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Université Sorbonne Paris Cité
Université Paris Diderot

École doctorale de sciences mathématiques de Paris centre (ED 386)

Institut de mathématiques de Jussieu – Paris rive gauche (UMR 7586)

Mémoire présenté pour l'obtention du

Diplôme d'habilitation à diriger des recherches

Discipline : mathématiques

par

Patrick Le Meur

Dualité homologique et théorie des représentations des algèbres non commutatives

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Date de soutenance : 28 juin 2019

Composition du jury :

Julien Bichon	Université Clermont Auvergne	Rapporteur
Claude Cibils	Université de Montpellier	Examinateur
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Marc Rosso	Université Paris Diderot	Rapporteur
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Alexander Zimmermann	Université de Picardie Jules Verne	Examinateur

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Introduction

This thesis presents the research work that I did after my PhD ([LM06]). It is written in view of obtaining the diploma *habilitation à diriger des recherches* from Université Paris Diderot. The research work presented here is published in the articles [LM08, LM10a, ABLM10, LM10b, LM11, CLMT11, ABLM11, LM19, LLM18, ALMM17, CLMT19b, CLMT19a, ABD$^+$19][1][2]. The presentation is divided in two parts (Part 2 and Part 3) according to the following topics.

[1]For various reasons including a matter of place and of topic, [ABD$^+$19] is not presented here.

[2]The recent preprint [LM18] is not presented in detail either. See Section 5.8 for information on the questions addressed by this article.

(1) Calabi-Yau and related dualities in non commutative algebra ([LM10a, LM19, LLM18]).
(2) Fundamental tools of the representation theory of finite dimensional algebras ([LM08, ABLM10, LM10b, LM11, CLMT11, ABLM11, ALMM17, CLMT19b, CLMT19a]).

Both parts 2 and 3 feature open questions which are directly related to the results presented there. Perspectives for future research related to the above mentioned topics are presented in part 4.

Overview of Part 2

Part 2 deals with homological properties of non commutative algebras related to Calabi-Yau duality. These properties are rooted in the Grothendieck duality of commutative algebra and they are based on the concept of inverse dualising complex $\mathrm{RHom}_{A^e}(A, A^e)$ of the (dg) algebra A under consideration. The homological duality may have specific features depending on the particular shape of the inverse dualising complex. According to these features, A is said to have Van den Bergh duality ([vdB98]), to be skew Calabi-Yau ([RRZ14]), or to be Calabi-Yau ([Gin06]). In this framework, I have computed inverse dualising complexes for various contexts of particular interest in non commutative algebra in view of determining homological duality of the considered algebras.

In [LM10a] I computed the inverse dualising complex of the skew group dg algebra $A * G$, where A is a dg algebra acted on by a finite group G (with order not divisible by the characteristic of the field). Using this description I proved the following.

- If A is the Calabi-Yau completion (in the sense of [Kel11]) of a homologically smooth dg algebra R, then $A * G$ is the Calabi-Yau completion of $R * G$, and hence is Calabi-Yau (see below for a generalisation to actions of Hopf algebras).
- If A is the Ginzburg dg algebra (in the sense of [Gin06]) of a quiver with potential, where the path algebra of the quiver is acted on by G and the potential is G-invariant, then $A * G$ is Morita equivalent to a Ginzburg dg algebra (see Section 4.3.2).

In a recent update of that work I applied these results to a detailed comparison of the generalised cluster categories of A and of $A * G$ (in the sense of [Ami09] and assuming that these are defined, see Section 5.6). The problem of making such a comparison has been worked on by many researchers interested in actions of finite groups on cluster structures (see [Dem11, PS17, AP17, GP19]). I also applied these results to higher representation theoretical considerations (in the sense of [Iya11, HIO14]). More precisely, assuming that A is a finite dimensional algebra of finite global dimension, then, for all non negative integers n (see Section 5.7),

- A is n-representation finite if and only if so is $A * G$,
- A is n-representation infinite if and only if so is $A * G$.

In [LM19] I computed the inverse dualising complexes of the smash products $A\sharp H$ of homologically smooth dg algebras A by Hopf algebras H with invertible antipodes. Note that skew group dg algebras are particular cases of this construction. On one hand, I used this description to express the (deformed) Calabi-Yau completions of $A\sharp H$ in terms of the ones of A when H has Van den Bergh duality (see Section 5.4.1). On the other hand, I applied this description to the duality of $A\sharp H$ when A is an algebra (see Section 5.5.2). This matter has been investigated deeply since 2010 mainly because of the problem of determining the homological

properties of the invariant subalgebra A^H (see for instance [KKZ09]); the corresponding investigations have led to the definition of skew Calabi-Yau algebras ([RRZ14]). In this context, I have proved the following results.

- $A\sharp H$ has Van den Bergh duality if and only if so have A and H.
- If A and H are skew Calabi-Yau then so is $A\sharp H$ and a Nakayama automorphism of $A\sharp H$ may be described in terms of ones of A and H. This description involves the concept of weak homological determinant which replaces the classical homological determinant whose definition makes sense only when A is $\mathbb{N}$-graded and connected (see Section 5.5.2).

In [LLM18], Thierry Lambre and I investigated the inverse dualising complexes of algebras U of differential operators (or, enveloping algebras) of Lie-Rinehart algebras (S, L) (in the sense of [Rin63]) over commutative rings R. Previously, Chemla made such a description ([Che99]) for the algebras of differential operators of Lie algebroids over smooth affine complex varieties. Huebschmann defined in [Hue99] the concept of duality for such an (S, L) and he established a fairly general condition for such a duality to occur. Under this sufficient condition, we proved that if S is flat over R and if S has Van den Bergh duality, then so has U (see Section 6.2); assuming, in addition, that S is Calabi-Yau and the highest non zero exterior power of L over S is isomorphic to S as an S-module, we proved that U is skew Calabi-Yau and we computed explicitly a Nakayama automorphism of it. These results specialise even more precisely to Lie-Rinehart algebras arising from Poisson brackets on S (see Section 6.3).

Overview of Part 3

Part 3 deals with fundamental invariants and tools in the representation theory of finite dimensional algebras: the Galois coverings; the strong global dimension; and the Auslander-Reiten structure.

During my PhD, I investigated the existence of universal Galois coverings for (basic and connected) finite dimensional algebras over algebraically closed fields. After finishing my PhD, I made a study ([LM08, LM10b, ABLM10, LM11, ABLM11]) of the simple connectedness of these algebras. This was motivated by the following question by Skowroński ([Sko92]) to which an abundant literature had been devoted previously: *Is a tame triangular algebra A simply connected if and only if its Hochschild cohomology group* $\mathrm{HH}^1(A)$ *is zero?* Taking into account that $\mathrm{HH}^1(A)$ is invariant under derived equivalences, I answered this question for several classes of (not necessarily tame) algebras related to tilting theory. These answers are based on the studies of the behaviour of simple connectedness under tilting (see Section 10.2) and of the extent to which the considered classes of algebras are closed under Galois coverings (see Section 10.3). For a finite dimensional algebra A, I proved the equivalence between A being simply connected and $\mathrm{HH}^1(A)$ being zero in the following cases (see Section 10.4):

- when A is piecewise hereditary, which includes the case where A is tilted,
- when A is weakly shod (where *shod* stands for *of small homological dimension*),
- when A is laura standard (in collaboration with Ibrahim Assem and Juan Carlos Bustamante).

With different (combinatorial) techniques, Ibrahim Assem, Juan Carlos Bustamante and I proved that the same equivalence holds true when A is special biserial.

In [ALMM17], Edson Ribeiro Alvares, Eduardo N. Marcos and I investigated the strong global dimension of piecewise hereditary algebras. Skowroński defined ([Sko87]) the strong global dimension of a finite dimensional algebra as an invariant

lying in $\mathbb{N}\cup\{+\infty\}$, and Happel and Zacharia proved ([HZ08]) that it is finite if and only if the algebra under consideration is piecewise hereditary. On one hand, the Auslander-Reiten structure of the derived category $\mathcal{D}^b(\mathrm{mod}(A))$ was described by Happel in [Hap88]; we have characterised the strong global dimension of A in terms of the Auslander-Reiten components of $\mathcal{D}^b(\mathrm{mod}(A))$ containing indecomposable direct summands of A (see Section 11.1). On the other hand, by repeating a process called (generalised) mutation, a piecewise hereditary algebra A may be turned into a quasitilted algebra (that is, the endomorphism algebra of a tilting object in a hereditary abelian category); we have characterised the strong global dimension of A in terms of the minimal number of mutations needed to make such a transformation (see Section 11.2).

Finally, in [CLMT11, CLMT19a, CLMT19b], Claudia Chaio, Sonia Trepode and I have developed covering techniques in the Auslander-Reiten theory of finite dimensional algebras over fields. The motivation of this work is to understand better the filtration of the module category $\mathrm{mod}(A)$ of such an algebra A by the powers rad^ℓ ($\ell\in\mathbb{N}$) of its radical rad. The covering techniques are based on the concept of well-behaved functors defined by Riedtmann [Rie80]. These are functors $F\colon \Bbbk(\widetilde{\Gamma})\to \mathrm{ind}\,\Gamma$ where Γ is an Auslander-Reiten component of A and $\widetilde{\Gamma}$ is a covering of Γ with mesh category denoted by $\Bbbk(\widetilde{\Gamma})$. In this context, the main problem is to determine whether such a functor exists for a given Γ and to establish its properties. We have proved that such functors always exist provided that the ground field is perfect and we have established the general properties of these functors (see Section 9.1). Using these results, we have characterised the degree of irreducible morphisms (in the sense of [Liu92]), which is the main tool to understand the behaviour of the composition of morphisms in relationship with the filtration $(\mathrm{rad}^\ell)_{\ell\geqslant 0}$ (see Section 9.2). As an application we have established a new characterisation of the algebras of finite representation type in terms of the finiteness of the degrees of specific irreducible morphisms (see Section 9.3).

Part 1. Publications

1. Publications on the research work done after finishing the PhD

Articles in peer-reviewed international journals

[1] Patrick Le Meur. On Galois coverings and tilting modules. *J. Algebra*, 319(12):4961–4999, 2008.

[2] Ibrahim Assem, Juan Carlos Bustamante, and Patrick Le Meur. Coverings of Laura algebras: the standard case. *J. Algebra*, 323(1):83–120, 2010.

[3] Patrick Le Meur. Galois coverings of weakly shod algebras. *Comm. Algebra*, 38(4):1291–1318, 2010.

[4] Ibrahim Assem, Juan Carlos Bustamante, and Patrick Le Meur. Special biserial algebras with no outer derivations. *Colloq. Math.*, 125(1):83–98, 2011.

[5] Claudia Chaio, Patrick Le Meur, and Sonia Trepode. Degrees of irreducible morphisms and finite representation type. *J. Lond. Math. Soc. (2)*, 84(1):35–57, 2011.

[6] Patrick Le Meur. Topological invariants of piecewise hereditary algebras. *Trans. Amer. Math. Soc.*, 363(4):2143–2170, 2011.

[7] Edson Ribeiro Alvares, Patrick Le Meur, and Eduardo N. Marcos. The strong global dimension of piecewise hereditary algebras. *J. Algebra*, 481:36–67, 2017.

[8] Thierry Lambre and Patrick Le Meur. Duality for Differential Operators of Lie-Rinehart Algebras. *Pacific Journal of Mathematics*, 297(2):405–454, 2018.

[9] Ibrahim Assem, Juan Carlos Bustamante, Julie Dionne, Patrick Le Meur, and David Smith. Representation theory of partial relation extensions. *Colloq. Math.*, 155(2):157–186, 2019.

[10] Claudia Chaio, Patrick Le Meur, and Sonia Trepode. Covering techniques in Auslander-Reiten theory. *J. Pure Appl. Algebra*, 223(2):641–659, 2019.

[11] Claudia Chaio, Patrick Le Meur, and Sonia Trepode. Degrees of Irreducible Morphisms over Perfect Fields. *Algebr. Represent. Theory*, 22(2):495–515, 2019.

[12] Patrick Le Meur. Smash Products of Calabi-Yau Algebras by Hopf Algebras. *Journal of Noncommutative Geometry*, 2019. To appear, arXiv:1512.01045 [math.RA].

Texts on arXiv, texts in preparation

[13] Thierry Lambre, Patrick Le Meur, and Anne Pichereau. BV Isomorphisms for Calabi-Yau and Poisson Algebras. In preparation.
[14] Patrick Le Meur. Crossed-products of Calabi-Yau algebras by finite groups. arXiv:1006.1082v2 [math.RA], 2010.
[15] Edson Ribeiro Alvares, Diane Castonguay, Patrick Le Meur, and Tanise Carnieri Pierin. (m, n)-Quasitilted and (m, n)-Almost Hereditary Algebras. arXiv:1709.07086 [math.RT], 2017.
[16] Patrick Le Meur. On the Morita Reduced Version of Skew Group Algebras of Path Algebras. arXiv:1810.12612 [math.RT], 2018.

2. Publications on the research work during the PhD

Articles in peer-reviewed international journals

[17] Patrick Le Meur. The fundamental group of a triangular algebra without double bypasses. *C. R. Math. Acad. Sci. Paris*, 341(4):211–216, 2005.
[18] Patrick Le Meur. The universal cover of an algebra without double bypass. *J. Algebra*, 312(1):330–353, 2007.
[19] Patrick Le Meur. The universal cover of a monomial triangular algebra without multiple arrows. *J. Algebra Appl.*, 7(4):443–469, 2008.
[20] Patrick Le Meur. On maximal diagonalizable Lie subalgebras of the first Hochschild cohomology. *Comm. Algebra*, 38(4):1325–1340, 2010.

Part 2. Homological duality in non commutative algebra

This part presents the main results of the articles [LM10a, LM19, LLM18] which investigate the homological properties related to Calabi-Yau duality for non commutative algebras or dg algebras obtained from the following constructions,

(1) smash products with Hopf algebras and in particular skew group algebras
(2) and enveloping algebras of Lie-Rinehart algebras.

The problems addressed in these articles and the context in which these problems are considered are presented in section 3. The needed definitions are recalled in section 4. The results relative to point (1) above are presented in section 5. The results relative to point (2) are presented in section 6.

3. Historical context and addressed problems

3.1. **Regular algebras and dualising complexes.** The emergence of Calabi-Yau duality in non commutative algebra is rooted in the study of regular algebras and dualising complexes.

Regular algebras, which are considered as non commutative analogues of polynomial algebras, appeared with the classification by Artin and Schelter ([AS87a]) of regular algebras of dimension 3 which was completed by Artin, Tate, and Van den Bergh in [ATVdB90]. Regular algebras are now known as Artin-Schelter regular algebras.

Dualising complexes appeared in non commutative algebra with the study of the Poincaré duality of the local cohomology of graded modules in analogy with the Grothendieck duality in the commutative setting. In this context, a crucial problem is to determine whether the algebra under consideration has a dualising complex and to describe it. A *dualising* complex M over a (left and right Noetherian) algebra A is a bounded complex of A-bimodules with the following properties: the cohomology groups of M are finitely generated and of finite injective dimensions on both sides, and the canonical morphisms $A \to \mathrm{RHom}_A(M, M)$ and

$A \to \mathrm{RHom}_{A^{\mathrm{op}}}(M, M)$ are isomorphisms in $\mathcal{D}(\mathrm{Mod}(A^e))$. For all such complexes M, the functor $\mathrm{RHom}_A(-, M)$ induces a duality between the full subcategories of $\mathcal{D}(\mathrm{Mod}(A))$ and $\mathcal{D}(\mathrm{Mod}(A^{\mathrm{op}}))$ consisting of objects with finitely generated total cohomology.

- Noetherian Artin-Schelter regular algebras have balanced dualising complexes ([Yek92]);
- a filtered algebra whose associated graded algebra has a balanced dualising complex has a rigid dualising complex ([vdB97]);
- Van den Bergh proved in [vdB97] that a connected graded Noetherian algebra A has a balanced dualising complex if and only if both A and A^{op} satisfy the χ-condition of Artin and Zhang ([AZ94]) and their local cohomology functors have finite cohomological dimension;
- Artin-Schelter Gorenstein algebras have rigid dualising complexes ([YZ99]).

Here, being balanced or being rigid are properties of dualising complexes introduced in [Yek92] and [vdB97], respectively, in order to deal with uniqueness issues. A dualising complex M over a connected graded algebra A is called balanced when the local cohomologies of ${}_AM$ and M_A are concentrated in degree 0 and isomorphic to the Matlis dual A'. And a dualising complex M over an algebra A is called rigid when $\mathrm{RHom}_{A^e}(A, M \otimes M) \simeq M$ in $\mathcal{D}(\mathrm{Mod}(A^e))$.

In particular, the non commutative counterpart of invariant theory provides large classes of algebras which are Artin-Schelter Gorenstein, and hence have rigid dualising complexes. Indeed, Watanabe proved in [Wat74] that if G is a finite subgroup of $\mathrm{SL}(V)$, where V is a finite dimensional vector space over a field of characteristic zero, then the invariant subalgebra $S(V)^G$ is Artin-Schelter Gorenstein. In [KKZ09], Kirkman, Kuzmanovich, and Zhang made a similar investigation for the actions of finite dimensional and semi-simple Hopf algebras H on Artin-Schelter regular algebras A; they introduced the homological determinant $\mathrm{hdet}\colon H \to \Bbbk$, which takes into account the structure of H-module on $\mathrm{Ext}^\bullet_A(\Bbbk, A)$, and they proved that if hdet is equal to the counit of H, then the invariant subalgebra A^H is Artin-Schelter Gorenstein. This work is a generalisation of a previous work by Jørgensen and Zhang in [JZ00], where they proved a stronger connection between the homological determinant being trivial and the invariant subalgebra being Artin-Schelter Gorenstein when H is the group algebra of a finite group.

3.2. **Van den Bergh duality.** For many of the algebras with a rigid dualising complex mentioned previously, this complex is quasi-isomorphic to a stalk complex of an invertible bimodule. Van den Bergh proved in [vdB98] that if a homologically smooth algebra A has such a property, then for all A-bimodules X, there is an isomorphism between the Hochschild cohomology and the Hochschild homology,

$$\mathrm{HH}^\bullet(A; X) \simeq \mathrm{HH}_{d-\bullet}(A; \mathrm{Ext}^n_{A^e}(A, A \otimes A) \otimes_A X)\,.$$

Actually, $\Sigma^n \mathrm{Ext}^n_{A^e}(A, A^e)^{-1}$ is the rigid dualising complex of A. The algebras satisfying the above sufficient conditions are now called algebras with Van den Bergh duality in dimension n. In particular, the universal enveloping algebras of the finite dimensional Lie algebras ([Yek00]), the algebras of differential operators of smooth affine varieties over a field ([Yek00]), and the sheaf of algebras of differential operators on a Lie algebroid over a smooth affine complex variety ([Che99]) all have Van den Bergh duality.

3.3. **Calabi-Yau algebras.** If an algebra A with Van den Bergh duality has $\Sigma^n A$ as a rigid dualising complex, then A is said to be Calabi-Yau. Recently, these algebras have been investigated and used intensively. After Kontsevich's definition of Calabi-Yau triangulated categories ([Kon98]), Ginzburg defined the Calabi-Yau

(dg) algebras in [Gin06]. Among the reasons for the strong development of Calabi-Yau algebras, here are three of them.

Firstly, when giving his definition of Calabi-Yau algebras, Ginzburg gave a general recipe to construct algebras which are good candidates for being Calabi-Yau starting from a quiver with potential. These are the Jacobian algebras and the Ginzburg dg algebras. The former are the 0-th cohomology algebras of the latter. On one hand, Keller and Van den Bergh proved in [Kel11] that Ginzburg dg algebras are Calabi-Yau. Actually, Keller's proof explains that these dg algebras are instances of a more general construction of (possibly) Calabi-Yau dg algebras and called (deformed) Calabi-Yau completions. On the other hand, in certain situations, there are criterions to determine whether a Jacobian algebra is Calabi-Yau (see for instance [VdB05]), and conversely, for certain classes of algebras it was proved that being Calabi-Yau entails being given by a potential. This is the case of connected graded algebras with generators in degree one ([Boc08]), of connected graded algebras with suitable finiteness conditions on the associated Yoneda algebra ([Seg08]), and for complete Calabi-Yau dg algebras ([VdB15]).

Secondly, the Calabi-Yau algebras show up easily in the study of non commutative crepant resolutions. For instance, Auslander's theorem asserts that, for all finite subgroups G of $\mathrm{SL}_2(\mathbb{C})$, the skew group algebra $\mathbb{C}[x,y] * G$ is isomorphic to $\mathrm{End}_{\mathbb{C}[x,y]^G}(\mathbb{C}[x,y])$. Here, $\mathbb{C}[x,y] * G$ is Calabi-Yau in dimension 2. Actually, it is proved in [BSW10] that for all finite dimensional complex vector spaces V and for all finite subgroups G of $\mathrm{SL}(V)$, the algebra $S(V) * G$ is Calabi-Yau. Note that the assumption on G also entails that $S(V)^G$ is Artin-Schelter Gorenstein.

Thirdly, the Calabi-Yau dg algebras are involved in the development of several branches of representation theory related to cluster tilting theory. On one hand, they are key ingredients in the construction of generalised cluster categories ([Ami09]). In particular, Ginzburg dg algebras and generalised cluster categories may be associated to marked surfaces with triangulations ([LF09]); note that recent attempts to extend this theory to orbifold surfaces are based on skew group dg algebras ([PS17, AP17]). On the other hand, in the higher Auslander-Reiten theory initiated by Iyama ([Iya11]), the higher representation infinite algebras are characterised in terms of their Calabi-Yau completions which are called higher derived preprojective algebras in this context.

3.4. **Skew Calabi-Yau algebras.** The success of Calabi-Yau algebras has shed a new light on regular algebras. Reyes, Rogalski, and Zhang proved in [RRZ14] that a connected graded algebra A is Artin-Schelter regular of global dimension n if and only if it has Van den Bergh duality in dimension n and there exists an automorphism σ of A such that $\mathrm{Ext}^n_{A^e}(A, A^e) \simeq A^\sigma$ for some algebra automorphism σ of A. The algebras having Van den Bergh duality and satisfying this last condition are now called skew Calabi-Yau, and such a σ is called a Nakayama automorphism of the algebra. For instance, the universal enveloping algebras of finite dimensional Lie algebras are skew Calabi-Yau.

It is worth noticing that many standard constructions used in non commutative algebra behave fairly well with respect to the class of skew Calabi-Yau algebras. In particular, one of the main results of the work of Reyes, Rogalski, and Zhang ([RRZ14]) is the computation of the Nakayama automorphisms of the smash product of any Artin-Schelter regular algebra A by the action of any finite dimensional Hopf algebra H in terms of Nakayama automorphisms of A and of H and using the homological determinant of the action of H on A.

3.5. **Addressed problems.** In the previous context, the articles [LM10a, LM19, LLM18] which are presented here deal with the following questions.

Considering a Calabi-Yau dg algebra A in relationship with cluster tilting theory, and given a finite group G acting on A by dg automorphisms, when is the skew group dg algebra $A * G$ Calabi-Yau? To what extent are the constructions of Calabi-Yau dg algebras used in cluster tilting theory compatible with taking skew group algebras? When both A and $A * G$ are Calabi-Yau and have an associated generalised cluster category, how are these categories related?

In view of the above mentioned results in non commutative invariant theory for actions of Hopf algebras on regular algebras, when does the smash product of an algebra by a Hopf algebra have Van den Bergh duality, and when is it skew Calabi-Yau? In the latter case, is it possible to compute a Nakayama automorphism? In particular, when is the smash product a Calabi-Yau algebra?

The contexts of the above mentioned results of Yekutieli ([Yek00]) and Chemla ([Che99]) fit in the one of Lie-Rinehart algebras ([Rin63]). Given a Lie-Rinehart algebra (S, L) over a commutative ring R, under which conditions does its enveloping R-algebra have Van den Bergh duality? When this is the case, how is this duality related to the duality of Lie-Rinehart algebras in the sense of [Hue99]?

4. Basic definitions, conventions, notation, and known results

4.1. **Conventions and notation.** $\Bbbk$ denotes a field. The tensor product over $\Bbbk$ is denoted by $\otimes$. In all triangulated categories, Σ denotes the suspension functor. It is implicit that (associative and unital) dg algebras are over $\Bbbk$ and that $\Bbbk$-algebras as considered as dg algebras concentrated in degree 0. Given a dg algebra A, the following associated notation is used,

- A^e is the enveloping dg algebra, $A \otimes A^{\mathrm{op}}$;
- $\mathrm{Diff}(A)$ is dg category of left dg A-modules;
- $\mathrm{Hom}_A(M, N)$ is the complex of morphisms $M \to N$ of dg A-modules, where $M, N \in \mathrm{Diff}(A)$;
- $\mathcal{D}(A)$ is derived category of A, that is, the localisation of $H^0(\mathrm{Diff}(A))$ at the class of quasi-isomorphisms;
- $\mathrm{per}(A)$ is perfect derived category of A;
- $\mathcal{D}_{\mathrm{fd}}(A)$ is thick subcategory of $\mathcal{D}(A)$ whose objects are the dg modules $M \in \mathcal{D}(A)$ such that $\sum_{i\in\mathbb{Z}} \dim H^i(M) < \infty$.

Dg modules over A^e, whether left or right, are viewed as dg A-bimodules (which are symmetric as $\Bbbk$-bimodules). In particular, A^e is a dg A^e-bimodule. Accordingly, for all $M \in \mathrm{Diff}(A^e)$, the complex $\mathrm{Hom}_{A^e}(M, A^e)$, which is a right dg A^e-module in a natural way, is also considered as a dg A-bimodule, and hence as a left dg A^e-module. In this sense, $\mathrm{RHom}_{A^e}(-, A^e)$ is a triangle functor from $\mathcal{D}(A^e)$ to $\mathcal{D}(A^e)$.

For an algebra R, the category of left R-modules is denoted by $\mathrm{Mod}(R)$ and the full subcategory of finitely generated modules is denoted by $\mathrm{mod}(R)$.

If H is a Hopf algebra, then S and ε denote the antipode and the counit, respectively. The comultiplication is denoted using Sweedler's convention and omitting the sum sign, $h \mapsto h_1 \otimes h_2$. When H acts on a vector space M on the left (or on the right), the action of $h \in H$ on $m \in M$ is denoted by $h \rightharpoonup m$ (or, $m \leftharpoonup h$, respectively).

4.2. **Homological duality.**

Definition 4.2.1. A dg algebra A is *Calabi-Yau* in dimension n, where n is an integer, if A is *homologically smooth*, that is $A \in \mathrm{per}(A^e)$, and $\Sigma^n \mathrm{RHom}_{A^e}(A, A^e) \simeq A$ in $\mathcal{D}(A^e)$.

In general, a dg A-bimodule isomorphic to $\mathrm{RHom}_{A^e}(A, A^e)$ is called an *inverse dualising complex* of A.

Definition 4.2.2. Let A be an algebra. Let n be an integer.

(1) A is said to have *Van den Bergh duality* in dimension n if A is homologically smooth and the A-bimodule $\mathrm{Ext}^i_{A^e}(A, A^e)$ is invertible if $i = n$ and is zero otherwise. If so, the *inverse dualising bimodule* of A is $\mathrm{Ext}^n_{A^e}(A, A^e)$.
(2) A is *skew* Calabi-Yau in dimension n if A has Van den Bergh duality in dimension n and there exists $\mu \in \mathrm{Aut}_{\Bbbk-\mathrm{alg}}(A)$ such that $\mathrm{Ext}^n_{A^e}(A, A^e) \simeq A^\mu$, such a μ is called a *Nakayama automorphism* of A.

Recall that A^μ denotes the A-bimodule whose underlying vector space is A and whose structure of A-bimodule is given by $a \otimes x \otimes b \mapsto ax\mu(b)$. If A is skew Calabi-Yau in dimension n, then there is a Nakayama automorphism associated to each free generator of the left A-module $\mathrm{Ext}^n_{A^e}(A, A^e)$. Moreover, A is Calabi-Yau if and only if at least one Nakayama automorphism is inner, equivalently, all Nakayama automorphisms are inner.

Example 4.2.3 ([Yek00]). The universal enveloping algebra $\mathcal{U}_\Bbbk(\mathfrak{g})$ of a finite dimensional Lie algebra $\mathfrak{g}$ over $\Bbbk$ is skew Calabi-Yau in dimension $\dim \mathfrak{g}$. Moreover, there is a Nakayama automorphism whose restriction to $\mathfrak{g}$ is given by $X \mapsto X + \mathrm{Tr}(\mathrm{ad}_X)$.

Definition 4.2.4. Let A be an augmented algebra. Let n be an integer.

(1) A is *Artin-Schelter Gorenstein* in dimension n if
   - the A-modules ${}_AA$ and A_A have finite (and equal) injective dimensions,
   - $\mathrm{Ext}^i_A({}_A\Bbbk, A)$ is one dimensional if $i = n$ and zero otherwise,
   - and $\mathrm{Ext}^i_{A^{\mathrm{op}}}(\Bbbk_A, A)$ is one dimensional if $i = n$ and zero otherwise.
(2) A is *Artin-Schelter regular* in dimension n if it is Artin-Schelter Gorenstein and $\mathrm{gl.dim.}\, A = n$.

4.3. **Deformed Calabi-Yau completions and Ginzburg dg algebras.** The exposition of the definitions follows [Kel11].

4.3.1. *Deformed Calabi-Yau completions.* Let A be a homologically smooth dg algebra, n be an integer, and D_A be a cofibrant replacement of $\mathrm{RHom}_{A^e}(A, A^e)$ in $\mathrm{Diff}(A^e)$.

Definition 4.3.1 ([Kel11])**.** The tensor dg algebra $T_A(\Sigma^{n-1}D_A)$ is called the *Calabi-Yau completion* in dimension n of A. It is denoted by $\mathbf{\Pi}_n(A)$.

Following [Kel11], the dg algebra $\mathbf{\Pi}_n(A)$ is Calabi-Yau in dimension n and it does not depend on the choice of D_A up to a quasi-isomorphism of dg algebras.

Now, let α be a Hochschild homology class in $\mathrm{HH}_{n-2}(A)$. Since A is homologically smooth, there is an isomorphism of vector spaces,

$$\mathrm{HH}_{n-2}(A) \simeq H^0\mathrm{Hom}_{A^e}(\Sigma^{n-1}D_A, \Sigma A)\,.$$

Let $\delta \in \mathrm{Hom}_{A^e}(\Sigma^{n-1}D_A, \Sigma A)$ be a 0-cocycle representing α. A unique square-zero skew derivation, d_δ, of the underlying graded algebra of $T_A(\Sigma^{n-1}D_A)$ is such that

- its restriction to A is induced by the differential of A,
- its restriction to $\Sigma^{n-1}D_A$ takes its values in $\Sigma A \oplus \Sigma^n D_A$ and
  - the induced mapping $\Sigma^{n-1}D_A \to \Sigma A$ is δ,
  - the induced mapping $\Sigma^{n-1}D_A \to \Sigma^n D_A$ is the differential of $\Sigma^{n-1}D_A$.

Definition 4.3.2 ([Kel11])**.** The *deformed Calabi-Yau completion* of A (given by α) is the dg algebra $(T_A(\Sigma^{n-1}D_A), d_\delta)$. It is denoted by $\mathbf{\Pi}_n(A, \alpha)$.

As proved in [Kel11], the dg algebra $\mathbf{\Pi}_n(A, \alpha)$ is Calabi-Yau in dimension n and does not depend on D_A nor on α, up to a quasi-isomorphism of dg algebras.

4.3.2. *Ginzburg dg algebras.* Let Q be a graded $\Bbbk$-quiver, that is, Q is quiver with set of vertices denoted by Q_0, with set of arrows denoted by Q_1, and such that the path algebra $\Bbbk Q$ is endowed with a $\mathbb{Z}$-grading such that trivial paths are homogeneous of degree 0 and all arrows are homogeneous.

The following vector space is by definition the space of potentials,

$$\frac{\Bbbk Q}{\operatorname{span}(uv - (-1)^{\deg(u)\cdot\deg(v)}vu \ ; \ u, v \in \Bbbk Q \text{ homogeneous})}$$

A potential is hence a linear combination of oriented cycles of Q considered up to cyclic permutation with signs. *In this report, homogeneous potentials of degree n are considered as elements of* $\mathrm{HH}_n(\Bbbk Q)$. For all oriented cycles p and for all arrows a in Q, define the *cyclic derivative* $\partial_a p$ by

$$\partial_a p = \sum_{p=p_1 a p_2} (-1)^{\deg(p_1)\cdot\deg(ap_2)} p_2 p_1 \,,$$

where the sum runs over all decompositions $p = p_1 a p_2$ with paths p_1 and p_2. The definition extends linearly to cyclic derivatives $\partial_a W$ for all $a \in \Bbbk Q_1$ and all potentials W. The definition of the Ginzburg dg algebra was given in [Gin06] when $n = 3$ using ideas of Kontsevich who defined cyclic derivatives in [Kon93] and it was given in [Kel11] in general.

Definition 4.3.3. Let n be an integer. Let W be a potential which is homogeneous of degree $n-3$. The *Ginzburg dg algebra* of (Q, W) is the following dg algebra denoted by $\mathcal{A}(Q, W)$. As a graded algebra, it is equal to $\Bbbk\widetilde{Q}$, where $\widetilde{Q}$ is the graded $\Bbbk$-quiver with the same vertices as Q and whose arrows are as follows,

- each arrow of Q is an arrow of $\widetilde{Q}$,
- for all arrows $a\colon i \to j$ of Q, there is an additional arrow $a^*\colon j \to i$ in $\widetilde{Q}$, its degree is $\deg(a) - n + 2$, and
- for all $i \in Q_0$, there is an additional loop $c_i\colon i \to i$ in $\widetilde{Q}$ of degree $-n+1$.

