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Normandie Université

## THÈSE

Pour obtenir le diplôme de doctorat

Spécialité MATHÉMATIQUES

Préparée au sein de l'Université de Caen Normandie

**Quantum transformation groupoids: An algebraic and analytical approach**

Présentée et soutenue par  
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## *Introduction*

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### **Motivation**

La motivation principale de ce travail de thèse est la construction de groupoïdes quantiques dans le contexte des algèbres d'opérateurs, en particulier dans le contexte des  $C^*$ -algèbres. L'un des exemples les plus importants de groupoïde usuel est un *groupoïde de transformations* associé à une action d'un groupe sur un ensemble [Ren80]. Sa structure contient l'information sur le groupe, sur l'action et sur l'ensemble sur lequel le groupe agit. C'est pourquoi l'objectif de cette thèse est d'introduire la notion de *groupoïde quantique de transformations* à la fois dans un contexte purement algébrique et dans celui de  $C^*$ -algèbres pouvant être considéré comme une quantification des groupoïdes de transformations ordinaires.

Au début du travail sur cette thèse, nous avons distingué deux axes de recherche importants qui existaient déjà dans la littérature que nous avons essayé de développer et de généraliser:

- Le premier axe a été réalisé dans un contexte purement algébrique, dans lequel un groupe usuel a été remplacé par une algèbre de Hopf, l'ensemble a été remplacé par une algèbre et l'action a été remplacée par une coaction de l'algèbre de Hopf sur l'algèbre et dans lequel le groupoïde quantique de transformations construit est un exemple particulier d'un objet appelé *algébroïde de Hopf*. Dans ce travail, nous pouvons mettre en évidence les constructions algébriques explicites qui utilisent la théorie des actions et coactions des algèbres de Hopf. Plus spécifiquement, l'algébroïde de Hopf mentionnée ci-dessus a été construite à partir d'une action du double de Drinfeld d'une algèbre de Hopf sur une algèbre ou de manière équivalente à partir d'une *algèbre de Yetter-Drinfeld tressée commutative*



sur une algèbre de Hopf.

- Le deuxième axe de recherche a été développé dans le contexte des algèbres de von Neumann, dans lesquelles un groupe usuel a été remplacé par un *groupe quantique localement compact au sens de Kustermans et Vaes version von Neumann* agissant sur une algèbre de von Neumann et dans lequel le groupoïde quantique de transformations était un exemple particulier d'un objet connu sous le nom de *groupoïde quantique mesurable au sens de Enock et Lesieur*. Dans ce travail, nous pouvons mettre en évidence la construction d'un *objet dual* qui est aussi, lui, un groupoïde quantique mesurable. Plus précisément, ces deux objets ont été construits à partir d'une *algèbre de Yetter-Drinfeld-von Neumann tressée commutative* donnée.

Chacune des œuvres citées nous fournit des idées et des constructions utiles, mais avec certaines limites. Par exemple, dans le contexte purement algébrique, il est difficile de construire le dual d'un groupoïde quantique de transformations, car le dual d'une algèbre de Hopf n'est pas un objet très convenable pour travailler avec, à l'exception du cas de la dimension finie. Et dans le contexte von Neumann, de nombreuses constructions dépendent fortement de la théorie des algèbres de von Neumann, y compris la théorie modulaire qui n'admet pas une traduction complète au niveau des  $C^*$ -algèbres, ce qui est notre objectif principal. Notre approche est de donner une telle construction algébrique qui permettrait une complétion  $C^*$ -algébrique.

C'est pourquoi, au début de notre travail de recherche, nous aimerions trouver un objet mathématique qui pourrait servir comme une quantification d'un groupe usuel et qui satisferait les conditions suivantes: il devrait avoir une structure algébrique claire, admettre un objet dual de la même nature et devrait également avoir de bonnes propriétés analytiques qui pourraient être exprimées en termes d'algèbres d'opérateurs. Il était donc naturel de tester comme candidat possible le bien connu *groupe quantique compact au sens de Woronowicz* qui est une algèbre de Hopf involutive unifère en général de dimension infinie dotée d'une fonctionnelle de Haar qui est une généralisation de la mesure de Haar sur un groupe. Un groupe quantique compact admet un objet dual appelé *groupe quantique discret*, qui n'est pas une algèbre de Hopf involutive usuelle en général car n'est pas unifère, mais elle est dotée d'une fonctionnelle de Haar non borné qui est une généralisation de la mesure de Haar sur un groupe discret. Les groupes quantiques compacts et discrets sont bien compris à la fois en termes purement algébriques et au niveau des algèbres d'opérateurs mais, malheureusement, ils ne satisfont pas les conditions mentionnées ci-dessus selon laquelle le dual doit être du même type que l'objet initial.

Il existe une classe plus large de groupes quantiques appelée *groupes quantiques  $C^*$ -algébriques au sens de Van Daele* qui contient à la fois les groupes quantiques compacts et discrets et répond à nos exigences. Un groupe quantique  $C^*$ -algébrique est en général une algèbre de Hopf de multiplicateurs involutive de dimension infinie et non-unifère équipée d'une intégrale de Haar fidèle et positive. Cela permet, en utilisant la construction de Gelfand-Naimark-Segal, de développer une bonne théorie, très riche du point de vue purement algébrique et de celle des algèbres d'opérateurs. En particulier, les groupes quantiques algébriques admettent une complétion  $C^*$ - et von Neumann algébrique naturelle. Nous obtenons ainsi une vaste gamme d'exemples de *groupes quantiques localement compacts au sens de Kustermans et Vaes*. Une autre caractéristique importante de cette théorie est son auto-dualité, c'est-à-dire la possibilité de construire pour tout objet son dual qui est aussi de la même classe.

Cette thèse est divisée en deux parties. Dans la première partie, nous transformons les idées et les constructions de Lu [Lu96] et de Enock et Timmermann [ET16] afin de les rendre compatibles avec les actions des groupes quantiques algébriques. En conséquence, nous obtenons un nouvel objet mathématique que nous appelons un *groupoïde quantique algébrique de transformations*. Et dans la deuxième partie, nous associons à un groupoïde quantique algébrique de transformations un objet que nous appelons un *groupoïde quantique  $C^*$ -algébrique de transformations* et expliquons sa relation avec le groupoïde quantique de transformation de Enock-Timmermann. Dans les deux cas, nous sommes en mesure de construire un objet dual de même nature et d'expliquer les relations de nos résultats avec les résultats déjà connus dans la littérature. En particulier, dans le contexte purement algébrique, notre nouvel objet est un exemple d'*algébroïde de Hopf de multiplicateurs* introduit dans [TV18] et dans le contexte von Neumann algébrique, notre nouvel objet est un exemple d'*groupoïde quantique de transformations au sens de Enock et Timmermann* introduit dans [ET16].

## Contexte de cette thèse

### 1) Le groupoïde de transformations usuel [Ren80]

Un groupoïde de transformations est une structure construite à partir d'une action d'un groupe sur un ensemble. Plus explicitement, étant donné une action (à gauche) d'un groupe  $G$  sur un ensemble  $S$ ,  $\cdot : G \times S \rightarrow S$ , le produit cartésien  $G \curvearrowright S := G \times S$  avec

les applications

$$\begin{aligned} \cdot : (G \curvearrowright S)^{(2)} &\rightarrow G \curvearrowright S & -1 : G \curvearrowright S &\rightarrow G \curvearrowright S \\ ((g, s), (h, t)) &\mapsto (gh, t) & (g, s) &\mapsto (g^{-1}, g \cdot s) \end{aligned} ,$$

où  $(G \curvearrowright S)^{(2)} = \{((g, s), (h, t)) \in (G \curvearrowright S) \times (G \curvearrowright S) : s = h \cdot t\}$ , admet une structure de groupoïde, avec son espace d'unités donné par  $(G \curvearrowright S)^{(0)} = S$  et avec les applications *source* et *but* données par

$$\begin{aligned} d : G \curvearrowright S &\rightarrow S & r : G \curvearrowright S &\rightarrow S \\ (g, s) &\mapsto s & (g, s) &\mapsto g \cdot s \end{aligned} ,$$

respectivement.

## 2) Groupoïdes quantiques de transformations au sens de Lu

Motivée par la notion de groupe de Poisson dans la géométrie de Poisson, J.H. Lu [Lu96] a introduit une généralisation d'un algébroïde de Hopf avec une base commutative et, comme exemple important d'un tel objet, elle a construit un analogue non commutatif d'un groupoïde associé à un groupe de Poisson-Lie. Pour être précis, étant donné une certaine action du double de Drinfeld d'une algèbre de Hopf sur une algèbre, Lu a construit une algébroïde de Hopf qu'elle considère comme une quantification d'un groupoïde de transformations usuel. Quelques années plus tard, dans [BM02], la notion d'une algèbre de Yetter-Drinfeld tressée commutative a été utilisée comme l'équivalent du principal ingrédient dans la construction de Lu. On rappelle le résultat de J.H. Lu trouvé dans [BM02] :

**Théorème.** Soit  $H$  une bigèbre,  $(A, \triangleright)$  une algèbre de  $H$ -module à gauche et  $(A, \rho)$  un  $H$ -comodule à droite. Alors,  $(A, \triangleright, \rho)$  est une algèbre tressée commutative dans la catégorie des algèbres de Yetter-Drinfeld  ${}^H\mathcal{YD}^H$  si et seulement si  $(A \# H, s, t, \Delta, \varepsilon)$  est un  $A$ -bigébroïde avec applications *source* et *but*

$$\begin{aligned} s : A &\rightarrow A \# H & t : A &\rightarrow A \# H \\ a &\mapsto a \# 1_H & a &\mapsto a_{[0]} \# a_{[1]} \end{aligned}$$

*comultiplication*

$$\begin{aligned} \Delta : A \# H &\rightarrow A \# H \overset{A}{\times} A \# H \\ a \# h &\mapsto a \# h_{(1)} \overset{A}{\times} 1_A \# h_{(2)} \end{aligned}$$

et counité

$$\begin{aligned} \varepsilon : A \# H &\rightarrow A \\ a \# h &\mapsto \varepsilon_H(h)a \end{aligned}$$

En plus, si  $H$  possède une antipode  $S$ , alors  $A \# H$  est un algébroïde de Hopf avec antipode donnée par

$$\begin{aligned} \tilde{S} : A \# H &\rightarrow A \# H \\ a \# h &\mapsto (S(h_{(2)})S^2(a_{[1]})) \triangleright a_{[0]} \# S(h_{(1)})S^2(a_{[2]}) \end{aligned}$$

### 3) Groupoïdes quantiques de transformations au sens de Enock et Timmermann

Dans [ET16], M. Enock et T. Timmermann ont construit une version d'un groupoïde de transformations dans le contexte des groupoïdes quantiques mesurés précédemment introduits par F. Lesieur et M. Enock. Comme dans le cas purement algébrique de Lu, ils ont utilisé une algèbre de Yetter-Drinfeld-von Neuman tressée commutative sur un groupe quantique localement compact version von Neumann comme l'ingédient principal de leur construction. On rappelle le resultat de M. Enock et T. Timmermann trouvé dans [ET16] :

**Théorème** Soit  $\mathbb{G}$  un groupe quantique localement compact version von Neumann,  $(N, \alpha, \widehat{\alpha})$  une algèbre de Yetter-Drinfeld-von Neumann tressée commutative sur  $\mathbb{G}$  et  $\nu$  un poids fidèle semi-finie normale sur  $N$ ,  $D_t$  la dérivée de Radon-Nikodym de  $\nu$  par rapport à l'action  $\alpha$ ,  $\tilde{\nu}$  le poids dual de  $\nu$  sur le produit croisé  $\mathbb{G} \rtimes_{\alpha} N$ . Alors, il existe un groupe d'automorphismes à un paramètre de  $\mathbb{G} \rtimes_{\alpha} N$ ,  $\tilde{\tau}_t$ , un groupe d'automorphismes à un paramètre de  $N$ ,  $\gamma_t$ , et une famille d'automorphismes  $\Phi_t := \tau_t \circ \text{Ad}\delta^{-it}$  tels que, les assertions suivantes sont équivalentes

(i)  $(\Phi_t \otimes \gamma_t)(D_s) = D_s$  pour tout  $s, t \in \mathbb{R}$ .

(ii)  $\sigma_t^{\tilde{\nu}}$  et  $\tilde{\tau}_s$  commutes pour tout  $s, t \in \mathbb{R}$ .

(iii)  $\sigma_t^{\tilde{\nu}}$  et  $\sigma_s^{\tilde{\nu} \circ \widehat{R}}$  commutes pour tout  $s, t \in \mathbb{R}$ .

(iv)  $\mathfrak{G}(N, \alpha, \widehat{\alpha}, \nu)$  est un groupoïde quantique mesuré.

## Description des objets

### 1) Groupes quantiques algébriques au sens de Van Daele

Dans [Van97], A. Van Daele a développé la théorie des algèbres de Hopf de multiplicateurs réguliers en tant qu'une généralisation de la théorie classique des algèbres de Hopf. Dans cette théorie, la condition selon laquelle l'algèbre sous-jacente est unifiée ou de dimension finie a été remplacée par la propriété d'être non-dégénérée. Dans ce cadre, il était donc possible de travailler avec des algèbres de dimension infinie et non-unifiées. En particulier, il a été possible d'étendre certaines constructions associées aux algèbres de Hopf usuelles, par exemple: la théorie des actions et des coactions développée dans [DVZ99], le couplage entre deux algèbres de Hopf de multiplicateurs réguliers développé dans [DV01], la construction du double de Drinfeld [DV04], les produits bicroisés des algèbres de Hopf de multiplicateurs [DVW11, DVW13], ainsi que la théorie de Hopf Galois [VZ99b, De 09].

De plus, dans [Van98], A. Van Daele a également introduit la notion des *groupes quantiques algébriques*, qui sont des algèbres de Hopf de multiplicateurs munies d'une fonctionnelle invariante à gauche pour la comultiplication du groupe quantique algébrique. Il a montré de manière élégante que pour ce type d'algèbre de Hopf de multiplicateurs régulière, il était possible de construire un objet dual qui est aussi un groupe quantique algébrique. Cette dualité est une généralisation de la dualité des algèbres de Hopf de dimension finie et de la dualité entre les groupes quantiques compacts et discrets.

Si un groupe quantique algébrique admet une structure involutive compatible et si la fonctionnelle invariante à gauche est positive, A. Van Daele et J. Kustermans dans [KV97] ont montré qu'ils étaient capables de produire un groupe quantique localement compact version  $C^*$ -algébrique. Plus tard, J. Kustermans dans [Kus03], De Commer et Van Daele dans [DV10], ont montré comment il était possible d'étendre divers objets associés à un groupe quantique  $*$ -algébrique, au contexte des algèbres d'opérateurs.

### 2) Algèbroïdes de Hopf comme langage pour les groupoïdes quantiques algébriques

La première définition formelle d'un algèbroïde de Hopf a été donnée dans [Rav86] afin de généraliser l'espace des fonctions sur un groupoïde usuel. Dans [Mal92], la notion d'un groupoïde quantique a été introduite et étudiée comme une généralisation simultanée des groupoïdes usuels et des groupes quantiques de Drinfeld. Dans ce travail,

utilisant la notion de coalgèbre déjà définie dans [Del90], la notion de bialgèbre de Hopf a été donnée. Puis, dans [Mal00], G. Maltsiniotis a généralisé son travail précédent et introduit la notion de groupoïde quantique qu'il considérait comme un analogue non commutatif d'un groupe de Lie et d'un groupe algébrique. Il convient également de noter le travail de L. Vainerman [Vai92] qui a construit une sorte de groupoïde quantique de transformations.

Dans le cas de dimension finie, dans [BNS99], les auteurs ont donné la définition d'une algèbre de Hopf faible qui a servi pour le développement de la théorie des groupoïdes quantiques finis, étudiée par la suite et reliée avec d'autres domaines des mathématiques par D. Nikshych et L. Vainerman [NV00, NV02] et par D. Nikshych, V. Turaev et L. Vainerman [NTV03].

Nous devons également dire ici que A. Van Daele et S. Wang, dans [VW12, VW15], ont généralisé la notion d'une algèbre de Hopf faible au cas de la dimension infinie et non unifiée. Ils ont également développé la théorie des intégrales et de la dualité pour les algèbres de Hopf faible de multiplicateurs dans [VW14, VW17].

Dans [TV18, TV15], T. Timmermann et A. Van Daele ont généralisé la notion d'algèbre de Hopf afin d'élargir simultanément les notions d'algèbre de Hopf faible de multiplicateurs et d'algèbres de Hopf. Ensuite, T. Timmermann [Tim16, Tim17] a développé la théorie des groupoïdes quantiques algébriques et de leur dualité, prolongeant ainsi les travaux de Van Daele sur les groupes quantiques algébriques.

### **3) Bimodules de Hopf-von Neumann et bimodules de Hopf- $C^*$ comme langage pour les groupoïdes quantiques analytiques**

Les premières étapes de la théorie des groupoïdes quantiques dans le cadre des algèbres de von Neumann ont été effectuées par J-M. Vallin et M. Enock, qui ont généralisé le concept d'un bigèbre de von Neumann et d'un unitaire multiplicatif dans [EV00a, EV00b, Val96, Val00]. Plus tard, en utilisant ces travaux, F. Lesieur dans sa thèse de doctorat [Les03] a développé une théorie générale des groupoïdes quantiques mesurés dans laquelle il suit les idées de la théorie des groupes quantiques localement compacts de J. Kustermans et S. Vaes. Une caractéristique principale de cette théorie est une généralisation de la dualité de Pontrjagin: avec chaque groupoïde quantique mesuré, Lesieur associe un dual de la même nature, de sorte que le second dual est naturellement isomorphe à l'objet initial.

Contrairement à la théorie des groupoïdes quantiques mesurés, une théorie générale des groupoïdes quantiques localement compacts dans le contexte des  $C^*$ -algèbres n'existe pas encore. Mais on peut mentionner deux tentatives pour cela : La première est la thèse de T. Timmermann [Tim05], où il introduit la théorie des modules de Hopf- $C^*$  que nous utilisons dans notre thèse, la seconde est la théorie en développement de B.-J. Kahng et A. Van Daele [KV17a, KV17b] d'une classe de groupoïdes quantiques localement compacts, motivée par la notion purement algébrique d'une algèbre de Hopf faible de multiplicateurs.

## Principales contributions

- Étude du produit croisé lié à une action d'un groupes quantique algébrique. Ici, nous utilisons systématiquement l'algèbre involutive de Heisenberg d'un groupe quantique algébrique. Etant donnée une action d'un groupe quantique algébrique  $\mathbb{G} = (A, \Delta, \varphi)$  sur une algèbre involutive non-dégénérée,  $\alpha : X \rightarrow \mathcal{M}(A \otimes X)$ , il est possible de construire une sous-algèbre involutive non-dégénérée de  $\mathcal{M}(H(\mathbb{G}) \otimes X)$ , à savoir  $\mathbb{G} \times_{\alpha} X$ , appelée le produit croisé de  $\mathbb{G}$  et  $X$  par rapport à l'action  $\alpha$ . Ici  $H(\mathbb{G})$  est l'algèbre de Heisenberg de  $\mathbb{G}$ . Ce produit croisé joue un rôle important dans la construction de l'algèbre totale de notre groupoïde quantique. Nous soulignons également l'importance de l'utilisation de l'algèbre de Heisenberg.
- Introduction d'une nouvelle définition d'une algèbre involutive de Yetter-Drinfeld en utilisant l'unitaire multiplicatif algébrique dans le cadre purement algébrique modélisé à partir de la définition développée dans le contexte des algèbres d'opérateurs. On donne aussi des nouvelles preuves de nombreuses propriétés connues pour ces algèbres. Plus explicitement, on introduit la définition suivante :

**Définition 2.8.1.** Soit  $\mathbb{G} = (A, \Delta, \varphi)$  un groupe quantique algébrique et  $U$  son unitaire multiplicatif algébrique. Une algèbre involutive de Yetter-Drinfeld sur  $\mathbb{G}$  est un triple  $(X, \alpha, \hat{\alpha})$ , où  $X$  est une algèbre non-dégénérée,  $\alpha$  est une action à gauche de  $\mathbb{G}$  sur  $X$  et  $\hat{\alpha}$  est une action de  $\widehat{\mathbb{G}}$  sur  $X$ , telles que

$$(id_{\mathcal{M}(\widehat{A})} \otimes \alpha) \circ \hat{\alpha} = (\Sigma \otimes id_X) \circ (Ad(U) \otimes id_X) \circ (id_{\mathcal{M}(A)} \otimes \hat{\alpha}) \circ \alpha.$$

De manière équivalente, le diagramme

$$\begin{array}{ccccc}
 X & \xrightarrow{\hat{\alpha}} & \mathcal{M}(\hat{A} \otimes X) & \xrightarrow{id_{\mathcal{M}(\hat{A})} \otimes \alpha} & \mathcal{M}(\hat{A} \otimes A \otimes X) \\
 \downarrow \alpha & & & & \uparrow \Sigma \otimes id_X \\
 \mathcal{M}(A \otimes X) & \xrightarrow{id_{\mathcal{M}(A)} \otimes \hat{\alpha}} & \mathcal{M}(A \otimes \hat{A} \otimes X) & \xrightarrow{Ad(U) \otimes id_X} & \mathcal{M}(A \otimes \hat{A} \otimes X)
 \end{array}$$

est commutatif. Une algèbre involutive de Yetter-Drinfeld  $(X, \alpha, \hat{\alpha})$  est appelée réduite si  $\alpha$  et  $\hat{\alpha}$  sont des actions réduites.

## Résultats principaux

- Construction d'un algébroïde de Hopf de multiplicateurs mesuré à partir d'une algèbre involutive de Yetter-Drinfeld tressée commutative munie d'une fonctionnelle invariante positive et fidèle :

**Théorème 3.2.3** Soit  $(X, \alpha, \hat{\alpha})$  une algèbre involutive de Yetter-Drinfeld tressée commutative sur un groupe quantique algébrique  $\mathbb{G}$  et  $\mu : X \rightarrow \mathbb{C}$  une intégrale de Yetter-Drinfeld. Alors, le tuple

$$(\mathbb{G} \ltimes_{\alpha} X, X, \alpha, \beta, \Delta_{\alpha}, \tilde{S}, \tilde{\varepsilon}, \mu, \Omega),$$

où

$$\begin{aligned}
 \Delta_{\alpha} : \quad \mathbb{G} \ltimes_{\alpha} X &\rightarrow (\mathbb{G} \ltimes_{\alpha} X) \overset{\alpha}{\times} (\mathbb{G} \ltimes_{\alpha} X) \\
 \omega \otimes 1_{\mathcal{M}(X)} &\mapsto (\omega_{(2)} \otimes 1_{\mathcal{M}(X)}) \overset{\alpha}{\times} (\omega_{(1)} \otimes 1_{\mathcal{M}(X)}) \\
 \alpha(x) &\mapsto \alpha(x) \overset{\alpha}{\times} 1_{\mathcal{M}(\mathbb{G} \ltimes_{\alpha} X)}
 \end{aligned}$$

$$\begin{aligned}
 \tilde{S} : \quad \mathbb{G} \ltimes_{\alpha} X &\rightarrow \mathbb{G} \ltimes_{\alpha} X \\
 \alpha(x)(\omega \otimes 1_{\mathcal{M}(X)}) &\mapsto (\hat{S}(x^{[-1]} \cdot \omega) \otimes 1_{\mathcal{M}(X)}) \alpha(x^{[0]})
 \end{aligned}$$

$$\begin{aligned}
 \tilde{\varepsilon} : \quad \mathbb{G} \ltimes_{\alpha} X &\rightarrow \alpha(X) \\
 \alpha(x)(\omega \otimes 1_{\mathcal{M}(X)}) &\mapsto \hat{\psi}(\omega) \alpha(x)
 \end{aligned}$$

et

$$\begin{aligned}
 \Omega : \quad \mathbb{G} \ltimes_{\alpha} X &\rightarrow \alpha(X) \\
 (\omega \otimes 1_{\mathcal{M}(X)}) \alpha(x) &\mapsto \hat{\varphi}(\omega) \alpha(x)
 \end{aligned}$$

est un algébroïde de Hopf de multiplicateurs mesuré (Définition 1.3.38).



- Construction d'un groupoïde quantique  $C^*$ -algébrique de transformations à partir d'une algèbre involutive de Yetter-Drinfeld tressée commutative munie d'une intégrale de Yetter-Drinfeld :

**Théorème 5.2.1.** *Soit  $(X, \alpha, \hat{\alpha})$  une algèbre involutive de Yetter-Drinfeld tressée commutative sur un groupe quantique algébrique  $\mathbb{G}$  et  $\mu : X \rightarrow \mathbb{C}$  une intégrale de Yetter-Drinfeld. Considérons la  $C^*$ -base  $\mathfrak{b} = (L_\mu^2 X, \mathfrak{B}, \mathfrak{B}^\dagger)$  où  $\mathfrak{B} := \overline{\pi_\mu(X)}^{\|\cdot\|}$  et  $\mathfrak{B}^\dagger := \overline{\pi_\mu^\circ(X^{op})}^{\|\cdot\|}$ , l'espace Hilbertien  $\mathcal{H} := L^2 \mathbb{G} \otimes L_\mu^2 X$  et la complétion  $C^*$ -algébrique réduit du produit croisé algébrique  $\mathcal{A} := C_0(\mathbb{G} \rtimes_\alpha^{red} X)$ . Alors, il existe des sous-espaces  $E := E(\alpha, \hat{\alpha}), F := F(\alpha, \hat{\alpha}) \in \mathcal{B}(L_\mu^2 X, L^2 \mathbb{G} \otimes L_\mu^2 X)$ , un  $*$ -homomorphisme*

$${}^E \Delta^F : \mathcal{A}_{\mathcal{H}}^{F,E} \rightarrow \mathcal{A}_{\mathcal{H}}^{F,E} \underset{\mathfrak{b}}{*} \mathcal{A}_{\mathcal{H}}^{F,E},$$

un anti-automorphisme  ${}^E R^F : \mathcal{A} \rightarrow \mathcal{A}$  et un poids complètement positive à valeurs opératorielles  ${}^E \Phi^F : \mathcal{A} \rightarrow \rho_F(\mathfrak{B})$  telles que

- la paire  $(\mathcal{A}_{\mathcal{H}}^{F,E}, {}^E \Delta^F)$  est un  $C^*$ -bimodule de Hopf sur la  $C^*$ -base  $\mathfrak{b}$ ,
- ${}^E R^F$  est une co-involution pour le  $C^*$ -bimodule de Hopf  $(\mathcal{A}_{\mathcal{H}}^{F,E}, {}^E \Delta^F)$ ,
- ${}^E \Phi^F$  est un poids de Haar à gauche borné pour le  $C^*$ -bimodule de Hopf  $(\mathcal{A}_{\mathcal{H}}^{F,E}, {}^E \Delta^F)$ .

## Relations entre les groupoïdes quantiques $C^*$ -algébriques de transformations et les groupoïdes quantiques de transformations mesurés

Nous faisons des commentaires précis sur la relation entre notre construction  $C^*$ -algébrique et la construction d'un groupoïde quantique de transformations mesuré dans le cadre des algèbres de von Neumann faite par Enock et Timmermann dans [ET16].

### Ingrédients principaux

- En utilisant les objets algébriques étudiés au debut du chapitre 2, nous obtenons des constructions similaires à celles de la théorie de groupes quantiques localement compacts version von Neumann. Plus précisément :

- (1) L'unitaire multiplicatif algébrique d'un groupe quantique algébrique  $\mathbb{G}$  (Définition 2.4.1) remplace l'unitaire multiplicatif d'un groupe quantique locale-

ment compact version von Neumann et possède des propriétés similaires permettant d'obtenir des relations algébriques entre celui-ci et la comultiplication du groupe quantique algébrique  $\mathbb{G}$  ainsi qu'entre celui-ci et la comultiplication de  $\widehat{\mathbb{G}}$ . Il sert également comme élément important dans la construction du double de Drinfeld (Définition 2.5.5).

- (2) Nous utilisons la théorie des actions des groupes quantiques algébriques afin de construire des objets similaires à ceux de [ET16]. Par exemple, la construction du produit croisé d'une action d'un groupe quantique algébrique (Définition 2.7.4), l'existence de l'implémentation standard lorsque nous munissons notre algèbre involutive d'une fonctionnelle positive invariante pour l'action (Définition 2.6.8), la construction de l'action duale (Définition 2.7.7) qui nous donne également le moyen de construire la fonctionnelle duale de la fonctionnelle invariante initiale (Définition 2.7.10).
- De la même façon que dans la construction faite dans [ET16], nos ingrédients principaux sont une algèbre involutive de Yetter-Drinfeld tressée commutative munie d'une fonctionnelle invariante mais dans notre cas sur un groupe quantique algébrique.
    - (1) Notre définition d'une algèbre involutive de Yetter-Drinfeld sur un groupe quantique algébrique (Définition 2.8.1) est similaire à celle de la section 2.4 de [ET16], mais elle utilise des actions du groupe quantique algébrique et de son dual et de l'unitaire multiplicatif algébrique du groupe quantique algébrique. Nous introduisons également la définition d'une intégrale de Yetter-Drinfeld (Définition 2.8.6) qui nous permet de construire l'implémentation standard de nos actions et constitue un objet important pour le passage au niveau  $C^*$ -algébrique.
    - (2) Notre définition d'une algèbre involutive de Yetter-Drinfeld tressée commutative (Définition 2.8.12) est équivalente à celle de la Définition 2.5.3 dans [ET16] (Théorème 2.5.4, [ET16]).

## Relations entre les deux structures de bimodule de Hopf : La $C^*$ -algébrique et la von Neumann algébrique

Dans [ET16], les auteurs utilisent les produits tensoriels relatifs des espaces Hilbertiens et les produits fibrés des algèbres de von Neumann afin de construire les objets

nécessaires pour la construction d'une structure de bimodule de Hopf-von Neumann au sens de Vallin [Val96]. Puisque notre objectif est le cadre  $C^*$ -algébrique, nous devons remplacer ces objets par des objets similaires dans la langage des  $C^*$ -algèbres. Pour cela, nous utilisons la théorie des  $C^*$ -algèbres sur  $C^*$ -bases développées par T. Timmermann [Tim09, Tim12b, Tim12a]. Il a été montré dans la section 2.3 de [Tim07] que, en utilisant une  $C^*$ -base associée avec un poids KMS, nous pouvons obtenir le lien entre le produit tensoriel relatif sur une  $C^*$ -base et le produit tensoriel relatif algébrique des espaces Hilbertiens sur une algèbre de von Neumann (qui utilise dans son définition des éléments bornés). Plus précisément:

- Le théorème 5.2.9, la proposition 5.2.10 et la proposition 5.2.11 remplacent la proposition 4.3 [ET16] afin de construire une structure de  $C^*$ -bimodule de Hopf.
- Le théorème 5.2.13 est prouvé de manière similaire que le théorème 4.4 [ET16], ce qui permet de construire une comultiplication sur la  $C^*$ -complétion du produit croisé, ce qui nous donne un  $C^*$ -bimodule de Hopf.
- Le théorème 5.2.15 est prouvé d'une manière semblable au théorème 4.6 [ET16], pour la construction d'une co-involution pour notre  $C^*$ -bimodule de Hopf.

## Perspectives

- Dans un premier temps il s'agira de continuer à construire d'autres exemples d'algèbres involutives de Yetter-Drinfeld tressées commutatives sur des groupes quantiques algébriques, par exemple :
  - \* En utilisant un *décalage partiel de Bernoulli sur un groupe quantique discret*  $\mathbb{G}$ , défini dans [KQT17], T. Timmermann indique la possibilité de construire une  $C^*$ -algèbre de Yetter-Drinfeld tressée commutative sur le groupe quantique discret  $\mathbb{G}$  qui peut être restreint au niveau algébrique et ainsi on obtient une algèbre de Yetter-Drinfeld tressée commutative sur l'algèbre de Hopf  $\mathbb{C}[\mathbb{G}]$ . Cela nous donne une façon de construire des exemples non-triviaux que je voudrais étudier.
  - \* D'après l'article [YZC17], on sait que l'algèbre de Heisenberg d'un groupe quantique algébrique  $\mathbb{G} = (A, \Delta, \varphi)$ , peut être muni d'une action de l'algèbre de Hopf de multiplicateurs  $(A, \Delta)$  et une coaction de son dual  $(\widehat{A}, \widehat{\Delta})$  qui donnent une structure d'algèbre de Yetter-Drinfeld tressée commutative au

sens algébrique. On voudrait transporter cet exemple dans notre cadre afin de l'étudier dans le contexte algébrique ainsi que dans le contexte des algèbres d'opérateurs.

- Suite à la question posée par A. Van Daele et du fait qu'il existe des travaux en développement [KV17a, KV17b] sur une théorie d'une famille de groupoïdes quantiques localement compacts, nous voudrions étudier et regarder plus en détail le lien entre nos constructions et celles construites dans le cadre d'algèbres de Hopf faibles de multiplicateurs.
- De la même façon que dans le cadre de groupes quantiques compacts et discrets au sens de Woronowicz, nous voudrions étudier la théorie de représentations des groupoïdes quantiques de transformations de type compact et discrets et obtenir de cette manière le lien avec les  $C^*$ -catégories tensorielles.

## Structure de la thèse

La structure de cette thèse est la suivante:

- La partie I ne traite que des aspects purement algébriques. Dans sa présentation, nous avons essayé de la rendre la plus complète possible sans renvoyer à d'autres ouvrages. Cette partie contient des commentaires utiles pour la compréhension de divers autres résultats.
  - \* Le chapitre 1 est destiné à rappeler les définitions et les constructions de la théorie des algèbres de Hopf de multiplicateurs involutives et de la théorie des algébroïdes de Hopf de multiplicateurs involutives mesuré. Nous n'avons aucun résultat original dans ce chapitre.
  - \* Le chapitre 2 vise à développer la théorie des algèbres involutives de Yetter-Drinfeld dans le contexte des groupes quantiques algébriques. Pour cela, nous fixons les notations et les propriétés des objets associés à la théorie des groupes quantiques algébriques au sens de Van Daele. À la fin du chapitre, nous donnons les principaux exemples qui seront utilisés dans les chapitres suivants de cette première partie.
  - \* Le chapitre 3 est le plus important de cette première partie. Ici, nous construisons une famille des algébroïdes de Hopf de multiplicateurs involutifs

mesurés associés aux algèbres involutives de Yetter-Drinfeld tréssées commutatives sur les groupes quantiques algébriques. Nous terminons ce chapitre en présentant des exemples de groupoïdes quantiques algébriques de transformations, en montrant que les groupoïdes de transformations classique entrent dans notre cadre.

- La partie II traite des aspects analytiques et, tout comme la première partie, nous avons essayé de la rendre la plus complète possible sans renvoyer à d'autres ouvrages.

\* Le chapitre 4 a pour but de rappeler les définitions et les constructions associées aux groupes quantiques dans le cadre des  $C^*$ -algèbres. Ici, nous présentons également les objets nécessaires pour la construction des  $C^*$ -complétions d'un groupoïde quantique algébrique de transformations.

\* Le chapitre 5 est le plus important de cette partie. Ici, nous construisons la complétion d'un groupoïde quantique algébrique de transformations dans le contexte des  $C^*$ -algèbres. Ces nouveaux objets préservent la dualité et nous expliquons ici que la complétion von Neumann donne un exemple d'un groupoïde quantique de transformations au sens de Enock et Timmermann.

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## *Introduction*

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### **Motivation**

The main motivation of this work is the construction of quantum groupoids in the context of operator algebras, in particular, in the context of  $C^*$ -algebras. One of the most important examples of a usual groupoid is a *transformation groupoid* associated with an action of a group on a space [Ren80]. Its structure contains the information about the group, about the action and about the space on which the group acts. That is why the objective of this thesis is to introduce the notion of a *quantum transformation groupoid* and to investigate the duality for these objects both in purely algebraic context and in the context of  $C^*$ -algebras that can be considered as a quantization of classical transformation groupoids.

At the beginning of the work on this thesis, we have distinguished two important research lines already existed in the literature that we tried to develop and to generalize:

- The first one has been done in the purely algebraic context where a usual group was replaced by a Hopf algebra and in which the constructed quantum transformation groupoid is a particular example of an object known as *Lu's Hopf algebroid*. In this work, we can highlight the explicit algebraic constructions that make use of the theory of actions and coactions of Hopf algebras on algebras. More specifically, the above mentioned Hopf algebroid was constructed from an action of the Drinfeld double of a Hopf algebra on an algebra or equivalently from a *braided commutative Yetter-Drinfeld algebra*.
- The second research line has been developed in the context of von Neumann algebras where a usual group was replaced by a *Kustermans-Vaes's von Neumann algebraic locally compact quantum group* and in which the quantum transformation

groupoid was a particular example of an object known as a *Enock-Lesieur's measurable quantum groupoid*. In this work, we can highlight the construction of a *dual object* which was again a measurable quantum groupoid. More specifically, both these objects were constructed from a given *braided commutative Yetter-Drinfeld von Neumann algebra*.

Each of the mentioned works provides us with useful ideas and constructions but with some limitations. For example, in the purely algebraic context, it is difficult to construct a dual quantum transformation groupoid because already a dual Hopf algebra is not a very convenient object, except for the finite dimensional case. And in von Neumann algebraic context, many constructions heavily depend on the von Neumann algebra theory including the modular theory that does not admit a complete translation to the  $C^*$ -algebraic setting which is our main goal. And our approach is to give such an algebraic construction that would allow a passage to the  $C^*$ -algebraic setting.

That is why at the beginning of our research, we wanted to find a mathematical object that could serve as a quantization of a usual group and that would satisfy the following conditions: it should have a clear algebraic structure, admit a dual object of the same nature and also should have some nice analytical properties that could be expressed in terms of operator algebras. It would be natural to try as a possible candidate the well known *Woronowicz's compact quantum group* which are unitary involutive Hopf algebras (in general, infinite dimensional) endowed with a Haar functional that is a generalization of the Haar measure on a compact group. A compact quantum group admits a dual object called a *discrete quantum group*, this is not a usual Hopf algebra, in general, because it is not unitary, but endowed with an unbounded Haar functional that is a generalization of the Haar measure on a discrete group. Compact and discrete quantum groups are well understood both on purely algebraic and on operator algebraic level but unfortunately, they do not satisfy the above mentioned condition that the dual must be of the same type as the initial object.

There is a wider class of quantum groups called *Van Daele's \*-algebraic quantum groups* that contains both compact and discrete quantum groups and meets our requirements. These objects are in general non-unitary infinite dimensional multiplier Hopf \*-algebras equipped with a Haar positive functional. This allows, using the Gelfand-Naimark-Segal construction, to develop a nice theory very rich both from purely algebraic and operator algebraic point of view. In particular, algebraic quantum groups admit natural  $C^*$ - and von Neumann algebraic completion, this way we obtain a wide class of examples of *locally compact quantum groups* in the sense of Kustermans and Vaes. Another important

feature of this theory is its self-duality, that is, a possibility to construct for any object the dual within the same class.

This thesis is divided into two parts. In the first part we transform the ideas and constructions by Lu [Lu96] and by Enock and Timmermann [ET16] in order to make them compatible with actions of the algebraic quantum groups. As a result, we get a new mathematical object which we call an *algebraic quantum transformation groupoid*. And in the second part we associate with any algebraic quantum transformation groupoid an object that we call a quantum transformation groupoid in  $C^*$ -algebraic setting and study its relation with the Enock-Timmermann's quantum transformation groupoid. In both cases we are able to construct a dual object of exactly same nature and to explain the relations of our results with the results already known in the literature. In particular, in the purely algebraic context, our new object is an example of a *multiplier Hopf algebra* [TV18] and in the von Neumann algebraic context, our new object is an example of an *Enock-Timmermann's quantum transformation groupoid* [ET16].

## Context

### 1) Classical Transformation Groupoids [Ren80]

A transformation groupoid is a structure constructed from an action of a group on a set. More explicitly, given a (left) action of a group  $G$  in a set  $S$ ,  $\cdot : G \times S \rightarrow S$ , the cartesian product  $G \curvearrowright S := G \times S$  together with the maps

$$\begin{aligned} \cdot : (G \curvearrowright S)^{(2)} &\rightarrow G \curvearrowright S & \cdot^{-1} : G \curvearrowright S &\rightarrow G \curvearrowright S \\ ((g, s), (h, t)) &\mapsto (gh, t) & (g, s) &\mapsto (g^{-1}, g \cdot s) \end{aligned}$$

where  $(G \curvearrowright S)^{(2)} = \{((g, s), (h, t)) \in (G \curvearrowright S) \times (G \curvearrowright S) : s = h \cdot t\}$ , admits a groupoid structure, with space of units given by  $(G \curvearrowright S)^{(0)} = S$  and with the domain and range maps given by

$$\begin{aligned} d : G \curvearrowright S &\rightarrow S & r : G \curvearrowright S &\rightarrow S \\ (g, s) &\mapsto s & (g, s) &\mapsto g \cdot s \end{aligned}$$

respectively.



## 2) Lu's Quantum Transformation Groupoids

Motivated by the notion of a Poisson group from the Poisson geometry, J.H. Lu [Lu96] has introduced a generalization of a Hopf algebroid with a commutative basis and as an important example of such an object has constructed a non-commutative analogue of a groupoid associated with a Poisson-Lie group. To be precise, given a certain action of the Drinfeld double of some Hopf algebra on an algebra, Lu has constructed a Hopf algebroid that she considers as a quantization of a classical transformation groupoid. Some years later, in [BM02], the notion of a braided commutative Yetter-Drinfeld algebra over a Hopf algebra was used as the equivalent of the main ingredient in the construction of Lu.

## 3) Enock-Timmermann's Quantum Transformation Groupoids

In [ET16], M. Enock and T. Timmermann have constructed a version of a transformation groupoid in the context of measurable quantum groupoids previously introduced by F. Lesieur and M. Enock. Similarly to Lu's purely algebraic case, they have considered a braided commutative Yetter-Drinfeld von Neumann algebra over a locally compact quantum group as the main ingredient for their construction.

# Background

## 1) Van Daele's Algebraic quantum groups

In [Van97], A. Van Daele has developed the theory of regular multiplier Hopf algebras as a generalization of the classical theory of Hopf algebras. In this theory, the condition that the underlying algebra is unitary or finite dimensional was replaced by a non-degeneracy property, so in this framework it was possible to work with infinite dimensional and non-unitary algebras. In particular, it was possible to extend some constructions associated with usual Hopf algebras, for example: the theory of actions and coactions developed in [DVZ99], pairing between regular multiplier Hopf algebras developed in [DV01], the construction of the Drinfeld double [DV04], bicrossed products of multiplier Hopf algebras [DVW11, DVW13], and also the Hopf Galois theory, see [VZ99b] and [De 09].

Moreover, in [Van98], A. Van Daele has also introduced the notion of *algebraic quantum groups*, which is multiplier Hopf algebra endowed with a left invariant functional. He has shown in an elegant way, that for this kind of regular multiplier Hopf algebras it was possible to construct a dual object which is also an algebraic quantum group. This

duality is a generalization of the duality of finite dimensional Hopf algebras and the duality between compact and discrete quantum groups.

If an algebraic quantum group admits a compatible  $*$ -structure and its left invariant functional is positive, A. Van Daele and J. Kustermans in [KV97] have completed it to a locally compact quantum group in the  $C^*$ -algebra setting. Later on, J. Kustermans in [Kus03], De Commer and Van Daele in [DV10], have shown how it was possible to extend various objects associated with a  $*$ -algebraic quantum group, to the operator algebraic framework.

## 2) Hopf algebroids as a language for algebraic quantum groupoids

The first formal definition of a Hopf algebroid was given in [Rav86] in order to generalize the function space on a usual groupoid. In [Mal92], the notion of quantum groupoid had been introduced and studied as a simultaneous generalization of usual groupoids and of the Drinfeld's quantum groups. In this work, using the notion of a coalgebroid already defined in [Del90], the notion of a Hopf bialgebroid was given. Then, in [Mal00], G. Maltsiniotis generalized his previous work and introduced the notion of a quantum groupoid that he considered as a non-commutative analogue of a Lie group and of an algebraic group. One should also note the work of L. Vainerman [Vai92] who has constructed a kind of a quantum transformation groupoid.

In the finite dimensional case, in [BNS99] the authors have given the definition of a weak Hopf algebra that served for the development of the theory of finite quantum groupoids, later studied and related to other domains of mathematics by D. Nikshych and L. Vainerman [NV00, NV02] and by D. Nikshych, V. Turaev, and L. Vainerman [NTV03].

We need to say here also that A. Van Daele and S. Wang, in [VW12, VW15], have generalized the notion of a weak Hopf algebra to the infinite dimensional and non-unitary case. They also developed the theory of integrals and duality for the weak multiplier Hopf algebras in [VW14, VW17].

In [TV18, TV15], T. Timmermann and A. Van Daele have generalized the notion of a Hopf algebroid in order to extend simultaneously the notions of a weak multiplier Hopf algebra and of a Hopf algebroid. Later, T. Timmermann [Tim16, Tim17] has developed the theory of algebraic quantum groupoids and their duality, extending the Van Daele's work on algebraic quantum groups.

### 3) Hopf-von Neumann bimodules and Hopf- $C^*$ bimodules as a language for analytical quantum groupoids

The first steps of the theory of quantum groupoids in the setting of von Neumann algebras were made by J-M. Vallin and M. Enock, who have generalized the concept of a von Neumann bialgebra and of a multiplicative unitary in [EV00a, EV00b, Val96, Val00]. Later, using these works, F. Lesieur in his PhD thesis [Les03] has developed a general theory of measured quantum groupoids. He follows the ideas from the theory of locally compact quantum groups by J. Kustermans and S. Vaes. A main feature of this theory is a generalization of the Pontrjagin duality: with every measured quantum groupoid, Lesieur associates a dual of the same nature such that the second dual is naturally isomorphic to the initial object.

In contrast to the theory of measured quantum groupoids, a general theory of locally compact quantum groupoids in the setting of  $C^*$ -algebras does not exist yet. But we can distinguish two attempts for that. The first one, the PhD thesis of T. Timmermann [Tim05], where he introduces the theory of Hopf- $C^*$ -modules that we use in this thesis. The second is the theory in development by B.-J. Kahng and A. Van Daele [KV17a, KV17b] of a class of locally compact quantum groupoids which is motivated by the purely algebraic notion of a weak multiplier Hopf algebra.

## Main contributions

- The study of the properties of crossed products related to actions of algebraic quantum groups. Here we use systematically the Heisenberg  $*$ -algebra of an algebraic quantum group.
- The new definition of a Yetter-Drinfeld  $*$ -algebra using the algebraic multiplicative unitary in the purely algebraic setting modelled from the definition developed in the context of operator algebras. New proofs of many known properties of these algebras.

## Principal results

- The construction of a measured multiplier Hopf  $*$ -algebroid from a braided commutative Yetter-Drinfeld  $*$ -algebra endowed with a positive invariant faithful functional (Theorem 3.1.9).

- The passage to the reduced  $C^*$ -completion of an algebraic quantum transformation groupoid (Theorem 5.2.1).

## Outline of the thesis

The structure of this thesis is as follows:

- Part I deals only with purely algebraic aspects, in its presentation we tried to be as self-contained as possible. This part contains some comments useful for the understanding of various further results.
  - \* Chapter 1 is intended to remind the definitions and constructions from the theory of multiplier Hopf  $*$ -algebras and from the theory of measurable multiplier Hopf  $*$ -algebroids. We do not have any results in this paragraph.
  - \* Chapter 2 is intended to develop the theory of Yetter-Drinfeld  $*$ -algebras in the context of algebraic quantum groups. For that, we fix notations and properties of the objects associated with the Van Daele's theory of algebraic quantum groups. At the end of the chapter, we give the main examples that will be used in the following chapters of this first part.
  - \* Chapter 3 is the most important of this first part. Here we construct a family of measurable multiplier Hopf  $*$ -algebroids associated with braided commutative Yetter-Drinfeld  $*$ -algebras over algebraic quantum groups. We finish this chapter by presenting examples of algebraic quantum transformation groupoids, we show that classical transformation groupoids enter in our picture.
- Part II deals with analytical aspects, and similarly to the first part, we tried to be as self-contained as possible.
  - \* Chapter 4 is intended to remind the definitions and constructions associated with quantum groups in the setting of  $C^*$ - and von Neumann algebras. Here we also introduce the objects necessary for the construction of the  $C^*$ - and von Neumann completions of an algebraic quantum transformation groupoid.
  - \* Chapter 5 is the most important in this part. Here we construct the completion of an algebraic quantum transformation groupoid in the  $C^*$ -algebraic

setting. These new object preserve duality and we want to show here that the von Neumann completion gives an example of the Enock-Timmermann's quantum transformation groupoid.

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## *Conventions*

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In the present manuscript:

- We use the term an *algebraic quantum group* to refer to what A. Van Daele calls a  $*$ -algebraic quantum group and we will abbreviate it by *AQG*.
- We use the term "analytic" to refer to objects whose origin or construction is in the context of operator algebras.
- We consider only idempotent algebras (see Definition [1.1.2](#)).
- We use the covering technique and in particular the Sweedler type leg notation for multiplier Hopf  $*$ -algebras and their coactions, for details we refer to Appendix [A](#).



# **Part I**

## **Purely algebraic setting**





## Chapter N° 1

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### Preliminaries

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#### 1.1 Basic definitions and notations

Let  $A$  be a algebra over complex numbers, not necessarily unital. We denote by  $A^{op}$  the opposite algebra and by  $A^\vee$  the vector space of linear functionals on  $A$ . We denote by  ${}_A M$  a left  $A$ -module  $M$  with a module map  $\triangleright : A \otimes M \rightarrow M$ .

**Remark 1.**

- (1) We write  ${}_A A$  when we regard  $A$  as a left module over itself with respect of the multiplication on the left.
- (2) We denote by  $\text{Hom}({}_A M, {}_A N)$  the vector space of  $A$ -linear maps between the left modules  $M$  and  $N$ , i.e. linear maps  $T : M \rightarrow N$  such that  $T(a \triangleright m) = a \triangleright T(m)$  for all  $a \in A$  and  $m \in M$ . In particular, we denote  $({}_A M)^\vee := \text{Hom}({}_A M, {}_A A)$ .

**Definition 1.1.1.** A module  ${}_A M$  is called

- *faithful*, if for each non-zero  $a \in A$  there exists  $m \in M$  such that  $a \triangleright m$  is non-zero.
- *non-degenerate*, if for each non-zero  $m \in M$  there exists  $a \in A$  such that  $a \triangleright m$  is non-zero.
- *unital*, if  $A \triangleright M := \text{span}\{a \triangleright m : a \in A, m \in M\} = M$ .
- *firm*, if the module map induces an isomorphism  $\triangleright : A_A \otimes {}_A M \cong M$ .
- *locally projective*, if for every finite number of elements  $m_1, m_2, \dots, m_k \in M$ , there exist finitely many  $d_i \in \text{Hom}({}_A M, {}_A A)$  and  $e_i \in \text{Hom}({}_A A, {}_A M)$  such that

$$m_j = \sum_i e_i(d_i(m_j))$$

for all  $j \in \{1, \dots, k\}$ .

Moreover, we say that  ${}_A M$  admits local units in  $A$  if for every finite subset  $F \subset M$  there exists  $a \in A$  with  $a \triangleright m = m$  for all  $m \in F$ .

**Remark 2.**

- (1) If  ${}_A M$  admits local units, then it is non-degenerate and unital.
- (2) For right modules, we use similar notations and terminology by identifying right  $A$ -modules with left  $A^{op}$ -modules.

**Definition 1.1.2.** An algebra  $A$  is called

- *involutive or  $*$ -algebra* if there exists an anti-linear map  $*$  :  $A \rightarrow A$  called *the involution of  $A$*  such that  $(a^*)^* = a$  for all  $a \in A$  and  $(ab)^* = b^*a^*$  for all  $a, b \in A$ .
- *non-degenerate* if the modules  ${}_A A, A_A$  both are non-degenerates.
- *idempotent*, if one of the modules  ${}_A A, A_A$  is unital.
- *firm*, if one of the modules  ${}_A A, A_A$  is firm.

We say that  $A$  *admits local units* if the modules  ${}_A A, A_A$  both admit local units in  $A$ .

**Remark 3.**  ${}_A A$  (or  $A_A$ ) is non-degenerate if and only if  $A_A$  (or  ${}_A A$ ) is faithful, respectively.

**Notation 1.** For any algebra  $A$ , we will use the following notations

$$L(A) := \text{End}(A_A), \quad R(A) := \text{End}({}_A A)^{op}.$$

**Proposition 1.1.3.** *The module  $A_A$  is non-degenerate, if and only if the natural map*

$$\begin{array}{ccc} A & \rightarrow & L(A) \\ a & \mapsto & [L_a : b \mapsto ab] \end{array}$$

*is injective. Similarly, the module  ${}_A A$  is non-degenerate, if and only if*

$$\begin{array}{ccc} A & \rightarrow & R(A) \\ a & \mapsto & [R_a : b \mapsto ba] \end{array}$$

*is injective.*

**Definition 1.1.4.** If the module  $A_A$  is non-degenerate, the *left multiplier algebra of  $A$*  is the sub-algebra

$$\mathcal{M}_l(A) := \{T \in L(A) : AT \subseteq A\} \subseteq L(A),$$

where we identify  $A$  with its image in  $L(A)$ . If the module  ${}_A A$  is non-degenerate, likewise the *right multiplier algebra of  $A$*  is the sub-algebra

$$\mathcal{M}_r(A) := \{T \in R(A) : TA \subseteq A\} \subseteq R(A),$$

where we identify  $A$  with its image in  $R(A)$ .

**Definition 1.1.5.** Let  $A$  be an algebra, not necessarily unital. We define *the multiplier algebra of  $A$*  to be the algebra

$$\mathcal{M}(A) := \{(L, R) \in L(A) \times R(A) : R(a)b = aL(b) \text{ for all } a, b \in A\}.$$

with the structure given by

$$(L, R) \cdot (L', R') = (L \circ L', R' \circ R),$$

for all  $L, L' \in L(A)$  and  $R, R' \in R(A)$ .

**Remark 4.** It follows from Remark 1.1.3 that if  $A$  is non-degenerate then  $A$  sits in  $\mathcal{M}(A)$  as a sub-algebra with the identification  $a \mapsto (L_a, R_a)$  for all  $a \in A$ . In particular, if  $A$  admits a unity, then  $\mathcal{M}(A) \cong A$ .

**Proposition 1.1.6** ([Tim17]). *If  $A$  is non-degenerate, then the left and right multiplier algebras coincide up to a natural identification. They also identify with the algebra  $\mathcal{M}(A)$ .*

**Remark 5.**

- If  $A$  is a  $*$ -algebra, then  $\mathcal{M}(A)$  is also a  $*$ -algebra with involution given by

$$(L, R)^* := (* \circ R \circ *, * \circ L \circ *),$$

for all  $(L, R) \in \mathcal{M}(A)$ .

**Remark 6.** Given  $P = (P_1, P_2) \in \mathcal{M}(A)$  and  $a \in A$ , we have

$$P \cdot (L_a, R_a) = (L_{P_1(a)}, R_{P_1(a)}) \quad \text{and} \quad (L_a, R_a) \cdot P = (L_{P_2(a)}, R_{P_2(a)})$$

in  $\mathcal{M}(A)$ , then we will write simply

$$Pa = P_1(a) \quad \text{and} \quad aP = P_2(a).$$

**Definition 1.1.7.** Let  $A, B$  be two non-degenerate  $*$ -algebras. A  $*$ -homomorphism  $\theta : A \rightarrow \mathcal{M}(B)$  is called *non-degenerate* if  $\theta(A)B = B$ .

**Remark 7.** Sometimes, we will refer to a homomorphism  $\theta : A \rightarrow \mathcal{M}(B)$  as a *representation of  $A$  in  $B$* . The set of non-degenerate homomorphism from  $A$  to  $\mathcal{M}(B)$  will be denoted by  $\text{Hom}(A, B)$ .

**Proposition 1.1.8.** *If  $\theta : A \rightarrow \mathcal{M}(B)$  is a non-degenerate  $*$ -homomorphism, there exists a unique  $*$ -homomorphism  $\bar{\theta} : \mathcal{M}(A) \rightarrow \mathcal{M}(B)$  such that  $\bar{\theta}(a) = \theta(a)$  for all  $a \in A$ , called the extension of  $\theta$  to multipliers and usually still denoted by  $\theta$ .*

**Notation 2.** Let  $A, B$  be two non-degenerate algebras. Consider the flip map  $\chi_{A,B} : A \otimes B \rightarrow B \otimes A$  with inverse denoted by  $\chi_{B,A} : B \otimes A \rightarrow A \otimes B$ , then we have the isomorphism

$$\begin{aligned} \Sigma_{A,B} : \mathcal{M}(A \otimes B) &\rightarrow \mathcal{M}(B \otimes A) \\ (L, R) &\mapsto (\chi_{A,B} \circ L \circ \chi_{B,A}, \chi_{A,B} \circ R \circ \chi_{B,A}) \end{aligned}$$

For each invertible element  $P \in \mathcal{M}(A \otimes B)$ , we will denote by  $\text{Ad}(P)$  the isomorphism  $S \in \mathcal{M}(A \otimes B) \mapsto PSP^{-1} \in \mathcal{M}(A \otimes B)$ , by  $\text{Ad}(P^{-1})$  its inverse. We also consider the isomorphism  $\Sigma_P : \mathcal{M}(A \otimes B) \rightarrow \mathcal{M}(B \otimes A)$ , defined by

$$\Sigma_P := \Sigma_{A,B} \circ \text{Ad}(P) = \text{Ad}(\Sigma_{A,B}(P)) \circ \Sigma_{A,B},$$

with inverse given by the map  $\Sigma_P^{-1} = \text{Ad}(P^{-1}) \circ \Sigma_{B,A} = \Sigma_{B,A} \circ \text{Ad}(\Sigma_{A,B}(P^{-1}))$ .

**Definition 1.1.9.** Let  $\omega$  be a linear functional on an algebra  $A$ , then we say that  $\omega$  is *faithful* if

$$\forall b \in A : \omega(ab) = 0 \Rightarrow a = 0 \quad \text{and} \quad \forall b \in A : \omega(ba) = 0 \Rightarrow a = 0.$$

**Definition 1.1.10.** Let  $\omega$  be a linear functional on a  $*$ -algebra  $A$ , then we say that  $\omega$  is

- *positive*, if  $\omega(a^*a) \geq 0$  for all  $a \in A$ .
- *self-adjoint*, if  $\omega(a^*) = \overline{\omega(a)}$  for all  $a \in A$ .

**Proposition 1.1.11.** *If  $\omega$  is a positive linear functional on a  $*$ -algebra  $A$ , then  $\omega$  is faithful if and only if for every  $a \in A$  such that  $\omega(a^*a) = 0$ , we have  $a = 0$ .*

**Notation 3.** For any linear functional  $\omega$  on a non-degenerate algebra  $A$  and any element  $P \in \mathcal{M}(A)$ , we define the linear functionals  $\omega_P : A \rightarrow \mathbb{C}$  and  ${}_P\omega : A \rightarrow \mathbb{C}$  by

$$\omega_P(a) := \omega(aP), \quad {}_P\omega(a) := \omega(Pa)$$

for every  $a \in A$ .

**Proposition 1.1.12.** *The map  $\mathcal{M}(A) \otimes A^\vee \rightarrow A^\vee$ ,  $P \otimes \omega \mapsto \omega_P$  defines a left  $\mathcal{M}(A)$ -module structure on  $A^\vee$ . Similarly,  $A^\vee \otimes \mathcal{M}(A) \rightarrow A^\vee$ ,  $\omega \otimes P \mapsto {}_P\omega$  defines a right  $\mathcal{M}(A)$ -module structure on  $A^\vee$ .*

**Notation 4.** For any  $\omega \in A^\vee$ , by the proposition above, we use the notations

$$A \cdot \omega := \{ \omega_a : a \in A \}, \quad \omega \cdot A := \{ {}_a\omega : a \in A \}.$$

**Definition 1.1.13.** Let  $\omega$  be a faithful linear functional on an algebra  $A$ . We say that  $\omega$  *admits a modular automorphism*, if there exists an automorphism  $\sigma : A \rightarrow A$  such that

$$\omega(ab) = \omega(b\sigma(a))$$

for all  $a, b \in A$ . If there exist this automorphism it is unique and sometimes it is denoted by  $\sigma^\omega$ .

**Remark 8.** One can easily verify that a faithful linear functional  $\omega \in A^\vee$  admits a modular automorphism if and only if  $A \cdot \omega = \omega \cdot A$ . In this case, the modular automorphism  $\sigma$  is characterized by the relation  $\omega_{\sigma(a)} = {}_a\omega$  for all  $a \in A$ .

**Proposition 1.1.14.** *Let  $A$  be a non-degenerate algebra which admits local units. If  $X$  is a unital left  $A$ -module through the module map  $\triangleright : A \otimes X \rightarrow X$ , then there exists a unique module map  $\triangleright : \mathcal{M}(A) \otimes X \rightarrow X$  turning  $X$  into a left  $\mathcal{M}(A)$ -module.*

*Proof* Let  $x \in X$ , because  $\triangleright$  is unital then there exist finite elements  $a_i \in A$  and  $y_i \in X$  such that  $x = \sum_i a_i \triangleright y_i$ . As  $A$  admits local units, there exists an element  $a \in A$  such that  $aa_i = a_i$  for every  $i$ . Then  $a \triangleright x = a \triangleright (\sum_i a_i \triangleright y_i) = \sum_i aa_i \triangleright y_i = \sum_i a_i \triangleright y_i = x$ . This implies that every element of  $X$  is of the form  $a \triangleright x$  with  $a \in A$  and  $x \in X$ .

Now, fix  $P \in \mathcal{M}(A)$  and  $x \in X$ . Suppose we have  $a, a' \in A$  such that  $a \triangleright x = a' \triangleright x$ , then because  $A$  admits local units we have  $a'' \in A$  such that  $a''a = a$  and  $a''a' = a'$ , and

$$Pa \triangleright x = Pa''a \triangleright x = Pa'' \triangleright (a \triangleright x) = Pa'' \triangleright (a' \triangleright x) = Pa''a' \triangleright x = Pa' \triangleright x.$$

We can define now the action of an element  $P \in \mathcal{M}(A)$  on  $X$  by

$$P \triangleright (a \triangleright x) := Pa \triangleright x,$$

for all  $a \in A, x \in X$ . It is easy to show that this defines a structure of left  $\mathcal{M}(A)$ -module on  $X$ .  $\square$

**Remark 9.** The module map  $\triangleright : \mathcal{M}(A) \otimes X \rightarrow X$  as above is called *the extension of  $\triangleright$  to multipliers*.

