa forme quadratique associée q définie dans $H^1(\Omega)$ par

$$q(h) = \int_{\Omega} |\nabla h|^2 - \int_{\partial\Omega} \frac{\partial f}{\partial t}(x, u)h^2.$$

On a le résultat suivant :

Théorème 0.2.1. *Soit $(u_n)_n$ une suite de solutions de (P) . Sous les hypothèses (0.2.4) et (0.2.5), $(u_n)_n$ est bornée dans $L^\infty(\bar{\Omega})$ si et seulement si $(i(u_n))_n$ est bornée.*

La preuve du premier sens de l'équivalence est basée sur un argument spectral, à savoir les valeurs propres du $\partial./\partial\nu$ (défini sur $H := \{h \in H^1(\Omega) : \Delta h = 0\}$) forment une suite croissante tendant vers ∞ .

Pour prouver le deuxième sens, on a besoin d'un théorème de type Liouville après une analyse de "blow up".

On raisonne par l'absurde et ceci en considérant la suite $(u_n)_n \subset H^1(\Omega) \cap \mathcal{C}(\bar{\Omega})$ de solutions de (P) telles que $\|u_n\|_{L^\infty(\Omega)} \rightarrow +\infty$ et $i(u_n)$ reste borné. Soient $M_n = \max_{\bar{\Omega}} |u_n|$ et x_n un point maximum de $|u_n|$. Etant donné que u_n est harmonique, d'après le principe du maximum, x_n est situé sur $\partial\Omega$. Quitte à considérer une sous-suite, x_n converge vers un certain $\bar{x} \in \partial\Omega$. Puis on utilise un argument classique de "blow up" similaire à celui utilisé par Gidas-Spruck [14] et Bahri-Lions [6]. Cette technique consiste à un changement d'échelle obtenu par dilatation, translation et

normalisation par le maximum absolu. Ce changement nous permet d'obtenir une nouvelle suite \tilde{v}_n d'un autre problème (\tilde{P}_n) définie sur un domaine $\tilde{\Omega}_n$. Ce nouveau domaine convergera vers un demi-espace qu'on note par Π et $\tilde{v}_n \rightarrow \tilde{v}$ dans $\mathcal{C}_{loc}^1(\bar{\Pi})$. Par conséquent, on se ramène à étudier la solution \tilde{v} du problème de type Liouville suivant

$$(PL) \quad \begin{cases} \Delta u = 0 & \text{dans } \Pi \\ \frac{\partial u}{\partial \nu} = b|u|^{p-1}u & \text{sur } \partial\Pi. \end{cases}$$

Cette solution limite vérifie de plus $|\tilde{v}(0)| = 1$, $|\tilde{v}| \leq 1$ sur $\bar{\Pi}$ et $i(\tilde{v}) < \infty$ où $i(\tilde{v})$ est l'indice de Morse de la solution \tilde{v} , défini par la dimension de l'espace de négativité de sa forme quadratique associée q donnée par

$$q(h) = \int_{\Pi} |\nabla h|^2 - p \int_{\partial\Pi} |\tilde{v}|^{p-1} h^2, \quad \forall h \in H^1(\Pi).$$

0.2.2 Théorème de type Liouville

Théorème 0.2.2. *Sous l'hypothèse (0.2.4), soit $u \in \mathcal{C}^1(\bar{\Pi})$ une solution bornée de (PL) satisfaisant $i(u) < \infty$. Alors, $u \equiv 0$ dans $\bar{\Pi}$.*

Ce résultat sera utile pour prouver que le phénomène de "blow up" ne se produit pas sous l'hypothèse de la bornitude de l'indice de Morse. En effet, Théorème 0.2.2 implique que $\tilde{v} \equiv 0$, en contradiction avec $|\tilde{v}(0)| = 1$, ce qui achève la démonstration de Théorème 0.2.1.

Pour prouver Théorème 0.2.2, on s'est inspiré de la démarche de Bahri et Lions développée dans [6]. Notre méthode est basée sur des estimations prudentes de l'énergie $\int |u|^{p+1}$ sur des boules ou des anneaux du bord de Π . On a combiné l'information spectrale contenue dans la finitude de l'indice de Morse de u avec certaines identités (dont notamment l'identité de Pohozaev) vérifiées par les solutions du problème limite (PL).

On signale que dans [6], le terme non linéaire $|u|^{p-1}u$ apparaît à l'intérieur du domaine, par contre dans notre cas, la non-linéarité est au bord. Par conséquent, la démarche de Bahri et Lions n'est pas intégralement applicable pour notre problème (PL).

Pour contourner cette difficulté, on a essayé de dégager quelques informations concernant la solution u sur $\partial\Pi$. Nous avons besoin de choisir des fonctions tests appropriées pour ne faire apparaître que les intégrales sur $\partial\Pi$. Nous introduisons alors une nouvelle

fonction test $\phi_{r,R}$ extension harmonique bornée dans Π de la fonction $\varphi_{r,R}$ (fonction test introduite dans [17]) i. e. la solution bornée du système suivant

$$\begin{cases} \Delta\phi = 0 & \text{dans } \Pi, \\ \phi = \varphi_{r,R} & \text{sur } \partial\Pi. \end{cases}$$

Notons que l'information sur l'indice de Morse fini pour une solution de (PL) permet d'établir un certain genre de stabilité de la solution u à l'extérieur d'un compact, à savoir

Lemme 0.2.2. *Soit u une solution de (PL) telle que $i(u) < \infty$. Alors, il existe $r_0 > 0$ tel que $\forall R > 2r_0$, on a $q(u\phi_{r_0,R}) \geq 0$.*

Compte tenu de Lemme 0.2.2 et en utilisant un argument "boot strap", on montre

Proposition 0.2.3. *Soit u une solution bornée de (PL) vérifiant $i(u) < \infty$. Alors on a $\int_{\partial\Pi} |u|^{p+1} < \infty$.*

La preuve de Théorème 0.2.2 est basée sur la proposition précédente. En effet, si u est solution de (PL) vérifiant $\int_{\partial\Pi} |u|^{p+1} < \infty$ alors $\int_{\Pi} |\nabla u|^2 = \int_{\partial\Pi} |u|^{p+1} < \infty$ et l'identité de Pohozaev [24] entraîne que $u \equiv 0$.

Chapitre 1

Préliminaires

Dans ce chapitre, nous collectons plusieurs outils de base qui seront nécessaires tout au long de ce travail. Le lien commun entre tous les résultats de ce chapitre, c'est qu'ils sont préparatoires pour les principaux résultats, qui sont contenus dans les chapitres suivants.

1.1 Espaces fonctionnels

Dans cette section nous définissons les espaces de Sobolev qui sont les espaces "naturels" de fonctions permettant de résoudre les formulations variationnelles des équations aux dérivées partielles. Physiquement les espaces de Sobolev s'interprètent comme des espaces de fonctions d'énergie finie. Il s'avère donc judicieux d'en faire une brève présentation avant d'aborder ces équations. Nous reprenons dans cette section certains énoncés de Brézis [10] et de Kavian [19], pour une présentation plus complète des espaces de Sobolev, on pourra aussi voir Adams [1].

Soit Ω un domaine ouvert de \mathbb{R}^N , $\mathcal{D}(\Omega)$ désigne l'ensemble des fonctions de classe \mathcal{C}^∞ et à support compact dans Ω .

Pour $1 \leq p < +\infty$, l'espace de Lebesgue $L^p(\Omega)$ est défini par :

$$L^p(\Omega) = \left\{ u : \Omega \rightarrow \mathbb{R} \text{ mesurable ; } \int_{\Omega} |u|^p dx < \infty \right\}$$

muni de la norme

$$\|u\|_{L^p} = \left(\int_{\Omega} |u|^p dx \right)^{\frac{1}{p}}.$$

Pour $p = \infty$, on note

$$L^\infty(\Omega) = \left\{ u : \Omega \rightarrow \mathbb{R} \text{ mesurable ; } \operatorname{ess-sup}_{\Omega} |u| < \infty \right\}$$

avec

$$\text{ess-sup}_{\Omega} |u| = \inf\{C > 0; |u(x)| \leq C \text{ p.p. dans } \Omega\}.$$

$L^{\infty}(\Omega)$ est muni de la norme suivante : $\|u\|_{L^{\infty}} = \text{ess-sup}_{\Omega} |u|$.

1.1.1 Les espaces de Sobolev $W^{m,p}(\Omega)$

Soit $\Omega \in \mathbb{R}^N$ un ouvert borné et régulier et soit $p \in \mathbb{R}$ avec $1 \leq p < \infty$:

Définition 1.1.1. Soit $m > 1$ un entier et p un réel tel que $1 \leq p < \infty$: On définit

$W^{m,p}(\Omega) = \{u \in L^p(\Omega); \forall \alpha \text{ multi-indice avec } |\alpha| \leq m, \exists g_{\alpha} \in L^p(\Omega) \text{ telle que}$

$$\int_{\Omega} u D^{\alpha} \varphi = (-1)^{|\alpha|} \int_{\Omega} g_{\alpha} \varphi \quad \forall \varphi \in \mathcal{D}(\Omega)\}.$$

On pose $D^{\alpha} u = g_{\alpha}$. Notons que par récurrence, on a

$$W^{m,p}(\Omega) = \{u \in W^{m-1,p}(\Omega); \frac{\partial u}{\partial x_i} \in W^{m-1,p}(\Omega) \quad \forall i = 1, 2, 3, \dots, N\}.$$

L'espace $W^{m,p}(\Omega)$ est muni de la norme

$$\|u\|_{W^{m,p}(\Omega)} = \sum_{0 \leq |\alpha| \leq m} \|D^{\alpha} u\|_{L^p(\Omega)}.$$

Proposition 1.1.2 (19). *L'espace $W^{m,p}(\Omega)$ muni de la norme $\|\cdot\|_{W^{m,p}(\Omega)}$ est un espace de Banach pour $1 \leq p < \infty$.*

On pose $H^m(\Omega) = W^{m,2}(\Omega)$. Ainsi définie, $H^m(\Omega)$ muni du produit scalaire

$$(u, v)_{H^m(\Omega)} = \sum_{0 \leq |\alpha| \leq m} (D^{\alpha} u, D^{\alpha} v)_{L^2(\Omega)}$$

est un espace de Hilbert.

1.1.2 Injections de Sobolev

Dans la manipulation des espaces de Sobolev, très souvent on fait appel à certaines injections dites de Sobolev. Nous rappelons quelques injections données par le Théorème de Rellich-Kondrachov.

Théorème 1.1.1. (Rellich-Kondrachov) *On suppose que Ω est un ouvert borné et régulier de \mathbb{R}^N , $m \in \mathbb{N}^*$ et $p \in [1; +\infty[$. On a*

Si $N - mp > 0$ alors $W^{m,p}(\Omega) \hookrightarrow L^q(\Omega)$; avec injection compacte pour tout $q \in [1; q^[$ tel que $\frac{1}{q^*} = \frac{1}{p} - \frac{m}{N}$.*

Si $N - mp = 0$ alors $W^{m,p}(\Omega) \hookrightarrow L^q(\Omega)$; avec injection compacte pour tout $q \in [1; \infty[$.

Si $N - mp < 0$ alors $W^{m,p}(\Omega) \hookrightarrow \mathcal{C}^k(\overline{\Omega})$; avec injection continue où $k = E(m - \frac{N}{p})$, avec E désigne la partie entière. Si de plus $m - \frac{N}{p}$ n'est pas entier, alors l'injection devient compacte.

1.1.3 Les espaces $W_0^{1,p}(\Omega)$

Définition 1.1.3. Soit $1 \leq p < \infty$, on désigne par $W_0^{1,p}(\Omega)$ l'adhérence de $\mathcal{D}(\Omega)$,

$$\text{i. e. } W_0^{1,p}(\Omega) = \overline{\mathcal{D}(\Omega)}^{W^{1,p}(\Omega)}.$$

On note $H_0^1(\Omega) = W_0^{1,2}(\Omega)$.

L'espace $W_0^{1,p}(\Omega)$ muni de la norme induite par celle de $W^{1,p}(\Omega)$ est un espace de Banach séparable ; et il est réflexif si $1 < p < \infty$. $H_0^1(\Omega)$ est un espace de Hilbert pour le produit scalaire de $H^1(\Omega)$.

Proposition 1.1.4. *On suppose que Ω est borné régulier (de classe \mathcal{C}^1). Soit $u \in W^{1,p}(\Omega) \cap \mathcal{C}(\overline{\Omega})$ avec $1 \leq p < \infty$, alors les propriétés suivantes sont équivalentes :*

1. $u = 0$ sur $\partial\Omega$.
2. $u \in W_0^{1,p}(\Omega)$.

Voici maintenant la plus importante inégalité de cette partie.

Théorème 1.1.2. *(Inégalité de Poincaré)*

On suppose que l'ouvert Ω est borné (ou juste borné dans une direction), alors il existe une constante C dépendante de Ω et de p telle que

$$\|u\|_{L^p} \leq C \|\nabla u\|_{L^p}, \quad \forall u \in W_0^{1,p}(\Omega) \text{ avec } 1 \leq p < \infty.$$

En particulier l'application $u \mapsto \|\nabla u\|_{L^p}$ définit une norme sur $W_0^{1,p}(\Omega)$ équivalente à celle induite par $W^{1,p}(\Omega)$. Sur $H_0^1(\Omega)$ l'application $(u, v) \mapsto \int_{\Omega} \nabla u \nabla v$ définit un produit scalaire qui induit la norme $\|\nabla u\|_{L^2}$ équivalente à la norme $\|u\|_{H^1}$.

1.2 Problèmes elliptiques

Nous allons étudier dans cette partie une catégorie très intéressante d'EDP qui sont les EDP elliptiques. Soit l'opérateur différentiel suivant :

$$L = - \sum_{i,j=1}^N \frac{\partial}{\partial x_j} (a_{ij} \frac{\partial}{\partial x_i}) + \sum_{i=1}^N b_i \frac{\partial}{\partial x_i} + c$$

où a_{ij} , b_i , c sont des fonctions mesurables bornées.

Définition 1.2.1. Soit f une application de Ω dans \mathbb{R} . L'équation $L(u) = f$ est dite elliptique s'il existe $\alpha > 0$ vérifiant :

$$\sum_{i,j=1}^N a_{ij}(x) \xi_i \xi_j \geq \alpha \|\xi\|^2 \quad \forall \xi \in \mathbb{R}^N, \quad \forall x \in \Omega.$$

Exemple 1.2.2. Lorsque les fonctions $a_{i,i} = 1$, $b_i = 0 \forall i$, $a_{i,j} = 0 \forall i \neq j$ et $c = 0$, l'opérateur $L = -\sum_{i=1}^N \frac{\partial^2}{\partial x^2} = -\Delta$ où Δ est le Laplacien.

On s'intéresse maintenant au problème de Dirichlet : trouver une fonction $u : \Omega \rightarrow \mathbb{R}$ telle que

$$(P) \quad \begin{cases} -\Delta u = f & \text{dans } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

avec f une application de Ω dans \mathbb{R} .

Définition 1.2.3. (Solution classique-solution faible)

- Une solution classique de (P) est une application $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}(\overline{\Omega})$ vérifiant (P) .
- Soit $f \in H^{-1}(\Omega)$ (l'espace dual de $H_0^1(\Omega)$). Une solution faible de (P) est une fonction $u \in H_0^1(\Omega)$ vérifiant

$$\int_{\Omega} \nabla u \nabla v = \int_{\Omega} f v, \quad \forall v \in H_0^1(\Omega).$$

Généralement, on commence par justifier l'existence des solutions faibles par des méthodes variationnelles, puis pour certains problèmes, on prouve que ces solutions sont aussi classiques en utilisant des résultats de régularité.

1.3 Principe du maximum

Le principe du maximum consiste à donner des informations sur le signe de la solution u dans Ω en connaissant ceux de $L(u)$ dans Ω et de $u|_{\partial\Omega}$. Il constitue un outil très puissant pour étudier l'unicité des solutions et retrouver des estimations a priori. Dans toute la suite, on se limitera aux résultats concernant l'opérateur Laplacien. Pour une étude plus approfondie des opérateurs elliptiques dans le cas général, on pourra consulter [15].

Définition 1.3.1. Une fonction u est dite harmonique sur Ω si $\Delta u = 0$ dans Ω .

Théorème 1.3.1. Soit $u \in \mathcal{C}^2(\Omega)$ satisfaisant $\Delta u = 0$ (≥ 0 , ≤ 0). Alors pour toute boule $B_R(y) \subset\subset \Omega$, on a

$$u(y) = (\leq, \geq) \frac{1}{N\omega_N R^{N-1}} \int_{\partial B_R(y)} u \, d\sigma,$$

$$u(y) = (\leq, \geq) \frac{1}{\omega_N R^N} \int_{B_R(y)} u \, dx,$$

où ω_n désigne la mesure de la boule unité.

Commençons par une première version du principe du maximum pour les fonctions régulières c'est-à-dire les solutions classiques.

Théorème 1.3.2. *Soit $u \in \mathcal{C}(\overline{\Omega}) \cap \mathcal{C}^2(\Omega)$ avec $\Delta u \geq 0$ (≤ 0) dans Ω . Alors*

$$\sup_{\Omega} u = \sup_{\partial\Omega} u \quad (\inf_{\Omega} u = \inf_{\partial\Omega} u).$$

En particulier, si u est harmonique, alors

$$\inf_{\partial\Omega} u \leq u \leq \sup_{\partial\Omega} u, \quad \forall x \in \Omega.$$

Comme application immédiate, le principe du maximum entraîne l'unicité de la solution du problème du (P) dans la classe $\mathcal{C}(\overline{\Omega}) \cap \mathcal{C}^2(\Omega)$.

Lemme 1.3.2. *(Inégalité de Harnack). Soit $u \in \mathcal{C}^2(\Omega)$ une fonction positive et harmonique. Alors pour tout compact $\Omega' \subset\subset \Omega$, il existe une constante C (qui dépend uniquement de Ω et Ω') telle que :*

$$\sup_{\Omega'} u \leq C \inf_{\Omega'} u.$$

On peut généraliser l'inégalité de Harnack pour une fonction non harmonique vérifiant certaines conditions (pour plus de détails voir [15] page 199).

Définition 1.3.3. On dit que Ω vérifie la condition de la sphère intérieure au point $x_0 \in \Omega$, s'il existe une boule $B \subset \Omega$ avec $x_0 \in \partial B$. (On peut supposer que $\partial B \cap \partial\Omega = \{x_0\}$. Il suffit de prendre une autre petite boule tangente à la première).

Ainsi tout domaine de classe \mathcal{C}^1 satisfait cette condition.

Lemme 1.3.4. *(Lemme de Hopf). Soit $u \in \mathcal{C}^2(\Omega)$ vérifiant :*

$$\Delta u \geq 0 \text{ dans } \Omega \quad \text{et}$$

soit $x_0 \in \partial\Omega$ tel que :

- u est continue au point x_0 ,
- $u(x_0) > u(x) \quad \forall x \in \Omega$,
- $\partial\Omega$ vérifie la condition de la sphère intérieure au point x_0 .

Alors

$$\frac{\partial u}{\partial \nu}(x_0) > 0.$$

Théorème 1.3.3. *(Principe du maximum fort) Soit $u \in \mathcal{C}(\overline{\Omega}) \cap \mathcal{C}^2(\Omega)$ telle que $\Delta u \geq 0$ (≤ 0). On suppose qu'il existe un point $y \in \Omega$ tel que $u(y) = \sup_{\Omega} u$ ($\inf_{\Omega} u$). Alors u est constante. Par conséquent, une fonction harmonique admettant un extrémum dans Ω est une fonction constante.*

Théorème 1.3.4. *Soient u une fonction harmonique dans Ω et un compact $\Omega' \subset\subset \Omega$. Alors, pour tout multi-indice α , on a*

$$\sup_{\Omega'} |D^\alpha u| \leq \left(\frac{N|\alpha|}{d} \right)^{|\alpha|} \sup_{\Omega} |u|$$

où $d = \text{dist}(\Omega', \partial\Omega)$.

1.4 Représentation intégrale

Théorème 1.4.1. *(Formules de Green [15]). On suppose que Ω est un ouvert borné et régulier de \mathbb{R}^N . Soient $u, v \in \mathcal{C}^2(\overline{\Omega})$. Alors on a :*

$$\begin{aligned} \int_{\Omega} v \Delta u dx &= - \int_{\Omega} \nabla v \nabla u dx + \int_{\partial\Omega} v \frac{\partial u}{\partial \nu} d\sigma \\ \int_{\Omega} (v \Delta u - u \Delta v) dx &= \int_{\partial\Omega} \left(v \frac{\partial u}{\partial \nu} - u \frac{\partial v}{\partial \nu} \right) d\sigma. \end{aligned}$$

Notons que, si Ω n'est pas borné, par exemple $\Omega = \mathbb{R}^N$, on peut appliquer ce théorème sur des boules B_R puis tendre R vers ∞ . Ceci implique que ce résultat reste vrai pour $\Omega = \mathbb{R}^N$ sous certaines hypothèses sur le comportement des fonctions u et v à l'infini. De plus, ce théorème reste valable lorsque $u \in H^2(\Omega)$ et $v \in H^1(\Omega)$ pour la première formule et $u, v \in H^2(\Omega)$ pour la deuxième.

Soient Ω ouvert borné de \mathbb{R}^N ($N \geq 3$), G et H respectivement la fonction de Green du Laplacien sur Ω avec condition de Dirichlet au bord, et sa partie régulière, i.e.

$$\begin{cases} G(x, y) = \frac{1}{|x - y|^{N-2}} - H(x, y) & (x, y) \in \Omega \times \Omega, \\ \Delta_x H = 0 \text{ in } \Omega \times \Omega & G = 0 \text{ on } \partial(\Omega \times \Omega). \end{cases}$$

On a le résultat suivant :

Théorème 1.4.2. *Pour tout $u \in \mathcal{C}^2(\overline{\Omega})$, on a :*

$$u(y) = \int_{\Omega} G(x, y)(-\Delta u(x)) dx - \int_{\partial\Omega} \frac{\partial G(x, y)}{\partial \nu} u(x) d\sigma, \quad \forall y \in \Omega.$$

On a aussi des résultats analogues lorsque le domaine Ω n'est pas borné. Par exemple, on peut trouver une représentation intégrale pour une fonction u harmonique dans un demi espace (qu'on note par Π), en utilisant le noyau de Poisson et la restriction de u sur le bord. Pour présenter ce résultat, on utilise par la suite [2] et [8].

On note les points du demi espace Π par (x, t) , où $x \in \mathbb{R}^{N-1}$ et $t > 0$. On considère le problème de Dirichlet suivant :

$$(P_\varphi) : \begin{cases} \phi \in \mathcal{C}^2(\Pi) \cap \mathcal{C}(\overline{\Pi}) \\ \Delta\phi = 0 \quad \text{dans } \Pi \\ \phi(x, 0) = \varphi(x) \in \mathcal{C}(\mathbb{R}^{N-1}) \cap L^\infty(\mathbb{R}^{N-1}). \end{cases}$$

Une solution ϕ du problème (P_φ) est dite une extension harmonique de φ dans le demi espace Π . Le noyau de Poisson du demi espace est défini, pour tout $N \geq 2$, par

$$K(x, y) = c_N \frac{t}{(t^2 + |x - y|^2)^{N/2}}, \quad \text{où } c_N := 2/(N \text{mes}(B_1))$$

et $\text{mes}(B_1)$ est la mesure de la boule unité B_1 .

Théorème 1.4.3. *Pour toute fonction $\varphi \in \mathcal{C}(\mathbb{R}^{N-1}) \cap L^\infty(\mathbb{R}^{N-1})$, le problème (P_φ) admet une solution unique ϕ , donnée par la formule suivante :*

$$\phi(x, t) = c_N \int_{\mathbb{R}^{N-1}} \frac{t\varphi(y)}{(t^2 + |x - y|^2)^{N/2}} dy.$$

La solution ϕ est l'extension harmonique bornée de φ dans Π .

1.5 Régularité

Pour plusieurs EDP, la preuve de l'existence des solutions faibles est plus simple que les solutions classiques. Plusieurs études sont faites pour montrer que les solutions faibles, pour certaines EDP, sont en effet des solutions classiques. Parmi les théorèmes intéressants, on présente :

Théorème 1.5.1. *[15] Soit Ω un ouvert borné de \mathbb{R}^N , $u \in W^{2,p}(\Omega)$ avec $1 < p < +\infty$, et $f \in L^p(\Omega)$ vérifiant :*

$$\Delta u = f.$$

Alors pour tout compact $\Omega' \subset\subset \Omega$, on a

$$\|u\|_{W^{2,p}(\Omega')} \leq c(\|u\|_{L^p(\Omega)} + \|f\|_{L^p(\Omega)}).$$

On remarque que si on impose une condition sur le bord, on peut améliorer la conclusion du dernier théorème.

Théorème 1.5.2. *[15] Soit Ω un domaine borné de \mathbb{R}^N ($N \geq 3$) de classe \mathcal{C}^∞ et soit $u \in H_0^1(\Omega)$ une solution de*

$$-\Delta u = g(x, u) \text{ dans } \Omega, \quad u = 0 \text{ sur } \partial\Omega.$$

Notons $f(x) = g(x, u(x))$.

- Si $f \in L^p(\Omega)$, avec $p \geq \frac{2N}{N+2}$, alors $u \in W^{2,p}(\Omega)$.
- Si $f \in W^{m,p}(\Omega)$, alors $u \in W^{m+2,p}(\Omega)$.
- Si $f \in C^m(\Omega)$, alors $u \in C^{m+2}(\overline{\Omega})$.

Une application des théorèmes de régularité (combinés avec les injections de Sobolev) est la preuve que toute solution faible du problème

$$-\Delta u = |u|^{q-1}u \text{ dans } \Omega, \quad u = 0 \text{ sur } \partial\Omega \quad (1 < q < \frac{N+2}{N-2})$$

est une solution de classe C^∞ sur $\overline{\Omega}$.

1.6 Identité de Pohozaev

Théorème 1.6.1. (*Identité de Pohozaev [20][24]*). Soient Ω un ouvert connexe borné de \mathbb{R}^N ($N \geq 3$), $f : \mathbb{R} \rightarrow \mathbb{R}$ une fonction continue et $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ une solution de l'équation

$$-\Delta u = f(u).$$

Alors

$$\begin{aligned} \int_{\Omega} \left(NF(u) - \frac{N-2}{2} uf(u) \right) &= \int_{\partial\Omega} \left(\sum_{i=1}^N x_i \nu_{x_i} \left(F(u) - \frac{1}{2} |\nabla u|^2 \right) \right. \\ &\quad \left. + \frac{\partial u}{\partial \nu_x} \left(\sum_{i=1}^N x_i \frac{\partial u}{\partial x_i} + \frac{N-2}{2} u \right) \right), \end{aligned}$$

où $F(u) = \int_0^u f(t)dt$, $\nu = \nu_x$ est la normale extérieure au point $x \in \partial\Omega$.

L'application la plus importante de ce théorème est la non-existence de solution du problème

$$-\Delta u = u^{\frac{N+2}{N-2}}, \quad u > 0 \text{ dans } \Omega, \quad u = 0 \text{ sur } \partial\Omega$$

où Ω est un ouvert borné étoilé de \mathbb{R}^N et $N \geq 3$. L'hypothèse que Ω est étoilé est nécessaire pour avoir $(x-a) \cdot \nu_x \geq 0, \forall x \in \partial\Omega$ où a est un point bien choisi (qu'on peut supposer $a = 0$ dans le théorème). En effet, si on suppose qu'il existe une solution, alors la formule précédente dérive :

$$\int_{\partial\Omega} \sum_{i=1}^N \left(\frac{\partial u}{\partial \nu_x} \right)^2 \cdot x_i \nu_i = 0$$

Ce qui entraîne une contradiction en utilisant le lemme de Hopf.

1.7 Théorie de Morse

Un problème est dit variationnel si ces solutions peuvent être vues comme des points critiques de certaines fonctionnelles définies sur des espaces appropriés.

Exemple 1.7.1. *Soit le problème suivant*

$$(P') \quad \begin{cases} \Delta u = 0 & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} = f(x, u) & \text{on } \partial\Omega, \end{cases}$$

$u \in H^1(\Omega)$ est une solution de (P') si et seulement si u est un point critique de la fonctionnelle

$$I(v) := \frac{1}{2} \int_{\Omega} |\nabla v|^2 dx - \int_{\partial\Omega} F(x, v) d\sigma$$

où $F(x, v) := \int_0^v f(x, t) dt$.

Etant donnée la structure variationnelle de ces problèmes, les solutions cherchées correspondent aux points critiques d'une fonctionnelle J . Pour résoudre l'équation $J'(u) = 0$, on peut commencer par minimiser J . La difficulté principale commune à tous ces problèmes est le manque de compacité. En fait, l'espace de base n'étant pas compact, l'infimum n'est pas nécessairement atteint. Pour remédier à cette difficulté, on utilise une forme déguisée de compacité : la condition de Palais-Smale (PS).

Définition 1.7.2. Soit J une fonction de classe \mathcal{C}^1 sur un Banach E . On dit que J vérifie la condition (PS) si pour toute suite (u_n) telle que $J(u_n)$ reste bornée et $\|J'(u_n)\| \rightarrow 0$, alors (u_n) est relativement compacte.

Cette condition entraîne que si J est minorée alors son minimum est atteint. Cependant, la condition (PS) est trop forte, c'est-à-dire, les fonctionnelles associées à certains problèmes (issus de la physique et de la géométrie différentielle etc ...) ne vérifient pas cette condition. Plusieurs études sont faites pour comprendre le comportement asymptotique des suites violant la condition (PS) pour certains problèmes, on peut citer le problème de Yamabe pour lequel ces suites sont devenues explicites. La détection d'un point critique d'une fonctionnelle J est bien développée dans le cas de la dimension finie. Par exemple lorsqu'on détecte un changement de topologie des ensembles de niveau $J_c = \{u : J(u) \leq c\}$, ce changement est dû à l'existence d'un point critique. En fait, des résultats en théorie de Morse expliquent ce phénomène [5].

Lemme 1.7.3. (*Lemme de Morse*) Soit $f : M \rightarrow \mathbb{R}$ de classe C^∞ sur une variété M , une fonction ayant un point critique x_0 non dégénéré. Alors il existe U voisinage de x_0 et une carte locale $y = \varphi(x)$ où $\varphi : U \rightarrow \mathbb{R}^N$, $\varphi(x_0) = 0$ et

$$f \circ \varphi^{-1}(y) = f(x_0) + y_{r+1}^2 + \dots + y_N^2 - y_1^2 - \dots - y_r^2$$

où r est l'indice de Morse de f en x_0 .

Définition 1.7.4. Soit $X \subset Y$ espace topologique. On dit que Y se rétracte par déformation sur X s'il existe une application continue $U : [0, 1] \times Y \rightarrow Y$ qui satisfait les propriétés suivantes :

1. $U(0, y) = y, \forall y \in Y$.
2. $U(t, x) = x, \forall x \in X, \forall t \in [0, 1]$.
3. $U(1, y) \in X, \forall y \in Y$.

Lemme 1.7.5. (*Lemme de déformation 1*) Soit $f : M \rightarrow \mathbb{R}$ de classe C^∞ sur une variété compacte M de dimension finie m . On suppose que f n'a pas de valeurs critiques dans $[a, b]$. Alors l'ensemble de niveau f_b se rétracte par déformation sur f_a .

Lemme 1.7.6. (*Lemme de déformation 2*) Soit $f : M \rightarrow \mathbb{R}$ de classe C^∞ sur une variété compacte M de dimension finie m . On suppose que f admet une seule valeur critique $c \in]a, b[$ correspondant à un seul point critique x_0 non dégénéré d'indice de Morse r . Alors pour $\epsilon > 0$ assez petit on a : l'ensemble de niveau $f_{c+\epsilon}$ se rétracte par déformation sur $f_{c-\epsilon} \cup B_{\sqrt{\epsilon}}(x_0)$ (boule incluse dans \mathbb{R}^r).

On note que les lemmes de déformation 1 et 2 restent valables si J est définie sur un espace de dimension infini, sous la condition : J vérifie (PS). D'où l'importance de cette propriété.