The differential of $\mathcal{A}(Q, W)$ is the unique one such that

- the elements of $\Bbbk Q$ have zero differential,
- for all arrows a of Q, the differential of a^* is equal to $\partial_a W$, and
- for all vertices i, the differential of c_i is equal to

$$(-1)^n \left(\sum_{a\colon i \to \cdot} aa^* - \sum_{a\colon \cdot \to i} (-1)^{\deg(a)\cdot\deg(a^*)} a^* a \right) ,$$

  where the first sum runs over all the arrows of Q starting in i and the second sum runs over all the arrows of Q arriving in i.

Denote by α the image under Connes' boundary $\mathrm{HH}_{n-3}(\Bbbk Q) \to \mathrm{HH}_{n-2}(\Bbbk Q)$ of W. It is proved in [Kel11] that that the dg algebras $\mathcal{A}(Q, W)$ and $\mathbf{\Pi}_n(\Bbbk Q, \alpha)$ are quasi-isomorphic. In particular, $\mathcal{A}(Q, W)$ is Calabi-Yau in dimension n.

4.4. **Duality of Hopf algebras.** Let H be a Hopf algebra. Let d be an integer.

The vector space $\mathrm{Ext}^d_H({}_H\Bbbk, H)$ (or $\mathrm{Ext}^d_{H^{\mathrm{op}}}(\Bbbk_H, H)$) is a right H-module (or, a left H-module, respectively) and, if it is one dimensional, then its H-module structure is determined by an algebra homomorphism $H \to \Bbbk$.

Definition 4.4.1. Assume that H is Artin-Schelter Gorenstein in dimension d.

(1) $\mathrm{Ext}^d_H({}_H\Bbbk, H)$ is called a *left homological integral* of H and is denoted by $\int_\ell$. By abuse of notation, $\int_\ell$ also denotes the algebra homomorphism $H \to \Bbbk$ such that, for all $h \in H$ and $e \in \mathrm{Ext}^d_H({}_H\Bbbk, H)$,

$$e \leftharpoonup h = \int_\ell (h) e \,.$$

(2) The *right homological integral* $\int_r$ of H is defined analogously by considering $\mathrm{Ext}^d_{H^{\mathrm{op}}}(\Bbbk_H, H)$ instead of $\mathrm{Ext}^d_H({}_H\Bbbk, H)$.

Homological integrals may be used to express Nakayama automorphisms of H. The expression uses winding automorphisms.

Definition 4.4.2 ([BZ08])**.** Let $\pi\colon H \to \Bbbk$ be an algebra homomorphism.

(1) The *left winding* automorphism associated to π is the automorphism of H denoted by Ξ^ℓ_π and defined such that, for all $h \in H$,
$$\Xi^\ell_\pi(h) = \pi(h_1)h_2\,.$$
(2) The *right winding* automorphism of π is the automorphism of H denoted by Ξ^r_π and defined such that, for all $h \in H$,
$$\Xi^r_\pi(h) = h_1\pi(h_2)\,.$$

Actually, $(\Xi^\bullet_\pi)^{-1} = \Xi^\bullet_{\pi\circ S}$ for all $\bullet \in \{\ell, r\}$. All winding automorphisms commute with S^2. Brown and Zhang proved the following result.

Theorem 4.4.3 ([BZ08])**.** *Assume that H is Noetherian and S is invertible.*

(1) If H is Artin-Schelter Gorenstein in dimension d, then $\Sigma^d\left({}^{S^2\circ\Xi^\ell_{\int_\ell}}H\right)$ is a rigid dualising complex of H.
(2) If, in addition, gl.dim. $H < \infty$*, then H is skew Calabi-Yau in dimension d and $S^{-2}\circ\Xi^\ell_{\int_\ell\circ S}$ is a Nakayama automorphism of H.*

4.5. **Smash products.**

Definition 4.5.1. Let H be a Hopf algebra. An *H-module* dg algebra is a dg algebra A endowed with a morphism of complexes $H \otimes A \to A$, $h \otimes a \mapsto h \rightharpoonup a$ such that, for all $a, b \in A$ and $h, k \in H$,
$$\begin{aligned} h \rightharpoonup (ab) &= (h_1 \rightharpoonup a)(h_2 \rightharpoonup b), & h \rightharpoonup 1_A &= \varepsilon(h),\\ (hk) \rightharpoonup a &= h \rightharpoonup (k \rightharpoonup a), & 1_H \rightharpoonup a &= a\,. \end{aligned}$$
The *smash product* dg algebra $A\sharp H$ is then the dg algebra whose underlying complex of vector spaces is $A \otimes H$ and whose product is such that $(a \otimes h) \times (b \otimes k) = a(h_1 \rightharpoonup b) \otimes h_2k$ for all $h, k \in H$ and $a, b \in A$. If $H = \Bbbk G$ for some finite group G, then $A\sharp H$ is called *skew group* dg algebra, it is denoted by $A * G$.

In the smash product, $A \otimes 1_H$ and $1_A \otimes H$ are dg subalgebras isomorphic to A and H respectively. In all places of this report, a and h are identified with the elements $a \otimes 1$ and $1 \otimes h$ of $A\sharp H$, for all $a \in A$ and $h \in H$.

4.6. **Equivariant dg modules.** In the framework of Calabi-Yau duality, equivariant bimodules have been introduced formally by Reyes, Rogalski, and Zhang. Consider a Hopf algebra and an H-module dg algebra A. Denote $A\sharp H$ by Λ.

Definition 4.6.1 ([RRZ14, Definition 2.2])**.** Let i be an integer. If i is negative, then assume that the antipode S is invertible. An *$H_{S^{2i}}$-equivariant* (or, H-equivariant when $i = 0$) dg A-bimodule is a dg A-bimodule M endowed with a morphism of complexes $H \otimes M \to M$, $h \otimes m \mapsto h \rightharpoonup m$ such that

- M is a left dg H-module,
- for all $m \in M$, $h \in H$, and $a, b \in A$,

$$h \rightharpoonup (amb) = (h_1 \rightharpoonup a)(h_2 \rightharpoonup m)(S^{2i}(h_3) \rightharpoonup b)\,. \tag{4.6.0.1}$$

For instance, A is an H-equivariant dg A-bimodule. If $H = \Bbbk G$ for some group G, then being H-equivariant is also said being G-equivariant. Equivariant dg A-bimodules yield dg Λ-bimodules. Here is the needed data for their definition.

Definition 4.6.2. An *equivariant triple* is a triple (i, M, σ) where

- i is an integer, assuming that the antipode S is invertible if $i < 0$,
- M is an $H_{S^{2i}}$-equivariant dg A-bimodule,
- and $\sigma \in \mathrm{Aut}_{\Bbbk-\mathrm{alg}}(H)$ is an automorphism such that, for all $h \in H$,

$$\sigma(h)_1 \otimes \sigma(h)_2 = S^{2i}(h_1) \otimes \sigma(h_2)\,, \tag{4.6.0.2}$$

Every equivariant triple defines a dg Λ-bimodule as follows.

Definition 4.6.3 ([RRZ14, Lemma 2.5])**.** Let (i, M, σ) be an equivariant triple. Denote by $M\sharp^{\sigma}H$ the dg Λ-bimodule whose underlying complex of vector spaces is $M \otimes H$ and which is such that, for all $m \in M$, $a, b \in A$, and $h, k \in H$,

$$\begin{cases} a(m \otimes \ell)b = am(\ell_1 \rightharpoonup b) \otimes \ell_2 \\ h(m \otimes \ell)k = h_1 \rightharpoonup m \otimes \sigma(h_2)\ell k\,. \end{cases}$$

For instance, $\Lambda = A\sharp^{\mathrm{Id}_H}H$, taking $i = 0$. If $H = \Bbbk G$ for some group G, then $M\sharp^{\mathrm{Id}_H}H$ is denoted by $M * G$.

4.7. **Lie-Rinehart algebras.** Let R be a commutative ring. Lie-Rinehart algebras and their enveloping algebras where defined formally by Rinehart.

Definition 4.7.1 ([Rin63])**.** A *Lie-Rinehart* algebra over R is a couple (S, L) where S is a commutative R-algebra and L is a Lie R-algebra which is also a left S-module, endowed with a homomorphism of R-Lie algebras, where $\mathrm{Der}_R(S)$ is the space of R-linear derivations of S,

$$\begin{array}{ccc} L & \to & \mathrm{Der}_R(S) \\ \alpha & \mapsto & \partial_\alpha := \alpha(-) \end{array}$$

such that, for all $\alpha, \beta \in L$ and $s \in S$,

$$[\alpha, s\beta] = s[\alpha, \beta] + \alpha(s)\beta\,.$$

For instance, (R, L) is a Lie-Rinehart algebra for all R-Lie algebras L, for the zero action of L on R. So is $(S, \mathrm{Der}_R(S))$, where $\mathrm{Der}_R(S)$ acts naturally on S, for all commutative R-algebras S.

Given a Lie-Rinehart algebra (S, L) over R, its enveloping algebra was defined by Rinehart in [Rin63]. The following convenient reformulation is due to Huebschmann. Note that the action of L on S endows S with a structure of $\mathcal{U}_R(L)$-module R-algebra.

Definition 4.7.2 ([Hue90])**.** The *enveloping algebra* is the quotient R-algebra $(S\sharp\mathcal{U}_R(L))/J$ where J is the (right and automatically two-sided) ideal generated by $\{st \otimes \alpha - s \otimes t \cdot \alpha \mid s, t \in S,\ \alpha \in L\}$.

If $S = R$ and the action of L on S is the zero one, then this R-algebra is the universal enveloping algebra of L. If $L = \mathrm{Der}_R(S)$ and if L acts naturally on S, then this R-algebra is the R-algebra of differential operators of S.

A left module over this R-algebra is an S-module M endowed with a structure of left module over the R-Lie algebra L such that, for all $m \in M$, $\alpha \in L$, and $s \in S$,

$$\alpha \cdot (sm) = \alpha(s)m + s(\alpha m) \text{ and } s(\alpha \cdot m) = (s\alpha) \cdot m\,.$$

In particular, S is a left module over the enveloping algebra of (S, L). The right modules over the enveloping algebra are characterised by similar considerations.

The duality for Lie-Rinehart algebras was introduced by Huebschmann.

Definition 4.7.3 ([Hue99])**.** The Lie-Rinehart algebra (S, L) with enveloping algebra denoted by U is said to have *duality* in dimension d if there exists a right

U-module C, called the *dualising* module of (S, L), such that, for all left U-modules M, there is an isomorphism

$$\mathrm{Ext}^{\bullet}_U(S, M) \cong \mathrm{Tor}^U_{d-\bullet}(C, M)\,.$$

Huebschmann proved a general existence theorem of duality for (S, L) in [Hue99]. Denoting $-^{\vee}$ the functor $\mathrm{Hom}_S(-, S)$, then every $\alpha \in L$ acts on $\Lambda^{\bullet}_S L^{\vee}$ by the unique R-linear skew derivation

$$\lambda_\alpha \colon \Lambda^{\bullet}_S L^{\vee} \to \Lambda^{\bullet}_S L^{\vee}$$

such that, for all $s \in S$, $\varphi \in L^{\vee}$, and $\beta \in L$,

$$\lambda_\alpha(s) = \alpha(s) \text{ and } \lambda_\alpha(\varphi)(\beta) = \alpha(\varphi(\beta)) - \varphi([\alpha, \beta])\,.$$

If the S-module L is finitely generated and projective with constant rank d, then the S-module $\Lambda^d_S L^{\vee}$ is a right U-module for the action of L such that, for all $s \in S$, $\varphi \in L^{\vee}$, and $\beta \in L$,

$$\varphi \cdot \alpha = -\lambda_\alpha(\varphi)\,.$$

Theorem 4.7.4 ([Hue99]). *Let d be an integer. If L is finitely generated and projective with constant rank d as an S-module, then (S, L) has duality in dimension d with dualising module $\Lambda^d_S L^{\vee}$.*

5. Homological duality of smash products

Let H be a Hopf algebra and A be an H-module dg algebra. Denote $A\sharp H$ by Λ.

This section presents the results of [LM10a, LM19]. On one hand, [LM10a] studies when skew group algebras of Calabi-Yau dg algebras are Calabi-Yau with a view towards the generalised cluster categories and the higher representation theory of finite dimensional algebras. On the other hand, [LM19] extends part of the results of [LM10a] to smash products by Hopf algebras. As a byproduct it establishes fairly general results comparing the homological properties of $\Bbbk$-algebras and those of their smash products. These results are based on the description of $\mathrm{RHom}_{\Lambda^e}(\Lambda, \Lambda^e)$ in terms of $\mathrm{RHom}_{A^e}(A, A^e)$ and $\mathrm{RHom}_{H^e}(H, H^e)$. They are presented as follows.

- Section 5.1 presents properties of H regarding $\mathrm{RHom}_{H^e}(H, H^e)$ and the invertibility of the antipode.
- Section 5.2 presents dg algebras Δ_0, Δ_1 used to describe $\mathrm{RHom}_{\Lambda^e}(\Lambda, \Lambda^e)$.
- Section 5.3 presents a general description of $\mathrm{RHom}_{\Lambda^e}(\Lambda, \Lambda^e)$ on which the subsequent results are based.
- Section 5.4 presents the extent to which known constructions of Calabi-Yau dg algebras are compatible with taking smash products.
- Section 5.5 applies section 5.3 to determine when skew group dg algebras are Calabi-Yau and when smash products of $\Bbbk$-algebras have Van den Bergh duality, or are (skew) Calabi-Yau, or are Artin-Schelter Gorenstein.
- Section 5.6 presents tools to compare the generalised cluster categories of Calabi-Yau dg algebras such that one is a skew group dg algebra of the other.
- Section 5.7 applies section 5.4 to compare the behaviour of the higher representation type (finite or infinite) of finite dimensional algebras under taking skew group algebras.

5.1. Invertible antipodes in Van den Bergh duality.

Brown and Zhang made in [BZ08] a systematic study of Hopf algebras from the point of view of Van den Bergh and related dualities. This was followed by many works on this topic. Many of these are based on technical assumptions like H being Noetherian or having an invertible antipode and which are preferable to avoid when considering smash

products. This section aims at determining the extent to which these assumptions are related to the homological properties of H.

First, there is a relationship between $\mathrm{RHom}_{H^e}(H, H^e)$ and $\mathrm{RHom}_{H^{\mathrm{op}}}(\Bbbk_H, H)$. It involves an exact functor denoted by $\bullet \uparrow^{H^e}$,

$$\bullet \uparrow^{H^e} : \mathrm{Mod}(H) \to \mathrm{Mod}(H^e)\,.$$

For all $N \in \mathrm{Mod}(H)$, the H-bimodule $N \uparrow^{H^e}$ is the vector space $H \otimes N$ endowed with the actions of H such that, for all $h, k, \ell \in H$ and $n \in N$,

$$h(\ell \otimes n)k = S^2(h_1)\ell k \otimes (h_2 \rightharpoonup n)\,.$$

Proposition 5.1.1. *If $\Bbbk_H$ has a (possibly unbounded) resolution by finitely generated projectives, then $\mathrm{RHom}_{H^e}(H, H^e) \simeq \mathrm{RHom}_{H^{\mathrm{op}}}(\Bbbk_H, H) \uparrow^{H^e}$ in $\mathcal{D}(H^e)$.*

Bibliographical note. The isomorphism of Proposition 5.1.1 was proved for Noetherian Artin-Schelter Gorenstein Hopf algebras in [HVOZ10, Corollary 2.2] and previously in [BZ08, Section 4.5] assuming in addition that S is invertible.

Next, H being homologically smooth may be characterised in terms of $\Bbbk_H$.

Proposition 5.1.2. *H is homologically smooth if and only if $\Bbbk_H \in \mathrm{per}(H^{\mathrm{op}})$.*

Bibliographical note. When the antipode is invertible, Proposition 5.1.2 is a consequence of [Neg15, Lemma 1.3].

Using Propositions 5.1.1 and 5.1.2, it is possible to prove that the invertibility of the antipode is a consequence of having Van den Bergh duality.

Proposition 5.1.3. *If H has Van den Bergh duality, then its antipode is invertible.*

Bibliographical note. Proposition 5.1.3 was proved previously for Noetherian Calabi-Yau Hopf algebras in [HVOZ10, Theorem 2.3].

This property allows to refine Theorem 4.4.3 of Brown and Zhang.

Theorem 5.1.4. *Let H be a Hopf algebra. The following assertions are equivalent*

(i) H has Van den Bergh duality,
(ii) $\Bbbk_H \in \mathrm{per}(H^{\mathrm{op}})$, the antipode is invertible and the graded $\Bbbk$-vector space $\mathrm{Ext}^{\bullet}_{H^{\mathrm{op}}}(\Bbbk_H, H)$ is finite-dimensional and concentrated in one degree,
(iii) H is skew Calabi-Yau.

Under these conditions, $S^{-2} \circ \Xi^r_{\int_r \circ S}$ is a Nakayama automorphism of H. Hence, H is Calabi-Yau in dimension d if and only if the following assertions hold true.

(a) $\Bbbk_H \in \mathrm{per}(H^{\mathrm{op}})$.
(b) S^2 is an inner automorphism of H.
(c) For all integers i, the H-module $\mathrm{Ext}^i_{H^{\mathrm{op}}}(\Bbbk_H, H)$ is isomorphic to ${}_H\Bbbk$ if $i = d$ and is zero otherwise.

Bibliographical note. For Noetherian Hopf algebras with invertible antipodes, it is proved in [RRZ14, Lemma 1.3] that being skew Calabi-Yau is equivalent to being Artin-Schelter regular. Moreover, [HVOZ10, Theorem 2.3] shows that a Noetherian Hopf algebra is Calabi-Yau if and only if it is Artin-Schelter regular, its left homological integral is trivial, and the square of its antipode is an inner automorphism.

5.2. **The dg algebras Δ_i.** These algebras, which are not smash products in general, are used to describe $\mathrm{RHom}_{\Lambda^e}(\Lambda, \Lambda^e)$.

Definition 5.2.1. Let i be an integer. If i is negative, then assume that the antipode S is invertible. Define a dg algebra Δ_i as follows.

- Its underlying complex of vector spaces is $A^e \otimes H$.

- Its product is such that, for all $a, b \in A$ and $h, k \in H$, then

$$(a \otimes b \otimes h) \times (a' \otimes b' \otimes k) = (-1)^{\deg(b)\cdot\deg(a')} \cdot (a(h_1 \rightharpoonup a')) \otimes ((S^{2i}(h_3) \rightharpoonup b')b) \otimes h_2 k\,.$$

This does define a dg algebra. The canonical mapping $A^e \to \Delta_i$ is an injective homomorphism of dg algebras. If S is invertible, then so does the mapping $\Delta_i \to \Lambda^e$ given by $a \otimes b \otimes h \mapsto (a \otimes b) \times (h_1 \otimes S^{2i+1}(h_2))$.

Kaygun defined Δ_0 in [Kay07, Definition 3.1]. If $S^2 = \mathrm{Id}_H$, then Δ_i does not depend on i and is a smash product of A^e by H. If $H = \Bbbk G$ for some finite group G, then $\Delta_i = A^e * G$ for the natural *diagonal* action of G on A^e.

By restriction of scalars to A^e and to H, a left dg Δ_i-module is an $H_{S^{2i}}$-equivariant dg A-bimodule. All $H_{S^{2i}}$-equivariant dg A-bimodules arise in this way.

5.3. **Inverse dualising complexes of smash products.** Section 5 is based on answers to the following questions.

(1) When is Λ homologically smooth?
(2) How is $\mathrm{RHom}_{\Lambda^e}(\Lambda, \Lambda^e)$ related to $\mathrm{RHom}_{A^e}(A, A^e)$?

Answers to these questions are presented below in the following setting.

Setting 5.3.1. H is a Hopf algebra with invertible antipode and A is an H-module dg algebra.

These answers are based on the following facts.

- A is an H-equivariant dg A-bimodule and $\Lambda^e \otimes_{\Delta_0} A \simeq \Lambda$ in $\mathrm{Diff}(\Lambda^e)$.
- There exists $D_A \in \mathrm{Diff}(\Delta_1)$ such that $D_A \simeq \mathrm{RHom}_{A^e}(A, A^e)$ in $\mathcal{D}(A^e)$.

More precisely, since all cofibrant objects in $\mathrm{Diff}(\Delta_0)$ are cofibrant in $\mathrm{Diff}(A^e)$, then there exists a morphism $P \to A$ in $\mathrm{Diff}(\Delta_0)$ which is a quasi-isomorphism in $\mathrm{Diff}(A^e)$. Denote by D_A the following complex,

$$D_A := \mathrm{Hom}_{A^e}(P, A^e)\,.$$

It is isomorphic to $\mathrm{RHom}_{A^e}(A, A^e)$ in $\mathcal{D}(A^e)$ and it is an H_{S^2}-equivariant dg A-bimodule, and hence a left dg Δ_1-module, for the action of H such that, for all $f \in D_A$, $h \in H$, and $p \in P$,

$$(h \rightharpoonup f)(p) = S^2(h_3) \rightharpoonup \underbrace{f(S(h_2) \rightharpoonup p)}_{\in A \otimes A} \leftharpoonup S(h_1)\,,$$

where $a \leftharpoonup k$ denotes $S^{-1}(k) \rightharpoonup a$ for all $a \in A$ and $k \in H$.

Example 5.3.2. Assume that $A = \Bbbk[x_1, \dots, x_n]$ for some positive integer n and that H is the universal enveloping algebra of some finite dimensional Lie algebra $\mathfrak{g}$.

- For all $X \in \mathfrak{g}$, the action of X on A is a $\Bbbk$-linear derivation, denote this derivation by ∂_X.
- For all monomials $m := x_{j_1} \cdots x_{j_r}$, where $1 \leqslant j_1 \leqslant \cdots \leqslant j_r \leqslant n$, denote $\sum_{t=1}^{r} x_{j_1} \cdots x_{j_{t-1}} \otimes x_{j_t} \otimes x_{j_{t+1}} \cdots x_{j_r}$ symbolically by $\sum_i m_i' \otimes x_i \otimes m_i''$.

Take P to be the Koszul resolution of A as an A-bimodule,

$$P := \Lambda^{-\bullet}_{A^e}(A \otimes \Sigma V \otimes A) \to A\,.$$

It is a graded commutative A^e-algebra. For all $X \in \mathfrak{g}$, denote by ∂_X the unique derivation of P such that, for all $a, b \in A$ and $v \in V$,

- $\partial_X(a \otimes b) = \partial_X(a) \otimes b + a \otimes \partial_X(b)$ and
- $\partial_X(1 \otimes v \otimes 1) = \sum_i \partial_X(v)_i' \otimes x_i \otimes \partial_X(v)_i''$.

Then, P is uniquely a left dg Δ_0-module such that the action $X \rightharpoonup \bullet$ of X is given by ∂_X, for all $X \in \mathfrak{g}$. Now, $D_A := \mathrm{Hom}_{A^e}(P, A^e)$ is a left dg Δ_1-module for the action of H such that, for all $f \in D_A$ and $X \in \mathfrak{g}$, the cochain $X \rightharpoonup f$ is given by $\omega \mapsto \partial_X(f(\omega)) - f(\partial_X(\omega))$.

First, here is when Λ is homologically smooth.

Proposition 5.3.3. *Keep setting 5.3.1. If both A and H are homologically smooth, then so is Λ. If, moreover, $H = \Bbbk G$ for some finite group G whose order is not divisible by* $\mathrm{char}(\Bbbk)$*, then the converse holds true.*

Bibliographical note. When A is an algebra, the first assertion of Proposition 5.3.3 was proved in [LWZ12, Proposition 2.11].

Next, here is the description of $\mathrm{RHom}_{\Lambda^e}(\Lambda, \Lambda^e)$.

Proposition 5.3.4. *Keep setting 5.3.1. Assume that both A and H are homologically smooth. For all morphisms $P \to A$ in* $\mathrm{Diff}(\Delta_0)$ *which are cofibrant resolutions in* $\mathrm{Diff}(A^e)$ *and for all cofibrant resolutions $E_H \to \mathrm{RHom}_H({}_H\Bbbk, H)$ in* $\mathrm{Diff}(H^{\mathrm{op}})$*, there is an isomorphism in $\mathcal{D}(\Lambda^e)$,*

$$D_A \otimes H \otimes E_H \simeq \mathrm{RHom}_{\Lambda^e}(\Lambda, \Lambda^e)\,,$$

where $D_A = \mathrm{Hom}_{A^e}(P, A^e)$ and where the left-hand side is considered as a dg Λ-bimodule such that, for all $a, b \in A$, $h, k, \ell \in H$, $d \in D_A$, and $e \in E_H$,

$$\begin{array}{rcl} a(d \otimes \ell \otimes e)b & = & ad(\ell_1 \rightharpoonup b) \otimes \ell_2 \otimes e \\ h(d \otimes \ell \otimes e)k & = & h_1 \rightharpoonup d \otimes S^2(h_2)\ell k \otimes e \leftharpoonup S^{-1}(h_3)\,. \end{array}$$

In particular,

- *if H has Van den Bergh duality, say in dimension d, then the automorphism $\sigma := (S^{-2} \circ \Xi^r_{\int_\ell})^{-1}$ of H is such that $\sigma(h)_1 \otimes \sigma(h)_2 = S^2(h_1) \otimes \sigma(h_2)$ for all $h \in H$, and $\mathrm{RHom}_{\Lambda^e}(\Lambda, \Lambda^e) \simeq \Sigma^{-d} D_A \sharp^{(S^{-2} \circ \Xi^r_{\int_\ell})^{-1}} H$ in $\mathcal{D}(\Lambda^e)$;*
- *if $H = \Bbbk G$ for G a finite group whose order is not divisible by* $\mathrm{char}(\Bbbk)$*, then $\mathrm{RHom}_{\Lambda^e}(\Lambda, \Lambda^e) \simeq \mathrm{RHom}_{A^e}(A, A^e) * G$ in $\mathcal{D}(\Lambda^e)$.*

5.4. **Smash products and constructions of Calabi-Yau dg algebras.** Using section 5.3, it is possible to present general situations where taking smash products is an operation which commutes with taking a deformed Calabi-Yau completions.

5.4.1. *Deformed Calabi-Yau completions of A and of Λ.* Take the following setting.

Setting 5.4.1. H is a Hopf algebra, A is an H-module dg algebra, and n, d are integers such that

- A is homologically smooth and
- H has Van den Bergh duality in dimension d, denote by σ the automorphism $(S^{-2} \circ \Xi^r_{\int_\ell})^{-1}$ of H.

Moreover,

- let D_A be a cofibrant replacement of $\mathrm{RHom}_{A^e}(A, A^e)$ in $\mathrm{Diff}(\Delta_1)$ and take $\mathbf{\Pi}_n(A)$ to be equal to $T_A(\Sigma^{n-1} D_A)$ and
- $\Sigma^{-d} D_A \sharp^\sigma H$ is denoted by D_Λ, take $\mathbf{\Pi}_{n+d}(\Lambda)$ to be equal to $T_\Lambda(\Sigma^{n+d-1} D_\Lambda)$.

All this makes sense because, following Proposition 5.3.4, the dg Λ-bimodule D_Λ is a cofibrant replacement of $\mathrm{RHom}_{\Lambda^e}(\Lambda, \Lambda^e)$.

The description of $\mathbf{\Pi}_{n+d}(\Lambda)$ is based on the following well-defined construction, which is itself based on the study of tensor products of equivariant dg A-bimodules.

Definition 5.4.2. Let $(1, M, \sigma)$ be an equivariant triple. Assume that σ commutes with S^2. Define a dg algebra denoted by $T_A(M) \sharp^{\sigma^*} H$ as follows.

- Its underlying complex of vector spaces is $T_A(M) \otimes H$.
- $T_A(M) \otimes 1_H$ is a dg subalgebra of $T_A(M) \sharp^{\sigma^*} H$.
- $1_{T_A(M)} \otimes H$ is a dg subalgebra of $T_A(M) \sharp^{\sigma^*} H$.

- For all $h \in H$ and $m_1, \dots, m_n \in M$,

$$(1_A \otimes h) \times ((m_1 \otimes_A \cdots \otimes_A m_n) \otimes 1_H) = \\ ((h_1 \rightharpoonup m_1) \otimes_A (S^2(h_2) \rightharpoonup m_2) \otimes_A \cdots \otimes_A (S^{2(n-1)}(h_n) \rightharpoonup m_n)) \otimes \sigma^n(h_n)\,.$$

If $S^2 = \sigma = \mathrm{Id}_H$, then $T_A(M)\sharp^{\sigma^*}H$ is equal to a smash product of $T_A(M)$ by H. In general, $T_A(M)\sharp^{\sigma^*}H$ is not a smash product, but it is isomorphic to $T_A(M\sharp^{\sigma}H)$. Whence the following description of $\mathbf{\Pi}_{n+d}(\Lambda)$.

Proposition 5.4.3. *Keep setting 5.4.1.*

(1) The identity mappings $\Lambda \to A \otimes H$ and $\Sigma^{n+d-1}D_\Lambda \to \Sigma^{n-1}D_A\sharp^{\sigma}H$ extend to an isomorphism of dg algebras

$$\mathbf{\Pi}_{n+d}(\Lambda) \xrightarrow{\sim} \mathbf{\Pi}_n(A)\sharp^{\sigma^*}H\,.$$

(2) If H is Calabi-Yau and $S^2 = \mathrm{Id}_H$, then the actions of H on A and on $\Sigma^{n-1}D_A$ extend uniquely to a structure of H-module dg algebra on $\mathbf{\Pi}_n(A)$; for this structure the isomorphism of (1) is an isomorphism of dg algebras

$$\mathbf{\Pi}_{n+d}(\Lambda) \xrightarrow{\sim} \mathbf{\Pi}_n(A)\sharp H\,.$$

The statements of Proposition 5.4.3 behave well under deformation. For a precise statement, several preliminary remarks are needed. First, there are isomorphisms of vector spaces which describe the spaces used to deform $\mathbf{\Pi}_n(A)$ and $\mathbf{\Pi}_{n+d}(\Lambda)$, respectively,

$$\left\{\begin{array}{l} \mathrm{HH}_{n-2}(A) \simeq H^0\mathrm{Hom}_{A^e}(\Sigma^{n-1}D_A, \Sigma A) \\ \mathrm{HH}_{n+d-2}(\Lambda) \simeq H^0\mathrm{Hom}_{\Lambda^e}(\Sigma^{n+d-1}D_\Lambda, \Sigma\Lambda)\,. \end{array}\right. \tag{5.4.1.1}$$

Next, it is worth explaining how $\mathbf{\Pi}_n(A)\sharp^{\sigma^*}H$ is to be deformed. Its deformations are defined using $\mathrm{Hom}_{A^e}(D_A, \Lambda)$. The following mapping is a morphism of complexes,

$$\begin{array}{ccc} \mathrm{Hom}_{\Lambda^e}(D_A\sharp^{\sigma}H, \Lambda) & \to & \mathrm{Hom}_{A^e}(D_A, \Lambda) \\ c & \mapsto & c_{|D_A\otimes 1_H}\,. \end{array} \tag{5.4.1.2}$$

It induces an isomorphism from $\mathrm{Hom}_{\Lambda^e}(D_A\sharp^{\sigma}H, \Lambda)$ to the subcomplex of those $c \in \mathrm{Hom}_{A^e}(D_A, \Lambda)$ such that $\mathcal{H}_c$, where $\mathcal{H}_c$ is the following assertion,

$$\mathcal{H}_c : \text{"}(\forall d \in D_A)\ (\forall h \in H)\ \ c(h \rightharpoonup d) = h_1 c(d) \int_\ell (h_2)S^3(h_3)\text{"}\,. \tag{5.4.1.3}$$

Hence, (5.4.1.1) and (5.4.1.2) yield an isomorphism of vector spaces,

$$\mathrm{HH}_{n+d-2}(\Lambda) \xrightarrow{\sim} H^0\{c \in \mathrm{Hom}_{A^e}(\Sigma^{n-1}D_A, \Sigma\Lambda) \mid \mathcal{H}_c\}\,. \tag{5.4.1.4}$$

This explains why the first part in the following definition makes sense, and so does the second one which introduces the deformations of $\mathbf{\Pi}_n(A)\sharp^{\sigma^*}H$ to consider here.

Definition 5.4.4. Let $\alpha \in \mathrm{HH}_{n+d-2}(\Lambda)$. Let $c \in Z^0\mathrm{Hom}_{A^e}(\Sigma^{n-1}D_A, \Sigma\Lambda)$ be such that $\mathcal{H}_c$.

(1) If the image of α under (5.4.1.4) is equal to the cohomology class of c, then c is said to *represent* α or to be a *representative* of α.

(2) Define ∂_c to be the unique square-zero skew derivation of the underlying graded algebra of $\mathbf{\Pi}_n(A)\sharp^{\sigma^*}H$ such that
   (a) the restriction to $A\sharp H$ $(= \Lambda)$ is equal to the differential of Λ and
   (b) the restriction to $\Sigma^{n-1}D_A\sharp^{\sigma}H$ takes its values in $(\Sigma A\sharp H)\oplus(\Sigma^n D_A\sharp^{\sigma}H)$ and is such that
      - the component $\Sigma^{n-1}D_A\sharp^{\sigma}H \to \Sigma A\sharp H$ $(= \Sigma\Lambda)$ is given by $d\otimes h \mapsto c(d)h$ and
      - the component $\Sigma^{n-1}D_A\sharp^{\sigma}H \to \Sigma^n D_A\sharp^{\sigma}H$ is the differential of $\Sigma^{n+d-1}D_\Lambda$ $(= \Sigma^{n-1}D_A\sharp^{\sigma}H)$.