### 1.1.1 The multiplier algebra of a non-degenerate \*-algebra

Let  $A$  be a non-degenerate \*-algebra and consider the map

$$\langle \cdot, \cdot \rangle_A : A \times A \rightarrow A, \quad a \times b \mapsto a^*b.$$

This map satisfies the following properties:

- $\langle a, \lambda_1 b_1 + \lambda_2 b_2 \rangle_A = \lambda_1 \langle a, b_1 \rangle_A + \lambda_2 \langle a, b_2 \rangle_A$ ,
- $\langle \lambda_1 a_1 + \lambda_2 a_2, b \rangle_A = \bar{\lambda}_1 \langle a_1, b \rangle_A + \bar{\lambda}_2 \langle a_2, b \rangle_A$ ,
- $\langle a, bc \rangle_A = \langle a, b \rangle_{Ac}$ ,
- $\langle ac, b \rangle_A = c^* \langle a, b \rangle_A$ ,
- $\langle a, b \rangle_A^* = \langle b, a \rangle_A$ ,

for all  $a, a_1, a_2, b, b_1, b_2 \in A$  and  $\lambda_1, \lambda_2 \in \mathbb{C}$ . Moreover, if  $\langle a, b \rangle_A = 0$  for all  $b \in A$ , then  $a = 0$ .

**Definition 1.1.15** ([Pal18]). Let  $A$  be a \*-algebra. A map  $T : A \rightarrow A$  is called *adjointable on  $A$*  if there is a map  $T' : A \rightarrow A$  such that

$$\langle T(a), b \rangle_A = \langle a, T'(b) \rangle_A$$

for all  $a, b \in A$ . The map  $T'$  is called an *adjoint map for  $T$* .

The following results are similar to the ones in the case of Hilbert  $C^*$ -modules [Lan95].

**Proposition 1.1.16** ([Pal18]). *Let  $A$  be a non-degenerate \*-algebra. If  $T : A \rightarrow A$  is an adjointable map, then  $T$  is a right  $A$ -linear map. Moreover, its adjoint map is unique and denoted by  $T^\dagger$ , which is also an adjointable map with  $(T^\dagger)^\dagger = T$ .*

We will denote by  $\mathcal{L}_A(A)$  the  $*$ -algebra of all adjointable maps on  $A$  with involution given by  $T \mapsto T^\dagger$ . For every map  $T : A \rightarrow A$ , we define the map  $T^\sharp : A \rightarrow A$  by  $a \mapsto T(a^*)^*$ . We observe that  $T$  is a left (right)  $A$ -linear map if and only if  $T^\sharp$  is a right (left)  $A$ -linear map, respectively. Recall that for a non-degenerate  $*$ -algebra  $A$ , its multiplier algebra is given by

$$\mathcal{M}(A) = \{(L, R) \in \text{Hom}(A_A) \times \text{Hom}({}_A A) : R(a)b = aL(b) \text{ for all } a, b \in A\}.$$

The next proposition characterizes the multiplier algebras of non-degenerate  $*$ -algebras.

**Proposition 1.1.17.** *There is a  $*$ -isomorphism*

$$\mathfrak{F} : \mathcal{M}(A) \rightarrow \mathcal{L}_A(A), \quad (L, R) \mapsto L$$

with inverse given by

$$\mathfrak{F}^{-1} : \mathcal{L}_A(A) \rightarrow \mathcal{M}(A), \quad T \mapsto (T, (T^\dagger)^\sharp)$$

*Proof* Take  $(L, R) \in \mathcal{M}(A)$ , then

$$\langle L(a), b \rangle_A = L(a)^*b = (b^*L(a))^* = (R(b^*)a)^* = a^*(R(b^*)^*) = \langle a, R^\sharp(b) \rangle_A$$

for all  $a, b \in A$ , then  $L$  is an adjointable map. Conversely, if  $T : A \rightarrow A$  is an adjointable map, we have

$$(T^\dagger)^\sharp(a)b = (T^\dagger(a^*))^*b = (\langle b, T^\dagger(a^*) \rangle_A)^* = (\langle T(b), a^* \rangle_A)^* = \langle a^*, T(b) \rangle_A = aT(b)$$

for all  $a, b \in A$ . Then  $(T, \widehat{(T^*)}) \in \mathcal{M}(A)$ . Moreover,

$$\mathfrak{F}((L, R)^*) = \mathfrak{F}((R^\sharp, L^\sharp)) = R^\sharp = L^\dagger.$$

□

## 1.1.2 Factorizable functionals on bimodules

**Definition 1.1.18** (Definition 3.5, [Tim16]). Let  $B$  and  $C$  be non-degenerate, idempotent algebras with faithful functionals  $\mu_B$  and  $\mu_C$ , respectively, and let  $M$  be a unital  $B$ - $C$ -bimodule. We call a functional  $\omega \in M^\vee$  *factorizable with respect to*  $(\mu_B, \mu_C)$ , if there exist maps  ${}_B\omega \in \text{Hom}({}_B M, {}_B B)$  and  $\omega_C \in \text{Hom}(M_C, C_C)$  such that

$$\mu_B \circ {}_B\omega = \omega = \mu_C \circ \omega_C.$$

We denote by  $({}_B M_C)^\sqcup \subseteq M^\vee$  the vector sub-space of all such factorizable functionals.

**Remark 10.**

- A functional  $\omega \in M^\vee$  is factorizable if and only if there exist linear maps  ${}_B\omega : M \rightarrow B$  and  $\omega_C : M \rightarrow C$  such that for all  $x \in B, y \in C, m \in M$ ,

$$\omega(xm) = \mu_B(x{}_B\omega(m)) \quad \text{and} \quad \omega(my) = \mu_C(\omega_C(m)y).$$

- If  $\omega \in M^\vee$  is factorizable, the maps  ${}_B\omega$  and  $\omega_C$  are uniquely determined.
- Evidently, the assignment  ${}_B M_C \rightarrow ({}_B M_C)^\sqcup$  is functorial.

The important property of factorizable functionals is that one can form slice maps and tensor products for such functionals as follows.

**Lemma 1.1.19** (Lemma 3.4, [Tim16]). *Let  ${}_B M_C$  be a unital  $B$ - $C$ -bimodule and  ${}_C N_D$  a unital  $C$ - $D$ -bimodule, where  $B, C, D$  are non-degenerate, idempotent algebras with faithful functionals  $\mu_B, \mu_C, \mu_D$ , respectively. Take  $\theta \in ({}_B M_C)^\sqcup$  and  $\omega \in ({}_C N_D)^\sqcup$ . Then, the following slice maps*

$$\begin{aligned} \theta_C \bar{\otimes} id &: M_C \otimes {}_C N_D \rightarrow N_D, & m \otimes n &\mapsto \theta_C(m)n, \\ id \bar{\otimes} {}_C \omega &: {}_B M_C \otimes {}_C N \rightarrow {}_B M, & m \otimes n &\mapsto m {}_C \omega(n) \end{aligned}$$

are morphisms of modules, and the map

$$\theta \otimes_{\mu_C} \omega := \theta \circ (id \bar{\otimes} {}_C \omega) = \omega \circ (\theta_C \bar{\otimes} id)$$

is a factorizable functional which belongs to  $({}_B M_C \otimes {}_C N_D)^\sqcup$  and satisfies

$$(\theta \otimes_{\mu_C} \omega)(m \otimes n) = \mu_C(\theta_C(m) {}_C \omega(n))$$

for all  $m \in M, n \in N$ .

**Notation 5.** Let  $M$  be a  $B$ - $C$ -bimodule. If  $\omega \in M^\vee$ , we can define the functional

$$y \cdot \omega \cdot x : M \rightarrow \mathbb{C}, \quad m \mapsto \omega(x \cdot m \cdot y),$$

for all  $x \in B$  and  $y \in C$ .

**Lemma 1.1.20** (Lemma 3.5, [Tim16]). *Suppose that the functionals  $\mu_B$  and  $\mu_C$  admit modular automorphisms  $\sigma_B$  and  $\sigma_C$ , respectively. Then, the vector space  $({}_B M_C)^\sqcup$  is an  $\mathcal{M}(B)$ - $\mathcal{M}(C)$ -sub-bimodule of  $M^\vee$  and*

$${}_B(y \cdot \omega \cdot x)(m) = {}_B \omega(my) \sigma_B(x), \quad (y \cdot \omega \cdot x)_C(m) = \sigma_C^{-1}(y) \omega_C(mx)$$

for all  $\omega \in ({}_B M_C)^\sqcup, x \in B$  and  $y \in C$ .

## 1.2 Theory of multiplier Hopf \*-algebras

### 1.2.1 Multiplier Hopf \*-algebras

**Definition 1.2.1** (Definition 2.2, [Van94]). Let  $A$  be a non-degenerate \*-algebra. A co-multiplication on  $A$  is a \*-homomorphism  $\Delta : A \rightarrow \mathcal{M}(A \otimes A)$  such that

- $\Delta(a)(1 \otimes b)$  and  $(a \otimes 1)\Delta(b)$  belong to  $A \otimes A$  for all  $a, b \in A$ ,



- $\Delta$  is coassociative in the following sense

$$(a \otimes 1 \otimes 1)(\Delta \otimes id)(\Delta(b)(1 \otimes c)) = (id \otimes \Delta)((a \otimes 1)\Delta(b))(1 \otimes 1 \otimes c)$$

for all  $a, b, c \in A$ .

**Definition 1.2.2** (Definition 2.3, [Van94]). Let  $A$  be a non-degenerate  $*$ -algebra and  $\Delta$  a comultiplication on  $A$ . We say that  $(A, \Delta)$  is a *multiplier Hopf  $*$ -algebra* if the linear maps from  $A \otimes A$  to itself, defined by

$$T_1 : a \otimes b \mapsto \Delta(a)(1 \otimes b), \quad T_2 : a \otimes b \mapsto (a \otimes 1)\Delta(b)$$

are bijective.

**Remark 11.**

- Any unital Hopf  $*$ -algebra is a particular case of a multiplier Hopf  $*$ -algebra.
- The map  $\Delta^{cop} := \Sigma_{A,A} \circ \Delta$  is also a comultiplication on  $A$ . Then  $(A, \Delta^{cop})$  is also a multiplier Hopf  $*$ -algebra.

**Proposition 1.2.3** (Proposition 2.4, [Van97]). Let  $(A, \Delta)$  be a multiplier Hopf  $*$ -algebra. There exists a unique  $*$ -homomorphism  $\varepsilon : A \rightarrow \mathbb{C}$ , named the counit, satisfying

$$(\varepsilon \otimes id)(\Delta(a)(1 \otimes b)) = ab, \quad \text{and} \quad (id \otimes \varepsilon)((a \otimes 1)\Delta(b)) = ab$$

for all  $a, b \in A$ . There exists also a unique anti- $*$ -isomorphism  $S : A \rightarrow A$ , named the antipode, satisfying

$$m(S \otimes id)(\Delta(a)(1 \otimes b)) = \varepsilon(a)b, \quad \text{and} \quad m(id \otimes S)((a \otimes 1)\Delta(b)) = \varepsilon(b)a$$

for all  $a, b \in A$ , where  $m$  denotes the multiplication map from  $A \otimes A$  to  $A$ , and

$$S \circ * \circ S \circ * = id_A.$$

**Proposition 1.2.4** (Proposition 2.5, [Van97]). Let  $(A, \Delta)$  be a multiplier Hopf  $*$ -algebra. If  $A$  is a unital  $*$ -algebra, then  $(A, \Delta)$  is a Hopf  $*$ -algebra.

**Example 1.2.5.** Let  $G$  be a group and let  $\text{Fun}_f(G)$  be the  $*$ -algebra of complex finitely supported functions on  $G$ . In this case,  $\mathcal{M}(\text{Fun}_f(G))$  is the  $*$ -algebra of all complex functions on  $G$ ,  $\text{Fun}_f(G) \otimes \text{Fun}_f(G)$  can be naturally identified with the  $*$ -algebra of complex finitely supported functions on  $G \times G$ , and  $\mathcal{M}(\text{Fun}_f(G) \otimes \text{Fun}_f(G))$  can be identified with the space of all complex functions on  $G \times G$ . We define the  $*$ -homomorphism

$$\begin{aligned} \Delta : \text{Fun}_f(G) &\rightarrow \mathcal{M}(\text{Fun}_f(G) \otimes \text{Fun}_f(G)) \\ p &\mapsto [\Delta(p) : (g, h) \mapsto p(gh)] \end{aligned}$$

For all  $p, q \in \text{Fun}_f(G)$ , the functions

$$\begin{aligned} \Delta(p)(1 \otimes q) : G \times G &\rightarrow \mathbb{C} \\ (g, h) &\mapsto p(gh)q(h) \end{aligned}$$

and

$$\begin{aligned} (p \otimes 1)\Delta(q) : G \times G &\rightarrow \mathbb{C} \\ (g, h) &\mapsto p(g)q(gh) \end{aligned}$$

have finite support, so belong to  $\text{Fun}_f(G) \otimes \text{Fun}_f(G)$ ; the coassociativity of  $\Delta$  is an immediate consequence of the associativity of the multiplication on  $G$ . So  $\Delta$  is a comultiplication on  $\text{Fun}_f(G)$ . The map  $T_1$ , given by  $(T_1p)(g, h) = p(gh, h)$ , is bijective and its inverse  $R_1$  is given by  $(R_1p)(g, h) = p(gh^{-1}, h)$ . Similarly, the map  $T_2$ , given by  $(T_2p)(g, h) = p(g, gh)$ , is bijective and its inverse  $R_2$  is given by  $(R_2p)(g, h) = p(g, g^{-1}h)$ . Then  $(\text{Fun}_f(G), \Delta)$  is a commutative multiplier Hopf\*-algebra. In this case, it is easy to show that its counit is given by

$$\begin{aligned} \varepsilon : \text{Fun}_f(G) &\rightarrow \mathbb{C} \\ p &\mapsto p(e) \end{aligned}$$

and its antipode is given by

$$\begin{aligned} S : \text{Fun}_f(G) &\rightarrow \text{Fun}_f(G) \\ p &\mapsto [S(p) : g \mapsto p(g^{-1})] \end{aligned} .$$

Let  $\mathbb{C}[G]$  be the group\*-algebra of  $G$ , i.e., as a vector space it is generated by formal elements  $\{U_g\}_{g \in G}$ , the algebra structure is given by the multiplication  $U_g U_h = U_{gh}$  for all  $g, h \in G$  and the \*-structure is given by  $(U_g)^* = U_{g^{-1}}$ . For each element  $x \in \mathbb{C}[G]$ , there exists a unique complex finitely supported function  $p_x : G \rightarrow \mathbb{C}$ , such that  $x = \sum_{g \in G} p_x(g) U_g$ . We define the \*-homomorphism

$$\begin{aligned} \Delta' : \mathbb{C}[G] &\rightarrow \mathbb{C}[G] \otimes \mathbb{C}[G] \\ U_g &\mapsto U_g \otimes U_g \end{aligned}$$

The pair  $(\mathbb{C}[G], \Delta')$  is a Hopf\*-algebra, its counit is given by

$$\begin{aligned} \varepsilon' : \mathbb{C}[G] &\rightarrow \mathbb{C} \\ U_g &\mapsto 1 \end{aligned}$$

and its antipode is given by

$$\begin{aligned} S' : \mathbb{C}[G] &\rightarrow \mathbb{C}[G] \\ U_g &\mapsto U_{g^{-1}} \end{aligned} .$$

**Definition 1.2.6** ([Van98]). Let  $(A, \Delta_A), (B, \Delta_B)$  be two multiplier Hopf\*-algebras. A non-degenerate \*-homomorphism  $T : A \rightarrow \mathcal{M}(B)$  is called a *morphism of multiplier Hopf\*-algebras* if

$$\Delta_B \circ T = (T \otimes T) \circ \Delta_A$$

It is easy to prove the following

**Lemma 1.2.7.** *Let  $T_1 : A \rightarrow \mathcal{M}(B)$ ,  $T_2 : B \rightarrow \mathcal{M}(A)$  be two morphisms of multiplier Hopf\*-algebras. We have*

- (i) *If  $T_2 \circ T_1 = id_A$ , then  $A \subseteq T_2(B) \subseteq \mathcal{M}(A)$ .*
- (ii) *If  $T_1 \circ T_2 = id_B$ , then  $B \subseteq T_1(A) \subseteq \mathcal{M}(B)$ .*
- (iii) *If  $T_2 \circ T_1 = id_A$  and  $T_1 \circ T_2 = id_B$ , then  $T_1(A) = B$  and  $T_2(B) = A$ .*

*Proof*

- (i) For all  $a \in A$  and  $b \in B$ , because  $T_2(T_1(a)) = a$ , we have

$$T_2(T_1(a)b)a' = T_2(T_1(a))T_2(b)a' = aT_2(b)a'$$

for all  $a' \in A$ , and the non-degeneracy of  $A$ , implies  $T_2(T_1(a)b) = aT_2(b)$ . Finally, the non-degeneracy of the morphism  $T_2$  implies  $A = AT_2(B) \subseteq T_2(B)$ .

- (ii) Similar to item (i).

- (iii) If  $T_2 \circ T_1 = id_A$  and  $T_1 \circ T_2 = id_B$ , by item (i),  $A \subseteq T_2(B)$ , and by item (ii),  $B \subseteq T_1(A)$ . For conclude, we have  $B \subseteq T_1(A) \subseteq T_1(T_2(B)) = B$ . The other equality is similar.

□

**Remark 12.** Let  $a \in A$  and  $\omega$  be a non-zero linear functional on  $A$ . We will denote by  $(id \otimes \omega)(\Delta(a))$  the element in  $\mathcal{M}(A)$  such that

$$[(id \otimes \omega)(\Delta(a))]a' := (id \otimes \omega)(\Delta(a)(a' \otimes 1)),$$

$$a'[(id \otimes \omega)(\Delta(a))] := (id \otimes \omega)((a' \otimes 1)\Delta(a))$$

for every  $a' \in A$ . Of course, we will use a similar notation for  $(\omega \otimes id)(\Delta(a))$ .

**Definition 1.2.8** (Definition 3.2, [Van98]). Let  $(A, \Delta)$  be a multiplier Hopf\*-algebra and  $\varphi, \psi$  be non-zero linear functionals on  $A$ . We say that  $\varphi$  is *left invariant* if

$$(id \otimes \varphi)(\Delta(a)) = \varphi(a)1_{\mathcal{M}(A)}$$

for every  $a \in A$ . Similarly, we say that  $\psi$  is *right invariant* if

$$(\psi \otimes id)(\Delta(a)) = \psi(a)1_{\mathcal{M}(A)}$$

for every  $a, b \in A$ .

**Proposition 1.2.9** (Proposition 3.11, [Van98]). *If  $\varphi$  is a non-zero left invariant functional with respect to  $\Delta$ , then we also have the next relation*

$$(id \otimes \varphi)((1 \otimes a)\Delta(b)) = S((id \otimes \varphi)(\Delta(a)(1 \otimes b))),$$

for all  $a, b \in A$ . This equality is called the strong left invariance of  $\varphi$ . Similarly, if  $\psi$  is a non-zero right invariant functional with respect to  $\Delta$ , then we have the next relation

$$(\psi \otimes id)(\Delta(a)(b \otimes 1)) = S((\psi \otimes id)((a \otimes 1)\Delta(b))),$$

for all  $a, b \in A$ . This equality is called the strong right invariance of  $\psi$ .

**Theorem 1.2.10** (Theorem 3.7, [Van98]). *Let  $(A, \Delta)$  be a multiplier Hopf\*-algebra. If  $(A, \Delta)$  admits a non-zero left invariant functional, then it is unique (up to a scalar) and there exists also a unique right invariant functional. These functionals are faithful.*

**Remark 13.** Usually, non-zero invariant functionals are called integrals in the Hopf algebra theory and Haar measures in the quantum group setting. We will, throughout this work, speak about integrals.

**Proposition 1.2.11** (Proposition 3.8 – Proposition 3.10, [Van98]). *Let  $(A, \Delta)$  be a multiplier Hopf\*-algebra. If there is a left integral  $\varphi$  on  $A$  and we consider the right integral  $\psi := \varphi \circ S$  on  $A$ , then*

- There exists an invertible element  $\delta \in \mathcal{M}(A)$  such that

$$\Delta(\delta) = \delta \otimes \delta, \quad \varepsilon(\delta) = 1, \quad S(\delta) = \delta^{-1}$$

and

$$(\varphi \otimes id)(\Delta(a)) = \varphi(a)\delta, \quad (id \otimes \psi)(\Delta(a)) = \psi(a)\delta^{-1},$$

for all  $a \in A$ . Moreover,

$$\varphi(S(a)) = \varphi(a\delta), \quad \psi(S^{-1}(a)) = \psi(a\delta^{-1}),$$

for all  $a \in A$ .

- There exists an automorphism  $\rho : A \rightarrow A$  such that  $\varphi(ab) = \varphi(b\rho(a))$  for all  $a, b \in A$ ,  $\varphi$  is  $\rho$ -invariant, i.e.  $\varphi(\rho(a)) = \varphi(a)$  for all  $a \in A$  and

$$\Delta(\rho(a)) = (S^2 \otimes \rho)(\Delta(a))$$

for all  $a \in A$ . Similarly, there is an automorphism  $\rho' : A \rightarrow A$  such that  $\psi(ab) = \psi(b\rho'(a))$  for all  $a, b \in A$ ,  $\psi$  is  $\rho'$ -invariant and  $\Delta \circ \rho' = (\rho' \otimes S^{-2}) \circ \Delta$ . Moreover, we have the relations

$$S \circ \rho' = \rho^{-1} \circ S, \quad \rho'(\cdot) = \delta\rho(\cdot)\delta^{-1}.$$

**Proposition 1.2.12** (Proposition 1.2 [VZ99a] and Lemma 2.2 - Lemma 2.11 [Ver06]). *If  $(A, \Delta)$  is a multiplier Hopf\*-algebra, then given  $a_1, \dots, a_n \in A$ , there exist an element  $e \in A$  such that  $ea_i = a_i e = a_i$  for every  $i \in \{1, \dots, n\}$ .*

**Example 1.2.13.** Let  $G$  be a group. If we define the non-zero linear functional

$$\begin{aligned} \varphi : \text{Fun}_f(G) &\rightarrow \mathbb{C} \\ p &\mapsto \sum_{g \in G} p(g) \end{aligned}$$

then one can see that, for all  $p, q \in \text{Fun}_f(G)$ ,  $g \in G$ ,

$$[(id \otimes \varphi)(\Delta(p)(q \otimes 1))](g) = [\varphi(p)q](g), \quad [(id \otimes \varphi)((p \otimes 1)\Delta(q))](g) = [\varphi(q)p](g),$$

$$(\varphi \circ S)(p) = \sum_{g \in G} p(g^{-1}) = \varphi(p),$$

then  $\varphi$  is a faithful left and right integral for  $(\text{Fun}_f(G), \Delta)$ . Also, we define the non-zero linear functional

$$\begin{aligned} \varphi' : \mathbb{C}[G] &\rightarrow \mathbb{C} \\ U_g &\mapsto \delta_{g,e} \end{aligned}$$

and we have

$$(id \otimes \varphi')(\Delta(U_g)(U_h \otimes 1)) = (\varphi' \otimes id)(\Delta(U_g)(1 \otimes U_h)) = \delta_{g,e} U_{gh} = \varphi'(U_g) U_h$$

$$(id \otimes \varphi')((U_g \otimes 1)\Delta(U_h)) = (\varphi' \otimes id)((1 \otimes U_g)\Delta(U_h)) = \delta_{h,e} U_{gh} = \varphi'(U_h) U_g$$

for all  $g, h \in G$ , then  $\varphi'$  is a faithful left and right integral for  $(\mathbb{C}[G], \Delta')$ .

**Remark 14.** Let  $(A, \Delta)$  be a multiplier Hopf \*-algebra.

- When we write  $(A, \Delta, \varphi)$ , we say that  $\varphi$  is a left integral for  $(A, \Delta)$ .
- When we write  $(A, \Delta, \varphi, \psi)$ , we say that  $\varphi$  is a left integral for  $(A, \Delta)$  and  $\psi$  is a right integral for  $(A, \Delta)$ .

## 1.2.2 Duality for multiplier Hopf \*-algebras with integrals

Let  $(A, \Delta)$  be a multiplier Hopf \*-algebra with a positive faithful left integral  $\varphi$ . We fix  $\psi := \varphi \circ S$  as its positive faithful right integral. We denote by  $\widehat{A}$  the space of linear functionals on  $A$  of the form  $\varphi_a$  where  $a \in A$ . By Proposition 1.2.11, we have

$$\widehat{A} = \{\varphi_a : a \in A\} = \{{}_a\varphi : a \in A\} = \{\psi_a : a \in A\} = \{{}_a\psi : a \in A\}$$

**Remark 15.** By Proposition 1.2.11, we have  $\varphi_a = \psi_{a\delta^{-1}}$  and  ${}_a\varphi = \varphi_{\rho(a)}$  for all  $a \in A$ .

**Proposition 1.2.14** (Proposition 4.2, [Van98]). Let  $\omega_1, \omega_2$  be elements of  $\widehat{A}$ . Then, we can define a linear functional  $\omega_1 \cdot \omega_2$  on  $A$  by

$$(\omega_1 \cdot \omega_2)(a) := (\omega_1 \otimes \omega_2)(\Delta(a))$$

and  $\omega_1 \cdot \omega_2 \in \widehat{A}$ . This multiplication is non-degenerate and turns  $\widehat{A}$  into an associative algebra.

**Remark 16.** In [Kus97a], Kustermans shows that the multiplier algebra  $\mathcal{M}(\widehat{A})$  can be identified with the space

$$\{\omega \in A^\vee : (\omega \otimes id)(\Delta(a)) \in A, (id \otimes \omega)(\Delta(a)) \in A \text{ for all } a \in A\}$$

**Proposition 1.2.15** (Proposition [Van98]). *For every  $\omega \in \widehat{A}$ , the functional  $\omega^* : a \mapsto \overline{\omega(S(a)^*)}$  is in  $\widehat{A}$ . Moreover, the map  $\omega \mapsto \omega^*$  is an anti-linear anti-homomorphism on  $\widehat{A}$ , then  $\widehat{A}$  is a  $*$ -algebra.*

**Proposition 1.2.16** (Proposition 4.6, [Van98]). *There exists a  $*$ -homomorphism  $\widehat{\Delta} : \widehat{A} \rightarrow \mathcal{M}(\widehat{A} \otimes \widehat{A})$  such that*

$$(\widehat{\Delta}(\omega_1) \cdot (1 \otimes \omega_2))(a \otimes b) = (\omega_1 \otimes \omega_2)((a \otimes 1)\Delta(b)),$$

$$((\omega_1 \otimes 1) \cdot \widehat{\Delta}(\omega_2))(a \otimes b) = (\omega_1 \otimes \omega_2)(\Delta(a)(1 \otimes b))$$

for all  $\omega_1, \omega_2 \in \widehat{A}$  and  $a, b \in A$ .

**Proposition 1.2.17** (Proposition 4.7, [Van98]). *The pair  $(\widehat{A}, \widehat{\Delta})$  is a multiplier Hopf $*$ -algebra. Its antipode  $\widehat{S}$  is given for all  $\omega \in \widehat{A}$  and  $a \in A$  by:*

$$\widehat{S}(\omega)(a) := \omega(S(a))$$

and the counit  $\widehat{\varepsilon}$  is given by

$$\widehat{\varepsilon}(\omega) := \omega(1_{\mathcal{M}(A)}).$$

**Remark 17.** For any  $a \in A$ , the map  $\mathcal{F} : A \rightarrow \widehat{A} : a \mapsto \varphi_a$  is a bijection, called the Fourier transform.

**Proposition 1.2.18** (Proposition 4.8 - 4.9. [Van98]). *The maps from  $\widehat{A}$  to  $\mathbb{C}$  given by*

$$\widehat{\varphi}(a\psi) := \varepsilon(a) \quad \text{and} \quad \widehat{\psi}(\varphi_a) := \varepsilon(a),$$

for all  $a \in A$ , define left and right integrals for  $(\widehat{A}, \widehat{\Delta})$ , respectively.

**Lemma 1.2.19** (Lemma 4.11, [Van98]). *For all  $a \in A$  and  $\omega \in \widehat{A}$ , we have that*

$$\widehat{\psi}(\omega \cdot \varphi_a) = \omega(S^{-1}(a)).$$

In particular, we have the identity  $\widehat{\psi}((\varphi_b)^* \cdot \varphi_a) = \varphi(b^*a)$  for all  $a, b \in A$ .

**Theorem 1.2.20** (Theorem 4.12, [Van98]). *If  $(A, \Delta)$  is a multiplier Hopf $*$ -algebra with integrals, then  $(\widehat{A}, \widehat{\Delta})$  is a  $*$ -multiplier Hopf algebra with integrals. Moreover, if we define*

$$\Gamma(a)(\omega) := \omega(a)$$

whenever  $a \in A$  and  $\omega \in \widehat{A}$ , then  $\Gamma(a) \in \widehat{A}$  and  $\Gamma$  is an isomorphism of multiplier Hopf $*$ -algebras between  $(A, \Delta)$  and the dual of  $(\widehat{A}, \widehat{\Delta})$ .

**Example 1.2.21.** Let  $G$  be a group, by the Example 1.2.5 and Example 1.2.13, we have two multiplier Hopf\*-algebras with integrals  $(\text{Fun}_f(G), \Delta, \varphi)$  and  $(\mathbb{C}[G], \Delta', \varphi')$ . The bilinear map defined by

$$\begin{aligned} (\cdot, \cdot) : \text{Fun}_f(G) \times \mathbb{C}[G] &\rightarrow \mathbb{C} \\ (p, U_g) &\mapsto p(g) \end{aligned}$$

allows to identify  $\mathbb{C}[G]$  with  $\widehat{\text{Fun}_f(G)}$  as algebras, i.e., for each element  $U_g \in \mathbb{C}[G]$ , we have

$$U_g(p) := (p, U_g) = p(g) = \sum_{h \in G} p(h) \delta_g(h) = \varphi(p \cdot \delta_g) = \varphi_{\delta_g}(p),$$

for all  $p \in \text{Fun}_f(G)$ , here  $\delta_g$  denoted the Dirac function  $h \mapsto \delta_{g,h}$ .

$$(p \cdot q, U_g) = p(g)q(g) = (p, U_g)(q, U_g) = (p \otimes q, \Delta'(U_g)),$$

$$(p, U_g U_h) = p(gh) = (U_g \otimes U_h) \Delta(p) = (\Delta(p), U_g \otimes U_h),$$

for all  $p, q \in \text{Fun}_f(G)$ ,  $g, h \in G$ . Moreover, we see that

$$\varepsilon'(U_g) = 1 = U_g \left( \sum_{h \in G} \delta_h \right) = U_g(1),$$

$$S'(U_g)(p) = U_{g^{-1}}(p) = p(g^{-1}) = S(p)(g) = U_g(S(p)),$$

$$\varphi'(\varphi_{\delta_g}) = \varphi'(U_g) = \delta_{g,e} = \varepsilon(\delta_g),$$

for all  $p \in \text{Fun}_f(G)$ ,  $g \in G$ . Then, the two multiplier Hopf\*-algebras with integrals  $(\text{Fun}_f(G), \Delta, \varphi)$ ,  $(\mathbb{C}[G], \Delta', \varphi')$  are in duality.

### 1.2.3 Corepresentations of multiplier Hopf\*-algebras

Let  $(A, \Delta)$  be a multiplier Hopf\*-algebra.

**Definition 1.2.22** (Definition 7.5, [Kus97a]). Let  $X$  be a non-degenerate \*-algebra. An unitary element  $W \in \mathcal{M}(A \otimes X)$  is called a *unitary corepresentation* of  $(A, \Delta)$  on  $X$  if

$$(\Delta \otimes id)(W) = W_{13}W_{23}$$

in  $\mathcal{M}(A \otimes A \otimes X)$ .

**Proposition 1.2.23** (Proposition 5.8, [Kus97a] - Proposition 2.1, [KVZ00]). If  $W$  is a unitary corepresentation of  $(A, \Delta)$  on a non-degenerate \*-algebra  $X$ , then we have the following equations in  $\mathcal{M}(A \otimes X)$ :

$$(A \otimes 1_{\mathcal{M}(X)})W(1_{\mathcal{M}(A)} \otimes X) = A \otimes X, \quad (1_{\mathcal{M}(A)} \otimes X)W(A \otimes 1_{\mathcal{M}(X)}) = A \otimes X,$$

$$(A \otimes 1_{\mathcal{M}(X)})W^*(1_{\mathcal{M}(A)} \otimes X) = A \otimes X, \quad (1_{\mathcal{M}(A)} \otimes X)W^*(A \otimes 1_{\mathcal{M}(X)}) = A \otimes X.$$

### 1.2.4 Actions of multiplier Hopf $*$ -algebras

**Notation 6.** Let  $(A, \Delta)$  be a multiplier Hopf  $*$ -algebra. From now on, given  $a \in A$ , we will use the notation

$$\Delta(a) = a_{(1)} \otimes a_{(2)}$$

in the following sense:

$$\begin{aligned} a_{(1)}b \otimes a_{(2)} &:= \Delta(a)(b \otimes 1), & a_{(1)} \otimes a_{(2)}b &:= \Delta(a)(1 \otimes b) \\ ba_{(1)} \otimes a_{(2)} &:= (b \otimes 1)\Delta(a), & a_{(1)} \otimes ba_{(2)} &:= (1 \otimes b)\Delta(a) \end{aligned}$$

for all  $b \in B$ .

**Remark 18.** Let  $(A, \Delta)$  be a multiplier Hopf  $*$ -algebra and  $X$  a left unital  $A$ -module with module map  $\triangleright : A \otimes X \rightarrow X$ . Fix  $a \in A$  and  $x, y \in X$ , then there exist  $b, c \in A$  such that  $b \triangleright x = x$  and  $c \triangleright y = y$ . We define

$$(a_{(1)} \triangleright x)(a_{(2)} \triangleright y) := (a_{(1)}b \triangleright x)(a_{(2)}c \triangleright y),$$

as element in  $X$ , because  $a_{(1)}b \otimes a_{(2)}c = \Delta(a)(b \otimes c) \in A \otimes A$ . To see that this is well defined, take two elements  $b', c' \in A$  such that  $b' \triangleright x = x$  and  $c' \triangleright y = y$ . Because,  $A$  has local units, there exist  $b'', c'' \in A$  such that  $b''b = b$ ,  $b''b' = b'$ ,  $c''c = c$  and  $c''c' = c'$ , then

$$\begin{aligned} (a_{(1)}b' \triangleright x)(a_{(2)}c' \triangleright y) &= (a_{(1)}b''b' \triangleright x)(a_{(2)}c''c' \triangleright y) \\ &= (a_{(1)}b'' \triangleright (b' \triangleright x))(a_{(2)}c'' \triangleright (c' \triangleright y)) \\ &= (a_{(1)}b'' \triangleright (b \triangleright x))(a_{(2)}c'' \triangleright (c \triangleright y)) \\ &= (a_{(1)}b''b \triangleright x)(a_{(2)}c''c \triangleright y) \\ &= (a_{(1)}b \triangleright x)(a_{(2)}c \triangleright y). \end{aligned}$$

**Definition 1.2.24** (Definition, [DVZ99]). Let  $(A, \Delta)$  be a multiplier Hopf  $*$ -algebra. A non-degenerate  $*$ -algebra  $X$  is called a *left  $(A, \Delta)$ -module  $*$ -algebra*, if  $X$  is a left unital  $A$ -module with module map  $\triangleright : A \otimes X \rightarrow X$  and satisfies the conditions

$$a \triangleright (xy) = (a_{(1)} \triangleright x)(a_{(2)} \triangleright y), \quad a^* \triangleright x^* = (S^{-1}(a) \triangleright x)^*, \quad (1.1)$$

for all  $a \in A$  and  $x, y \in X$ . Similarly, a non-degenerate  $*$ -algebra  $X$  is called a *right  $(A, \Delta)$ -module  $*$ -algebra*, if  $X$  is a right unital  $A$ -module with module map  $\triangleleft : X \otimes A \rightarrow X$  and satisfies the conditions

$$(xy) \triangleleft a = (x \triangleleft a_{(1)})(y \triangleleft a_{(2)}), \quad x^* \triangleleft a^* = (x \triangleleft S^{-1}(a))^*$$

for all  $a \in A$  and  $x, y \in X$ .

**Definition 1.2.25** ([DVZ99]). Let  $X$  be a non-degenerate left  $(A, \Delta)$ -module  $*$ -algebra. We define the  $*$ -algebra  $X \# A$ , called *the smash product associated with the left action of  $(A, \Delta)$  on  $X$* , as the vector space  $X \otimes A$  with the structure of a  $*$ -algebra given by

$$(x \# a)(y \# b) = x(a_{(1)} \triangleright y) \# a_{(2)}b, \quad (x \# a)^* = (S^{-1}(a_{(1)}) \triangleright x)^* \# a_{(2)}^*$$



for all  $x, y \in X$  and  $a, b \in A$ . This  $*$ -algebra  $X \# A$  is associative and non-degenerate. Similarly, if  $X$  is a non-degenerate right  $(A, \Delta)$ -module  $*$ -algebra, we define the  $*$ -algebra  $A \# X$ , called *the smash product associated with the right action of  $(A, \Delta)$  on  $X$* , as the vector space  $A \otimes X$  with the structure of a  $*$ -algebra given by

$$(a \# x)(b \# y) = ab_{(1)} \# (x \triangleleft b_{(2)})y, \quad (a \# x)^* = a_{(1)}^* \# (x \triangleleft S^{-1}(a_{(2)}))^*$$

for all  $x, y \in X$  and  $a, b \in A$ . This  $*$ -algebra  $A \# X$  is also associative and non-degenerate.

**Remark 19.** The above multiplications is well defined because

$$a_{(1)} \otimes a_{(2)}b = \Delta(a)(1 \otimes b) \in A \otimes A, \quad ab_{(1)} \otimes b_{(2)} = (a \otimes 1)\Delta(b) \in A \otimes A,$$

for all  $a, b \in A$ . Also, using the fact that for every  $x \in X$ , there exists  $a_x \in A$  such that  $a_x \triangleright x = x$ , the next expression make sense

$$(x \# a)^* = (S^{-1}(a_{(1)}) \triangleright x)^* \# a_{(2)}^* := (S^{-1}(a_{(1)})a_x \triangleright x)^* \# a_{(2)}^*.$$

**Proposition 1.2.26** (Proposition 6.2, [DVZ99]). *Let  $\triangleright : A \otimes X \rightarrow X$  be a left action of the multiplier Hopf  $*$ -algebra  $(A, \Delta)$  on a non-degenerate  $*$ -algebra  $X$ . Then the linear map*

$$\begin{aligned} \bar{\triangleright} : X \# A \otimes X &\rightarrow X \\ x \# a \otimes y &\mapsto x(a \triangleright y) \end{aligned}$$

*is a faithful left action. Similarly, let  $\triangleleft : X \otimes A \rightarrow X$  be a right action of the multiplier Hopf  $*$ -algebra  $(A, \Delta)$  on a non-degenerate  $*$ -algebra  $X$ . Then the linear map*

$$\begin{aligned} \bar{\triangleleft} : X \otimes A \# X &\rightarrow X \\ y \otimes a \# x &\mapsto (y \triangleleft a)x \end{aligned}$$

*is a faithful right action.*

**Definition 1.2.27** (Definition 4.13, [DVZ99]). A left action  $\triangleright : A \otimes X \rightarrow X$  of a multiplier Hopf  $*$ -algebra  $(A, \Delta)$  on a non-degenerate  $*$ -algebra  $X$  is called *a inner* if there exists a non-degenerate  $*$ -representation  $\gamma : A \rightarrow \mathcal{M}(X)$  of  $A$  on  $X$  such that

$$\gamma(a)x = (a_{(1)} \triangleright x)\gamma(a_{(2)}) \quad \text{or equivalently} \quad a \triangleright x = \gamma(a_{(1)})x\gamma(S(a_{(2)}))$$

for all  $a \in A, x \in X$ . Similarly, a right action  $\triangleleft : X \otimes A \rightarrow X$  of the multiplier Hopf  $*$ -algebra  $(A, \Delta)$  on a non-degenerate  $*$ -algebra  $X$  is called *a inner* if there exists a non-degenerate  $*$ -representation  $\gamma : A \rightarrow \mathcal{M}(X)$  of  $A$  on  $X$  such that

$$\gamma(a)x = (x \triangleleft a_{(2)})\gamma(a_{(1)}) \quad \text{or equivalently} \quad x \triangleleft a = \gamma(S(a_{(1)}))x\gamma(a_{(2)})$$

for all  $a \in A$  and  $x \in X$ . In both cases, we say that *the action is implemented by  $\gamma$* .

**Remark 20.** Fix  $a \in A$  and  $x \in X$ . Because  $\gamma$  is non-degenerate, there exists  $e \in A$  such that  $x = \gamma(e)x$ , then  $a_{(1)}e \otimes a_{(2)} = \Delta(a)(e \otimes 1) \in A \otimes A$  and we see that the expression

$$\gamma(a_{(1)})x\gamma(S(a_{(2)})) = \gamma(a_{(1)}e)x\gamma(S(a_{(2)}))$$

makes sense.

We have the next important result.

**Proposition 1.2.28** (Proposition 5.12, [DVZ99]). *If the left action of a multiplier Hopf  $*$ -algebra  $(A, \Delta)$  on a non-degenerate  $*$ -algebra  $X$  is inner, then the linear map*

$$X \# A \rightarrow X \otimes A, \quad x \# a \mapsto x\gamma(a_{(1)}) \otimes a_{(2)}$$

is an isomorphism of  $*$ -algebras with inverse given by

$$X \otimes A \rightarrow X \# A, \quad x \otimes a \mapsto x\gamma(S(a_{(1)})) \# a_{(2)}.$$

Similarly, if the right action of a multiplier Hopf  $*$ -algebra  $(A, \Delta)$  on  $X$  is inner, then the linear map

$$A \# X \rightarrow A \otimes X, \quad a \# x \mapsto a_{(1)} \otimes \gamma(a_{(2)})x$$

is an isomorphism of  $*$ -algebras with inverse given by

$$A \otimes X \rightarrow A \# X, \quad a \otimes x \mapsto a_{(1)} \# \gamma(S(a_{(2)}))x.$$

**Proposition 1.2.29** (Proposition 4.2. [KVZ00]). *Let  $\triangleright : A \otimes X \rightarrow X$  be a left action of a multiplier Hopf  $*$ -algebra  $(A, \Delta)$  on a non-degenerate  $*$ -algebra  $X$ . The action  $\triangleright$  is inner if and only if there is an algebra  $*$ -isomorphism  $f : X \# A \rightarrow X \otimes A$  such that  $f(x \# 1_{\mathcal{M}(A)}) = x \otimes 1$  for all  $x \in X$ .*

## 1.2.5 Pairing of multiplier Hopf $*$ -algebras

Let  $(A, \Delta_A), (B, \Delta_B)$  be two multiplier Hopf  $*$ -algebras and  $p : A \times B \rightarrow \mathbb{C}$  be a bilinear map. For all  $a \in A$  and  $b \in B$ , we define the linear functionals  ${}_a p := p(a, \cdot) \in B^\vee$  and  $p_b := p(\cdot, b) \in A^\vee$ .

**Definition 1.2.30** (Definition 2.1, [DV01]). *A pre-pairing for the pair  $(A, B)$  is a bilinear map  $p : A \times B \rightarrow \mathbb{C}$  such that*

- (i)  $({}_a p \otimes id)\Delta_B(b) \in B$  and  $(id \otimes {}_a p)\Delta_B(b) \in B$ ,
- (ii)  $(p_b \otimes id)\Delta_A(a) \in A$  and  $(id \otimes p_b)\Delta_A(a) \in A$ ,
- (iii)  ${}_a p(id \otimes {}_{a'} p)\Delta_B(b) = {}_{a'} p({}_{a'} p \otimes id)\Delta_B(b) = {}_{aa'} p(b)$ ,
- (iv)  $p_b(id \otimes p_{b'})\Delta_A(a) = p_{b'}(p_b \otimes id)\Delta_A(a) = p_{bb'}(a)$ ,

for all  $a, a' \in A$  and  $b, b' \in B$ .

**Remark 21.** By the conditions above, a pre-pairing  $p : A \times B \rightarrow \mathbb{C}$  induces linear maps

$$\begin{array}{ll} \blacktriangleright_A: A \otimes B \rightarrow B & \blacktriangleleft_A: B \otimes A \rightarrow B \\ a \otimes b \mapsto (id \otimes {}_a p)\Delta_B(b) & b \otimes a \mapsto ({}_a p \otimes id)\Delta_B(b) \\ \\ \blacktriangleright_B: B \otimes A \rightarrow A & \blacktriangleleft_B: A \otimes B \rightarrow A \\ b \otimes a \mapsto (id \otimes p_b)\Delta_A(a) & a \otimes b \mapsto (p_b \otimes id)\Delta_A(a) \end{array}$$

such that  $(B, \blacktriangleright_A)$  is a left  $A$ -module,  $(B, \blacktriangleleft_A)$  is a right  $A$ -module,  $(A, \blacktriangleright_B)$  is a left  $B$ -module and  $(A, \blacktriangleleft_B)$  is a right  $B$ -module, respectively.

**Lemma 1.2.31** (Lemma 2.3, [DV01]). *We have*

$$\begin{aligned} p(a, a' \blacktriangleright_A b) &= p(aa', b), & p(a, b \blacktriangleleft_A a') &= (a'a, b), \\ p(b \blacktriangleright_B a, b') &= p(a, b'b), & p(a \blacktriangleleft_B b, b') &= p(a, bb'), \end{aligned}$$

for all  $a, a' \in A$  and  $b, b' \in B$ .

**Definition 1.2.32** (Definition 2.8, [DV01]). A *pairing for the pair*  $(A, B)$  is a pre-pairing  $p : A \times B \rightarrow \mathbb{C}$  such that the induced actions are unital and satisfies the conditions

$$p(a^*, b) = \overline{p(a, S_B(b)^*)}, \quad p(a, b^*) = \overline{p(S_A(a)^*, b)}$$

for all  $a \in A$  and  $b \in B$ .

**Remark 22.** It is shown in [DV01], that it suffices to demand the unital property only for one of the actions defined in Remark 21.

**Remark 23.** If  $p : A \times B \rightarrow \mathbb{C}$  is a pairing for  $(A, B)$ , then the following extensions are well defined :

$$\begin{array}{ccc} p : & \mathcal{M}(A) \times B & \rightarrow & \mathbb{C} \\ & (L, R) \times (a \blacktriangleright_A b) & \mapsto & p(L(a), b) \\ & (L, R) \times (b \blacktriangleleft_A a) & \mapsto & p(R(a), b) \end{array} \quad \begin{array}{ccc} p : & A \times \mathcal{M}(B) & \rightarrow & \mathbb{C} \\ & (b \blacktriangleright_B a) \times (L, R) & \mapsto & p(a, L(b)) \\ & (a \blacktriangleleft_B b) \times (L, R) & \mapsto & p(a, R(b)) \end{array}$$

**Proposition 1.2.33** (Proposition, [DV01]). *Let  $p, p'$  be two pairings for the pair  $(A, B)$ . Then the bilinear map*

$$\begin{aligned} (p \otimes p') : & (A \otimes A) \times (B \otimes B) & \rightarrow & \mathbb{C} \\ & (a \otimes a') \times (b \otimes b') & \mapsto & p(a, b)p'(a', b') \end{aligned}$$

is a pairing for  $(A \otimes A, B \otimes B)$ . It is called the tensor product pairing of  $p$  and  $p'$ .

**Remark 24.** Let  $p$  be a pairing for the pair  $(A, B)$ . Then, it is obvious that the bilinear map  $\bar{p} : B \times A \rightarrow \mathbb{C}$ ,  $\bar{p}(b, a) = p(a, b)$ , is a pairing for the pair  $(B, A)$ . This pairing is called the *flipped pairing* of  $p$ .

**Proposition 1.2.34** (Proposition 1.2, [Van18]). *Let  $p$  be a pairing for the pair  $(A, B)$ . Consider the tensor product pairing for  $(A \otimes A, B \otimes B)$ ,  $p^2 := p \otimes p$ , and its extension to multiplier algebras as in Remark 23. Then, we have*

$$p^2(\Delta_A(a), b \otimes b') = p(a, bb'), \quad p^2(a \otimes a', \Delta_B(b)) = p(aa', b),$$

for all  $a, a' \in A$  and  $b, b' \in B$ .

Let  $p : A \times B \rightarrow \mathbb{C}$  be a pairing for  $(A, B)$ . Take the non-degenerate unital right action of  $B$  on  $A$ ,  $\blacktriangleleft_B : A \otimes B \rightarrow A$ , induced by this pairing.

**Remark 25.** Fix  $a \in A$  and  $b \in B$ , because we have

$$a'(a \blacktriangleleft_B b) = a'(p_b \otimes id)(\Delta_A(a)) = (p_b \otimes id)((1 \otimes a')\Delta_A(a)) = a'p_b(a_{(1)})a_{(2)} = a'p(a_{(1)}, b)a_{(2)},$$

for all  $a' \in A$ , we can write simply  $a \blacktriangleleft_B b = p(a_{(1)}, b)a_{(2)}$ .

The next lemma shows that  $A$  is a right  $(B, \Delta_B)$ -module  $*$ -algebra for the module map  $\blacktriangleleft_B: A \otimes B \rightarrow A$ . It can be proved by direct calculations.

**Lemma 1.2.35** ([Van18]). *We have*

$$(aa') \blacktriangleleft_B b = (a \blacktriangleleft_B b_{(1)})(a' \blacktriangleleft_B b_{(2)}), \quad a^* \blacktriangleleft_B b^* = (a \blacktriangleleft_B S_B^{-1}(b))^*$$

for all  $a, a' \in A$  and  $b \in B$ . Here, we make use of the Sweedler type leg notation,  $b_{(1)} \otimes b_{(2)} = \Delta_B(b)$ .

**Definition 1.2.36** ([Van18]). *The Heisenberg  $*$ -algebra associated with the pairing  $p: A \times B \rightarrow \mathbb{C}$ , is the smash product  $B \# A$  associated with the unital right action  $\blacktriangleleft_B: A \otimes B \rightarrow A$ .*

**Remark 26.** There exist two non-degenerate  $*$ -embeddings  $j_A: A \rightarrow \mathcal{M}(B \# A)$ ,  $a \mapsto 1 \# a$ , and  $j_B: B \rightarrow \mathcal{M}(B \# A)$ ,  $b \mapsto b \# 1$ , such that we have

$$b \# a = j_B(b)j_A(a),$$

for  $a \in A$  and  $b \in B$ . Then, when we will identify  $A$  and  $B$  with their images in  $\mathcal{M}(B \# A)$ , we will denote the algebra  $B \# A$  by  $BA$ .

**Proposition 1.2.37** (Proposition 1.8, [Van18]). *We have*

$$(b \# a)(b' \# a') = bb'_{(1)} \# p(a_{(1)}, b'_{(2)})a_{(2)}a' \quad (1.2)$$

for all  $a, a' \in A$  and  $b, b' \in B$ . The algebra  $BA$  acts faithfully on  $A$  on the right by

$$\begin{aligned} \blacktriangleleft_{BA}: \quad A \otimes BA &\rightarrow A \\ a' \otimes (b \# a) &\mapsto (a' \blacktriangleleft_B b)a \end{aligned}$$

Consider now, the multiplier Hopf  $*$ -algebras with integrals  $(A, \Delta, \varphi)$  and its dual  $(\widehat{A}, \widehat{\Delta}, \widehat{\varphi})$ . Using the canonical pairing

$$p_A: A \times \widehat{A} \rightarrow \mathbb{C}, \quad a \times \varphi_b \mapsto \varphi(ab)$$

we have the next important results.

**Proposition 1.2.38** ([Van18]). *Consider the extension to the multiplier algebras of the tensor product of the canonical pairing  $p_A$  and its flipped pairing  $\bar{p}_A$ . There exists a unique unitary element  $V \in \mathcal{M}(\widehat{A} \otimes A)$  such that*

$$(\bar{p}_A \otimes p_A)(V, a \otimes \omega) = p_A(a, \omega)$$

for all  $a \in A$  and  $\omega \in \widehat{A}$ . Moreover, we have the relations

- (i)  $(id \otimes \Delta)(V) = V_{12}V_{13}$  in  $\mathcal{M}(\widehat{A} \otimes A \otimes A)$ ,
- (ii)  $(\widehat{\Delta} \otimes id)(V) = V_{13}V_{23}$  in  $\mathcal{M}(\widehat{A} \otimes \widehat{A} \otimes A)$ ,
- (iii)  $V^* = (\widehat{S} \otimes id)(V) = (id \otimes S)(V)$ ,
- (iv)  $V_{12}V_{13}V_{23} = V_{23}V_{12}$  in  $\mathcal{M}(\widehat{A} \otimes \widehat{A}A \otimes A)$  (pentagon equation)

**Definition 1.2.39** ([Van18]). The above element  $V \in \mathcal{M}(\widehat{A} \otimes A)$  is called *the duality of the pairing*  $p_A$ .

**Proposition 1.2.40** (Proposition 1.11, [Van18]). *The map*

$$\begin{aligned} T : \widehat{A}A &\rightarrow \text{End}_1(A) \\ \omega \# a &\mapsto [\omega \# a : b \mapsto b \blacktriangleleft_{\widehat{A}A} (\omega \# a)] \end{aligned}$$

is an anti-isomorphism of algebras, where

$$\text{End}_1(A) := \text{Span}\{[\omega T_a : b \mapsto a p_A(b, \omega)] : a \in A, \omega \in \widehat{A}\} \subseteq \text{End}(A).$$

**Proposition 1.2.41** (Proposition 1.2, [Van18]). *Consider the extension to multiplier algebras of the non-degenerate faithful right action*

$$\begin{aligned} \triangleleft : (A \otimes A) \otimes (\widehat{A} \otimes A) &\rightarrow A \otimes A \\ (b \otimes b') \otimes (\omega \otimes a) &\mapsto (b \blacktriangleleft_{\widehat{A}} \omega) \otimes b'a \end{aligned}$$

Then :

$$(a \otimes b) \triangleleft V = (1 \otimes b) \Delta^{cop}(a),$$

for all  $a, b \in A$ .

**Corollary 1.2.42.** *We have*

$$(a \otimes b) \triangleleft V^* = (1 \otimes b)(id \otimes S)(\Delta^{cop}(a))$$

for all  $a, b \in A$ .

*Proof* By calculation, we have

$$\begin{aligned} ((1 \otimes b)(id \otimes S)(\Delta^{cop}(a))) \triangleleft V &= (a_{(2)} \otimes bS(a_{(1)})) \triangleleft V \\ &= (1 \otimes bS(a_{(1)})) \Delta^{cop}(a_{(2)}) \\ &= a_{(3)} \otimes bS(a_{(1)})a_{(2)} \\ &= a \otimes b, \end{aligned}$$

then

$$\begin{aligned} (a \otimes b) \triangleleft V^* &= (((1 \otimes b)(id \otimes S)(\Delta^{cop}(a))) \triangleleft V) \triangleleft V^* \\ &= (1 \otimes b)(id \otimes S)(\Delta^{cop}(a)), \end{aligned}$$

for all  $a, b \in A$ . □

**Proposition 1.2.43** (Proposition 1.11, [Van18]). *For any  $a \in A$  and  $\omega \in \widehat{A}$ , we have*

- (i)  $\Delta^{cop}(a) = V^*(a \otimes 1)V$  in  $\mathcal{M}(\widehat{A}A \otimes A)$ .
- (ii)  $\widehat{\Delta}^{cop}(\omega) = V(1 \otimes \omega)V^*$  in  $\mathcal{M}(\widehat{A} \otimes \widehat{A}A)$ .

**Remark 27.** Fix  $a \in A$  and  $\omega, \theta \in \widehat{A}$ . Because

$$\begin{aligned}
 p_A^2(b \otimes b', (\theta \otimes 1) \cdot \widehat{\Delta}(\omega_a)) &= p_A^2((b \otimes b') \triangleleft (\theta \otimes 1), \widehat{\Delta}(\omega_a)) \\
 &= p_A^2((b \triangleleft_{\widehat{A}} \theta) \otimes b', \widehat{\Delta}(\omega_a)) \\
 &= p_A((b \triangleleft_{\widehat{A}} \theta)b', \omega_a) \\
 &= p_A((b \triangleleft_{\widehat{A}} \theta)b'a, \omega) \\
 &= p_A^2((b \triangleleft_{\widehat{A}} \theta) \otimes b'a, \widehat{\Delta}(\omega)) \\
 &= p_A^2(b \otimes b'a, (\theta \otimes 1) \cdot \widehat{\Delta}(\omega)) \\
 &= p_A^2(b \otimes b'a, \theta \cdot \omega_{(1)} \otimes \omega_{(2)}) \\
 &= p_A^2(b \otimes b', \theta \cdot \omega_{(1)} \otimes (\omega_{(2)})_a)
 \end{aligned}$$

for all  $b, b' \in A$ , then

$$\theta \cdot (\omega_a)_{(1)} \otimes (\omega_a)_{(2)} = (\theta \otimes 1) \cdot \widehat{\Delta}(\omega_a) = \theta \cdot \omega_{(1)} \otimes (\omega_{(2)})_a.$$

That implies that we can formally write

$$(\omega_a)_{(1)} \otimes (\omega_a)_{(2)} = \omega_{(1)} \otimes (\omega_{(2)})_a.$$

## 1.2.6 Quantum double for multiplier Hopf \*-algebras

Let  $(A, \Delta_A), (B, \Delta_B)$  be two multiplier Hopf \*-algebras and  $p : A \times B \rightarrow \mathbb{C}$  be a pairing for the pair  $(A, B)$ .

**Definition 1.2.44** (Definition 3.1, [DV01]). *The twist map associated with the pairing  $p$  is the bijective linear map*

$$\begin{aligned}
 T_p : B \otimes A &\rightarrow B \otimes A \\
 b \otimes a &\mapsto b_{(1)} \blacktriangleright_B a \triangleleft_B S_B^{-1}(b_{(3)}) \otimes b_{(2)},
 \end{aligned}$$

with inverse map given by

$$\begin{aligned}
 T_p^{-1} : A \otimes B &\rightarrow B \otimes A \\
 a \otimes b &\mapsto b_{(2)} \otimes S_B^{-1}(b_{(1)}) \blacktriangleright_B a \triangleleft_B b_{(3)}.
 \end{aligned}$$

**Remark 28.** Because the actions  $\blacktriangleright_B$  and  $\triangleleft_B$  are unital, given  $a \in A$  there exist elements  $e, f \in B$  such that  $e \blacktriangleright_B a = a$  and  $a \triangleleft_B f = a$ , then for all  $b \in B$ , the expressions

$$b_{(1)} \blacktriangleright_B a \triangleleft_B S_B^{-1}(b_{(3)}) \otimes b_{(2)} := b_{(1)}e \blacktriangleright_B a \triangleleft_B fS_B^{-1}(b_{(3)}) \otimes b_{(2)}$$

and

$$b_{(2)} \otimes S_B^{-1}(b_{(1)}) \blacktriangleright_B a \triangleleft_B b_{(3)} := b_{(2)} \otimes S_B^{-1}(b_{(1)})e \blacktriangleright_B a \triangleleft_B fb_{(3)}$$

are well defined.

**Definition - Proposition 1.2.45** (Definition 3.4 - Proposition 3.5, [DV01]). The *quantum double*  $A \bowtie B$  associated with the pairing  $p : A \times B \rightarrow \mathbb{C}$  is the non-degenerate  $*$ -algebra  $(A \otimes B, m_D)$  with multiplication map defined by

$$m_D := (m_A \otimes m_B) \circ (id_A \otimes T_p \otimes id_B) : (A \bowtie B) \otimes (A \bowtie B) \rightarrow A \bowtie B.$$

**Remark 29** (Proposition 3.7 - Corollary 3.8, [DV01]). There are natural non-degenerate  $*$ -algebra embeddings

$$\begin{array}{ccc} i_A : A & \rightarrow & \mathcal{M}(A \bowtie B) \\ a & \mapsto & a \bowtie 1 \end{array} \quad , \quad \begin{array}{ccc} i_B : B & \rightarrow & \mathcal{M}(A \bowtie B) \\ b & \mapsto & 1 \bowtie b \end{array}$$

**Theorem 1.2.46** (Theorem 3.16, [DV01]). The non-degenerate  $*$ -algebra  $A \bowtie B$  and the map

$$\begin{array}{ccc} \Delta_{A \bowtie B} : A \bowtie B & \rightarrow & \mathcal{M}(A \bowtie B \otimes A \bowtie B) \\ a \bowtie b & \mapsto & (i_A \otimes i_A)(\Delta_A^{cop}(a))(i_B \otimes i_B)(\Delta_B(b)) \end{array}$$

define a multiplier Hopf  $*$ -algebra. The counit and the antipode are given by

$$\begin{array}{ccc} \varepsilon_{A \bowtie B} : A \bowtie B & \rightarrow & \mathbb{C} \\ a \bowtie b & \mapsto & \varepsilon_A(a)\varepsilon_B(b) \end{array} \quad S_{A \bowtie B} : A \bowtie B \rightarrow A \bowtie B \\ a \bowtie b \mapsto T_p(S_B(b) \otimes S_A^{-1}(a)).$$

Moreover, if  $A$  and  $B$  have non-zero positive faithful integrals, then  $A \bowtie B$  has non-zero positive faithful integrals. The left integral is given by

$$\varphi_{A \bowtie B} : A \bowtie B \rightarrow \mathbb{C}, \quad a \bowtie b \mapsto \varphi_A(a)\psi_B(b).$$

**Notation 7.** We can formally write  $\Delta_{A \bowtie B}(a \bowtie b) = a_{(2)} \bowtie b_{(1)} \otimes a_{(1)} \bowtie b_{(2)}$ , for all  $a \in A$  and  $b \in B$ .

## 1.2.7 Coactions of multiplier Hopf $*$ -algebras

**Definition 1.2.47** (Definition 2.1, [VZ99b]). Let  $(A, \Delta)$  be a multiplier Hopf  $*$ -algebra and  $X$  a non-degenerate  $*$ -algebra. By a *left coaction* of  $(A, \Delta)$  on  $X$  we mean an injective  $*$ -homomorphism  $\alpha : X \rightarrow \mathcal{M}(A \otimes X)$  satisfying

- $\alpha(X)(A \otimes 1) = A \otimes X$ , and using the involution also  $(A \otimes 1)\alpha(X) = A \otimes X$ ,
- $(id \otimes \alpha) \circ \alpha = (\Delta \otimes id) \circ \alpha$ , i.e.

$$(id \otimes \alpha)(\alpha(x)(a \otimes 1)) = (\Delta \otimes id)(\alpha(x))(a \otimes 1 \otimes 1)$$

for all  $x \in X$  and  $a \in A$ .

