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Patrick Martinez

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Mémoire en vue de l'obtention de
L'Habilitation à Diriger des Recherches

Université Paul Sabatier, Toulouse 3

Spécialité: Mathématiques appliquées

présenté par

Patrick Martinez

le 3 octobre 2019

Document de synthèse:

**Contributions à l'étude de la contrôlabilité, du
comportement asymptotique et de la stabilisation
d'équations aux dérivées partielles**

Composition du jury

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CHAPTER 1

Introduction and publications

1.1. Introduction

1.1.1. Outline of the document.

The document contains my main contributions after my doctoral thesis (december 1998). I have organized into 5 different thematic parts:

- Part 1, the main one, is devoted to our contributions concerning *controllability properties of degenerate parabolic equations*,
- Part 2 is devoted to our contributions concerning *reaction-diffusion equations*,
- Part 3 is devoted to our contributions concerning *stabilization of second-order evolution equations* (related to my Ph. D. subject),
- Part 4 is a short part devoted to our contributions concerning *population dynamics*,
- Part 5 is a short part devoted to our contributions concerning *some "blow-up" property for bounded solutions of ordinary differential equations*.

Most of the results have been obtained in collaboration, and I will always mention my co-authors in the following (and I take here once again the opportunity to thank them for all these collaborations).

In the following, I briefly describe every part, and I list the related personal publications (organized in a thematic way).

1.1.2. Introduction to Part 1.

1.1.2.1. Null controllability of a class of degenerate parabolic equations.

Motivated by several models coming from physics (a boundary layer model: the so-called Crocco equation), genetics (the Fleming-Viot model), climate dynamics (the Sellers and Budyko models), we study the controllability properties of some classes of degenerate parabolic equations. We develop two methods:

- the first one is based on Carleman estimates,
- the second one is based on the moment method.

They are complementary:

- the Carleman estimate approach allows us to obtain null controllability results for a large class of degenerate parabolic equations in 1D and 2D, of the type: null controllability holds if and only if some degeneracy parameter is less than some explicit *critical value*; and we prove such Carleman estimates
 - adapting the classical weights appearing in the Carleman estimates to the degeneracy,
 - and improving some classical inequalities, in particular of Hardy type; this provides also results for models close to the motivating ones;
- the moment method approach allows us to *analyze what happens near this critical value* for simple and typical 1D models; we estimate the null controllability cost with respect to various parameters appearing in the problem (the time T and the degeneracy parameter), and we obtain *precise*

upper and lower estimates on the null controllability cost; to obtain these estimates,

- we prove general and precise estimates on the biorthogonal family to the set of some exponentials related to the eigenvalues of the problem, using Hilbertian and complex analysis techniques,
- we study the associated spectral problem, and we prove several properties of the eigenvalues and eigenfunctions of the problem, using ordinary differential techniques.

1.1.2.2. *Inverse problems results.*

Finally, we investigate inverse problems questions for some degenerate and semilinear parabolic equations close to the climate dynamics models, the first one set on a manifold (the Earth), and a second one with a memory effect (which is natural for such models). We obtain uniqueness results and Lipschitz stability results for some coefficients appearing in these models, using our Carleman estimates and suitable maximum principles (in order to deal with the nonlinearity).

1.1.3. **Introduction to Part 2.**

This part is devoted to reaction-diffusion equations. We study several questions, mainly

- the existence of pulsating waves for a model of solid combustion, when the distribution of the reactant is periodic at the unburnt end; the proof is based on the study of a scalar reaction-diffusion with nonstandard (infinite) boundary conditions;
- we consider a general reaction-diffusion equation of the KPP type, posed on an infinite cylinder; such a model will have a family of pulsating waves of constant speed, larger than a critical speed c_* ; the family of all supercritical waves attract a large class of initial data, and we try to understand how; we describe the fate of an initial datum trapped between two supercritical waves of the same velocity: the solution will converge to a whole set of translates of the same wave, and we identify the convergence dynamics as that of an effective drift, around which an effective diffusion process occurs; in several nontrivial particular cases, we are able to describe the dynamics by an effective equation.

(The major part of this work has been done in collaboration with M. Bages during his Ph. D.)

1.1.4. **Introduction to Part 3.**

The following part consists of several results concerning mainly the stabilization of the wave equation, damped by different feedbacks laws:

- under nonlinear feedbacks: we prove the optimality of the classical decay estimates; the proofs are based on the d'Alembert formula and the study of the asymptotic behaviour of some real sequences defined by induction;
- under time-dependent (and in particular on-off) feedbacks: we prove precise positive/negative results, for linear and semilinear wave equations; the proofs are based on new observability inequalities, and classical properties of rational and irrational numbers;
- stabilization by nonlinear feedbacks of viscoelastic problems: we prove precise decay estimates, comparing the influence of the feedback term and of the memory term.

(Some of these results complete several results obtained during my Ph. D.)

1.1.5. Introduction to Part 4.

We study a new kind of models of population dynamics describing pregnancy, inspired by the recent theory of delay equations with nonautonomous past. We perform a complete analysis of the qualitative properties of a first model, whose novelty is to take into account the events that may happen during the pregnancy. This finally leads us to introduce and study a new model, more natural than the previous one.

1.1.6. Introduction to Part 5.

The object of this part is to give counterexamples to elementary results in analysis that are well-known in finite dimension. By the classical Cauchy-Lipschitz theory of ordinary differential equations, no maximal solution of $x' = f(t, x)$ can belong to some compact subset of the domain of definition D of f . In the finite dimensional case it follows that the maximal solutions are defined up to the boundary of D . Dieudonné and later Deimling gave counterexamples in some infinite dimensional spaces: the maximal solution can remain bounded while it blows up in finite time. We give a complete, elementary and natural proof of this result for *all* infinite dimensional Banach spaces.

1.1.7. Some perspectives.

There are many open problems that I would like to study. I detail some of them in Chapter 8 (related to Part 1) and in Chapter 13 (related to Part 2), in particular

- concerning the Crocco boundary layer model, see section 8.1: this model has been a major motivation for us, and many questions remain open concerning controllability and stabilization properties, due to the presence of a critical degeneracy and nonlinear terms; answers to these questions would be interesting from a theoretical and also an applied point of view;
- in climatology, see section 8.4: involved climate dynamics are described with nonlinear (and/or degenerate) parabolic equations with memory terms and set-valued operators, and many interesting problems are still open (among others: controllability, inverse problems questions, phase transition problems...)
- concerning reaction-diffusion equations, see section 13.2: several recent papers study the influence of roads in the propagation of invasive species (in particular mosquitos); I would like to study related questions from the point of view of controllability and nverse problems; from a mathematical point of view, the problems are particularly interesting, since they are described by systems coupling 1D and 2D equations (the propagation along the road and the diffusion in the field).

1.2. Publications, by theme

1.2.1. Personal papers related to Part 1.

[A1] P. Martinez, J.-P. Raymond, J. Vancostenoble, *Regional null controllability of a Crocco type linearized equation*, SIAM J. Control Optim. 42 (2003), 2, 709-728.

[A2] P. Cannarsa, P. Martinez, J. Vancostenoble, *Null controllability of the heat equation in unbounded domains by a finite measure control region*, ESAIM: Control, Optim, Calc. Var. 10 (2004), 381-408.

[A3] P. Cannarsa, P. Martinez, J. Vancostenoble, *Persistent regional null controllability for a class of degenerate parabolic equations*, Commun. Pure Appl. Anal. 3 (2004),4, 607-635.

[A4] P. Cannarsa, P. Martinez, J. Vancostenoble, *Null Controllability of degenerate heat equations*, Adv. Differential Equations 10 (2005), 2, 153-190.

[A5] P. Cannarsa, P. Martinez, J. Vancostenoble, *Carleman estimates for a class of degenerate parabolic operators*, SIAM J. Control Optim. 47 (2008), no. 1, 1-19.

[A6] P. Martinez, J. Vancostenoble, *Carleman estimates for one-dimensional degenerate heat equations*, J. Evol. Equ. 6 (2006), no. 2, 325-362.

[A7] P. Cannarsa, P. Martinez, J. Vancostenoble, *Global Carleman estimates for degenerate parabolic operators with applications*, mémoire AMS, Vol 239, No 1133 (2016) 209 p.

[A8] P. Cannarsa, P. Martinez, J. Vancostenoble, *The cost of controlling weakly degenerate parabolic equations*, Mathematical Control and Related Fields, Volume 7, Number 2, June 2017, p. 171-211.

[A9] P. Cannarsa, P. Martinez, J. Vancostenoble, *Precise estimates for biorthogonal families under asymptotic gap conditions*, Discrete Contin. Dyn. Syst. Ser. S, to appear.

[A10] P. Cannarsa, P. Martinez, J. Vancostenoble, *The cost of controlling strongly degenerate parabolic equations*, ESAIM Control Optim. Calc. Var., to appear.

[A11] P. Martinez, J. Vancostenoble, *The cost of boundary controllability for a parabolic equation with inverse square potential*, Evol. Eq. Control Theory, to appear.

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Part 1

Controllability of degenerate
parabolic equations and
applications

Presentation

2.1. Motivations to study degenerate parabolic equations

Among various interesting topics in partial differential equations, the theory of parabolic equations could be singled out for at least one feature: it provides a ground for the interaction of very different mathematical objects, either abstract in nature such as evolution equations, harmonic analysis, stochastic processes, or application-oriented like fluid models, population dynamics and mathematical finance.

One of the most fascinating aspects of the theory of parabolic operators is the role they play in the typical issues of control theory, from approximate and exact (null) controllability to optimality conditions.

Pioneering works on the controllability of parabolic equations date back by about half a century being mainly due to Fattorini and Russell [143, 311, 144, 124]. Essentially based on Riesz basis expansion techniques, their approach was well suited to treat operators with constant coefficients.

After the initial progress, new substantial results were obtained in the nineties by the systematic use of Carleman type estimates. Such estimates in weighted Sobolev norms, first introduced by Carleman [85] to study unique continuation problems for elliptic operators in dimension two, were extended to large classes of partial differential operators in arbitrary space dimensions by Hörmander [205, 206] and other authors (see, e.g., [363]), again in a unique continuation context. The application of Carleman estimates to control problems for parabolic operators initiated with the work by Lebeau and Robbiano [234], who combined local (i.e., for solutions with compact support) estimates with Riesz basis techniques. Then, Fursikov and Imanuvilov [209, 166] obtained global estimates, and null controllability results as a direct consequence.

More recently, controllability theory for parabolic equations has grown in various directions, such as:

- semilinear parabolic problems (see for example [10], [11], [100], [125, 126], [138], [149, 154, 155], [361]),
- problems in unbounded domains (see [69], [112, 113, 265, 269] and also [A2] and [251]),
- fluid models such as Euler, Stokes, and Navier-Stokes equations (see, for instance, [21], [97, 98], [136, 137], [211, 210, 151, 153, 166] and the references therein),
- parabolic systems (see, for instance, [150, 7, 176]) and heat or fluid/solid structure models (see, for instance, [303, 304]),
- equations with discontinuous coefficients (see [33, 232]), and
- singular transport-diffusion equations (see [99, 174, 177, 178]).

While, as it is clear from the above discussion, the analysis of the controllability properties of *uniformly parabolic* equations has been developed in various directions, a few results were known for *degenerate parabolic* operators, even though such a class

of equations appears in many problems. We describe below typical examples where degenerate parabolic equations arise in a natural way.

2.1.1. Stochastic invariance.

Degenerate parabolic operators related to stochastic processes are well-known since Feller's investigations [147], [148]. In recent years, several authors have singled out the class of degenerate elliptic operators we discuss below, in connection with the study of *invariant sets for diffusion processes*. Given Lipschitz continuous maps $b : \mathbb{R}^n \rightarrow \mathbb{R}^n$ and $\sigma : \mathbb{R}^n \rightarrow \mathcal{L}(\mathbb{R}^n; \mathbb{R}^m)$, with $n, m \in \mathbb{N}$, let $X(\cdot, x)$ denote the unique solution of

$$\begin{cases} dX(t) = b(X(t))dt + \sigma(X(t)) dW(t) & t \geq 0 \\ X(0) = x \in \mathbb{R}^n \end{cases}$$

where $W(t)$ is a standard m -dimensional Brownian motion on a complete filtered probability space. We recall that a set $S \subset \mathbb{R}^n$ is said to be *invariant* for $X(\cdot, \cdot)$ iff

$$x \in S \implies X(t, x) \in S \quad \mathbb{P} - \text{a.s.} \quad \forall t \geq 0.$$

The problem of finding conditions for the invariance of a closed domain $\bar{\Omega}$ for the stochastic flow $X(\cdot, \cdot)$ has been extensively studied and several results were obtained even for more general problems such as stochastic differential inclusions and control systems (see [165], [14, 15], [65], [22], [108, 109, 110]). Invariance properties were derived using the distance function and the elliptic operator

$$(2.1) \quad Lu(x) = \frac{1}{2} \text{Tr} [a(x)\nabla^2 u(x)] + \langle b(x), \nabla u(x) \rangle,$$

where $a(x) = \sigma(x)\sigma^*(x)$. More precisely, one can show that, when $\Gamma := \partial\Omega$ is regular, $\bar{\Omega}$ is invariant if and only if

$$(2.2) \quad \begin{cases} (i) & Ld_{\Gamma, \Omega}(x) \geq 0 \\ (ii) & \langle a(x)\nabla d_{\Gamma, \Omega}(x), \nabla d_{\Gamma, \Omega}(x) \rangle = 0 \end{cases} \quad \forall x \in \Gamma,$$

where

$$d_{\Gamma, \Omega}(x) = \begin{cases} d(x, \Gamma) & \text{if } x \in \Omega \\ -d(x, \partial\Gamma) & \text{if } x \in \Omega^c \end{cases}$$

is the so-called oriented distance from Γ . Note that condition (ii) in (2.2) above implies that $a(x)$ is a *singular* matrix for all $x \in \Gamma$, and $\nabla d_{\Gamma, \Omega}(x)$ (the inward unit *normal* to $\bar{\Omega}$ at x) is an *eigenvector* of $a(x)$ associated with the zero eigenvalue. Furthermore, conditions (2.2) are necessary and sufficient for the invariance of the open set Ω . Using the invariance of Ω , one can then show that, for any sufficiently smooth function $\varphi : \bar{\Omega} \rightarrow \mathbb{R}$, the *transition semigroup*

$$u(t, x) = \mathbb{E}[\varphi(X(t, x))]$$

is the unique solution of the parabolic equation

$$\begin{cases} u_t = Lu & \text{in } (0, T) \times \Omega, \\ \langle a\nabla u, \nabla d_{\Gamma, \Omega} \rangle = 0 & \text{on } (0, T) \times \Gamma, \\ u(0, x) = \varphi(x) & x \in \Omega, \end{cases}$$

where the above boundary condition is a direct consequence of (2.2).

2.1.2. Laminar flow.

Another example of a degenerate parabolic operator arising from a completely different domain, is related to fluid dynamics models. It is known that the velocity field of a laminar flow on a flat plate can be described by the Prandtl equations, see, e.g., [290]. By using the so-called ‘‘Crocco change of variables’’, these equations are transformed into a nonlinear degenerate parabolic equation—the Crocco equation—in the plane domain $\Omega = (0, L) \times (0, 1)$. Although the latter equation is still hard to study, it is well-known that important properties of a given dynamical systems at an equilibrium state are captured by the linearization of such a system. Now, the linearization of the Crocco equation at a stationary solution takes the form

$$(2.3) \quad \begin{cases} u_t + bu_x - au_{yy} + cu = f, & (t, x, y) \in (0, T) \times \Omega, \\ u_y(t, x, 0) = u(t, x, 1) = 0, & (t, x) \in (0, T) \times (0, L), \\ u(t, 0, y) = u_1(t, y), & (t, y) \in (0, T) \times (0, 1), \\ u(0, x, y) = u_0(x, y), & (x, y) \in \Omega, \end{cases}$$

where f and u_1 depend on the incident velocity of the flow. Here, the coefficients a , b and c are regular but *degenerate* at the boundary, since one can show that

$$0 < b_1 \leq \frac{b(y)}{y} \leq b_2, \quad 0 < a_1 \leq \frac{a(x, y)}{-(y-1)^2 \ln(\mu(1-y))} \leq a_2, \quad c(x, y) \geq 0,$$

for suitable constants a_i, b_i ($i = 1, 2$) and $\mu \in (0, 1)$ (see [62]). Clearly, another source of degeneracy for problem (2.3) is the fact that the second derivative u_{xx} is missing throughout the whole domain.

2.1.3. Budyko-Sellers climate model.

One of the first attempts to model the interaction between large ice masses and solar radiation on climate is the one due, independently, to Budyko [66, 67] and Sellers [317] (for the mathematical analysis of those models see for example [116, 202, 122] and the references therein). Such a model studies the sea level mean zonally averaged temperature $u(t, x)$ on the Earth, where t denotes time and x the sine of the latitude. The heat-balance equation for u is given by

$$(2.4) \quad cu_t - (k(1-x^2)u_x)_x = \frac{1}{4}S_0s(x)\alpha(x, u) - I(u), \quad (x, t) \in (0, T) \times (-1, 1)$$

where c is the thermal capacity of the Earth, k the horizontal thermal conductivity which may be a function of x , S_0 the solar constant, $s(x)$ the normalized distribution of solar input, α the coalbedo and $I(u)$ the outgoing infrared radiation which, in Budyko’s model, is an affine function, that is, $I(u) = a + bu$. Notice that (2.4) degenerates at the boundary of the space domain. The boundary conditions associate with the above equation are the following ones

$$(2.5) \quad (1-x^2)u_x = 0 \quad \text{at} \quad x = \pm 1.$$

2.1.4. Fleming-Viot gene frequency model.

Other interesting situations where degenerate parabolic models occur concern population genetics. More precisely, we are interested in gene frequency models that describe the genetic evolution of a population. They may take the form of degenerate parabolic equations with a diffusion term in a non-divergence form. Let us for example describe the d -dimensional Fleming-Viot model [133, 134, 90, 72].

The evolution problems associated to Fleming-Viot operators are diffusion approximations of gene frequency models in population genetics. More precisely, the Fleming-Viot operator corresponds to some diffusion model in population dynamics in which each individual is of some *type* and the *type space* is given by a finite

number d of elements. In this case the state space is the following d -dimensional simplex in \mathbb{R}^d :

$$K_d = \left\{ (x_1, \dots, x_d) \in (\mathbb{R}_+)^d \mid \sum_{i=1}^d x_i \leq 1 \right\}.$$

Here x_i denotes the proportion of the population that is of type i . We consider the differential operator

$$Au(x) = \sum \frac{x_i(\delta_{ij} - x_j)}{2} \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum b_i(x) \frac{\partial u}{\partial x_i}(x).$$

The evolution problem associated to this operator is a diffusion approximation of gene frequency models in population genetics. The first order term $b \cdot \nabla u$ corresponds to mutation, migration and selection phenomena. Therefore, in general, some of the functions b_i are not identically equal to zero.

The difficulty in studying this operator relies in particular on the fact A is a second-order elliptic operator that degenerates at the boundary and that the boundary of K_d is non smooth due to the presence of edges and corners.

2.2. Presentation of our main results

An easy model to explain the main points in which degenerate parabolic operators differ from uniformly parabolic one—as for null controllability—is provided by equations in one space dimension. We studied in several directions typical 1D and 2D degenerate parabolic equations, and we also investigated some of the motivating examples. In the following:

- Chapter 3: we present our (optimal) regional null controllability result on the Crocco equation (based on the combination of classical properties of diffusion equations and transport phenomena);
- Chapter 4: we study in a general way 1D degenerate parabolic equations, providing positive and negative results, the transition between them being some critical degeneracy parameter; (this study will be based on Carleman estimates, adapted to the degenerate equation under consideration);
- Chapter 5: we study in a similar way 2D degenerate parabolic equations; computations are much more tedious, but that allows us to obtain results for models close to the Fleming-Viot one;
- Chapter 6: we study precisely what happens for degeneracy parameter close to the critical one, extending some classical properties of biorthogonal families, under suitable gap conditions on the eigenvalues, these gap conditions being satisfied in the models we have in mind; we apply our general results to several problems:
 - parabolic equations with degeneracy and control acting at the same boundary point,
 - parabolic equations with degeneracy at the boundary and control acting in a localized way,
 - parabolic equations with degeneracy inside and control acting in a localized way,
 - heat equation with inverse square potential;
- Chapter 7: we will study inverse problem questions related to the Budyko-Sellers model arising in climatology; we will consider two kinds of problem:
 - the first one: set on a manifold (the Earth),
 - the second one: a 1D version, but with an additional memory term (that modelizes the long response time of ice sheets);
- Chapter 8: we present some open problems in this thematic.

**The motivating example of the Crocco equation
(Joint work with J.-P. Raymond and J.
Vancostenoble [A1])**

3.1. Introduction: motivation and related paper

3.1.1. Motivation.

The velocity field of a laminar flow on a flat plate can be described by the Prandtl equations [290]. For a two dimensional flow, these equations are stated in an unbounded domain $(0, L) \times (0, \infty)$, where $(0, L)$ represents the part of the plate where the flow is laminar, and $(0, \infty)$ represents the thickness of the boundary layer. The matching conditions with the external flow are stated at $+\infty$. By using the so-called Crocco transformation, these equations are transformed into a nonlinear degenerate parabolic equation (the Crocco equation; see [290]) which is stated in a bounded domain $\Omega = (0, L) \times (0, 1)$. The linearization of the Crocco equation around a stationary solution is an equation of the form

$$(3.1) \quad \begin{cases} u_t + au_x - bu_{yy} + cu = f, & (x, y, t) \in \Omega \times (0, T), \\ u_y(x, 0, t) = u(x, 1, t) = 0, & (x, t) \in (0, L) \times (0, T), \\ u(0, y, t) = u_1(y, t), & (y, t) \in (0, 1) \times (0, T), \\ u(x, y, 0) = u_0(x, y), & (x, y) \in \Omega, \end{cases}$$

where f and u_1 depend on the incident velocity of the flow, and where the coefficients a , b and c are regular, but degenerate, and have the following behavior [64, 62]:

$$0 < a_1 \leq \frac{a(y)}{y} \leq a_2, \quad 0 < b_1 \leq \frac{b(x, y)}{-(y-1)^2 \ln(\mu(1-y))} \leq b_2, \quad c(x, y) \geq 0,$$

with $0 < \mu < 1$. Since the coefficient b is degenerate, the Dirichlet boundary condition at $y = 1$ has to be correctly interpreted (see [62]).

This linearized model has been used to study stabilization problems of boundary layers in [61]. The perturbations of the velocity field in the boundary layer are controlled by a suction velocity through the plate, localized on a slot (x_0, x_1) . Here we are interested in the null controllability problem for an equation of the type (3.1), but with constant coefficients.

3.1.2. The related paper [A1].

For simplicity, we first study a problem with homogeneous Dirichlet boundary conditions and a locally distributed control. The boundary condition at $y = 0$ of (3.1) will be replaced by a Dirichlet condition, in order to simplify the functional setting. The cases of the other kinds of boundary conditions (Neumann or mixed Dirichlet-Neumann conditions) together with the case of a boundary control are treated in a second part of [A1].

Consider $\omega = (x_0, x_1) \times \omega_y$ where $0 < x_0 < x_1 < L$ and ω_y is an open subdomain of $(0, 1)$; denote χ_ω the characteristic function of ω .

Given $u_0 \in L^2(\Omega)$, $u_1 \in L^2((0, 1) \times (0, T))$ and $f \in L^2(\omega \times (0, T))$, we consider the following control problem:

$$(3.2) \quad \begin{cases} u_t + u_x - u_{yy} = \chi_\omega f, & (x, y, t) \in \Omega \times (0, T), \\ u(x, 0, t) = u(x, 1, t) = 0, & (x, t) \in (0, L) \times (0, T), \\ u(0, y, t) = u_1(y, t), & (y, t) \in (0, 1) \times (0, T), \\ u(x, y, 0) = u_0(x, y), & (x, y) \in \Omega. \end{cases}$$

First, one can prove that the problem is well-posed: given $u_0 \in L^2(\Omega)$, $u_1 \in L^2((0, 1) \times (0, T))$ and $f \in L^2(\omega \times (0, T))$, the problem (3.2) has a unique solution $u \in \mathcal{C}([0, T]; L^2(\Omega)) \cap \mathcal{C}([0, L]; L^2((0, T) \times (0, 1))) \cap L^2((0, T) \times (0, L); H_0^1(0, 1))$.

Next we study the following *regional* null controllability problem: given $u_0 \in L^2(\Omega)$, $u_1 \in L^2((0, 1) \times (0, T))$, does there exist $f \in L^2(\omega \times (0, T))$ such that the solution u of (3.2) satisfies $u(x, y, T) = 0$ for all $(x, y) \in \Omega_C$, where Ω_C denotes a part of Ω ?

Indeed, in the case studied here, due to transport phenomenon, the influence domain of the control $\chi_\omega f$ is not the whole domain Ω at time $T > 0$. Thus (global) null controllability does not occur. For this reason, we introduce the notion of regional null controllability. As a first step, we give a geometric characterization of the influence domain of the control $\chi_\omega f$ in order to determine the region Ω_C of Ω on which it will be possible to control u at time T .

3.2. Influence domain of the control

Take $T > 0$ and consider

$$\Omega_C(T) := \begin{cases} (x_0, x_1 + T) \times (0, 1) & \text{if } T < L - x_1, \\ (x_0, L) \times (0, 1) & \text{if } T > L - x_1. \end{cases}$$

Using spectral decomposition of the solution of (3.2), one can prove (see **[A1]**) that the domain of influence of $\chi_\omega f$ at time T is the domain $\Omega_C(T)$. Indeed, due to the phenomenon of diffusion in the direction y , the region of influence in y at time T of a control supported in y in ω_y is the whole interval $(0, 1)$. On the other hand, due to the transport phenomenon (at speed equal to 1) in the x -direction, the region of influence in x at time T of a control supported in x in (x_0, x_1) is only $(x_0, x_1 + T)$ in the case $T < L - x_1$ and is only (x_0, L) in the case $T > L - x_1$. This means that a control localized in $\omega \times (0, T)$ has no influence at time T on the solution in $\Omega \setminus \Omega_C(T)$.

Next we prove a result of regional null controllability in a domain slightly smaller than $\Omega_C(T)$, that will follow from an observability inequality for the associated adjoint problem.

3.3. Observability inequality

Assume that $0 < T < L - x_1$ and define

$$\Omega_C(\delta, T) := (x_0 + \delta, x_1 + T - \delta) \times (0, 1)$$

where $0 < \delta < (x_1 - x_0)/2$. Then we prove the following observability inequality:

THEOREM 3.1. (**[A1]**) *Under the previous assumptions, there exists $C(\delta, \omega_y) > 0$ such that all solution $v \in \mathcal{C}([0, T]; L^2((0, L) \times (0, 1))) \cap \mathcal{C}([0, L]; L^2((0, T) \times (0, 1))) \cap L^2((0, L) \times (0, T); H_0^1(0, 1))$ of the problem*

$$(3.3) \quad v_t + v_x + v_{yy} = 0, \quad (x, y, t) \in \Omega \times (0, T),$$

verifies

$$(3.4) \quad \iint_{(0,L) \times (0,1)} v(x,y,0)^2 dydx + \iint_{(0,1) \times (0,T)} v(0,y,t)^2 dt dy \\ \leq C(\delta, \omega_y) \left(\iiint_{\omega \times (0,T)} v(x,y,t)^2 dt dy dx \right. \\ \left. + \iint_{\Omega \setminus \Omega_C(\delta,T)} v(x,y,T)^2 dydx + \iint_{(0,1) \times (0,T)} v(L,y,t)^2 dt dy \right).$$

Idea of the proof: along the characteristic lines, v is solution of a nondegenerate parabolic equation: indeed, fix $\xi \in (-T, L)$ and consider $w^\xi(y, t) := v(\xi + t, y, t)$; then w^ξ is solution of

$$(3.5) \quad \begin{cases} w_t + w_{yy} = 0, & (y, t) \in (0, 1) \times (t_1^\xi, t_2^\xi), \\ w(1, t) = w(0, t) = 0, & t \in (t_1^\xi, t_2^\xi), \end{cases}$$

where $t_1^\xi = \max(0, -\xi)$ and $t_2^\xi = \min(L, L - \xi)$. Then the proof of the observability inequality (3.4) follows from a suitable decomposition of the domain $\Omega \times (0, T)$, combining on the characteristic lines classical Carleman estimates [166, 234]) with the nondecreasingness of the energy of solutions of (3.5).

3.4. Null controllability result

Then we derive the following result:

THEOREM 3.2. ([A1]) *Under the previous assumptions, given $u_0 \in L^2(\Omega)$ and $u_1 \in L^2((0, 1) \times (0, T))$, there exists $f \in L^2(\omega \times (0, T))$ such that the solution u of (3.2) verifies*

$$u(x, y, T) = 0 \quad \text{a.e. } (x, y) \in \Omega_C(\delta, T).$$

Idea of the proof: we introduce the following penalized problem

$$(3.6) \quad \inf \{ J_\varepsilon(f) \mid f \in L^2(\omega \times (0, T)) \},$$

where

$$J_\varepsilon(f) := \frac{1}{2} \iiint_{\omega \times (0,T)} f^2 dt dy dx + \frac{1}{2\varepsilon} \iint_{\Omega_C(\delta)} u_f(x, y, T)^2 dy dx,$$

with u_f the solution of (3.2) associated to f .

If $(u_\varepsilon, f_\varepsilon)$ is solution of (3.6), then consider v_ε the solution of

$$(3.7) \quad \begin{cases} v_t + v_x + v_{yy} = 0, & (x, y, t) \in \Omega \times (0, T), \\ v(x, 0, t) = v(x, 1, t) = 0, & (x, t) \in (0, L) \times (0, T), \\ v(L, y, t) = 0, & (y, t) \in (0, 1) \times (0, T), \\ v(x, y, T) = \frac{1}{\varepsilon} \chi_{\Omega_C(\delta)} u_\varepsilon(x, y, T), & (x, y) \in \Omega, \end{cases}$$

and

$$f_\varepsilon := -v_\varepsilon \chi_\omega.$$

Then, thanks to (3.4), one can show that $(f_\varepsilon, \frac{1}{\varepsilon} \chi_{\Omega_C(\delta,T)} u_\varepsilon(T))_\varepsilon$ is bounded in $L^2(\omega \times (0, T)) \times L^2(\Omega)$. This a priori estimate allows to pass to the limit in (3.7) letting $\varepsilon \rightarrow 0$, and we obtain a solution of the null controllability problem.

Positive and negative results for 1-D degenerate parabolic equations

(Joint works with P. Cannarsa and J. Vancostenoble ([A2]-[A6] and a part of [A7]))

4.1. Introduction: motivation and related papers

4.1.1. Motivation.

Motivated by several examples, in particular the Crocco equation, we studied the null controllability properties of the typical degenerate parabolic equation

$$u_t - (x^\alpha u_x)_x$$

under the action of a control acting either on the boundary of the domain or on a subdomain, the goal being to be able to go closer and closer to natural problems. Even in this simple case, it appears that the results depend on the value of α , and the value $\alpha = 2$ is a threshold value, as we explain in the following.

4.1.2. Related papers [A2]-[A6] and a part of [A7].

- [A5] contains our main results concerning degenerate parabolic equations: the Carleman estimate, with weight functions adapted to the degeneracy, and the positive results when $\alpha < 2$;
- [A3], [A4] contain partial results when $\alpha < 2$ (the steps that drove us to the ones of [A5]), the negative result when $\alpha \geq 2$ (and the associated "persistent" notion);
- [A6] is an extension of [A5] (which was published 2 years before, the mystery of publication dates...);
- [A7] is devoted to the space dimension 2 but contains some material useful in 1D (generalized Hardy type inequalities), that allows to avoid some technical complications; we include them in this chapter devoted to the 1D case;
- in [A2], we study the nondegenerate parabolic equation in an unbounded case, a question which is close but somewhat different; we provide positive results under suitable assumptions on the control region (of finite measure, but unbounded).

4.2. Degenerate parabolic equations in a bounded interval([A3]-[A7])

4.2.1. Statement of the controllability problem.

Given $0 \leq \alpha < 2$, define

$$\forall x \in [0, 1], \quad a(x) := x^\alpha,$$

and let ω be a nonempty subinterval of $(0, 1)$. For $T > 0$, set

$$Q_T = (0, T) \times (0, 1),$$

and consider the initial-boundary value problem

$$(4.1) \quad \begin{cases} u_t - (au_x)_x = h \chi_\omega, & (t, x) \in Q_T, \\ u(t, 1) = 0, & t \in (0, T), \\ \text{and } \begin{cases} u(t, 0) = 0, & \text{for } 0 \leq \alpha < 1, \\ (au_x)(t, 0) = 0, & \text{for } 1 \leq \alpha < 2, \end{cases} & t \in (0, T), \\ u(0, x) = u_0(x), & x \in (0, 1), \end{cases}$$

where u_0 is given in $L^2(0, 1)$ and $h \in L^2(Q_T)$.

The controllability problem will be: given u_0 and $T > 0$, find a control h such that $u(T) = 0$.

4.2.2. Well-posedness.

Let us recall that the above problem is well-posed in appropriate weighted spaces. For $0 \leq \alpha < 1$, define the Hilbert space $H_a^1(0, 1)$ as

$$H_a^1(0, 1) := \{u \in L^2(0, 1) \mid u \text{ absolutely continuous in } [0, 1], \\ \sqrt{a}u_x \in L^2(0, 1) \text{ and } u(0) = u(1) = 0\},$$

and the unbounded operator $A : D(A) \subset L^2(0, 1) \rightarrow L^2(0, 1)$ by

$$\begin{cases} \forall u \in D(A), & Au := (au_x)_x, \\ D(A) := \{u \in H_a^1(0, 1) \mid au_x \in H^1(0, 1)\}. \end{cases}$$

Notice that, if $u \in D(A)$ (or even $u \in H_a^1(0, 1)$), then u satisfies the Dirichlet boundary conditions $u(0) = u(1) = 0$.

For $1 \leq \alpha < 2$, let us change the definition of $H_a^1(0, 1)$ to

$$H_a^1(0, 1) := \{u \in L^2(0, 1) \mid u \text{ locally absolutely continuous in } (0, 1], \\ \sqrt{a}u_x \in L^2(0, 1) \text{ and } u(1) = 0\}.$$

Then, the operator $A : D(A) \subset L^2(0, 1) \rightarrow L^2(0, 1)$ will be defined by

$$\begin{cases} \forall u \in D(A), & Au := (au_x)_x, \\ D(A) := \{u \in H_a^1(0, 1) \mid au_x \in H^1(0, 1)\}, \\ \quad = \{u \in L^2(0, 1) \mid u \text{ locally absolutely continuous in } (0, 1], \\ \quad \quad au \in H_0^1(0, 1), au_x \in H^1(0, 1) \text{ and } (au_x)(0) = 0\}. \end{cases}$$

Notice that, if $u \in D(A)$, then u satisfies the Neumann boundary condition $(au_x)(0) = 0$ at $x = 0$ and the Dirichlet boundary condition $u(1) = 0$ at $x = 1$.

In both cases, the following results hold, (see, e.g., [71] and [A4]).

PROPOSITION 4.1. $A : D(A) \subset L^2(0, 1) \rightarrow L^2(0, 1)$ is a closed self-adjoint negative operator with dense domain.

Hence, A is the infinitesimal generator of a strongly continuous semigroup e^{tA} on $L^2(0, 1)$. Consequently, we have the following well-posedness result.

THEOREM 4.2. Let h be given in $L^2(Q_T)$. For all $u_0 \in L^2(0, 1)$, problem (4.1) has a unique solution

$$(4.2) \quad u \in \mathcal{C}^0([0, T]; L^2(0, 1)) \cap L^2(0, T; H_a^1(0, 1)).$$

Moreover, if $u_0 \in D(A)$, then

$$(4.3) \quad u \in \mathcal{C}^0([0, T]; H_a^1(0, 1)) \cap L^2(0, T; D(A)) \cap H^1(0, T; L^2(0, 1)).$$

4.2.3. Carleman estimates for degenerate problems.

In order to study the controllability properties of (4.1), we need to derive a Carleman estimate for the adjoint problem. Keeping the notations

$$a(x) := x^\alpha, \quad \text{with } 0 \leq \alpha < 2, \quad \text{and } Q_T = (0, T) \times (0, 1) \quad \text{for } T > 0,$$

let us consider the parabolic problem

$$(4.4) \quad \begin{cases} w_t + (aw_x)_x = f, & (t, x) \in Q_T, \\ w(t, 1) = 0, & t \in (0, T), \\ \text{and } \begin{cases} w(t, 0) = 0, & \text{for } 0 \leq \alpha < 1, \\ (aw_x)(t, 0) = 0, & \text{for } 1 \leq \alpha < 2, \end{cases} & t \in (0, T), \\ w(T, x) = w_T(x), & x \in (0, 1), \end{cases}$$

where $w_T \in L^2(0, 1)$ and $f \in L^2(Q_T)$. Our main result is the following:

THEOREM 4.3. ([A5]) *Let $0 \leq \alpha < 2$ and $T > 0$ be given. Then, there exists $\sigma : (0, T) \times [0, 1] \rightarrow \mathbb{R}_+^*$ of the form $\sigma(t, x) = \theta(t)p(x)$, with*

$$p(x) > 0 \quad \forall x \in [0, 1] \quad \text{and} \quad \theta(t) \rightarrow \infty \quad \text{as } t \rightarrow 0^+, T^-,$$

and two positive constants, C and R_0 , such that, for all $w_T \in L^2(0, 1)$ and $f \in L^2(Q_T)$, the solution w of (4.4) satisfies, for all $R \geq R_0$,

$$\begin{aligned} \iint_{Q_T} \left(R\theta x^\alpha w_x^2 + R^3 \theta^3 x^{2-\alpha} w^2 \right) e^{-2R\sigma} dx dt \\ \leq C \iint_{Q_T} e^{-2R\sigma} f^2 dx dt + C \int_0^T \left\{ R\theta e^{-2R\sigma} w_x^2 \right\}_{|x=1}. \end{aligned}$$

REMARK 4.4. We will choose

$$\begin{cases} p(x) := \frac{2-x^{2-\alpha}}{(2-\alpha)^2} \\ \forall x \in [0, 1] \end{cases} \quad \text{and} \quad \begin{cases} \theta(t) = \left(\frac{1}{t(T-t)} \right)^4 \\ \forall t \in (0, T) \end{cases}.$$

This weight function θ satisfies the following essential properties :

$$\theta(t) \rightarrow 0 \quad \text{as } t \rightarrow 0^+ \text{ or } T^- \quad \text{and} \quad |\theta_t| \leq c\theta^{5/4}, \quad |\theta_{tt}| \leq c\theta^{3/2},$$

for some constant $c > 0$ depending on T , and p is positive and $(x^\alpha p x)_{xx} = 0$, which is interesting in the computations leading to Carleman estimates.

4.2.4. Observability inequalities.

As it is well-known, very useful tools to study controllability are provided by observability inequalities for the adjoint problem

$$(4.5) \quad \begin{cases} v_t + (av_x)_x = 0, & (t, x) \in Q_T, \\ v(t, 1) = 0, & t \in (0, T), \\ \text{and } \begin{cases} v(t, 0) = 0, & \text{for } 0 \leq \alpha < 1, \\ (av_x)(t, 0) = 0, & \text{for } 1 \leq \alpha < 2, \end{cases} & t \in (0, T), \\ v(T, x) = v_T(x), & x \in (0, 1), \end{cases}$$

where v_T is given in $L^2(0, 1)$. From the Carleman estimate of Theorem 4.3, we obtain the following observability inequalities for (4.5):

THEOREM 4.5. ([A5]) *Let $0 \leq \alpha < 2$ and $T > 0$ be given, and let ω be a nonempty subinterval of $(0, 1)$. Then there exists $C > 0$ such that, for all $v_T \in L^2(0, 1)$, the solution v of (4.5) satisfies*

$$(4.6) \quad \int_0^1 x^\alpha v_x(0, x)^2 dx \leq C \int_0^T \int_\omega v(t, x)^2 dx dt$$

and

$$(4.7) \quad \int_0^1 v(0, x)^2 dx \leq C \int_0^T \int_{\omega} v(t, x)^2 dx dt.$$

4.2.5. Application to controllability.

For any $0 \leq \alpha < 2$, the following observability inequalities follows from Theorem 4.5 and Hardy's inequalities, and yield, by standard arguments (see, e.g., [124, 166]), a null controllability result for degenerate heat equations with initial data in $L^2(0, 1)$.

THEOREM 4.6. ([A5]) *Let $0 \leq \alpha < 2$ and $T > 0$ be given, and let ω be a nonempty subinterval of $(0, 1)$. Then, for all $u_0 \in L^2(0, 1)$, there exists $h \in L^2((0, T) \times \omega)$ such that the solution of the degenerate problem (4.1) satisfies $u(T) \equiv 0$ in $(0, 1)$.*

4.2.6. Hardy type inequalities.

A major ingredient for the proofs of Theorems 4.3 and 4.5 is the following well-known Lemma:

LEMMA 4.7. ([291])

(i) *Let $0 \leq \alpha^* < 1$. Then, for all locally absolutely continuous function z on $(0, 1)$ satisfying*

$$z(x) \rightarrow 0 \quad \text{and} \quad \int_0^1 x^{\alpha^*} z_x^2 dx < \infty,$$

the following inequality holds

$$(4.8) \quad \int_0^1 x^{\alpha^* - 2} z^2 dx \leq \frac{4}{(1 - \alpha^*)^2} \int_0^1 x^{\alpha^*} z_x^2 dx.$$

(ii) *Let $1 < \alpha^* < 2$. Then, the above inequality (4.8) still holds for all locally absolutely continuous function z on $(0, 1)$ satisfying*

$$z(x) \rightarrow 0 \quad \text{and} \quad \int_0^1 x^{\alpha^*} z_x^2 dx < +\infty.$$

REMARK 4.8. Notice that (4.8) is false for $\alpha^* = 1$.