Using this construction, it is possible to deduce from Proposition 5.4.3 the following description of the deformed Calabi-Yau completions of Λ.

Proposition 5.4.5. *Keep setting 5.4.1. Let $\alpha \in \mathrm{HH}_{n+d-2}(\Lambda)$. For all representatives $c \in Z^0\mathrm{Hom}_{A^e}(D_A, \Sigma\Lambda)$ of α (such that $\mathcal{H}_c$), the identity mappings $\Lambda \to A\sharp H$ and $\Sigma^{n+d-1}D_\Lambda \to \Sigma^{n-1}D_A\sharp^\sigma H$ extend to an isomorphism of dg algebras,*

$$\mathbf{\Pi}_{n+d}(A\sharp H, \alpha) \xrightarrow{\sim} (\mathbf{\Pi}_n(A)\sharp^{\sigma^*} H, \partial_c)\,. \tag{5.4.1.5}$$

Remark 5.4.6. In Proposition 5.4.5, assume that H is Calabi-Yau, $S^2 = \mathrm{Id}_H$, and c takes its values in ΣA, that is, $c \in Z^0\mathrm{Hom}_{A^e}(\Sigma^{n-1}D_A, \Sigma A)$. Hence, c defines a Hochschild homology class in $\mathrm{HH}_{n-2}(A)$, denote it by φ. For such a c, the assertion $\mathcal{H}_c$ is equivalent to $c\colon \Sigma^{n-1}D_A \to \Sigma A$ being H-linear. The actions of H on A and on D_A extend uniquely to a structure of H-module dg algebra on $\mathbf{\Pi}_n(A, \varphi)$ and the isomorphism (5.4.1.5) is actually as follows,

$$\mathbf{\Pi}_{n+d}(A\sharp H, \alpha) \xrightarrow{\sim} \mathbf{\Pi}_n(A, \varphi)\sharp H\,.$$

If, in addition, $H = \Bbbk G$ for some finite group G whose order is not divisible by $\mathrm{char}(\Bbbk)$, then $d = 0$ and $\Bbbk G$ is semi-simple. This is an even more particular case in which Proposition 5.4.5 may be reformulated as follows. For all $\varphi \in \mathrm{HH}_{n-2}(A)^G$, the actions of G on A and on D_A extend to an action on $\mathbf{\Pi}_n(A, \varphi)$ by dg automorphisms and the identity mappings $\Lambda \to A \otimes H$ and $\Sigma^{n-1}D_\Lambda \to \Sigma^{n-1}D_A \otimes H$ extend to an isomorphism of dg algebras,

$$\mathbf{\Pi}_n(A * G, \alpha) \xrightarrow{\sim} \mathbf{\Pi}_n(A, \varphi) * G\,,$$

α being the image of φ under the natural mapping $\mathrm{HH}_{n-2}(A) \to \mathrm{HH}_{n-2}(\Lambda)$.

5.4.2. *Skew group algebras of Ginzburg dg algebras.* The results of section 5.4.1 apply to skew group algebras of Ginzburg dg algebras. Actually, the resulting statements can be refined. Take the following setting.

Setting 5.4.7. Let Q, G, n, and W be as follows.

- Q is a finite graded $\Bbbk$-quiver.
- G is a finite group with order not divisible by $\mathrm{char}(\Bbbk)$ and acting on $\Bbbk Q$ by graded algebra automorphisms in such a way that the set Q_0 of vertices and the vector space $\Bbbk Q_1$ generated by the arrows of Q are stable under this action.
- $n \in \mathbb{Z}$ and W is a potential on Q which is homogeneous of degree $n-3$ and invariant under the induced action of G on the space of potentials of Q.

With this setting, the following cyclic derivation mapping is G-equivariant,

$$\begin{array}{ccc} \Bbbk Q_1 & \longrightarrow & \Bbbk Q \\ a & \longmapsto & \partial_a W\,. \end{array}$$

In the very particular case where the set of arrows Q_1 is stable under the action of G, this is straightforward to check. In general, this may be proved using the non commutative differential calculus developed by Crawley-Boevey, Etingof, Ginzburg, and Van den Bergh ([CBEG07]). Indeed, it is possible to prove that all the operations of this calculus are compatible with actions of groups; and this includes the reduced contraction from which the cyclic derivation of potentials may be recovered. Hence, the action of G on $\Bbbk Q$ extends uniquely to an action on the Ginzburg dg algebra $\mathcal{A}(Q, W)$ by dg automorphisms.

On one hand, Remark 5.4.6 and Keller's interpretation of Ginzburg dg algebras as deformed Calabi-Yau completions yields a quasi-isomorphism of dg algebras $\mathcal{A}(Q, W) * G \to \mathbf{\Pi}_n(\Bbbk Q * G, \alpha)$, where α is the image of W under the composition of Connes' boundary $\mathrm{HH}_{n-3}(\Bbbk Q) \to \mathrm{HH}_{n-2}(\Bbbk Q)$ with the natural mapping $\mathrm{HH}_{n-2}(\Bbbk Q) \to \mathrm{HH}_{n-2}(\Bbbk Q * G)$.

On the other hand, Demonet determined in [Dem10] a graded $\Bbbk$-quiver Q' and a homomorphism of graded algebras $\Bbbk Q' \to \Bbbk Q * G$ whose restriction-of-scalars functors is an equivalence from $\mathrm{Diff}(\Bbbk Q * G)$ to $\mathrm{Diff}(\Bbbk Q')$. Since there exists a potential W' homogeneous of degree $n-3$ on Q' whose image under the composition of Connes' boundary $\mathrm{HH}_{n-3}(\Bbbk Q') \to \mathrm{HH}_{n-2}(\Bbbk Q')$ with the natural isomorphism $\mathrm{HH}_{n-2}(\Bbbk Q') \to \mathrm{HH}_{n-2}(\Bbbk Q * G)$ is equal to α, then the dg algebra $\mathbf{\Pi}_n(\Bbbk Q * G, \alpha)$ is quasi-isomorphic to $\mathcal{A}(Q', W')$. This yields the following result.

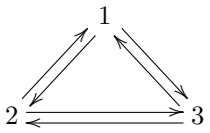
Corollary 5.4.8. *Keep setting 5.4.7. For all quivers Q' and for all potentials W' such as in the previous discussion, the graded algebra homomorphism $\Bbbk Q' \to \Bbbk Q * G$ extends to a (non unital) dg algebra homomorphism*

$$\mathcal{A}(Q', W') \to \mathcal{A}(Q, W) * G$$

whose image is equal to $e(\mathcal{A}(Q, W) * G)e$ for some idempotent $e \in \Bbbk Q$ and whose restriction-of-scalars functor is an equivalence

$$\mathrm{Diff}(\mathcal{A}(Q, W) * G) \xrightarrow{\sim} \mathrm{Diff}(\mathcal{A}(Q', W'))\,.$$

Example 5.4.9. Assume that (Q, W) is the following quiver with potential



$$W = x_{12}x_{23}x_{31} - x_{13}x_{32}x_{21}\,,$$

where $x_{i,j}$ denotes the unique arrow from i to j, for all distinct vertices i, j. Let G be the symmetric group on $\{1, 2, 3\}$. It acts on $\Bbbk Q$ as follows.

- The action on the set of vertices is the natural one.
- ${}^{\sigma}x_{i,j} = (-1)^{\sigma} x_{\sigma(i)\sigma(j)}$ for all $\sigma \in G$ and $i, j \in \{1, 2, 3\}$, where $(-1)^{\sigma}$ is the sign of the permutation σ.

Note that W is invariant for this action. Actually, $W = \frac{1}{3}\sum_{\sigma \in G} {}^{\sigma}(x_{12}x_{23}x_{31})$. Apply Corollary 5.4.8. The quiver Q' is

and W' is twice the sum of all the oriented cycles of length 3 of Q'.

5.5. **Homological duality of smash products.** Section 5.3 has several consequences regarding criterions for a smash product to have a homological duality. This section presents these consequences under two settings,

- for skew group algebras of dg algebras and
- for smash products of algebras by Hopf algebras with Van den Bergh duality.

5.5.1. *When is a skew group dg algebra Calabi-Yau?* It is assumed here that $H = \Bbbk G$ for some finite group G whose order is not divisible by $\mathrm{char}(\Bbbk)$. Note that, in Proposition 5.4.3, Remark 5.4.6, and Corollary 5.4.8, the respective dg algebras $\mathbf{\Pi}_n(A) * G$, $\mathbf{\Pi}_n(A, \alpha) * G$ and $\mathcal{A}(Q, W) * G$ are Calabi-Yau in dimension n. These facts are particular cases of the following consequence of Proposition 5.3.4.

Theorem 5.5.1. *Let G be a finite group acting on a dg algebra A by dg automorphisms. Assume that* $\mathrm{char}(\Bbbk)$ *does not divide the order of G. Let n be an integer.*

(1) If A is homologically smooth and $\Sigma^n \mathrm{RHom}_{A^e}(A, A^e) \simeq A$ in $\mathcal{D}(A^e * G)$, then A and $A * G$ are Calabi-Yau in dimension n.

(2) Assume that $\mathrm{HH}^0(A)$ is a local algebra. If $A * G$ is Calabi-Yau in dimension n, then A is homologically smooth and there exists $g \in G$ such that $\Sigma^n \mathrm{RHom}_{A^e}(A, A^e) \simeq A^g$ in $\mathcal{D}(A^e)$.

5.5.2. *Duality for smash products of algebras by Hopf algebras with Van den Bergh duality.* Take the following setting.

Setting 5.5.2. Assume that H is a Hopf algebra with an invertible antipode, that A is an H-module algebra, and that both A and H are homologically smooth.

Given an equivariant triple (i, M, σ) such that M is concentrated in degree 0, the A-bimodule M is invertible if and only if so is the Λ-bimodule $M\sharp^{\sigma}H$. Hence, taking the cohomology in the isomorphism of Proposition 5.3.4 yields the following characterisation.

Theorem 5.5.3. *Keep setting 5.5.2. The following assertions are equivalent.*

(i) A and H have Van den Bergh duality,

(ii) Λ has Van den Bergh duality.

When these conditions are satisfied and n, d are the corresponding homological dimensions of A and H, respectively, then Λ has homological dimension $n+d$ and

$$\mathrm{Ext}^{n+d}_{\Lambda^e}(\Lambda, \Lambda^e) \simeq \mathrm{Ext}^{n}_{A^e}(A, A^e)\sharp^{(S^{-2}\circ\Xi^r_{\int_\ell})^{-1}}H\,.$$

Bibliographical note. It was proved in [Far05, Theorem 17] that, if H is Calabi-Yau and A has Van den Bergh duality, then Λ has Van den Berg duality.

Theorem 5.5.3 can be used to determine whether Λ is skew Calabi-Yau and to describe a Nakayama automorphism. The description uses a weak homological determinant which replaces the homological determinant when A is no longer connected graded.

Definition 5.5.4. Let H be a Hopf algebra with invertible antipode. Let A be an H-module algebra.

(1) The action of H on A is said to *have a weak homological determinant* when, for some integer n,
   - $\mathrm{Ext}^{i}_{A^e}(A, A^e)$ is zero for all $i \neq n$
   - and $\mathrm{Ext}^{n}_{A^e}(A, A^e) \simeq A$ as left A-modules.

(2) Assume that the action of H on A has a weak homological determinant. Let e_A be a free generator of the left A-module $\mathrm{Ext}^{n}_{A^e}(A, A^e)$. Denote by λ the mapping $H \to A$ such that, for all $h \in H$,
$$h \rightharpoonup e_A = \lambda(h)e_A\,.$$
Define the *weak homological determinant* of the action of H on A and associated to e_A as the following mapping denoted by whdet,
$$\begin{array}{rccl} \mathrm{whdet}\colon & H & \longrightarrow & A \\ & h & \longmapsto & S^{-2}(h_2) \rightharpoonup \lambda(S^{-3}(h_1))\,. \end{array}$$

(3) In the same setting, define the *winding mapping* of the action of H on A and associated to e_A as the following mapping denoted by θ_{whdet},
$$\begin{array}{rccl} \theta_{\mathrm{whdet}}\colon & H & \longrightarrow & \Lambda \\ & h & \longmapsto & \mathrm{whdet}(S^2(h_1))h_2\,. \end{array}$$

(4) The action of H on A is said to *have a homological determinant* if λ takes its values in $\Bbbk$ for at least one generator e_A, which implies that so does whdet.

Note that θ_{whdet} is an algebra homomorphism because, for all $h, k \in H$,
$$\mathrm{whdet}(hk) = \mathrm{whdet}(h_1)\,(S^{-2}(h_2) \rightharpoonup \mathrm{whdet}(k))\,.$$
The action of H on A has a weak homological determinant if A is skew Calabi-Yau. When there is one, whdet depends on the choice of e_A. When A is skew Calabi-Yau, it is more relevant to choose the same generator e_A for the weak homological

determinant and for the Nakayama automorphism. Finally, if A is connected graded and skew Calabi-Yau and if a generator e_A is chosen so that it is homogeneous, then whdet is equal to the homological determinant hdet and θ_{whdet} is equal to the winding automorphism $\Xi^{\ell}_{\mathrm{hdet}}$.

Example 5.5.5. This is the continuation of Example 5.3.2. Then, $\Bbbk[x_1, \dots, x_n]$ is Calabi-Yau, and hence has a weak homological determinant. Let $e_A \in \mathrm{Ext}^n_{A^e}(A, A^e)$ be represented by the cocycle mapping $1 \otimes x_1 \wedge \cdots \wedge x_n \otimes 1$ to $1 \otimes 1$, then

- whdet is given by $\mathrm{whdet}(1) = 1$ and $\mathrm{whdet}(X) = \mathrm{div}(\partial_X)$,
- and θ_{whdet} is given by $\theta_{\mathrm{whdet}}(1) = 1$ and $\theta_{\mathrm{whdet}}(X) = X + \mathrm{div}(\partial_X)$.

Here, div denotes the canonical divergence.

Theorem 5.5.3 yields the following characterisation, where, for given mappings $\alpha\colon A \to A$ and $\beta\colon H \to \Lambda$, the mapping $\alpha\sharp\beta\colon \Lambda \to \Lambda$ is defined by $a \otimes h \mapsto \alpha(a)\beta(h)$.

Corollary 5.5.6. *Keep setting 5.5.2.*

(1) If A and H are skew Calabi-Yau, then so is Λ.
(2) If Λ is skew Calabi-Yau, then so is H and the action of H on A has a weak homological determinant. If, moreover, a homological determinant exists, then A is skew Calabi-Yau.

In the setting of (1), then Λ admits as a Nakayama automorphism

$$\mu_\Lambda = \mu_A \sharp (\theta_{\mathrm{whdet}} \circ \mu_H)$$

where μ_A is a Nakayama automorphism of A, where $\mathrm{whdet}\colon H \to A$ *is an associated weak homological determinant, and $\mu_H = S^{-2} \circ \Xi^r_{\int_\ell}$.*

Bibliographical note. Part (1) as well as the computation of a Nakayama automorphism of Λ in Corollary 5.5.6 were done first in [RRZ14] in the case H is finite dimensional and A is Noetherian, connected graded, and Artin-Schelter regular. The proof given in [RRZ14] is based on the analysis of the local cohomology over A and over $A\sharp H$.

Example 5.5.7. This is the continuation of Example 5.5.5. The smash product of $\Bbbk[x_1, \dots, x_n]$ by the universal enveloping algebra of $\mathfrak{g}$ is hence skew Calabi-Yau in dimension $n + \dim(\mathfrak{g})$ and it has a Nakayama automorphism μ such that $\mu(a) = a$ for all $a \in \Bbbk[x_1, \dots, x_n]$ and $\mu(X) = X + \mathrm{div}(\partial_X) + \mathrm{Tr}(\mathrm{ad}_X)$ for all $X \in \mathfrak{g}$.

Example 5.5.8. Corollary 5.5.6 also applies to the computation of Nakayama automorphisms of smash products of the quantum plane $\mathbb{C}_q[x, y]$ by the quantum enveloping algebra $\mathcal{U}_q(\mathfrak{sl}_2)$, where $q \in \mathbb{C}^\times$ is not a root of unity. Indeed, $\mathbb{C}_q[x, y]$ is skew Calabi-Yau in dimension 2, with a Nakayama automorphism μ_A such that $\mu_A(x) = q^{-1}x$ and $\mu_A(y) = qy$ ([Neg15]) and $\mathcal{U}_q(\mathfrak{sl}_2)$ is Calabi-Yau in dimension 2 ([Che99]). The possible actions of $\mathcal{U}_q(\mathfrak{sl}_2)$ are classified in [DS10] into six families, see Table 1. Although $\mathbb{C}_q[x, y]$ is connected graded, not all of these actions preserve the grading. Following Corollary 5.5.6, all smash products $\mathbb{C}_q[x, y]\sharp\mathcal{U}_q(\mathfrak{sl}_2)$ are skew Calabi-Yau in dimension 4. Table 2 describes a Nakayama automorphism of these smash products. There is a homological determinant in cases 0 and 5 only.

Since a skew Calabi-Yau algebra is Calabi-Yau if and only if a Nakayama automorphism is inner, Corollary 5.5.6 applies to prove the following characterisation.

Corollary 5.5.9. *Keep setting 5.5.2. Assume that A is a connected graded and that the grading is preserved by the action of H.*

(1) Assume that H is Calabi-Yau. Let $h_0 \in H^\times$ be such that S^{-2} is the inner automorphism of h_0. The algebra $A\sharp H$ is Calabi-Yau if and only if A is skew Calabi-Yau, $\mathrm{hdet} = \varepsilon$, *and there exists $k_A \in Z(H)^\times$ such that $\mu_A(a) = (h_0 k_A) \rightharpoonup a$ for all $a \in A$.*

case	actions of E, F, K on x, y	parameters
0	$K \rightharpoonup x = \pm x$ $\quad K \rightharpoonup y = \pm y$ $E \rightharpoonup x = 0$ $\quad E \rightharpoonup y = 0$ $F \rightharpoonup x = 0$ $\quad F \rightharpoonup y = 0$	$\emptyset$
1	$K \rightharpoonup x = qx$ $\quad K \rightharpoonup y = q^{-2}y$ $E \rightharpoonup x = 0$ $\quad E \rightharpoonup y = b_0$ $F \rightharpoonup x = b_0^{-1}xy$ $\quad F \rightharpoonup y = -qb_0^{-1}y^2$	$b_0 \in \mathbb{C}^\times$
2	$K \rightharpoonup x = q^2x$ $\quad K \rightharpoonup y = q^{-1}y$ $E \rightharpoonup x = -qc_0^{-1}x^2$ $\quad E \rightharpoonup y = c_0^{-1}xy$ $F \rightharpoonup x = c_0$ $\quad F \rightharpoonup y = 0$	$c_0 \in \mathbb{C}^\times$
3	$K \rightharpoonup x = q^{-2}x$ $\quad K \rightharpoonup y = q^{-1}y$ $E \rightharpoonup x = a_0$ $\quad E \rightharpoonup y = 0$ $F \rightharpoonup x = -qa_0^{-1}x^2 + ty^4$ $\quad F \rightharpoonup y = -qa_0^{-1}xy + sy^3$	$a_0 \in \mathbb{C}^\times$ $s, t \in \mathbb{C}$
4	$K \rightharpoonup x = qx$ $\quad K \rightharpoonup y = q^2y$ $E \rightharpoonup x = -qd_0^{-1}xy + sx^3$ $\quad E \rightharpoonup y = -qd_0^{-1}y^2 + tx^4$ $F \rightharpoonup x = 0$ $\quad F \rightharpoonup y = d_0$	$d_0 \in \mathbb{C}^\times$ $s, t \in \mathbb{C}$
5	$K \rightharpoonup x = qx$ $\quad K \rightharpoonup y = q^{-1}y$ $E \rightharpoonup x = 0$ $\quad E \rightharpoonup y = \tau x$ $F \rightharpoonup x = \tau^{-1}y$ $\quad F \rightharpoonup y = 0$	$\tau \in \mathbb{C}^\times$

TABLE 1. Classification ([DS10]) of the actions of $\mathcal{U}_q(\mathfrak{sl}_2)$ on $\mathbb{C}_q[x, y]$

Action of Table 1	Values of the Nakayama automorphism on K, E and F
case 0	$K \mapsto K$ $E \mapsto E$ $F \mapsto F$
case 1	$K \mapsto q^{-1}K$ $E \mapsto q^{-2}E$ $F \mapsto q^3F - b_0^{-1}qyK$
case 2	$K \mapsto qK$ $E \mapsto q^{-2}E - c_0^{-1}q^{-1}xK$ $F \mapsto qF$
case 3	$K \mapsto q^{-3}K$ $E \mapsto q^{-2}E$ $F \mapsto q^5F + s(1 + q^2 + q^4)y^2K - a_0^{-1}(q + q^3 + q^4)xK$
case 4	$K \mapsto q^3K$ $E \mapsto q^{-2}E + s(q^2 + 1 + q^{-2})x^2K - d_0^{-1}(q^2 + q + q^{-1})yK$ $F \mapsto q^{-1}F$
case 5	$K \mapsto K$ $E \mapsto q^{-2}E$ $F \mapsto q^2F$

TABLE 2. Nakayama automorphism of $\mathbb{C}_q[x, y]\sharp\mathcal{U}_q(\mathfrak{sl}_2)$

(2) *Assume that both A and H are Calabi-Yau. The algebra $A\sharp H$ is Calabi-Yau if and only if* $\mathrm{hdet} = \varepsilon$.

Bibliographical note. The equivalence of part (2) in Corollary 5.5.9 has been investigated in numerous situations. On one hand, it was proved to hold true

- in [WZ11] assuming that A is p-Koszul Calabi-Yau and that $H = \Bbbk G$, for any finite subgroup G of $\mathrm{Aut}_{\Bbbk-\mathrm{alg}}(A)$ with order not divisible by $\mathrm{car}(\Bbbk)$,
- in [LWZ12] assuming that A is p-Koszul and Calabi-Yau and that $S^2 = \mathrm{Id}_H$,
- in [HVOZ11, Corollary 3.4] assuming that A is p-Koszul and Calabi-Yau and $H = (\Bbbk G)^*$, for any finite group G.

On the other hand, it was proved in [HVOZ10, Theorem 3.4] that, for all finite dimensional Lie algebras $\mathfrak{g}$ and for all finite subgroups G of $\mathrm{GL}(\mathfrak{g})$ with order not divisible by $\mathrm{char}(\Bbbk)$, then $\mathcal{U}(\mathfrak{g}) * G$ is Calabi-Yau if and only $G \subseteq \mathrm{SL}(\mathfrak{g})$ and $\mathcal{U}(\mathfrak{g})$ is Calabi-Yau. Note that in this situation, $\mathcal{U}(\mathfrak{g})$ need not be a graded algebra.

Goodman and Krähmer proved in [GK14] that if an algebra A is skew Calabi-Yau, then the smash product $A\sharp\Bbbk Z$ defined by the action of a Nakayama automorphism of A is Calabi-Yau. Whence the following consequence of Corollary 5.5.9.

Corollary 5.5.10. *Let A be an augmented skew Calabi-Yau algebra with Nakayama automorphism μ_A. Then,*

(1) $p \circ \mathrm{whdet}(\mu_A) = 1$, where $p\colon A \to \Bbbk$ is the augmentation;
(2) if A is connected graded and μ_A is also graded, then $\mathrm{hdet}(\mu_A) = 1$.

Note that, in part (2), there always exists a graded Nakayama automorphism.

Bibliographical note. The equality in part (2) of Corollary 5.5.10 was conjectured to hold true for the algebras which are Noetherian, connected graded, and Artin-Schelter Gorenstein ([RRZ14, Conjecture 6.4]). It was proved in the following cases,

- in [RRZ14, Theorem 0.4], for Noetherian connected graded Koszul Artin-Schelter regular algebras;
- in [RRZ17, Corollary 5.4], for Noetherian connected graded Artin-Schelter Gorenstein algebras with at least one of the following properties,
  - the algebra is a graded twist of an algebra which is finite over its affine centre; or
  - the algebra is a quotient of a Noetherian Artin-Schelter regular algebra;
- in [MS16, Theorem 1.6], for m-Koszul Artin-Schelter regular algebras;
- in [CKS17, Theorem 3.11], for a class of 4 dimensional connected graded Artin-Schelter regular algebras constructed as normal extensions of 3 dimensional ones.

This section ends with sufficient conditions for Λ to be Artin-Schelter Gorenstein. For this to make sense, A is assumed to be augmented and the action of H on A is assumed to respect the augmentation ideal. The following proposition is independent of the preceding results. Rather it completes them. It is proved using a spectral sequence similar to the one used by Stefan in [Şte95, Theorem 3.3]. More precisely, for all $M \in \mathrm{Mod}(\Lambda)$, there is a natural left H-module structure on $\mathrm{Ext}^q_A(M, \Lambda)$, for all q, in such a way that there is a spectral sequence

$$\mathrm{Ext}^p_H({}_H\Bbbk, \mathrm{Ext}^q_A(M, \Lambda)) \Rightarrow \mathrm{Ext}^{p+q}_\Lambda(M, \Lambda)\,.$$

Combining it with the results of section 5.1 yields the proposition.

Proposition 5.5.11. *Let H be a Hopf algebra. Let A be an augmented H-module algebra that is moreover Noetherian. Assume that A is Artin-Schelter Gorenstein in dimension n.*

(1) If $\mathrm{gl.dim.}\, A < \infty$ and H has Van den Bergh duality in dimension d, then Λ is Artin-Schelter regular in dimension $n + d$.
(2) If $\dim_\Bbbk H < \infty$, then Λ is Artin-Schelter Gorenstein in dimension n.
(3) If H has Van den Bergh duality in dimension d, then $A\sharp H$ is Artin-Schelter Gorenstein in dimension $n + d$.

Bibliographical note. Part (2) of Proposition 5.5.11 was proved in [RRZ14, Theorem 4.1] assuming, in addition, that A is connected graded. It was also proved in [HVOZ11, Proposition 3.8] assuming that A is an Artin-Schelter Gorenstein dg algebra concentrated in non positive degrees and such that $A^0 = \Bbbk$.

5.6. **Application to generalised cluster categories.** Section 5.5.1 provides situations where the action of a finite group on a Calabi-Yau dg algebra yields a Calabi-Yau skew group dg algebra. The present section compares the associated generalised cluster categories in the following setting.

Setting 5.6.1. Let A be a dg algebra. Assume that A is Calabi-Yau in dimension 3, that it is concentrated in non positive degrees, and that $\dim(H^0(A)) < \infty$. Let G be a finite group acting on A by dg automorphisms. Assume that the order of G is not divisible by $\mathrm{char}(\Bbbk)$ and that $A * G$ is Calabi-Yau. The neutral element of G is denoted by e.

- Section 5.6.1 is a reminder on generalised cluster categories.
- Section 5.6.2 presents a biadjoint pair of functors between the generalised cluster categories of A and of $A * G$ and which are used in the subsequent sections.
- Section 5.6.3 presents a correspondence between the cluster tilting objects of these categories.
- Section 5.6.4 presents some consequences in terms of acyclicity.

In the unpublished version [LM07a] of [LM11], a similar study is made in a slightly different context where the generalised cluster category of $A * G$ is replaced by the cluster category of a finite dimensional algebra R derived equivalent to the path algebra of a finite quiver and where the generalised cluster category of A is replaced by the cluster category of any Galois covering of R.

5.6.1. *Reminder on generalised cluster categories and cluster tilting objects.* Following [Ami09] the generalised cluster categories of A and $A * G$ are, respectively,

$$\begin{aligned} \mathcal{C}_A &= \mathrm{per}(A)/\mathcal{D}_{\mathrm{fd}}(A)\,, \\ \mathcal{C}_{A*G} &= \mathrm{per}(A*G)/\mathcal{D}_{\mathrm{fd}}(A*G)\,. \end{aligned}$$

They are Hom-finite, Krull-Schmidt and Calabi-Yau in dimension 2, that is, the square of their suspensions are Serre functors. Denote by π_A and π_{A*G} the canonical functors

$$\begin{aligned} \pi_A &\colon \mathrm{per}(A) \to \mathcal{C}_A\,, \\ \pi_{A*G} &\colon \mathrm{per}(A*G) \to \mathcal{C}_{A*G}\,. \end{aligned}$$

Denote by $\overline{A}$ and $\overline{A*G}$ the objects $\pi_A(A)$ and $\pi_{A*G}(A*G)$, respectively. For simplicity, given $T \in \mathcal{C}_A$ (or $T \in \mathcal{C}_{A*G}$), its endomorphism algebra is denoted by $\mathrm{End}(T)$.

A *cluster tilting subcategory* of $\mathcal{C}_A$ is a full and functorially finite subcategory $\mathcal{T}$ such that

$$\mathcal{T} \quad = \quad \{X \in \mathcal{C}_A \mid (\forall M \in \mathcal{T})\ \mathcal{C}_A(X, \Sigma M) = 0\}\,,$$

and a *cluster tilting object* of $\mathcal{C}_A$ is an object $T \in \mathcal{C}_A$ such that its closure $\mathrm{add}(T)$ under direct sums and direct summands is a cluster tilting subcategory of $\mathcal{C}_A$. Recall also that a *maximal rigid subcategory* of $\mathcal{C}_A$ is a full and functorially finite subcategory $\mathcal{T}$ of $\mathcal{C}_A$ such that

- $\mathcal{T}$ is rigid, that is, $\mathcal{C}_A(X, \Sigma Y) = 0$ for all $X, Y \in \mathcal{T}$
- and there is no rigid functorially finite full subcategory of $\mathcal{C}_A$ which contains $\mathcal{T}$ as a proper subcategory.

A *maximal rigid object* of $\mathcal{C}_A$ is an object T such that $\mathrm{add}(T)$ is a maximal rigid subcategory of $\mathcal{T}$. The same concepts are defined for $\mathcal{C}_{A*G}$.

In particular, it is proved in [Ami09] that $\overline{A}$ and $\overline{A*G}$ are cluster tilting objects of $\mathcal{C}_A$ and $\mathcal{C}_{A*G}$, respectively.

5.6.2. *Functors between $\mathcal{C}_A$ and $\mathcal{C}_{A*G}$.* For all $g \in G$, the restriction-of-scalars-functor $\mathcal{D}(A) \to \mathcal{D}(A)$ along the action $A \to A$, $a \mapsto {}^g a$ of g on A induces a strict automorphism of the triangulated category $\mathcal{C}_A$. This defines a strict action $(g, M) \mapsto M^g$ of G on $\mathcal{C}_A$ on the right by strict automorphisms of triangulated category. In particular, for all $M \in \mathcal{C}_A$, the stabiliser $\{g \in G \mid M \simeq M^g\}$ is denoted by G_M.

The restriction-of-scalars functor $\mathcal{D}(A * G) \to \mathcal{D}(A)$ along the dg algebra homomorphism $A \to A * G$, $a \mapsto a \otimes e$ induces a triangle functor $\mathcal{C}_{A*G} \to \mathcal{C}_A$. This functor is denoted by $F_.$.

The extension-of-scalars functor $\mathcal{D}(A) \to \mathcal{D}(A * G)$ along the same homomorphism induces a triangle functor $\mathcal{C}_A \to \mathcal{C}_{A*G}$. This functor is denoted by F_λ.

The comparison of the cluster tilting theories of $\mathcal{C}_A$ and $\mathcal{C}_{A*G}$ is based on the properties of $F_.$ and F_λ presented below. They are analogous to the ones of the extension- and restriction-of-scalars functors arising from skew group algebras or from Galois coverings of finite dimensional algebras ([Gab81, RR85]).

Proposition 5.6.2. *The functors $F_.$ and F_λ have the following properties.*

(1) $F_\lambda \overline{A} = \overline{A * G}$.
(2) The pairs $(F_., F_\lambda)$ and $(F_\lambda, F_.)$ are adjoint with split units.
(3) For all $M \in \mathcal{C}_A$, there is a functorial isomorphism $F_. F_\lambda M \simeq \oplus_{g \in G} M^g$.
(4) For all indecomposable objects $M \in \mathcal{C}_A$, if $G_M = \{e\}$, then $F_\lambda M$ is indecomposable.

5.6.3. *Comparison of cluster tilting objects.* The functors F_λ and $F_.$ yield the following correspondence between cluster tilting subcategories of $\mathcal{C}_A$ and $\mathcal{C}_{A*G}$.

Proposition 5.6.3. *The two following assignments*

$$\begin{cases} \mathcal{T} \mapsto \mathrm{add}(F_\lambda \mathcal{T}) \\ \mathcal{T}' \mapsto \mathrm{add}(F_. \mathcal{T}') \end{cases}$$

induce mutually inverse bijections between

(1) the cluster tilting subcategories $\mathcal{T}$ of $\mathcal{C}_A$ which are stable under the action of G, and
(2) the cluster tilting subcategories $\mathcal{T}'$ of $\mathcal{C}_{A*G}$ which are stable under $F_\lambda F_.$.