If, in addition,  $\alpha(X)(1 \otimes X) \subseteq A \otimes X$ , then  $\alpha$  is called a *reduced coaction*. The definition of *right coaction* is given in a similar way.

**Notation 8.** Given a left coaction  $\alpha$  of  $(A, \Delta)$  on  $X$ , we will use the symbolic notation  $x_{[-1]} \otimes x_{[0]} := \alpha(x)$ , called the *Sweedler type leg notation* (when  $A$  is unital, we have the

usual Sweedler leg notation for coactions, [Swe69]). This notation is useful for calculations and we use it in the following sense : for all  $a \in A$  and  $x \in X$ ,

$$x_{[-1]}a \otimes x_{[0]} := \alpha(x)(a \otimes 1) \in A \otimes X, \quad ax_{[-1]} \otimes x_{[0]} := (a \otimes 1)\alpha(x) \in A \otimes X.$$

**Remark 30.** It is obvious that the set  $X^\alpha := \{x \in \mathcal{M}(X) : \alpha(x) = 1 \otimes x\}$  is a unital  $*$ -algebra. It will be called *the fixed point  $*$ -algebra of the left coaction  $\alpha$* .

**Remark 31.** We do not suppose the non-degeneracy for the coaction. But, because the  $*$ -algebra  $X$  is idempotent, i.e.  $X^2 = X$ , we have  $\alpha(X)(A \otimes X) = A \otimes X^2 = A \otimes X$  and  $(A \otimes X)\alpha(X) = A \otimes X$ , this mean that automatically the coaction  $\alpha$  is non-degenerate.

**Proposition 1.2.48** (Proposition 2.3, [VZ99b]). *We have*

$$(\varepsilon \otimes id) \circ \alpha = id_X.$$

**Remark 32.** The relation  $(\varepsilon \otimes id) \circ \alpha = id_X$  implies easily that  $\alpha$  is an injective map.

**Definition 1.2.49** ([VZ99b]). Let  $(A, \Delta, \varphi)$  be a multiplier Hopf  $*$ -algebra equipped with integrals. If  $\alpha : X \rightarrow \mathcal{M}(A \otimes X)$  is a left coaction of  $(A, \Delta)$  on a non-degenerate  $*$ -algebra  $X$ . Then, the map

$$\begin{aligned} \triangleleft_\alpha : X \otimes \widehat{A} &\rightarrow X \\ x \otimes \omega &\mapsto (\omega \otimes id)(\alpha(x)) \end{aligned}$$

is a right action of the multiplier Hopf  $*$ -algebra  $(\widehat{A}, \widehat{\Delta})$  on  $X$  called the *dual action associated with  $\alpha$* .

**Remark 33.** The definition above makes sense, because for all  $x \in X$  and  $\omega = \varphi_a \in \widehat{A}$ , we have

$$(\omega \otimes id)(\alpha(x)) = (\varphi \otimes id)(\alpha(x)(a \otimes 1)) \in X$$

## 1.3 Measured multiplier Hopf $*$ -algebroids [TV17, TV18, Tim16, Tim17]

### 1.3.1 Regular multiplier Hopf algebroids

#### 1.3.1.1 Left multiplier bialgebroids

**Definition 1.3.1.** A *left quantum graph* is a tuple  $(A, L, s, t)$ , where

- (1)  $A$  and  $L$  are algebras and  $A_A$  has local units in  $A$  (in particular, is non-degenerate and idempotent as right  $A$ -module),
- (2)  $s : L \rightarrow \mathcal{M}(A) \subseteq L(A)$  is a homomorphism and  $t : L \rightarrow \mathcal{M}(A) \subseteq L(A)$  is an anti-homomorphism such that their images commute,



- (3) the  $L$ -modules  $A^L$ ,  ${}_L A$  are faithful, non-degenerate and idempotent (here the modules  $A^L$  and  ${}_L A$  are the vector space  $A$  with the module structure of the left multiplication by  $t$  and  $s$  respectively, i.e.,  $a \triangleleft_t x = t(x)a$  and  $x \triangleright_s a = s(x)a$  respectively),
- (4) the vector space  $A^L \otimes {}_L A$ , named the *balanced algebraic tensor product of  $A^L$  and  ${}_L A$* , is non-degenerate as a right module over  $A \otimes 1_{\mathcal{M}(A)}$  and over  $1_{\mathcal{M}(A)} \otimes A$  with actions defined by

$$(a' \otimes b') \cdot (a \otimes 1_{\mathcal{M}(A)}) := a'a \otimes b', \quad (a' \otimes b') \cdot (1_{\mathcal{M}(A)} \otimes b) := a' \otimes b'b,$$

respectively.

**Remark 34.**

- The vector space  $A^L \otimes {}_L A$  is a right module over  $\mathcal{M}(A) \otimes \mathcal{M}(A)$  with the action

$$(a \otimes b) \cdot (P \otimes Q) = aP \otimes bQ.$$

Moreover, we have for all  $a, a', b, b' \in A$

$$(a \otimes b) \cdot (a' \otimes b') = ((a \otimes b) \cdot (1_{\mathcal{M}(A)} \otimes b')) \cdot (a' \otimes 1_{\mathcal{M}(A)}) = ((a \otimes b) \cdot (a' \otimes 1_{\mathcal{M}(A)})) \cdot (1_{\mathcal{M}(A)} \otimes b') = aa' \otimes bb'.$$

- The vector space  $A^L \otimes {}_L A$  with the map

$$\begin{aligned} \triangleright_s : L \otimes A^L \otimes {}_L A &\rightarrow A^L \otimes {}_L A \\ x \otimes (a \otimes b) &\mapsto s(x)a \otimes b \end{aligned}$$

is a left  $L$ -module, because, by condition (2), we have

$$x \triangleright_s ((a \triangleleft y) \otimes b) = s(x)t(y)a \otimes b = t(y)s(x)a \otimes b = s(x)a \otimes s(y)b = x \triangleright_s (a \otimes (y \triangleright b)),$$

for all  $x, y \in L$ . This module will be denoted by  ${}_L(A^L \otimes {}_L A)$ , and we will write

$$(s(x) \otimes 1_{\mathcal{M}(A)}) \cdot (a \otimes b) := x \triangleright_s (a \otimes b)$$

when we work with this left structure.

- The vector space  $A^L \otimes {}_L A$  with the map

$$\begin{aligned} \triangleright_t : L \otimes A^L \otimes {}_L A &\rightarrow A^L \otimes {}_L A \\ x \otimes (a \otimes b) &\mapsto a \otimes bt(x) \end{aligned}$$

is a left  $L$ -module, because we have

$$x \triangleright_t ((a \triangleleft y) \otimes b) = t(y)a \otimes bt(x) = a \otimes s(y)bt(x) = x \triangleright_t (a \otimes (y \triangleright b)),$$

for all  $x, y \in L$ . This module will be denoted by  ${}^L(A^L \otimes {}_L A)$ , and we will write

$$(a \otimes b) \cdot (1_{\mathcal{M}(A)} \otimes t(x)) := x \triangleright_t (a \otimes b)$$

when we work with this left structure.

- The vector space  $A^L \otimes_L A$  with the map

$$\begin{aligned} \triangleleft_s : A^L \otimes_L A \otimes L &\rightarrow A^L \otimes_L A \\ (a \otimes b) \otimes x &\mapsto as(x) \otimes b \end{aligned}$$

is a right  $L$ -module, because we have

$$((a \triangleleft y) \otimes b) \triangleleft_s x = t(y)as(x) \otimes b = as(x) \otimes s(y)b = (a \otimes (y \triangleright b)) \triangleleft_s x,$$

for all  $x, y \in L$ . This module will be denoted by  $(A^L \otimes_L A)_L$ , and we will write

$$(a \otimes b) \cdot (s(x) \otimes 1_{\mathcal{M}(A)}) := (a \otimes b) \triangleleft_s x$$

when we work with this right action.

- The vector space  $A^L \otimes_L A$  with the map

$$\begin{aligned} \triangleleft_t : A^L \otimes_L A \otimes L &\rightarrow A^L \otimes_L A \\ (a \otimes b) \otimes x &\mapsto a \otimes t(x)b \end{aligned}$$

is a right  $L$ -module, because, by condition (2), we have

$$((a \triangleleft y) \otimes b) \triangleleft_t x = t(y)a \otimes t(x)b = a \otimes s(y)t(x)b = a \otimes t(x)s(y)b = (a \otimes (y \triangleright b)) \triangleleft_t x,$$

for all  $x, y \in L$ . This module will be denoted by  $(A^L \otimes_L A)^L$ , and we will write

$$(1_{\mathcal{M}(A)} \otimes t(x)) \cdot (a \otimes b) := (a \otimes b) \triangleleft_t x$$

when we work with this right action.

**Notation 9.** Let  $(A, L, s, t)$  be a left quantum graph. Denote by  $A^L \overline{\times}_L A$ , the vector space formed by all endomorphisms  $T$  of  $A^L \otimes_L A$  satisfying the following condition: For every  $a, b \in A$ , there exist elements  $T(a \otimes 1_{\mathcal{M}(A)}) \in A^L \otimes_L A$  and  $T(1_{\mathcal{M}(A)} \otimes b) \in A^L \otimes_L A$  such that

$$T(a \otimes b) = (T(a \otimes 1_{\mathcal{M}(A)})) \cdot (1_{\mathcal{M}(A)} \otimes b) = (T(1_{\mathcal{M}(A)} \otimes b)) \cdot (a \otimes 1_{\mathcal{M}(A)}).$$

**Remark 35.** Given  $T \in A^L \overline{\times}_L A$ . It follows from condition (4) above, that for all  $a, b \in A$ , the elements  $T(a \otimes 1_{\mathcal{M}(A)})$  and  $T(1_{\mathcal{M}(A)} \otimes b)$  are unique.

We prove the next proposition in order to justify the algebra structure on the space  $A^L \overline{\times}_L A$ .

**Proposition 1.3.2.**

(i) We have

$$A^L \overline{\times}_L A \subseteq \text{End}((A^L \otimes_L A)_{(A \otimes 1)}) \cap \text{End}((A^L \otimes_L A)_{(1 \otimes A)}) \subseteq \text{End}((A^L \otimes_L A)_{(A \otimes A)}).$$

In particular

$$A^L \overline{\times}_L A \subseteq \text{End}^L(A^L \otimes_L A)_L.$$

(ii) The vector space  $A^L \overline{\times}_L A$  is a sub-algebra of  $\text{End}(A^L \otimes_L A)$ .

*Proof*

(i) Let  $T \in A^L \overline{\times}_L A$ . By simple calculations, we have

$$\begin{aligned} T((a \otimes b) \cdot (a' \otimes 1_{\mathcal{M}(A)})) &= T(aa' \otimes b) \\ &= T(1_{\mathcal{M}(A)} \otimes b) \cdot (aa' \otimes 1_{\mathcal{M}(A)}) \\ &= T(1_{\mathcal{M}(A)} \otimes b) \cdot ((a \otimes 1_{\mathcal{M}(A)})(a' \otimes 1_{\mathcal{M}(A)})) \\ &= (T(1_{\mathcal{M}(A)} \otimes b) \cdot (a \otimes 1_{\mathcal{M}(A)})) \cdot (a' \otimes 1_{\mathcal{M}(A)}) \\ &= T(a \otimes b) \cdot (a' \otimes 1_{\mathcal{M}(A)}), \end{aligned}$$

$$\begin{aligned} T((a \otimes b) \cdot (1_{\mathcal{M}(A)} \otimes b')) &= T(a \otimes bb') \\ &= T(a \otimes 1_{\mathcal{M}(A)}) \cdot (1_{\mathcal{M}(A)} \otimes bb') \\ &= (T(a \otimes 1_{\mathcal{M}(A)}) \cdot (1_{\mathcal{M}(A)} \otimes b)) \cdot (1_{\mathcal{M}(A)} \otimes b') \\ &= T(a \otimes b) \cdot (1_{\mathcal{M}(A)} \otimes b') \end{aligned}$$

and

$$\begin{aligned} T((a \otimes b) \cdot (a' \otimes b')) &= T(aa' \otimes bb') \\ &= T((a \otimes bb') \cdot (a' \otimes 1_{\mathcal{M}(A)})) \\ &= T(a \otimes bb') \cdot (a' \otimes 1_{\mathcal{M}(A)}) \\ &= T((a \otimes b) \cdot (1_{\mathcal{M}(A)} \otimes b')) \cdot (a' \otimes 1_{\mathcal{M}(A)}) \\ &= (T(a \otimes b) \cdot (1_{\mathcal{M}(A)} \otimes b')) \cdot (a' \otimes 1_{\mathcal{M}(A)}) \\ &= T(a \otimes b) \cdot (a' \otimes b') \end{aligned}$$

for all  $a, b, a', b' \in A$ .

(ii) Let  $T_1, T_2 \in A^L \overline{\times}_L A$ . Fixed  $a, b \in A$ , because

$$\begin{aligned} (T_1 \circ T_2)(a \otimes b) &= T_1(T_2(a \otimes b)) \\ &= T_1(T_2(a \otimes 1_{\mathcal{M}(A)}) \cdot (1_{\mathcal{M}(A)} \otimes b)) \\ &= T_1(T_2(a \otimes 1_{\mathcal{M}(A)})) \cdot (1_{\mathcal{M}(A)} \otimes b) \end{aligned}$$

and

$$\begin{aligned} (T_1 \circ T_2)(a \otimes b) &= T_1(T_2(a \otimes b)) \\ &= T_1(T_2(b \otimes 1_{\mathcal{M}(A)}) \cdot (a \otimes 1_{\mathcal{M}(A)})) \\ &= T_1(T_2(b \otimes 1_{\mathcal{M}(A)})) \cdot (a \otimes 1_{\mathcal{M}(A)}), \end{aligned}$$

if we define

$$(T_1 \circ T_2)(a \otimes 1_{\mathcal{M}(A)}) := T_1(T_2(a \otimes 1_{\mathcal{M}(A)})) \in A^L \otimes_L A$$

$$(T_1 \circ T_2)(1_{\mathcal{M}(A)} \otimes b) := T_1(T_2(1_{\mathcal{M}(A)} \otimes b)) \in A^L \otimes_L A,$$

then we can see that  $T_1 \circ T_2$  belongs to  $A^L \overline{\times}_L A$ .

□

**Definition 1.3.3.**  $A^L \overline{\times}_L A$  is called *the generalized left Takeuchi product of  $A^L$  and  ${}_L A$* .

**Corollary 1.3.4.** *It is easy to show that if  $A$  has a unit  $1_A$ , then the image of the injective homeomorphism*

$$A^L \overline{\times}_L A \rightarrow A^L \otimes_L A, \quad T \mapsto T(1_A \otimes 1_A)$$

*is the classical left Takeuchi product  $A^L \times_L A$ , i.e.*

$$A^L \overline{\times}_L A \cong A^L \times_L A := \{a \otimes b \in A^L \otimes_L A : at(x) \otimes b = a \otimes bs(x) \text{ for all } x \in L\}$$

**Definition 1.3.5.** A *left comultiplication* on a left quantum graph  $(A, L, s, t)$  is a homomorphism  $\Delta_L : A \rightarrow A^L \overline{\times}_L A$  satisfying the following conditions:

(1) for all  $x, x', y, y' \in L$  and  $a \in A$ ,

$$\Delta_L(s(x)t(y)as(x')t(y')) = (s(x) \otimes t(y))\Delta_L(a)(s(x') \otimes t(y'));$$

(2) for all  $a, b, c \in A$ ,

$$((\Delta_L \otimes id)(\Delta_L(b)(1_{\mathcal{M}(A)} \otimes c))) \cdot (a \otimes 1_{\mathcal{M}(A)} \otimes 1_{\mathcal{M}(A)}) = ((id \otimes \Delta_L)(\Delta_L(b)(a \otimes 1_{\mathcal{M}(A)}))) \cdot (1_{\mathcal{M}(A)} \otimes 1_{\mathcal{M}(A)} \otimes c).$$

**Definition 1.3.6.** A *left multiplier bialgebroid* is a tuple  $(A, L, s, t, \Delta_L)$ , where  $(A, L, s, t)$  is a left quantum graph and  $\Delta_L$  is a left comultiplication on  $(A, L, s, t)$ .

We prove the following lemma in order to get a convenient characterization of the comultiplication  $\Delta_L$ .

**Lemma 1.3.7.** *Let  $F : A \rightarrow A^L \overline{\times}_L A$  be a linear map. Then the linear maps*

$$\tilde{T}_{1,F} : A \otimes A \rightarrow A^L \otimes_L A, \quad a \otimes b \mapsto F(a)(1_{\mathcal{M}(A)} \otimes b)$$

$$\tilde{T}_{2,F} : A \otimes A \rightarrow A^L \otimes_L A, \quad a \otimes b \mapsto F(b)(a \otimes 1_{\mathcal{M}(A)})$$

*satisfy*

(i) *For all  $a, b \in A$  and  $x \in L$*

$$\tilde{T}_{1,F}(a \otimes s(x)b) = \tilde{T}_{1,F}(a \otimes b) \cdot (t(x) \otimes 1_{\mathcal{M}(A)}),$$

$$\tilde{T}_{2,F}(t(x)a \otimes b) = \tilde{T}_{2,F}(a \otimes b) \cdot (1_{\mathcal{M}(A)} \otimes s(x)).$$

(ii) *The following diagrams are commutative,*

$$\begin{array}{ccc} A \otimes A \otimes A & \xrightarrow{id \otimes \tilde{T}_{1,F}} & A \otimes A^L \otimes_L A \\ \tilde{T}_{2,F} \otimes id \downarrow & & \downarrow m \circ \Sigma \otimes id \\ A^L \otimes_L A \otimes A & \xrightarrow{id \otimes m} & A^L \otimes_L A \end{array}$$

$$\begin{array}{ccc}
A \otimes A \otimes A & \xrightarrow{\tilde{T}_{1,F} \otimes id} & A^L \otimes_L A \otimes A \\
\downarrow id \otimes m & & \downarrow id \otimes m \\
A \otimes A & \xrightarrow{\tilde{T}_{1,F}} & A^L \otimes_L A
\end{array}
\quad
\begin{array}{ccc}
A \otimes A \otimes A & \xrightarrow{id \otimes \tilde{T}_{2,F}} & A \otimes A^L \otimes_L A \\
\downarrow m \circ \Sigma \otimes id & & \downarrow m \circ \Sigma \otimes id \\
A \otimes A & \xrightarrow{\tilde{T}_{2,F}} & A^L \otimes_L A
\end{array}$$

where  $m$  denotes the multiplication and  $\Sigma$  the flip map.

*Proof*

(i) Fix  $a, b \in A$  and  $x \in L$ . We have

$$\begin{aligned}
\tilde{T}_{1,F}(a \otimes s(x)b) \cdot (a' \otimes 1_{\mathcal{M}(A)}) &= F(a)(1_{\mathcal{M}(A)} \otimes s(x)b) \cdot (a' \otimes 1_{\mathcal{M}(A)}) \\
&= F(a)(a' \otimes s(x)b) \\
&= F(a)(t(x)a' \otimes b) \\
&= F(a)(1_{\mathcal{M}(A)} \otimes b) \cdot (t(x)a' \otimes 1_{\mathcal{M}(A)}) \\
&= (\tilde{T}_{1,F}(a \otimes b) \cdot (t(x) \otimes 1_{\mathcal{M}(A)})) \cdot (a' \otimes 1_{\mathcal{M}(A)})
\end{aligned}$$

and

$$\begin{aligned}
\tilde{T}_{2,F}(t(x)a \otimes b) \cdot (1_{\mathcal{M}(A)} \otimes a') &= F(b)(t(x)a \otimes 1_{\mathcal{M}(A)}) \cdot (1_{\mathcal{M}(A)} \otimes a') \\
&= F(b)(t(x)a \otimes a') \\
&= F(b)(a \otimes s(x)a') \\
&= F(b)(a \otimes 1_{\mathcal{M}(A)}) \cdot (1_{\mathcal{M}(A)} \otimes s(x)a') \\
&= (\tilde{T}_{2,F}(a \otimes b) \cdot (1_{\mathcal{M}(A)} \otimes s(x))) \cdot (1_{\mathcal{M}(A)} \otimes a')
\end{aligned}$$

for all  $a' \in A$ . By non-degeneracy of the right module  $A^L \otimes_L A$ , we have the result.

(ii) Fix  $a, b, c \in A$ , we have

$$\begin{aligned}
(id \otimes m) \circ (\tilde{T}_{2,F} \otimes id)(a \otimes b \otimes c) &= (id \otimes m)(F(b)(a \otimes 1_{\mathcal{M}(A)}) \otimes c) \\
&= F(b)(a \otimes 1_{\mathcal{M}(A)}) \cdot (1_{\mathcal{M}(A)} \otimes c) \\
&= F(b)(1_{\mathcal{M}(A)} \otimes c) \cdot (a \otimes 1_{\mathcal{M}(A)}) \\
&= (m \circ \Sigma \otimes id)(a \otimes F(b)(1_{\mathcal{M}(A)} \otimes c)) \\
&= (m \circ \Sigma \otimes id) \circ (id \otimes \tilde{T}_{1,F})(a \otimes b \otimes c)
\end{aligned}$$

$$\begin{aligned}
[\tilde{T}_{1,F} \circ (id \otimes m)(a \otimes b \otimes c)] \cdot (d \otimes 1_{\mathcal{M}(A)}) &= \tilde{T}_{1,F}(a \otimes bc) \cdot (d \otimes 1_{\mathcal{M}(A)}) \\
&= F(a)(1_{\mathcal{M}(A)} \otimes bc) \cdot (d \otimes 1_{\mathcal{M}(A)}) \\
&= F(a)(d \otimes 1_{\mathcal{M}(A)}) \cdot (1_{\mathcal{M}(A)} \otimes bc) \\
&= (F(a)(d \otimes 1_{\mathcal{M}(A)}) \cdot (1_{\mathcal{M}(A)} \otimes b)) \cdot (1_{\mathcal{M}(A)} \otimes c) \\
&= (F(a)(1_{\mathcal{M}(A)} \otimes b) \cdot (d \otimes 1_{\mathcal{M}(A)})) \cdot (1_{\mathcal{M}(A)} \otimes c) \\
&= (F(a)(1_{\mathcal{M}(A)} \otimes b) \cdot (1_{\mathcal{M}(A)} \otimes c)) \cdot (d \otimes 1_{\mathcal{M}(A)}) \\
&= [(id \otimes m)(F(a)(1_{\mathcal{M}(A)} \otimes b) \otimes c)] \cdot (d \otimes 1_{\mathcal{M}(A)}) \\
&= [(id \otimes m) \circ (\tilde{T}_{1,F} \otimes id)(a \otimes b \otimes c)] \cdot (d \otimes 1_{\mathcal{M}(A)})
\end{aligned}$$

$$\begin{aligned}
[\tilde{T}_{2,F} \circ (m \circ \Sigma \otimes id)(a \otimes b \otimes c)] \cdot (1_{\mathcal{M}(A)} \otimes d) &= \tilde{T}_{2,F}(ba \otimes c) \cdot (1_{\mathcal{M}(A)} \otimes d) \\
&= F(c)(ba \otimes 1_{\mathcal{M}(A)}) \cdot (1_{\mathcal{M}(A)} \otimes d) \\
&= F(c)(1_{\mathcal{M}(A)} \otimes d) \cdot (ba \otimes 1_{\mathcal{M}(A)}) \\
&= (F(c)(1_{\mathcal{M}(A)} \otimes d) \cdot (b \otimes 1_{\mathcal{M}(A)})) \cdot (a \otimes 1_{\mathcal{M}(A)}) \\
&= (F(c)(b \otimes 1_{\mathcal{M}(A)}) \cdot (1_{\mathcal{M}(A)} \otimes d)) \cdot (a \otimes 1_{\mathcal{M}(A)}) \\
&= (F(c)(b \otimes 1_{\mathcal{M}(A)}) \cdot (a \otimes 1_{\mathcal{M}(A)})) \cdot (1_{\mathcal{M}(A)} \otimes d) \\
&= [(m \circ \Sigma \otimes id)(a \otimes F(c)(b \otimes 1_{\mathcal{M}(A)}))] \cdot (1_{\mathcal{M}(A)} \otimes d) \\
&= [(m \circ \Sigma \otimes id) \circ (id \otimes \tilde{T}_{2,F})(a \otimes b \otimes c)] \cdot (1_{\mathcal{M}(A)} \otimes d)
\end{aligned}$$

for all  $d \in A$ .

□

**Remark 36.** Let  $(A, L, s, t, \Delta_L)$  be a left multiplier bialgebroid. By Lemma (1.3.7), associated to  $\Delta_L$ , we have linear maps denoted by

$$\begin{aligned}
\tilde{T}_{1,L} : A \otimes A &\rightarrow A^L \otimes_L A, & a \otimes b &\mapsto \Delta_L(a)(1_{\mathcal{M}(A)} \otimes b), \\
\tilde{T}_{2,L} : A \otimes A &\rightarrow A^L \otimes_L A, & a \otimes b &\mapsto \Delta_L(b)(a \otimes 1_{\mathcal{M}(A)}).
\end{aligned}$$

**Definition 1.3.8.** A *left counit* for a left multiplier bialgebroid  $(A, L, s, t, \Delta_L)$  is a linear map  ${}_L\varepsilon : A \rightarrow L$  satisfying

- (i)  ${}_L\varepsilon(s(x)a) = x{}_L\varepsilon(a)$  and  ${}_L\varepsilon(t(x)a) = {}_L\varepsilon(a)x$  for all  $a \in A$  and  $x \in L$ , i.e.  ${}_L\varepsilon$  is a bimodule map from  ${}_L A^L$  to  ${}_L L_L$ .
- (ii)  $({}_L\varepsilon \bar{\otimes} id)(\Delta_L(a)(1_{\mathcal{M}(A)} \otimes b)) = ab = (id \bar{\otimes} {}_L\varepsilon)(\Delta_L(a)(b \otimes 1_{\mathcal{M}(A)}))$  for all  $a, b \in A$ , whence

$$\begin{aligned}
{}_L\varepsilon \bar{\otimes} id : A^L \otimes_L A &\rightarrow A, & a \otimes b &\mapsto s({}_L\varepsilon(a))b, \\
id \bar{\otimes} {}_L\varepsilon : A^L \otimes_L A &\rightarrow A, & a \otimes b &\mapsto t({}_L\varepsilon(b))a,
\end{aligned}$$

are well-defined slice maps by item (i).

### 1.3.1.2 Right multiplier bialgebroids

In a similar way, we have the right-handed analogue to a left multiplier bialgebroid. Here, we give only the corresponding definitions.

**Definition 1.3.9.** A *right quantum graph* is a tuple  $(A, R, s, t)$ , where

- (1)  $A$  and  $R$  are algebras and  ${}_A A$  has local units in  $A$  (in particular, is non-degenerate and idempotent as left  $A$ -module),
- (2)  $s : R \rightarrow \mathcal{M}(A) \subseteq R(A)$  is a homomorphism and  $t : R \rightarrow \mathcal{M}(A) \subseteq R(A)$  is an anti-homomorphism such that their images commute,
- (3) the  $R$ -modules  $A_R$  and  ${}^R A$  are faithful, non-degenerate and idempotent (here the modules  $A_R$  and  ${}^R A$  are the vector space  $A$  with the module structure of the right multiplication by  $s$  and  $t$  respectively, i.e.,  $a \triangleleft x = as(x)$  and  $x \triangleright a = at(x)$  respectively),

- (4) the  $A_R \otimes {}^R A$  is non-degenerate as a left module over  $A \otimes 1$  and over  $1 \otimes A$  with actions defined by

$$(a \otimes 1) \cdot (a' \otimes b') := aa' \otimes b', \quad (1 \otimes b) \cdot (a' \otimes b') := a' \otimes bb',$$

respectively.

**Notation 10.** Let  $(A, R, s, t)$  be a right quantum graph. Denote by  $A_R \overline{\times}^R A$ , the vector space formed by all endomorphisms  $T$  of  $A_R \otimes {}^R A$  satisfying the following condition: For every  $a, b \in A$ , there exist elements  $(a \otimes 1)T \in A_R \otimes {}^R A$  and  $(1 \otimes b)T \in A_R \otimes {}^R A$  such that

$$(a \otimes b)T = (1 \otimes b) \cdot ((a \otimes 1)T) = (a \otimes 1) \cdot ((1 \otimes b)T),$$

where  $(a \otimes b)T$  denotes the evaluation of  $T$  at  $a \otimes b$ .

**Remark 37.** Using the notation  $(a \otimes b)T$  for the evaluation of  $T$  at  $a \otimes b$ , we have

$$(a \otimes b)(T_1 \overset{op}{\circ} T_2) = ((a \otimes b)T_1)T_2$$

for all  $a, b \in A$  and  $T_1, T_2 \in \text{End}(A_R \otimes {}^R A)^\circ$ .

**Definition 1.3.10.**  $A_R \overline{\times}^R A$  is an algebra called *the generalized right Takeuchi product of  $A_R$  and  ${}^R A$* .

**Corollary 1.3.11.** *If  $A$  has a unit  $1_A$ , then the imagen of the injective homeomorphism*

$$\begin{aligned} A_R \overline{\times}^R A &\rightarrow A_R \otimes {}^R A \\ T &\mapsto (1_A \otimes 1_A)T \end{aligned}$$

*is the classical right Takeuchi product  $A_R \times {}^R A$ , i.e.*

$$A_R \overline{\times}^R A \cong A_R \times {}^R A := \{a \otimes b \in A_R \otimes {}^R A : s(x)a \otimes b = a \otimes t(x)b \text{ for all } x \in R\}$$

**Definition 1.3.12.** A *right comultiplication* on a right quantum graph  $(A, R, s, t)$  is a homomorphism  $\Delta_R : A \rightarrow A_R \overline{\times}^R A$  satisfying the following conditions:

- (1) for all  $x, x', y, y' \in R$  and  $a \in A$ ,

$$\Delta_R(s(x)t(y)as(x')t(y')) = (t(y) \otimes s(x))\Delta_R(a)(t(y') \otimes s(x'));$$

- (2) for all  $a, b, c \in A$ ,

$$(a \otimes 1 \otimes 1) \cdot ((\Delta_R \otimes id)((1 \otimes c)\Delta_R(b))) = (1 \otimes 1 \otimes c) \cdot ((id \otimes \Delta_R)((a \otimes 1)\Delta_R(b))).$$

**Definition 1.3.13.** A *right multiplier bialgebroid* is a tuple  $(A, R, s, t, \Delta_R)$ , where  $(A, R, s, t)$  is a right quantum graph and  $\Delta_R$  is a right comultiplication on  $(A, R, s, t)$ .

**Remark 38.** Let  $(A, R, s, t, \Delta_R)$  be a right multiplier bialgebroid. In a similar way to the left quantum graphs, we can associate to  $\Delta_R$  the following linear maps

$$\tilde{T}_{1,R} : A \otimes A \rightarrow A_R \otimes {}^R A, \quad a \otimes b \mapsto (1 \otimes b)\Delta_R(a),$$

$$\tilde{T}_{2,R} : A \otimes A \rightarrow A_R \otimes {}^R A, \quad a \otimes b \mapsto (a \otimes 1)\Delta_R(b).$$

**Definition 1.3.14.** A *right counit* for a right multiplier bialgebroid  $(A, R, s, t, \Delta_R)$  is a linear map  $\varepsilon_B : A \rightarrow R$  satisfying

- (i)  $\varepsilon_R(as(x)) = \varepsilon_R(a)x$  and  $\varepsilon_R(at(x)) = x\varepsilon_R(a)$  for all  $a \in A$  and  $x \in R$ , i.e.  $\varepsilon_B$  is a bimodule map from  ${}^R A_R$  to  ${}_R R_R$ .
- (ii)  $(\varepsilon_R \bar{\otimes} id)((1 \otimes b)\Delta_R(a)) = ba = (id \bar{\otimes} \varepsilon_R)((b \otimes 1)\Delta_R(a))$  for all  $a, b \in A$ , whence

$$\varepsilon_R \bar{\otimes} id : A_R \otimes {}^R A \rightarrow A, \quad a \otimes b \mapsto bs(\varepsilon_R(a)),$$

$$id \bar{\otimes} \varepsilon_R : A_R \otimes {}^R A \rightarrow A, \quad a \otimes b \mapsto at(\varepsilon_R(b)),$$

are well-defined slice maps by item (i).

### 1.3.1.3 Multiplier bialgebroids

**Definition 1.3.15.** A *multiplier bialgebroid* is a tuple  $\mathcal{A} = (A, L, R, t_L, t_R, \Delta_L, \Delta_R)$  where

- (i)  $A$  is a non-degenerate and idempotent algebra,
- (ii)  $L$  and  $R$  are sub-algebras of  $\mathcal{M}(A)$  with anti-isomorphisms  $t_L : L \rightarrow R$  and  $t_R : R \rightarrow L$ ,
- (iii)  $(A, L, id_L, t_L)$  is a left quantum graph and  $(A, R, id_R, t_R)$  is a right quantum graph,
- (iv)  $\Delta_L : A \rightarrow A^L \bar{\times}_L A$  and  $\Delta_R : A \rightarrow A_R \bar{\times}^R A$  are homomorphism such that the tuple  $\mathcal{A}_L := (A, L, id_L, t_L, \Delta_L)$  is a left multiplier bialgebroid and the tuple  $\mathcal{A}_R := (A, R, id_R, t_R, \Delta_R)$  is a right multiplier bialgebroid,
- (v) the following mixed co-associativity conditions hold

$$((\Delta_L \otimes id)((1 \otimes c)\Delta_R(b))) \cdot (a \otimes 1 \otimes 1) = (1 \otimes 1 \otimes c) \cdot ((id \otimes \Delta_R)(\Delta_L(b)(a \otimes 1))),$$

$$(a \otimes 1 \otimes 1) \cdot ((\Delta_R \otimes id)(\Delta_L(b)(1 \otimes c))) = ((id \otimes \Delta_L)((a \otimes 1)\Delta_R(b))) \cdot (1 \otimes 1 \otimes c)$$

for all  $a, b, c \in A$ .

**Remark 39.** By item (iv), we have

$$\Delta_L(xyax'y') = (x \otimes y) \cdot \Delta_L(a) \cdot (x' \otimes y')$$

$$\Delta_R(xyax'y') = (x \otimes y) \cdot \Delta_R(a) \cdot (x' \otimes y')$$

for all  $x, x' \in L, y, y' \in R$  and  $a \in A$ . Then, the maps  $\tilde{T}_{1,L}, \tilde{T}_{2,L}, \tilde{T}_{1,R}, \tilde{T}_{2,R}$  can be factorized into maps

$$T_{1,L} : A_R \otimes {}^R A \rightarrow A^L \otimes {}_L A \quad T_{2,L} : A^R \otimes {}^R A \rightarrow A^L \otimes {}_L A,$$

$$T_{1,R} : A^L \otimes {}_L A \rightarrow A_R \otimes {}^R A \quad T_{2,R} : A_L \otimes {}_L A \rightarrow A^R \otimes {}^R A,$$

which will be called *the canonical maps of the multiplier bialgebroid  $\mathcal{A}$* .



**Definition 1.3.16** (Definition 2.1.4, [Tim17]). Let  $\mathcal{A} = (A, L, R, t_L, t_R, \Delta_L, \Delta_R)$  be a multiplier bialgebroid. We say that  $\mathcal{A}$  is called *locally projective* if the algebras  $L$  and  $R$  are firm and the modules  ${}_L A, A_L, {}_R A, A_R$  are locally projectives.

**Definition 1.3.17.** Let  $\mathcal{A} = (A, L, R, t_L, t_R, \Delta_L, \Delta_R)$  be a multiplier bialgebroid. We say that  $\mathcal{A}$  is a *regular multiplier Hopf algebroid* if the following conditions hold

- there exists an anti-automorphism  $S$  of  $A$ , called the *antipode* of  $\mathcal{A}$ , such that

$$S(t_L(x)t_R(y)at_L(x')t_R(y')) = x'y'S(a)xy \quad (1.3)$$

for all  $x, x' \in L, y, y' \in R$  and  $a \in A$ .

- there exist a left counit  ${}_L \varepsilon$  and a right counit  $\varepsilon_R$  for  $\mathcal{A}$  such that the following diagrams are commutative:

$$\begin{array}{ccc} A_R \otimes {}_R A & \xrightarrow{\varepsilon_R \bar{\otimes} id} & A \\ \downarrow T_{1,L} & & \uparrow m \\ A^L \otimes {}_L A & \xrightarrow{S \otimes id} & A_L \otimes {}_L A \end{array} \quad \begin{array}{ccc} A_L \otimes {}_L A & \xrightarrow{id \bar{\otimes} {}_L \varepsilon} & A \\ \downarrow T_{2,R} & & \uparrow m \\ A_R \otimes {}_R A & \xrightarrow{id \otimes S} & A_R \otimes {}_R A \end{array}$$

**Remark 40.** It was proof in Theorem 5.8 ([TV17]) that the two conditions in the definition above are equivalent.

**Proposition 1.3.18** ([TV17]). *A multiplier bialgebroid  $\mathcal{A} = (A, L, R, t_L, t_R, \Delta_L, \Delta_R)$  is a regular multiplier Hopf algebroid if and only if*

- The canonical maps of  $\mathcal{A}$ ,  $T_{1,L}, T_{2,L}, T_{1,R}, T_{2,R}$ , are bijectives, and
- we have

$$A = t_R({}_R I) \cdot A = A \cdot {}^L I = I^R \cdot A = A \cdot t_L(I_L),$$

where

$$I_L := \{ \omega(a) : \omega \in (A_L)^\vee, a \in A \} \subseteq L,$$

$${}^L I := \{ \omega(a) : \omega \in ({}^L A)^\vee, a \in A \} \subseteq L,$$

$${}_R I := \{ \omega(a) : \omega \in ({}_R A)^\vee, a \in A \} \subseteq R,$$

$$I^R := \{ \omega(a) : \omega \in (A^R)^\vee, a \in A \} \subseteq R.$$

**Remark 41.** It is possible to extend the antipode  $S$  to the multiplier algebra  $\mathcal{M}(A)$ , by the condition of 1.3, this extension restricts to the inverses of  $t_L$  and  $t_R$ , we write then  $S_R := S|_R = t_L^{-1} : R \rightarrow L$  and  $S_L := S|_L = t_R^{-1} : L \rightarrow R$ . We will use the maps  $S_R$  and  $S_L$  instead of their inverses  $t_L$  and  $t_R$ , respectively. Moreover, with this notation, we have

$$xa \otimes b = a \otimes S_R(x)b \text{ in } A^L \otimes {}_L A \text{ for all } a, b \in A, x \in R,$$

$$a \otimes by = aS_L(y) \otimes b \text{ in } A_R \otimes {}_R A \text{ for all } a, b \in A, y \in L.$$

The antipode and the counits are related by

$$S_L \circ {}_L\varepsilon = \varepsilon_R \circ S, \quad S_R \circ \varepsilon_R = {}_L\varepsilon \circ S,$$

and we also have the following multiplicativity of the counits,

$$\begin{aligned} {}_L\varepsilon(ab) &= {}_L\varepsilon(a{}_L\varepsilon(b)) = {}_L\varepsilon(aS_R^{-1}({}_L\varepsilon(b))), \\ \varepsilon_R(ab) &= \varepsilon_R(\varepsilon_R(a)b) = \varepsilon_R(S_L^{-1}(\varepsilon_R(a))b) \end{aligned}$$

for all  $a, b \in A$ .

**Definition 1.3.19.** Consider a regular multiplier Hopf algebroid

$$\mathcal{A} = (A, L, R, t_L, t_R, \Delta_L, \Delta_R)$$

with faithful functionals  $\mu_L : L \rightarrow \mathbb{C}$  and  $\mu_R : R \rightarrow \mathbb{C}$ . A functional  $\omega \in A^\vee$  is called *factorizable with respect to*  $(\mu_L, \mu_R)$  if and only if

$$\omega \in A^\sqcup := ({}_L A_L)^\sqcup \cap ({}_R A_R)^\sqcup \subseteq A^\vee.$$

The set  $A^\sqcup$  it is call the set of all factorizable functionals with respect to  $(\mu_L, \mu_R)$ .

**Definition 1.3.20.** A *multiplier Hopf\*-algebroid* is a regular multiplier Hopf algebroid  $\mathcal{A} = (A, L, R, t_L, t_R, \Delta_L, \Delta_R)$  such that

- (1)  $A$  is a \*-algebra, and  $L, R$  are sub-\*-algebras of  $\mathcal{M}(A)$ ;
- (2)  $t_L \circ * \circ t_R \circ * = id_R$  and  $t_R \circ * \circ t_L \circ * = id_L$ ;
- (3)  $\Delta_L(a^*)(b^* \otimes c^*) = ((b \otimes c)\Delta_R(a))^{(-)* \otimes (-)*}$  for all  $a, b, c \in A$ .

**Remark 42.** The condition (2) ensures that the map

$$(-)^* \otimes (-)^* : A_R \otimes {}^R A \rightarrow A^L \otimes {}_L A, \quad a \otimes b \mapsto a^* \otimes b^*$$

is well-defined. For a regular multiplier Hopf\*-algebroid, the antipode  $S$  and its counits  ${}_L\varepsilon, \varepsilon_R$  satisfy the relations

$$S \circ * \circ S \circ * = id_A, \quad \varepsilon_R \circ * = * \circ S_L \circ {}_L\varepsilon, \quad {}_L\varepsilon \circ * = * \circ S_R \circ \varepsilon_R.$$

**Example 1.3.21.** We recall the example constructed by Lu [Lu96] in the form given by Timmermann [Tim16]. Let  $(H, \Delta)$  be a regular Hopf algebra, i.e. its antipode  $S_H$  is bijective, and let  $X$  be a unital braided-commutative (right-left) Yetter-Drinfeld algebra, that is  $X$  is a right  $H$ -module algebra with structural map given by  $\triangleleft : X \otimes H \rightarrow X$  and a left  $H^{op}$ -comodule algebra with structural map given by  $X \rightarrow H^{op} \otimes X, x \mapsto x_{[-1]} \otimes x_{[0]}$ , which satisfy the Yetter-Drinfeld condition

$$x_{[-1]}h_{(1)} \otimes x_{[0]} \triangleleft h_{(2)} = h_{(2)}(x \triangleleft h_{(1)})_{[-1]} \otimes (x \triangleleft h_{(1)})_{[0]} \quad (1.4)$$

and the braided-commutativity condition

$$xy = (y \triangleleft x_{[-1]})x_{[0]}. \quad (1.5)$$

Denote by  $A := H \# X$  the smash product  $*$ -algebra associated with the action  $\triangleleft$  whose algebra structure is given by  $(h \# x)(g \# y) = hg_{(1)} \# (x \triangleleft g_{(2)})y$ , and let  $L := 1_H \# X$ . The equation 1.5, implies that

$$R := \{x_{[-1]} \# x_{[0]} : x \in X\} \subseteq H \# X$$

is a sub-algebra which commutes elementwise with  $L$ , and that the maps

$$\begin{aligned} t_L : L &\rightarrow R & t_R : R &\rightarrow L \\ 1 \# x &\mapsto x_{[-1]} \# x_{[0]} & x_{[-1]} \# x_{[0]} &\mapsto 1 \# (x_{[0]} \triangleleft S_H^{-1}(x_{[-1]})) \end{aligned}$$

are anti-isomorphisms. We remark here, that the equation 1.4 takes the form

$$\begin{aligned} t_L(1 \# x)(h \# 1) &= (x_{[-1]} \# x_{[0]})(h \# 1) \\ &= x_{[-1]} h_{(1)} \# x_{[0]} \triangleleft h_{(2)} \\ &= h_{(2)}(x \triangleleft h_{(1)})_{[-1]} \# (x \triangleright h_{(1)})_{[0]} \\ &= (h_{(2)} \# 1)((x \triangleleft h_{(1)})_{[-1]} \# (x \triangleright h_{(1)})_{[0]}) \\ &= (h_{(2)} \# 1)t_L(1 \# (x \triangleleft h_{(1)})). \end{aligned}$$

With the maps

$$\begin{aligned} \Delta_L : A &\rightarrow A^L \times_L A & \Delta_R : A &\rightarrow A_R \times^R A \\ h \# x &\mapsto h_{(1)} \# x \otimes h_{(2)} \# 1 & xh &\mapsto xh_{(1)} \# x \otimes h_{(2)} \# 1 & , \\ \varepsilon_L : A &\rightarrow L & \varepsilon_R : A &\rightarrow R \\ h \# x &\mapsto 1 \# \varepsilon_H(h)x & hx_{[-1]} \# x_{[0]} &\mapsto \varepsilon_H(h)x_{[-1]} \# x_{[0]} & , \\ S : A &\rightarrow A \\ h \# x &\mapsto x_{[0]} S_H^2(x_{[-1]}) S_H(h) \end{aligned}$$

the tuple  $(A, L, R, t_L, t_R, \Delta_L, \Delta_R, \varepsilon_L, \varepsilon_R, S)$  is a regular (multiplier) Hopf algebroid.

**Remark 43.** For the example above, we recall the notation for the elements in the smash product  $H \# X$ :

$$hx := (h \# 1)(1 \# x), \quad xh := (1 \# x)(h \# 1) = h_{(1)} \# x \triangleleft h_{(2)}$$

### 1.3.2 Measured multiplier Hopf $*$ -algebroids

Let  $\mathcal{A} = (A, L, R, t_L, t_R, \Delta_L, \Delta_R)$  be a multiplier Hopf  $*$ -algebroid with left counit  $\varepsilon_L$ , right counit  $\varepsilon_R$  and antipode  $S$ .

**Proposition 1.3.22** (Proposition 3.1, [Tim16]). *The following conditions on a bimodule map  ${}_L\varphi_L \in \text{Hom}({}_L A_L, {}_L L_L)$  are equivalent:*

(LI1) For all  $a, b \in A$ ,

$$(id \bar{\otimes} {}_L\varphi_L)(\Delta_L(b)(a \otimes 1)) = {}_L\varphi_L(b)a,$$

where  $id \bar{\otimes} {}_L\varphi_L : A^L \otimes {}_L A \rightarrow A$  is defined by  $(id \bar{\otimes} {}_L\varphi_L)(a \otimes b) := a_L\varphi_L(b)$ .

(LI2) For all  $a, b \in A$

$$(id \bar{\otimes} (S_L \circ {}_L\varphi_L))((a \otimes 1)\Delta_R(b)) = a_L\varphi_L(b),$$

where  $id \bar{\otimes} (S_L \circ {}_L\varphi_L) : A_R \otimes {}^R A \rightarrow A$  is defined by

$$(id \bar{\otimes} (S_L \circ {}_L\varphi_L))(a \otimes b) := a_L S(\varphi_L(b)).$$

(LI3) the following diagram is commutative:

$$\begin{array}{ccc} A_R \otimes {}_R A & \xrightarrow{T_{1,L}} & A^L \otimes {}_L A & , \\ \downarrow T_{1,R} \circ {}_R \Sigma^L & & \downarrow S \circ (id \bar{\otimes} {}_L\varphi_L) & \\ A_R \otimes {}^R A & \xrightarrow{id \bar{\otimes} (S_L \circ {}_L\varphi_L)} & A & \end{array}$$

where  ${}_R \Sigma^L : A_R \otimes {}_R A \rightarrow A^L \otimes {}_L A$  denotes the flip map.

Likewise, for every  ${}_R\psi_R \in \text{Hom}({}_R A_R, {}_R R_R)$ , the following conditions are equivalent:

(RI1)  $({}_R\psi_R \bar{\otimes} id)((1 \otimes b)\Delta_R(a)) = b{}_R\psi_R(a)$  for all  $a, b \in A$ ,

(RI2)  $((S_R \circ {}_R\psi_R) \bar{\otimes} id)(\Delta_L(a)(1 \otimes b)) = {}_R\psi_R(a)b$  for all  $a, b \in A$ ,

(RI3) the following diagram is commutative:

$$\begin{array}{ccc} A_L \otimes {}_L A & \xrightarrow{T_{2,R}} & A_R \otimes {}^R A & , \\ \downarrow T_{2,L} \circ {}_L \Sigma^R & & \downarrow S \circ ({}_R\psi_R \bar{\otimes} id) & \\ A^L \otimes {}_L A & \xrightarrow{({}_R\psi_R \bar{\otimes} id)} & A & \end{array}$$

where  ${}_L \Sigma^R : A_L \otimes {}_L A \rightarrow A^R \otimes {}^R A$  denotes the flip map.

**Definition 1.3.23.** A measured multiplier Hopf\*-algebroid consists of a multiplier Hopf\*-algebroid

$$\mathcal{A} = (A, L, R, t_L, t_R, \Delta_L, \Delta_R, {}_L\varepsilon, \varepsilon_R, S),$$

faithful positive functionals  $\mu_L$  on  $L$  and  $\mu_R$  on  $R$ , and maps

$${}_L\varphi_L \in \text{Hom}({}_L A_L, {}_L L_L) \quad \text{and} \quad {}_R\psi_R \in \text{Hom}({}_R A_R, {}_R R_R)$$

satisfying the following conditions

- (1)  $\mu_R \circ t_L = \mu_L$ ,  $\mu_L \circ t_R = \mu_R$ , and  $\mu_L \circ_L \varepsilon = \mu_R \circ \varepsilon_R$ ;
- (2)  $(LI1) - (LI3)$  holds for  ${}_L\varphi_L$  and  $(RI1) - (RI3)$  holds for  ${}_R\psi_R$ ;
- (3) the compositions  $\varphi := \mu_L \circ_L \varphi_L$  and  $\psi := \mu_R \circ_R \psi_R$  are positive and faithful;
- (4) there exist surjective maps  ${}_R\varphi \in ({}_R A)^\vee$ ,  $\varphi_R \in (A_R)^\vee$ ,  ${}_L\psi \in ({}_L A)^\vee$ ,  $\psi_L \in (A_L)^\vee$  such that

$$\varphi = \mu_R \circ_R \varphi = \mu_R \circ \varphi_R \quad \text{and} \quad \psi = \mu_L \circ_L \psi = \mu_L \circ \psi_L.$$

We call  $(\mu_L, \mu_R)$  the *base weight*,  ${}_L\varphi_L$  the *partial left integral*,  ${}_R\psi_R$  the *partial right integral*, and the compositions  $\varphi$  and  $\psi$  the *total left* and the *total right integral*, respectively.

**Example 1.3.24.** Consider the regular (multiplier) Hopf algebroid associated with a unital braided (right-left) Yetter-Drinfel'd algebra  $X$  over a regular Hopf algebra  $(H, \Delta)$ , see Example 1.3.21. If, we suppose that  $\phi_H$  is a normalized integral of  $H$ , i.e.  $\phi_H(1_H) = 1$  and  $\phi_H(h_{(1)})h_{(2)} = \phi_H(h)1_H = h_{(1)}\phi_H(h_{(2)})$  for all  $h \in H$ . Then, the maps

$$\begin{aligned} {}_L\phi_L : H\#X &\rightarrow L & {}_L\psi_L : H\#X &\rightarrow L \\ h\#x &\mapsto x\phi_H(h) & hx_{[-1]}\#x_{[0]} &\mapsto x\phi_H(h) \end{aligned}$$

are partial left and right integrals for the regular (multiplier) Hopf algebroid, respectively.

**Theorem 1.3.25** (Theorem 2.2.3, [Tim16]). Let  $(\mathcal{A}, \mu_L, \mu_R, {}_L\varphi_L, {}_R\psi_R)$  be a measured multiplier Hopf\*-algebroid. Then, the functional  $\mu_L, \mu_R$  and the total integrals  $\varphi, \psi$  admit modular automorphisms  $\sigma_L, \sigma_R, \sigma^\varphi, \sigma^\psi$ , respectively, and

$$\begin{aligned} \sigma_L = \sigma^\varphi|_L &= S_R \circ S_L, & \sigma_R = \sigma^\psi|_R &= S_R^{-1} \circ S_L^{-1}, \\ \Delta_L \circ \sigma^\varphi &= (S^2 \otimes \sigma^\varphi) \circ \Delta_L, & \Delta_L \circ \sigma^\psi &= (\sigma^\psi \otimes S^{-2}) \circ \Delta_L, \\ \Delta_R \circ \sigma^\varphi &= (S^2 \otimes \sigma^\varphi) \circ \Delta_R, & \Delta_R \circ \sigma^\psi &= (\sigma^\psi \otimes S^{-2}) \circ \Delta_R. \end{aligned}$$

If  $\mathcal{A}$  is locally projective, then

$$\begin{aligned} \sigma^\psi(L) &= L, & \mu_L \circ \sigma^\psi|_L &= \mu_L, & \sigma^\psi|_L \circ {}_L\psi &= \sigma_L \circ \psi_L, \\ \sigma^\varphi(R) &= R, & \mu_R \circ \sigma^\varphi|_R &= \mu_R, & \sigma^\varphi|_R \circ {}_R\varphi &= \sigma_R \circ \varphi_R. \end{aligned}$$

**Theorem 1.3.26** (Theorem 2.2.4, [Tim16]). Let  $(\mathcal{A}, \mu_L, \mu_R, {}_L\varphi_L, {}_R\psi_R)$  be a measured multiplier Hopf\*-algebroid. Then there exist invertible multipliers  $\delta, \delta^+, \delta^- \in \mathcal{M}(A)$  such that  $\psi = \varphi_\delta, \varphi \circ S = \varphi_{\delta^+}$  and  $\varphi \circ S^{-1} = \delta^- \varphi$ .

### 1.3.3 Duality for measured multiplier Hopf\*-algebroids

Let  $\mathcal{A} = (A, L, R, t_L, t_R, \Delta_L, \Delta_{R \cdot L} \varepsilon, \varepsilon_R, S)$  be a multiplier Hopf\*-algebroid with base weight  $(\mu_L, \mu_R)$  and partial integrals  $({}_L\varphi_L, {}_R\psi_R)$ . In this section we recall the construction of its dual and we will give the theorem of duality for measured multiplier Hopf\*-algebroids.

As before, we write  $\varphi = \mu_L \circ_L \varphi_L$  and  $\psi = \mu_R \circ_R \psi_R$  for the total left and right integral, respectively. The set

$$\widehat{A} := \{\varphi_a : a \in A\} = \{\varphi_a : a \in A\} = \{\psi_a : a \in A\} = \{\psi_a : a \in A\}$$

is a sub-space of the factorizable functionals vectorial space

$$A^\sqcup = ({}_L A^L)^\sqcup \cap ({}^R A_R)^\sqcup = ({}_L A_L)^\sqcup \cap ({}^R A_R)^\sqcup = ({}^L A_L)^\sqcup \cap ({}^R A^R)^\sqcup.$$

**Remark 44.** A functional  $\omega : A \rightarrow \mathbb{C}$  belongs to  $A^\sqcup$  if and only if there exist unique module maps

$$\begin{aligned} {}_L \omega &\in ({}_L A)^\vee, & \omega_L &\in (A_L)^\vee, & {}_R \omega &\in ({}^R A)^\vee, & \omega_R &\in (A_R)^\vee, \\ {}^L \omega &\in ({}^L A)^\vee, & \omega^L &\in (A^L)^\vee, & {}^R \omega &\in ({}^R A)^\vee, & \omega^R &\in (A^R)^\vee, \end{aligned}$$

such that

$$\omega = \mu_L \circ_L \omega = \mu_L \circ \omega_L = \mu_R \circ_R \omega = \mu_R \circ \omega_R = \mu_L \circ^L \omega = \mu_L \circ \omega^L = \mu_R \circ^R \omega = \mu_R \circ \omega^R.$$

In this case, by unicity, we have

$${}^L \omega = S_R \circ \omega_R, \quad \omega^L = S_R \circ_R \omega, \quad {}^R \omega = S_L \circ \omega_L, \quad \omega^R = S_L \circ_L \omega,$$

because

$$\mu_L \circ S_R = \mu_R, \quad \text{and} \quad \mu_R \circ S_L = \mu_L.$$

**Lemma 1.3.27** (Lemma 3.1.1, [Tim17]). *Using the linear maps,*

$$\begin{aligned} L \otimes \widehat{A} &\rightarrow \widehat{A} & \widehat{A} \otimes L &\rightarrow \widehat{A} \\ y \otimes \varphi_a &\mapsto \varphi_{ya} & \varphi_a \otimes y &\mapsto \varphi_{t_L(y)a} \\ R \otimes \widehat{A} &\rightarrow \widehat{A} & \widehat{A} \otimes R &\rightarrow \widehat{A} \\ x \otimes {}_a \varphi &\mapsto {}_{at_R(x)} \varphi & {}_a \varphi \otimes x &\mapsto {}_{ax} \varphi \end{aligned}$$

the vector space  $\widehat{A}$  is idempotent and faithful as  $L$ -bimodule and as  $R$ -bimodule. Moreover, if  $A$  is locally projective, then also  $\widehat{A}$  is locally projective as a right and as a left module over  $L$  and over  $R$ , respectively.

**Proposition 1.3.28** (Proposition 6.1, [Tim16] - Lemma 3.2.3, [Tim17]). *Let  $\omega \in A^\sqcup$  and  $a \in A$ . Then, there exist unique multipliers  $\omega \triangleright a \in \mathcal{M}(A)$  and  $a \triangleleft \omega \in \mathcal{M}(A)$  such that*

$$\begin{aligned} (\omega \triangleright a)b &:= (id \overline{\otimes} {}_L \omega)(\Delta_L(a)(b \otimes 1)), & b(\omega \triangleright a) &:= (id \overline{\otimes} {}^R \omega)((b \otimes 1)\Delta_R(a)), \\ (a \triangleleft \omega)b &:= (\omega^L \overline{\otimes} id)(\Delta_L(a)(1 \otimes b)), & b(a \triangleleft \omega) &:= (\omega_R \overline{\otimes} id)((1 \otimes b)\Delta_R(a)), \end{aligned}$$

for all  $b \in A$ . Moreover,

- For all  $\omega, \theta \in \widehat{A}$  and  $a \in A$ , we have  $\omega \triangleright a, a \triangleleft \theta \in A$  and

$$\omega \triangleright (a \triangleleft \theta) = (\omega \triangleright a) \triangleleft \theta, \quad \varepsilon(\omega \triangleright a) = \omega(a), \quad \varepsilon(a \triangleleft \theta) = \theta(a).$$

- For all  $a, b \in A$ ,

$${}_a\varphi \triangleright b = S(\varphi_b \triangleright a), \quad \text{and} \quad b \triangleleft \psi_a = S(a \triangleleft {}_b\psi).$$

- (i) The vector space  $\widehat{A}$  is a non-degenerate idempotent algebra with multiplication given by

$$\omega * \theta := (\omega \underset{\mu_L}{\otimes} \theta) \circ \Delta_L = (\omega \underset{\mu_R}{\otimes} \theta) \circ \Delta_R.$$

- (ii) We have

$$\mathcal{M}(\widehat{A}) = \{\omega \in A^\sqcup : \omega \triangleright a \in A \text{ and } a \triangleleft \omega \in A \text{ for all } a \in A\}$$

- (iii) There exist an injective homomorphism

$$\begin{aligned} j_L : L &\rightarrow \mathcal{M}(\widehat{A}) \\ y &\mapsto \begin{aligned} j_L(y)\omega &:= \omega_y \\ \omega j_L(y) &:= \omega_{t_L(y)} \end{aligned} \end{aligned}$$

and an injective anti-homomorphism

$$\begin{aligned} j_R : R &\rightarrow \mathcal{M}(\widehat{A}) \\ x &\mapsto \begin{aligned} j_R(x)\omega &:= t_R(x)\omega \\ \omega j_R(x) &:= {}_x\omega \end{aligned} \end{aligned}$$

such that

$$\begin{aligned} (\omega j_R(x)) * \theta &= \omega * (j_R(x)\theta) \\ (\theta j_R(x)) * \omega &= \omega * (j_R(x)\theta) \end{aligned}$$

Consider the bilinear maps

$$\begin{aligned} \langle \quad | \quad \rangle : (A^\sqcup \otimes A^\sqcup) \times (A^L \otimes_L A) &\rightarrow \mathbb{C} \\ (\omega \otimes \theta) \times (a \otimes b) &\mapsto (\omega \underset{\mu_L}{\otimes} \theta)(a \otimes b), \end{aligned}$$

$$\begin{aligned} \langle \quad | \quad \rangle : (A^\sqcup \otimes A^\sqcup) \times (A_R \otimes^R A) &\rightarrow \mathbb{C} \\ (\omega \otimes \theta) \times (a \otimes b) &\mapsto (\omega \underset{\mu_r}{\otimes} \theta)(a \otimes b), \end{aligned}$$

then, we have the next important theorem

**Theorem 1.3.29** (Theorem 3.3.6, [Tim17]). *There exist unique maps*

$$\widehat{\Delta}_L : \widehat{A} \rightarrow \widehat{A}_L \overline{\times}^L \widehat{A}, \quad \widehat{\Delta}_R : \widehat{A} \rightarrow \widehat{A}^R \overline{\times}^R \widehat{A},$$

such that for all  $\omega, \theta \in \widehat{A}$  and  $a, b \in A$

$$\langle (\omega \otimes 1) \widehat{\Delta}_L(\theta) | a \otimes b \rangle = \langle \omega \otimes \theta | \Delta_L(a)(1 \otimes b) \rangle,$$

$$\langle \widehat{\Delta}_R(\omega)(1 \otimes \theta) | a \otimes b \rangle = \langle \omega \otimes \theta | (a \otimes 1) \Delta_R(b) \rangle,$$

and the tuple  $\widehat{\mathcal{A}} := (\widehat{A}, R, L, t_L^{-1}, t_R^{-1}, \widehat{\Delta}_R, \widehat{\Delta}_L)$  is a multiplier bialgebroid which is locally projective.

**Theorem 1.3.30.** Consider the linear maps

$$\begin{array}{ccc} \widehat{S} : \widehat{A} & \rightarrow & \widehat{A} \\ \omega & \mapsto & \omega \circ S \end{array}, \quad \begin{array}{ccc} {}_R\widehat{\varepsilon} : \widehat{A} & \rightarrow & R \\ {}_a\psi & \mapsto & \varepsilon_R(a) \end{array}, \quad \begin{array}{ccc} \widehat{\varepsilon}_L : \widehat{A} & \rightarrow & L \\ \varphi_a & \mapsto & {}_L\varepsilon(a) \end{array}.$$

Then,  $(\widehat{A}, R, L, t_L^{-1}, t_R^{-1}, \widehat{\Delta}_R, \widehat{\Delta}_L, {}_R\widehat{\varepsilon}, \widehat{\varepsilon}_L, \widehat{S})$  is a regular multiplier Hopf algebroid.