In [A7], we proved in particular the following generalized Hardy type inequality, useful in several situations:

LEMMA 4.9. ([A7]) *Given $L > 0$, $\alpha \in [0, 2)$, $\beta > 0$, $n > 0$, there exists $C_1 = C_1(L, \alpha, \beta, n) > 0$ and $x_1 = x_1(L, \beta, n) \in (0, L)$ such that the following inequality holds: for all $z \in \mathcal{D}(0, L)$,*

$$(4.9) \quad \frac{(1 - \alpha)^2}{4} \int_0^L x^{\alpha - 2} z(x)^2 dx + n \int_0^L x^{\alpha - 2 + \beta} z(x)^2 dx \\ \leq \int_0^L x^{\alpha} z_x(x)^2 dx + C_1 \int_{x_1}^L z(x)^2 dx,$$

Note that Lemma 4.9 allows us to bound terms of the form $\int_0^L x^{\alpha - 2 + \beta} z(x)^2 dx$ uniformly with respect to $\alpha \in [0, 1)$, thus yielding the uniform-in- α bound considering the null controllability cost as $\alpha \rightarrow 1^-$. The proof of Lemma 4.9 gives an explicit formula for C_1 and x_1 .

4.2.7. Extensions.

In [A6] we extended these results to more general 1D equations:

Let us assume that

$$(4.10) \quad a \in \mathcal{C}^0([0, 1]) \cap \mathcal{C}^1((0, 1)), \quad a > 0 \text{ on } (0, 1) \text{ and } \frac{1}{\sqrt{a}} \in L^1(0, 1).$$

Of course, we are mainly interested in the situation of a degenerate equation, i.e. in the case where

$$a(0) = 0 \text{ and/or } a(1) = 0.$$

4.2.7.1. *Definition 1: Weakly-weakly degenerate problem (WW).*

If, in addition of (4.10), a satisfies

$$(4.11) \quad \frac{1}{a} \in L^1(0, 1),$$

we consider the weakly-weakly degenerate problem (WW):

$$(4.12) \quad \begin{cases} u_t - (a(x)u_x)_x = h(t, x)\chi_\omega(x), & (t, x) \in (0, T) \times (0, 1), \\ u(t, 0) = u(t, 1) = 0, & t \in (0, T), \\ u(0, x) = u_0(x), & x \in (0, 1). \end{cases}$$

Notice that nondegenerate problems ($a > 0$ on $[0, 1]$) are contained in this case.

4.2.7.2. *Definition 2: Strongly-weakly degenerate problem (SW).*

If, in addition of (4.10), a satisfies

$$(4.13) \quad a \in \mathcal{C}^1([0, 1]), \quad a(0) = 0, \quad \text{and} \quad \frac{1}{a} \in L^1(1/2, 1),$$

we consider the strongly-weakly degenerate problem (SW):

$$(4.14) \quad \begin{cases} u_t - (a(x)u_x)_x = h(t, x)\chi_\omega(x), & (t, x) \in (0, T) \times (0, 1), \\ (au_x)(t, 0) = u(t, 1) = 0, & t \in (0, T), \\ u(0, x) = u_0(x), & x \in (0, 1). \end{cases}$$

4.2.7.3. *Definition 3: Strongly-strongly degenerate problem (SS).*

If, in addition of (4.10), a satisfies

$$(4.15) \quad a \in \mathcal{C}^1([0, 1]), \quad \text{and} \quad a(0) = 0 = a(1),$$

we consider the strongly-strongly degenerate problem (SS):

$$(4.16) \quad \begin{cases} u_t - (a(x)u_x)_x = h(t, x)\chi_\omega(x), & (t, x) \in (0, T) \times (0, 1), \\ (au_x)(t, 0) = (au_x)(t, 1) = 0, & t \in (0, T), \\ u(0, x) = u_0(x), & x \in (0, 1). \end{cases}$$

4.2.7.4. *The general problem.*

Summing up, keeping the notation

$$Q_T = (0, T) \times (0, 1) \text{ for } T > 0,$$

we consider the problem

$$(4.17) \quad \begin{cases} u_t - (au_x)_x = h\chi_\omega, & \text{in } Q_T, \\ \left[\begin{array}{ll} u(t, 0) = 0 = u(t, 1) & \text{in case (WW),} \\ (au_x)(t, 0) = 0 = u(t, 1) & \text{in case (SW),} \\ (au_x)(t, 0) = 0 = (au_x)(t, 1) & \text{in case (SS),} \end{array} \right. & \text{for } t \in (0, T), \\ u(0, x) = u_0(x), & \text{for } x \in (0, 1). \end{cases}$$

We also assume the following: Let $a : [0, 1] \rightarrow \mathbb{R}_+$ be a function satisfying (4.10) and satisfying either (4.11) or (4.13) or (4.15). Moreover assume that

$$(4.18) \quad \frac{xa'(x)}{a(x)} \xrightarrow{x \rightarrow 0^+} \alpha \quad \text{and} \quad \frac{(1-x)a'(x)}{a(x)} \xrightarrow{x \rightarrow 1^-} -\beta \quad \text{with} \quad \alpha, \beta \in [0, 2].$$

And we prove the following

THEOREM 4.10 (Null controllability). ([A6]) *Let us assume that (4.18) holds. Then, given $T > 0$, ω a nonempty subinterval of $(0, 1)$ and $u_0 \in L^2(0, 1)$, there exists $h \in L^2((0, T) \times \omega)$ such that the solution of the degenerate problem (4.17) satisfies $u(T) \equiv 0$ in $(0, 1)$.*

4.2.8. Negative result when $\alpha \geq 2$.

It is important to note that the above results are optimal in the following sense: null controllability is not possible if $\alpha > 2$. First consider the typical case $a(x) = x^\alpha$ with $\alpha > 0$, $\alpha \neq 2$. Then $xa'(x)/a(x) = \alpha$, and the standard change of variable given by

$$X := \int_x^1 \frac{1}{\sqrt{a}}, \quad U(t, X) := a(x)^{-1/4} u(t, x)$$

transforms the degenerate parabolic equation

$$u_t - (au_x)_x = f\chi_\omega$$

set in the space domain $(0, 1)$ with $\omega = (x_0, x_1) \subset\subset (0, 1)$ into the following heat equation with a potential term:

$$U_t - U_{XX} + b(X)U = F\chi_{\tilde{\omega}}$$

set in the space domain $(0, 1/(1 - \alpha/2))$ if $\alpha < 2$, and in $(0, +\infty)$ if $\alpha > 2$. The potential term is given by

$$b(X) = \frac{\alpha}{4} \left(\frac{3\alpha}{4} - 1 \right) \frac{1}{\left(1 + \left(\frac{\alpha}{2} - 1 \right) X \right)^2},$$

hence it is *singular* if $\alpha < 2$, since $b(X) \rightarrow +\infty$ as $X \rightarrow 1/(1 - \alpha/2)$, and *bounded* if $\alpha > 2$.

Consider now only the case $\alpha > 2$: the new control region is now $\tilde{\omega} = (X_1, X_0)$ with $0 < X_1 < X_0 < \infty$, and remains a relatively compact subset of the space domain $(0, +\infty)$. Using a cut-off argument and standard controllability results, it is easy to see that any initial condition compactly supported in $[0, X_0)$ can be controlled to zero with a control F acting on (X_1, X_0) . On the other hand, a result of Escauriaza, Seregin and Šverák [131, 132], that generalizes a result of Micu and Zuazua [265], says that the initial conditions that can be controlled to zero in finite time have to be supported in $[0, X_0)$. Coming back to the natural variables, it says that when $a(x) = x^\alpha$ and $\alpha > 2$, then the initial conditions u_0 that can be controlled to zero in finite time are the ones that are compactly supported in $(x_0, 1]$. More generally, if $\int_0^1 1/\sqrt{a} = +\infty$ and if the function $x \in (0, 1) \mapsto a''(x) - \frac{1}{4} \frac{a'(x)^2}{a(x)}$ (that gives the potential term in the transformed equation) is bounded on $(0, 1)$, then the same conclusion holds: the only initial conditions that can be controlled to zero in finite time are the ones that are compactly supported in $(x_0, 1]$. Concerning these questions of null controllability in unbounded domains, we also refer the reader to Miller [274].

4.3. Controllability properties of the heat equation in unbounded domains ([A2])

Motivated by the negative results for the heat equation on unbounded domains, we study in [A2] the heat equation in an unbounded domain, and we obtain a

positive null controllability result when the control acts on an unbounded but 'small' subdomain:

4.3.1. Introduction.

We consider the usual heat equation in the unbounded domain $\Omega := (0, +\infty)$:

$$(4.19) \quad \begin{cases} u_t - u_{xx} = f\chi_\omega, & x \in (0, \infty), t \in (0, T), \\ u(t, x = 0) = 0, & t \in (0, T), \\ u(t = 0, x) = u_0(x), & x \in (0, +\infty). \end{cases}$$

The control is localized on the subregion ω . When Ω is bounded, ω is an open subdomain and $u_0 \in L^2(\Omega)$, there exists $f \in L^2(\omega \times (0, T))$ such that $u(T) = 0$. But when Ω is unbounded, the answer is not always positive:

- *negative result when ω is bounded:* when the control region is bounded, Micu-Zuazua [265] proved that, within the class of solutions defined by transposition, there is no smooth compactly supported initial datum that might be driven to 0 in finite time, due to the fact that an unbounded region is left without control; Escauriaza-Seregin-Sverak [131, 132] generalized this in several directions (in particular adding a potential term);
- *positive result when $\Omega \setminus \omega$ is bounded:* in this case, Cabanillas-de Menezes-Zuazua [69] proved that given $u_0 \in L^2(\Omega)$, there exists $f \in L^2(\omega \times (0, T))$ such that $u(T) = 0$.

In [A2] we considered an unbounded control region of the form $\omega = \cup_n (a_n, b_n)$, where the intervals (a_n, b_n) are disjoint; under some assumptions on the lengths $b_n - a_n$ of the controlled parts and the lengths $a_{n+1} - b_n$ of the uncontrolled parts, we obtain two positive null controllability results. In the following we precise our results. Miller [269] obtained positive and negative results in the same spirit.

4.3.2. Null controllability by a control supported in a 'small' region.

Main assumption: Consider two positive functions ρ_1 and ρ_2 that satisfy the following differential inequality: there exists positive constants α_1 and α_2 such that

$$(4.20) \quad \forall x, |\rho_{1,x}| \leq \alpha_1 \rho_1, \quad \text{and} \quad |\rho_{2,x}| \leq \alpha_2 \rho_2.$$

Let us introduce the space

$$L^2_{\rho_2}(\Omega) := \{v : \Omega \rightarrow \mathbb{R}, \int_{\Omega} |v|^2 \rho_2(x) dx < \infty\}.$$

Of course, endowed with the scalar product

$$\langle u, v \rangle_{\rho_2} := \int_{\Omega} uv \rho_2(x) dx,$$

$L^2_{\rho_2}(\Omega)$ is an Hilbert space. We introduce also the space

$$H^2_{\rho_2}(\Omega) := \{v \in L^2_{\rho_2}(\Omega) \text{ s.t. } u_x, u_{xx} \in L^2_{\rho_2}(\Omega)\}.$$

Denote \tilde{A} the unbounded operator $\frac{d^2}{dx^2}$ with the Dirichlet boundary condition $u(x = 0) = 0$. The domain of \tilde{A} is

$$D(\tilde{A}) := \{u \in H^2_{\rho_2}(\Omega) \text{ s.t. } u(x = 0) = 0\}.$$

Standard arguments can be used to justify that, if ρ_2 satisfies the differential inequality (4.20), then \tilde{A} is the generator of an analytic semi-group in $L^2_{\rho_2}(\Omega)$. Therefore, given $u_0 \in L^2_{\rho_2}(\Omega)$ and $f \in L^2(0, T; L^2_{\rho_2}(\Omega))$, problem (4.19) has a unique mild solution $u^f \in C([0, T]; L^2_{\rho_2}(\Omega))$.

THEOREM 4.11. ([A2]) *Fix $T > 0$. Under assumption (4.20), the following holds:*

(i) *if $u_0 \rho_1^{-1/2} \in L^2(\Omega)$ (or equivalently if $u_0 \in L^2_{1/\rho_1}(\Omega)$), then there exists a control $f \in L^2(0, T; L^2(\Omega))$ such that the solution u of (4.19) satisfies $u^f(T) = 0$.*

(ii) *if $u_0 \in L^2(\Omega)$, there exists a control $f \in L^2(0, T; L^2_{\rho_2}(\Omega))$ such that the solution u of (4.19) satisfies $u^f(T) = 0$ (in the space $L^2_{\rho_2}(\Omega)$).*

Now it is interesting to exhibit examples where the technical assumption (4.20) is satisfied.

Example 1. Assume that the lengths of the uncontrolled parts are bounded from above:

$$a_{n+1} - b_n \leq M < \infty,$$

and that the lengths of the controlled parts $\omega_n = (a_n, b_n)$ are bounded from below:

$$\forall n, b_n - a_n \geq m > 0;$$

then the functions ρ_1 and ρ_2 can be chosen constant, and Theorem 4.11 applies, and gives "usual" null controllability results: given $u_0 \in L^2(\Omega)$, there exists $f \in L^2(0, T; L^2(\Omega))$ such that the solution u of (4.19) satisfies $u(T) = 0$. This refines the result of [69] in the linear case. (However note that the total measure of the control region ω is still infinite in this case.)

Now we are interested in finding finite measure control sets. Hence in Examples 2 and 3, we assume that the lengths of the uncontrolled parts are bounded below. In Example 2 we assume that they are also bounded above, in Example 3 we assume that they go to infinity at infinity.

Example 2. Property (4.20) holds true for instance if the lengths of the uncontrolled parts are bounded below and above and if the lengths of the controlled parts $\omega_n = (a_n, b_n)$ go to zero at most exponentially, that is

$$0 < m \leq a_{n+1} - b_n \leq M < \infty,$$

$$b_n - a_n \geq ce^{-c'n},$$

for every n and for some positive constants m, M, c, c' . Hence this provides a class of situations for which the control region has a finite measure (for example if $b_n - a_n = e^{-n}$) and the two null controllability results hold.

Example 3. The other interesting situation is when the lengths of the uncontrolled parts go to infinity at infinity: estimating the constants that appear in the Carleman estimates, we prove that if

$$mn^\alpha \leq a_{n+1} - b_n \leq Mn^\beta$$

with $0 \leq \alpha \leq \beta \leq 1$, and $2\beta \leq 1 + \alpha$, then the technical assumption that we need is satisfied if

$$\forall n, |\omega_n| \geq ce^{-c'n^{1+\alpha}}.$$

For example if $\alpha = 1 = \beta$, then under the very weak condition

$$\forall n, |\omega_n| \geq ce^{-c'n^2},$$

the two null controllability hold (and the control set ω may have a finite measure).

These three examples satisfy the following general conditions:

$$(4.21) \quad \begin{cases} (a_{n+1} - b_n)^2 \leq ra_n, \\ |\omega_n| \geq r^{-1}e^{-ra_n} \end{cases}$$

for some constant r . We prove that under these conditions, our assumption (4.20) is satisfied, and thus our null controllability results hold. Note that the first condition gives a bound on the length of the uncontrolled parts (they must be "not too large"),

the second gives a bound on the length of the control parts (they must be "not too small"), in both cases the bound is given by the position of the control part (a_n, b_n) .

REMARK 4.12. *Note that recently null controllability was studied in unbounded domains, see in particular Le Rousseau-Moyano [233] and a recent paper of K. Beauchard and K. Pravda-Starov [30], where essentially it is proved that null controllability in the whole space holds if and only if the control region is sufficiently spread out.*

**Carleman estimates and null controllability
properties for 2-D degenerate parabolic equations
(Joint work with P. Cannarsa and J.
Vancostenoble ([A7]))**

5.1. Introduction: motivation and related paper

5.1.1. Motivation.

The goal of this chapter is to extend the previous results to space domains of dimension 2, a natural mathematical question, motivated in particular by the Fleming-Viot example.

5.1.2. The related paper [A7].

We will assume that Ω is a bounded open set of \mathbb{R}^2 with boundary Γ is of class \mathcal{C}^4 (an assumption that clearly is too strong to consider Fleming-Viot problems, but which was necessary for us).

[A7] extends the 1D results to this 2D case: we consider the parabolic problem

$$u_t - \operatorname{div}(A(x)\nabla u) = h\chi_\omega;$$

under natural boundary conditions, and assuming

- that the matrix A has an eigenvalue that behaves as some power of $d(x, \partial\Omega)$ (hence that is equal to 0 at the boundary),
- for which the associated eigenvector behaves as the normal to the boundary,
- while the other eigenvalue remains positive and bounded from below by a positive constant,

we obtain

- positive results when $\alpha < 2$ (as in the 1D case),
- negative results when $\alpha \geq 2$ (as in the 1D case).

The positive results follow from

- suitable Carleman estimates,
- which follow from improved (and 2D) Hardy inequalities.

The proof of the Carleman estimates are quite natural but require long computations. We conclude by estimating the blow-up of the null controllability cost as $\alpha \rightarrow 2^-$, considering the first eigenvalue of the problem. This opened a new direction of research, and will be explained in the following chapter.

In the following, we precise the assumptions and the results.

5.2. Weakly degenerate case

5.2.1. Assumptions on degeneracy.

In this section, we state the assumptions on matrix $A(x)$ for the “weakly degenerate” case. We denote by $\mathcal{S}_2(\mathbb{R})$ the space of all 2×2 real symmetric matrices.

HYPOTHESIS 5.1. *We assume that A satisfies the following properties:*

- (1) **Smoothness:** $A = (a_{i,j})_{i,j=1,2}$ with $a_{i,j} \in \mathcal{C}^0(\overline{\Omega}; \mathbb{R}) \cap \mathcal{C}^3(\Omega; \mathbb{R})$ for all $i, j = 1, 2$.
(2) **Symmetry:** $A(x) \in \mathcal{S}_2(\mathbb{R})$ for all $x \in \overline{\Omega}$.
(3) **Positivity:** $A(x)$ is positive definite for all $x \in \Omega$, that is,

$$\forall x \in \Omega, \quad A(x) > 0.$$

Notice that, by continuity,

$$\forall x \in \Gamma, \quad A(x) \geq 0.$$

For all $x \in \overline{\Omega}$, the spectrum of $A(x)$ is given by

$$\sigma(A(x)) = \{\lambda_1(x), \lambda_2(x)\} \quad \text{with } \lambda_1(x) \leq \lambda_2(x) \quad \forall x \in \overline{\Omega}.$$

Hence, by our positivity assumption on $A(x)$,

$$\forall x \in \Omega, \quad \lambda_1(x) > 0.$$

- (4) **Eigenvalues:** there exist $0 < \eta_1 < \eta_0$ and

$$(5.1) \quad \alpha \in [0, 1)$$

such that

- λ_1 : for some positive function $\lambda_1^* \in \mathcal{C}^3(\overline{C(\Gamma, \eta_1)} \times [0, \eta_1^\alpha])$, and for some function $\lambda_1^{**} \in \mathcal{C}^3(C(\Gamma, \eta_1) \times [0, \eta_1^\alpha])$, we have

$$(5.2) \quad \forall x \in C(\Gamma, \eta_1), \quad \lambda_1(x) = d_\Gamma(x)^\alpha \lambda_1^*(x, d_\Gamma(x)^\alpha) + d_\Gamma(x) \lambda_1^{**}(x, d_\Gamma(x)^\alpha),$$

- λ_2 : for some function $\lambda_2^* \in \mathcal{C}^3(\overline{C(\Gamma, \eta_1)} \times [0, \eta_1^\alpha])$, positive on $\overline{C(\Gamma, \eta_1)} \times [0, \eta_1^\alpha]$, we have

$$(5.3) \quad \forall x \in C(\Gamma, \eta_1), \quad \lambda_2(x) = \lambda_2^*(x, d_\Gamma(x)^\alpha).$$

- (5) **Eigenvectors:** we denote by $\varepsilon_1(x)$ the unit eigenvector of $A(x)$ associated to $\lambda_1(x)$, and by $\varepsilon_2(x)$ the unit eigenvector of $A(x)$ associated to $\lambda_2(x)$; since $A(x)$ is symmetric, $\varepsilon_1(x)$ and $\varepsilon_2(x)$ are orthogonal, hence we can decide that

$$\varepsilon_2(x) = \mathcal{R} \varepsilon_1(x) \quad \text{where} \quad \mathcal{R} := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Concerning $\varepsilon_1(x)$, we make the following assumptions:

$$(5.4) \quad \forall x \in \Gamma, \quad \varepsilon_1(x) = \nu(x),$$

and the perturbation v_1 defined by

$$(5.5) \quad \forall x \in C(\Gamma, \eta_1), \quad v_1(x) := \varepsilon_1(x) - \nu(p_\Gamma(x))$$

(which vanishes identically on Γ) is such that: for some function $v_1^* \in \mathcal{C}^3(\overline{C(\Gamma, \eta_1)} \times [0, \eta_1^\alpha])$, we have

$$(5.6) \quad \forall x \in C(\Gamma, \eta_1), \quad v_1(x) = v_1^*(x, d_\Gamma(x)^\alpha).$$

Observe that $\lambda_1 \equiv 0$ on Γ for $\alpha \neq 0$. So, A fails to be uniformly positive on Ω . We refer to [A7] for several examples where these assumptions are satisfied.

5.2.2. Statement of the controllability problem and main results.

Let A be a matrix-valued function on Ω satisfying Hypothesis 5.1. For any time $T > 0$ and open set $\omega \subset \Omega$ let us define

$$(5.7) \quad \Omega_T := (0, T) \times \Omega, \quad \Gamma_T := (0, T) \times \Gamma, \quad \text{and} \quad \omega_T := (0, T) \times \omega.$$

Herafter, χ_ω will denote the characteristic function of ω .

We shall here address the *null controllability problem* in arbitrary time T for the weakly degenerate parabolic operator by a locally distributed control with support in ω , that is, we ask whether for all $u_0 \in L^2(\Omega)$ there exists $h \in L^2(\Omega_T)$ such that the solution of

$$(5.8) \quad \begin{cases} u_t - \operatorname{div}(A(x)\nabla u) = h\chi_\omega & (t, x) \in \Omega_T, \\ u(t, x) = 0 & (t, x) \in \Gamma_T, \\ u(0, x) = u_0(x) & x \in \Omega, \end{cases}$$

satisfies $u(T, x) \equiv 0$ for a.e. $x \in \Omega$.

THEOREM 5.2. ([A7]) *Assume that A satisfies Hypothesis 5.1. Let $T > 0$ be given and let ω be a nonempty open subset of Ω . Then for all $u_0 \in L^2(\Omega)$ there exists $h \in L^2(\Omega_T)$ such that the solution u of (5.8) satisfies $u(T, \cdot) = 0$ in $L^2(\Omega)$. Moreover, there exists some $C(\Omega, \omega, T, \alpha)$ such that*

$$(5.9) \quad \|h\|_{L^2((0, T) \times \Omega)} \leq C(\Omega, \omega, T, \alpha) \|u_0\|_{L^2(\Omega)}.$$

REMARK 5.3. Hardy-type inequalities appear to be essential tools to prove Carleman estimates and observability properties for degenerate parabolic operators. For one dimensional problems, this key idea was first introduced in [A4, A5], and then adapted to more general situations (in one space dimension) in [4, 75], [A6]. Similar ideas were also used in [338, 340, 333] to treat nondegenerate operators with singular potentials.

5.2.3. Estimates with respect to the degeneracy parameter α . It is interesting to know the behavior of the controllability cost as $\alpha \rightarrow 1^-$. Hence now we are going to consider that the matrix $A(x)$ depends also on the degeneracy parameter α , in the following way: we assume that λ_1^* and λ_1^{**} satisfy the following additional assumptions:

HYPOTHESIS 5.4.

- (1) $(x, t, \alpha) \mapsto \lambda_1^*(x, t, \alpha)$ is smooth (C^3) on $\overline{\Omega} \times [0, \eta_1^\alpha] \times [0, 1]$, and positive, and bounded from below by a positive constant,
- (2) λ_1^{**} satisfies:

$$(5.10) \quad \lambda_1^{**}(x, t, \alpha) = (1 - \alpha)\zeta_1(x, t, \alpha) + t\zeta_2(x, t, \alpha),$$

where ζ_1 and ζ_2 are C^3 on $\overline{C(\Gamma, \eta_1)} \times [0, \eta_1^\alpha] \times [0, 1]$.

Then, using the new Hardy-type inequalities introduced in [333], we obtain the following results:

PROPOSITION 5.5. ([A7]) *Assume that λ_1^* and λ_1^{**} satisfy the previous additional assumptions. Then the controllability cost remains bounded when $\alpha \rightarrow 1^-$: there exists a constant $C(\Omega, \omega, T)$, independent of $\alpha \in [0, 1)$ such that, given $u_0 \in L^2(\Omega)$, there exists a control $h \in L^2(\Omega_T)$ that drives the solution of (5.8) to 0 in time T , and that satisfies*

$$\|h\|_{L^2((0, T) \times \Omega)} \leq C(\Omega, \omega, T) \|u_0\|_{L^2(\Omega)}.$$

5.2.4. Global Carleman estimate.

The above results are consequences of the following Carleman estimate for the degenerate problem:

THEOREM 5.6. ([A7]) *Let A be a matrix-valued function satisfying Hypothesis 5.1 for some $\alpha \in [0, 1)$. Let $T > 0$ be given and let ω be a nonempty open subset of Ω . Then consider the weight function ϕ (explicitly constructed in [A7]) and define the following weights σ and ρ by*

$$\sigma(t, x) := \theta(t)(e^{2S\|\phi\|_\infty} - e^{S\phi(x)}) \quad \text{and} \quad \rho(t, x) := RS\theta(t)e^{S\phi(x)},$$

where

$$\forall t \in (0, T), \quad \theta(t) := \left(\frac{1}{t(T-t)} \right)^k,$$

and k is a sufficiently large real number.

Then there are universal constants $C = C(\Omega, \omega, T, \alpha)$ and $S_0(\alpha) \geq 1$, and an increasing function $S \mapsto R_0(S, \alpha)$ such that, for all $S \geq S_0(\alpha)$, all $R \geq R_0(S, \alpha)$, all $w_T \in L^2(\Omega)$ and all $f \in L^2(\Omega_T)$ the solution w of (10.8) satisfies the following Carleman estimates:

(i) for w

$$(5.11) \quad \begin{aligned} & \iint_{\Omega_T} 2S|A(x)\nabla\phi \cdot \nabla\phi|^2 \rho^3 w^2 e^{-2R\sigma} \\ & + (2-\alpha) \iint_{C(\Gamma, \eta)_T} d(x, \Gamma)^{2-\alpha} \rho^3 w^2 e^{-2R\sigma} \\ & + (2-\alpha) \iint_{\Omega_T} \rho^{3/2} w^2 e^{-2R\sigma} \\ & + (2-\alpha)(\alpha-1)^2 \iint_{\Omega_T} \rho^2 w^2 e^{-2R\sigma} \\ & \leq C \left[\iint_{\Omega_T} f^2 e^{-2R\sigma} + \iint_{\omega_T} \rho^3 w^2 e^{-2R\sigma} \right], \end{aligned}$$

(ii) for the first order spatial derivatives of w

$$(5.12) \quad \begin{aligned} & (2-\alpha) \iint_{\Omega_T} \rho^{1/4} A(x) \nabla w \cdot \nabla w e^{-2R\sigma} \\ & + (2-\alpha)(\alpha-1)^2 e^{-2S\|\phi\|_\infty} \iint_{\Omega_T} \frac{\rho}{\theta} A(x) \nabla w \cdot \nabla w e^{-2R\sigma} \\ & + (2-\alpha) \iint_{C(\Gamma, \eta)_T} \rho d(x, \Gamma)^\alpha (\nabla w, \varepsilon_1)^2 e^{-2R\sigma} \\ & + \iint_{\Omega_T} S\rho(A\nabla\phi \cdot \nabla w)^2 e^{-2R\sigma} \\ & \leq C \left[\iint_{\Omega_T} f^2 e^{-2R\sigma} + \iint_{\omega_T} \rho^3 w^2 e^{-2R\sigma} \right], \end{aligned}$$

(iii) for the time derivative of w

$$(5.13) \quad (2-\alpha) \iint_{\Omega_T} \frac{1}{\rho} w_t^2 e^{-2R\sigma} \leq C \left[\iint_{\Omega_T} f^2 e^{-2R\sigma} + \iint_{\omega_T} \rho^3 w^2 e^{-2R\sigma} \right],$$

and

(iv) for the second order spatial derivatives of w

$$(5.14) \quad (2 - \alpha) \iint_{\Omega_T} \left| \frac{\operatorname{div}(A(x)\nabla w)}{\sqrt{\rho}} \right|^2 e^{-2R\sigma} + \left| \operatorname{div} \left(A(x)\nabla \left(\frac{w}{\sqrt{\rho}} \right) \right) \right|^2 e^{-2R\sigma} \\ \leq C \left[\iint_{\Omega_T} f^2 e^{-2R\sigma} + \iint_{\omega_T} \rho^3 w^2 e^{-2R\sigma} \right].$$

Moreover, under Hyp. 5.4, C , S_0 and R_0 can be chosen independent of $\alpha \in [0, 1)$.

5.3. Strongly degenerate case

5.3.1. Geometric assumptions and properties of the domain.

5.3.2. Assumptions on degeneracy. In this section, we specify the assumptions on matrix $A(x)$ for the “strongly degenerate” case that we intend to study. We keep the notation of section 5.2.1.

HYPOTHESIS 5.7. *We assume that A has all the properties stated in Hypothesis 5.1 except for (5.1) that is now replaced by*

$$(5.15) \quad \alpha \geq 1,$$

and that the assumption (5.2) on λ_1 is satisfied with additionally the assumption that

$$(5.16) \quad \forall x_\Gamma \in \Gamma, \quad \lambda_1^{**}(x_\Gamma, 0) = 0$$

(which is natural in order to have $\lambda_1(x) \sim d(x, \Gamma)^\alpha \lambda_1^*(x)$ near the boundary).

5.3.3. Statement of the controllability problem and main results. Let A be a matrix-valued function satisfying Hypothesis 5.7 and consider the “strongly” degenerate parabolic operator

$$(5.17) \quad Pu := u_t - \operatorname{div}(A(x)\nabla u) = 0 \quad \text{in } \mathbb{R}_+ \times \Omega.$$

The natural boundary conditions to associate with (5.17) are the following Neumann-type conditions:

$$(5.18) \quad A(x)\nabla u(t, x) \cdot \nu(x) = 0 \quad (t, x) \in \mathbb{R}_+ \times \Gamma.$$

We now proceed to state the null controllability problem for (5.17)-(5.18) in arbitrary time $T > 0$. As previously, we shall consider locally distributed controls, supported by a nonempty open set $\omega \subset \Omega$, and use the notation Ω_T , Γ_T and ω_T introduced in (5.7). The problem consists in finding, for all $u_0 \in L^2(\Omega)$, a control $h \in L^2(\Omega_T)$ such that the solution of

$$(5.19) \quad \begin{cases} u_t - \operatorname{div}(A(x)\nabla u) = h\chi_\omega & (t, x) \in \Omega_T, \\ A(x)\nabla u(t, x) \cdot \nu(x) = 0 & (t, x) \in \Gamma_T, \\ u(0, x) = u_0(x) \in L^2(\Omega) & x \in \Omega. \end{cases}$$

satisfies $u(T, x) = 0$ for a.e. $x \in \Omega$. We give below a positive answer to the problem for $\alpha \in [1, 2)$, and a negative one for $\alpha \geq 2$.

5.3.3.1. A null controllability result for $\alpha \in [1, 2)$. In the following result, we recall that v_1 denotes the vector-valued function defined in (5.5).

THEOREM 5.8. ([A7]) *Assume A satisfies Hypothesis 5.7 for some $\alpha \in [1, 2)$, let $T > 0$ be given, and let ω be a nonempty open subset of Ω . Then, for all $u_0 \in L^2(\Omega)$ there is a control $h \in L^2(\Omega_T)$ such that the solution u of (5.19) satisfies $u(T, \cdot) = 0$ in $L^2(\Omega)$. Moreover,*

$$(5.20) \quad \|h\|_{L^2((0,T)\times\Omega)} \leq e^{e^{C_1(\Omega, \omega, T)/(2-\alpha)^2}} \|u_0\|_{L^2(\Omega)}$$

for some constant $C_1(\Omega, \omega, T) > 0$ independent of α .

REMARK 5.9.

- As we did for the weakly degenerate case (see remark 5.3), let us point out the novelties of Theorem 5.8 with respect to the literature:
 - this is the first null controllability result for strongly degenerate parabolic operators in two space dimensions as previous theory was limited to one dimensional problems (see [A5]), and
 - we show that the minimum norm control remains bounded with respect to the degeneracy parameter α when $\alpha \rightarrow 1^+$, which is new even in one space dimension (compare with the main result of [A5]).
- The constant appearing in (5.20), $e^{e^{C_1(\Omega, \omega, T)/(2-\alpha)^2}}$, gives an estimate from above of the controllability cost. This estimate is bounded as $\alpha \rightarrow 1^+$, but explodes as $\alpha \rightarrow 2^-$, suggesting that null controllability may fail for $\alpha = 2$. In fact, Proposition 5.10 will show that the result of Theorem 5.8 is no longer true for $\alpha \geq 2$. Moreover, in the following, we will consider the analogous 1-dimensional degenerate problem, and we will provide an estimate *from below* of the controllability cost; this estimate is unbounded as $\alpha \rightarrow 2^-$. It is a very interesting open problem whether the minimum norm control blows up in L^2 as $\alpha \rightarrow 2^-$ in space dimension 2, and to evaluate precisely the blow-up rate of h from below, in 1- and 2- space dimension.
- Our result applies in particular to smooth operators that satisfy
 - $x \mapsto A(x)$ of class C^3 on $\bar{\Omega}$,
 - $A(x)$ symmetric, nonnegative,
 - $\det A(x) > 0$ on Ω , bounded from below on compact subsets of Ω , $\det A(x) = 0$ on Γ , and

$$C_1 d_\Gamma(x) \leq \det A(x) \leq C_2 d_\Gamma(x),$$

- $\text{Tr} A(x) \geq m_0 > 0$ on Ω ,
- $A(x)\nu(x) = 0$ for all $x \in \Gamma$.

Then it is easy to see that Hyp. 5.7 is satisfied. This follows from the explicit expression of the eigenvalues of $A(x)$ and the explicit expression of the associated eigenvectors. The system of Fleming-Viot satisfy several of these assumptions, except the fact that the operator is not written in divergence form and overall that the natural geometrical domain is a triangle, and vertices are points where both eigenvalues are zero. It is possible to produce a smooth version of the Fleming-Viot operator, acting on a smoothed triangle. However the most interesting question is to know what can be said when the geometrical domain is polygonal, moreover when vertices are points where both eigenvalues are zero.

The proof of Theorem 5.8 follows from suitable Carleman estimates for the strongly degenerate operator (similar to the ones given before, with additionally

$$S_0(\alpha) = \frac{S_0}{2-\alpha} \quad \text{and} \quad R_0(S) = S^{14} + e^{12S\|\phi\|_\infty}.$$

5.3.3.2. Counterexample for $\alpha \in [2, +\infty)$.

The following example shows that, in general, null controllability fails for $\alpha \geq 2$.

PROPOSITION 5.10. ([A7]) *Let $\Omega = D(O, 1)$ be the unit disc of \mathbb{R}^2 centered at O . Consider a smooth nondecreasing function $\psi : [0, 1] \rightarrow [0, 1]$ such that $\psi(s) = 0$*

for all $s \in [0, 1/3]$, and $\psi(s) = 1$ for all $s \in [2/3, 1]$. Consider the matrix

$$(5.21) \quad A(x, y) = \begin{pmatrix} \tilde{a}(x, y) \frac{x^2}{x^2 + y^2} + \frac{y^2}{x^2 + y^2} & (\tilde{a}(x, y) - 1) \frac{xy}{x^2 + y^2} \\ (\tilde{a}(x, y) - 1) \frac{xy}{x^2 + y^2} & \frac{x^2}{x^2 + y^2} + \tilde{a}(x, y) \frac{y^2}{x^2 + y^2} \end{pmatrix},$$

with $\tilde{a}(x, y) = (1 - \sqrt{x^2 + y^2} \psi(x^2 + y^2))^\alpha$. Then the following properties hold true:

- For all (x, y) such that $x^2 + y^2 \geq 2/3$, the eigenvalues of $A(x, y)$ are $\tilde{a}(x, y)$ and 1, associated, respectively, to eigenvectors

$$\varepsilon_1(x, y) = \begin{pmatrix} \frac{x}{\sqrt{x^2 + y^2}} \\ \frac{y}{\sqrt{x^2 + y^2}} \end{pmatrix} \quad \text{and} \quad \varepsilon_2(x, y) = \begin{pmatrix} \frac{-y}{\sqrt{x^2 + y^2}} \\ \frac{x}{\sqrt{x^2 + y^2}} \end{pmatrix}$$

so that, in particular, $\varepsilon_1(x, y) = \nu(p_\Gamma(x, y))$.

- For $\alpha \geq 2$, given an open set ω with $\bar{\omega} \subset D(O, 1)$, a number $T > 0$, and a positive function $u_0 \in H_A^1(D(O, 1))$, there is no control

$$h \in L^2((0, T) \times D(O, 1))$$

such that the solution of

$$(5.22) \quad \begin{cases} u_t - \operatorname{div}(A(x, y) \nabla u) = h \chi_\omega, & t \in (0, T), (x, y) \in D(0, 1) \\ A \nabla u \cdot \nu = 0, & t \in (0, T), (x, y) \in \partial D(0, 1) \\ u(0, x, y) = u_0, & (x, y) \in D(0, 1) \end{cases}$$

satisfies $u(T, \cdot) = 0$.

REMARK 5.11. The above example suggests that null controllability fails in a general way when degeneracy is too strong. Observe that this phenomenon is somewhat in line with the probabilistic interpretation of a parabolic problem as the Kolmogorov equation of a given diffusion process: the regions where diffusion degenerates are unlikely to be reached by the stochastic process. So, the effect of control becomes negligible.

5.4. Extensions

The previous positive results can be extended to operators of the form

$$u_t - \operatorname{div}(A(x) \nabla u) + b(x) \cdot \nabla u + c(t, x)u$$

with $c \in L^\infty((0, T) \times \Omega)$, $b \in L^\infty(\Omega)$ and $b \cdot \varepsilon_1 = O(d(x, \Gamma)^{\alpha/2})$, see [A7].

Biorthogonal families and the cost of null controllability with respect to the degeneracy parameter

(Joint works with P. Cannarsa and J. Vancostenoble ([A7, A8, A9, 10]), with J. Vancostenoble ([A11]), with P. Cannarsa and R. Ferretti ([A12]))

6.1. Introduction: motivation and related papers

6.1.1. Motivation.

The aim of this chapter is to study the null controllability cost for several typical problems, when some parameter comes into play, in particular related to the typical 1D degenerate parabolic operator

$$(6.1) \quad Pu = u_t - (x^\alpha u_x)_x$$

under the action of a boundary control or a locally distributed control, acting either at the non degeneracy point or at the degeneracy point.

For example, consider $\alpha \in [1, 2)$ and the action of a locally distributed control h :

$$(6.2) \quad \begin{cases} u_t - (x^\alpha u_x)_x = h(x, t)\chi_{[a, b]}(x) & x \in (0, 1), t > 0, \\ (x^\alpha u_x)(0, t) = 0, & t > 0, \\ u(1, t) = 0, & t > 0, \\ u(x, 0) = u_0(x), & x \in (0, 1). \end{cases}$$

In [A5], we established the following property:

given $\alpha \geq 1$, $T > 0$, $0 < a < b < 1$, then, for any $u_0 \in L^2(0, 1)$, problem (6.2) admits a control $h \in L^2((a, b) \times (0, T))$ that drives the solution to 0 in time $T > 0$ if and only if $\alpha < 2$.

Defining as usually the "cost of null controllability" by

$$C_{ctr-loc}(\alpha, T) := \sup_{\|u_0\|_{L^2(0,1)}=1} \inf\{\|h\|_{L^2((0,1) \times ((0,T)))}, u^{(h)}(T) = 0\},$$

the aim is to understand the behavior of $C_{ctr-loc}(\alpha, T)$:

- as $\alpha \rightarrow 2^-$ (since $\alpha = 2$ is the threshold for null controllability) ,
- and, additionally, as $T \rightarrow 0^+$, an issue related to the so-called 'fast control problem'.

It is well-known that $C_{ctr-loc}(\alpha, T)$ blows up when $T \rightarrow 0^+$ (at least for nondegenerate parabolic equations), and it is expected to blow up when $\alpha \rightarrow 2^-$. In this work, we will prove precise upper and lower bounds for this blow-up. This

quite innocent question (the behavior as $\alpha \rightarrow 2^-$) has revealed to be very interesting, allowing us to make connections with several earlier problems studied in the literature, and then to prove general tools in order to obtain precise results.

6.1.2. The related papers [A7-A12].

Our goal was to obtain suitable upper and lower estimates of the null controllability cost. In general, Carleman estimates provide upper estimates, but not sharp ones. On the other hand, the moment method allows us to obtain precise estimates, and moreover lower estimates. Since some parameter comes into play (for example the degeneracy parameter), we had to refine the ones of the literature.

- [A8], [A9] contain precise and general results concerning biorthogonal families, under some gap conditions on the associated family of exponentials (the exponents being in fact the eigenvalues of the considered problem);
- [A8], [A10] contain the results concerning the null controllability cost: upper and lower estimates, under various types of control (acting at the degeneracy point, or at the other boundary point, or locally distributed);
- [A11], [A12] concern the same type of natural questions, for other degenerate equations; we managed to use the tools developed in [A8], [A9] to solve them.

6.2. Relation to literature

6.2.1. Several close problems.

This question of the cost of null controllability when some parameter comes into play has been studied for several equations and in several situations:

- the 'fast control problem', that is, the cost of null controllability with respect to time T as $T \rightarrow 0^+$, has been investigated for the heat operator

$$(6.3) \quad Pu = u_t - \Delta u$$

(with a boundary or localized control) and the Schrödinger equation by several authors, see, in particular, the works by Seidman et al [315, 316], Güichal [180], Miller [270, 271, 272, 275], Tenenbaum and Tucsnak [330, 331], and the more recent papers by Lissy [242] (for dispersive equations) and Benabdallah et al [35] (for parabolic systems);

- the 'vanishing viscosity limit', that is the cost of null controllability of a heat operator with the addition of a transport term when the diffusion coefficient goes to zero:

$$(6.4) \quad P_\varepsilon u = u_t - \varepsilon u_{xx} + Mu_x$$

(again with a boundary or localized control) has been investigated by Coron and Guerrero [99], Guerrero and Lebeau [177], Glass [173], Glass and Guerrero [174], and Lissy [243];

- the 1D degenerate parabolic equation, controlled by a boundary control acting at the degeneracy point (and $\alpha \rightarrow 1^-$, 1 being the threshold value of well-posedness in this case, see [A8]).