Bibliographical note. The correspondence of Proposition 5.6.3 is similar to the one of [Dem11, Section 3]. In this work a Hom-finite, Krull-Schmidt, and stably Calabi-Yau in dimension 2 Frobenius category is used instead of $\mathcal{C}_A$ and an equivariantisation of it is used instead of $\mathcal{C}_{A*G}$.

The endomorphism algebra of a cluster tilting object T, say of $\mathcal{C}_A$, has a representation theory equivalent to the factor category $\mathcal{C}_A/\mathrm{add}(\Sigma T)$ ([BMR07]), which makes it worth investigating the representation theory of such endomorphism algebras. In the framework of the correspondence of Proposition 5.6.3, the endomorphism algebras of corresponding cluster tilting objects are related as follows.

Proposition 5.6.4. *Let $\mathcal{T}$ be a cluster tilting subcategory of $\mathcal{C}_A$ stable under the action of G. Then there exists a cluster tilting object $T \in \mathcal{C}_A$ such that $\mathrm{add}(T) = \mathcal{T}$ and $T^g = T$ for all $g \in G$. Moreover, for any such T,*

- $\mathrm{add}(F_\lambda \mathcal{T}) = \mathrm{add}(F_\lambda T)$,
- *the action of G on $\mathcal{C}_A$ induces an action on the algebra $\mathrm{End}(T)$,*
- *and $\mathrm{End}(F_\lambda T)$ is Morita equivalent to $\mathrm{End}(T) * G$.*

In certain circumstances, the correspondence of Proposition 5.6.3 can be refined from the following point of view. Call an object of $\mathcal{C}_{A*G}$ of the *first kind* if all its indecomposable direct summands lie in the essential image of F_λ.

For instance, $F_\lambda \overline{A} = \overline{A * G}$. Hence, $\overline{A * G}$ is of the first kind if and only if the stabilisers of the indecomposable direct summands of $\overline{A}$ are all trivial, which is equivalent to the induced action of G on the isomorphism classes of indecomposable projective $H^0(A)$-modules being free. In the particular case where $A = \mathcal{A}(Q, W)$ with (Q, W) such as in setting 5.4.7, if W is a linear combination of oriented cycles

of length at least 3, this freeness condition is equivalent to the action of G on the set of vertices of Q being free.

It is possible to adapt to cluster tilting objects the tools developed for tilting objects in section 10.1 of part 3. As a consequence, a cluster tilting object of $\mathcal{C}_{A*G}$ is of the first kind if and only if any of its mutations is so. Recall that two cluster tilting objects T, T' are said to be *mutation* of one another when their respective sets S, S' of isomorphism classes of indecomposable direct summands are such that both $S \backslash S'$ and $S' \backslash S$ consist of exactly one element. With this definition, the *mutation class* of a cluster tilting object T is the transitive closure of $\{T\}$ under mutation and isomorphisms. These considerations yield a refinement of Proposition 5.6.4.

Proposition 5.6.5. *Assume that G acts freely on the isomorphism classes of indecomposable projective $H^0(A)$-modules. Then, for all basic cluster tilting objects T of $\mathcal{C}_{A*G}$ in the mutation class of $\overline{A * G}$,*

- *T is of the first kind and*
- *there exists a basic cluster tilting object $\overline{T}$ of $\mathcal{C}_A$ such that $\overline{T}^g = \overline{T}$ for all $g \in G$, such that $\mathrm{add}(T) = \mathrm{add}(F_\lambda \overline{T})$, and such that F_λ yields a Galois covering (see section 8.3) with group G from $\mathrm{End}(\overline{T})$ to $\mathrm{End}(T)$.*

Mutatis mutandis, Propositions 5.6.3, 5.6.4 and 5.6.5 still hold true when cluster tilting objects (or subcategories) are replaced by maximal rigid ones. The resulting statements yield the following characterisation of indecomposable rigid objects of $\mathcal{C}_{A*G}$ when its maximal rigid objects form a mutation class.

Corollary 5.6.6. *Assume that G acts freely on the isomorphism classes of indecomposable projective $H^0(A)$-modules and that the basic maximal rigid objects of $\mathcal{C}_{A*G}$ form a mutation class. Let $N \in \mathcal{C}_{A*G}$ be indecomposable. The following assertions are equivalent.*

(i) N is rigid, that is, $\mathcal{C}_{A*G}(N, \Sigma N) = 0$.
(ii) There exists a maximal rigid object $T \in \mathcal{C}_A$ such that $T^g \simeq T$ for all $g \in G$ and there exists an indecomposable direct summand M of T such that $N \simeq F_\lambda M$.

5.6.4. *Applications to acyclicity and representation type of cluster categories.* It is possible to compare $\mathcal{C}_A$ and $\mathcal{C}_{A*G}$ beyond a correspondence between their cluster tilting subcategories. Recall that a Hom-finite, Krull-Schmidt and Calabi-Yau in dimension 2 triangulated category is called *acyclic* when it is equivalent, as a triangulated category, to the cluster category of a finite quiver without oriented cycles, in the sense of [BMR$^+$06]. Following [KR08], this is the case if and only if the triangulated category is algebraic and contains a cluster tilting object whose endomorphism algebra is hereditary.

Following [Ami07, Corollary 6.6], the category $\mathcal{C}_A$ (or $\mathcal{C}_{A*G}$) has finitely many isomorphism classes of indecomposable objects if and only if it is equivalent (as a triangulated category) to the cluster category of a Dynkin quiver. This fact and the splitting of the unit of the adjoint pair $(F_\lambda, F_\cdot)$ entail the following result.

Corollary 5.6.7. *The following assertions are equivalent.*

(i) $\mathcal{C}_A$ is equivalent to the cluster category of a Dynkin quiver.
(ii) $\mathcal{C}_{A*G}$ is equivalent to the cluster category of a Dynkin quiver.

Bibliographical note. Corollary 5.6.7 was proved in [PS17, Proposition 7.2] assuming that A is the Ginzburg dg algebra of a quiver with potential (Q, W) and that the action of G on A is induced by a free action of G on Q by quiver automorphisms.

It is also possible to relate $\mathcal{C}_A$ being acyclic to $\mathcal{C}_{A*G}$ being acyclic.

On one hand, if $\mathcal{C}_A$ is acyclic and has infinitely many isomorphism classes of indecomposable objects, then, adapting ideas from [DLS09, Section 4.1], the unique

transjective component of the Auslander-Reiten quiver of $\mathcal{C}_A$ has a section S which is stable under the action of G; in particular, $\mathrm{End}(S)$ is hereditary. Hence, $\mathcal{T} := \mathrm{add}(S)$ satisfies the hypotheses of Proposition 5.6.4. Accordingly, $\mathcal{C}_{A*G}$ has a cluster tilting object whose endomorphism algebra is hereditary, and hence it is acyclic.

On the other hand, if $\mathcal{C}_{A*G}$ is acyclic, then its basic cluster tilting objects form a mutation class and at least one of them, say T, has a hereditary endomorphism algebra. If, in addition, G acts freely on the indecomposable projective $H^0(A)$-modules, then Proposition 5.6.5 applies to T and yields a cluster tilting object of $\mathcal{C}_A$ whose endomorphism algebra is hereditary. Therefore, $\mathcal{C}_A$ is acyclic.

Corollary 5.6.8. *If $\mathcal{C}_A$ is acyclic and has infinitely many isomorphism classes of indecomposable objects, then $\mathcal{C}_{A*G}$ is acyclic and has infinitely many isomorphism classes of indecomposable objects. The converse holds true if the action of G on the set of isomorphism classes of indecomposable projective $H^0(A)$-modules is free.*

5.7. **Applications to higher Auslander-Reiten theory.** The techniques and results of the previous sections apply to compare the higher representation theory of a finite dimensional algebra and the one of its skew group algebras.

Setting 5.7.1. Let n be a positive integer. Let A be a finite dimensional algebra of global dimension n. Let G be a finite group with order not divisible by $\mathrm{char}(\Bbbk)$ and acting on A by algebra automorphisms.

A finite dimensional algebra R of global dimension at most n is *n-representation finite* ([Iya11]) when its has an n-cluster tilting module, that is, when there exists $M \in \mathrm{mod}(R)$ such that

$$\begin{aligned} \mathrm{add}(M) &= \{N \in \mathrm{mod}(R) \mid (\forall i = 1, \dots, n-1)\ \mathrm{Ext}^i_R(M, N) = 0\} \\ &= \{N \in \mathrm{mod}(R) \mid (\forall i = 1, \dots, n-1)\ \mathrm{Ext}^i_R(N, M) = 0\}\,. \end{aligned}$$

When such an M exists and when R has global dimension n, then $\mathrm{add}(M)$ is necessarily equal to the closure under direct sums and direct summands of the orbit of the injective cogenerator $\mathrm{Hom}_{\Bbbk}(R, \Bbbk)$ under the higher Auslander-Reiten translation τ_n ([HI11]).

Since both the extension-of-scalars functor $\mathrm{mod}(A) \to \mathrm{mod}(A * G)$ and the restriction-of-scalars functor $\mathrm{mod}(A * G) \to \mathrm{mod}(A)$ commute with the higher Auslander-Reiten translations in $\mathrm{mod}(A)$ and $\mathrm{mod}(A * G)$, respectively, then the fact of being n-representation finite is invariant under taking skew group algebras.

Proposition 5.7.2. *Keep setting 5.7.1. The following assertions are equivalent.*

(1) A is n-representation finite.
(2) $A * G$ is n-representation finite.

The higher representation infinite algebras were defined in [HIO14]. A finite dimensional algebra R of global dimension n is *n-representation infinite* if and only if the cohomology of the Calabi-Yau completion $\mathbf{\Pi}_{n+1}(R)$ is concentrated in degree 0. In general, $\mathbf{\Pi}_{n+1}(R)$ is a dg algebra concentrated in non positive degrees, it is called the *derived $(n+1)$-preprojective algebra* of R, and its 0-th cohomology algebra is called the *$(n+1)$-preprojective algebra* of R and denoted by $\Pi_{n+1}(R)$. When $n = 1$ and R is the path algebra of a quiver, then the 2-preprojective algebra of R is isomorphic to the usual preprojective algebra of that quiver. Taking $H = \Bbbk G$ in Proposition 5.4.3 yields the following result.

Corollary 5.7.3. *Keep setting 5.7.1.*

(1) G acts on the derived $(n+1)$-preprojective algebra $\mathbf{\Pi}_{n+1}(A)$ by dg automorphisms and $\mathbf{\Pi}_{n+1}(A) * G$ is quasi-isomorphic to the derived $(n+1)$-preprojective algebra $\mathbf{\Pi}_{n+1}(A * G)$.

(2) *G acts on the $(n+1)$-preprojective algebra $\Pi_{n+1}(A)$ by algebra automorphisms and the skew group algebra $\Pi_{n+1}(A)*G$ is isomorphic to the $(n+1)$-preprojective algebra $\Pi_{n+1}(A*G)$.*
(3) *The following assertions are equivalent.*
  (i) *A is n-representation infinite.*
  (ii) *$A*G$ is n-representation infinite.*

Bibliographical note. For $n=1$, part (2) of Corollary 5.7.2 was proved in [Dem10].

5.8. **Open questions.**

5.8.1. *Skew Calabi-Yau smash product algebras.* Corollaries 5.5.6, 5.5.9, and 5.5.10 have seemingly incomplete conclusions and restrictive assumptions. Hence, it would be interesting to have complete answers to the following questions.

Question 1. Keep setting 5.5.2.

- Is A skew Calabi-Yau if $A\sharp H$ is so?
- Assuming that both A and H are Calabi-Yau, is it possible to characterise $A\sharp H$ being Calabi-Yau in terms of weak homological determinants?
- What is the value of $\mathrm{whdet}(\mu_A)$ when A is skew Calabi-Yau?

5.8.2. *Computation of weak homological determinants.* Let H be a skew Calabi-Yau Hopf algebra and A be a skew Calabi-Yau H-module algebra. The concept of weak homological determinant, whdet, is essential in the description of a Nakayama automorphism of $A\sharp H$ in Corollary 5.5.6. However, the computation of whdet from its definition is rather long because it requires describing the $\mathrm{Ext}^{\mathrm{top}}_{A^e}(A, A^e)$ as an H-equivariant A-bimodule.

Question 2. Is there a direct way to compute whdet? If A is Calabi-Yau in dimension 3 and given by generators and relations which derive from a potential, is it possible to express whdet in terms of the action of H on the generators of A?

5.8.3. *Skew group algebras of Ginzburg dg algebras.* This question is now answered, see below.

Theorem 5.4.8 gives a general setting under which a skew group dg algebra $\mathcal{A}(Q,W)*G$ of a Ginzburg dg algebra $\mathcal{A}(Q,W)$ is Morita equivalent to a Ginzburg dg algebra $\mathcal{A}(Q',W')$. The proof of this theorem gives a direct theoretical link between W and W'. But computing W' in examples is uneasy.

Question 3. Is it possible to characterise W' so as to make its computation easier?

This computation was done in specific cases.

- When $\Bbbk=\mathbb{C}$, Q consists of one loop and three vertices x, y, and z, and $W=xyz-xzy$ (so that the resulting Jacobian algebra is isomorphic to $\mathbb{C}[x,y,z]$), Ginzburg expressed W' in [Gin06] in terms of the tensor category of representations of G. This description was later extended by Bocklandt, Schedler, and Wemyss to the case where Q consists of one loop and n vertices $x_1,\dots,x_n$, and W is the following potential (see [BSW10])
$$\sum_{\sigma\in\mathscr{S}_n}\varepsilon(\sigma)x_{\sigma(1)}\cdots x_{\sigma(n)}\,.$$
- When G is of order 2, Amiot and Plamondon described W' in [AP17].
- When G is cyclic and each vertex of Q has a stabiliser which is either trivial or equal to G, Giovannini and Pasquali made a combinatorial description of W' in [GP19] under specific assumptions on W.

In [LM18] I made a description of W' in full generality. The quiver Q', which was described by Demonet in [Dem10], is determined by the representation theory of the stabilisers of the vertices of Q. Forming the group algebras of these stabilisers and next the direct product of these group algebras yields a finite-dimensional and semi-simple algebra A. The description of W' that I made expresses it as a linear combination of oriented cycles in Q' where the scalars of the linear combination are obtained by performing specific operations and pairings in the monoidal category $(\mathrm{mod}(A^e), \otimes_A)$ of finite-dimensional A-bimodules.

6. Homological duality of enveloping algebras of Lie-Rinehart algebras

Let R be a commutative ring. Let (S, L) be a Lie-Rinehart algebra over R. Denote by U the enveloping algebra of (S, L).

This section presents the results of [LLM18]. In a setting which ensures that the R-algebra S has Van den Bergh duality and that (S, L) has duality in the sense of Definition 4.7.3, these results prove that the R-algebra U has Van den Bergh duality and give an explicit description of its inverse dualising bimodule. With additional assumptions, it appears that U is skew Calabi-Yau and a Nakayama automorphism may be described explicitly.

In this section, S and U are considered as R-algebras. The field $\Bbbk$ used in the previous sections is replaced by R, in particular,

- S^e and U^e denote the R-algebras $S \otimes_R S^{\mathrm{op}}$ $(= S \otimes_R S)$ and $U \otimes_R U^{\mathrm{op}}$, respectively
- and having Van den Bergh duality or being skew Calabi-Yau is understood according to the definitions 4.2.1 and 4.2.2, replacing $\Bbbk$ by R.

The results are presented according to the following structure.

- Section 6.1 presents the material of differential calculus used to describe the inverse dualising bimodule of U when it exists.
- Section 6.2 presents the main result about the Van den Bergh duality of U.
- Section 6.3 presents the application to the case of Lie-Rinehart algebras arising from Poisson structures on S.

6.1. **Differential calculus and duality over arbitrary ground rings.** Since R is not assumed to be a field and S need not be projective in $\mathrm{Mod}(R)$, it is necessary to clarify or adapt certain aspects relative to Van den Bergh and Calabi-Yau dualities and to the differential calculus of Lie derivations and volume forms, which are used in the proofs and statements of the main theorems. Indeed, the bar construction need not yield a projective resolution of S in $\mathrm{Mod}(S^e)$. Take the following setting.

Setting 6.1.1. S is a commutative algebra over a commutative ring R.

Let $P^\bullet$ be an arbitrary projective resolution of S in $\mathrm{Mod}(S^e)$,

$$\cdots \to P^{-2} \to P^{-1} \to P^0 \xrightarrow{\varepsilon} S\,.$$

6.1.1. *Lie derivations.* These where defined on $\mathrm{Ext}^\bullet_{S^e}(S, S)$ and $\mathrm{Tor}^{S^e}_\bullet(S, S)$ by Rinehart in [Rin63] when S is projective in $\mathrm{Mod}(R)$. For algebras over fields, they are part of the Tamarkin-Tsygan calculus ([TT05]) on Hochschild (co)homology. Here, it is necessary to define Lie derivations on $\mathrm{Ext}^\bullet_{S^e}(S, S^e)$ and on $\mathrm{Tor}^{S^e}_n(S, S)$ under setting 6.1.1 in order to describe the inverse dualising complex of enveloping algebras of Lie-Rinehart algebras based on S.

Definition 6.1.2. Let $\partial \in \mathrm{Der}_R(S)$. A *derivation* of $P^\bullet$ relative to ∂ is a morphism of complexes of R-modules $\partial^\bullet \colon P^\bullet \to P^\bullet$ such that

- $\partial^\bullet(sps') = \partial(s)ps' + s\partial^\bullet(p)s' + sp\partial(s')$ for all $s, s' \in S$ and $p \in P^\bullet$ and
- up to the identification of $H^0(P^\bullet)$ with S by means of ε, then $H^0(\partial^\bullet) = \partial$.

Such derivations always exist. For all $\partial \in \mathrm{Der}_R(S)$ and for all derivations $\partial^\bullet$ of $P^\bullet$ relative to ∂, the following mappings are morphisms of complexes of R-modules and the induced mappings in cohomology do not depend the choice of $\partial^\bullet$,

$$\begin{array}{rcl} \mathrm{Hom}_{S^e}(P^\bullet, S^e) & \longrightarrow & \mathrm{Hom}_{S^e}(P^\bullet, S^e) \\ \psi & \longmapsto & (\partial \otimes \mathrm{Id}_S + \mathrm{Id}_S \otimes \partial) \circ \psi - \psi \circ \partial^\bullet \\ S \otimes_{S^e} P^\bullet & \longrightarrow & S \otimes_{S^e} P^\bullet \\ s \otimes p & \longmapsto & \partial(s) \otimes p + s \otimes \partial^\bullet(p)\,. \end{array} \tag{6.1.1.1}$$

Definition 6.1.3. For all $\partial \in \mathrm{Der}_R(S)$, the *Lie derivation* of ∂ on $\mathrm{Ext}^\bullet_{S^e}(S, S^e)$ (or, on $\mathrm{Tor}^{S^e}_\bullet(S, S)$) is the linear mapping $\mathrm{Ext}^\bullet_{S^e}(S, S^e) \to \mathrm{Ext}^\bullet_{S^e}(S, S^e)$ (or, the linear mapping $\mathrm{Tor}^{S^e}_\bullet(S, S) \to \mathrm{Tor}^{S^e}_\bullet(S, S)$, respectively) induced by (6.1.1.1) in cohomology. It is denoted by $\mathcal{L}_\partial$.

Proposition 6.1.4. *The actions of* $\mathrm{Der}_R(S)$ *on* $\mathrm{Ext}^\bullet_{S^e}(S, S^e)$ *and on* $\mathrm{Tor}^{S^e}_\bullet(S, S)$ *by Lie derivations are structures of left module over the Lie algebra* $\mathrm{Der}_R(S)$ *such that, for all* $\partial \in \mathrm{Der}_R(S)$, $s \in S$, $e \in \mathrm{Ext}^\bullet_{S^e}(S, S^e)$, *and* $\omega \in \mathrm{Tor}^{S^e}_\bullet(S, S)$,

$$\left\{ \begin{array}{l} \mathcal{L}_\partial(se) = \partial(s)e + s\mathcal{L}_\partial(e) \\ \mathcal{L}_{s\partial}(e) = s\mathcal{L}_\partial(e) - \partial(s)e \end{array} \right. \quad \textit{and} \quad \left\{ \begin{array}{l} \mathcal{L}_\partial(s\omega) = \partial(s)\omega + s\mathcal{L}_\partial(\omega) \\ \mathcal{L}_{s\partial}(\omega) = s\mathcal{L}_\partial(\omega)\,. \end{array} \right.$$

6.1.2. *Van den Bergh and Calabi-Yau duality.* The results of [vdB98], which are written for algebras over fields, still hold true for R-algebras provided that the Hochschild (co)homologies $H^\bullet(S, -)$ and $H_\bullet(S, -)$ are replaced by $\mathrm{Ext}^\bullet_{S^e}(S, -)$ and $\mathrm{Tor}^{S^e}_\bullet(S, -)$, respectively: if the R-algebra S has Van den Bergh duality in dimension n and if N is any S-bimodule, then there is a functorial isomorphism,

$$\mathrm{Ext}^\bullet_{S^e}(S, N) \simeq \mathrm{Tor}^{S^e}_{n-\bullet}(S, \mathrm{Ext}^n_{S^e}(S, S^e) \otimes_S N)\,. \tag{6.1.2.1}$$

When R is a field, Van den Bergh duality for commutative R-algebras is closely related to smoothness, which is the relevant context in which to consider volume form. The precise relationship is due to Krähmer ([Krä07]) when R is a perfect field. It adapts as follows in the present context. Here, $\Omega_{S/R}$ denotes the S-module of Kähler forms relative to R.

Proposition 6.1.5. *Keep setting 6.1.1. If R is Noetherian and S finitely generated as an R-algebra and projective as an R-module, then the following assertions are equivalent.*

(i) S has Van den Bergh duality in dimension n.
(ii) $\mathrm{gl.dim}(S^e) < \infty$ *and* $\Omega_{S/R}$, *which is then projective in* $\mathrm{Mod}(S)$, *has constant rank n.*

If (i) holds true, then $\mathrm{gl.dim}(S) < \infty$ *and* $\mathrm{Ext}^n_{S^e}(S, S^e) \simeq \Lambda^n_S \mathrm{Der}_R(S)$ *as S-modules.*

When S is Calabi-Yau in dimension n, there is an isomorphism $S \to \mathrm{Ext}^n_{S^e}(S, S^e)$ in $\mathrm{Mod}(S)$. Combining such an isomorphism with the degree 0 part of (6.1.2.1), with $N = S$, yields an isomorphism $S \to \mathrm{Tor}^{S^e}_n(S, S)$ in $\mathrm{Mod}(S)$. In other words, to every free generator of the S-module $\mathrm{Ext}^n_{S^e}(S, S^e)$ is associated a volume form on S, and hence a divergence, in the following sense.

Definition 6.1.6. If S is Calabi-Yau in dimension n, a *volume form* on S is a free generator of the S-module $\mathrm{Tor}^{S^e}_n(S, S)$. The *divergence* associated to a volume form ω on S is then the mapping $\mathrm{div}\colon \mathrm{Der}_R(S) \to S$ such that, for all $\partial \in \mathrm{Der}_R(S)$,

$$\mathcal{L}_\partial(\omega) = \mathrm{div}(\partial)\omega\,.$$

The divergences control the behaviour of Lie derivations on $\mathrm{Ext}^n_{S^e}(S, S^e)$.

Proposition 6.1.7. *Assume that S is Calabi-Yau in dimension n. Let e_S be a free generator of the S-module $\mathrm{Ext}^n_{S^e}(S, S^e)$. Denote by* div *the divergence of the volume form on S associated to e_S. Then, for all $\partial \in \mathrm{Der}_R(S)$,*

$$\mathcal{L}_\partial(e_S) = -\mathrm{div}(\partial) e_S \,.$$

6.2. **Van den Bergh duality of the enveloping algebra.** The setting in which this duality is presented is the following.

Setting 6.2.1. Let R be a commutative ring. Let (S, L) be a Lie-Rinehart algebra over R. Denote by U the enveloping algebra of (S, L). Assume that

- S is flat as an R-module,
- S has Van den Bergh duality in dimension n, and
- L is finitely generated and projective with constant rank d as an S-module.

On one hand, the Lie derivations yield an action $(\alpha, e) \mapsto \alpha \cdot e$ of L on $\mathrm{Ext}^\bullet_{S^e}(S, S^e)$ such that, for all $\alpha \in L$ and $e \in \mathrm{Ext}^\bullet_{S^e}(S, S^e)$,

$$\alpha \cdot e = \mathcal{L}_{\partial_\alpha}(e) \,.$$

This does not yield a structure of U-module on $\mathrm{Ext}^\bullet_{S^e}(S, S^e)$. However, for all right U-modules M, the S-module $M \otimes_S \mathrm{Ext}^\bullet_{S^e}(S, S^e)$ has a structure of left U-module such that, for all $\alpha \in L$, $m \in M$, and $e \in \mathrm{Ext}^\bullet_{S^e}(S, S^e)$,

$$\alpha \cdot (m \otimes e) = -m \cdot \alpha \otimes e + m \otimes \alpha \cdot e \,.$$

This applies in particular to the right U-module $M = \Lambda^d_S L^\vee$ when L is finitely generated, projective, and with constant rank d as an S-module.

On the other hand, there exists an exact functor

$$F \colon \mathrm{Mod}(U) \to \mathrm{Mod}(U^e)$$

such that, if $N \in \mathrm{Mod}(U)$, then $F(N)$ equals $U \otimes_S N$ as a left U-module and has a structure of right U-module such that, for all $\alpha \in L$, $u \in U$, and $n \in N$,

$$(u \otimes n) \cdot \alpha = u\alpha \otimes n - u \otimes \alpha \cdot n \,.$$

Theorem 6.2.2. *Keep setting 6.2.1. Then, U has Van den Bergh duality in dimension $n + d$ and there is an isomorphism of U-bimodules*

$$\mathrm{Ext}^{n+d}_{U^e}(U, U^e) \simeq F(\Lambda^d_S L^\vee \otimes_S \mathrm{Ext}^n_{S^e}(S, S^e)) \,.$$

Here are a few words on the proof of Theorem 6.2.2.

- First, U being homologically smooth is based on the facts that $S \in \mathrm{per}(U)$ ([Rin63]), that F is exact, that $F(S) \simeq U$, and that $F(U) \in \mathrm{per}(U^e)$.
- Next, the description of $\mathrm{Ext}^\bullet_{U^e}(U, U^e)$ is obtained starting from the isomorphism $F(S) \simeq U$ and deriving the adjunction between F and its right adjoint which is induced by $\mathrm{Hom}_{S^e}(S, -)$.
- Finally, the reason why $\mathrm{Ext}^{n+d}_{U^e}(U, U^e)$ is invertible as a U-bimodule is that F takes left U-modules which are invertible as S-modules to invertible U-bimodules.

Example 6.2.3. If is L is free as an S-module and $(\alpha_1, \ldots, \alpha_d)$ is a basis of it, then the isomorphism of Theorem 6.2.2 may be reformulated as follows,

$$\mathrm{Ext}^{n+d}_{U^e}(U, U^e) \simeq U \otimes_S \mathrm{Ext}^n_S(S, S^e) \,,$$

where the right-hand side is a left U-module in a natural way and is a right U-module such that, for all $u \in U$, $e \in \mathrm{Ext}^n_{S^e}(S, S^e)$, and $\alpha \in L$,

$$(u \otimes e) \cdot \alpha = u\alpha \otimes e + u \otimes \mathrm{Tr}(\mathrm{ad}_\alpha) e - u \otimes \mathcal{L}_{\partial_\alpha}(e) \,.$$

Here, by abuse of notation, $\mathrm{Tr}(\mathrm{ad}_\alpha)$ stands for the trace of the matrix in the basis $(\alpha_1, \ldots, \alpha_d)$ of the R-linear mapping $\mathrm{ad}_\alpha \colon L \to L$. This matrix has coefficients in S and its trace does depend on the choice of the basis.

Bibliographical note. As mentioned previously, Chemla proved in [Che99] that for all Lie algebroids over smooth affine complex varieties, the $\mathbb{C}$-algebra of global sections of the sheaf of differential operators has Van den Bergh duality. She also described explicitly its rigid dualising complex.

Theorem 6.2.2 has an interesting reformulation in terms of skew Calabi-Yau algebras when both $\Lambda_S^d L^\vee$ and $\mathrm{Ext}^n_{S^e}(S, S^e)$ are isomorphic to S as S-modules.

On one hand, according to Proposition 6.1.7, the action of Lie derivations on $\mathrm{Ext}^n_{S^e}(S, S^e)$ are controlled by the divergence of any volume form on S.

On the other hand, if $\Lambda_S^d L^\vee$ is isomorphic to S, then every free generator of it defines a trace mapping in the following sense.

Definition 6.2.4. Assume that L is finitely generated and projective with constant rank d as an S-module and that $\Lambda_S^d L^\vee \simeq S$ in $\mathrm{Mod}(S)$. The *trace mapping on* $\Lambda_S^d L^\vee$ associated to a given free generator φ of $\Lambda_S^d L^\vee$ is the mapping $\lambda_L \colon L \to S$ such that, for all $\alpha \in L$,

$$\varphi \cdot \alpha = \lambda_L(\alpha)\varphi\,.$$

Example 6.2.5. If L is free as an S-module and $(\alpha_1, \ldots, \alpha_d)$ is a basis of L with dual basis denoted by $(\alpha_1^*, \ldots, \alpha_d^*)$, then the trace mapping of the generator $\alpha_1^* \wedge \cdots \wedge \alpha_d^*$ of $\Lambda_S^d L^\vee$ is given by $\alpha \mapsto \mathrm{Tr}(\mathrm{ad}_\alpha)$, with the same abuse of notation as in Example 6.2.3

Theorem 6.2.2 therefore specialises as follows.

Corollary 6.2.6. *Keep setting 6.2.1. Assume in addition that,*

- *S is Calabi-Yau in dimension n*
- *and $\Lambda_S^d L$, which is projective of rank one, is free as an S-module.*

Then, U is skew Calabi-Yau with a Nakayama automorphism $\nu \in \mathrm{Aut}_R(U)$ such that, for all $s \in S$ and $\alpha \in L$,

$$\begin{cases} \nu(s) &= s \\ \nu(\alpha) &= \alpha + \lambda_L(\alpha) + \mathrm{div}(\partial_\alpha), \end{cases}$$

where λ_L is any trace mapping on $\Lambda_S^d L^\vee$ and div *is any divergence.*

Example 6.2.7. This is the continuation of Example 6.2.3. The Nakayama automorphism in Corollary 6.2.6 is then such that $\nu(s) = s$ and $\nu(\alpha) = \alpha + \mathrm{Tr}(\mathrm{ad}_\alpha) + \mathrm{div}(\partial_\alpha)$, for all $s \in S$ and $\alpha \in L$.

6.3. **Case of Lie-Rinehart algebras of Poisson algebras.** Among all Lie-Rinehart algebras based on a commutative algebra S over a commutative ring R, those arising from Poisson structures on S play a special role because of the connection to Poisson (co)homology. Following Huebschmann ([Hue90]), an R-bilinear Poisson bracket $\{-,-\}$ on S defines a structure of Lie-Rinehart algebra on $(S, L) = (S, \Omega_{S/R})$. Here $\Omega_{S/R}$ denotes the S-module of Kähler differentials relative to R. The structure of Lie-Rinehart algebra is such that, for all $s, t \in S$,

- $\partial_{ds} = \{s, -\}$;
- $[ds, dt] = d\{s, t\}$.

The resulting enveloping algebra is the *Poisson enveloping* algebra of $(S, \{-,-\})$.

In this setting, several remarks are worth making when S has Van den Bergh duality in dimension n and when the assumptions of Proposition 6.1.5 are satisfied. On one hand, $\Omega_{S/R}$ is projective with constant rank n as an S-module. In addition, $\Lambda_S^n \Omega_{S/R}^\vee$ is canonically isomorphic to $\Lambda_S^n \mathrm{Der}_R(S)$ as an S-module. Hence,

$\Lambda^n_S \mathrm{Der}_R(S)$ inherits of a structure of right U-module. This structure is described by classical Lie derivations as follows, for all $\varphi \in \Lambda^n_S \mathrm{Der}_R(S)$ and $s \in S$,

$$\varphi \cdot ds = -\mathcal{L}_{\{s,-\}}(\varphi)\,. \tag{6.3.0.2}$$

On the other hand, the isomorphism $\mathrm{Ext}^n_{S^e}(S, S^e) \simeq \Lambda^n_S \mathrm{Der}_R(S)$ claimed in Proposition 6.1.5 is equivariant for the actions of Lie derivations on both sides.

From these considerations, it is possible to deduce the following result from Theorem 6.2.2 and Corollary 6.2.6.