**Definition 1.3.31.** The tuple  $\widehat{\mathcal{A}}$  is called the dual regular multiplier Hopf algebroid of  $\mathcal{A} = (\mathcal{A}, \mu_L, \mu_R, {}_L\varphi_L, {}_R\psi_R)$ .

**Proposition 1.3.32** (Proposition 3.2.6, [Tim17]). Let  $(\mathcal{A}, \mu_L, \mu_R, {}_L\varphi_L, {}_R\psi_R)$  be a measured multiplier Hopf  $*$ -algebroid. Then  $\widehat{A}$  is a  $*$ -algebra with respect to the involution given by

$$\omega^*(a) := \overline{\omega(S(a)^*)}$$

for all  $\omega \in \widehat{A}$  and  $a \in A$ .

Consider the linear maps

$$\begin{array}{ccc} {}_L\widehat{\psi}_L : \widehat{A} & \rightarrow & L \\ \varphi_a & \mapsto & {}_L\varepsilon(a) \end{array}, \quad \begin{array}{ccc} {}_R\widehat{\varphi}_R : \widehat{A} & \rightarrow & R \\ {}_a\psi & \mapsto & \varepsilon_R(a) \end{array}.$$

Then, we have

**Theorem 1.3.33** (Theorem, [Tim17]). Let  $(\mathcal{A}, \mu_L, \mu_R, {}_L\varphi_L, {}_R\psi_R)$  be a measured multiplier Hopf  $*$ -algebroid, where  $\mathcal{A}$  is locally projective. Then,  $(\widehat{\mathcal{A}}, \mu_R, \mu_L, {}_R\widehat{\varphi}_R, {}_L\widehat{\psi}_L)$  is a measured multiplier Hopf  $*$ -algebroid, where  $\widehat{\mathcal{A}} = (\widehat{A}, R, L, \widehat{t}_R, \widehat{t}_L, \widehat{\Delta}_R, \widehat{\Delta}_L)$ .

The next theorem establishes the duality in the case of measured multiplier Hopf  $*$ -algebroids.

**Theorem 1.3.34** ([Tim17]). Let  $(\mathcal{A}, \mu_L, \mu_R, {}_L\varphi_L, {}_R\psi_R)$  be a measured multiplier Hopf  $*$ -algebroid, where  $\mathcal{A}$  is locally projective and let  $(\widehat{\mathcal{A}}, \mu_R, \mu_L, {}_R\widehat{\varphi}_R, {}_L\widehat{\psi}_L)$  be its dual. Then, the map  $A \rightarrow (\widehat{A})^\vee$  given by  $a \mapsto a^\vee$  is a  $*$ -isomorphism from  $(\mathcal{A}, \mu_L, \mu_R, {}_L\varphi_L, {}_R\psi_R)$  to the bidual  $(\widehat{\widehat{\mathcal{A}}}, \mu_L, \mu_R, {}_L\widehat{\varphi}_L, {}_R\widehat{\psi}_R)$ , where  $\widehat{\widehat{\mathcal{A}}} = (\widehat{\widehat{A}}, L, R, t_L, t_R, \widehat{\widehat{\Delta}}_L, \widehat{\widehat{\Delta}}_R)$ .

### 1.3.4 A simplified definition of a multiplier Hopf $*$ -algebroid

Let  $A$  be an idempotent non-degenerate  $*$ -algebra,  $B$  a non-degenerate  $*$ -algebra,  $\alpha : B \rightarrow \mathcal{M}(A)$  be an injective  $*$ -homomorphism and  $\beta : B \rightarrow \mathcal{M}(A)$  be an injective anti- $*$ -homomorphism such that its images commute elementwise in  $\mathcal{M}(A)$ . Denote  $t = \beta \circ \alpha^{-1}$  the bijective anti- $*$ -homomorphism from  $\alpha(X)$  to  $\beta(X)$ . It is straightforward to prove the following lemma.



**Lemma 1.3.35** ([TV18]). *The formula  $a_1 \otimes a_2 \rightarrow a_1^* \otimes a_2^*$  defines invertible anti-linear maps*

$$A_{\alpha(B)} \otimes_{\alpha(B)} A \rightarrow A^{\beta(B)} \otimes_{\beta(B)} A, \quad A_{\beta(B)} \otimes_{\beta(B)} A \rightarrow A^{\alpha(B)} \otimes_{\alpha(B)} A,$$

$$A^{\alpha(B)} \otimes_{\alpha(B)} A \rightarrow A_{\beta(B)} \otimes_{\beta(B)} A,$$

that give rise to the following anti-linear bijective map:

$$\# : \text{End}(A^{\alpha(B)} \otimes_{\alpha(B)} A) \rightarrow \text{End}(A_{\beta(B)} \otimes_{\beta(B)} A), \quad T \mapsto (* \otimes *) \circ T \circ (* \otimes *).$$

**Lemma 1.3.36** ([TV18]).

- *The tuple  $(A, \alpha(B), id, t)$  is a left quantum graph if and only if the tuple  $(A, \beta(B), id, t^{-1})$  is a right quantum graph.*
- *The tuple  $(A, \alpha(B), id, t, \Delta_\alpha)$  is a left multiplier bialgebroid if and only if the tuple  $(A, \beta(B), id, t^{-1}, \Delta_\alpha^\#)$  is a right multiplier bialgebroid. Here we use the notation  $\Delta_\alpha^\#(a) := \Delta_\alpha^\#(a^*)$  for all  $a \in A$ . In this case, the tuple  $(A, \alpha(B), \beta(B), t, t^{-1}, \Delta_\alpha, \Delta_\alpha^\#)$  is a multiplier bialgebroid. Moreover, a map  $\varepsilon$  is a left counit for  $(A, \alpha(B), id, t, \Delta_\alpha)$  if and only if the map  $t \circ \varepsilon$  is a right counit for  $(A, \beta(B), id, t^{-1}, \Delta_\alpha^\#)$ .*

Then one can easily deduce

**Proposition 1.3.37.** *Suppose that the multiplier bialgebroid*

$$\mathcal{A} = (A, \alpha(B), \beta(B), t, t^{-1}, \Delta_\alpha, \Delta_\alpha^\#)$$

*is a multiplier Hopf\*-algebroid, with left counit  $\varepsilon$ , right counit  $t \circ \varepsilon$  and antipode  $S$ . Then, if  $\mu$  is a faithful positive functional over  $B$  and  $\Phi : \alpha(B)A_{\alpha(B)} \rightarrow \alpha(B)$  is a linear map which satisfies*

$$(id \bar{\otimes} \Phi)(\Delta_L(a')(a \otimes 1)) = \Phi(a')a,$$

*for all  $a, a' \in A$ , where*

$$id \bar{\otimes} \Phi : A^{\alpha(B)} \otimes_{\alpha(B)} A \rightarrow A, \quad a \otimes b \mapsto a\Phi(b)$$

*and  $\mu \circ \alpha^{-1} \circ \Phi : A \rightarrow \mathbb{C}$  is positive. Then,*

$$(\mathcal{A}, \mu \circ \alpha^{-1}, \mu \circ \beta^{-1}, \Phi, S \circ \Phi \circ S)$$

*is a measured multiplier Hopf\*-algebroid.*

Using this proposition and Lemma 1.3.36, we have the next more simple definition.

**Definition 1.3.38.** In the setting of the proposition above, we say that a tuple

$$(A, B, \alpha, \beta, \Delta_\alpha, \varepsilon, S, \mu, \Phi)$$

is a measured multiplier Hopf\*-algebroid with base weight  $\mu$  et left partial integral  $\Phi$ , if

$$(A, \alpha(B), \beta(B), t, t^{-1}, \Delta_\alpha, \Delta_\alpha^\#)$$

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is a measured multiplier Hopf  $*$ -algebroid with base weight  $(\mu \circ \alpha^{-1}, \mu \circ \alpha^{-1} \circ t^{-1})$  and with left and right partial integrals given by  $\Phi$  and  $S \circ \Phi \circ S$ , respectively.



## Chapter N° 2

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# Yetter-Drinfeld $*$ -algebras over algebraic quantum groups

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### 2.1 The category of algebraic quantum groups

**Definition 2.1.1.** An *algebraic quantum group* is a multiplier Hopf  $*$ -algebra  $(A, \Delta)$  equipped with a positive left integral  $\varphi$ . We will use the notation  $\mathbb{G} = (A, \Delta, \varphi)$ .

**Remark 45.** By Theorem 1.2.10, a left (or right) integral of an algebraic quantum group is always faithful.

**Example 2.1.2.** Let  $G$  be a group. By the Example 1.2.5 and Example 1.2.13, we have two algebraic quantum group  $(\text{Fun}_f(G), \Delta, \varphi)$  and  $(\mathbb{C}[G], \Delta', \varphi')$ . They are in duality as multiplier Hopf  $*$ -algebras with integrals.

**Remark 46.** Let  $\mathbb{G} = (A, \Delta, \varphi)$  be an algebraic quantum group. It was show [Kus03] (see 4.2.56) that there exist an anti- $*$ -isomorphism  $R : A \rightarrow A$  called the unitary antipode of  $\mathbb{G}$ . This object have similar algebraic properties as  $S$  and also allows have some analytic features close to framework of  $C^*$ -algebraic quantum groups. Then, instead of use  $S$ , we will use the unitary antipode  $R$  and fix the left integral of  $\mathbb{G}$ , we will use  $\psi := \varphi \circ R$  as its right integral from now on.

**Definition 2.1.3.** Let  $\mathbb{G} = (A, \Delta, \varphi)$  be an algebraic quantum group. By Theorem 1.2.20,  $(\widehat{A}, \widehat{\Delta}, \widehat{\varphi})$  is an algebraic quantum group, then the triple  $\widehat{\mathbb{G}} := (\widehat{A}, \widehat{\Delta}^{cop}, \widehat{\varphi} \circ \widehat{R})$  is also an algebraic quantum group, called *the dual algebraic quantum group of  $\mathbb{G}$* , where  $\widehat{\Delta}^{cop} := \Sigma_{\widehat{A}, \widehat{A}} \circ \widehat{\Delta}$ .

**Remark 47.** We define the dual algebraic quantum group in this sense, the comultiplication being the co-opposite, because it is more compatible with the theory of quantum groups in the operator algebraic setting.

**Definition 2.1.4.** Let  $\mathbb{G} = (A, \Delta, \varphi)$  be an algebraic quantum group. We define

- *the opposite algebraic quantum group of  $\mathbb{G}$*  as the triple  $(A, \Delta^{cop}, \varphi \circ R)$ , and we will denote it by  $\mathbb{G}^\circ$ .

- the commutant algebraic quantum group of  $\mathbb{G}$  as the triple  $(A^{op}, \Delta^{op}, \varphi^{op})$ , and we will denote it by  $\mathbb{G}^c$ . Here,

$$\Delta^{op}(a^{op})(b^{op} \otimes c^{op}) := ((b \otimes c)\Delta(a))^{op \otimes op},$$

$$(b^{op} \otimes c^{op})\Delta^{op}(a^{op}) := (\Delta(a)(b \otimes c))^{op \otimes op}$$

$$\text{and } \varphi^{op}(a^{op}) := \varphi(a).$$

**Definition 2.1.5.** An algebraic quantum group  $\mathbb{G} = (A, \Delta, \varphi)$  is said to be of *compact type* if the \*-algebra  $A$  is unital, and it is said to be of *discrete type* if its dual  $\widehat{\mathbb{G}}$  is of compact type, in other words, if the \*-algebra  $\widehat{A}$  is unital.

Let  $(A, \Delta)$  be an algebraic quantum group. Using the left Haar integral, we can define an inner product on  $A$  in the following way

$$\begin{aligned} \langle \cdot, \cdot \rangle_\varphi : A \otimes A &\rightarrow \mathbb{C} \\ a \otimes b &\mapsto \varphi(a^*b) \end{aligned}$$

**Lemma 2.1.6.** The algebra  $End(A)$  is an \*-algebra with the involution given by

$$\begin{aligned} * : End(A) &\rightarrow End(A) \\ T &\mapsto T^* \end{aligned}$$

where

$$\langle T(a), b \rangle_\varphi = \langle a, T^*(b) \rangle_\varphi$$

for all  $a, b \in A$ .

*Proof* It is straightforward. □

**Lemma 2.1.7.** For all  $a \in A$  and  $\theta \in \widehat{A}$ , we have the following relation

$$\theta \cdot \varphi_a = \varphi_{a \blacktriangleleft \widehat{S}^{-1}(\theta)},$$

*Proof* Fix  $a \in A$  and  $\theta \in \widehat{A}$ . By direct calculation and using the strong left invariance of  $\varphi$ .

$$\begin{aligned} \theta \cdot \varphi_a(c) &= (\theta \otimes \varphi)(\Delta(c)(1 \otimes a)) \\ &= \theta((id \otimes \varphi)(\Delta(c)(1 \otimes a))) \\ &= \widehat{S}^{-1}(\theta)(id \otimes \varphi)((1 \otimes c)\Delta(a)) \\ &= \widehat{S}^{-1}(\theta)(id \otimes \varphi)(a_{(1)} \otimes ca_{(2)}) \\ &= \varphi(c(\widehat{S}^{-1}(\theta) \otimes id)(\Delta(a))) \\ &= \varphi(c(a \blacktriangleleft \widehat{S}^{-1}(\theta))) \\ &= \varphi_{(a \blacktriangleleft \widehat{S}^{-1}(\theta))}(c) \end{aligned}$$

for all  $c \in A$ . □

We recall three linear maps associated with an algebraic quantum group  $(A, \Delta)$

$$\begin{array}{ccc} \pi : A & \rightarrow & \text{End}(A) \\ a & \mapsto & [b \mapsto ab] \end{array} \quad , \quad \begin{array}{ccc} \widehat{\pi} : \widehat{A} & \rightarrow & \text{End}(\widehat{A}) \\ \omega & \mapsto & [\theta \mapsto \omega \cdot \theta] \end{array}$$

$$\begin{array}{ccc} \lambda : \widehat{A} & \rightarrow & \text{End}(A) \\ \omega & \mapsto & [b \mapsto (\widehat{S}^{-1}(\omega) \otimes id)\Delta(b)] \end{array}$$

For the linear map  $\lambda$ , we know that it is relate to the right action associated with the canonical pairing  $p_A : A \times \widehat{A} \rightarrow \mathbb{C}$ ,  $a \times \varphi_b \mapsto \varphi(ab)$ , i.e.

$$\lambda(\omega)(a) = a \blacktriangleleft \widehat{S}^{-1}(\omega)$$

for every  $\omega \in \widehat{A}$  and  $a \in A$ .

**Lemma 2.1.8.** *We have*

$$\pi(a^*) = \pi(a)^*, \quad \widehat{\pi}(\omega^*) = \widehat{\pi}(\omega)^*, \quad \lambda(\omega^*) = \lambda(\omega)^*$$

for all  $a \in A$  and  $\omega \in \widehat{A}$ .

*Proof* The two first equalities are straightforward. The last equalities follows because we have

$$\begin{aligned} \langle \lambda(\omega)(a), b \rangle_\varphi &= \varphi((\lambda(\omega)(a))^*b) \\ &= \varphi((a \blacktriangleleft \widehat{S}^{-1}(\omega))^*b) \\ &= \varphi((a^* \blacktriangleleft \omega^*)b) \\ &= p_A(a^* \blacktriangleleft \omega^*, \varphi_b) \\ &= p_A(a^*, \omega^* \cdot \varphi_b) \\ &= p_A(a^*, \varphi_{b \blacktriangleleft \widehat{S}^{-1}(\omega^*)}) \\ &= p_A(a^*, \varphi_{\lambda(\omega^*)(b)}) \\ &= \varphi(a^* \lambda(\omega^*)b) \\ &= \langle a, \lambda(\omega^*)b \rangle_\varphi \end{aligned}$$

for all  $a, b \in A$  and  $\omega \in \widehat{A}$ . □

**Lemma 2.1.9.** *The maps defined by*

$$\begin{array}{ccc} \pi : A & \rightarrow & \text{End}(A) \\ a & \mapsto & [b \mapsto ab] \end{array} \quad , \quad \begin{array}{ccc} \widehat{\pi} : \widehat{A} & \rightarrow & \text{End}(\widehat{A}) \\ \omega & \mapsto & [\theta \mapsto \omega \cdot \theta] \end{array}$$

$$\begin{array}{ccc} \lambda : \widehat{A} & \rightarrow & \text{End}(A) \\ \omega & \mapsto & [b \mapsto (\widehat{S}^{-1}(\omega) \otimes id)\Delta(b)] \end{array}$$

are \*-homomorphisms.

*Proof* The maps  $\pi$  and  $\widehat{\pi}$  are obviously homomorphisms, because they are the left multiplication of  $A$  over  $A$  and of  $\widehat{A}$  over  $\widehat{A}$ , respectively. On the other hand, fix  $\omega$  and  $\theta$  in

$\widehat{A}$ , then

$$\begin{aligned}
 \lambda(\omega)(\lambda(\theta)(a)) &= (\widehat{S}^{-1}(\omega) \otimes id)\Delta((\widehat{S}^{-1}(\theta) \otimes id)\Delta(a)) \\
 &= (\widehat{S}^{-1}(\omega) \otimes id)(\widehat{S}^{-1}(\theta) \otimes \Delta)\Delta(a) \\
 &= (\widehat{S}^{-1}(\theta) \otimes \widehat{S}^{-1}(\omega) \otimes id)(id \otimes \Delta)\Delta(a) \\
 &= (\widehat{S}^{-1}(\theta) \otimes \widehat{S}^{-1}(\omega) \otimes id)(\Delta \otimes id)\Delta(a) \\
 &= (\widehat{S}^{-1}(\theta) \cdot \widehat{S}^{-1}(\omega) \otimes id)\Delta(a) \\
 &= (\widehat{S}^{-1}(\omega \cdot \theta) \otimes id)\Delta(a) \\
 &= \lambda(\omega \cdot \theta)(a),
 \end{aligned}$$

for all  $a \in A$ . It follows from Lemma 2.1.8 that this maps are \*-homomorphisms.  $\square$

**Lemma 2.1.10.** For  $\omega \in \widehat{A}$ , we have  $\lambda(\omega) = \mathcal{F}^{-1} \circ \widehat{\pi}(\omega) \circ \mathcal{F}$ , where  $\mathcal{F}$  is the Fourier transform defined in Remark 17.

*Proof* Fix  $a \in A$ , then

$$\begin{aligned}
 [\widehat{\pi}(\omega)(\mathcal{F}(a))](x) &= (\omega \cdot \varphi_a)(x) \\
 &= \omega((id \otimes \varphi)(\Delta(x)(1 \otimes a))) \\
 &= \omega((S^{-1} \otimes \varphi)((1 \otimes x)\Delta(a))) \\
 &= \varphi(x(\widehat{S}^{-1}(\omega) \otimes id)\Delta(a)) \\
 &= [\mathcal{F}((\widehat{S}^{-1}(\omega) \otimes id)\Delta(a))](x) \\
 &= [\mathcal{F}(\lambda(\omega)(a))](x),
 \end{aligned}$$

for all  $x \in A$ . This implies the result.  $\square$

**Lemma 2.1.11** (Heisenberg commutation relations). For any  $a, a' \in A$ , we have

$$\pi(a)\lambda(\varphi_{a'}) = \lambda(\varphi_{a_{(1)}a'})\pi(a_{(2)})$$

in  $End(A)$ .

*Proof* Fixed  $a, a' \in A$ , we have

$$\begin{aligned}
 \pi(a)\lambda(\varphi_{a'})(b) &= a(\widehat{S}^{-1}(\varphi_{a'}) \otimes id)(\Delta(b)) \\
 &= (\varphi_{a'} \circ S^{-1} \otimes id)((1 \otimes a)\Delta(b)) \\
 &= (\varphi_{a'} \circ S^{-1} \otimes id)((S(a_{(1)}) \otimes 1)\Delta(a_{(2)}b)) \\
 &= (\varphi_{a'} \otimes id)((S^{-1} \otimes id)(\Delta(a_{(2)}b))(a_{(1)} \otimes 1)) \\
 &= (\varphi_{a_{(1)}a'} \circ S^{-1} \otimes id)(\Delta(a_{(2)}b)) \\
 &= \lambda(\varphi_{a_{(1)}a'})(a_{(2)}b) \\
 &= \lambda(\varphi_{a_{(1)}a'})\pi(a_{(2)})(b),
 \end{aligned}$$

for all  $b \in A$ . Here, we use the equality

$$\begin{aligned}
 (1 \otimes a)\Delta(b) &= b_{(1)} \otimes ab_{(2)} \\
 &= b_{(1)} \otimes \varepsilon(a_{(1)})a_{(2)}b_{(2)} \\
 &= S(a_{(1)})a_{(2)}b_{(1)} \otimes a_{(3)}b_{(2)} \\
 &= S(a_{(1)})a_{(2)}b_{(1)} \otimes a_{(3)}b_{(2)} \\
 &= (S(a_{(1)}) \otimes 1)(a_{(2)}b_{(1)} \otimes a_{(3)}b_{(2)}) \\
 &= (S(a_{(1)}) \otimes 1)\Delta(a_{(2)}b).
 \end{aligned}$$

□

**Definition 2.1.12.** Let  $\mathbb{G} = (A, \Delta_A, \varphi^A)$  and  $\mathbb{H} = (B, \Delta_B, \varphi^B)$  be two algebraic quantum groups. A non-degenerate \*-homomorphism  $\Phi : A \rightarrow \mathcal{M}(B)$  is called a morphism from  $\mathbb{H}$  to  $\mathbb{G}$  if  $\Delta_B \circ \Phi = (\Phi \otimes \Phi) \circ \Delta^A$ .

**Definition 2.1.13.** We define the category  $\mathcal{AQG}$  called the category of algebraic quantum groups as follows

- Its objects are the algebraic quantum groups.
- Its morphisms are the morphisms of algebraic quantum groups in the sense of Definition 2.1.12.

The next result is well know in the case of Hopf algebras and locally compact quantum groups.

**Proposition 2.1.14.** *In the category  $\mathcal{AQG}$ , we have*

$$(\mathbb{G}^c)^\circ \cong \mathbb{G}, \quad \widehat{\mathbb{G}}^\circ \cong (\widehat{\mathbb{G}})^c, \quad \widehat{(\mathbb{G}^c)} \cong (\widehat{\mathbb{G}})^\circ,$$

**Definition 2.1.15.** Let  $\mathbb{G} = (A, \Delta_A, \varphi^A)$  be an algebraic quantum group. An algebraic quantum group  $\mathbb{H} = (B, \Delta_B, \varphi^B)$  is called a *closed quantum subgroup* of  $\mathbb{G}$  if there exists a surjective morphism from  $\mathbb{H}$  to  $\mathbb{G}$ .

**Remark 48.** We will justify the term *closed*, when we discuss  $C^*$ -algebraic quantum groups.

## 2.2 The Heisenberg algebra of an algebraic quantum group

Let  $\mathbb{G} = (A, \Delta, \varphi)$  be an algebraic quantum group.

**Definition 2.2.1.** Let  $p_A : A \times \widehat{A} \rightarrow \mathbb{C}$ ,  $a \times \omega \mapsto \omega(a)$ , be the canonical pairing. The Heisenberg \*-algebra of the algebraic quantum group  $\mathbb{G}$  is the Heisenberg \*-algebra associated with the canonical pairing  $p_A$ , it will be denoted by  $H(\mathbb{G}) := \widehat{A}A$ . The duality of the algebraic quantum group  $\mathbb{G}$  is the duality associated with the pairing  $p_A$ , it will be denoted by  $V^{\mathbb{G}}$  or simply by  $V$ .



We recall that

$$\begin{aligned} H(\mathbb{G}) = \widehat{A}A &:= \{ \omega \# a : (\omega \# a)(\theta \# b) = \omega \cdot \theta_{(1)} \# (a \blacktriangleleft_{\widehat{A}} \theta_{(2)})b \} \\ &= \{ \omega \# a : (\omega \# a)(\theta \# b) = \omega \cdot (a_{(1)} \blacktriangleright_A \theta) \# a_{(2)}b \}. \end{aligned}$$

and there exist two non-degenerate \*-embeddings

$$\begin{aligned} j_A : A &\rightarrow \mathcal{M}(H(\mathbb{G})) & j_{\widehat{A}} : \widehat{A} &\rightarrow \mathcal{M}(H(\mathbb{G})) \\ a &\mapsto 1 \# a & \omega &\mapsto \omega \# 1 \end{aligned}$$

such that

$$\omega \# a = j_{\widehat{A}}(\omega)j_A(a)$$

for all  $a \in A$  and  $\omega \in \widehat{A}$ .

**Remark 49.** Using the \*-embeddings of  $A$  and  $\widehat{A}$  in  $\mathcal{M}(H(\mathbb{G}))$ , we can consider  $U \in \mathcal{M}(A \otimes \widehat{A})$  as an element in  $\mathcal{M}(H(\mathbb{G}) \otimes \widehat{A})$  or as an element in  $\mathcal{M}(A \otimes H(\mathbb{G}))$ . Similarly, we can also consider  $\widehat{U} \in \mathcal{M}(\widehat{A} \otimes A)$  as an element in  $\mathcal{M}(H(\mathbb{G}) \otimes A)$  or as an element in  $\mathcal{M}(\widehat{A} \otimes H(\mathbb{G}))$ .

Now, the Heisemberg \*-algebra of the algebraic quantum group  $\widehat{\mathbb{G}}^\circ = (\widehat{A}, \widehat{\Delta}, \widehat{\varphi})$  is given by

$$\begin{aligned} H(\widehat{\mathbb{G}}^\circ) = A\widehat{A} &:= \{ a \# \omega : (a \# \omega)(b \# \theta) = ab_{(1)} \# (\omega \blacktriangleleft_A b_{(2)}) \cdot \theta \} \\ &= \{ a \# \omega : (a \# \omega)(b \# \theta) = a(\omega_{(1)} \blacktriangleright_{\widehat{A}} b) \# \omega_{(2)} \cdot \theta \}, \end{aligned}$$

and the next proposition shows its relations with  $H(\mathbb{G})$ .

**Proposition 2.2.2.** *The linear map*

$$\begin{aligned} \Phi : H(\mathbb{G}) &\rightarrow H(\widehat{\mathbb{G}}^\circ) \\ \omega \# a &\mapsto S(a) \# \widehat{S}^{-1}(\omega) \end{aligned}$$

is an anti-\* -isomorphism

*Proof* We have

$$\begin{aligned} \Phi((\omega \# a)(\theta \# b)) &= \Phi(\omega \cdot \theta_{(1)} \# (a \blacktriangleleft_{\widehat{A}} \theta_{(2)})b) \\ &= S((a \blacktriangleleft_{\widehat{A}} \theta_{(2)})b) \# \widehat{S}^{-1}(\omega \cdot \theta_{(1)}) \\ &= S(b)S(a \blacktriangleleft_{\widehat{A}} \theta_{(2)}) \# \widehat{S}^{-1}(\theta_{(1)}) \cdot \widehat{S}^{-1}(\omega) \\ &= S(b)(\widehat{S}^{-1}(\theta_{(2)}) \blacktriangleright_{\widehat{A}} S(a)) \# \widehat{S}^{-1}(\theta_{(1)}) \cdot \widehat{S}^{-1}(\omega) \\ &= S(b)(\widehat{S}^{-1}(\theta)_{(1)} \blacktriangleright_{\widehat{A}} S(a)) \# \widehat{S}^{-1}(\theta)_{(2)} \cdot \widehat{S}^{-1}(\omega) \\ &= (S(b) \# \widehat{S}^{-1}(\theta))(S(a) \# \widehat{S}^{-1}(\omega)) \\ &= \Phi(\theta \# b)\Phi(\omega \# a) \end{aligned}$$

□

## 2.3 Corepresentations of algebraic quantum groups

Let  $\mathbb{G} = (A, \Delta, \varphi)$  be an algebraic quantum group and  $X$  be a non-degenerate \*-algebra.

**Definition 2.3.1** (Definition 5.1, [Kus97a]). A *corepresentation of  $\mathbb{G}$  on  $X$*  is an element  $W$  in  $\mathcal{M}(A \otimes X)$  such that  $(\Delta \otimes id)(W) = W_{13}W_{23}$ . If also  $W$  is unitary in  $\mathcal{M}(A \otimes X)$ , we say that  $W$  is a *unitary corepresentation*.

**Notation 11.** For any  $W \in \mathcal{M}(A \otimes X)$ , we use the notations

$$W_{13} := (\Sigma_{A,A} \otimes id)(1_{\mathcal{M}(A)} \otimes W), \quad W_{23} := 1_{\mathcal{M}(A)} \otimes W,$$

both elements are  $\mathcal{M}(A \otimes A \otimes X)$ .

**Proposition 2.3.2** ([Kus97a]). For any unitary corepresentation  $W$  of  $\mathbb{G}$  on a non-degenerate \*-algebra  $X$ , we have

- $A \otimes X = \text{span}\{ (a \otimes 1)W(1 \otimes x) : a \in A, x \in X \}$
- $A \otimes X = \text{span}\{ (1 \otimes x)W(a \otimes 1) : a \in A, x \in X \}$
- $A \otimes X = \text{span}\{ (a \otimes 1)W^*(1 \otimes x) : a \in A, x \in X \}$
- $A \otimes X = \text{span}\{ (1 \otimes x)W^*(a \otimes 1) : a \in A, x \in X \}$

**Remark 50.** For any  $\omega = \varphi_a \in \widehat{A}$  and  $P \in \mathcal{M}(A \otimes X)$ , we define  $(\omega_a \otimes id)(P)$  as the element in  $\mathcal{M}(X)$ :

$$[(\omega \otimes id)(P)]x := (\varphi \otimes id)(P(a \otimes x)), \quad x[(\omega \otimes id)(P)] := (\varphi \otimes id)((\rho^{-1}(a) \otimes x)P),$$

for all  $x \in X$ . Moreover, we have

$$(\omega_1 \cdot \omega_2 \otimes id)(P) = (\omega_1 \otimes \omega_2 \otimes id)((\Delta \otimes id)(P))$$

and

$$(\omega_1 \otimes id)(P)(\omega_2 \otimes id)(Q) = (\omega_1 \otimes \omega_2 \otimes id)(P_{13}Q_{23})$$

for every  $\omega_1, \omega_2 \in \widehat{A}$  and  $P, Q \in \mathcal{M}(A \otimes B)$ .

**Proposition 2.3.3** ([Kus97a]). Let  $W$  be an element of  $\mathcal{M}(A \otimes X)$  and consider the linear map

$$\begin{aligned} \theta_W : \widehat{A} &\rightarrow \mathcal{M}(X) \\ \omega &\mapsto (\omega \otimes id)(W) \end{aligned}$$

Then,  $W$  is a corepresentation of  $\mathbb{G}$  on  $X$  if and only if  $\theta_W$  is an homomorphism.

**Proposition 2.3.4** ([Kus97a]). Let  $W$  be a corepresentation of  $\mathbb{G}$  on  $X$ . Then the following statements are equivalent:

- (i)  $W$  is unitary with  $W^* = (S \otimes id)(W)$ .
- (ii)  $(\varepsilon \otimes id)(W) = 1_{\mathcal{M}(X)}$ .

(iii)  $W(A \otimes X) = A \otimes X = (A \otimes X)W$ .

(iv)  $\theta_W$  is a non-degenerate \*-homomorphism.

**Proposition 2.3.5** ([Kus97a]). *Let  $Y$  be a non-degenerate \*-algebra and  $W$  a corepresentation of  $\mathbb{G}$  on  $X$ . If  $\theta \in \text{Hom}(X, Y)$ , then  $(\text{id} \otimes \theta)(W)$  is a corepresentation of  $\mathbb{G}$  on  $Y$  such that  $\theta_{(\text{id} \otimes \theta)(W)} = \theta \circ \theta_W$ . Moreover, if  $W$  is unitary, then  $(\text{id} \otimes \theta)(W)$  is also unitary.*

### 2.3.1 The universal corepresentation

Let  $\mathbb{G} = (A, \Delta, \varphi)$  be an algebraic quantum group. In this section we mainly recall the construction of the universal corepresentation of  $\mathbb{G}$  and the natural bijection between corepresentations of  $\mathbb{G}$  and representations of  $\widehat{A}$ .

**Remark 51.** Because  $\widehat{A}$  is a subset of  $A^\vee$ , we can regard  $A \otimes \widehat{A}$  in a natural way as a subspace of  $\text{End}(A)$ .

Consider the following linear maps

$$\begin{aligned} U_l : A \otimes \widehat{A} &\rightarrow \text{End}(A) \\ a \otimes \omega &\mapsto [b \mapsto (\text{id} \otimes \omega)(\Delta(b)(a \otimes 1))] \end{aligned}$$

$$\begin{aligned} U_r : A \otimes \widehat{A} &\rightarrow \text{End}(A) \\ a \otimes \omega &\mapsto [b \mapsto (\omega \otimes \text{id})((1 \otimes a)\Delta(b))] \end{aligned}$$

**Lemma 2.3.6** ([Kus97a]). *For any  $a \in A$  and  $\omega \in \widehat{A}$ , we have that  $U_l(a \otimes \omega)$  and  $U_r(a \otimes \omega)$  belong to  $A \otimes \widehat{A}$ . Moreover, we have*

$$U_l((a \otimes \omega)(a' \otimes \omega')) = U_l(a \otimes \omega)(a' \otimes \omega'), \quad U_r((a \otimes \omega)(a' \otimes \omega')) = (a \otimes \omega)U_r(a' \otimes \omega')$$

and

$$U_r(a \otimes \omega)(a' \otimes \omega') = (a \otimes \omega)U_l(a' \otimes \omega'),$$

for any  $a, a' \in A$  and  $\omega, \omega' \in \widehat{A}$ ,

**Theorem 2.3.7** ([Kus97a]). *There exists a unique element  $U$  in  $\mathcal{M}(A \otimes \widehat{A})$  such that*

$$[U(a \otimes \omega)](b) = (\text{id} \otimes \omega)(\Delta(b)(a \otimes 1)), \quad [(a \otimes \omega)U](b) = (\omega \otimes \text{id})((1 \otimes a)\Delta(b))$$

for all  $a, b \in A$  and  $\omega \in \widehat{A}$ . Moreover, we have the following properties:

(i)  $U$  is a unitary in  $\mathcal{M}(A \otimes \widehat{A})$ .

(ii)  $(\Delta \otimes \text{id})(U) = U_{13}U_{23}$ .

(iii)  $(\text{id} \otimes \widehat{\Delta})(U) = U_{12}U_{13}$ .

(iv)  $(\omega \otimes \text{id})(U) = \omega$  for all  $\omega \in \widehat{A}$ , this implies that  $\theta_U = \text{id}_{\widehat{A}}$ .

(v)  $(\text{id} \otimes a)(U) = a$  for all  $a \in A$  (here we use the identification of  $A$  and  $\widehat{A}$ ).

**Definition 2.3.8** ([Kus97a]). By the theorem above, we have that  $U \in \mathcal{M}(A \otimes \widehat{A})$  is a unitary corepresentation of  $\mathbb{G}$  on  $\widehat{A}$  and it is called *the universal corepresentation of  $\mathbb{G}$* .

**Remark 52.** Denote  $\widehat{U} := \Sigma(U^*) = (\Sigma(U))^* \in \mathcal{M}(\widehat{A} \otimes A)$ . In [Kus97a], it was prove that  $\widehat{U}^* = \Sigma(U)$  is the universal corepresentation of the algebraic quantum group  $(\widehat{A}, \widehat{\Delta}, \widehat{\varphi})$ . Now, follows from items (ii) and (iii) of Theorem 2.3.7, that

$$(\widehat{\Delta}^{cop} \otimes id)(\widehat{U}) = \widehat{U}_{13}\widehat{U}_{23}, \quad (id \otimes \Delta)(\widehat{U}) = \widehat{U}_{13}\widehat{U}_{12},$$

that means that  $\widehat{U}$  is the universal corepresentation of  $\widehat{\mathbb{G}}$ .

The next theorem shows the property of universality of the corepresentation  $U$ .

**Theorem 2.3.9** ([Kus97a]). *For any unitary corepresentation  $W$  of  $\mathbb{G}$  on a non-degenerate \*-algebra  $X$  and any  $\theta \in \text{Hom}(\widehat{A}, X)$ , we have*

$$(i) \quad (id \otimes \theta_W)(U) = W.$$

$$(ii) \quad \theta_{(id \otimes \theta)(U)} = \theta.$$

## 2.4 The algebraic multiplicative unitary

Let  $\mathbb{G} = (A, \Delta, \varphi)$  be an algebraic quantum group,  $\widehat{\mathbb{G}} = (\widehat{A}, \widehat{\Delta}^{cop}, \widehat{\varphi} \circ \widehat{S})$  be its dual,  $V$  the duality of  $\mathbb{G}$  and  $H(\mathbb{G})$  the Heisenberg algebra of  $\mathbb{G}$ .

**Definition 2.4.1.** By Theorem 2.3.7, for an algebraic quantum group  $\mathbb{G}$  we have a unique unitary element  $U \in \mathcal{M}(A \otimes \widehat{A})$  such that  $(\Delta \otimes id)(U) = U_{13}U_{23}$  and  $(id \otimes \widehat{\Delta}^{cop})(U) = U_{13}U_{12}$ . This element will be called the *algebraic multiplicative unitary of  $\mathbb{G}$* , and will denoted by  $U^{\mathbb{G}}$  or sometimes simply by  $U$ .

**Remark 53.** By Remark 52,  $\widehat{U} = \Sigma(U^*) \in \mathcal{M}(\widehat{A} \otimes A)$  is the algebraic multiplicative unitary of  $\widehat{\mathbb{G}}$ .

**Lemma 2.4.2.** *If  $U$  is the algebraic multiplicative unitary of  $\mathbb{G}$ , then*

$$U(a \otimes \varphi_b) = S^{-1}(b_{(1)})a \otimes \varphi_{b_{(2)}}, \quad (a \otimes \varphi_b)U = a\delta S(b_{(2)}) \otimes \varphi_{b_{(1)}},$$

for all  $a, b \in A$ .

*Proof* For all  $c \in A$

$$\begin{aligned} [U(a \otimes \varphi_b)](c) &= (id \otimes \varphi_b)(\Delta(c)(a \otimes 1)) \\ &= (id \otimes \varphi)((\Delta(c)(1 \otimes b))a) \\ &= S^{-1}((id \otimes \varphi)((1 \otimes c)\Delta(b)))a \\ &= \varphi(cb_{(2)})S^{-1}(b_{(1)})a \\ &= [S^{-1}(b_{(1)})a \otimes \varphi_{b_{(2)}}](c), \end{aligned}$$

and

$$\begin{aligned}
 [(a \otimes \varphi_b)U](c) &= (\varphi_b \otimes id)((1 \otimes a)\Delta(c)) \\
 &= (\psi_{b\delta^{-1}} \otimes id)((1 \otimes a)\Delta(c)) \\
 &= a(\psi \otimes id)(\Delta(c)(b\delta^{-1} \otimes 1)) \\
 &= aS((\psi \otimes id)((c \otimes 1)\Delta(b\delta^{-1}))) \\
 &= aS((\psi \otimes id)((c \otimes 1)\Delta(b)(\delta^{-1} \otimes \delta^{-1}))) \\
 &= aS(b_{(2)}\delta^{-1})\psi(cb_{(1)}\delta^{-1}) \\
 &= aS(b_{(2)}\delta^{-1})\varphi(cb_{(1)}) \\
 &= [a\delta S(b_{(2)}) \otimes \varphi_{b_{(1)}}](c)
 \end{aligned}$$

□

**Lemma 2.4.3.** *If  $U$  is the algebraic multiplicative unitary of  $\mathbb{G}$ , then*

$$U^*(a \otimes \varphi_b) = b_{(1)}a \otimes \varphi_{b_{(2)}}, \quad (a \otimes {}_b\varphi)U^* = a\delta^{-1}b_{(2)} \otimes {}_{b_{(1)}}\varphi$$

for all  $a, b \in A$ .

*Proof* Fix  $a, b \in A$ , if we write  $\Delta(b)(a \otimes 1) = p \otimes q \in A \otimes A$ , then

$$\begin{aligned}
 [U(p \otimes \varphi_q)](x) &= (id \otimes \varphi_q)(\Delta(x)(p \otimes 1)) \\
 &= (id \otimes \varphi)(\Delta(x)(p \otimes q)) \\
 &= (id \otimes \varphi)(\Delta(xb)(a \otimes 1)) \\
 &= a\varphi(xb) \\
 &= [(a \otimes \varphi_b)](x),
 \end{aligned}$$

for all  $x \in A$ . This implies that

$$U^*(a \otimes \varphi_b) = p \otimes \varphi_q = b_{(1)}a \otimes \varphi_{b_{(2)}}.$$

Similarly, if we write  $(1 \otimes a\delta^{-1})\Delta(b) = q \otimes p \in A \otimes A$ , then

$$\begin{aligned}
 [(p \otimes {}_q\varphi)U](x) &= ({}_q\varphi \otimes id)((1 \otimes p)\Delta(x)) \\
 &= (\varphi \otimes id)((q \otimes p)\Delta(x)) \\
 &= (\varphi \otimes id)((1 \otimes a\delta^{-1})\Delta(bx)) \\
 &= a\delta^{-1}\varphi(bx)\delta \\
 &= [(a \otimes {}_b\varphi)](x),
 \end{aligned}$$

for all  $x \in A$ . This implies that

$$(a \otimes {}_b\varphi)U^* = p \otimes {}_q\varphi = a\delta^{-1}b_{(2)} \otimes {}_{b_{(1)}}\varphi.$$

□

**Remark 54.** Using the equality  $\Delta(\rho(q)) = (S^2 \otimes \rho)(\Delta(q))$ , we have

$$\begin{aligned} (p \otimes_q \varphi)U &= (p \otimes \varphi_{\rho(q)})U \\ &= p\delta S(\rho(q)_{(2)}) \otimes \varphi_{\rho(q)_{(1)}} \\ &= p\delta S(\rho(q_{(2)})) \otimes \varphi_{S^2(q_{(1)})} \\ &= p\delta S(\rho(q_{(2)})) \otimes \rho^{-1}(S^2(q_{(1)}))\varphi \end{aligned}$$

for all  $p, q \in A$ . Then, put  $a := p\delta S(\rho(q_{(2)}))$  and  $b := \rho^{-1}(S^2(q_{(1)}))$ , we have

$$\begin{aligned} (1 \otimes a\delta^{-1})\Delta(b) &= (1 \otimes p\delta S(\rho(q_{(2)}))\delta^{-1})\Delta(\rho^{-1}(S^2(q_{(1)}))) \\ &= (1 \otimes p\rho^{-1}(S(q_{(3)})))(q_{(1)} \otimes \rho^{-1}(S^2(q_{(2)}))) \\ &= q_{(1)} \otimes p\rho^{-1}(S(S(q_{(2)})q_{(3)})) \\ &= \varepsilon(q_{(2)})q_{(1)} \otimes p \\ &= q \otimes p. \end{aligned}$$

**Example 2.4.4.** Let  $\mathbb{G} = (\text{Fun}_f(G), \Delta, \varphi)$  be the algebraic quantum group of complex finitely supported functions on a group  $G$ . The “formal” element

$$U^{\mathbb{G}} := \sum_{g \in G} \delta_g \otimes U_g \in \mathcal{M}(\text{Fun}_f(G) \otimes \mathbb{C}[G])$$

is the algebraic multiplicative unitary of  $\mathbb{G}$ , because by simple calculation

$$(id \otimes U_g)(\Delta(q)(p \otimes 1)) = [U^{\mathbb{G}}(p \otimes U_g)](q) \quad (U_g \otimes id)((1 \otimes p)\Delta(q)) = [(p \otimes U_g)U^{\mathbb{G}}](q),$$

for all  $p, q \in \text{Fun}_f(G)$  and  $g \in G$ . This also implies, that the “formal” element

$$\widehat{U}^{\mathbb{G}} = \Sigma((U^{\mathbb{G}})^*) := \sum_{g \in G} U_g \otimes \delta_{g^{-1}} \in \mathcal{M}(\mathbb{C}[G] \otimes \text{Fun}_f(G))$$

is the algebraic multiplicative unitary of  $\widehat{\mathbb{G}} = (\mathbb{C}[G], \widehat{\Delta}, \widehat{\varphi})$ , because

$$(U^{\mathbb{G}})^* = \sum_{g \in G} \delta_{g^{-1}} \otimes U_g$$

In order to say more about the algebraic multiplicative unitary  $U$ , we will establish its relationship with the duality  $V$ .

**Proposition 2.4.5.** We have

$$V = \Sigma(U)$$

*Proof* By Lemma 2.4.2, we have

$$\begin{aligned}
 (p_A \otimes \bar{p}_A)(\Sigma(U), (\varphi_a \blacktriangleright_{\hat{A}} a') \otimes (b \blacktriangleright_A \varphi_{b'})) &= (p_A \otimes \bar{p}_A)(\Sigma(U), (\varphi_a \otimes b) \triangleright (a' \otimes \varphi_{b'})) \\
 &= (p_A \otimes \bar{p}_A)(\Sigma(U)(\varphi_a \otimes b), a' \otimes \varphi_{b'}) \\
 &= (\bar{p}_A \otimes p_A)(U(b \otimes \varphi_a), \varphi_{b'} \otimes a') \\
 &= (\bar{p}_A \otimes p_A)(S^{-1}(a_{(1)})b \otimes \varphi_{a_{(2)}}, \varphi_{b'} \otimes a') \\
 &= \varphi_{b'}(S^{-1}(a_{(1)})b)\varphi(a'a_{(2)}) \\
 &= \varphi_{b'}(S^{-1}((id \otimes \varphi)((1 \otimes a')\Delta(a)))b) \\
 &= \varphi_{bb'}((id \otimes \varphi)(\Delta(a')(1 \otimes a))) \\
 &= (\varphi_{bb'} \otimes \varphi_a)(\Delta(a')) \\
 &= p_A^2(\Delta(a'), \varphi_{bb'} \otimes \varphi_a) \\
 &= p_A(a', \varphi_{bb'} \cdot \varphi_a) \\
 &= p_A(\varphi_a \blacktriangleright_{\hat{A}} a', \varphi_{bb'}) \\
 &= p_A(\varphi_a \blacktriangleright_{\hat{A}} a', b \blacktriangleright_A \varphi_{b'}),
 \end{aligned}$$

for all  $a, a', b, b' \in A$ . The result follows from the fact that the actions coming from the pairing are unital and from the unicity of the duality  $V$ .  $\square$

**Remark 55.** It is obvious that  $A$  is a left  $\hat{A}$ -module with the module map given by

$$\begin{aligned}
 \triangleright_\lambda : \hat{A} \otimes A &\rightarrow A \\
 \omega \otimes a &\mapsto \lambda(\omega)(a)
 \end{aligned}$$

Moreover,

$$\omega \triangleright_\lambda a = (\hat{S}^{-1}(\omega) \otimes id)\Delta(a) = a \blacktriangleleft_{\hat{A}} \hat{S}^{-1}(\omega),$$

for all  $a \in A$  and  $\omega \in \hat{A}$ , and then

$$\begin{aligned}
 \omega \triangleright_\lambda (ab) &= ab \blacktriangleleft_{\hat{A}} \hat{S}^{-1}(\omega) \\
 &= (a \blacktriangleleft_{\hat{A}} \hat{S}^{-1}(\omega_{(2)}))(b \blacktriangleleft_{\hat{A}} \hat{S}^{-1}(\omega_{(1)})) \\
 &= (\omega_{(2)} \triangleright_\lambda a)(\omega_{(1)} \triangleright_\lambda b),
 \end{aligned}$$

for all  $a, b \in A$  and  $\omega \in \hat{A}$ . This implies that  $\triangleright_\lambda$  is a left action of  $(\hat{A}, \hat{\Delta}^{cop})$  on  $A$ .

**Proposition 2.4.6.** *The map*

$$\begin{aligned}
 \pi \triangleright_\lambda : (A \otimes \hat{A}) \otimes (A \otimes A) &\rightarrow A \otimes A \\
 (a \otimes \omega) \otimes (b \otimes b') &\mapsto ab \otimes (\omega \triangleright_\lambda b')
 \end{aligned}$$

*defines a unital non-degenerate left action of the algebra  $A \otimes \hat{A}$  on  $A \otimes A$ , such that its extension to multiplier algebras  $\triangleright : \mathcal{M}(A \otimes \hat{A}) \otimes (A \otimes A) \rightarrow A \otimes A$  satisfies*

$$U_\pi \triangleright_\lambda (a \otimes b) = (S^{-1} \otimes id)(\Delta(b))(a \otimes 1),$$

*for all  $a, b \in A$ .*

*Proof* Fix  $a \in A$  and  $\omega \in \widehat{A}$ , then

$$\begin{aligned}
 (a \otimes \omega) \pi \triangleright_\lambda (b \otimes b') &= ab \otimes (\omega \triangleright_\lambda b') \\
 &= ab \otimes (b' \triangleleft_{\widehat{A}} \widehat{S}^{-1}(\omega)) \\
 &= \Sigma \circ (id \otimes S^{-1})[(b' \otimes S(b)) \triangleleft (\widehat{S}^{-1}(\omega) \otimes S(a))] \\
 &= \Sigma \circ (id \otimes S^{-1})[(b' \otimes S(b)) \triangleleft (\widehat{S}^{-1} \otimes S)(\omega \otimes a)],
 \end{aligned}$$

for all  $b, b' \in A$ . The last equality implies that for all  $T \in \mathcal{M}(\widehat{A} \otimes A)$  we have

$$\Sigma(T) \pi \triangleright_\lambda (a \otimes b) = \Sigma \circ (id \otimes S^{-1})[(b \otimes S(a)) \triangleleft (\widehat{S}^{-1} \otimes S)(T)],$$

for all  $a, b \in A$ . In particular, it follows from the lemma above that

$$\begin{aligned}
 U \pi \triangleright_\lambda (a \otimes b) &= \Sigma(V) \pi \triangleright_\lambda (a \otimes b) \\
 &= \Sigma \circ (id \otimes S^{-1})[(b \otimes S(a)) \triangleleft (\widehat{S}^{-1} \otimes S)(V)] \\
 &= \Sigma \circ (id \otimes S^{-1})[(b \otimes S(a)) \triangleleft V] \\
 &= \Sigma \circ (id \otimes S^{-1})[(1 \otimes S(a))\Delta^{cop}(b)] \\
 &= \Sigma(b_{(2)} \otimes S^{-1}(b_{(1)})a) \\
 &= (S^{-1} \otimes id)(\Delta(b))(a \otimes 1).
 \end{aligned}$$

□

**Corollary 2.4.7.** *We have*

$$U^* \pi \triangleright_\lambda (a \otimes b) = \Delta(b)(a \otimes 1),$$

for all  $a, b \in A$ .

*Proof* Fix  $a, b \in A$ . It follows from the equality

$$\begin{aligned}
 U \pi \triangleright_\lambda (\Delta(b)(a \otimes 1)) &= U \pi \triangleright_\lambda (b_{(1)}a \otimes b_{(2)}) \\
 &= (S^{-1} \otimes id)(\Delta(b_{(2)}))(b_{(1)}a \otimes 1) \\
 &= S^{-1}(b_{(2)})b_{(1)}a \otimes b_{(3)} \\
 &= \varepsilon(b_{(1)})a \otimes b_{(2)} \\
 &= a \otimes b
 \end{aligned}$$

that

$$U^* \pi \triangleright_\lambda (a \otimes b) = U^* \pi \triangleright_\lambda (U \pi \triangleright_\lambda (\Delta(b)(a \otimes 1))) = \Delta(b)(a \otimes 1)$$

□

**Proposition 2.4.8.** *We have*

$$\Delta(a) = U^*(1 \otimes a)U \in \mathcal{M}(A \otimes H(\mathbb{G}))$$

and

$$\widehat{\Delta}^{cop}(\omega) = \widehat{U}^*(1 \otimes \omega)\widehat{U} \in \mathcal{M}(\widehat{A} \otimes H(\mathbb{G})),$$



for all  $a \in A$  and  $\omega \in \widehat{A}$ .

*Proof* It follows from Proposition 1.2.43, that

$$\Delta(a) = \Sigma(\Delta^{cop}(a)) = \Sigma(V^*(a \otimes 1)V) = \Sigma(V)^*(1 \otimes a)\Sigma(V) = U^*(1 \otimes a)U,$$

and

$$\widehat{\Delta}^{cop}(\omega) = V(1 \otimes \omega)V^* = \Sigma(U^*)^*(1 \otimes \omega)\Sigma(U^*) = \widehat{U}^*(1 \otimes \omega)\widehat{U},$$

for all  $a \in A$  and  $\omega \in \widehat{A}$ . □

**Proposition 2.4.9.** *We have*

$$U_{12}U_{13}U_{23} = U_{23}U_{12}$$

in  $\mathcal{M}(A \otimes H(\mathbb{G}) \otimes \widehat{A})$ .

*Proof* As  $U = \Sigma(V)$ , the result follows from item (iv) of Proposition 1.2.38. □

A similar result to the next lemma has been proven in [KV97]. Here we give a purely algebraic prove necessary for our purpose.

**Lemma 2.4.10.** *For all  $a, b \in A$  and  $\omega \in \widehat{A}$ , we have*

$$(id \otimes a)(U(b \otimes \omega)U^*) = (id \otimes \omega)(\Delta(a_{(1)}))bS(a_{(2)})$$

*Proof* Fix  $a, b \in A$  and  $\omega \in \widehat{A}$ . If we denote  $U(b \otimes \omega) := \sum_i p_i \otimes \omega_i$ , then

$$\sum_i p_i \omega_i(c) = [U(b \otimes \omega)](c) = (id \otimes \omega)(\Delta(c)(b \otimes 1)) = (id \otimes \omega)(\Delta(c))b,$$

for all  $c \in A$ , this implies that

$$\begin{aligned} (U(b \otimes \omega)U^*)_{\pi} \triangleright_{\lambda} (d \otimes e) &= (U(b \otimes \omega))_{\pi} \triangleright_{\lambda} (U^*_{\pi} \triangleright_{\lambda} (d \otimes e)) \\ &= \left( \sum_i p_i \otimes \omega_i \right)_{\pi} \triangleright_{\lambda} (e_{(1)}d \otimes e_{(2)}) \\ &= \sum_i p_i e_{(1)}d \otimes (\widehat{S}^{-1}(\omega_i) \otimes id)\Delta(e_{(2)}) \\ &= \sum_i p_i e_{(1)}d \otimes \omega_i(S^{-1}(e_{(2)}))e_{(3)} \\ &= \sum_i p_i \omega_i(S^{-1}(e_{(2)}))e_{(1)}d \otimes e_{(3)} \\ &= (id \otimes \omega)(\Delta(S^{-1}(e_{(2)})))be_{(1)}d \otimes e_{(3)}, \end{aligned}$$

for all  $d, e \in A$ . Now, we denote  $U(b \otimes \omega)U^* := \sum_j q_j \otimes \theta_j$ , then

$$\begin{aligned}
 (id \otimes a)(U(b \otimes \omega)U^*)d &= \left( \sum_j q_j \theta_j(a) \right) d \\
 &= (id \otimes \varepsilon) \left( \sum_j q_j d \otimes \theta_j(a_{(2)}) S(a_{(1)}) \right) \\
 &= (id \otimes \varepsilon) \left( \sum_j q_j d \otimes (\widehat{S}^{-1}(\theta_j) \otimes id) \Delta(S(a)) \right) \\
 &= (id \otimes \varepsilon) \left( \left( \sum_j q_j \otimes \theta_j \right)_\pi \triangleright_\lambda (d \otimes S(a)) \right) \\
 &= (id \otimes \varepsilon) \left( (U(b \otimes \omega)U^{-1})_\pi \triangleright_\lambda (d \otimes S(a)) \right) \\
 &= (id \otimes \varepsilon) \left( (id \otimes \omega) \Delta(S^{-1}(S(a)_{(2)})) bS(a)_{(1)} d \otimes S(a)_{(3)} \right) \\
 &= (id \otimes \omega) \left( \Delta(S^{-1}(S(a)_{(2)})) \right) bS(a)_{(1)} d \\
 &= (id \otimes \omega) \left( \Delta(a_{(1)}) \right) bS(a_{(2)}) d,
 \end{aligned}$$

for all  $d \in A$ . As  $A$  is non-degenerate,

$$(id \otimes a)(U(b \otimes \omega)U^*) = (id \otimes \omega) \left( \Delta(a_{(1)}) \right) bS(a_{(2)}).$$

□

**Lemma 2.4.11.** For all  $\omega, \theta \in \widehat{A}$  and  $a \in A$ , we have

$$(id \otimes \omega) \left( \widehat{U}(\theta \otimes a) \widehat{U}^* \right) = \widehat{S}(\omega_{(1)}) \cdot \theta \cdot (id \otimes a) \left( \widehat{\Delta}^{cop}(\omega_{(2)}) \right)$$

*Proof* As  $\widehat{U}$  is the algebraic multiplicative unitary of  $\widehat{\mathbb{G}}$ , we can use Lemma 2.4.10 to conclude. □

**Example 2.4.12.** Let  $\mathbb{G} = (\text{Fun}_f(G), \Delta, \varphi)$  be the algebraic quantum group of complex finitely supported functions on a group  $G$ , with dual given by  $\widehat{\mathbb{G}} = (\mathbb{C}[G], \widehat{\Delta}, \widehat{\varphi})$ . Associated with the canonical pairing  $p : \text{Fun}_f(G) \times \mathbb{C}[G] \rightarrow \mathbb{C}$ ,  $\delta_g \times U_h \mapsto \delta_{g,h}$ , we have the right action of  $(\mathbb{C}[G], \widehat{\Delta})$  on the \*-algebra  $\text{Fun}_f(G)$  defined by

$$\begin{aligned}
 \blacktriangleleft : \text{Fun}_f(G) \otimes \mathbb{C}[G] &\rightarrow \text{Fun}_f(G) \\
 \delta_g \otimes U_h &\mapsto \delta_{h^{-1}g}
 \end{aligned}$$

Then, the Heisenberg algebra of  $\mathbb{G}$  is

$$H(\mathbb{G}) = \mathbb{C}[G] \# \text{Fun}_f(G) = \text{span}\{U_g \# \delta_h : (U_g \# \delta_h)(U_{\bar{g}} \# \delta_{\bar{h}}) = U_{g\bar{g}} \# \delta_{\bar{g}^{-1}h\bar{h}}\}.$$

Moreover, using the formal notations

$$U_g \delta_h := U_g \# \delta_h, \quad \delta_g U_h := U_h \# \delta_{h^{-1}g} = U_h \delta_{h^{-1}g},$$

we have

$$U^{\mathbb{G}} \Delta(\delta_g) = (1 \otimes \delta_g) U^{\mathbb{G}} \quad \text{and} \quad \widehat{U}^{\mathbb{G}} \widehat{\Delta}^{cop}(U_g) = (1 \otimes U_g) \widehat{U}^{\mathbb{G}},$$

for all  $g \in G$ . Recall that we have, for all  $g \in G$

$$\Delta(\delta_g) = \sum_{h \in G} \delta_h \otimes \delta_{h^{-1}g}, \quad \text{and} \quad \widehat{\Delta}(U_g) = U_g \otimes U_g.$$

## 2.5 Drinfeld double for an algebraic quantum group

In this section, we give the construction of the Drinfeld double for an algebraic quantum group, in the form close to [NV09, Nak96].

Let  $\mathbb{G} = (A, \Delta, \varphi)$  be an algebraic quantum group,  $\widehat{\mathbb{G}} = (\widehat{A}, \widehat{\Delta}^{cop}, \widehat{\varphi} \circ \widehat{R})$  be its dual and  $U$  be the algebraic multiplicative unitary of  $\mathbb{G}$ .