6.2.2. Description of the method and connection with the literature.

For the proof of our results, we follow the classical strategy, based on the moment method and the construction of suitable biorthogonal families (combining and adapting ideas of Seidman-Avdonin-Ivanov [316], Güichal [180] and Tenenbaum-Tucsnak [330]).

Hence a starting point is the study of the spectral problem. In the context of degenerate parabolic equations, it is classical (Kamke [217], Gueye [179]) that the

eigenfunctions of the problem are expressed in terms of Bessel functions of order $\nu_\alpha = \frac{\alpha-1}{2-\alpha}$, and the eigenvalues in terms of the zeros of these Bessel functions. For strongly degenerate parabolic equations, an additional source of difficulty is that the order of the useful Bessel functions blows up as $\alpha \rightarrow 2^-$. To cope with such difficulties, several classical results from Watson [345] and Qu-Wong [302] will be needed.

It turns out that there is a common phenomenon in the classical fast control problem ((6.3) when $T \rightarrow 0^+$), the vanishing viscosity problem ((6.4) when $\varepsilon \rightarrow 0^+$), and the null controllability of the degenerate parabolic equation (6.1) when the degeneracy parameter approaches its critical value: *the eigenvalues concentrate when parameters go to their critical values*. Such a concentration phenomenon can be observed:

- for the vanishing viscosity problem, in [99];
- for degenerate parabolic equations, once the eigenvalues have been computed;
- for the classical heat equation and the fast control problem, once time has been renormalized to a fixed value.

This common feature is the key point in understanding the behavior of the control cost in every context. Indeed, the construction of suitable biorthogonal families is strongly related to gap properties, and the control cost is related to the speed at which the gap $\lambda_{n+1} - \lambda_n$ goes to 0 with respect to the hidden parameter (in general the degeneracy parameter α).

6.3. General results concerning biorthogonal families

6.3.1. Motivation.

Biorthogonal families are a classical tool in analysis. In particular, they play a crucial role in the so-called moment method, which was developed by Fattorini-Russell [143, 144] to study controllability for parabolic equations.

Given any sequence of nonnegative real numbers, $(\lambda_n)_{n \geq 1}$, we recall that a sequence $(\sigma_m)_{m \geq 1}$ is biorthogonal to the sequence $(e^{\lambda_n t})_{n \geq 1}$ in $L^2(0, T)$ if

$$\forall m, n \geq 1, \quad \int_0^T \sigma_m(t) e^{\lambda_n t} dt = \begin{cases} 1 & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases}.$$

The goal of this part is to provide explicit and precise upper and lower bounds for the biorthogonal family $(\sigma_m)_{m \geq 1}$ under the following gap conditions:

- a 'global gap condition':

$$(6.5) \quad \forall n \geq 1, \quad 0 < \gamma_{min} \leq \sqrt{\lambda_{n+1}} - \sqrt{\lambda_n} \leq \gamma_{max},$$

- and, possibly, an additional 'asymptotic gap condition':

$$(6.6) \quad \forall n \geq N^*, \quad \gamma_{min}^* \leq \sqrt{\lambda_{n+1}} - \sqrt{\lambda_n} \leq \gamma_{max}^*,$$

where $\gamma_{max}^* - \gamma_{min}^* < \gamma_{max} - \gamma_{min}$.

Before explaining why we are interested in such a question, let us describe some of the main results of the literature on this subject.

6.3.1.1. The context.

Among the most important applications of biorthogonal families to control theory are those to the null controllability and sensitivity of control costs to parameters. Major contributions in such directions are the following:

- Fattorini-Russell [143, 144], Hansen [187], and Ammar Khodja-Benabdallah-González Burgos-de Teresa [8] studied the existence of biorthogonal sequences and their application to controllability for various equations;
- for nondegenerate parabolic equations and dispersive equations, Seidman [315], Güichal [180], Seidman-Avdonin-Ivanov [316], Miller [270], Tenenbaum-Tucsnak [330], and Lissy [242, 243] studied the dependence of the null controllability cost C_T with respect to the time T (as $T \rightarrow 0$, the so-called 'fast control problem') and with respect to the domain, proving that

$$e^{c(\Omega)/T} \leq C_T \leq e^{C(\Omega)/T},$$

and obtaining extremely sharp estimates of the constants $c(\Omega)$ and $C(\Omega)$;

- Coron-Guerrero [99], Glass [173], Lissy [243] investigated the vanishing viscosity problem:

$$\begin{cases} y_t + My_x - \varepsilon y_{xx} = 0, & x \in (0, L), \\ y(0, t) = f(t), \end{cases}$$

obtaining sharp estimates of the null controllability cost with respect to the time T , the transport coefficient M , the size of the domain L , and the diffusion coefficient ε .

There is a common feature in these works: they depend on some parameter p , and this parameter forces the eigenvalues to satisfy (6.5) (sometimes after normalization) with gap bounds $\gamma_{min}(p)$ and $\gamma_{max}(p)$ such that

$$\gamma_{min}(p) \rightarrow 0 \quad \text{and/or} \quad \gamma_{max}(p) \rightarrow \infty.$$

This fact makes it necessary to have general and precise estimates with respect to the main parameters that appear in the problem.

We will provide general results in this direction. Our proofs are based on complex analysis techniques and hilbertian methods developed by Seidman-Avdonin-Ivanov [316] and Güichal [180]. We have also used an idea from Tenenbaum-Tucsnak [330] and Lissy [242, 243], based on the introduction of an extra parameter depending on T and the gap conditions.

6.3.2. Under global gap conditions ([A8]).

6.3.2.1. A precise upper bound.

THEOREM 6.1. (*Existence and upper bound [A8]*) Assume that

$$\forall n \geq 0, \quad \lambda_n \geq 0,$$

and that there is some $\gamma_{min} > 0$ such that

$$(6.7) \quad \forall n \geq 0, \quad \sqrt{\lambda_{n+1}} - \sqrt{\lambda_n} \geq \gamma_{min}.$$

Then there exists a family $(\sigma_m^+)_{m \geq 0}$ which is biorthogonal to the family $(e^{\lambda_n t})_{n \geq 0}$ in $L^2(0, T)$:

$$(6.8) \quad \forall m, n \geq 0, \quad \int_0^T \sigma_m^+(t) e^{\lambda_n t} dt = \delta_{mn}.$$

Moreover, it satisfies: there is some universal constant C_u independent of T , γ_{min} and m such that, for all $m \geq 0$, we have

$$(6.9) \quad \|\sigma_m^+\|_{L^2(0, T)}^2 \leq C_u e^{-2\lambda_m T} e^{C_u \sqrt{\lambda_m} / \gamma_{min}} B(T, \gamma_{min}),$$

with

$$(6.10) \quad B(T, \gamma_{min}) = \begin{cases} \left(\frac{1}{T} + \frac{1}{T^2 \gamma_{min}^2} \right) e^{\frac{C_u}{\gamma_{min}^2 T}} & \text{if } T \leq \frac{1}{\gamma_{min}^2}, \\ C_u \gamma_{min}^2 & \text{if } T \geq \frac{1}{\gamma_{min}^2}. \end{cases}$$

Theorem 6.1 completes Theorem 1.5 of Fattorini-Russell [144], expliciting the dependence of the L^2 bound with respect to γ_{min} and in short time. It is useful in several problems, in which either $\gamma_{min} \rightarrow 0$ or $\gamma_{max} \rightarrow \infty$ with respect to some parameter, which occurs in several cases, see, e.g. [145], [27] and [A10].

6.3.2.2. *A general lower bound.*

On the other hand, we generalize a result of Guichal [180] to prove the following

THEOREM 6.2. (*Lower bound [A8]*) Assume that

$$\forall n \geq 0, \quad \lambda_n \geq 0,$$

and that there is some $0 < \gamma_{min} \leq \gamma_{max}$ such that

$$(6.11) \quad \forall n \geq 0, \quad \gamma_{min} \leq \sqrt{\lambda_{n+1}} - \sqrt{\lambda_n} \leq \gamma_{max}.$$

Then there exists some $C(m, \gamma_{max}, \lambda_0) > 0$ (given explicitly in [A8]) and $c_u > 0$ independent of T , α and m such that: any family $(\sigma_m^+)_{m \geq 0}$ which is biorthogonal to the family $(e^{\lambda_n t})_{n \geq 0}$ in $L^2(0, T)$ (hence that satisfies (6.8)) satisfies:

$$(6.12) \quad \|\sigma_m^+\|_{L^2(0, T)}^2 \geq e^{-2\lambda_m T} e^{\frac{1}{2\gamma_{max}^2 T}} b(T, \gamma_{max}, m),$$

with

$$(6.13) \quad b(T, \gamma_{max}, m) = \frac{c_u^2}{C(m, \gamma_{max}, \lambda_0)^2 T} \left(\frac{1}{2\gamma_{max}^2 T} \right)^{2m} \frac{1}{(4\gamma_{max}^2 T + 1)^2}.$$

6.3.3. Under asymptotic gap conditions ([A9]).

Even though the aforementioned results give a fairly good picture of the properties of the family $(\sigma_m)_m$, the eigenvalues of some motivating examples satisfy that: there exists some asymptotic gap ('asymptotic' meaning here 'after some rank') much better than the global gap. This motivated us to improve our general results:

6.3.3.1. *A precise upper bound.*

We will prove the following result, that in some sense precises results of Fattorini and Russell [143, 144] (in short time) and are in the spirit of results of Tenenbaum-Tucsak [330], Lissy [242, 243] (with a slightly different assumption on the eigenvalues).

THEOREM 6.3. (*Existence and upper bound [A9]*) Assume that

$$\forall n \geq 1, \quad \lambda_n \geq 0,$$

and that there is some $0 < \gamma_{min} < \gamma_{min}^*$ such that

$$(6.14) \quad \forall n \geq 1, \quad \sqrt{\lambda_{n+1}} - \sqrt{\lambda_n} \geq \gamma_{min},$$

and

$$(6.15) \quad \forall n \geq N^*, \quad \sqrt{\lambda_{n+1}} - \sqrt{\lambda_n} \geq \gamma_{min}^*.$$

Denote

$$(6.16) \quad M^* := \left(1 - \frac{\gamma_{min}}{\gamma_{min}^*}\right)(N^* - 1).$$

Then there exists a family $(\sigma_m^+)_{m \geq 1}$ which is biorthogonal to the family $(e^{\lambda_n t})_{n \geq 1}$ in $L^2(0, T)$:

$$(6.17) \quad \forall m, n \geq 1, \quad \int_0^T \sigma_m^+(t) e^{\lambda_n t} dt = \delta_{mn}.$$

Moreover, it satisfies: there is some universal constant C independent of T , γ_{min} , γ_{min}^* , N^* and m such that, for all $m \geq 1$, we have

$$(6.18) \quad \|\sigma_m^+\|_{L^2(0,T)}^2 \leq e^{-2\lambda_m T} e^{\frac{C}{T(\gamma_{min}^*)^2}} e^{C\frac{\sqrt{\lambda_m}}{\gamma_{min}^*}} B^*(T, \gamma_{min}, \gamma_{min}^*, N^*, m),$$

where

$$(6.19) \quad B^*(T, \gamma_{min}, \gamma_{min}^*, N^*, m) = \begin{cases} C_u \left(\frac{(8M^*)!}{(\lambda_m (\gamma_{min}^*)^{2T^2})^{4M^*}} + 1 \right) e^{C_u M^*} e^{C_u \frac{\lambda_{N^*}}{\gamma_{min} \sqrt{\lambda_m}}} \left(\frac{1}{T^{3/2}} + \frac{1}{(\gamma_{min}^*)^2 T^2} \right) & \text{if } T \leq \frac{1}{(\gamma_{min}^*)^2} \\ C_u \left(\left(\frac{(\gamma_{min}^*)^{8M^*}}{\lambda_m^{4M^*}} (8M^*)! \right) + 1 \right) e^{C_u M^*} e^{C_u \frac{\lambda_{N^*}}{\gamma_{min} \sqrt{\lambda_m}}} ((\gamma_{min}^*)^2 + (\gamma_{min}^*)^3) & \text{if } T \geq \frac{1}{(\gamma_{min}^*)^2} \end{cases}$$

6.3.3.2. *A general lower bound.*

We generalize a result of Güichal [180] to prove the following

THEOREM 6.4. (*Lower bound [A9]*) *Assume that*

$$\forall n \geq 1, \quad \lambda_n \geq 0,$$

and that there are $0 < \gamma_{min} \leq \gamma_{max}^* \leq \gamma_{max}$ such that

$$(6.20) \quad \forall n \geq 1, \quad \gamma_{min} \leq \sqrt{\lambda_{n+1}} - \sqrt{\lambda_n} \leq \gamma_{max},$$

and

$$(6.21) \quad \forall n \geq N_*, \quad \sqrt{\lambda_{n+1}} - \sqrt{\lambda_n} \leq \gamma_{max}^*.$$

Then any family $(\sigma_m^+)_{m \geq 1}$ which is biorthogonal to the family $(e^{\lambda_n t})_{n \geq 1}$ in $L^2(0, T)$ (hence that satisfies (6.8)) satisfies:

$$(6.22) \quad \|\sigma_m^+\|_{L^2(0,T)}^2 \geq e^{-2\lambda_m T} e^{\frac{2}{T(\gamma_{max}^*)^2}} b^*(T, \gamma_{max}, \gamma_{max}^*, N_*, \lambda_1, m)^2,$$

where b^* is rational in T (and explicitly given in [A9]).

6.3.3.3. *Remarks.*

Theorems 6.3 and Theorem 6.4 improve Theorems 6.1 and Theorem 6.2 when there is an additional asymptotic gap better than the global one. Roughly speaking, (6.9) and (6.12) hold true replacing γ_{min} by γ_{min}^* and γ_{max} by γ_{max}^* . Moreover, the fact the 'good' gap condition (6.6) holds true only after the N^* first eigenvalues has a cost, and we obtain a precise estimate for that cost.

Let us observe that the presence of the exponential factors $e^{\frac{C}{T(\gamma_{min}^*)^2}}$ and $e^{\frac{2}{T(\gamma_{max}^*)^2}}$ in (6.18) and (6.22) is quite natural and has already been pointed out by Seidman-Avdonin-Ivanov [316], Tenenbaum-Tucsnak [330], and Lissy [242, 243] (see also Haraux [191] and Komornik [225] for a closely related context). On the other hand, the precise estimate of the behavior of b_m^* and B_m^* with respect to parameters is crucial for the sensitivity analysis of control costs performed in [A10].

6.3.4. Ideas of the proofs of Theorems 6.1 and 6.3.

We use the construction that was used by Seidman, Avdonin and Ivanov in [316], which has the advantage to be completely explicit (which is not the case for the construction of [143, 144, 145, 187, 8], since there is a contradiction argument), combined with some ideas coming from the construction of Tenenbaum-Tucsnak [330] and Lissy [242], adding some parameter, in order to obtain precise results:

- a Weierstrass product:

$$F_m(z) := \prod_{k=1, k \neq m}^{\infty} \left(1 - \left(\frac{iz - \lambda_m}{\lambda_k - \lambda_m}\right)^2\right)$$

whose growth is estimated using the gap condition,

- a suitable mollifier $M_m(z)$ (slight change w.r.t. literature), and

$$f_m := F_m M_m \text{ satisfies } \begin{cases} \forall m, n \geq 1, & f_m(-i\lambda_n) = \delta_{mn}, \\ \forall z \in \mathbb{C}, & |f_m(-z)e^{-iz\frac{T}{2}}| \leq C_m e^{\frac{T}{2}|z|}, \\ \forall m \geq 1, & f_m \in L^2(\mathbb{R}) \end{cases},$$

- the Paley-Wiener theorem: $f_m(-z)e^{-iz\frac{T}{2}}$ is the inverse Fourier transform of some compactly supported function ϕ_m (properties 2 and 3), that will give the biorthogonal sequence (property 1).

6.3.5. Ideas of the proofs of Theorems 6.2 and 6.4.

Theorems 6.2 and 6.4 complete a result of Güichal [180], and follow by Hilbertian techniques:

- any biorthogonal families verify

$$\|\sigma_m^+\| \geq \frac{1}{d_{T,m}}, \quad \text{where } d_{T,m} := \text{dist}(e^{\lambda_m t}, \overline{\text{Vect}\{e^{\lambda_k t}, k \neq m\}});$$

- then:

$$d_{T,m} \leq \|e^{\lambda_m t} - \sum_{i=1, i \neq m}^{M+1} A_i e^{\lambda_i t}\| = \left\| \frac{-1}{\tilde{A}_m} \sum_{i=1}^{M+1} -\tilde{A}_i e^{\lambda_i t} \right\| = \left\| \frac{-1}{\tilde{A}_m} q(t) \right\|,$$

with a special choice of the coefficients \tilde{A}_i : chosen (Güichal [180]) such that

$$q(0) = \dots = q^{(M-1)}(0) = 0, \quad q^{(M)}(0) = 1$$

(in order to have q small);

- then (Vandermonde determinant)

$$d_{T,m} \leq \left(\prod_{i=1, i \neq m}^{M+1} |\lambda_i - \lambda_m| \right) \left(\int_0^T \frac{s^{2M}}{M!^2} e^{-2\lambda_1 s} ds \right)^{1/2} \leq \dots$$

It is to be noted that the behavior with respect to m can perhaps be improved, comparing with Theorem 1.1 of Hansen [187]. It would be interesting to investigate this.

6.4. The cost of controlling strongly degenerate parabolic equations with localized controls ([A10])

6.4.1. The controllability problems.

We study the cost of null controllability of a degenerate parabolic equation, using either a boundary control acting at the non degeneracy point or a locally distributed control. More precisely, we fix $\ell > 0$, $\alpha \geq 1$, $T > 0$, and for any $u_0 \in L^2(0, \ell)$, we wish to find a control H such that the solution of

$$(6.23) \quad \begin{cases} u_t - (x^\alpha u_x)_x = 0, & x \in (0, \ell), t > 0, \\ (x^\alpha u_x)(0, t) = 0, & t > 0, \\ u(\ell, t) = H(t), & t > 0, \\ u(x, 0) = u_0(x), & x \in (0, 1), \end{cases}$$

also satisfies $u(\cdot, T) = 0$.

Similarly, given $0 < a < b < \ell$ and for any $u_0 \in L^2(0, \ell)$, we wish to find a control h such that the solution of

$$(6.24) \quad \begin{cases} u_t - (x^\alpha u_x)_x = h(x, t)\chi_{[a,b]}(x) & x \in (0, \ell), t > 0, \\ (x^\alpha u_x)(0, t) = 0, & t > 0, \\ u(\ell, t) = 0, & t > 0, \\ u(x, 0) = u_0(x), & x \in (0, \ell), \end{cases}$$

also satisfies $u(\cdot, T) = 0$.

In space dimension 1, these two problems are very close. As already recalled ([A5]), we know that such controls exist if and only if $\alpha \in [1, 2)$.

Consider

$$(6.25) \quad C_{ctr-bd}(\alpha, T, \ell) := \sup_{\|u_0\|_{L^2(0, \ell)}=1} \inf\{\|H\|_{H^1(0, T)}, u^{(H)}(T) = 0\},$$

where $u^{(H)}$ is the solution of problem (6.23), and

$$(6.26) \quad C_{ctr-loc}(\alpha, T, \ell) := \sup_{\|u_0\|_{L^2(0, \ell)}=1} \inf\{\|h\|_{L^2((a,b) \times (0, T))}, u^{(h)}(T) = 0\},$$

where $u^{(h)}$ is the solution of problem (6.24). Then we prove the following estimates:

6.4.2. Null controllability cost for boundary control.

6.4.2.1. Lower bound of the null controllability cost for boundary control.

THEOREM 6.5. (Lower bound [A10]) *There exists a constant $C_u > 0$ independent of $\alpha \in [1, 2)$, of $\ell > 0$ and of $T > 0$ such that*

$$(6.27) \quad C_{ctr-bd}(\alpha, T, \ell) \geq C_u \frac{\ell^{2-\alpha}}{\sqrt{(2-\alpha)T\ell}} e^{-\pi^2 \frac{T}{\ell^{2-\alpha}}} e^{C_u \frac{\ell^{2-\alpha}}{T(2-\alpha)^2}} e^{-\frac{1}{C_u} \left(\frac{1}{(2-\alpha)^{4/3}} + \frac{\ell^{1-\alpha/2}}{2-\alpha} \right) \left(\ln \frac{\ell^{1-\alpha/2}}{2-\alpha} + \ln \frac{1}{T} \right)}.$$

This proves that the cost blows up when $T \rightarrow 0^+$, or $\alpha \rightarrow 2^-$, or $\ell \rightarrow +\infty$, and at least exponentially fast. When ℓ is fixed and $T \leq T_0$, this simplifies into

$$C_{ctr-bd}(\alpha, T, \ell) \geq C_u e^{\frac{C_u}{T(2-\alpha)^2}} e^{-\frac{1}{C_u} \frac{1}{(2-\alpha)^{4/3}} \left(\ln \frac{1}{2-\alpha} + \ln \frac{1}{T} \right)}.$$

6.4.2.2. Upper bound of the null controllability cost for boundary control.

THEOREM 6.6. (Upper bound [A10]) *There exists a constant $C_u > 0$ independent of $\alpha \in [1, 2)$, of $\ell > 0$ and of $T > 0$ such that*

$$(6.28) \quad C_{ctr-bd}(\alpha, T, \ell) \leq \frac{C_u}{\sqrt{(2-\alpha)T\ell}} e^{-\frac{1}{C_u} \frac{T}{\ell^{2-\alpha}}} e^{C_u \frac{\ell^{2-\alpha}}{T(2-\alpha)^2}}.$$

This proves that the cost blows up exactly exponentially fast as $T \rightarrow 0^+$, or $\alpha \rightarrow 2^-$, or $\ell \rightarrow +\infty$. When ℓ is fixed and $T \leq T_0$, this simplifies into

$$C_{ctr-bd}(\alpha, T, \ell) \leq \frac{C_u}{\sqrt{(2-\alpha)T}} e^{\frac{C_u}{T(2-\alpha)^2}}.$$

6.4.3. Null controllability results for the locally distributed control.

6.4.3.1. *Lower bound of the null controllability cost for the locally distributed control.*

THEOREM 6.7. (*Lower bound [A10]*) *Given $\ell > 0$, and $0 < a < b < \ell$, there exists a constant $\tilde{C} = \tilde{C}(a, b, \ell) > 0$ independent of $\alpha \in [1, 2)$ and of $T > 0$ such that*

$$(6.29) \quad C_{ctr-loc}(\alpha, T, \ell) \geq \tilde{C} e^{\frac{\tilde{C}}{T(2-\alpha)^2}} e^{-\frac{1}{\tilde{C}} \frac{1}{(2-\alpha)^{4/3}} (\ln \frac{1}{2-\alpha} + \ln \frac{1}{T}) - \frac{1}{\tilde{C}} T} - 1.$$

In the proof of Theorem 6.7 we obtain an explicit expression of $\tilde{C}(a, b, \ell)$. And of course Theorem 6.7 proves that the cost blows up (exponentially fast) when $T \rightarrow 0^+$, or $\alpha \rightarrow 2^-$: when $T \rightarrow 0$ and/or $\alpha \rightarrow 2^-$, this simplifies into

$$C_{ctr-loc}(\alpha, T, \ell) \geq \tilde{C} e^{\frac{\tilde{C}}{T(2-\alpha)^2}} e^{-\frac{1}{\tilde{C}} \frac{1}{(2-\alpha)^{4/3}} (\ln \frac{1}{2-\alpha} + \ln \frac{1}{T})}.$$

6.4.3.2. *Upper bound of the null controllability cost for the locally distributed control.*

THEOREM 6.8. (*Upper bound [A10]*) *There exists a constant $C_u = C_u > 0$ independent of $\alpha \in [1, 2)$, of $T > 0$ and of $0 < a < b < \ell$, and $\gamma_0^* = \gamma_0^*(a, b, \ell) > 0$ such that*

$$(6.30) \quad C_{ctr-loc}(\alpha, T, \ell) \leq \frac{C_u}{\gamma_0^*} e^{C_u \frac{\ell^{2-\alpha}}{T(2-\alpha)^2}} e^{-\frac{1}{C_u} \frac{T}{\ell^{2-\alpha}}} \max\left\{\frac{1}{\sqrt{T}(2-\alpha)}, \frac{1}{\ell^{1-\alpha/2}}\right\}.$$

This proves that the cost blows up exactly exponentially fast as $T \rightarrow 0^+$, or $\alpha \rightarrow 2^-$, or $\ell \rightarrow +\infty$. When ℓ is fixed and $T \leq T_0$, this simplifies into

$$C_{ctr-loc}(\alpha, T, \ell) \leq C_u e^{C_u \frac{1}{T(2-\alpha)^2}}.$$

6.4.4. The eigenvalue problem.

The knowledge of the eigenvalues and associated eigenfunctions of the degenerate diffusion operator $u \mapsto -(x^\alpha u)'$, i.e. the solutions (λ, Φ) of

$$(6.31) \quad \begin{cases} -(x^\alpha \Phi'(x))' = \lambda \Phi(x) & x \in (0, \ell), \\ (x^\alpha \Phi'(x))(0) = 0, \\ \Phi(\ell) = 0. \end{cases}$$

will be essential for our purposes.

6.4.4.1. Eigenvalues and eigenfunctions.

It is well-known that Bessel functions play an important role in this problem, see, e.g., Kamke [217]. For $\alpha \in [1, 2)$, let

$$\nu_\alpha := \frac{\alpha - 1}{2 - \alpha}, \quad \kappa_\alpha := \frac{2 - \alpha}{2}.$$

Given $\nu \geq 0$, we denote by J_ν the Bessel function of first kind and of order ν (see [345]) and denote $j_{\nu,1} < j_{\nu,2} < \dots < j_{\nu,n} < \dots$ the sequence of positive zeros of J_ν . Then we have the following:

PROPOSITION 6.9. ([A10]) *The eigenvalues λ for problem (6.31) are given by*

$$(6.32) \quad \forall n \geq 1, \quad \lambda_{\alpha,n} = \ell^{\alpha-2} \kappa_\alpha^2 j_{\nu_\alpha,n}^2$$

and the corresponding normalized (in $L^2(0, \ell)$) eigenfunctions takes the form

$$(6.33) \quad \Phi_{\alpha,n}(x) = \frac{\sqrt{2\kappa_\alpha}}{\ell^{\kappa_\alpha} |J'_{\nu_\alpha}(j_{\nu_\alpha,n})|} x^{(1-\alpha)/2} J_{\nu_\alpha}(j_{\nu_\alpha,n} (\frac{x}{\ell})^{\kappa_\alpha}), \quad x \in (0, \ell).$$

Moreover the family $(\Phi_{\alpha,n})_{n \geq 1}$ forms an orthonormal basis of $L^2(0, \ell)$.

REMARK 6.10. Gueye [179] proved Proposition 6.9 in the case $\alpha \in [0, 1)$ and when $\ell = 1$. The case $\alpha \in [1, 2)$ and $\ell \neq 1$ is very similar.

6.4.4.2. The eigenfunctions in the control region.

We will prove the following property:

PROPOSITION 6.11. ([A10]) *Given $0 < a < b < \ell$, there exists $\gamma_0^* = \gamma_0^*(a, b, \ell) > 0$ such that*

$$(6.34) \quad \forall \alpha \in [1, 2), \forall m \geq 1, \quad \int_a^b \Phi_{\alpha, m}(x)^2 dx \geq \gamma_0^*(2 - \alpha).$$

It is classical in the nondegenerate case (Lagnese [227]) that

$$\inf_m \int_a^b \Phi_{\alpha, m}^2 > 0;$$

but, in our purpose of estimating the cost of null controllability, it is necessary to have a lower bound of $\int_a^b \Phi_{\alpha, m}^2$ with respect to the degeneracy parameter α when $\alpha \rightarrow 2^-$, and the dependence is given in Proposition 6.11. This does not come easily, since $\Phi_{\alpha, m}$ is solution of a second-order differential equation depending on a large parameter. We overcome this difficulty with ODE techniques.

6.4.5. Ideas of the proofs of the main results.

6.4.5.1. The moment method for the boundary control problem.

- The moment problem: if $u(T) = 0$, then multiplying by $\Phi_{\alpha, n}(x)e^{\lambda_{\alpha, n}(t-T)}$:

$$\forall n \geq 1, \quad \int_0^T H(t)e^{\lambda_{\alpha, n}t} dt = \frac{(u_0, \Phi_{\alpha, n})}{r_{\alpha, n}},$$

with $r_{\alpha, n} = \Phi'_{\alpha, n}(1)$.

- If there is a control H_m for $u_0 := \Phi_{\alpha, m}$, then

$$\forall n \geq 1, \quad \int_0^T (r_{\alpha, m} H_m(t))e^{\lambda_{\alpha, n}t} dt = \delta_{mn},$$

hence $(r_{\alpha, m} H_m)_m$ is biorthogonal to $(e^{\lambda_{\alpha, n}t})_{n \geq 1}$ in $L^2(0, T)$, hence

null controllability holds \implies some biorthogonal family exists;

- and the converse is formally true: if $(\sigma_{\alpha, m}^+)_{m \geq 1}$ is biorthogonal to $(e^{\lambda_{\alpha, n}t})_{n \geq 1}$ in $L^2(0, T)$, then formally:

$$H_{expl}(t) := \sum_{m=1}^{\infty} \frac{(u_0, \Phi_{\alpha, m})}{\Phi'_{\alpha, m}(1)} \sigma_{\alpha, m}^+(t)$$

solves the moment problem and drives the solution to 0 in time T , hence:

\exists some biorthogonal family exists (+ bounds) \implies null controllability holds.

6.4.5.2. Ideas of the proof of Theorem 6.6.

When $\nu_{\alpha} = \frac{\alpha-1}{2-\alpha} \geq \frac{1}{2}$, the sequence $(j_{\nu_{\alpha, n+1}} - j_{\nu_{\alpha, n}})_n$ decays to π (Komornik-Loreti [225]), hence

$$\sqrt{\lambda_{\alpha, n+1}} - \sqrt{\lambda_{\alpha, n}} \geq \frac{\pi}{2}(2 - \alpha) =: \gamma_{min}(\alpha),$$

hence the existence of a biorthogonal family, and Theorem 6.6 follows from the moment method and Theorem 6.1.

6.4.5.3. *Ideas of the proof of Theorem 6.5.*

As already seen:

$$\pi \leq j_{\nu_\alpha, n+1} - j_{\nu_\alpha, n} \leq j_{\nu_\alpha, 2} - j_{\nu_\alpha, 1},$$

hence $\gamma_{max} = \kappa_\alpha (j_{\nu_\alpha, 2} - j_{\nu_\alpha, 1})$. But (classical)

$$j_{\nu, 2} - j_{\nu, 1} \sim a\nu^{1/3} \quad \text{as } \nu \rightarrow +\infty.$$

Hence $\gamma_{max} \sim c\nu_\alpha^{-2/3} \sim c'(2-\alpha)^{2/3}$ as $\alpha \rightarrow 2^-$. Then Theorem 6.2 implies that

$$C(\alpha, T) \geq e^{\frac{c}{(2-\alpha)^{4/3}T}}.$$

To obtain the power 2 at the place of $4/3$, we noted the following fact:

LEMMA 6.12. ([A10])

$$\forall \nu \geq \frac{1}{2}, \forall n > \nu, \quad j_{\nu_\alpha, n+1} - j_{\nu_\alpha, n} \leq 2\pi.$$

(The proof is based on a classical oscillation theorem of Sturm for second order ODE, in the spirit of Komornik-Loreti [225].) Then the eigenvalues satisfy the asymptotic gap condition of Theorem 6.4 with $\gamma_{max}^* = \frac{2-\alpha}{2}2\pi = \pi(2-\alpha)$ and $N_* = [\nu_\alpha] + 1$, and then Theorem 6.5 follows from the moment method and Theorem 6.4.

6.4.5.4. *Ideas of the proof of Proposition 6.11.*

Proposition 6.11 follows from extending some classical results on Bessel functions, estimating Bessel functions of large order (ν_α) on a zone depending on the order ('the transition zone').

6.5. The cost of controlling weakly degenerate parabolic equations from the degeneracy point ([A8])

We are interested in the controllability properties of the problem

$$(6.35) \quad \begin{cases} u_t - (x^\alpha u_x)_x = 0, \\ u(0, t) = G(t), \\ u(1, t) = 0, \\ u(x, 0) = u_0(x), \end{cases}$$

that is when the control acts at the degeneracy point 0 through a nonhomogeneous Dirichlet boundary condition. First, we recall that, in a general way, the well-posedness of degenerate parabolic equations is stated in weighted Sobolev spaces. We will consider the problem when $\alpha \in [0, 1)$; in this case, the Dirichlet boundary control makes sense.

Gueye [179] proved that null controllability holds for all $\alpha \in [0, 1)$ (using the transmutation method [129, 130]). Then it is interesting to measure the cost to drive any u_0 to 0 in time T , with respect to α .

We define the controllability costs in the following way: given $u_0 \in L^2(0, 1)$, we consider the set of admissible controls that drive the solution u of (6.35) to 0 in time T :

$$\mathcal{U}_{ad}(\alpha, T, u_0) := \{G \in H^1(0, T), u^{(G)}(T) = 0\},$$

where $u^{(G)}$ denotes the solution of (6.35); we consider the controllability cost

$$(6.36) \quad C_{bd}^{H^1}(\alpha, T, u_0) := \inf_{G \in \mathcal{U}_{ad}(\alpha, T, u_0)} \|G\|_{H^1(0, T)},$$

which is the minimal value to drive u_0 to 0. We also consider a global notion of controllability cost:

$$(6.37) \quad C_{bd}^{H^1}(\alpha, T) := \sup_{\|u_0\|=1} C_{bd}^{H^1}(\alpha, T, u_0).$$

Then we prove the following

THEOREM 6.13. ([A8])

a) *There exist two positive constants M_1, M_2 independent of $\alpha \in [0, 1)$ and $T > 0$ such that*

$$(6.38) \quad \frac{M_1}{1-\alpha} \frac{1}{T(T+1)} e^{1/(\pi^2 T)} e^{-9\pi^2 T/16} \leq C_{bd}^{H^1}(\alpha, T) \leq \frac{1}{1-\alpha} e^{M_2/T} e^{-T/M_2}.$$

b) *Given $u_0 \in L^2(0, 1)$, there exists $M_3(u_0)$ independent of $\alpha \in [0, 1)$ and $T > 0$ such that*

$$(6.39) \quad \frac{M_3(u_0)}{1-\alpha} \frac{1}{\sqrt{T}} e^{-9\pi^2 T/16} \leq C_{bd}^{H^1}(\alpha, T, u_0) \leq \frac{1}{1-\alpha} e^{M_2/T} e^{-T/M_2} \|u_0\|_{L^2(0,1)}.$$

This shows that the controllability cost blows up as $\alpha \rightarrow 1^-$, and that our upper estimate is optimal. Note that the cost of controlling *any* initial condition blows up as $\frac{1}{1-\alpha}$ as $\alpha \rightarrow 1^-$. We do not know if the cost $C^{H^1}(\alpha, T, u_0)$ blows up exponentially as $T \rightarrow 0^+$, as it is the case for $C^{H^1}(\alpha, T)$.

6.6. The cost of controlling degenerate parabolic equations with inverse square potential (Joint work with J. Vancostenoble ([A11]))

6.6.1. Introduction.

In this part, we are interested in the linear $1-D$ heat equation with an inverse square potential (that arises for example in the context of combustion theory and quantum mechanics):

$$(6.40) \quad \begin{cases} u_t - u_{xx} - \frac{\mu}{x^2} u = 0 & x \in (0, 1), t \in (0, T), \\ u(0, t) = 0 & t \in (0, T), \\ u(1, t) = H(t) & t \in (0, T), \\ u(x, 0) = u_0(x) & x \in (0, 1), \end{cases}$$

where $u_0 \in L^2(0, 1)$, $T > 0$ and μ is a real parameter. Here H represents some control term that aims to steer the solution to zero at time T . Our goal is not only to establish the existence of such control (which could be easily deduced from known results, see later) but also to provide sharp estimates of the cost of such control.

Since the works by Baras and Goldstein [18, 19], it is known that existence/non-existence of positive solutions is determined by the value of μ with respect to the constant $1/4$ appearing in the Hardy inequality [194, 291]:

$$(6.41) \quad \forall z \in H_0^1(0, 1), \quad \frac{1}{4} \int_0^1 \frac{z^2}{|x|^2} dx \leq \int_0^1 |z_x|^2 dx.$$

When $\mu < 1/4$, the operator $z \mapsto -z_{xx} - \mu x^{-2} z$ generates a coercive quadratic form in $H_0^1(0, 1)$. This allows showing the well-posedness in the classical variational setting of the linear heat equation with smooth coefficients. For the critical value $\mu = 1/4$, the space $H_0^1(0, 1)$ has to be slightly enlarged as shown in [341] but a similar result of well-posedness occurs. Finally, when $\mu > 1/4$, the problem is ill-posed (due to possible instantaneous blow-up) as proved in [18].

For these reasons, we concentrate on the two first cases and *we assume here that μ satisfies $\mu \leq 1/4$.*

Recently, the null controllability properties of (6.40) began to be studied. For any $\mu \leq 1/4$, it has been proved in [338] that such equations can be controlled (in any time $T > 0$) by a locally distributed control: $\forall \mu \leq 1/4, \forall u_0 \in L^2(0, 1), \forall T > 0, \forall 0 \leq a < b \leq 1$, there exists $h \in L^2((0, 1) \times (0, T))$ such that the solution of

$$(6.42) \quad \begin{cases} u_t - u_{xx} - \frac{\mu}{x^2}u = h(x, t)\chi_{(a,b)}(x) & x \in (0, 1), t \in (0, T), \\ u(0, t) = 0 & t \in (0, T), \\ u(1, t) = 0 & t \in (0, T), \\ u(x, 0) = u_0(x) & x \in (0, 1), \end{cases}$$

satisfies $u(\cdot, T) \equiv 0$.

The proof in [338] is based on Carleman estimates. It also concerns the case of the N -dimensional equation with some restricting geometric condition on the region of the control, condition that has been later erased in [128]. After those first results, several other works followed extending them in various situations. See for instance [335, 336, 88, 53, 159, 181].

Here we are interested in the study of null controllability using a boundary control acting at $x = 1$ (see problem (6.40)). More precisely, our aim is to provide sharp estimates of the cost of controllability in that case, analyzing its dependence with respect to the two parameters $T > 0$ and $\mu \in (-\infty, 1/4]$.

As seen before, the obtention of explicit and precise (upper and lower) estimates for such biorthogonal families is closely related to gap conditions on the eigenvalues, namely

$$\forall n, \quad \gamma_{min} \leq \sqrt{\lambda_{n+1}} - \sqrt{\lambda_n} \leq \gamma_{max}.$$

(Roughly speaking, the gap γ_{min} gives the upper estimates whereas the gap γ_{max} gives the lower estimates). Then here

- we solve the eigenvalue problem, and we express the eigenvalues and eigenfunctions using Bessel functions and their zeros;
- when $\mu \in [0, 1/4]$, the gap between the square root of successive eigenvalues satisfies some upper and lower bounds which are uniform with respect to the parameter μ ; this enables us to use the results of [A8];
- but when $\mu \leq 0$, the eigenvalues do not satisfy a good uniform gap condition from above; we use here the tools developed in [A9] in order to treat cases where the eigenvalues do not satisfy a good uniform gap condition but satisfy some better asymptotic gap condition; these new results have already been applied in the context of the degenerate heat equation with a strong degeneracy in [A10], and in the present case they help us to provide a suitable lower bound of the cost.

6.6.2. Main results and comments. Let us define the notion of cost of controllability. For any $T > 0$, $\mu \leq 1/4$ and $u_0 \in L^2(0, 1)$, we introduce the set of admissible controls:

$$\mathcal{U}_{ad}(\mu, T, u_0) := \{H \in H^1(0, T) \mid u^{(H)}(T) = 0\},$$

where $u^{(H)}$ denotes the solution of (6.40). Then we consider the controllability cost for any $u_0 \in L^2(0, 1)$

$$C^{H^1}(\mu, T, u_0) := \inf_{H \in \mathcal{U}_{ad}(\mu, T, u_0)} \|H\|_{H^1(0, T)}$$

which is the minimal value to drive u_0 to 0. Finally, we define the global notion of controllability cost:

$$C_{bd-ctr}^{H^1}(\mu, T) := \sup_{\|u_0\|_{L^2(0,1)}} C^{H^1}(\mu, T, u_0).$$

Then we prove the following results

THEOREM 6.14. ([A11]) *Given $\mu \in [0, \frac{1}{4}]$, $T > 0$, and $u_0 \in L^2(0, 1)$, there exists $H \in H^1(0, T)$ such that the solution of (6.40) satisfies $u(\cdot, T) \equiv 0$. Moreover, we have the following estimates of the controllability cost: there exists $0 < c < C$ both independent of $\mu \in [0, 1/4]$ and of $T > 0$ such that*

$$(6.43) \quad C_{ctr-bd}(\mu, T) \leq C e^{C/T} e^{-(1+\sqrt{\frac{1}{4}-\mu})^2 T/C} \left(1 + \sqrt{\frac{1}{4} - \mu}\right),$$

and

$$(6.44) \quad C_{ctr-bd}(\mu, T) \geq c e^{c/T} e^{-(1+\sqrt{\frac{1}{4}-\mu})^2 T/c}.$$

THEOREM 6.15. ([A11]) *Given $\mu \leq 0$, $T > 0$, and $u_0 \in L^2(0, 1)$, there exists $H \in H^1(0, T)$ such that the solution of (6.40) satisfies $u(\cdot, T) \equiv 0$. Moreover, we have the following estimates of the controllability cost: there exists $0 < c < C$ both independent of $\mu \leq 0$ and of $T > 0$ such that*

$$(6.45) \quad C_{ctr-bd}(\mu, T) \leq C e^{C/T} e^{-(1+\sqrt{\frac{1}{4}-\mu})^2 T/C} \left(1 + \sqrt{\frac{1}{4} - \mu}\right),$$

and

$$(6.46) \quad C_{ctr-bd}(\mu, T) \geq c e^{c/T} e^{-(1+\sqrt{\frac{1}{4}-\mu})^2 T/c} e^{-\sqrt{\frac{1}{4}-\mu}^{4/3} (\ln \sqrt{\frac{1}{4}-\mu} + \ln \frac{1}{T})/c}.$$

6.7. Null controllability of degenerate parabolic equations with interior degeneracy and one-sided control (Joint work with P. Cannarsa and R. Ferretti ([A12]))

Intrigued by numerical tests (section 6.7.3), we investigate the null controllability of a degenerate parabolic operator in one space dimension, which degenerates at a single point inside the space domain, under the action of a locally distributed control supported only on one side of the domain with respect to the point of degeneracy. In formulas, we consider the problem

$$(6.47) \quad \begin{cases} u_t - (|x|^\alpha u_x)_x = h(x, t) \chi_{(a,b)}(x), & x \in (-1, 1) \\ u(-1, t) = 0 = u(1, t), \\ u(x, 0) = u_0(x), \end{cases}$$

assuming either $0 < a < b < 1$ or $-1 < a < b < 0$.