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Scalar Curvature Type Problem On The Three Dimensional Bounded Domain

*(with M. Ben Ayed), accepted for publication in **Acta Mathematica Scientia**.*

Abstract : In this paper we prove an existence result for the nonlinear elliptic problem : $-\Delta u = Ku^5$, $u > 0$ in Ω , $u = 0$ on $\partial\Omega$, where Ω is a smooth bounded domain of \mathbb{R}^3 and K is a positive function in $\bar{\Omega}$. Our method relies on studying its corresponding subcritical approximation problem and then using a topological argument.

Key words : Nonlinear equations, scalar-curvature, critical point, limiting Sobolev exponent, variational problems with lack of compactness, blow up analysis.

Chapitre 2

Scalar Curvature Type Problem On The Three Dimensional Bounded Domain

2.1 Introduction and results

Let us consider the nonlinear elliptic problem

$$(P) \quad \begin{cases} -\Delta u = Ku^p, & u > 0 \text{ in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a smooth bounded domain in \mathbb{R}^N , $N \geq 3$, K is a positive function in $\overline{\Omega}$ and $p = \frac{N+2}{N-2}$ is the critical Sobolev exponent.

The interest in this equation grew up from its resemblance to the Scalar Curvature problem in the differential geometry, which is given by the question : assigned a function $K : M \rightarrow \mathbb{R}$, where M is a Riemannian manifold of dimension N without boundary, does there exist a metric g' conformally equivalent to g such that $R_{g'} = K$? So it is reduced to solve the problem :

$$(E_0) \quad \begin{cases} -4\frac{n-1}{n-2}\Delta_g u + R_g u = K(x)u^p & \text{on } M, \\ u > 0 & \text{on } M. \end{cases}$$

The problem (E_0) was deeply studied by several authors and with several methods. There have been many works devoted to the existence and multiplicity results, trying to understand under what conditions (E_0) is solvable.

For instance, in a 3-dimensional manifold which is not equivalent to a sphere, an optimal result was achieved by Escobar and Schoen in 1986. Precisely, they proved ;

if M^3 is not diffeomorphic to the sphere S^3 , then (E_0) has solution if and only if $\max_{M^3} K > 0$. Nevertheless in the case of the sphere, there is a topological obstruction given by Kazdan and Warner (1975), which provides us with an example of function K on S^N where the problem (E_0) has no solution. Hence it is not expectable to solve problem (E_0) for all functions K , and it is natural to impose some conditions on it. In the case of manifolds without boundary, this problem has been widely studied in various works (see for example [2] [4]). In [2], Bahri-Coron constructed a decreasing pseudo-gradient in the neighborhood of functions which are concentrated on two points or more. The main property of this pseudo-gradient is that the functional satisfies the Palais Smale condition along the flow lines. In other words, there is a kind of compactness outside of a neighborhood of functions which are concentrated at one point. Later, Ben Ayed et al [4] obtained an analogous result on M^4 where M^4 is a compact manifold without boundary. They proved that the lost of compactness can be also in a neighborhood of concentrated functions at many points (a_1, \dots, a_k) (the tuple (a_1, \dots, a_k) has to satisfy some conditions).

The same phenomenon was proved by Z. Djadli et al [8] when they studied the scalar curvature problem on the half sphere S_+^3 . They proved that the concentration points are on the boundary. However, for a bounded domain Ω , this phenomenon appears in dimension 4 (see Ben Ayed-Hammami [6]). Taking $u = 0$ on $\partial\Omega$, the authors proved that the concentration points have to be in a compact set of Ω . Our aim is to understand what happens in the case $N = 3$.

From now, we assume that $N = 3$.

Let us denote by G and H respectively the Green's function of the Laplacian with Dirichlet boundary conditions on Ω , and its regular part, i.e.

$$\begin{cases} G(x, y) = \frac{1}{|x - y|} - H(x, y) & (x, y) \in \Omega \times \Omega, \\ \Delta_x H = 0 \text{ in } \Omega \times \Omega & G = 0 \text{ on } \partial(\Omega \times \Omega). \end{cases}$$

For a positive integer k and $x = (x_1, \dots, x_k) \in \Omega^k$ with $x_i \neq x_j$ for $i \neq j$, we set $M(x) = (m_{ij})_{1 \leq i, j \leq k}$ the matrix defined as

$$m_{ii} = \frac{H(x_i, x_i)}{\sqrt{K(x_i)}}; \quad m_{ij} = -\frac{G(x_i, x_j)}{(K(x_i)K(x_j))^{\frac{1}{4}}} \quad (2.1.1)$$

and let $\rho(x)$ be the least eigenvalue of M .

In this paper we will assume the following :

(H₁) K is a positive \mathcal{C}^3 function in $\bar{\Omega}$ having only nondegenerate critical points y_1, \dots, y_m .

(H₂) For each $x \in \partial\Omega$, $\frac{\partial K(x)}{\partial \nu} < 0$ where ν is the outward normal to $\partial\Omega$.

(H₃) For any s -tuple $\tau_s = (i_1, \dots, i_s) \in (1, \dots, m)^s$ with $i_j \neq i_q$ for $j \neq q$, $M(\tau_s) = M(y_{i_1}, \dots, y_{i_s})$ (defined by (2.1.1)) is nondegenerate.

Our first result is the following.

Theorem 2.1.1. *Assume that assumptions (H₁), (H₂) and (H₃) hold. If*

$$1 \neq \sum_{s=1}^m \sum_{\tau_s=(i_1, \dots, i_s)/M(\tau_s)>0} (-1)^{4s-1-\sum_{j=1}^s k_{i_j}}$$

where $k_{i_j} = \text{index}(K, y_{i_j})$ and $m = \#\{y/\nabla K(y) = 0\}$, then (P) has at least a solution.

The proof of Theorem 2.1.1 relies on the study of the following subcritical approximation of equation (P) :

$$(P_\varepsilon) \quad \begin{cases} -\Delta u = Ku^{5-\varepsilon}, & u > 0 \text{ in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \quad (2.1.2)$$

where ε is a positive small real number. To (P_ε) is associated the functional

$$I_\varepsilon(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 - \frac{1}{6-\varepsilon} \int_{\Omega} K|u|^{6-\varepsilon}, \quad u \in H_0^1(\Omega),$$

which is compact, and whose positive critical points are solutions to (P_ε).

Such solutions exist and it follows from that, as ε goes to zero, either they converge to a solution of (P), or they blow up at finite number of points in Ω . Namely, (u_ε) being a bounded sequence in $H_0^1(\Omega)$ of solutions to (P_ε), up to a subsequence we have :

$$u_\varepsilon = u_0 + \sum_{i=1}^k \frac{1}{\sqrt[4]{K(a_i^\varepsilon)}} P\delta_{a_i^\varepsilon, \lambda_i^\varepsilon} + v^\varepsilon \quad (2.1.3)$$

with u_0 a solution of (P) or $u_0 = 0$, v^ε goes to zero in $H_0^1(\Omega)$ and $k \in \mathbb{N}$. For simplicity we shall write a_i (resp. λ_i and v) for a_i^ε (resp. λ_i^ε and v_ε).

The function $P\delta_{a,\lambda}$ is defined as follows. For $\lambda \in (0, +\infty)$ and $a \in \mathbb{R}^3$ we denote $\delta_{a,\lambda}$ the function defined in \mathbb{R}^3 by

$$\delta_{a,\lambda}(x) = \sqrt[4]{3} \frac{\lambda^{\frac{1}{2}}}{(1 + \lambda^2 |x - a|^2)^{\frac{1}{2}}}. \quad (2.1.4)$$

These functions are the only solutions of

$$-\Delta u = u^5, \quad u > 0 \text{ in } \mathbb{R}^3, \quad u \in L^6(\mathbb{R}^3), \quad \text{and } \nabla u \in (L^2(\mathbb{R}^3))^3. \quad (2.1.5)$$

$P\delta_{a,\lambda}$ is defined as the projection of $\delta_{a,\lambda}$ onto $H_0^1(\Omega)$, that is :

$$\Delta P\delta_{a,\lambda} = \Delta\delta_{a,\lambda} \text{ in } \Omega, \quad P\delta_{a,\lambda} = 0 \text{ on } \partial\Omega. \quad (2.1.6)$$

It is well known that we have, as ε goes to 0,

$$a_i \in \Omega, a_i \rightarrow \bar{a}_i \in \bar{\Omega}, \quad \forall i$$

and

$$\begin{cases} \varepsilon \log \lambda_i \rightarrow 0, \quad \lambda_i d(a_i, \partial\Omega) \rightarrow +\infty, \quad \forall i \\ \varepsilon_{ij} = \left(\frac{\lambda_i}{\lambda_j} + \frac{\lambda_j}{\lambda_i} + \lambda_i \lambda_j |a_i - a_j|^2 \right)^{-\frac{1}{2}} \rightarrow 0, \quad \forall i, j, \quad i \neq j. \end{cases} \quad (2.1.7)$$

Note that, the variable ε_{ij} comes from the scalar product $\langle P\delta_i, P\delta_j \rangle_{H_0^1}$ for $i \neq j$ which tells us that the functions are almost orthogonal.

Note that (2.1.3) and (2.1.7) ensure that we have

$$I_\varepsilon(u_\varepsilon) = I_0(u_0) + \sum_{i=1}^k \frac{S_3}{3\sqrt{K(\bar{a}_i)}} + o(1) \quad (2.1.8)$$

where $S_3 := \inf_{u \in H_0^1(\Omega), u \neq 0} \frac{|u|_{H_0^1(\Omega)}^2}{|u|_{L^6(\Omega)}^{-2}}$ is the best Sobolev constant.

For $x \in \Omega^k$, we define also

$$\begin{aligned} F_x &: (0, +\infty)^k \longrightarrow \mathbb{R} \\ \Lambda &= (\Lambda_1, \dots, \Lambda_k) \longrightarrow \frac{1}{2} \Lambda M(x)^t \Lambda - \sum_{i=1}^k \frac{\log(\Lambda_i)}{\sqrt{K(x_i)}}. \end{aligned}$$

Note that if (u_ε) is bounded in $L^\infty(\bar{\Omega})$ then it has to converge to a solution of (P_0) . For $K \equiv 1$, using a result of R. Schoen [19], we know that $u_0 \equiv 0$ or $k = 0$ that means when u_ε blows up it follows that $u_\varepsilon \rightarrow 0$ as $\varepsilon \rightarrow 0$. Moreover, this result [19] provides us more information about the concentration points, it tells us that these points are far away from the boundary and from each other. This result was used in [3] [18]. For $K \neq 1$, we lose this information in general. In fact, in [7], taking $\Omega = B(0, 1)$ and $K(x) = |x|^\gamma$, the authors proved that the ground state solution has to blow up at a point $\bar{a} \in \partial\Omega$. (See also [14]). In our case, using the assumption (H_2) , we are able to prove that the concentration points have to be in a compact set of Ω . This program is done using the blow up analysis introduced by R. Schoen and developed by Y. Y. Li and C. S. Lin and some ideas introduced by O. Rey in [18].

Since our goal is to prove the existence of solution of (P_0) , arguing by contradiction, we assume that (P_0) has no solution and therefore u_ε has to blow up. The aim of our second result is to give the asymptotic behavior of (u_ε) . In fact, we prove

Theorem 2.1.2. *Assume that assumptions (H_1) , (H_2) and (H_3) hold and (P_0) has no solution. Let A be a positive large constant. Assume that (u_ε) is a sequence of solutions of (P_ε) that satisfies $I_\varepsilon(u_\varepsilon) < A$. Then it has to blow up at k points a_1, \dots, a_k of $\bar{\Omega}$ with $k \leq \#\{y/\nabla K(y) = 0\}$. These solutions have to be written as :*

$$u_\varepsilon = \sum_{i=1}^k \frac{1}{\sqrt[4]{K(a_i^\varepsilon)}} P\delta_{a_i^\varepsilon, \lambda_i^\varepsilon} + v_\varepsilon, \text{ with } |v_\varepsilon|_{H_0^1} \rightarrow 0.$$

Moreover, we have

1. $(a_1^\varepsilon, \dots, a_k^\varepsilon) \in \Omega_{d_0}^k$, with $d_0 > 0$ and $\Omega_{d_0} = \{x \in \Omega/d(x, \partial\Omega) > d_0\}$.
2. $|a_i^\varepsilon - a_j^\varepsilon| \geq d'_0 \quad \forall i \neq j$, with $d'_0 > 0$.
3. $\forall i \neq j, c \leq \frac{\lambda_i^\varepsilon}{\lambda_j^\varepsilon} \leq C$ with c and C some positive constants.
4. $\forall i \in \{1, \dots, k\}$, $a_i^\varepsilon \rightarrow y_i$ critical point of K such that $\rho(y_1, \dots, y_k) > 0$.
5. $\varepsilon^{-\frac{1}{2}}\Lambda \rightarrow c\bar{\Lambda}$ where $\bar{\Lambda}$ is a critical point of F_y with $y = (y_1, \dots, y_k)$ defined in Claim 4, $\Lambda := (\frac{1}{\sqrt{\lambda_1^\varepsilon}}, \dots, \frac{1}{\sqrt{\lambda_k^\varepsilon}})$ and c is some positive constant.

Remark 2.1.3. *We note that if A is very large then the number of blow up points k is independent of A .*

Our next result provides a kind of converse to Theorem 2.1.2.

Theorem 2.1.4. *Let K be a C^2 positive function in $\bar{\Omega}$ and let y_1, \dots, y_k be different nondegenerate critical points of K such that $\rho(y_1, \dots, y_k) > 0$. There exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0]$, (P_ε) has a solution of the form*

$$u_\varepsilon = \sum_{i=1}^k \alpha_i^\varepsilon P\delta_{a_i^\varepsilon, \lambda_i^\varepsilon} + v_\varepsilon$$

which satisfies $|v_\varepsilon|_{H_0^1} \rightarrow 0$ when $\varepsilon \rightarrow 0$ and

$$\left| \alpha_i^\varepsilon - \frac{1}{\sqrt[4]{k(a_i^\varepsilon)}} \right| = O(\varepsilon |\log \varepsilon|), \quad \varepsilon \lambda_i^\varepsilon \rightarrow \bar{c}, \quad |a_i^\varepsilon - y_i| = O(\varepsilon |\log \varepsilon|).$$

Moreover, we have $|\nabla u_\varepsilon|^2 \rightharpoonup S_3^{3/2} \sum_{i=1}^k \frac{1}{\sqrt[4]{K(y_i)}} \delta_{y_i}$ in $\mathcal{D}'(\Omega)$ when $\varepsilon \rightarrow 0$, where δ_y denotes the Dirac mass at the point y .

A similar result as in Theorem 2.1.1 was proved by R. Ghoudi [10] using the approach of critical points at infinity. In his paper, he assumed that $K = 0$ on $\partial\Omega$. This assumption allows him to prove that the sequences which do not satisfy the Palais

Smale condition are concentrated far away from the boundary. This phenomenon is proved in our case for the solutions (u_ε) of (P_ε) in Lemma 2.2.6 (using the blow up analysis) and Lemma 2.2.8 (using some ideas introduced by O. Rey in [18]).

The paper is organized as follows. In Section 2.2, we collect some useful technical tools and then we investigate the behavior of the sequence u_ε as ε goes to 0. In Section 2.3, we construct a solution for the subcritical problem by the finite reduction method while Section 2.4 is devoted to the proof of Theorem 2.1.1 which is an existence result of the critical problem. In Section 2.5, we show that solution of subcritical approximation possesses only blow up points which are far away from each other compared with the distance to the boundary. Finally, Section 2.6 deals with the estimate of the odd part of v .

2.2 Asymptotic behavior of the solutions

According to (2.1.4) (2.1.5) (2.1.6) we may write

$$P\delta_{a,\lambda} = \delta_{a,\lambda} - \varphi_{a,\lambda} \quad \text{with} \quad \Delta\varphi_{a,\lambda} = 0 \text{ in } \Omega \quad \text{and} \quad \varphi_{a,\lambda} = \delta_{a,\lambda} \text{ on } \partial\Omega. \quad (2.2.1)$$

From the maximum principle, we deduce the following :

Lemma 2.2.1. *Assume that $\lambda d(a, \partial\Omega)$ is very large. Then*

$$\varphi_{a,\lambda} = \frac{\sqrt[4]{3}}{\lambda^{\frac{1}{2}}} H(a, x) + O\left(\frac{1}{\lambda^{\frac{5}{2}} d^3(a, \partial\Omega)}\right).$$

Let u_ε be a solution of (P_ε) . Arguing as in [13] [20], the blow up analysis leads to

$$u_\varepsilon = \sum_{i=1}^k \frac{1}{\sqrt[4]{K(a_i)}} P\delta_{a_i, \lambda_i} + v, \quad \text{with } |v|_{H_0^1} \rightarrow 0, \quad \lambda_i d(a_i, \partial\Omega) \rightarrow \infty \text{ and } \varepsilon_{ij} \rightarrow 0.$$

Using [1], the following problem

$$\min_{\alpha_i, a_i, \lambda_i, i=1, \dots, k} \left| u_\varepsilon - \sum_{i=1}^k \alpha_i P\delta_{a_i, \lambda_i} \right|_{H_0^1}$$

has a unique solution (up to permutation). Hence, in the sequel, the solution u_ε will be written as

$$u_\varepsilon = \sum_{i=1}^k \alpha_i P\delta_{a_i, \lambda_i} + v_\varepsilon,$$

with $\alpha_i \in \mathbb{R}$ such that $\alpha_i^4 K(a_i) \rightarrow 1$, v_ε satisfying $|v_\varepsilon|_{H_0^1} \rightarrow 0$ and $v_\varepsilon \in E_{a,\lambda}$ with

$$E_{a,\lambda} = \left\{ v \in H_0^1(\Omega) / \langle v, P\delta_{a_i,\lambda_i} \rangle_{H_0^1} = \langle v, \frac{\partial P\delta_{a_i,\lambda_i}}{\partial \lambda_i} \rangle_{H_0^1} = \langle v, \frac{\partial P\delta_{a_i,\lambda_i}}{\partial (a_i)_j} \rangle_{H_0^1} = 0, \right. \\ \left. \forall 1 \leq i \leq k, 1 \leq j \leq 3 \right\}.$$

For simplicity we shall write δ_i for δ_{a_i,λ_i} and v for v_ε . We begin by estimating $|v|_{H_0^1}$.

Proposition 2.2.2. *The following estimate holds;*

$$|v|_{H_0^1} = O \left(\varepsilon + \sum_{i=1}^k \left(\frac{1}{\lambda_i d_i} + \frac{|DK(a_i)|}{\lambda_i} \right) + \sum_{i \neq j} \varepsilon_{ij} \log^{\frac{1}{3}}(\varepsilon_{ij}^{-1}) \right),$$

where $d_i = d(a_i, \partial\Omega)$.

Proof. Multiplying the equation $-\Delta u = Ku^{5-\varepsilon}$ by v , integrating by parts over Ω and using $v \in E_{a,\lambda}$, we obtain

$$\int_{\Omega} |\nabla v|^2 = \int_{\Omega} K \left(\sum_{i=1}^k \alpha_i P\delta_i + v \right)^{5-\varepsilon} v. \quad (2.2.2)$$

On the one hand, expanding $(\sum_{i=1}^k P\delta_i + v)^{5-\varepsilon}$ in the usual way we get

$$\begin{aligned} & \int_{\Omega} K \left(\sum_{i=1}^k \alpha_i P\delta_i + v \right)^{5-\varepsilon} v \\ &= \int_{\Omega} K \left(\sum_{i=1}^k \alpha_i P\delta_i \right)^{5-\varepsilon} v + (5-\varepsilon) \int_{\Omega} K \left(\sum_{i=1}^k \alpha_i P\delta_i \right)^{4-\varepsilon} v^2 \\ & \quad + O \left(\sum_{i=1}^k \int_{\Omega} \delta_i^3 v^3 + \int_{\Omega} |v|^{6-\varepsilon} \right) \\ &= \int_{\Omega} K \left(\sum_{i=1}^k \alpha_i P\delta_i \right)^{5-\varepsilon} v + 5 \sum_{i=1}^k \alpha_i^4 K(a_i) \int_{\Omega} P\delta_i^4 v^2 + o(|v|_{H_0^1}^2). \end{aligned} \quad (2.2.3)$$

On the other hand, from [1], there exists a positive constant ρ such that

$$\int_{\Omega} |\nabla v|^2 - 5 \sum_{i=1}^k \int_{\Omega} P\delta_i^4 v^2 \geq \rho |v|_{H_0^1}^2, \quad \forall v \in E_{a,\lambda}. \quad (2.2.4)$$

Combining (2.2.2), (2.2.3), (2.2.4) and Hölder's inequality we have :

$$(\rho + o(1)) |v|_{H_0^1}^2 \leq \int_{\Omega} K \left(\sum_{i=1}^k \alpha_i P\delta_i \right)^{5-\varepsilon} v := f(v). \quad (2.2.5)$$

We claim, since $v \in E_{a,\lambda}$ and $\delta_i^{-\varepsilon} = \lambda_i^{-\frac{\varepsilon}{2}} + O(\varepsilon \log(1 + \lambda_i^2|x - a_i|^2))$, that

$$f(v) = O\left(|v|_{H_0^1}\left(\varepsilon + \sum_{i=1}^k\left(\frac{1}{\lambda_i d_i} + \frac{|DK(a_i)|}{\lambda_i}\right) + \sum_{i \neq j} \varepsilon_{ij} \log^{\frac{1}{3}}(\varepsilon_{ij}^{-1})\right)\right) \quad (2.2.6)$$

(for details, see [2] and [18]). (2.2.5) and (2.2.6) imply the assertion of Proposition 2.2.2. \square

Let $\beta_i := \alpha_i - \frac{1}{\sqrt[4]{K(a_i)}}$. Multiplying the equation (P_ε) by $P\delta_i$ and integrating on Ω , we find

$$\textbf{Proposition 2.2.3.} \quad \beta_i = O\left(\varepsilon \log \lambda_i + \sum_{j=1}^k \frac{1}{\lambda_j d_j} + \sum_{j \neq i} \varepsilon_{ij} \log^{\frac{1}{3}}(\varepsilon_{ij}^{-1})\right), \quad \forall i \leq k.$$

The following proposition is also, as in [2] [18], available in this framework and its proof, which we omit here, requires only minor modifications.

Multiplying Eq (2.1.2) by $\lambda_i \frac{\partial P\delta_i}{\partial \lambda_i}$ and integrating on Ω we find

Proposition 2.2.4. *For each $i \leq k$, we have*

$$\begin{aligned} & \frac{\Gamma_1}{\sqrt[4]{K(a_i)}} \varepsilon - \frac{\Gamma_2}{\sqrt[4]{K(a_i)}} \frac{H(a_i, a_i)}{\lambda_i} - \sum_{\ell \neq i} \frac{\Gamma_2}{\sqrt[4]{K(a_\ell)}} \left(\lambda_i \frac{\partial \varepsilon_{i\ell}}{\partial \lambda_i} + \frac{1}{2} \frac{H(a_i, a_\ell)}{\sqrt{\lambda_i \lambda_\ell}} \right) \\ & = O\left(\varepsilon^2 \log \lambda_i + \sum_{\ell=1}^k \left(\frac{1}{\lambda_\ell^2 d_\ell^2} + \frac{|DK(a_\ell)|^2}{\lambda_\ell^2}\right) + \sum_{\ell \neq j} \varepsilon_{\ell j}^2 \log^{\frac{2}{3}}(\varepsilon_{\ell j}^{-1}) + \beta_i \left(\varepsilon + \frac{1}{\lambda_i d_i}\right) \right. \\ & \quad \left. + \sum_{\ell \neq i} \beta_\ell \left(\varepsilon_{i\ell} + \frac{1}{\lambda_\ell d_\ell}\right)\right), \end{aligned}$$

where Γ_1, Γ_2 are some positive constants.

Note that in the above Proposition we use

$$\delta_i^{-\varepsilon} = \lambda_i^{-\frac{\varepsilon}{2}} \left(1 + \frac{\varepsilon}{2} \log(1 + \lambda_i^2|x - a_i|^2) + O(\varepsilon^2 \log^2(1 + \lambda_i^2|x - a_i|^2)) \right).$$

Now, multiplying Eq (2.1.2) by $\frac{1}{\lambda_i} \frac{\partial P\delta_i}{\partial (a_i)_j}$ (where $\frac{\partial}{\partial (a_i)_j}$ denotes the derivative with respect to the j -th component of the variable a_i) and integrating on Ω , we find

Proposition 2.2.5. *For each $i \leq k$, we have*

$$\begin{aligned} & \Gamma_3 \frac{1}{(K(a_i))^{\frac{5}{4}}} \frac{DK(a_i)}{\lambda_i} + \Gamma_4 \sum_{\ell \neq i} \frac{1}{\sqrt[4]{K(a_\ell)}} \frac{1}{\lambda_i} \frac{\partial \varepsilon_{i\ell}}{\partial a_i} \\ & = O\left(\frac{\varepsilon \log \lambda_i}{\lambda_i} + \sum_{\ell=1}^k \left(\frac{1}{\lambda_\ell^2 d_\ell^2} + \frac{|DK(a_\ell)|^2}{\lambda_\ell^2}\right) + \sum_{\ell \neq j} \varepsilon_{\ell j}^2 \log^{\frac{2}{3}}(\varepsilon_{\ell j}^{-1}) + \sum_{\ell \neq i} \varepsilon_{i\ell}^4 \lambda_\ell |a_i - a_\ell| \right. \\ & \quad \left. + \frac{\beta_i |DK(a_i)|}{\lambda_i} + \sum_{\ell \neq i} \beta_\ell \varepsilon_{i\ell}\right) \end{aligned}$$

where $d_\ell = d(a_\ell, \partial\Omega)$ and Γ_3, Γ_4 are some positive constants.

Observe that, in the proof of Proposition 2.2.4, the terms where the v appears are of order $O(|v|_{H_0^1}^2)$ or $O(|f(v)|)$ which are small with respect to the principal part (by using Proposition 2.2.2). Furthermore, these estimates are also good for Proposition 2.2.5 if the concentration points are in a compact set of Ω , since the principal part becomes $\frac{|DK(a)|}{\lambda}$. However if one (or more) of the concentration points, we denote it by a_i , is close to the boundary, we get that $|DK(a_i)|/\lambda_i \geq c/\lambda_i$ but the remainder term $(\lambda_i d_i)^{-2}$ can be very large with respect to c/λ_i (in the case where $\lambda_i d_i^2$ is very small). Hence we need to ameliorate the previous proposition to make appear $(\lambda_i d_i)^{-2}$ in the principal part. Note that in [18], O. Rey considered the case of $K \equiv 1$ and therefore, the principal part becomes $\frac{\partial H}{\partial a}(a_i, a_i) \lambda_i^{-2}$. For this reason, he studied these integrals (the integrals involving v) carefully and he proved that the even part does not have a contribution and the odd part has a better estimate. In [18], the author used the fact that the concentration points are far away from the boundary and from each other (by using the result of R. Schoen) and he considered some balls around each concentration point of a fixed radius $r > 0$. In our case, we need to use this idea. Hence, we require to introduce some balls around each point to ameliorate the estimate of these integrals. This is the goal of the following lemma.

Lemma 2.2.6. *There exists a positive constant c (independent of ε) such that*

$$|a_i - a_j| \geq c \max(d_i, d_j), \forall i \neq j.$$

Remark 2.2.7. *We remark that, Lemma 2.2.6 implies that*

$$\varepsilon_{ij} \leq \frac{C}{\sqrt{\lambda_i \lambda_j d_i d_j}} \leq \frac{C}{\lambda_i d_i} + \frac{C}{\lambda_j d_j}.$$

Following the idea of O. Rey in [18], the next estimate is concerned with the odd part of v .

Lemma 2.2.8. *With the notations introduced in page 47, we have the following.*

$$|v^o|_{H_0^1} = O \left(\varepsilon^{\frac{3}{2}} + \sum_{i=1}^k \left(\frac{1}{(\lambda_i d_i)^{\frac{3}{2}}} + \frac{|DK(a_i)|}{\lambda_i} \right) + \sum_{i \neq \ell} \varepsilon_{i\ell}^{\frac{3}{2}} \log^{\frac{1}{2}}(\varepsilon_{i\ell}^{-1}) \right).$$

The proofs of these Lemmas are postponed until Sections 5 and 6.

We are now in position to prove the following proposition.

Proposition 2.2.9. *Taking i such that a_i is close to the boundary (i.e. d_i is very small). We have*

$$\begin{aligned} & \frac{\Gamma_3}{(K(a_i))^{\frac{5}{4}}} \frac{DK(a_i)}{\lambda_i} (1 + o(1)) - \frac{\Gamma_4}{\sqrt[4]{K(a_i)\lambda_i^2}} \frac{\partial H}{\partial x}(a_i, a_i) \\ & + \sum_{\ell \neq i} \frac{\Gamma_4}{\sqrt[4]{K(a_\ell)\lambda_i^{\frac{3}{2}}\lambda_\ell^{\frac{1}{2}}}} \frac{\partial G}{\partial x}(a_i, a_\ell) = o\left(\varepsilon^2 + \sum_{\ell=1}^k \frac{1}{(\lambda_\ell d_\ell)^2}\right), \end{aligned} \quad (2.2.7)$$

where $\frac{\partial}{\partial x}$ (resp. $\frac{\partial}{\partial y}$) denotes the derivative with respect to the first (resp. second) variable, and Γ_3, Γ_4 are some positive constants.

Proof. We concentrate our attention on the proof of Proposition 2.2.9, that represent a sort of asymptotic expansion of the $\langle \partial I_\varepsilon, \frac{1}{\lambda_i} \frac{\partial P\delta_i}{\partial a_i} \rangle$. Concerning Propositions 2.2.3, 2.2.4 and 2.2.5, they may be obtained in the same way, with easier computations which do not differ with the case $N \geq 4$ (see for instance [1] [3][6][18]).

Let $1 \leq j \leq 3$. Multiplying Eq (2.1.2) by $\frac{1}{\lambda_i} \frac{\partial P\delta_i}{\partial (a_i)_j}$ and integrating on Ω we obtain

$$\int_{\Omega} -\Delta \left(\sum_{\ell=1}^k \alpha_\ell P\delta_\ell + v \right) \frac{1}{\lambda_i} \frac{\partial P\delta_i}{\partial (a_i)_j} = \int_{\Omega} K \left(\sum_{\ell=1}^k \alpha_\ell P\delta_\ell + v \right)^{5-\varepsilon} \frac{1}{\lambda_i} \frac{\partial P\delta_i}{\partial (a_i)_j}.$$

This may be written as

$$\begin{aligned} & \int_{\Omega} \left(\sum_{\ell=1}^k \alpha_\ell \delta_\ell^5 \right) \frac{1}{\lambda_i} \frac{\partial P\delta_i}{\partial (a_i)_j} \\ & = \int_{\Omega} K \left(\sum_{\ell=1}^k \alpha_\ell P\delta_\ell \right)^{5-\varepsilon} \frac{1}{\lambda_i} \frac{\partial P\delta_i}{\partial (a_i)_j} + (5-\varepsilon) \int_{\Omega} K \left(\sum_{\ell=1}^k \alpha_\ell P\delta_\ell \right)^{4-\varepsilon} v \frac{1}{\lambda_i} \frac{\partial P\delta_i}{\partial (a_i)_j} \\ & \quad + \frac{(5-\varepsilon)(4-\varepsilon)}{2} \int_{\Omega} K \left(\sum_{\ell=1}^k \alpha_\ell P\delta_\ell \right)^{3-\varepsilon} v^2 \frac{1}{\lambda_i} \frac{\partial P\delta_i}{\partial (a_i)_j} \\ & \quad + O \left(\int_{\Omega} \sum_{\ell=1}^k \delta_\ell^2 |v|^3 \delta_i + \int_{\Omega} |v|^{5-\varepsilon} \delta_i \right) \end{aligned} \quad (2.2.8)$$

since $\frac{1}{\lambda_i} \frac{\partial P\delta_i}{\partial (a_i)_j} = O(\delta_i)$ in Ω .