**Lemma 2.5.1.**

$$\begin{aligned} (\widehat{\Delta}^{cop} \otimes id_{\mathcal{M}(A)}) \circ \Sigma_U &= (id_{\mathcal{M}(\widehat{A})} \otimes \Sigma_U) \circ (\Sigma_U \otimes id_{\mathcal{M}(\widehat{A})}) \circ (id_{\mathcal{M}(A)} \otimes \widehat{\Delta}^{cop}) \\ (id_{\mathcal{M}(\widehat{A})} \otimes \Delta) \circ \Sigma_U &= (\Sigma_U \otimes id_{\mathcal{M}(A)}) \circ (id_{\mathcal{M}(A)} \otimes \Sigma_U) \circ (\Delta \otimes id_{\mathcal{M}(\widehat{A})}) \end{aligned}$$

*Proof* Easily, we have the relations

$$\begin{aligned} (id_{\mathcal{M}(\widehat{A})} \otimes \Sigma_U) &= \text{Ad}(\Sigma(U)_{23}) \circ \Sigma_{23} \\ (\Sigma_U \otimes id_{\mathcal{M}(\widehat{A})}) &= \text{Ad}(\Sigma(U)_{12}) \circ \Sigma_{12} \\ (id_{\mathcal{M}(\widehat{A})} \otimes \Delta) \circ \Sigma &= \Sigma_{12} \circ \Sigma_{23} \circ (\Delta \otimes id_{\mathcal{M}(\widehat{A})}) \\ (\widehat{\Delta}^{cop} \otimes id_{\mathcal{M}(A)}) \circ \Sigma &= \Sigma_{23} \circ \Sigma_{12} \circ (id_{\mathcal{M}(A)} \otimes \widehat{\Delta}^{cop}), \end{aligned}$$

then

$$\begin{aligned} (id_{\mathcal{M}(\widehat{A})} \otimes \Sigma_U) \circ (\Sigma_U \otimes id_{\mathcal{M}(\widehat{A})}) \circ (id_{\mathcal{M}(A)} \otimes \widehat{\Delta}^{cop}) &= \text{Ad}(\Sigma(U)_{23}) \circ \Sigma_{23} \circ \text{Ad}(\Sigma(U)_{12}) \circ \Sigma_{12} \circ (id_{\mathcal{M}(A)} \otimes \widehat{\Delta}^{cop}) \\ &= \text{Ad}(\Sigma(U)_{23}) \circ \text{Ad}(\Sigma(U)_{13}) \circ \Sigma_{23} \circ \Sigma_{12} \circ (id_{\mathcal{M}(A)} \otimes \widehat{\Delta}^{cop}) \\ &= \text{Ad}(\Sigma(U)_{23} \Sigma(U)_{13}) \circ (\widehat{\Delta}^{cop} \otimes id_{\mathcal{M}(A)}) \circ \Sigma \\ &= \text{Ad}((\widehat{\Delta}^{cop} \otimes id_{\mathcal{M}(\widehat{A})})(\Sigma(U))) \circ (\widehat{\Delta}^{cop} \otimes id_{\mathcal{M}(A)}) \circ \Sigma \\ &= (\widehat{\Delta}^{cop} \otimes id_{\mathcal{M}(\widehat{A})}) \circ \text{Ad}(\Sigma(U)) \circ \Sigma \\ &= (\widehat{\Delta}^{cop} \otimes id_{\mathcal{M}(\widehat{A})}) \circ \Sigma_U. \end{aligned}$$

Similarly,

$$\begin{aligned} (\Sigma_U \otimes id_{\mathcal{M}(A)}) \circ (id_{\mathcal{M}(A)} \otimes \Sigma_U) \circ (\Delta \otimes id_{\mathcal{M}(\widehat{A})}) &= \text{Ad}(\Sigma(U)_{12}) \circ \Sigma_{12} \circ \text{Ad}(\Sigma(U)_{23}) \circ \Sigma_{23} \circ (\Delta \otimes id_{\mathcal{M}(\widehat{A})}) \\ &= \text{Ad}(\Sigma(U)_{12}) \circ \text{Ad}(\Sigma(U)_{13}) \circ \Sigma_{12} \circ \Sigma_{23} \circ (\Delta \otimes id_{\mathcal{M}(\widehat{A})}) \\ &= \text{Ad}(\Sigma(U)_{12} \Sigma(U)_{13}) \circ (id_{\mathcal{M}(\widehat{A})} \otimes \Delta) \circ \Sigma \\ &= \text{Ad}((id_{\mathcal{M}(\widehat{A})} \otimes \Delta)(\Sigma(U))) \circ (id_{\mathcal{M}(\widehat{A})} \otimes \Delta) \circ \Sigma \\ &= (id_{\mathcal{M}(\widehat{A})} \otimes \Delta) \circ \text{Ad}(\Sigma(U)) \circ \Sigma \\ &= (id_{\mathcal{M}(\widehat{A})} \otimes \Delta) \circ \Sigma_U, \end{aligned}$$

and we can prove the second relation. □

Consider the map

$$\begin{aligned} \Delta_D : A \otimes \widehat{A} &\rightarrow \mathcal{M}(A \otimes \widehat{A} \otimes A \otimes \widehat{A}) \\ a \otimes \omega &\mapsto (id_{\mathcal{M}(A)} \otimes \Sigma_U \otimes id_{\mathcal{M}(\widehat{A})}) \circ (\Delta \otimes \widehat{\Delta}^{cop})(a \otimes \omega) \end{aligned}$$

**Proposition 2.5.2.** *The linear map  $\Delta_D$  is a non-degenerate \*-homomorphism and  $(A \otimes \widehat{A}, \Delta_D)$  is a multiplier Hopf\*-algebra.*

*Proof* We have

$$\begin{aligned} (\Delta_D \otimes id_{\mathcal{M}(A \otimes \hat{A})}) \circ \Delta_D &= [(id_{\mathcal{M}(A)} \otimes \Sigma_U \otimes id_{\mathcal{M}(\hat{A})}) \circ (\Delta \otimes \hat{\Delta}^{cop}) \otimes id_{\mathcal{M}(A \otimes \hat{A})}] \circ [(id \otimes \Sigma_U \otimes id) \circ (\Delta \otimes \hat{\Delta}^{cop})] \\ &= (id_{\mathcal{M}(A)} \otimes \Sigma_U \otimes id_{\mathcal{M}(\hat{A})} \otimes id_{\mathcal{M}(A \otimes \hat{A})}) \circ (\Delta \otimes (\hat{\Delta}^{cop} \otimes id_{\mathcal{M}(A)}) \circ \Sigma_U \otimes id_{\mathcal{M}(\hat{A})}) \circ (\Delta \otimes \hat{\Delta}^{cop}) \\ &= (id_{\mathcal{M}(A)} \otimes N \otimes id_{\mathcal{M}(\hat{A})}) \circ (\Delta \otimes \hat{\Delta}^{cop}), \end{aligned}$$

where

$$\begin{aligned} N &= (\Sigma_U \otimes id_{\mathcal{M}(\hat{A})} \otimes id_{\mathcal{M}(A)}) \circ (id_{\mathcal{M}(A)} \otimes (id_{\mathcal{M}(\hat{A})} \otimes \Sigma_U) \circ (\Sigma_U \otimes id_{\mathcal{M}(\hat{A})}) \circ (id_{\mathcal{M}(A)} \otimes \hat{\Delta}^{cop})) \circ (\Delta \otimes id_{\mathcal{M}(\hat{A})}) \\ &= (\Sigma_U \otimes \Sigma_U) \circ (id_{\mathcal{M}(A)} \otimes \Sigma_U \otimes id_{\mathcal{M}(\hat{A})}) \circ (\Delta \otimes \hat{\Delta}^{cop}) \\ &= (\Sigma_U \otimes \Sigma_U) \circ \Delta_D \end{aligned}$$

On the other hand, we also have

$$\begin{aligned} (id_{\mathcal{M}(A \otimes \hat{A})} \otimes \Delta_D) \circ \Delta_D &= [id_{\mathcal{M}(A \otimes \hat{A})} \otimes (id_{\mathcal{M}(A)} \otimes \Sigma_U \otimes id_{\mathcal{M}(\hat{A})}) \circ (\Delta \otimes \hat{\Delta}^{cop})] \circ [(id_{\mathcal{M}(A)} \otimes \Sigma_U \otimes id_{\mathcal{M}(\hat{A})}) \circ (\Delta \otimes \hat{\Delta}^{cop})] \\ &= (id_{\mathcal{M}(A \otimes \hat{A})} \otimes id_{\mathcal{M}(A)} \otimes \Sigma_U \otimes id_{\mathcal{M}(\hat{A})}) \circ (id_{\mathcal{M}(A)} \otimes (id_{\mathcal{M}(\hat{A})} \otimes \Delta) \circ \Sigma_U \otimes \hat{\Delta}^{cop}) \circ (\Delta \otimes \hat{\Delta}^{cop}) \\ &= (id_{\mathcal{M}(A)} \otimes M \otimes id_{\mathcal{M}(\hat{A})}) \circ (\Delta \otimes \hat{\Delta}^{cop}), \end{aligned}$$

where

$$\begin{aligned} M &= (id_{\mathcal{M}(\hat{A})} \otimes id_{\mathcal{M}(A)} \otimes \Sigma_U) \circ ((\Sigma_U \otimes id_{\mathcal{M}(A)}) \circ (id_{\mathcal{M}(A)} \otimes \Sigma_U) \circ (\Delta \otimes id_{\mathcal{M}(\hat{A})}) \otimes id_{\mathcal{M}(\hat{A})}) \circ (id_{\mathcal{M}(A)} \otimes \hat{\Delta}^{cop}) \\ &= (\Sigma_U \otimes \Sigma_U) \circ (id_{\mathcal{M}(A)} \otimes \Sigma_U \otimes id_{\mathcal{M}(\hat{A})}) \circ (\Delta \otimes \hat{\Delta}^{cop}) \\ &= (\Sigma_U \otimes \Sigma_U) \circ \Delta_D. \end{aligned}$$

Then

$$(\Delta_D \otimes id_{\mathcal{M}(A \otimes \hat{A})}) \circ \Delta_D = (id_{\mathcal{M}(A \otimes \hat{A})} \otimes \Delta_D) \circ \Delta_D.$$

□

Consider the map

$$\begin{aligned} \varphi_D : A \otimes \hat{A} &\rightarrow \mathbb{C} \\ a \otimes \omega &\mapsto (\varphi \otimes \hat{\varphi})(a \otimes \omega) \end{aligned}$$

**Lemma 2.5.3.** *We have*

$$\varphi_D = (\varphi \otimes \hat{\varphi} \circ \hat{R}) \circ Ad(U).$$

*Proof* If we denote  $b = \hat{\varphi} \circ \hat{R}$ , follows from the equality

$$\hat{\varphi}(\omega) = p_A(\hat{\varphi} \circ \hat{R}, \hat{R}^{-1}(\omega)) = p_A(b, \hat{R}^{-1}(\omega)) = p_A(R^{-1}(b), \omega) = \omega(R^{-1}(b)),$$

that

$$\varphi_D(a \otimes \omega) = \varphi(a)\hat{\varphi}(\omega) = \varphi(a)\omega(R^{-1}(b)).$$

On the other hand, we have

$$\begin{aligned} (\varphi \otimes \hat{\varphi} \circ \hat{R}) \circ Ad(U)(a \otimes \omega) &= (\varphi \otimes \hat{\varphi} \circ \hat{R})(U(a \otimes \omega)U^*) \\ &= \varphi((id \otimes b)(U(a \otimes \omega)U^*)) \\ &= \varphi((id \otimes \omega)(\Delta(b_{(1)}))aR(b_{(2)})) \\ &= \varphi(a)\omega(R^{-1}(b)). \end{aligned}$$

□

**Proposition 2.5.4.** *The map  $\varphi_D$  is a non-zero positive left and right invariant faithful functional on  $A \otimes \widehat{A}$ , and  $(A \otimes \widehat{A}, \Delta_D, \varphi_D, \varphi_D)$  is a unimodular algebraic quantum group.*

*Proof* The equation  $(id_{\mathcal{M}(A)} \otimes \widehat{\Delta}^{cop})(U) = U_{13}U_{12}$  implies

$$(id_{\mathcal{M}(A)} \otimes \widehat{\Delta}^{cop}) \circ \text{Ad}(U) = \text{Ad}(U_{13}U_{12}) \circ (id_{\mathcal{M}(A)} \otimes \widehat{\Delta}^{cop}),$$

then

$$\begin{aligned} (id_{\mathcal{M}(A \otimes \widehat{A})} \otimes \varphi_D) \circ \Delta_D &= [id_{\mathcal{M}(A)} \otimes id_{\mathcal{M}(\widehat{A})} \otimes (\varphi \otimes \widehat{\varphi} \circ \widehat{S}) \circ \text{Ad}(U)] \circ [(id_{\mathcal{M}(A)} \otimes \Sigma_U \otimes id_{\mathcal{M}(\widehat{A})}) \circ (\Delta \otimes \widehat{\Delta}^{cop})] \\ &= (id_{\mathcal{M}(A)} \otimes id_{\mathcal{M}(\widehat{A})} \otimes \varphi \otimes \widehat{\varphi} \circ \widehat{S}) \circ \text{Ad}(U_{34}) \circ \Sigma_{23} \circ \text{Ad}(U_{23}) \circ (\Delta \otimes \widehat{\Delta}^{cop}) \\ &= (id_{\mathcal{M}(A)} \otimes id_{\mathcal{M}(\widehat{A})} \otimes \varphi \otimes \widehat{\varphi} \circ \widehat{S}) \circ \Sigma_{23} \circ \text{Ad}(U_{24}U_{23}) \circ (\Delta \otimes \widehat{\Delta}^{cop}) \\ &= (id_{\mathcal{M}(A)} \otimes \varphi \otimes id_{\mathcal{M}(\widehat{A})} \otimes \widehat{\varphi} \circ \widehat{S}) \circ (id_{\mathcal{M}(A)} \otimes id_{\mathcal{M}(A)} \otimes \widehat{\Delta}^{cop}) \circ \text{Ad}(1_{\mathcal{M}(A)} \otimes U) \circ (\Delta \otimes id_{\mathcal{M}(\widehat{A})}) \\ &= (id_{\mathcal{M}(A)} \otimes \varphi \otimes (id_{\mathcal{M}(\widehat{A})} \otimes \widehat{\varphi} \circ \widehat{S}) \circ \widehat{\Delta}^{cop}) \circ \text{Ad}(1_{\mathcal{M}(A)} \otimes U) \circ (\Delta \otimes id_{\mathcal{M}(\widehat{A})}) \\ &= [id_{\mathcal{M}(A)} \otimes \varphi \otimes (\widehat{\varphi} \circ \widehat{S})(\cdot)1_{\mathcal{M}(A)}] \circ \text{Ad}(1_{\mathcal{M}(A)} \otimes U) \circ (\Delta \otimes id_{\mathcal{M}(\widehat{A})}) \\ &= [id_{\mathcal{M}(A)} \otimes \varphi_D(\cdot)1_{\mathcal{M}(\widehat{A})}] \circ [\Delta \otimes id_{\mathcal{M}(\widehat{A})}] \\ &= [id_{\mathcal{M}(A)} \otimes (\varphi \otimes \widehat{\varphi})(\cdot)1_{\mathcal{M}(\widehat{A})}] \circ [\Delta \otimes id_{\mathcal{M}(\widehat{A})}] \\ &= (id_{\mathcal{M}(A)} \otimes \varphi) \circ \Delta \otimes \widehat{\varphi}(\cdot)1_{\mathcal{M}(\widehat{A})} \\ &= \varphi(\cdot)1_{\mathcal{M}(A)} \otimes \widehat{\varphi}(\cdot)1_{\mathcal{M}(\widehat{A})} \\ &= (\varphi \otimes \widehat{\varphi})(\cdot)1_{\mathcal{M}(A)} \otimes 1_{\mathcal{M}(\widehat{A})} \\ &= \varphi_D(\cdot)1_{\mathcal{M}(A \otimes \widehat{A})}. \end{aligned}$$

The right invariance is proved similarly. □

**Definition 2.5.5.** The algebraic quantum group  $(A \otimes \widehat{A}, \Delta_D, \varphi_D)$ , is called *the Drinfeld double of  $\mathbb{G}$*  and is denoted by  $D(\mathbb{G})$ .

## 2.6 Actions of algebraic quantum groups

**Definition 2.6.1.** Let  $\mathbb{G} = (A, \Delta, \varphi)$  be an algebraic quantum group and  $X$  be a non-degenerate \*-algebra. A *left action of  $\mathbb{G}$  on  $X$*  is a left coaction  $\alpha : X \rightarrow \mathcal{M}(A \otimes X)$  of the multiplier Hopf \*-algebra  $(A, \Delta)$  on  $X$ . The pair  $(X, \alpha)$  is called a  $\mathbb{G}$ -\*-algebra if  $X$  is a non-degenerate \*-algebra and  $\alpha$  is a left action of the algebraic quantum group  $\mathbb{G}$  on  $X$ . We can define in the similar way a *right action of  $\mathbb{G}$  on  $X$* .

Let us recall that for every left action  $\alpha : X \rightarrow \mathcal{M}(A \otimes X)$  of  $\mathbb{G}$  on a non-degenerate \*-algebra  $X$ , there exists a right action

$$\begin{aligned} \triangleleft_\alpha : X \otimes \widehat{A} &\rightarrow X \\ x \otimes \omega &\mapsto (\omega \otimes id)(\alpha(x)) \end{aligned}$$

of the multiplier Hopf \*-algebra  $(\widehat{A}, \widehat{\Delta})$  on  $X$ , called the *dual action associated with  $\alpha$* .

**Remark 56.** For any  $x \in X$ , we have

$$\begin{aligned} \alpha((\varphi \circ R \otimes id)\alpha(x)) &= (\varphi \circ R \otimes id \otimes id)((id \otimes \alpha)(\alpha(x))) \\ &= (\varphi \circ R \otimes id \otimes id)((\Delta \otimes id)(\alpha(x))) \\ &= ((\varphi \circ R \otimes id) \circ \Delta \otimes id)(\alpha(x)) \\ &= ((\varphi \circ R(\cdot)1_{\mathcal{M}(A)} \otimes id)(\alpha(x))) \\ &= 1_{\mathcal{M}(A)} \otimes (\varphi \circ R \otimes id)(\alpha(x)), \end{aligned}$$

then

$$\begin{aligned} E_\alpha : X &\rightarrow X^\alpha \\ x &\mapsto (\varphi \circ R \otimes id)(\alpha(x)) \end{aligned}$$

is a well defined linear map, where  $X^\alpha = \{x \in \mathcal{M}(X) : \alpha(x) = 1 \otimes x\}$ .

**Example 2.6.2.** Let  $\cdot : G \times S \rightarrow S$  be a right action of  $G$  on the set  $S$ . Using the right action  $\alpha : G \rightarrow \text{Aut}(\text{Fun}_f(S))$ ,  $\alpha_g(f)(s) = f(s \cdot g^{-1})$ , we have

$$\begin{aligned} \alpha : \text{Fun}_f(S) &\rightarrow \mathcal{M}(\text{Fun}_f(G) \otimes \text{Fun}_f(S)) \cong \text{Fun}(G, \text{Fun}_f(S)) \\ f &\mapsto \alpha_f : g \mapsto \alpha_g(f) \end{aligned}$$

with the dual right action of the Hopf\*-algebra  $\mathbb{C}[G]$  on  $A$ , given by

$$\begin{aligned} \triangleleft_\alpha : A \otimes \mathbb{C}[G] &\rightarrow A \\ a \otimes U_g &\mapsto \alpha_g(a) \end{aligned}$$

**Example 2.6.3.** Let  $X$  be a non-degenerate \*-algebra and  $\mathbb{G} = (A, \Delta, \varphi)$  be an algebraic quantum group. The map  $\alpha : X \rightarrow \mathcal{M}(A \otimes X)$  given by  $\alpha(x) = 1_{\mathcal{M}(A)} \otimes x$  is a left action of  $\mathbb{G}$  on  $X$  called the trivial action.

**Example 2.6.4.** With every algebraic quantum group, we can always associate two non-trivial left actions. Let  $\mathbb{G} = (A, \Delta_A, \varphi_A)$  be an algebraic quantum group, and  $U$  be the algebraic multiplicative unitary of  $\mathbb{G}$ . It is obvious that we can see the comultiplication  $\Delta_A : A \rightarrow \mathcal{M}(A \otimes A)$  as a left (and right) action of  $\mathbb{G}$  on  $A$ . On the other hand, if we denote  $\hat{U} = \Sigma(U^*)$ , we can define the map

$$\begin{aligned} Ad_{\mathbb{G}} : A &\rightarrow \mathcal{M}(\hat{A} \otimes A) \\ a &\mapsto \hat{U}^*(1 \otimes a)\hat{U} \end{aligned} ,$$

then

$$\begin{aligned} (id \otimes Ad_{\mathbb{G}})(Ad_{\mathbb{G}}(a)) &= \hat{U}_{23}^* \Sigma_{12}(1 \otimes \hat{U}^*(1 \otimes a)\hat{U})\hat{U}_{23} \\ &= \hat{U}_{23}^* \Sigma_{12}(\hat{U}_{23}^*(1 \otimes 1 \otimes a)\hat{U}_{23})\hat{U}_{23} \\ &= \hat{U}_{23}^* \hat{U}_{13}^*(1 \otimes 1 \otimes a)\hat{U}_{13}\hat{U}_{23} \\ &= (\hat{\Delta}^{cop} \otimes id)(\hat{U}^*)(1 \otimes 1 \otimes a)(\hat{\Delta}^{cop} \otimes id)(\hat{U}) \\ &= (\hat{\Delta}^{cop} \otimes id)(\hat{U}^*(1 \otimes a)\hat{U}) \\ &= (\hat{\Delta}^{cop} \otimes id)(Ad_{\mathbb{G}}(a)), \end{aligned}$$

for all  $a \in A$ , and we know that

$$Ad_{\mathbb{G}}(A)(\widehat{A} \otimes 1) = \widehat{U}^*(1 \otimes A)\widehat{U}(\widehat{A} \otimes 1) = \widehat{A} \otimes A.$$

This implies that  $Ad_{\mathbb{G}}$  is a left action of  $\widehat{\mathbb{G}}$  on  $A$ , we will call it the left adjoint action of  $\mathbb{G}$ .

**Example 2.6.5.** Let  $W$  be a unitary corepresentation of  $\mathbb{G}$  on a non-degenerate \*-algebra  $X$ . Because  $W$  is unitary, the linear map defined by

$$\begin{aligned} \alpha_W : X &\rightarrow \mathcal{M}(A \otimes X) \\ x &\mapsto W^*(1 \otimes x)W \end{aligned}$$

is injective. Moreover, we have

$$\begin{aligned} (id \otimes \alpha_W)(\alpha_W(x)) &= W_{23}^* \Sigma_{12} (W_{23}^*(1 \otimes 1 \otimes x)W_{23})W_{23} \\ &= W_{23}^* W_{13}^* (1 \otimes 1 \otimes x)W_{13}W_{23} \\ &= (\Delta \otimes id)(W^*(1 \otimes x)W) \\ &= (\Delta \otimes id)(\alpha_W(x)), \end{aligned}$$

for all  $x \in X$  and it follows from the equality

$$(1 \otimes X)W(A \otimes 1) = A \otimes X$$

that

$$\alpha_W(X)(A \otimes 1) = W^*(1 \otimes X)W(A \otimes 1) = W^*(A \otimes X) = A \otimes X.$$

This implies that  $\alpha_W$  is a left action of  $\mathbb{G}$  on  $X$ .

**Remark 57.** Let  $\alpha : X \rightarrow \mathcal{M}(A \otimes X)$  be a left action of  $\mathbb{G}$  on a non-degenerate \*-algebra  $X$ . For any non-zero functional  $\mu$  on  $X$  and any  $x \in X$ , we define  $(id \otimes \mu)(\alpha(x))$  as the element in  $\mathcal{M}(A)$  such that

$$[(id \otimes \mu)(\alpha(x))]a := (id \otimes \mu)(\alpha(x)(a \otimes 1)), \quad a[(id \otimes \mu)(\alpha(x))] := (id \otimes \mu)((a \otimes 1)\alpha(x))$$

for every  $a \in A$ .

**Definition 2.6.6.** Let  $\alpha : X \rightarrow \mathcal{M}(A \otimes X)$  be a left action of  $\mathbb{G}$  on a non-degenerate \*-algebra  $X$  and  $\mu$  be a non-zero functional on  $X$ . We say that  $\mu$  is invariant under  $\alpha$  or  $\alpha$ -invariant if

$$(id \otimes \mu)(\alpha(x)) = \mu(x)1_{\mathcal{M}(A)},$$

for all  $x \in X$ .

**Remark 58.**

- Let  $\mathcal{V}$  be vector space over  $\mathbb{C}$  with inner product. A operator  $T : \mathcal{V} \rightarrow \mathcal{V}$  is said to be *adjointable* if there exists an operator  $T^* : \mathcal{V} \rightarrow \mathcal{V}$  such that

$$\langle T(\eta), \xi \rangle = \langle \eta, T^*(\xi) \rangle,$$

for all  $\xi, \eta \in \mathcal{V}$ . It is not hard to show that every adjointable operator  $T$  is necessarily linear and that  $T^*$  is unique and adjointable with  $(T^*)^* = T$ . We use the following notation

- $\mathcal{L}(\mathcal{V})$  denotes the \*-algebra of all adjointable operators on  $\mathcal{V}$ ,
- $\mathcal{B}(\mathcal{V})$  denotes the \*-algebra of all bounded adjointable operators on  $\mathcal{V}$ .
- $\mathcal{U}(\mathcal{V}) = \{T \in \mathcal{B}(\mathcal{V}) : T \circ T^* = T^* \circ T = id_{\mathcal{V}}\}$  denotes the \*-algebra of all unitary operators on  $\mathcal{V}$ .

We always have  $\mathcal{U}(\mathcal{V}) \subseteq \mathcal{B}(\mathcal{V}) \subseteq \mathcal{L}(\mathcal{V})$ , with  $\mathcal{B}(\mathcal{V})$  and  $\mathcal{L}(\mathcal{V})$  coinciding when  $\mathcal{V}$  is a Hilbert space.

- If we have a non-zero positive faithful functional  $\tau$  on a \*-algebra  $Y$ , the map

$$\begin{aligned} \langle \cdot, \cdot \rangle_{\tau} : Y \otimes Y &\rightarrow \mathbb{C} \\ y \otimes y' &\mapsto \tau(y^*y') \end{aligned}$$

defines an inner product on  $Y$ .

**Proposition 2.6.7.** *Let  $\alpha : X \rightarrow \mathcal{M}(A \otimes X)$  be a left action of  $\mathbb{G}$  on a non-degenerate \*-algebra  $X$  and  $\mu$  be a  $\alpha$ -invariant functional on  $X$ . Then, the linear map*

$$\begin{aligned} U_{\mu}^{\alpha} : A \otimes X &\rightarrow A \otimes X \\ a \otimes x &\mapsto \alpha(x)(a \otimes 1) \end{aligned}$$

satisfies

- (1)  $\langle U_{\mu}^{\alpha}(a \otimes x), U_{\mu}^{\alpha}(b \otimes y) \rangle_{\varphi \otimes \mu} = \langle a \otimes x, b \otimes y \rangle_{\varphi \otimes \mu}$  for all  $a, b \in A$  and  $x, y \in X$ .
- (2)  $U^{\alpha}$  is unitary with  $(U_{\mu}^{\alpha})^{\dagger} = (U_{\mu}^{\alpha})^{-1}$ .
- (3)  $\alpha(x) = U_{\mu}^{\alpha}(id_A \otimes \pi_X(x))(U_{\mu}^{\alpha})^{\dagger}$  in  $End(A \otimes X)$ , for all  $x \in X$ .

Here

$$\pi_X : X \rightarrow End(X), \quad x \mapsto [\pi_X(x) : y \mapsto xy].$$

*Proof*

- (1) The invariance of  $\mu$  implies that

$$\begin{aligned} \langle U^{\alpha}(a \otimes x), U^{\alpha}(b \otimes y) \rangle_{\varphi \otimes \mu} &= \langle \alpha(x)(a \otimes 1), \alpha(y)(b \otimes 1) \rangle_{\varphi \otimes \mu} \\ &= (\varphi \otimes \mu)((a^* \otimes 1)\alpha(x^*y)(b \otimes 1)) \\ &= \varphi(\mu(x^*y)a^*b) \\ &= \langle a \otimes x, b \otimes y \rangle_{\varphi \otimes \mu} \end{aligned}$$

for all  $a, b \in A$  and  $x, y \in X$ .

- (2) Because  $\alpha(x)(A \otimes 1_{\mathcal{M}(X)}) = A \otimes X$ , if  $a \otimes x = \alpha(y)(b \otimes 1) = U^{\alpha}(b \otimes y)$ , then it is not hard to show that  $R(b) \otimes y = (R(a) \otimes 1)\alpha(x)$ . This implies that

$$b \otimes y = (R^{-1} \otimes id)(\alpha(x))(a \otimes 1),$$



and we have that

$$\begin{aligned} (U^\alpha)^{-1} : A \otimes X &\rightarrow A \otimes X \\ a \otimes x &\mapsto (R^{-1} \otimes id)(\alpha(x))(a \otimes 1_{\mathcal{M}(X)}) \end{aligned} .$$

Moreover

$$\begin{aligned} \langle U_\mu^\alpha(a \otimes x), b \otimes y \rangle_{\varphi \otimes \mu} &= \langle U_\mu^\alpha(a \otimes x), U_\mu^\alpha((U_\mu^\alpha)^{-1}(b \otimes y)) \rangle_{\varphi \otimes \mu} \\ &= \langle a \otimes x, (U_\mu^\alpha)^{-1}(b \otimes y) \rangle_{\varphi \otimes \mu} \end{aligned}$$

for all  $a, b \in A$  and  $x, y \in X$ . This implies that  $(U_\mu^\alpha)^\dagger = (U_\mu^\alpha)^{-1}$ .

(3) Fix  $x \in X$ , the result follows because we have

$$\alpha(x)U^\alpha(a \otimes y) = \alpha(x)\alpha(y)(a \otimes 1) = \alpha(xy)(a \otimes 1) = U^\alpha(id_A \otimes \pi_X(x))(a \otimes y)$$

for all  $a \in A$  and  $y \in X$ .

□

**Definition 2.6.8.** Let  $\alpha : X \rightarrow \mathcal{M}(A \otimes X)$  be a left action of  $\mathbb{G}$  on a non-degenerate \*-algebra  $X$  and  $\mu$  be a  $\alpha$ -invariant functional on  $X$ . The linear map  $U_\mu^\alpha$  defines as above it will be call the *standard implementation of  $\alpha$  associated with  $\mu$* .

**Remark 59.** Let  $X^{op}$  be the opposite \*-algebra of  $X$ . We recall that if  $\mu$  is a non-zero faithful linear functional on  $X$ , the linear map  $\mu^{op} : X^{op} \rightarrow \mathbb{C}$ , defined by  $\mu^{op}(x^{op}) = \mu(x)$  for all  $x \in X$ , is a non-zero faithful linear functional on  $X^{op}$ , this functional will be called *the opposite functional of  $\mu$* .

**Definition 2.6.9.** Let  $\alpha : X \rightarrow \mathcal{M}(A \otimes X)$  be a left action of an algebraic quantum group  $\mathbb{G} = (A, \Delta, \varphi)$  on a non-degenerate \*-algebra  $X$ . Consider the map

$$\begin{aligned} \alpha^\circ : X^{op} &\rightarrow \mathcal{M}(A \otimes X^{op}) \\ x^{op} &\mapsto (R \otimes {}^{op})(\alpha(x)) \end{aligned} ,$$

i.e.,

$$\begin{aligned} \alpha^\circ(x^{op})(a \otimes y^{op}) &:= (R \otimes {}^{op})((R^{-1}(a) \otimes y)\alpha(x)), \\ (a \otimes y^{op})\alpha^\circ(x^{op}) &:= (R \otimes {}^{op})(\alpha(x)(R^{-1}(a) \otimes y)), \end{aligned}$$

for all  $a \in A$  and  $x, y \in X$ .

**Lemma 2.6.10.** *The map  $\alpha^\circ : X^{op} \rightarrow \mathcal{M}(A \otimes X^{op})$  is a left action of the algebraic quantum group  $\mathbb{G}^\circ := (A, \Delta^{cop}, \varphi \circ S)$  on  $X^{op}$ .*

*Proof* We have

$$\begin{aligned}
 (id_{\mathcal{M}(A)} \otimes \alpha^\circ) \circ \alpha^\circ &= (id_{\mathcal{M}(A)} \otimes (R \otimes {}^{op}) \circ \alpha \circ {}^{op}) \circ (R \otimes {}^{op}) \circ \alpha \circ {}^{op} \\
 &= (R \otimes R \otimes {}^{op}) \circ (id_{\mathcal{M}(A)} \otimes \alpha) \circ \alpha \circ {}^{op} \\
 &= ((R \otimes R) \circ \Delta \otimes {}^{op}) \circ \alpha \circ {}^{op} \\
 &= (\Delta^{cop} \circ R \otimes {}^{op}) \circ \alpha \circ {}^{op} \\
 &= (\Delta^{cop} \otimes id_{X^{op}}) \circ (R \otimes {}^{op}) \circ \alpha \circ {}^{op} \\
 &= (\Delta^{cop} \otimes id_{X^{op}}) \circ \alpha^\circ
 \end{aligned}$$

and

$$\begin{aligned}
 \alpha^\circ(X^{op})(A \otimes 1_{\mathcal{M}(X^{op})}) &= (R \otimes {}^{op})(\alpha(X))(A \otimes 1_{\mathcal{M}(X^{op})}) \\
 &= (R \otimes {}^{op})((R^{-1}(A) \otimes 1_{\mathcal{M}(X)})\alpha(X)) \\
 &= (R \otimes {}^{op})(A \otimes X) \\
 &= A \otimes X^{op}.
 \end{aligned}$$

□

It is easy to prove the next lemma.

**Lemma 2.6.11.** *Let  $\alpha$  be a left action of  $\mathbb{G}$  on  $X$  which is equipped with a  $\alpha$ -invariant functional and  $U_\mu^\alpha$  be the standard implementation of  $\alpha$  associated with  $\mu$ . Then*

- (i)  $(id \otimes {}^{op})((U_\mu^\alpha)^\dagger)$  is the standard implementation of the left action  $\alpha^\circ$  on  $X^{op}$ .
- (ii)  $\mathcal{M}(\mathbb{G} \ltimes_\alpha X)^\circ = U_\mu^\alpha \mathcal{M}(\mathbb{G}^\circ \ltimes_{\alpha^\circ} X^\circ)(U_\mu^\alpha)^\dagger$  in  $\mathcal{M}(H\mathbb{G}^\circ \otimes X^\circ)$ .

## 2.7 Crossed product for actions of algebraic quantum groups

Let  $\mathbb{G} = (A, \Delta, \varphi)$  be an algebraic quantum group and  $X$  be a  $\mathbb{G}$ -\*-algebra with left action  $\alpha : X \rightarrow \mathcal{M}(A \otimes X)$ . If we consider the dual action associated with  $\alpha$ ,

$$\begin{aligned}
 \triangleleft_\alpha : X \otimes \widehat{A} &\rightarrow X \\
 x \otimes \omega &\mapsto (\omega \otimes id)(\alpha(x))
 \end{aligned}$$

we can construct the non-degenerate \*-algebra  $\widehat{A} \# X$  called the *smash product of  $\widehat{A}$  and  $X$*  as the vector space  $\widehat{A} \otimes X$  with the following \*-algebra structure:

$$\begin{aligned}
 (\omega \# x)(\theta \# y) &= \omega \cdot \theta_{(1)} \# (x \triangleleft_\alpha \theta_{(2)})y = \omega \cdot \theta_{x_{[-1]}} \# x_{[0]}y. \\
 (\omega \# x)^* &= \omega_{(1)}^* \# (x^* \triangleleft_\alpha \omega_{(2)}^*)
 \end{aligned} \tag{2.1}$$

**Remark 60.** The equation 2.1 makes sense, because if  $\omega = \varphi_a \in \widehat{A}$  and  $x \in X$ , we have

$$\alpha(x)(a \otimes 1) = x_{[-1]}a \otimes x_{[0]} \in A \otimes X$$

and then the equality  $\omega_{x_{[-1]}}(b) := \varphi_{x_{[-1]}a}(b)$ , for all  $b \in A$ , shows that

$$\omega_{x_{[-1]}} \# x_{[0]} = \varphi_{x_{[-1]}a} \# x_{[0]} \in \widehat{A} \# X.$$

**Notation 12.** Sometimes, we will use the shorthand notations

$$x\omega := \omega \# x, \quad x\omega := \omega_{x_{[-1]}} \# x_{[0]}.$$

**Remark 61.** For each  $x \in X$ ,  $\omega \in \widehat{A}$ , we have  $\alpha(x) \in \mathcal{M}(A \otimes X) \subseteq \mathcal{M}(\widehat{A}A \otimes X)$ , and  $(\omega \otimes 1_{\mathcal{M}(X)}) \in \mathcal{M}(\widehat{A} \otimes X) \subseteq \mathcal{M}(\widehat{A}A \otimes X)$ . Moreover, there exists a unital faithful non-degenerate right action

$$\begin{aligned} \triangleleft : (A \otimes X) \otimes (\widehat{A}A \otimes X) &\rightarrow A \otimes X \\ (b \otimes y) \otimes (\omega a \otimes x) &\mapsto (b \triangleleft_{\widehat{A}} \omega) a \otimes yx \end{aligned}$$

such that

$$(a \otimes y) \triangleleft \alpha(x) = (a \otimes y)\alpha(x),$$

for all  $a \in A$  and  $x, y \in X$ .

**Lemma 2.7.1.** For all  $x \in X$ ,  $\omega \in \widehat{A}$ ,

$$\alpha(x)(\omega \otimes 1) = (\omega_{x_{[-1]}} \otimes 1)\alpha(x_{[0]})$$

in  $\mathcal{M}(A\widehat{A} \otimes X)$ .

*Proof* Fix  $\omega \in \widehat{A}$  and  $x \in X$ , then

$$\begin{aligned} (a \otimes y) \triangleleft \alpha(x)(\omega \otimes 1_{\mathcal{M}(X)}) &= (ax_{[-1]} \otimes yx_{[0]}) \triangleleft (\omega \otimes 1_{\mathcal{M}(X)}) \\ &= (\omega \otimes id)(\Delta(ax_{[-1]})) \otimes yx_{[0]} \\ &= \omega(a_{(1)}x_{[-1](1)})a_{(2)}x_{[-1](2)} \otimes yx_{[0]} \\ &= \omega(a_{(1)}x_{[-1]})a_{(2)}x_{[0]_{[-1]}} \otimes yx_{[0]_{[0]}} \\ &= ((\omega_{x_{[-1]}} \otimes id)(\Delta(a)) \otimes y) \triangleleft \alpha(x_{[0]}) \\ &= (a \otimes y) \triangleleft (\omega_{x_{[-1]}} \otimes 1_{\mathcal{M}(X)})\alpha(x_{[0]}), \end{aligned}$$

for all  $a \in A$  and  $y \in X$ . As the action of  $\mathcal{M}(\widehat{A}A \otimes X)$  on  $A \otimes X$  is faithful, we have the result.  $\square$

**Lemma 2.7.2.** For all  $x \in X$ ,  $\omega \in \widehat{A}$ ,

$$\alpha(x_{[0]})(\omega_{S^{-1}(x_{[-1]})} \otimes 1_{\mathcal{M}(X)}) = (\omega \otimes 1_{\mathcal{M}(X)})\alpha(x),$$

in  $\mathcal{M}(A\widehat{A} \otimes X)$ .

*Proof* Fix  $\omega = \varphi_a \in \widehat{A}$  with  $a \in A$  and  $x \in X$ , we have

$$\begin{aligned}
 \alpha(x_{[0]})(\omega_{S^{-1}(x_{[-1]})} \otimes 1_{\mathcal{M}(X)}) &= \alpha(x_{[0]})(\varphi_{S^{-1}(x_{[-1]})a} \otimes 1_{\mathcal{M}(X)}) \\
 &= ((\varphi_{S^{-1}(x_{[-1]})a})_{x_{[0]_{[-1]}}} \otimes 1_{\mathcal{M}(X)})\alpha(x_{[0]_{[0]}}) \\
 &= (\varphi_{(x_{[0]_{[-1]})}S^{-1}(x_{[-1]})a} \otimes 1_{\mathcal{M}(X)})\alpha(x_{[0]_{[0]}}) \\
 &= (\varphi_{(x_{[-1]_{(2)}})S^{-1}(x_{[-1]_{(1)}})a} \otimes 1_{\mathcal{M}(X)})\alpha(x_{[0]}) \\
 &= (\varphi_{\varepsilon(x_{[-1]})a} \otimes 1_{\mathcal{M}(X)})\alpha(x_{[0]}) \\
 &= (\varphi_a \otimes 1_{\mathcal{M}(X)})\alpha(x) \\
 &= (\omega \otimes 1_{\mathcal{M}(X)})\alpha(x)
 \end{aligned}$$

□

**Lemma 2.7.3.** *The vector subspace of  $\mathcal{M}(\widehat{A}A \otimes X)$  generated by*

$$\{ (\omega \otimes 1_{\mathcal{M}(X)})\alpha(x) : x \in X, \omega \in \widehat{A} \}$$

*is an associative non-degenerate \*-algebra. Moreover, if  $X$  is idempotent, then also this algebra is idempotent.*

*Proof* It is only necessary to show that the vector sub-space generated by elements of the form  $(\omega \otimes 1_{\mathcal{M}(X)})\alpha(x)$ , with  $x \in X$  and  $\omega \in \widehat{A}$ , is closed under multiplication. It follows from Lemma 2.7.1 that

$$\begin{aligned}
 (\omega \otimes 1_{\mathcal{M}(X)})\alpha(x)(\theta \otimes 1_{\mathcal{M}(X)})\alpha(y) &= (\omega \otimes 1_{\mathcal{M}(X)})(\theta_{x_{[-1]}} \otimes 1_{\mathcal{M}(X)})\alpha(x_{[0]})\alpha(y) \\
 &= (\omega \cdot \theta_{x_{[-1]}} \otimes 1_{\mathcal{M}(X)})\alpha(x_{[0]y}),
 \end{aligned}$$

for all  $\omega, \theta \in \widehat{A}$  and  $x, y \in X$ . If  $X$  is idempotent, because  $A$  is idempotent, then the algebra mentioned is idempotent. □

**Definition 2.7.4.** Let  $\alpha : X \rightarrow \mathcal{M}(A \otimes X)$  be a left action of  $\mathbb{G}$  on a non-degenerate \*-algebra  $X$ . The non-degenerate \*-algebra

$$\mathbb{G} \rtimes_{\alpha} X := \text{span}\{ (\omega \otimes 1_{\mathcal{M}(X)})\alpha(x) : x \in X, \omega \in \widehat{A} \} \subseteq \mathcal{M}(\widehat{A}A \otimes X).$$

is called *the crossed product of  $\mathbb{G}$  and  $X$* .

**Proposition 2.7.5.** *The map defined by*

$$\begin{aligned}
 \Theta : \widehat{A} \# X &\rightarrow \mathbb{G} \rtimes_{\alpha} X \\
 \omega x &\mapsto (\omega \otimes 1_{\mathcal{M}(X)})\alpha(x)
 \end{aligned}$$

*is a \*-isomorphism.*

*Proof* This map is well defined and it is bijective by the definition of the crossed product. It follows from the definition of the smash product, equality (2.1), and Lemma 2.7.3, that the map  $\tau$  is a homomorphism of \*-algebras. □

**Remark 62.** Fix  $x \in X$ . By simple calculations, we see that

$$\begin{aligned}\alpha(x)(\omega \otimes 1_{\mathcal{M}(X)})\alpha(y) &= (\omega_{x_{[-1]}} \otimes 1_{\mathcal{M}(X)})\alpha(x_{[0]}y), \\ (\omega \otimes 1_{\mathcal{M}(X)})\alpha(y)\alpha(x) &= (\omega \otimes 1_{\mathcal{M}(X)})\alpha(yx),\end{aligned}$$

for all  $\omega \in \widehat{A}$  and  $y \in X$ , then  $\alpha(x)$  can be regarded as an element of  $\mathcal{M}(\mathbb{G} \ltimes_{\alpha} X)$ .

### 2.7.1 Dual actions

Let  $\alpha : X \rightarrow \mathcal{M}(A \otimes X)$  be a left action of the algebraic quantum group  $\mathbb{G} = (A, \Delta, \varphi)$  on a non-degenerate \*-algebra  $X$  and  $V$  be the duality of  $\mathbb{G}$ . Consider the map

$$\begin{aligned}\tilde{\alpha} : \mathbb{G} \ltimes_{\alpha} X &\rightarrow \mathcal{M}(\widehat{A} \otimes (\mathbb{G} \ltimes_{\alpha} X)) \\ P &\mapsto V_{12}(1_{\mathcal{M}(\widehat{A})} \otimes P)V_{12}^*\end{aligned}$$

**Proposition 2.7.6.** *The map  $\tilde{\alpha}$  is a left action of the algebraic quantum group  $\widehat{\mathbb{G}}$  on the non-degenerate \*-algebra  $\mathbb{G} \ltimes_{\alpha} X$ , such that its extension to multiplier algebras satisfies*

- (i)  $\tilde{\alpha}(\alpha(x)) = 1_{\mathcal{M}(\widehat{A})} \otimes \alpha(x)$  for all  $x \in X$ ,
- (ii)  $\tilde{\alpha}(\omega \otimes 1_{\mathcal{M}(X)}) = \widehat{\Delta}^{cop}(\omega) \otimes 1_{\mathcal{M}(X)}$  for all  $\omega \in \widehat{A}$ ,
- (iii)  $(\mathbb{G} \ltimes_{\alpha} X)^{\tilde{\alpha}} = \alpha(X)$ .

*Proof* Fix  $P \in \mathbb{G} \ltimes_{\alpha} X$ , then

$$\begin{aligned}(id \otimes \tilde{\alpha})(\tilde{\alpha}(P)) &= V_{23}\Sigma_{12}(V_{23}(1 \otimes 1 \otimes P)V_{23}^*)V_{23}^* \\ &= V_{23}V_{13}(1 \otimes 1 \otimes P)V_{13}^*V_{23}^* \\ &= (\widehat{\Delta}^{cop} \otimes id \otimes id)(V_{12})(\widehat{\Delta}^{cop} \otimes id \otimes id)(1 \otimes P)(\widehat{\Delta}^{cop} \otimes id \otimes id)(V_{12}^*) \\ &= (\widehat{\Delta}^{cop} \otimes id \otimes id)(V_{12}(1 \otimes P)V_{12}^*) \\ &= (\widehat{\Delta}^{cop} \otimes id \otimes id)(\tilde{\alpha}(P)).\end{aligned}$$

(i) Consider the non-degenerate faithful left action

$$\begin{aligned}\lambda \triangleleft_{\pi} : (A \otimes A \otimes X) \otimes (\widehat{A} \otimes \widehat{A}A \otimes X) &\rightarrow A \otimes A \otimes X \\ (b \otimes c \otimes y) \otimes (\omega \otimes \theta a \otimes x) &\mapsto (b \triangleleft_{\widehat{A}} \omega) \otimes (c \triangleleft_{\widehat{A}} \theta)a \otimes yx\end{aligned}$$

and its extension to multiplier algebras. We have

$$(b \otimes c \otimes y) \lambda \triangleleft_{\pi} (\omega \otimes a \otimes 1_{\mathcal{M}(X)}) = (b \triangleleft_{\widehat{A}} \omega) \otimes ca \otimes y = ((b \otimes c) \triangleleft (\omega \otimes a)) \otimes y,$$

then

$$(b \otimes c \otimes y) \lambda \triangleleft_{\pi} V_{12} = ((b \otimes c) \triangleleft V) \otimes y = b_{(1)} \otimes b_{(2)}c \otimes y.$$

Fix  $x \in X$ , then for all  $a, b \in A$  and  $y \in X$  we have

$$\begin{aligned} (b \otimes a \otimes y) \lambda \triangleleft_{\pi} ((1_{\mathcal{M}(\widehat{A})} \otimes \alpha(x))V_{12}) &= (b \otimes ax_{[-1]} \otimes yx_{[0]}) \lambda \triangleleft_{\pi} V_{12} \\ &= b_{(1)} \otimes b_{(2)}ax_{[-1]} \otimes yx_{[0]} \end{aligned}$$

and

$$\begin{aligned} (b \otimes a \otimes y) \lambda \triangleleft_{\pi} (V_{12}(1_{\mathcal{M}(\widehat{A})} \otimes \alpha(x))) &= (b_{(1)} \otimes b_{(2)}a \otimes y) \lambda \triangleleft_{\pi} (1_{\mathcal{M}(\widehat{A})} \otimes \alpha(x)) \\ &= b_{(1)} \otimes b_{(2)}ax_{[-1]} \otimes yx_{[0]}. \end{aligned}$$

$$\begin{aligned} (b \otimes a \otimes y) \lambda \triangleleft_{\pi} ((1_{\mathcal{M}(\widehat{A})} \otimes \alpha(x))) &= (b_{(1)} \otimes b_{(2)}a \otimes y) \lambda \triangleleft_{\pi} (1_{\mathcal{M}(\widehat{A})} \otimes \alpha(x)) \\ &= b_{(1)} \otimes b_{(2)}ax_{[-1]} \otimes yx_{[0]}. \end{aligned}$$

This implies

$$\tilde{\alpha}(\alpha(x)) = V_{12}(1_{\mathcal{M}(\widehat{A})} \otimes \alpha(x))V_{12}^* = 1_{\mathcal{M}(\widehat{A})} \otimes \alpha(x).$$

(ii) We have

$$\begin{aligned} \tilde{\alpha}(\omega \otimes 1_{\mathcal{M}(X)}) &= V_{12}(1_{\mathcal{M}(\widehat{A})} \otimes \omega \otimes 1_{\mathcal{M}(X)})V_{12}^* \\ &= V(1_{\mathcal{M}(\widehat{A})} \otimes \omega)V^* \otimes 1_{\mathcal{M}(X)} \\ &= \widehat{\Delta}^{cop}(\omega) \otimes 1_{\mathcal{M}(X)}, \end{aligned}$$

for all  $\omega \in \widehat{A}$ .

(iii) Follows from the items (i) and (ii). □

**Definition 2.7.7.** The action  $\tilde{\alpha} : \mathbb{G} \ltimes_{\alpha} X \rightarrow \mathcal{M}(\widehat{A} \otimes (\mathbb{G} \ltimes_{\alpha} X))$  defined above is called *the dual action of  $\alpha$* .

**Remark 63.** We have

$$\begin{aligned} \tilde{\alpha}((\omega \otimes 1_{\mathcal{M}(X)})\alpha(x)) &= \tilde{\alpha}(\omega \otimes 1_{\mathcal{M}(X)})\tilde{\alpha}(\alpha(x)) \\ &= (\omega_{(2)} \otimes \omega_{(1)} \otimes 1_{\mathcal{M}(X)})(1 \otimes \alpha(x)) \\ &= \omega_{(2)} \otimes (\omega_{(1)} \otimes 1_{\mathcal{M}(X)})\alpha(x), \end{aligned}$$

for all  $\omega \in \widehat{A}$  and  $x \in X$ , then the left action of  $\widehat{\mathbb{G}}$  on  $\widehat{A} \# X$  is defined by

$$\begin{aligned} \bar{\alpha} : \widehat{A} \# X &\rightarrow \mathcal{M}(\widehat{A} \otimes \widehat{A} \# X) \\ \omega x &\mapsto \omega_{(2)} \otimes \omega_{(1)}x \end{aligned}$$

**Corollary 2.7.8.** *The dual action of  $\tilde{\alpha}$  is given by*

$$\begin{aligned} \triangleleft_{\tilde{\alpha}} : \mathbb{G} \ltimes_{\alpha} X \otimes A &\rightarrow \mathbb{G} \ltimes_{\alpha} X \\ (\omega \otimes 1_{\mathcal{M}(X)})\alpha(x) \otimes a &\mapsto (\omega_a \otimes 1_{\mathcal{M}(X)})\alpha(x) \end{aligned}$$

*Proof* We have

$$\begin{aligned}
 (\omega \otimes id_X)\alpha(x) \triangleleft_{\tilde{\alpha}} a &= (a \otimes id)[\tilde{\alpha}((\omega \otimes 1_{\mathcal{M}(X)})\alpha(x))] \\
 &= (a \otimes id)[\tilde{\alpha}((\omega \otimes 1_{\mathcal{M}(X)})\tilde{\alpha}((\alpha(x))))] \\
 &= (a \otimes id)[(\widehat{\Delta}^{cop}(\omega) \otimes 1_{\mathcal{M}(X)})(1_{\mathcal{M}(\widehat{A})} \otimes \alpha(x))] \\
 &= ((a \otimes id)(\widehat{\Delta}^{cop}(\omega)) \otimes 1_{\mathcal{M}(X)})\alpha(x) \\
 &= (\omega_a \otimes 1_{\mathcal{M}(X)})\alpha(x),
 \end{aligned}$$

for all  $a \in A$ ,  $\omega \in \widehat{A}$  and  $x \in X$ . Here, because

$$p_A(b, \omega_a) = p_A(ba, \omega) = p_A(b, a \blacktriangleright_A \omega)$$

for all  $b \in A$ , we have the equality

$$\omega_a = a \blacktriangleright_A \omega = (id \otimes a)(\widehat{\Delta}(\omega)) = (a \otimes id)(\widehat{\Delta}^{cop}(\omega)).$$

□

**Proposition 2.7.9.** *The map  $T_{\tilde{\alpha}} := (\widehat{\varphi} \otimes id) \circ \tilde{\alpha} : \mathbb{G} \rtimes_{\alpha} X \rightarrow \alpha(X)$  is an injective map of  $\alpha(X)$ -bimodules and the map  $\phi_{\alpha} := \alpha^{-1} \circ T_{\tilde{\alpha}}$  is an injective linear map from  $\mathbb{G} \rtimes_{\alpha} X$  to  $X$ .*

*Proof* The injectivity of  $T_{\tilde{\alpha}}$  follows from the injectivity of  $\tilde{\alpha}$  and the faithfulness of  $\widehat{\varphi}$ . Fix  $x, y \in X$ , because

$$\begin{aligned}
 T_{\tilde{\alpha}}(\alpha(x)P\alpha(y)) &= (\widehat{\varphi} \otimes id)(\tilde{\alpha}(\alpha(x))\tilde{\alpha}(P)\tilde{\alpha}(\alpha(y))) \\
 &= (\widehat{\varphi} \otimes id)((1 \otimes \alpha(x))\tilde{\alpha}(P)(1 \otimes \alpha(y))) \\
 &= \alpha(x)(\widehat{\varphi} \otimes id)(\tilde{\alpha}(P))\alpha(y) \\
 &= \alpha(x)T_{\tilde{\alpha}}(P)\alpha(y),
 \end{aligned}$$

for all  $P \in \mathbb{G} \rtimes_{\alpha} X$ , we can extend this equality to the algebra  $\mathcal{M}(\mathbb{G} \rtimes_{\alpha} X)$ . Finally, because  $\alpha$  is an injective homomorphism, we have that  $\phi_{\alpha}$  is a injective linear map. □

**Definition 2.7.10.** Let  $\mu : X \rightarrow \mathbb{C}$  be a non-zero invariant faithful positive functional under the left action  $\alpha : X \rightarrow \mathcal{M}(A \otimes X)$  of  $\mathbb{G}$  on  $X$ . The map  $\tilde{\mu} : \mathbb{G} \rtimes_{\alpha} X \rightarrow \mathbb{C}$  defined by  $\tilde{\mu} := \mu \circ \phi_{\alpha}$  is a non-zero faithful positive linear functional on  $\mathbb{G} \rtimes_{\alpha} X$  called *the dual functional of  $\mu$  by the action  $\alpha$* .

## 2.8 Yetter-Drinfeld \*-algebras over algebraic quantum groups

In this section, we give a definition of a Yetter-Drinfeld \*-algebras close to the well known definition in the operator algebra context, for example see [NV09, ET16]. Using the algebraic multiplicative unitary associated with an algebraic quantum group, we prove that this definition is equivalent to the purely algebraic one, for example see [BM02, NY14], and so we have similar results as in this framework.

**Definition 2.8.1.** Let  $\mathbb{G} = (A, \Delta, \varphi)$  be an algebraic quantum group and  $U$  its algebraic multiplicative unitary. A  $\mathbb{G}$ -Yetter-Drinfeld \*-algebra is a triple  $(X, \alpha, \hat{\alpha})$ , where  $X$  is a non-degenerate \*-algebra,  $\alpha$  is a left action of  $\mathbb{G}$  on  $X$  and  $\hat{\alpha}$  is a left action of  $\hat{\mathbb{G}}$  on  $X$ , such that

$$(id_{\mathcal{M}(\hat{A})} \otimes \alpha) \circ \hat{\alpha} = (\Sigma \otimes id_X) \circ (\text{Ad}(U) \otimes id_X) \circ (id_{\mathcal{M}(A)} \otimes \hat{\alpha}) \circ \alpha. \quad (2.2)$$

Equivalently, the diagram

$$\begin{array}{ccccc} X & \xrightarrow{\hat{\alpha}} & \mathcal{M}(\hat{A} \otimes X) & \xrightarrow{id_{\mathcal{M}(\hat{A})} \otimes \alpha} & \mathcal{M}(\hat{A} \otimes A \otimes X) \\ \downarrow \alpha & & & & \uparrow \Sigma \otimes id_X \\ \mathcal{M}(A \otimes X) & \xrightarrow{id_{\mathcal{M}(A)} \otimes \hat{\alpha}} & \mathcal{M}(A \otimes \hat{A} \otimes X) & \xrightarrow{\text{Ad}(U) \otimes id_X} & \mathcal{M}(A \otimes \hat{A} \otimes X) \end{array}$$

must be commutative. A  $\mathbb{G}$ -Yetter-Drinfeld \*-algebra  $(X, \alpha, \hat{\alpha})$  is called *reduced* if  $\alpha$  and  $\hat{\alpha}$  are reduced actions.

**Proposition 2.8.2.** *We have that,  $(X, \alpha, \hat{\alpha})$  is a  $\mathbb{G}$ -Yetter-Drinfeld \*-algebra if and only if  $(X, \hat{\alpha}, \alpha)$  is a  $\hat{\mathbb{G}}$ -Yetter-Drinfeld \*-algebra.*

*Proof* We know that

$$\text{Ad}(U^*) \circ \Sigma^{-1} = \Sigma^{-1} \circ \text{Ad}(\Sigma(U^*)) = \Sigma^{-1} \circ \text{Ad}(\hat{U}),$$

then

$$(id_{\mathcal{M}(\hat{A})} \otimes \alpha) \circ \hat{\alpha} = (\Sigma \otimes id_X) \circ (\text{Ad}(U) \otimes id_X) \circ (id_{\mathcal{M}(A)} \otimes \hat{\alpha}) \circ \alpha.$$

if and only if

$$(\Sigma^{-1} \otimes id_X) \circ (\text{Ad}(\hat{U}) \otimes id_X) \circ (id_{\mathcal{M}(\hat{A})} \otimes \alpha) \circ \hat{\alpha} = (id_{\mathcal{M}(A)} \otimes \hat{\alpha}) \circ \alpha.$$

□

**Proposition 2.8.3.** *Let  $\mathbb{G}$  be an algebraic quantum group. The following conditions are equivalent:*

- (i)  $(X, \alpha, \hat{\alpha})$  is a  $\mathbb{G}$ -Yetter-Drinfeld \*-algebra;
- (ii)  $\alpha_D := (id_{\mathcal{M}(A)} \otimes \hat{\alpha}) \circ \alpha$  is a left action of  $D(\mathbb{G})$  on the \*-algebra  $X$ .

*Proof*



- (i)  $\Rightarrow$  (ii): We have

$$\begin{aligned}
 (id \otimes \alpha_D) \circ \alpha_D &= (id \otimes id \otimes id \otimes \hat{\alpha}) \circ (id \otimes id \otimes \alpha) \circ (id \otimes \hat{\alpha}) \circ \alpha \\
 &= (id \otimes id \otimes id \otimes \hat{\alpha}) \circ (id \otimes \Sigma \otimes id) \circ \text{Ad}(U_{23}) \circ (id \otimes id \otimes \hat{\alpha}) \circ (id \otimes \alpha) \circ \alpha \\
 &= (id \otimes \Sigma \otimes id \otimes id) \circ \text{Ad}(U_{23}) \circ (id \otimes id \otimes \hat{\alpha}) \circ (id \otimes id \otimes \hat{\alpha}) \circ (\Delta \otimes id) \circ \alpha \\
 &= (id \otimes \Sigma \otimes id \otimes id) \circ \text{Ad}(U_{23}) \circ (\Delta \otimes \hat{\Delta}^{cop} \otimes id) \circ (id \otimes \hat{\alpha}) \circ \alpha \\
 &= (id \otimes \Sigma \otimes id \otimes id) \circ \text{Ad}(U_{23}) \circ (\Delta \otimes \hat{\Delta}^{cop} \otimes id) \circ \alpha_D \\
 &= (\Delta_D \otimes id) \circ \alpha_D.
 \end{aligned}$$

This means that  $\alpha_D$  is a left coaction of  $D(\mathbb{G})$  on  $X$ .

- (ii)  $\Rightarrow$  (i): If we denote

$$\lambda := (id_{\mathcal{M}(A)} \otimes \Sigma_U) \circ (\Delta \otimes id_{\mathcal{M}(\hat{A})}) = (id_{\mathcal{M}(A)} \otimes \Sigma) \circ \text{Ad}(U_{23}) \circ (\Delta \otimes id_{\mathcal{M}(\hat{A})})$$

and  $\hat{\lambda} := id_{\mathcal{M}(A)} \otimes \hat{\Delta}^{cop}$ , then

$$\begin{aligned}
 (id_{\mathcal{M}(A \otimes \hat{A})} \otimes \alpha) \circ \alpha_D &= (id_{\mathcal{M}(A_D)} \otimes \alpha) \circ (id_{\mathcal{M}(A)} \otimes \hat{\alpha}) \circ \alpha \\
 &= (id_{\mathcal{M}(A)} \otimes (id_{\mathcal{M}(\hat{A})} \otimes \alpha) \circ \hat{\alpha}) \circ \alpha \\
 &= (id_{\mathcal{M}(A)} \otimes (\Sigma \otimes id_X) \circ (\text{Ad}(U) \otimes id_X) \circ (id_{\mathcal{M}(A)} \otimes \hat{\alpha}) \circ \alpha) \circ \alpha \\
 &= (id_{\mathcal{M}(A)} \otimes \Sigma_U \otimes id_X) \circ (id_{\mathcal{M}(A)} \otimes id_{\mathcal{M}(A)} \otimes \hat{\alpha}) \circ (id_{\mathcal{M}(A)} \otimes \alpha) \circ \alpha \\
 &= (id_{\mathcal{M}(A)} \otimes \Sigma_U \otimes id_X) \circ (id_{\mathcal{M}(A)} \otimes id_{\mathcal{M}(A)} \otimes \hat{\alpha}) \circ (\Delta \otimes id_X) \circ \alpha \\
 &= (id_{\mathcal{M}(A)} \otimes \Sigma_U \otimes id_X) \circ (\Delta \otimes id_{\mathcal{M}(\hat{A})} \otimes id_X) \circ (id_{\mathcal{M}(A)} \otimes \hat{\alpha}) \circ \alpha \\
 &= (\lambda \otimes id_X) \circ \alpha_D
 \end{aligned}$$

and

$$\begin{aligned}
 (id_{\mathcal{M}(A)} \otimes \hat{\Delta}^{cop} \otimes id_X) \circ \alpha_D &= (id_{\mathcal{M}(A)} \otimes \hat{\Delta}^{cop} \otimes id_X) \circ (id_{\mathcal{M}(A)} \otimes \hat{\alpha}) \circ \alpha \\
 &= (id_{\mathcal{M}(A)} \otimes id_{\mathcal{M}(\hat{A})} \otimes \hat{\alpha}) \circ (id_{\mathcal{M}(A)} \otimes \hat{\alpha}) \circ \alpha \\
 &= (id_{\mathcal{M}(A \otimes \hat{A})} \otimes \hat{\alpha}) \circ \alpha_D.
 \end{aligned}$$

Now we have

$$\begin{aligned}
 \text{Ad}(U_{23}) \circ (id \otimes \alpha_D) \circ \alpha_D &= \text{Ad}(U_{23}) \circ (id \otimes id \otimes \hat{\alpha}) \circ (\lambda \otimes id) \circ \alpha_D \\
 &= \text{Ad}(U_{23}) \circ (\lambda \otimes id \otimes id) \circ (id \otimes \hat{\alpha}) \circ \alpha_D \\
 &= \text{Ad}(U_{23}) \circ (\lambda \otimes id \otimes id) \circ (id \otimes \Delta^{cop} \otimes id) \circ \alpha_D \\
 &= \text{Ad}(U_{34}) \circ (id \otimes \Sigma \otimes id \otimes id) \circ \text{Ad}(U_{23}) \circ (\Delta \otimes \hat{\Delta}^{cop} \otimes id) \circ \alpha_D \\
 &= (id \otimes \Sigma \otimes id \otimes id) \circ \text{Ad}(U_{24}U_{23}) \circ (\Delta \otimes \hat{\Delta}^{cop} \otimes id) \circ \alpha_D \\
 &= (id \otimes \Sigma \otimes id \otimes id) \circ \text{Ad}((id \otimes \hat{\Delta}^{cop})(U)_{234}) \circ (\Delta \otimes \hat{\Delta}^{cop} \otimes id) \circ \alpha_D \\
 &= (id \otimes id \otimes \Sigma^{-1} \otimes id) \circ (id \otimes \hat{\Delta}^{cop} \otimes id \otimes id) \circ (id \otimes \Sigma_U \otimes id) \circ (\Delta \otimes id \otimes id) \circ \alpha_D \\
 &= (id \otimes \Sigma^{-1} \otimes id) \circ (id \otimes \hat{\Delta}^{cop} \otimes id \otimes id) \circ (\lambda \otimes id) \circ \alpha_D \\
 &= (id \otimes \Sigma^{-1} \otimes id) \circ (id \otimes \hat{\Delta}^{cop} \otimes id \otimes id) \circ (id \otimes \alpha) \circ \alpha_D \\
 &= (id \otimes \Sigma^{-1} \otimes id) \circ (id \otimes id \otimes \alpha) \circ (id \otimes \hat{\Delta}^{cop} \otimes id) \circ \alpha_D \\
 &= (id \otimes \Sigma^{-1} \otimes id) \circ (id \otimes id \otimes \alpha) \circ (id \otimes \hat{\alpha}) \circ \alpha_D,
 \end{aligned}$$

this implies

$$(\text{Ad}(U) \otimes \text{id}) \circ (\text{id} \otimes \hat{\alpha}) \circ \alpha = (\Sigma^{-1} \otimes \text{id}) \circ (\text{id} \otimes \alpha) \circ \hat{\alpha},$$

then  $(X, \alpha, \hat{\alpha})$  is a  $\mathbb{G}$ -Yetter-Drinfeld \*-algebra. □

**Proposition 2.8.4.**  $X^{\alpha_D} = X^\alpha \cap X^{\hat{\alpha}}$

*Proof* As  $\alpha_D = (\text{id}_{\mathcal{M}(A)} \otimes \hat{\alpha}) \circ \alpha$ , we get that  $X^\alpha \cap X^{\hat{\alpha}} \subseteq X^{\alpha_D}$ . On the other hand, using the equation

$$(\text{id}_{\mathcal{M}(A)} \otimes \text{id}_{\mathcal{M}(\hat{A})} \otimes \hat{\alpha}) \circ \alpha_D = (\text{id}_{\mathcal{M}(A)} \otimes \hat{\Delta}^{\text{cop}} \otimes \text{id}_X) \circ \alpha_D,$$

we get that  $X^{\alpha_D} \subseteq X^{\hat{\alpha}}$ , and using the relation

$$(\text{id}_{\mathcal{M}(A)} \otimes \text{id}_{\mathcal{M}(\hat{A})} \otimes \alpha) \circ \alpha_D = (\text{id}_{\mathcal{M}(A)} \otimes \Sigma_U \otimes \text{id}_X) \circ (\Delta \otimes \text{id}_{\mathcal{M}(\hat{A})} \otimes \text{id}_X) \circ \alpha_D,$$

we get that  $X^{\alpha_D} \subseteq X^\alpha$ . □

**Theorem 2.8.5.** Let  $\mathbb{G}$  be an algebraic quantum group,  $(X, \alpha, \hat{\alpha})$  be a  $\mathbb{G}$ -Yetter-Drinfeld \*-algebra, and  $\alpha_D = (\text{id}_{\mathcal{M}(A)} \otimes \hat{\alpha}) \circ \alpha$  be the action of  $D(\mathbb{G})$  on  $X$ . If  $\mu$  is a non-zero faithful linear functional on  $X$ , then the following conditions are equivalent:

- (i)  $\mu$  is invariant under each of the actions  $\alpha$  and  $\hat{\alpha}$ ;
- (ii)  $\mu$  is invariant under the action  $\alpha_D$ .