The analysis of problem (6.47) allows us to discover some interesting properties, both positive and negative from the point of view of null controllability. More precisely, we obtain:

- *Negative results for $\alpha \in [1, 2)$.* The negative result we prove means that, when $\alpha \in [1, 2)$, the degeneracy is too strong to allow the control to act on the other side of the domain with respect to the point of degeneracy. However, null controllability still holds true for those initial condition that are supported in the same region as the control.
- *Positive results for $\alpha \in [0, 1)$.* The proof of the fact that the control is sufficiently strong to cross the degeneracy point does require to use fine properties of Bessel functions. We also give a sharp estimate of the blow-up rate of the null controllability cost as $\alpha \rightarrow 1^-$.

Degenerate parabolic equations with one (or more) degeneracy point inside the domain have also been studied by the flatness method developed by Martin-Rosier-Rouchon in [257, 258, 259] (see also Moyano [277] for some strongly degenerate equations). More specifically, one can use the null controllability result with boundary control derived in [259] to construct a locally distributed control which steers the initial datum to 0 for $\alpha \in [0, 1)$. On the other hand, neither our analysis of the cost in the weakly degenerate case, nor our negative result for the strongly degenerate case seem to be attainable by the flatness approach.

Parabolic equations with interior degeneracy were also considered by Fragnelli-Mugnai in [162, 163], where positive null controllability results were obtained for a general class of coefficients. Their approach, based on Carleman estimates, gives the controllability result when the control region is on both sides of the space domain with respect to the degeneracy point. Indeed, as our negative result shows, strongly degenerate problems ($\alpha \in [1, 2)$) fail to be null controllable otherwise. On the other hand, for weakly degenerate problems, Carleman estimates do not seem to lead to null controllability results with the same generality as we obtain here for problem (6.47) (see Proposition 6.19).

Our method is based on a careful analysis of the spectral problem associated with (6.47). When degeneracy occurs inside, the problem looks like a simplified version of the one studied in Zhang-Zuazua [356] in the case of a 1-D fluid-structure model: we solve the problem on both sides of the degeneracy, and we study the transmission conditions. Once the spectral problem is solved, negative results come quite immediately when $\alpha \in [1, 2)$. For the positive part, we combine the moment method with general results obtained in [A8] concerning the existence of biorthogonal families under general gap conditions on the square roots of the eigenvalues. Then, we complete the analysis with some L^2 lower bounds for the eigenfunctions on the control region.

The theoretical results of our findings are completed with a final numerical work which concludes the paper (see section 6.7.3). Starting from the pioneering works of J.-L. Lions (see, e.g., [240]), numerical approximation of controllability problems for parabolic equations has become an established matter. Among the rich literature, we quote here the basic results in [252], devoted to the nondegenerate heat equation with boundary control, along with the more recent studies in [57] and the general framework provided in [226].

Most of the literature is concerned with semi-discretized problems and aims at constructing an approximation of the stabilizing control. We will rather use here a fully discrete approximation, and avoid the problem of convergence of approximate stabilizing controls to a limit solution, which is known to be a very ill-conditioned problem (see [58] for a study of the full discretization, as well as a review of the relevant literature). Therefore, the last section should be understood as a numerical illustration of the theoretical results, while a rigorous study of numerical approximations for this problem will be postponed to future works.

6.7.1. The strongly degenerate case: $\alpha \in [1, 2)$.

6.7.1.1. The eigenvalue problem when $\alpha \in [1, 2)$.

The knowledge of the eigenvalues and associated eigenfunctions of the degenerate diffusion operator $u \mapsto -(|x|^\alpha u)'$, i.e., the nontrivial solutions (λ, Φ) of

$$(6.48) \quad \begin{cases} -(|x|^\alpha \Phi'(x))' = \lambda \Phi(x) & x \in (-1, 1), \\ \Phi(-1) = 0 = \Phi(1), \end{cases}$$

will be essential for our purposes. For $\alpha \in [1, 2)$, let

$$\nu_\alpha := \frac{|\alpha - 1|}{2 - \alpha} = \frac{\alpha - 1}{2 - \alpha}, \quad \kappa_\alpha := \frac{2 - \alpha}{2}.$$

When $\alpha \in [1, 2)$, we have the following description of the spectrum of the associated operator:

PROPOSITION 6.16. ([A12]) *The admissible eigenvalues λ for problem (6.48) are given by*

$$(6.49) \quad \forall n \geq 1, \quad \lambda_{\alpha,n} = \kappa_\alpha^2 j_{\nu_\alpha,n}^2.$$

The associated eigenspace is of dimension 2, and an orthonormal basis (in $L^2(-1, 1)$) is given by the following eigenfunctions

$$(6.50) \quad \tilde{\Phi}_{\alpha,n}^{(r)}(x) := \begin{cases} \frac{\sqrt{2\kappa_\alpha}}{|J'_{\nu_\alpha}(j_{\nu_\alpha,n})|} x^{(1-\alpha)/2} J_{\nu_\alpha}(j_{\nu_\alpha,n} x^{\kappa_\alpha}) & \text{if } x \in (0, 1) \\ 0 & \text{if } x \in (-1, 0) \end{cases},$$

and

$$(6.51) \quad \tilde{\Phi}_{\alpha,n}^{(l)}(x) := \begin{cases} 0 & \text{if } x \in (0, 1) \\ \frac{\sqrt{2\kappa_\alpha}}{|J'_{\nu_\alpha}(j_{\nu_\alpha,n})|} |x|^{(1-\alpha)/2} J_{\nu_\alpha}(j_{\nu_\alpha,n} |x|^{\kappa_\alpha}) & \text{if } x \in (-1, 0) \end{cases}.$$

Moreover $\{\tilde{\Phi}_{\alpha,n}^{(r)}, \tilde{\Phi}_{\alpha,n}^{(l)}, n \geq 1\}$ forms an orthonormal basis of $L^2(-1, 1)$.

6.7.1.2. *Null controllability when $\alpha \in [1, 2)$.*

The following controllability result is a direct consequence of the above proposition.

PROPOSITION 6.17. ([A12]) *Assume that $\alpha \in [1, 2)$ and let $0 < a < b < 1$. Then null controllability fails, and the initial conditions that can be steered to 0 in time T are exactly those which are supported in $[0, 1)$.*

6.7.2. The weakly degenerate case: $\alpha \in [0, 1)$.

6.7.2.1. *Eigenvalues and eigenfunctions when $\alpha \in [0, 1)$.*

Once again, the knowledge of the eigenvalues and associated eigenfunctions of the degenerate diffusion operator $u \mapsto -(|x|^\alpha u)'$, i.e. the solutions (λ, Φ) of

$$(6.52) \quad \begin{cases} -(|x|^\alpha \Phi'(x))' = \lambda \Phi(x) & x \in (-1, 1), \\ \Phi(-1) = 0 = \Phi(1) \end{cases}$$

will be essential for our purposes. When $\alpha \in [0, 1)$, let

$$\nu_\alpha := \frac{|\alpha - 1|}{2 - \alpha} = \frac{1 - \alpha}{2 - \alpha}, \quad \kappa_\alpha := \frac{2 - \alpha}{2}.$$

Now we will need the zeros of the Bessel function J_{ν_α} , and also the zeros of the Bessel function of negative order $J_{-\nu_\alpha}$.

We prove the following description for (6.52):

PROPOSITION 6.18. ([A12]) *When $\alpha \in [0, 1)$, we have exactly two sub-families of eigenvalues and associated eigenfunctions for problem (6.52), that is:*

- *the eigenvalues of the form $\kappa_\alpha^2 j_{\nu_\alpha,n}^2$, associated with the odd functions*

$$(6.53) \quad \Phi_{\alpha,n}^{(o)}(x) = \begin{cases} x^{\frac{1-\alpha}{2}} J_{\nu_\alpha}(j_{\nu_\alpha,n} x^{\kappa_\alpha}) & \text{if } x \in (0, 1) \\ -|x|^{\frac{1-\alpha}{2}} J_{\nu_\alpha}(j_{\nu_\alpha,n} |x|^{\kappa_\alpha}) & \text{if } x \in (-1, 0) \end{cases},$$

- *the eigenvalues of the form $\kappa_\alpha^2 j_{-\nu_\alpha,n}^2$, associated with the even functions*

$$(6.54) \quad \Phi_{\alpha,n}^{(e)}(x) = \begin{cases} x^{\frac{1-\alpha}{2}} J_{-\nu_\alpha}(j_{-\nu_\alpha,n} x^{\kappa_\alpha}) & \text{if } x \in (0, 1) \\ |x|^{\frac{1-\alpha}{2}} J_{-\nu_\alpha}(j_{-\nu_\alpha,n} |x|^{\kappa_\alpha}) & \text{if } x \in (-1, 0) \end{cases}.$$

Moreover, the family $\{\Phi_{\alpha,n}^{(o)}, \Phi_{\alpha,n}^{(e)}, n \geq 1\}$ forms an orthogonal basis of $L^2(-1, 1)$.

6.7.2.2. *Null controllability when $\alpha \in [0, 1)$.*

PROPOSITION 6.19. ([A12]) *Assume that $\alpha \in [0, 1)$ and that $0 < a < b < 1$. Then null controllability holds: given $u_0 \in L^2(-1, 1)$, there exists a control h that drives the solution u to 0 in time T .*

6.7.2.3. *Blow-up of the control cost as $\alpha \rightarrow 1^-$.*

Given $\alpha \in [0, 1)$, $T > 0$ and $u_0 \in L^2(-1, 1)$, consider $\mathcal{U}^{ad}(\alpha, T; u_0)$ the set of admissible controls:

$$\mathcal{U}^{ad}(\alpha, T; u_0) := \left\{ h \in L^2((a, b) \times (0, T)) \mid u^{(h)}(T) = 0 \right\}.$$

Since null controllability holds if and only if $\alpha < 1$, it is natural to expect that the null controllability cost

$$(6.55) \quad C_{NC}(\alpha, T) := \sup_{\|u_0\|_{L^2(-1,1)}} \inf_{h \in \mathcal{U}^{ad}(\alpha, T; u_0)} \|h\|_{L^2((a,b) \times (0,T))}$$

blows up when $\alpha \rightarrow 1^-$. This is the object of the following result:

THEOREM 6.20. ([A12]) *a) Estimate from above: there exists some $C > 0$ independent of $\alpha \in [0, 1)$ and of $T > 0$ such that*

$$(6.56) \quad C_{NC}(\alpha, T) \leq \frac{C}{(1-\alpha)^2} e^{-T/C} e^{C/T};$$

b) Estimate from below: there exists some $C' > 0$ independent of $\alpha \in [0, 1)$ and of $T > 0$ such that

$$(6.57) \quad C_{NC}(\alpha, T) \geq \frac{C'}{(1-\alpha)\sqrt{T}} e^{-T/C'}.$$

Note that Theorem 6.20 proves that the null controllability cost blows up when $\alpha \rightarrow 1^-$, and when $T \rightarrow 0^+$. Moreover:

- with respect to α : we have a good estimate of the behavior when $\alpha \rightarrow 1^-$, but the upper and lower estimates are not of the same order; maybe the moment method by blocks, recently introduced by Benabdallah-Boyer-Morancey [36] could help to obtain optimal results;
- with respect to T : in [A10] we prove that the blow-up of the null controllability cost is of the order $e^{C/T}$; here we only obtain a weak blow-up estimate, of the order $1/\sqrt{T}$; we conjecture that the blow-up rate is of the form $e^{C/T}$.

It would be interesting to have better blow-up estimates, with respect to $\alpha \rightarrow 1^-$ and $T \rightarrow 0^+$.

6.7.3. Numerical tests.

In this test, we take an initial state supported in \mathbb{R}_- :

$$(6.58) \quad u_0(x) = \chi_{(-1/2, -1/4)}.$$

The two typical situations may be seen in Fig. 1–2, where the former shows the final state at $T = 0.5$ for $\alpha = 0.8$ (left) and $\alpha = 1.2$ (right), while the latter shows the corresponding control h for $(x, t) \in [a, b] \times [0, T]$ (the plots of h have been scaled by a factor of respectively 10^{-5} for $\alpha = 0.8$ and $2 \cdot 10^{-8}$ for $\alpha = 1.2$). Note that, although the qualitative behaviour in the two cases is similar, the scales are very different: the final state has an L^2 norm of about $4.3 \cdot 10^{-7}$ for $\alpha = 0.8$ versus a norm of $2.4 \cdot 10^{-4}$ for $\alpha = 1.2$. The optimal control maintains the known feature of being highly oscillating, but its norm is considerably higher in the second case – this reflects the transition between a zero-controllable situation and a non-controllable one.

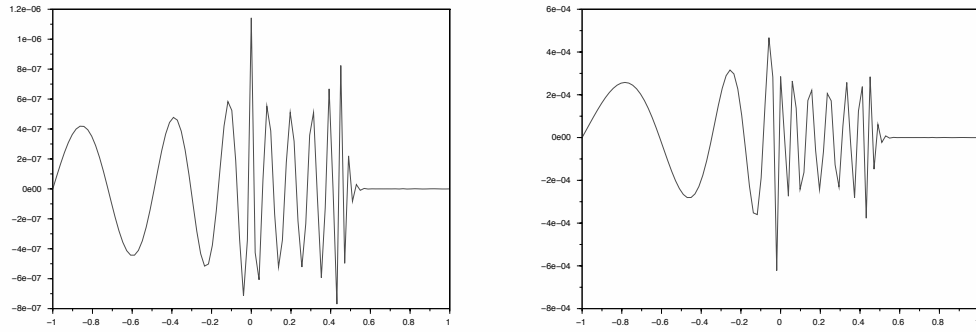


FIGURE 1. Test 1, final state for $\alpha = 0.8$ (left) and $\alpha = 1.2$ (right).

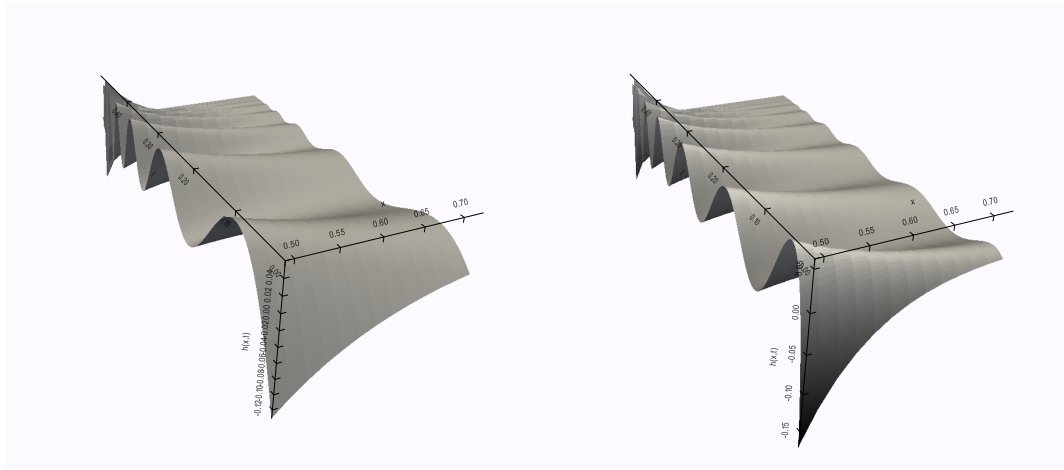


FIGURE 2. Test 1, control $h(x, t)$ for $\alpha = 0.8$ (left) and $\alpha = 1.2$ (right). The scaling factor is respectively 10^{-5} and $2 \cdot 10^{-8}$.

Parameter estimation for Energy Balance Models (joint works with J. Tort and J. Vancostenoble ([A13]) and P. Cannarsa and M. Malfitana ([A14]))

7.1. Introduction: motivation and related papers

7.1.1. Motivation.

We are interested in a problem arising in climatology, coming more specifically from the classical Energy Balance models introduced independently by Budyko [67] and Sellers [317]. These models describe the evolution of temperature as the effect of the balance between the amount of energy received from the Sun and radiated from the Earth, and were developed in order to understand the past and future climate and its sensitivity to some relevant parameters on large time scales (centuries). They take the form of the following nonlinear parabolic equation:

$$u_t - \operatorname{div}(k\nabla u) = R_a - R_e$$

where

- u is the surface temperature,
- R_a represents the fraction of solar energy absorbed by the Earth,
- R_e represents the energy emitted by the Earth,

A crucial role in the analysis will be played by the absorbed energy R_a , which is a fraction of the incoming solar flux $Q(t, x)$. Sellers and Budyko have chosen different assumptions on the coefficients appearing in the parabolic equation describing the evolution of the temperature, in particular, R_a is chosen to be smooth in Sellers' models while it is a monotone graph in Budyko's models (which brings, of course, interesting mathematical difficulties).

The mathematical analysis of quasilinear EBM problems of the form

$$\partial_t u - \operatorname{div}(\rho(x)|\nabla u|^{p-2}\nabla u) = f(t, x, u, H(t, x, u)),$$

where H describes a memory effect of the form

$$H(t, x, u) = \int_{-\tau}^0 k(s)u(t+s, x) ds,$$

has been the subject of many deep works for a long time. Questions such as well-posedness, uniqueness, asymptotic behavior, existence of periodic solutions, bifurcation, free boundary, numerical approximation were investigated for:

- 1-D models without memory by Ghil in the seminal paper [170],
- 0-D models in Fraedrich [160, 161],
- 1-D models with memory in Bhattacharya-Ghil-Vulis [52] and Diaz [118, 116],
- 2-D models (on a manifold without boundary, typically representing the Earth's surface) in Hetzer [197], Diaz-Tello [119], Diaz-Hetzer [120], Hetzer [199], Diaz [121], Diaz-Hetzer-Tello [122], and Hetzer [200].

In this part, we are interested in the following inverse problem: is it possible to recover the "insolation function" (which is a part of R_a) from measurements of the solution? Our motivation comes from the fact that, with suitable tuning of their parameters, EBMs have shown to mimic the observed zonal temperatures for the observed present climate [288], and can be used to estimate the temporal response patterns to various forcing scenarios, which is of interest in particular in the detection of climate change. Unfortunately, in practice, the model coefficients cannot be measured directly, but are quantified through the measures of the solution, see Roques-Checkroun-Cristofol-Ghil [309]. Hence, results proving that measuring the solution in some specific (small) part of the space and time domain is sufficient to recover a specified coefficient are of practical interest.

7.1.2. The related papers [A13, A14].

- [A13]: we study the 2D Sellers type problem; for this,
 - we prove a Carleman estimate on a manifold without boundary,
 - and suitable maximum principles to deal with the nonlinearity,
 - and we use them to obtain a Lipschitz stability result (under localized observation) concerning the determination of the insolation function.
- [A14]:
 - we study the 1D Sellers type problem with degenerate diffusion and memory effects; more precisely, we prove regularity results and use them to study the determination of the insolation function, obtaining
 - * a uniqueness result, under pointwise observation,
 - * a Lipschitz stability result, under localized observation;
 - at last, we address 1D Budyko type problems with degenerate diffusion and memory effects, for which we obtain precise existence results.

Our analysis will be based on several works, in particular Imanuvilov-Yamamoto in their seminal paper [212] for the use of Carleman estimates to obtain Lipschitz stability results in inverse problems questions, Roques-Checkroun-Cristofol-Soubeyrand-Ghil [309] for uniqueness results and Diaz-Hetzer [120] for general techniques to study Budyko's type models.

7.2. The 2D Sellers model (joint work with J. Tort and J. Vancostenoble [A13])

In this part, we are interested in some inverse problem that consists in recovering the so-called insolation function q in the nonlinear Sellers climate model. The case of the 1-D Sellers model has been considered in [326]. Here we focus on the 2-D Sellers model on the Earth's surface:

$$(7.1) \quad \begin{cases} u_t - \Delta_{\mathcal{M}} u = \overbrace{r(t)q(x)\beta(u)}^{\mathcal{R}_a(t,x,u)} - \overbrace{\varepsilon(u)u|u|^3}^{\mathcal{R}_e(u)} & x \in \mathcal{M}, t > 0, \\ u(0, x) = u^0(x) & x \in \mathcal{M}. \end{cases}$$

The Earth's surface is materialized by a sub-manifold \mathcal{M} of \mathbb{R}^3 which is assumed to be of dimension 2, compact, connected, oriented, and without boundary. The function u represents the mean annual or seasonal temperature, and $\Delta_{\mathcal{M}}$ is the Laplace-Beltrami operator on \mathcal{M} . The right hand side of the equation corresponds to

- the mean radiation flux depending on the solar radiation $\mathcal{R}_a = r(t)q(x)\beta(u)$, where
 - β is the coalbedo function,
 - $q(x)$ is the insolation function, the one that we want to recover using measures of the solution,
 - $r(t)$ describes the seasonal variations,
- and the radiation \mathcal{R}_e emitted by the Earth.

For more details on the model, we refer the reader to [116, 117] and the references therein. In the following we precise our assumptions.

7.2.1. Geometrical and regularity assumptions.

Consider a sub-manifold \mathcal{M} of \mathbb{R}^3 which is assumed to be of dimension 2, compact, connected, oriented, and without boundary.

Throughout this paper, we make the following assumptions (that are compatible with applications, see [326]):

HYPOTHESIS 7.1.

$$(7.2) \quad \beta \in C^1(\mathbb{R}) \cap L^\infty(\mathbb{R}), \beta' \in L^\infty(\mathbb{R}), \beta' \text{ is } k\text{-Lipschitz } (k > 0),$$

$$(7.3) \quad \exists \beta_{min} > 0, \forall u \in \mathbb{R}, \beta(u) \geq \beta_{min},$$

$$(7.4) \quad q \in L^\infty(\mathcal{M}), q \geq 0,$$

$$(7.5) \quad r \in C^1(\mathbb{R}) \text{ is } \tau\text{-periodic } (\tau > 0),$$

$$(7.6) \quad \exists r_{min} > 0, \forall t \in \mathbb{R}, r(t) \geq r_{min},$$

$$(7.7) \quad \varepsilon \in C^1(\mathbb{R}) \cap L^\infty(\mathbb{R}), \varepsilon' \text{ is } K\text{-Lipschitz } (K > 0),$$

$$(7.8) \quad \exists \varepsilon_{min} > 0, \forall u \in \mathbb{R}, \varepsilon(u) > \varepsilon_{min}.$$

We also make the following geometrical assumption:

HYPOTHESIS 7.2. *Let ω be a non empty open subset of \mathcal{M} . We assume that there exists a weight function $\psi \in C^\infty(\mathcal{M})$ that satisfies:*

$$(7.9) \quad \nabla \psi(m) = 0 \quad \implies \quad m \in \omega.$$

(Here ∇ stands for the usual gradient associated to the Riemannian structure.)

7.2.2. Main results.

Our main result is the stability estimated stated in Theorem 7.4, which follows from a Carleman estimate ([A13]) and maximum principles stated in Theorem 7.3.

7.2.2.1. *Maximum principles and global existence of the solution of the nonlinear problem.*

First, in a classical way, we obtain that the nonlinear problem 7.1 has a unique maximal solution, defined on some interval $[0, T^*(u^0))$, and we obtain that $T^*(u^0) = +\infty$ using the following maximum principle:

THEOREM 7.3. ([A13]) *Let $u^0 \in D(\Delta) \cap L^\infty(\mathcal{M})$. We denote*

$$(7.10) \quad M := \max \left\{ \|u^0\|_{L^\infty(\mathcal{M})}, \left(\frac{\|q\|_{L^\infty(\mathcal{M})} \|r\|_{L^\infty(\mathbb{R})} \|\beta\|_{L^\infty(\mathbb{R})}}{\varepsilon_{\min}} \right)^{1/4} \right\}.$$

Then the solution u of problem (7.1) satisfies $\|u\|_{L^\infty((0, T^(u^0)) \times \mathcal{M})} \leq M$.*

7.2.2.2. *A stability estimate.*

As in [326], our aim is to prove some Lipschitz stability result for the inverse problem that consists in recovering the insulation function q in (7.1) from partial measurements. We introduce

- the set of admissible initial conditions: given $A > 0$, we consider \mathcal{U}_A :

$$(7.11) \quad \mathcal{U}_A := \{u^0 \in D(\Delta_{\mathcal{M}}) \cap L^\infty(\mathcal{M}) : \Delta_{\mathcal{M}} u^0 \in L^\infty(\mathcal{M}), \\ \|u_0\|_{L^\infty(\mathcal{M})} + \|\Delta_{\mathcal{M}} u_0\|_{L^\infty(\mathcal{M})} \leq A\},$$

where $D(\Delta_{\mathcal{M}})$ is the domain of the Laplace-Beltrami operator in $L^2(\mathcal{M})$,

- and the set of admissible coefficients: given $B > 0$, we consider

$$(7.12) \quad \mathcal{Q}_B := \{q \in L^\infty(\mathcal{M}) : \|q\|_{L^\infty(\mathcal{M})} \leq B\}.$$

The main result is the following one:

THEOREM 7.4. ([A13]) *Consider*

- $t_0 \in [0, T)$ and $T' \in (t_0, T)$,
- $A > 0$ and $u_1^0, u_2^0 \in \mathcal{U}_A$ (defined in (7.11)),
- $B > 0$ and $q_1, q_2 \in \mathcal{Q}_B$ (defined in (7.12)),
- u_1 the solution of (7.1) associated to q_1 and the initial condition u_1^0 , and u_2 the solution of (7.1) associated to q_2 and the initial condition u_2^0 ,
- $\omega \subset \mathcal{M}$ such that Assumption 7.2 holds.

Then there exists $C(t_0, T', T, A, B) > 0$ such that, for all $u_1^0, u_2^0 \in \mathcal{U}_A$, for all $q_1, q_2 \in \mathcal{Q}_B$, the corresponding solutions u_1, u_2 of problem (7.1) satisfy

$$(7.13) \quad \|q_1 - q_2\|_{L^2(\mathcal{M})}^2 \leq C(\|u_1(T') - u_2(T')\|_{D(\Delta_{\mathcal{M}})}^2 + \|u_{1,t} - u_{2,t}\|_{L^2((t_0, T) \times \omega)}^2).$$

We complete Theorem 7.4 by the following remark: the geometrical assumption 7.2 is satisfied when \mathcal{M} is simply connected (hence in particular for the sphere \mathbb{S}^2):

PROPOSITION 7.5. ([A13]) *Additionally, assume that \mathcal{M} is simply connected. Consider any ω non empty open set of \mathcal{M} . Then Assumption 7.2 is fulfilled: there exists some smooth function ψ that satisfies (7.9).*

The proof of Theorem 7.4 is based on

- a global Carleman estimate for the heat equation (see [A13]),
- maximum principles, useful to study this nonlinear problem (see Theorem 7.3),
- and Riemannian geometry tools, since we are in the manifold setting.

The proof of Proposition 7.5 is based on

- a direct construction when \mathcal{M} is the sphere \mathbb{S}^2 , using the stereographic projection,

- the celebrated uniformisation theorem ([1, 346]) when \mathcal{M} is simply connected.

7.2.3. Relation to literature.

A similar problem is considered in [326], where stability estimates for the insolation function are obtained combining Carleman estimates with maximum principles, the main difference with the present paper being that the problem in [326] is stated and studied in the interval $(-1, 1)$ and with a degenerate diffusion coefficient.

Global Carleman estimates have proved their usefulness in the context of null controllability, unique continuation properties, we refer in particular to [234] for the seminal paper on the null controllability of the heat equation on compact manifolds, to [208, 166] for Carleman estimates in a general setting, to [273] for unique continuation properties for the heat equation on non compact manifolds, to [300, 301] for uniqueness results for manifolds with poles, to [87] for stabilization results of the wave equation on manifolds.

Concerning inverse problems, Isakov [215] provided many results for elliptic, hyperbolic and parabolic problems. Imanuvilov-Yamamoto [212] developed a general method to solve some standard inverse source problem for the linear heat equation, using global Carleman estimates. In the context of semilinear parabolic equations in bounded domains of \mathbb{R}^n , we can also mention in particular [308, 309], where uniqueness results are obtained under analyticity assumptions, [105], that combines also Carleman estimates with maximum principles to obtain stability estimates (for two coefficients but under rather strong assumptions on the time interval of observation).

Concerning global existence results and asymptotic behaviour, Diaz-Hetzer-Tello [122], Hetzer [203, 204] studied more general problems, involving memory terms and Volterra operators.

7.3. Parameter determination for Energy Balance Models with Memory (joint work with P. Cannarsa and M. Malfitana [A14])

In this part, we study two Energy Balance Models with Memory arising in climatology, which consist in a 1D degenerate nonlinear parabolic equation involving a memory term, and possibly a set-valued reaction term (of Sellers type and of Budyko type, in the usual terminology). We provide existence and regularity results, and obtain uniqueness and stability estimates that are useful for the determination of the insolation function in Sellers' model with memory.

7.3.1. The 1D Energy Balance Model with memory.

It was noted (see Bhattacharya-Ghil-Vulis [52]) that, in order to take into account the long response times that cryosphere exhibits (for instance, the expansion or retreat of huge continental ice sheets occurs with response times of thousands of years), it is useful to let the coalbedo function depend not only on u , but also on the history function, which can be represented by the integral term

$$H(t, x, u) := \int_{-\tau}^0 k(s, x)u(t + s, x)ds \quad \forall t > 0, x \in I,$$

where k is the memory kernel (and $\tau \sim 10^4$ years, in real problems). As in Roques-Checkroun-Cristofol-Soubeyrand-Ghil [309], we will assume a nonlinear response to memory in the form

$$f(H(t, x, u)).$$

Hence, here we are interested in the following Energy Balance Model with Memory (EBBM) problem, set in the space domain $I := (-1, 1)$:

$$(7.14) \quad \begin{cases} u_t - (\rho_0(1-x^2)u_x)_x = r(t)q(x)\beta(u) + f(H(t, x, u)) - R_e(u), \\ \rho_0(1-x^2)u_x(t, x) = 0, \quad t > 0, x \in \partial I \\ u(s, x) = u_0(s, x), \quad s \in [-\tau, 0]. \end{cases},$$

Concerning the function β , we will assume, as it is classical for such problems, that

- either β is positive and at least Lipschitz continuous (the classical assumption for Sellers type models),
- or β is positive, monotone and discontinuous (the classical assumption for Budyko type models).

7.3.2. Presentation of our main results.

We study, first, the 1D Sellers type problem with degenerate diffusion and memory effects. More precisely, we prove regularity results and use them to study the determination of the insolation function, obtaining

- a uniqueness result, under pointwise observation,
- a Lipschitz stability result, under localized observation,

in the spirit of the above mentioned references. Then, we address 1D Budyko type problems with degenerate diffusion and memory effects, for which we obtain precise existence results as in Diaz-Hetzer [120]. For this, we need to regularize the coalbedo and use the existence results obtained in the first part of the paper.

7.3.3. Mathematical assumptions for these climate models.

7.3.3.1. Budyko type models with memory.

We make the following assumptions:

- concerning R_a : we assume that
 - the insolation function and r are such that:

$$(7.15) \quad \begin{cases} q \in L^\infty(I), \\ r \in C^1(\mathbb{R}_+) \text{ and } r, r' \in L^\infty(\mathbb{R}_+); \end{cases}$$

- β is the classical Budyko type coalbedo function usually it is considered roughly constant for temperatures far enough from the ice-line, that is a circle of constant latitude that separates the polar ice caps from the lower ice-free latitudes; the classical Budyko type coalbedo is:

$$(7.16) \quad \beta(u) = \begin{cases} a_i, & u < \bar{u}, \\ [a_i, a_f], & u = \bar{u}, \\ a_f, & u > \bar{u}, \end{cases}$$

- where $a_i < a_f$ (and the threshold temperature $\bar{u} := -10^\circ$);
- H is the history function; it is assumed to be given by

$$(7.17) \quad H(t, x, u) = \int_{-\tau}^0 k(s, x) u(t+s, x) ds$$

where the kernel k is such that:

$$(7.18) \quad k \in C^1([-\tau, 0] \times [-1, 1]; \mathbb{R});$$

- f : the nonlinearity that describes the memory effects; we assume that $f : \mathbb{R} \rightarrow \mathbb{R}$ is C^1 and such that

$$(7.19) \quad \begin{cases} f, f' \in L^\infty(\mathbb{R}) \\ f, f' \text{ are } L\text{-Lipschitz;} \end{cases}$$

- concerning R_e : the classical Budyko type assumption is

$$(7.20) \quad R_e(t, x, u, H) = a + bu,$$

where a, b are constants;

- the initial condition: since we define H over a past temperature, the initial condition in such models has to be of the form

$$(7.21) \quad u(s, x) = u_0(s, x) \quad \forall s \in [-\tau, 0], \quad x \in I$$

for some $u_0(s, x)$ defined on $[-\tau, 0] \times I$, for which we will precise our assumptions in our different results.

Sometimes we will only add positivity assumptions on q and r ; these assumptions are natural with respect to the model, but only useful in the inverse problems results.

7.3.3.2. Sellers type models with memory.

The differences concern the assumptions on the coalbedo and on the emitted energy:

- β : in Sellers type models, we assume that

$$(7.22) \quad \beta \in C^2(\mathbb{R}), \quad \beta, \beta', \beta'' \in L^\infty(\mathbb{R})$$

(typically, β is C^2 and takes values between the lower value for the coalbedo a_i and higher value a_f (even if there is a sharp transition between these two values around the threshold temperature \bar{u})).

- R_e is assumed to follow a Stefan-Boltzmann type law (assuming that the Earth radiates as a black body):

$$(7.23) \quad R_e = \varepsilon(u)|u|^3,$$

where the function ε represents the emissivity; we assume that

$$(7.24) \quad \begin{cases} \varepsilon \in C^1(\mathbb{R}) \text{ and } \varepsilon, \varepsilon' \in L^\infty(\mathbb{R}), \\ \exists \varepsilon_1 > 0, \text{ s.t. } \forall u, \quad \varepsilon(u) \geq \varepsilon_1 > 0. \end{cases}$$

7.3.4. Main results for the Sellers type model.

As usually, we consider

$$V := \{w \in L^2(I) : w \in AC_{loc}(I), \sqrt{\rho}w_x \in L^2(I)\},$$

and

$$(7.25) \quad \begin{cases} D(A) := \{u \in V : \rho u_x \in H^1(I)\} \\ Au := (\rho u_x)_x \quad u \in D(A) \end{cases}$$

7.3.4.1. Global existence and uniqueness result for the Sellers model.

We obtain the following global existence result of the integrodifferential problem:

THEOREM 7.6. ([A14]) *Consider u_0 such that*

$$u_0 \in C([-\tau, 0]; V) \quad \text{and} \quad u_0(0) \in D(A) \cap L^\infty(I).$$

Then, for all $T > 0$, the problem (7.14) has a unique mild solution u on $[0, T]$.

(See [A14] for the definition of a mild solution.)

(Note that existence and uniqueness of a global regular solution to (7.14) without the memory term has been proved in [326], that the local existence of our model without the boundary degeneracy has been studied in [309], and that the global existence of a similar 2D-model with memory (hence on a manifold but without the boundary degeneracy), has been investigated in [120].)

7.3.4.2. *Inverse problem results: determination of the insolation function.*

Here we prove that the insolation function $q(x)$ can be determined in the whole space domain I by using only local information about the temperature.

To achieve this goal, we add the following extra assumptions, as in [309]: the very recent past temperatures are not taken into account in the history function:

$$(7.26) \quad \exists \delta > 0 \text{ s.t. } k(s, \cdot) \equiv 0 \quad \forall s \in [-\delta, 0]$$

where $\delta < \tau$.

Hence, we have the following situation: consider two insolation functions q and \tilde{q} , two initial conditions u_0 and \tilde{u}_0 , and the associated solutions: u satisfying (7.14) and \tilde{u} satisfying

$$(7.27) \quad \begin{cases} \tilde{u}_t - (\rho(x)\tilde{u}_x)_x = r(t)\tilde{q}(x)\beta(\tilde{u}) + f(\tilde{H}) - R_e(\tilde{u}), \\ \rho(x)\tilde{u}_x = 0, \quad x \in \partial I, \\ \tilde{u}(s, x) = \tilde{u}_0(s, x), \quad s \in [-\tau, 0], x \in I, \end{cases}$$

where we denote

$$\tilde{H} := H(t, x, \tilde{u}) = \int_{-\tau}^{-\delta} k(s, x)\tilde{u}(t+s, x) ds.$$

In the following, we state two inverse problems results, according to different assumptions on the control region.

7.3.4.3. *Pointwise observation and uniqueness result.*

Let us choose suitable regularity assumptions on the initial conditions and on the insolation functions, in order to have sufficient regularity on the time derivative of the associated solutions: we consider

- the set of admissible initial conditions: we consider

$$(7.28) \quad \mathcal{U}^{(pt)} = C^{1,2}([-\tau, 0] \times [-1, 1]),$$

- and the set of admissible coefficients: we consider

$$(7.29) \quad \mathcal{Q}^{(pt)} := \{q \text{ is Lipschitz-continuous and piecewise analytic on } I\},$$

where we recall the following

DEFINITION 7.7. *A continuous function ψ is called piecewise analytic if there exist $n \geq 1$ and an increasing sequence $(p_j)_{1 \leq j \leq n}$ such that $p_1 = -1$, $p_n = 1$, and*

$$\psi(x) = \sum_{j=1}^{n-1} \chi_{[p_j, p_{j+1})}(x) \varphi_j(x) \quad \forall x \in I,$$

where φ_j are analytic functions defined on the intervals $[p_j, p_{j+1})$ and $\chi_{[p_j, p_{j+1})}$ is the characteristic function of the interval $[p_j, p_{j+1})$ for $j = 1, \dots, n-1$.

Then we prove the following uniqueness result:

THEOREM 7.8. ([A14]) *Consider*

- two insolation functions $q, \tilde{q} \in \mathcal{Q}^{(pt)}$ (defined in (7.29))
- an initial condition $u_0 = \tilde{u}_0 \in \mathcal{U}^{(pt)}$ (defined in (7.28))

and let u be the solution of (7.14) and \tilde{u} the solution of (7.27).

Assume that

- the memory kernel satisfies (7.26),
- r and β are positive,
- there exists $x_0 \in I$ and $T > 0$ such that

$$(7.30) \quad \forall t \in (0, T), \quad \begin{cases} u(t, x_0) = \tilde{u}(t, x_0), \\ u_x(t, x_0) = \tilde{u}_x(t, x_0). \end{cases}$$

Then $q \equiv \tilde{q}$ on I .

This result means that the insolation function $q(x)$ is uniquely determined on I by any measurement of u and u_x at a single point x_0 during the time period $(0, T)$. Theorem 7.8 is a natural extension of [309] to the degenerate problem.

7.3.4.4. *Localized observation and stability result.*

Let us choose suitable regularity assumptions on the initial conditions and on the insolation functions, in order to have sufficient regularity on the time derivative of the associated solutions: we consider

- the set of admissible initial conditions: given $M > 0$, we consider $\mathcal{U}_M^{(loc)}$:

$$(7.31) \quad \mathcal{U}_M^{(loc)} := \{u_0 \in C([- \tau, 0]; V \cap L^\infty(-1, 1)), u_0(0) \in D(A), Au_0(0) \in L^\infty(I), \\ \sup_{t \in [- \tau, 0]} (\|u_0(t)\|_V + \|u_0(t)\|_{L^\infty}) + \|Au_0(0)\|_{L^\infty(I)} \leq M\},$$

- and the set of admissible coefficients: given $M' > 0$, we consider

$$(7.32) \quad \mathcal{Q}_{M'}^{(loc)} := \{q \in L^\infty(I) : \|q\|_{L^\infty(I)} \leq M'\}.$$

Now we are ready to state our Lipschitz stability result:

THEOREM 7.9. ([A14]) *Assume that*

- the memory kernel satisfies (7.26),
- r and β are positive.

Consider

- $0 < T' < \delta$,
- $t_0 \in [0, T']$, $T > T'$,
- $M, M' > 0$.

Then there exists $C(t_0, T', T, M, M') > 0$ such that, for all $u_0, \tilde{u}_0 \in \mathcal{U}_M^{(loc)}$ (defined in (7.31)), for all $q, \tilde{q} \in \mathcal{Q}_{M'}^{(loc)}$ (defined in (7.32)), the solution u of (7.14) and the solution \tilde{u} of (7.27) satisfy

$$(7.33) \quad \|q - \tilde{q}\|_{L^2(I)}^2 \leq C \left(\|u(T') - \tilde{u}(T')\|_{D(A)}^2 + \|u_t - \tilde{u}_t\|_{L^2((t_0, T) \times (a, b))}^2 + \|u_0 - \tilde{u}_0\|_{C([- \tau, 0]; V)}^2 \right).$$

Theorem 7.9 is a natural extension of [326].

7.3.5. Main result for the Budyko type model.

Now we treat the global existence of regular solutions for the Budyko model. In a classical way (see, e.g. Diaz [116]), we study the set valued problem

- first regularizing the coalbedo, hence transforming the Budyko type problem into a Sellers one, for which we have a (unique) regular solution,

- and then passing to the limit with respect to the regularization parameter.