The terms where v does not occur may be computed explicitly, using (2.1.4), (2.1.6),

Lemma 2.2.1, and one finds- see the integral estimates in [1], [2] and [17] :

$$\begin{aligned}
 & \int_{\Omega} \left(\sum_{\ell=1}^k \alpha_{\ell} \delta_{\ell}^5 \right) \frac{1}{\lambda_i} \frac{\partial P \delta_i}{\partial (a_i)_j} - \int_{\Omega} K \left(\sum_{\ell=1}^k \alpha_{\ell} P \delta_{\ell} \right)^{5-\varepsilon} \frac{1}{\lambda_i} \frac{\partial P \delta_i}{\partial (a_i)_j} \\
 &= -\frac{\Gamma_3}{(K(a_i))^{5/4}} \frac{DK(a_i)_j}{\lambda_i} (1 + o(1)) + \frac{\Gamma_4}{\sqrt[4]{K(a_i)} \lambda_i^2} \frac{\partial H}{\partial x_j}(a_i, a_i) \\
 & \quad - \sum_{\ell \neq i} \frac{\Gamma_4}{\sqrt[4]{K(a_{\ell})} \lambda_i^{3/2} \lambda_{\ell}^{1/2}} \frac{\partial G}{\partial x_j}(a_i, a_{\ell}) + o\left(\varepsilon^2 + \sum_{\ell=1}^k \frac{1}{(\lambda_{\ell} d_{\ell})^2}\right). \tag{2.2.9}
 \end{aligned}$$

It only remains to control the integrals involving v . In particular, we require some small estimates with respect to $1/(\lambda d)^2$ as ε goes to 0, in order to keep the previous term as dominating term. The two last integrals in (2.2.8) are easy to treat. Namely, using Hölder inequality and Sobolev embedding theorem, we have

$$\int_{\Omega} \sum_{\ell=1}^k \delta_{\ell}^2 |v|^3 \delta_i + \int_{\Omega} |v|^{5-\varepsilon} \delta_i = O(|v|_{H_0^1}^3). \tag{2.2.10}$$

Let us now consider the linear and the quadratic terms in v . From Lemma 2.2.6, we can choose a fixed constant $\bar{c} > 0$ such that $B_{\ell} := B(a_{\ell}, \bar{c} d_{\ell}) \subset \Omega$ for each ℓ and $B_{\ell} \cap B_k = \emptyset, \forall \ell \neq k$ (we ask that $2\bar{c} \leq \min(1, c)$ where c is the constant introduced in Lemma 2.2.6). Note that, with this choice, for $x \in B_{\ell}$, we have $|x - a_i| \geq \bar{c} \max(d_i, d_{\ell})$ for $i \neq \ell$. Outside of $\cup_{1 \leq \ell \leq k} B_{\ell}$, we have $\int_{B_{\ell}^c} \delta_{\ell}^6 = O(1/(\lambda_{\ell} d_{\ell})^3)$. Therefore

$$\int_{(\cup_{\ell=1}^k B_{\ell})^c} K \left(\sum_{\ell=1}^k \alpha_{\ell} P \delta_{\ell} \right)^{4-\varepsilon} \frac{1}{\lambda_i} \frac{\partial P \delta_i}{\partial (a_i)_j} v = O\left(|v|_{H_0^1} \sum_{\ell=1}^k \frac{1}{(\lambda_{\ell} d_{\ell})^{5/2}}\right) \text{ and} \tag{2.2.11}$$

$$\int_{(\cup_{\ell=1}^k B_{\ell})^c} K \left(\sum_{\ell=1}^k \alpha_{\ell} P \delta_{\ell} \right)^{3-\varepsilon} \frac{1}{\lambda_i} \frac{\partial P \delta_i}{\partial (a_i)_j} v^2 = O\left(|v|_{H_0^1}^2 \sum_{\ell=1}^k \frac{1}{(\lambda_{\ell} d_{\ell})^2}\right) \tag{2.2.12}$$

using the fact that K is bounded in $\bar{\Omega}$, $\frac{1}{\lambda_i} \frac{\partial P \delta_i}{\partial (a_i)_j} = O(\delta_i)$, Hölder's inequality and Sobolev embedding theorem. On B_q , $q \neq i$, $\delta_i = O(\frac{1}{\sqrt{\lambda_i} d_i})$ and $\frac{1}{\lambda_i} \frac{\partial \delta_i}{\partial (a_i)_j} = O(\frac{\delta_i^2}{\sqrt{\lambda_i}}) = O(\frac{\delta_i}{\lambda_i d_i})$, so that

$$\int_{B_q} K \left(\sum_{\ell=1}^k \alpha_{\ell} P \delta_{\ell} \right)^{4-\varepsilon} \frac{1}{\lambda_i} \frac{\partial P \delta_i}{\partial (a_i)_j} v = O\left(|v|_{H_0^1} \left(\frac{1}{\lambda_i d_i} |\delta_i|_{L^6(B_q^c)} + \left| \frac{1}{\lambda_i} \frac{\partial \varphi_i}{\partial (a_i)_j} \right|_{L^6} \right)\right) \tag{2.2.13}$$

$$\int_{B_q} K \left(\sum_{\ell=1}^k \alpha_{\ell} P \delta_{\ell} \right)^{3-\varepsilon} \frac{1}{\lambda_i} \frac{\partial P \delta_i}{\partial (a_i)_j} v^2 = O\left(|v|_{H_0^1}^2 \left(\frac{1}{\lambda_i d_i} |\delta_i|_{L^6(B_q^c)} + \left| \frac{1}{\lambda_i} \frac{\partial \varphi_i}{\partial (a_i)_j} \right|_{L^6} \right)\right) \tag{2.2.14}$$

still using Hölder's inequality and Sobolev embedding theorem. A simple computation leads to

$$|\delta_i|_{L^6(B_i^c)} = O\left(\frac{1}{\sqrt{\lambda_i d_i}}\right) \quad \text{and} \quad \left|\frac{1}{\lambda_i} \frac{\partial \varphi_i}{\partial (a_i)_j}\right|_{L^6} = O\left(\frac{1}{(\lambda_i d_i)^{\frac{3}{2}}}\right) \quad (\text{see [17]}). \quad (2.2.15)$$

From (2.2.11) – (2.2.15) we derive

$$\begin{cases} \int_{B_i^c} K \left(\sum_{\ell=1}^k \alpha_\ell P \delta_\ell \right)^{4-\varepsilon} \frac{1}{\lambda_i} \frac{\partial P \delta_i}{\partial (a_i)_j} v = O\left(|v|_{H_0^1} \sum_{\ell=1}^k \frac{1}{(\lambda_\ell d_\ell)^{\frac{3}{2}}}\right), \\ \int_{B_i^c} K \left(\sum_{\ell=1}^k \alpha_\ell P \delta_\ell \right)^{3-\varepsilon} \frac{1}{\lambda_i} \frac{\partial P \delta_i}{\partial (a_i)_j} v^2 = O\left(|v|_{H_0^1}^2 \sum_{\ell=1}^k \frac{1}{(\lambda_\ell d_\ell)^{\frac{3}{2}}}\right). \end{cases} \quad (2.2.16)$$

We turn now to the last and most delicate part, the integral on B_i . We first note that (as in (2.2.13) and (2.2.14) by using (2.2.15))

$$\begin{cases} \int_{B_i} K \left(\sum_{\ell=1}^k \alpha_\ell P \delta_\ell \right)^{4-\varepsilon} \frac{1}{\lambda_i} \frac{\partial \varphi_i}{\partial (a_i)_j} v = O\left(\frac{|v|_{H_0^1}}{(\lambda_i d_i)^{\frac{3}{2}}}\right), \\ \int_{B_i} K \left(\sum_{\ell=1}^k \alpha_\ell P \delta_\ell \right)^{3-\varepsilon} \frac{1}{\lambda_i} \frac{\partial \varphi_i}{\partial (a_i)_j} v^2 = O\left(\frac{|v|_{H_0^1}^2}{(\lambda_i d_i)^{\frac{3}{2}}}\right). \end{cases} \quad (2.2.17)$$

In order to estimate the terms involving $\frac{\partial \delta_i}{\partial (a_i)_j}$, we can write on B_i

$$\sum_{\ell=1}^k \alpha_\ell P \delta_\ell = \alpha_i \delta_i + \left(\sum_{\ell \neq i} \alpha_\ell P \delta_\ell - \alpha_i \varphi_i \right)(a_i) + O\left(\left(\frac{1}{\sqrt{\lambda_i d_i^2}} + \sum_{\ell \neq i} \frac{1}{\sqrt{\lambda_\ell d_\ell d_i^{\frac{3}{2}}}}\right) |x - a_i|\right) \quad (2.2.18)$$

since $|\nabla P \delta_\ell| = O\left(\frac{1}{\sqrt{\lambda_\ell d_\ell d_i^{\frac{3}{2}}}}\right)$ on B_i , for $\ell \neq i$, and $|\nabla \varphi_i| = O\left(\frac{1}{\sqrt{\lambda_i d_i^2}}\right)$ on B_i (it is a consequence of Lemma 2.2.6 and the choice of \bar{c}).

So, using $P \delta_\ell = O\left(\frac{1}{\sqrt{\lambda_\ell d_\ell d_i}}\right)$ on B_i , for $\ell \neq i$ and $\varphi_i = O\left(\frac{1}{\sqrt{\lambda_i d_i}}\right)$ on B_i , we have

$$\begin{aligned} \left(\sum_{\ell=1}^k \alpha_\ell P \delta_\ell \right)^{4-\varepsilon} &= \alpha_i^{4-\varepsilon} \delta_i^{4-\varepsilon} + (4-\varepsilon) \alpha_i^{3-\varepsilon} \left(\sum_{\ell \neq i} \alpha_\ell P \delta_\ell - \alpha_i \varphi_i \right)(a_i) \delta_i^{3-\varepsilon} \\ &\quad + O\left(\sum_{\ell=1}^k \left(\frac{\delta_i^2}{\lambda_\ell d_\ell d_i} + \frac{\delta_i^3 |x - a_i|}{\sqrt{\lambda_\ell d_\ell d_i^{\frac{3}{2}}}} + \frac{1}{(\lambda_\ell d_\ell)^2 d_i^2} \right) \right). \end{aligned} \quad (2.2.19)$$

Concerning the last quantities in (2.2.19), we have

$$\begin{aligned} \int_{B_i} \delta_i^2 \frac{1}{\lambda_i} \left| \frac{\partial \delta_i}{\partial (a_i)_j} \right| |v| &\leq C \int_{B_i} \delta_i^3 |v| = O\left[\left(\int_{B_i} \delta_i^{\frac{18}{5}} \right)^{\frac{5}{6}} |v|_{H_0^1} \right] \\ \int_{B_i} \delta_i^3 |x - a_i| \frac{1}{\lambda_i} \left| \frac{\partial \delta_i}{\partial (a_i)_j} \right| |v| &\leq C \int_{B_i} \delta_i^4 |x - a_i| |v| = O\left[\left(\int_{B_i} \delta_i^{\frac{24}{5}} |x - a_i|^{\frac{6}{5}} \right)^{\frac{5}{6}} |v|_{H_0^1} \right] \\ \int_{B_i} \frac{1}{\lambda_i} \left| \frac{\partial \delta_i}{\partial (a_i)_j} \right| |v| &\leq C \int_{B_i} \delta_i |v| = O\left[d_i^2 \left(\int_{B_i} \delta_i^6 \right)^{\frac{1}{6}} |v|_{H_0^1} \right] \end{aligned}$$

since $\frac{1}{\lambda_i} \frac{\partial \delta_i}{\partial (a_i)_j} = O(\delta_i)$ in B_i . Straightforward computations yield

$$\left(\int_{B_i} \delta_i^6 \right)^{\frac{1}{6}} = O(1), \quad \left(\int_{B_i} \delta_i^{\frac{18}{5}} \right)^{\frac{5}{6}} = O\left(\frac{1}{\lambda_i}\right) \quad \text{and} \quad \left(\int_{B_i} \delta_i^{\frac{24}{5}} |x - a_i|^{\frac{6}{5}} \right)^{\frac{5}{6}} = O\left(\frac{1}{\lambda_i^{\frac{3}{2}}}\right)$$

so that the contribution of the last term in (2.2.19), multiplied by $\frac{1}{\lambda_i} \frac{\partial \delta_i}{\partial (a_i)_j} v$, to the integral on B_i , is dominated by $|v|_{H_0^1} \sum_{\ell=1}^k 1/(\lambda_\ell d_\ell)^2$. Let us now compute the contribution of the first term. We recall that

$$-\Delta P \delta_i = \delta_i^5 \text{ in } \Omega$$

so that

$$-\Delta \frac{\partial P \delta_i}{\partial (a_i)_j} = 5 \delta_i^4 \frac{\partial \delta_i}{\partial (a_i)_j}.$$

As $v \in E_{a,\lambda}$, we know that

$$\int_{\Omega} \nabla \frac{\partial P \delta_i}{\partial (a_i)_j} \cdot \nabla v = - \int_{\Omega} \Delta \frac{\partial P \delta_i}{\partial (a_i)_j} v = 5 \int_{\Omega} \delta_i^4 \frac{\partial \delta_i}{\partial (a_i)_j} v = 0$$

and

$$\int_{\Omega} \delta_i^{4-\varepsilon} \frac{\partial \delta_i}{\partial (a_i)_j} v = \int_{\Omega} \delta_i^4 \left(\delta_i^{-\varepsilon} - \frac{1}{\lambda_i^{\frac{\varepsilon}{2}}} \right) \frac{\partial \delta_i}{\partial (a_i)_j} v.$$

Now expanding K around a_i ; $K(x) = K(a_i) + DK(a_i) \cdot (x - a_i) + O(|x - a_i|^2)$ and using the previous estimates, we get

$$\begin{aligned} & \int_{\Omega} K \left(\sum_{\ell=1}^k \alpha_\ell P \delta_\ell \right)^{4-\varepsilon} \frac{1}{\lambda_i} \frac{\partial P \delta_i}{\partial (a_i)_j} v = \alpha_i^{4-\varepsilon} K(a_i) \int_{B_i} \delta_i^4 \left(\delta_i^{-\varepsilon} - \frac{1}{\lambda_i^{\frac{\varepsilon}{2}}} \right) \frac{1}{\lambda_i} \frac{\partial \delta_i}{\partial (a_i)_j} v \\ & + (4-\varepsilon) \alpha_i^{3-\varepsilon} \left(\sum_{\ell \neq i} \alpha_\ell P \delta_\ell - \alpha_i \varphi_i \right)(a_i) K(a_i) \int_{B_i} \delta_i^{3-\varepsilon} \frac{1}{\lambda_i} \frac{\partial \delta_i}{\partial (a_i)_j} v \\ & + o\left(\frac{|DK(a_i)|}{\lambda_i}\right) + O\left(|v|_{H_0^1} \sum_{\ell=1}^k \frac{1}{(\lambda_\ell d_\ell)^{\frac{3}{2}}}\right). \end{aligned} \quad (2.2.20)$$

Concerning the quadratic term in v , we write

$$\left(\sum_{\ell=1}^k \alpha_\ell P \delta_\ell \right)^{3-\varepsilon} = \alpha_i^{3-\varepsilon} \delta_i^{3-\varepsilon} + O\left(\sum_{\ell=1}^k \left(\frac{\delta_i^2}{\sqrt{\lambda_\ell d_\ell} \sqrt{d_i}} + \frac{1}{(\lambda_\ell d_\ell d_i)^{\frac{3}{2}}} \right) \right), \text{ in } B_i.$$

Observe that

$$\begin{aligned} & \int_{B_i} K \frac{\delta_i^2}{\sqrt{\lambda_\ell d_\ell} \sqrt{d_i}} \frac{1}{\lambda_i} \frac{\partial P \delta_i}{\partial (a_i)_j} v^2 \leq C \int_{B_i} \frac{\delta_i^3}{\sqrt{\lambda_\ell d_\ell} \sqrt{d_i}} v^2 = O\left(\frac{|v|_{H_0^1}^2}{\sqrt{\lambda_\ell d_\ell} \sqrt{\lambda_i d_i}}\right) \\ & \frac{1}{(\lambda_\ell d_\ell d_i)^{\frac{3}{2}}} \int_{B_i} K \frac{1}{\lambda_i} \frac{\partial P \delta_i}{\partial (a_i)_j} v^2 \leq C \frac{1}{(\lambda_\ell d_\ell d_i)^{\frac{3}{2}}} \int_{B_i} \delta_i v^2 = O\left(\frac{|v|_{H_0^1}^2}{(\lambda_\ell d_\ell)^{\frac{3}{2}}}\right) \end{aligned}$$

$$\int_{B_i} K \delta_i^{3-\varepsilon} \frac{1}{\lambda_i} \frac{\partial \varphi_i}{\partial (a_i)_j} v^2 \leq C \left| \frac{1}{\lambda_i} \frac{\partial \varphi_i}{\partial (a_i)_j} \right|_{L^6} |v|_{H_0^1}^2 = O\left(\frac{|v|_{H_0^1}^2}{(\lambda_i d_i)^{\frac{3}{2}}}\right)$$

and, for each $x \in B_i$ we have $K(x) = K(a_i) + O(|x - a_i|)$, so

$$\begin{aligned} \int_{\Omega} K \left(\sum_{\ell=1}^k \alpha_{\ell} P \delta_{\ell} \right)^{3-\varepsilon} \frac{1}{\lambda_i} \frac{\partial P \delta_i}{\partial (a_i)_j} v^2 &= \alpha_i^{3-\varepsilon} K(a_i) \int_{B_i} \delta_i^{3-\varepsilon} \frac{1}{\lambda_i} \frac{\partial \delta_i}{\partial (a_i)_j} v^2 \\ &+ O\left(\frac{|v|_{H_0^1}^2}{\lambda_i} + \sum_{\ell=1}^k \frac{|v|_{H_0^1}^2}{\lambda_{\ell} d_{\ell}}\right). \end{aligned} \quad (2.2.21)$$

Using, as previously, Hölder's inequality and Sobolev embedding theorem, and the fact that $\frac{1}{\lambda_i} \frac{\partial \delta_i}{\partial \lambda_i} = O(\delta_i)$, $\delta_i^{-\varepsilon} - \frac{1}{\lambda_i^{\varepsilon/2}} = O(\varepsilon \log(1 + \lambda_i^2 |x - a_i|^2))$ we find

$$\begin{aligned} \left| \int_{B_i} \delta_i^4 \left(\delta_i^{-\varepsilon} - \frac{1}{\lambda_i^{\frac{\varepsilon}{2}}} \right) \frac{1}{\lambda_i} \frac{\partial \delta_i}{\partial (a_i)_j} v \right| &\leq C \varepsilon \left(\int_{B_i} \delta_i^6 (\log(1 + \lambda_i^2 |x - a_i|^2))^{\frac{6}{5}} \right)^{\frac{5}{6}} |v|_{H_0^1} = O(\varepsilon |v|_{H_0^1}), \\ \int_{B_i} \delta_i^{3-\varepsilon} \frac{1}{\lambda_i} \frac{\partial \delta_i}{\partial (a_i)_j} v &\leq C \left(\int_{B_i} \delta_i^{\frac{24}{5}} \right)^{\frac{5}{6}} |v|_{H_0^1} = O\left(\frac{|v|_{H_0^1}}{\sqrt{\lambda_i}}\right), \\ \int_{B_i} \delta_i^{3-\varepsilon} \frac{1}{\lambda_i} \frac{\partial \delta_i}{\partial (a_i)_j} v^2 &\leq C \left(\int_{B_i} \delta_i^6 \right)^{\frac{2}{3}} |v|_{H_0^1}^2 = O(|v|_{H_0^1}^2). \end{aligned}$$

In view of Proposition 2.2.2, we obtain for the v -part in the expansion of $\langle \partial I_{\varepsilon}, \frac{1}{\lambda_i} \frac{\partial P \delta_i}{\partial (a_i)_j} \rangle$ a quantity dominated by $1/(\lambda d)^{\frac{3}{2}} + \varepsilon^{\frac{3}{2}}$. This is sufficient to get result in the next section devoted to the construction of a solution (since the concentration points are assumed to be in a compact set of Ω). However, we need sharper estimates in (2.2.20) and (2.2.21) to prove Proposition 2.2.9. Precisely, when we want to prove that the concentration points are far away from the boundary.

For these purposes, following [18], we remark that δ_i is even and $\frac{\partial \delta_i}{\partial (a_i)_j}$ is odd with respect to the variable $(x - a_i)_j$. Splitting v in an even part and an odd one with respect to this variable in a neighborhood of a_i , we are able to control the contribution of the even part by using the oddness of $\partial \delta_i / \partial (a_i)_j$ and the evenness of v^e . Furthermore we obtain a better estimate for the odd part v^o . In fact, we will prove that $|v^o|_{H_0^1} = o(|v|_{H_0^1})$. This method will provide us with a suitable control of the quantities involving v and v^2 .

Let us make this precise. Firstly, we set

$$v = \sum_{i=1}^k v_i + w \quad (2.2.22)$$

with v_i the projection of v onto $H_0^1(B_i)$, that is

$$\Delta v_i = \Delta v \text{ in } B_i; \quad v_i = 0 \text{ on } \partial B_i \quad (2.2.23)$$

v_i being continued by 0 in $\Omega \setminus B_i$. Note that $w \in H_0^1(\Omega)$ is harmonic in B_i , and is orthogonal to v_i , that is

$$\Delta w = 0 \text{ in } B_i; \quad \int_{\Omega} \nabla w \cdot \nabla v_i = 0 \quad \forall i, \quad 1 \leq i \leq k. \quad (2.2.24)$$

As consequence, we have

$$\int_{\Omega} |\nabla v|^2 = \sum_{i=1}^k \int_{\Omega} |\nabla v_i|^2 + \int_{\Omega} |\nabla w|^2. \quad (2.2.25)$$

We split v_i in an even part v^e and an odd one v^o with respect to $(x - a_i)_j$.

On B_i , $v = v^e + v^o + w$ whence, in view of (2.2.20) and (2.2.21)

$$\begin{aligned} \int_{B_i} \delta_i^4 \left(\delta_i^{-\varepsilon} - \frac{1}{\lambda_i^{\frac{\varepsilon}{2}}} \right) \frac{1}{\lambda_i} \frac{\partial \delta_i}{\partial (a_i)_j} v &= \int_{B_i} \delta_i^4 \left(\delta_i^{-\varepsilon} - \frac{1}{\lambda_i^{\frac{\varepsilon}{2}}} \right) \frac{1}{\lambda_i} \frac{\partial \delta_i}{\partial (a_i)_j} (v^o + w) \\ &= \int_{B_i} \delta_i^4 \left(\delta_i^{-\varepsilon} - \frac{1}{\lambda_i^{\frac{\varepsilon}{2}}} \right) \frac{1}{\lambda_i} \frac{\partial \delta_i}{\partial (a_i)_j} w + O(\varepsilon |v^o|_{H_0^1}) \end{aligned}$$

$$\int_{B_i} \delta_i^{3-\varepsilon} \frac{1}{\lambda_i} \frac{\partial \delta_i}{\partial (a_i)_j} v = \int_{B_i} \delta_i^{3-\varepsilon} \frac{1}{\lambda_i} \frac{\partial \delta_i}{\partial (a_i)_j} (v^o + w) = \int_{B_i} \delta_i^{3-\varepsilon} \frac{1}{\lambda_i} \frac{\partial \delta_i}{\partial (a_i)_j} w + O\left(\frac{|v^o|_{H_0^1}}{\sqrt{\lambda_i}}\right)$$

$$\begin{aligned} \int_{B_i} \delta_i^{3-\varepsilon} \frac{1}{\lambda_i} \frac{\partial \delta_i}{\partial (a_i)_j} v^2 &= \int_{B_i} \delta_i^{3-\varepsilon} \frac{1}{\lambda_i} \frac{\partial \delta_i}{\partial (a_i)_j} (2v^e v^o + 2(v - w)w + w^2) \\ &= \int_{B_i} \delta_i^{3-\varepsilon} \frac{1}{\lambda_i} \frac{\partial \delta_i}{\partial (a_i)_j} (2v - w)w + O(|v|_{H_0^1} |v^o|_{H_0^1}). \end{aligned}$$

The estimate of $|v^o|_{H_0^1}$ will be sufficient to conclude. Concerning the integrals $\int_{B_i} \delta_i^4 \left(\delta_i^{-\varepsilon} - \frac{1}{\lambda_i^{\frac{\varepsilon}{2}}} \right) \frac{1}{\lambda_i} \frac{\partial \delta_i}{\partial (a_i)_j} w$, $\int_{B_i} \delta_i^{3-\varepsilon} \frac{1}{\lambda_i} \frac{\partial \delta_i}{\partial (a_i)_j} w$ and $\int_{B_i} \delta_i^{3-\varepsilon} \frac{1}{\lambda_i} \frac{\partial \delta_i}{\partial (a_i)_j} (2v - w)w$ i.e. where w occurs, we use the fact that w is harmonic in B_i . We will adapt the method of O. Rey [18]. In his work, the balls are of constant radius while in ours, the radius is $\bar{c}d_i$. We will give the details for the first integral to mention the influence of d_i in the estimates. The others can be obtained by a similar argument.

Let ψ be the solution of

$$\Delta \psi = \delta_i^4 \left(\delta_i^{-\varepsilon} - \frac{1}{\lambda_i^{\frac{\varepsilon}{2}}} \right) \frac{1}{\lambda_i} \frac{\partial \delta_i}{\partial (a_i)_j} \text{ in } B_i, \quad \psi = 0 \text{ on } \partial B_i.$$

Thus we have

$$\int_{B_i} \delta_i^4 (\delta_i^{-\varepsilon} - \frac{1}{\lambda_i^{\frac{\varepsilon}{2}}}) \frac{1}{\lambda_i} \frac{\partial \delta_i}{\partial (a_i)_j} w = \int_{B_i} \Delta \psi \cdot w = \int_{\partial B_i} \frac{\partial \psi}{\partial n} w. \quad (2.2.26)$$

Let G_i be the Green's function for the Laplacian in B_i , that is

$$G_i(x, y) = \frac{1}{|x - y|} - \frac{\bar{c}d_i}{|x||y - \frac{(\bar{c}d_i)^2 x}{|x|^2}|}, \quad (x, y) \in B_i^2.$$

ψ is given by

$$\psi(y) = \int_{B_i} G_i(x, y) \delta_i^4 (\delta_i^{-\varepsilon} - \frac{1}{\lambda_i^{\frac{\varepsilon}{2}}}) \frac{1}{\lambda_i} \frac{\partial \delta_i}{\partial (a_i)_j} dx, \quad y \in B_i$$

and its normal derivative by

$$\frac{\partial \psi}{\partial n}(y) = \int_{B_i} \frac{\partial G_i}{\partial n_y}(x, y) \left(\delta_i^4 (\delta_i^{-\varepsilon} - \frac{1}{\lambda_i^{\frac{\varepsilon}{2}}}) \frac{1}{\lambda_i} \frac{\partial \delta_i}{\partial (a_i)_j} \right) (x) dx, \quad y \in \partial B_i$$

with

$$\frac{\partial G_i}{\partial n_y}(x, y) = \frac{x \cdot y - (\bar{c}d_i)^2}{\bar{c}d_i |x - y|^3} + (\bar{c}d_i)^2 \frac{|x|^2 - x \cdot y}{|x|^3 |y - \frac{(\bar{c}d_i)^2 x}{|x|^2}|^3} = O\left(\frac{1}{|x - y|^2}\right).$$

For $x \in B_i \setminus B(y, \frac{\bar{c}d_i}{2})$

$$\frac{\partial G_i}{\partial n_y}(x, y) = O\left(\frac{1}{d_i^2}\right), \quad \delta_i^4 (\delta_i^{-\varepsilon} - \frac{1}{\lambda_i^{\frac{\varepsilon}{2}}}) \frac{1}{\lambda_i} \frac{\partial \delta_i}{\partial (a_i)_j}(x) = O(\varepsilon \delta_i^5 \log(1 + \lambda_i^2 |x - a_i|^2)).$$

For $x \in B_i \cap B(y, \frac{\bar{c}d_i}{2})$

$$\frac{\partial G_i}{\partial n_y}(x, y) = O\left(\frac{1}{|x - y|^2}\right), \quad \delta_i^4 (\delta_i^{-\varepsilon} - \frac{1}{\lambda_i^{\frac{\varepsilon}{2}}}) \frac{1}{\lambda_i} \frac{\partial \delta_i}{\partial (a_i)_j}(x) = O\left(\frac{\varepsilon \log(\lambda_i d_i)}{\lambda_i^{\frac{5}{2}} d_i^5}\right).$$

Therefore

$$\begin{aligned} \frac{\partial \psi}{\partial n}(y) &= O\left(\frac{\varepsilon}{d_i^2} \int_{B_i} \delta_i^5 \log(1 + \lambda_i^2 |x - a_i|^2) + \frac{\varepsilon \log(\lambda_i d_i)}{\lambda_i^{\frac{5}{2}} d_i^5} \int_{|x| \leq \frac{\bar{c}d_i}{2}} \frac{1}{|x|^2} dx\right) \\ &= O\left(\frac{\varepsilon}{\sqrt{\lambda_i} d_i^2}\right). \end{aligned} \quad (2.2.27)$$

We need the following result.

Lemma 2.2.10. *Let $f \in H_0^1(\Omega)$ and $B(a, d) \subset \Omega$, then*

$$\int_{\partial B(a, d)} |f| = O(d^{\frac{3}{2}} |f|_{H_0^1}).$$

Proof. Let \tilde{f} be defined as : $\tilde{f}(X) = \sqrt{d}f(a + dX)$, for $X \in B(0, 1)$. we have

$$\begin{aligned}
 \int_{\partial B(a,d)} |f(x)|dx &= \int_{\partial B(0,1)} |f(a + dX)|d^2 dX \\
 &= d^{\frac{3}{2}} \int_{\partial B(0,1)} |\tilde{f}(X)| dX \\
 &\leq Cd^{\frac{3}{2}}|\tilde{f}|_{H^1(B(0,1))} \\
 &\leq Cd^{\frac{3}{2}}(|\tilde{f}|_{L^2(B(0,1))} + |\nabla\tilde{f}|_{L^2(B(0,1))}) \\
 &\leq Cd^{\frac{3}{2}}(|\tilde{f}|_{L^6(B(0,1))} + |\nabla\tilde{f}|_{L^2(B(0,1))}) \\
 &\leq Cd^{\frac{3}{2}}(|\tilde{f}|_{L^6(\tilde{\Omega})} + |\nabla\tilde{f}|_{L^2(\tilde{\Omega})}) \quad (\text{where } \tilde{\Omega} := \frac{1}{d}(\Omega - a)) \\
 &\leq Cd^{\frac{3}{2}}|\tilde{f}|_{H_0^1(\tilde{\Omega})} \\
 &\leq Cd^{\frac{3}{2}}|f|_{H_0^1(\Omega)}
 \end{aligned}$$

where we used the continuous embeddings $L^6(B(0, 1)) \hookrightarrow L^2(B(0, 1))$ and $H_0^1(\tilde{\Omega}) \hookrightarrow L^6(\tilde{\Omega})$. Note that the constant in the inequality given by the continuity in the last embedding is independent of the domain. \square

Using (2.2.25), (2.2.27) and Lemma 2.2.10, we have

$$\int_{\partial B_i} \frac{\partial\psi}{\partial n} w = O\left(\frac{\varepsilon}{\sqrt{\lambda_i d_i^2}} \int_{\partial B_i} |w|\right) = O\left(\frac{\varepsilon}{\sqrt{\lambda_i d_i}} |v|_{H_0^1}\right)$$

so that, from (2.2.26) we deduce

$$\int_{B_i} \delta_i^4 \left(\delta_i^{-\varepsilon} - \frac{1}{\lambda_i^{\frac{\varepsilon}{2}}}\right) \frac{1}{\lambda_i} \frac{\partial\delta_i}{\partial(a_i)_j} w = O\left(\frac{\varepsilon}{\sqrt{\lambda_i d_i}} |v|_{H_0^1}\right). \quad (2.2.28)$$

Concerning the second and the third integrals in which w occurs, we have

$$\int_{B_i} \delta_i^{3-\varepsilon} \frac{1}{\lambda_i} \frac{\partial\delta_i}{\partial(a_i)_j} w = O\left(\frac{1}{\lambda_i d_i^2} \int_{\partial B_i} |w|\right) = O\left(\frac{|v|_{H_0^1}}{\lambda_i \sqrt{d_i}}\right) \quad (2.2.29)$$

and

$$\int_{B_i} \delta_i^{3-\varepsilon} \frac{1}{\lambda_i} \frac{\partial\delta_i}{\partial(a_i)_j} (2v - w)w = O\left(\frac{|v|_{H_0^1}}{\sqrt{\lambda_i d_i^2}} \int_{\partial B_i} |w|\right) = O\left(\frac{|v|_{H_0^1}^2}{\sqrt{\lambda_i d_i}}\right). \quad (2.2.30)$$

(2.2.20), (2.2.21), (2.2.28) - (2.2.30), Remark 2.2.7 and Lemma 2.2.8 , together with Proposition 2.2.2, show that

$$\left\{ \begin{array}{l} \int_{\Omega} K \left(\sum_{\ell=1}^k \alpha_{\ell} P \delta_{\ell} \right)^{4-\varepsilon} \frac{1}{\lambda_i} \frac{\partial\delta_i}{\partial a_i} v = o\left(\frac{|DK(a_i)|}{\lambda_i} + \varepsilon^2 + \sum_{\ell=1}^k \frac{1}{(\lambda_{\ell} d_{\ell})^2}\right), \\ \int_{\Omega} K \left(\sum_{\ell=1}^k \alpha_{\ell} P \delta_{\ell} \right)^{3-\varepsilon} \frac{1}{\lambda_i} \frac{\partial\delta_i}{\partial a_i} v^2 = o\left(\varepsilon^2 + \sum_{\ell=1}^k \frac{1}{(\lambda_{\ell} d_{\ell})^2}\right). \end{array} \right. \quad (2.2.31)$$

Then, from (2.2.8) - (2.2.10), (2.2.16), (2.2.31) and Proposition 2.2.2, we get the desired estimate of Proposition 2.2.9. \square

Proof of Theorem 2.1.2

We start by proving the first point i.e. $d_i > d_0, \forall i \in \{1, \dots, k\}$. We argue by contradiction. So we suppose that $d_1 := \min d_i$ has limit zero when ε goes to zero. From Proposition 2.2.4 and Remark 2.2.7, we get

$$\varepsilon = O\left(\sum_{j=1}^k \frac{1}{\lambda_j d_j}\right). \quad (2.2.32)$$

From an arrangement of the d_i , we can suppose that $\forall i \in \{1, \dots, q\}$ we have d_i goes to zero and the others do not with some integer q between 1 and k . So we can write : $d_1 \leq \dots \leq d_q \ll d_{q+1} \leq \dots \leq d_k$.