*Proof*

- **(i)  $\Rightarrow$  (ii):** If  $\mu$  is invariant under  $\alpha$  and  $\hat{\alpha}$ , then

$$\begin{aligned} (\text{id}_{\mathcal{M}(A)} \otimes \text{id}_{\mathcal{M}(\hat{A})} \otimes \mu) \circ (\text{id}_{\mathcal{M}(A)} \otimes \hat{\alpha}) \circ \alpha &= (\text{id}_{\mathcal{M}(A)} \otimes (\text{id}_{\mathcal{M}(\hat{A})} \otimes \mu) \circ \hat{\alpha}) \circ \alpha \\ &= (\text{id}_{\mathcal{M}(A)} \otimes \mu(\cdot)1_{\mathcal{M}(\hat{A})}) \circ \alpha \\ &= \mu(\cdot)1_{\mathcal{M}(A \otimes \hat{A})} \end{aligned}$$

- **(ii)  $\Rightarrow$  (i):** Let  $\omega$  a functional on  $\hat{A}$  and define the functional  $\mu' = (\omega \otimes \mu) \circ \hat{\alpha}$ . The equality  $(\text{id}_{\mathcal{M}(A)} \otimes \text{id}_{\mathcal{M}(\hat{A})} \otimes \mu) \circ \alpha_D = \mu(\cdot)1_{\mathcal{M}(A \otimes \hat{A})}$  implies that  $(\text{id}_{\mathcal{M}(A)} \otimes \mu') \circ \alpha = \mu(\cdot)1_{\mathcal{M}(A)}$ . For every linear functional  $\omega'$  on  $\hat{A}$ , we have

$$\begin{aligned} (\text{id}_{\mathcal{M}(A)} \otimes \omega' \otimes \mu') \circ \alpha_D &= (\text{id}_{\mathcal{M}(A)} \otimes \omega' \otimes (\omega \otimes \mu) \circ \hat{\alpha}) \circ (\text{id}_{\mathcal{M}(A)} \otimes \hat{\alpha}) \circ \alpha \\ &= (\text{id}_{\mathcal{M}(A)} \otimes \omega' \otimes \omega \otimes \mu) \circ (\text{id}_{\mathcal{M}(A)} \otimes (\text{id}_{\mathcal{M}(\hat{A})} \otimes \hat{\alpha}) \circ \hat{\alpha}) \circ \alpha \\ &= (\text{id}_{\mathcal{M}(A)} \otimes \omega' \otimes \omega \otimes \mu) \circ (\text{id}_{\mathcal{M}(A)} \otimes (\hat{\Delta}^{\text{cop}} \otimes \text{id}_X) \circ \hat{\alpha}) \circ \alpha \\ &= (\text{id}_{\mathcal{M}(A)} \otimes \omega' \otimes \omega \otimes \mu) \circ (\text{id}_{\mathcal{M}(A)} \otimes \hat{\Delta}^{\text{cop}} \otimes \text{id}_X) \circ \alpha_D \\ &= (\text{id}_{\mathcal{M}(A)} \otimes (\omega' \otimes \omega) \circ \hat{\Delta}^{\text{cop}} \otimes \mu) \circ \alpha_D \\ &= (\text{id}_{\mathcal{M}(A)} \otimes (\omega' \otimes \omega) \circ \hat{\Delta}^{\text{cop}}) \circ (\text{id}_{\mathcal{M}(A)} \otimes \text{id}_{\mathcal{M}(\hat{A})} \otimes \mu) \circ \alpha_D \\ &= \mu(\cdot)1_{\mathcal{M}(A)} \end{aligned}$$

and, then  $(id_{\mathcal{M}(A)} \otimes id_{\mathcal{M}(\widehat{A})} \otimes \mu') \circ \alpha_D = \mu(\cdot)1_{\mathcal{M}(A \otimes \widehat{A})}$ . On the other hand

$$\begin{aligned} (id_{\mathcal{M}(A)} \otimes id_{\mathcal{M}(\widehat{A})} \otimes \mu') \circ \alpha_D &= (\text{Ad}(U^*) \circ \Sigma) \circ (id_{\mathcal{M}(A)} \otimes id_{\mathcal{M}(\widehat{A})} \otimes \mu') \circ (id_{\mathcal{M}(A)} \otimes \alpha) \circ \widehat{\alpha} \\ &= (\text{Ad}(U^*) \circ \Sigma) \circ (id_{\mathcal{M}(A)} \otimes (id_{\mathcal{M}(\widehat{A})} \otimes \mu') \circ \alpha) \circ \widehat{\alpha} \\ &= (\text{Ad}(U^*) \circ \Sigma) \circ (id_{\mathcal{M}(A)} \otimes \mu) \circ \widehat{\alpha} \end{aligned}$$

and, we get that  $(id_{\mathcal{M}(A)} \otimes \mu) \circ \widehat{\alpha} = \mu(\cdot)1_{\mathcal{M}(\widehat{A})}$ , therefore  $\mu$  is invariant under  $\widehat{\alpha}$ . Then

$$\mu' = \omega \circ (id \otimes \mu) \circ \widehat{\alpha} = \mu(\cdot)\omega(1_{\mathcal{M}(\widehat{A})}) = \mu,$$

and  $\mu$  is invariant under  $\alpha$ .

□

**Definition 2.8.6.** Let  $\mathbb{G}$  be an algebraic quantum group and  $(X, \alpha, \widehat{\alpha})$  be a  $\mathbb{G}$ -Yetter-Drinfeld \*-algebra. A non-zero positive faithful linear functional on  $X$  will be called a *Yetter-Drinfeld integral* for  $(X, \alpha, \widehat{\alpha})$  if it satisfies one of the equivalent conditions of Theorem 2.8.5 and it admits a modular automorphism.

**Corollary 2.8.7.** Let  $(X, \alpha, \widehat{\alpha})$  be a  $\mathbb{G}$ -Yetter-Drinfeld \*-algebra and  $\mu$  a Yetter-Drinfeld integral on  $X$ .

(i) the standard implementations of the action  $\alpha, \widehat{\alpha}$  and  $\alpha_D$  associated with  $\mu$  are linked by the relation

$$U_{\mu}^{\alpha_D} = (U_{\mu}^{\widehat{\alpha}})_{23}(U_{\mu}^{\alpha})_{13};$$

(ii)  $(U_{\mu}^{\alpha})_{13}(U_{\mu}^{\widehat{\alpha}})_{23} = U_{12}(U_{\mu}^{\widehat{\alpha}})_{23}(U_{\mu}^{\alpha})_{13}U_{12}^*$ ;

(iii)  $U_{12}(U_{\mu}^{\alpha})_{13}^* = (U_{\mu}^{\alpha})_{23}^*U_{12}(U_{\mu}^{\alpha})_{23}$ ;

(iv)  $\text{Ad}(1 \otimes U_{\mu}^{\widehat{\alpha}}(U_{\mu}^{\alpha})^*)(U \otimes 1) = (U_{\mu}^{\alpha})_{13}^*U_{12}$ .

*Proof*

(i) On one hand, because

$$\begin{aligned} (id_{\mathcal{M}(A)} \otimes \widehat{\alpha})(a \otimes x) &= a \otimes U_{\mu}^{\widehat{\alpha}}(1_{\mathcal{M}(\widehat{A})} \otimes x)(U_{\mu}^{\widehat{\alpha}})^* \\ &= (1_{\mathcal{M}(A)} \otimes U_{\mu}^{\widehat{\alpha}})(a \otimes 1_{\mathcal{M}(\widehat{A})} \otimes x)(1_{\mathcal{M}(A)} \otimes U_{\mu}^{\widehat{\alpha}})^* \\ &= (U_{\mu}^{\widehat{\alpha}})_{23}(\Sigma_{\widehat{A}, A} \otimes id_{\mathcal{M}(A)})(1_{\mathcal{M}(\widehat{A})} \otimes a \otimes x)(U_{\mu}^{\widehat{\alpha}})_{23}^*, \end{aligned}$$

for all  $x \in X$  and  $a \in A$ , we have

$$(id_{\mathcal{M}(A)} \otimes \widehat{\alpha})(K) = (U_{\mu}^{\widehat{\alpha}})_{23}(\Sigma_{\widehat{A}, A} \otimes id_{\mathcal{M}(A)})(1 \otimes K)(U_{\mu}^{\widehat{\alpha}})_{23}^*,$$

for all  $K \in \mathcal{M}(A \otimes X)$ . On the other hand,

$$\begin{aligned} (\Sigma_{\widehat{A}, A} \otimes id_{\mathcal{M}(A)})(1_{\mathcal{M}(\widehat{A})} \otimes \alpha(x)) &= (\Sigma_{\widehat{A}, A} \otimes id_{\mathcal{M}(A)})(1_{\mathcal{M}(\widehat{A})} \otimes U_{\mu}^{\alpha}(1_{\mathcal{M}(A)} \otimes x)(U_{\mu}^{\alpha})^*) \\ &= (U_{\mu}^{\alpha})_{13}(1_{\mathcal{M}(A)} \otimes 1_{\mathcal{M}(\widehat{A})} \otimes x)(U_{\mu}^{\alpha})_{13}^*, \end{aligned}$$

because

$$(U_\mu^\alpha)_{13} = (\Sigma_{\hat{A}, A} \otimes id_{\mathcal{M}(A)})(1_{\mathcal{M}(\hat{A})} \otimes U_\mu^\alpha).$$

Now,

$$\begin{aligned} \alpha_D(x) &= (id_{\mathcal{M}(A)} \otimes \hat{\alpha})(\alpha(x)) \\ &= (U_\mu^{\hat{\alpha}})_{23}(\Sigma_{\hat{A}, A} \otimes id_{\mathcal{M}(A)})(1_{\mathcal{M}(\hat{A})} \otimes \alpha(x))(U_\mu^{\hat{\alpha}})_{23}^* \\ &= (U_\mu^{\hat{\alpha}})_{23}(U_\mu^\alpha)_{13}(1_{\mathcal{M}(A)} \otimes 1_{\mathcal{M}(\hat{A})} \otimes x)(U_\mu^\alpha)_{13}^{-1}(U_\mu^{\hat{\alpha}})_{23}^* \\ &= (U_\mu^{\hat{\alpha}})_{23}(U_\mu^\alpha)_{13}(1_{\mathcal{M}(A)} \otimes 1_{\mathcal{M}(\hat{A})} \otimes x)((U_\mu^{\hat{\alpha}})_{23}(U_\mu^\alpha)_{13})^*. \end{aligned}$$

(ii) By item (i), we get that  $(U_\mu^{\hat{\alpha}})_{23}(U_\mu^\alpha)_{13}$  is a corepresentation of  $D(\mathbb{G})$ , i.e

$$\begin{aligned} (U_\mu^{\hat{\alpha}})_{45}(U_\mu^\alpha)_{35}(U_\mu^{\hat{\alpha}})_{25}(U_\mu^\alpha)_{15} &= (U^{\alpha_D})_{23}(U^{\alpha_D})_{13} = (\Delta_D \otimes id)(U^{\alpha_D}) \\ &= (id \otimes \Sigma_U \otimes id \otimes id)(\Delta \otimes \hat{\Delta}^{cop} \otimes id)((U_\mu^{\hat{\alpha}})_{23}(U_\mu^\alpha)_{13}) \\ &= (id \otimes \Sigma_U \otimes id \otimes id)((U_\mu^{\hat{\alpha}})_{45}(U_\mu^{\hat{\alpha}})_{35}(U_\mu^\alpha)_{25}(U_\mu^\alpha)_{15}) \\ &= (U_\mu^{\hat{\alpha}})_{45}(id \otimes \Sigma_U \otimes id \otimes id)((U_\mu^{\hat{\alpha}})_{35}(U_\mu^\alpha)_{25})(U_\mu^\alpha)_{15}, \end{aligned}$$

this implies

$$\begin{aligned} (U_\mu^\alpha)_{25}(U_\mu^{\hat{\alpha}})_{35} &= (id \otimes \Sigma \otimes id \otimes id)((U_\mu^\alpha)_{35}(U_\mu^{\hat{\alpha}})_{25}) \\ &= (id \otimes \Sigma \circ \Sigma_U \otimes id \otimes id)((U_\mu^{\hat{\alpha}})_{35}(U_\mu^\alpha)_{25}) \\ &= (id \otimes \text{Ad}(U) \otimes id \otimes id)((U_\mu^{\hat{\alpha}})_{35}(U_\mu^\alpha)_{25}) \\ &= U_{23}(U_\mu^{\hat{\alpha}})_{35}(U_\mu^\alpha)_{25}U_{23}^* \end{aligned}$$

and by renumbering the legs, we obtain

$$(U_\mu^\alpha)_{13}(U_\mu^{\hat{\alpha}})_{23} = U_{12}(U_\mu^{\hat{\alpha}})_{23}(U_\mu^\alpha)_{13}U_{12}^*.$$

(iii) We have

$$(\Delta \otimes id)(a \otimes x) = U^*(1_{\mathcal{M}(A)} \otimes a)U \otimes x = U_{12}^*(1_{\mathcal{M}(A)} \otimes a \otimes x)U_{12},$$

for all  $a \in A$  and  $x \in X$ , then

$$(\Delta \otimes id)(K) = U_{12}^*(1_{\mathcal{M}(A)} \otimes K)U_{12} = U_{12}^*K_{23}U_{12}$$

for all  $K \in \mathcal{M}(A \otimes X)$ . Now, the relation

$$\begin{aligned} U_{12}^*(U_\mu^\alpha)_{23}^*U_{12} &= (U_{12}^*(U_\mu^\alpha)_{23}U_{12})^* \\ &= [(\Delta \otimes id)(U_\mu^\alpha)]^* \\ &= [(U_\mu^\alpha)_{23}(U_\mu^\alpha)_{13}]^* \\ &= (U_\mu^\alpha)_{13}^*(U_\mu^\alpha)_{23}^* \end{aligned}$$

implies

$$U_{12}(U_\mu^\alpha)^*_{13} = (U_\mu^\alpha)^*_{23}U_{12}(U_\mu^\alpha)_{23}.$$

(iv) Item (ii) and (iii) imply

$$\begin{aligned} U_{12}^*(U_\mu^\alpha)_{13} &= U_{12}^*(U_\mu^\alpha)_{13}(U_\mu^{\hat{\alpha}})_{23}(U_\mu^{\hat{\alpha}})^*_{23} \\ &= U_{12}^*U_{12}(U_\mu^{\hat{\alpha}})_{23}(U_\mu^\alpha)_{13}U_{12}^*(U_\mu^{\hat{\alpha}})^*_{23} \\ &= (U_\mu^{\hat{\alpha}})_{23}[U_{12}(U_\mu^\alpha)^*_{13}]^*(U_\mu^{\hat{\alpha}})^*_{23} \\ &= (U_\mu^{\hat{\alpha}})_{23}[(U_\mu^\alpha)^*_{23}U_{12}(U_\mu^\alpha)_{23}]^*(U_\mu^{\hat{\alpha}})^*_{23} \\ &= (U_\mu^{\hat{\alpha}})_{23}(U_\mu^\alpha)^*_{23}U_{12}^*(U_\mu^\alpha)_{23}(U_\mu^{\hat{\alpha}})^*_{23}, \end{aligned}$$

then

$$\begin{aligned} \text{Ad}(1 \otimes U_\mu^{\hat{\alpha}}(U_\mu^\alpha)^*)(U \otimes 1) &= [\text{Ad}(1 \otimes U_\mu^{\hat{\alpha}}(U_\mu^\alpha)^*)(U^* \otimes 1)]^* \\ &= [(U_\mu^{\hat{\alpha}})_{23}(U_\mu^\alpha)^{-1}_{23}U_{12}^*(U_\mu^\alpha)_{23}(U_\mu^{\hat{\alpha}})^*_{23}]^* \\ &= [U_{12}^*(U_\mu^\alpha)_{13}]^* \\ &= (U_\mu^\alpha)^*_{13}U_{12}. \end{aligned}$$

□

**Definition 2.8.8** (cf [NY14]). Let  $\mathbb{G} = (A, \Delta, \varphi)$  be an algebraic quantum group and  $X$  a non-degenerate \*-algebra. Assume that we have a left coaction  $\alpha : X \rightarrow \mathcal{M}(A \otimes X)$  of the multiplier Hopf \*-algebra  $(A, \Delta)$  on  $X$  and a left action  $\triangleright : A \otimes X \rightarrow X$  of the multiplier Hopf \*-algebra  $(A, \Delta)$  on  $X$ , then we say that  $X$  is a  $\mathbb{G}$ -Yetter-Drinfeld algebra in the algebraic sense if

$$\alpha(a \triangleright x) = a_{(1)}x_{[-1]}S(a_{(3)}) \otimes (a_{(2)} \triangleright x_{[0]}),$$

for all  $a \in A$  and  $x \in X$ .

The following proposition gives the equivalence of the two definitions of a Yetter-Drinfeld algebra.

**Proposition 2.8.9.** *The triple  $(X, \alpha, \hat{\alpha})$  is a  $\mathbb{G}$ -Yetter-Drinfeld algebra if and only if  $X$  is a  $\mathbb{G}$ -Yetter-Drinfeld algebra in the algebraic sense with respect to the coaction  $\alpha$  and the dual left action associated with  $\hat{\alpha}$ .*

*Proof* We have two coactions  $\alpha : X \rightarrow \mathcal{M}(A \otimes X)$ ,  $\hat{\alpha} : X \rightarrow \mathcal{M}(\hat{A} \otimes X)$ , the dual action associated with the left action  $\kappa := \Sigma_{\hat{A}, X} \circ \hat{\alpha}$  of  $\hat{\mathbb{G}}^\circ$  on  $X$  is given by

$$\begin{aligned} \triangleright_\kappa : A \otimes X &\rightarrow X \\ a \otimes x &\mapsto (a \otimes id)(\hat{\alpha}(x)) \end{aligned} .$$

Fix  $a \in A$  and  $x \in X$ , it is obvious that

$$\alpha(a \triangleright_\kappa x) = \alpha \circ (a \otimes id)(\hat{\alpha}(x)) = (a \otimes id \otimes id)((id \otimes \alpha)\hat{\alpha}(x)).$$

Now, if we use the symbolic expressions  $x_{[-1]} \otimes x_{[0]} = \alpha(x)$  and  $y \otimes z = \widehat{\alpha}(x_{[0]})$ , then by Lemma 2.4.10

$$\begin{aligned}
 a_{(1)}x_{[-1]}S(a_{(3)}) \otimes a_{(2)} \triangleright_{\kappa} x_{[0]} &= a_{(1)}x_{[-1]}S(a_{(3)}) \otimes (a_{(2)} \otimes id)(y \otimes z) \\
 &= a_{(1)}y(a_{(2)})x_{[-1]}S(a_{(3)}) \otimes z \\
 &= (id \otimes y)(\Delta(a_{(1)}))x_{[-1]}S(a_{(2)}) \otimes z \\
 &= (id \otimes a)(U(x_{[-1]} \otimes y)U^*) \otimes z \\
 &= (a \otimes id \otimes id)(\Sigma \otimes id)(Ad(U) \otimes id)(x_{[-1]} \otimes y \otimes z) \\
 &= (a \otimes id \otimes id)(\Sigma \otimes id)(Ad(U) \otimes id)((id \otimes \widehat{\alpha})\alpha(x)).
 \end{aligned}$$

This implies that the equality

$$\alpha(a \triangleright_{\kappa} x) = a_{(1)}x_{[-1]}S(a_{(3)}) \otimes a_{(2)} \triangleright_{\kappa} x_{[0]}$$

holds for all  $a \in A$  and  $x \in X$  if and only if

$$(id \otimes \alpha) \circ \widehat{\alpha} = (\Sigma \otimes id) \circ (Ad(U) \otimes id) \circ (id \otimes \widehat{\alpha}) \circ \alpha.$$

□

### 2.8.1 Braided commutative Yetter-Drinfeld \*-algebras

Let us recall the unital faithful non-degenerate right and left actions

$$\begin{aligned}
 \triangleleft : \quad (A \otimes X) \otimes (\widehat{A}A \otimes X) &\rightarrow A \otimes X \\
 (b \otimes y) \otimes (\omega a \otimes x) &\mapsto (b \triangleleft_{\widehat{A}} \omega)a \otimes yx \\
 \\
 \triangleleft : \quad (A\widehat{A} \otimes X^{op}) \otimes (A \otimes X) &\rightarrow A \otimes X \\
 (a\omega \otimes x^{op}) \otimes (b \otimes y) &\mapsto a(\omega \triangleright_{\widehat{A}} b) \otimes yx
 \end{aligned}$$

**Lemma 2.8.10.** For every  $a, b \in A, \omega \in \widehat{A}$  and  $x, y \in X$ , we have

$$(S \otimes id)[(b \otimes y) \triangleleft (\omega a \otimes x)] = (S(a)\widehat{S}^{-1}(\omega) \otimes x^{op}) \triangleright (S(b) \otimes y)$$

*Proof* Fix  $a, \in A, \omega \in \widehat{A}$  and  $x \in X$ , then by direct calculation, we have

$$\begin{aligned}
 (S \otimes id)[(b \otimes y) \triangleleft (\omega a \otimes x)] &= (S \otimes id)[(b \triangleleft_{\widehat{A}} \omega)a \otimes yx] \\
 &= S(a)S(b \triangleleft_{\widehat{A}} \omega) \otimes yx \\
 &= S(a)(\widehat{S}^{-1}(\omega) \triangleright_{\widehat{A}} S(b)) \otimes yx \\
 &= (S(a)\widehat{S}^{-1}(\omega) \otimes x^{op}) \triangleright (S(b) \otimes y),
 \end{aligned}$$

for all  $b \in A$  and  $y \in X$ .

□

Now, we can consider the injective linear maps

$$\begin{aligned} \mathfrak{J}_1 : \mathcal{M}(H(G) \otimes X) &\rightarrow \text{End}(A \otimes X) \\ P &\mapsto [\mathfrak{J}_1(P) : a \otimes x \mapsto (a \otimes x) \triangleleft P] \end{aligned}$$

$$\begin{aligned} \mathfrak{J}_2 : \mathcal{M}(H(\widehat{\mathbb{G}}^\circ) \otimes X^{op}) &\rightarrow \text{End}(A \otimes X) \\ P &\mapsto [\mathfrak{J}_2(P) : a \otimes x \mapsto P \triangleright (a \otimes x)] \end{aligned}$$

**Lemma 2.8.11.** *We have*

$$\mathfrak{J}_2(\widehat{\alpha}^\circ(x^{op}))(a \otimes y)(b \otimes 1_{\mathcal{M}(X)}) = a_{(1)}b \otimes y(S(a_{(2)}) \otimes id)(\widehat{\alpha}(x))$$

for all  $a, b \in A, x, y \in X$

*Proof* Follows directly from Lemma 2.8.10. □

**Definition 2.8.12.** Let  $(X, \alpha, \widehat{\alpha})$  be a  $\mathbb{G}$ -Yetter-Drinfeld \*-algebra equipped with a Yetter-Drinfeld integral  $\mu$  and  $U_\mu^\alpha$  the standard implementation of  $\alpha$ . The linear map

$$\begin{aligned} \beta : X &\rightarrow \text{End}(A \otimes X) \\ x &\mapsto U_\mu^\alpha \mathfrak{J}_2(\widehat{\alpha}^\circ(x^\circ))(U_\mu^\alpha)^* \end{aligned}$$

is an injective anti\*-homomorphism. We say that the  $\mathbb{G}$ -Yetter-Drinfeld \*-algebra  $(X, \alpha, \widehat{\alpha})$  is *braided commutative* if

$$\beta(X) \subseteq \mathfrak{J}_1(\mathcal{M}(\mathbb{G} \rtimes_\alpha X)).$$

**Remark 64.** If  $(X, \alpha, \widehat{\alpha})$  is braided commutative, we have

$$\beta(x) = \mathfrak{J}_1((\widehat{S}(x^{[-1]}) \otimes 1_{\mathcal{M}(X)})\alpha(x^{[0]})),$$

for all  $x \in X$ , where we use the Sweedler type leg notation

$$x^{[-1]} \otimes x^{[0]} := \widehat{\alpha}(x) \in \mathcal{M}(\widehat{A} \otimes X)$$

for the action  $\widehat{\alpha}$ .

**Proposition 2.8.13.** *Writing  $\beta^\circ(x^\circ) := \beta(x)$  for all  $x \in X$ , we have*

$$Ad(U \otimes 1)(1 \otimes \beta(x)) = (id \otimes \beta^\circ)(\alpha^\circ(x^\circ)),$$

for all  $x \in X$ .

*Proof* We have

$$\beta^\circ(x^\circ) = Ad(U_\mu^\alpha (U_\mu^{\widehat{\alpha}})^*)(1 \otimes x^\circ),$$

for all  $x \in X$ , and by relation (iv) of Corollary 2.8.7

$$U_{12}(U_\mu^\alpha)_{23}(U_\mu^{\widehat{\alpha}})^*_{23} = (U_\mu^\alpha)_{23}(U_\mu^{\widehat{\alpha}})^{-1}_{23}(U_\mu^\alpha)^{-1}_{13}U_{12}.$$

Then

$$\begin{aligned}
 \text{Ad}(U \otimes 1)(1 \otimes \beta(x)) &= \text{Ad}(U_{12}(U_\mu^\alpha)_{23}(U_\mu^{\hat{\alpha}})_{23}^*)(1 \otimes 1 \otimes x^\circ) \\
 &= \text{Ad}((U_\mu^\alpha)_{23}(U_\mu^{\hat{\alpha}})_{23}^*(U_\mu^\alpha)_{13}^*U_{12})(1 \otimes 1 \otimes x^\circ) \\
 &= \text{Ad}((U_\mu^\alpha)_{23}(U_\mu^{\hat{\alpha}})_{23}^*(U_\mu^\alpha)_{13}^*)(1 \otimes 1 \otimes x^\circ) \\
 &= \text{Ad}((U_\mu^\alpha)_{23}(U_\mu^{\hat{\alpha}})_{23}^*)(\alpha^\circ(x^\circ)) \\
 &= (id \otimes \beta^\circ)(\alpha^\circ(x^\circ)),
 \end{aligned}$$

for all  $x \in X$ . □

**Proposition 2.8.14.** *For every  $x, y \in X$ , we have*

$$\mathfrak{J}_1(\alpha(x))\beta(y) = \beta(y)\mathfrak{J}_1(\alpha(x))$$

in  $\text{End}(A \otimes X)$ .

*Proof* Fix  $x, y \in X$ . Because

$$\mathfrak{J}_1(\alpha(x)) = (U_\mu^\alpha)\mathfrak{J}_1(1 \otimes x)(U_\mu^\alpha)^*$$

and

$$\beta(x) = U_\mu^\alpha\mathfrak{J}_2(\hat{\alpha}^\circ(x^\circ))(U_\mu^\alpha)^*.$$

We only need to prove that

$$\mathfrak{J}_1(1 \otimes x)\mathfrak{J}_2(\hat{\alpha}^\circ(y^\circ)) = \mathfrak{J}_2(\hat{\alpha}^\circ(y^\circ))\mathfrak{J}_1(1 \otimes x),$$

which true by Lemma 2.8.11. □

**Definition 2.8.15** (cf [NY14]). Let  $\mathbb{G} = (A, \Delta, \varphi)$  be an algebraic quantum group and  $X$  a unital  $\mathbb{G}$ -Yetter-Drinfeld algebra in the algebraic sense, as Definition 2.8.8, i.e.

$$\alpha(a \triangleright x) = a_{(1)}x_{[-1]}S(a_{(3)}) \otimes (a_{(2)} \triangleright x_{[0]}),$$

for all  $a \in A$  and  $x \in X$ . We say that  $X$  is *braided commutative in the algebraic sense* if

$$xy = y_{[0]}(S^{-1}(y_{[-1]}) \triangleright x), \tag{2.3}$$

for all  $x, y \in X$ .

We will show now that our definition of a braided commutative Yetter-Drinfeld algebra is equivalent to Definition 2.8.15 in the unital case.

**Remark 65.** If  $\hat{\alpha} : X \rightarrow \mathcal{M}(\hat{A} \otimes X)$  is a left action of  $\mathbb{G}$  on  $X$ , then the linear map  $\beta = (\hat{S}^{-1} \otimes id) \circ \hat{\alpha} : X \rightarrow \mathcal{M}(\hat{A} \otimes X)$  is a left action of  $\mathbb{G}^\circ$  on  $X$ . Moreover, we can regard  $\beta(x)$  as an element in  $\mathcal{M}(H(\mathbb{G}) \otimes X)$  for all  $x \in X$ .

**Proposition 2.8.16.** *In the unital case, the relation (2.3) holds if and only if for all  $x, y \in X$ ,*

$$\alpha(y)\beta(x) = \beta(x)\alpha(y)$$



in  $H(\mathbb{G}) \otimes X$ .

*Proof* In the unital case, using the right faithful action

$$\begin{aligned} \triangleleft : (A \otimes X) \otimes (\widehat{A}A \otimes X) &\rightarrow A \otimes X \\ (b \otimes y) \otimes (\omega a \otimes x) &\mapsto (b \triangleleft_{\widehat{A}} \omega) a \otimes yx \end{aligned}$$

we have

$$\begin{aligned} (b \otimes y) \triangleleft \alpha(x) &= (b \otimes y)\alpha(x) = bx_{[-1]} \otimes yx_{[0]} \\ (b \otimes y) \triangleleft \beta(x) &= (b \triangleleft_{\widehat{A}} \widehat{S}^{-1}(x^{[-1]})) \otimes yx^{[0]} = \widehat{S}^{-1}(x^{[-1]})(b_{(2)})b_{(1)} \otimes yx^{[0]} \end{aligned}$$

for all  $b \in A$  and  $x, y \in X$ . Here, we used the Sweedler type leg notation

$$\alpha(x) = x_{[-1]} \otimes x_{[0]}, \quad \widehat{\alpha}(x) = x^{[-1]} \otimes x^{[0]}.$$

( $\Rightarrow$ ) Fix  $x, y \in X$ , we have

$$\begin{aligned} (a \otimes z) \triangleleft (\alpha(y)\beta(x)) &= (ay_{[-1]} \otimes zy_{[0]}) \triangleleft \beta(x) \\ &= (ay_{[-1]} \triangleleft_{\widehat{A}} \widehat{S}^{-1}(x^{[-1]})) \otimes zy_{[0]}x^{[0]} \\ &= \widehat{S}^{-1}(x^{[-1]})(a_{(2)}y_{[-1]_{(2)}})a_{(1)}y_{[-1]_{(1)}} \otimes zy_{[0]}x^{[0]} \\ &= \widehat{S}^{-1}(x^{[-1]})(a_{(2)}y_{[0]_{[-1]}})a_{(1)}y_{[-1]} \otimes zy_{[0]_{[0]}}x^{[0]} \\ &= \widehat{S}^{-1}(x_{(2)}^{[-1]})(a_{(2)})\widehat{S}^{-1}(x_{(1)}^{[-1]})(y_{[0]_{[-1]}})a_{(1)}y_{[-1]} \otimes zy_{[0]_{[0]}}x^{[0]} \\ &= \widehat{S}^{-1}(x_{(2)}^{[-1]})(a_{(2)})a_{(1)}y_{[-1]} \otimes zy_{[0]_{[0]}}\widehat{S}^{-1}(x_{(1)}^{[-1]})(y_{[0]_{[-1]}})x^{[0]} \\ &= \widehat{S}^{-1}(x_{(2)}^{[-1]})(a_{(2)})a_{(1)}y_{[-1]} \otimes zy_{[0]_{[0]}}(S^{-1}(y_{[0]_{[-1]}}) \otimes id)(x_{(1)}^{[-1]} \otimes x^{[0]}) \\ &= \widehat{S}^{-1}(x_{(2)}^{[-1]})(a_{(2)})a_{(1)}y_{[-1]} \otimes zy_{[0]_{[0]}}(S^{-1}(y_{[0]_{[-1]}}) \otimes id)(x_{(1)}^{[-1]} \otimes x^{[0]}) \\ &= \widehat{S}^{-1}(x^{[-1]})(a_{(2)})a_{(1)}y_{[-1]} \otimes zy_{[0]_{[0]}}(S^{-1}(y_{[0]_{[-1]}}) \otimes id)(x^{[0]_{[-1]}} \otimes x^{[0]_{[0]}}) \\ &= \widehat{S}^{-1}(x^{[-1]})(a_{(2)})a_{(1)}y_{[-1]} \otimes zy_{[0]_{[0]}}(S^{-1}(y_{[0]_{[-1]}}) \triangleright_{\kappa} x^{[0]}) \\ &= \widehat{S}^{-1}(x^{[-1]})(a_{(2)})a_{(1)}y_{[-1]} \otimes zx^{[0]}y_{[0]} \\ &= ((a \triangleleft_{\widehat{A}} \widehat{S}^{-1}(x^{[-1]})) \otimes zx^{[0]}) \triangleleft \alpha(y) \\ &= (a \otimes z) \triangleleft (\beta(x)\alpha(y)) \end{aligned}$$

for all  $a \in A$  and  $z \in X$ . The last equality follows from the relation 2.3, because we have

$$y_{[0]_{[0]}}(S^{-1}(y_{[0]_{[-1]}}) \triangleright_{\kappa} x^{[0]}) = x^{[0]}y_{[0]}.$$

Using the fact that the action is faithful, we have

$$\alpha(y)\beta(x) = \beta(x)\alpha(y).$$

( $\Leftarrow$ ) Fix  $x, y \in X$ , if  $\alpha(y)\beta(x) = \beta(x)\alpha(y)$ , then

$$\begin{aligned}
 y_{[-1]} \otimes xy_{[0]} &= (1 \otimes \widehat{\varepsilon}(\widehat{S}^{-1}(x^{[-1]}))x^{[0]}) \triangleleft \alpha(y) \\
 &= ((1 \triangleleft_{\widehat{A}} \widehat{S}^{-1}(x^{[-1]})) \otimes x^{[0]}) \triangleleft \alpha(y) \\
 &= (1 \otimes 1) \triangleleft \beta(x)\alpha(y) \\
 &= (1 \otimes 1) \triangleleft \alpha(y)\beta(x) \\
 &= (y_{[-1]} \triangleleft_{\widehat{A}} \widehat{S}^{-1}(x^{[-1]})) \otimes y_{[0]}x^{[0]} \\
 &= \widehat{S}^{-1}(x^{[-1]})(y_{[-1]_{(2)}})y_{[-1]_{(1)}} \otimes y_{[0]}x^{[0]} \\
 &= y_{[-1]_{(1)}} \otimes y_{[0]}(S^{-1}(y_{[-1]_{(2)}}) \otimes id)(x^{[-1]} \otimes x^{[0]}) \\
 &= y_{[-1]_{(1)}} \otimes y_{[0]}(S^{-1}(y_{[-1]_{(2)}}) \triangleright_{\kappa} x).
 \end{aligned}$$

Applying the counit  $\varepsilon : A \rightarrow \mathbb{C}$  in the first leg of the last equality, we obtain

$$\begin{aligned}
 xy &= (\varepsilon \otimes id)((1 \otimes x)\alpha(y)) \\
 &= (\varepsilon \otimes id)(y_{[-1]} \otimes xy_{[0]}) \\
 &= (\varepsilon \otimes id)(y_{[-1]_{(1)}} \otimes y_{[0]}(S^{-1}(y_{[-1]_{(2)}}) \triangleright_{\kappa} x)) \\
 &= y_{[0]}(S^{-1}(\varepsilon(y_{[-1]_{(1)}})y_{[-1]_{(2)}}) \triangleright_{\kappa} x) \\
 &= y_{[0]}(S^{-1}(y_{[-1]}) \triangleright_{\kappa} x).
 \end{aligned}$$

□

## 2.8.2 Examples of braided commutative Yetter-Drinfeld \*-algebras

### 2.8.2.1 Usual transformation groupoids

Consider a left action of a group  $G$  on a set  $S$ ,  $\cdot : G \times S \rightarrow S$ , and define the set  $G \curvearrowright S := G \times S$  and the maps

$$\begin{aligned}
 \cdot : (G \curvearrowright S)^{(2)} &\rightarrow G \curvearrowright S & \quad \quad \quad \cdot^{-1} : G \curvearrowright S &\rightarrow G \curvearrowright S \\
 ((g, s), (h, t)) &\mapsto (gh, t) & \quad \quad \quad (g, s) &\mapsto (g^{-1}, g \cdot s)
 \end{aligned}$$

where  $(G \curvearrowright S)^{(2)} = \{((g, s), (h, t)) \in (G \curvearrowright S) \times (G \curvearrowright S) : s = h \cdot t\}$ . Then  $G \curvearrowright S$  is a groupoid called the *transformation groupoid* [Ren80]. For this groupoid, we have  $(G \curvearrowright S)^{(0)} = \{e\} \times S \simeq S$  as the unit space of the grupoid and  $(G \curvearrowright S)^{(2)}$  as the set of composable pairs. Moreover, the maps given by

$$\begin{aligned}
 d : G \curvearrowright S &\rightarrow S & \quad \quad \quad r : G \curvearrowright S &\rightarrow S \\
 (g, s) &\mapsto s & \quad \quad \quad (g, s) &\mapsto g \cdot s
 \end{aligned}$$

are the domain and range maps, respectively.

**Example 2.8.17.** Let  $\cdot : G \times S \rightarrow S$  be a left action of a group  $G$  on a set  $S$ . By Example 1.2.21, we have the algebraic quantum group  $\mathbb{G} = (\text{Fun}_f(G), \Delta, \varphi)$  with  $\widehat{\mathbb{G}} = (\mathbb{C}[G], \widehat{\Delta}, \widehat{\varphi})$  as its dual algebraic quantum group. Consider the left action of  $\mathbb{G}$ , induced

by the action of  $G$  on  $S$ , and the trivial left action of  $\widehat{\mathbb{G}}$ , respectively

$$\begin{aligned} \alpha : \text{Fun}_f(S) &\rightarrow \mathcal{M}(\text{Fun}_f(G) \otimes \text{Fun}_f(S)) & \widehat{\alpha} : \text{Fun}_f(S) &\rightarrow \mathcal{M}(\mathbb{C}[G] \otimes \text{Fun}_f(S)) \\ p &\mapsto [\alpha(p) : (g, s) \mapsto p(g \cdot s)] & p &\mapsto U_e \otimes p \end{aligned}$$

For any  $g, \bar{g} \in G$  and  $s, \bar{s} \in S$ ,

$$\begin{aligned} [\alpha(p)(\delta_g \otimes \delta_s)](\bar{g}, \bar{s}) &= p(\bar{g} \triangleright \bar{s})\delta_{g, \bar{g}}\delta_{s, \bar{s}} \\ &= \left[ \sum_{h \in G, t \in S} p(h \triangleright t)\delta_{g, h}\delta_h \otimes \delta_{s, t}\delta_t \right] (\bar{g}, \bar{s}) \\ &= \left[ \left( \sum_{h \in G, t \in S} p(h \triangleright t)\delta_h \otimes \delta_t \right) (\delta_g \otimes \delta_s) \right] (\bar{g}, \bar{s}). \end{aligned}$$

then, we have

$$\alpha(p) = \sum_{g \in G, s \in S} p(g \cdot s)\delta_g \otimes \delta_s \in \text{Fun}(G \times S)$$

for all  $p \in \text{Fun}_f(S)$ . In particular,

$$\alpha(\delta_s) = \sum_{g \in G} \delta_g \otimes \delta_{g^{-1} \cdot s}$$

for all  $s \in S$ .

**Claim:** The triple  $(\text{Fun}_f(S), \alpha, \widehat{\alpha})$  is a  $\mathbb{G}$ -Yetter-Drinfeld \*-algebra.

By direct calculation, for all  $s \in S$ , we have

$$\begin{aligned} (\text{Ad}(U^{\mathbb{G}}) \otimes id) \circ (id \otimes \widehat{\alpha}) \circ \alpha(\delta_s) &= (\text{Ad}(U^{\mathbb{G}}) \otimes id) \left( \sum_{g \in G} \delta_g \otimes U_e \otimes \delta_{g^{-1} \cdot s} \right) \\ &= \sum_{g \in G} \text{Ad}(U^{\mathbb{G}})(\delta_g \otimes U_e) \otimes \delta_{g^{-1} \cdot s} \\ &= \sum_{g, h \in G} (\delta_h \otimes U_h)(\delta_g \otimes U_e)(\delta_h \otimes U_{h^{-1}}) \otimes \delta_{g^{-1} \cdot s} \\ &= (\Sigma \otimes id) \left( \sum_{g \in G} U_e \otimes \delta_g \otimes \delta_{g^{-1} \cdot s} \right) \\ &= (\Sigma \otimes id) \circ (id \otimes \alpha)(U_e \otimes \delta_s) \\ &= (\Sigma \otimes id) \circ (id \otimes \alpha) \circ \widehat{\alpha}(\delta_s). \end{aligned}$$

**Claim:** The  $\mathbb{G}$ -Yetter-Drinfeld \*-algebra  $(\text{Fun}_f(S), \alpha, \widehat{\alpha})$  is braided commutative.

The dual right action of  $\alpha$  is given by

$$\begin{aligned} \triangleleft_{\alpha} : \text{Fun}_f(S) \otimes \mathbb{C}[G] &\rightarrow \text{Fun}_f(S) \\ p \otimes U_g &\mapsto (U_g \otimes id)(\alpha(p)) = \sum_{s \in S} p(g \cdot s)\delta_s \end{aligned}$$

and its crossed product is given by the identification

$$\begin{aligned} \mathbb{G} \rtimes_{\alpha} \text{Fun}_f(S) &\cong \mathbb{C}[G] \#_{\triangleleft \alpha} \text{Fun}_f(S) \cong \mathbb{C}[G \curvearrowright S] \\ (U_g \otimes 1_{\mathcal{M}(\text{Fun}_f(S))})\alpha(p) &\mapsto U_g \# p \mapsto \sum_{s \in S} p(s)U_{(g,s)}, \end{aligned}$$

where  $\mathbb{C}[G \curvearrowright S]$  denotes the \*-algebra of the groupoid  $G \curvearrowright S$ , i.e. the \*-algebra generated by formal elements  $\{U_{(g,s)}\}_{(g,s) \in G \curvearrowright S}$  and relations

$$U_{(g,s)}U_{(h,t)} = \begin{cases} U_{(gh,t)} & \text{if } s = h \cdot t \\ 0 & \text{if not} \end{cases}, \quad U_{(g,s)}^* = U_{(g,s)}^{-1} = U_{(g^{-1},g \cdot s)}.$$

Using this identification, the map  $\alpha$  can be regarded as the map

$$\begin{aligned} \alpha : \text{Fun}_f(S) &\rightarrow \mathcal{M}(\mathbb{G} \rtimes_{\alpha} \text{Fun}_f(S)) \\ p &\mapsto \sum_{s \in S} p(s)U_{(e,s)} \end{aligned}$$

and because

$$\beta_{\hat{\alpha}}(\delta_s) = (\hat{S}^{-1}(U_e) \otimes 1_{\mathcal{M}(\text{Fun}_f(S))})\alpha(\delta_s) \cong \sum_{t \in S} \delta_s(t)U_{e,t} = U_{e,s}$$

the map  $\beta_{\hat{\alpha}}$  is given by

$$\begin{aligned} \beta_{\hat{\alpha}} : \text{Fun}_f(S) &\rightarrow \mathcal{M}(\mathbb{G} \rtimes_{\alpha} \text{Fun}_f(S)) \\ p &\mapsto \sum_{s \in S} p(s)U_{(e,s)}. \end{aligned}$$

**Example 2.8.18.** Let  $\cdot : G \times S \rightarrow S$  be a left action of a group  $G$  on a set  $S$ . By Example 1.2.21, we have the algebraic quantum group  $\mathbb{H} = (\mathbb{C}[G], \hat{\Delta}, \hat{\varphi})$  with  $\hat{\mathbb{H}} = (\text{Fun}_f(G), \Delta^{\text{cop}}, \varphi)$  as its dual algebraic quantum group. Consider the trivial left coaction of  $\mathbb{H}$ , and the left coaction of  $\hat{\mathbb{H}}$ , induced by the action of  $G$  on  $S$ , respectively,

$$\begin{aligned} \lambda : \text{Fun}_f(S) &\rightarrow \mathcal{M}(\mathbb{C}[G] \otimes \text{Fun}_f(S)) & \hat{\lambda} : \text{Fun}_f(S) &\rightarrow \mathcal{M}(\text{Fun}_f(G) \otimes \text{Fun}_f(S)) \\ p &\mapsto U_e \otimes p & p &\mapsto [\hat{\lambda}(p) : (g, s) \mapsto p(g^{-1} \cdot s)] \end{aligned}$$

For any  $g, \bar{g} \in G$  and  $s, \bar{s} \in S$ ,

$$\begin{aligned} [\hat{\lambda}(p)(\delta_g \otimes \delta_s)](\bar{g}, \bar{s}) &= p(\bar{g}^{-1} \triangleright \bar{s})\delta_{g, \bar{g}}\delta_{s, \bar{s}} \\ &= \left[ \sum_{h \in G, t \in S} p(h^{-1} \triangleright t)\delta_{g, h}\delta_h \otimes \delta_{s, t}\delta_t \right] (\bar{g}, \bar{s}) \\ &= \left[ \left( \sum_{h \in G, t \in S} p(h^{-1} \triangleright t)\delta_h \otimes \delta_t \right) (\delta_g \otimes \delta_s) \right] (\bar{g}, \bar{s}). \end{aligned}$$

then, we have

$$\widehat{\lambda}(p) = \sum_{g \in G, s \in S} p(g^{-1} \cdot s) \delta_g \otimes \delta_s \in \text{Fun}(G \times S)$$

for all  $p \in \text{Fun}_f(S)$ . In particular,

$$\widehat{\lambda}(\delta_s) = \sum_{g \in G} \delta_g \otimes \delta_{g \cdot s}$$

for all  $s \in S$ .

**Claim:** The triple  $(\text{Fun}_f(S), \lambda, \widehat{\lambda})$  is a  $\mathbb{H}$ -Yetter-Drinfeld \*-algebra.

By direct calculation, for all  $s \in S$ , we have

$$\begin{aligned} (Ad(U^{\mathbb{H}}) \otimes id) \circ (id \otimes \widehat{\lambda}) \circ \lambda(\delta_s) &= (Ad(U^{\mathbb{H}}) \otimes id) \left( \sum_{g \in G} U_e \otimes \delta_g \otimes \delta_{g \cdot s} \right) \\ &= \sum_{g \in G} Ad(U^{\mathbb{H}})(U_e \otimes \delta_g) \otimes \delta_{g \cdot s} \\ &= \sum_{g, h \in G} (U_{h^{-1}} \otimes \delta_h)(U_e \otimes \delta_g)(U_h \otimes \delta_h) \otimes \delta_{g \cdot s} \\ &= (\Sigma \otimes id) \left( \sum_{g \in G} \delta_g \otimes U_e \otimes \delta_{g \cdot s} \right) \\ &= (\Sigma \otimes id) \circ (id \otimes \lambda) \left( \sum_{g \in G} \delta_g \otimes \delta_{g \cdot s} \right) \\ &= (\Sigma \otimes id) \circ (id \otimes \lambda) \circ \widehat{\lambda}(\delta_s). \end{aligned}$$

**Claim:** The  $\mathbb{H}$ -Yetter-Drinfeld algebra  $(\text{Fun}_f(S), \lambda, \widehat{\lambda})$  is braided commutative.

The dual right action of  $\lambda$  is given by

$$\begin{aligned} \triangleleft_{\lambda} : \text{Fun}_f(S) \otimes \text{Fun}_f(G) &\rightarrow \text{Fun}_f(S) \\ p \otimes \delta_g &\mapsto (\delta_g \otimes id)(\lambda(p)) = \varepsilon(\delta_g)p = \delta_{g,e}p \end{aligned}$$

and its crossed product is given by the identification

$$\begin{aligned} \mathbb{H} \times_{\lambda} \text{Fun}_f(S) &\cong \text{Fun}_f(G) \otimes \text{Fun}_f(S) \cong \text{Fun}_f(G \curvearrowright S) \\ (\delta_g \otimes 1_{\mathcal{M}(\text{Fun}_f(S))})\lambda(p) &\mapsto \delta_g \otimes p \mapsto \sum_{s \in S} p(s)\delta_{(g,s)} \end{aligned} \quad ,$$

where  $\text{Fun}_f(G \curvearrowright S)$  is the \*-algebra of complex finitely supported functions on the groupoid  $G \curvearrowright S$ . Using this identification, the map  $\lambda$  can be regarded as the map

$$\begin{aligned} \lambda : \text{Fun}_f(S) &\rightarrow \mathcal{M}(\widehat{\mathbb{G}} \times_{\lambda} \text{Fun}_f(S)) \\ p &\mapsto \sum_{g \in G, s \in S} p(s)\delta_{(g,s)} \end{aligned}$$

and because

$$\begin{aligned}
 \beta_{\tilde{\lambda}}(\delta_s) &= \sum_{g \in G} (S(\delta_g) \otimes 1_{\mathcal{M}(\text{Fun}_f(S))}) \lambda(\delta_{g \cdot s}) \\
 &= \sum_{g \in G} (\delta_g \otimes 1_{\mathcal{M}(\text{Fun}_f(S))}) \lambda(\delta_{g^{-1} \cdot s}) \\
 &\cong \sum_{g \in G, t \in S} \delta_{g^{-1} \cdot s}(t) \delta_{(g,t)} \\
 &= \sum_{g \in G, t \in S} \delta_s(g \cdot t) \delta_{(g,t)},
 \end{aligned}$$

for all  $s \in S$ , the map  $\beta_{\tilde{\lambda}}$  is given by

$$\begin{aligned}
 \beta_{\tilde{\lambda}} : \text{Fun}_f(S) &\rightarrow \mathcal{M}(\mathbb{H} \rtimes_{\lambda} \text{Fun}_f(S)) \\
 p &\mapsto \sum_{g \in G, s \in S} p(g \cdot s) \delta_{(g,s)}.
 \end{aligned}$$

**Remark 66.** By simple calculation, for any  $p \in \text{Fun}_f(S)$ , we have

$$p \circ d = \lambda(p) \quad \text{and} \quad p \circ r = \beta_{\tilde{\lambda}}(p),$$

as elements in  $\text{Fun}(G \curvearrowright S) \cong \mathcal{M}(\text{Fun}_f(G \curvearrowright S))$ .

### 2.8.2.2 Quotient type coideals

In this subsection, we adapt to the algebraic quantum group case a result from [ET16].

Let  $\mathbb{G} = (A, \Delta_A, \varphi_A)$  be an algebraic quantum group. If  $\mathbb{H} = (B, \Delta_B, \varphi_B)$  is a closed quantum subgroup of  $\mathbb{G}$ , as Definition 2.1.15, with non-degenerate surjective \*-homomorphism  $\Phi : A \rightarrow \mathcal{M}(B)$ , then it is obvious that the map  $\alpha^{\Phi} : A \rightarrow \mathcal{M}(A \otimes B)$  defined by

$$\alpha^{\Phi}(a) = (id \otimes \Phi)(\Delta_A(a)),$$

for all  $a \in A$ , is a right action of  $\mathbb{H}$  on  $A$  such that

$$(\Delta_A \otimes id) \circ \alpha^{\Phi} = (id \otimes \alpha^{\Phi}) \circ \Delta_A.$$

**Proposition 2.8.19.** Let  $I$  be the sub- $*$ -algebra  $A^{\alpha^{\Phi}} = \{a \in A : \alpha^{\Phi}(a) = a \otimes 1_{\mathcal{M}(B)}\}$ .

- (i) If  $\alpha_I$  denotes the restriction of  $\Delta_A$  to  $I$ , then  $(I, \alpha_I)$  is a  $\mathbb{G}$ - $*$ -algebra.
- (ii) If  $\lambda_I$  denotes the restriction of  $Ad_{\mathbb{G}}$  to  $I$ , then  $(I, \lambda_I)$  is a  $\widehat{\mathbb{G}}$ - $*$ -algebra.
- (iii)  $(I, \alpha_I, \lambda_I)$  is a  $\mathbb{G}$ -Yetter-Drinfeld \*-algebra.
- (iv)  $(I, \alpha_I, \lambda_I)$  is braided commutative.

*Proof*

(i) For all  $a \in I$ , we have

$$\begin{aligned} (id \otimes \alpha^\Phi)(\Delta_A(a)) &= (\Delta_A \otimes id)(\alpha^\Phi(a)) \\ &= \Delta_A(a) \otimes 1, \end{aligned}$$

then  $\Delta_A(a) \in \mathcal{M}(A \otimes I)$ .

(ii) Let  $W = (id \otimes \Phi)(\hat{U})$ , for all  $x \in I$ , then

$$\begin{aligned} (id \otimes \alpha^\Phi)(\text{Ad}_G(a)) &= (id \otimes (id \otimes \Phi) \circ \Delta_A)(\hat{U}^*(1 \otimes a)\hat{U}) \\ &= (id \otimes id \otimes \Phi)((id \otimes \Delta_A)(\hat{U}^*)(1 \otimes \Delta_A(a))(id \otimes \Delta_A)(\hat{U})) \\ &= (id \otimes id \otimes \Phi)(\hat{U}_{12}^* \hat{U}_{13}^*(1 \otimes \Delta_A(a))\hat{U}_{13}\hat{U}_{12}) \\ &= \hat{U}_{12}^* W_{13}^*(1 \otimes (id \otimes \Phi)(\Delta_A(a)))W_{13}\hat{U}_{12} \\ &= \hat{U}_{12}^* W_{13}^*(1 \otimes \alpha^\Phi(a))W_{13}\hat{U}_{12} \\ &= \hat{U}_{12}^* W_{13}^*(1 \otimes a \otimes 1)W_{13}\hat{U}_{12} \\ &= \text{Ad}_G(a) \otimes 1, \end{aligned}$$

so  $\text{Ad}_G(a) \in \mathcal{M}(\hat{A} \otimes I)$ .

(iii) For all  $a \in I$ , we have

$$\begin{aligned} \text{Ad}(U_{12})((id \otimes \lambda_I)(\alpha_I(a))) &= U_{12}\hat{U}_{23}^*\Sigma_{12}(1 \otimes \Delta(a))\hat{U}_{23}U_{12}^* \\ &= U_{12}\hat{U}_{23}^*\Sigma_{12}(U_{23}^*(1 \otimes 1 \otimes a)U_{23})\hat{U}_{23}U_{12}^* \\ &= U_{12}\hat{U}_{23}^*U_{13}^*(1 \otimes 1 \otimes a)U_{13}\hat{U}_{23}U_{12}^* \\ &= \text{Ad}(\Sigma_{23}(U_{13}U_{23}U_{12}^*))(1 \otimes 1 \otimes a) \\ &= \text{Ad}(\Sigma_{23}(U_{12}^*U_{23}))(1 \otimes 1 \otimes a) \\ &= U_{13}^*\hat{U}_{23}^*(1 \otimes 1 \otimes a)\hat{U}_{23}U_{13} \\ &= U_{13}^*(1 \otimes \lambda_I(a))U_{13} \\ &= (\Sigma^{-1} \otimes id)((id \otimes \alpha_I)(\lambda_I(a))), \end{aligned}$$

then  $(I, \alpha_I, \lambda_I)$  is a  $\mathbb{G}$ -Yetter-Drinfeld \*-algebra.

(iv) In the following, we work with the next well known relations

$$\begin{aligned} R(a) &= \hat{J}a^*\hat{J} \in \text{End}(A \otimes X) \\ U^* &= J_{\hat{\varphi}}(\hat{J} \otimes J) \Rightarrow U = (\hat{J} \otimes J)J_{\hat{\varphi}} \\ U^{cop} &= (\hat{J} \otimes \hat{J})U(\hat{J} \otimes \hat{J}) \\ \alpha_I(a) &:= \Delta(a) = U^*(1 \otimes a)U \\ \lambda_I(a) &:= \text{Ad}_G(a) = \hat{U}^*(1 \otimes a)\hat{U} \\ U^{\lambda_I} &= \hat{U}^*(J \otimes J)\hat{U}(J \otimes J) \end{aligned}$$

$$\lambda^\circ(x^\circ) = (J \otimes J)\lambda(x^*)(J \otimes J)$$

$$\begin{aligned} \Sigma(\Delta(a)) &= ((R \otimes R) \circ \Delta \circ R)(a) = (\hat{J} \otimes \hat{J})U^*(1 \otimes \hat{J}a\hat{J})U(\hat{J} \otimes \hat{J}) \\ &= (\hat{J} \otimes \hat{J})U^*(\hat{J} \otimes \hat{J})(1 \otimes a)(\hat{J} \otimes \hat{J})U(\hat{J} \otimes \hat{J}) \end{aligned}$$

We have that  $\alpha_I^\circ(A^\circ)$  commutes with  $\lambda_I^\circ(A^\circ)$ , because for all  $a \in A$

$$\begin{aligned} \Sigma(\alpha_I^\circ(a^\circ)) &= \Sigma((\hat{J} \otimes J)\alpha_I(a^*)(\hat{J} \otimes J)) \\ &= (J \otimes \hat{J})\Sigma(\alpha_I(a^*))(\hat{J} \otimes \hat{J}) \\ &= (J \otimes \hat{J})(\hat{J} \otimes \hat{J})U^*(\hat{J} \otimes \hat{J})(1 \otimes a^*)(\hat{J} \otimes \hat{J})U(\hat{J} \otimes \hat{J})(J \otimes \hat{J}) \\ &= (J\hat{J} \otimes 1)U^*(1 \otimes \hat{J}a^*\hat{J})U(\hat{J}J \otimes 1) \\ &= (J \otimes J)(\hat{J} \otimes J)U^*(1 \otimes \hat{J}a^*\hat{J})U(\hat{J} \otimes J)(J \otimes J) \\ &= (J \otimes J)U(\hat{J} \otimes J)(1 \otimes \hat{J}a^*\hat{J})(\hat{J} \otimes J)U^*(J \otimes J) \\ &= (J \otimes J)U(1 \otimes J\hat{J}a^*\hat{J}J)U(J \otimes J) \end{aligned}$$

$$\begin{aligned} \Sigma(\lambda_I^\circ(a^\circ)) &= \Sigma((J \otimes J)\lambda_I(a^*)(J \otimes J)) \\ &= \Sigma((J \otimes J)\hat{U}^*(1 \otimes a^*)\hat{U}(J \otimes J)) \\ &= (J \otimes J)U(a^* \otimes 1)U^*(J \otimes J) \end{aligned}$$

We define the injective \*-homomorphism

$$\begin{aligned} \Psi : \widehat{\mathbb{G}} \times_{\lambda_I} I &\rightarrow \text{End}(A \otimes I) \\ X &\mapsto \text{Ad}((J\hat{J} \otimes 1)\hat{U})(X) \end{aligned}$$

then, for all  $a \in A, b \in I$

$$\begin{aligned} \Psi((a \otimes 1_{\mathcal{M}(X)})\lambda_I(b)) &= \text{Ad}(J\hat{J} \otimes 1)(\hat{U}((a \otimes 1)\lambda_I(b))\hat{U}^*) \\ &= \text{Ad}(J\hat{J} \otimes 1)(\hat{U}(a \otimes 1)\hat{U}^*\hat{U}\lambda_I(b)\hat{U}^*) \\ &= \text{Ad}(J\hat{J} \otimes 1)(\Sigma(\Delta(a))(1 \otimes b)) \\ &= \text{Ad}(J\hat{J} \otimes 1)(\Sigma(\Delta(a))(b \otimes 1)) \end{aligned}$$

this implies that  $\Psi(\widehat{\mathbb{G}} \times_{\lambda_I} I) = \text{Ad}(J\hat{J})(I) \otimes A$ , and then

$$\Psi(\mathcal{M}(\widehat{\mathbb{G}} \times_{\lambda_I} I)) = \mathcal{M}(\text{Ad}(J\hat{J})(I) \otimes A).$$



We have the relation

$$\begin{aligned}
 \text{Ad}((J\hat{J} \otimes 1)\hat{U})(\beta(a)) &= \text{Ad}((J\hat{J} \otimes 1)\hat{U}U_\varphi^{\lambda_I})(\alpha_I^\circ(a^\circ)) \\
 &= \text{Ad}((J\hat{J} \otimes 1)\hat{U}U_\varphi^{\lambda_I}(\hat{J} \otimes J))(\alpha_I(a^*)) \\
 &= \text{Ad}((J\hat{J} \otimes 1)(J \otimes J)\hat{U}(J \otimes J)(\hat{J} \otimes J))(\alpha_I(a^*)) \\
 &= \text{Ad}((J\hat{J} \otimes J\hat{J})(J \otimes \hat{J})\Sigma(\tilde{J}_{\alpha_I})(J \otimes \hat{J})(J \otimes J)(\hat{J} \otimes J))(\alpha_I(a^*)) \\
 &= \text{Ad}((J\hat{J} \otimes J\hat{J})\hat{U}^*(\hat{J} \otimes \hat{J}))(\alpha_I(a^*)) \\
 &= \text{Ad}((J\hat{J} \otimes J\hat{J})\hat{U}^*)((R \otimes R)\Delta(a)) \\
 &= \text{Ad}((J\hat{J} \otimes J\hat{J})\hat{U}^*)(\Sigma(\Delta(R(a)))) \\
 &= \text{Ad}(J\hat{J} \otimes J\hat{J})(\Sigma(U\Delta(R(a))U^*)) \\
 &= \text{Ad}(J\hat{J} \otimes J\hat{J})(R(a) \otimes 1),
 \end{aligned}$$

for all  $a \in I$ , then  $\Psi(\beta(I)) \subseteq \mathcal{M}(\text{Ad}(J\hat{J})(I) \otimes A)$ . This implies that  $\beta(I) \subseteq \mathcal{M}(\widehat{\mathbb{G}} \rtimes_{\lambda_I} I)$ , i.e., the  $\widehat{\mathbb{G}}$ -Yetter-Drinfeld algebra  $(I, \alpha_I, \lambda_I)$  is braided commutative.  $\square$

**Definition 2.8.20.** The sub- $*$ -algebra  $I$  above is called a *quotient type left coideal* of  $\widehat{\mathbb{G}}$ .