Corollary 6.3.1. *Let R be a Noetherian ring. Let $(S, \{-,-\})$ be a finitely generated Poisson algebra over R. Denote by U the enveloping algebra of the associated Lie Rinehart algebra $(S, \Omega_{S/R})$. Assume that*

- *S is projective in $\mathrm{Mod}(R)$,*
- *$S \in \mathrm{per}(S^e)$,*
- *and $\Omega_{S/R}$, which is then projective in $\mathrm{Mod}(S)$, has constant rank n.*

Then, U has Van den Bergh duality in dimension $2n$ and there is an isomorphism of U-bimodules

$$\mathrm{Ext}^{2n}_{U^e}(U, U^e) \simeq U \otimes_S \Lambda^n_S \mathrm{Der}_R(S) \otimes_S \Lambda^n_S \mathrm{Der}_R(S)\,,$$

where the right-hand side term is a left U-module in a natural way and a right U-module such that, for all $u \in U$, $\varphi, \varphi' \in \Lambda^n_S \mathrm{Der}_R(S)$, and $s \in S$,

$$(u \otimes \varphi \otimes \varphi') \cdot ds = u\, ds \otimes \varphi \otimes \varphi' - u \otimes (\mathcal{L}_{\{s,-\}}(\varphi) \otimes \varphi' + \varphi \otimes \mathcal{L}_{\{s,-\}}(\varphi'))\,.$$

In particular, if S is Calabi-Yau in dimension n, then U is skew Calabi-Yau with a Nakayama automorphism $\nu \colon U \to U$ such that, for all $s \in S$,

$$\left\{ \begin{array}{rcl} \nu(s) & = & s \\ \nu(ds) & = & ds + 2\,\mathrm{div}(\{s,-\})\,, \end{array} \right.$$

where div *is any divergence.*

Example 6.3.2. Assume that $S = R[x,y]$. Let $\{-,-\}$ be a Poisson bracket on S. Denote $\{x,y\}$ by P. Then, the Poisson enveloping algebra of $(S, \{-,-\})$ is skew Calabi-Yau in dimension 4 with a Nakayama automorphism ν such that

$$\left\{ \begin{array}{rclrcl} \nu(x) & = & x\,, & \nu(dx) & = & dx + 2\frac{\partial P}{\partial y} \\ \nu(y) & = & y\,, & \nu(dy) & = & dy - 2\frac{\partial P}{\partial x}\,. \end{array} \right.$$

Example 6.3.3. Assume that $S = R[x,y,z]$. Let $P_x, P_y, P_z \in S$ be such that

$$\overrightarrow{P} \wedge \mathrm{curl}(\overrightarrow{P}) = 0\,,$$

where $\overrightarrow{P}$ denotes $\begin{pmatrix} P_x \\ P_y \\ P_z \end{pmatrix}$. Hence, the following defines a Poisson bracket on S,

$$\{x,y\} = P_z\,,\ \{y,z\} = P_x\,,\ \{z,x\} = P_y\,.$$

The resulting Poisson enveloping algebra is hence skew Calabi-Yau in dimension 6 and has a Nakayama automorphism $\nu \in \mathrm{Aut}_R(S)$ such that

$$\begin{pmatrix} \nu(x) \\ \nu(y) \\ \nu(z) \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \nu(dx) \\ \nu(dy) \\ \nu(dz) \end{pmatrix} = \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} + 2\,\mathrm{curl}(\overrightarrow{P})\,.$$

Bibliographical note. Using the results of Chemla in [Che99], Corollary 6.3.1 was proved in [LWZ17] when R is an algebraically closed field of characteristic zero.

Part 3. Representation theory of finite dimensional algebras

This part presents the main results of the articles [LM08, LM10b, ABLM10, LM11, ABLM11, CLMT11, ALMM17, CLMT19a, CLMT19b] which deal with problems of representation theory of finite dimensional algebras related to the following topics,

(1) the composition of irreducible morphisms in relationship with the filtration of the module category by the powers of its radical,
(2) the characterisation of the simple connectedness of finite dimensional algebras in terms of their Hochschild cohomologies,
(3) and the strong global dimension of piecewise hereditary algebras.

The context of these results is presented in section 7. The needed definitions are recalled in section 8. The results relative to the points (1), (2), and (3) above are presented in the sections 9, 10, and 11, respectively.

7. Context and addressed problems

The representation theory of finite dimensional algebras, and more generally of Artin algebras, deals with the description of module categories of Artin algebras over Artin commutative rings. Its main purposes are, on one hand, the classification of the indecomposable finitely generated modules over these algebras up to isomorphism and the description of the morphisms between these modules; and, on the other hand, the classification of these algebras up to Morita or derived equivalence.

Let A be a connected Artin algebra over an Artin commutative ring $\Bbbk$.

7.1. **Auslander-Reiten theory.** This theory ([ARS97]) is central in the study of the category $\mathrm{mod}(A)$ of (finitely generated right) modules over A. It is based on

- the radical ideal of $\mathrm{mod}(A)$, denoted by rad, and its powers rad^n, for $n \in \mathbb{N}$;
- the irreducible morphisms of $\mathrm{mod}(A)$;
- the almost split sequences of $\mathrm{mod}(A)$;
- the Auslander-Reiten translation τ_A;
- and the Auslander-Reiten quiver $\Gamma(\mathrm{mod}(A))$ of $\mathrm{mod}(A)$.

The use of $\Gamma(\mathrm{mod}(A))$, as a combinatorial object, has been so successful in the study of $\mathrm{mod}(A)$ that the information carried by this quiver is considered as a first approximation of the module category. For example, the Riedtmann quivers played an important role in the study of algebras of finite representation type; and for many classes of algebras such as the quasitilted algebras, the shapes of the Auslander-Reiten components are known. However, this combinatorial information is not enough to understand $\mathrm{mod}(A)$ completely. For instance, it does not render properly the composition of morphisms. Of course, in standard Auslander-Reiten components the composition of morphisms is encoded by the mesh categories. But not all Auslander-Reiten components are standard. In general, many aspects of the composition of morphisms relatively to the Auslander-Reiten structure are unknown. For instance, Igusa and Todorov proved in [IT84b] that the composite morphism of a sectional path of n irreducible morphisms between indecomposable A-modules lies in $\mathrm{rad}^n \backslash \mathrm{rad}^{n+1}$, but little is known when the path is no longer sectional. Also, when A is of finite representation type or $\Bbbk$ is an algebraically closed field, they determined the kernel of the following natural transformation induced by the composition with an irreducible morphism $f\colon X \to Y$ such that X or Y is indecomposable, for all integers n,

$$\frac{\mathrm{rad}^n(-,X)}{\mathrm{rad}^{n+1}(-,X)} \to \frac{\mathrm{rad}^{n+1}(-,Y)}{\mathrm{rad}^{n+2}(-,Y)}\,. \tag{7.1.0.3}$$

Again, little is known on this kernel in general. These considerations are rather fundamental than trivial. Indeed, Liu defined in [Liu92] the left degree of f as the least integer n, if any, such that the kernel of (7.1.0.3) is non zero; by considerations on irreducible morphisms with infinite left degrees, he obtained structural results on the possible shapes of the Auslander-Reiten components of Artin algebras.

The results of this thesis related to this context provide new information on the composition of morphisms relatively to the filtration of $\mathrm{mod}(A)$ by the powers of the radical with a particular emphasis on (7.1.0.3) and on the left degree. Assuming that $\Bbbk$ is a perfect field, these results are based on existence theorems of functors from appropriate mesh categories to $\mathrm{mod}(A)$ and called well-behaved.

These results are presented in section 9.

7.2. **The strong global dimension of piecewise hereditary algebras.** Assume that $\Bbbk$ is an algebraically closed field. The concept of strong global dimension is rooted in the development of tilting theory. Here are some milestones of this development, the precise definitions are given in section 8.2. The class of hereditary Artin algebras is the first one to have been fairly well understood in the representation theory of Artin algebras (see [Gab72]). When tilting theory emerged with the study of reflection functors ([BGP73]) and with the Brenner-Butler theorem ([BB80]), it was applied successfully to hereditary algebras, giving rise to tilted algebras ([HR82]). And, in view of setting a common framework for tilted algebras and for canonical algebras ([Rin84]), Happel, Reiten, and Smalø defined quasitilted algebras in [HRS96a] by replacing the use of module categories of hereditary algebras by the use of hereditary abelian categories. Quasitilted algebras are particular cases of piecewise hereditary algebras, which, by definition, are the algebras whose bounded derived categories of complexes of finitely generated modules are triangle equivalent to the bounded derived categories of hereditary abelian categories. Recently, the piecewise hereditary algebras have played an increasingly important role in representation theory. On one hand, this is due to Happel's result on the Auslander-Reiten structure of $\mathcal{D}^b(\mathrm{mod}(H))$, where H is a hereditary algebra. On the other hand, this is due to the connections with other areas of mathematics among which,

- the geometry of weighted projective lines ([GL87]), whose categories of coherent sheaves appear in the classification of hereditary abelian categories by Happel and Reiten ([HR02])
- and the acyclic cluster categories, which are categorified by orbit categories of $\mathcal{D}^b(\mathrm{mod}(H))$ for hereditary algebras H ([BMR$^+$06]).

It is hence desirable to have characterisations of piecewise hereditary algebras. On one hand, the piecewise hereditary algebras are exactly the algebras obtained from the quasitilted ones by taking repeatedly opposites of endomorphism algebras of tilting or cotilting modules ([HRS88, HRS96b]). On the other hand, Skowroński introduced in [Sko87] a new invariant of finite dimensional algebras called the strong global dimension, which is defined like the global dimension except that the indecomposable objects of the homotopy category of bounded complexes of finitely generated projective modules replace the modules; Happel and Zacharia proved that an algebra is piecewise hereditary if and only if its strong global dimension is finite ([HZ08]).

The work that I did in this context is a starting collaboration with Alvares and Marcos. Considering piecewise hereditary algebras as opposites of endomorphism algebras of tilting objects T in specific triangulated categories $\mathcal{T}$, this work characterises the strong global dimensions of these algebras from the following viewpoints: the Auslander-Reiten structure of $\mathcal{T}$; the hereditary abelian generating

subcategories of $\mathcal{T}$ related to the indecomposable direct summands of T; and the sequences of tilting mutations in $\mathcal{T}$ which transform T into a tilting object whose endomorphism algebra is quasitilted.

These results are presented in section 11.

7.3. **Simple connectedness and Hochschild cohomologies.** Assume that $\Bbbk$ is an algebraically closed field. Assume that A is basic and connected, which is equivalent to the fact that there exists a unique connected quiver Q and at least one admissible ideal I of the path algebra $\Bbbk Q$ such that $A \simeq \Bbbk Q/I$.

The Galois coverings of algebras emerged from the work of Riedtmann and of Bongartz and Gabriel on the computation of the indecomposable modules of finite dimensional algebras of finite representation type. Their definition is due to Gabriel ([Gab81]). Given a Galois covering $F\colon A' \to A$, where A' is a locally bounded $\Bbbk$-category, the extension-of-scalars functor $F_\lambda\colon \mathrm{mod}(A') \to \mathrm{mod}(A)$ reduces the study of $\mathrm{mod}(A)$ to the easier one of $\mathrm{mod}(A')$. For instance, Gabriel proved that A being of finite representation type is tightly related to A' being locally of finite representation type; he also proved that, if the induced action of G on the isomorphism classes of indecomposable A'-modules is free, then the Auslander-Reiten components of $\mathrm{mod}(A')$ are Galois coverings of certain Auslander-Reiten components of $\mathrm{mod}(A)$. More recently, the work of Skowroński and his collaborators on the classification of selfinjective algebras of quasitilted type is heavily based on Galois coverings ([SY08]). This raises the question of characterising and classifying the algebras for which there is no such reduction. Assem and Skowroński defined simply connected algebras in [AS88], the algebra A is simply connected if it is triangular and if, for all bound quivers (Q, I) such that $A \simeq \Bbbk Q/I$, the fundamental group $\pi_1(Q, I)$, which was defined by Martinez-Villa and de la Peña in [MVdlPn83] and which does depend on the choice of (Q, I), is trivial. Equivalently, A is triangular and it has no proper Galois covering. Skowroński asked the following question, when A is tame and triangular ([Sko92]).

(Q1) Is A simply connected if and only if the Hochschild cohomology group $\mathrm{HH}^1(A)$ is zero?

On one hand, strongly simply connected algebras, which are the triangular algebras all of whose convex full subcategories are simply connected, are easily characterised in terms of their Hochschild cohomologies; and striking results have been obtained in the classification of tame strongly simply connected algebras (see [BdlPnS11]). On the other hand, the analysis of the fundamental groups $\pi_1(Q, I)$ has led to a positive answer to (Q1) for certain classes of tame algebras like the tame quasitilted algebras (see section 10.4). These classes of algebras have the following specificities, they are related to tilting theory and their Auslander-Reiten structures are well-known.

The work that I made in this context answers (Q1) for several classes of algebras with such specificities and which have been deeply investigated recently: piecewise hereditary algebras and certain classes of algebras determined by their left and right parts. This is done by involving more deeply the representation theory of the considered algebras. Instead of considering the fundamental groups $\pi_1(Q, I)$, the answers to (Q1) are based on a general study of Galois coverings with the following questions in mind.

(Q2) Given a group G and derived equivalent algebras A and B, does A have a Galois covering with group G if and only if so does B?

(Q3) In which circumstances does $\mathrm{mod}(A)$ have an Auslander-Reiten component Γ such that, for all groups G, the algebra A has Galois covering with group G if and only if Γ has a translation quiver Galois covering with group G?

These results are presented in section 10.

8. Recollection of definitions

Let A be an Artin algebra over an Artin commutative ring $\Bbbk$. The category of finitely generated (right) A-modules is denoted by $\operatorname{mod}(A)$ and $\operatorname{ind} A$ denotes a full subcategory of $\operatorname{mod}(A)$ whose objects form a complete set of representatives of the isomorphism classes of indecomposable A-modules. In all Krull-Schmidt categories, if S is a class of objects, then its closure under direct sums and direct summands is denoted by $\operatorname{add}(S)$.

8.1. **Auslander-Reiten theory.** This thesis follows the definitions and conventions on Auslander-Reiten theory of the textbook [ARS97] to which the reader is referred. In particular,

- the Auslander-Reiten quiver of $\operatorname{mod}(A)$ is denoted by $\Gamma(\operatorname{mod}(A))$; by definition, its vertices are the objects of $\operatorname{ind} A$;
- as soon as $\Bbbk$ is not an algebraically closed field, it is assumed that translation quivers have no multiple arrows;
- the *radical* of $\operatorname{mod}(A)$, which is the ideal generated by the non invertible morphisms between indecomposables, is denoted by rad;
- for all indecomposable modules $X, Y \in \operatorname{mod}(A)$, then $\frac{\operatorname{rad}(X,Y)}{\operatorname{rad}^2(X,Y)}$ is denoted by $\operatorname{irr}(X,Y)$ and called the *space of irrreducible morphisms* from X to Y; given $f \in \operatorname{rad}(X,Y)$, its residue class in $\operatorname{irr}(X,Y)$ is denoted by $\overline{f}$;
- an *Auslander-Reiten component* of $\operatorname{mod}(A)$ is a connected component Γ of $\Gamma(\operatorname{mod}(A))$; the full subcategory of $\operatorname{ind} A$ given by the vertices of Γ is denoted by $\operatorname{ind}\Gamma$;
- a *path of irreducible morphisms* is a sequence of irreducible morphisms of the shape $X_0 \xrightarrow{f_1} \cdots \xrightarrow{f_n} X_n$, where $X_0, \ldots, X_n$ are indecomposable modules.

To every Auslander-Reiten component Γ of $\operatorname{mod}(A)$ is associated a $\Bbbk$-category $\Bbbk(\Gamma)$ called the mesh category. When $\Bbbk$ is a perfect field, part of the work presented in section 9 consists in proving the existence of suitable functors $\Bbbk(\widetilde{\Gamma}) \to \operatorname{ind}\Gamma$ for adequate translation quiver coverings $\widetilde{\Gamma} \to \Gamma$. The definition of mesh categories and of translation quiver coverings is recalled below.

8.1.1. *Modulated translation quivers and their mesh categories.* Modulated translation quivers where defined by Igusa and Todorov.

Definition 8.1.1 ([IT84a])**.** A *modulation* on a translation quiver (Γ, τ) is the following data,

(1) a division $\Bbbk$-algebra κ_x for every vertex $x \in \Gamma$,
(2) a non-zero $\kappa_x - \kappa_y$ bimodule $M(x,y)$ for every arrow $x \to y$ in Γ,
(3) a $\Bbbk$-algebra isomorphism $\tau_* \colon \kappa_x \xrightarrow{\sim} \kappa_{\tau x}$ for every vertex $x \in \Gamma$,
(4) a non-degenerate $\kappa_y - \kappa_y$-linear map $\sigma_* \colon M(y,x) \otimes_{\kappa_x} M(\tau x, y) \to \kappa_y$ (the left κ_x-module structure on $M(\tau x, y)$ is defined using its structure of left $\kappa_{\tau x}$-module and $\tau_* \colon \kappa_x \to \kappa_{\tau x}$).

A *modulated* translation quiver is a translation quiver endowed with a modulation.

The Auslander-Reiten quiver, and hence every Auslander-Reiten component, of $\operatorname{mod}(A)$ can be endowed with a modulation in the following way. For every non projective $X \in \operatorname{ind} A$ fix an almost split sequence $0 \to \tau_A X \to E \to X \to 0$ in $\operatorname{mod}(A)$. The associated modulation of $\Gamma(\operatorname{mod}(A))$ is as follows.

- $\kappa_X = \operatorname{End}_A(X)/\operatorname{rad}(X,X)$ for all $X \in \operatorname{ind} A$.
- $M(X,Y) = \operatorname{irr}(X,Y)$ for all arrows $X \to Y$ of $\Gamma(\operatorname{mod}(A))$.

- For all non projective $X \in \operatorname{ind} A$ and all morphisms $u\colon X \to X$ defining the residue class $\overline{u} \in \kappa_X$, let $\tau_* \overline{u}\colon \tau_A X \to \tau_A X$ be the residue class $\overline{v}$ where $v\colon \tau_A X \to \tau_A X$ is a morphism fitting into a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \tau_A X \arrow[r] \arrow[d, "v"'] & E \arrow[r] \arrow[d] & X \arrow[r] \arrow[d, "u"] & 0 \\ 0 \arrow[r] & \tau_A X \arrow[r] & E \arrow[r] & X \arrow[r] & 0\,. \end{tikzcd}$$

- For all arrows $X \to Y$ of $\Gamma(\operatorname{mod}(A))$, let $\overline{u} \in M(\tau_A X, Y)$ and $\overline{v} \in M(Y, X)$ be residue classes of morphisms $u\colon \tau_A X \to Y$ and $v\colon Y \to X$, respectively, and define $\sigma_*(\overline{v} \otimes \overline{u})$ as the composition $\overline{v'u'}$ where u', v' are morphisms fitting into a commutative diagram

$$\begin{tikzcd} & & & Y \arrow[dl, "v'"'] \arrow[d, "v"] & \\ 0 \arrow[r] & \tau_A X \arrow[r] \arrow[d, "u"'] & E \arrow[r] \arrow[dl, "u'"] & X \arrow[r] & 0 \\ & Y & & & \end{tikzcd}\,.$$

The modulations of the Auslander-Reiten components of $\operatorname{mod}(A)$ arising in this way are said to be *standard*.

Modulations are used to define mesh categories when $\Bbbk$ is no longer an algebraically closed field. The definition below is due to Igusa and Todorov.

Given a modulated translation quiver (Γ, τ), let S be the semi-simple category whose object set is the set of vertices in Γ and such that $S(x, y) = \kappa_x$ if $x = y$ and $S(x, y) = 0$ otherwise. The collection $\{M(x, y)\}_{x \to y \text{ in } \Gamma}$ naturally defines an $S - S$-bimodule denoted by M. The *path category* is the tensor category $T_S(M)$ also denoted by $\Bbbk\Gamma$. The *mesh ideal* is the ideal in $\Bbbk\Gamma$ generated by a collection of morphisms $\gamma_x\colon \tau x \to x$ indexed by the non projective vertices $x \in \Gamma$. Given a non projective vertex $x \in \Gamma$, the morphism $\gamma_x\colon \tau x \to x$ in $\Bbbk\Gamma$ is defined as follows. For every arrow $y \to x$ ending in x, fix a basis $(u_1, \dots, u_d)$ of the κ_y-vector space $M(y, x)$. Let $(u_1^*, \dots, u_d^*)$ be the associated dual basis of the κ_y-vector space $M(\tau x, y)$ under the pairing σ_* (that is, $\sigma_*(u_i \otimes u_j^*)$ is 1 if $i = j$ and 0 otherwise). Then $\gamma_x = \sum_{y \to x \text{ in } \Gamma} \sum_i u_i^* u_i \in \Bbbk\Gamma(\tau x, x)$. This morphism does not depend on the choice of the basis $(u_1, \dots, u_d)$.

Definition 8.1.2 ([IT84a])**.** Let (Γ, τ) be a modulated translation quiver. The *mesh category* is the quotient category of $\Bbbk\Gamma$ by the mesh ideal. It is denoted by $\Bbbk(\Gamma)$. The ideal of $\Bbbk(\Gamma)$ generated by the non invertible morphisms is denoted by $\mathcal{R}\Bbbk(\Gamma)$ and its successive powers are denoted by $\mathcal{R}^\ell \Bbbk(\Gamma)$ for $\ell \geqslant 0$.

For all Auslander-Reiten components Γ of $\operatorname{mod}(A)$, all standard modulations yield the same mesh category and $\Bbbk(\Gamma)$ is understood for any of these modulations.

8.1.2. *Translation quiver coverings.* These were introduced by Riedtmann and by Bongartz and Gabriel in their study of algebras of finite representation type.

Definition 8.1.3 ([Rie80, BG82])**.** A *translation quiver covering* is a quiver morphism $p\colon \widetilde{\Gamma} \to \Gamma$ such that

(1) (Γ, τ) and $(\widetilde{\Gamma}, \tau)$ are translation quivers and Γ is connected,
(2) x is projective if and only if so is px, for all vertices $x \in \widetilde{\Gamma}$,
(3) $p \circ \tau = \tau \circ p$,
(4) as a graph morphism, p is a covering.

Connected translation quivers have universal coverings which are Galois coverings in the following sense.

Definition 8.1.4. A *translation quiver Galois covering* with group G is a translation quiver covering $p\colon \widetilde{\Gamma} \to \Gamma$ together with an action of G on $\widetilde{\Gamma}$ by translation quiver automorphisms such that

- the induced action of G on the vertices of $\widetilde{\Gamma}$ is free,
- p is G-invariant,
- Γ and $\widetilde{\Gamma}$ are connected,
- $p^{-1}(x)$ is a G-orbit, for all vertices x of Γ.

Theorem 8.1.5 ([BG82])**.** *Let Γ be a connected translation quiver. There exists a translation quiver Galois covering $\widetilde{\Gamma} \to \Gamma$ which factors through all translation quiver coverings of Γ.*

The translation quiver Galois covering $\widetilde{\Gamma} \to \Gamma$ of Theorem 8.1.5 is called the *universal cover* of Γ. Its group is denoted by $\pi_1(\Gamma)$. When Γ is an Auslander-Reiten component of $\operatorname{mod}(A)$ with only finitely many τ-orbits, this group is free.

8.2. **Reminder on algebras related to tilting theory.** Here, $\Bbbk$ is an algebraically closed field.

8.2.1. *Tilted algebras.* A *tilting* A-module is a module $T \in \operatorname{mod}(A)$ of projective dimension at most one, such that $\operatorname{Ext}^1_A(T,T) = 0$ and such that there exists an exact sequence $0 \to A \to T_0 \to T_1 \to 0$ in $\operatorname{mod}(A)$, where $T_0, T_1 \in \operatorname{add}(A)$.

The *tilted* algebras are the algebras isomorphic to $\operatorname{End}_H(T)^{\mathrm{op}}$ for some finite dimensional and hereditary $\Bbbk$-algebra H and some tilting module $T \in \operatorname{mod}(H)$. They are characterised as follows by their Auslander-Reiten structures.

Theorem 8.2.1 ([Liu93, Sko93])**.** *The algebra A is tilted if and only if* $\operatorname{mod}(A)$ *has an Auslander-Reiten component Γ with a full subquiver Δ such that*

- *Δ is faithful,*
- $\operatorname{Hom}_A(U, \tau_A V) = 0$ *for all $U, V \in \Delta$,*
- *and Δ is a* section *of Γ, that is, it contains no oriented cycles, it is convex as a subquiver of Γ and it intersects each τ_A-orbit of Γ exactly once.*

When A is tilted, an Auslander-Reiten component Γ such as in the previous theorem is called a *connecting component.*

8.2.2. *Quasitilted algebras.* A *tilting object* in a skeletally small abelian $\Bbbk$-linear category $\mathcal{A}$ with finite dimensional Ext spaces is an object $T \in \mathcal{A}$ such that $\operatorname{Ext}^n(T,T) = 0$ for all integers n, such that $\operatorname{Ext}^n(T,-) = 0$ for all large enough integers n, and such that 0 is the only object X of $\mathcal{A}$ such that $\operatorname{Ext}^n(T,X) = 0$ for all integers n.

The *quasitilted* algebras are the algebras isomorphic to $\operatorname{End}(T)^{\mathrm{op}}$ for some tilting object T in a skeletally small hereditary abelian $\Bbbk$-linear category $\mathcal{H}$ with finite dimensional Ext-spaces. They are characterised as follows.

Theorem 8.2.2 ([HRS96a])**.** *The algebra A is quasitilted if and only if*

(a) the injective dimension or the projective dimension of X is at most one, for all $X \in \operatorname{ind} A$

(b) and $\operatorname{gl.dim.} A \leqslant 2$.

Quasitilted algebras are also characterised in terms of their left parts, whose definition is given below. Given $X, Y \in \operatorname{ind} A$, then X is called a *predecessor* of Y in $\operatorname{ind} A$ and Y is called a *successor* of X in $\operatorname{ind} A$ if there exists a sequence $X = X_0 \to \cdots \to X_n = Y$ of non zero morphisms in $\operatorname{ind} A$.

Definition 8.2.3 ([HRS96a])**.** The *left part* of A is the full subcategory of $\operatorname{ind} A$ denoted by $\mathcal{L}_A$ and consisting of the modules $X \in \operatorname{ind} A$ all of whose predecessors have projective dimension at most one. The *right part* of A is defined dually and denoted by $\mathcal{R}_A$.

Theorem 8.2.4 ([HRS96a])**.** *The following assertions are equivalent.*

(i) A is quasitilted.
(ii) for all $P \in \operatorname{ind} A$, if P is projective, then $P \in \mathcal{L}_A$.
(iii) for all $I \in \operatorname{ind} A$, if I is injective, then $I \in \mathcal{R}_A$.

They imply that $\operatorname{ind} A = \mathcal{L}_A \cup \mathcal{R}_A$.

8.2.3. *Piecewise hereditary algebras.*

Definition 8.2.5. The algebra A is *piecewise hereditary* if there exist an equivalence of triangulated categories $\mathcal{D}^b(\operatorname{mod}(A)) \xrightarrow{\sim} \mathcal{D}^b(\mathcal{H})$, where $\mathcal{H}$ is a hereditary abelian $\Bbbk$-linear category $\mathcal{H}$.

If A is piecewise hereditary and $\mathcal{H}$ is as in Definition 8.2.5, then both $\mathcal{H}$ and A are said to be of *quiver type* or of *canonical type* if $\mathcal{H}$ is equivalent to $\operatorname{mod}(\Bbbk Q)$ for some finite quiver Q without oriented cycles or to $\operatorname{coh}(\mathbb{X})$ for some weighted projective line $\mathbb{X}$ ([GL87]), respectively. Since $\mathcal{D}^b(\operatorname{mod}(A))$ determines the representation type of Q (Dynkin, Euclidean, or wild) or of $\mathbb{X}$ (domestic, tubular, or wild), see [Hap88], then $\mathcal{D}^b(\operatorname{mod}(A))$ is said to have the corresponding type of Q or $\mathbb{X}$, respectively.

The work presented in section 11 characterises the strong global dimension of piecewise hereditary algebras. The following definition is taken for this invariant. It differs from the original definition of [Sko87] by one.

Definition 8.2.6 ([HZ08])**.** The *strong global dimension* of A is the element of $\mathbb{N} \cup \{+\infty\}$ denoted by $\operatorname{s.gl.dim.} A$ and defined as the supremum of the differences $s - t$ over all minimal projective resolutions, where P^t and P^s are non zero,

$$\cdots \to 0 \to P^t \to \cdots \to P^s \to 0 \to \cdots$$

of indecomposable objects of the homotopy category of bounded complexes of finitely generated projective A-modules.

In particular, the strong global dimension is not smaller than the global dimension. It was conjectured in [KSYZ04] that the piecewise hereditary algebras are characterised by the finiteness of their strong global dimensions.

Theorem 8.2.7 ([HZ08])**.** *The algebra A is piecewise hereditary if and only if* $\operatorname{s.gl.dim.} A$ *is finite. In particular, A is hereditary if and only if* $\operatorname{s.gl.dim.} A \leqslant 1$. *Also, A is quasitilted if and only if* $\operatorname{s.gl.dim.} A \leqslant 2$.

8.2.4. *Algebras determined by their left and right parts.* The characterisation of the quasitilted algebras by their left and right parts has given rise to new classes of algebras: the shod, the weakly shod and the laura algebras.

Definition 8.2.8. The algebra A is

- *shod* (or, of *small homological dimension*, [CL99]) if every $X \in \operatorname{ind} A$ has projective dimension at most one or injective dimension at most one,
- *weakly shod* ([CL03]) if there is an upper bound on the length of sequences $X_0 \to \cdots \to X_n$ of non zero and non invertible morphisms such that $X_0, \ldots, X_n \in \operatorname{ind} A$, such that X_0 is injective and such that X_n is projective,
- *laura* ([AC03, RS04]) if $\mathcal{L}_A \cup \mathcal{R}_A$ is cofinite in $\operatorname{ind} A$.

The arrows in the following diagram specify the inclusions between all these classes of algebras.

$$\begin{tikzcd}
 & & \left\{\begin{matrix}\text{repres.}\\ \text{directed}\end{matrix}\right\} \arrow[r] \arrow[d] & \left\{\begin{matrix}\text{finite}\\ \text{repres.}\\ \text{type}\end{matrix}\right\} \arrow[d] \\
\{\text{tilted}\} \arrow[r] & \{\text{quasitilted}\} \arrow[r] & \{\text{shod}\} \arrow[r] & \left\{\begin{matrix}\text{weakly}\\ \text{shod}\end{matrix}\right\} \arrow[r] & \{\text{laura}\}\,.
\end{tikzcd}$$

The laura algebras are characterised by their Auslander-Reiten structures.

Theorem 8.2.9 ([AC03, RS04])**.** *Assume that A is not quasitilted. The following assertions are equivalent.*

(i) A is laura.
(ii) There exists an Auslander-Reiten component Γ of $\operatorname{mod}(A)$ which contains both a projective and an injective and which is faithful, generalised standard, and such that at most finitely many vertices lie on oriented cycles.

If A is laura, then this Auslander-Reiten component is the unique one which contains both an injective and a projective.

When A is laura and not quasitilted, the *connecting component* of A is the unique Auslander-Reiten component of $\operatorname{mod}(A)$ which contains both an injective and a projective. In particular, A is weakly shod if and only if the connecting component of A contains no oriented cycles ([CL03]).

Thus, a laura algebra with a connecting component is a tilted algebra or a laura algebra which is not quasitilted.

8.2.5. *The oriented graph of generalised tilting modules.* Section 10.2 presents invariance properties of the data of Galois coverings under tilting. This is proved using inductions along the oriented graph of generalised tilting modules.

Tilting theory was extended to arbitrary projective dimensions by Miyashita. Following [Miy86], a *generalised tilting module* over A is a module $T \in \operatorname{mod}(A)$ of finite projective dimension, such that $\operatorname{Ext}^n_A(T,T) = 0$ for all integers n, and such that there exists an exact sequence in $\operatorname{mod}(A)$, where $T_0, \dots, T_d \in \operatorname{add}(T)$,

$$0 \to A \to T_0 \to \cdots \to T_d \to 0\,.$$

Such a module fits in Rickard's theory of derived equivalences: $\operatorname{RHom}_A(T,-)$ is a triangle equivalence from $\mathcal{D}^b(\operatorname{mod}(A))$ to $\mathcal{D}^b(\operatorname{mod}(\operatorname{End}_A(T)^{\mathrm{op}}))$. In other words, T is a tilting complex, or a tilting object of $\mathcal{D}^b(\operatorname{mod}(A))$.

Happel and Unger introduced in [HU05a] an ordering and an oriented graph on the (set of isomorphism classes of) generalised tilting modules. Given such a module T, denote by $T^\perp$ the class $\{X \in \operatorname{mod}(A) \mid (\forall n > 0)\ \operatorname{Ext}^n_A(T,X) = 0\}$. The ordering is defined such that, for all generalised tilting A-modules T and T',

$$T \leqslant T' \iff T^\perp \subseteq T'^\perp\,.$$

By definition, the *oriented graph of generalised tilting A-modules* is the Hasse diagram of the resulting poset. It is denoted by $\overrightarrow{\mathcal{K}}_A$. Its arrows are characterised as follows.

Theorem 8.2.10 ([HU05a])**.** *Let T, T' be non isomorphic generalised tilting A-modules. There is an arrow in $\overrightarrow{\mathcal{K}}_A$ from the isomorphism class of T to the one of T' if and only if there exist modules $X, Y, M \in \operatorname{mod}(A)$ such that*

- *X is indecomposable and $T \simeq X \oplus M$,*
- *Y is indecomposable and $T' \simeq Y \oplus M$,*

- *and there exists a short exact sequence* $0 \to X \to \widetilde{M} \to Y \to 0$ *in* $\mathrm{mod}(A)$ *with* $\widetilde{M} \in \mathrm{add}(M)$.