On one hand, since Ω is bounded, the boundary of Ω is a compact set, and therefore the assumption (H_2) implies that there exists a positive constant c such that

$$\frac{\partial K(x)}{\partial \nu} < -c < 0 \text{ on } \partial\Omega. \quad (2.2.33)$$

On the other hand, we have $\forall i \in \{1, \dots, q\}$

$$\frac{\partial H}{\partial x}(a_i, a_i) \sim \frac{c}{d_i^2} \text{ and } \frac{\partial G}{\partial x}(a_i, a_r) = O\left(\frac{1}{|a_i - a_r|^2} + \left|\frac{\partial H}{\partial x}(a_i, a_r)\right|\right) = O\left(\frac{1}{d_i^{\frac{3}{2}} \sqrt{d_r}}\right) \quad (2.2.34)$$

by using Lemma 2.2.6 and the fact that H is a harmonic function.

Let $\Omega_{d_i} := \{x \in \Omega : d(x, \partial\Omega) > d_i\}$. Taking the scalar product in \mathbb{R}^3 of (2.2.7) with ν_i the outward normal to $\partial\Omega_{d_i}$ at a_i , using (2.2.32) - (2.2.34) and the fact that $\frac{\partial G}{\partial \nu_i}(a_i, a_r) \leq 0$ for $d_r \geq d_i$ (see Lemma B.3 of [5]), we get for $i = 1$ (we recall that $d_1 \leq d_r \quad \forall r$)

$$-\frac{c}{\lambda_1} - \frac{c}{(\lambda_1 d_1)^2} = o\left(\sum_{j \geq 1} \frac{1}{(\lambda_j d_j)^2}\right).$$

Hence we derive that

$$\frac{1}{(\lambda_1 d_1)^2} = o\left(\sum_{j \geq 2} \frac{1}{(\lambda_j d_j)^2}\right). \quad (2.2.35)$$

For $i = 2$, we repeat the same computation. However, since $d_1 \leq d_2$, we lose the information about the sign of $\partial G / \partial \nu_2(a_2, a_1)$. Note that we have, by using (2.2.34) and (2.2.35)

$$\frac{1}{\lambda_2^{\frac{3}{2}} \lambda_1^{\frac{1}{2}}} \left| \frac{\partial G}{\partial \nu_2}(a_2, a_1) \right| \leq \frac{C}{(\lambda_2 d_2)^{\frac{3}{2}} (\lambda_1 d_1)^{\frac{1}{2}}} = o\left(\sum_{j \geq 2} \frac{1}{(\lambda_j d_j)^2}\right).$$

Hence we get that

$$\frac{1}{(\lambda_2 d_2)^2} = o\left(\sum_{j \geq 3} \frac{1}{(\lambda_j d_j)^2}\right).$$

By iteration, step by step, we obtain

$$\frac{1}{\lambda_i d_i} = o\left(\sum_{j=q+1}^k \frac{1}{\lambda_j d_j}\right), \quad \forall i \in \{1, \dots, q\}. \quad (2.2.36)$$

We distinguish two cases : $q = k$ or $q \neq k$. It is clear that in the first case we have a contradiction. Concerning the second one i.e. $q < k$ we start by proving this result :

Lemma 2.2.11. *Recall that for $j \geq q + 1$, we have $d_j \rightarrow 0$.*

1. *All the λ_j 's, $j \geq q + 1$, are comparable.*
2. *Let $j \geq q + 1$. The concentration point a_j converges to a critical point y_j of K .*

Remark 2.2.12. *Lemma 2.2.11 holds also if $q = 0$, that is all the concentration points are in a compact set of Ω .*

Proof. Note that if we have the first point i.e. λ_j 's are comparable, from Proposition 2.2.5, (2.2.32) and (2.2.36) we get the second point. In fact, Lemma 2.2.6 implies $|a_i - a_j| \geq c > 0$ for each $j \geq q + 1$ and $i \neq j$. Hence, an easy computation implies that

$$\frac{1}{\lambda_j} \left| \frac{\partial \varepsilon_{ij}}{\partial a_j} \right| = \varepsilon_{ij}^3 \lambda_i |a_i - a_j| = O\left(\frac{1}{\lambda_j^{\frac{3}{2}} \lambda_i^{\frac{1}{2}}}\right).$$

Concerning the remainder term in Proposition 2.2.5, using (2.2.32), (2.2.36) and Remark 2.2.7, we get

$$\frac{|DK(a_j)|}{\lambda_j} = o\left(\frac{1}{\lambda_j}\right).$$

Thus a_j has to be close to a critical point of K . Hence the second claim follows from the first one.

It remains to prove the Claim 1. We will argue by contradiction, we suppose that $\lambda_{\min} \ll \lambda_{\max}$. We can order the λ_i 's such that

$$\lambda_{\min} = \lambda_{q+1} \leq \dots \leq \lambda_s \ll \lambda_{s+1} \leq \dots \leq \lambda_k$$

with s some integer between $q + 1$ and k .

In the first step, assuming that $\lambda_{\min} \ll \lambda_{\max}$, we can ameliorate the estimate (2.2.32).

In fact it is easy to get :

$$\frac{H(a_k, a_j)}{\sqrt{\lambda_k \lambda_j}} = o\left(\frac{1}{\lambda_{q+1}} + \frac{1}{\lambda_j}\right), \quad \forall j \quad \text{and} \quad -\lambda_k \frac{\partial \varepsilon_{kj}}{\partial \lambda_k} = \frac{1}{2} \varepsilon_{kj} + o\left(\frac{1}{\lambda_{q+1}}\right).$$

Now, using Proposition 2.2.4, for $i = k$ we obtain

$$\varepsilon + \sum_{j \neq k} \varepsilon_{kj} + o\left(\frac{1}{\lambda_{q+1}} + \sum_{j \neq k} \frac{1}{\lambda_j d_j}\right) = O(\varepsilon^2 \log \lambda_k + \sum_j \frac{1}{(\lambda_j d_j)^{\frac{3}{2}}}).$$

So

$$\varepsilon = o\left(\sum_{j=1}^k \frac{1}{\lambda_j d_j}\right) \quad (2.2.37)$$

which implies with (2.2.36)

$$\varepsilon = o\left(\frac{1}{\lambda_{min}}\right) = o\left(\frac{1}{\lambda_{q+1}}\right). \quad (2.2.38)$$

In a second step, since we have assumed that the λ_j 's, for $q+1 \leq j \leq s$ are comparable, it is easy to see (as in the proof of the second claim (done before) by using Proposition 2.2.5) that a_j for $q+1 \leq j \leq s$ has to converge to a critical point of K .

We are now able to analyze what the $s-q$ equations $(E_{\lambda_j})_{q+1 \leq j \leq s}$ mean. This estimate will be useful to accomplish the proof of Lemma 2.2.11. Using Proposition 2.2.4, (2.2.38) and $\frac{H(a_j, a_j)}{\sqrt[4]{K(a_j)}} = \frac{H(y_j, y_j)}{\sqrt[4]{K(y_j)}} + o(1)$, $\frac{G(a_j, a_r)}{\sqrt[4]{K(a_r)}} = \frac{G(y_j, y_r)}{\sqrt[4]{K(y_r)}} + o(1)$, the equation (E_{λ_j}) is equivalent to

$$o\left(\frac{1}{\lambda_{q+1}}\right) + \frac{\Gamma_2}{\lambda_j} \left(\frac{H(y_j, y_j)}{\sqrt[4]{K(y_j)}} + o(1)\right) - \sum_{r \neq j} \frac{\Gamma_2}{\sqrt{\lambda_j \lambda_r}} \left(\frac{G(y_j, y_r)}{\sqrt[4]{K(y_r)}} + o(1)\right) = O\left(\sum_{j \geq q+1} \frac{1}{\lambda_j^2}\right).$$

We prove that in the sum $\sum_{r \neq j}$ we will find only the indices r that satisfy

$q+1 \leq r \leq s$ since $\frac{G(a_j, a_r)}{\sqrt{\lambda_j \lambda_r}} = o\left(\frac{1}{\lambda_{min}}\right)$ when $r \notin \{q+1, \dots, s\}$ (by using (2.2.36) for $i \leq q$ and $\lambda_r \gg \lambda_{min}$ for $r > s$).

For $j \in \{q+1, \dots, s\}$, multiplying the last equation by $\frac{\sqrt{\lambda_j}}{\sqrt[4]{K(y_j)}}$ it follows

$$\frac{H(y_j, y_j)}{\sqrt{K(y_j)}} \frac{1}{\sqrt{\lambda_j}} - \sum_{r \neq j/q+1 \leq r \leq s} \frac{G(y_j, y_r)}{\sqrt[4]{K(y_j)} \sqrt[4]{K(y_r)}} \frac{1}{\sqrt{\lambda_r}} = o\left(\frac{1}{\sqrt{\lambda_j}}\right). \quad (2.2.39)$$

The system composed of the $s-q$ equations (2.2.39) is equivalent to

$$M(y_{q+1}, \dots, y_s)^t \Upsilon = o(\Upsilon) \text{ where } \Upsilon = \left(\frac{1}{\sqrt{\lambda_{q+1}}}, \dots, \frac{1}{\sqrt{\lambda_s}}\right)$$

and M is the matrix defined by (2.1.1).

It follows that $M(y_{q+1}, \dots, y_s)^t (\sqrt{\lambda_{min}} \Upsilon) = o(1)$. Tending ε to zero we obtain

$$M(y_{q+1}, \dots, y_s)^t \bar{\Upsilon} = 0 \text{ where } \bar{\Upsilon} = \lim_{\varepsilon \rightarrow 0} \sqrt{\lambda_{min}} \Upsilon \text{ and } \bar{\Upsilon} \neq 0$$

which contradicts that the matrix $M(y_{q+1}, \dots, y_s)$ is nondegenerate (see assumption (H_3)). So Lemma 2.2.11 is proved. \square

We return now to the proof of Theorem 2.1.2. We recall that we have assumed that $d_1 := \inf d_i \rightarrow 0$. From Proposition 2.2.4 for $i = 1$ and (2.2.36), we find $\varepsilon = o(\frac{1}{\lambda_{min}})$ since $\frac{H(a_1, a_1)}{\lambda_1} = o(\frac{1}{\lambda_{min}})$, $\frac{G(a_1, a_j)}{\sqrt{\lambda_1 \lambda_j}} = o(\frac{1}{\lambda_{min}})$ and using Lemma 2.2.6. By the same way, the system composed of the $k - q$ equations (2.2.39) is equivalent to

$$M(y_{q+1}, \dots, y_k)^t \Upsilon' = o({}^t \Upsilon') \text{ where } \Upsilon' = \left(\frac{1}{\sqrt{\lambda_{q+1}}}, \dots, \frac{1}{\sqrt{\lambda_k}} \right)$$

and we get a contradiction as in the end of the proof of Lemma 2.2.11. We conclude that $d_1 := \min_{1 \leq i \leq k} d_i \not\rightarrow 0$. Hence, the first claim of Theorem 2.1.2 is proved.

Concerning the second claim, it is a consequence of the first one and Lemma 2.2.6.

The third claim and the fact that each concentration point converges to a critical point of K follow from Lemma 2.2.11. By using Claim 2, these critical points are different.

To conclude this part, it remains to prove the two last claims of Theorem 2.1.2 (i.e. $\rho(y) > 0$ and $\Lambda \rightarrow \bar{\Lambda}$ critical point of F_y). We recall that the estimate of β_i is given in Proposition 2.2.3. However, since we have proved that $|a_i - a_j| > c > 0$, $\forall i \neq j$ and $d(a_i, \partial\Omega) > c > 0 \forall i$, we can improve the remainder term in (2.6) and in Proposition 2.2 which is $\varepsilon_{ij} \log^{1/3}(\varepsilon_{ij}^{-1})$. In fact, this remainder term becomes $\frac{1}{\lambda_i} + \frac{1}{\lambda_j}$. Hence we obtain

$$\beta_i = O\left(\varepsilon \log \lambda + \frac{1}{\lambda}\right), \quad (2.2.40)$$

$$\begin{aligned} & \frac{\Gamma_1}{\sqrt{K(y_i)}} \varepsilon - \Gamma_2 \frac{H(y_i, y_i)}{\sqrt{K(y_i)}} \frac{1}{\lambda_i} + \Gamma_2 \sum_{j \neq i} \frac{G(y_i, y_j)}{\sqrt{K(y_i)K(y_j)}} \frac{1}{\sqrt{\lambda_i \lambda_j}} \\ & = O\left(\varepsilon^2 \log \lambda + \frac{1}{\lambda^2} + |\beta|(\varepsilon + \lambda)\right) \end{aligned} \quad (2.2.41)$$

where $O(f(\lambda))$ denotes any quantity dominated by $\sum_{i=1}^k f(\lambda_i)$.

Let us perform the change of variables

$$\Lambda_i = \left(\frac{\Gamma_2}{\Gamma_1 \varepsilon \lambda_i} \right)^{\frac{1}{2}}.$$

Note that the properties

$$\lambda_i \rightarrow +\infty, \quad \varepsilon \log \lambda_i \rightarrow 0, \quad \frac{\lambda_i}{\lambda_j} < c$$

are translated in

$$\Lambda_i \varepsilon^{\frac{1}{2}} \rightarrow 0, \quad \varepsilon \log \Lambda_i \rightarrow 0, \quad \frac{\Lambda_i}{\Lambda_j} < c$$

and that (2.2.41) reads as

$$\frac{1}{\sqrt{K(y_i)}} - \frac{H(y_i, y_i)}{\sqrt{K(y_i)}} \Lambda_i^2 + \sum_{j \neq i} \frac{G(y_i, y_j)}{\sqrt[4]{K(y_i)} \sqrt[4]{K(y_j)}} \Lambda_i \Lambda_j = o(|\Lambda|^2 + 1). \quad (2.2.42)$$

Dividing each of these equalities by Λ_i we get

$$M(y)\Lambda + o(\Lambda) = \Lambda' + o\left(\frac{1}{|\Lambda|}\right) \quad (2.2.43)$$

with $y = (y_1, \dots, y_k)$, ${}^t\Lambda = (\Lambda_1, \dots, \Lambda_k)$ and ${}^t\Lambda' = \left(\frac{1}{\sqrt{K(y_1)}\Lambda_1}, \dots, \frac{1}{\sqrt{K(y_k)}\Lambda_k}\right)$. Let $r(y)$ be the eigenvector associated to $\rho(y)$. (Notice that all the components of $r(y)$ are non zero and they are of same sign (see [3]). Hence, we will choose them to be positive). The scalar product of (2.2.43) with $r(y)$ yields

$$\rho(y)r(y).\Lambda + o(\Lambda) = r(y).\Lambda' + o\left(\frac{1}{|\Lambda|}\right). \quad (2.2.44)$$

Three cases may occur :

$$i)\Lambda \rightarrow 0 \quad ii)\Lambda \rightarrow \bar{\Lambda} \in (0, +\infty)^k \quad iii)\Lambda \rightarrow +\infty \text{ (i.e. } \Lambda_i \rightarrow +\infty, \forall i).$$

The first case and the last one are impossible as shows (2.2.44) and (H_3) . So only the second case may occur. From (2.2.44) we deduce

$$\rho(y)r(y).{}^t\bar{\Lambda} = r(y).{}^t\bar{\Lambda}'$$

hence $\rho(y)$ is positive. Moreover, the limit in (2.2.43) provides us with

$$M(y){}^t\bar{\Lambda} = {}^t\bar{\Lambda}'.$$

This equality means that $\bar{\Lambda}$ is a critical point of F_y . □

2.3 Construction of a solution

To construct a family of solutions of (P_ε) , we will follow the ideas introduced in [3]. The method becomes well known and adapted in many works. In our case, we will adapt the proof [18] where he studied the same problem with $K \equiv 1$. We will repeat some proofs to establish the contribution of the function K in the formulas.

Let y_1, \dots, y_k be different critical points of K such that $\rho(y_1, \dots, y_k) > 0$, we define the subset

$$M_\varepsilon = \left\{ m = (\alpha, \lambda, a, v) \in \mathbb{R}^k \times (0, +\infty)^k \times \Omega^k \times H_0^1(\Omega) / \left| \alpha_i - \frac{1}{\sqrt[4]{K(a_i)}} \right| < \nu_0, \right. \\ \left. \lambda_i > \frac{1}{\nu_0}, \quad \varepsilon \log \lambda_i < \nu_0, \quad \frac{\lambda_i}{\lambda_j} < c_0, \quad |a_i - y_i| < \nu_0, \quad 1 \leq i, j \leq k; \right. \\ \left. v \in E_{a, \lambda}, |v|_{H_0^1(\Omega)} < \nu_0 \right\}$$

with ν_0, c_0 being some suitably chosen positive constants.

Note that, for $\nu_0 \leq \frac{1}{4} \min_{i \neq j} (d(y_i, \partial\Omega), |y_i - y_j|)$, we get that $|a_i - a_j| \geq c > 0$ for $i \neq j$ and $d(a_i, \partial\Omega) \geq c > 0$ for each i . Defining on M_ε the functional

$$K_\varepsilon : M_\varepsilon \longrightarrow \mathbb{R}, \quad m = (\alpha, \lambda, a, v) \longmapsto I_\varepsilon \left(\sum_{i=1}^k \alpha_i P\delta_{a_i, \lambda_i} + v \right).$$

In order to prove Theorem 2.1.4, we introduce the next result.

Proposition 2.3.1. *$m = (\alpha, \lambda, a, v) \in M_\varepsilon$ is a critical point of K_ε if and only if $u = \sum_{i=1}^k \alpha_i P\delta_{a_i, \lambda_i} + v$ is a critical point of I_ε . This means that there exist (A, B, C) in $\mathbb{R}^k \times \mathbb{R}^k \times (\mathbb{R}^3)^k$ such that*

$$\left\{ \begin{array}{l} (E_{\alpha_i}) \quad \frac{\partial K_\varepsilon}{\partial \alpha_i} = 0, \quad 1 \leq i \leq k \\ (E_{\lambda_i}) \quad \frac{\partial K_\varepsilon}{\partial \lambda_i} = B_i \langle v, \lambda_i \frac{\partial^2 P\delta_i}{\partial \lambda_i^2} \rangle + \sum_{j=1}^3 C_{ij} \langle v, \frac{1}{\lambda_i} \frac{\partial^2 P\delta_i}{\partial \lambda_i \partial (a_i)_j} \rangle, \quad 1 \leq i \leq k \\ (E_{(a_i)_j}) \quad \frac{\partial K_\varepsilon}{\partial (a_i)_j} = B_i \langle v, \lambda_i \frac{\partial^2 P\delta_i}{\partial \lambda_i \partial (a_i)_j} \rangle + \sum_{\ell=1}^3 C_{i\ell} \frac{1}{\lambda_i} \langle v, \frac{\partial^2 P\delta_i}{\partial (a_i)_j \partial (a_i)_\ell} \rangle, \\ \hspace{20em} 1 \leq i \leq k, \quad 1 \leq j \leq 3 \\ (E_v) \quad \frac{\partial K_\varepsilon}{\partial v} = \sum_{i=1}^k A_i P\delta_i + B_i \lambda_i \frac{\partial P\delta_i}{\partial \lambda_i} + \sum_{j=1}^3 C_{ij} \frac{1}{\lambda_i} \frac{\partial P\delta_i}{\partial (a_i)_j}. \end{array} \right. \quad (E)$$

We first look at the last equation of (E). The result is the following :

Proposition 2.3.2. *There exists a smooth map which to any $(\varepsilon, \alpha, \lambda, a)$ such that ε small enough and $(\alpha, \lambda, a, 0) \in M_\varepsilon$; associates $\bar{v} \in E_{a, \lambda}$, such that $|\bar{v}|_{H_0^1} < \nu_0$ and (E_v) is satisfied for some $(A, B, C) \in \mathbb{R}^k \times \mathbb{R}^k \times (\mathbb{R}^3)^k$.*

Such a \bar{v} is unique, minimizes $K_\varepsilon(\alpha, \lambda, a, v)$ with respect to v in $\{v \in E_{a, \lambda}, |v|_{H_0^1} < \nu_0\}$, and we have the estimates

$$|\bar{v}|_{H_0^1} = O\left(\varepsilon + \frac{1}{\lambda}\right) \quad (2.3.1)$$

$$\begin{cases} A = O(|\beta| + \frac{1}{\lambda} + \varepsilon \log \lambda) \\ B = O(\frac{1}{\lambda} + \varepsilon) \\ C = O(\frac{1}{\lambda}) \end{cases} \quad (2.3.2)$$

with

$$\beta = (\beta_1, \dots, \beta_k) = (\alpha_1 - \frac{1}{\sqrt[4]{K(a_1)}}, \dots, \alpha_k - \frac{1}{\sqrt[4]{K(a_k)}}). \quad (2.3.3)$$

For sake of simplicity, $\frac{\lambda_i}{\lambda_j}$ being bounded for any i, j , $O(f(\lambda))$ denotes any quantity dominated by $\sum_{i=1}^k f(\lambda_i)$. We sketch the proof of Proposition 2.3.2. Expanding K_ε with respect to v , we obtain

$$\begin{aligned} K_\varepsilon(\alpha, \lambda, a, v) &= K_\varepsilon(\alpha, \lambda, a, 0) - \int_{\Omega} K \left(\sum_{i=1}^k \alpha_i P \delta_{a_i, \lambda_i} \right)^{5-\varepsilon} v + \frac{1}{2} \int_{\Omega} |\nabla v|^2 \\ &\quad - \frac{5}{2} \int_{\Omega} \left(\sum_{i=1}^k \alpha_i P \delta_{a_i, \lambda_i} \right)^4 v^2 + R_{\varepsilon, \alpha, \lambda}(v) \end{aligned} \quad (2.3.4)$$

where $R_{\varepsilon, \alpha, \lambda, a}$ is a \mathcal{C}^2 function satisfying

$$R_{\varepsilon, \alpha, \lambda, a}(v) = o(|v|_{H_0^1}^2), \quad R'_{\varepsilon, \alpha, \lambda, a}(v) = o(|v|_{H_0^1}), \quad R''_{\varepsilon, \alpha, \lambda, a}(v) = o(1)$$

uniformly with respect to $\varepsilon, \alpha, \lambda, a$, $(\alpha, \lambda, a, 0) \in M_\varepsilon$ and ε small enough.

Moreover, we know that the quadratic term in v is coercive, with a modulus of coercivity bounded from below as $(\alpha, a, \lambda, 0) \in M_\varepsilon$ and ε is sufficiently small- for a proof of this fact, see [1] [17]. Since $|a_i - a_j| > c > 0 \forall i \neq j$ and $d(a_i, \partial\Omega) > c > 0 \forall i$, we can be more precise than the previous section and we claim that

$$\int_{\Omega} K \left(\sum_{i=1}^k \alpha_i P \delta_{a_i, \lambda_i} \right)^{5-\varepsilon} v = O\left(|v|_{H_0^1} \left(\varepsilon + \frac{1}{\lambda} \right)\right). \quad (2.3.5)$$

Consequently, the implicit function theorem yields the conclusion of Proposition 2.3.2, together with estimate (2.3.1).

To complete the proof of Proposition 2.3.2, it only remains to show that estimate (2.3.2) holds. We proceed as follows : we take the scalar product in $H_0^1(\Omega)$ of E_v with $P\delta_i, \lambda_i \frac{\partial P\delta_i}{\partial \lambda_i}, \frac{1}{\lambda_i} \frac{\partial P\delta_i}{\partial (a_i)_j}$ respectively, $1 \leq i \leq k, 1 \leq j \leq 3$. On the right hand side, we get a linear system involving the quantities A_i, B_i, C_{ij} , which is nearly diagonal, invertible, and whose coefficients are given by

$$\begin{cases} \int_{\Omega} \nabla P \delta_i \nabla P \delta_j = C_1 \delta_{ij} + O\left(\frac{1}{\lambda}\right), \\ \int_{\Omega} \nabla \lambda_i \frac{\partial P \delta_i}{\partial \lambda_i} \cdot \nabla \lambda_j \frac{\partial P \delta_j}{\partial \lambda_j} = C_2 \delta_{ij} + O\left(\frac{1}{\lambda}\right), \\ \int_{\Omega} \nabla \frac{1}{\lambda_i} \frac{\partial P \delta_i}{\partial (a_i)_a} \cdot \nabla \frac{1}{\lambda_j} \frac{\partial P \delta_j}{\partial (a_j)_b} = C_3 \delta_{ij} \delta_{ab} + O\left(\frac{1}{\lambda^3}\right), \end{cases} \quad (2.3.6)$$

δ_{ij}, δ_{ab} denoting the kröonecker symbol, the C_i 's being positive constants, and

$$\begin{cases} \int_{\Omega} \nabla P \delta_i \cdot \nabla \lambda_j \frac{\partial P \delta_j}{\partial \lambda_j} = O\left(\frac{1}{\lambda}\right), \\ \int_{\Omega} \nabla P \delta_i \cdot \nabla \frac{1}{\lambda_j} \frac{\partial P \delta_j}{\partial (a_j)_a} = O\left(\frac{1}{\lambda^2}\right), \\ \int_{\Omega} \nabla \lambda_i \frac{\partial P \delta_i}{\partial \lambda_i} \cdot \nabla \frac{1}{\lambda_j} \frac{\partial P \delta_j}{\partial (a_j)_a} = O\left(\frac{1}{\lambda^2}\right). \end{cases} \quad (2.3.7)$$

These estimates follow easily from (2.1.4) (2.1.6) and Lemma 2.2.1, and may also be found in [1] [17]. On the left hand side we find

$$\left\langle \frac{\partial K_{\varepsilon}}{\partial v}, P \delta_i \right\rangle = \frac{\partial K_{\varepsilon}}{\partial \alpha_i}; \quad \lambda_i \frac{\partial K_{\varepsilon}}{\partial \lambda_i} = \alpha_i \left\langle \frac{\partial K_{\varepsilon}}{\partial v}, \lambda_i \frac{\partial P \delta_i}{\partial \lambda_i} \right\rangle; \quad \frac{1}{\lambda_i} \frac{\partial K_{\varepsilon}}{\partial (a_i)_j} = \alpha_i \left\langle \frac{\partial K_{\varepsilon}}{\partial v}, \frac{1}{\lambda_i} \frac{\partial P \delta_i}{\partial (a_i)_j} \right\rangle$$

and we have

Proposition 2.3.3. *For ε small enough and $(\alpha, \lambda, a, 0) \in M_{\varepsilon}$, the following estimates hold*

$$\frac{\partial K_{\varepsilon}}{\partial \alpha_i}(\alpha, \lambda, a, \bar{v}) = -\Gamma_0 \beta_i + V_{\alpha_i}(\varepsilon, \alpha, \lambda, a) \quad (2.3.8)$$

$$\begin{aligned} \lambda_i \frac{\partial K_{\varepsilon}}{\partial \lambda_i}(\alpha, \lambda, a, \bar{v}) &= \frac{\Gamma_1}{\sqrt{K(a_i)}} \varepsilon - \frac{\Gamma_2}{\sqrt{K(a_i)}} \frac{H(a_i, a_i)}{\lambda_i} \\ &\quad - \sum_{j \neq i} \frac{\Gamma_2}{\sqrt[4]{K(a_i)K(a_j)}} \left(\lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} + \frac{1}{2} \frac{H(a_i, a_j)}{\sqrt{\lambda_i \lambda_j}} \right) + V_{\lambda_i}(\varepsilon, \alpha, \lambda, a) \end{aligned} \quad (2.3.9)$$

$$\begin{aligned} \frac{1}{\lambda_i} \frac{\partial K_{\varepsilon}}{\partial (a_i)_j}(\alpha, \lambda, a, \bar{v}) &= -\frac{\Gamma_3}{(K(a_i))^{\frac{3}{2}}} \frac{1}{\lambda_i} \frac{\partial K}{\partial x_j}(a_i) - \sum_{\ell \neq i} \frac{\Gamma_4}{\lambda_i \sqrt[4]{K(a_i)K(a_{\ell})}} \frac{\partial \varepsilon_{i\ell}}{\partial (a_i)_j} \\ &\quad + V_{(a_i)_j}(\varepsilon, \alpha, \lambda, a) \end{aligned} \quad (2.3.10)$$

where $\Gamma_0, \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4$ are positive constants and $\frac{\partial}{\partial (a_i)_j}$ denotes the derivative with respect to the j -th component of the variable a_i . The functions $V_{\alpha_i}, V_{\lambda_i}$ and $V_{(a_i)_j}$ are smooth functions satisfying

$$\begin{cases} V_{\alpha_i}(\varepsilon, \alpha, \lambda, a) = O\left(\beta_i^2 + \frac{1}{\lambda} + \varepsilon \log \lambda\right); \\ V_{\lambda_i}(\varepsilon, \alpha, \lambda, a) = O\left(\varepsilon^2 \log \lambda_i + \frac{1}{\lambda^2} + |\beta|(\varepsilon + \frac{1}{\lambda})\right); \\ V_{(a_i)_j}(\varepsilon, \alpha, \lambda, a) = O\left(\frac{\varepsilon \log \lambda_i}{\lambda_i} + \frac{1}{\lambda^2} + \frac{|\beta|}{\lambda}\right). \end{cases}$$

The proof of this proposition is similar up to a minor modification to Propositions 2.2.3 - 2.2.5 .