**Remark 67.** Let  $\mathbb{G} = (A, \Delta, \varphi)$  be an algebraic quantum group. If we consider the non-degenerate morphism  $\varepsilon : A \rightarrow \mathbb{C}$ , then  $(\mathbb{C}, id_{\mathbb{C}}, id_{\mathbb{C}})$  is a closed quantum subgroup of  $\mathbb{G}$ . Moreover,  $A^{\alpha^\varepsilon} = A^{id_A} = A$  is a quotient type left coideal of  $\mathbb{G}$ , this implies that  $(A, \Delta, \text{Ad}_{\mathbb{G}})$  is a braided commutative  $\mathbb{G}$ -Yetter-Drinfeld algebra.

**Remark 68.** Let  $\lambda_I : I \rightarrow \mathcal{M}(\widehat{A} \otimes I)$  be the left action of  $\widehat{\mathbb{G}}$  on  $I$  defined in the last proposition. Then dual left action associated with  $\lambda_I$  is given by

$$\begin{aligned}
 \triangleright_I : A \otimes I &\rightarrow I \\
 a \otimes x &\mapsto (id \otimes a)(U(x \otimes 1)U^*)
 \end{aligned}$$

and, by Lemma 2.4.10, we have

$$a \triangleright_I x = (id \otimes a)(U(x \otimes 1)U^*) = (id \otimes \varepsilon)\Delta(a_{(1)})xS(a_{(2)}) = a_{(1)}xS(a_{(2)}).$$

for all  $a \in A$  and  $x \in I$ . The sub- $*$ -algebra  $I$  with the left coaction  $\alpha_I$  and the left action  $\triangleright_I$  is a braided commutative  $\widehat{\mathbb{G}}$ -Yetter-Drinfeld algebra in the algebraic sense, because by simple calculation, we have

$$\alpha_I(a \triangleright_I x) = \alpha_I(a_{(1)}xS(a_{(2)})) = a_{(1)}x_{(1)}S(a_{(3)}) \otimes (a_{(2)} \triangleright_I x_{(2)})$$

and

$$y_{(2)}(S^{-1}(y_{(1)}) \triangleright_I x) = y_{(3)}S^{-1}(y_{(2)})xS(S^{-1}(y_{(1)})) = xy,$$

for all  $x, y \in X$  and  $a \in A$ .

## Chapter N° 3

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### *Algebraic quantum transformation groupoids*

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Having prepared in the previous chapters all the necessary results, here we give a direct construction of a measured multiplier Hopf  $*$ -algebroid called an *algebraic quantum transformation groupoid*. The main ingredient in this construction is a braided commutative Yetter-Drinfeld  $*$ -algebra over an algebraic quantum group.

Fix a braided commutative  $\mathbb{G}$ -Yetter-Drinfeld  $*$ -algebra  $(X, \alpha, \hat{\alpha})$ .

### 3.1 The multiplier Hopf $*$ -algebroid structure

In this section, we will construct an structure of left multiplier bialgebroid. Then using Proposition 1.3.37, we will have a multiplier Hopf  $*$ -algebroid. As a first step, it is no hard to see that the tuple  $(\mathbb{G} \ltimes_{\alpha} X, X, \alpha, \beta)$  is a left quantum graph. The following subsection aims to show that we can equip with a left comultiplication to our left quantum graph.

#### 3.1.1 The left multiplier bialgebroid structure

We recall the modules necesaries for the formulation of the comultiplication of our left quantum graph.

- The module  $(\mathbb{G} \ltimes_{\alpha} X)^{\alpha(X)}$  is the algebra  $\mathbb{G} \ltimes_{\alpha} X$  with module structure given by

$$\alpha(y)(\varphi_a \otimes 1_{\mathcal{M}(X)}) \triangleleft \alpha(x) = \beta(x)\alpha(y)(\varphi_a \otimes 1_{\mathcal{M}(X)})$$

for all  $x, y \in X$  and  $a \in A$ .

- The module  ${}_{\alpha(X)}(\mathbb{G} \ltimes_{\alpha} X)$  is the algebra  $\mathbb{G} \ltimes_{\alpha} X$  with module structure given by

$$\alpha(x) \triangleright \alpha(y)(\varphi_a \otimes 1_{\mathcal{M}(X)}) = \alpha(xy)(\varphi_a \otimes 1_{\mathcal{M}(X)})$$

for all  $x, y \in X$  and  $a \in A$ .

**Notation 13.** From now on, we denote

$$(\mathbb{G} \ltimes_{\alpha} X) \overset{\alpha}{\otimes} (\mathbb{G} \ltimes_{\alpha} X)$$

for the balanced algebraic tensor product between  $(\mathbb{G} \ltimes_{\alpha} X)^{\alpha(X)}$  and  ${}_{\alpha(X)}(\mathbb{G} \ltimes_{\alpha} X)$ .

### 3.1.2 The comultiplication

Fix  $x \in X$  and  $\omega \in \widehat{A}$ . If  $P = (\theta \otimes 1_{\mathcal{M}(X)})\alpha(y)$  with  $\theta \in \widehat{A}$  and  $y \in X$ . We write

$$\alpha(x)(\omega_{(2)} \otimes 1_{\mathcal{M}(X)})P \otimes (\omega_{(1)} \otimes 1_{\mathcal{M}(X)}) := \alpha(x)(\omega_{(2)} \cdot \theta \otimes 1_{\mathcal{M}(X)})\alpha(y') \otimes (\omega_{(1)} \otimes 1_{\mathcal{M}(X)})$$

and

$$\alpha(x)(\omega_{(2)} \otimes 1_{\mathcal{M}(X)}) \otimes (\omega_{(1)} \otimes 1_{\mathcal{M}(X)})P := \alpha(x)(\omega_{(2)} \otimes 1_{\mathcal{M}(X)}) \otimes (\omega_{(1)} \cdot \theta \otimes 1_{\mathcal{M}(X)})\alpha(y).$$

Both are elements of  $(\mathbb{G} \ltimes_{\alpha} X) \otimes (\mathbb{G} \ltimes_{\alpha} X)$ . Here, we use the Sdeedler notation

$$\omega_{(1)} \otimes \omega_{(2)} \cdot \theta = \widehat{\Delta}(\omega)(1_{\mathcal{M}(\widehat{A})} \otimes \theta) \in \widehat{A} \otimes \widehat{A},$$

$$\omega_{(1)} \cdot \theta \otimes \omega_{(2)} = \widehat{\Delta}(\omega)(\theta \otimes 1_{\mathcal{M}(\widehat{A})}) \in \widehat{A} \otimes \widehat{A},$$

We define the linear map

$$\begin{aligned} \Delta_{\alpha}(\alpha(x)(\omega \otimes 1_{\mathcal{M}(X)})) : (\mathbb{G} \ltimes_{\alpha} X) \otimes (\mathbb{G} \ltimes_{\alpha} X) &\rightarrow (\mathbb{G} \ltimes_{\alpha} X) \overset{\alpha}{\otimes} (\mathbb{G} \ltimes_{\alpha} X) \\ P \otimes Q &\mapsto \alpha(x)(\omega_{(2)} \otimes 1_{\mathcal{M}(X)})P \overset{\alpha}{\otimes} (\omega_{(1)} \otimes 1_{\mathcal{M}(X)})Q \end{aligned}$$

**Lemma 3.1.1.** For all  $y \in X$  and  $P, Q \in \mathbb{G} \ltimes_{\alpha} X$ , we have

$$\Delta_{\alpha}(\alpha(x)(\omega \otimes 1_{\mathcal{M}(X)}))(P \triangleleft \alpha(y) \otimes Q) = \Delta_{\alpha}(\alpha(x)(\omega \otimes 1_{\mathcal{M}(X)}))(P \otimes \alpha(y) \triangleright Q).$$

*Proof* The result follows from the equality

$$\begin{aligned} \Delta_{\alpha}(\alpha(x)(\omega \otimes 1_{\mathcal{M}(X)}))(P \triangleleft \alpha(y) \otimes Q) &= \Delta_{\alpha}(\alpha(x)(\omega \otimes 1_{\mathcal{M}(X)}))(\beta(y)P \otimes Q) \\ &= \alpha(x)(\omega_{(2)} \otimes 1)\beta(y)P \overset{\alpha}{\otimes} (\omega_{(1)} \otimes 1)Q \\ &= \alpha(x)\beta(y')(\omega' \otimes 1) \overset{\alpha}{\otimes} (\omega_{(1)} \otimes 1)Q \\ &= \beta(y')\alpha(x)(\omega' \otimes 1) \overset{\alpha}{\otimes} (\omega_{(1)} \otimes 1)Q \\ &= \alpha(x)(\omega' \otimes 1) \overset{\alpha}{\otimes} \alpha(y')(\omega_{(1)} \otimes 1)Q \\ &= \alpha(x)(\omega' \otimes 1) \overset{\alpha}{\otimes} (\omega'' \otimes 1)\alpha(y)Q \\ &= \alpha(x)(\omega_{(2)} \otimes 1) \overset{\alpha}{\otimes} (\omega_{(1)} \otimes 1)\alpha(y)Q \\ &= \Delta_{\alpha}(\alpha(x)(\omega \otimes 1_{\mathcal{M}(X)}))(P \otimes \alpha(y) \triangleright Q) \end{aligned}$$

□

Using the lemma above, we have

$$\Delta_\alpha(\alpha(x)(\omega \otimes 1_{\mathcal{M}(X)})) \in \text{End}((\mathbb{G} \ltimes_\alpha X) \overset{\alpha}{\otimes} (\mathbb{G} \ltimes_\alpha X)).$$

Also, for simplify our notations, we denote

$$(\mathbb{G} \ltimes_\alpha X) \overset{\alpha}{\times} (\mathbb{G} \ltimes_\alpha X)$$

for the generalized left Takeuchi product of the modules  $(\mathbb{G} \ltimes_\alpha X)^{\alpha(X)}$  and  ${}_{\alpha(X)}(\mathbb{G} \ltimes_\alpha X)$  (Notation 9 and Definition 1.3.3).

**Lemma 3.1.2.** *We have*

$$\Delta_\alpha(\alpha(x)(\omega \otimes 1_{\mathcal{M}(X)})) \in (\mathbb{G} \ltimes_\alpha X) \overset{\alpha}{\times} (\mathbb{G} \ltimes_\alpha X).$$

*Proof* Fix  $P, Q \in \mathbb{G} \ltimes_\alpha X$ , if we define

$$\Delta_\alpha(\alpha(x)(\omega \otimes 1_{\mathcal{M}(X)}))(P \otimes 1) := \alpha(x)(\omega_{(2)} \otimes 1_{\mathcal{M}(X)})P \overset{\alpha}{\otimes} (\omega_{(1)} \otimes 1_{\mathcal{M}(X)}) \in (\mathbb{G} \ltimes_\alpha X) \overset{\alpha}{\otimes} (\mathbb{G} \ltimes_\alpha X),$$

$$\Delta_\alpha(\alpha(x)(\omega \otimes 1_{\mathcal{M}(X)}))(1 \otimes Q) := \alpha(x)(\omega_{(2)} \otimes 1_{\mathcal{M}(X)}) \overset{\alpha}{\otimes} (\omega_{(1)} \otimes 1_{\mathcal{M}(X)})Q \in (\mathbb{G} \ltimes_\alpha X) \overset{\alpha}{\otimes} (\mathbb{G} \ltimes_\alpha X),$$

then

$$\begin{aligned} \Delta_\alpha(\alpha(x)(\omega \otimes 1_{\mathcal{M}(X)}))(P \otimes Q) &= \alpha(x)(\omega_{(2)} \otimes 1_{\mathcal{M}(X)})P \overset{\alpha}{\otimes} (\omega_{(1)} \otimes 1_{\mathcal{M}(X)})Q \\ &= (\alpha(x)(\omega_{(2)} \otimes 1_{\mathcal{M}(X)})P \overset{\alpha}{\otimes} (\omega_{(1)} \otimes 1_{\mathcal{M}(X)}))(1 \otimes Q) \\ &= (\Delta_\alpha(\alpha(x)(\omega \otimes 1_{\mathcal{M}(X)}))(P \otimes 1))(1 \otimes Q) \end{aligned}$$

and

$$\begin{aligned} \Delta_\alpha(\alpha(x)(\omega \otimes 1_{\mathcal{M}(X)}))(P \otimes Q) &= \alpha(x)(\omega_{(2)} \otimes 1_{\mathcal{M}(X)})P \overset{\alpha}{\otimes} (\omega_{(1)} \otimes 1_{\mathcal{M}(X)})Q \\ &= (\alpha(x)(\omega_{(2)} \otimes 1_{\mathcal{M}(X)}) \overset{\alpha}{\otimes} (\omega_{(1)} \otimes 1_{\mathcal{M}(X)})Q)(P \otimes 1) \\ &= (\Delta_\alpha(\alpha(x)(\omega \otimes 1_{\mathcal{M}(X)}))(1 \otimes Q))(P \otimes 1) \end{aligned}$$

□

The next lemma shows that  $\Delta_\alpha$  is well defined

**Lemma 3.1.3.** *Let  $x, x' \in X$  and  $\omega, \omega' \in \widehat{A}$  such that*

$$\alpha(x)(\omega \otimes 1_{\mathcal{M}(X)}) = \alpha(x')(\omega' \otimes 1_{\mathcal{M}(X)})$$

*as elements in  $\mathbb{G} \ltimes_\alpha X$ , then*

$$\Delta_\alpha(\alpha(x)(\omega \otimes 1_{\mathcal{M}(X)})) = \Delta_\alpha(\alpha(x')(\omega' \otimes 1_{\mathcal{M}(X)})).$$

*Proof* Because  $\alpha(x)(\omega \otimes 1_{\mathcal{M}(X)}) = \alpha(x')(\omega' \otimes 1_{\mathcal{M}(X)})$  in  $\mathbb{G} \ltimes_\alpha X$ , then applying the dual action  $\tilde{\alpha}$ , we have

$$\omega_{(1)} \otimes \alpha(x)(\omega_{(2)} \otimes 1_{\mathcal{M}(X)}) = \omega'_{(1)} \otimes \alpha(x')(\omega'_{(2)} \otimes 1_{\mathcal{M}(X)})$$

as elements in  $\mathcal{M}(\widehat{A} \otimes \mathbb{G} \ltimes_{\alpha} X)$ , then given  $P \in \mathbb{G} \ltimes_{\alpha} X$ , we have

$$\omega_{(1)} \otimes \alpha(x)(\omega_{(2)} \otimes 1_{\mathcal{M}(X)})P = \omega'_{(1)} \otimes \alpha(x')(\omega'_{(2)} \otimes 1_{\mathcal{M}(x)})P$$

in  $\widehat{A} \otimes \mathbb{G} \ltimes_{\alpha} X$ . Finally we can see that

$$\begin{aligned} \Delta_{\alpha}(\alpha(x)(\omega \otimes 1_{\mathcal{M}(X)}))(P \overset{\alpha}{\otimes} Q) &= \alpha(x)(\omega_{(2)} \otimes 1_{\mathcal{M}(X)})P \overset{\alpha}{\otimes} (\omega_{(1)} \otimes 1_{\mathcal{M}(X)})Q \\ &= \alpha(x')(\omega'_{(2)} \otimes 1_{\mathcal{M}(x)})P \overset{\alpha}{\otimes} (\omega'_{(1)} \otimes 1_{\mathcal{M}(X)})Q \\ &= \Delta_{\alpha}(\alpha(x')(\omega' \otimes 1_{\mathcal{M}(X)}))(P \overset{\alpha}{\otimes} Q) \end{aligned}$$

for every  $P, Q \in \mathbb{G} \ltimes_{\alpha} X$ . □

**Proposition 3.1.4.** *The map*

$$\Delta_{\alpha} : \mathbb{G} \ltimes_{\alpha} X \rightarrow (\mathbb{G} \ltimes_{\alpha} X) \overset{\alpha}{\times} (\mathbb{G} \ltimes_{\alpha} X)$$

is a \*-homomorphism, such that

(1) *For all  $x, x', y, y' \in X$  and  $P \in \mathbb{G} \ltimes_{\alpha} X$ , we have*

$$\Delta_{\alpha}(\alpha(x)\beta(y)P\alpha(x')\beta(y')) = (\alpha(x) \otimes \beta(y))\Delta_{\alpha}(P)(\alpha(x') \otimes \beta(y')).$$

(2) *For all  $P, Q, N \in \mathbb{G} \ltimes_{\alpha} X$ , we have*

$$((\Delta_{\alpha} \otimes id)(\Delta_{\alpha}(Q)(1 \otimes N)))(P \otimes 1 \otimes 1) = ((id \otimes \Delta_{\alpha})(\Delta_{\alpha}(Q)(P \otimes 1)))(1 \otimes 1 \otimes N)$$

.

*Proof* The map  $\Delta_{\alpha}$  is a \*-homomorphism because  $\widehat{\Delta}^{cop}$  is a \*-homomorphism, and we have the equalities

$$\Delta_{\alpha}(\omega \otimes 1_{\mathcal{M}(X)})(\theta \otimes 1_{\mathcal{M}(X)}) \overset{\alpha}{\otimes} 1_{\mathcal{M}(\mathbb{G} \ltimes_{\alpha} X)} = (\omega_{(2)} \cdot \theta \otimes 1_{\mathcal{M}(X)}) \overset{\alpha}{\otimes} (\omega_{(1)} \otimes 1_{\mathcal{M}(X)}),$$

$$\Delta_{\alpha}(\alpha(x)) = \alpha(x) \overset{\alpha}{\otimes} 1_{\mathcal{M}(\mathbb{G} \ltimes_{\alpha} X)}$$

and

$$\begin{aligned}
\Delta_\alpha(\beta(x)) &= \Delta_\alpha(\widehat{S}(x^{[-1]}) \otimes 1_{\mathcal{M}(X)})\alpha(x^{[0]}) \\
&= \Delta_\alpha(\widehat{S}(x^{[-1]}) \otimes 1_{\mathcal{M}(X)})\Delta_\alpha(\alpha(x^{[0]})) \\
&= ((\widehat{S}(x^{[-1]})_{(2)} \otimes 1_{\mathcal{M}(X)}) \otimes ((\widehat{S}(x^{[-1]})_{(1)} \otimes 1_{\mathcal{M}(X)}))(\alpha(x^{[0]}) \otimes 1) \\
&= (\widehat{S}(x_{(1)}^{[-1]}) \otimes 1_{\mathcal{M}(X)})\alpha(x^{[0]}) \otimes (\widehat{S}(x_{(2)}^{[-1]}) \otimes 1_{\mathcal{M}(X)}) \\
&= (\widehat{S}(x^{[0]^{[-1]}}) \otimes 1_{\mathcal{M}(X)})\alpha(x^{[0]^{[0]}}) \otimes (\widehat{S}(x^{[-1]}) \otimes 1_{\mathcal{M}(X)}) \\
&= \beta(x^{[0]}) \otimes (\widehat{S}(x^{[-1]}) \otimes 1_{\mathcal{M}(X)}) \\
&= 1 \overset{\alpha}{\otimes} \alpha(x^{[0]})(\widehat{S}(x^{[-1]}) \otimes 1_{\mathcal{M}(X)}) \\
&= 1 \overset{\alpha}{\otimes} \beta(x).
\end{aligned}$$

Then:

(1) We have

$$\begin{aligned}
\Delta_\alpha(\alpha(x)\alpha(y)(\omega \otimes 1_{\mathcal{M}(X)})\alpha(x'))[P \overset{\alpha}{\otimes} Q] &= \Delta_\alpha(\alpha(xy)\alpha(x'_{[0]})(\omega_{S^{-1}(x'_{[-1]})} \otimes 1_{\mathcal{M}(X)})[P \overset{\alpha}{\otimes} Q] \\
&= \Delta_\alpha(\alpha(xy x'_{[0]}) (\omega_{S^{-1}(x'_{[-1]})} \otimes 1_{\mathcal{M}(X)}) [P \overset{\alpha}{\otimes} Q] \\
&= \alpha(xy x'_{[0]}) ((\omega_{S^{-1}(x'_{[-1]})})_{(2)} \otimes 1_{\mathcal{M}(X)}) P \overset{\alpha}{\otimes} ((\omega_{S^{-1}(x'_{[-1]})})_{(1)} \otimes 1_{\mathcal{M}(X)}) Q \\
&= \alpha(x)\alpha(y)\alpha(x'_{[0]}) ((\omega_{(2)})_{S^{-1}(x'_{[-1]})} \otimes 1_{\mathcal{M}(X)}) P \overset{\alpha}{\otimes} (\omega_{(1)} \otimes 1_{\mathcal{M}(X)}) Q \\
&= \alpha(x)\alpha(y)(\omega_{(2)} \otimes 1_{\mathcal{M}(X)})\alpha(x') P \overset{\alpha}{\otimes} (\omega_{(1)} \otimes 1_{\mathcal{M}(X)}) Q \\
&= (\alpha(x) \otimes 1)[\alpha(y)(\omega_{(2)} \otimes 1_{\mathcal{M}(X)})\alpha(x') P \overset{\alpha}{\otimes} (\omega_{(1)} \otimes 1_{\mathcal{M}(X)}) Q] \\
&= (\alpha(x) \otimes 1)\Delta_\alpha(\alpha(y)(\omega \otimes 1_{\mathcal{M}(X)}))[\alpha(x') P \overset{\alpha}{\otimes} Q] \\
&= (\alpha(x) \otimes 1)\Delta_\alpha(\alpha(y)(\omega \otimes 1_{\mathcal{M}(X)}))(\alpha(x') \otimes 1)[P \overset{\alpha}{\otimes} Q]
\end{aligned}$$

and

$$\Delta_\alpha(\beta(x)P\beta(x')) = \Delta_\alpha(\beta(x))\Delta_\alpha(P)\Delta_\alpha(\beta(x')) = (1 \otimes \beta(x))\Delta_\alpha(P)(1 \otimes \beta(x')),$$

for all  $P, Q \in \mathbb{G} \times_\alpha X$ ,  $x, x', y \in X$  and  $\omega \in \widehat{A}$ .

(2) Follows from the coassociativity of the comultiplication of the multiplier Hopf \*-algebra  $(\widehat{A}, \widehat{\Delta}^{cop})$ . □

**Proposition 3.1.5.** *The tuple  $\mathcal{A} = (\mathbb{G} \times_\alpha X, X, \alpha, \beta, \Delta_\alpha)$  is a multiplier bialgebroid.*

*Proof* Follows directly from Corollary 3.1.4. □

### 3.1.3 The counit and the antipode

Define the linear maps

$$\begin{aligned} \tilde{\varepsilon} : \quad \mathbb{G} \ltimes_{\alpha} X &\rightarrow \alpha(X) \\ \alpha(x)(\omega \otimes 1_{\mathcal{M}(X)}) &\mapsto \widehat{\psi}(\omega)\alpha(x) \end{aligned}$$

$$\begin{aligned} \tilde{S} : \quad \mathbb{G} \ltimes_{\alpha} X &\rightarrow \mathbb{G} \ltimes_{\alpha} X \\ \alpha(x)(\omega \otimes 1_{\mathcal{M}(X)}) &\mapsto (\widehat{S}(x^{[-1]} \cdot \omega) \otimes 1_{\mathcal{M}(X)})\alpha(x^{[0]}) \end{aligned} \quad .$$

**Remark 69.** Obviously,

$$\tilde{S}(\omega \otimes 1_{\mathcal{M}(X)}) = \widehat{S}(\omega) \otimes 1_{\mathcal{M}(X)}, \quad \tilde{S}(\alpha(x)) = (\widehat{S}(x^{[-1]}) \otimes 1_{\mathcal{M}(X)})\alpha(x^{[0]}) = \beta(x)$$

**Lemma 3.1.6.** We have  $\tilde{S}(\beta(x)) = \alpha(x)$  for all  $x \in X$ .

*Proof*

$$\begin{aligned} \tilde{S}(\beta(x)) &= \tilde{S}((\widehat{S}(x^{[-1]}) \otimes 1_{\mathcal{M}(X)})\alpha(x^{[0]})) \\ &= \tilde{S}(\alpha((x^{[0]})_{[0]}))(\widehat{S}(x^{[-1]})_{S^{-1}((x^{[0]})_{[-1]})} \otimes 1_{\mathcal{M}(X)}) \\ &= \alpha(x) \end{aligned}$$

□

**Proposition 3.1.7.** The linear map  $\tilde{S}$  is an anti-isomorphism such that

$$\tilde{S}(\alpha(x)\beta(y)P\alpha(x')\beta(y')) = \alpha(y')\beta(x')\tilde{S}(P)\alpha(y)\beta(x),$$

for all  $x, x', y, y' \in X$  and  $P \in \mathbb{G} \ltimes_{\alpha} X$ .

*Proof* We know that  $\mathbb{G} \ltimes_{\alpha} X$  is generated by elements of the form  $(\omega \otimes 1_{\mathcal{M}(X)})\alpha(x)$  with  $\omega \in \widehat{A}$  and  $x \in X$ . Then, the equalities

$$\begin{aligned} \tilde{S}(\omega \otimes 1_{\mathcal{M}(X)})\tilde{S}(\alpha(x)) &= (\widehat{S}(\omega) \otimes 1_{\mathcal{M}(X)})(\widehat{S}(x^{[-1]}) \otimes 1_{\mathcal{M}(X)})\alpha(x^{[0]}) \\ &= (\widehat{S}(\omega) \cdot \widehat{S}(x^{[-1]}) \otimes 1_{\mathcal{M}(X)})\alpha(x^{[0]}) \\ &= \tilde{S}(\alpha(x)(\omega \otimes 1_{\mathcal{M}(X)})) \end{aligned}$$

and

$$\begin{aligned} \tilde{S}(\alpha(x))\tilde{S}(\omega \otimes 1_{\mathcal{M}(X)}) &= (\widehat{S}(x^{[-1]}) \otimes 1_{\mathcal{M}(X)})\alpha(x^{[0]})(\widehat{S}(\omega) \otimes 1_{\mathcal{M}(X)}) \\ &= (\widehat{S}(x^{[-1]}) \cdot \widehat{S}(\omega)_{(x^{[0]})_{[-1]}} \otimes 1_{\mathcal{M}(X)})\alpha((x^{[0]})_{[0]}) \\ &= (\widehat{S}(x^{[-1]}) \cdot \widehat{S}(\omega)_{(x^{[0]})_{[-1]}} \otimes 1_{\mathcal{M}(X)})\alpha((x^{[0]})_{[0]}) \\ &= (\widehat{S}(\omega_{S^{-1}(x_{[-1]})}) \cdot \widehat{S}((x_{[0]})^{[-1]}) \otimes 1_{\mathcal{M}(X)})\alpha((x_{[0]})^{[0]}) \\ &= \tilde{S}(\alpha(x_{[0]})(\omega_{S^{-1}(x_{[-1]})} \otimes 1_{\mathcal{M}(X)})) \\ &= \tilde{S}((\omega \otimes 1_{\mathcal{M}(X)})\alpha(x)) \end{aligned}$$

for all  $\omega \in \widehat{A}$  and  $x \in X$ , imply that  $\tilde{S}$  is an anti-isomorphism. Moreover, take  $P = \alpha(x)(\omega \otimes 1_{\mathcal{M}(X)})$ , then we have

$$\begin{aligned}
\tilde{S}(\alpha(y)P) &= \tilde{S}(\alpha(y)\alpha(x)(\omega \otimes 1_{\mathcal{M}(X)})) \\
&= \tilde{S}(\alpha(yx)(\omega \otimes 1_{\mathcal{M}(X)})) \\
&= (\widehat{S}(\omega) \cdot \widehat{S}(y^{[-1]}x^{[-1]}) \otimes 1_{\mathcal{M}(X)})\alpha(y^{[0]}x^{[0]}) \\
&= (\widehat{S}(\omega) \cdot \widehat{S}(x^{[-1]}) \otimes 1_{\mathcal{M}(X)})(\widehat{S}(y^{[-1]}) \otimes 1_{\mathcal{M}(X)})\alpha(y^{[0]})\alpha(x^{[0]}) \\
&= (\widehat{S}(\omega) \cdot \widehat{S}(x^{[-1]}) \otimes 1_{\mathcal{M}(X)})\beta(y)\alpha(x^{[0]}) \\
&= (\widehat{S}(\omega) \cdot \widehat{S}(x^{[-1]}) \otimes 1_{\mathcal{M}(X)})\alpha(x^{[0]})\beta(y) \\
&= \tilde{S}(P)\beta(y),
\end{aligned}$$

Similar, we can show the next equalities

$$\tilde{S}(\beta(y)P) = \tilde{S}(P)\tilde{S}(\beta(y)) = \tilde{S}(P)\alpha(y),$$

$$\tilde{S}(P\alpha(y)) = \tilde{S}(\alpha(y))\tilde{S}(P) = \beta(y)\tilde{S}(P), \quad \tilde{S}(P\beta(y)) = \tilde{S}(\beta(y))\tilde{S}(P) = \alpha(y)\tilde{S}(P).$$

□

**Proposition 3.1.8.**

(i) We have

$$\tilde{\varepsilon}(\alpha(y)\alpha(x)(\omega \otimes 1_{\mathcal{M}(X)})) = \alpha(y)\tilde{\varepsilon}(\alpha(x)(\omega \otimes 1_{\mathcal{M}(X)}))$$

$$\tilde{\varepsilon}(\beta(y)\alpha(x)(\omega \otimes 1_{\mathcal{M}(X)})) = \tilde{\varepsilon}(\alpha(x)(\omega \otimes 1_{\mathcal{M}(X)}))\alpha(y)$$

(ii)

$$(\varepsilon \overline{\otimes} id)(\Delta_\alpha(\alpha(x)(\omega \otimes 1_{\mathcal{M}(X)}))(1 \otimes \alpha(y)(\theta \otimes 1_{\mathcal{M}(X)}))) = \alpha(x)(\omega \otimes 1_{\mathcal{M}(X)})\alpha(y)(\theta \otimes 1_{\mathcal{M}(X)})$$

*Proof*

(i) Fix  $x, y \in X$  and  $\omega \in \widehat{A}$ . We have

$$\begin{aligned}
\tilde{\varepsilon}(\alpha(y)\alpha(x)(\omega \otimes 1_{\mathcal{M}(X)})) &= \tilde{\varepsilon}(\alpha(yx)(\omega \otimes 1_{\mathcal{M}(X)})) \\
&= \widehat{\varphi}(\omega)\alpha(yx) \\
&= \alpha(y)\widehat{\varphi}(\omega)\alpha(x) \\
&= \alpha(y)\tilde{\varepsilon}(\alpha(x)(\omega \otimes 1_{\mathcal{M}(X)})).
\end{aligned}$$

and

$$\tilde{\varepsilon}(\beta(y)\alpha(x)(\omega \otimes 1_{\mathcal{M}(X)})) = \tilde{\varepsilon}(\alpha(x)(\omega \otimes 1_{\mathcal{M}(X)}))\alpha(y)$$

(ii) Follows from the item (i) of Proposition 3.1.4 and Proposition 1.2.3

□

Now, we can conclude the principal result of this section



**Theorem 3.1.9.** *Let  $(X, \alpha, \widehat{\alpha})$  be a braided commutative  $\mathbb{G}$ -Yetter-Drinfeld\*-algebra. Then, the tuple*

$$(\mathbb{G} \ltimes_{\alpha} X, X, \alpha, \beta, \Delta_{\alpha}),$$

where

$$\begin{aligned} \Delta_{\alpha} : \quad \mathbb{G} \ltimes_{\alpha} X &\rightarrow (\mathbb{G} \ltimes_{\alpha} X) \overset{\alpha}{\times} (\mathbb{G} \ltimes_{\alpha} X) \\ \omega \otimes 1_{\mathcal{M}(X)} &\mapsto (\omega_{(2)} \otimes 1_{\mathcal{M}(X)}) \overset{\alpha}{\times} (\omega_{(1)} \otimes 1_{\mathcal{M}(X)}) \\ \alpha(x) &\mapsto \alpha(x) \overset{\alpha}{\times} 1_{\mathcal{M}(\mathbb{G} \ltimes_{\alpha} X)} \end{aligned}$$

is a multiplier Hopf\*-algebroid with antipode given by

$$\begin{aligned} \widetilde{S} : \quad \mathbb{G} \ltimes_{\alpha} X &\rightarrow \mathbb{G} \ltimes_{\alpha} X \\ \alpha(x)(\omega \otimes 1_{\mathcal{M}(X)}) &\mapsto (\widehat{S}(x^{[-1]} \cdot \omega) \otimes 1_{\mathcal{M}(X)})\alpha(x^{[0]}) \end{aligned}$$

and counit given by

$$\begin{aligned} \widetilde{\varepsilon} : \quad \mathbb{G} \ltimes_{\alpha} X &\rightarrow \alpha(X) \\ \alpha(x)(\omega \otimes 1_{\mathcal{M}(X)}) &\mapsto \widehat{\psi}(\omega)\alpha(x). \end{aligned}$$

*Proof* Follows from Propositions 3.1.5 and 3.1.7. □

## 3.2 The measured structure

Let  $\mu : X \rightarrow \mathbb{C}$  be a Yetter-Drinfeld integral for the braided commutative  $\mathbb{G}$ -Yetter-Drinfeld\*-algebra  $(X, \alpha, \widehat{\alpha})$  with modular automorphism  $\sigma_{\mu} : X \rightarrow X$ . In this section, we give the base weight and the partial integral for the multiplier Hopf\*-algebroid found above.

### 3.2.1 The base weight, the partial and total integral

Recall the injective  $\alpha(X)$ -bimodule map constructed in Proposition 2.7.9,

$$\begin{aligned} T_{\widehat{\alpha}} : \quad \mathbb{G} \ltimes_{\alpha} X &\rightarrow \alpha(X) \\ (\omega \otimes 1_{\mathcal{M}(X)})\alpha(x) &\mapsto \widehat{\varphi}(\omega)\alpha(x) \end{aligned}$$

Then, take  $\Omega := T_{\widehat{\alpha}}$ .

**Lemma 3.2.1.** *For any  $x, y \in X$  and  $\omega, \theta \in \widehat{A}$ ,*

$$(id \overline{\otimes} \Omega)(\Delta_L(\alpha(x)(\omega \otimes 1_{\mathcal{M}(X)}))(\alpha(y)(\theta \otimes 1_{\mathcal{M}(X)}) \otimes 1)) = \Omega(\alpha(x)(\omega \otimes 1_{\mathcal{M}(X)}))\alpha(y)(\theta \otimes 1_{\mathcal{M}(X)}).$$

*Proof* Fix  $x, y \in X$  and  $\omega, \theta \in \widehat{A}$ . By direct calculation, we have

$$(id \overline{\otimes} \Omega)(\Delta_L(\alpha(x))(\alpha(y)(\theta \otimes 1_{\mathcal{M}(X)}) \otimes 1)) = (id \overline{\otimes} \Omega)(\alpha(xy)(\theta \otimes 1_{\mathcal{M}(X)}) \otimes 1)$$

and

$$\begin{aligned} (id \otimes \bar{\Omega})(\Delta_L(\omega \otimes 1_{\mathcal{M}(X)})(\alpha(y)(\theta \otimes 1_{\mathcal{M}(X)}) \otimes 1)) &= (\omega_{(2)} \otimes 1_{\mathcal{M}(X)})\alpha(y)(\theta \otimes 1_{\mathcal{M}(X)})\widehat{\psi}(\omega_{(1)}) \\ &= \widehat{\psi}(\omega)\alpha(y)(\theta \otimes 1_{\mathcal{M}(X)}) \\ &= \Omega(\omega \otimes 1_{\mathcal{M}(X)})\alpha(y)(\theta \otimes 1_{\mathcal{M}(X)}). \end{aligned}$$

□

**Proposition 3.2.2.** *The map  $\mu : X \rightarrow \mathbb{C}$  is a base weight and the map  $\Omega$  is a partial integral for Hopf\*-algebroid*

$$(\mathbb{G} \ltimes_{\alpha} X, X, \alpha, \beta, \Delta_{\alpha}, \tilde{\varepsilon}, \tilde{S})$$

following Definition 1.3.38. Moreover, the dual action  $\tilde{\mu} : \mathbb{G} \ltimes_{\alpha} X \rightarrow \mathbb{C}$  is a total integral for the multiplier Hopf\*-algebroid

$$(\mathbb{G} \ltimes_{\alpha} X, X, \alpha, \beta, \Delta_{\alpha}, \tilde{\varepsilon}, \tilde{S}).$$

*Proof* It is not hard to show that  $\mu \circ \alpha^{-1} : X \rightarrow \mathbb{C}$  is a base weight, as Definition 1.3.38, for  $(\mathbb{G} \ltimes_{\alpha} X, X, \alpha, \beta, \Delta_{\alpha}, \tilde{\varepsilon}, \tilde{S})$ . The rest of the proof follows from Proposition 1.3.22, Lemma 3.2.1 and because the positivity of  $\mu$  implies the positivity of  $\tilde{\mu} = \mu \circ \alpha^{-1} \circ \Omega = \mu \circ \alpha^{-1} \circ T_{\tilde{\alpha}}$  (Definition 2.7.10). □

We can now enunciate the most important theorem of this chapter.

**Theorem 3.2.3.** *Let  $(X, \alpha, \hat{\alpha})$  be a braided commutative  $\mathbb{G}$ -Yetter-Drinfeld\*-algebra and  $\mu$  a Yetter-Drinfeld integral on  $X$ . Then, the tuple*

$$(\mathbb{G} \ltimes_{\alpha} X, X, \alpha, \beta, \Delta_{\alpha}, \tilde{\varepsilon}, \tilde{S}, \mu, \Omega)$$

where

$$\begin{aligned} \Delta_{\alpha} : \quad \mathbb{G} \ltimes_{\alpha} X &\rightarrow (\mathbb{G} \ltimes_{\alpha} X) \overset{\alpha}{\times} (\mathbb{G} \ltimes_{\alpha} X) \\ \omega \otimes 1_{\mathcal{M}(X)} &\mapsto (\omega_{(2)} \otimes 1_{\mathcal{M}(X)}) \overset{\alpha}{\times} (\omega_{(1)} \otimes 1_{\mathcal{M}(X)}) \\ \alpha(x) &\mapsto \alpha(x) \overset{\alpha}{\times} 1_{\mathcal{M}(\mathbb{G} \ltimes_{\alpha} X)} \\ \tilde{S} : \quad \mathbb{G} \ltimes_{\alpha} X &\rightarrow \mathbb{G} \ltimes_{\alpha} X \\ \alpha(x)(\omega \otimes 1_{\mathcal{M}(X)}) &\mapsto (\widehat{S}(x^{[-1]} \cdot \omega) \otimes 1_{\mathcal{M}(X)})\alpha(x^{[0]}) \\ \tilde{\varepsilon} : \quad \mathbb{G} \ltimes_{\alpha} X &\rightarrow \alpha(X) \\ \alpha(x)(\omega \otimes 1_{\mathcal{M}(X)}) &\mapsto \widehat{\psi}(\omega)\alpha(x) \end{aligned}$$

and

$$\begin{aligned} \Omega : \quad \mathbb{G} \ltimes_{\alpha} X &\rightarrow \alpha(X) \\ (\omega \otimes 1_{\mathcal{M}(X)})\alpha(x) &\mapsto \widehat{\varphi}(\omega)\alpha(x) \end{aligned}$$

is a measured multiplier Hopf\*-algebroid, as Definition 1.3.38.

*Proof* Follows from Theorem 3.1.9 and Proposition 3.2.2. □

**Definition 3.2.4.** The measured multiplier Hopf  $*$ -algebroid associated with the braided commutative  $\mathbb{G}$ -Yetter-Drinfeld algebra  $(X, \alpha, \widehat{\alpha})$  eand with the Yetter-Drinfeld integral  $\mu$  is called an *algebraic quantum transformation groupoid* and is denoted by  $\mathfrak{G}(X, \alpha, \widehat{\alpha}, \mu)$ .

### 3.3 Examples

The construction presented in this chapter shows that any example of a braided commutative Yetter-Drinfeld  $*$ -algebra over an algebraic quantum group (see 2.8.2) gives rise to an example of an algebraic quantum transformation groupoid. The simplest of these examples if as follows.

#### 3.3.1 Usual transformation groupoids

Let  $\cdot : G \times S \rightarrow S$  be a left action of a group  $G$  on a set  $S$ . Consider the braided commutative  $\mathbb{H}$ -Yetter-Drinfeld  $*$ -algebra  $(\text{Fun}_f(S), \lambda, \widehat{\lambda})$ , where  $\mathbb{H} = (\mathbb{C}[G], \Delta', \varphi')$  and recall the maps

$$\begin{aligned} \lambda : \text{Fun}_f(S) &\rightarrow \mathcal{M}(\mathbb{H} \rtimes_{\lambda} \text{Fun}_f(S)) & \beta_{\widehat{\lambda}} : \text{Fun}_f(S) &\rightarrow \mathcal{M}(\mathbb{H} \rtimes_{\lambda} \text{Fun}_f(S)) \\ p &\mapsto \sum_{g \in G, s \in S} p(s) \delta_{(g,s)} & p &\mapsto \sum_{g \in G, s \in S} p(g \cdot s) \delta_{(g,s)}. \end{aligned}$$

Then the crossed product is given by the identification

$$\begin{aligned} \mathbb{H} \rtimes_{\lambda} \text{Fun}_f(S) &\cong \text{Fun}_f(G) \otimes \text{Fun}_f(S) \cong \text{Fun}_f(G \curvearrowright S) \\ (\delta_g \otimes 1_{\mathcal{M}(\text{Fun}_f(S))}) \lambda(p) &\mapsto \delta_g \otimes p \mapsto \sum_{s \in S} p(s) \delta_{(g,s)}, \end{aligned}$$

where  $\text{Fun}_f(G \curvearrowright S)$  is the  $*$ -algebra of complex finitely supported functions on the groupoid  $G \curvearrowright S$ . For any  $p \in \text{Fun}_f(S)$ , we have

$$p \circ d = \lambda(p) \quad \text{and} \quad p \circ r = \beta_{\widehat{\lambda}}(p),$$

as elements in  $\text{Fun}(G \curvearrowright S) \cong \mathcal{M}(\text{Fun}_f(G \curvearrowright S))$ . Denote  $L = \text{Fun}_f(S)$  and consider now, the linear maps

$$\begin{aligned} \Delta_L : \text{Fun}_f(G \curvearrowright S) &\rightarrow \text{Fun}_f(G \curvearrowright S)^{\text{Fun}_f(S)} \overline{\times}_{\text{Fun}_f(S)} \text{Fun}_f(G \curvearrowright S) \\ f &\mapsto [\widetilde{\Delta}(f)((g, s), (h, t)) \mapsto f(gh, t)] \end{aligned}$$

$$\begin{aligned} \tilde{\varepsilon} : \text{Fun}_f(G \curvearrowright S) &\rightarrow \text{Fun}_f(S) \\ f &\mapsto \sum_{s \in S} f(e, s) \delta_{(e,s)} \end{aligned}$$

$$\begin{aligned} S : \text{Fun}_f(G \curvearrowright S) &\rightarrow \text{Fun}_f(G \curvearrowright S) \\ f &\mapsto \sum_{g \in G, s \in S} f(g, g^{-1} \cdot s) \delta_{(g^{-1}, s)} \end{aligned}$$

where

$$\mathcal{M}(\text{Fun}_f((G \curvearrowright S)^{(2)})) \cong \text{Fun}_f(G \curvearrowright S)^{\text{Fun}_f(S)} \overline{\times}_{\text{Fun}_f(S)} \text{Fun}_f(G \curvearrowright S),$$

Then by direct calculations,  $\mathcal{A} = (\widehat{\mathbb{G}} \times_{\widehat{\alpha}} \text{Fun}_f(S), \text{Fun}_f(S), \lambda, \beta_{\widehat{\lambda}}, \Delta_L, \widehat{\varepsilon}, S)$  is a multiplier Hopf  $*$ -algebroid. Moreover, if we consider the linear map

$$\begin{aligned} \Omega : \text{Fun}_f(G \curvearrowright S) &\rightarrow \text{Fun}_f(S) \\ f &\mapsto (\varphi \otimes id) \left( \sum_{g \in G, s \in S} f(g, s) \delta_{(g, s)} \right) \end{aligned}$$

where  $\varphi$  is the left and right integral for  $\widehat{\mathbb{H}}$ , and a function  $\mu : G \rightarrow \mathbb{C}$  such that we have the functional

$$\begin{aligned} \mu : \text{Fun}_f(G) &\rightarrow \mathbb{C} \\ p &\mapsto \sum_{g \in G} p(g) \mu(g). \end{aligned}$$

Then  $\mu$  is a base weight for  $\mathcal{A}$  and  $\Omega$  is a partial integral for  $\mathcal{A}$ , that implies that  $\mathcal{A}$  is commutative algebraic quantum transformation groupoid (because all its algebras are commutative).

### 3.3.2 Quotient type left coideals

Let  $\mathbb{G} = (A, \Delta_A, \varphi_A)$  be a unimodular algebraic quantum group with algebraic multiplicative unitary  $U$  and  $\mathbb{H} = (B, \Delta_B, \varphi_B)$  an algebraic quantum group of compact type which is a closed quantum subgroup of  $\mathbb{G}$ , as in Definition 2.1.15, with non-degenerate surjective  $*$ -homomorphism  $\Phi : A \rightarrow B$ . Denote by  $I$  the quotient type left coideal defined as in Proposition 2.8.19.

By this Proposition, the triple  $(I, \alpha_I := \Delta_A|_I, \lambda_I := \text{Ad}_{\mathbb{G}}|_I)$  is a braided commutative Yetter-Drinfeld  $*$ -algebra. Now, we want to equip it with a Yetter-Drinfeld integral in order to construct an algebraic quantum transformation groupoid.

**Lemma 3.3.1.** *The restriction of the functional  $\varphi_A$  to  $I$  is a Yetter-Drinfeld integral.*

*Proof* Denote  $\mu := \varphi_A|_I$ . Because  $\varphi_A$  is a left and right integral for  $\mathbb{G}$ , it follows that  $(id \otimes \mu) \circ \alpha_I = \mu(\cdot) 1_{\mathcal{M}(A)}$ . Now, fix  $a \in A$ , then we have

$$\begin{aligned} (b \otimes \mu)(\lambda_I(a)) &= (b \otimes \mu)(\widehat{U}^*(1 \otimes a)\widehat{U}) \\ &= \mu((id \otimes b)(U(a \otimes 1)U^*)) \\ &= \mu(b_{(1)} a S(b_{(2)})) \\ &= \varepsilon(b) \mu(a) \\ &= (b \otimes id)(\mu(a) 1_{\mathcal{M}(\widehat{A})}) \end{aligned}$$

for all  $b \in A$ . This implies

$$(id \otimes \mu)(\lambda_I(a)) = \mu(a) 1_{\mathcal{M}(\widehat{A})}.$$

□

**Proposition 3.3.2.** *Using the Yetter-Drinfeld integral  $\varphi_A|_I$ , we have two algebraic quantum transformation groupoids  $\mathfrak{G}(I, \Delta_A|_I, Ad_G|_I, \varphi_A|_I)$  and  $\mathfrak{G}(I, Ad_G|_I, \Delta_A|_I, \varphi_A|_I)$ .*

*Proof* Using the lemma above and Theorem 3.1.9, we have all ingredients to construct these two algebraic quantum transformation groupoids.  $\square$

## **Part II**

### **Analytical setting**



## Chapter N° 4

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### Preliminaries

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#### Notations and conventions:

- Given a subset  $Y$  of a normed space  $X$ , we denote by  $[Y] \subseteq X$  the closed linear span of  $Y$ , i.e.

$$[Y] := \overline{\text{span}(Y)}^{\|\cdot\|_X} \subseteq X.$$

- Given a Hilbert space  $\mathcal{H}$  and a subset  $X \subseteq \mathcal{B}(\mathcal{H})$ , we denote by  $X'$  the commutant of  $X$ , i.e.

$$X' := \{T \in \mathcal{B}(\mathcal{H}) : Tx = xT \text{ for all } x \in X\}.$$

- All sesquilinear maps like inner products on vector spaces over  $\mathbb{C}$  are assumed to be antilinear in the first component and linear in the second one.
- Given a Hilbert space  $\mathcal{H}$  and an element  $\xi \in \mathcal{H}$ , we use the ket-bra notation

$$\begin{aligned} |\xi\rangle &: \mathbb{C} \rightarrow \mathcal{H}, & \lambda &\mapsto \lambda\xi, \\ \langle\xi| = |\xi\rangle^* &: \mathcal{H} \rightarrow \mathbb{C}, & \eta &\mapsto \langle\xi|\eta\rangle. \end{aligned}$$

For every subset  $X \subseteq \mathcal{H}$ , we will use the notation

$$|X\rangle := \{ |x\rangle : x \in X \} \subseteq \mathcal{B}(\mathbb{C}, \mathcal{H}).$$

- Given a Hilbert spaces  $\mathcal{H}$  and elements  $\xi, \eta \in \mathcal{H}$ , we denote by  $\omega_{\xi, \eta}$  the linear functional on  $\mathcal{B}(\mathcal{H})$  defined by  $\omega_{\xi, \eta}(T) = \langle\xi|T|\eta\rangle$ .
- We use the symbol  $\otimes$  for the spatial tensor product of  $C^*$ -algebras and for the tensor product of Hilbert spaces.



- We shall make extensive use of right Hilbert  $C^*$ -modules instead the left ones.
- Given Hilbert  $C^*$ -modules  $E$  and  $F$  over  $B$ , we denote by  $\mathcal{L}_B(E, F)$  the space of all adjointable operators from  $E$  to  $F$ .

**Definition 4.0.1** ([Kus03]). Let  $\omega$  be a positive linear functional on a non-degenerate  $*$ -algebra  $A$ . A *GNS-pair associated with  $\omega$*  is a pair  $(\mathcal{H}_\omega, \Lambda_\omega)$ , where  $\mathcal{H}_\omega$  is a Hilbert space and  $\Lambda_\omega$  is a linear map from  $A$  into  $\mathcal{H}$  such that

- $\Lambda_\omega(A)$  is dense in  $\mathcal{H}_\omega$ ,
- $\langle \Lambda_\omega(a) | \Lambda_\omega(b) \rangle = \omega(a^*b)$  for all  $a, b \in A$ .

It is clear that such a pair exists and that is unique up to a unitary transformation.

## 4.1 Basics of $C^*$ -algebras

**Remark 70.** As any  $C^*$ -algebra  $A$  is a non-degenerate  $*$ -algebra, we can consider its multiplier  $*$ -algebra  $\mathcal{M}(A)$  that is also a  $C^*$ -algebra.

**Definition 4.1.1.** Let  $A, B$  be  $C^*$ -algebras. A  $*$ -homomorphism  $f : A \rightarrow \mathcal{M}(B)$  is called *non-degenerate* if

$$[f(A)B] = B.$$

**Proposition 4.1.2.** Let  $f : A \rightarrow \mathcal{M}(B)$  be a non-degenerate  $*$ -homomorphism of  $C^*$ -algebras. There exists a  $*$ -homomorphism  $\mathcal{M}(A) \rightarrow \mathcal{M}(B)$ , which is the unique extension of  $f$ , and that we will denote with the same symbol.

### 4.1.1 KMS-weights on $C^*$ -algebras

**Remark 71.** Let  $A$  be a  $C^*$ -algebra. Denote by  $A_+$  the cone of all positive elements of  $A$ .

**Definition 4.1.3.** Let  $A$  be a  $C^*$ -algebra. A *weight on  $A$*  is a function  $\phi : A_+ \rightarrow [0, \infty]$  satisfying

- (i)  $\phi(a + b) = \phi(a) + \phi(b)$  for all  $a, b \in A_+$ ;
- (ii)  $\phi(ra) = r\phi(a)$  for all  $r \in [0, \infty)$  and  $a \in A_+$ .

The weight  $\phi$  is called *faithful* if  $\phi(a) \neq 0$  for each non-zero  $a \in A_+$ .

**Notation 14.** Let  $\phi$  be a weight on a  $C^*$ -algebra  $A$ . We use the following standard notation:

- $\mathcal{M}_\phi^+ := \{a \in A_+ : \phi(a) < \infty\}$  is the set of all positive  $\phi$ -integrable elements;
- $\mathcal{N}_\phi := \{a \in A : \phi(a^*a) < \infty\}$  is the set of all  $\phi$ -square-integrable elements;
- $\mathcal{M}_\phi = \text{span } \mathcal{M}_\phi^+ = \text{span } \mathcal{N}_\phi^* \mathcal{N}_\phi$  is the set of all  $\phi$ -integrable elements.

**Definition 4.1.4.** A GNS-construction for a weight  $\phi$  on a  $C^*$ -algebra  $A$  is a triple  $(\mathcal{H}_\phi, \Lambda_\phi, \pi_\phi)$ , where  $\mathcal{H}_\phi$  is a Hilbert space,  $\Lambda_\phi : \mathcal{N}_\phi \rightarrow \mathcal{H}_\phi$  is a linear map with dense image, and  $\pi_\phi : A \rightarrow \mathcal{B}(\mathcal{H}_\phi)$  is a  $*$ -representation such that  $\langle \Lambda_\phi(a) | \Lambda_\phi(b) \rangle = \phi(a^*b)$  and  $\pi_\phi(a)\Lambda_\phi(b) = \Lambda_\phi(ab)$  for all  $a \in A$  and  $b \in \mathcal{N}_\phi$ . It is easy to see that for each weight, there is a GNS-construction, and this construction is unique up to a unitary transformation.

**Definition 4.1.5.** A weight  $\phi$  on a  $C^*$ -algebra  $A$  is called

- *lower semi-continuous* if the subset  $\{a \in A_+ : \phi(a) \leq \lambda\} \subset A$  is closed for every  $\lambda \in \mathbb{R}^+$ ;
- *densely defined* if  $\mathcal{M}_\phi^+$  is dense in  $A_+$ , or equivalently, if  $\mathcal{N}_\phi$  or  $\mathcal{M}_\phi$  is dense in  $A$ ;
- *proper* if  $\phi$  is non-zero, densely defined, and lower semi-continuous.

**Definition 4.1.6.** A one-parameter group of  $*$ -automorphisms on a  $C^*$ -algebra  $A$  is a family  $\sigma = (\sigma_t)_{t \in \mathbb{R}}$  of  $*$ -automorphism of  $A$  that satisfies  $\sigma_s \circ \sigma_t = \sigma_{s+t}$  for all  $s, t \in \mathbb{R}$ . The one-parameter group  $\sigma$  is called *norm-continuous* if for every  $a \in A$ , the map  $t \in \mathbb{R} \mapsto \sigma_t(a) \in A$  is norm continuous.

We will be interested in analytic extensions of one-parameter groups. These extensions involve analytic functions that are defined on horizontal strips on the complex plane of the form

$$I(z) := \{y \in \mathbb{C} : |\operatorname{Im}(y)| \leq |\operatorname{Im}(z)|\} \subset \mathbb{C},$$

where  $z \in \mathbb{C}$ . Given a  $C^*$ -algebra  $A$ , let us call a function  $f : I(z) \rightarrow A$  *norm-regular* if it is

- analytic on the interior of  $I(z)$ , that is, for every  $y_0$  in the interior of  $I(z)$ , the limit  $\lim_{y \rightarrow y_0} \frac{f(y) - f(y_0)}{y - y_0}$  exists w.r.t. the norm-topology;
- norm-bounded on  $I(z)$ ;
- norm-continuous on  $I(z)$ .

**Definition 4.1.7.** Let  $\sigma$  be a norm-continuous one-parameter group of  $*$ -automorphisms of a  $C^*$ -algebra  $A$ . For each  $z \in \mathbb{C} - \mathbb{R}$ , put

$$\operatorname{Dom}(\sigma_z) := \{x \in A : \text{there exists } f : I(z) \rightarrow A \text{ norm-regular s.t. } f(t) = \sigma_t(x) \text{ for all } t \in \mathbb{R}\}$$

and define  $\sigma_z : \operatorname{Dom}(\sigma_z) \rightarrow A$  by  $\sigma_z(x) = f(z)$ , where  $f$  is as above. The function  $f$  is uniquely determined by the Identity Theorem of complex analysis and  $\sigma_z$  is well defined. The family  $(\sigma_z)_{z \in \mathbb{C}}$  is called the *analytic extension of  $\sigma$*  and the elements of  $\bigcup_{z \in \mathbb{C}} \operatorname{Dom}(\sigma_z)$  are called *analytic*.

**Definition 4.1.8.** A proper weight  $\phi$  on a  $C^*$ -algebra  $A$  is called a *KMS-weight* if there exists a norm-continuous one-parameter group  $\sigma$  on  $A$  such that

- $\sigma$  leaves  $\phi$  invariant. i.e,  $\phi \circ \sigma_t = \phi$  for all  $t \in \mathbb{R}$ ,
- $\phi(x^*x) = \phi(\sigma_{i/2}(x)\sigma_{i/2}(x)^*)$  for all  $x \in \operatorname{Dom}(\sigma_{i/2})$ .

Moreover,  $\phi(xy) = \phi(y\sigma_{-i}(x))$  in the following two situations:

- $x \in \text{Dom}(\sigma_{-i})$  and  $y \in \mathcal{M}_\phi$ .
- $y \in \mathcal{N}_\phi \cap \mathcal{N}_\phi^*$ ,  $x \in \mathcal{N}_\phi^* \cap \text{Dom}(\sigma_{-i})$  and  $\sigma_{-i}(x) \in \mathcal{N}_\phi$ .

The one-parameter group  $\sigma$  is called a *modular automorphism group* for  $\phi$ .

**Theorem 4.1.9.** *Let  $\phi$  be a KMS-weight on a  $C^*$ -algebra  $A$  with GNS-construction  $(\mathcal{H}_\phi, \Lambda_\phi, \pi_\phi)$  and modular automorphism group  $\sigma$ . Then*

- *There exists a unique closed antilinear operator  $T$  on  $\mathcal{H}_\phi$  such that  $\Lambda_\phi(\mathcal{N}_\phi \cap \mathcal{N}_\phi^*)$  is a core for  $T$  and  $T\Lambda_\phi(x) = \Lambda_\phi(x^*)$  for all  $x \in \mathcal{N}_\phi \cap \mathcal{N}_\phi^*$ .*
- *The operator  $\nabla_\phi := T^*T$  is strictly positive. There exists a unique antiunitary  $J_\phi$  on  $\mathcal{H}_\phi$  such that  $T = J_\phi \nabla_\phi^{1/2}$ . One has  $J_\phi = J_\phi^*$ ,  $J_\phi^2 = 1_{\mathcal{B}(\mathcal{H}_\phi)}$  and  $J_\phi \nabla_\phi^{it} J_\phi = \nabla_\phi^{it}$ ,  $J_\phi \nabla_\phi^t J_\phi = \nabla_\phi^{-t}$  for all  $t \in \mathbb{R}$ .*
- $J_\phi \pi_\phi(A) J_\phi \subseteq \pi_\phi(A)'$ .
- $\nabla_\phi^{it} \Lambda_\phi(x) = \Lambda_\phi(\sigma_t(x))$  for all  $x \in \mathcal{N}_\phi$  and  $t \in \mathbb{R}$ .
- $J_\phi \Lambda_\phi(x) = \Lambda_\phi(\sigma_{i/2}(x)^*)$  for all  $x \in \mathcal{N}_\phi \cap \text{Dom}(\sigma_{i/2})$ .
- *If  $x \in \mathcal{N}_\phi$  and  $y \in \text{Dom}(\sigma_{i/2})$ , then  $xy \in \mathcal{N}_\phi$  and  $J_\phi \pi_\phi(\sigma_{i/2}(x))^* J_\phi \Lambda_\phi(y) = \Lambda_\phi(yx)$ .*

**Definition 4.1.10.** The operators  $J_\phi$  and  $\nabla_\phi$  defined in the theorem above are called the *modular conjugation* and the *modular operator* of  $\phi$ , respectively.