In general, $\overrightarrow{\mathcal{K}}_A$ is not connected. However,

- if $\overrightarrow{\mathcal{K}}_A$ contains a finite connected component $\mathcal{C}$, then $\overrightarrow{\mathcal{K}}_A = \mathcal{C}$ (see [HU05a]),
- if A is piecewise hereditary of quiver type, then there exists a hereditary abelian $\Bbbk$-linear category with at least one tilting object and such that,
  - $\mathcal{D}^b(\mathrm{mod}(A))$ and $\mathcal{D}^b(\mathcal{H})$ are equivalent as triangulated categories
  - and $\overrightarrow{\mathcal{K}}_{\mathcal{H}}$, which is defined analogously as $\overrightarrow{\mathcal{K}}_A$ from the tilting objects of $\mathcal{H}$, is connected (see [HU05b]).

8.3. **Galois coverings of algebras.** Section 10 is all about Galois coverings. The main definitions of this theory are recalled below. Here, $\Bbbk$ is an algebraically closed field and A is assumed to be basic.

8.3.1. *Locally bounded categories.* A *locally bounded* category is a small $\Bbbk$-category $\mathcal{C}$ where distinct objects are not isomorphic, all endomorphism algebras are local, and the direct sum of the morphism spaces with domain or codomain x is finite dimensional for all $x \in \mathrm{ob}(\mathcal{C})$.

The module category $\mathrm{mod}(\mathcal{C})$ of such a locally bounded category is the abelian category of contravariant $\Bbbk$-linear functors $M\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{mod}(\Bbbk)$ such that $M(x) \neq 0$ for all but finitely many objects $x \in \mathrm{ob}(\mathcal{C})$. It is Krull-Schmidt and has an Auslander-Reiten structure defined like the one of $\mathrm{mod}(A)$.

In order to consider Galois coverings of A, it is necessary to consider it as a locally bounded $\Bbbk$-category as follows. Let $e_1, \dots, e_n \in A$ be primitive and pairwise orthogonal idempotents such that $1_A = \sum_{i=1}^n e_i$. The algebra A is viewed as a locally bounded $\Bbbk$-category whose object set is $\{e_1, \dots, e_n\}$, whose space of morphisms $e_i \to e_j$ is equal to $A(e_i, e_j) = e_i A e_j$ for all $i, j \in \{1, \dots, n\}$, and whose composition of morphisms is induced by the multiplication in A.

8.3.2. *Covering functors and Galois coverings.* Covering functors were defined by Bongartz and Gabriel in [BG82]. A *covering functor* is a $\Bbbk$-linear functor $F\colon \mathcal{C} \to \mathcal{B}$ between locally bounded $\Bbbk$-categories such that the induced mapping $\mathrm{ob}(\mathcal{C}) \to \mathrm{ob}(\mathcal{B})$ is surjective and such that the two following mappings induced by F are bijective, for all $x, y \in \mathrm{ob}(\mathcal{C})$,

$$\bigoplus_{Fx'=Fx} \mathcal{C}(x', y) \longrightarrow \mathcal{B}(Fx, Fy) \text{ and } \bigoplus_{Fy'=Fy} \mathcal{C}(x, y') \longrightarrow \mathcal{B}(Fx, Fy)\,.$$

Galois coverings were defined by Gabriel in [Gab81] using quotients $\mathcal{C}/G$ of locally bounded $\Bbbk$-categories $\mathcal{C}$ by free actions of groups G. They are particular cases of covering functors. A covering functor $F\colon \mathcal{C} \to \mathcal{B}$ is a *Galois covering* with group G if and only G acts on $\mathcal{C}$ by $\Bbbk$-category automorphisms in such a way that F is G-invariant and the induced action of G on $F^{-1}(x)$ is transitive and free for all $x \in \mathrm{ob}(\mathcal{B})$. A *connected* Galois covering is a Galois covering $\mathcal{C} \to \mathcal{B}$ such that the $\Bbbk$-category $\mathcal{C}$ is connected, which entails that so is $\mathcal{B}$.

8.3.3. *The push down functor.* Let $F\colon \mathcal{C} \to \mathcal{B}$ be a covering functor. The *push down* functor ([BG82]) is isomorphic to the extension-of-scalars functor $\mathrm{mod}(\mathcal{C}) \to \mathrm{mod}(\mathcal{B})$. For the sake of simplicity, these two functors are identified and denoted by F_λ. It is exact and preserves simpleness and projectiveness. The induced functor $\mathcal{D}^b(\mathrm{mod}(\mathcal{C})) \to \mathcal{D}^b(\mathrm{mod}(\mathcal{B}))$ is also denoted by F_λ.

Assume, in addition, that F is a Galois covering with group G. For all $g \in G$, the restriction-of-scalars along the $\Bbbk$-category automorphism $\mathcal{C} \to \mathcal{C}$, $x \mapsto g \cdot x$ is automorphism of the triangulated category $\mathcal{D}^b(\mathrm{mod}(\mathcal{C}))$. This yields a strict

action $(g, M) \mapsto M^g$ of G on $\mathcal{D}^b(\mathrm{mod}(\mathcal{C}))$ on the right by strict automorphisms of triangulated category. For this action, F_λ is G-invariant and the following mapping induced by F_λ is bijective, for all $M, N \in \mathcal{D}^b(\mathrm{mod}(\mathcal{C}))$,

$$\bigoplus_{g \in G} \mathcal{D}^b(\mathrm{mod}(\mathcal{C}))(M, N^g) \to \mathcal{D}^b(\mathrm{mod}(\mathcal{B}))(F_\lambda M, F_\lambda N)\,.$$

Given $M \in \mathcal{D}^b(\mathrm{mod}(\mathcal{C}))$, the *stabiliser* of M is the subgroup $\{g \in G \mid M^g \simeq M\}$ denoted by G_M.

9. Covering techniques in Auslander-Reiten theory and applications

This section presents the results of the articles [CLMT11, CLMT19a, CLMT19b] written in collaboration with Chaio and Trepode. These articles deal with properties of the Auslander-Reiten theory and of the filtrations of the module categories by the powers of their radicals. Let A be a finite dimensional algebra over a perfect field $\Bbbk$. The starting points of these articles are the two following observations, where m and n are integers,

(1) the composition of a morphism lying in rad^n with a morphism lying in rad^m lies in rad^{m+n} and in some situations it can lie in a higher power of rad;
(2) any morphism is equal modulo rad^{n+1} to a sum of compositions of paths of irreducible morphisms.

These articles provide solutions to the following problems.

- To determine more precisely the powers of the radical to which the composition of point (1) above belongs and
- to have a better control on the irreducible morphisms used in point (2).

This leads to a new characterisation of algebras of finite representation type.

These tools consist first in existence results of well-behaved functors using the covering theory of translation quivers and next in the application of their existence to determining the kernel of the natural transformation

$$\bigoplus_{n \geqslant 0} \frac{\mathrm{rad}^n(-, X)}{\mathrm{rad}^{n+1}(-, X)} \xrightarrow{-\cdot f} \bigoplus_{n \geqslant 0} \frac{\mathrm{rad}^{n+d}(-, Y)}{\mathrm{rad}^{n+d+1}(-, Y)}\,, \tag{9.0.3.1}$$

where $f\colon X \to Y$ is a morphism in $\mathrm{mod}(A)$ lying in rad^d and not lying in rad^{d+1} for some integer d. When f is irreducible, the smallest possible degree of a non zero homogeneous component of the kernel of (9.0.3.1) is actually the left degree of f. Accordingly, the above tools also provide a characterisation of the degrees of irreducible morphisms defined by Liu in [Liu92].

The results of the articles [CLMT11, CLMT19a, CLMT19b] are presented in the subsequent sections as follows.

- Section 9.1 presents the properties of well-behaved functors including their existence.
- Section 9.2 presents the application of these results to the description of the kernel of (9.0.3.1) and to the characterisation of the left and right degrees of morphisms.
- Section 9.3 presents an application of the results of section 9.2 to a new characterisation of algebras of finite representation type.
- Section 9.4 presents results relative to the compositions of paths of irreducible morphisms and to the powers of the radical to which these compositions belong.

In this section, the following convention is used: for all composable mappings $f\colon X \to Y$ and $g\colon Y \to Z$, their composition $g \circ f$ is denoted by fg.

9.1. **Well-behaved functors.** This section discusses the existence and properties of these functors in the following setting.

Setting 9.1.1. Let $\Bbbk$ be a perfect field. Let A be a finite dimensional $\Bbbk$-algebra. Let Γ be an Auslander-Reiten component of $\operatorname{mod}(A)$. Let $\pi\colon \widetilde{\Gamma} \to \Gamma$ be a translation quiver covering.

The translation quiver covering π and the standard $\Bbbk$-modulation of Γ yield a $\Bbbk$-modulation on $\widetilde{\Gamma}$ such that $\kappa_x = \kappa_{\pi x}$ for all vertices x of $\widetilde{\Gamma}$ and $M(x,y) = M(\pi x, \pi y)$ for all arrows $x \to y$ in $\widetilde{\Gamma}$. This modulation yields the mesh category $\Bbbk(\widetilde{\Gamma})$ and well-behaved functors are understood in the following sense.

Definition 9.1.2. Keep setting 9.1.1. A *well-behaved* functor associated with π is a $\Bbbk$-linear functor $F\colon \Bbbk(\widetilde{\Gamma}) \to \operatorname{ind}\Gamma$ such that,

(a) $Fx = \pi x$ for all vertices x of $\widetilde{\Gamma}$,
(b) for all vertices x of $\widetilde{\Gamma}$, the $\Bbbk$-algebra mapping from κ_x to $\operatorname{End}_A(\pi x)$ induced F is a section of the natural surjection $\operatorname{End}_A(\pi x) \twoheadrightarrow \kappa_{\pi x}$, and
(c) for all arrows $x \to y$ of $\widetilde{\Gamma}$, the following $\Bbbk$-linear composite mapping is a $\kappa_x - \kappa_y$-linear section

$$\operatorname{irr}(\pi x, \pi y) = M(x,y) \hookrightarrow \Bbbk(\widetilde{\Gamma})(x,y) \xrightarrow{F} \operatorname{Hom}_A(\pi x, \pi y)\,.$$

When $\Bbbk$ is algebraically closed, this definition coincides with the one used by Riedtmann and by Bongartz and Gabriel in [Rie80] and [BG82], respectively, for algebras of finite representation type. Well-behaved functors do not always exist. For instance, if $\Bbbk$ is algebraically closed, A is of finite representation type and not standard, and $\widetilde{\Gamma} = \Gamma$, then F does not exist. In the setting under consideration, well-behaved functors exist as soon as $\widetilde{\Gamma}$ is *with length,* that is, there exists a mapping ℓ from the vertex set of $\widetilde{\Gamma}$ to $\mathbb{Z}$ such that $\ell(y) = \ell(x) + 1$ for all arrows $x \to y$ in $\widetilde{\Gamma}$.

Example 9.1.3 ([BG82]). If $\pi\colon \widetilde{\Gamma} \to \Gamma$ is the universal translation quiver covering, then $\widetilde{\Gamma}$ is with length.

The results of Section 9 are based on the existence of well-behaved functors.

Theorem 9.1.4. *Keep setting 9.1.1. If $\widetilde{\Gamma}$ is with length, then there exists a well-behaved functor $F\colon \Bbbk(\widetilde{\Gamma}) \to \operatorname{ind}\Gamma$.*

Here are some explanations on how this existence is proved. A well-behaved functor $\Bbbk(\widetilde{\Gamma}) \to \operatorname{ind}\Gamma$ is defined by the following data,

- a $\Bbbk$-algebra section $\kappa_x \to \operatorname{End}_A(\pi x)$ of the quotient $\operatorname{End}_A(\pi x) \twoheadrightarrow \kappa_{\pi x}$, for all vertices x of $\widetilde{\Gamma}$,
- a $\kappa_x - \kappa_y$-linear section $M(x,y) \to \operatorname{rad}(\pi x, \pi y)$ of the canonical surjection $\operatorname{rad}(\pi x, \pi y) \to \operatorname{irr}(\pi x, \pi y)$, for all arrows $x \to y$ of $\widetilde{\Gamma}$,

such that, for all non injective vertices x of $\widetilde{\Gamma}$, the image of γ_x (see section 8.1.1) under the mapping

$$\bigoplus_{?\to x \text{ in } \widetilde{\Gamma}} \operatorname{irr}(\pi x, \pi ?) \otimes_{\kappa_{\pi ?}} \operatorname{irr}(\pi ?, \tau_A^{-1}\pi x) \longrightarrow \operatorname{rad}^2(\pi x, \tau_A^{-1}\pi x)$$

induced by these sections is equal to zero. The existence of a collection of all such sections is proved by induction on the full subquivers of $\widetilde{\Gamma}$. The inductive step is based on the following facts.

- By the work of Wedderburn and Malcev, for all finite dimensional $\Bbbk$-algebras E, there exists at least one $\Bbbk$-algebra section $E/\operatorname{rad}(E) \to E$ of the canonical surjection $E \to E/\operatorname{rad}(E)$ because $\Bbbk$ is perfect.

- For all arrows $x \to y$ of $\widetilde{\Gamma}$ the tensor product $\kappa_x \otimes_{\Bbbk} \kappa_y^{\mathrm{op}}$ of division $\Bbbk$-algebras is semi-simple because $\Bbbk$ is perfect.
- For all meshes in $\widetilde{\Gamma}$,

$$\begin{tikzcd} & x_1 \arrow[dr] \arrow[d, dotted, no head] & \\ x \arrow[ur] \arrow[dr] & & \tau^{-1}x\,, \\ & x_r \arrow[ur] & \end{tikzcd}$$

  for all collections $\{\kappa_x \to \mathrm{End}_A(\pi x)\} \cup \{\kappa_{x_i} \to \mathrm{End}_A(\pi x_i)\}_{i \in \{1,\dots,r\}}$ of $\Bbbk$-algebra sections, and for all collections $\{M(x, x_i) \to \mathrm{rad}(\pi x, \pi x_i)\}_i$ of bimodule sections, there exists a $\Bbbk$-algebra section $\kappa_{\tau^{-1}x} \to \mathrm{End}_A(\tau_A^{-1}\pi x)$ and a collection $\{M(x_i, \tau^{-1}x) \to \mathrm{rad}(\pi x_i, \tau_A^{-1}\pi x)\}_i$ of bimodule sections such that γ_x is mapped onto zero, in the sense expressed previously.
- The dual version of the preceding point.

The assumption that $\widetilde{\Gamma}$ is with length ensures that the induction abuts.

The main reason for considering well-behaved functors is the following property. Its proof is based on the axioms defining these functors and on the fact that $\pi\colon \widetilde{\Gamma} \to \Gamma$ is a translation quiver covering.

Proposition 9.1.5. *Keep setting 9.1.1. Let $F\colon \Bbbk(\widetilde{\Gamma}) \to \mathrm{ind}\,\Gamma$ be a well-behaved functor. For all vertices x, y of $\widetilde{\Gamma}$ and all integers n,*

(a) the two following maps induced by F are bijective

$$\bigoplus_{Fz=Fy} \mathfrak{R}^n\Bbbk(\widetilde{\Gamma})(x,z)/\mathfrak{R}^{n+1}\Bbbk(\widetilde{\Gamma})(x,z) \quad \to \quad \mathrm{rad}^n(Fx,Fy)/\mathrm{rad}^{n+1}(Fx,Fy)$$

$$\bigoplus_{Fz=Fy} \mathfrak{R}^n\Bbbk(\widetilde{\Gamma})(z,x)/\mathfrak{R}^{n+1}\Bbbk(\widetilde{\Gamma})(z,x) \quad \to \quad \mathrm{rad}^n(Fy,Fx)/\mathrm{rad}^{n+1}(Fy,Fx) \quad ,$$

(b) the two following maps induced by F are injective

$$\bigoplus_{Fz=Fy} \Bbbk(\widetilde{\Gamma})(x,z) \to \mathrm{Hom}_A(Fx,Fy) \quad \textit{and} \quad \bigoplus_{Fz=Fy} \Bbbk(\widetilde{\Gamma})(z,x) \to \mathrm{Hom}_A(Fy,Fx)\,,$$

(c) and Γ is generalised standard if and only if the two maps of (b) are bijective.

Bibliographical note. Assume that $\Bbbk$ is algebraically closed. When A is selfinjective and of finite representation type and when $\mathrm{mod}(A)$ is replaced by $\underline{\mathrm{mod}(A)}$, Theorem 9.1.4 and the fact that the mappings of part (b) in Proposition 9.1.5 are bijective were proved by Riedtmann in [Rie80]. When A is of finite representation type, the same statements were proved by Bongartz and Gabriel in [BG82].

9.2. **Compositions of irreducible morphisms.** The results of section 9.1 apply to the problems mentioned in the introduction of section 9 using the following general idea. Letting Γ be an Auslander-Reiten component of $\mathrm{mod}(A)$ containing the indecomposable modules involved in one of these problems and letting $F\colon \Bbbk(\widetilde{\Gamma}) \to \mathrm{ind}\,\Gamma$ be a well-behaved functor associated with the universal translation quiver covering $\pi\colon \widetilde{\Gamma} \to \Gamma$, the considered problem, which is formulated in $\mathrm{ind}\,\Gamma$, may be lifted to a problem in $\Bbbk(\widetilde{\Gamma})$ by means of Proposition 9.1.5; it may be solved in $\Bbbk(\widetilde{\Gamma})$, and the resulting solution may yield a solution to the original problem in $\mathrm{ind}\,\Gamma$ using the functor $F\colon \Bbbk(\widetilde{\Gamma}) \to \mathrm{ind}\,\Gamma$. The problem in $\Bbbk(\widetilde{\Gamma})$ is indeed easier to solve than the one in $\mathrm{ind}\,\Gamma$ because of the following property.

Lemma 9.2.1. *Let $\widetilde{\Gamma}$ be a modulated translation quiver with length. For all integers ℓ and all vertices x, y of $\widetilde{\Gamma}$, if there exists a path of length ℓ from x to y in $\widetilde{\Gamma}$, then*

(a) $\Bbbk(\widetilde{\Gamma})(x,y) = \mathfrak{R}\Bbbk(\widetilde{\Gamma})(x,y) = \mathfrak{R}^2\Bbbk(\widetilde{\Gamma})(x,y) = \cdots = \mathfrak{R}^\ell\Bbbk(\widetilde{\Gamma})(x,y)$

(b) and $\mathfrak{R}^i\Bbbk(\widetilde{\Gamma})(x,y)=0$ *for all* $i>\ell$.

These considerations apply to the kernel of (9.0.3.1) when f is homogeneous up to rad^ℓ and has finite left degree. Homogeneous morphisms are defined as follows.

Definition 9.2.2. Let A be a finite dimensional algebra over a perfect field $\Bbbk$. Let Γ be an Auslander-Reiten component of $\mathrm{mod}(A)$. Let $f\colon \oplus_s X_s \to \oplus_t Y_t$ be a morphism in $\mathrm{mod}(A)$ such that $X_s, Y_t \in \Gamma$ for all s,t.

(1) The morphism f is *homogeneous* if there exists a well-behaved functor $F\colon \Bbbk(\widetilde{\Gamma}) \to \mathrm{ind}\,\Gamma$ associated with some translation quiver covering $\widetilde{\Gamma} \to \Gamma$ where $\widetilde{\Gamma}$ is with length, there exist tuples $(x_s)_s$ and $(y_t)_t$ of vertices of $\widetilde{\Gamma}$, and there exist morphisms $(\varphi_{s,t})_{s,t} \in \oplus_{s,t}\Bbbk(\widetilde{\Gamma})(x_s,y_t)$ such that $X=\oplus_x Fx_s$, $Y=\oplus_t Fy_t$, and, for all s,t, the component $Fx_s \to Fy_t$ of f is equal to $F(\varphi_{s,t})$.
(2) The morphism f is *homogeneous up to* rad^d, where d is an integer, if f is the sum of a homogeneous morphism and a morphism lying in rad^{d+1}.

Example 9.2.3. It follows from Proposition 9.1.5 that, if $\oplus_s X_s$ or $\oplus_t Y_t$ is indecomposable, then f is homogeneous up to rad^d for all integers d. If in addition f is *freely irreducible,* then f is homogeneous. Here is an explanation of this concept. Given a morphism $[f_{i,j}\ ;\ i,j]\colon X \to \oplus_{i=1}^r X_i^{n_i}$, where $X, X_1, \dots, X_r$ are pairwise non isomorphic indecomposable A-modules, this morphism is irreducible if and only if, for all $i \in \{1,\dots,r\}$, the family $(\overline{f_{i,j}})_{j=1,\dots,n_i}$ of $\mathrm{irr}(X,X_i)$ is free over the division algebra $\kappa_{X_i}^{\mathrm{op}}$. By definition, the morphism $[f_{i,j}\ ;\ i,j]$ is freely irreducible if, for all i, the family $(\overline{f_{i,j}})_j$ of $\mathrm{irr}(X,X_i)$ is free over the semi-simple algebra $\kappa_X \otimes_\Bbbk \kappa_{X_i}^{\mathrm{op}}$. In particular, this morphism is freely irreducible, and hence homogeneous, in any of the following situations,

- $\Bbbk$ is algebraically closed, or
- $\kappa_X = \Bbbk$, or
- $\kappa_{X_i} = \Bbbk$ for all i.

Freely irreducible morphisms with indecomposable codomains are defined dually.

The definition of the left and right degrees of a morphism is just an extension of the one of Liu for irreducible morphisms.

Definition 9.2.4. Let $f\colon X \to Y$ be a morphism in $\mathrm{mod}(A)$ lying in rad^d and not lying in rad^{d+1} for some integer d. The *left degree* of f is the least integer n such that the following mapping is not injective for at least one indecomposable A-module Z,

$$\frac{\mathrm{rad}^n(Z,X)}{\mathrm{rad}^{n+1}(Z,X)} \xrightarrow{-\cdot f} \frac{\mathrm{rad}^{n+d}(Z,Y)}{\mathrm{rad}^{n+d+1}(Z,Y)}\ ;$$

it is ∞ if no such integer n exists. It is denoted by $d_\ell(f)$. The right degree of f is defined dually and denoted by $d_r(f)$.

With these definitions, the kernel of (9.0.3.1) is described as follows.

Theorem 9.2.5. *Let A be a finite dimensional algebra over a perfect field. Let Γ be an Auslander-Reiten component of* $\mathrm{mod}(A)$. *Let $f\colon X \to Y$ be a morphism in* $\mathrm{mod}(A)$ *such that $X, Y \in \mathrm{add}(\Gamma)$. Assume that f is homogeneous up to* rad^d *for some integer d and that its left degree is finite, and denoted by n. Then, there exist*

- *a morphism $f'\colon X \to Y$ such that $f - f' \in \mathrm{rad}^{d+1}$ and such that the inclusion mapping $i\colon \mathrm{Ker}(f') \to X$ lies in* rad^n *and does not lie in* rad^{n+1}

- *and direct sum decompositions*

$$\begin{cases} \operatorname{Ker}(f') = K^{(\infty)} \oplus \oplus_{m \geqslant n} K^{(m)} \\ K^{(m)} = \oplus_r K_r^{(m)} & \text{for all } m \geqslant n, \end{cases}$$

such that

- *the inclusion mapping* $K^{(\infty)} \to X$ *lies in* $\operatorname{rad}^\infty$,
- $K_r^{(m)}$ *lies in* Γ *and the inclusion mapping* $K_r^{(m)} \to X$ *lies in* rad^m *and does not lie in* rad^{m+1}, *for all* r *and all* $m \geqslant n$,
- *and for all integers* $\ell \geqslant n$ *and all indecomposable* A*-modules* Z, *the following sequence is exact,*

$$0 \to \bigoplus_{n \leqslant m \leqslant \ell} \frac{\operatorname{rad}^{\ell-m}}{\operatorname{rad}^{\ell-m+1}}(Z, K^{(m)}) \xrightarrow{-\cdot i} \frac{\operatorname{rad}^{\ell}}{\operatorname{rad}^{\ell+1}}(Z, X) \xrightarrow{-\cdot f} \frac{\operatorname{rad}^{\ell+d}}{\operatorname{rad}^{\ell+d+1}}(Z, Y)\,.$$

If, in addition, f *is homogeneous, then the same conclusion holds true if* f' *is replaced by* f.

Bibliographical note. In [IT84b], Igusa and Todorov proved Theorem 9.2.2 for irreducible morphisms with indecomposable domains or indecomposable codomains when A is an Artin algebra of finite representation type or a finite dimensional algebra over an algebraically closed field. In this situation, f' may be replaced by f. Their proof does not involve homogeneous morphisms or well-behaved functors. It is based on diagrams in $\operatorname{ind} A$ made of irreducible morphisms and forming almost split sequences. The existence of such diagrams is one of the problems addressed in that article (see [IT84b, Section 1]).

Theorem 9.2.5 yields the following characterisation of when the left degree of an irreducible morphism is finite.

Corollary 9.2.6. *Let* A *be a finite dimensional algebra over a perfect field. Let* $f\colon X \to Y$ *be an irreducible morphism such that* $X \in \operatorname{ind} A$ *or* $Y \in \operatorname{ind} A$. *Let* n *be an integer. There exists an irreducible morphism* $f'\colon X \to Y$ *such that* $f - f' \in \operatorname{rad}^2$ *and such that the following assertions are equivalent,*

(i) $d_\ell(f) = n$,
(ii) the inclusion mapping $\operatorname{Ker}(f') \to X$ *lies in* rad^n *and does not lie in* rad^{n+1}.

If, in addition, f *is freely irreducible, then* f' *may be replaced by* f.

Bibliographical note. This characterisation was proved by Chaio in [Cha10] assuming that $\Bbbk$ is algebraically closed and that A is of finite representation type and standard. Previously, it was also proved by Chaio, Platzeck and Trepode in [CPT04] assuming that A is an Artin algebra and that the involved Auslander-Reiten component of $\operatorname{mod}(A)$ is generalised standard and with length.

Corollary 9.2.6 has the following consequences, keeping the same context.

(1) If $d_\ell(f) < \infty$, then f is not a monomorphism.
(2) If f is a left minimal almost split morphism then $d_\ell(f) < \infty$.
(3) If A is of finite representation type, the following are equivalent,
  (i) $d_\ell(f) < \infty$,
  (ii) $d_r(f) = \infty$,
  (iii) f is an epimorphism.

Finally, the exact sequence of Theorem 9.2.5 provides new information on the behaviour of the degrees of irreducible morphisms in almost split sequences and on the kernels of irreducible morphisms.

Corollary 9.2.7. *Let* A *be a finite dimensional algebra over a perfect field* $\Bbbk$.

(1) For all almost split sequences in $\operatorname{mod}(A)$

$$\begin{tikzcd} & & X' \arrow[dr, "f'"] & & \\ 0 \arrow[r] & \tau_A Y \arrow[ur, "g"] \arrow[dr, "g'"'] & & Y \arrow[r] & 0\,, \\ & & X \arrow[ur, "f"'] & & \end{tikzcd}$$

if both X *and* X' *are non zero, then* $d_\ell(g) = d_\ell(f) - 1$ *in* $\mathbb{N} \cup \{\infty\}$.

(2) Let $f_1, f_2 \colon X \to Y$ *be irreducible morphisms in* $\operatorname{mod}(A)$. *Assume that* $X \in \operatorname{ind} A$ *and* $\kappa_X \simeq \Bbbk$, *or* $Y \in \operatorname{ind} A$ *and* $\kappa_Y \simeq \Bbbk$. *If* $d_\ell(f_1) < \infty$, *then* $d_\ell(f_1) = d_\ell(f_2)$ *and* $\operatorname{Ker}(f_1) \simeq \operatorname{Ker}(f_2)$.

In this corollary, part (1) is proved using that f and g have isomorphic kernels and part (2) is proved applying twice the exact sequence of Theorem 9.2.5: first to the composite morphism $\operatorname{Ker}(f_1) \hookrightarrow X \xrightarrow{f_2} Y$ taking $f = f_2$; next to the composite morphism $\operatorname{Ker}(f_2) \hookrightarrow X \xrightarrow{f_1} Y$ taking $f = f_1$.

Bibliographical note. In the more general situation where A is an Artin algebra, Liu proved in [Liu92] that $d_\ell(g) \leqslant d_\ell(f) - 1$ and that $d_\ell(f_1) = d_\ell(f_2)$ for all parallel irreducible morphisms f_1 and f_2 between indecomposable modules.

The dual formulations of all these results also hold true.

9.3. **Application to algebras of finite representation type.** The characterisation of the left and right degrees of irreducible morphisms presented in section 9.2 yields the following characterisation of algebras of finite representation type.

Theorem 9.3.1. *Let* A *be a finite dimensional algebra over a perfect field* $\Bbbk$. *The following assertions are equivalent.*

(i) A *is of finite representation type,*

(ii) All left minimal almost split morphisms with injective domains have finite left degrees.

(iii) All irreducible epimorphisms with indecomposable domains or indecomposable codomains have finite left degrees.

If A *is of finite representation type, then there exists a left minimal almost split morphism with injective domain and whose left degree is the supremum of the left degrees of all irreducible epimorphisms with indecomposable domains or indecomposable codomains.*

Here are some explanations on the implication $(ii) \Longrightarrow (i)$, which is the main part of Theorem 9.3.1. Given an indecomposable and non injective $X \in \operatorname{ind} A$, there exists an indecomposable injective $I \in \operatorname{ind} A$ such that the embedding $\operatorname{soc}(I) \to I$ factors through X. Following section 9.2, if $I \to I/\operatorname{soc}(I)$ has finite left degree, n, then there exists a path in Γ from X to I and with length at most n. Since every vertex of Γ is attached to finitely many arrows only, then (ii) implies (i).

9.4. **Paths of irreducible morphisms.** Section 9.2 also applies to paths of irreducible morphisms. On one hand, under setting 9.1.1, all paths in Γ lift to paths in $\widetilde{\Gamma}$. Proposition 9.1.5 and Lemma 9.2.1 can therefore be used to prove the following.

Proposition 9.4.1. *Let* A *be a finite dimensional algebra over a perfect field* $\Bbbk$. *Let* $X_1, \dots, X_{n+1} \in \operatorname{ind} A$. *The following assertions are equivalent*

(a) There exist irreducible morphisms $X_1 \xrightarrow{h_1} \cdots \xrightarrow{h_n} X_{n+1}$ *such that* $h_1 \cdots h_n \in \operatorname{rad}^{n+1} \setminus \{0\}$.

(b) There exist an irreducible morphism $f_i\colon X_i \to X_{i+1}$ and a morphism $\varepsilon_i\colon X_i \to X_{i+1}$, for all i, such that $f_1 \cdots f_n = 0$, such that $\varepsilon_1 \cdots \varepsilon_n \neq 0$ and such that, for all i, either $\varepsilon_i \in \mathrm{rad}^2$ or else $\varepsilon_i = f_i$.

Bibliographical note. In previous articles ([CCT08, CCT07, CT10]) Chaio and her collaborators proved the characterisation of Proposition 9.4.1 in various specific cases.

On the other hand, the characterisation of the left and right degrees of irreducible morphisms provide a refinement of Proposition 9.4.1.

Proposition 9.4.2. *Let A be a finite dimensional $\Bbbk$-algebra over a perfect field. Let $X_0 \xrightarrow{f_1} X_1 \to \cdots \to X_{n-1} \xrightarrow{f_n} X_n$ be a chain of irreducible morphisms between indecomposable A-modules. For each t, let $f'_t\colon X_{t-1} \to X_t$ be such as f' in Theorem 9.2.5 when $f = f_t$. Consider the following assertions.*

(i) $f_1 \cdots f_n \in \mathrm{rad}^{n+1}$.
(ii) There exists $t \in \{1, \ldots, n\}$ such that $d_\ell(f_t) \leqslant t-1$, and there exists $h \in \mathrm{rad}^{t-1-d_\ell(f_t)}(X_0, \mathrm{Ker}(f'_t))$ not lying in $\mathrm{rad}^{t-d_\ell(f_t)}$, and such that $f_1 \cdots f_{t-1} - hi \in \mathrm{rad}^t$ (where $i\colon \mathrm{Ker}(f'_t) \to X_{t-1}$ is the inclusion morphism).
(iii) There exists $t \in \{1, \ldots, n\}$ such that $d_l(f_t) \leqslant t-1$, there exists a path of irreducible morphisms of length $t-1-d_\ell(f_t)$, from X_0 to $\mathrm{Ker}(f'_t)$, and with non zero composition, and there exists a path $X_0 \to X_1 \to \cdots \to X_{t-1} \to X_t$ of irreducible morphisms with zero composition.

Then (i) and (ii) are equivalent and imply (iii). If, moreover, $\dim_\Bbbk \mathrm{irr}(X_{s-1}, X_s) = 1$ for every $s \in \{1, \ldots, t\}$, then (iii) implies (i) and (ii).

9.5. **Open questions.** Let $\Bbbk$ be an Artin commutative ring, A be an Artin algebra, and Γ be an Auslander-Reiten component of $\mathrm{mod}(A)$. Two remarks can be made on the results presented previously.

The first remark is about Theorem 9.1.5, which deals with the existence of well-behaved functors. This theorem assumes that $\Bbbk$ is a perfect field. This is because this hypothesis entails the following assertions.

(1) The $\Bbbk$-algebra surjection $\mathrm{End}_A(X) \to \kappa_X$ has a section, for all $X \in \Gamma$.
(2) The $\kappa_X - \kappa_Y$-bimodule $\mathrm{irr}(X, Y)$ is semi-simple, for all arrows $X \to Y$ of Γ.

These assertions do not hold true in general. For instance, if A is of finite representation type, then (1) implies the existence of a semi-simple subalgebra S of A such that $A = S \oplus \mathrm{rad}(A)$. This yields the following question.