Now, since $|a_i - a_j| \geq c \forall i \neq j$ and the λ_i 's are of the same order, we derive that $\varepsilon_{ij} = O(1/\lambda)$. Thus, from Proposition 2.3.3, we deduce that

$$\begin{cases} \frac{\partial K_\varepsilon}{\partial \alpha_j} = O\left(|\beta| + \frac{1}{\lambda} + \varepsilon \log \lambda\right), \\ \lambda_i \frac{\partial K_\varepsilon}{\partial \lambda_j} = O\left(\frac{1}{\lambda} + \varepsilon\right), \\ \frac{1}{\lambda_i} \frac{\partial K_\varepsilon}{\partial (a_i)_j} = O\left(\frac{1}{\lambda}\right), \end{cases}$$

hence (2.3.2) follows, using (2.3.6) and (2.3.7) to invert the linear system involving A_i, B_i, C_{ij} . \square

Proof of Theorem 2.1.4

Once (E_v) is solved, we are left with a finite dimensional system of equations $(E_{\alpha_i}), (E_{\lambda_i}), (E_{(a_i)_j}), 1 \leq i \leq k, 1 \leq j \leq 3$, whose left hand side is given by Proposition 2.3.3, and whose right hand side may be estimated using (2.3.2) , namely

$$\begin{aligned} & B_i \left\langle \lambda_i \frac{\partial^2 P \delta_i}{\partial \lambda_i^2}, \bar{v} \right\rangle_{H_0^1} + \sum_{j=1}^3 C_{ij} \left\langle \frac{1}{\lambda_i} \frac{\partial^2 P \delta_i}{\partial \lambda_i \partial (a_i)_j}, \bar{v} \right\rangle_{H_0^1} \\ &= O\left(\left(\frac{|B_i|}{\lambda} + \sum_{j=1}^3 |C_{ij}| \frac{1}{\lambda}\right) |\bar{v}|_{H_0^1}\right) = O\left(\left(\frac{1}{\lambda^2} + \frac{\varepsilon}{\lambda}\right) |\bar{v}|_{H_0^1}\right) \end{aligned}$$

since, as straight forward computations show

$$\left| \frac{\partial^2 P \delta_i}{\partial \lambda_i^2} \right|_{H_0^1} = O\left(\frac{1}{\lambda^2}\right) \quad \text{and} \quad \left| \frac{\partial^2 P \delta_i}{\partial \lambda_i \partial (a_i)_j} \right|_{H_0^1} = O(1). \quad (2.3.11)$$

In the same way

$$\begin{aligned} & B_i \left\langle \lambda_i \frac{\partial^2 P \delta_i}{\partial \lambda_i \partial (a_i)_j}, \bar{v} \right\rangle_{H_0^1} + \sum_{\ell=1}^3 C_{i\ell} \left\langle \frac{1}{\lambda_i} \frac{\partial^2 P \delta_i}{\partial (a_i)_j \partial (a_i)_\ell}, \bar{v} \right\rangle_{H_0^1} \\ &= O\left(\left(\lambda_i |B_i| + \sum_{\ell=1}^3 \lambda |C_{i\ell}|\right) |\bar{v}|_{H_0^1}\right) = O\left((1 + \varepsilon \lambda) |\bar{v}|_{H_0^1}\right) \end{aligned}$$

since

$$\left| \frac{\partial^2 P \delta_i}{\partial (a_i)_j \partial (a_i)_\ell} \right|_{H_0^1} = O(\lambda^2). \quad (2.3.12)$$

These estimates, together with Proposition 2.3.2 and Proposition 2.3.3, show that with $v = \bar{v}$, (E) is equivalent to a new system (E')

$$(E') \begin{cases} \beta_i = \tilde{V}_{\alpha_i}, \quad 1 \leq i \leq k, \\ \frac{\Gamma_1}{\sqrt{K(a_i)}} \varepsilon - \frac{\Gamma_2}{\sqrt{K(a_i)}} \frac{H(a_i, a_i)}{\lambda_i} + \sum_{\ell \neq i} \frac{\Gamma_2}{\sqrt[4]{K(a_i)K(a_\ell)}} \frac{G(a_i, a_\ell)}{\lambda_i^{\frac{1}{2}} \lambda_\ell^{\frac{1}{2}}} = \tilde{V}_{\lambda_i}, \quad 1 \leq i \leq k, \\ \frac{1}{(K(a_i))^{\frac{3}{2}}} \frac{1}{\lambda_i} \frac{\partial K}{\partial x_j}(a_i) = \tilde{V}_{(a_i)_j}, \quad 1 \leq i \leq k \quad 1 \leq j \leq 3, \end{cases}$$

where $\tilde{V}_{\alpha_i}, \tilde{V}_{\lambda_i}, \tilde{V}_{(a_i)_j}$ are smooth functions on the variables $(\varepsilon, \alpha, \lambda, a)$ which satisfy

$$\begin{cases} \tilde{V}_{\alpha_i}(\varepsilon, \alpha, \lambda, a) = O(|\beta|^2 + \frac{1}{\lambda} + \varepsilon \log \lambda), \\ \tilde{V}_{\lambda_i}(\varepsilon, \alpha, \lambda, a) = O(\frac{1}{\lambda^2} + \varepsilon^2 \log \lambda + |\beta|(\varepsilon + \frac{1}{\lambda})), \\ \tilde{V}_{(a_i)_j}(\varepsilon, \alpha, \lambda, a) = O(\frac{1}{\lambda^2} + \frac{\varepsilon \log \lambda}{\lambda} + \frac{|\beta|}{\lambda}). \end{cases}$$

We recall that $y = (y_1, \dots, y_k) \in \Omega^k$ such that (y_1, \dots, y_k) are critical points of K and $\rho(y_1, \dots, y_k) > 0$. We perform the changes of variables

$$\frac{1}{\lambda_i} = \frac{\Gamma_1}{\Gamma_2}(\bar{\Lambda}_i + \zeta_i)^2 \varepsilon \quad \text{and} \quad a_i = y_i + \xi_i,$$

$\zeta_i \in \mathbb{R}$ and $\xi_i \in \mathbb{R}^3$ assumed to be small. With $v = \bar{v}(\varepsilon, \alpha, \lambda, a)$, the previous estimates show that solving (E) is equivalent to finding $(\beta, \zeta, \xi) \in \mathbb{R}^k \times \mathbb{R}^k \times (\mathbb{R}^3)^k$ such that

$$\beta_i = V_{\beta_i}(\varepsilon, \beta, \zeta, \xi) \tag{2.3.13}$$

$$\frac{1}{\sqrt{K(y_i)}} - \frac{H(y_i, y_i)}{\sqrt{K(y_i)}}(\bar{\Lambda}_i + \zeta_i)^2 + \sum_{j \neq i} \frac{G(y_i, y_j)}{\sqrt[4]{K(y_i)}\sqrt[4]{K(y_j)}}(\bar{\Lambda}_i + \zeta_i)(\bar{\Lambda}_j + \zeta_j) = V_{\zeta_i}(\varepsilon, \beta, \zeta, \xi) \tag{2.3.14}$$

$$\frac{1}{(K(y_i))^{\frac{3}{2}}} D^2 K(y_i) \xi_i = V_{\xi_i}(\varepsilon, \beta, \zeta, \xi) \tag{2.3.15}$$

with $V_{\beta_i}, V_{\zeta_i}, V_{\xi_i}$ smooth functions which satisfy

$$\begin{cases} V_{\beta_i} = O(|\beta|^2 + \varepsilon |\log \varepsilon|), \\ V_{\zeta_i} = O(\varepsilon |\log \varepsilon| + \zeta^2 + |\xi|^2 + |\beta|), \\ V_{\xi_i} = O(\varepsilon |\log \varepsilon| + \zeta^2 + |\xi|^2 + |\beta|). \end{cases} \tag{2.3.16}$$

Equations (2.3.13) – (2.3.15) may be written as

$$\begin{cases} \beta = V(\varepsilon, \beta, \zeta, \xi) \\ L(\zeta, \xi) = W(\varepsilon, \beta, \zeta, \xi) \end{cases} \tag{2.3.17}$$

L being a fixed linear operator in $\mathbb{R}^k \times (\mathbb{R}^3)^k$ and V, W smooth functions which satisfy

$$\begin{cases} V(\varepsilon, \beta, \zeta, \xi) = O(|\beta|^2 + \varepsilon |\log \varepsilon|) \\ W(\varepsilon, \beta, \zeta, \xi) = O(\varepsilon |\log \varepsilon| + |\beta| + \zeta^2 + |\xi|^2). \end{cases} \tag{2.3.18}$$

y_1, \dots, y_k being assumed to be nondegenerate critical points of K and $M(y)$ a nondegenerate matrix, L is invertible, and Brouwer's fixed point theorem ensures, provided that ε is small enough, the existence of a solution $(\beta^\varepsilon, \zeta^\varepsilon, \xi^\varepsilon)$ to (2.3.18), such that

$$\beta^\varepsilon = O(\varepsilon |\log \varepsilon|), \quad \zeta^\varepsilon = O(\varepsilon |\log \varepsilon|) \quad \text{and} \quad \xi^\varepsilon = O(\varepsilon |\log \varepsilon|).$$

By construction, $u_\varepsilon = \sum_{i=1}^k \alpha_i^\varepsilon P\delta_{a_i^\varepsilon, \lambda_i^\varepsilon} + \bar{v}(\varepsilon, \alpha^\varepsilon, \lambda^\varepsilon, a^\varepsilon)$ with

$$\alpha_i^\varepsilon = \frac{1}{\sqrt[4]{K(a_i^\varepsilon)}} - \beta_i^\varepsilon \quad \frac{1}{\lambda_i^\varepsilon} = \frac{\Gamma_1}{\Gamma_2} (\bar{\Lambda}_i + \zeta_i^\varepsilon)^2 \varepsilon \quad a_i^\varepsilon = y_i + \xi_i^\varepsilon$$

is a critical point of I_ε , whence

$$-\Delta u_\varepsilon = K|u_\varepsilon|^{4-\varepsilon} u_\varepsilon \text{ in } \Omega.$$

Now, let $u_\varepsilon^- = \max(0, -u_\varepsilon)$. Since $|u_\varepsilon^-|_{L^6}$ is small, arguing as in [18], the solution u_ε has to be positive in Ω . Hence, we have the desired result. \square

2.4 Proof of Theorem 2.1.1

The index of the critical point u_ε of the functional I_ε is the same as the index of the critical point of

$$\tilde{K}_\varepsilon(\alpha, \lambda, a) = I_\varepsilon\left(\sum_{i=1}^k \alpha_i P\delta_{a_i, \lambda_i} + v(\varepsilon, \alpha, \lambda, a)\right).$$

Therefore, we get the following result :

Lemma 2.4.1.

$$\text{index}(I_\varepsilon, u_\varepsilon) = 4k - \sum_{i=1}^k \text{index}(K, y_i).$$

Proof. We will limit ourselves, in the proof of this Lemma, to the case where $u_\varepsilon = \alpha P\delta_{a, \lambda} + v(\varepsilon, \alpha, \lambda, a)$ (i.e. $k = 1$). We recall that $\text{index}(I_\varepsilon, u_\varepsilon) = \text{index}(\tilde{K}_\varepsilon, (\alpha, \lambda, a))$. Elementary computations imply that $D^2\tilde{K}_\varepsilon(\alpha, \lambda, a)$ is equal to :

$$\begin{pmatrix} -4S_3 + O(\frac{1}{\lambda}) & O(\frac{1}{\lambda^2}) & O(\frac{1}{\lambda}) & O(\frac{1}{\lambda}) & O(\frac{1}{\lambda}) \\ O(\frac{1}{\lambda^2}) & \frac{c}{\lambda^3} + O(\frac{1}{\lambda^4}) & O(\frac{1}{\lambda^2}) & O(\frac{1}{\lambda^2}) & O(\frac{1}{\lambda^2}) \\ O(\frac{1}{\lambda}) & O(\frac{1}{\lambda^2}) & & & \\ O(\frac{1}{\lambda}) & O(\frac{1}{\lambda^2}) & \left(\begin{array}{c} -D^2K(y) + O(\frac{1}{\lambda}) \\ \end{array} \right) & & \\ O(\frac{1}{\lambda}) & O(\frac{1}{\lambda^2}) & & & \end{pmatrix}$$

where c is a positive constant and $y = \lim_{\varepsilon \rightarrow 0} a$. Note that y is a critical point of K . We recall that the index of the critical point (α, λ, a) of \tilde{K}_ε is equal to the number

of negative eigenvalues of the matrix $D^2\tilde{K}_\varepsilon(\alpha, \lambda, a)$. Multiplying the second row and second column of this matrix by $\lambda^{\frac{3}{2}}$ we find the following matrix :

$$N := \begin{pmatrix} -4S_3 + o(1) & o(1) & o(1) & o(1) & o(1) \\ o(1) & c + o(1) & o(1) & o(1) & o(1) \\ o(1) & o(1) & \begin{pmatrix} -D^2K(y) + o(1) \end{pmatrix} & & \\ o(1) & o(1) & & & \end{pmatrix}$$

which is equivalent to $D^2\tilde{K}_\varepsilon(\alpha, \lambda, a)$ i.e. it represents the quadratic form associated to $D^2\tilde{K}_\varepsilon(\alpha, \lambda, a)$ in another basis. In fact, we have $N = P \times D^2\tilde{K}_\varepsilon(\alpha, \lambda, a) \times^t P$ with $P := \text{diag}(1, \lambda^{\frac{3}{2}}, 1, 1, 1)$ is a scaling matrix of ratio $\lambda^{\frac{3}{2}}$. So they have the same number of negative eigenvalues. Then, we deduce

$$\text{index}(I_\varepsilon, u_\varepsilon) = 1 + (3 - \text{index}(K, y)) = 4 - \text{index}(K, y).$$

Following the same argument, we can prove the result for the case $k > 1$. □

Let $\Sigma = \{u \in H_0^1 : |u|_{H_0^1} = 1\}$. We can see that the functional

$$J_\varepsilon(u) := \frac{1}{\left(\int_\Omega K|u|^{6-\varepsilon}\right)^{\frac{1}{3-\frac{\varepsilon}{2}}}}, \quad u \in \Sigma$$

is also related to our problem (P_ε) (that is the positive critical points of J_ε are the solutions of (P_ε) up to multiplicative constant). We have a relation between critical points of I_ε and those of J_ε which implies

$$\text{index}(J_\varepsilon, \bar{u}_\varepsilon) = \text{index}(I_\varepsilon, u_\varepsilon) - 1 = 4k - 1 - \sum_{i=1}^k \text{index}(K, y_i).$$

For $\tau_s = (i_1, \dots, i_s)$ such that $M(\tau_s)$ is positive definite, using Theorem 2.1.4, we can construct a solution u_ε of (P_ε) of the form $u_\varepsilon = \sum_{j=1}^s \alpha_j P\delta_{a_j, \lambda_j} + v$. This solution corresponds to a critical point of J_ε of index $4s - 1 - \sum_{j=1}^s \text{index}(K, y_{i_j})$. Furthermore, as in [12] (see Theorem 2.2'), we know that this solution u_ε is unique.

Let $c(\tau_s)$ be the critical value associated to this critical point. (For simplicity, we will assume that all the critical values are different). Thus for $\theta > 0$ small enough (we choose θ so that the only critical value between $c(\tau_s) + \theta$ and $c(\tau_s) - \theta$ is $c(\tau_s)$), $J^{c(\tau_s)+\theta} := \{w : J(w) \leq c(\tau_s) + \theta\}$ retracts by deformation onto $J^{c(\tau_s)-\theta} \cup W_u(u_\varepsilon)$. (see [15]). Let χ be the Euler -Poincaré characteristic, we have

$$\chi(J_\varepsilon^{c(\tau_s)+\theta}) = \chi(J_\varepsilon^{c(\tau_s)-\theta}) + (-1)^{4s-1-\sum_{j=1}^s \text{index}(K, y_{i_j})}.$$

Since $\Sigma^+ := \{u \in \Sigma : u \geq 0\}$ is contractible, therefore $\chi(\Sigma^+) = 1$. Thus since J_ε is lower bounded, we get

$$1 = \chi(\Sigma^+) = \sum_{s=1}^m \sum_{\tau_s(i_1, \dots, i_s)/M(\tau_s) > 0} (-1)^{4s-1-\sum_{j=1}^s k_{i_j}}$$

where $k_{i_j} = \text{index}(K, y_{i_j})$ and $m = \#\{y/\nabla K(y) = 0\}$ which is in contradiction with the assumption of Theorem 2.1.1. Therefore, (P) has a solution u_0 . \square

2.5 Proof of Lemma 2.2.6

Recall that (u_ε) is a family of solutions of (P_ε) such that $I_\varepsilon(u_\varepsilon)$ is less than a large fixed constant A . (see Theorem 2.1.2). Note that, in our case, we are in dimension 3 and K is a positive \mathcal{C}^2 function in $\bar{\Omega}$. Hence, it is easy to see that the $(*)_\beta$ (where $\beta = 2$), introduced in [11], is well satisfied.

Let $q = 5 - \varepsilon$. Arguing as in [11], we have

Lemma 2.5.1. *There exists $\mathcal{S} = \{a_1^\varepsilon, \dots, a_p^\varepsilon\}$ such that*

$$d(x, \mathcal{S})^{2/(q-1)} u_\varepsilon(x) \leq C \text{ in } \Omega.$$

In fact, since u_ε is a \mathcal{C}^∞ function on $\bar{\Omega}$, taking a_1^ε such that $u_\varepsilon(a_1^\varepsilon) = \max u_\varepsilon(x)$ and defining

$$\tilde{u}_\varepsilon(X) := \frac{1}{u_\varepsilon(a_1^\varepsilon)} u_\varepsilon\left(a_1^\varepsilon + \frac{X}{(u_\varepsilon(a_1^\varepsilon))^{q-1}}\right), \quad X \in \Omega_\varepsilon := (u_\varepsilon(a_1^\varepsilon))^{\frac{q-1}{2}} (\Omega - a_1^\varepsilon).$$

Since \tilde{u}_ε is bounded in L^∞ norm, we prove that \tilde{u}_ε will converge (in \mathcal{C}_{loc}^1) in each compact set to $\delta_{0, \frac{1}{\sqrt{3}}}$. In a second step, let $h_\varepsilon := |\cdot - a_1^\varepsilon|^{q-1} u_\varepsilon$ and assume that $|h_\varepsilon|_\infty \rightarrow +\infty$, we define a_2^ε such that $h_\varepsilon(a_2^\varepsilon) := \max h_\varepsilon$ and we prove that, as in the first step, \tilde{u}_ε , with a_2^ε instead of a_1^ε and defined in $B(a_2^\varepsilon, \frac{1}{2}|a_1^\varepsilon - a_2^\varepsilon|)$ is bounded in L^∞ norm and therefore it will converge also to $\delta_{b, \gamma}$.

By induction, since $|u_\varepsilon|_{H_0^1}$ is bounded, we have to stop after finite steps. Hence we constructed a finite set $\mathcal{S} = \{a_1^\varepsilon, \dots, a_p^\varepsilon\}$ such that $d(x, \mathcal{S})^{\frac{2}{q-1}} u_\varepsilon(x)$ is bounded. In the sequel, we will denote a_i instead of a_i^ε . As consequence we have

Lemma 2.5.2. $\lambda_i |a_i - a_j| \rightarrow +\infty, \forall i \neq j$, where $\lambda_i := (u(a_i))^{\frac{q-1}{2}}$.

Proof. Remark that, to construct the k^{th} point a_k , we had

$$|d(\cdot, \{a_1, \dots, a_{k-1}\})^{2/(q-1)} u_\varepsilon|_\infty \rightarrow +\infty.$$

Hence, it is easy to see that

$$\lambda_k |a_k - a_j| \rightarrow +\infty, \quad \forall j < k. \quad (2.5.1)$$

Now let $j > k$. Note that if $\lambda_k \geq \frac{1}{2}\lambda_j$, then we get

$$\lambda_k |a_k - a_j| \geq \frac{1}{2}\lambda_j |a_k - a_j| \rightarrow +\infty \text{ (by using (2.5.1))}.$$

It remains to see the case where $\lambda_k \leq \frac{1}{2}\lambda_j$. Let $\mathcal{S}_{k-1} = \{a_1, \dots, a_{k-1}\}$ and recall that in the k^{th} step, we have

$$\lambda_k d(a_k, \mathcal{S}_{k-1}) \geq \lambda_j d(a_j, \mathcal{S}_{k-1}) \quad (\text{since } \lambda_j^{\frac{2}{q-1}} = u(a_j)).$$

Thus, we get that $\frac{1}{2}d(a_k, \mathcal{S}_{k-1}) \geq d(a_j, \mathcal{S}_{k-1})$ which implies that

$$|a_j - a_\ell| \leq \frac{1}{2}|a_k - a_\ell| \text{ (for some } \ell \leq k-1).$$

So $|a_k - a_j| \geq |a_k - a_\ell| - |a_\ell - a_j| \geq \frac{1}{2}|a_k - a_\ell|$ and we obtain

$$\lambda_k |a_k - a_j| \geq \frac{1}{2}\lambda_k |a_k - a_\ell| \rightarrow +\infty \text{ (by using (2.5.1) since } \ell \leq k-1).$$

Thus our lemma follows. \square

Let (ε_n) be a sequence which goes to 0 when $n \rightarrow +\infty$ and denote $u_n := u_{\varepsilon_n}$.

In this part we will prove the following result which implies Lemma 2.2.6.

Proposition 2.5.3. *Let $\mu_n = \inf \left\{ \frac{|a_i - a_j|}{\max(d_i, d_j)}, i \neq j \right\}$. Then $\mu_n \rightarrow 0$.*

Proof. Let i, j be such that $\mu_n = \frac{|a_i - a_j|}{\max(d_i, d_j)}$.

We argue by contradiction, we suppose that $\mu_n \rightarrow 0$. So we can neglect the distance between the two points via the distances to $\partial\Omega$. Hence these two points are so near and then $d_i \sim d_j$. Let

$$\tilde{u}_n(X) := |a_i - a_j|^{\frac{2}{q-1}} u_n(a_i + |a_i - a_j|X), \quad \text{in } \tilde{\Omega}_n := \frac{1}{|a_i - a_j|}(\Omega - a_i).$$

A simple computation shows that \tilde{u}_n satisfies

$$-\Delta \tilde{u}_n = \tilde{K} \tilde{u}_n^q, \quad \tilde{u}_n > 0 \quad \text{in } \tilde{\Omega}_n, \quad \text{where } \tilde{K}(X) := K(a_i + |a_i - a_j|X). \quad (2.5.2)$$

We can see that $x \in B(a_i, d_i)$ is equivalent to $X \in B(0, \frac{d_i}{|a_i - a_j|})$ and $\tilde{\Omega}_n \rightarrow \mathbb{R}^3$. Let $\tilde{\mathcal{S}}_i = \{\tilde{a}_{i,k} := \frac{a_k - a_i}{|a_i - a_j|} / |\tilde{a}_{i,k}| \rightarrow +\infty\}$. So we can write $\tilde{\mathcal{S}}_i = \{0 = \tilde{a}_{i,0}, \dots, \tilde{a}_{i,m}\}$. For simplicity we shall write \tilde{a}_k for $\tilde{a}_{i,k}$. For a subsequence these \tilde{a}_k 's have to converge. Let $\bar{a}_k = \lim_{n \rightarrow +\infty} \tilde{a}_k$ and we set $\bar{\mathcal{S}}_i = \{0 = \bar{a}_0, \dots, \bar{a}_m\}$. In the following, we will use some ideas introduced by R. Schoen and developed by Y. Y. Li and C. S. Lin. Hence, we need to coat some definitions from [11] (page 322) to be self contained.

Definition 2.5.4. [11] A point \bar{y} is called a blow up point of (\tilde{u}_n) if there exists a sequence y_n tending to \bar{y} , such that $\tilde{u}_n(y_n) \rightarrow +\infty$.

Definition 2.5.5. [11] A point \bar{y} is called an isolated blow up point of (\tilde{u}_n) if there exist $0 < \bar{r}, \bar{C} > 0$, and a sequence (y_n) converging to \bar{y} , such that y_n is a local maximum of \tilde{u}_n , $\tilde{u}_n(y_n) \rightarrow +\infty$ and

$$\tilde{u}_n(y) \leq \bar{C}|y - y_n|^{-2/(q-1)} \text{ for all } y \in B(y_n, \bar{r}) \setminus \{y_n\}.$$

To describe the behavior of blowing up solutions near an isolated blow up point, we define spherical averages of \tilde{u}_n centered at y_n as follows

$$\bar{u}_n(r) = \frac{1}{|\partial B(y_n, r)|} \int_{B(y_n, r)} \tilde{u}_n, \quad 0 < r < \bar{r}.$$

Now we define the notion of isolated simple blow up point.

Definition 2.5.6. [11] \bar{y} is called an isolated simple blow up point of (\tilde{u}_n) if \bar{y} is an isolated blow up point, such that, for some positive constant ρ (independent of n) the function $r^{2/(q-1)}\bar{u}_n(r)$ has exactly one critical point in $(0, \rho)$ for n large.

Now, we are in position to prove the next result

Lemma 2.5.7. *If $0 \leq k \leq m$ then \bar{a}_k is an isolated blow up point for \tilde{u}_n .*

Proof. Let $0 \leq k \leq m$. From the definition of the set $\tilde{\mathcal{S}}_i$ there exists a positive constant M such that $|\frac{a_k - a_i}{a_i - a_j}| \leq M$. Then

$$\tilde{u}_n(\tilde{a}_k) = (\lambda_k |a_i - a_j|)^{\frac{2}{q-1}} \geq \left(\frac{1}{M} \lambda_k |a_k - a_i|\right)^{\frac{2}{q-1}}. \quad (2.5.3)$$

From Lemma 2.5.2 we conclude that \bar{a}_k is a blow up point of \tilde{u}_n .

We can also see that if $\tilde{a}_k \in \tilde{\mathcal{S}}_i$ then a_k is close to a_i and $d_k \sim d_i$. Now, let prove that $B(\tilde{a}_k, \frac{1}{4}) \cap B(\tilde{a}_r, \frac{1}{4}) = \emptyset$.

If $\tilde{a}_k \in \tilde{\mathcal{S}}_i$ and $\tilde{a}_r \notin \tilde{\mathcal{S}}_i$, \tilde{a}_r is far from \tilde{a}_k so we get the result.

If $0 \leq k, r \leq m$ i.e. \tilde{a}_k and \tilde{a}_r are in $\tilde{\mathcal{S}}_i$, we have $d_k \sim d_r$ and $\frac{|a_r - a_k|}{\max(d_r, d_k)} \geq \frac{|a_i - a_j|}{\max(d_i, d_j)}$ which implies $|a_r - a_k| \geq |a_i - a_j|(1 + o(1))$ and $|\tilde{a}_r - \tilde{a}_k| \geq 1 + o(1)$. Then $B(\tilde{a}_k, \frac{1}{4}) \cap B(\tilde{a}_r, \frac{1}{4}) = \emptyset$.

We deduce that for each $k \leq m$ and $r \leq m$ we have $B(\tilde{a}_k, \frac{1}{4}) \cap B(\tilde{a}_r, \frac{1}{4}) = \emptyset$.

From Lemma 2.5.1 (which is also true for \tilde{u}_n) and the balls are disjoint, we have for each $k \leq m$ and $X \in B(\tilde{a}_k, \frac{1}{4})$

$$|X - \tilde{a}_k|^{\frac{2}{q-1}} \tilde{u}_n(X) \leq C. \quad (2.5.4)$$

Results (2.5.3) and (2.5.4) imply that \bar{a}_k is an isolated blow up point of \tilde{u}_n . \square

Using Proposition 3.1 of [11], we get

Consequence 2.5.8. \bar{a}_k is an isolated simple blow up point of \tilde{u}_n for each $0 \leq k \leq m$.

Now we are able to prove Proposition 2.5.3.

Let $W = \bar{B}(0, R) \setminus (\cup_{k=0}^m B(\tilde{a}_k, \eta))$ be a compact set of $B(0, R+1) \setminus \tilde{S}_i$ with $\eta > 0$.

We set $\tilde{w}_n(X) := \tilde{u}_n(0)\tilde{u}_n(X)$ defined on W .

Let $\eta' > \eta$. From Consequence 2.5.8 and Proposition 2.3 of [11], we deduce that

$$\tilde{u}_n(X) \leq \frac{C}{\tilde{u}_n(\tilde{a}_k)} \frac{1}{|X - \tilde{a}_k|} \text{ for each } 0 < |X - \tilde{a}_k| \leq \eta'. \quad (2.5.5)$$

Therefore, we derive that

$$\tilde{u}_n(X) \leq \frac{C}{\tilde{u}_n(\tilde{a}_k)} \text{ for } \eta < |X - \tilde{a}_k| \leq \eta'. \quad (2.5.6)$$

Besides, from Lemma 2.5.1 (which is also true for \tilde{u}_n), we get that

$$0 \leq \tilde{u}_n(X) \leq C, \forall X \in W. \quad (2.5.7)$$

Taking account of (2.5.7), Harnack inequality (see [9] page 199) provides us with

$$\max_W \tilde{u}_n \leq C \min_W \tilde{u}_n. \quad (2.5.8)$$

Combining this argument with the fact that 0 is an isolated simple blow up point and (2.5.6), we find, for a fixed point X_0 such that $\eta < |X_0| < \eta'$

$$\tilde{u}_n(X) \leq \max_W \tilde{u}_n \leq C \min_W \tilde{u}_n \leq C\tilde{u}_n(X_0) \leq \frac{C}{\tilde{u}_n(0)} \text{ for } X \in W$$

which implies,

$$\tilde{w}_n(X) := \tilde{u}_n(0)\tilde{u}_n(X) \leq C_W.$$

Using again that 0 is an isolated simple blow up point of \tilde{u}_n and applying Proposition 2.2 of [11] and the Harnack inequality, we get, for a fixed point X_0 satisfying $\eta < |X_0| < \eta'$

$$\frac{c}{\tilde{u}_n(0)} \leq \tilde{u}_n(X_0) \leq \max_W \tilde{u}_n \leq c \min_W \tilde{u}_n \leq c\tilde{u}_n(X) \quad \forall X \in W.$$

We obtain

$$c_W \leq \tilde{w}_n(X) = \tilde{u}_n(0)\tilde{u}_n(X).$$

So for any compact set $W \subset B(0, R) \setminus \tilde{S}_i$ we have

$$\forall X \in W, c_W \leq \tilde{w}_n(X) \leq C_W.$$

We recall that \tilde{u}_n satisfies $-\Delta\tilde{u}_n = \tilde{K}\tilde{u}_n^q$. Then \tilde{w}_n is solution of the equation

$$-\Delta\tilde{w}_n = \frac{1}{\tilde{u}_n(0)^{q-1}}\tilde{K}\tilde{w}_n^q, \quad 0 < c_W \leq \tilde{w}_n \leq C_W \text{ in } W. \quad (2.5.9)$$

It follows from (2.5.9), the maximum principle and some standard elliptic theories that

$$\lim_{n \rightarrow \infty} \tilde{w}_n(y) = h^*(y) \text{ in } C_{loc}^0(\mathbb{R}^3 \setminus \bar{\mathcal{S}}_i).$$

This function h^* satisfies

$$h^*(y) > 0, \quad \forall y \in \mathbb{R}^3 \setminus \bar{\mathcal{S}}_i \quad \text{and} \quad \Delta h^*(y) = 0, \quad \forall y \in \mathbb{R}^3 \setminus \bar{\mathcal{S}}_i.$$

We claim that $\bar{\mathcal{S}}_i$ is the set of singularities of h^* . In fact, from (2.5.6), (2.5.8) and Proposition 2.2 of [11], it is easy to see that by choosing two points $X_k \in B(\tilde{a}_k, \eta') \setminus B(\tilde{a}_k, \eta)$ for $0 \leq k = \ell, r \leq m$

$$\frac{c}{\tilde{u}_n(\tilde{a}_\ell)} \leq \tilde{u}_n(X_\ell) \leq c\tilde{u}_n(X_r) \leq \frac{c}{\tilde{u}_n(\tilde{a}_r)}$$

and therefore the concentration speeds are of the same order. Furthermore, for $k \leq m$ since \tilde{a}_k is an isolated simple blow up point of \tilde{u}_n , we get that from Proposition 2.2 of [11]

$$\tilde{u}_n(X) \geq \frac{c}{\tilde{u}_n(\tilde{a}_k)|X - \tilde{a}_k|} \text{ for } X \in B(\tilde{a}_k, \eta') \setminus B(\tilde{a}_k, \eta)$$

and therefore

$$\tilde{w}_n \geq C \frac{\tilde{u}_n(0)}{\tilde{u}_n(\tilde{a}_k)} \frac{1}{|X - \tilde{a}_k|} \geq \frac{C}{|X - \tilde{a}_k|}.$$

Hence our claim follows.