Since β is the graph given in (7.16), the Budyko type problem has to be understood as the following differential inclusion problem:

$$(7.34) \quad \begin{cases} u_t - (\rho(x)u_x)_x \in r(t)q(x)\beta(u) - (a + bu) + f(H(u)), & t > 0, x \in I, \\ \rho(x)u_x = 0, & x = \pm 1, \\ u(s, x) = u_0(s, x), & s \in [-\tau, 0], x \in I. \end{cases}$$

7.3.5.1. *The notion of mild solutions for the Budyko model (7.34).*

Let us define a mild solution for this kind of problem.

DEFINITION 7.10. *Given $u_0 \in C([-\tau, 0]; V)$, a function*

$$u \in H^1(0, T; L^2(I)) \cap L^2(0, T; D(A)) \cap C([-\tau, T]; V)$$

is called a mild solution of (7.34) on $[-\tau, T]$ iff

- $u(s) = u_0(s)$ for all $s \in [-\tau, 0]$;
- there exists $g \in L^2([0, T]; L^2(I))$ such that
 - u satisfies

$$(7.35) \quad \forall t \in [0, T], \quad u(t) = e^{tA}u_0(0) + \int_0^t e^{(t-s)A}g(s) ds,$$

– and g satisfies the inclusion

$$g(t, x) \in r(t)q(x)\beta(u(t, x)) - (a + bu(t, x)) + f(H(t, x, u)) \quad \text{a.e. } (t, x) \in (0, T) \times I.$$

7.3.5.2. *Global existence for the Budyko model (7.34).*

THEOREM 7.11. ([A14]) *Assume that*

$$u_0 \in C([-\tau, 0], V) \quad \text{and} \quad u_0(0) \in D(A) \cap L^\infty(I).$$

Then (7.34) has a mild solution u , which is global in time (i.e. defined in $[0, +\infty)$) and mild on $[-\tau, T]$ for all $T > 0$).

Perspectives

A lot has been done in the last years concerning degenerate parabolic equations, but the motivating examples deserve a special attention. We describe several open problems in the following.

8.1. The Crocco boundary layer model

This study (see Chapter 3) was initially motivated by a question in which the ONERA (Office National d'Etude et de Recherches Aérospatiales, Toulouse) was interested. A two dimensional flow over a flat plate can be described by the Prandtl equations (Oleinik-Samokhin [290]). Using the so-called Crocco transformation, the system turns into a nonlinear degenerate parabolic equation stated in a 2-dimensional bounded domain, the Crocco equation [64, 63]:

$$w_t - \nu w^2 w_{yy} + U_\infty y w_x + \frac{U'_\infty}{U_\infty} (1-y) w_y + \frac{U'_\infty}{U_\infty} w = 0 \quad (x, y) \in \Omega, \quad t \in (0, T).$$

Here $\Omega = (0, L) \times (0, 1)$ where $(0, L)$ represents a part of the plate where the flow is laminar and $(0, 1)$ is the 'thickness' of the boundary layer in the Crocco variables. The positive constant ν is the viscosity of the fluid and U_∞ is the incident velocity.

In view of obtaining local controllability results for the Crocco equation, a first step is to study the null controllability properties of the linearized Crocco equation. The picture is by now quite clear for simplified models: [A1], Beauchard et al [31, 26, 29], however, it appears that the degeneracy of the Crocco equation is exactly critical (see [A5]), and hence creates here a real difficulty from the controllability viewpoint.

Since the original problem comes from applications and could have real applied impact, it remains important to bring answers.

- The linearized problem:
 - even with a weakly degenerate diffusion coefficient, the question of null controllability with the natural boundary conditions is open;
 - and of course, nothing is known with a violently degenerate diffusion coefficient, which is the case in the real model. Here, recent results and techniques of [28], where the null controllable data are characterized even when global null controllability fails for the Grushin model could bring some ideas.
- The nonlinear problem: we do not know if some null controllability results hold for the nonlinear model (combining usual techniques for the linearized problem with some classical nonlinear strategies: fixed point or return method of J.-M. Coron ? or with new tools ?)
- The stabilization problem of the flow around an unstable stationary solution: this was the original problem coming from aeronautics; classical methods are based on null controllability results, but here null controllability will hold only on a weak sense: on some part of the spatial domain, and for a subclass of initial conditions; does the equation offer a sufficiently adapted structure to be able to compute suitable feedback laws,

even if null controllability holds in a very weak sense here ? This would have to be compared from the numerical methods developed by J.-M. Buchot [64].

8.2. Control of combustion models around singular stationary solutions

Consider the following nonlinear combustion models [32, 267]:

$$u_t - \Delta u = \lambda f(u).$$

Equations like this appear in a number of applications in combustion theory, like the description of a ball of isothermal gas in gravitational equilibrium. Existence, uniqueness, blow-up, asymptotic behavior or stability for this problem have been actively studied, see for instance [60, 341]. Typical examples are $f(u) = e^u$ and $f(u) = (1 + u)^p$ for some $p \geq 1$. In both cases (see [267, 60]), there exist explicit singular stationary weak solutions $u_{\#}$ (associated to some values of the parameter $\lambda_{\#}$). The issue here is to obtain local controllability results around such singular stationary solutions. When linearizing the operator (around those solutions), one obtain

$$L_{\#} = -\Delta - \lambda_{\#} f'(u_{\#}) = -\Delta - \frac{\mu}{|X|^2},$$

for some explicit constant μ . From the works of Baras-Goldstein [19] and Vázquez-Zuazua [341], inverse-square singular potential are known to generate interesting phenomena: there exists a critical value of the parameter arising in the potential term that changes radically the well-posedness of the equation; when the parameter is sub-critical, the problem is well-posed (by standard theory) whereas, for a super-critical parameter, the problem is ill-posed (with instantaneous and complete blow-up of positive solutions).

In the linearized case, the question of null controllability has been solved by Vancostenoble-Zuazua [338], Ervedoza [128], and completed in [88, 53] and [A11]. However, many questions remain to be solved, in particular to obtain a result of local controllability for the original nonlinear equation; the result that has been proved for the linearized problem is a first step in the treatment of the nonlinear original models; however, the treatment of the nonlinear problem remains non trivial: even if the operator is formally invertible, the difficulty comes from a lack of suitable functional setting that would allow to develop a proof based on the standard fixed point arguments.

8.3. Inverse problems in genetics

Another interesting situation where degenerate parabolic models occur concerns population genetics. More precisely, we are interested in the gene frequency Fleming-Viot model that describes the genetic evolution of a population, see for example [72, 90].

As recalled in the introduction, it corresponds to some diffusion model in population dynamics in which each individual is of some *type* and the *type space* is given by a finite number d of elements. In this case the state space is the following d -dimensional simplex in \mathbb{R}^d :

$$K_d = \{(x_1, \dots, x_d) \in (R_+)^d \mid \sum_{i=1}^d x_i \leq 1\}.$$

Here x_i denotes the proportion of the population that is of type i . We consider the differential operator

$$Au(x) = \frac{1}{2} \sum_{i,j=1}^d (x_i(\delta_{ij} - x_j)) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^d b_i(x) \frac{\partial u}{\partial x_i}(x).$$

The evolution problem associated to this operator is a diffusion approximation of gene frequency models in population genetics. The first order term $b \cdot \nabla u$ corresponds to mutation, migration and selection phenomena. Therefore, in general, some of the functions b_i are not identically equal to zero. The issue here is to recover those coefficients from some measurement of the solution. The difficulty in studying this operator resides in the fact that the boundary of K_d is not smooth due to the presence of edges and corners, and A is a second-order elliptic operator that degenerates at the boundary.

In [A7], we studied such degenerate operators in space dimension 2, with similar degeneracy on the boundary, but only on smooth domains, and with only one eigenvalue equal to zero on the boundary. In the case of the Fleming-Viot model, the natural space domain is a simplex, and 0 is a multiple eigenvalue at the vertices of the simplex. This creates new difficulties for establishing null controllability results, which are strongly connected with uniqueness and stability results of the coefficients:

- obtain null controllability results for the Fleming-Viot model, taking into account the transport term, and above all the fact that the spatial domain is a simplex; this could be connected to [27, 26, 99], where the influence and importance of the transport term is put into light, and also to a very recent work [12], where a typical degenerate operator is studied on a rectangle;
- recover the transport coefficients from some measurement of the solution; once again, identify the influence of the transport term in the possible stability estimate.

Some recent results of well-posedness of Albanese-Mangino [5, 6] coupled with the tools developed in [A7] and new ideas ([29]) based on the Lebeau-Robbiano technique could be promising, but a lot remains to be done.

We mention also the connection with recent probabilistic works, in particular of S. Méléard et al [54], where an eco-evolutionary model is proposed and studied, in order to understand the dynamics of an adaptive trait and a neutral marker. Using the fact that the neutral marker mutates faster than the trait under selection, S. Méléard et al [54] connect the dynamics of such model to the ones given by a Fleming-Viot distribution between two trait substitutions. Once again, to be able to recover the coefficients of Fleming-Viot models could lead to the recovery of some of the coefficients in such models, and so to fit the parameters, in order to have accurate predictions of the future behavior.

8.4. Inverse problems in climatology

As recalled in Chapter 7, the coalbedo function represents the fraction of the incoming radiation flux which is absorbed by the surface. In *Sellers* models, β is assumed to be 'regular' (locally Lipschitz) whereas, in *Budyko* models, it is modeled as a discontinuous function of the temperature. Indeed a main change occurs in a neighborhood of a critical temperature for which ice become white. Therefore, in *Budyko* models, β takes the form of a maximal monotone graph which leads to a multi-valued functional \mathcal{R}_a .

In these models, the effect of the oceans is only considered in an implicit and empirical way in the spatial dependence of the coefficients. Hence the problematic is to recover some coefficients appearing in the equation from some measurement of the solution, and if possible with stability estimates, which would allow one to perform numerical predictions of the behavior in the future, with estimates of the error.

Several recent works go in that direction, in particular, in [309]. However, for mathematical reasons and to get closer to possible applications, it seems interesting to consider *Budyko* models. Existence results have been studied in particular in the works of Diaz [116] and Hetzer [199] in this setting, where a multi-valued functional appears in the equation, but to the best of our knowledge there is no result of recovery of the coefficient in this setting, and it would be interesting to study what can be done, in order to obtain mathematical results on this physical class of models.

Concerning problems involving memory terms, motivated by [52], recent papers ([256, 91, 357]) concerning null controllability properties could provide new techniques and ideas for inverse problems.

Part 2

Reaction-diffusion equations: travelling waves and asymptotic behaviour

Presentation

9.1. Motivations to study reaction-diffusion equations

9.1.1. Some problems where reaction-diffusion equations appear.

Reaction-diffusion equations appear in several models of combustion ([347]) and population dynamics ([280]). They allow to better understand propagation phenomena, thanks to some particular solutions (in particular the "travelling waves"). The basic equation

$$u_t - u_{xx} = f(u), \quad x \in \mathbb{R},$$

was introduced by Fisher [157] in population genetics, and by Zeldovich [354] in combustion. The fonction u represents a dominant gene in the first case and the temperature in the second one.

9.1.2. Types of nonlinearities and travelling waves.

The dynamics have then been extensively studied since the work of Kolmogorov, Petrovsky and Piskunov [222]. The nonlinearities satisfy

$$f \geq 0, \quad f(0) = 0 = f(1),$$

and are classified in several types, we will be in particular interested in the following ones: the nonlinearity f is said to be

- "of KPP type", if

$$f > 0 \text{ on } (0, 1), \quad f'(1) < 0 < f'(0), \quad f(u) \leq f'(0)u \text{ on } [0, 1],$$

- "of ignition type" if there exists $\theta \in (0, 1)$ such that

$$f = 0 \text{ on } [0, \theta], f > 0 \text{ on } (\theta, 1), f'(1) < 0.$$

Under these assumptions, it is interesting to investigate the existence of particular solutions: the "travelling waves"; they will be solutions of the type

$$u(t, x) = \phi_c(x + ct),$$

where ϕ_c solves

$$\begin{cases} -\phi_c'' + c\phi_c' = f(\phi_c), \\ \phi_c(-\infty) = 0, \phi_c(+\infty) = 1, \end{cases}$$

and they represent a solution of constant profile (ϕ_c) that propagates at the speed c to the left, and invades the state 0. Then,

- if f is of KPP type, [222] proves that this problem has a solution if and only if $c \geq c^* = 2\sqrt{f'(0)}$, and that the profile ϕ_c is increasing and unique up to translations;
- if f is of ignition type, [216] proves that there is a unique speed of propagation c .

9.1.3. Asymptotic behaviour.

Of course, such a function $u(t, x) = \phi_c(x + ct)$ is solution of the Cauchy problem

$$\begin{cases} u_t - u_{xx} = f(u), \\ u(0, c) = u_0(x) \end{cases}$$

with $u_0(x) = \phi_c(x)$. To describe the dynamics of the Cauchy problem, a first question is then to investigate the local stability of these particular solutions: what happens for initial conditions close to some ϕ_c ? and what happens for more general initial conditions? This has been studied in several cases:

- if f is of ignition type, [218, 156, 313] investigate the asymptotic behaviour for initial conditions exponentially decreasing at $-\infty$;
- if f is of KPP type, the situation is more complex, because of the half-line of admissible propagation speeds; if $c > c^*$, then the associated travelling wave is locally stable in some weighted space ([313]); and given an initial condition, the behaviour of the associated solution will depend on the behaviour of the initial condition at $-\infty$; indeed, [13] precises the asymptotic behaviour at $-\infty$ of the travelling waves:

– if $c > c^*$, then

$$\phi_c(x) = Ae^{\lambda_c x} + O(e^{(\lambda_c + \delta)x})$$

with $A > 0$, $\delta > 0$, and λ_c the smallest (and positive) root of

$$-\lambda^2 + c\lambda - f'(0) = 0,$$

which is the characteristic equation of the linearized (at $-\infty$) problem

$$-\phi'' + c\phi' - f'(0)\phi = 0;$$

and then of course the associated solution

$$\phi_c(x + ct) = Ae^{\lambda_c(x+ct)} + O(e^{(\lambda_c + \delta)(x+ct)}) = Ae^{\lambda_c ct} e^{\lambda_c x} + O(e^{(\lambda_c + \delta)x})$$

behaves as $e^{\lambda_c x}$ as $x \rightarrow -\infty$;

– if $c = c^*$, then

$$\phi_{c^*}(x) = (Ax + B)e^{\lambda_{c^*} x} + O(e^{(\lambda_{c^*} + \delta)x}),$$

with $A < 0$, $\delta > 0$, and $\lambda_{c^*} = \frac{c^*}{2}$ the double root of

$$-\lambda^2 + c^*\lambda - f'(0) = 0,$$

which is the characteristic equation of the linearized (at $-\infty$) problem

$$-\phi'' + c^*\phi' - f'(0)\phi = 0;$$

and then of course the associated solution has a similar behaviour as $x \rightarrow -\infty$.

Hence it is clear that the asymptotic behaviour as $x \rightarrow -\infty$ of the solution depends on the one of the initial condition. Uchiyama [332] and Bramson [59] proved that if the initial condition u_0 satisfies

$$u_0(x) \sim_{x \rightarrow -\infty} e^{\lambda_c x} \quad \text{and} \quad u_0(x) \rightarrow_{x \rightarrow +\infty} 1,$$

then the associated solution of the Cauchy problem converges to a travelling waves propagating at the speed c . The case where u_0 decays fast at $-\infty$ (for example for compactly supported initial data) has also been studied ([332, 59, 167]), and the associated solution converges to some shifted travelling wave:

$$u(t, x - m(t)) \rightarrow \phi_{c^*}(x) \quad \text{as } t \rightarrow +\infty,$$

with the shift m of the form $m(t) = c^*t + \frac{3}{2}c^* \ln t + o(1)$, and the convergence being rational (in $t^{-3/2}$, [167]).

9.1.4. Generalized travelling waves.

The notion of travelling wave has been generalized in several directions:

- in several space dimensions, for problems of the type

$$u_t - \Delta u = f(u), \quad x \in \mathbb{R}^n,$$

where here it is interesting to look for solutions of the form

$$u(t, x) = \phi_c(x \cdot e + ct)$$

propagating in the direction $-e$ at the speed c ;

- for inhomogeneous problems of the form

$$u_t - \Delta u + \alpha(y)u_x = f(u), \quad x \in \mathbb{R}, y \in \omega,$$

where here it is interesting to look for solutions of the form

$$u(t, x) = \phi_c(x + ct, y)$$

(see in particular [46, 253]);

- in periodic media, for problems of the form

$$u_t - u_{xx} = f(x, u), \quad x \in \mathbb{R},$$

with a source term f which is periodic in x , a problem appearing in particular in ecology, see [318]; in this case, it is interesting to look for solutions of the form

$$u(t, x) = \phi(t, x + ct)$$

where $\phi(t, y)$ is $1/c$ periodic in t ; the existence and the properties of such "pulsating waves" have been studied in particular in [350, 42];

- under general assumptions, in order to unify all these concepts, in particular [41];
- for reaction-diffusion systems, in particular [324, 96, 188].

(For more references, we refer to the thesis of M. Bages [17].)

9.2. Presentation of our main results

- Chapter 10 contains a new proof, more natural in some sense, of the existence of pulsating waves in a cylinder: we consider a new change of variables (new with respect to the existing literature), that transforms the problem into the existence of a periodic solution to a reaction-diffusion equation, and we solve it using the implicit function theorem in some suitable weighted spaces;
- Chapter 11 contains the study of a system appearing in solid combustion; we prove the existence of pulsating waves (with a half-line of admissible speeds, as in the KPP problem), the pulsating property coming from periodic boundary conditions imposed at one end of the cylinder; one of the interesting intermediate result is to study a reaction-diffusion equation with infinite boundary conditions;
- Chapter 12 contains results concerning
 - the asymptotic behaviour of pulsating waves (which completes general results of Hamel [182]): we extend to the periodic setting the well-known results concerning the asymptotic behaviour of travelling waves (recalled before), using a natural approach;

- then we study the problem of convergence of the solutions, and we describe the fate of an initial datum trapped between the translates of a supercritical wave
 - * in the classical setting (where pulsating waves are "just" travelling waves),
 - * in the periodic setting,
 - * and in the inhomogeneous setting;
- the general idea is to realize a change of variables allowed by the pulsating wave, and to investigate the solution of this new problem
- * linearizing it (using what is known for pulsating waves),
 - * and then studying the difference between the solution of the approximate problem (the linearized one), and the solution of the initial problem.

**Existence of pulsating waves of
advection-reaction-diffusion equations of ignition
type by a new method
(Joint work with Michael Bages ([B1]))**

10.1. Introduction: motivation and the related paper

10.1.1. Motivation.

Here we are interested in the following advection-reaction-diffusion equation:

$$(10.1) \quad \begin{cases} u_t - \Delta u + q(x, y) \cdot \nabla u = f(u), & (x, y) \in \Sigma, \\ \partial_\nu u = 0, & (x, y) \in \partial\Sigma, \end{cases}$$

stated on the infinite cylinder $\Sigma = \{(x, y) \in \mathbb{R} \times \omega\}$, whose cross section $\omega \subset \mathbb{R}^{N-1}$ is a bounded, smooth and connected open set. Equation (10.1) is classical in combustion theory: the thermodiffusive model (with Lewis number equal to 1) of propagation of a premixed flame (see [347]). The function u represents the normalized temperature of the reactant, and q is its velocity field. The function f is assumed to be sufficiently smooth, and satisfies $f(0) = 0 = f(1)$, which implies that 0 and 1 are equilibrium of (10.1).

The main issue is the understanding of the dynamics of (10.1). This question has been extensively studied since the pioneering works of Kolmogorov, Petrovskii, and Piskunov [222]. Let us recall some well-known results.

- In the nonadvective case $q = 0$: under several possible assumptions on the nonlinearity f (monostable, bistable, ignition type), there exist some special solutions: travelling waves, that are solutions of the type $u(t, x, y) = \phi(x + ct)$; ϕ is the profile of the wave u and c is its speed of propagation; their existence, uniqueness, stability and influence on the dynamics of the problem have been studied in [167, 216, 218, 222, 237, 313, 332, 344, 348, 350] (among others).
- When the velocity field q is a shear flow, that is $q(x, y) \nabla u = \alpha(y) u_x$, there exist travelling waves of the form $u(t, x, y) = \phi(x + ct, y)$; the previous results have been generalized in [45, 46, 48, 253, 305, 306].

When the velocity field q is periodic with respect to x , the notion of travelling waves has to be replaced by the notion of pulsating waves [318, 348, 349]:

DEFINITION 10.1. *A pulsating wave of (10.1) propagating at the speed $c \neq 0$ is a classical solution $u \in C^{1,2}(\overline{\Sigma})$ of (10.1), satisfying the following condition of propagation*

$$u\left(t + \frac{L}{c}, x, y\right) = u(t, x + L, y) \quad \forall t \in \mathbb{R}, (x, y) \in \overline{\Sigma},$$

and has the following limits as $x \rightarrow \pm\infty$:

$$\forall t \in \mathbb{R}, \quad u(t, -\infty, y) = 0, \quad u(t, +\infty, y) = 1, \quad \text{uniformly with respect to } y.$$

To sum up, a pulsating wave of (10.1) is a solution (c, u) of the problem

$$(10.2) \quad \begin{cases} u_t - \Delta u + q(x, y) \cdot \nabla u = f(u), & \forall t \in \mathbb{R}, (x, y) \in \Sigma, \\ \partial_\nu u = 0, & t \in \mathbb{R}, (x, y) \in \partial\Sigma, \\ u(t, -\infty, y) = 0, \quad u(t, +\infty, y) = 1, & t \in \mathbb{R}, y \in \omega, \\ u(t + \frac{L}{c}, x, y) = u(t, x + L, y) & \forall t \in \mathbb{R}, (x, y) \in \bar{\Sigma} \end{cases}$$

In this paper we concentrate on the case where

- the nonlinear source term is assumed to be of ignition type: $f \in C^2([0, 1], \mathbb{R})$, and there exists $\theta \in (0, 1)$ such that

$$(10.3) \quad f = 0 \text{ on } [0, \theta], \quad f > 0 \text{ on }]\theta, 1[, \quad f(1) = 0, \quad f'(1) < 0,$$

- the velocity field q is assumed to be of class C^2 on $\bar{\Sigma}$, and satisfies the assumptions: there exists some $L > 0$ such that

$$(10.4) \quad \begin{cases} \operatorname{div} q = 0 & \text{in } \bar{\Sigma}, \\ \forall (x, y) \in \bar{\Sigma}, \quad q(x + L, y) = q(x, y), \\ \int_{(0, L) \times \omega} q_1(x, y) dx dy = 0, \\ q \cdot \nu = 0 & \text{on } \partial\Sigma. \end{cases}$$

The first assumption means that the medium is incompressible; the velocity field represents a turbulent fluctuation with respect to a mean velocity field.

Then Xin [349], Berestycki-Hamel [40] proved the following existence result:

THEOREM 10.2. [40, 349] *Assume that the velocity field q satisfies (10.4), and that f is of ignition type ((10.3)). Then Problem (10.2) has at least a solution (c, u) .*

The proofs of Theorem 10.2 of Xin [349] and of Berestycki and Hamel [40] are based on the following change of variables:

$$u(t, x, y) = \phi(x + ct, x, y) = \phi(\tau, x, y);$$

using this change of variables, they prove that

- (c, u) is solution of (10.2) if and only if ϕ is solution of some degenerate elliptic equation,
- and that this degenerate equation has a solution, using regularization, regularity and continuation techniques.

Existence, uniqueness, qualitative properties, stability properties of pulsating waves have been recently extensively studied for several types of equations and systems, appearing in biology, population dynamics, combustion theory, see, among others, [40, 43, 182, 276, 281, 286], [B3]. The existence of particular solutions (travelling waves, pulsating waves, generalized pulsating waves) and their qualitative properties (in particular their asymptotic behavior) is always the starting (and main) point of the study of the dynamics of the Cauchy problem associated to (10.1).

Hence Theorem 10.2 is a major result when f is of ignition type; but it was also used to prove the existence of pulsating waves when f is of KPP type, approximating f by a sequence of ignition type functions and passing to the limit in the associated sequence of pulsating waves, see [40]. (Concerning this question, we refer the reader to [B3] for an elementary proof of the existence of pulsating waves with precised asymptotic properties when f is of KPP type, and its consequence on the dynamics of the associated Cauchy problem.)

Due to the importance of Theorem 10.2, we wanted to give another proof of it, closer to the spirit of the theory of existence of travelling waves, and avoiding

the difficulties encountered dealing with a degenerate elliptic equation. This is the goal of [B1].

10.1.2. The related paper [B1].

Using the change of variables:

$$u(t, x, y) = v(t, x + ct, y), \quad v(t, \xi, y) := u(t, \xi - ct, y),$$

the existence of a solution (c, u) of (10.2) is transformed into the existence of a periodic solution to a nonlinear parabolic equation with periodic coefficients. Then we prove the existence of a periodic solution of this parabolic equation using a continuation method, based on the implicit function theorem. In the next section, we give a precise description of our method.

10.2. Precise description of our method

We want to give a rather natural proof of the existence of a pulsating wave of (10.1). Assume that (c, u) is a pulsating wave solution of (10.2). Consider the frame moving with the speed c , and the function v defined by

$$(10.5) \quad v(t, \xi, y) = u(t, \xi - ct, y) = u(t, x, y).$$

Then the function v is $\frac{L}{c}$ -periodic in time, since

$$v\left(t + \frac{L}{c}, \xi, y\right) = u\left(t + \frac{L}{c}, \xi - ct - L, y\right) = u(t, \xi - ct, y) = v(t, \xi, y),$$

and satisfies the problem

$$(10.6) \quad \begin{cases} v_t - \Delta v + q(\xi - ct, y) \cdot \nabla v + cv_\xi = f(v) & \text{in } \Sigma, \\ \partial_\nu v = 0 & \text{on } \partial\Sigma, \\ v(t, -\infty, y) = 0, v(t, +\infty, y) = 1, & \forall t, \text{ uniformly in } y, \\ v\left(t + \frac{L}{c}, \xi, y\right) = v(t, \xi, y). \end{cases}$$

Of course, if (c, v) is solution of the above problem, then defining u by

$$u(t, x, y) := v(t, x + ct, y)$$

(c, u) is solution of (10.2). Hence the existence of pulsating waves of (10.2) is equivalent to the existence of solutions of (10.6) (hence $\frac{L}{c}$ -periodic in time).

Since the speed c is one of the unknown of the problem, we prefer to work with 1-periodic functions using the following change of variables:

$$(10.7) \quad w(\tau, \xi, y) = v\left(\frac{L}{c}\tau, \xi, y\right) = v(t, \xi, y).$$

Therefore, if v is solution of (10.6), then the function w is 1-periodic in time and satisfies the problem

$$(10.8) \quad \begin{cases} \frac{c}{L}w_\tau - \Delta w + q(\xi - L\tau, y) \cdot \nabla w + cw_\xi = f(w) & \text{in } \Sigma, \\ \partial_\nu w = 0 & \text{on } \partial\Sigma, \\ w(\tau, -\infty, y) = 0, w(\tau, +\infty, y) = 1, & \forall \tau, \text{ uniformly in } y, \\ w(\tau + 1, \xi, y) = w(\tau, \xi, y). \end{cases}$$

Reciprocally, if (c, w) solution of (10.8), then defining v by

$$v(t, \xi, y) = w\left(\frac{c}{L}t, \xi, y\right),$$

(c, v) is solution of (10.6).

Hence, (c, u) is solution of (10.2) if and only if (c, w) is solution of (10.8); hence we have to investigate the existence of 1-periodic solutions of a nonlinear parabolic equation. We are going to prove the following

THEOREM 10.3. *Assume that q satisfies (10.4) and that f is of ignition type. Then the problem (10.8) has a solution (c, w) .*

The main difficulty come from the advection term $q(\xi - L\tau, y) \cdot \nabla w$: without this term, and coming back to the variables (t, x, y) , the equation would be the classical reaction-diffusion one:

$$u_t - \Delta u = f(u),$$

which has a (unique up to translations) travelling wave of the one-dimensional type: $u(t, x, y) = \phi(x + ct)$ (see [216]). This motivates us to use a continuation method: we introduce the following parametrized problems: given $s \in [0, 1]$, consider

$$(Pbm)_s \begin{cases} \frac{c^{(s)}}{L} w_\tau^{(s)} - \Delta w^{(s)} + sq(\xi - L\tau, y) \cdot \nabla w^{(s)} + c^{(s)} w_\xi^{(s)} = f(w^{(s)}) & \text{in } \Sigma, \\ \partial_\nu w^{(s)} = 0 & \text{on } \partial\Sigma, \\ w^{(s)}(\tau, -\infty, y) = 0, \quad w^{(s)}(\tau, +\infty, y) = 1, \\ w^{(s)}(\tau + 1, \xi, y) = w^{(s)}(\tau, \xi, y). \end{cases}$$

To prove Theorem 10.3, we need to prove that $(Pbm)_1$ has a solution. In fact we prove that $(Pbm)_s$ has a solution for all $s \in [0, 1]$, in the following way:

- (1) for $s = 0$, problem $(Pbm)_0$ has a solution $(c^{(0)}, w^{(0)})$, that is given by the travelling wave of (10.1) when $q = 0$.
- (2) Applying in a suitable way the implicit function theorem, we prove that there exists some $\delta > 0$ such that problem $(Pbm)_s$ has a solution $(c^{(s)}, w^{(s)})$ $\forall s \in [0, \delta]$.
- (3) Then we consider $s_{max} = \sup \{ \tau \in [0, 1] \text{ such that problem } (Pbm)_s \text{ has a solution for all } s \in [0, \tau] \}$, and we prove that problem $(Pbm)_{s_{max}}$ has a solution. (Hence if $s_{max} = 1$, $(Pbm)_1$ has a solution, and Theorem 10.3 is proved.)
- (4) Finally, we prove that $s_{max} = 1$, reasoning by contradiction: if $s_{max} < 1$, the implicit function theorem helps us to prove that there exists some $\delta > 0$ such that problem $(Pbm)_s$ has a solution for all $s \in [s_{max}, s_{max} + \delta]$, which is absurd, comparing to the definition of s_{max} . Hence $s_{max} = 1$, and $(Pbm)_1$ has a solution by step 3, and Theorem 10.3 is proved.

Let us make some comments:

- Step 1 follows from the well-known theory of travelling waves of reaction diffusion equations of ignition type: indeed, $(c^{(0)}, w^{(0)})$ is solution of (10.8) if and only if the function $u^{(0)}$ defined by

$$u^{(0)}(t, x, y) = w^{(0)}\left(\frac{c^{(0)}}{L}t, x + c^{(0)}t, y\right)$$

is solution of the classical reaction-diffusion problem of ignition type

$$(10.10) \quad \begin{cases} u_t^{(0)} - \Delta u^{(0)} = f(u^{(0)}) & \text{in } \Sigma, \\ \partial_\nu u^{(0)} = 0 & \text{on } \partial\Sigma, \\ u^{(0)}(t, -\infty, y) = 0, u^{(0)}(t, +\infty, y) = 1, & \forall t, \text{ uniformly in } y, \\ u^{(0)}\left(t + \frac{L}{c^{(0)}}, x, y\right) = u^{(0)}(t, x + L, y); \end{cases}$$

but it is easy to see that this problem has a solution: indeed, consider the one dimensional reaction-diffusion problem of ignition type

$$(10.11) \quad \begin{cases} u_t - u_{xx} = f(u) & \text{in } \mathbb{R}, \\ u(t, -\infty) = 0, u(t, +\infty) = 1, & \forall t \end{cases}$$

it is well-known that there exists a unique speed of propagation c_0 and a profile ϕ_0 (unique up to translations) such that $u(t, x) = \phi_0(x + c_0 t)$ is solution of (10.11). Then $u^{(0)}(t, x, y) := \phi_0(x + c_0 t)$ is a travelling wave solution of (10.10), hence a pulsating solution propagating at the speed $c^{(0)} = c_0$.

- Steps 2 and 4 are identical: given a solution $(c^{(s)}, w^{(s)})$ of problem $(Pbm)_s$, we prove the existence of a solution of problem $(Pbm)_{s+\delta}$ for all δ small enough. The key point is to linearize the problem around $(c^{(s)}, w^{(s)})$, and to apply the implicit function theorem in suitable weighted spaces.
- Step 3 consists in studying a sequence $(c^{(s_n)}, w^{(s_n)})_n$ of solutions of problems $(Pbm)_{s_n}$, and let $s_n \rightarrow s_{max}$: first we prove a priori estimates on the speeds $c^{(s_n)}$, and on the functions $w^{(s_n)}$, then we prove that, up to a subsequence, the sequence $(c^{(s_n)}, w^{(s_n)})_n$ converges to a solution of problem $(Pbm)_{s_{max}}$ as $s_n \rightarrow s_{max}$, the main difficulty being to prove that the boundary conditions are satisfied.

Existence of pulsating waves in a monostable reaction-diffusion system in solid combustion (Joint work with Michael Bages ([B2]))

11.1. Introduction: motivation and related paper

11.1.1. Motivation.

This part is devoted to the following one-dimensional reaction-diffusion system

$$(11.1) \quad \begin{cases} T_t - T_{xx} = TY, \\ Y_t = -TY, \end{cases}$$

where T is the renormalized temperature, Y the renormalized concentration of the reactant and TY represents the reaction kinetics. This kind of model has been extensively used in solid combustion (see [248]) to analyze in particular the process of Self-propagating High-temperature Synthesis (SHS) (see, e.g., [264]).

Solid combustion differs from usual combustion by the absence of gaseous products involved in the reaction; the reaction takes place directly in the reactant. A classical way to get a model for this type of combustion is to take the limit $Le \rightarrow +\infty$ in the classical thermal-diffusive model ([44, 347]):

$$(11.2) \quad \begin{cases} T_t - T_{xx} = f(T)Y, & \forall t, x \in \mathbb{R} \\ Y_t - \frac{1}{Le}Y_{xx} = -f(T)Y & \forall t, x \in \mathbb{R}. \end{cases}$$

The Lewis number Le is the quotient of thermal diffusivity by molecular diffusivity, and the latter is equal to zero in solid combustion. The nonlinearity $f(T)Y$ represents the reaction rate and satisfies the Arrhenius law. To avoid the cold boundary problem (i.e. the reaction rate is not null in the unburnt zone), the nonlinearity f is often modified using an ignition temperature assumption ($f = 0$ on $[0, \theta]$) or a KPP assumption ($f(0) = 0$, $f > 0$ on $]0, +\infty[$). Here we consider the second case, and take more precisely $f(T) = T$.

When the density of reactant is uniform in the unburnt region, interesting (and global in time) solutions are travelling waves of the type $T(t, x) = \tilde{T}(x + ct)$, $Y(t, x) = \tilde{Y}(x + ct)$, where $(c, \tilde{T}, \tilde{Y})$ solve

$$(11.3) \quad \begin{cases} -\tilde{T}'' + c\tilde{T}' = \tilde{T}\tilde{Y}, \\ c\tilde{Y}' = -\tilde{T}\tilde{Y}, \\ \tilde{T}(-\infty) = 0, \quad \tilde{T}(+\infty) = 1, \\ \tilde{Y}(-\infty) = 1, \quad \tilde{Y}(+\infty) = 0. \end{cases}$$

Existence and uniqueness of travelling waves with an half line of admissible speeds have been proved by Logak [249] for general nonlinearities as $f(T)Y$ with f of KPP type. For the gaseous model (11.2) with ignition nonlinearity, the problem has been solved by Berestycki, Nicolaenko and Scheurer [47].

The goal of [B2] is to study the existence of particular solutions (pulsating waves) when the distribution of reactant is periodic in the fresh zone.

11.1.2. The related paper [B2].

In this paper, we consider a periodic distribution Y_p of reactant in the fresh zone, represented by the following boundary condition as $x \rightarrow -\infty$:

$$(11.4) \quad \forall t \in \mathbb{R}, \quad T(t, -\infty) = 0$$

and

$$(11.5) \quad \forall t \in \mathbb{R}, \quad \lim_{x \rightarrow -\infty} (Y(t, x) - Y_p(x)) = 0,$$

where $Y_p(x)$ is a 1-periodic and positive function. We impose then the following boundary conditions in the burnt zone (at $+\infty$):

$$(11.6) \quad \forall t \in \mathbb{R}, \quad Y(t, +\infty) = 0,$$

which means that the combustion is complete at $+\infty$. The periodic limit condition (11.5) leads to search for pulsating wave solutions ([40, 349]):

DEFINITION 11.1. (Pulsating waves of the system (11.1)) *A pulsating wave of the problem (11.1) propagating at the speed $c \neq 0$, and with a periodic distribution Y_p of reactant at $-\infty$, is a classical solution*

$$(T, Y) : \mathbb{R}^2 \rightarrow \mathbb{R}^2, \quad (t, x) \mapsto (T(t, x), Y(t, x))$$

of the problem (11.1) satisfying the pulsating relation

$$(11.7) \quad \forall t, x \in \mathbb{R}, \quad T(t + \frac{1}{c}, x) = T(t, x + 1), \quad Y(t + \frac{1}{c}, x) = Y(t, x + 1),$$

and the boundary conditions (11.4), (11.5) and (11.6).

It can be proved (see [B2]) that the pulsating relation (11.7) and the boundary conditions (11.4), (11.5) and (11.6) set the temperature at the end of combustion at the value

$$(11.8) \quad \forall t \in \mathbb{R}, \quad T(t, +\infty) = \langle Y_p \rangle := \int_0^1 Y_p(x) dx.$$

Thus pulsating waves of the problem (11.1) propagating at the speed $c \neq 0$ are classical solutions of

$$(11.9) \quad \begin{cases} T_t - T_{xx} = TY, & \forall t, x \in \mathbb{R} \\ Y_t = -TY, & \forall t, x \in \mathbb{R} \\ T(t, -\infty) = 0, & \lim_{x \rightarrow -\infty} (Y(t, x) - Y_p(x)) = 0, \quad \forall t \in \mathbb{R}, \\ T(t, +\infty) = \langle Y_p \rangle, & Y(t, +\infty) = 0, \quad \forall t \in \mathbb{R}, \\ T(t + \frac{1}{c}, x) = T(t, x + 1), & Y(t + \frac{1}{c}, x) = Y(t, x + 1), \forall t, x \in \mathbb{R}. \end{cases}$$

The purpose of this paper is to study the existence and the properties of pulsating waves of the problem (11.1). Moreover, with respect to the physical background of the problem, it is natural to study the existence of solutions (T, Y) such that Y is nonnegative (since it is the concentration of the reactant), and T is also nonnegative (since 0 is the normalized temperature in the unburnt zone). Our main result is stated in Theorem 11.2: there is some explicit critical speed $c^* > 0$ such that

- given $c \geq c^*$, there exists a pulsating wave of the problem (11.1) propagating at the speed c , and such that T and Y are positive everywhere,
- given $c < c^*$, there does not exist a pulsating wave of the problem (11.1) propagating at the speed c and such that T is nonnegative everywhere.

Hence, the problem we consider is of KPP type.

The main novelties of our work are the following:

- first, we consider a system, and not a scalar equation; there are very few results on systems (see subsection 11.2.1);

- next, we prove the existence of pulsating waves of problem (11.1) by a new method of reduction to a nonlinear parabolic scalar equation (see (11.14)); this reduction to a scalar equation allows us to precise the complete set of the admissible speeds of the pulsating waves of (11.1) as well as to get a variational formula on the minimal speed;
- finally, the parabolic scalar equation (11.14) is of new type : the nonlinearity has only one finite steady state (instead of two states, classically); that leads to consider unusual infinite boundary conditions.

11.2. Main result: existence of pulsating waves

Before stating our main result, we need to introduce some notations: consider the elliptic operator

$$(11.10) \quad L_\lambda v = v'' + 2\lambda v' + (\lambda^2 + Y_p(x))v,$$

acting on the set of $C^2(\mathbb{R})$ and 1-periodic functions; we denote k_λ its principal eigenvalue, and

$$(11.11) \quad c^* = \min_{\lambda > 0} \frac{k_\lambda}{\lambda};$$

then $c^* > 0$ (see [42]).

Now we are ready to state our main result on the existence of pulsating waves of the system (11.9):

THEOREM 11.2. ([B2]) *(i) If $c < c^*$, system (11.9) has no pulsating wave (T, Y) propagating at the speed c and such that $T \geq 0$ everywhere.*

(ii) If $c \geq c^$, there exists (T, Y) such that T and Y are positive everywhere, and (T, Y) is a pulsating wave of (11.9) propagating at the speed c .*

Hence there is a family of pulsating waves and (11.9) is of KPP type.

11.2.1. Relation to literature.

Existence of travelling waves for reaction-diffusion equations has been extensively studied since the pionnering work of Kolmogorov, Petrovsky and Piskunov [222]. The existence of travelling waves for the gaseous combustion model (11.2) has been considered first by Berestycki, Nicolaenko and Scheurer [47] for an ignition nonlinearity and then by Marion [255] for a KPP nonlinearity. The solid combustion model has been studied then by Logak and Loubeau [249, 248] for ignition and KPP nonlinearities. We refer to the monograph [344] for a review on travelling waves for parabolic systems.

The study of pulsating waves is more recent and has begun with the works of Xin ([349, 350]). The subject has been developed afterwards, especially by Berestycki and Hamel ([40, 42, 182]). These works deal with scalar equation and use a change of variable which leads a degenerate elliptic equation.

Nevertheless, there are very few results about the existence of pulsating waves for systems. Most part of the works use a bifurcation approach (see for example [189]) and the only nonperturbative result was, in our knowledge, the one of Constantin, Domelevo, Roquejoffre and Ryzhik [96] on the existence of pulsating waves for a one-dimensional model of dyphasic combustion. The limit of SHS in the high activation energy scaling has been studied in detail by matched asymptotic expansions (Matkowsky, Sivashinsky [262]); concerning the existence of pulsating waves, the only existing work to our knowledge is that of Monneau and Weiss [276] on a limit problem - the one-phase Stefan equation.