Question 4. When do (1) and (2), and hence the conclusion of Theorem 9.1.5, hold true without assuming that $\Bbbk$ is a perfect field?

It might be useful to restrict first the framework of this question to Artin algebras of finite representation type.

The second remark is about the morphism f' in Theorem 9.2.5 and Corollary 9.2.6. This morphism is the homogeneous morphism mentioned in part (2) of Definition 9.2.2. In particular, there is no control on $\mathrm{Ker}(f')$, which might be source of problems in applications. In view of Corollary 9.2.7, it is hence worth having a clear answer to the following question.

Question 5. Let $X, Y \in \mathrm{mod}(A)$ be such that at least one of X and Y is indecomposable. Let $f_1, f_2\colon X \to Y$ be irreducible morphisms such that $d_\ell(f_1) < \infty$. Is it true that $d_\ell(f_2) = d_\ell(f_1)$ and that $\mathrm{Ker}(f_1) \simeq \mathrm{Ker}(f_2)$?

10. Simple connectedness and Hochschild cohomology

Setting 10.0.1. Let $\Bbbk$ be an algebraically closed field. Let A be a finite dimensional, basic, and connected $\Bbbk$-algebra.

This section presents the results of the articles [LM08, LM11, LM10b, ABLM10] whose aim is to answer the questions (Q1), (Q2), and (Q3) of section 7.3 about Galois coverings, simple connectedness, and Hochschild cohomology. The provided answers concern several classes of algebras related to tilting theory and to algebras of finite representation type, namely, the piecewise hereditary algebras of quiver type and the standard laura algebras (see section 8.2). Here, simple connectedness is understood in the following broader sense.

Definition 10.0.2. The algebra A is *simply connected* if it has no proper Galois covering.

In this sense, A is simply connected if and only if, for all bound quivers (Q, I) such that $A \simeq \Bbbk Q/I$, the fundamental group $\pi_1(Q, I)$ defined by Martinez-Villa and de la Peña in [MVdlPn83] is trivial.

Given a Galois covering $F\colon A' \to A$ and a basic tilting complex $T \in \mathcal{D}^b(\mathrm{mod}(A))$, if every indecomposable direct summand of T is isomorphic to the image under $F_\lambda\colon \mathcal{D}^b(\mathrm{mod}(A')) \to \mathcal{D}^b(\mathrm{mod}(A))$ of an object whose stabiliser is trivial, then F_λ yields a Galois covering of $\mathrm{End}_A(T)^{\mathrm{op}}$. This construction, which is explained below and is clearly a major step towards an answer to (Q2), yields the following question.

(Q4) Are all indecomposable direct summands of T isomorphic to the image under F_λ of objects of $\mathcal{D}^b(\mathrm{mod}(A'))$ with trivial stabilisers?

The presentation of the results of the articles [LM08, LM11, LM10b, ABLM10] is made according to the following structure.

- Section 10.1 presents sufficient conditions for (Q4) to have a positive answer.
- Section 10.2 details how to transfer the Galois coverings of A to Galois coverings of $\mathrm{End}_A(T)^{\mathrm{op}}$, it also explains when this transfer is a bijective correspondence. This gives partial answers to (Q2).
- Section 10.3 presents classes of algebras for which (Q3) has a positive answer.
- Section 10.4 explains how all this information can be used to answer (Q1) for the above mentioned classes of algebras.

10.1. Tilting objects of the first kind.

Definition 10.1.1. Keep setting 10.0.1. Let $F\colon A' \to A$ be a Galois covering with group denoted by G. Let $T \in \mathcal{D}^b(\mathrm{mod}(A))$ be a tilting object. Denote by $\mathcal{H}_{T,F}$ the following assertion: "for all indecomposable direct summands X of T, there exists $\widetilde{X} \in \mathcal{D}^b(\mathrm{mod}(A'))$ such that $F_\lambda \widetilde{X} \simeq X$ and $G_{\widetilde{X}} = \{e\}$".

It is not known whether $\mathcal{H}_{F,T}$ holds true for all F and T. However, it is possible to prove that, for all Galois coverings F of A, then $\mathcal{H}_{F,\bullet}$ being true is invariant within a mutation class of tilting objects. This is a consequence of a more general phenomenon which is valid in the following setting.

Setting 10.1.2. Let $\Bbbk$ be a field. Let $\mathcal{C}$ and $\mathcal{C}'$ be Hom-finite, Krull-Schmidt, and triangulated $\Bbbk$-linear categories. Let G be a group acting strictly on $\mathcal{C}'$ by strict triangle automorphisms. Let $\varphi\colon \mathcal{C}' \to \mathcal{C}$ be a G-invariant triangle functor such that the following mapping induced by φ is bijective, for all $U, V \in \mathcal{C}'$,

$$\bigoplus_{g \in G} \mathcal{C}'(U, V^g) \longrightarrow \mathcal{C}(\varphi U, \varphi V)\,. \tag{10.1.0.2}$$

For all $g \in G$, a morphism in $\mathcal{C}(\varphi U, \varphi V)$ lying in the image of $\mathcal{C}'(U, V^g)$ under (10.1.0.2) is said to be *homogeneous* of degree g.

Here is the general phenomenon mentioned previously. Its dual version holds true.

Lemma 10.1.3. *Keep setting 10.1.2. Let $X \xrightarrow{u} \oplus_i M_i \xrightarrow{v} Y \to \Sigma X$ be a distinguished triangle of $\mathcal{C}$, where M_i is indecomposable for all i. Denote $\oplus_i M_i$ by M.*

(1) If $\mathcal{C}(Y, \Sigma M) = 0$, if there exists $\widetilde{X} \in \mathcal{C}'$ such that $\varphi\widetilde{X} \simeq X$, and if there exists $\widetilde{M}_i \in \mathcal{C}'$ such that $\varphi\widetilde{M}_i \simeq M_i$ for all i, then there exists $\widetilde{Y} \in \mathcal{C}'$ such that $\varphi\widetilde{Y} \simeq Y$.

(2) If, in addition, $\mathcal{C}(M, \Sigma X) = 0$, $G_{\widetilde{X}} = \{e\}$, and $G_{\widetilde{M}_i} = \{e\}$ for all i, then $G_{\widetilde{Y}} = \{e\}$.

The whole section 10 is based on this lemma. Hence, here are some explanations. On one hand, there exists an automorphism $\lambda\colon M \xrightarrow{\sim} M$ such that the number of non zero homogeneous components of $X \xrightarrow{\lambda \circ u} M \twoheadrightarrow M_i$ is minimal for all i. Using the equality $\mathcal{C}(Y, \Sigma M) = 0$, this entails that $X \xrightarrow{\lambda \circ u} M \twoheadrightarrow M_i$ is homogeneous or zero, for all i. Hence, $\lambda \circ u$ is the image under φ of a morphism in $\mathcal{C}'$ of the shape $\widetilde{X} \to \oplus_i \widetilde{M}_i^{g_i}$, where $g_i \in G$ for all i. Letting $\widetilde{Y}$ be the cone of this morphism of $\mathcal{C}'$ yields part (1). On the other hand, using a resulting distinguished triangle $\widetilde{X} \to \oplus_i \widetilde{M}_i^{g_i} \to \widetilde{Y} \to \Sigma\widetilde{X}$ of $\mathcal{C}'$, the proof of part (2) consists in associating to each isomorphism $\widetilde{Y} \to \widetilde{Y}^g$, where $g \in G$, an isomorphism $\widetilde{X} \to \widetilde{X}^g$.

Lemma 10.1.3 applies to mutations of tilting objects taking $\mathcal{C}' = \mathcal{D}^b(\mathrm{mod}(A'))$, $\mathcal{C} = \mathcal{D}^b(\mathrm{mod}(A))$, and $\varphi = F_\lambda$, for some Galois covering F of A. This yields the following result.

Proposition 10.1.4. *Keep setting 10.0.1. Let $F\colon A' \to A$ be a Galois covering with group denoted by G.*

(1) Both $\mathcal{H}_{F,A}$ and $\mathcal{H}_{F,DA}$ hold true.

(2) Let T and T' be generalised tilting A-modules. If they lie in the same connected component of $\overrightarrow{\mathcal{K}}_A$, then

$$\mathcal{H}_{F,T} \iff \mathcal{H}_{F,T'}\,.$$

(3) Assume, in addition, that $\mathcal{D}^b(\mathrm{mod}(A))$ is equivalent as a triangulated category to $\mathcal{D}^b(\mathcal{H})$ where $\mathcal{H}$ is a hereditary abelian category with at least one tilting object and such that $\overrightarrow{\mathcal{K}}_{\mathcal{H}}$ is connected (see section 8.2.5). Then, $\mathcal{H}_{F,T}$ holds true for all tilting objects T of $\mathcal{D}^b(\mathrm{mod}(A))$.

10.2. **Invariance of Galois coverings under tilting.** The contents of this section appear in [LM08, LM11] for generalised tilting modules and for tilting complexes over piecewise hereditary algebras. Actually, they are valid for tilting complexes of A-modules. Transferring Galois coverings of A to Galois coverings of endomorphism algebras of tilting complexes is possible in the following setting.

Setting 10.2.1. Keep setting 10.0.1. Let $F\colon A' \to A$ be a Galois covering with group denoted by G. Let $T = T_1 \oplus \cdots \oplus T_n$ be a basic tilting object of $\mathcal{D}^b(\mathrm{mod}(A))$, where $T_1, \ldots, T_n$ are indecomposable, such that $\mathcal{H}_{F,T}$ holds true.

Under this setting, there exists an isomorphism $\lambda_i\colon F_\lambda\widetilde{T}_i \to T_i$ in $\mathcal{D}^b(\mathrm{mod}(A))$, where $\widetilde{T}_i \in \mathcal{D}^b(\mathrm{mod}(A'))$, for all $i \in \{1, \ldots, n\}$. The full small subcategories of $\mathcal{D}^b(\mathrm{mod}(A))$ and $\mathcal{D}^b(\mathrm{mod}(A'))$ whose object sets are equal to $\{T_1, \ldots, T_n\}$ and $\{\widetilde{T}_i^g \mid i \in \{1, \ldots, n\},\ g \in G\}$, respectively, are locally bounded. By abuse of

notation, they are denoted by $\mathrm{End}_A(T)$ and $\mathrm{End}_{A'}(\oplus_{i,g}\widetilde{T}_i^g)$, respectively. The G-invariant functor $F_\lambda\colon \mathcal{D}^b(\mathrm{mod}(A')) \to \mathcal{D}^b(\mathrm{mod}(A))$ induces a functor between these categories,

$$\begin{array}{ccc} \mathrm{End}_{A'}(\oplus_{i,g}\widetilde{T}_i^g) & \longrightarrow & \mathrm{End}_A(T) \\ \widetilde{T}_i^g & \longmapsto & T_i \\ \widetilde{T}_i^g \xrightarrow{u} \widetilde{T}_j^g & \longmapsto & T_i \xrightarrow{\lambda_j \circ F_\lambda u \circ \lambda_i^{-1}} T_j\,. \end{array} \tag{10.2.0.3}$$

This is a Galois covering with group G and the category $\mathrm{End}_{A'}(\oplus_{i,g}\widetilde{T}_i^g)$ is connected if and only if so is A'. This yields a correspondence between equivalence classes of Galois coverings of A and equivalence of Galois coverings of $\mathrm{End}_A(T)^{\mathrm{op}}$. These equivalence classes were introduced in [LM07b]. Here is a reminder.

Definition 10.2.2 ([LM07b])**.** Keep setting 10.0.1.

(1) Two Galois coverings $F_1\colon A_1' \to A$ and $F_2\colon A_2' \to A$ are *equivalent* if there exists a commutative diagram of $\Bbbk$-categories

$$\begin{tikzcd} A_1' \arrow[r] \arrow[d, "F_1"'] & A_2' \arrow[d, "F_2"] \\ A \arrow[r] & A \end{tikzcd}$$

such that the horizontal arrows are isomorphisms and $A \to A$ maps x to x for all objects x of A. The equivalence class of a Galois covering F is denoted by $[F]$.

(2) For all groups G, the set of equivalence classes of connected Galois coverings with group G of A is denoted by $\mathrm{Gal}_G(A)$.

(3) A *universal Galois covering* of A is a connected Galois covering $A' \to A$ such that, for all connected Galois coverings $A'' \to A$, there exists a commutative diagram of $\Bbbk$-categories,

$$\begin{tikzcd} A' \arrow[dr] \arrow[dd] & \\ & A'' \arrow[d] \\ A \arrow[r] & A\,, \end{tikzcd}$$

where $A \to A$ is an isomorphism which maps x to x for all objects x of A.

Proposition 10.2.3. *Keep setting 10.0.1. Let $T = T_1 \oplus \cdots \oplus T_n$ be a basic tilting object of $\mathcal{D}^b(\mathrm{mod}(A))$, where $T_1, \ldots, T_n$ are indecomposable. Let G be a group. Assume that*

(a) $\mathcal{H}_{F,T}$ holds true for all Galois coverings F with group G of A

(b) and, for all automorphisms of $\Bbbk$-category $\psi\colon A \to A$ which map x to x for all objects x of A, then $\psi_\lambda T_i \simeq T_i$ for all $i \in \{1, \ldots, n\}$.

Then, there exists a bijective mapping

$$\begin{array}{ccc} \mathrm{Gal}_G(A) & \longrightarrow & \mathrm{Gal}_G(\mathrm{End}_A(T)) \\ \alpha & \longmapsto & \alpha_T \end{array}$$

such that, for all $\alpha \in \mathrm{Gal}_G(A)$, for all $F \in \alpha$, then α_T is the equivalence class of (10.2.0.3).

Assertion (b) in Proposition 10.2.3 seems technical. However, when the statement of Proposition 10.1.4 is rewritten so as to replace $\mathcal{H}_{F,\bullet}$ by this assertion (b), then the reformulation also holds true (and $F\colon A' \to A$ plays no role). In view of Proposition 10.1.4, Proposition 10.2.3 has the following corollary.

Corollary 10.2.4. *Keep setting 10.0.1.*

(1) *Let T be a generalised tilting A-module which lies in the connected component of $\overrightarrow{\mathcal{K}}_A$ containing A. For all groups G, there exists a bijective mapping*

$$\mathrm{Gal}_G(A) \xrightarrow{\sim} \mathrm{Gal}_G(\mathrm{End}_A(T))\,.$$

Moreover, A has universal Galois covering with group G if and only if so has $\mathrm{End}_A(T)^{\mathrm{op}}$. In particular, A is simply connected if and only if so is $\mathrm{End}_A(T)^{\mathrm{op}}$.

(2) *Assume that A is derived equivalent to $\Bbbk Q$ for some finite quiver Q without oriented cycles. Then, A has a universal Galois covering with group $\pi_1(Q)$. In particular, A is simply connected if and only if Q is a tree.*

Example 10.2.5. Assume that A is given by quiver and relations as follows,

$$\begin{tikzcd} & 2 \arrow[dl, "b"'] & \\ 1 & & 3 \arrow[ll, "a"] \arrow[ul, "c"'] \end{tikzcd}\,, \qquad \langle bc\rangle\,.$$

Let T be the APR-tilting module $e_1A \oplus e_2A \oplus \tau_A^{-1}(e_3A)$. Then, $\mathrm{End}_A(T)^{\mathrm{op}}$ is given by quiver and relations as follows,

$$\begin{tikzcd} 1 & 2 \arrow[l, "a"'] \arrow[r, shift left, "b"] & 3 \arrow[l, shift left, "c"] \end{tikzcd}, \qquad \langle bc\rangle\,.$$

There is an arrow $A \to T$ in $\overrightarrow{\mathcal{K}_A}$ and both A and $\mathrm{End}_A(T)^{\mathrm{op}}$ have a universal Galois covering with group $\mathbb{Z}$.

Example 10.2.6. Assume that A is given by quiver and relations as follows,

$$\begin{tikzcd} & & 3 \arrow[dl, "c"'] & \\ 1 & 2 \arrow[l, "a"'] & & 4 \arrow[ll, "b"'] \arrow[ul, "d"'] & 5 \arrow[l, "e"'] \end{tikzcd}\,, \qquad \langle ab - acd,\, be\rangle\,.$$

Let T be the APR-tilting module $e_1A \oplus e_2A \oplus e_3A \oplus e_4A \oplus \tau_A^{-1}(e_5A)$. Then, $\mathrm{End}_A(T)^{\mathrm{op}}$ is given by quiver and relations as follows,

$$\begin{tikzcd} & 2 \arrow[dl, "\alpha"'] & & \\ 1 & & 4 \arrow[ll, "\beta"'] \arrow[dl, "\delta"] & 5 \arrow[l, "\varepsilon"'] \\ & 3 \arrow[ul, "\gamma"] & & \end{tikzcd}\,, \qquad \langle \alpha\beta\varepsilon - \gamma\delta\varepsilon\rangle\,.$$

There is an arrow $A \to T$ in $\overrightarrow{\mathcal{K}_A}$ and both A and $\mathrm{End}_A(T)^{\mathrm{op}}$ are simply connected.

Bibliographical note. Let A be a simply connected standard algebra of finite representation type and T be a tilting A-module. Assem proved in [Ass83] that, if T is splitting as a tilting module, then $\mathrm{End}_A(T)^{\mathrm{op}}$ is also a simply connected standard algebra of finite representation type. Besides, Assem and Skowroński proved in [AS94] that $\mathrm{End}_A(T)^{\mathrm{op}}$ satisfies the separation condition, and hence it is simply connected.

10.3. **Auslander-Reiten components controlling Galois coverings.** The results in section 10.2 provide several situations in which the Galois coverings of A are controlled by those of a particular Auslander-Reiten component of $\mathrm{mod}(A)$ or of $\mathcal{D}^b(\mathrm{mod}(A))$ (when $\mathrm{gl.dim.}\, A < \infty$). As a byproduct, the corresponding classes of algebras are closed under Galois coverings. The control is understood in the following sense.

Definition 10.3.1. Let Γ be an Auslander-Reiten component of $\mathrm{mod}(A)$. This Auslander-Reiten component *controls* the Galois coverings of A if

(1) for all connected Galois coverings $F\colon A' \to A$ with groups denoted by G, the class of objects $\{X \in \operatorname{mod}(A') \mid (\exists Y \in \Gamma)\ F_\lambda X \simeq Y\}$ forms an Auslander-Reiten component, Γ', of $\operatorname{mod}(A')$ and F_λ induces a translation quiver Galois covering $\Gamma' \to \Gamma$ with group G,

(2) and for all translation quiver Galois coverings $p\colon \widetilde{\Gamma} \to \Gamma$ with groups denoted by G, there exists a connected Galois covering $F\colon A' \to A$ with group G and a commutative diagram of translation quivers whose horizontal arrow is a G-equivariant isomorphism,

$$\begin{tikzcd} \widetilde{\Gamma} \arrow[rr, "\sim"] \arrow[dr, "p"'] & & \Gamma' \arrow[dl, "F_\lambda"] \\ & \Gamma\,. & \end{tikzcd}$$

When A has finite global dimension, the control of the Galois coverings of A by an Auslander-Reiten component of $\mathcal{D}^b(\operatorname{mod}(A))$ is defined similarly up to replacing $\operatorname{mod}(A)$ by $\mathcal{D}^b(\operatorname{mod}(A))$ and $F_\lambda\colon \operatorname{mod}(A') \to \operatorname{mod}(A)$ by $F_\lambda\colon \mathcal{D}^b(\operatorname{mod}(A')) \to \mathcal{D}^b(\operatorname{mod}(A))$ in all places.

Note that, for such a control to occur, it is necessary, yet not sufficient, that $\mathcal{H}_{F,X}$ holds true for all $X \in \Gamma$ and all Galois coverings F of A.

When A is piecewise hereditary of quiver type, a vertex X of a transjective Auslander-Reiten component of $\mathcal{D}^b(\operatorname{mod}(A))$ is a direct summand of a tilting object; accordingly, $\mathcal{H}_{F,X}$ holds true for all Galois coverings F of A. Using both this fact and Corollary 10.2.4, it is possible to deduce the following result.

Proposition 10.3.2. *Keep setting 10.0.1. Assume that there exists an equivalence of triangulated categories $\mathcal{D}^b(\operatorname{mod}(A)) \to \mathcal{D}^b(\operatorname{mod}(\Bbbk Q))$, where Q is a finite quiver without oriented cycles.*

(1) For all connected Galois coverings $F\colon A' \to A$ with groups denoted by G, there exists a quiver Galois covering $Q' \to Q$ with group G and an equivalence of triangulated categories $\mathcal{D}^b(\operatorname{mod}(A')) \to \mathcal{D}^b(\operatorname{mod}(\Bbbk Q'))$ which is G-equivariant and such that the following diagram commutes up to an isomorphism of functors,

$$\begin{tikzcd} \mathcal{D}^b(\operatorname{mod}(A')) \arrow[r, "\sim"] \arrow[d, "F_\lambda"'] & \mathcal{D}^b(\operatorname{mod}(\Bbbk Q')) \arrow[d, "p_\lambda"] \\ \mathcal{D}^b(\operatorname{mod}(A)) \arrow[r, "\sim"] & \mathcal{D}^b(\operatorname{mod}(\Bbbk Q))\,, \end{tikzcd}$$

where $p\colon \Bbbk Q' \to \Bbbk Q$ is induced by $Q' \to Q$.

(2) The Galois coverings of A are controlled by any transjective Auslander-Reiten component of $\mathcal{D}^b(\operatorname{mod}(A))$.

Bibliographical note. Dionne, Lanzilotta, and Smith proved in [DLS09] that, given a finite dimensional, basic, connected, and piecewise hereditary algebra A', if it is acted on by algebra automorphisms by a finite group G whose order is not divisible by $\operatorname{char}(\Bbbk)$, then $A' * G$ is piecewise hereditary. The relation with Proposition 10.3.2 is the following (see [CM06]): if, in addition, G acts freely on A', then A' is a Galois covering with group G of the quotient category A'/G and $A' * G$ is Morita equivalent to A'/G.

When A is a tilted algebra of infinite representation type, any connecting component of A is a full subquiver of a transjective Auslander-Reiten component, Γ, of $\mathcal{D}^b(\operatorname{mod}(A))$; it is actually equal to $\Gamma \cap \operatorname{mod}(A)$. Therefore, Proposition 10.3.2

also works if piecewise hereditary algebras and transjective Auslander-Reiten components are replaced by tilted algebras and connecting components, respectively. Taking iterated one-point extensions or coextensions yields the following result on weakly shod algebras.

Proposition 10.3.3. *Keep setting 10.0.1. Assume that A is weakly shod.*

(1) If A has a connecting component, Γ, then the Galois coverings of A are controlled by Γ. Besides, A has a universal Galois covering with group $\pi_1(\Gamma)$, which is a free group.

(2) Let $A' \to A$ be a Galois covering with finite group. Then, A is tilted (or quasitilted, or weakly shod) if and only if so is A'.

Here are some explanations. If A has a connecting component, Γ, and if it is not quasitilted, then A is isomorphic to a one-point extension $B[M]$ or to a one-point coextension $[M]B$, where B is a product of connected weakly shod algebras with connecting components. The translation quiver Γ is then isomorphic to a glueing of these connecting components. The Galois coverings of A (or, the translation quiver Galois coverings of Γ) can hence be expressed in terms of the ones of the connected components of B (or, of the connecting components of these connected components, respectively). Using these expressions, it is possible to prove (1) by induction on $\mathrm{rk}(K_0(A))$. The initialisation deals with the case where A is tilted. Point (2) is proved by using the known characterisations of tilted algebras in terms of their connecting components, of quasitilted algebras in terms of their left parts, and of weakly shod and not quasitilted algebras in terms of their connecting components.

It is not possible, however, to extend further the statements of Proposition 10.3.3 to the class of laura algebras because these statements do not hold true for non standard algebras of finite representation type. Instead, they can be extended to standard laura algebras which are defined as follows.

Definition 10.3.4. A *standard* laura algebra over an algebraically closed field is a laura algebra with a connecting component, Γ, such that Γ is standard.

Example 10.3.5. The following classes of algebras consist of standard laura algebras,

- the tilted algebras;
- the algebras which are weakly shod and not quasitilted;
- and the laura algebras with connecting components and with maximal filtrations whose first term is a product of standard algebras of finite representation type.

The maximal filtrations of laura algebras are defined as follows using the indecomposable projective modules lying in their connecting components.

Definition 10.3.6. Keep setting 10.0.1. Assume that A is laura with a connecting component.

(1) An indecomposable projective A-module is *maximal* if it has an injective predecessor and no proper projective successor in $\operatorname{ind} A$.

(2) A is a *maximal extension* of B if there exists a primitive idempotent $e \in A$ such that the indecomposable projective A-module eA is maximal, $B = (1-e)A(1-e)$, and $A = B[M]$ where $M = \mathrm{rad}(eA)$. *Maximal coextensions* are defined dually.

(3) A *maximal filtration* of A is a sequence of full convex subcategories,

$$A_0 \subsetneq A_1 \subsetneq \cdots \subsetneq A_m = A \tag{10.3.0.4}$$

such that A_0 is of finite representation type and A_{i+1} is a maximal extension or coextension of A_i for all $i \in \{0, \ldots, m-1\}$.

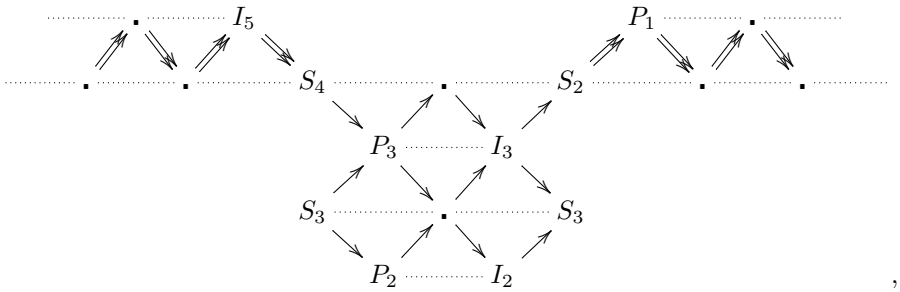
Example 10.3.7. Let A be the radical square-zero algebra given by the quiver

$$\begin{tikzcd} 1 \arrow[r, bend right] & 2 \arrow[l, bend right] & 3 \arrow[l] \arrow[loop above] & 4 \arrow[l] \arrow[r, bend right] & 5 \arrow[l, bend right] \end{tikzcd} .$$

Following [AC03], this is a laura algebra. For all vertices x of the quiver, denote by I_x, P_x, and S_x the corresponding indecomposable injective, projective and simple A-module, respectively. Denote by C the full convex subcategory with object set $\{2, 3, 4\}$. This is an algebra of finite representation type and A has a maximal filtration as follows

$$C \subsetneq [S_4 \oplus S_4]C \subsetneq A\,.$$

Since C is standard, so is A. The connecting component of A is as follows,



,

where the two copies of S_3 are identified.

It is possible to prove the following result on laura algebras which is analogous to Proposition 10.3.2.

Proposition 10.3.8. *Keep setting 10.0.1. Assume that A is laura with a connecting component, Γ.*

(1) Let $\Gamma' \to \Gamma$ be a translation quiver Galois covering such that there exists an associated well-behaved functor $\Bbbk(\Gamma') \to \operatorname{ind}\Gamma$ (see Theorem 9.1.4). Then, there exists a covering functor $F\colon A' \to A$ and a faithful functor $\Phi\colon \Bbbk(\Gamma') \to \operatorname{mod}(A')$ such that the following diagram commutes,

$$\begin{tikzcd} \Bbbk(\Gamma') \arrow[r, "\Phi"] \arrow[d] & \operatorname{mod}(A') \arrow[d, "F_\lambda"] \\ \operatorname{ind}\Gamma \arrow[r, hook] & \operatorname{mod}(A)\,. \end{tikzcd}$$

(2) If, in addition, A is standard and $\Gamma' \to \Gamma$ is a translation quiver Galois covering with group denoted by G, then there exist F and Φ like in (1), such that F is a Galois covering with group G and Φ is G-equivariant, full, and with essential image forming an Auslander-Reiten component of $\operatorname{mod}(A')$. When G is finite, A' is moreover laura.

Here are some explanations on how to construct the covering functor $A' \to A$ using the well-behaved functor $\Bbbk(\Gamma') \to \operatorname{ind}\Gamma$. By the general theory of laura algebras, there is an isomorphism of $\Bbbk$-categories $A \simeq \left(\begin{smallmatrix} B & M \\ 0 & C \end{smallmatrix}\right)$ where

- B is the full subcategory of $\operatorname{ind} A$ generated by the projectives in $\operatorname{ind} A$ lying in $\mathcal{L}_A$, this is actually a product of tilted algebras
- and C is the full subcategory of $\operatorname{ind} A$ generated by the projectives in $\operatorname{ind} A$ not lying in $\mathcal{L}_A$, this is a full subcategory of $\operatorname{ind}\Gamma$.

On one hand, the full subcategory, Δ, of the Ext-injectives objects of $\mathcal{L}_A$ is contained in $\operatorname{ind}\Gamma$ and is equal to the disjoint union of sections of the connecting components of the connected components of B. Besides, the well-behaved functor $\Bbbk(\Gamma') \to \operatorname{ind}\Gamma$ induces a covering functor of Δ by the preimage of Δ in $\Bbbk(\Gamma')$. Since

the Galois coverings of tilted algebras are controlled by their connecting components, there is a covering functor of B associated to the above mentioned covering functor of Δ,

$$B' \to B\,. \tag{10.3.0.5}$$

On the other hand, since C is a full subcategory of $\operatorname{ind}\Gamma$, the well-behaved functor $\Bbbk(\Gamma') \to \operatorname{ind}\Gamma$ induces a covering functor of C by its preimage in $\Bbbk(\Gamma')$,

$$C' \to C\,. \tag{10.3.0.6}$$

Now, let M' be the $B' - C'$-bimodule obtained from the $B - C$-module M by restriction of scalars along (10.3.0.5) and (10.3.0.6). The resulting covering functor $\left(\begin{smallmatrix} B' & M' \\ 0 & C' \end{smallmatrix}\right) \to \left(\begin{smallmatrix} B & M \\ 0 & C \end{smallmatrix}\right) \simeq A$ is the one claimed by Proposition 10.3.8.

10.4. **Simple connectedness.** The previous sections provide tools to relate the simple connectedness of A to the vanishing of $\mathrm{HH}^1(A)$ when A is piecewise hereditary of quiver type and when A is a standard laura algebra. The characterisation in the former case is the following.

Theorem 10.4.1. *Keep setting 10.0.1. Assume that A is piecewise hereditary of type Q, where Q is a finite quiver without oriented cycles. The following assertions are equivalent.*

(i) A is simply connected.
(ii) Q is a tree.
(iii) $\mathrm{HH}^1(A) = 0$.

Theorem 10.4.1 follows from part (2) of Corollary 10.2.4 using the following facts.

- When an algebra has a universal Galois covering with group denoted by G, this algebra is simply connected if and only if $G = \{e\}$.
- If a given group G is the group of a connected Galois covering of A, then there exists an injective $\Bbbk$-linear mapping $\mathrm{Hom}_{\mathrm{group}}(G, (\Bbbk, +)) \to \mathrm{HH}^1(A)$.
- The algebra $\Bbbk Q$ is simply connected if and only if Q is a tree, which is equivalent to $\mathrm{HH}^1(\Bbbk Q) = 0$ ([Hap89]).

Bibliographical note. The equivalence between (i) and (iii) in Theorem 10.4.1 was proved previously for certain subclasses of piecewise hereditary algebras in relationship with tame algebras.

- This equivalence was proved for tame tilted algebras by Assem, Marcos, and de la Peña in [AMdlPn01] and for tame quasitilted algebras by Assem, Coelho, and Trepode in [ACT02].
- Assem and Skowroński proved in [AS87b, AS88] that $\mathcal{D}^b(\mathrm{mod}(A))$ is cycle finite if and only if A is piecewise hereditary and $\mathcal{D}^b(\mathrm{mod}(A))$ is of Dynkin type, or of Euclidean type, or of tubular canonical type; they proved moreover that such an algebra is simply connected if and only if it is not piecewise hereditary of type $\widetilde{\mathbb{A}}$, which is equivalent to $\mathrm{HH}^1(A)$ being zero.

In a recent preprint ([FG18]), Fu and Geng claimed a proof of the connectedness of $\overrightarrow{\mathcal{K}}_{\mathrm{coh}(\mathbb{X})}$ for all weighted projective lines $\mathbb{X}$ of wild type. Recall that Barot, Kussin, and Lenzing already proved the same result for weighted projective lines of tubular type (see [BKL10]). Should the validity of the proof of Fu and Geng be confirmed, the methods used to prove Theorem 10.4.1 would apply to all piecewise hereditary algebras. Accordingly, for all piecewise hereditary algebras A, the following assertions would be equivalent.

(i) A is simply connected.
(ii) $\mathrm{HH}^1(A) = 0$.

(iii) A is piecewise hereditary of type Q where Q is a tree quiver, of tubular canonical type, or of wild canonical type.

With similar considerations, it is possible to prove the following characterisation of simple connectedness for standard laura algebras.

Theorem 10.4.2. *Keep setting 10.0.1. Assume that A is a standard laura algebra with a connecting component denoted by Γ. The following assertions are equivalent.*

(i) A is simply connected.
(ii) The orbit graph of Γ, in the sense of [BG82]*, is a tree.*
(iii) Γ is simply connected.
(iv) $\mathrm{HH}^1(A) = 0$.

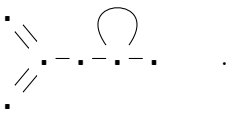
If A is simply connected, then it is triangular and weakly shod.