Note that $\{0, \bar{a}_1\} \subset \bar{\mathcal{S}}_i$. As in [11], it follows from the maximum principle that there exists some non-negative function $b^*(y)$ with

$$b^*(y) \geq 0, \quad \forall y \in \mathbb{R}^3 \setminus (\bar{\mathcal{S}}_i \setminus \{0, \bar{a}_1\}) \quad \text{and} \quad \Delta b^*(y) = 0, \quad \forall y \in \mathbb{R}^3 \setminus (\bar{\mathcal{S}}_i \setminus \{0, \bar{a}_1\}),$$

and some positive constants $c_1, c_2 > 0$, such that

$$h^*(y) = c_1|y|^{-1} + c_2|y - \bar{a}_1|^{-1} + b^*(y), \quad y \in \mathbb{R}^3 \setminus (\bar{\mathcal{S}}_i \setminus \{0, \bar{a}_1\}). \quad (2.5.10)$$

For $0 < \sigma < 1$, applying Corollary 1.1 of [11] (which is a Pohozaev-type identity, see [16]) to (2.5.2), we find

$$\begin{aligned} \int_{\partial B_\sigma} B(\sigma, x, \tilde{u}_n, \nabla \tilde{u}_n) &= \frac{1}{q+1} \sum_j \int_{B_\sigma} x_j \frac{\partial \tilde{K}}{\partial x_j} \tilde{u}_n^{q+1} + \frac{\varepsilon}{2(q+1)} \int_{B_\sigma} \tilde{K} \tilde{u}_n^{q+1} \\ &\quad - \frac{\sigma}{q+1} \int_{\partial B_\sigma} \tilde{K} \tilde{u}_n^{q+1} \\ &\geq \frac{1}{q+1} \sum_j \int_{B_\sigma} x_j \frac{\partial \tilde{K}}{\partial x_j} \tilde{u}_n^{q+1} - \frac{\sigma}{q+1} \int_{\partial B_\sigma} \tilde{K} \tilde{u}_n^{q+1}, \end{aligned} \quad (2.5.11)$$

where $B(\sigma, x, \tilde{u}_n, \nabla \tilde{u}_n) = \frac{1}{2} \tilde{u}_n \frac{\partial \tilde{u}_n}{\partial \nu} - \frac{\sigma}{2} |\nabla \tilde{u}_n|^2 + \sigma \left(\frac{\partial \tilde{u}_n}{\partial \nu} \right)^2$.

Multiplying (2.5.11) by $(\tilde{u}_n(0))^2$ and sending n to ∞ , we obtain

$$\begin{aligned} \int_{\partial B_\sigma} B(\sigma, x, h^*, \nabla h^*) &= \lim_{n \rightarrow \infty} \int_{\partial B_\sigma} B(\sigma, x, w_n, \nabla w_n) \\ &= \lim_{n \rightarrow \infty} (\tilde{u}_n(0))^2 \int_{\partial B_\sigma} B(\sigma, x, \tilde{u}_n, \nabla \tilde{u}_n). \end{aligned} \quad (2.5.12)$$

Note that, using the fact that 0 is an isolated simple blow up point and (2.5.5),

$$(\tilde{u}_n(0))^2 \int_{\partial B_\sigma} \tilde{K} |\tilde{u}_n|^{q+1} \leq (\tilde{u}_n(0))^2 \frac{C \text{mes}(\partial B_\sigma)}{|\tilde{u}_n(0)|^{q+1} \sigma^{q+1}} \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (2.5.13)$$

Combining (2.5.11) - (2.5.13) and applying Lemma 2.8 of [11] to (2.5.2) with taking account that \tilde{K} is \mathcal{C}^3 function on $\bar{\Omega}$, we deduce

$$\int_{\partial B_\sigma} B(\sigma, x, h^*, \nabla h^*) \geq 0. \quad (2.5.14)$$

On the other hand, we use (2.5.10) and apply Proposition 1.1 of [11] to obtain that

$$\int_{\partial B_\sigma} B(\sigma, x, h^*, \nabla h^*) < 0$$

for $\sigma > 0$ sufficiently small, which contradicts (2.5.14). The proof of Proposition 2.5.3 is thereby completed. \square

2.6 Proof of Lemma 2.2.8

In [18], O. Rey proved that the odd part of v satisfies the following estimate

$$\|v^o\| = O\left(\frac{|v|_{H_0^1}}{\lambda^{\frac{1}{2}}}\right).$$

In his case the points are far away from each other and from the boundary (i.e. $|a_i - a_j| > d_0$ and $d_i > d'_0$). Hence he is able to use some disjoint balls with a fixed radius. However in our case, using the previous section we know that $|a_i - a_j| > c \max(d_i, d_j)$ with c is universal constant. We recall that $B_i = B(a_i, \bar{c}d_i)$ (\bar{c} is the constant introduced in page 43) and from Lemma 2.2.6, these balls are disjoint.

In the following, we will follow the proof of [18] to get the influence of the new radius (which depend on the distances d_i) in the estimate of the v^o part of v .

The estimate of $|v^o|_{H_0^1}$ will make the proof of Proposition 2.2.9 complete. We recall that v^o is the odd part of v_i with respect to $(x - a_i)_j$ (see page 47). For sake of simplicity, we may assume that $i = j = 1$ and, up to a translation, that $a_1 = 0$. We write

$$v^o = \tilde{v}^o + aP\delta_1 + b\lambda_1 \frac{\partial P\delta_1}{\partial \lambda_1} + \sum_{\ell=1}^3 c_\ell \frac{1}{\lambda_1} \frac{\partial P\delta_1}{\partial (a_1)_\ell} \quad \text{with} \quad (2.6.1)$$

$$\langle P\delta_1, \tilde{v}^o \rangle_{H_0^1} = \left\langle \frac{\partial P\delta_1}{\partial \lambda_1}, \tilde{v}^o \right\rangle_{H_0^1} = \left\langle \frac{\partial P\delta_1}{\partial (a_1)_\ell}, \tilde{v}^o \right\rangle_{H_0^1} = 0 \quad 1 \leq \ell \leq 3.$$

Taking the scalar product in $H_0^1(\Omega)$ of (2.6.1) with $P\delta_1, \lambda_1 \frac{\partial P\delta_1}{\partial \lambda_1}, \frac{1}{\lambda_1} \frac{\partial P\delta_1}{\partial (a_1)_\ell}, 1 \leq \ell \leq 3$, provides us an invertible linear system in a, b, c_ℓ whose coefficients are given by (2.3.6) (2.3.7) (Here the variables λ_i 's are not assumed to be of same order. Hence, the remainder terms in (2.3.6) and (2.3.7) will be $\sum (\lambda_i d_i)^{-\gamma}$ instead of $\lambda^{-\gamma}$). On the left hand side, we find

$$\int_{B_1} \nabla P\delta_1 \cdot \nabla v^o = 0 \quad \text{where } B_1 = B(0, \bar{c}d_1) \quad (2.6.2)$$

since

$$\int_{B_1} \nabla \delta_1 \cdot \nabla v^o = 0 \quad \text{and} \quad \int_{B_1} \nabla \varphi_1 \cdot \nabla v^o = 0$$

because of evenness of δ_1 and oddness of v^o with respect to the first variable for the first integral, and harmonicity of φ_1 and nullity of v^o on ∂B_1 for the second one. In the same way

$$\int_{B_1} \nabla \lambda_1 \frac{\partial P\delta_1}{\partial \lambda_1} \cdot \nabla v^o = \int_{B_1} \nabla \frac{1}{\lambda_1} \frac{\partial P\delta_1}{\partial (a_1)_\ell} \cdot \nabla v^o = 0 \quad \ell = 2, 3. \quad (2.6.3)$$

Lastly, we have

$$\begin{aligned} \int_{B_1} \nabla \frac{1}{\lambda_1} \frac{\partial P\delta_1}{\partial (a_1)_1} \cdot \nabla v^o &= \int_{B_1} \nabla \frac{1}{\lambda_1} \frac{\partial P\delta_1}{\partial (a_1)_1} \cdot \nabla (v - v^e - w) \\ &= - \int_{\Omega \setminus B_1} \nabla \frac{1}{\lambda_1} \frac{\partial P\delta_1}{\partial (a_1)_1} \cdot \nabla v - \int_{B_1} \nabla \frac{1}{\lambda_1} \frac{\partial P\delta_1}{\partial (a_1)_1} \cdot \nabla w \end{aligned}$$

since $v \in E_{a,\lambda}$ and v^e is even with respect to the first variable, zero on ∂B_1 , and $\frac{\partial \varphi}{\partial a_1}$ is harmonic in B_1 . On one hand, using $|\nabla \frac{1}{\lambda_1} \frac{\partial \delta_1}{\partial (a_1)_1}| = O(\delta_1^3)$ in B_1^c and Hölder inequality, we find

$$\int_{\Omega \setminus B_1} \nabla \frac{1}{\lambda_1} \frac{\partial \delta_1}{\partial (a_1)_1} \cdot \nabla v \leq C \int_{\Omega \setminus B_1} \delta_1^3 |\nabla v| = O\left(\frac{|v|_{H_0^1}}{(\lambda_1 d_1)^{\frac{3}{2}}}\right).$$

Using that $\frac{\partial \varphi_1}{\partial (a_1)_1}$ is a harmonic function, $\frac{\partial}{\partial \nu} \left(\frac{1}{\lambda_1} \frac{\partial \varphi_1}{\partial (a_1)_1}\right) = O\left(\frac{1}{\lambda_1^{3/2} d_1^3}\right)$ on ∂B_1 and Lemma 2.2.10, we have

$$\int_{\Omega \setminus B_1} \nabla \frac{1}{\lambda_1} \frac{\partial \varphi_1}{\partial (a_1)_1} \cdot \nabla v = \int_{\partial B_1} \frac{\partial}{\partial \nu} \left(\frac{1}{\lambda_1} \frac{\partial \varphi_1}{\partial (a_1)_1}\right) v = O\left(\frac{1}{\lambda_1^{\frac{3}{2}} d_1^3} \int_{\partial B_1} |v|\right) = O\left(\frac{|v|_{H_0^1}}{(\lambda_1 d_1)^{\frac{3}{2}}}\right).$$

So we deduce that

$$\int_{\Omega \setminus B_1} \nabla \frac{1}{\lambda_1} \frac{\partial P \delta_1}{\partial (a_1)_1} \cdot \nabla v = O\left(\frac{|v|_{H_0^1}}{(\lambda_1 d_1)^{\frac{3}{2}}}\right).$$

On the other hand, let $\psi^{(1)}$ be such that

$$\Delta \psi^{(1)} = \Delta \frac{1}{\lambda_1} \frac{\partial P \delta_1}{\partial (a_1)_1} = -5\delta_1^4 \frac{1}{\lambda_1} \frac{\partial \delta_1}{\partial (a_1)_1} \text{ in } B_1; \quad \psi^{(1)} = 0 \text{ on } \partial B_1.$$

Writing

$$\psi^{(1)} = \frac{1}{\lambda_1} \frac{\partial P \delta_1}{\partial (a_1)_1} + \theta$$

we have

$$\int_{B_1} \nabla \frac{1}{\lambda_1} \frac{\partial P \delta_1}{\partial (a_1)_1} \cdot \nabla w = \int_{B_1} \nabla (\psi^{(1)} - \theta) \cdot \nabla w = - \int_{\partial B_1} \frac{\partial \theta}{\partial n} \cdot w \quad (2.6.4)$$

since w and θ are harmonic in B_1 , and $\psi^{(1)}$ is zero on ∂B_1 . Using, as previously, an integral representation for $\psi^{(1)}$, we obtain for $y \in \partial B_1$

$$\begin{aligned} \frac{\partial \psi^{(1)}}{\partial n}(y) &= -5 \int_{B_1} \frac{\partial G_1}{\partial n_y}(x, y) \left(\delta_1^4 \frac{1}{\lambda_1} \frac{\partial \delta_1}{\partial (a_1)_1} \right)(x) dx \\ &= -5 \int_{B(0, \frac{\bar{c}d_1}{2})} \left(\frac{\partial G_1}{\partial n_y}(0, y) + O\left(\frac{|x|}{d_1^3}\right) \right) \left(\delta_1^4 \frac{1}{\lambda_1} \frac{\partial \delta_1}{\partial (a_1)_1} \right)(x) dx \\ &\quad + O\left(\int_{B^c(0, \frac{\bar{c}d_1}{2})} \frac{\delta_1^5(x)}{|x-y|^2} dx \right) \end{aligned}$$

since $\nabla_x \frac{\partial G_1}{\partial n_y}(x, y) = O\left(\frac{1}{d_1^3}\right)$ in $B(0, \frac{\bar{c}d_1}{2}) \times \partial B_1$, $\frac{1}{\lambda_1} \frac{\partial \delta_1}{\partial (a_1)_1} = O(\delta_1)$ and $\frac{\partial G_1}{\partial n_y}(x, y) = O\left(\frac{1}{|x-y|^2}\right)$ in $B_1 \times \partial B_1$.

For the last quantity, using $\delta_1^5 = O\left(\lambda_1^{-\frac{5}{2}}d_1^{-5}\right)$ in $B^c(0, \frac{\bar{c}d_1}{2})$ and $\frac{1}{|x-y|^2} = O\left(\frac{1}{d_1^2}\right)$ in $B^c(0, \frac{\bar{c}d_1}{2}) \setminus B(y, \frac{\bar{c}d_1}{2})$, we have

$$\begin{aligned} \int_{B^c(0, \frac{\bar{c}d_1}{2})} \frac{\delta_1^5(x)}{|x-y|^2} dx &= \int_{B^c(0, \frac{\bar{c}d_1}{2}) \cap B(y, \frac{\bar{c}d_1}{2})} \frac{\delta_1^5(x)}{|x-y|^2} dx + \int_{B^c(0, \frac{\bar{c}d_1}{2}) \setminus B(y, \frac{\bar{c}d_1}{2})} \frac{\delta_1^5(x)}{|x-y|^2} dx \\ &= O\left(\frac{1}{\lambda_1^{\frac{5}{2}}d_1^5} \int_{B^c(0, \frac{\bar{c}d_1}{2}) \cap B(y, \frac{\bar{c}d_1}{2})} \frac{1}{|x-y|^2} dx \right. \\ &\quad \left. + \int_{B^c(0, \frac{\bar{c}d_1}{2}) \setminus B(y, \frac{\bar{c}d_1}{2})} \frac{\delta_1^5(x)}{d_1^2} dx\right) \\ &= O\left(\frac{1}{\lambda_1^{\frac{5}{2}}d_1^4}\right). \end{aligned}$$

$\delta_1^4 \frac{\partial \delta_1}{\partial (a_1)_1}$ being odd with respect to the first variable, and

$$\int_{B(0, \frac{\bar{c}d_1}{2})} \frac{|x|}{d_1^3} \delta_1^4 \left| \frac{1}{\lambda_1} \frac{\partial \delta_1}{\partial (a_1)_1} \right| dx = O\left(\frac{1}{\lambda_1^{\frac{3}{2}}d_1^3}\right)$$

we find

$$\frac{\partial \psi^{(1)}}{\partial n}(y) = O\left(\frac{1}{\lambda_1^{\frac{3}{2}}d_1^3}\right).$$

Notice that $\frac{\partial}{\partial n} \left(\frac{1}{\lambda_1} \frac{\partial P \delta_1}{\partial (a_1)_1} \right) = O\left(\lambda_1^{-\frac{3}{2}}d_1^{-3}\right)$ on ∂B_1 , we get

$$\frac{\partial \theta}{\partial n}(y) = O\left(\frac{1}{\lambda_1^{\frac{3}{2}}d_1^3}\right) \text{ on } \partial B_1. \quad (2.6.5)$$

It follows using (2.2.25), (2.6.4), (2.6.5) and Lemma 2.2.10 that

$$\begin{aligned} \int_{B_1} \nabla \frac{1}{\lambda_1} \frac{\partial P \delta_1}{\partial (a_1)_1} \cdot \nabla w &= O\left(\frac{|v|_{H_0^1}}{(\lambda_1 d_1)^{\frac{3}{2}}}\right) \text{ and} \\ \int_{B_1} \nabla \frac{1}{\lambda_1} \frac{\partial P \delta_1}{\partial (a_1)_1} \cdot \nabla v^o &= O\left(\frac{|v|_{H_0^1}}{(\lambda_1 d_1)^{\frac{3}{2}}}\right). \end{aligned} \quad (2.6.6)$$

Inverting the linear system involving a, b, c_ℓ , whose coefficients are given by (2.3.6) (2.3.7) and whose left hand side is given by (2.6.2) (2.6.3) (2.6.6), the following estimates are obtained :

$$a, b = O\left(\frac{|v|_{H_0^1}}{(\lambda_1 d_1)^{\frac{5}{2}}}\right), \quad c_1 = O\left(\frac{|v|_{H_0^1}}{(\lambda_1 d_1)^{\frac{3}{2}}}\right), \quad c_2, c_3 = O\left(\frac{|v|_{H_0^1}}{(\lambda_1 d_1)^{\frac{5}{2}}}\right). \quad (2.6.7)$$

Using again (2.3.6) and (2.3.7), this implies through (2.6.1)

$$|v^o - \tilde{v}^o|_{H_0^1} = O\left(\frac{|v|_{H_0^1}}{(\lambda_1 d_1)^{\frac{3}{2}}}\right), \quad |v^o|_{H_0^1}^2 = |\tilde{v}^o|_{H_0^1}^2 + O\left(\frac{|v|_{H_0^1}^2}{(\lambda_1 d_1)^3}\right). \quad (2.6.8)$$

We turn now to the last step, which consists in estimating \tilde{v}^o in $H_0^1(\Omega)$. Multiplying the equation $-\Delta u = Ku^{5-\varepsilon}$ by v^o and integrating by parts over Ω , we obtain

$$\int_{\Omega} -\Delta \left(\sum_{i=1}^k \alpha_i P \delta_i + v \right) v^o - \int_{\Omega} K \left(\sum_{i=1}^k \alpha_i P \delta_i + v \right)^{5-\varepsilon} v^o = 0. \quad (2.6.9)$$

Concerning the first integral, we know that

$$\int_{\Omega} -\Delta \left(\sum_{i=1}^k \alpha_i P \delta_i + v \right) v^o = \sum_{i=2}^k \alpha_i \int_{B_1} \delta_i^5 v^o + \int_{B_1} |\nabla v^o|^2$$

since $-\Delta P \delta_i = \delta_i^5$ in Ω , v^o is zero in $\Omega \setminus B_1$, $v = v^e + v^o + w$ in B_1 with v^e even and v^o odd with respect to the first variable, and w harmonic in B_1 . Therefore, using Hölder inequality and taking account of (2.6.8), we find :

$$\int_{\Omega} -\Delta \left(\sum_{i=1}^k \alpha_i P \delta_i + v \right) v^o = \int_{B_1} |\nabla \tilde{v}^o|^2 + O\left(|v|_{H_0^1} \sum_{i=1}^k \frac{1}{(\lambda_i d_i)^{\frac{5}{2}}}\right). \quad (2.6.10)$$

Let us consider the second integral, which may be restricted to B_1 , since v^o is zero in $\Omega \setminus B_1$. We expand

$$\begin{aligned} & \int_{\Omega} K \left(\sum_{i=1}^k \alpha_i P \delta_i + v \right)^{5-\varepsilon} v^o \\ &= \alpha_1^{5-\varepsilon} \int_{B_1} K P \delta_1^{5-\varepsilon} v^o + (5-\varepsilon) \alpha_1^{4-\varepsilon} \int_{B_1} K P \delta_1^{4-\varepsilon} \left(\sum_{i=2}^k \alpha_i P \delta_i + v \right) v^o \\ &+ O\left[\int_{B_1} \left(\delta_1^3 \left(\sum_{i=2}^k \delta_i^2 + |v|^2 \right) + \sum_{i=2}^k \delta_i^5 + |v|^{5-\varepsilon} \right) |v^o| \right]. \end{aligned}$$

Estimating the last term is easy, namely

$$\begin{aligned} & \int_{B_1} \left(\delta_1^3 \left(\sum_{i=2}^k \delta_i^2 + |v|^2 \right) + \sum_{i=2}^k \delta_i^5 + |v|^{5-\varepsilon} \right) |v^o| \\ &= O\left(|v^o|_{H_0^1} \left(\sum_{i=2}^k \left(\int_{B_1} (\delta_1 \delta_i)^3 \right)^{\frac{2}{3}} \left(\int_{B_1} \delta_1^6 \right)^{\frac{1}{6}} + |v|_{H_0^1}^2 + \sum_{i=2}^k \left(\int_{B_1} \delta_i^6 \right)^{\frac{5}{6}} + |v|_{H_0^1}^{5-\varepsilon} \right)\right) \\ &= O\left(|v^o|_{H_0^1} \left(\sum_{i=2}^k \left(\frac{1}{(\lambda_i d_i)^{\frac{5}{2}}} + \varepsilon_{1,i}^2 \log^{\frac{2}{3}}(\varepsilon_{1,i}^{-1}) \right) + |v|_{H_0^1}^2 \right)\right). \end{aligned}$$

Concerning the remaining terms, we write (since $\varphi_1(x) = O(\frac{1}{\sqrt{\lambda_1 d_1}})$ in B_1)

$$\int_{B_1} KP\delta_1^{5-\varepsilon}v^o = \int_{B_1} K\delta_1^{5-\varepsilon}v^o + (5-\varepsilon) \int_{B_1} K\delta_1^{4-\varepsilon}\varphi_1v^o + O\left(\frac{1}{\lambda_1 d_1^2} \int_{B_1} \delta_1^3|v^o|\right).$$

Using the evenness of δ_1 and the oddness of v^o with respect to the first variable, and noticing that

$$K(x) = K(0) + DK(0).x + O(|x|^2) \text{ in } B_1 \text{ and } \left(\int_{B_1} \delta_1^{\frac{24}{5}}|x|^{\frac{6}{5}}\right)^{\frac{5}{6}} = O\left(\frac{1}{\lambda_1^{\frac{3}{2}}}\right),$$

we obtain

$$\begin{aligned} \int_{B_1} KP\delta_1^{5-\varepsilon}v^o &= (5-\varepsilon)K(0) \int_{B_1} \delta_1^{4-\varepsilon}\varphi_1v^o + O\left(|v^o|_{H_0^1}\left(\frac{1}{\lambda_1^2 d_1^2} + \frac{|DK(0)|}{\lambda_1}\right)\right) \\ &= (5-\varepsilon)K(0) \int_{B_1} \delta_1^{4-\varepsilon}\left(\varphi_1(0) + O\left(\frac{|x|}{\sqrt{\lambda_1 d_1^2}}\right)\right)v^o \\ &\quad + O\left(|v^o|_{H_0^1}\left(\frac{1}{\lambda_1^2 d_1^2} + \frac{|DK(0)|}{\lambda_1}\right)\right) \\ &= O\left(|v^o|_{H_0^1}\left(\frac{1}{\lambda_1^2 d_1^2} + \frac{|DK(0)|}{\lambda_1}\right)\right). \end{aligned} \tag{2.6.11}$$

In the same way and using $|\nabla P\delta_i| = O(1/\sqrt{\lambda_i d_i} d_1^{\frac{3}{2}})$ in $B_1 \forall i \neq 1$, we have

$$\begin{aligned} &\int_{B_1} KP\delta_1^{4-\varepsilon}\left(\sum_{i=2}^k \alpha_i P\delta_i\right)v^o \\ &= K(0) \int_{B_1} P\delta_1^{4-\varepsilon}\left(\sum_{i=2}^k \alpha_i P\delta_i\right)v^o + O\left(\int_{B_1} \delta_1^4\left(\sum_{i=2}^k \delta_i\right)|x||v^o|\right) \\ &= K(0) \int_{B_1} \delta_1^{4-\varepsilon}\sum_{i=2}^k \left(\alpha_i P\delta_i(0) + O\left(\frac{|x|}{\sqrt{\lambda_i d_i} d_1^{\frac{3}{2}}}\right)\right)v^o \\ &\quad + O\left(\int_{B_1} \frac{\delta_1^3}{\sqrt{\lambda_1 d_1}}\left(\sum_{i=2}^k \delta_i\right)|v^o|\right) + O\left(|v^o|_{H_0^1}\sum_{i=2}^k |\delta_i|_{L^\infty(B_1)}\left(\int_{B_1} \delta_1^{\frac{24}{5}}|x|^{\frac{6}{5}}\right)^{\frac{5}{6}}\right) \end{aligned}$$

whence

$$\int_{B_1} KP\delta_1^{4-\varepsilon}\left(\sum_{i=2}^k \alpha_i P\delta_i\right)v^o = O\left(\sum_{i=1}^k \frac{|v^o|_{H_0^1}}{(\lambda_i d_i)^2}\right). \tag{2.6.12}$$

Expanding K around 0, the last term to consider is written as

$$\int_{B_1} KP\delta_1^{4-\varepsilon}vv^o = K(0) \int_{B_1} P\delta_1^{4-\varepsilon}(v^e + v^o + w)v^o + O\left(\frac{|v|_{H_0^1}|v^o|_{H_0^1}}{\lambda_1}\right).$$

On one hand, using the oddness of v^o and the evenness of v^e

$$\begin{aligned}
 & \int_{B_1} P\delta_1^{4-\varepsilon}(v^e + v^o)v^o \\
 &= \int_{B_1} \left(\delta_1^{4-\varepsilon} - (4-\varepsilon)\delta_1^{3-\varepsilon}(\varphi_1(0) + O(\frac{|x|}{\sqrt{\lambda_1 d_1^2}})) + O(\frac{\delta_1^2}{\lambda_1 d_1^2}) \right) (v^e + v^o)v^o \\
 &= \int_{B_1} \delta_1^{4-\varepsilon}(v^o)^2 + O\left(\frac{1}{\sqrt{\lambda_1 d_1}} \int_{B_1} \delta_1^3(v^o)^2 + \frac{1}{\sqrt{\lambda_1 d_1^2}} \int_{B_1} \delta_1^3|x||v_1||v^o| \right. \\
 &\quad \left. + \frac{1}{\lambda_1 d_1^2} \int_{B_1} \delta_1^2|v_1||v^o| \right) \\
 &= \int_{B_1} \delta_1^4(v^o)^2 + O\left(\varepsilon \log(\lambda_1)|v^o|_{H_0^1}^2 + \frac{1}{\sqrt{\lambda_1 d_1}} \left(\int_{B_1} \delta_1^{\frac{9}{2}}\right)^{\frac{2}{3}}|v^o|_{H_0^1}^2 \right. \\
 &\quad \left. + \frac{1}{\sqrt{\lambda_1 d_1^2}} \left(\int_{B_1} \delta_1^{\frac{9}{2}}|x|^{\frac{3}{2}}\right)^{\frac{2}{3}}|v|_{H_0^1}|v^o|_{H_0^1} + \frac{1}{\lambda_1 d_1^2} \left(\int_{B_1} \delta_1^3\right)^{\frac{2}{3}}|v|_{H_0^1}|v^o|_{H_0^1} \right) \\
 &= \int_{B_1} \delta_1^4(v^o)^2 + o(|v^o|_{H_0^1}^2) + O\left(\frac{\log^{\frac{2}{3}}(\lambda_1 d_1)}{(\lambda_1 d_1)^2}|v|_{H_0^1}|v^o|_{H_0^1}\right) \tag{2.6.13}
 \end{aligned}$$

because of the following computations

$$\int_{B_1} \delta_1^{\frac{9}{2}} = O\left(\frac{1}{\lambda_1^{\frac{3}{4}}}\right) \quad , \quad \int_{B_1} \delta_1^{\frac{9}{2}}|x|^{\frac{3}{2}} = O\left(\frac{\log(\lambda_1 d_1)}{\lambda_1^{\frac{9}{4}}}\right) \quad , \quad \int_{B_1} \delta_1^3 = O\left(\frac{\log(\lambda_1 d_1)}{\lambda_1^{\frac{3}{2}}}\right).$$

On the other hand

$$\int_{B_1} P\delta_1^{4-\varepsilon}wv^o = \int_{B_1} \Delta\psi^{(2)}.w = \int_{\partial B_1} \frac{\partial\psi^{(2)}}{\partial n}w$$

with $\psi^{(2)}$ defined as

$$\Delta\psi^{(2)} = P\delta_1^{4-\varepsilon}v^o \text{ in } B_1; \quad \psi^{(2)} = 0 \text{ on } \partial B_1.$$

The normal derivative of $\psi^{(2)}$ at $y \in \partial B_1$ is given by

$$\begin{aligned}
 \frac{\partial\psi^{(2)}}{\partial n}(y) &= \int_{B_1} \frac{\partial G_1}{\partial n_y}(x, y)P\delta_1^{4-\varepsilon}v^o \\
 &= O\left(\int_{B_1 \cap \{x: |x-y| \geq \frac{\bar{c}d_1}{2}\}} \frac{\delta_1^4|v^o|}{d_1^2}dx + \int_{B_1 \cap \{x: |x-y| \leq \frac{\bar{c}d_1}{2}\}} \frac{|v^o|}{\lambda_1^2 d_1^4 |x-y|^2}dx\right) \\
 &= O\left(\frac{1}{d_1^2} \left(\int_{B_1} \delta_1^{\frac{24}{5}}\right)^{\frac{5}{6}}|v^o|_{H_0^1} + \frac{1}{\lambda_1^2 d_1^4}|v^o|_{H_0^1} \left(\int_{|x-y| \leq \frac{\bar{c}d_1}{2}} \frac{1}{|x-y|^{\frac{12}{5}}}dx\right)^{\frac{5}{6}}\right) \\
 &= O\left(\frac{|v^o|_{H_0^1}}{\sqrt{\lambda_1 d_1^2}}\right)
 \end{aligned}$$

since $(\int_{B_1} \delta_1^{\frac{24}{5}})^{\frac{5}{6}} = O(\frac{1}{\sqrt{\lambda_1}})$. Consequently

$$\int_{B_1} P\delta_1^{4-\varepsilon}wv^o = O\left(\frac{|v^o|_{H_0^1}}{\sqrt{\lambda_1 d_1^2}} \int_{\partial B_1} |w|\right)$$

and so, using (2.2.25) and Lemma 2.2.10, we find

$$\int_{B_1} P\delta_1^{4-\varepsilon} wv^o = O\left(\frac{|v^o|_{H_0^1}|v|_{H_0^1}}{\sqrt{\lambda_1 d_1}}\right). \quad (2.6.14)$$

This yields finally, taking account of (2.6.10) - (2.6.14)

$$\begin{aligned} & \int_{\Omega} -\Delta\left(\sum_{i=1}^k \alpha_i P\delta_i + v\right)v^o - \int_{\Omega} K\left(\sum_{i=1}^k \alpha_i P\delta_i + v\right)^{5-\varepsilon} v^o \\ &= \int_{B_1} |\nabla v^o|^2 - 5 \int_{B_1} \delta_1^4 (v^o)^2 + o(|v^o|_{H_0^1}^2) + O\left(|v^o|_{H_0^1} \left(\frac{|DK(0)|}{\lambda_1} + \sum_{i=1}^k \frac{1}{\lambda_i^2 d_i^2} + \frac{|v|_{H_0^1}}{\sqrt{\lambda_1 d_1}}\right)\right. \\ & \quad \left. + \sum_{i=1}^k \frac{|v|_{H_0^1}}{\lambda_i^{\frac{5}{2}} d_i^{\frac{5}{2}}}\right) \\ &= \int_{\Omega} |\nabla \tilde{v}^o|^2 - 5 \int_{\Omega} \delta_1^4 (\tilde{v}^o)^2 + o(|\tilde{v}^o|_{H_0^1}^2) + O\left(\frac{|v|_{H_0^1}^2}{(\lambda_1 d_1)^2} + \sum_{i=1}^k \frac{|v|_{H_0^1}}{(\lambda_i d_i)^{\frac{5}{2}}}\right. \\ & \quad \left. + \frac{|v|_{H_0^1} |\tilde{v}^o|_{H_0^1}}{\sqrt{\lambda_1 d_1}} + |\tilde{v}^o|_{H_0^1} \left(\sum_{i=1}^k \frac{1}{\lambda_i^2 d_i^2} + \frac{|DK(0)|}{\lambda_1}\right)\right) \end{aligned}$$

because of (2.6.8). Comparing with (2.6.8) and (2.6.9), and the quadratic form

$$v \mapsto \int_{\Omega} |\nabla v|^2 - 5 \int_{\Omega} \delta_1^4 v^2$$

being coercive on the subset $[Span(P\delta_1, \frac{\partial P\delta_1}{\partial \lambda_1}, \frac{\partial P\delta_1}{\partial (x_1)_j})]_{H_0^1}^{\perp}$, the estimate of $|v^o|_{H_0^1}$ follows.

□

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Harmonic functions with nonlinear Neumann boundary condition and their Morse indices

(with M. Ben Ayed and A. Selmi).

"Dedicated to the memory of Professor Abbas Bahri."