11.2.2. Strategy to prove Theorem 11.2.

- The first key argument in the proof of Theorem 11.2 is to work in the moving frame of the wave to transform the pulsating relation (11.7) into a periodicity property. More precisely, if (T, Y) is a pulsating wave of the problem (11.9), then the functions \tilde{T} and \tilde{Y} defined by

$$\tilde{T}(t, \xi) = T(t, \xi - ct), \quad \tilde{Y}(t, \xi) = Y(t, \xi - ct)$$

satisfy the problem

$$(11.12) \quad \begin{cases} \tilde{T}_t - \tilde{T}_{\xi\xi} + c\tilde{T}_\xi = \tilde{T}\tilde{Y}, & \forall t, \xi \in \mathbb{R} \\ \tilde{Y}_t + c\tilde{Y}_\xi = -\tilde{T}\tilde{Y}, & \forall t, \xi \in \mathbb{R} \\ \tilde{T}(t, -\infty) = 0, \quad \lim_{\xi \rightarrow -\infty} (\tilde{Y}(t, \xi) - Y_p(\xi - ct)) = 0, & \forall t \in \mathbb{R}, \\ \tilde{T}(t, +\infty) = \langle Y_p \rangle, \quad \tilde{Y}(t, +\infty) = 0, & \forall t \in \mathbb{R}, \\ \tilde{T}(t + \frac{1}{c}, \xi) = \tilde{T}(t, \xi), \quad \tilde{Y}(t + \frac{1}{c}, \xi) = \tilde{Y}(t, \xi), & \forall t, \xi \in \mathbb{R}. \end{cases}$$

In particular, \tilde{T} and \tilde{Y} are $1/c$ -periodic in time functions.

- Then we study (11.12), and we prove some qualitative properties of the pulsating waves of the system (11.9):

PROPOSITION 11.3. ([B2]) *Assume that $(c, \tilde{T}, \tilde{Y})$ is solution of (11.12) and such that $\tilde{T} \geq 0$ everywhere. Then $c > 0$, \tilde{T} and \tilde{Y} are positive everywhere, and \tilde{T} decays exponentially to 0 as $\xi \rightarrow -\infty$: there exists $C > 0$ and $\alpha > 0$ such that for all t and all $\xi \leq 0$,*

$$(11.13) \quad \tilde{T}(t, \xi) \leq Ce^{\alpha\xi}.$$

These properties are interesting in themselves, and useful in the following.

- Then we show that pulsating waves of the system (11.9), *if they exist*, generate pulsating waves of the following scalar nonlinear parabolic equation:

$$(11.14) \quad u_t - u_{xx} = Y_p(x)(1 - e^{-u}),$$

defined by the following

DEFINITION 11.4. (Pulsating waves of the scalar equation (11.14))
A pulsating wave $u(t, x)$ solution of (11.14), and propagating at the speed c , is a classical solution defined for all $t, x \in \mathbb{R}$, such that for some $c \neq 0$

$$\forall t, x \in \mathbb{R}, \quad u(t + \frac{1}{c}, x) = u(t, x + 1),$$

and that satisfies the limit conditions

$$\forall t \in \mathbb{R}, \quad u(t, -\infty) = 0, \quad \text{and} \quad u(t, +\infty) = +\infty,$$

hence the couple (c, u) is solution of the problem

$$(11.15) \quad \begin{cases} u_t - u_{xx} = Y_p(x)(1 - e^{-u}), \\ u(t, -\infty) = 0, \quad u(t, +\infty) = +\infty, \\ u(t + \frac{1}{c}, x) = u(t, x + 1). \end{cases}$$

Note that the nonlinearity $1 - e^{-u}$ is not classical. The classical nonlinearities have two finite zeros and the associated pulsating waves are heteroclinic orbits connecting these two stationary states. Since the nonlinearity $1 - e^{-u}$ is positive for $u > 0$ and has only one zero, it is natural to search solutions connecting 0 and $+\infty$.

The link between the pulsating waves of the system (11.9) and the pulsating waves of the scalar equation (11.15) comes from the following

PROPOSITION 11.5. ([B2]) *Let $(c, \tilde{T}, \tilde{Y})$ be a solution of (11.12). Moreover, assume that $\tilde{T} \geq 0$ everywhere. Consider the function \tilde{u} defined by*

$$(11.16) \quad \tilde{u}(t, \xi) = \frac{1}{c} \int_{-\infty}^{\xi} \tilde{T}\left(t + \frac{s - \xi}{c}, s\right) ds = \int_{-\infty}^0 \tilde{T}(t + \sigma, \xi + c\sigma) d\sigma,$$

and u defined by $u(t, x) = \tilde{u}(t, x + ct)$. Then u is a positive pulsating wave of (11.15) propagating at the speed c .

- Then we study the pulsating waves of this scalar parabolic equation:
 - first a classical property:

PROPOSITION 11.6. ([B2]) *There is no pulsating wave $u > 0$ of (11.15) propagating at the speed c if $c < c^*$.*

- next we study the asymptotic properties of the solutions of (11.15), and we obtain an asymptotic development: considering $\tilde{u}(t, \xi) := u(t, \xi - ct)$ (hence working in the moving frame, we prove that

PROPOSITION 11.7. ([B2]) *Assume that (c, \tilde{u}) satisfies: for all $\alpha > 0$, there exists $C_\alpha > 0$ such that*

$$\forall t \in \mathbb{R}, \forall \xi \geq 0, \quad |\tilde{u}(t, \xi)| \leq C_\alpha e^{\alpha\xi}.$$

Then \tilde{u} has the following expansion as $\xi \rightarrow +\infty$: there exists some $\alpha \in \mathbb{R}$ and $\beta > 0$, such that \tilde{u} satisfies

$$\tilde{u}(t, \xi) = \alpha + \frac{\langle Y_p \rangle}{c} \xi + Z_p(\xi - ct) + O(e^{-\beta\xi}), \quad \text{as } \xi \rightarrow +\infty,$$

where Z_p is a smooth and 1-periodic function (explicitly known).

- At this point, we are able to prove the following: given $c \geq c^*$, there exists a pulsating wave (c, u) solution of (11.15), and we precise its asymptotic behavior; we separate the critical case $c = c^*$ from the noncritical case $c > c^*$ since our results are slightly different.
- And finally, we prove that these pulsating waves of the scalar equation generate pulsating waves of the system (11.9) through some algebraic relations:

PROPOSITION 11.8. ([B2]) *Given $c \geq c^*$, consider \tilde{u} the solution constructed previously. Consider*

$$(11.17) \quad \tilde{T}(t, \xi) = \tilde{u}_t(t, \xi) + c\tilde{u}_\xi(t, \xi) \quad \text{and} \quad \tilde{Y}(t, \xi) = Y_p(\xi - ct)e^{-\tilde{u}(t, \xi)}.$$

Then $(c, \tilde{T}, \tilde{Y})$ is a solution of (11.12), hence (T, Y) is a pulsating solution of (11.9) propagating at the speed c , for which $T > 0$ everywhere.

In all these steps, proofs are based on the weak and strong parabolic maximum principles for scalar equations.

11.2.3. Comments and open questions.

11.2.3.1. *Uniqueness and monotonicity of the pulsating waves.* Travelling and pulsating waves of scalar equations are in general unique (up to shifts) and have monotonicity properties: travelling waves often exhibit uniqueness and monotonicity with respect to the space variable, whereas pulsating waves are unique and monotonous with respect to the time.

It would be interesting to study the questions of uniqueness and monotonicity for our system (11.1). Note that the question is still open for the scalar equation (11.14): the results of Hamel and Roques [185] let us think that these pulsating waves $u(t, x)$ are unique up to shifts in time (and monotonous with respect to t), but their results do not apply readily to our problem.

11.2.3.2. *Generalizations.* A last and delicate question is to study more realistic reaction terms of the type $f(T)Y$, as studied, e.g., in Logak and Loubeau [248] concerning travelling waves. Our approach seems to be specific to the linear term $f(T) = T$. We do not know if such a reduction to a scalar equation is possible in the general case.

**How travelling waves attract the solutions of
KPP-type equations**
(Joint works with M. Bages and J.-M. Roquejoffre
([B3])
and with J.-M. Roquejoffre ([B4]))

12.1. Introduction: motivation and related papers

12.1.1. Motivation.

We consider a general reaction-diffusion equation of the KPP type, posed on an infinite cylinder. Such a model will have a family of pulsating waves of constant speed, larger than a critical speed c_* . The family of all supercritical waves attract a large class of initial data, and we try to understand how. We describe the fate of an initial datum trapped between two supercritical waves of the same velocity: the solution will converge to a whole set of translates of the same wave, and we identify the convergence dynamics as that of an effective drift, around which an effective diffusion process occurs. In several nontrivial particular cases, we are able to describe the dynamics by an effective equation.

12.1.2. The related papers [B3, B4].

- [B3]: We study the asymptotic behaviour of the solutions in the following situations:
 - first we study the most basic 1D model;
 - next, thanks to the informations given by this basic model, we will consider the problem in the full generality (obtaining, of course, little less precise results);
 - at last, we will consider a periodic 1D case, for which we will be able to give precise answers.
- [B4]: we consider here the thermo-diffusive model for flame propagation; (however, the presence of a nonconstant shear flow brings new difficulties).

12.2. The basic one dimensional model ([B3])

What motivated this study is the following, seemingly innocent question. Consider the most basic 1D model, namely

$$(12.1) \quad \begin{cases} u_t - u_{xx} = f(u) & (x \in \mathbb{R}) \\ \lim_{x \rightarrow -\infty} u(t, x) = 0, & \lim_{x \rightarrow +\infty} u(t, x) = 1, \end{cases}$$

with f concave, $f(0) = f(1) = 0$. Then (see, for instance, [222]) (12.1) has a family of travelling waves; that is, for every $c \geq c_* := 2\sqrt{f'(0)}$, there is a unique (up to translation in x) ϕ_c , solving

$$(12.2) \quad \begin{cases} c\phi' - \phi'' = f(\phi), & (x \in \mathbb{R}) \\ \lim_{x \rightarrow -\infty} \phi(x) = 0, & \lim_{x \rightarrow +\infty} \phi(x) = 1. \end{cases}$$

In other words $\phi_c(x + ct)$ solves (12.1). We ask the question of the stability, under large perturbations, of the *supercritical* waves (those whose speed is $> c_*$). Much is known in this direction; let us extract the two following results.

Theorem 0.1 (*Uchiyama [332], Bramson [59]*) *Let $u_0(x)$ be a Cauchy datum for (12.1) such that there is $c > c_*$ and $r > 0$ for which we have*

$$u_0(x) = \phi_c(x)(1 + O(e^{rx})) \text{ as } x \rightarrow -\infty.$$

Assume moreover that $\limsup_{x \rightarrow +\infty} u_0(x) = 1$. Then there is some $\omega > 0$ such that, we have, as $t \rightarrow +\infty$:

$$u(t, x) = \phi_c(x + ct) + O(e^{-\omega t}).$$

A much related, and more recent theorem is

Theorem 0.2 (*Berestycki-Hamel [41]*) *Let $u(t, x)$ be a time-global (that is, defined for $t \in \mathbb{R}$) solution to (12.1) such that there is $c > c_*$ and $M > 0$ for which we have*

$$\phi_c(x + ct - M) \leq u(t, x) \leq \phi_c(x + ct + M).$$

Then there is $m \in [-M, M]$ such that: $u(t, x) = \phi_c(x + ct + m)$.

In fact, the assumption that $u(t, x)$ is trapped between two translates of a wave can be considerably weakened under our assumptions on f , see [183].

At first sight, Theorem 0.2 seems to imply the convergence to a wave for any solution starting from a datum trapped between two waves. Not quite, though: Theorem 0.1 only allows perturbations which decay faster than the wave. And this seemingly insignificant gap between the two results signals in fact that initial data which are merely sandwiched between two waves - and which do not select an asymptotic wave at $-\infty$ have a wilder behavior. Here is what we can prove.

THEOREM 12.1. (**[B3]**) *Let $u_0(x)$ be a Cauchy datum for (12.1). Assume the existence of $c > c_*$ and $M > 0$ such that $\phi_c(x - M) \leq u_0(x) \leq \phi_c(x + M)$. Denote $r_-(c)$ the smallest characteristic exponent at $-\infty$ of (12.2), i.e*

$$r_-(c) = \frac{c - \sqrt{c^2 - 4f'(0)}}{2}.$$

Then there is some initial condition m_0 (given explicitly in the proof), bounded between $-M$ and M , and such that, considering the solution $s(t, \xi)$ of

$$(12.3) \quad s_t - s_{\xi\xi} + \sqrt{c^2 - c_*^2} s_{\xi} = 0, \quad (t > 0, \xi \in \mathbb{R}), \quad s(0, \xi) = e^{r_-(c)m_0(\xi)}$$

and setting

$$m^{app}(t, \xi) = \frac{1}{r_-(c)} \ln s(t, \xi) = \frac{1}{r_-(c)} \ln \left(\frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} e^{(\xi - \sqrt{c^2 - c_*^2}t - y)/4t} e^{r_-(c)m_0(y)} dy \right),$$

we have, as $t \rightarrow +\infty$,

$$\sup_{x \in \mathbb{R}} |u(t, x) - \phi_c(x + ct + m^{app}(t, x + ct))| = O\left(\frac{1}{\sqrt{t}}\right).$$

In subsection 12.5.6, we give examples and applications of Theorem 12.1, that allow us to extend Theorem 0.1 above in several directions Theorem 0.1:

- the case where the initial shift m_0 is periodic, hence when the initial condition “oscillates” between two translates of the travelling wave;
- the case where the initial shift m_0 converges to some constant at $-\infty$: then we prove the convergence of the solution u to a travelling wave, with a precise convergence rate;

- a case where the solution does not converge to any travelling wave; indeed, it is known - see [94], [342] - that very simple equations like (12.3) can exhibit complex behaviours; in particular, the ω -limit set (in the sense of uniform convergence on every compact set) - can be a whole interval; a related phenomenon for *two-dimensional* bistable equations was noticed in [307], the mechanism is somehow different; see [295] for related results in nonlinear supercritical heat equations.

12.3. A general model ([B3])

Consider the general model

$$(12.4) \quad \begin{cases} u_t - \operatorname{div} (A(x, y) \nabla u) + B(x, y) \cdot \nabla u = f(x, y, u), & (x, y) \in \mathbb{R} \times \mathbb{T}^{N-1}, \\ \lim_{x \rightarrow -\infty} u(t, x, y) = 0, & \lim_{x \rightarrow +\infty} u(t, x, y) = 1. \end{cases}$$

We assume that A , B and f satisfy the following additional assumptions:

- A is symmetric, uniformly positive, 1-periodic with respect to x , and $C^3(\mathbb{R} \times \mathbb{T}^{N-1})$,
- B is 1-periodic with respect to x , $C^{1+\delta}(\mathbb{R} \times \mathbb{T}^{N-1})$, and moreover $\operatorname{div} B = 0$ and $\int_{(0,1) \times \mathbb{T}^{N-1}} B_1 = 0$ (where B_1 is the first component of B),
- f is 1-periodic with respect to x .

Let us first state, under the form of a theorem, the basic result that we shall need: **Theorem 0.3** (Berestycki-Hamel [40], Hamel-Roques [185]) *There is c_* such that (12.4) has no pulsating wave solution if $c < c_*$, and a unique - up to translation in t - pulsating wave solution if $c \geq c_*$. Moreover, for a pulsating wave ϕ_c we have $\partial_t \phi_c > 0$.*

Existence and monotonicity come from [40], uniqueness comes from [182] and [185]. The theorem corresponding to Theorem 12.1 is:

THEOREM 12.2. ([B3]) *Let $u_0(x, y)$ be a Cauchy datum for (12.4). Assume the existence of $c > c_*$ and $M > 0$ such that $\phi_c(-M, x, y) \leq u_0(x, y) \leq \phi_c(M, x, y)$. Then there exists a smooth function $m(t, x, y)$, solution a nonlinear parabolic equation with periodic coefficients, such that $\lim_{t \rightarrow +\infty} \|(m_t, \nabla m, D^2 m)(t, \cdot, \cdot)\|_\infty = 0$, and such that*

$$\sup_{(x,y) \in \mathbb{R} \times \mathbb{T}^{N-1}} |u(t, x, y) - \phi_c(t + m(t, x, y), x, y)| \rightarrow 0 \quad \text{as } t \rightarrow +\infty.$$

As we will see, the shift $m(t, x, y)$ will satisfy - up to a Hopf-Cole transform - a linear diffusion equation (with periodic coefficients). In order to have more insight into its dynamics, we will interpret it in the light of general heat kernel estimates for operators with periodic coefficients, that were proved by Norris [287] at this level of generality. We will see that the underlying processes at work are

- an effective drift $V_*(c)$ which can be computed explicitly (and which is, fortunately, consistent with the 1D expression $\sqrt{c^2 - c_*^2}$!),
- an effective diffusion process around the drift.

Apart from Theorem 12.2, the only multi-dimensional stability results are those of [253] - there we have $A = I$ and $B(x, y) = (\alpha(y), 0)$ - and [185] - general A and B , which prove the asymptotic stability of all the waves under fastly decaying perturbations.

12.3.1. More precise results concerning the general model (12.4) in 1D ([B3]).

We are able to push Theorem 12.2 further for the 1D version of Problem (12.4). It reads - for simplicity, the matrix $A(x)$ has been set to identity, but our result

would undoubtedly hold without this assumption:

$$(12.5) \quad \begin{cases} u_t - u_{xx} = f(x, u), & (x \in \mathbb{R}) \\ \lim_{x \rightarrow -\infty} u(t, x) = 0, & \lim_{x \rightarrow +\infty} u(t, x) = 1. \end{cases}$$

Of course the pulsating wave solutions of (12.4) specialize to (12.5); Theorem 0.3 applies and we denote by c_* the minimal speed. The additional information of this section is an optimal convergence rate of a solution to (12.5), initially trapped between two waves, to the shifted wave.

THEOREM 12.3. ([B3]) *Let $u_0(x, y)$ be a Cauchy datum for (12.5). Assume the existence of $c > c_*$ and $M > 0$ such that $\phi_c(-M, x, y) \leq u_0(x, y) \leq \phi_c(M, x, y)$. Then the smooth function $m(t, x, y)$ solution of the nonlinear parabolic equation with periodic coefficients studied in Theorem 12.2 satisfies $\lim_{t \rightarrow +\infty} \|(m_t, m_x, m_{xx})(t, \cdot)\|_\infty = O(\frac{1}{\sqrt{t}})$, and*

$$\sup_{x \in \mathbb{R}} |u(t, x) - \phi_c(t + m(t, x), x)| = O\left(\frac{1}{\sqrt{t}}\right) \quad \text{as } t \rightarrow +\infty.$$

The proof of this result is long and nontrivial; the best part of it consists in retrieving the precise expression of the heat kernel. But it is worth the effort, because it really gives an insight into the heat kernel, and a precise description of the mechanisms at work. It is therefore of independent interest. Needless to say, the effective drift is present here, and $V_*(c) = \sqrt{c^2 - c_*^2}$ when $f(x, u)$ does not depend on x .

12.4. The rate of attraction of super-critical waves in a Fisher-KPP type model with shear flow ([B4])

Now we estimate how fast the solutions of:

$$(12.6) \quad \begin{cases} u_t - \Delta u + \alpha(y)u_x = f(u) & ((x, y) \in \mathbb{R} \times \mathbb{T}^{N-1}) \\ \lim_{x \rightarrow -\infty} u(t, x, y) = 0, & \lim_{x \rightarrow +\infty} u(t, x, y) = 1, \end{cases}$$

with suitable initial data, will converge to travelling wave profiles. Here α is a sufficiently smooth function (C^∞ to avoid technical difficulties), and \mathbb{T}^{N-1} denotes the $(N-1)$ -dimensional torus. The function f will always be supposed to be smooth enough, and positive on $(0, 1)$; moreover it will be assumed to be concave in u , and

$$f(0) = f(1) = 0, \quad f'(0) > 0, \quad f'(1) < 0.$$

This model is sometimes known, in the mathematical theory of flame propagation, as the 'thermo-diffusive model'. It is indeed the simplest model with nontrivial flow that may be derived from the reacting fluid dynamics equations; the (shear flow) field $\vec{V}(x, y) = (0, \alpha(y))$ is imposed and only the chemical and heat transfer processes are conserved. The model was the object of numerical studies - see for instance [38], [254] - as a relevant preliminary account of the wrinkling of a flame front. See also [44] for its mathematical justification, and [16] for large shear asymptotics. The model remains an important tool to understand the interplay between a flow field and reaction-diffusion processes, see for instance estimates on the burning rate (see [219]) or existence theorems for systems (see [186]).

Travelling waves propagating at the speed c are solutions of the form $\phi(x+ct, y)$, where the function $\phi(\xi, y)$ solves

$$(12.7) \quad \begin{cases} -\Delta \phi + (c + \alpha(y))\phi_\xi = f(\phi) & ((\xi, y) \in \mathbb{R} \times \mathbb{T}^{N-1}) \\ \lim_{\xi \rightarrow -\infty} \phi(\xi, y) = 0, & \lim_{\xi \rightarrow +\infty} \phi(\xi, y) = 1. \end{cases}$$

Their existence and qualitative properties are given by the following

THEOREM 12.4. (*Berestycki-Nirenberg [48]*) *There is c_* such that (12.7) has no solution if $c < c_*$, and a unique - up to translation in ξ - solution if $c \geq c_*$. Moreover we have $\partial_\xi \phi_c > 0$.*

Some stability results have been proved before by Mallordy-Roquejoffre [253]:

THEOREM 12.5. (*Mallordy-Roquejoffre [253]*) *Consider $c > c_*$ and ϕ_c a solution of (12.7). Choose an initial datum u_0 satisfying*

$$\frac{u_0(x, y)}{\phi_c(x, y)} = 1 + O(e^{rx}) \quad \text{as } x \rightarrow -\infty, \quad u_0(+\infty, y) = 1$$

for some $r > 0$. Consider the solution u of the Cauchy problem

$$(12.8) \quad \begin{cases} u_t - \Delta u + \alpha(y)u_x = f(u) & ((x, y) \in \mathbb{R} \times \mathbb{T}^{N-1}), \\ u(0, x, y) = u_0(x, y). \end{cases}$$

Then $u(t, x, y) = \phi_c(x+ct, y) + O(e^{-\omega t})$ as $t \rightarrow +\infty$, uniformly in $(x, y) \in \mathbb{R} \times \mathbb{T}^{N-1}$.

The goal of this work is to study what happens when the initial datum u_0 is trapped between two supercritical waves of the same velocity, a slightly more general assumption than that of Theorem 12.5. We are going to prove the following (rather drastic) change in asymptotic behaviour:

THEOREM 12.6. (**[B4]**) *Let $u_0(x, y)$ be a Cauchy datum for (12.8). Assume the existence of $c > c_*$ and $M > 0$ such that*

$$\forall (x, y) \in \mathbb{R} \times \mathbb{T}^{N-1}, \quad \phi_c(x - M, y) \leq u_0(x, y) \leq \phi_c(x + M, y).$$

Define the initial shift $m_0(x, y)$ as

$$\forall (x, y) \in \mathbb{R} \times \mathbb{T}^{N-1}, \quad \phi_c(x + m_0(x, y), y) = u_0(x, y).$$

Then there exist $D_*(c) \geq 1$ and $V_*(c) > 0$ such that: if $s^{pp}(t, \xi)$ is the solution of

$$(12.9) \quad s_t^{pp} - D_*(c)s_{\xi\xi}^{pp} + V_*(c)s_\xi^{pp} = 0, \quad s^{pp}(0, \xi, y) = \int_{y'} e^{r_-(c)m_0(\xi, y')} \psi_{r_c}(y')^2 dy',$$

where $r_-(c) > 0$ and ψ_{r_c} is the (positive) principal eigenfunction associated to some explicit elliptic operator, and if

$$m^{pp}(t, \xi) := \frac{1}{r_-(c)} \ln s^{pp}(t, \xi),$$

then we have

$$\sup_{(x, y) \in \mathbb{R} \times \mathbb{T}^{N-1}} |u(t, x, y) - \phi_c(x + ct + m^{pp}(t, x + ct), y)| = O\left(\frac{1}{t^{1/4}}\right).$$

It is known - see [94], [342] - that very simple equations like (12.9) can exhibit complex behaviours. In particular, the ω -limit set (in the sense of uniform convergence on every compact - can be a whole interval.

Our result extends and completes in several directions some of our earlier results. Before explaining this, let us add some comments.

- The advection-diffusion (12.9) can be solved explicitly. Hence we may find reasonably sharp conditions ensuring the convergence of u to some translate of the travelling wave ϕ_c .
- We have $D_*(c) > 1$ as soon as α is nonconstant; this is a manifestation of the well-known 'convection-enhanced'-diffusion - see [140].
- When $\alpha = 0$, $V_*(c) = \sqrt{c^2 - c_*^2}$; and the proof breaks down when $c = c_*$ (the reason being that we have $V_*(c_*) = 0$).

- Our result completes that of [185], which proves the asymptotic stability of all the waves under fatly decaying perturbations, and [B3] described in the previous chapter, where we considered general models but without being able to provide an effective one dimensional equation, nor an estimate of the rate of attraction of the family of translates of the pulsating waves, and simple 1D models without shear flow. The goal of the present work is to prove such results in the context of the thermo-diffusive model.

The main step of Theorem 12.6 will be the computation of the effective dynamics (12.9), by a Fourier argument combined with some classical functional analysis.

12.5. The basic 1D model: ideas of the proof of Theorem 12.1

12.5.1. The travelling wave of speed c .

We consider the classical change of variables $(t, x) \mapsto (t, \xi = x + ct)$: if $u(t, x)$ is a solution of (12.1), the function \tilde{u} defined by

$$\tilde{u}(t, \xi) := u(t, \xi - ct)$$

satisfies $u(t, x) = \tilde{u}(t, x + ct)$, and thus is solution of

$$(12.10) \quad \begin{cases} \tilde{t}u_t + c\tilde{u}_\xi - \tilde{u}_{\xi\xi} = f(\tilde{u}), & t > 0, \xi \in \mathbb{R}, \\ \tilde{u}(0, \xi) = u(0, \xi) = u_0(\xi). \end{cases}$$

In the whole section we assume that $c > c_* = 2\sqrt{f'(0)}$. Let $r_\pm(c)$ be the characteristic exponents at $-\infty$ of (12.2), i.e.

$$r_\pm(c) = \frac{c \pm \sqrt{c^2 - 4f'(0)}}{2}.$$

We recall that, for a given wave ϕ_c , there is $q > 0$ and $\delta > 0$ such that, as $\xi \rightarrow -\infty$,

$$\begin{aligned} \phi_c(\xi) &= qe^{r_-(c)\xi} + O(e^{(r_-(c)+\delta)\xi}), \quad \phi_c'(\xi) = qr_-(c)e^{r_-(c)\xi} + O(e^{(r_-(c)+\delta)\xi}), \\ \phi_c''(\xi) &= qr_-^2(c)e^{r_-(c)\xi} + O(e^{(r_-(c)+\delta)\xi}). \end{aligned}$$

12.5.2. Exact local shift.

Given a sufficiently smooth function $m : (0, +\infty) \times \mathbb{R} \rightarrow \mathbb{R}$, consider

$$T^{(m)}\phi_c(t, \xi) := \phi_c(\xi + m(t, \xi)).$$

Of course when m is identically zero, we have $T^{(0)}\phi_c(t, \xi) := \phi_c(\xi)$. Since ϕ_c is strictly increasing, we can consider the exact shift

$$m^*(t, \xi) := \phi_c^{-1}(\tilde{u}(t, \xi)) - \xi;$$

some computations show that m^* satisfies the following nonlinear parabolic equation

$$\begin{cases} (m_t^* - m_{\xi\xi}^* - c(m_\xi^* + m_{\xi^2}^*))\phi_c'(\xi + m^*(t, \xi)) + (2m_\xi^* + m_{\xi^2}^*)f(\phi_c(\xi + m^*(t, \xi))) = 0, \\ m^*(0, \xi) = \phi_c^{-1}(u_0(\xi)) - \xi =: m_0^*(\xi). \end{cases}$$

To study the solution of this problem seems difficult, hence our strategy will be:

- to find a parabolic problem that will be: as close as possible of the previous one, but simpler; this will allow us to study the properties of its solution m , that we will call the ‘‘approximate shift’’;
- then to consider the difference $\tilde{u} - T^{(m)}\phi_c$, and to estimate its asymptotic behavior as $t \rightarrow +\infty$.

Even if we cannot say many things on the exact shift $m^*(t, \xi) = \phi_c^{-1}(\tilde{u}(t, \xi)) - \xi$, we can see that it has the following property: for all $t > 0$, $m^*(t, \cdot)$ is of class $C^1(\mathbb{R})$ and is bounded in the natural C^1 -norm. This comes from classical parabolic PDEs arguments, and will be useful later.

12.5.3. Approximate shift.

Linearizing the equation at $-\infty$, studying the behavior of the coefficients $\phi'_c(\xi + m(t, \xi))$ and $f(\phi_c(\xi + m(t, \xi)))$ as $\xi \rightarrow -\infty$, and using that

$$-c + 2\frac{f'(0)}{r_-(c)} = \sqrt{c^2 - c_*^2} \quad \text{and} \quad -c + \frac{f'(0)}{r_-(c)} = -r_-(c),$$

it appears that it is natural to consider the solution m of

$$m_t - m_{\xi\xi} + \sqrt{c^2 - c_*^2}m_\xi - r_-(c)m_\xi^2 = 0,$$

12.5.4. The properties of the approximate shift.

There is an easy expression for m : its Hopf-Cole transform $s(t, \xi) = e^{r_-(c)m(t, \xi)}$ solves

$$\begin{cases} s_t - s_{\xi\xi} + \sqrt{c^2 - c_*^2}s_\xi = 0, & t > 0, \xi \in \mathbb{R}, \\ s(0, \xi) = e^{r_-(c)m_0^*(\xi)} =: s_0(\xi), & \xi \in \mathbb{R}, \end{cases}$$

and thus the function S defined by

$$S(t, \xi) := s(t, \xi + \sqrt{c^2 - c_*^2}t)$$

is solution of the heat equation

$$\begin{cases} S_t - S_{\xi\xi} = 0, & t > 0, \xi \in \mathbb{R}, \\ S(0, \xi) = s_0(\xi), & \xi \in \mathbb{R}. \end{cases}$$

Hence, if $G(t, \xi)$ is the heat kernel $\frac{1}{\sqrt{4\pi t e^{-\frac{\xi^2}{4t}}}}$, then

$$(12.11) \quad S(t, \xi) = \int_{\mathbb{R}} G(t, \xi - y) s_0(y) dy,$$

$$(12.12) \quad s(t, \xi) = S(t, \xi - \sqrt{c^2 - c_*^2}t) = \int_{\mathbb{R}} G(t, \xi - \sqrt{c^2 - c_*^2}t - y) s_0(y) dy,$$

and finally

$$(12.13) \quad m(t, \xi) = \frac{1}{r_-(c)} \ln s(t, \xi) = \frac{1}{r_-(c)} \ln \left(\int_{\mathbb{R}} G(t, \xi - \sqrt{c^2 - c_*^2}t - y) e^{r_-(c)m_0^*(y)} dy \right).$$

This is exactly the expression in Theorem 12.1, choosing $m^{app} = m$. We deduce the following properties, useful in the sequel:

- first, m is bounded, and more precisely, $m(t, \xi) \in [-M, M]$ for all $t \geq 0$ and all $\xi \in \mathbb{R}$: (indeed, this is true at $t = 0$, and remains true thanks to the weak maximum principle);
- its spatial derivative satisfy: for all $t > 0$, $\|m_\xi(t, \cdot)\|_\infty = O(\frac{1}{1+\sqrt{t}})$; indeed,

$$m_\xi(t, \xi) = \frac{1}{r_-(c)} \frac{S_\xi(t, \xi - \sqrt{c^2 - c_*^2}t)}{S(t, \xi - \sqrt{c^2 - c_*^2}t)},$$

and, for all $t > 0$ and $\xi \in \mathbb{R}$, we have

$$|S_\xi(t, \xi)| \leq C \|s'_0\|_\infty \quad \text{and} \quad |S_\xi(t, \xi)| \leq C \frac{1}{\sqrt{t}} \|s_0\|_\infty,$$

that gives what we claimed;

- note that, in the same way, $\|m_t(t, \cdot)\|_\infty = O(\frac{1}{\sqrt{t}})$ and $\|m_{\xi\xi}(t, \cdot)\|_\infty = O(\frac{1}{t})$ for all $t > 0$.

Now we denote \tilde{u}^{app} the associated shifted wave:

$$(12.14) \quad \tilde{u}^{app}(t, \xi) := T^{(m)}\phi_c = \phi_c(\xi + m(t, \xi)).$$

It remains to prove that we have grabbed the correct shift, that is that $\|\tilde{u}(t, \cdot) - \tilde{u}^{app}(t, \cdot)\|_\infty = O(\frac{1}{\sqrt{t}})$ as $t \rightarrow +\infty$, and Theorem 12.1 will be proved.

12.5.5. The difference between the solution and the shifted wave.

12.5.5.1. *Solutions decaying sufficiently fast in space will decay exponentially in time.* The following (quite standard) lemma is useful to estimate the difference between the solution and the shifted wave.

LEMMA 12.7. *Let $v(t, \xi)$ solve*

$$v_t - v_{\xi\xi} + cv_\xi - f'(0)v = 0 \quad (t > 0, \xi \in \mathbb{R}), \quad v(0, \xi) = v_0(\xi)$$

with v_0 bounded, uniformly continuous on \mathbb{R} . Assume additionally the existence of $\delta \in (0, r_+(c) - r_-(c))$ such that

$$v_0(\xi) = O(e^{(r_-(c)+\delta)\xi}) \quad \text{as } \xi \rightarrow -\infty.$$

Then there is $\omega(\delta) > 0$ such that

$$|v(t, \xi)| \leq e^{(r_-(c)+\delta)\xi} e^{-\omega(\delta)t} \sup_{z \in \mathbb{R}} |e^{-(r_-(c)+\delta)z} v_0(z)|.$$

Its proof is based on the weak maximum principle.

12.5.5.2. *Application: proof of Theorem 12.1.* With this in hand, we may complete the proof of Theorem 12.1, proving that $\|\tilde{u}(t, \cdot) - \tilde{u}^{app}(t, \cdot)\|_\infty = O(\frac{1}{\sqrt{t}})$ as $t \rightarrow +\infty$.

Introduce

$$a(t, \xi) := -\frac{f(\tilde{u}(t, \xi)) - f(\tilde{u}^{app}(t, \xi))}{\tilde{u}(t, \xi) - \tilde{u}^{app}(t, \xi)} \in [-f'(0), -f'(1)],$$

and

$$g_1(t, \xi) := (\tilde{t}u - \tilde{u}^{app})_t + c(\tilde{u} - \tilde{u}^{app})_\xi - (\tilde{u} - \tilde{u}^{app})_{\xi\xi} + a(t, \xi)(\tilde{u} - \tilde{u}^{app}).$$

Then, let $\tilde{w}(t, \xi)$ solve

$$(12.15) \quad \begin{cases} \tilde{w}_t + c\tilde{w}_\xi - \tilde{w}_{\xi\xi} - f'(0)\tilde{w} = |g_1(t, \xi)|, \\ \tilde{w}(0, \xi) = |\tilde{u} - \tilde{u}^{app}|(0, \xi) = |u_0(\xi) - \phi_c(\xi + m_0^*(\xi))| = 0. \end{cases}$$

The proof of Theorem 12.1 follows from the following facts:

- for all $t > 0$ and all $\xi \in \mathbb{R}$, we have $|\tilde{u} - \tilde{u}^{app}|(t, \xi) \leq \tilde{w}(t, \xi)$;
- for all $\xi_0 \in \mathbb{R}$, $\sup_{\xi \leq \xi_0} \tilde{w}(t, \xi) = O(\frac{1}{1+\sqrt{t}})$;
- there exists $\xi_0 \in \mathbb{R}$ such that $\sup_{\xi \geq \xi_0} |\tilde{u} - \tilde{u}^{app}|(t, \xi) = O(\frac{1}{1+\sqrt{t}})$.

It is clear that these three facts imply Theorem 12.1; and the first follows from the weak maximum principle, the second from Lemma 12.7 and Duhamel's formula, and the third from the second and the weak maximum principle.

12.5.6. Examples and comparison with the literature. We have closely looked to the following situations:

- The case where m_0 is periodic, which is not covered by the existing literature; we are able to prove that the associated solution u of (12.1) converges to a translate of the travelling wave, but not the one that could be expected:

PROPOSITION 12.8. *Assume that the initial shift is T -periodic, and denote by $\langle m_0 \rangle$ its mean value. The solution u of (12.1) satisfies*

$$(12.16) \quad \sup_{x \in \mathbb{R}} |u(t, x) - \phi_c(x + ct + \mu_0)| = O\left(\frac{1}{\sqrt{t}}\right),$$

where

$$\mu_0 = \frac{1}{r_-(c)} \ln \langle e^{r_-(c)m_0} \rangle.$$

(One could have expected the convergence of the solution u of (12.1) to the the travelling wave $\phi_c(x + ct + \langle m_0 \rangle)$. However this is not the case, since in general $\mu_0 \neq \langle m_0 \rangle$.)

- A case where m_0 oscillates between two values: we prove that m_0 can be chosen so that the associated solution u of (12.1) does not converge to any translate of the travelling wave.
- The typical case where m_0 converges to some constant $m_0(-\infty)$ as $x \rightarrow -\infty$: in this case, Theorem 0.1 applies when additionally the convergence is exponentially fast. We prove that the associated solution u of (12.1) converges to a translate of the travelling wave, and more precisely the one that is expected: $\phi_c(x + ct + m_0(-\infty))$, with a precise rate of convergence, roughly speaking the rate of convergence of m_0 to its limit $m_0(-\infty)$.

Everything is based on the fact that we have an explicit formula for the approximate shift $m(t, \xi)$, obtained in (12.13).

12.6. Additional comments and open questions

We hope that the ideas developed here will not only provide a better understanding of the dynamics of super-critical KPP waves, but will also help to understand how the critical wave is attained from fastly decaying initial data. The general case is an important issue that goes far beyond scalar reaction-diffusion equations, see [127].

12.6.1. Idea of the proof of 12.6.

The strategy is the same: since the travelling wave ϕ_c is strictly increasing in its first variable, we are able to define

$$n^*(t, x, y) := (\phi_c(\cdot, y))^{-1}(u(t, x, y)),$$

which satisfies

$$u(t, x, y) = \phi_c(n^*(t, x, y), y),$$

hence informations on n^* give informations on u . And to have informations on n^* , we write the differential problem that it satisfies. Unfortunately, this problem is fully nonlinear, but it admits an ‘‘approximate’’ solution $n(t, x, y)$, obtained linearizing the problem at $-\infty$. The final task is then

- to obtain informations on the approximate solution n ,
- to study the difference between u and its ‘‘approximation’’ $u^{app}(t, x, y) := \phi_c(n(t, x, y), y)$.

These two things are in fact closely related.

12.6.2. Open questions.

There are several questions close to this work whose answers would be very interesting:

- Concerning the general model: we could not provide a decay rate estimate about the derivatives of approximate shift, but just the fact that $\lim_{t \rightarrow +\infty} \|(m_t, \nabla m, D^2 m)(t)\|_\infty = 0$, using a contradiction argument. Any decay rate estimate would immediately provide also a decay rate of the

uniform convergence as $t \rightarrow +\infty$ of the shifted wave $\phi_c(t+m(t, x, y), x, y)$ to the solution u of the Cauchy problem.

- Hence the problem of the convergence rate remains open in the general N -dimensional case. In [B4], we have studied the case $A = I$, $B(x, y) = (\alpha(y), 0)$ - thermo-diffusive model for flame propagation - where the techniques are related to the ones used in this paper. We point out that, except in the case of self-adjoint operators - where the heat kernel is known with a lot of precision, see [292] - these are the only cases where we can go that far.
- We concentrated our study in the case where the initial condition of the Cauchy problem is trapped between two translates of the same pulsating wave. It would be very interesting to investigate the behavior of the solutions under weakened assumptions on the initial condition. An important first step is taken in Hamel-Nadirashvili [183], where it is proved that (almost) every time-global solution of the N -dimensional homogeneous model

$$u_t - \Delta u = u(1 - u), \quad t > 0, \quad x \in \mathbb{R}^N, \quad 0 < u < 1$$

is a (possibly uncountably infinite) convex combination of one-dimensional waves. See also the later reference [352].

Perspectives

There are some questions about our works, and another perspective.

13.1. About solid combustion

In [B2], we investigated the existence of a pulsating wave for the system of solid combustion. It remains to study the dynamics of the problem, hence to understand how these pulsating waves attract the solutions of the problem.

13.2. About the influence of 1D structures in the phenomenon of propagation

In recent papers ([49, 50, 51, 172]), the phenomenon of spreading of epidemics has been studied when a road (where fast diffusion occurs) drives the diffusion. This phenomenon has been observed for a long time, for example the spread of the Black death” plague in the middle of the 14th century, driven by the silk road, or recently the propagation of invasive species such as the Processionary caterpillar of the pine tree in Europe, or the invasion of the *Aedes albopictus* mosquito in Europe, where also the propagation is driven by roads.

This phenomenon has been studied in a series of papers (in particular [49, 50, 51, 172, 238]), using mathematical models of the form

$$\begin{cases} \partial_t u - D\partial_{xx}u = \nu v(x, 0, t) - \mu u, \\ \partial_t v - d\Delta v = f(v), \\ -d\partial_y v(x, 0, t) = \mu u(x, t) - \nu v(x, 0, t), \end{cases}$$

where $u(x, t)$ is the density of population on the road (described by the line $x = 0$ in the (x, y) plane), $v(x, y, t)$ is the density of population on the field (which is described by the plane (x, y)); $\nu v(x, 0, t)$ is the fraction of individuals from the field at the road that join the road, $\mu u(x, t)$ is the fraction of individuals on the road that goes in the field, $f(v)$ is a logistic type of growth, of Fisher KPP type, modelling the reproduction on the field, and d and D are the diffusion coefficients respectively on the field and on the road (typically $d < D$).

This invasion phenomenon has been studied from the point of view of the asymptotic behavior, describing what is the behavior of $u(x, t)$ ($v(x, y, t)$) when $t \rightarrow +\infty$, what is the influence of a large diffusion coefficient D on the propagation of the invasion. We would like to change the point of view, and study this phenomenon from the point of view of controllability:

- first, considering the problem in a bounded domain (and not on the all plane), which has more practical signification,
- next, considering a linearized problem: then the problem appears as a system of coupled parabolic equations, but the coupling is unusual, since it couples an equation stated naturally on a 1D spatial domain with an equation stated on a 2D spatial domain,
- and looking to controllability issues; many questions appear to be natural:

- geometrical control conditions: is it possible to control the invasion using a localized control ? supported only on the field ? supported both on the field and on the road ? with a minimal time of propagation, depending on the values of d and D ?
- possible constraints: what can be said if one uses only L^∞ controls ? what can be said if one uses only controls that keep nonnegative the solution (which is natural when one studies population problems) ?
- next, considering the nonlinear problem: is it possible to obtain local controllability results ?
- and also from a numerical point of view.