In addition to the facts explaining the proof of Theorem 10.4.1, the following ones are used to prove Theorem 10.4.2.

On one hand, following arguments due to Bongartz and Gabriel in [BG82], the fundamental groups of Γ and of the orbit graph of Γ are isomorphic, which explains the equivalence $(ii) \Leftrightarrow (iii)$.

On the other hand, if Γ is simply connected, then it has no oriented cycles, which entails that A is weakly shod. In view of the considerations following the statement of Proposition 10.3.3, this entails that A is a tilted algebra or, by induction, a one-point extension of a direct product of simply connected weakly shod algebras. Starting from these considerations, the implication $(iii) \Rightarrow (iv)$ is proved by induction on the number objects of A. In the initialising step the algebra under consideration is tilted, and hence Theorem 10.4.1 applies. The inductive step applies Happel's long exact sequence of Hochschild cohomology for one-point (co)extensions ([Hap89]).

Example 10.4.3. This is the continuation of Example 10.3.7. The orbit graph of the connecting component of A is as follows



Hence, the fundamental group of the connecting component is free of rank 3 and A has a connected Galois covering with group the free group of rank 3. In particular, A is not simply connected and $\dim \mathrm{HH}^1(A) \geqslant 3$. Actually, $\dim \mathrm{HH}^1(A) = 7$.

Bibliographical note. The equivalence between (i), (iii) and (iv) in Theorem 10.4.2 was proved for standard algebras of finite representation type by Buchweitz and Liu in [BL03]. It was also proved by Assem and Lanzilotta in [AL04] for tame weakly shod algebras.

Theorem 10.4.2 is part of the collaboration [ABLM10] with Assem and Bustamante. As a sequel, have investigated in [ABLM11] the simple connectedness of special biserial algebras ([SW83]) with completely different tools. The combinatorial analysis of the fundamental groups $\pi_1(Q, I)$ of the bound quivers (Q, I) presenting a special biserial algebra yields the following result, where $\chi(Q)$ denotes the Euler characteristic of the underlying graph of Q.

Theorem 10.4.4. *Keep setting 10.0.1. Assume that A is special biserial. The following are equivalent.*

(i) A is of finite representation type, standard, and simply connected.
(ii) $\mathrm{HH}^1(A) = 0$.

(iii) $\mathrm{HH}^i(A) = 0$ *for all positive integers* i.
(iv) $\chi(Q) = \dim S$, *where* S *is direct sum of the socles of the indecomposable projective-injective* A*-modules.*

10.5. **Open questions.** The following questions are related to the results presented in section 10.4.

10.5.1. *Rigid modules of the first kind.* The proof of these results are based on the results of section 10.1, which presents sufficient conditions for a generalised tilting module to be of the first kind with respect to a given Galois covering. This suggest the following question.

Question 6. Let $F\colon A' \to A$ be a Galois covering. Let M be a finitely generated A-module such that $\mathrm{Ext}^1_A(M, M) = 0$. Is M of the first kind with respect to F?

In view of correspondence of [CM06] between Galois coverings and smash products by group gradings, the question is equivalent to the following one. Given a grading of A by a group G and a finitely generated A-module M, is M gradable? When the group of F is finite, a possible approach to answer the question is to relate the varieties of representations of A' and the ones of A by means of F_λ and to use the fact that the orbit of M in a corresponding variety of representations of A is open.

10.5.2. *Simple connectedness and Hochschild cohomology of tame triangular algebras.* Despite the results of section 10.4, the original Skowroński's question whether the simple connectedness is equivalent to the vanishing of the Hochschild cohomology group HH^1 for tame triangular algebras remains unanswered. Clearly, answering this question requires an approach different from the one presented in section 10.4. Here is a possible strategy using Drozd's theorem on the tame-wild dichotomy of boxes ([Dro80]). On one hand, the definition of the Hochschild cohomology seems to carry out to boxes. On the other hand, the concept of Galois coverings of boxes was defined by Drozd and Ovsienko in [DO]. The strategy hence consists in considering the box (A, A) associated to A and in examining whether the simple connectedness of boxes and the vanishing of the Hochschild cohomology group HH^1 of boxes are properties which are invariant under the reduction algorithms involved in Drozd's theorem.

11. Strong global dimension

This section presents the results of the article [ALMM17] on various characterisations of the strong global dimension of piecewise hereditary algebras in the following setting. This is a collaboration with Alvares and Marcos.

Setting 11.0.1. Let $\Bbbk$ be an algebraically closed field. Let $\mathcal{T}$ be a triangulated $\Bbbk$-linear category equivalent to the bounded derived category of complexes in a hereditary abelian $\Bbbk$-linear category which is Hom-finite, is Krull-Schmidt, and has at least one tilting object. Let T be a tilting object of $\mathcal{T}$.

Recall that T being tilting means that $\mathcal{T}(T, \Sigma^n T) = 0$ for all non zero integers n and that 0 is the only object X of $\mathcal{T}$ such that $\mathcal{T}(T, \Sigma^n X) = 0$ for all integers n. In view of determining the strong global dimension of the piecewise hereditary algebra $\mathrm{End}(T)^{\mathrm{op}}$, the work presented here starts from two simple facts using the following alternative description of this invariant due to Angeleri, Koenig, and Liu.

Proposition 11.0.2 ([AHKL12])**.** *Keep setting 11.0.1. Then,* $\mathrm{s.gl.dim.}\,\mathrm{End}(T)^{\mathrm{op}}$ *is the largest possible integer* ℓ *such that the morphism spaces* $\mathcal{T}(T, X)$ *and* $\mathcal{T}(X, \Sigma^\ell T)$ *are both non zero for at least one indecomposable object* X *of* $\mathcal{T}$.

The first simple fact is an upper bound for $\mathrm{s.gl.dim.}\,\mathrm{End}(T)^{\mathrm{op}}$ relative the hereditary abelian generating subcategories of $\mathcal{T}$, which are defined as follows.

Definition 11.0.3. Keep setting 11.0.1. A *hereditary abelian generating* subcategory of $\mathcal{T}$ is a full additive subcategory $\mathcal{H}$ which is abelian, hereditary, and such that the natural functor $\mathcal{H} \to \mathcal{T}$ extends to an equivalence of triangulated categories $\mathcal{D}^b(\mathcal{H}) \to \mathcal{T}$.

As shown by Happel ([Hap88]), if $\mathcal{H}$ is as in Definition 11.0.3, then every object of $\mathcal{T}$ lies in $\Sigma^i\mathcal{H} \vee \Sigma^{i+1}\mathcal{H} \vee \cdots \vee \Sigma^{i+j}\mathcal{H}$ for suitable integers i, j such that $j \geqslant 0$. Here, $\mathcal{A} \vee \mathcal{B}$ denotes $\mathrm{add}(\mathcal{A} \cup \mathcal{B})$ for all subcategories $\mathcal{A}$ and $\mathcal{B}$.

Proposition 11.0.4. *Keep setting 11.0.1. Let $\mathcal{H} \subseteq \mathcal{T}$ be a hereditary abelian generating subcategory. Assume that $T \in \mathcal{H} \vee \Sigma\mathcal{H} \vee \cdots \vee \Sigma^\ell\mathcal{H}$ for some non negative integer ℓ. Then,* $\mathrm{s.gl.dim.}\,\mathrm{End}(T)^{\mathrm{op}} \leqslant \ell + 2$.

The second simple fact is the comparison of the strong global dimensions arising from tilting objects of $\mathcal{T}$ related by a mutation in the following sense (see [AI12]).

Definition 11.0.5. Keep setting 11.0.1. A tilting object T' of $\mathcal{T}$ is a *mutation* of T if the following assertion or its dual holds true. There exist $T_1, T_2, T_2' \in \mathcal{T}$ such that

- $T \simeq T_1 \oplus T_2$ and $\mathrm{add}(T_1) \cap \mathrm{add}(T_2) = \{0\}$,
- $T' \simeq T_1 \oplus T_2'$ and $\mathrm{add}(T_1) \cap \mathrm{add}(T_2') = \{0\}$,
- and there exists a distinguished triangle $T_2' \to M \to T_2 \to \Sigma T_2'$ of $\mathcal{T}$ such that $M \to T_2$ is a minimal right $\mathrm{add}(T_1)$-approximation.

Note that T_2 need not be indecomposable. By *right* $\mathrm{add}(T_1)$*-approximation* is meant that any morphism from an object of $\mathrm{add}(T_1)$ to T_2 factors through $M \to T_2$. By *minimal* is meant that the morphism $M \to T_2$ is radical.

Proposition 11.0.6. *Keep setting 11.0.1. Let T' be a tilting object of $\mathcal{T}$ which is a mutation of T. Then,*

$$|\mathrm{s.gl.dim.}\,\mathrm{End}(T)^{\mathrm{op}} - \mathrm{s.gl.dim.}\,\mathrm{End}(T')^{\mathrm{op}}| \leqslant 1.$$

The results presented below are characterisations of $\mathrm{s.gl.dim.}\,\mathrm{End}(T)^{\mathrm{op}}$ from the points of view of Propositions 11.0.4 and 11.0.6. These characterisations are based on a description of this invariant presented in section 11.1 according to the shapes of the Auslander-Reiten components of $\mathcal{T}$ containing specific indecomposable direct summands of T. Section 11.2 presents these characterisations.

11.1. **Strong global dimension *via* Auslander-Reiten theory.** It is possible to express $\mathrm{s.gl.dim.}\,\mathrm{End}(T)^{\mathrm{op}}$ according to the subcategories of $\mathcal{T}$ in which T starts or ends in. These subcategories are defined as follows.

Definition 11.1.1. Keep setting 11.0.1. Let $\mathcal{C}$ be a full and convex additive subcategory of $\mathcal{T}$ closed under taking direct summands. The object T *starts* in $\mathcal{C}$ if the following assertions hold true,

- $\mathcal{C}$ contains at least one indecomposable direct summand of T and
- for all indecomposable direct summands X of T, there exists a sequence $X_0 \to X_1 \to \cdots \to X_n$ of non zero morphisms between indecomposable objects of $\mathcal{T}$ such that $X = X_n$ and $X_0 \in \mathcal{C}$.

If the dual assertion holds true, then T *ends* in $\mathcal{C}$.

Before going further, here is a reminder of some known facts and definitions.

First, given a hereditary abelian generating subcategory $\mathcal{H}$ of $\mathcal{T}$, if an equivalence of additive categories $\mathcal{H} \xrightarrow{\sim} \mathrm{coh}(\mathbb{X})$ is given for some weighted projective line $\mathbb{X}$, then

$\mathcal{H}_0$ denotes the torsion class of $\mathcal{H}$ consisting of finite length objects. It corresponds to the class of torsion sheaves on $\mathbb{X}$. And $\mathcal{H}_+$ denotes the torsion free class of $\mathcal{H}$ associated to $\mathcal{H}_0$. It corresponds to the class of vector bundles on $\mathbb{X}$ (see [GL87]).

Next, under setting 11.0.1, $\mathcal{T}$ has an Auslander-Reiten structure which is described in [Hap88]. Denoting by τ the Auslander-Reiten translation, a *transjective* component is an Auslander-Reiten component of $\mathcal{T}$ with finitely many τ-orbits only. If $\mathcal{T}$ has a transjective component Γ, then $\Gamma \simeq \mathbb{Z}\Delta$ for some finite graph Δ; the transjective components of $\mathcal{T}$ are $\Sigma^i\Gamma$ for $i \in \mathbb{Z}$; and $\oplus_{X\in\Delta} X$ is a tilting object of $\mathcal{T}$ whose endomorphism algebra is hereditary (see [Hap88]). By abuse of terminology, T is said to *start in a transjective* component Γ of $\mathcal{T}$ if T starts in $\operatorname{add}(\Gamma)$.

The description of $\operatorname{s.gl.dim.}\operatorname{End}(T)^{\mathrm{op}}$ in terms of the subcategories in which T starts or ends in is based on the following lower bounds. Let $\mathcal{H}$ be a hereditary abelian generating subcategory of $\mathcal{T}$ of quiver type or of canonical type, let ℓ be a non negative integer, and let M_0 and M_1 be indecomposable direct summands of T such that $M_0 \in \mathcal{H}$ and $M_1 \in \Sigma^\ell\mathcal{H}$. Compare with Proposition 11.0.4.

If $\mathcal{H}$ is of canonical type, $M_0 \in \mathcal{H}_+$, and $M_1 \in \Sigma^\ell\mathcal{H}_0$, then there exists an indecomposable object $X \in \Sigma^{\ell+1}\mathcal{H}_0$, lying in the $(\ell+1)$-th suspension of the tube which contains M_1, such that $\mathcal{T}(M_1, X) \neq 0$ and $\mathcal{T}(X, \Sigma^{\ell+2}M_0) \neq 0$. In particular, $\operatorname{s.gl.dim.}\operatorname{End}(T)^{\mathrm{op}} \geqslant \ell+2$.

If $\mathcal{H}$ is of canonical type, $M_0 \in \mathcal{H}_0$, and $M_1 \in \Sigma^\ell\mathcal{H}_0$, then there exists an indecomposable object $X \in \Sigma^{\ell+1}\mathcal{H}_+$ such that $\mathcal{T}(M_1, X) \neq 0$ and $\mathcal{T}(X, \Sigma^{\ell+1}M_0) \neq 0$. In particular, $\operatorname{s.gl.dim.}\operatorname{End}(T)^{\mathrm{op}} \geqslant \ell+1$.

If $\mathcal{H}$ is of canonical type and there exists a tube $\mathcal{U} \subseteq \mathcal{H}_0$ such that $M_0 \in \mathcal{U}$ and $M_1 \in \Sigma^\ell\mathcal{U}$, then there exists an indecomposable object $X \in \Sigma^{\ell+1}\mathcal{U}$ such that $\mathcal{T}(M_1, X) \neq 0$ and $\mathcal{T}(X, \Sigma^{\ell+2}M_1) \neq 0$. In particular, $\operatorname{s.gl.dim.}\operatorname{End}(T)^{\mathrm{op}} \geqslant \ell+2$.

If both M_0 and M_1 lie in Auslander-Reiten components of $\mathcal{T}$ of shape $\mathbb{Z}A_\infty$, then there exists an indecomposable object $X \in \Sigma^{\ell+1}\mathcal{H}$, lying in an Auslander-Reiten component of $\mathcal{T}$ of shape $\mathbb{Z}A_\infty$, such that $\mathcal{T}(M_1, X) \neq 0$ and $\mathcal{T}(X, \Sigma^{\ell+2}M_0) \neq 0$. In particular, $\operatorname{s.gl.dim.}\operatorname{End}(T)^{\mathrm{op}} \geqslant \ell+2$.

If $\mathcal{H}$ is of quiver type and all simple projective objects of $\mathcal{H}$ lie in $\operatorname{add}(T)$, then there exist an indecomposable object $X \in \tau^{-1}\Sigma^{\ell+1}\mathcal{H}$ and an indecomposable projective object M_0' of $\mathcal{H}$ such that $\mathcal{T}(M_1, X) \neq 0$ and $\mathcal{T}(X, \Sigma^{\ell+2}M_0') \neq 0$. In particular, $\operatorname{s.gl.dim.}\operatorname{End}(T)^{\mathrm{op}} \geqslant \ell+2$.

It is possible to prove that, since T is tilting, then it fits in exactly one of the following four situations. It is worth noticing that, in each case, at least one of the lower bounds presented previously applies.

Case 1 - T starts in a transjective component.

Case 2 - There exists a hereditary abelian generating subcategory $\mathcal{H}$ of $\mathcal{T}$ which is of canonical type and such that T starts in $\mathcal{H}_+$ and ends in $\Sigma^\ell\mathcal{H}_0$ for some non negative integer ℓ.

Case 3 - There exists a hereditary abelian generating subcategory $\mathcal{H}$ of $\mathcal{T}$ which is of canonical type and such that T starts in $\mathcal{H}_0$ and ends in $\Sigma^\ell\mathcal{H}_0$ for some non negative integer ℓ.

Case 4 - T does not end in a transjective component. There exists a hereditary abelian generating subcategory $\mathcal{H}$ of $\mathcal{T}$ which is of wild quiver type, such that T starts the additive closure of the union of the Auslander-Reiten components of $\mathcal{H}$ of shape $\mathbb{Z}A_\infty$, and such that T ends in $\Sigma^\ell\mathcal{H}$.

Proposition 11.1.2. *Keep setting 11.0.1. Assume that* $\operatorname{End}(T)^{\mathrm{op}}$ *is not hereditary. The value of* $\operatorname{s.gl.dim.}\operatorname{End}(T)^{\mathrm{op}}$ *depends on the case in which T fits as follows.*

(1) In case 1, there exists a hereditary abelian generating subcategory $\mathcal{H}$ of $\mathcal{T}$ which is of quiver type, whose simple projective objects belong to $\operatorname{add}(T)$,

and such that T starts in $\mathcal{H}$. Denote by ℓ the least integer such that $T \in \mathcal{H} \vee \Sigma\mathcal{H} \vee \cdots \vee \Sigma^{\ell}\mathcal{H}$. Then, $\mathrm{s.gl.dim.}\,\mathrm{End}(T)^{\mathrm{op}} = \ell + 2$.

(2) *In cases 2 and 4, the strong global dimension of* $\mathrm{End}(T)^{\mathrm{op}}$ *is* $\ell + 2$.

(3) *In case 3, denote by $\mathcal{E}(T)$ the set of tubes $\mathcal{U}$ of $\mathcal{H}_0$ such that T has at least one indecomposable direct summand in $\mathcal{U}$ and at least one indecomposable direct summand in $\Sigma^{\ell}\mathcal{U}$. Then,*
   - $\mathcal{E}(T) \neq \emptyset \Longrightarrow \mathrm{s.gl.dim.}\,\mathrm{End}(T)^{\mathrm{op}} = \ell + 2$,
   - $\mathcal{E}(T) = \emptyset \Longrightarrow \mathrm{s.gl.dim.}\,\mathrm{End}(T)^{\mathrm{op}} = \ell + 1$.

Here are explanations in case 1. Let Γ the transjective component in which T starts. Let $\mathcal{S}$ be the set of vertices of Γ consisting of pairwise Hom-orthogonal indecomposable direct summands of T such that $\mathcal{T}(-, M)_{|\mathcal{S}} \neq 0$ for all $M \in \Gamma \cap \mathrm{add}(T)$. Let Δ the set of vertices X of Γ such that there exists a path $S \rightsquigarrow X$ in Γ with $S \in \mathcal{S}$ and with the additional requirement that all such paths are sectional. Then, $\Gamma \simeq \mathbb{Z}\Delta$ and $\mathrm{add}(\Delta)$ is the class of projective objects of $\mathcal{H}$.

Example 11.1.3. Assume that $\mathcal{T} = \mathcal{D}^b(\mathrm{coh}(\mathbb{X}))$, where $\mathbb{X}$ is a weighted projective line of type $(2,2,2,2,2,2)$. This example follows the conventions of notation of [Mel04]. Let T_1 and T_2 be the following tilting objects of $\mathcal{T}$,

$$\begin{cases} T_1 &= \Sigma^{-1}S_{1,0} \oplus \mathcal{O} \oplus \mathcal{O}(\overrightarrow{x_2}) \oplus \mathcal{O}(\overrightarrow{x_3}) \oplus \mathcal{O}(\overrightarrow{x_4}) \oplus \mathcal{O}(\overrightarrow{x_5}) \oplus \mathcal{O}(\overrightarrow{x_6}) \oplus S_{1,1} \\ T_2 &= \Sigma^{-1}S_{3,0} \oplus \Sigma^{-1}S_{4,0} \oplus \mathcal{O} \oplus \mathcal{O}(\overrightarrow{x}_1) \oplus \mathcal{O}(\overrightarrow{x}_1) \oplus \mathcal{O}(2\overrightarrow{x}_1) \oplus S_{5,1} \oplus S_{6,1}\,. \end{cases}$$

With the notation used in Proposition 11.1.2,

- both T_1 and T_2 fit in case 3 with $\mathcal{H} = \Sigma^{-1}\mathrm{coh}(\mathbb{X})$,
- $\mathcal{E}(T_1)$ consists of a single tube of $\mathrm{coh}(\mathbb{X})_0$, namely, the one containing both $\Sigma^{-1}S_{1,0}$ and $\Sigma^{-1}S_{1,1}$, and $\mathcal{E}(T_2) = \emptyset$,
- $\mathrm{s.gl.dim.}\,\mathrm{End}(T_1)^{\mathrm{op}} = 3$ and $\mathrm{s.gl.dim.}\,\mathrm{End}(T_2) = 2$.

11.2. **Characterisation of the strong global dimension.** The description of $\mathrm{s.gl.dim.}\,\mathrm{End}(T)^{\mathrm{op}}$ presented in section 11.1 provides two characterisations of this invariant. First, the analysis of that section yields a hereditary abelian generating subcategory $\mathcal{H}$ of $\mathcal{T}$ such that the inequality of Proposition 11.0.4 is an equality.

Theorem 11.2.1. *Keep setting 11.0.1. Assume that* $\mathrm{End}(T)^{\mathrm{op}}$ *is not hereditary. Let ℓ be the integer* $\mathrm{s.gl.dim.}\,\mathrm{End}(T)^{\mathrm{op}} - 2$. *Then, ℓ is the least possible integer such that there exists a hereditary abelian generating subcategory $\mathcal{H}$ of $\mathcal{T}$ with the following properties,*

- $T \in \mathcal{H} \vee \Sigma\mathcal{H} \vee \cdots \vee \Sigma^{\ell}\mathcal{H}$ *and*
- $\mathrm{add}(T) \cap \mathcal{H} \neq \{0\}$ *and* $\mathrm{add}(T) \cap \Sigma^{\ell}\mathcal{H} \neq \{0\}$.

Example 11.2.2. Assume that $\mathcal{T} = \mathcal{D}^b(\mathrm{mod}(\Bbbk Q))$ where Q is the quiver

$$1 \longleftarrow 2 \longleftarrow 3 \longleftarrow 4 \longleftarrow 5$$

and that T is the tilting object $3 \oplus \begin{smallmatrix}2\\3\end{smallmatrix} \oplus \begin{smallmatrix}1\\2\\3\end{smallmatrix} \oplus \Sigma 4 \oplus \Sigma^2 5$. Then, T lies in $\mathrm{mod}(\Bbbk Q) \vee \Sigma\mathrm{mod}(\Bbbk Q) \vee \Sigma^2\mathrm{mod}(\Bbbk Q)$, and hence Proposition 11.0.4 yields that

$$\mathrm{s.gl.dim.}\,\mathrm{End}(T)^{\mathrm{op}} \leqslant 4\,.$$

Now, Theorem 11.2.1 applies with $\mathcal{H}$ being the hereditary abelian generating subcategory of $\mathcal{T}$ whose class indecomposable projective objects is given by

$$\{3,\ \begin{smallmatrix}2\\3\end{smallmatrix},\ \begin{smallmatrix}1\\2\\3\end{smallmatrix},\ \Sigma\begin{smallmatrix}4\\5\end{smallmatrix},\ \Sigma 4\}\,.$$

Since $T \in \mathcal{H}$, then $\mathrm{s.gl.dim.}\,\mathrm{End}(T)^{\mathrm{op}} = 2$.

Next, the situations of the cases 1, 2, 3, and 4 introduced in section 11.1 behave well under mutation. Indeed, considering $\mathcal{H}$ as in Theorem 11.2.1, there exists a direct sum decomposition $T = T_1 \oplus T_2$, where $T_2 \in \Sigma^{\ell}\mathcal{H}$ and $T_1 \in \mathcal{H} \vee \Sigma\mathcal{H} \vee$

$\cdots \vee \Sigma^{\ell-1}\mathcal{H}$. For all distinguished triangles $T_2' \to M \to T_2 \to \Sigma T_2'$ of $\mathcal{T}$, where $M \to T_2$ is a minimal right $\mathrm{add}(T_1)$-approximation, the object $T_1 \oplus T_2'$ is tilting, it is a mutation of T, it starts in $\mathcal{H}$ and ends in $\Sigma^{\ell-1}\mathcal{H}$, and it fits in the same case 1, 2, 3, or 4 of section 11.1 as T. Repeating this procedure yields the following characterisation of the strong global dimension in terms of mutations.

Theorem 11.2.3. *Keep setting 11.0.1. Assume that* $\mathrm{End}(T)^{\mathrm{op}}$ *is not hereditary. Let* ℓ *be the integer* $\mathrm{s.gl.dim.}\,\mathrm{End}(T)^{\mathrm{op}} - 2$. *Then,* ℓ *is the least possible integer in a sequence* $T^{(0)}, T^{(1)}, \ldots, T^{(\ell)}$ *of tilting objects of* $\mathcal{T}$ *such that*

- $\mathrm{End}(T^{(0)})^{\mathrm{op}}$ *is quasitilted and* $T^{(\ell)} = T$,
- $\mathrm{s.gl.dim.}\,\mathrm{End}(T^{(i)})^{\mathrm{op}} = i + 2$ *for all* $i \in \{0, \ldots, \ell\}$,
- $T^{(i+1)}$ *is a mutation of* $T^{(i)}$ *for all* $i \in \{0, \ldots, \ell - 1\}$.

Part 4. Perspectives

Let $\Bbbk$ be a field. The results presented in this thesis provide many perspectives for future research. Here are some of them.

12. Skew Calabi-Yau algebras

12.1. **Relationship to minimal models.** Let A be an augmented $\Bbbk$-algebra. The Yoneda algebra $E := \mathrm{Ext}^{\bullet}_A(\Bbbk, \Bbbk)$ is an A_∞-algebra and, in many circumstances, it is possible to recover A from E using the bar-cobar formalism. Such circumstances include the case where A is $\mathbb{N}$-graded and connected (up to an adequate completion). The questions addressed below aim at understanding how being skew Calabi-Yau is encoded in E.

- When A is $\mathbb{N}$-graded and connected, Lu, Palmieri, Wu and Zhang proved in [LPWZ08, Corollary D] that A is Artin-Schelter regular if and only if the algebra E is Frobenius;
- Recall that, under the same condition, Reyes, Rogalski and Zhang proved in [RRZ14] that A is Artin-Schelter regular if and only if it is skew Calabi-Yau.
- Finally, when A is $\mathbb{N}$-graded, connected, generated in degree 1, and Artin-Schelter regular, Reyes, Rogalski and Zhang described in [RRZ17] the Nakayama automorphism of the Frobenius algebra E in terms of the one of the skew Calabi-Yau algebra A. Since A is generated in degree 1, this description also allows one to recover the latter in terms of the former.

These considerations motivate the following questions for an augmented skew Calabi-Yau $\Bbbk$-algebra A.

Question 7. Is it possible to describe the Nakayama automorphisms of A in terms of E? If, in addition, A is an H-module algebra for some skew Calabi-Yau Hopf algebra H, is it possible to describe a weak homological determinant of the action of H on A in terms of E?

The bar-cobar formalism is likely to give information on the structure of augmented skew Calabi-Yau algebras. On one hand, Van den Bergh proved that all complete Calabi-Yau algebras derive from potentials and the same result holds true in the setting of Calabi-Yau $\mathbb{N}$-graded and connected algebras. On the other hand, Bocklandt, Wemyss and Schedler proved in ([BSW10]) that, if A is Koszul and skew Calabi-Yau, then it derives from a twisted potential.

Question 8. Under which condition does a skew Calabi-Yau algebra derive from a twisted potential?

12.2. **Representation theoretical aspects.** Let A be a skew Calabi-Yau augmented $\Bbbk$-algebra. Denote by E its Yoneda algebra. As mentioned previously, E is a (finite dimensional) Frobenius algebra. In the representation theory of finite dimensional algebras, the Frobenius algebras, and more particularly the selfinjective algebras, have been studied intensively both in terms of their classification and in terms of their Auslander-Reiten structures. Recall that when $\Bbbk$ is an algebraically closed field, then a basic finite dimensional algebra is Frobenius if and only if it is selfinjective.

Under appropriate assumptions, A and E are related by the bar-cobar formalism, and hence their representation theories are related by Koszul duality. This motivates the following question.

Question 9. How do the representation theoretic specificities like the Auslander-Reiten structure of E express in terms of A?

13. Homological properties of Lie-Rinehart algebras and beyond

Let R be a commutative ring, let (S, L) be a Lie-Rinehart algebra over R, and denote by U its enveloping algebra.

13.1. **Lie-Rinehart algebras and Hochschild cohomology.** Let (S, L) be a Lie-Rinehart algebra over a commutative ring R. Denote by U its enveloping algebra. Under the setting of Theorem 6.2.2, the proof that U has Van den Bergh duality is made by expressing $\mathrm{Ext}^\bullet_{U^e}(U, U^e)$ in terms of $\mathrm{Ext}^\bullet_{S^e}(S, S^e)$ and $\mathrm{Ext}^\bullet_U(S, U)$. Note that, when S is projective as an R-module, these spaces are the Hochschild cohomology of U with coefficients in U^e, the Hochschild cohomology of S with coefficients in S^e, and the cohomology of the Lie-Rinehart algebra (S, L) with coefficients in U ([Rin63]), respectively. The reformulation of Theorem 6.2.2 in terms of these cohomologies raises the question of replacing U^e, as the coefficient U-bimodule, by U. Note that Huebschmann described in [Hue99] a differential calculus on the cohomology and on the homology of (S, L) with properties that are similar to the Tamarkin-Tsygan calculus ([TT05]) on the Hochschild cohomology and on the Hochschild homology of associative algebras.

Question 10. Keep the setting of Theorem 6.2.2. Assume that S is projective as an R-module. How are the Hochschild cohomology and the Hochschild homology of U together with their Tamarkin-Tsygan calculus related to the differential calculus on the cohomology and on the homology of (S, L)?[3]

As proved in [Gin06], the Gerstenhaber bracket on the Hochschild cohomology of any Calabi-Yau algebra has a Batalin-Vilkoviskiĭ(BV) generator.

Question 11. Keep the setting of Corollary 6.2.6. Assume that S is projective as an R-module and let Δ be a BV generator on the Hochschild cohomology $\mathrm{HH}^\bullet(S)$. How is Δ related to the differential calculus on the (co)homology of (S, L)?

13.2. **Extension of the results to bialgebroids.** Assume that R is a field. The couple (S, U) of R-algebras is an instance of bialgebroid ([Sch00]) and, from this point of view, Huebschmann's result [Hue99, Theorem 2.10] on the duality on (S, L) when L is finitely generated and with constant rank as an S-module has been extended by Kowalzig and Krähmer ([KK10]) to bialgebroids (A, V) such that $A \in \mathrm{per}(V)$ and V is projective in $\mathrm{Mod}(A)$ and in $\mathrm{Mod}(A^{\mathrm{op}})$. Since the proofs of the results of section 6 are based on a pair of functors between $\mathrm{Mod}(U)$ and $\mathrm{Mod}(U^e)$

[3]During the preparation of this thesis, Kordon published the preprint [Kor18] presenting a spectral sequence relating the Hochschild cohomology of U and the cohomology of (S, L), which is a first step towards an answer to this question.

such that the functor $\mathrm{Mod}(U) \to \mathrm{Mod}(U^e)$ is exact and transforms S into U it is natural to expect that the results of sections 6.2 and 6.3 should still hold true if the couple (S, U) is replaced by a general bialgebroid.

Question 12. Let (A, V) be a bialgebroid over R such that $A \in \mathrm{per}(V)$ and V is projective in $\mathrm{Mod}(A)$ and in $\mathrm{Mod}(A^{\mathrm{op}})$. Is it true that, if A has Van den Bergh duality, then so does V?

14. Characterisation of piecewise hereditary algebras

Assume that $\Bbbk$ is algebraically closed. Despite the descriptions of the strong global dimension of piecewise hereditary $\Bbbk$-algebras made in section 11, it is uneasy to determine effectively whether a given finite dimensional $\Bbbk$-algebra is piecewise hereditary. On one hand, Happel, Reiten, and Smalø proved ([HRS96a]) that a finite dimensional $\Bbbk$-algebra is quasitilted if and only if its global dimension is at most 2 and any of its indecomposable finitely generated modules has projective dimension at most 1 or injective dimension at most 1. On the other hand, Happel and Zacharia investigated in [HZ10] the homological properties of the *pieces* of piecewise hereditary algebras, which are specific subcategories of the module categories of these algebras. This raises the following question.

Question 13. Is it possible to characterise when a finite dimensional $\Bbbk$-algebra A is piecewise hereditary by the existence of subcategories of $\mathrm{mod}(A)$ defined by homological conditions?

The preparatory text [ACLMP17] is a starting point in this direction.

15. Equivariantisation in cluster tilting and higher Auslander-Reiten theories

Sections 5.6 and 5.7 show that various facets of cluster tilting theory behave well with actions of finite groups. Besides, the recent studies [PS17, AP17] of marked orbifold surfaces from the viewpoint of this theory is likely to develop further. It is hence desirable to have a deeper understanding of the above mentioned behaviour.

Question 14. Let G be a finite group whose order is not divisible by $\mathrm{char}(\Bbbk)$. Let n be an integer not smaller than 2. Let $\mathcal{C}$ be a $\Bbbk$-linear category from one of the following classes: Calabi-Yau in dimension n triangulated categories; n-abelian categories ([Jas16]); n-exact categories ([Jas16]); and n-angulated categories ([GKO13]). If G acts on $\mathcal{C}$ by automorphisms which preserve the underlying structure of $\mathcal{C}$, does the resulting G-equivariant category belong the same class as $\mathcal{C}$?

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