Abstract : We consider the solutions of a nonlinear Neumann elliptic equation $\Delta u = 0$ in Ω , $\partial u / \partial \nu = f(x, u)$ on $\partial\Omega$, where Ω is a bounded open smooth domain in \mathbb{R}^N , $N \geq 2$ and f satisfies super-linear and subcritical growth conditions. We prove that L^∞ -bounds on solutions are equivalent to bounds on their Morse indices.

Key words : Liouville type theorem, Blow up Analysis, harmonic function, Elliptic equation, Morse index.

Chapitre 3

Harmonic functions with nonlinear Neumann boundary condition and their Morse indices

3.1 Introduction and main results

The purpose of this paper is to study the nonlinear Neumann elliptic problem

$$(P) \quad \begin{cases} \Delta u = 0 & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} = f(x, u) & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded open smooth domain in \mathbb{R}^N , $\partial/\partial\nu$ denotes the derivative with respect to the outward normal to $\partial\Omega$ ($N \geq 2$), $f(x, t)$ is continuous on $\partial\Omega \times \mathbb{R}$, differentiable with respect to t ; and $\partial f/\partial t$ is continuous on $\partial\Omega \times \mathbb{R}$.

Elliptic problem with nonlinear boundary condition like (P) has been widely studied in the past by many authors and it is still an area of intensive research. This kind of boundary condition appears in a rather natural way in some physical models. For example, problem (P) can be thought of as a model for heat propagation. In this case u stands for the stationary temperature and the normal derivative $\frac{\partial u}{\partial \nu}$ that appears in the boundary condition represents the heat flux. Hence the boundary condition represents a nonlinear radiation law at the boundary.

Here, we will deal with the super-linear case, that is,

$$\lim_{|t| \rightarrow +\infty} \frac{f(x, t)}{t} = +\infty, \quad \text{uniformly on } x \in \partial\Omega, \quad (3.1.1)$$

with a "subcritical" growth

$$|f(x, t)| \leq a(1 + |t|^p), \quad a > 0, \quad \text{for all } (x, t) \in \partial\Omega \times \mathbb{R}, \quad (3.1.2)$$

where p satisfies

$$1 < p < N/(N - 2) \text{ if } N \geq 3 \text{ and } p \in (1, \infty) \text{ if } N = 2. \quad (3.1.3)$$

We remark that the assumptions (3.1.1) – (3.1.3) imply that we are in a nonlinear and subcritical case. Hence, using the compactness of the embedding of $H^1(\Omega)$ into $L^{p+1}(\partial\Omega)$, the variational methods imply that the problem (P) has solutions.

In this paper, we assume that f satisfies

$$\lim_{|t| \rightarrow +\infty} f(x, t)t^{-1}|t|^{-(p-1)} = b \text{ uniformly on } x \in \partial\Omega, \text{ where } b > 0 \quad (3.1.4)$$

and p satisfies (3.1.3). Note that (3.1.4) implies (3.1.1) and (3.1.2).

Our main goal is to study the properties of the solutions of (P). We will prove that their boundedness is equivalent to the finiteness of their Morse indices.

There have been many works devoted to the relationship between the properties of solutions and Morse indices. The earliest and the most known result is due to Bahri and Lions [5], they studied the following problem :

$$-\Delta u = f(x, u) \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega. \quad (3.1.5)$$

Under similar conditions on f , they proved that bounds on solutions of the above problem are equivalent to bounds on their Morse indices. Bahri and Lions needed this useful information when they apply topological methods to establish existence and multiplicity result for (3.1.5) (see for instance [3] and [4]). To get the a priori estimate, they used a classical blow up argument which leads to treat those systems :

$$\begin{cases} -\Delta u = |u|^{q-1}u \text{ in } \mathbb{R}^N \\ u \in \mathcal{C}_b^2(\mathbb{R}^N), \quad i(u) < \infty \end{cases}$$

and

$$\begin{cases} -\Delta u = |u|^{q-1}u \text{ in } \mathbb{R}_+^N, \quad u = 0 \text{ on } \partial\mathbb{R}_+^N \\ u \in \mathcal{C}_b^2(\mathbb{R}_+^N), \quad i(u) < \infty \end{cases}$$

where $i(u)$ denotes the Morse index of u , \mathcal{C}_b^2 denotes the set of bounded and \mathcal{C}^2 functions, $1 < q < \frac{N+2}{N-2}$ when $N \geq 3$ and $q \in (1, \infty)$ if $N = 2$. They proved that the problems below have only the trivial solution $u \equiv 0$. This result extended the nonexistence result of positive solutions in [11] to finite Morse index solution.

Later, Harrabi et al [13, 14] generalize the work of Bahri and Lions to other nonlinearities. Recently, Harrabi et al [12] studied the corresponding Neumann boundary

value problem and X.Yu studied the mixed boundary problems in [22]. Concerning the supercritical case, one can refer to the following works [7], [15] and [20].

In the present work, we will show that a similar result holds for problem (P). The proof relies on a blow up argument and a Liouville type theorem. This is in the spirit of similar ideas carried out for interior reaction by Bahri and Lions [5]. We mention that Liouville type theorems in unbounded domains play a crucial role to obtain a priori L^∞ -bounds for solutions of semilinear boundary value problems in bounded domain (see [11] for the case of positive solutions and [5, 7, 10, 13] for sign changing solutions and having finite Morse index). Also, nonlinear Liouville type theorems combined with the degree type arguments, are useful to obtain the existence of solutions of (3.1.5) (see for instance [6]).

We define the Morse index of a solution u of (P) as the dimension of the negative space of its associated quadratic form q defined on $H^1(\Omega)$ by

$$q(h) = \int_{\Omega} |\nabla h|^2 - \int_{\partial\Omega} \frac{\partial f}{\partial t}(x, u) h^2.$$

We denote it by $i(u)$. We begin with the simple

Proposition 3.1.1. *Assume that f satisfies : for all t , $x \in \partial\Omega$,*

$$\frac{\partial f}{\partial t}(x, t) > \frac{1}{t} f(x, t), \quad f(x, 0) = 0 \tag{3.1.6}$$

and let u be a solution of (P) say in $H^2(\Omega) \cap C(\bar{\Omega})$. Then we have

$$i(u) \geq n(u)$$

where $n(u)$ is the number of connected components of $\{x \in \Omega, \quad u(x) \neq 0\}$.

Proof. Let us denote by \mathcal{C}_j ($1 \leq j \leq n(u)$) the connected components of $\{x \in \Omega, \quad u(x) \neq 0\}$, \mathcal{C}_j is an open subset of Ω and $\partial\mathcal{C}_j \cap \partial\Omega$ is a nonempty subset in $\partial\Omega$. In fact, if $\bar{\mathcal{C}}_j \subset \Omega$ then from the maximum principle $u \equiv 0$ in \mathcal{C}_j since u is a harmonic function, which contradicts the definition of \mathcal{C}_j .

Observe that $u = 0$ on $\partial\mathcal{C}_j \setminus \partial\Omega$. Hence, since u is a harmonic function, we get

$$\int_{\mathcal{C}_j} |\nabla u|^2 = \int_{\partial\mathcal{C}_j \cap \partial\Omega} f(x, u) u dx < \int_{\partial\mathcal{C}_j \cap \partial\Omega} \frac{\partial f}{\partial t}(x, u) u^2 dx$$

by (3.1.6). Now, let u_j be defined as : $u_j = u$ in \mathcal{C}_j and $u_j = 0$ in $\Omega \setminus \mathcal{C}_j$. It is easy to see that for each $j \leq n(u)$, $u_j \in H^1(\Omega)$ and it satisfies $q(u_j) < 0$. Thus, since the supports of the u_j 's are disjoint, we derive that $i(u) \geq n(u)$. The result follows. \square

We turn now to our main result, we consider $u_n \in H^1(\Omega) \cap C(\bar{\Omega})$ a solution of (P) . Here we study the relationship between the solutions of (P) and their Morse indices.

Theorem 3.1.2. *Under the assumptions (3.1.3)-(3.1.4), let (u_n) be a sequence of solutions of (P) then, (u_n) is bounded in $L^\infty(\Omega)$ if and only if $i(u_n)$ is bounded.*

To prove the a priori estimate, we used a blow-up argument. Since u is a harmonic function, then by the maximum principle, the maximum of u is located on the boundary and therefore, when using the blow up analysis, the new domain will converge to a half space. Hence, we are left to study the following Liouville type problem

$$(PL) \quad \begin{cases} \Delta u = 0 & \text{in } \Pi \\ \frac{\partial u}{\partial \nu} = b|u|^{p-1}u & \text{on } \partial\Pi, \end{cases}$$

where Π denotes the half space $\{x_N > 0\}$ and b is the positive constant introduced in (3.1.4). For each solution u of (PL) we denote by $i(u)$ the dimension of the negative space of its associated quadratic form q defined on $H^1(\Pi)$ by

$$q(h) = \int_{\Pi} |\nabla h|^2 - p \int_{\partial\Pi} |u|^{p-1} h^2.$$

Our second result provides a Liouville type theorem.

Theorem 3.1.3. *Under assumption (3.1.3), let $u \in C^1(\bar{\Pi})$ be a bounded solution of (PL) which satisfies $i(u) < \infty$. Then, $u \equiv 0$ in $\bar{\Pi}$.*

This result will be useful to prove that the blowing up phenomenon does not appear under the boundedness of the Morse index. To get Theorem 3.1.3, we prove some integrable conditions on the solution, then we use the Pohozaev identity to prove the nonexistence result (as in [5]). Since the nonlinear term is on the boundary, it is much more complicated to prove the integral conditions than the usual nonlinear elliptic equation (3.1.5).

There are many works concerning the positive solutions of Laplace equation in a half space with nonlinear boundary condition, see, for example, [1], [16], [17], [18]. In [16], B. Hu proves that if $1 < p < N/(N - 2)$ there is no nontrivial positive classical solution of (PL) . Otherwise, Theorem 3.1.3 extended the nonexistence result of a nontrivial positive solution of (PL) in [16] to finite Morse index solution.

The paper is organized as follows. In Section 3.2, we study the bounded solutions of the Liouville type problem (PL) with finite Morse index and we prove Theorem 3.1.3. Section 3.3 is devoted to the proof of Theorem 3.1.2 using a blow up analysis. Finally, Section 3.4 deals with a kind of stability of the solution outside a compact set.

3.2 A Liouville Type Theorem

As previously described, the important step is to get Theorem 3.1.3 which corresponds to a Liouville Theorem for the boundary problem (PL) .

Note that if u is a solution of (PL) then $v = b^{1/(p-1)}u$ is a solution of (PL) with $b = 1$. Hence, for simplicity, we will consider $b = 1$ in the sequel.

We start by the following

Proposition 3.2.1. *Let u be a solution of the problem (PL) such that $\int_{\partial\Pi} |u|^{p+1} < \infty$ and p satisfies (3.1.3). Then $\int_{\partial\Pi} |u|^{p+1} = \int_{\Pi} |\nabla u|^2$.*

Proof. Let us recall some preliminaries (see for example [2] and [8]) useful to our framework. Denote points in Π by (x, t) , where $x \in \mathbb{R}^{N-1}$ and $t > 0$. Consider the Dirichlet problem

$$(P_\varphi) : \begin{cases} \phi \in C^2(\Pi) \cap C(\bar{\Pi}) \\ \Delta\phi = 0 \quad \text{in } \Pi \\ \phi(x, 0) = \varphi(x) \in C(\mathbb{R}^{N-1}) \cap L^\infty(\mathbb{R}^{N-1}). \end{cases}$$

A solution ϕ to (P_φ) is called a harmonic extension of φ in the upper half-space Π . The Poisson kernel for the half-space is defined for all $N \geq 2$ by

$$K(x, y) = c_N \frac{t}{(t^2 + |x - y|^2)^{N/2}}, \quad \text{where } c_N := 2/(N \text{mes}(B_1))$$

and $\text{mes}(B_1)$ is the measure of the unit ball B_1 . Using the fact

$$c_N \int_{\mathbb{R}^{N-1}} \frac{t}{(t^2 + |x - y|^2)^{N/2}} dy = 1 \quad \text{for every } (x, t) \in \Pi, \quad (3.2.1)$$

we prove that every $\varphi \in C(\mathbb{R}^{N-1}) \cap L^\infty(\mathbb{R}^{N-1})$ has a unique bounded harmonic extension ϕ in Π , given by

$$\phi(x, t) = c_N \int_{\mathbb{R}^{N-1}} \frac{t\varphi(y)}{(t^2 + |x - y|^2)^{N/2}} dy. \quad (3.2.2)$$

We define the cut-off functions

$$\varphi(x) = 1 \text{ for } |x| < 1, \quad \varphi(x) = 0 \text{ for } |x| \geq 2 \quad \text{and} \quad \varphi_R(x) = \varphi\left(\frac{x}{R}\right).$$

Easy computations imply that $\phi_R(x, t) = \phi\left(\frac{1}{R}(x, t)\right)$ where ϕ_R and ϕ denote respectively the bounded harmonic extension of φ_R and φ .

In the sequel, we denote the ball of center 0 and radius R in \mathbb{R}^N by $B_R := B_R(0)$ and the ball in $\partial\Pi$ by $D_R := B_R \cap \partial\Pi$. We claim that

$$\int_{\Pi} |\nabla u|^2 \phi_R - \int_{D_{2R}} |u|^{p+1} \varphi_R = -\frac{1}{2} \int_{\partial\Pi} u^2 \frac{\partial \phi_R}{\partial \nu}. \quad (3.2.3)$$

In fact, let $\ell > 4R$. Multiplying the first equation of (PL) by $u\phi_R$ and integrating by part over $B_\ell^+ := B_\ell \cap \Pi$, we find

$$\int_{B_\ell^+} |\nabla u|^2 \phi_R - \int_{D_{2R}} |u|^{p+1} \varphi_R = -\frac{1}{2} \int_{D_\ell} u^2 \frac{\partial \phi_R}{\partial \nu} - \frac{1}{2} \int_{S_\ell} u^2 \frac{\partial \phi_R}{\partial \nu} + \int_{S_\ell} u \frac{\partial u}{\partial \nu} \phi_R, \quad (3.2.4)$$

where $D_R := B_R \cap \partial\Pi$ and $S_R := \partial B_R \cap \Pi$. Next, we will show that the integrals over S_ℓ go to 0 as $\ell \rightarrow \infty$.

For each $(x, t) \in S_\ell$ and $(y, 0) \in D_{2R}$, we have

$$|(x, t) - (y, 0)| = \sqrt{t^2 + |x - y|^2} \geq \frac{1}{2}\ell. \quad (3.2.5)$$

In the sequel, We use C to denote various positive constants.

Using (3.2.2), (3.2.5) and the fact that $\text{supp } \varphi_R \subset D_{2R}$, we get

$$\left| \frac{\partial \phi_R}{\partial t}(x, t) \right| \leq C \int_{D_{2R}} \frac{1}{(t^2 + |x - y|^2)^{\frac{N}{2}}} dy \quad (3.2.6)$$

$$\leq C \frac{R^{N-1}}{\ell^N}, \quad \text{for each } (x, t) \in S_\ell. \quad (3.2.7)$$

As previously, we show that

$$\left| \frac{\partial \phi_R}{\partial x_i}(x, t) \right| \leq C \frac{R^{N-1}}{\ell^N}, \quad \text{for each } 1 \leq i \leq N - 1 \text{ and } (x, t) \in S_\ell. \quad (3.2.8)$$

(3.2.7) and (3.2.8) imply

$$|\nabla \phi_R(x, t)| \leq C \frac{R^{N-1}}{\ell^N}, \quad \text{for each } (x, t) \in S_\ell. \quad (3.2.9)$$

Using (3.2.9) and the fact that u is bounded, we get

$$\int_{S_\ell} u^2 \left| \frac{\partial \phi_R}{\partial \nu} \right| \leq C \frac{R^{N-1}}{\ell^N} \text{mes}(S_\ell) \leq C \frac{R^{N-1}}{\ell}. \quad (3.2.10)$$

On the other hand, (3.2.2) and (3.2.5) imply

$$|\phi_R(x, t)| \leq Ct \int_{D_{2R}} \frac{1}{(t^2 + |x - y|^2)^{\frac{N}{2}}} dy \leq C \frac{tR^{N-1}}{\ell^N}, \quad \text{for each } (x, t) \in S_\ell.$$

Being a harmonic function, u satisfies

$$|\nabla u(x, t)| \leq \frac{C}{t}, \text{ for each } (x, t) \in S_\ell.$$

Using the fact that u is bounded and the two last estimates, we derive

$$\left| \int_{S_\ell} u \frac{\partial u}{\partial \nu} \phi_R \right| \leq C \frac{R^{N-1}}{\ell}. \quad (3.2.11)$$

From (3.2.4), (3.2.10) and (3.2.11), tending ℓ to ∞ , we get the identity (3.2.3).

Let $I(R) = -\frac{1}{2} \int_{\partial \Pi} u^2 \frac{\partial \phi_R}{\partial \nu}$ which represents the right hand side of (3.2.3). In the following, we prove that $I(R)$ tends to 0 when R tends to ∞ .

$$I(R) = -\frac{1}{2} \int_{D_{3R}} u^2 \frac{\partial \phi_R}{\partial \nu} - \frac{1}{2} \int_{D_{3R}^c} u^2 \frac{\partial \phi_R}{\partial \nu} := I_1(R) + I_2(R).$$

Using $|\nabla \phi_R| \leq C/R$, Hölder's inequality and $\int_{\partial \Pi} |u|^{p+1} < \infty$, we obtain

$$\begin{aligned} |I_1(R)| &\leq \frac{C}{R} \left(\int_{D_{3R}} u^{p+1} \right)^{2/(p+1)} \cdot \left(\int_{D_{3R}} 1 \right)^{(p-1)/(p+1)} \\ &\leq CR^{(N-1)\frac{p-1}{p+1}-1} \longrightarrow 0 \text{ when } R \text{ tends to } \infty, \end{aligned}$$

since p satisfies (3.1.3). On the other hand, from (3.2.2), we get

$$\begin{aligned} |I_2(R)| &\leq C \int_{D_{3R}^c} u^2(x) \int_{D_{2R}} \frac{1}{|x-y|^N} dy dx \\ &\leq C \int_{D_R^c} \int_{D_{2R}} \frac{u^2(z+y)}{|z|^N} dy dz \quad (\text{taking } z = x-y) \\ &\leq C \int_{D_R^c} \frac{1}{|z|^N} \left(\int_{D_{2R}} u^2(z+y) dy \right) dz \\ &\leq CR^{(N-1)\frac{p-1}{p+1}} \int_{D_R^c} \left(\int_{D_{2R}} |u|^{p+1}(z+y) dy \right)^{2/(p+1)} \frac{1}{|z|^N} dz \\ &\leq CR^{(N-1)\frac{p-1}{p+1}-1} \longrightarrow 0 \text{ when } R \text{ tends to } \infty. \end{aligned}$$

So $I(R)$ goes to 0 as $R \rightarrow \infty$. Hence

$$\int_{\Pi} |\nabla u|^2 \phi_R - \int_{D_{2R}} |u|^{p+1} \varphi_R \rightarrow 0 \quad \text{as } R \rightarrow \infty.$$

Since $0 \leq \varphi \leq 1$ and $\varphi \equiv 1$ in D_R , we get that $\int_{D_{2R}} |u|^{p+1} \varphi_R \rightarrow \int_{\partial \Pi} |u|^{p+1}$. Moreover, it is easy to prove that ϕ_R converges to 1 a.e. in Π . Hence, using the monotone convergence Theorem, we derive $\int_{\Pi} |\nabla u|^2 \phi_R \rightarrow \int_{\Pi} |\nabla u|^2$ as $R \rightarrow \infty$ and therefore the desired result follows. \square

To use Proposition 3.2.1, we need the next result

Proposition 3.2.2. *Let u be a bounded solution of (PL) which satisfies $i(u) < \infty$. Then, it holds $\int_{\partial\Pi} |u|^{p+1} < \infty$.*

Proof. We will adopt the method developed in [5], based on careful estimates of the energy $\int |u|^{p+1}$ on balls or spherical annuli on the boundary, mixing the spectral information contained in the finiteness of $i(u)$ and an identity satisfied by solutions of (PL). So we need to choose suitable test functions to bring out only the integral over $\partial\Pi$.

Let now $\varphi_{r,R}$ be a cut-off function such that

$$\varphi_{r,R}(x) = \begin{cases} 1 & \text{for } 2r \leq |x| \leq R, \\ 0 & \text{for } |x| < r \text{ or } |x| > 2R \end{cases} \quad (3.2.12)$$

and $|\nabla^\alpha \varphi_{r,R}(x)| \leq C/R^\alpha$ for each $|x| \geq 2r$. It is easy to check that it is possible to construct these test functions under the above assumptions.

Denote by $\phi_{r,R}$ the bounded harmonic extension of $\varphi_{r,R}$ i.e. $\phi_{r,R}$ is the bounded solution of $(P_{\varphi_{r,R}})$. We need the following result.

Lemma 3.2.3. *Let u be a solution of (PL) with $i(u) < \infty$. Then, there exists r_0 such that $\forall R > 2r_0$, we have $q(u\phi_{r_0,R}) \geq 0$.*

The proof of this lemma is postponed until Section 4.

Remark 3.2.4. *Let $\tilde{\varphi}_{r,R}$ be a cut-off function with support included in $\{x \in \mathbb{R}^{N-1} : r \leq |x| \leq 2R\}$ and $\tilde{\phi}_{r,R}$ its bounded harmonic extension. We note that Lemma 3.2.3 holds also for $\tilde{\phi}_{r,R}$ instead of $\phi_{r,R}$.*

Return back to the proof of Theorem 3.1.3. Taking $R > 8r_0$, using Lemma 3.2.3 and integrating by part we obtain

$$q(u\phi_{r_0,R}) = \int_{\Pi} |\nabla u|^2 \phi_{r_0,R}^2 - p \int_{\partial\Pi} |u|^{p+1} \phi_{r_0,R}^2 + \frac{1}{2} \int_{\partial\Pi} u^2 \frac{\partial(\phi_{r_0,R})^2}{\partial\nu} \geq 0. \quad (3.2.13)$$

On the other hand, multiplying the equation (PL) by $u\psi_{r_0,R}$ where $\psi_{r_0,R}$ is the bounded harmonic extension of $\varphi_{r_0,R}^2$ i.e. the bounded solution of $(P_{\varphi_{r_0,R}^2})$ and integrating by part we get

$$\int_{\Pi} |\nabla u|^2 \psi_{r_0,R} - \int_{\partial\Pi} |u|^{p+1} \varphi_{r_0,R}^2 + \frac{1}{2} \int_{\partial\Pi} u^2 \frac{\partial\psi_{r_0,R}}{\partial\nu} = 0. \quad (3.2.14)$$

To obtain the two above identities, we proceed as in the proof of (3.2.3).

From (3.2.13) and (3.2.14), we get

$$\int_{\Pi} |\nabla u|^2 (\psi_{r_0,R} - \phi_{r_0,R}^2) + (p-1) \int_{\partial\Pi} |u|^{p+1} \varphi_{r_0,R}^2 + \frac{1}{2} \int_{\partial\Pi} u^2 \left(\frac{\partial \psi_{r_0,R}}{\partial \nu} - \frac{\partial (\phi_{r_0,R}^2)}{\partial \nu} \right) \leq 0. \quad (3.2.15)$$

We start by proving that $\psi_{r_0,R} - \phi_{r_0,R}^2 \geq 0$ on Π . We recall that we can write $\phi_{r_0,R}(x, t)$ by using (3.2.2). From Hölder's inequality and (3.2.1), we get $\phi_{r_0,R} \leq \sqrt{\psi_{r_0,R}}$ and therefore the desired inequality holds.

Hence we obtain

$$\int_{\partial\Pi} |u|^{p+1} \varphi_{r_0,R}^2 \leq C \left\{ \int_{\partial\Pi} u^2 \left| \frac{\partial \phi_{r_0,R}^2}{\partial \nu} \right| + \int_{\partial\Pi} u^2 \left| \frac{\partial \psi_{r_0,R}}{\partial \nu} \right| \right\} \quad (3.2.16)$$

and we are left to estimate $|\frac{\partial \phi}{\partial \nu}(x, t)|$ and $|\frac{\partial \psi}{\partial \nu}(x, t)|$ on $\partial\Pi$.

Lemma 3.2.5. *For each $|x| \geq 3r_0$, we have*

$$\left| \frac{\partial \phi_{r_0,R}}{\partial \nu}(x, 0) \right|, \quad \left| \frac{\partial \psi_{r_0,R}}{\partial \nu}(x, 0) \right| \leq c \left(\frac{1}{|x|^N} + \frac{1}{R} \right).$$

In particular, for each $|x| \geq R/2$, we have

$$\left| \frac{\partial \phi_{r_0,R}}{\partial \nu}(x, 0) \right|, \quad \left| \frac{\partial \psi_{r_0,R}}{\partial \nu}(x, 0) \right| \leq \frac{c}{R}.$$

Proof. We will concentrate on the estimate of $\partial \phi_{r_0,R} / \partial \nu$ and the same arguments hold for the case of $\partial \psi_{r_0,R} / \partial \nu$.

We recall that, from (3.2.1) and (3.2.2), we have

$$\frac{\phi_{r_0,R}(x, t) - \phi_{r_0,R}(x, 0)}{t} = c_N \int_{\mathbb{R}^{N-1}} \frac{\varphi_{r_0,R}(y) - \varphi_{r_0,R}(x)}{(t^2 + |x - y|^2)^{N/2}} dy. \quad (3.2.17)$$

We distinguish two cases.

First case : $3r_0 \leq |x| \leq R/2$. In this case, we have $\varphi_{r_0,R}(x) = 1$ and using (3.2.17), we deduce

$$\begin{aligned} \left| \frac{\partial \phi}{\partial \nu}(x, 0) \right| &\leq C \int_{\mathbb{R}^{N-1}} \frac{|\varphi_{r_0,R}(y) - 1|}{|x - y|^N} dy \\ &\leq C \int_{D_{r_0, 2r_0} \cup D_R^c} \frac{1}{|x - y|^N} dy \quad (\text{where } D_{r,R} := D_R \setminus D_r) \\ &\leq C \int_{D_{2r_0}} \frac{1}{|x - y|^N} dy + \int_{D_R^c} \frac{1}{|x - y|^N} dy \\ &\leq C \left\{ \frac{1}{|x|^N} + \int_{D_R^c} \frac{1}{|y|^N} dy \right\} \\ &\leq C \left\{ \frac{1}{|x|^N} + \frac{1}{R} \right\}. \end{aligned}$$

Indeed, To obtain the fourth inequality, we have used $|x - y| \geq |x|/3$ for the first integral and $|x - y| \geq |y|/2$ for the second one.

Second case : $|x| \geq R/2$. Using again (3.2.17), it follows

$$\begin{aligned} \frac{\phi_{r_0,R}(x,t) - \phi_{r_0,R}(x,0)}{t} &= c_N \int_{D_{R/4}(x)} \frac{\varphi_{r_0,R}(y) - \varphi_{r_0,R}(x)}{(t^2 + |x - y|^2)^{N/2}} dy \\ &\quad + c_N \int_{D_{R/4}^c(x)} \frac{\varphi_{r_0,R}(y) - \varphi_{r_0,R}(x)}{(t^2 + |x - y|^2)^{N/2}} dy \\ &:= I_1 + I_2. \end{aligned} \quad (3.2.18)$$

It is easy to check that

$$|I_2| \leq \int_{D_{R/4}^c(0)} \frac{1}{|z|^N} dz \leq \frac{C}{R}. \quad (3.2.19)$$

To estimate the integral I_1 , we expand $\varphi_{r_0,R}$ around x . Recall that $\varphi_{r_0,R}$ satisfies $|\nabla^\alpha \varphi_{r_0,R}| \leq C/R^\alpha$ in $D_{2r_0}^c$. Note that, taking $R > 8r_0$, for each $y \in D_{R/4}(x)$ we have $|y| > 2r_0$ and therefore

$$\varphi_{r_0,R}(y) = \varphi_{r_0,R}(x) + D\varphi_{r_0,R}(x)(y - x) + O\left(\frac{|y - x|^2}{R^2}\right), \quad \text{for each } y \in D_{R/4}(x).$$

Then

$$\begin{aligned} I_1 &= c_N D\varphi_{r_0,R}(x) \int_{D_{R/4}(x)} \frac{(y - x)}{(t^2 + |x - y|^2)^{N/2}} dy \\ &\quad + O\left(\frac{1}{R^2} \int_{D_{R/4}(x)} \frac{|y - x|^2}{(t^2 + |x - y|^2)^{N/2}} dy\right) \\ &= O\left(\frac{1}{R^2} \int_{D_{R/4}(x)} \frac{1}{|x - y|^{N-2}} dy\right) = O\left(\frac{1}{R}\right). \end{aligned} \quad (3.2.20)$$

Using (3.2.18)-(3.2.20) and tending t to 0, we get the desired estimate for the second case. So the lemma is proved. \square

Now, we return to the proof of Proposition 3.2.2. Using (3.2.16), Lemma 3.2.5 and Hölder's inequality, we obtain

$$\begin{aligned} &\int_{D_{2r_0,R}(0)} |u|^{p+1} \\ &\leq C \left\{ \int_{D_{r_0,2R}(0)} u^2 \varphi_{r_0,R} \left| \frac{\partial \phi_{r_0,R}}{\partial \nu} \right| + \int_{D_{3R}(0)} u^2 \left| \frac{\partial \psi_{r_0,R}}{\partial \nu} \right| + \int_{D_{3R}^c(0)} u^2 \left| \frac{\partial \psi_{r_0,R}}{\partial \nu} \right| \right\} \\ &\leq C_0 + C \left\{ \int_{D_{3r_0,3R}(0)} u^2 \left(\frac{1}{|x|^N} + \frac{1}{R} \right) + \int_{D_{3R}^c(0)} u^2 \left| \frac{\partial \psi_{r_0,R}}{\partial \nu} \right| \right\} \\ &\leq C_0 + C \left\{ \frac{1}{R} \left(\int_{D_{3R}(0)} u^{p+1} \right)^{\frac{2}{p+1}} \cdot \left(\int_{D_{3R}(0)} 1 \right)^{\frac{p-1}{p+1}} + \int_{D_{3R}^c(0)} u^2 \left| \frac{\partial \psi_{r_0,R}}{\partial \nu} \right| \right\} \\ &\leq C_0 + C \left\{ \left(\int_{D_{3R}(0)} u^{p+1} \right)^{\frac{2}{p+1}} \cdot R^{(N-1)\frac{p-1}{p+1}-1} + \int_{D_{3R}^c(0)} u^2 \left| \frac{\partial \psi_{r_0,R}}{\partial \nu} \right| \right\} \end{aligned} \quad (3.2.21)$$

where $D_{r,R}(0) := \{x \in \mathbb{R}^{N-1} : r < |x| < R\}$.

In fact, in the third inequality, we have used the following estimate :

$$\int_{D_{3r_0}^c(0)} u^2 \frac{1}{|x|^N} dx \leq \int_{D_{3r_0}^c(0)} \frac{dx}{|x|^N} \leq c_0.$$

Concerning the last integral, we have

$$\begin{aligned} & \int_{D_{3R}^c(0)} u^2 \left| \frac{\partial \psi_{r_0,R}}{\partial \nu} \right| \\ & \leq C \int_{D_{3R}^c(0)} u^2(x) \int_{D_{2R}(0)} \frac{1}{|x-y|^N} dy dx \\ & \leq C \int_{D_R^c(0)} \int_{D_{2R}(0)} \frac{u^2(z+y)}{|z|^N} dy dz \quad (\text{taking } z = x - y) \\ & \leq C \int_{D_R^c(0)} \frac{1}{|z|^N} \left(\int_{D_{2R}(0)} u^2(z+y) dy \right) dz \\ & \leq CR^{(N-1)\frac{p-1}{p+1}} \int_{D_R^c(0)} \frac{1}{|z|^N} \left(\int_{D_{2R}(z)} |u|^{p+1}(y) dy \right)^{2/(p+1)} dz. \end{aligned} \quad (3.2.22)$$

From (3.2.21) and (3.2.22), we get

$$\begin{aligned} \int_{D_{2r_0,R}(0)} |u|^{p+1} & \leq C_0 + CR^{(N-1)\frac{p-1}{p+1}} \left\{ \frac{1}{R} \left(\int_{D_{3R}(0)} |u|^{p+1} \right)^{2/(p+1)} \right. \\ & \quad \left. + \int_{D_R^c(0)} \frac{1}{|z|^N} \left(\int_{D_{2R}(z)} |u|^{p+1}(y) dy \right)^{2/(p+1)} dz \right\}. \end{aligned} \quad (3.2.23)$$

We shall then conclude by a bootstrap argument.