We believe these questions to be exciting from a theoretical point of view, and to have potential practical applications, of the following form:

- a description of the geometrical control regions that allows one to control the invasion, and those which are not able to control it,
- an estimate of the minimal time needed to control the invasion, at least in function of the parameters when one applies controls bounded by some uniform bound.

Note that a close problem (concerning the proteins localisation in stem cell division) has been considered in a bounded domain (see [196]), but not from the point of view of controllability or inverse problems. And also that very interesting new works [250, 328, 296] on the controllability of reaction-diffusion equations under state or control constraints will probably bring ideas and techniques.

Part 3

Stabilization of second order evolution equations

Presentation

The following part consists of several results concerning mainly the stabilization of the wave equation, damped by different feedback laws:

- under nonlinear feedbacks: the *optimality of the classical decay estimates*, under general assumptions on the nonlinear stabilization law,
- under time-dependent (and in particular on-off) feedbacks: precise positive/negative results, for linear and semilinear wave equations,
- stabilization for viscoelastic problems.

These results complete several results obtained during my thesis, in particular based on [260].

Optimality of energy estimates for time dependent damped wave equations (Joint work with J. Vancostenoble [C1], and [C2])

15.1. Introduction: motivation and related papers

15.1.1. Motivation.

We consider the wave equation damped by a nonlinear time independent (and next time dependent) velocity feedback, and we study the decay rate of the energy. When the feedback is strong enough to ensure strong stability ($E(t) \rightarrow 0$), there were many results concerning upper estimates, of the form

$$E(t) \leq f(t)$$

where the function $f(t)$ decays to 0, its asymptotic behaviour as $t \rightarrow \infty$ being linked to the behaviour of the feedback law. The goal of our works was to obtain *lower bounds* of the energy, and, if possible, *optimal lower bounds*, at least for some particular solutions (if not for all). For example: consider the following

$$\begin{cases} u_{tt} - u_{xx} = 0, & x \in (0, 1), t > 0, \\ u(0, t) = 0, & t > 0, \\ u_x(1, t) = -q(u_t(1, t)), & t > 0, \\ u(x, 0) = u_0(x), u_t(x, 0) = v_0(x), & x \in (0, 1), \end{cases}$$

where $q(y)$ is odd, increasing and behaves as y^p (with some $p > 1$) as $y > 0$ is close to 0. Then it is well known (Zuazua [358], Komornik [223]) that the energy satisfies

$$E(t) \leq \frac{C}{1+t}{}^{2/(p-1)}.$$

Is it true that the reverse inequality

$$E(t) \geq \frac{C'}{1+t}{}^{2/(p-1)}$$

holds true, at least for some solutions ?

15.1.2. The related papers [C1, C2].

- [C1]: we prove optimal lower estimates of the energy of particular solutions of the form

$$E(t) \geq C_1 g(t),$$

'optimal' in the sense that the reverse inequality

$$E(t) \leq C_2 g(t)$$

is known to be true for all solutions.

- [C2]: we extend these results to the case of time dependent feedbacks of the form $\sigma(t)g(u_t)$, where σ decays slowly to 0 (in order to have $\int_0^{+\infty} \sigma = +\infty$).

15.2. Optimality of energy estimates for damped wave equations (Joint work with J. Vancostenoble [C1])

15.2.1. Introduction.

We consider the 1D wave equation, damped by a boundary velocity feedback:

$$(15.1) \quad \begin{cases} u_{tt} - u_{xx} = 0, & x \in (0, 1), t > 0, \\ u(0, t) = 0, & t > 0, \\ u_x(1, t) = -q(u_t(1, t)), & t > 0, \\ u(x, 0) = u_0(x), u_t(x, 0) = v_0(x), & x \in (0, 1), \end{cases}$$

where $(u_0, v_0) \in V \times L^2(0, 1)$ with $V = \{v \in H^1(0, 1), v(0) = 0\}$. We consider its energy

$$E_u(t) = \frac{1}{2} \int_0^1 u_x(t, x)^2 + u_t(t, x)^2 dx.$$

Let us first recall some well-known results concerning the asymptotic behaviour of the energy:

- when q is continuous, nondecreasing function with a polynomial behaviour near 0 and a linear growth at infinity:

$$(15.2) \quad \begin{cases} \forall |s| \leq 1, & C_1 |s|^p \leq |q(s)| \leq C_2 |s|^{1/p} \quad (\text{with } p > 1), \\ \forall |s| \geq 1, & C_3 |s| \leq |q(s)| \leq C_4 |s|, \end{cases}$$

then E_u satisfies

$$(15.3) \quad \forall t \geq 0, \quad E_u(t) \leq \frac{C(E_u(0))}{(1+t)^{2/(p-1)}},$$

where $C(E_u(0))$ is a constant depending on $E_u(0)$, see Zuazua [358], Komornik [223] and the references therein, for similar results valid for several cases (locally distributed damping, N dimensional problems...)

- under more general conditions:

$$(15.4) \quad \begin{cases} \forall |s| \leq 1, & g(|s|) \leq |q(s)| \leq g^{-1}(|s|) \quad (\text{with } p > 1), \\ \forall |s| \geq 1, & C_3 |s| \leq |q(s)| \leq C_4 |s|, \end{cases}$$

for example when $g(s) = e^{-1/s}$, upper estimates of the energy were provided, see Lasiecka-Tataru [230] (where the energy is indirectly estimated through the solution of some differential equation), [260] (based on new integral inequalities), and Liu-Zuazua [244].

In [C1], we investigate the optimality of these upper estimates:

15.2.2. Main results on the optimality question.

15.2.2.1. Polynomial behaviour near 0.

There were very few results, although it was commonly thought that the upper estimates were optimal. Haraux [193] provided a lower estimate for smooth solutions of the 1D wave equation damped by the uniformly distributed damping $q(u_t)$:

$$\limsup_{t \rightarrow +\infty} E(t)(1+t)^{3/(p-1)} > 0$$

(even if, to be optimal, the exponent 3 should be 2). We obtained the following

THEOREM 15.1. ([C1]) *Assume that*

$$\forall s \in (-s_0, s_0), \quad q(s) = s|s|^{p-1} \quad \text{with some } p > 1,$$

and that q has a linear growth at infinity. Choose $u_0(x) = 2A_0x$ with some $A_0 \neq 0$, and $v_0(x) = 0$. Then the solution of (15.1) satisfies

$$E_u(t) \sim_{t \rightarrow +\infty} \frac{C_p}{t^{2/(p-1)}} \quad \text{with} \quad C_p = \frac{1}{2(p-1)^{2/(p-1)}}.$$

(The same estimate holds true if $q(s) = s^{1/p}$ on $(0, s_0)$ and is odd.) The proof is based on the d'Alembert formula and the study of a sequence satisfying some induction formula.

15.2.2.2. *General behaviour near 0.*

Under (15.4), Lasićka-Tataru [230] proved that the energy decays as fast as the solution of some ordinary differential equation (related to g), and I proved ([260]) that

$$E_u(t) \leq C(E_u(0)) \left(g^{-1}\left(\frac{1}{t}\right) \right)^2$$

if $g(s) = o(s)$ near 0 and $s \mapsto \frac{g(s)}{s}$ in increasing near 0. For example,

$$g(s) = e^{-1/s} \quad \implies \quad E_u(t) \leq \frac{C}{(\ln t)^2}.$$

We proved the following:

THEOREM 15.2. ([C1]) *Assume that $g(0) = 0 = g'(0)$ and odd. Assume that $q \leq g$ or $q \geq g^{-1}$ near 0. Choose $u_0(x) = 2A_0x$ with some $A_0 \neq 0$, and $v_0(x) = 0$. Then the solution of (15.1) satisfies*

$$\exists n_0, n_1 \forall n \geq n_0, \quad E_u(2n) \geq \frac{1}{2} \left((g')^{-1}\left(\frac{1}{2(n+n_1)}\right) \right)^2$$

if $s \mapsto s(\frac{1}{2}g^{-1})'(s) - 1$ is increasing in a neighborhood of 0. And also

$$E_u(2n) \geq \frac{2}{\alpha^2} \left((g)^{-1}\left(\frac{1}{Mn+C}\right) \right)^2$$

if $2\alpha \frac{g(2s)g'(\alpha s)}{g(\alpha s)^2} \leq M$.

We also provided a result in the spirit of Lasićka-Tataru [230]: the energy is bounded from below by the solution of some differential equation related to g . All these estimates provide the optimality of the classical upper estimates.

15.2.2.3. *Damping "weak" at infinity.*

Finally we studied the case where

$$\frac{q(s)}{s} \rightarrow 0 \quad \text{as } |s| \rightarrow +\infty.$$

In [261], we completed earlier results of Komornik [224] and Nakao [283] obtaining that: when the damping is uniformly distributed, the energy of strong solutions ($u_0 \in H^2(\Omega)$, $v_0 \in H_0^1(\Omega)$) decays exponentially (but not uniformly) to 0. We proved that in fact weak solutions can decay as slowly as we want:

THEOREM 15.3. ([C1]) *Consider*

$$\forall s \in [-2, 2], q(s) = \frac{s}{2}, \quad \text{and} \quad \forall |s| \geq 2, q(s) = \operatorname{sgn}(s).$$

Then

- if $(u_0, v_0) \in W^{1,\infty}(0,1) \times L^\infty(0,1)$, then the energy of the solution of (15.1) decays exponentially but not uniformly to 0;

- consider the sequence of iterated logarithms:

$$\ln_{p+1}(t) = \ln(\ln_p(t)), \quad \ln_1(t) = \ln t;$$

then, given $p \geq 1$, there exists $(u_0, v_0) \in V \times L^2(0, 1)$ such that the solution of (15.1) satisfies

$$E_u(t) \geq \frac{1}{\ln_p(t)}$$

for t large enough.

This implies that the decay really depends on the regularity of the initial condition, and that weak solutions can decay very slowly to 0. The proof is in the same spirit: the d'Alembert formula and the study of an associated sequence.

15.3. Precise decay rate estimates for time-dependent dissipative systems ([C2])

In [C2], I extended all the previous results to the case of time dependent feedbacks:

15.3.1. Upper estimates.

Consider

$$(15.5) \quad \begin{cases} u_{tt} - \Delta u + \rho(t, u_t) = 0 & x \in \Omega, t > 0, \\ u = 0, & x \in \partial\Omega, t > 0 \\ u(0, x) = u_0(x), u_t(0, x) = v_0(x), & x \in \Omega, \end{cases}$$

where

$$\sigma(t)g(|v|) \leq |\rho(t, v)| \leq g^{-1}\left(\frac{|v|}{\sigma(t)}\right)$$

with σ positive and nonincreasing.

THEOREM 15.4. ([C2]) *Assume that*

$$\int_0^{+\infty} \sigma(t) dt = +\infty,$$

and that g has a linear growth at infinity. Then the energy of the solution of (15.5) satisfies:

$$\begin{aligned} E(t) &\leq E(0)e^{1-\omega \int_0^t \sigma} && \text{if } g(v) = v \text{ near } 0, \\ E(t) &\leq \left(\frac{C(E(0))}{\int_0^t \sigma}\right)^{2/(p-1)} && \text{if } g(v) = v^p \text{ near } 0, \\ E(t) &\leq C(E(0)) \left[g^{-1}\left(\frac{1}{1 + \int_0^t \sigma}\right) \right]^2 && \text{if } \frac{g(v)}{v} \text{ decays to } 0 \text{ as } v \rightarrow 0^+. \end{aligned}$$

Hence, roughly speaking, the adjonction of $\sigma(t)$ in the equation is translated by the change of t into $\int_0^t \sigma$ in the upper estimates of the energy. This improved several results of Nakao [284] and Pucci-Serrin [298] among others.

15.3.2. Lower estimates.

The same property holds; considering the 1D wave equation damped by the boundary feedback $\sigma(t)q(v)$, the energy of some particular solutions satisfies lower bounds (the ones of the previous section concerning the time-independent case, changing t into $\int_0^t \sigma$), proving the optimality of the previous upper bounds, see [C2].

In the following, we will investigate what can be said when σ is no more positive and nonincreasing.

On-off stabilization of the wave equation (Joint works with J. Vancostenoble [C3, C4] and additionnally with A. Haraux [C5])

16.1. Introduction: motivation and related papers

16.1.1. Motivation.

Motivated by several works on the stabilization of the oscillator by on-off feedbacks, we study the related problem for the 1D wave equation. We obtain radically different results from those known for ordinary differential equations.

16.1.2. The related papers [C3-C5].

- [C3]: we study the stabilization of a 1D wave equation damped by
 - an on-off boundary damping of the form $a(t)u_t$, with a periodic,
 - next of a positive-negative boundary damping of the form $a(t)u_t$, with a periodic,
 - and finally of a locally distributed on-off feedback.

In every case, we determine the countable exceptional values of T for which stability does not hold, and we prove the exponential stability in the other situations; proofs are based on arithmetic properties and refined observability inequalities.

- [C4]: we study the controllability of a 1D wave equation, using a locally distributed control; using the previous refined observability inequalities, we prove that an arbitrary small amount of time is sufficient for controllability, if the control acts on suitably well chosen small intervals.
- [C5]: we study the stabilization of general second order evolution equations by on-off feedbacks; we extend the classical results on second order ODEs damped by on-off feedbacks, estimating in particular the decay of the energy during any given interval of time.

16.2. On-off boundary stabilization of the 1D wave equation (Joint work with J. Vancostenoble [C3])

16.2.1. The typical stabilization problem and the relation to literature.

We consider the following time-dependent feedback law:

$$(16.1) \quad \begin{cases} u_{tt} - u_{xx} = 0, & x \in (0, 1), t > 0, \\ u(0, t) = 0, & t > 0, \\ u_x(1, t) = -a(t)u_t(1, t), & t > 0, \\ u(x, 0) = u_0(x), u_t(x, 0) = v_0(x), & x \in (0, 1), \end{cases}$$

where a is nonnegative. The energy is defined by

$$E_u(t) = \frac{1}{2} \int_0^1 u_x(t, x)^2 + u_t(t, x)^2 dx,$$

and satisfies $E'(t) = -a(t)u_t(1,t)^2$, hence is nonincreasing but constant on any interval where $a = 0$. If a is constant: $a(t) = a_0$, then

- if $a_0 = 1$, then $E_u(2) = 0$ (and remains constant equal to zero),
- and if $a_0 \neq 1$, then the energy decays exponentially fast to 0:

$$E_u(t) \leq E_u(0)e^{-\omega(t/2-1)} \quad \text{with} \quad \omega = 2 \ln \left| \frac{a_0 + 1}{a_0 - 1} \right| > 0.$$

This derives from the d'Alembert formula, but the exponential decay can also be proved using the multiplier method (Komornik [223]), and is of course also a consequence of the general 'optic rays condition' of Bardos-Lebeau-Rauch [24].

If a decays to 0, then the results of [C2] give the decay estimate

$$E_u(t) \leq CE_u(0)e^{-\int_0^t a}.$$

The case where a is sometimes equal to 0 has been also largely studied for the damped oscillator, of the form

$$(16.2) \quad u'' + a(t)u' + u = 0,$$

see in particular [320, 195, 297] and the references therein. In this case, the energy decays to zero if the damping is "sufficiently active". We will come back on this later, but for example if a takes the value 1 on the intervals I_n and is always nonnegative, then the energy of the solutions of (16.2) decays to 0 if

$$\sum_n |I_n|^3 = \infty$$

(and the power 3 is the best possible ([297]). In particular, the location of the intervals is not important, only their length is important.

In the following, we study the case of the wave equation, damped by an on-off damping term (applied at the boundary or locally distributed in the domain). We will see that the situation is radically different from the case of ordinary differential equations.

16.2.2. Main results for boundary feedbacks.

16.2.2.1. On-off feedbacks.

Here we study in detail the case

$$(16.3) \quad a(t) = a_0 > 0 \text{ on } [0, T), a(t) = 0 \text{ on } [T, qT), \text{ and } a \text{ is } qT \text{ periodic.}$$

THEOREM 16.1. ([C3]) *Assume (16.3). For all $(u^0, u^1) \in V \times L^2(0, 1)$, there exists a unique u solution of (16.1). Moreover,*

(i) *if*

$$(16.4) \quad \frac{1}{T} \in \bigcup_{p=1}^{q-1} \frac{q}{2p} \mathbb{N},$$

there exists some $(u^0, u^1) \in V \times L^2(0, 1)$ such that $E_u(t)$ remains constant with time : $E_u(t) = E_u(0) > 0$ for all $t \geq 0$;

(ii) *if*

$$(16.5) \quad \frac{1}{T} \notin \bigcup_{p=1}^{q-1} \frac{q}{2p} \mathbb{N},$$

then for all $(u^0, u^1) \in V \times L^2(0, 1)$, the energy $E_u(t)$ of the solutions of (16.1) decays uniformly exponentially to 0 (or achieves zero in finite time in the particular case $a_0 = 1$).

Our proofs are based on d'Alembert formula and on congruence properties, which is equivalent in this simple case to the study the optic rays propagation: we prove that if T is not one of the exceptional values, each ray touches the boundary point $x = 1$ (where the dissipative condition is applied) in time at most $2N_T + 2$ where N_T depends on T and at an instant where the damping is effective, which is crucial for the decay of the energy.

Note also that we are also able to study the case of nonlinear feedbacks $a(t)q(u_t(1, t))$, combining with the tools developed in the previous chapter.

16.2.2.2. *Positive-negative feedbacks.*

In the same spirit, we are able to study the case of

$$(16.6) \quad a(t) = a_0 > 0 \text{ on } [0, T), a(t) = -b_0 < 0 \text{ on } [T, 2T), \text{ and } a \text{ is } 2T \text{ periodic.}$$

There are some works when the feedback is of the type $b(x)u'$, where the function b depends on x (and not on t) and is of indefinite sign, but "more positive than negative" (see, e.g., Freitas-Zuazua [164], Benaddi-Rao [37]) But to our knowledge, such time dependent positive-negative feedback laws have never being studied. We prove the following

THEOREM 16.2. ([C3]) *Assume (16.6). Assume that $b_0 \neq 1$. Then for all $(u^0, u^1) \in V \times L^2(0, 1)$, there exists a unique u solution of (16.1). Moreover,*

(i) *if T satisfies (16.4), that is simply $1/T \notin \mathbb{N}$ in this case (since $q = 2$), then there exists some $(u^0, u^1) \in V \times L^2(0, 1)$ such that $E_u(t)$ goes exponentially to infinity as $t \rightarrow \infty$;*

(ii) *if $1/T = p'/q'$ where p' and q' are relatively primes, denote*

$$(16.7) \quad \begin{cases} K_T := \left(\frac{a_0-1}{a_0+1}\right)^{1/2} \left(\frac{b_0+1}{b_0-1}\right)^{1/2} & \text{if } q' \text{ is even,} \\ K_T := \left(\frac{a_0-1}{a_0+1}\right)^{(q'-1)/2q'} \left(\frac{b_0+1}{b_0-1}\right)^{1-(q'-1)/2q'} & \text{if } q' \text{ is odd;} \end{cases}$$

then if $K_T < 1$, the energy of all solutions goes exponentially to zero (as K_T^t), and if $K_T > 1$, the energy of some solutions goes exponentially to infinity (as K_T^t);

(iii) *if $1/T \notin \mathbb{Q}$, denote*

$$(16.8) \quad K_0 := \left(\frac{a_0-1}{a_0+1}\right)^{1/2} \left(\frac{b_0+1}{b_0-1}\right)^{1/2};$$

then if $K_0 < 1$, the energy of all solutions goes exponentially to zero (as K_0^t), and if $K_0 > 1$, the energy of some solutions goes exponentially to infinity (as K_0^t).

(Part (iii) relies on a theorem of Weyl: if $\theta \notin \mathbb{Q}$, then the sequence $(\{n\theta\})_n$ is not only dense but also equidistributed in $[0, 1)$. (As usual, $\{x\}$ denotes the fractional part of x .) Consequently, the critical value K_0 that appears in (iii) does not depend on T .)

16.2.3. Main results for locally distributed feedbacks.

Now we consider the wave equation in one space dimension, damped by a *locally distributed on-off feedback* $a(t)\chi_\omega(x)u_t$, where $\omega \subset (0, 1)$:

$$(16.9) \quad \begin{cases} u_{tt} - u_{xx} = -a(t)\chi_\omega(x)u_t, & x \in (0, 1), t \geq 0, \\ u(0, t) = u(1, t) = 0, & t \geq 0, \\ (u(x, 0), u_t(x, 0)) = (u^0(x), u^1(x)), & x \in (0, 1), \end{cases}$$

where (u^0, u^1) is given in $H_0^1(0, 1) \times L^2(0, 1)$.

16.2.3.1. *Stabilization results.*

We choose ω the open nonempty subset $((1/2) - \lambda, (1/2) + \lambda)$ of $(0, 1)$, and a is the time periodic function (16.3) and (u^0, u^1) is given in $H_0^1 \times L^2(0, 1)$. We prove the following:

THEOREM 16.3. **[C3]** *Assume (16.3) and assume that $0 < \lambda \leq 1/2$.*

(i) *If*

$$\frac{1}{T} \in \bigcup_{p=1}^{q-1} \frac{q}{p} \mathbb{N} \quad \text{and} \quad (q-1)T > 2\lambda,$$

then there exist initial conditions $(u^0, u^1) \in H_0^1 \times L^2(0, 1)$ such that the energy of the solutions of (16.9) remains constant with time : $E_u(t) = E_u(0) > 0$ for all $t \geq 0$.

(ii) *If*

$$(16.10) \quad \left(\frac{1}{T} \in \bigcup_{p=1}^{q-1} \frac{q}{p} \mathbb{N} \quad \text{and} \quad (q-1)T < 2\lambda \right), \quad \text{or} \quad \left(\frac{1}{T} \notin \bigcup_{p=1}^{q-1} \frac{q}{p} \mathbb{N} \right),$$

then the energy of the solutions of (16.9) decays uniformly exponentially to 0.

Once again, this is *radically different* from what happens for ordinary differential equations, or even for the damped wave equation when the function a decreases to zero remaining always positive. In the case (16.10), we prove the uniform decay of the energy thanks to *new observability inequalities*:

16.2.3.2. *Refined observability inequalities.*

Considering the undamped problem

$$(16.11) \quad \begin{cases} \phi_{tt} - \phi_{xx} = 0, & x \in (0, 1), \quad t \geq 0, \\ \phi(0, t) = \phi(1, t) = 0, & t \geq 0, \\ (\phi(x, 0), \phi_t(x, 0)) = (\phi^0(x), \phi^1(x)), & x \in (0, 1), \end{cases}$$

it is well known that if $0 < a < b < 1$ and $T^* > 2 \max(a, 1 - b)$, then the solutions of (16.11) satisfy the following observability inequality

$$(16.12) \quad E_\phi(0) \leq C \int_0^{T^*} \int_a^b \phi_t^2(x, t) \, dx dt$$

for some positive constant $C = C(T^*)$ (see Haraux [192] and Zuazua [359]). This is optimal in the sense that you cannot have this inequality with some $T^* < 2 \max(a, 1 - b)$. In our case, for example, if $\lambda > 1/8$, then we can apply it with $T^* = 3/4$ and we obtain

$$(16.13) \quad E_\phi(0) \leq C \int_0^{3/4} \int_{(1/2)-\lambda}^{(1/2)+\lambda} \phi_t^2(x, t) \, dx dt.$$

We improve this inequality showing that

$$(16.14) \quad E_\phi(0) \leq C \int_0^{1/4} \int_{(1/2)-\lambda}^{(1/2)+\lambda} \phi_t^2(x, t) \, dx dt + C \int_{1/2}^{3/4} \int_{(1/2)-\lambda}^{(1/2)+\lambda} \phi_t^2(x, t) \, dx dt.$$

This is coherent with the fact that each optic ray touches the damping region during the time intervals $(0, 1/4)$ or $(1/2, 3/4)$.

More generally, we prove the following

THEOREM 16.4. **[C3]** *Assume $1/T \in \bigcup_{p=1}^{q-1} \frac{q}{p} \mathbb{N}$ and $(q-1)T < 2\lambda$. Then there exists $C > 0$ such that, for all solution ϕ of (16.11),*

$$(16.15) \quad E_\phi(0) \leq C \int_0^{q-1} a(t) \int_\omega \phi_t^2(x, t) \, dx dt.$$

(Note that this obviously also gives improved exact controllability results, applying the method H.U.M. of J.-L. Lions [241], we will develop this in the next section.)

16.2.3.3. Comments.

- Note that there are some values of T and some values of λ for which some rays cross the damping region when the feedback is non active. For example, take $q = 2$, $T = \frac{1}{2}$, $\lambda < \frac{T}{2} = \frac{1}{4}$, and consider the optic rays that leaves the point $x = \frac{T}{2} = \frac{1}{4}$ and that goes to the left (towards the point $x = 0$) at time $t = 0$: this ray describes the segment $[\frac{1}{4}; \frac{3}{4}]$ (that contains the dissipative region) in direct sens or in the other sens during the time intervals $[T; 2T]$, $[3T; 4T]$, ..., thus during periods when $a(t) = 0$. The same situation occurs if $\frac{1}{T} \in 2\mathbb{N}$ with $2\lambda < T$. We obtain negative results of exponential stabilization in all these cases, and positive results in the other cases, which is coherent with the optic ray condition known for time independent feedbacks [24].
- Note also that the situation is more complex than the case of boundary damping: even when T takes some "exceptional" values, we can still have exponential decay of the energy of the solutions, provided that the damping region is large enough. Note also that when the damping region is "large enough" (in particular when $\omega = (0, 1)$), then we find a result analogous to the one related to ordinary differential equation, since we obtain stabilization for all $T > 0$.
- The proof of the refined observability inequalities are based on spectral decomposition (Fourier series), congruence properties, and some usual tricks for the 1D wave equation (in particular: exchanging the role of t and x).

16.3. Controllability of the 1D wave equation in "arbitrarily short time"

(Joint work with J. Vancostenoble [C4])

16.3.1. Introduction.

We consider the problem of exact controllability of the linear wave equation by means of a locally distributed control force h :

$$(16.16) \quad \begin{cases} u_{tt} - u_{xx} = h(x, t), & (x, t) \in (0, 1) \times \mathbb{R}_+, \\ u(0, t) = u(1, t) = 0, & t \in \mathbb{R}_+, \\ u(x, 0) = u^0(x), \quad u_t(x, 0) = u^1(x), & x \in (0, 1). \end{cases}$$

Lagnese [227] proved that given $(a, b) \subset (0, 1)$, $T > 2$, and $(u_0, u_1) \in H_0^1(0, 1) \times L^2(0, 1)$, there exists $h \in L^2((0, 1) \times (0, T))$ such that

$$(16.17) \quad \text{supp}(h) \subset (a, b) \times (0, T),$$

and such that the solution u of (16.16) also satisfies

$$(16.18) \quad u(x, T) = u_t(x, T) = 0, \quad x \in (0, 1).$$

Later Haraux [192] improved this result, proving that, given $(a, b) \subset (0, 1)$, exact controllability holds in time $T > T_0(a, b) := 2 \max(a, 1 - b)$. This value is optimal: Haraux [192] proved that exact controllability fails in time $T \leq T_0(a, b)$.

However, here, we give several results of exact controllability where the condition (16.17) is *strongly weakened*. More precisely, given $(a, b) \subset (0, 1)$ and $T > T_0(a, b)$, we study the two following related problems :

- *Problem 1* : let $J \subset (0, T)$, $J \neq (0, T)$. Does there exist $h \in L^2((0, 1) \times (0, T))$ such that

$$(16.19) \quad \text{supp}(h) \subset (a, b) \times J,$$

and such that the solution u of (16.16) also satisfies (16.18) ?

- *Problem 2* : let $A \subset (a, b)$, $A \neq (a, b)$. Does there exist $h \in L^2((0, 1) \times (0, T))$ such that

$$(16.20) \quad \text{supp}(h) \subset A \times (0, T),$$

and such that the solution u of (16.16) also satisfies (16.18) ?

In the two cases, we give optimal conditions on J and A for which the answer is positive:

16.3.2. Main results.

We obtain the following results of exact controllability :

THEOREM 16.5. ([C4]) Exact controllability in "arbitrarily small time".

Let $(a, b) \subset (0, 1)$ and $T > T_0(a, b)$. Assume that

$$(16.21) \quad \begin{aligned} [0, T] &= [t_0, t_1] \cup [t_1, t_2] \cup [t_2, t_3] \cup \cdots \cup [t_{2n-1}, t_{2n}] \cup [t_{2n}, t_{2n+1}] \\ &= J_0 \cup I_1 \cup J_1 \cup \cdots \cup I_n \cup J_n, \end{aligned}$$

with $t_0 = 0$, $t_{2n+1} = T$ and where, for $j \in \{0, \dots, n\}$, $J_j = [t_{2j}, t_{2j+1}]$ are **nonempty** intervals and where, for $j \in \{1, \dots, n\}$, $I_j = [t_{2j-1}, t_{2j}]$ are (possibly empty) intervals that satisfy the following condition :

$$(16.22) \quad \forall j \in \{1, \dots, n\}, \quad |I_j| < b - a.$$

Then for any $(u_0, u_1) \in H_0^1(0, 1) \times L^2(0, 1)$, there exists $h \in L^2((0, 1) \times (0, T))$ satisfying (16.19) and such that the solution u of (16.16) also satisfies (16.18).

THEOREM 16.6. ([C4]) Exact controllability in "arbitrarily small region".

Let $(a, b) \subset (0, 1)$ and $T > T_0(a, b)$. Assume that

$$(16.23) \quad \begin{aligned} [a, b] &= [a_0, a_1] \cup [a_1, a_2] \cup [a_2, a_3] \cup \cdots \cup [a_{2n-1}, a_{2n}] \cup [a_{2n}, a_{2n+1}] \\ &= A_0 \cup B_1 \cup A_1 \cup \cdots \cup B_n \cup A_n, \end{aligned}$$

with $a_0 = a$, $a_{2n+1} = b$ and where, for $j \in \{0, \dots, n\}$, $A_j = [a_{2j}, a_{2j+1}]$ are **nonempty** intervals and where, for $j \in \{1, \dots, n\}$, $B_j = [a_{2j-1}, a_{2j}]$ are (possibly empty) intervals that satisfy the following condition :

$$(16.24) \quad \forall j \in \{1, \dots, n\}, \quad |B_j| < T.$$

Then for any $(u_0, u_1) \in H_0^1(0, 1) \times L^2(0, 1)$, there exists $h \in L^2((0, 1) \times (0, T))$ satisfying (16.20) and such that the solution u of (16.16) also satisfies (16.18).

16.3.3. Comments and extensions to semilinear problems.

Theorem 16.5 provides a result of *exact controllability in "arbitrarily short time" in any given space interval of observation (a, b)* . Indeed for any given space interval (a, b) , we consider $T > 2 \max(a, 1 - b)$ and we can construct a decomposition (16.21) of $(0, T)$ such that the time support $|J|$ of "real action" of the control is arbitrarily small. Note that such a decomposition can be constructed if $n > T/(b - a)$ (hence the number of time intervals that we need does not increase while their size goes to zero).

On the same way, Theorem 16.6 provides a result of *exact controllability in "arbitrarily small region" in any given time $T > 0$* . Indeed for any given $T > 0$, we can construct a space interval (a, b) such that $T > 2 \max(a, 1 - b)$ and we can

construct a decomposition (16.23) of (a, b) such that the measure $|A|$ of the region of "real action" of the control is arbitrarily small.

The proofs are reduced to observability inequalities (see Theorem 16.9), and then the result follows from classical exact controllability theory. Note that in case of Problem 2, the observability inequality follows from a careful rewriting of the proof of A. Haraux [192] and that it is also a consequence of the general condition on optic rays of Bardos-Lebeau-Rauch [23, 24] for time independent problems. However, our method is enough robust to let us study the problem of exact controllability of the *semilinear* wave equation (with no assumption on the sign of the nonlinearity):

$$(16.25) \quad \begin{cases} u_{tt} - u_{xx} + f(u) = h(x, t), & (x, t) \in (0, 1) \times \mathbb{R}_+, \\ u(0, t) = u(1, t) = 0, & t \in \mathbb{R}_+, \\ u(x, 0) = u^0(x), \quad u_t(x, 0) = u^1(x), & x \in (0, 1), \end{cases}$$

assuming that $f \in C^1(\mathbb{R})$ and that

$$(16.26) \quad \exists \beta_0 > 0 \text{ small enough such that } \limsup_{|s| \rightarrow \infty} \frac{|f(s)|}{|s| \ln^2 |s|} < \beta_0.$$

This growth assumption is sufficient to avoid blow up, see Zuazua [360] (and Cannarsa-Komornik-Loreti [79] for an optimal growth condition on f). Under (16.26), Zuazua [360] proved the following result: let $(a, b) \subset (0, 1)$, and $T > T_0 := 2 \max(a, 1 - b)$. Then for any (u_0, u_1) given $H_0^1(0, 1) \times L^2(0, 1)$, there exists h satisfying (16.17) and such that the solution u of (16.25) also satisfies (16.18) (using a fixed point argument, for which the technical condition " $\beta_0 > 0$ small enough" is needed).

In the same spirit, we obtain the two following results:

THEOREM 16.7. ([C4]) *Let $(a, b) \subset (0, 1)$ and $T > T_0(a, b)$. Assume (16.21)-(16.22), and the growth condition (16.26).*

Then for any $(u_0, u_1) \in H_0^1(0, 1) \times L^2(0, 1)$, there exists $h \in L^2((0, 1) \times (0, T))$ satisfying (16.19) and such that the solution u of (16.25) also satisfies (16.18).

THEOREM 16.8. ([C4]) *Let $(a, b) \subset (0, 1)$ and $T > T_0(a, b)$. Assume (16.23)-(16.24), and the growth condition (16.26).*

Then for any $(u_0, u_1) \in H_0^1(0, 1) \times L^2(0, 1)$, there exists $h \in L^2((0, 1) \times (0, T))$ satisfying (16.20) and such that the solution u of (16.25) also satisfies (16.18).

16.3.4. The key tool: observability inequality in "arbitrarily short time".

Consider $p \in L^\infty((0, 1) \times (0, T))$, and ϕ the solution of

$$(16.27) \quad \begin{cases} \phi_{tt} - \phi_{xx} + p(x, t)\phi = 0, & (x, t) \in (0, 1) \times \mathbb{R}_+, \\ \phi(0, t) = \phi(1, t) = 0, & t \in \mathbb{R}_+. \end{cases}$$

Theorems 16.5 and 16.7 derive from the following result:

THEOREM 16.9. ([C4]) *Let $(a, b) \subset (0, 1)$ and $T > T_0(a, b)$. Assume (16.21)-(16.22). Then there exists $C_1, C_2 > 0$ such that, for all ϕ solution of (16.27) (with initial conditions $(\phi_0, \phi_1) \in L^2(0, 1) \times H^{-1}(0, 1)$),*

$$(16.28) \quad \|\phi_0\|_{L^2(0,1)}^2 + \|\phi_1\|_{H^{-1}(0,1)}^2 \leq C_1 e^{C_2 \sqrt{\|p\|_\infty}} \int_J \int_a^b \phi^2(x, t) dx dt,$$

where $J = \cup_{j=0}^n J_j$.

16.4. On-off stabilization of second order evolution equations (Joint work with A. Haraux and J. Vancostenoble [C5])

16.4.1. Introduction.

Motivated by several works on ordinary differential equations, we are interested in the asymptotic stability of *intermittently controlled* partial differential equations. This question has been widely studied in the case of ordinary differential equations, (see for example [320, 195, 297]). The typical problem is the oscillator damped by an *on-off* damping :

$$(16.29) \quad u'' + u + a(t)u' = 0, \quad t > 0,$$

where $a : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is continuous nonnegative. For each solution u of (16.29), we define its energy by

$$\forall t \geq 0, \quad E_u(t) = \frac{1}{2}u(t)^2 + \frac{1}{2}u'(t)^2.$$

The derivative of the energy is

$$E'(t) = u(t)u'(t) + u'(t)u''(t) = -a(t)u'(t)^2,$$

hence the energy is always nonincreasing, but remains constant on the time intervals for which $a = 0$, and the decay is "very small" if a is "very small". Denote $\ell := \lim_{t \rightarrow \infty} E(t)$. Many authors (see in particular [320, 195, 297]) investigated the links between the distribution of sets where a is positive and the property $\ell = 0$.

Assume that there exists a sequence $(I_n)_{n \geq 0}$ of disjoint open intervals in $(0, +\infty)$ such that

$$\forall t \in I_n, \quad 0 < m_n \leq a(t) \leq M_n < \infty.$$

Roughly speaking, the energy is strictly decreasing on the time intervals I_n and just nonincreasing elsewhere. It is natural to wonder whether the decay on the time intervals I_n is sufficient to drive the energy to zero. Obviously some condition on the length of the intervals I_n has to be imposed to ensure $\ell = 0$. Smith [320] proved the following *sufficient condition of asymptotic stability* :

THEOREM 16.10. (Smith [320]) *Assume that*

$$(16.30) \quad \sum_{n=0}^{\infty} m_n T_n \delta_n^2 = +\infty,$$

where m_n and M_n are the minimum and the maximum values of $a(t)$ in I_n , T_n is the length of I_n and $\delta_n = \min(T_n, (1 + M_n)^{-1})$. Then equation (16.29) is asymptotically stable, i.e. every solution u of (16.29) satisfies $E_u(t) \rightarrow 0$ as $t \rightarrow \infty$.

For example, in the case of a damping such that $0 < m \leq a(t) \leq M$ for all $t \in I_n$ for all $n \in \mathbb{N}$, the condition (16.30) reduces to

$$(16.31) \quad \sum_{n=0}^{\infty} T_n^3 = +\infty.$$

It is noteworthy that (1.3) is also necessary in the following sense: given $\varepsilon > 0$ as small as we want, Pucci-Serrin [297] constructed an example for which the sequence $(T_n)_n$ satisfies

$$\sum_{n=0}^{\infty} T_n^{3-\varepsilon} = +\infty, \quad \text{while} \quad \sum_{n=0}^{\infty} T_n^3 < +\infty,$$

and suitable initial conditions such that the energy decays to some $\ell > 0$.

Note also that, under condition (16.30), the distribution of the intervals I_n has no importance. Only their size is important.

Condition (16.30) also requires that the damping coefficient a is not "too small" or "too large", in order to prevent "underdamping" or "overdamping". These phenomena are also a source of lack of strong stability (see [298, 284], [C2]), where the stability is studied for the wave equation, but always under the condition that the function a remains positive).

To our knowledge, stability properties for such "intermittently controlled" systems have not yet been studied in the case of *partial* differential equations.

In [C3, C4], we studied the effect of an on-off feedback on the wave equation. In [C4], the only case for which the situation was not different from the situation of the ordinary differential equations was the wave equation damped by an *uniformly distributed* on-off feedback. In that case, asymptotic stability occurs for any value of T . Thus the distribution of the intervals damping has no importance.

Here, we study the wave equation *uniformly damped* by a general on-off feedback (in particular not necessarily periodic). We prove that the uniformly damped wave equation behaves exactly like the oscillator, in the sense that Theorem 16.10 is still true. More generally, we prove this result in an abstract setting that includes both the oscillator and wave-like or plate-like equations and that also includes *bounded or unbounded* and *linear and nonlinear* damping operators.

In particular, this gives for the result of Smith a new proof quite different from the original one relying on monotonicity properties of the solutions of (16.29). Our method is based on a preliminary result which is interesting in itself: we provide an estimate of the energy decay on a *short* time interval (see Theorem 3.2). This estimate is true both for ordinary and partial differential equations.

16.4.2. Main results.

16.4.2.1. Abstract setting.

Let H be a real Hilbert space endowed with the scalar product $(\cdot, \cdot)_H$ and the norm $|\cdot|_H$.

Assume that $A : D(A) \subset H \rightarrow H$ is a linear self-adjoint and coercive operator on H with dense domain. We define $V = D(A^{1/2})$ endowed with the scalar product $((\cdot, \cdot))_V$ and the norm $\|\cdot\|_V$ defined by

$$\forall v \in V, \quad \|v\|_V^2 = |A^{1/2}v|_H^2 = \langle \tilde{A}v, v \rangle_{V', V},$$

where $\tilde{A} \in \mathcal{L}(V, V')$ represents the extension of A .

Let also W be a Hilbert space endowed with the norm $\|\cdot\|_W$ and such that

$$V \hookrightarrow W \hookrightarrow H \equiv H' \hookrightarrow W' \hookrightarrow V'.$$

with dense imbeddings. We also assume that A satisfies the following property:

$$(16.32) \quad \exists \lambda_0, C_0 > 0, \quad \text{such that: } \forall \lambda \in]0, \lambda_0], \quad (I + \lambda A)^{-1} \in \mathcal{L}(W) \\ \text{and } \|(I + \lambda A)^{-1}\|_{\mathcal{L}(W)} \leq C_0.$$

Next we consider a *time-dependent* operator B such that

$$(16.33) \quad B \in L^\infty(J, \text{Lip}(W, W')),$$

$$(16.34) \quad \forall t \in J, \forall w, z \in W, \quad \langle B(t)w - B(t)z, w - z \rangle_{W', W} \geq 0,$$

$$(16.35) \quad \forall t \in J, \forall w \in W, \quad \langle B(t)w, w \rangle_{W', W} \geq b^2(t) \|w\|_W^2,$$

$$(16.36) \quad \forall t \in J, \forall w, z \in W, \quad \|B(t)w - B(t)z\|_{W'} \leq Cb(t)^2 \|w - z\|_W,$$

where $J = [0, T]$ with $T > 0$ and where $b(t) \geq 0$ with $b \in L^2(J)$. Note that $B(t)$ is a *priori unbounded and nonlinear*. (The choice $W = H$ corresponds to the particular case of a bounded operator).

Now we consider the following second order evolution equation

$$(16.37) \quad u'' + Au + B(t)u' = 0, \quad t > 0,$$

with the initial conditions

$$(16.38) \quad u(0) = u_0 \in V, \quad u'(0) = u_1 \in H.$$

This problem is well-posed, see [C5].