Indeed, in the first step, since $|u| \leq 1$ on $\partial\Pi$ we get

$$\int_{D_R(z)} |u|^{p+1} \leq CR^{N-1}, \quad (3.2.24)$$

where C is some positive constant independent of z and R . Hence, from (3.2.23) and (3.2.24) we deduce the second estimate :

$$\int_{D_R(0)} |u|^{p+1} \leq C_0 + CR^{N-2}, \forall R > 8r_0. \quad (3.2.25)$$

Note that, for $N = 2$, this bound completes the proof.

Furthermore, for $N \geq 3$, it is easy to see that $C_0 + CR^{N-2}$ is less than CR^{N-2} for R large enough. But, we will conserve it as $C_0 + CR^{N-2}$, to be able to deduce a general estimate at the k^{th} step.

We remark that (3.2.25) holds for each bounded solution v of (PL) that satisfies $q(v\phi_{r_0,R}) \geq 0$.

This bound is not sufficient to conclude. So we need the following estimate

Lemma 3.2.6. *There exists a positive constant C such that for all $z \in \partial\Pi$, we have*

$$\int_{D_R(z)} |u|^{p+1} \leq C_0 + CR^{N-2}, \forall R > 8r_0.$$

Proof. Let z be in $\partial\Pi$. If $|z| \leq 3R$, then $D_R(z) \subset D_{4R}(0)$ and therefore our lemma follows from (3.2.25). Now, we will focus on the case $|z| > 3R$. Let us define the function v by

$$v(x) := u(x + z).$$

It is easy to see that v is a bounded solution of (PL) and we have $\int_{D_R(z)} |u|^{p+1} = \int_{D_R(0)} |v|^{p+1}$. Hence, it remains to verify that v satisfies $q(v\phi_{r_0,R}) \geq 0$ to obtain (3.2.25) with v , which implies the lemma. In this step, we need to use Remark 3.2.4.

Let R' be a large constant ($R' \geq |z| + 2R$) and let us denote $\tilde{\varphi}_{r,R'}$ a cut-off function such that

$$\tilde{\varphi}_{r,R'}(x) = \begin{cases} 1 & \text{for } 2r \leq |x - z| \leq R, \\ 0 & \text{for } |x - z| < r \text{ or } |x - z| > 2R, \end{cases} \quad \text{for } x \in \mathbb{R}^{N-1} \quad (3.2.26)$$

and $\tilde{\phi}_{r,R'}$ its bounded harmonic extension on Π . We remark that the support of $\tilde{\varphi}_{r,R'}$ is included in $\{x \in \mathbb{R}^{N-1} : r \leq |x| \leq 2R'\}$. Furthermore, we have $\tilde{\varphi}_{r,R'}(x) = \varphi_{r,R}(x - z)$. Using Remark 3.2.4, we get $q(u\tilde{\phi}_{r_0,R'}) \geq 0$.

On the other hand we have

$$\begin{aligned} q(u\tilde{\phi}_{r_0,R'}) &= \int_{\Pi} |\nabla(u\tilde{\phi}_{r_0,R'})|^2 - p \int_{\partial\Pi} |u|^{p+1} \tilde{\varphi}_{r_0,R'}^2 \\ &= \int_{\Pi} |\nabla(u(x)\tilde{\phi}_{r_0,R'}(x))|^2 dx - p \int_{\partial\Pi} |u(x)|^{p+1} \varphi_{r_0,R}^2(x - z) dx \\ &= \int_{\Pi} |\nabla(u(y+z)\tilde{\phi}_{r_0,R'}(y+z))|^2 dy - p \int_{\partial\Pi} |u(y+z)|^{p+1} \varphi_{r_0,R}^2(y) dy \\ &= \int_{\Pi} |\nabla(v(y)\phi_{r_0,R}(y))|^2 dy - p \int_{\partial\Pi} |v(y)|^{p+1} \varphi_{r_0,R}^2(y) dy. \end{aligned}$$

So we obtain $q(v\phi_{r_0,R}) = q(u\tilde{\phi}_{r_0,R'}) \geq 0$. Therefore, we get (3.2.25) with v . That is $\int_{D_R(0)} |v|^{p+1} \leq C_0 + CR^{N-2}$. The proof of the lemma follows. \square

Now, using Lemma 3.2.6, we can improve the estimate (3.2.25) by inserting the last bound in (3.2.23). The third step gives us the following estimate

$$\int_{D_{2r_0,R}(0)} |u|^{p+1} \leq C_0 + CR^{N-2-2/(p+1)}, \forall R > 8r_0.$$

As in Lemma 3.2.6, we prove that this estimate still holds for each $D_R(z)$, with $z \in \partial\Pi$. By induction, in the k^{th} step, $k \geq 2$, we obtain

$$\int_{D_{2r_0,R}(0)} |u|^{p+1} \leq C_0 + CR^{N-1-\sum_{i=0}^{k-2} (2/(p+1))^i}, \forall R > 8r_0. \quad (3.2.27)$$

Observe that, we have $\sum_{i=0}^{\infty} (2/(p+1))^i = (p+1)/(p-1) > N-1$ from (3.1.3). Hence, there exists k_0 such that $N-1 - \sum_{i=0}^{k_0-2} (2/(p+1))^i < 0$. Thus, tending R to infinity, we deduce from (3.2.27), by taking $k = k_0$, that $\int_{D_{2r_0}^c(0)} |u|^{p+1} \leq C_0$, which implies the result. \square

Proof of Theorem 3.1.3

We start by using Propositions 3.2.1 and 3.2.2. We find

$$\int_{\Pi} |\nabla u|^2 = \int_{\partial\Pi} |u|^{p+1} < \infty. \quad (3.2.28)$$

So

$$\lim_{R \rightarrow +\infty} \int_{B_{2R}^+ \setminus B_R^+} |\nabla u|^2 + \int_{D_{2R} \setminus D_R} |u|^{p+1} = 0. \quad (3.2.29)$$

From the mean formula, there exists $\bar{R} \in (R, 2R)$ such that

$$\int_{B_{2R}^+ \setminus B_R^+} |\nabla u|^2 + \int_{D_{2R} \setminus D_R} |u|^{p+1} = R \left\{ \int_{S_{\bar{R}}} |\nabla u|^2 + \int_{\partial D_{\bar{R}}} |u|^{p+1} \right\}. \quad (3.2.30)$$

The next step is to use the well-known Pohozaev identity (see [19]). Multiplying the equation (PL) by $\sum_i x_i \partial u / \partial x_i$ and integrating on B_R^+ , standard computations imply the following identity

Lemma 3.2.7. *Let u be a solution of the problem (PL), then, for all positive R , we have*

$$\frac{N-2}{2} \int_{B_R^+} |\nabla u|^2 = \frac{R}{2} \int_{S_R} |\nabla u|^2 - R \int_{S_R} \left(\frac{\partial u}{\partial \nu} \right)^2 + \frac{N-1}{p+1} \int_{D_R} |u|^{p+1} - \frac{R}{p+1} \int_{\partial D_R} |u|^{p+1}.$$

Using Lemma 3.2.7 with $R = \bar{R}$, we get

$$\left| \frac{N-2}{2} \int_{B_{\bar{R}}^+} |\nabla u|^2 - \frac{N-1}{p+1} \int_{D_{\bar{R}}} |u|^{p+1} \right| \leq C \bar{R} \left\{ \int_{S_{\bar{R}}} |\nabla u|^2 + \int_{\partial D_{\bar{R}}} |u|^{p+1} \right\}. \quad (3.2.31)$$

So from (3.2.29), (3.2.30) and (3.2.31) we obtain

$$\frac{N-2}{2} \int_{\Pi} |\nabla u|^2 = \frac{N-1}{p+1} \int_{\partial\Pi} |u|^{p+1}. \quad (3.2.32)$$

We recall that p satisfies (3.1.3), then (3.2.28) and (3.2.32) imply $\int_{\Pi} |\nabla u|^2 = \int_{\partial\Pi} |u|^{p+1} = 0$. This completes the proof of Theorem 3.1.3. \square

3.3 Blow-up analysis and proof of Theorem 3.1.2

The "only if" part follows immediately from standard considerations. Indeed, if u_n is bounded in $L^\infty(\Omega)$, then $\frac{\partial f}{\partial t}(x, u_n)$ is also bounded in $L^\infty(\partial\Omega)$ say by $C_0 \in (0, +\infty)$ and thus $i(u_n)$ is bounded by k where k is the smallest integer such that the k^{th} eigenvalue of $\frac{\partial}{\partial \nu}$ (defined on $H := \{h \in H^1(\Omega) : \Delta h = 0\}$) is bigger than C_0 .

The "if" part is rather delicate and will be a consequence of the Liouville type theorem after a blow up analysis which will be done. We argue by contradiction and thus consider a sequence $(u_n)_n \subset H^1(\Omega) \cap C(\bar{\Omega})$ of solutions of (P) such that $\|u_n\|_{L^\infty(\Omega)} \rightarrow +\infty$ and $i(u_n)$ remains bounded. We next use a classical blow-up argument.

Let $M_n = \max_{\bar{\Omega}} |u_n|$ and let x_n be a maximum point of $|u_n|$. From the maximum principle, $x_n \in \partial\Omega$ since u_n is a harmonic function. Up to a subsequence, x_n converges to some $\bar{x} \in \partial\Omega$. Without loss of generality, we may assume that $\bar{x} = 0 \in \partial\Omega$ and the unit outward normal to $\partial\Omega$ at 0 is $(-e_N)$ where e_N is the last element of a canonical basis in \mathbb{R}^N . It will be convenient to work in fixed half balls. For this reason, we need some change of coordinates. This program was done in many works (see for example [9] and [21]).

Since $\partial\Omega$ is a C^2 surface, we know that there is an $R > 0$ and a $C^2(\mathbb{R}^{N-1})$ function ρ such that (after a possible renumbering and reorientation of coordinates)

$$\partial\Omega \cap B_R(0) = \{x \in B_R(0) : x_N = \rho(x_1, x_2, \dots, x_{N-1})\}$$

$$\Omega \cap B_R(0) = \{x \in B_R(0) : x_N > \rho(x_1, x_2, \dots, x_{N-1})\}$$

and moreover, the mapping

$$B_R(0) \ni x \mapsto y = \Psi(x) \in \mathbb{R}^N$$

defined by

$$\begin{cases} y_i & := x_i & i = 1, \dots, N-1, \\ y_N & := x_N - \rho(x_1, x_2, \dots, x_{N-1}), \end{cases}$$

is one-to-one. Define $\Phi := \Psi^{-1}$. Note that Ψ is a C^2 function that transforms the set $\Omega' := \Omega \cap B_R(0)$ (in what we refer to as x space) into a set Ω'' in the half-space $y_N > 0$ (of y space). Note also that the point $\bar{x} = 0$ is mapped to the origin of y space.

Our task now is changing the partial differential equation (P) satisfied by u_n in Ω' into y coordinates. We define

$$v_n(y) := u_n(\Phi(y)), \quad \text{for all } y \in \Omega''.$$

Let $\varphi \in \mathcal{D}(B_R(0) \cap \bar{\Omega})$. Multiplying (P) by φ , integrating by part over $B_R(0) \cap \Omega$ and using the change of variable $x = \Phi(y)$, we find

$$\int_{B_R^+(0)} \nabla u_n(\Phi(y)) \cdot \nabla \varphi(\Phi(y)) dy = \int_{D_R(0)} f(\Phi(y'), 0, u_n(\Phi(y'), 0)) \varphi(\Phi(y'), 0) dy', \quad (3.3.1)$$

where $y' = (y_1, \dots, y_{N-1})$.

Let $\varphi_1(y) := \varphi(\Phi(y))$, for each $y \in B_R^+(0)$. A simple computation shows that

$$\nabla u_n(\Phi(y)) = \nabla v_n(y) - \left(\frac{\partial \rho}{\partial y_1}(y') \frac{\partial v_n}{\partial y_N}(y), \dots, \frac{\partial \rho}{\partial y_{N-1}}(y') \frac{\partial v_n}{\partial y_N}(y), 0 \right). \quad (3.3.2)$$

The above relation holds also for φ and φ_1 .

Using (3.3.1), (3.3.2) and Green's formula, we can prove that the functions v_n satisfy the following problem

$$\begin{cases} \Delta v_n - 2 \sum_{i=1}^{N-1} \frac{\partial \rho}{\partial y_i}(y') \frac{\partial^2 v_n}{\partial y_i \partial y_N} - \sum_{i=1}^{N-1} \frac{\partial^2 \rho}{\partial y_i^2}(y') \frac{\partial v_n}{\partial y_N} + |\nabla \rho(y')|^2 \frac{\partial^2 v_n}{\partial y_N^2} = 0 & \text{in } B_R^+, \\ \frac{\partial v_n}{\partial \nu} + \sum_{i=1}^{N-1} \frac{\partial \rho}{\partial y_i}(y') \frac{\partial v_n}{\partial y_N} - |\nabla \rho(y')|^2 \frac{\partial v_n}{\partial y_N} = f(\phi(y), v_n) & \text{on } D_R. \end{cases} \quad (3.3.3)$$

Now we perform a classical blow up argument. We then consider

$$\tilde{v}_n(y) = M_n^{-1} v_n(M_n^{-\alpha} y), \quad \forall y \in \bar{B}_n := M_n^\alpha \bar{B}_R^+ = \overline{B_{M_n^\alpha R}^+} \quad \text{with } \alpha = p - 1,$$

which satisfies in $\overline{B_{M_n^\alpha R}^+}$ the following equation

$$\begin{cases} \Delta \tilde{v}_n - 2 \sum_{i=1}^{N-1} \frac{\partial \rho}{\partial y_i}(M_n^{-\alpha} y') \frac{\partial^2 \tilde{v}_n}{\partial y_i \partial y_N} \\ \quad - M_n^{-\alpha} \sum_{i=1}^{N-1} \frac{\partial^2 \rho}{\partial y_i^2}(M_n^{-\alpha} y') \frac{\partial \tilde{v}_n}{\partial y_N} + |\nabla \rho(M_n^{-\alpha} y')|^2 \frac{\partial^2 \tilde{v}_n}{\partial y_N^2} = 0 & \text{in } B_{M_n^\alpha R}^+, \\ \frac{\partial \tilde{v}_n}{\partial \nu} + \sum_{i=1}^{N-1} \frac{\partial \rho}{\partial y_i}(M_n^{-\alpha} y') \frac{\partial \tilde{v}_n}{\partial y_N} \\ \quad - |\nabla \rho(M_n^{-\alpha} y')|^2 \frac{\partial \tilde{v}_n}{\partial y_N} = M_n^{-p} f(\phi(M_n^{-\alpha} y), M_n \tilde{v}_n) & \text{on } D_{M_n^\alpha R}. \end{cases} \quad (3.3.4)$$

Notice also that $|\tilde{v}_n| \leq 1$ in \bar{B}_n and $\lim_{n \rightarrow \infty} |\tilde{v}_n(0)| = 1$.

By the regularity theory of elliptic equations, we derive that $\tilde{v}_n \rightarrow \tilde{v}$ in $C_{loc}^1(\bar{\Pi})$. Since $\nabla \rho(0) = 0$, $M_n^{-1} \rightarrow 0$ and f satisfies (3.1.4), we conclude that \tilde{v} satisfies the problem (PL) introduced in Section 1. We verify that the limit solution \tilde{v} satisfies also

$|\tilde{v}(0)| = 1$, $|\tilde{v}| \leq 1$ on $\bar{\Pi}$ with $i(\tilde{v}) < \infty$.

From Theorem 3.1.3, we get $\tilde{v} \equiv 0$ in $\bar{\Pi}$, a contradiction with $|\tilde{v}(0)| = 1$. This contradiction implies that $\|u_n\|_\infty$ has to be bounded if $i(u_n)$ is bounded. This completes the proof of Theorem 3.1.2.

3.4 Proof of Lemma 3.2.3

Arguing by contradiction, we assume that for all r there exists $R > 2r$ such that $q(u\phi_{r,R}) < 0$ and therefore we are able to construct $m := i(u) + 1$ functions ϕ_{r_i, R_i} such that their supports on $\partial\Pi$ are disjoint and $q(u\phi_{r_i, R_i}) < 0$, for each $i \leq m$.

For simplicity, we shall write ϕ_i and φ_i for ϕ_{r_i, R_i} and φ_{r_i, R_i} .

Furthermore, we can also choose these functions such that $b(u\phi_i, u\phi_j)$ is very small with respect to $\max |q(u\phi_k)|$, that is

$$b(u\phi_i, u\phi_j) = o\left(\max_{1 \leq k \leq m} |q(u\phi_k)|\right) \text{ for all } i \neq j \quad (3.4.1)$$

where $b(., .)$ is the bilinear form associated to q defined by

$$b(h_1, h_2) = \int_{\Pi} \nabla h_1 \cdot \nabla h_2 - p \int_{\partial\Pi} |u|^{p-1} h_1 h_2 \quad \text{for each } h_1, h_2 \in H^1(\Pi).$$

This claim will be useful to prove that the dimension of the negative space of the quadratic form is greater than $i(u)$ which contradicts the definition of the Morse index of the solution. In fact, for all $(\alpha_1, \dots, \alpha_m) \in \mathbb{R}^m \setminus (0, \dots, 0)$ we have

$$\begin{aligned} q\left(\sum_{i=1}^m \alpha_i u\phi_i\right) &= \sum_{i=1}^m \alpha_i^2 q(u\phi_i) + \sum_{i \neq j} \alpha_i \alpha_j b(u\phi_i, u\phi_j) \\ &= \sum_{i=1}^m \alpha_i^2 q(u\phi_i) + \sum_{i \neq j} o\left(|\alpha_i \alpha_j| \max_{1 \leq k \leq m} |q(u\phi_k)|\right) \quad (\text{by (3.4.1)}) \\ &= \sum_{i=1}^m \alpha_i^2 (q(u\phi_i) + o(1)) < 0 \quad (\text{since } q(u\phi_i) < 0). \end{aligned}$$

Let us now construct m functions satisfying (3.4.1). Let $r_1 > 0$. We recall that there exists R_1 such that $q(u\phi_1) < 0$. We need the next result

Lemma 3.4.1. *There exists $\beta_1 \geq 4R_1 + r_1$ such that for all φ_2 with support in $D_{\beta_1}^c(0)$, we have*

$$b(u\phi_1, u\phi_2) = o(|q(u\phi_1)|) \quad \text{where } \phi_2 \text{ is the bounded harmonic extension of } \varphi_2.$$

Proof. We recall that

$$b(u\phi_1, u\phi_2) = \int_{\Pi} \nabla(u\phi_1) \cdot \nabla(u\phi_2) - p \int_{\partial\Pi} |u|^{p+1} \varphi_1 \varphi_2.$$

Following the Green's formula, the functions being harmonic and their supports on $\partial\Pi$ being disjoint imply

$$\begin{aligned} b(u\phi_1, u\phi_2) &= \int_{\Pi} |\nabla u(x, t)|^2 \phi_1(x, t) \phi_2(x, t) d(x, t) + \frac{1}{2} \int_{\partial\Pi} u^2(x) \frac{\partial(\phi_1 \phi_2)}{\partial\nu}(x) dx \\ &:= I + J. \end{aligned}$$

In the following, we will estimate $b(u\phi_1, u\phi_2)$ depending on the parameter β_1 . It is more delicate to estimate I . Let us denote by Γ_i the support of φ_i which is included in $\{x \in \mathbb{R}^{N-1} : r_i \leq |x| \leq 2R_i\}$.

From the integral representation (3.2.2) of ϕ_i and since the φ_i 's have compact supports, one can exchange the order of the integration to derive

$$\begin{aligned} I &\leq C \int_{\Pi} \int_{\mathbb{R}^{N-1}} \int_{\mathbb{R}^{N-1}} \frac{|\nabla u(x, t)|^2 t^2 \varphi_1(y) \varphi_2(z)}{(t^2 + |x - y|^2)^{N/2} (t^2 + |x - z|^2)^{N/2}} dz dy d(x, t) \\ &\leq C \int_{\Pi} \int_{\Gamma_1} \int_{\Gamma_2} \frac{|\nabla u(x, t)|^2 t^2}{(t^2 + |x - y|^2)^{N/2} (t^2 + |x - z|^2)^{N/2}} dz dy d(x, t) \\ &\leq C \int_{\Gamma_1} \int_{\Pi} \int_{\Gamma_2} \frac{|\nabla u(x, t)|^2 t^2}{(t^2 + |x - y|^2)^{N/2} (t^2 + |x - z|^2)^{N/2}} dz d(x, t) dy \\ &\leq C \left\{ \int_{\Gamma_1} \int_{B_1^+(y, 0)} \int_{\Gamma_2} \frac{|\nabla u(x, t)|^2 t^2}{(t^2 + |x - y|^2)^{N/2} (t^2 + |x - z|^2)^{N/2}} dz d(x, t) dy \right. \\ &\quad \left. + \int_{\Gamma_1} \int_{B_1^{+c}(y, 0)} \int_{\Gamma_2} \frac{|\nabla u(x, t)|^2 t^2}{(t^2 + |x - y|^2)^{N/2} (t^2 + |x - z|^2)^{N/2}} dz d(x, t) dy \right\} \\ &\leq I_1 + I_2. \end{aligned} \tag{3.4.2}$$

Here we have divided the integral over the region Π into two parts : one inside the half-ball $B_1^+(y, 0)$ and one outside the ball for each $y \in \Gamma_1$.

We recall that u is a bounded harmonic function in Π so $|\nabla u| \leq C$ and choosing $\beta_1 \geq 4R_1 + r_1$, we get $|x - z| \geq \frac{1}{2}|z|$ for each $x \in D_1(y)$ and $z \in \Gamma_2$. We obtain

$$\begin{aligned} I_1 &\leq C \int_{\Gamma_1} \int_{B_1^+(y, 0)} \frac{t^2}{(t^2 + |x - y|^2)^{N/2}} \int_{\Gamma_2} \frac{1}{|x - z|^N} dz d(x, t) dy \\ &\leq C \int_{\Gamma_1} \int_{B_1^+(y, 0)} \frac{1}{(t^2 + |x - y|^2)^{(N-2)/2}} \left(\int_{\Gamma_2} \frac{dz}{|z|^N} \right) d(x, t) dy \\ &\leq C \int_{\Gamma_1} \int_{B_1^+(0)} \frac{1}{|X|^{N-2}} dX \int_{r_2}^{2R_2} \frac{r^{N-2}}{r^N} dr dy \quad \text{using } X = (x - y, t) \\ &\leq C \frac{\text{mes}(\Gamma_1)}{\beta_1} = C \frac{R_1^{N-1}}{\beta_1} \quad \text{because } r_2 \geq \beta_1 \text{ and } \Gamma_1 \subset D_{2R_1}. \end{aligned} \tag{3.4.3}$$

To estimate the next integral I_2 , we split the integral over $B_1^{+c}(y, 0)$ to $B_1^{+c}(y, 0) \cap A$ and A^c where $A = \{(x, t) \in \Pi : |x| \leq 3\beta_1/4\}$. We have

$$\begin{aligned} I_2 &= C \left\{ \int_{\Gamma_1} \int_{B_1^{+c}(y,0) \cap A} \int_{\Gamma_2} \frac{|\nabla u(x, t)|^2 t^2}{(t^2 + |x - y|^2)^{N/2} (t^2 + |x - z|^2)^{N/2}} dz d(x, t) dy \right. \\ &\quad \left. + \int_{\Gamma_1} \int_{A^c} \int_{\Gamma_2} \frac{|\nabla u(x, t)|^2 t^2}{(t^2 + |x - y|^2)^{N/2} (t^2 + |x - z|^2)^{N/2}} dz d(x, t) dy \right\} \\ &= I_{21} + I_{22}. \end{aligned} \quad (3.4.4)$$

Observe that, for $(x, t) \in B_1^{+c}(y, 0) \cap A$, $y \in \Gamma_1$ and $z \in \Gamma_2$, we have $|x - z| \geq |x - y|/6$ and $|x - z| \geq |z|/4$. Furthermore, since u is a bounded harmonic function, we derive that $|\nabla u(x, t)| \leq C/t$ for each $(x, t) \in \Pi$. Hence, we obtain

$$\begin{aligned} I_{21} &\leq C \int_{\Gamma_1} \int_{B_1^{+c}(y,0) \cap A} \frac{1}{(t^2 + |x - y|^2)^{N/2}} \int_{\Gamma_2} \frac{1}{(t^2 + |x - z|^2)^{N/2}} dz d(x, t) dy \\ &\leq C \int_{\Gamma_1} \int_{B_1^{+c}(y,0) \cap A} \frac{1}{(t^2 + |x - y|^2)^{N/2+1/4}} \left(\int_{\Gamma_2} \frac{dz}{|z|^{N-\frac{1}{2}}} \right) d(x, t) dy \\ &\leq C \int_{\Gamma_1} \int_{B_1^{+c}(0)} \frac{1}{|X|^{N+1/2}} dX \int_{r_2}^{R_2} \frac{r^{N-2}}{r^{N-1/2}} dr \quad dy \\ &\leq C \frac{R_1^{N-1}}{\sqrt{\beta_1}}. \end{aligned} \quad (3.4.5)$$

On the other hand

$$\begin{aligned} I_{22} &= C \left\{ \int_{\Gamma_1} \int_{A^c \cap \{t \geq 1\}} \int_{\Gamma_2} \frac{|\nabla u(x, t)|^2 t^2}{(t^2 + |x - y|^2)^{N/2} (t^2 + |x - z|^2)^{N/2}} dz d(x, t) dy \right. \\ &\quad \left. + \int_{\Gamma_1} \int_{A^c \cap \{t \leq 1\}} \int_{\Gamma_2} \frac{|\nabla u(x, t)|^2 t^2}{(t^2 + |x - y|^2)^{N/2} (t^2 + |x - z|^2)^{N/2}} dz d(x, t) dy \right\} \\ &= I_{221} + I_{222}. \end{aligned} \quad (3.4.6)$$

Using $|x - y| \geq \beta_1/4$ and $|\nabla u(x, t)| \leq C/t$, we obtain

$$\begin{aligned} I_{221} &\leq C \int_{\Gamma_1} \int_{A^c \cap \{t \geq 1\}} \int_{\Gamma_2} \frac{1}{(t^2 + |x - y|^2)^{N/2}} \frac{1}{(t^2 + |x - z|^2)^{N/2}} dz d(x, t) dy \\ &\leq \frac{C}{\sqrt{\beta_1}} \int_{\Gamma_1} \int_{A^c \cap \{t \geq 1\}} \int_{\Gamma_2} \frac{1}{(t^2 + |x - y|^2)^{N/2-1/4}} \frac{1}{(t^2 + |x - z|^2)^{N/2}} dz d(x, t) dy \\ &\leq \frac{C}{\sqrt{\beta_1}} \int_{\Gamma_1} \int_{A^c \cap \{t \geq 1\}} \frac{1}{t^{9/8}} \frac{1}{(1 + |x - y|^2)^{N/2-3/8}} \int_{\mathbb{R}^{N-1}} \frac{dX}{(1 + |X|^2)^{N/2-7/16}} d(x, t) dy \\ &\leq \frac{C}{\sqrt{\beta_1}} \int_{\Gamma_1} \int_1^{+\infty} \frac{1}{t^{9/8}} dt \int_{\mathbb{R}^{N-1}} \frac{1}{(1 + |x - y|^2)^{N/2-3/8}} dx \quad dy \\ &\leq C \frac{R_1^{N-1}}{\sqrt{\beta_1}}. \end{aligned} \quad (3.4.7)$$

From $|\nabla u| \leq C$, (3.2.1) and $|x - y| \geq |x|/3$ since $|x| \geq 3\beta_1/4$ we get

$$\begin{aligned}
 & \int_{\Gamma_1} \int_{A^c \cap \{t \leq 1\}} \int_{\Gamma_2} \frac{|\nabla u(x, t)|^2 t^2}{(t^2 + |x - y|^2)^{N/2} (t^2 + |x - z|^2)^{N/2}} dz d(x, t) dy \\
 & \leq C \int_{\Gamma_1} \int_{A^c \cap \{t \leq 1\}} \frac{t}{(t^2 + |x - y|^2)^{N/2}} \int_{\mathbb{R}^{N-1}} \frac{t}{(t^2 + |x - z|^2)^{N/2}} dz d(x, t) dy \\
 & \leq C \int_{\Gamma_1} \int_0^1 \int_{\{|x| \geq 3\beta_1/4\}} \frac{t}{(t^2 + |x - y|^2)^{N/2}} dx dt dy \\
 & \leq C \int_{\Gamma_1} \int_{\{|x| \geq 3\beta_1/4\}} \frac{1}{|x|^N} dx dy \\
 & \leq C \frac{R_1^{N-1}}{\beta_1}.
 \end{aligned} \tag{3.4.8}$$

Thus, (3.4.2)-(3.4.8) give the estimate of I . Concerning the estimate of J , using (3.2.6) and the fact that $|u| \leq 1$, we derive

$$\begin{aligned}
 \left| \int_{\partial \Pi} u^2(x, 0) \varphi_2(x) \frac{\partial \phi_1}{\partial \nu}(x, 0) dx \right| & \leq \int_{\Gamma_2} \left(\int_{\Gamma_1} \frac{c}{|x - y|^N} dy \right) dx \\
 & \leq C \int_{\Gamma_1} \left(\int_{\Gamma_2} \frac{1}{|x - y|^N} dx \right) dy \\
 & \leq C \int_{\Gamma_1} \left(\int_{\Gamma_2} \frac{1}{|x|^N} dx \right) dy \\
 & \leq C \frac{\text{mes}(\Gamma_1)}{\beta_1} = C \frac{R_1^{N-1}}{\beta_1}.
 \end{aligned} \tag{3.4.9}$$

Note that in the third inequality, we have used that $|x - y| \geq c|x|$ for each $y \in \Gamma_1$ and $x \in \Gamma_2$ since we have $\beta_1 \geq 4R_1 + r_1$.

By the same way, we obtain

$$\left| \int_{\partial \Pi} u^2(x, 0) \varphi_1(x) \frac{\partial \phi_2}{\partial \nu}(x, 0) dx \right| \leq C \frac{R_1^{N-1}}{\beta_1}. \tag{3.4.10}$$

(3.4.9) and (3.4.10) imply the estimate of J . Finally, we get

$$|b(u\phi_1, u\phi_2)| \leq C \frac{R_1^{N-1}}{\sqrt{\beta_1}}.$$

We choose $\beta_1 \geq 4R_1 + r_1$ such that $\frac{R_1^{N-1}}{\sqrt{\beta_1}} = o(|q(u\phi_1)|)$ and Lemma 3.4.1 follows. \square

Now, since we assumed that for each r , there exists $R > 2r$ such that $q(u\phi_{r,R}) < 0$, then choosing $r_2 > \beta_1$, there exists R_2 such that $q(u\phi_{r_2, R_2}) < 0$. We derive the existence of two functions φ_1 and φ_2 such that $q(u\phi_i) < 0$ and $b(u\phi_1, u\phi_2) = o(q(u\phi_1))$.

By induction, we assume that we have constructed ℓ functions ϕ_i as above (that is $b(u\phi_i, u\phi_{i-1}) = o(q(u\phi_{i-1}))$ and $\beta_j \geq 4R_j + r_j$ for each $j \leq i - 1$ and $i \leq \ell$).

Concerning the function $\phi_{\ell+1}$, choosing β_ℓ such that :

$$\beta_\ell \geq 4R_\ell + r_\ell \quad \text{and} \quad \frac{R_\ell^{N-1}}{\sqrt{\beta_\ell}} = o(|q(u\phi_\ell)|),$$

the function $\varphi_{\ell+1}$ will be constructed exactly as the construction of φ_2 . Hence we get $q(u\phi_{\ell+1}) < 0$ and $b(u\phi_j, u\phi_{\ell+1}) = o(q(u\phi_j))$ for each $j \leq \ell$, by using Lemma 3.4.1 since the support of $\varphi_{\ell+1}$ is in $D_{\beta_j}^c$ for each $j \leq \ell$.

Observe that the construction of the ϕ_i 's guarantees that the functions $u\phi_i$'s satisfy (3.4.1) which conclude the proof of Lemma 3.2.3. □

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