16.4.2.2. *An energy decay estimate on a short time interval.*

Assume that (16.32)-(16.36) hold. In order to study the asymptotic behavior of the energy, we first prove the following result, interesting in itself, concerning the estimate of energy decay on a short interval of time :

THEOREM 16.11. ([C5]) *Let $T > 0$ be fixed and assume that there exist $M, m > 0$ such that*

$$(16.39) \quad \forall t \in (0, T), \quad \forall v \in W, \quad \langle B(t)v, v \rangle_{W', W} \geq m \|v\|_W^2,$$

and

$$(16.40) \quad \forall t \in (0, T), \quad \forall v \in W, \quad \|B(t)v\|_{W'}^2 \leq M \langle B(t)v, v \rangle_{W', W}.$$

Then there exists $c > 0$ (independent of T) such that, for all $(u_0, u_1) \in V \times H$, the solution u of (16.37)-(16.38) satisfies

$$(16.41) \quad E(T) \leq \frac{1}{1 + c \frac{m}{T^{-3} + T^{-1} + MmT^{-1}}} E(0).$$

Theorem 16.11 provides an estimate of the decay of the energy that is valid for t small. It has to be noted that, in general, estimates of the decay of the energy is provided for t large enough, even in the case of uniformly distributed damping terms.

16.4.2.3. *A condition for asymptotic stability.*

Assume that (16.32)-(16.36) hold for any $T > 0$. Then it follows from Theorem 16.11 that the result of Smith [320] may be extended to the case of problem (16.37)-(16.38):

THEOREM 16.12. ([C5]) *Consider a sequence $(I_n)_{n \geq 0}$ of disjoint open intervals in $(0, +\infty)$ and assume that, for all $n \geq 0$, there exist $M_n, m_n > 0$ such that*

$$(16.42) \quad \forall t \in I_n, \quad \forall v \in W, \quad \langle B(t)v, v \rangle_{W', W} \geq m_n \|v\|_W^2,$$

and

$$(16.43) \quad \forall t \in I_n, \quad \forall v \in W, \quad \|B(t)v\|_{W'}^2 \leq M_n \langle B(t)v, v \rangle_{W', W}.$$

Assume that the following condition holds :

$$(16.44) \quad \sum_{n=0}^{\infty} m_n T_n \min(T_n^2, \frac{1}{1 + m_n M_n}) = +\infty,$$

where T_n denotes the length of I_n . Then equation (16.37)-(16.38) is asymptotically stable, i.e. for all $(u_0, u_1) \in V \times H$, the solution u of (16.37)-(16.38) satisfies $E_u(t) \rightarrow 0$ as $t \rightarrow \infty$.

(Note we also provide in [C5] an explicit estimate of the energy decay, and we study the case of positive-negative feedbacks.)

16.4.2.4. *Examples.*

Our results apply in particular to the following nonlinear wave equation:

$$(16.45) \quad \begin{cases} u'' - \Delta u + a_1(t)f(u') - a_2(t)\Delta u' = 0, & x \in \Omega, t > 0, \\ u = 0, & x \in \partial\Omega, t > 0, \\ u(t=0) \in H_0^1(\Omega), \quad u'(t=0) \in L^2(\Omega), \end{cases}$$

and the following plate equation

$$(16.46) \quad \begin{cases} u'' + \Delta^2 u + a_1(t)f(u') - a_2(t)\Delta u' + a_3(t)\Delta g(\Delta u') = 0, & x \in \Omega, t > 0, \\ u = 0, \quad \frac{\partial u}{\partial \nu} = 0, & x \in \partial\Omega, t > 0, \\ u(t=0) = u_0 \in H_0^2(\Omega), \quad u'(t=0) = u_1 \in L^2(\Omega), \end{cases}$$

see [C5] for the details.

**Additional results on the stabilization of wave
type equations**
(Joint works with M. M. Cavalcanti and V. N. Cavalcanti [C6, C7])

17.1. Introduction: motivation and related papers

17.1.1. Motivation. The motivation was to see if the method (developed during my Ph. D.) giving stabilization results could provide (good) results concerning other problems, namely:

- a damped viscoelastic equation,
- a damped nonlinear wave equation.

17.1.2. The related papers [C6, C7].

- [C7]: we study the stabilization of a damped viscoelastic equation; we extend the classical stability estimates, being able to provide explicit (and new) estimates, known ([C1]) to be optimal without the memory term, for a large class of memory terms;
- [C6]: we study a damped wave equation with a nonlinear source term; once again, we extend the classical stability results.

17.2. General decay estimates for viscoelastic dissipative systems
(Joint work with M. M. Cavalcanti and V. N. Cavalcanti [C7])

17.2.1. Introduction.

This work is concerned with the uniform decay rates of solutions of the viscoelastic problem with nonlinear boundary damping

$$(17.1) \quad \left\{ \begin{array}{l} y_{tt} - \Delta y + \int_0^t h(t - \tau) \Delta y(\tau) d\tau = 0 \quad \text{in } \Omega \times (0, \infty) \\ y = 0 \quad \text{on } \Gamma_1 \times (0, \infty) \\ \frac{\partial y}{\partial \nu} - \int_0^t h(t - \tau) \frac{\partial y}{\partial \nu}(\tau) d\tau + g(y_t) = 0 \quad \text{on } \Gamma_0 \times (0, \infty) \\ y(x, 0) = y^0(x); \quad y_t(x, 0) = y^1(x) \quad \text{in } \Omega, \end{array} \right.$$

where Ω is a bounded domain of \mathbb{R}^n , $n \geq 1$, with a smooth boundary $\Gamma = \Gamma_0 \cup \Gamma_1$. Here, Γ_0 and Γ_1 are closed and disjoint and ν represents the unit outward normal to Γ . We assume the following geometrical condition holds true: there exists some $x^0 \in \mathbb{R}^n$ and $\delta > 0$ such that

$$\forall x \in \Gamma_0, (x - x^0) \cdot \nu(x) \geq \delta > 0, \quad \text{and} \quad \forall x \in \Gamma_1, (x - x^0) \cdot \nu(x) \leq 0.$$

When $n = 1$ and $\Omega = (0, L)$, for instance, problem (17.1) describes the motion's equation of a body made of viscoelastic material, with long memory, which occupies

the interval $[0, L]$ and such that one of its end is clamped while the other one is free and is subject to the action of a nonlinear dissipation.

When $h = 0$, problem (17.1) was widely studied; under quite strong assumptions on the geometry and on the feedback, some of the most important papers are those of Nakao [282], Haraux [190], Chen and Wong [92], Conrad et al. [95], Zuazua [358], Komornik [223]; using deeper techniques, Lasiecka and Tataru [230] studied the problem under very general conditions on the geometry and the feedback term.

Now, when $g = 0$, we refer the reader to the work of Barbosa Sobrinho and Muñoz Rivera [20] who consider the viscoelastic equation under Signorini's contact conditions. Let us also mention other papers in connection with viscoelastic effects such as Dafermos [106, 107], Muñoz Rivera and Jian [279], Lagnese [228], among others.

In [86], the authors studied the global existence of strong and weak solutions of (17.1) under the classical assumption

$$\int_0^\infty h(t) dt < 1.$$

Next they studied the uniform decay of the energy related with this problem:

$$E(t) = \frac{1}{2} \int_\Omega y_t(x, t)^2 + |\nabla y(x, t)|^2 dx$$

under quite restrictive assumptions on both the damping function g and the kernel function h : the function h had to behave exactly like e^{-mt} , and the function g had a polynomial behavior near zero.

The goal of this part is to generalize the results of [86] *under strongly weakened assumptions*: we prove that the energy goes uniformly to zero at infinity

- *without imposing a specific assumption on the behavior of g near zero* (as in my paper [260]),
- *and moreover under very weak conditions on the kernel function h .*

Our decay estimates depend both on the behavior of g near zero, *and* on the behavior of the relaxation function h at infinity.

17.2.2. Main results.

We will work under the following

- **(Hyp. 16.1) Assumptions on the nonlinearity g .** consider $g : \mathbb{R} \rightarrow \mathbb{R}$ a nondecreasing C^1 function such that $g(s)s > 0$ for all $s \neq 0$, and suppose that there exist a function $g_0 : \mathbb{R} \rightarrow \mathbb{R}$ strictly increasing, odd and of class C^1 , and $c > 0$ such that

$$(17.2) \quad \begin{cases} g_0(|s|) \leq g(|s|) \leq g_0^{-1}(|s|) \text{ for all } |s| \leq 1, \\ c|s| \leq g(|s|) \leq \frac{1}{c}|s| \text{ for all } |s| \geq 1; \end{cases}$$

- **(Hyp. 16.2) Assumptions on the kernel h .** assume that $h : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a strictly decreasing C^1 function such that

$$(17.3) \quad 1 - \int_0^\infty h(s) ds = \ell > 0.$$

Moreover assume that $h'(t) < 0$ for all $t \geq 0$.

The global existence for strong and weak solutions has been studied in [86].

The energy related to problem (17.1) is given by

$$(17.4) \quad E(t) = \frac{1}{2} \int_\Omega y_t(x, t)^2 + |\nabla y(x, t)|^2 dx.$$

Now we are in a position to state our results:

THEOREM 17.1. ([C7]) *Assume (Hyp. 16.1) and (Hyp. 16.2). Moreover assume that $h(0)$ and $\|h\|_{L^1(0,\infty)}$ are sufficiently small.*

1. *If $-h'/h$ is bounded from below by some positive constant m , then the energy is nonincreasing and goes to zero as fast as in the case of the wave equation (that corresponds to $h = 0$).*

2. *Assume that $-h'/h$ decays to zero at infinity. Then there exists a nondecreasing concave function $\phi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that $\phi(t) \rightarrow \infty$ as $t \rightarrow \infty$ and such that the energy of every weak solution satisfies the following decay rate estimate*

$$(17.5) \quad E(t) \leq \frac{C(E(0))}{\phi(t)}.$$

17.2.3. Examples and comments.

It is not easy to exhibit in the general case such a function ϕ . The problem can be summed up that way: the kernel and the nonlinearity give some constraints that the suitable function ϕ has to verify. However in very general situations, we are able to construct explicitly such a function ϕ , and we can compute the associated decay rate:

Case 1: g_0 linear. If $-h'/h$ is bounded from below by some positive constant m , then the energy decays exponentially to zero. If $-h'/h$ decays to zero at infinity, then the energy decays as

$$E(t) \leq CE(0) h(t)^\omega$$

for some $\omega > 0$. For example if $h(t) = h(0)/(1+t)^q$ with $q > 1$ then the energy decays at least polynomially to zero. If $h(t) = e^{-t^p}$ with $0 < p \leq 1$, then $E(t) \leq CE(0) e^{-\omega t^p}$.

Case 2: g_0 polynomial near 0: $g_0(s) = s^p$ for some $p > 1$. Then if $-h'/h$ is bounded from below by some positive constant m , then the energy decays as

$$E(t) \leq \frac{C(E(0))}{(1+t)^{2/(p-1)}}.$$

If $-h'/h$ decays to zero, the energy decays as

$$E(t) \leq \frac{C(E(0))}{(-\ln h(t))^{2/(p-1)}}.$$

Case 3: Assume that the function $G_0 : u \rightarrow g_0(u)/u$ is nondecreasing on a neighborhood of $(0, 1)$ and define for $t \geq 1$

$$(17.6) \quad \phi_0^{-1}(t) = 1 + \int_1^t \frac{1}{G_0(\frac{1}{\tau})} d\tau :$$

then ϕ_0 is a concave nondecreasing function that satisfies $\phi_0(t) \rightarrow \infty$ as $t \rightarrow \infty$ (see [260]) and can be easily extended on $[0, 1)$ such that it remains concave nondecreasing. Then, if $-h'/h \geq \xi \phi_0'$ for some $\xi > 0$, the energy decays as in the case $h = 0$:

$$E(t) \leq C(E(0)) g_0\left(\frac{1}{t}\right)^2.$$

All these estimates are optimal in the case of the wave equation ($h = 0$) (see [C1]). When h is not equal to zero, our estimates seem to say that the more slowly the function h decays to zero, the more slowly the energy decays. However since the memory term has also a damping effect, the problem that we consider has two damping terms, and it would be rather natural that the combination of these two dampings damps the system faster than just one feedback. In fact, *this is wrong*, as proved in Fabrizio and Polidoro [139]: roughly speaking, the energy

of the viscoelastic system cannot decay faster than the relaxation function. In the same spirit as [139], we provide in [C7] a simple example of this fact, using the Laplace transform.

It has to be noted that in such problems considering such weak assumptions on the kernel h is not usual. Indeed, the usual assumptions on the kernel function h are of the type

$$h(t) = h(0)e^{-t} \quad \text{or} \quad h(t) = \frac{h(0)}{(1+t)^q} \quad \text{with } q > 2$$

see the works of Muñoz-Rivera et al, e. g. [278]. In the literature, these quite restrictive assumptions on h come from technical differential inequalities imposed on h since the unique damping mechanism acting on the system is given by the memory term. We do not apply these techniques here, and we generalize the method introduced in [260]. The main idea is the following: the kernel function h and the damping function g bring constraints (of course of a different type), and the problem of the uniform stabilization of the energy turns to be an optimization problem: construct some weight function ϕ that satisfies at the same time these constraints. The construction is based on the following general lemma about convergent and divergent series:

LEMMA 17.2. ([C7]) *Given $u_n \geq 0$ such that $\sum u_n = \infty$, and $\varepsilon_n \geq 0$ such that $\varepsilon_n \rightarrow 0$ as $n \rightarrow \infty$, then there exists a nonincreasing sequence $(\theta_n)_n$ such that $\theta_n \leq 1$, $\sum \theta_n u_n = \infty$ while $\sum \theta_n u_n \varepsilon_n < \infty$.*

17.3. Decay rates for the wave equation with nonlinear damping and source term

(Joint work with M. M. Cavalcanti and V. N. Cavalcanti [C6])

We studied the problem of the wave equation with polynomial source term and boundary damping:

$$(17.7) \quad \begin{cases} u_{tt} - \Delta u = |u|^p u, & x \in \Omega, t > 0, \\ u = 0, & x \in \Gamma_0, t > 0, \\ \partial_\nu u + q(u_t) = 0, & x \in \Gamma_1, t > 0, \\ u(0, x) = u(x), u_t(0, x) = v_0(x), & x \in \Omega, \end{cases}$$

under the same classical geometrical assumptions on Ω . There is a competition between the possible blow-up generated by the source term, and the damping term. This has been studied in many papers, see, e.g., Georgiev-Todorova [169], Ikehata [207], Vitillaro [343].

In [C6], we prove a result of global existence and decay of the energy for strong and weak solutions of (17.7), under general assumptions on the nonlinear feedback (in the spirit of [260]) and initial conditions 'close' to the origin ('close' being precised in terms of the nonlinear source).

Part 4

Population dynamics

**Qualitative properties of two population dynamics
systems describing pregnancy
(Joint works with G. Fragnelli and J.
Vancostenoble [D1, D2])**

18.1. Introduction: motivation and related papers

18.1.1. Motivation.

The goal of this part is to study two models describing the evolution of pregnant individuals in some population. These models are described by a system of parabolic and nonlocal equations. In order to be close to what is observed (individuals have a limited expectance of life), the models contain a non integrable coefficient (which, of course, will bring some mathematical difficulties). We will focus on the asymptotic behavior and also some qualitative properties.

18.1.2. The related papers [D1, D2].

- [D1]: we study a coupled system describing the evolution (in time and space) of the total population and of the sub-population of pregnant individuals, taking care of the gestation age; the coupling comes from two elementary facts:
 - the quantity of individuals becoming pregnant (hence at the age of gestation equal to 0) is related to the quantity of total population,
 - the variation of the total population depends on the number of newborns (that is related to the number of pregnant individuals);
 we prove several qualitative properties (maximum principles), and we describe the asymptotic behaviour of these two populations, with respect of the natural parameters (fertility and mortality rates).
- [D2]: we modify the previous model taking care of the fact that the quantity of individuals becoming pregnant depends only of the quantity of the total population that is not already pregnant; this gives a more natural problem, having more natural qualitative properties, and we repeat the same analysis.

**18.2. A first model concerning the evolution of pregnant individuals
([D1])**

18.2.1. Introduction.

The goal of this part is to analyze a model of population dynamics describing pregnancy and diffusion phenomena. We consider a spatially distributed population where individuals are characterized by their position. (In particular, no sex or age differences are allowed). A special attention is paid to the mechanism of pregnancy, that leads to a delay in the renewal of the population. One novelty of the model is that it takes into account the events that may happen during the gestation : the pregnant individuals may move, die or bear before the term. For this reason, within the total population,

- we distinguish pregnant individuals and we refer to them by considering their 'age of gestation' a , ranging in $[0, r]$ where $r > 0$ is fixed;
- individuals are supposed to die at a given death rate d , whereas the pregnant individuals are supposed to die at a greater death rate $d' \geq d$;
- individuals are also supposed to be fecundated at a rate f_0 and to bear according to a rate $b = b(a)$;
- moreover, we assume that the dispersal of the population through the environment is realized by the Laplace operator.

Hence, summing up, let $\Omega \subset \mathbb{R}^n$ be open, connected and bounded with smooth boundary, and denote

- $u(t, x)$ the total population at time t and position x
- and $v(t, a, x)$ the subpopulation of pregnant individuals at time t and position x with a time of gestation a .

Then the dynamics of the two populations is governed by the following equations:

$$(18.1) \quad \begin{cases} u_t(t, x) - \Delta u(t, x) + du(t, x) = \int_0^r b(a)v(t, a, x) da, \\ v_t(t, a, x) + v_a(t, a, x) - \Delta v(t, a, x) + d'v(t, a, x) = -b(a)v(t, a, x), \\ v(t, 0, x) = f_0u(t, x), \\ u(t, x)|_{\partial\Omega} = 0, \\ v(t, a, x)|_{\partial\Omega} = 0, \\ u(0, x) = u_0(x), \\ v(0, a, x) = v_0(a, x), \end{cases}$$

where $(t, a, x) \in \mathbb{R}_+^* \times (0, r) \times \Omega$. Let us give some additional explanation on the model:

- the term $b(a)v(t, a, x)$ represents the density of pregnant individuals that bear at time t , at place x and after a time of gestation denoted by a . The minus sign comes from the fact that the pregnant individual that bears is no more pregnant, and hence this term behaves like a mortality one;
- the related increase of the total population is

$$\int_0^r b(a)v(t, a, x) da :$$

this represents the newborn individuals at time t and place x ;

- the boundary condition

$$v(t, 0, x) = f_0u(t, x)$$

is another coupling condition: it says that the number of fecundated individuals at time t and place x is a fixed proportion of the whole population that is present at that place and time. This is rather natural if the pregnant individuals population is rather small with respect to the total population;

- at last, the boundary conditions $u(t, x) = 0 = v(t, a, x)$ for $t > 0$, $x \in \partial\Omega$, $a \in (0, r)$ mean that no individual and no pregnant individual reaches the borderline.

18.2.2. Main results : the diffusive model.

Before mentioning some important related papers, let us detail our main results concerning the diffusive model (18.1). In this section, we shall make the following assumptions:

$$(18.2) \quad r > 0, \quad d' \geq d > 0 \quad \text{and} \quad f_0 > 0,$$

$$(18.3) \quad \begin{cases} b \in L_{\text{loc}}^{\infty}([0, r)) & \text{nonnegative nondecreasing such that } \int_0^r b(a) da = +\infty, \\ b = 0 & \text{on } \mathbb{R} \setminus (0, r), \end{cases}$$

$$(18.4) \quad \begin{cases} u_0 \in L^2(\Omega) & \text{nonnegative,} \\ v_0 \in L^2((0, r) \times \Omega) & \text{nonnegative such that } \sqrt{b}v_0 \in L^2((0, r) \times \Omega). \end{cases}$$

In the following, we set

$$\forall s \in (0, r), \quad \tilde{b}(s) := \int_0^s b(\sigma) d\sigma;$$

18.2.2.1. *Well-posedness.*

For any characteristic line

$S := \{(t, a) \in (0, T) \times (0, r) \mid a - t = a_0 - t_0\} = \{(t_0 + s, a_0 + s) \mid s \in (0, r - a_0)\}$, with $(t_0, a_0) \in (0, T) \times \{0\} \cup \{0\} \times (0, r)$, we denote by $W^{1,1}(S; L^2(\Omega))$ the space of functions $v : S \rightarrow L^2(\Omega)$ such that $v(t_0 + \cdot, a_0 + \cdot) : (0, r - a_0) \rightarrow L^2(\Omega)$ belongs to $W^{1,1}((0, r - a_0); L^2(\Omega))$. Then we prove

THEOREM 18.1. ([D1]) *Assume (18.2), (18.3) and (18.4). For any $T > 0$, problem (18.1) has a unique solution (u, v) on $(0, T)$ such that*

$$(18.5) \quad u \in \mathcal{C}([0, T]; L^2(\Omega)) \cap W^{1,1}(0, T; L^2(\Omega)) \cap L^2(0, T; H_0^1(\Omega)) \cap L_{\text{loc}}^2(0, T; H^2(\Omega)),$$

$$(18.6) \quad v \in \mathcal{C}(\bar{S}; L^2(\Omega)) \cap W^{1,1}(S; L^2(\Omega)) \cap L^2(S; H_0^1(\Omega)) \cap L_{\text{loc}}^2(S; H^2(\Omega)),$$

for almost any characteristic line S of the equation $a - t = a_0 - t_0$,

$$(18.7) \quad \sqrt{b}v \in L^2((0, T) \times (0, r) \times \Omega),$$

$$(18.8) \quad \int_0^r b(a)v(a) da \in L^2((0, T) \times \Omega).$$

18.2.2.2. *Qualitative properties.*

Here we define

$$V(t, x) := \int_0^r v(t, a, x) da,$$

which represents the total population of pregnant individuals at time t and place x , and we prove some important qualitative properties.

THEOREM 18.2. ([D1]) ([D1]) *Assume (18.2), (18.3) and (18.4). Then the solution (u, v) of (18.1) satisfies:*

$$(18.9) \quad u \geq 0 \text{ a.e. on } (0, T) \times \Omega \text{ and } v \geq 0 \text{ a.e. on } (0, T) \times (0, r) \times \Omega,$$

$$(18.10) \quad \forall t, \quad v(t - \varepsilon, r - \varepsilon, \cdot) \rightarrow 0 \text{ in } L^2(\Omega) \text{ as } \varepsilon \rightarrow 0.$$

Moreover, if there exists some $\theta \in [0, 1[$ such that

$$(18.11) \quad V_0(x) \leq \left(1 - \theta - f_0 \int_0^r e^{-\tilde{b}(\tau)} e^{-(d' - d)\tau} d\tau\right) u_0(x)$$

for almost all $x \in \Omega$, then the solution satisfies

$$(18.12) \quad \forall t \geq 0, \forall x \in \Omega, \quad V(t, x) \leq (1 - \theta)u(t, x).$$

In particular, if $\theta = 0$, (18.11) is a condition that insures that $V(t, x) \leq u(t, x)$.

We also prove the following comparison principle which expresses the fact that the population densities u and v decrease when the mortality rates d and d' increase.

PROPOSITION 18.3. ([D1]) *We assume (18.3), (18.4) and $r > 0$, $f_0 > 0$. Let $d'_1 \geq d_1 > 0$ and $d'_2 \geq d_2 > 0$ be such that $d'_2 \geq d'_1$ and $d_2 \geq d_1$. Consider (u^1, v^1) the solution of (18.1) with $(d, d') = (d_1, d'_1)$ and (u^2, v^2) the solution of (18.1) with $(d, d') = (d_2, d'_2)$. Then*

$$\begin{cases} u^2(t, x) \leq u^1(t, x) & \text{for } (t, x) \in \mathbb{R}_+ \times \Omega, \\ v^2(t, a, x) \leq v^1(t, a, x) & \text{for } (t, a, x) \in \mathbb{R}_+ \times (0, r) \times \Omega. \end{cases}$$

18.2.2.3. Asymptotic behavior.

Finally we also completely study the asymptotic behavior of the solution: denote by λ_0 the smallest eigenvalue of the Laplace operator with Dirichlet boundary conditions, φ_0 the associated eigenfunction, and

$$\tilde{R}_0 := f_0 \int_0^r b(s) e^{-(\tilde{b}(s) + (d' + \lambda_0)s)} ds.$$

Then we may prove

THEOREM 18.4. ([D1]) *Assume (18.2), (18.3) and (18.4). Assume that $(u_0, v_0) \neq (0, 0)$ (otherwise the solution is identically equal to zero). Consider α^* (given in [D1]). Then there exists some explicit positive constant c_0^0 (defined later) such that the solution of (18.1) satisfies*

(i) *if $\tilde{R}_0 < d + \lambda_0$, then $\alpha^* < d + \lambda_0$ and u goes to zero exponentially fast as $t \rightarrow \infty$. More precisely, there exists $C > 0$ such that*

$$\|u(t, \cdot)\|_{L^2(\Omega)} \leq C e^{(\alpha^* - d - \lambda_0)t} (\|u_0\|_{L^2(\Omega)} + \|v_0\|_{L^2(\Omega \times (0, r))}).$$

(ii) *if $\tilde{R}_0 = d + \lambda_0$, then $\alpha^* = d + \lambda_0$ and u converges exponentially fast to $c_0^0 \varphi_0(\cdot)$ as $t \rightarrow \infty$: there exists $C > 0$ and $\omega > 0$ such that*

$$\|u(t, \cdot) - c_0^0 \varphi_0(\cdot)\|_{L^2(\Omega)} \leq C e^{-\omega t} (\|u_0\|_{L^2(\Omega)} + \|v_0\|_{L^2(\Omega \times (0, r))}).$$

(iii) *if $\tilde{R}_0 > d + \lambda_0$, then $\alpha^* > d + \lambda_0$ and u goes exponentially fast to infinity as $t \rightarrow \infty$: there exists a function $u^0(t)$ such that*

$$\|u(t, \cdot)\|_{L^2(\Omega)} \geq u^0(t) \sim c_0^0 e^{(\alpha^* - d - \lambda_0)t} \text{ as } t \rightarrow \infty.$$

(The constants C and ω do not depend on the initial conditions (u_0, v_0) .)

Note that the function $u^0(t)$ in (iii) is completely determined in the proof. Concerning the behavior of v , it is then easy to see that $v(t, \cdot, \cdot)$ goes exponentially fast (in $L^2(\Omega \times (0, r))$) to zero, to a steady state or to infinity as $t \rightarrow \infty$ respectively if $\tilde{R}_0 < d + \lambda_0$, $\tilde{R}_0 = d + \lambda_0$ or $\tilde{R}_0 > d + \lambda_0$.

18.2.3. Tools and comments.

18.2.3.1. *On the assumptions.* Note that in Nickel-Rhandi [285], the authors assume that $b \in L^1(0, r)$ so that some delay operator is well-defined. Here we prefer to assume

$$\int_0^r b(a) da = +\infty$$

as in S. Anița [9]. Indeed it is a "natural" assumption that insures $v(\cdot, r, \cdot) = 0$, which means that r is the maximal time of gestation (see Theorem 18.2 below). Of course, this induces several technical difficulties, and the wellposedness of (18.1) is proved when the initial conditions belong to some suitable weighted space (see assumption (18.4)).

18.2.3.2. *Ideas of the proofs.* Following the approach of [9], our proofs are based on a complete preliminary study of the nondiffusive problem, in particular using the Laplace transform to obtain integral formulas, and then we go back to the diffusive problem thanks to spectral decomposition.

18.2.3.3. *On the qualitative properties.* To our point of view, the more interesting qualitative property is (18.12), that means (when $\theta = 0$) that the total population $u(t)$ is larger than the total population of pregnant individuals $V(t)$.

Even under condition (18.11), the proof of (18.12) is not obvious since we want to compare u with V whereas we do not have a system of equations in (u, V) but in (u, v) . To prove (18.12), we first establish a similar property for the nondiffusive model which is also not obvious. Then (18.12) follows mainly from this result and arguments of spectral decomposition.

Note that, even if the result is very natural, it does not follow easily from the equations and the assumption (18.11) is really needed to prove it: indeed, we may construct a 'counter-example' to this property (see [D1]): if

$$d' = d \quad u_0 = \int_0^r v_0(a) da \quad \text{and} \quad f_0 u_0 > 2 \int_0^r b(a) v_0(a) da,$$

then $u(t) < V(t)$ for some $t > 0$. Hence, we need to assume that the fecundation rate is "not too large" and that the pregnant individuals population is "small enough" with respect to the total population in order to ensure that $u(t) \geq V(t)$. We will come back on this question in the following section.

18.2.3.4. *On the asymptotic behavior.* This kind of result is classical for such models, we refer in particular to [9, 293], and the references listed in [D1].

18.3. An improved pregnancy model ([D2])

We have seen that the previous model is realistic only when the pregnant individuals population is "small enough". This comes directly from the coupling condition

$$v(t, 0, x) = f_0 u(t, x),$$

which is natural only when the pregnant individuals population is "small". A more natural condition is clearly

$$v(t, 0, x) = f_0 (u(t, x) - V(t, x)) :$$

the population $v(t, 0, x)$ that is fecundated at time t and place x is a fraction of the total population that *has not been fecundated* at time t and place x . This leads us to study the following model:

$$(18.13) \quad \begin{cases} u_t(t, x) - \Delta u(t, x) + du(t, x) = \int_0^r b(a) v(t, a, x) da, \\ v_t(t, a, x) + v_a(t, a, x) - \Delta v(t, a, x) + d'v(t, a, x) = -b(a)v(t, a, x), \\ v(t, 0, x) = f_0 \left(u(t, x) - \int_0^r v(t, a, x) da \right), \\ u(t, x)|_{\partial\Omega} = 0, \quad v(t, a, x)|_{\partial\Omega} = 0, \\ u(0, x) = u_0(x), \quad v(0, a, x) = v_0(a, x), \end{cases}$$

where $(t, a, x) \in \mathbb{R}_+^* \times (0, r) \times \Omega$. Note that there is a nonlocal local term in the first equation and now also in the (coupling) boundary condition.

18.3.1. Main results.

18.3.1.1. *Assumptions and well-posedness.* Let us make the following assumptions:

$$(18.14) \quad r > 0, \quad d' \geq d > 0 \quad \text{and} \quad f_0 > 0,$$

$$(18.15) \quad b \in L_{\text{loc}}^\infty([0, r)) \text{ such that } b \geq 0, \quad b \text{ nondecreasing,} \quad \int_0^r b(a) da = +\infty,$$

$$(18.16) \quad \begin{cases} u_0 \in L^2(\Omega), \quad u_0 \geq 0, \\ v_0 \in L^2((0, r) \times \Omega), \quad v_0 \geq 0 \text{ such that } \sqrt{b}v_0 \in L^2((0, r) \times \Omega). \end{cases}$$

Then we have a well-posedness result similar to Theorem 18.1, see [D2].

18.3.1.2. *Qualitative properties.*

THEOREM 18.5. ([D2]) *Assume (18.14), (18.15), (18.16) are satisfied.*

a) *Assume also that*

$$(18.17) \quad V_0(x) \leq u_0(x) \quad \text{in } \Omega.$$

Then the solution (u, v) of (18.13) satisfies:

$$(18.18) \quad u \geq 0 \quad \text{in } (0, T) \times \Omega \quad \text{and} \quad v \geq 0 \quad \text{in } (0, T) \times (0, r) \times \Omega,$$

and

$$(18.19) \quad \forall t \geq 0, \quad V(t, \cdot) \leq u(t, \cdot) \quad \text{in } \Omega.$$

b) *Assume that there is some $\theta \in [0, 1)$ such that*

$$(18.20) \quad V_0(x) \leq (1 - \theta)u_0(x) \quad \text{in } \Omega \quad \text{and} \quad d' - d \geq f_0 \frac{\theta}{1 - \theta}.$$

Then the solution (u, v) of (18.13) satisfies (18.18) and

$$(18.21) \quad \forall t \geq 0, \quad V(t, \cdot) \leq (1 - \theta)u(t, \cdot) \quad \text{in } \Omega.$$

18.3.1.3. *Asymptotic behaviour.* Finally we set

$$(18.22) \quad \begin{cases} \hat{d} = d + f_0, & \hat{d}' = \frac{d + d'}{2}, \\ \hat{f}_0 = 2f_0, & \hat{b} = b + \frac{d' - d}{2}. \end{cases}$$

We also denote by λ_0 the smallest eigenvalue of the Laplace operator with Dirichlet boundary conditions and

$$(18.23) \quad \hat{R}_0 := \hat{f}_0 \int_0^r \hat{b}(s) e^{-(\hat{b}(s) + (\hat{d}' + \lambda_0)s)} ds$$

where

$$\forall s \in (0, r), \quad \tilde{b}(s) := \int_0^s \hat{b}(\sigma) d\sigma.$$

Then we prove the following

THEOREM 18.6. **Asymptotic behaviour.** *Assume (18.14), (18.15), (18.16). Assume that $(u_0, v_0) \neq (0, 0)$ (otherwise the solution is identically equal to zero). Then the solution of (18.13) satisfies*

(i) *if $\hat{R}_0 < \hat{d} + \lambda_0$, then u goes to zero exponentially fast in $L^2(\Omega)$ as $t \rightarrow \infty$, and v goes to zero exponentially fast in $L^2((0, r) \times \Omega)$.*

(ii) *if $\hat{R}_0 = \hat{d} + \lambda_0$, then u converges exponentially fast (in $L^2(\Omega)$) to some stationary state $u^* \in L^2(\Omega)$ as $t \rightarrow \infty$, and v goes to some stationary state $v^* \in L^2((0, r) \times \Omega)$.*

(iii) *if $\hat{R}_0 > \hat{d} + \lambda_0$, then u goes exponentially fast to infinity in $L^2(\Omega)$ -norm as $t \rightarrow \infty$, and v goes exponentially fast to infinity in $L^2((0, r) \times \Omega)$ -norm.*

18.4. Perspectives

18.4.1. Refined problems.

It would be interesting to investigate more general models, for example related to the classical Mc Kendrick-Von Foerster model dealing with age-structured populations: one could investigate systems structured in space and time of gestation.

18.4.2. Controllability questions.

From the point of view of controllability: there are several works that study the controllability of parabolic problems concerning age-structured populations, see, e.g., [9, 2, 3]. At least from a mathematical point of view, it would be interesting to investigate the controllability properties of systems concerning pregnancy. Note that the equation in v has the same structure than the Crocco equation with constant coefficients, but of course the coupling and the nonlocal terms bring new difficulties.

Part 5

Blow-up of bounded solutions for ordinary differential equations

Blow up of bounded solutions for ordinary differential equations

(Joint works with J. Vancostenoble [E2] and with V. Komornik, M. Pierre and J. Vancostenoble [E1])

19.1. Introduction: motivation and related papers

19.1.1. Motivation.

This part was motivated by teaching questions: given a locally Lipschitz continuous function $f : X \rightarrow X$ in a Banach space X with norm denoted by $\|\cdot\|$ and given a point $e_1 \in X$, by the Cauchy-Lipschitz theorem the initial-value problem

$$(19.1) \quad u' = f(u), \quad u(0) = e_1$$

has a unique *maximal* continuously differentiable solution $u : I \rightarrow X$, defined on some open interval $I = (S, T)$; see, e.g., [93].

We also recall that if X is finite-dimensional and T is finite, then $\|u(t)\| \rightarrow \infty$ as $t \rightarrow T$; an analogous result holds for S . This blow-up phenomenon is proved usually by exploiting the fact that, in finite dimensional normed spaces, a locally Lipschitz continuous function is bounded on every bounded set.

When X is infinite dimensional, the same proof holds if f is locally Lipschitz continuous *and bounded on every bounded set*. However, this last additional assumption is no more a consequence of the Lipschitz property of f . Hence, what happens for an infinite dimensional Banach space? Is it possible that in some cases the associated solution

- exists only on a bounded interval of time
- and remains bounded on this interval?

19.1.2. The related papers [E1, E2].

- [E2]: we answer positively to that question, providing several examples of such f , E and u_0 for which the associated solution exists only on a bounded interval of time and remains bounded on this interval.
- [E1]: we give a general result: this exists in all infinite dimensional Banach space. (Due to publication mysteries, this general result was published before [E2], while being submitted later).

The idea is simple: we use the infinite number of free directions to make blow-up the solution.

19.2. An example in $c_0(\mathbb{N})$ (joint work with J. Vancostenoble [E2])

Consider $X_0 = c_0(\mathbb{N})$ the space of real sequences $(u_n)_n$ that go to 0 when $n \rightarrow \infty$, endowed with the usual norm

$$\|u\| := \max_n |u_n|.$$

$(X_0, \|\cdot\|)$ is a Banach space.

19.2.1. A locally Lipschitz function which is unbounded on some bounded sets.

Let $(h_n)_n$ be a sequence of real numbers such that $h_n \rightarrow +\infty$ when $n \rightarrow \infty$. Consider

$$(19.2) \quad \begin{aligned} f : X_0 &\rightarrow X_0, & f((u_n)_n) &= (v_n)_n, \text{ where} \\ & & \begin{cases} v_0 = 1, \\ \forall n \geq 0, v_{n+1} = h_n(u_n - 1)^+, \end{cases} \end{aligned}$$

where $x^+ := \max(0, x)$. Then

LEMMA 19.1. *The function f is well-defined and locally Lipschitz. However f is not Lipschitz on all the balls, and is unbounded on some balls. More precisely, given $u \in X_0$ and $r, r' \in \mathbb{R}$ such that $0 < r < 1 < r'$, then f is Lipschitz on the ball $B(u, r)$ and unbounded on the ball $B(u, r')$ (hence f is non Lipschitz on $B(u, r')$).*

19.2.2. The associated ordinary differential equation.

We study the problem

$$(19.3) \quad u'(t) = f(u(t)), \quad u(0) = 0$$

in the space $X_0 = c_0(\mathbb{N})$, where the function f has been defined previously. Since f is locally Lipschitz, the problem (19.3) has a unique maximal solution u , defined on the time interval $[0, T^*[$, with $0 < T^* \leq +\infty$. For $t < T^*$, $u(t)$ is an element of X_0 : $u(t) = ((u_n(t))_n$. Using the definition (19.2) of f , we see that, for $t < T^*$, the components of u verify the following system of ordinary differential equations:

$$(19.4) \quad \begin{cases} u'_0 = 1, & u_0(0) = 0, \\ u'_1 = h_0(u_0 - 1)^+, & u_1(0) = 0, \\ u'_2 = h_1(u_1 - 1)^+, & u_2(0) = 0, \\ \vdots \end{cases}$$

Of course, the solution (and its time interval of existence) depend on the constants h_n . For example if we choose $h_n = 0$ for all n , we obtain $u_0(t) = t$, $u_n(t) = 0$ for all $n \geq 1$ and all $t \geq 0$.

We proved the following result:

PROPOSITION 19.2. ([E2]) *One can choose positive constants h_n such that the maximal solution u of (19.3) is exactly defined on $[0, 2[$, stays bounded on $[0, 2[$, but has no limit in X_0 as $t \rightarrow 2^-$.*

We also provided similar examples in $\ell^\infty(\mathbb{N})$ and $C^0(\mathbb{R}_+)$, see [E2].

19.3. An example in a general setting (joint work with V. Komornik, M. Pierre and J. Vancostenoble [E1])

19.3.1. Main result.

We prove the following:

THEOREM 19.3. [E1] *Let X be an arbitrary infinite dimensional Banach space and (S, T) a bounded interval such that $S < 0 < T$. Then there exists a locally Lipschitz continuous function $f : X \rightarrow X$ and $e_1 \in X$ such that the maximal solution of (19.1) is exactly defined on (S, T) , although it remains bounded on (S, T) .*

Note that the maximal solution constructed in Theorem 1 is continuously differentiable on (S, T) but does not have any limit as $t \rightarrow T$ or as $t \rightarrow S$.

19.3.2. Comparison with the existing literature.

- This kind of counterexample is not completely new: probably the first example of this type is due to Dieudonné [123]. He constructed such a counterexample in the Banach space $X = c_0(\mathbb{N})$ of real sequences converging to zero and for a non autonomous equation $u' = f(t, u)$ with f continuous in (t, u) and locally Lipschitz in u .
- Deimling [114, 115] extended this construction to all Banach spaces having a Schauder basis. Since, any Banach space X has a closed subspace X_0 with a Schauder basis and since any continuous map of $\mathbb{R} \times X_0$ into $\mathbb{R} \times X_0$ may be extended to a continuous map of $\mathbb{R} \times X$ into $\mathbb{R} \times X_0$ (see again [115]), then this provides also a blow-up example for the non autonomous equation $u' = f(t, u)$ with a continuous f and for any Banach space.
- Note that several authors constructed dynamical systems with bounded but noncompact trajectories on *infinite* intervals $[0, +\infty)$. We refer, for instance, to [201] where such an example is given, in a specific Banach space, with a function f which is even Lipschitz continuous on bounded sets. We know that this last property would prevent bounded solutions from blowing up on a *finite* interval and this confirms the main difference between the two questions. However, one may relate them, at least formally, in the following way: if $x(\cdot)$ is a bounded noncompact solution on $[0, +\infty)$ of $x' = F(x)$, one may look for a function $\lambda : X \rightarrow (0, +\infty)$ such that $\int_0^{+\infty} \lambda(x(s)) ds < +\infty$. Then, the change of time $t \rightarrow \tau(t) = \int_0^t \lambda(x(s)) ds$ leads to the new system

$$\frac{dy}{d\tau} = F(y)/\lambda(y),$$

where blow up occurs in finite time. But, the construction of λ so that, moreover, F/λ be locally Lipschitz continuous, does not seem obvious. Note that, in particular, $1/\lambda$ is certainly not bounded on all bounded sets.

- In [123], Dieudonné also gave an example in $X = c_0(\mathbb{N})$ showing that Peano's existence theorem does not remain valid in infinite-dimensional Banach spaces under the weaker assumption that f is merely continuous. Later on, Godunov [175] constructed counterexamples for *all* infinite-dimensional Banach spaces. Different counterexamples with additional pathological properties were also constructed by Garay [198], [168]; see also the survey paper [247] of Lobanov and Smolyanov.
- Our example is elementary and works for all infinite-dimensional Banach spaces. Unlike the previous constructions, we consider *autonomous* differential equations. We still rely on the original idea of Dieudonné, but we avoid the difficulties related to the existence and non-existence of Schauder bases. Moreover, we construct a locally Lipschitz continuous function f directly on the whole space (this relies on a simple geometric argument).

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