**Definition 4.1.11.** A  $C^*$ -bialgebra is a  $C^*$ -algebra  $A$  equipped with a non-degenerate  $*$ -homomorphism  $\Delta : A \rightarrow \mathcal{M}(A \otimes A)$  called *comultiplication* such that

- $\Delta$  is *coassociative* in the sense  $(\Delta \otimes id) \circ \Delta = (id \otimes \Delta) \circ \Delta$ ,
- $\Delta(A)(1 \otimes A)$  and  $\Delta(A)(A \otimes 1)$  are contained in  $A \otimes A$ .

A  $C^*$  bialgebra  $(A, \Delta)$  is called *bisimplifiable* if the linear span  $\Delta(A)(A \otimes 1)$  and  $\Delta(A)(1 \otimes A)$  are dense subsets of  $A \otimes A$ . A *morphism of  $C^*$ -bialgebras*  $(A, \Delta_A)$  and  $(B, \Delta_B)$ , is a non-degenerate  $*$ -homomorphism  $f : A \rightarrow \mathcal{M}(B)$  that satisfies  $\Delta_B \circ f = (f \otimes f) \circ \Delta_A$ .

**Definition 4.1.12.** Let  $(A, \Delta)$  be a bisimplifiable  $C^*$ -bialgebra. A weight on  $A$  is called

- *left-invariant*, if  $\phi((\omega \otimes id)(\Delta(a))) = \omega(1)\phi(a)$  for all  $\omega \in A_+^\vee$  and  $a \in \mathcal{M}_\phi^+$ .
- *right-invariant*, if  $\phi((id \otimes \omega)(\Delta(a))) = \omega(1)\phi(a)$ , for all  $\omega \in A_+^\vee$  and  $a \in \mathcal{M}_\phi^+$ .

### 4.1.2 $C^*$ -valued weights

**Definition 4.1.13** (Definition 1.1, [Kus00]). Consider two  $C^*$ -algebras  $A$  and  $B$  and a hereditary cone  $P$  in  $A_+$ . Put  $\mathcal{N} = \{a \in A : a^*a \in P\}$  and  $\mathcal{M} = \text{span } P = \mathcal{N}^*\mathcal{N}$ . Suppose that  $T$  is a linear mapping from  $\mathcal{M}$  to  $\mathcal{M}(B)$  such that

$$\sum_{i,j=1}^n b_j^* T(a_j^* a_i) b_i \geq 0$$

for every  $n \in \mathbb{N}$  and  $a_i \in \mathcal{N}$ ,  $b_i \in B$  for all  $1 \leq i \leq n$ . Then we call  $T$  a  $C^*$ -valued weight from  $A$  into  $\mathcal{M}(B)$ .

**Remark 72.** We recall that a cone  $P$  in  $A_+$  is a subset closed under addition and scalar multiplication with positive elements. A cone  $P$  in  $A_+$  is called *hereditary* if given  $a \in A$  and  $b \in P$  such that  $a \leq b$  then we have that  $a$  also belongs to  $P$ .

**Notation 15.** Consider two  $C^*$ -algebras  $A$  and  $B$  and a  $C^*$ -valued weight  $T$  from  $A$  into  $\mathcal{M}(B)$ . We will use the following notation

- (i) the domain of  $T$  will be denoted by  $\mathcal{M}_T$ .
- (2) we define  $\mathcal{N}_T = \{a \in A : a^*a \in \mathcal{M}_T^+\}$ .

We have that  $\mathcal{M}_T$  is a sub- $*$ -algebra of  $A$ ,  $\mathcal{M}_T^+$  is a hereditary cone in  $A_+$  and  $\mathcal{M}_T = \text{span } \mathcal{M}_T^+$ . We will say that  $T$  is *densely defined* if  $\mathcal{M}_T$  is dense in  $A$ , which is equivalent to  $\mathcal{N}_T$  being dense in  $A$  or to  $\mathcal{M}_T^+$  being dense in  $A_+$ .

### 4.1.3 Hilbert $C^*$ -modules

We use [Tim08] as reference for the followings definitions and results relate to right Hilbert  $C^*$ -modules.

Let  $A$  and  $B$  be  $C^*$ -algebras. Given right Hilbert  $C^*$ -modules  $E$  and  $F$  over  $A$  and  $B$ , respectively, and a  $*$ -homomorphism  $\pi : A \rightarrow \mathcal{L}_B(F)$ , we can define the internal tensor product  $E \otimes_{\pi} F$ . This Hilbert  $C^*$ -module is the closed linear span of elements of the form  $e \otimes_{\pi} f$ , where  $e \in E$ ,  $f \in F$ , and the structure of a right Hilbert  $B$ -module is given by

$$\langle e \otimes_{\pi} f | e' \otimes_{\pi} f' \rangle_B := \langle f | \pi(\langle e, e' \rangle_A) f' \rangle_B, \quad (e \otimes_{\pi} f) \cdot b := e \otimes_{\pi} f \cdot b$$

for all  $e, e' \in E$ ,  $f, f' \in F$  and  $b \in B$ .

In a similar way, we can define the flipped internal tensor product  $F \otimes_{\pi} E$ , this Hilbert  $C^*$ -module is the closed linear span of elements of the form  $f \otimes_{\pi} e$ , where  $e \in E$ ,  $f \in F$ , and the structure of a right Hilbert  $B$ -module is given by

$$\langle f \otimes_{\pi} e | f' \otimes_{\pi} e' \rangle_B := \langle f | \pi(\langle e, e' \rangle_A) f' \rangle_B, \quad (f \otimes_{\pi} e) \cdot b := f \cdot b \otimes_{\pi} e$$

for all  $e, e' \in E, f, f' \in F$  and  $b \in B$ . Moreover, the usual and the flipped internal tensor product are related by a unitary map  $\Sigma : E \otimes_{\pi} F \rightarrow F \otimes_{\pi} E, e \otimes_{\pi} f \mapsto f \otimes_{\pi} e$ . If the representation  $\pi$  is clear, we write simply  $\otimes$  and  $\otimes$  instead of  $\otimes_{\pi}$  and  $\otimes$ , respectively.

Let  $E_1, E_2$  be Hilbert  $C^*$ -modules over  $A, F_1, F_2$  be Hilbert  $C^*$ -modules over  $B$  with  $*$ -homomorphisms  $\pi_i : A \rightarrow \mathcal{L}_B(F_i)$  for  $i = 1, 2$ , and  $S \in \mathcal{L}_A(E_1, E_2), T \in \mathcal{L}_B(F_1, F_2)$  such that  $T\pi_1(a) = \pi_2(a)T$  for all  $a \in A$ . Then, there exists a unique operator  $S \otimes T \in \mathcal{L}_B(E_1 \otimes_{\pi_1} F_1, E_2 \otimes_{\pi_2} F_2)$  such that  $(S \otimes T)(e \otimes_{\pi_1} f) = Se \otimes_{\pi_2} Tf$  for all  $e \in E_1, f \in F_1$  and  $(S \otimes T)^* = S^* \otimes T^*$ .

## 4.2 $C^*$ -algebraic quantum groups

### 4.2.1 Basic definitions and results

**Definition 4.2.1** ([KV00]). A *reduced  $C^*$ -algebraic quantum group* is a bisimplifiable  $C^*$ -bialgebra  $(A, \Delta)$  such that there exist a left invariant KMS-weight  $\phi$  on  $(A, \Delta)$  and a right invariant KMS-weight  $\psi$  on  $(A, \Delta)$ .

Let  $(\mathcal{H}_\phi, \Lambda_\phi, \pi_\phi)$  denote a GNS-construction for  $\phi$ . There exists a unitary operator  $W \in \mathcal{B}(\mathcal{H}_\phi \otimes \mathcal{H}_\phi)$  such that

$$W^*(\Lambda_\phi(a) \otimes \Lambda_\phi(b)) = (\Lambda_\phi \otimes \Lambda_\phi)(\Delta(b)(a \otimes 1))$$

for all  $a, b \in \mathcal{N}_\phi$ . This operator is called the multiplicative unitary of  $(A, \Delta)$ , and satisfies the pentagonal equation

$$W_{12}W_{13}W_{23} = W_{23}W_{12}.$$

**Theorem 4.2.2** ([KV00]). Let  $(A, \Delta)$  be a reduced  $C^*$ -algebraic quantum group with left invariant KMS-weight  $\phi$  and right invariant KMS-weight  $\psi$ . Denote by  $(\mathcal{H}_\phi, \Lambda_\phi, \pi_\phi)$  a GNS-construction for  $\phi$ .

- There exists a unique closed densely defined antilinear operator  $G$  on  $\mathcal{H}_\phi$  such that

$$\text{span}\{\Lambda_\phi((\psi \otimes id)(\Delta(b^*)(a \otimes 1))) : a, b \in \mathcal{N}_\phi^* \mathcal{N}_\psi\} \subseteq \mathcal{H}_\phi$$

is a core for  $G$  and

$$G\Lambda_\phi((\psi \otimes id)(\Delta(b^*)(a \otimes 1))) = \Lambda_\phi((\psi \otimes id)(\Delta(a^*)(b \otimes 1)))$$

for all  $a, b \in \mathcal{N}_\phi^* \mathcal{N}_\psi$ . This operator satisfies the relation  $G^2 = 1_{\mathcal{B}(\mathcal{H}_\phi)}$ .

- The operator  $N := G^*G$  is strictly positive. There exists a unique anti-unitary  $I$  on  $\mathcal{H}_\phi$  such that  $G = IN^{1/2}$ . Moreover,  $I = I^*, I^2 = 1$  and  $INI = N^{-1}$ .
- There exists a unique norm-continuous one-parameter group  $\tau$  of  $*$ -automorphisms of  $A$  such that  $\pi_\phi(\tau_t(a)) = N^{-it}\pi_\phi(a)N^{it}$  for all  $t \in \mathbb{R}$  and  $a \in A$ .
- There exists a unique anti- $*$ -automorphism  $R$  of  $A$  such that  $\pi_\phi(R(a)) = I\pi_\phi(a)^*I$  for all  $a \in A$ .

- The operator  $S := R\tau_{-i/2}$  satisfies the relations

- $\text{span}\{(id \otimes \phi)(\Delta(b^*)(1 \otimes a)) : a, b \in \mathcal{N}_\phi\} \subseteq A$  is a core for  $S$  and

$$S((id \otimes \phi)(\Delta(b^*)(1 \otimes a))) = (id \otimes \phi)((1 \otimes b^*)\Delta(a))$$

for all  $a, b \in \mathcal{N}_\phi$ .

- $\text{span}\{(\psi \otimes id)((b^* \otimes 1)\Delta(a)) : a, b \in \mathcal{N}_\psi\} \subseteq A$  is a core for  $S$  and

$$S((\psi \otimes id)((b^* \otimes 1)\Delta(a))) = (\psi \otimes id)(\Delta(b^*)(a \otimes 1))$$

for all  $a, b \in \mathcal{N}_\psi$ .

- $\{(id \otimes \omega)(W) : \omega \in \mathcal{B}_0(\mathcal{H}_\phi)^*\} \subseteq \pi_\phi(A) \cong A$  is a core for  $S$  and

$$S((id \otimes \omega)(W)) = (id \otimes \omega)(W^*)$$

for all  $\omega \in \mathcal{B}_0(\mathcal{H}_\phi)^*$ .

**Definition 4.2.3** ([KV00]). The one-parameter group  $\tau$  and the maps  $R$  and  $S$  are called the *scaling group*, the *unitary antipode* and the *antipode* of  $(A, \Delta)$ , respectively.

**Proposition 4.2.4** ([KV00]). Here, we collect relations between the maps associated with a reduced  $C^*$ -algebraic quantum group. Let  $\sigma^\phi$  and  $\sigma^\psi$  the modular automorphism groups of  $\phi$  and  $\psi$ , respectively :

- $S \circ R = R \circ S, \tau_t \circ R = R \circ \tau_t, \tau_t \circ S = S \circ \tau_t$  for all  $t \in \mathbb{R}$ .
- $\sigma_t^\phi \circ \sigma_t^\psi = \sigma_t^\psi \circ \sigma_t^\phi, \tau_t \circ \sigma_t^\phi = \sigma_t^\phi \circ \tau_t, \tau_t \circ \sigma_t^\psi = \sigma_t^\psi \circ \tau_t$  for all  $t \in \mathbb{R}$ .
- $R^2 = 1$  and  $S^2 = \tau_{-i}$ .
- $S$  is injective and  $S^{-1} = \tau_{i/2} \circ R$ .
- For all  $x, y \in \text{Dom}(S)$ , we have  $xy \in \text{Dom}(S)$  and  $S(xy) = S(y)S(x)$ .
- For all  $x \in \text{Dom}(S)$ , we have  $S(x)^* \in \text{Dom}(S)$  and  $S(S(x)^*)^* = x$ .
- $\Delta \circ R = \Sigma \circ (R \otimes R) \circ \Delta$ .
- For all  $t \in \mathbb{R}$

$$\Delta \circ \sigma_t^\phi = (\tau_t \otimes \sigma_t^\phi) \circ \Delta, \quad \Delta \circ \sigma_t^\psi = (\sigma_t^\psi \otimes \tau_{-t}) \circ \Delta,$$

$$(\tau_t \otimes \tau_t) \circ \Delta = \Delta \circ \tau_t = (\sigma_t^\phi \otimes \sigma_{-t}^\psi) \circ \Delta$$

- There exists a scaling constant  $\nu > 0$  such that

$$\phi \circ \tau_t = \nu^{-t}\phi, \quad \psi \circ \tau_t = \nu^{-t}\psi, \quad \phi \circ \sigma_t^\psi = \nu^t\phi, \quad \psi \circ \sigma_t^\phi = \nu^{-t}\psi.$$

**Definition 4.2.5** ([Wor91]). Let  $A$  be a  $C^*$ -algebra and  $T : \text{Dom}(T) \subseteq A \rightarrow A$  a densely defined linear map. We say that  $T$  is *affiliated with  $A$*  if there exists  $z_T \in \mathcal{M}(A)$  such that  $\|z_T\| \leq 1$  and for all  $x, y \in A$ ,

$x \in \text{Dom}(T)$  and  $y = T(x) \Leftrightarrow$  There exists  $a \in A$  with  $x = (1 - z_T^* z_T)^{1/2} a$  and  $y = z_T a$

**Proposition 4.2.6** ([KV00]). *There exists an affiliated element  $\delta$  called the modular element of  $(A, \Delta)$ , such that*

- $\psi = \phi(\delta^{1/2} \cdot \delta^{1/2})$ .
- $\Delta(\delta) = \delta \otimes \delta$ ,  $R(\delta) = \delta^{-1}$ .
- For all  $t \in \mathbb{R}$ ,  $x \in A$ ,

$$\tau_t(\delta) = \delta, \quad \sigma_t^\phi(\delta) = \nu^t \delta = \sigma_t^\psi(\delta), \quad \sigma_t^\psi(x) = \delta^{it} \sigma_t^\phi(x) \delta^{-it}.$$

**Definition 4.2.7** ([KV00]). We define

- The set  $\widehat{A} := [(\omega \otimes id)(W) \mid \omega \in \mathcal{B}_0(\mathcal{H}_\phi)^*] \subset \mathcal{B}(\mathcal{H}_\phi)$ .
- The injective linear map  $\widehat{\Delta} : \widehat{A} \rightarrow \mathcal{B}(\mathcal{H}_\phi \otimes \mathcal{H}_\phi)$  such that  $\widehat{\Delta}(x) = \Sigma W(x \otimes 1) W^* \Sigma$  for all  $x \in \widehat{A}$ . Here  $\Sigma \in \mathcal{B}(\mathcal{H}_\phi \otimes \mathcal{H}_\phi)$  denotes the flip map.

**Theorem 4.2.8** ([KV00]). *The set  $\widehat{A}$  is a non-degenerate sub- $C^*$ -algebra of  $\mathcal{B}(\mathcal{H}_\phi)$  and the map  $\widehat{\Delta}$  is a non-degenerate  $*$ -homomorphism from  $\widehat{A}$  to  $\mathcal{M}(\widehat{A} \otimes \widehat{A})$  such that*

- $(\widehat{\Delta} \otimes id) \circ \widehat{\Delta} = (id \otimes \widehat{\Delta}) \circ \widehat{\Delta}$
- *the linear span of  $\widehat{\Delta}(\widehat{A})(\widehat{A} \otimes 1)$  and  $\Delta(\widehat{A})(1 \otimes \widehat{A})$  are dense subsets of  $\widehat{A} \otimes \widehat{A}$ .*

Moreover, there exist

- a left invariant KMS-weight  $\widehat{\varphi}$  on  $(\widehat{A}, \widehat{\Delta})$ ,
- a right invariant KMS-weight  $\widehat{\psi}$  on  $(\widehat{A}, \widehat{\Delta})$ .

**Definition 4.2.9.** The reduced  $C^*$ -algebraic quantum group  $(\widehat{A}, \widehat{\Delta})$  is called the dual of  $(A, \Delta)$ .

**Remark 73.** We have that  $\widehat{W} := \text{Ad}(\Sigma)(W^*)$  is the multiplicative unitary of the dual  $(\widehat{A}, \widehat{\Delta})$  in the GNS-construction  $(\mathcal{H}_\phi, \widehat{\Lambda}, i)$  for  $\widehat{\varphi}$ .

## 4.2.2 Drinfeld double for $C^*$ -algebraic quantum groups ([NV09])

Let  $(A, \Delta)$  be a  $C^*$ -algebraic quantum group. We fix its left and right invariant KMS-weights  $\varphi, \psi$ , its multiplicative unitary  $W$ , and its dual  $C^*$ -algebraic quantum group  $(\widehat{A}, \widehat{\Delta})$  with left and right invariant KMS-weights  $\widehat{\varphi}, \widehat{\psi}$  respectively.

**Theorem 4.2.10.** *The linear map  $\Delta_D = (id \otimes \Sigma \circ \text{Ad}(W) \otimes id) \circ (\Delta \otimes \widehat{\Delta})$  is a non-degenerate  $*$ -homomorphism from  $A \otimes \widehat{A}$  to  $\mathcal{M}(A \otimes \widehat{A} \otimes A \otimes \widehat{A})$ . Moreover,  $(A \otimes \widehat{A}, \Delta_D)$  is a  $C^*$ -algebraic quantum group with  $\varphi_D = \varphi \otimes \widehat{\psi}$  as left and right invariant KMS-weight.*

**Definition 4.2.11.** The  $C^*$ -algebraic quantum group  $(A \otimes \widehat{A}, \Delta_D)$  is called the *Drinfeld double* of  $(A, \Delta)$ .

### 4.2.3 Coactions of $C^*$ -algebraic quantum groups ([Tim08])

**Definition 4.2.12.** A left coaction of a  $C^*$ -bialgebra  $(A, \Delta)$  on a  $C^*$ -algebra  $C$  is a non-degenerate  $*$ -homomorphism  $\alpha : C \rightarrow \mathcal{M}(A \otimes C)$  which satisfies

- $\alpha(C)(A \otimes 1) \subseteq A \otimes C$
- $(id_A \otimes \alpha) \circ \alpha = (\Delta \otimes id_C) \circ \alpha$ , that is, the following diagram commutes

$$\begin{array}{ccc} C & \xrightarrow{\alpha} & \mathcal{M}(A \otimes C) \\ \alpha \downarrow & & \downarrow \Delta \otimes id_C \\ \mathcal{M}(A \otimes C) & \xrightarrow{id_A \otimes \alpha} & \mathcal{M}(A \otimes A \otimes C) \end{array}$$

A  $C^*$ -algebra  $C$  equipped with a left coaction  $\alpha$  of a  $C^*$ -bialgebra  $(A, \Delta)$  will be called a  $(A, \Delta)$ - $C^*$ -algebra if

- $\alpha$  is injective,
- the linear span of  $\alpha(C)(A \otimes 1)$  is dense in  $A \otimes C$ .

### 4.2.4 $C^*$ -algebraic quantum groups associated with AQG

Fix an algebraic quantum group  $\mathbb{G} = (A, \Delta, \varphi)$  and a GNS-pair  $(L^2\mathbb{G}, \Lambda)$  for the faithful functional  $\varphi$ , i.e

- $L^2\mathbb{G}$  is a Hilbert space with scalar product  $\langle \cdot | \cdot \rangle_\varphi$ .
- $\Lambda : A \rightarrow L^2\mathbb{G}$  is an injective linear map.
- $\Lambda(A)$  is dense in  $L^2\mathbb{G}$ .
- $\langle \Lambda(a) | \Lambda(b) \rangle_\varphi = \varphi(a^*b)$  for all  $a, b \in A$ .

#### 4.2.4.1 The reduced version

**Proposition 4.2.13** (Proposition 2.1, [KV97]). *There exists a unique unitary operator  $W \in \mathcal{B}(L^2\mathbb{G} \otimes L^2\mathbb{G})$  such that*

$$W^*(\Lambda(a) \otimes \Lambda(b)) = (\Lambda \otimes \Lambda)(\Delta(b)(a \otimes 1)),$$

and

$$W(\Lambda(a) \otimes \Lambda(b)) = (\Lambda \otimes \Lambda)((S^{-1} \otimes id)(\Delta(b))(a \otimes 1)),$$

for all  $a, b \in A$ .

**Lemma 4.2.14** (Lemma 2.3, [KV97]). *For every  $a, b, c \in A$ ,*

$$(id \otimes \omega_{\Lambda(a), \Lambda(b)})(W)\Lambda(c) = \Lambda((id \otimes \varphi)(\Delta(a^*)(1 \otimes b))c)$$



**Lemma 4.2.15** (Lemma 2.4, [KV97]). *For each  $a \in A$ , there exists a unique bounded operator  $T_a$  on  $L^2\mathbb{G}$  such that  $T_a(\Lambda(b)) = \Lambda(ab)$  for all  $b \in A$ .*

**Definition 4.2.16** (Definition 2.5, [KV97]). We define the map  $\pi_r : A \rightarrow \mathcal{B}(L^2\mathbb{G})$  such that

$$\pi_r(a)(\Lambda(b)) = \Lambda(ab)$$

for all  $a, b \in A$ .

**Remark 74.** Because  $A$  is an idempotent, in the sense of 1.1.2, and non-degenerate  $*$ -algebra, then  $\pi_r$  is an injective  $*$ -homomorphism such that  $\pi_r(A)(L^2(\mathbb{G}))$  is dense in  $L^2\mathbb{G}$ .

Using Lemma 4.2.14, we have for all  $a, b \in A$

$$(id \otimes \omega_{\Lambda(a), \Lambda(b)})(W) = \pi_r((id \otimes \varphi)(\Delta(a^*)(1 \otimes b))) \quad (4.1)$$

**Definition 4.2.17** (Definition 2.5, [KV97]). We define  $A_r$  as the norm closure of  $\pi_r(A)$  in  $\mathcal{B}(L^2\mathbb{G})$ . So,  $A_r$  is a non-degenerate sub- $C^*$ -algebra of  $\mathcal{B}(L^2\mathbb{G})$ . By the equation (4.1), we have that

$$A_r = \overline{\{(id \otimes \omega)(W) : \omega \in \mathcal{B}(L^2\mathbb{G})_*\}}^{\|\cdot\|} \subseteq \mathcal{B}(L^2\mathbb{G})$$

**Theorem 4.2.18** (Theorem 2.11, [KV97]). *The map  $\Delta_r : A_r \rightarrow \mathcal{B}(L^2\mathbb{G} \otimes L^2\mathbb{G})$  defined by*

$$\Delta_r(x) = W^*(1 \otimes x)W$$

for all  $x \in A_r$ , is a non-degenerate injective  $*$ -homomorphism from  $A_r$  to  $\mathcal{M}(A_r \otimes A_r)$  such that

- $(\Delta_r \otimes id) \circ \Delta_r = (id \otimes \Delta_r) \circ \Delta_r$
- The vector spaces  $\Delta_r(A_r)(A_r \otimes 1)$  and  $\Delta_r(A_r)(1 \otimes A_r)$  are dense in  $A_r \otimes A_r$ .

**Proposition 4.2.19** (Lemma 2.8, [KV97]). *For any  $a \in A$  and  $x \in A \otimes A$ , we have*

$$\Delta_r(\pi_r(a))(\pi_r \otimes \pi_r)(x) = (\pi_r \otimes \pi_r)(\Delta(a)x), \quad (\pi_r \otimes \pi_r)(x)\Delta_r(\pi_r(a)) = (\pi_r \otimes \pi_r)(x\Delta(a))$$

There exists a left invariant KMS-weight  $\varphi_r$  on the  $C^*$ -bialgebra  $(A_r, \Delta_r)$  with GNS-construction  $(L^2\mathbb{G}, \Lambda_r, id)$  such that  $\pi_r(A)$  is a core for  $\Lambda_r$  and  $\Lambda_r(\pi_r(a)) = \Lambda(a)$  for all  $a \in A$ . This implies that  $\pi_r(A) \subseteq \mathcal{M}_{\varphi_r}$  and  $\varphi_r(\pi_r(a)) = \varphi(a)$  for all  $a \in A$ .

**Definition 4.2.20.** We denote the  $C^*$ -algebraic quantum group  $(A_r, \Delta_r)$  by  $\mathbb{G}_r$ . It is called *the reduced  $C^*$ -algebraic quantum group associated with  $\mathbb{G}$* .

We denote: the antipode of  $(A_r, \Delta_r)$  by  $S_r$ , the scaling group by  $\tau^r$  and the unitary antipode by  $R_r$ ; the modular group of  $\varphi_r$  by  $\sigma^r$  and the modular group of  $\psi_r := \varphi_r \circ R_r$  by  $'\sigma^r$ ; the modular element of  $(A_r, \Delta_r)$  by  $\delta_r$  and the scaling constant of  $(A_r, \Delta_r)$  by  $\nu$ . Recall that  $\varphi_r \circ \tau_t^r = \nu^{-t}\varphi_r$  for all  $t \in \mathbb{R}$ .

**Proposition 4.2.21** (Proposition 2.12, [KV97]). *The operator  $W$  is the multiplicative unitary of  $(A_r, \Delta_r)$  acting on  $L^2\mathbb{G}$ , i.e.  $W_{12}W_{13}W_{23} = W_{23}W_{12}$ .*

We define the linear map  $\widehat{\Lambda} : \widehat{A} \rightarrow L^2\mathbb{G}$  such that  $\widehat{\Lambda}(\varphi_a) = \Lambda(a)$  for every  $a \in A$ . Then, using the identity of the Lemma 1.2.19, we can consider  $(L^2\mathbb{G}, \widehat{\Lambda})$  as the GNS-pair for  $\widehat{\psi}$ . With this new GNS-pair, we have a new expression for  $W$ .

**Lemma 4.2.22** (Lemma 2.13, [KV97]). *For every  $\omega_1, \omega_2 \in \widehat{A}$ ,*

$$W(\widehat{\Lambda}(\omega_1) \otimes \widehat{\Lambda}(\omega_2)) = (\widehat{\Lambda} \otimes \widehat{\Lambda})(\widehat{\Delta}(\omega_1) \cdot (1 \otimes \omega_2))$$

**Definition 4.2.23** (Definition 2.14, [KV97]). We define the map  $\widehat{\pi}_r : \widehat{A} \rightarrow \mathcal{B}(L^2\mathbb{G})$  such that

$$\widehat{\pi}_r(\omega)(\widehat{\Lambda}(\theta)) = \widehat{\Lambda}(\omega \cdot \theta)$$

for all  $\omega, \theta \in \widehat{A}$ .

**Remark 75.** It follows from the idempotence and non-degeneracy of  $\widehat{A}$  that  $\widehat{\pi}_r$  is an injective  $*$ -homomorphism such that  $\widehat{\pi}_r(\widehat{A})(L^2\mathbb{G})$  is dense in  $L^2\mathbb{G}$ .

**Lemma 4.2.24** (Lemma 2.13, [KV97]). *For every  $\theta_1, \theta_2 \in \widehat{A}$ ,*

$$(\omega_{\widehat{\Lambda}(\theta_1), \widehat{\Lambda}(\theta_2)} \otimes id)(W) = \widehat{\pi}_r((\widehat{\psi} \otimes id)((\theta_1^* \otimes 1) \cdot \widehat{\Delta}(\theta_2))). \quad (4.2)$$

**Definition 4.2.25** (Definition 2.15, [KV97]). We define  $\widehat{A}_r$  as the norm closure of  $\widehat{\pi}_r(\widehat{A}) \in \mathcal{B}(L^2\mathbb{G})$ . So,  $\widehat{A}_r$  is a non-degenerate sub- $C^*$ -algebra of  $\mathcal{B}(L^2\mathbb{G})$ . By the equation (4.2), we have that

$$\widehat{A}_r = \overline{\{(\omega \otimes id)(W) : \omega \in \mathcal{B}(L^2\mathbb{G})_*\}}^{\|\cdot\|} \subseteq \mathcal{B}(L^2\mathbb{G})$$

**Theorem 4.2.26** (Theorem 2.18, [KV97]). *We define the map  $\widehat{\Delta}_r : \widehat{A}_r \rightarrow \mathcal{B}(L^2\mathbb{G} \otimes L^2\mathbb{G})$  by*

$$\widehat{\Delta}_r(x) = W(x \otimes 1)W^*$$

for all  $x \in \widehat{A}_r$ . Then  $\widehat{\Delta}_r$  is a non-degenerate injective  $*$ -homomorphism from  $\widehat{A}_r$  to  $\mathcal{M}(\widehat{A}_r \otimes \widehat{A}_r)$  such that

- $(\widehat{\Delta}_r \otimes id) \circ \widehat{\Delta}_r = (id \otimes \widehat{\Delta}_r) \circ \widehat{\Delta}_r$
- The vector spaces  $\widehat{\Delta}_r(\widehat{A}_r)(\widehat{A}_r \otimes 1)$  and  $\widehat{\Delta}_r(\widehat{A}_r)(1 \otimes \widehat{A}_r)$  are dense subsets of  $\widehat{A}_r \otimes \widehat{A}_r$ .

**Lemma 4.2.27** (Lemma 2.17, [KV97]). *For any  $\omega \in \widehat{A}$  and  $\theta \in \widehat{A} \otimes \widehat{A}$ , we have*

$$\widehat{\Delta}_r(\widehat{\pi}_r(\omega))(\widehat{\pi}_r \otimes \widehat{\pi}_r)(\theta) = (\widehat{\pi}_r \otimes \widehat{\pi}_r)(\Delta(\omega)\theta), \quad (\widehat{\pi}_r \otimes \widehat{\pi}_r)(\theta)\widehat{\Delta}_r(\widehat{\pi}_r(\omega)) = (\widehat{\pi}_r \otimes \widehat{\pi}_r)(\theta\widehat{\Delta}(\omega))$$

**Definition 4.2.28** (Lemma 2.19, [KV97]). We denote the  $C^*$ -algebraic quantum group  $(\widehat{A}_r, \widehat{\Delta}_r)$  by  $\widehat{\mathbb{G}}_r$ . It is called *the reduced  $C^*$ -algebraic quantum group associated with  $\widehat{\mathbb{G}}$* .

The next lemma gives another formula for  $\widehat{\pi}_r$ .

**Lemma 4.2.29** (Lemmas 2.19-2.20, [KV97]). *We have*

- For any  $\omega \in \widehat{A}$  and  $a \in A$ ,

$$\widehat{\pi}_r(\omega)(\Lambda(a)) = \Lambda((\widehat{S}^{-1}(\omega) \otimes id)\Delta(a)).$$

- For any  $a, b \in A$ ,

$$(\omega_{\Lambda(a), \Lambda(b)} \otimes id)(W) = \widehat{\pi}_r(a^* \varphi_b).$$

**Lemma 4.2.30** (Lemma 2.21, [KV97]). *Let  $\omega \in \widehat{A}$  and  $a \in A$ . We define the linear mapping  $F$  from  $A$  to  $A$  such that  $F(x) = (id \otimes \omega)(\Delta(x)(a \otimes 1))$ . Then  $F$  can be viewed as an element of  $A \otimes \widehat{A}$  such that  $W(\pi_r(a) \otimes \widehat{\pi}_r(\omega)) = (\pi_r \otimes \widehat{\pi}_r)(F)$ .*

**Proposition 4.2.31** (Proposition 2.22, [KV97]). *We have*

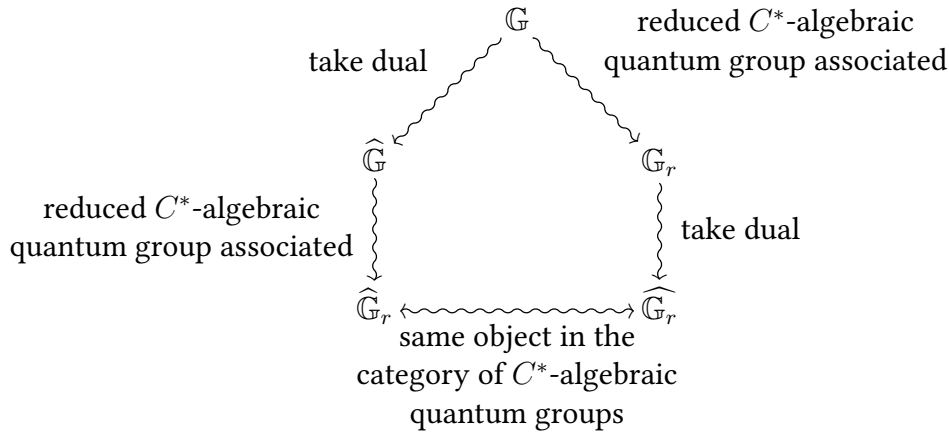
$$W(\pi_r(A) \otimes \widehat{\pi}_r(\widehat{A})) = \pi_r(A) \otimes \widehat{\pi}_r(\widehat{A}), \quad (\pi_r(A) \otimes \widehat{\pi}_r(\widehat{A}))W = \pi_r(A) \otimes \widehat{\pi}_r(\widehat{A}).$$

*In particular,  $W$  belongs to  $\mathcal{M}(A_r \otimes \widehat{A}_r)$ .*

**Proposition 4.2.32.** *The dual of the reduced  $C^*$ -algebraic quantum group associated with  $\mathbb{G}$  is the same object as the reduced  $C^*$ -algebraic quantum group associated with the dual  $C^*$ -algebraic quantum group of  $\mathbb{G}$ , i.e.,  $\widehat{\widehat{\mathbb{G}}} = \widehat{\mathbb{G}}$ .*

*Proof* It is enough to see that the multiplicative unitary  $W$  of  $(A_r, \Delta_r)$  is the extension of the algebraic multiplicative unitary of  $\mathbb{G}$ .  $\square$

**Remark 76.** The previous proposition gives the commutative diagram



#### 4.2.4.2 The universal version

We define a dense sub-algebra of  $A_r$ : consider

$$\mathcal{B}_r = \{(id \otimes \omega)(W^*) : \omega \in \mathcal{B}(L^2\mathbb{G})_*\},$$

and then we take  $\mathcal{A}_r := \mathcal{B}_r \cap \mathcal{B}_r^*$ . The  $*$ -algebra  $\mathcal{A}_r$  is a dense sub- $*$ -algebra of  $A_r$  (in [Kus03], it is called *the coefficient  $*$ -algebra of  $\mathbb{G}_r$* ). It is known that

$$(id \otimes \omega_{\Lambda(a), \Lambda(b)})(W^*) = \pi_r((id \otimes \varphi)((1 \otimes a^*)\Delta(b)))$$

for all  $a, b \in A$ , this implies that  $\pi_r(A) \subseteq \mathcal{B}_r$  and then  $\pi_r(A) \subseteq \mathcal{A}_r$ . Let  $A_u$  be a universal enveloping  $C^*$ -algebra of  $\mathcal{A}_r$  through a  $*$ -homomorphism  $\phi_u : \mathcal{A}_r \rightarrow A_u$  and  $\Lambda_{\mathbb{G}} : A_u \rightarrow A_r$  be a unique surjective  $*$ -homomorphism such that  $\Lambda_{\mathbb{G}}(\phi_u(a)) = a$  for

all  $a \in \mathcal{A}_r$ . We denote also by  $\pi_u : A \rightarrow A_u$  the injective  $*$ -homomorphism such that  $\pi_u(a) = \phi_u(\pi_r(a))$  for all  $a \in A$ . Then  $\Lambda_{\mathbb{G}} \circ \pi_u = \pi_r$ . There exists a multiplicative unitary  $W \in \mathcal{M}(A_u \otimes \mathcal{B}_0(L^2\mathbb{G}))$  called the universal left regular corepresentation of  $\mathbb{G}_r$  with respect to  $(A_u, \pi_u)$  and  $(L^2\mathbb{G}, \Lambda_r, \pi_r)$ .

**Proposition 4.2.33** ([Kus97b]). *The  $C^*$ -algebra  $A_u$  is the universal enveloping  $C^*$ -algebra of  $A$  through the  $*$ -homomorphism  $\pi_u$ .*

**Remark 77.** We have the next universal property for the  $C^*$ -algebra  $A_u$  : if  $C$  is a  $C^*$ -algebra and  $f : A \rightarrow C$  is a  $*$ -homomorphism, then there exists a unique morphism  $f_u : A_u \rightarrow C$  such that  $f = f_u \circ \pi_u$ .

**Theorem 4.2.34** (Theorem 3.15, [Kus97b]). *There is a unique non-degenerate  $*$ -homomorphism*

$$\Delta_u : A_u \rightarrow \mathcal{M}(A_u \otimes A_u)$$

such that  $(\Delta_u \otimes id)(W) = W_{13}W_{23}$ . Moreover, we have that

- $(\Delta_u \otimes id) \circ \Delta_u = (id \otimes \Delta_u) \circ \Delta_u$
- the linear span of  $\Delta_u(A_u)(A_u \otimes 1)$  and  $\Delta_u(A_u)(1 \otimes A_u)$  are dense subspaces of  $A_u \otimes A_u$ .

**Proposition 4.2.35** ([Kus97b]). *Consider  $a \in A$  and  $x \in A \otimes A$ , then*

$$\Delta_u(\pi_u(a))(\pi_u \otimes \pi_u)(x) = (\pi_u \otimes \pi_u)(\Delta(a)x), \quad (\pi_u \otimes \pi_u)(x)\Delta_u(\pi_u(a)) = (\pi_u \otimes \pi_u)(x\Delta(a))$$

**Proposition 4.2.36** ([Kus97b]). *The following identities hold*

- $(\Lambda_{\mathbb{G}} \otimes \Lambda_{\mathbb{G}}) \circ \Delta_u = \Delta_r \circ \Lambda_{\mathbb{G}}$ ,
- $(id \otimes \Lambda_{\mathbb{G}})(\Delta_u(a)) = W^*(1 \otimes \Lambda_{\mathbb{G}}(a))W$  for all  $a \in A_u$ .

**Proposition 4.2.37** ([Kus97b]). *There exists a unique non-zero  $*$ -homomorphism  $\varepsilon_u : A_u \rightarrow \mathbb{C}$  such that*

$$(\varepsilon_u \otimes id) \circ \Delta_u = (id \otimes \varepsilon_u) \circ \Delta_u = id.$$

Moreover,  $(\varepsilon_u \otimes id)(W) = 1$ . The map  $\varepsilon_u$  is called the counit of  $(A_u, \Delta_u)$ .

**Proposition 4.2.38** ([Kus97b]). *There exists a unique anti- $*$ -automorphism  $R_u : A_u \rightarrow A_u$  called the unitary antipode of  $(A_u, \Delta_u)$  such that  $(R_u \otimes \widehat{R}_r)(W) = W$ . Moreover, we have that  $\Lambda_{\mathbb{G}} \circ R_u = R \circ \Lambda_{\mathbb{G}}$ ,  $R_u^2 = id$  and  $\Sigma \circ (R_u \otimes R_u) \circ \Delta_u = \Delta_u \circ R_u$ .*

**Proposition 4.2.39** ([Kus97b]). *The map  $\varphi_u := \varphi_r \circ \Lambda_{\mathbb{G}}$  is a proper weight on  $A_u$  which has a GNS-construction  $(L^2\mathbb{G}, \Lambda_u, \Lambda_{\mathbb{G}})$  such that  $\Lambda_u = \Lambda_r \circ \Lambda_{\mathbb{G}}$ . Moreover, the map  $\psi_u := \varphi_u \circ R_u$  is also a proper weight on  $A_u$ .*

**Definition 4.2.40.** We denote the  $C^*$ -algebraic quantum group  $(A_u, \Delta_u)$  by  $\mathbb{G}_u$ . It is called the universal  $C^*$ -algebraic quantum group associated with  $\mathbb{G}$ .

**Proposition 4.2.41** ([Kus97b]). *We define the antipode  $S_u := R_u \circ \tau_{-i/2}^u = \tau_{-i/2}^u \circ R_u$ . It satisfies the following basic properties*

- $S_u$  is densely defined and has dense range.
- $S_u$  is injective and  $S_u^{-1} = R_u \circ \tau_{i/2}^u = \tau_{i/2}^u \circ R_u$ .
- $S_u$  is antimultiplicative.
- We have for all  $x \in D(S_u)$  that  $S_u(x)^* \in D(S_u)$  and  $S_u(S_u(x)^*)^* = x$ .
- $S_u^2 = \tau_{-i}^u$ .
- $S_u \circ R_u = R_u \circ S_u$  and  $S_u \circ \tau_t^u = \tau_t^u \circ S_u$  for all  $t \in \mathbb{R}$ .

**Proposition 4.2.42** ([Kus97b]). *There exists a unique strictly positive element  $\delta_u$  affiliated with  $A_u$  such that  $\Lambda_{\mathbb{G}}(\delta_u) = \delta_r$ . Moreover,  $\delta_u \otimes \delta_r = \mathbb{W}^*(1 \otimes \delta_r)\mathbb{W}$ .*

The element  $\delta_u$  is called the modular element of the  $C^*$ -algebraic quantum group  $\mathbb{G}_u$ . We have its basic properties:

**Proposition 4.2.43** ([Kus97b]). *The following properties hold*

- $\tau_t^u(\delta_u) = \delta_u$  for  $t \in \mathbb{R}$  and  $R_u(\delta_u) = \delta_u^{-1}$ .
- Let  $t \in \mathbb{R}$ , then  $\delta_u^{it}$  belongs to  $D(\overline{S_u})$  and  $S_u(\delta_u^{it}) = \delta_u^{-it}$ .
- $\sigma_t^u(\delta_u) = \nu^t \delta_u$  for all  $t \in \mathbb{R}$ .
- $'\sigma_t^u(a) = \delta_u^{it} \sigma_t^u(a) \delta_u^{-it}$  for all  $t \in \mathbb{R}$  and  $a \in A_u$ .
- $\psi_u = \delta_u(\varphi_u)$ .

Let  $\varphi_u$  denote the left Haar weight on  $\mathbb{G}_u$  with GNS-construction  $(L^2\mathbb{G}, \Lambda_u, \pi_u)$  defined in such a way that  $\varphi_u \circ \pi_u = \varphi_r$  and  $\Lambda_u \circ \pi_u = \Lambda$ . We denote the antipode of  $\mathbb{G}_u$  by  $S_u$ , the scaling group by  $\tau^u$  and the unitary antipode by  $R_u$ . The canonical modular group of  $\varphi_u$  by  $\sigma^u$  and the canonical modular group of  $\varphi_u \circ R_u$  by  $'\sigma^u$ . We denote the modular element of  $\mathbb{G}_u$  by  $\delta_u$ .

**Proposition 4.2.44** ([Kus97b]). *The following properties hold*

- We have that  $\varepsilon_u \circ \pi_u = \varepsilon$
- Let  $a \in A$ . Then  $\pi_u(a) \in D(S_u)$  and  $S_u(\pi_u(a)) = \pi_u(S(a))$ .

**Proposition 4.2.45** ([Kus97b]). *Consider  $a \in A$ , then  $\pi_u(a)$  is analytic with respect to  $\sigma^u$  and to  $'\sigma^u$ .*

### 4.2.4.3 Analytical structure associated with AQG

It was shown in [Kus03] that the analytical structure of the reduced and universal  $C^*$ -algebraic quantum group associated with an algebraic quantum group can be completely pulled down to the algebraic level. We recall in this section, what is the analytical structure of an algebraic quantum group. In the following, let  $\mathbb{G} = (A, \Delta, \varphi)$  be an algebraic quantum group.

**Definition 4.2.46** (Definition 3.1, [Kus03]). We say that a function  $\alpha : \mathbb{C} \rightarrow \text{End}(A, A)$  is an *analytic one-parameter group of automorphisms of  $\mathbb{G}$* , if

- For every  $t \in \mathbb{R}$ ,  $\alpha_t$  is a  $*$ -automorphism of  $A$ .
- For every  $s, t \in \mathbb{R}$ ,  $\alpha_{s+t} = \alpha_s \circ \alpha_t$ .
- For every  $t \in \mathbb{R}$ , there is a strictly positive number  $m_t$  such that  $\varphi \circ \alpha_t = m_t \varphi$ .
- For every  $a \in A$  and  $\omega \in \widehat{A}$ , the function  $\mathbb{C} \rightarrow \mathbb{C} : z \mapsto \omega(\alpha_z(a))$  is analytic.

**Lemma 4.2.47** (Lemma 3.2, [Kus03]). *Let  $\alpha$  be an analytic one-parameter group of automorphisms of  $\mathbb{G}$ . The following properties hold*

- (1) Consider  $z \in \mathbb{C}$ , then  $\alpha_z(a)^* = \alpha_{\bar{z}}(a^*)$  for all  $a \in A$ .
- (2) For all  $y, z \in \mathbb{C}$ ,  $\alpha_{y+z} = \alpha_y \circ \alpha_z$ .
- (3) Consider  $z \in \mathbb{C}$ , then  $\alpha_z$  is an automorphism of  $A$  and  $(\alpha_z)^{-1} = \alpha_{-z}$ .

**Definition 4.2.48** (Definition 4.1, [Kus03]). We say that a function  $u : \mathbb{C} \rightarrow \mathcal{M}(A)$  is an *analytic one-parameter group of unitary multipliers of  $A$* , if

- For every  $t \in \mathbb{R}$ ,  $u_t$  is unitary in  $M(A)$ .
- For every  $s, t \in \mathbb{R}$ ,  $u_{s+t} = u_s u_t$ .
- For every  $\omega \in \widehat{A}$ , the function  $\mathbb{C} \rightarrow \mathbb{C} : z \mapsto \omega(u_z)$  is analytic.

**Lemma 4.2.49** (Lemma 4.2, [Kus03]). *Let  $u$  be an analytic one-parameter group of unitary multipliers of  $A$ . The following properties hold*

- (1) For all  $z \in \mathbb{C}$ ,  $u_z^* = u_{-\bar{z}}$ .
- (2) For all  $y, z \in \mathbb{C}$ ,  $u_{y+z} = u_y u_z$ .
- (3) For all  $z \in \mathbb{C}$ ,  $u_z$  is invertible in  $\mathcal{M}(A)$  and  $(u_z)^{-1} = u_{-z}$ .

**Proposition 4.2.50** (Proposition 5.1, [Kus03]). *There exists a unique analytic one-parameter group of unitary multipliers  $w$  of  $A$  such that  $w_{-i} = \delta$ , and we define  $\delta^z := w_{-iz}$  for  $z \in \mathbb{C}$ .*

**Proposition 4.2.51** (Proposition 5.6 - Corollary 5.7, [Kus03]). *We have the following equalities*

$$\begin{aligned} \sigma_z^u(\pi_u(A)) &= \pi_u(A), \quad \prime\sigma_z^u(\pi_u(A)) = \pi_u(A), \\ \tau_z^u(\pi_u(A)) &= \pi_u(A), \quad R_u(\pi_u(A)) = \pi_u(A), \end{aligned}$$

for every  $z \in \mathbb{C}$ .

These last results imply that all the objects associated with the universal  $C^*$ -algebraic quantum group  $\mathbb{G}_u$  can be pulled down to the algebraic level.

**Proposition 4.2.52** (Proposition 5.8, [Kus03]). *There exists a unique analytic one-parameter group of automorphisms  $\sigma$  of  $A$  such that  $\sigma_{-i} = \rho$ . We call  $\sigma$  the modular group of  $\varphi$ . Moreover,  $\varphi \circ \sigma_z = \varphi$  and  $\pi_u(\sigma_z(a)) = \sigma_z^u(\pi_u(a))$  for every  $a \in A$  and  $z \in \mathbb{C}$ .*

Denote the modular operator of  $\varphi_u$  by  $\nabla$  and the modular conjugation of  $\varphi_u$  by  $J$ , both with respect to the GNS-construction  $(L^2\mathbb{G}, \Lambda_u, \Lambda_{\mathbb{G}})$ . These objects are completely characterized by the next lemma

**Lemma 4.2.53** (Lemma 5.9, [Kus03]). *The following properties hold*

- (1) Consider  $z \in \mathbb{C}$ . Then  $\Lambda(A)$  is a core for  $\nabla^{iz}$  and  $\nabla^{iz}(\Lambda(a)) = \Lambda(\sigma_z(a))$  for every  $a \in A$ .
- (2) We have for every  $a \in A$  that  $J(\Lambda(a)) = \Lambda(\sigma_{i/2}(a)^*)$ .

**Proposition 4.2.54** (Proposition 5.10, [Kus03]). *There exists a unique analytic one-parameter group of automorphisms  $\sigma'$  of  $A$  such that  $\sigma'_{-i} = \rho'$ . We call  $\sigma'$  the modular group of the right Haar integral. We have moreover that  $\varphi \circ \sigma'_z = \nu^z \varphi$  and  $\pi_u(\sigma'_z(a)) = \sigma'_z{}^u(\pi_u(a))$  for every  $a \in A$  and  $z \in \mathbb{C}$ .*

**Proposition 4.2.55** (Proposition 5.11, [Kus03]). *There exists a unique analytic one-parameter group of automorphisms  $\tau$  of  $A$  such that  $\tau_{-i} = S^2$ . We call  $\tau$  the scaling group of  $\mathbb{G}$ . We have moreover that  $\varphi \circ \tau_z = \nu^{-z} \varphi$  and  $\pi_u(\tau_z(a)) = \tau_z^u(\pi_u(a))$  for every  $a \in A$  and  $z \in \mathbb{C}$ .*

**Definition 4.2.56** (Definition 5.11, [Kus03]). We define the mapping  $R$  from  $A$  to  $A$  such that  $\pi_u(R(a)) = R_u(\pi_u(a))$  for all  $a \in A$ . Then  $R$  is a anti- $*$ -automorphism of  $A$  such that  $R^2 = id$ . We call  $R$  the unitary antipode of  $\mathbb{G}$ .

We give the principals relations between this objects.

**Proposition 4.2.57** ([Kus03]).

- $R \circ \tau_z = \tau_z \circ R$  and  $\tau_z \circ S = S \circ \tau_z$ , for every  $z \in \mathbb{C}$ .
- $S = R \circ \tau_{-i/2}$  and  $R \circ S = S \circ R$ .
- $\Delta \circ R = \Sigma \circ (R \otimes R) \circ \Delta$ .
- For every  $z \in \mathbb{C}$ ,

$$\begin{aligned} (\tau_z \otimes \tau_z) \circ \Delta &= \Delta \circ \tau_z, & (\tau_z \otimes \sigma_z) \circ \Delta &= \Delta \circ \sigma_z, \\ (\sigma'_z \otimes \tau_{-z}) \circ \Delta &= \Delta \circ \sigma'_z, & (\sigma_z \otimes \sigma'_{-z}) \circ \Delta &= \Delta \circ \tau_z. \end{aligned}$$

- For every  $z \in \mathbb{C}$ ,  $\Delta(\delta^z) = \delta^z \otimes \delta^z$ .

### 4.3 Hopf $C^*$ -bimodules

#### 4.3.1 $C^*$ -modules and $C^*$ -bimodules over $C^*$ -bases

**Definition 4.3.1** (Definition 2.2, [Tim12b]). A  $C^*$ -base  $\mathfrak{b} = (\mathfrak{K}, \mathfrak{B}, \mathfrak{B}^\dagger)$  consists of a Hilbert space  $\mathfrak{K}$  and two commuting non-degenerate  $C^*$ -algebras  $\mathfrak{B}, \mathfrak{B}^\dagger \subseteq \mathcal{B}(\mathfrak{K})$ . The *opposite* of  $\mathfrak{b}$  is the  $C^*$ -base  $\mathfrak{b}^\dagger := (\mathfrak{K}, \mathfrak{B}^\dagger, \mathfrak{B})$ . Two  $C^*$ -bases  $(\mathfrak{K}, \mathfrak{B}, \mathfrak{B}^\dagger), (\mathfrak{K}', \mathfrak{C}, \mathfrak{C}^\dagger)$  are called *equivalent*, if there exists a unitary  $V \in \mathcal{B}(\mathfrak{K}, \mathfrak{K}')$  such that  $\text{Ad}_V(\mathfrak{B}) = \mathfrak{C}$  and  $\text{Ad}_V(\mathfrak{B}^\dagger) = \mathfrak{C}^\dagger$ . We will use the notation  $(\mathfrak{K}, \mathfrak{B}, \mathfrak{B}^\dagger) \equiv (\mathfrak{K}', \mathfrak{C}, \mathfrak{C}^\dagger)$ .

**Notation 16.** Given a unitary operator  $V : \mathfrak{K} \rightarrow \mathfrak{K}'$ , we use the notation

$$\text{Ad}_V(\mathfrak{B}) := \{VTV^* : T \in \mathfrak{B} \subseteq \mathcal{B}(\mathfrak{K})\} \subseteq \mathcal{B}(\mathfrak{K}').$$

**Example 4.3.2.** The Hilbert space  $\mathbb{C}$  and twice the  $C^*$ -algebra  $\mathcal{B}(\mathbb{C}) \cong \mathbb{C}$  form a  $C^*$ -base  $\mathfrak{t} = (\mathbb{C}, \mathbb{C}, \mathbb{C})$  called the *trivial  $C^*$ -base*.

**Example 4.3.3.** Let  $\mu$  be a proper, faithful KMS-weight on a  $C^*$ -algebra  $A$  with GNS-space  $\mathcal{H}_\mu$ , GNS-representation  $\pi_\mu : A \rightarrow \mathcal{B}(\mathcal{H}_\mu)$ , modular conjugation  $J_\mu : \mathcal{H}_\mu \rightarrow \mathcal{H}_\mu$ , and opposite GNS-representation  $\pi_\mu^{\text{op}} : A^{\text{op}} \rightarrow \mathcal{B}(\mathcal{H}_\mu)$ ,  $a \mapsto J_\mu \pi_\mu(a^*) J_\mu$ . The triple  $(\mathcal{H}_\mu, \pi_\mu(A), \pi_\mu^{\text{op}}(A^{\text{op}}))$  is a  $C^*$ -base and it is called the  $C^*$ -base associated with the weight  $\mu$  and denote by  $\mathfrak{b}_\mu$ . Its opposite is equivalent to the  $C^*$ -base associated with the opposite weight  $\mu^{\text{op}}$  on  $A^{\text{op}}$ . Indeed,  $\mathfrak{b}_\mu^\dagger \equiv \mathfrak{b}_{\mu^{\text{op}}} = (\mathcal{H}_\mu, \pi_{\mu^{\text{op}}}(A^{\text{op}}), \pi_\mu(A))$ .

**Definition 4.3.4.** Let  $N$  be a von Neumann algebra with a normal, faithful, semi-finite weight  $\nu$ . Using the GNS-space  $\mathcal{H}_\nu$  and the GNS-representation  $\pi_\nu : N \rightarrow \mathcal{B}(\mathcal{H}_\nu)$  we can identify  $N$  and  $\pi_\nu(N)$ . A  $C^*$ -base for  $(N, \nu)$  is a  $C^*$ -base of the form  $(\mathcal{H}_\nu, \mathfrak{C}, \mathfrak{C}^\dagger)$ , where  $\mathfrak{C}'' = N$  and  $(\mathfrak{C}^\dagger)'' = N'$ . The opposite  $C^*$ -base  $(\mathcal{H}_\nu, \mathfrak{C}^\dagger, \mathfrak{C})$  is a  $C^*$ -base for  $(N^{\text{op}}, \nu^{\text{op}})$ .

Let  $\mathfrak{b} = (\mathfrak{K}, \mathfrak{B}, \mathfrak{B}^\dagger)$  be a  $C^*$ -base.

**Definition 4.3.5** (Definition 2.4, [Tim12b]). A  $C^*$ - $\mathfrak{b}$ -module  $\mathcal{H}_E = (\mathcal{H}, E)$  is a Hilbert space  $\mathcal{H}$  with a closed subspace  $E \subseteq \mathcal{B}(\mathfrak{K}, \mathcal{H})$  satisfying  $[E(\mathfrak{K})] = \mathcal{H}$ ,  $[E\mathfrak{B}] = E$  and  $[E^*E] = \mathfrak{B} \subseteq \mathcal{B}(\mathfrak{K})$ . A *morphism between  $C^*$ - $\mathfrak{b}$ -modules  $\mathcal{H}_E$  and  $\mathcal{K}_F$*  is an operator  $T \in \mathcal{B}(\mathcal{H}, \mathcal{K})$  satisfying  $TE \subseteq F$  and  $T^*F \subseteq E$ . We denote the set of such morphisms by  $\mathcal{L}(\mathcal{H}_E, \mathcal{K}_F)$ .

**Remark 78.**

- For the trivial  $C^*$ -base  $\mathfrak{t} = (\mathbb{C}, \mathbb{C}, \mathbb{C})$ , every  $C^*$ - $\mathfrak{t}$ -module is of the form  $(\mathcal{H}, | \mathcal{H})$ .
- The class of all  $C^*$ - $\mathfrak{b}$ -modules with its morphisms is a  $C^*$ -category.

**Lemma 4.3.6** (Lemma 2.6, [Tim12b]). Let  $\mathcal{H}_E$  be a  $C^*$ - $\mathfrak{b}$ -module.

- The vector space  $E$  is a right Hilbert  $C^*$ -module over  $\mathfrak{B}$  with inner product given by  $\langle e | e' \rangle_{\mathfrak{B}} := e^* e'$  for all  $e, e' \in E$ .
- There exist unitary isomorphisms  $E \underset{id}{\otimes} \mathfrak{K} \rightarrow \mathcal{H}$ ,  $e \otimes \eta \mapsto e(\eta)$  and  $\mathfrak{K} \underset{id}{\otimes} E \rightarrow \mathcal{H}$ ,  $\eta \otimes e \mapsto e(\eta)$ .



- There exists a unique normal, unital and faithful representation  $\rho_E : \mathfrak{B}^\dagger \rightarrow \mathcal{B}(\mathcal{H})$  such that  $\rho_E(b^\dagger)(e(\eta)) = (eb^\dagger)(\eta)$  for all  $b^\dagger \in \mathfrak{B}^\dagger$ ,  $e \in E$  and  $\eta \in \mathfrak{K}$ .
- Let  $\mathcal{K}_F$  be a  $C^*$ - $\mathfrak{b}$ -module and  $T \in \mathcal{L}(\mathcal{H}_E, \mathcal{K}_F)$ . Then  $T\rho_E(b^\dagger) = \rho_F(b^\dagger)T$  for all  $b^\dagger \in \mathfrak{B}^\dagger$ . Moreover, the map  $E \rightarrow F$ ,  $e \mapsto Te$  defines an operator in  $\mathcal{L}_{\mathfrak{B}}(E, F)$ , again denoted by  $T$ .

**Example 4.3.7.** For a  $C^*$ - $\mathfrak{a}$ -module  $\mathcal{H}_E$ ,  $E = |\mathcal{H}\rangle \cong \mathcal{H}$  is a Hilbert  $C^*$ -module over  $\mathbb{C}$ , i.e. a Hilbert space, and the inner product as Hilbert  $C^*$ -module and Hilbert space coincide, because

$$\langle |\xi\rangle || \xi'\rangle \rangle_{\mathbb{C}} = |\xi\rangle^* |\xi'\rangle = \langle \xi || \xi'\rangle = \langle \xi | \xi'\rangle_{\mathcal{H}}$$

for all  $\xi, \xi' \in \mathcal{H}$ . The representation  $\rho_E : \mathbb{C} \rightarrow \mathcal{B}(\mathcal{H})$  is given by  $\rho_E(\lambda)(\xi) = \lambda\xi$ .

Let also  $\mathfrak{a} = (\mathfrak{H}, \mathfrak{A}, \mathfrak{A}^\dagger)$  be a  $C^*$ -base.

**Definition 4.3.8** (Definition 2.8. [Tim12b]). A  $C^*$ - $(\mathfrak{a}^\dagger, \mathfrak{b})$ -bimodule is a triple  ${}_E\mathcal{H}_F = (\mathcal{H}, E, F)$ , where  $\mathcal{H}$  is a Hilbert space,  $(\mathcal{H}, E)$  a  $C^*$ - $\mathfrak{a}^\dagger$ -module,  $(\mathcal{H}, F)$  a  $C^*$ - $\mathfrak{b}$ -module, and  $[\rho_E(\mathfrak{A})F] = F$  and  $[\rho_F(\mathfrak{B}^\dagger)E] = E$ . The set of morphisms between  $C^*$ - $(\mathfrak{a}^\dagger, \mathfrak{b})$ -bimodules  ${}_E\mathcal{H}_F$  and  ${}_{E'}\mathcal{K}_{F'}$  is

$$\mathcal{L}({}_E\mathcal{H}_F, {}_{E'}\mathcal{K}_{F'}) := \mathcal{L}({}_E\mathcal{H}, {}_{E'}\mathcal{K}) \cap \mathcal{L}(\mathcal{H}_F, \mathcal{K}_{F'}).$$

**Remark 79.**

- By Lemma 4.3.6,  $[\rho_E(\mathfrak{A}), \rho_F(\mathfrak{B}^\dagger)] = 0$  for every  $C^*$ - $(\mathfrak{a}^\dagger, \mathfrak{b})$ -bimodule  ${}_E\mathcal{H}_F$ .
- The class of all  $C^*$ - $(\mathfrak{a}^\dagger, \mathfrak{b})$ -bimodules with its morphisms is a  $C^*$ -category [Mit02].

#### 4.3.1.1 Relative tensor product of $C^*$ -modules

**Definition 4.3.9** ([Tim12b]). Let  $\mathfrak{b} = (\mathfrak{K}, \mathfrak{B}, \mathfrak{B}^\dagger)$  be a  $C^*$ -base,  $\mathcal{H}_E$  a  $C^*$ - $\mathfrak{b}$ -module, and  $\mathcal{K}_F$  a  $C^*$ - $\mathfrak{b}^\dagger$ -module. The relative tensor product of  $\mathcal{H}_E$  and  $\mathcal{K}_F$  is the Hilbert space

$$\mathcal{H}_E \otimes_{\mathfrak{b}} \mathcal{K}_F := E \otimes \mathfrak{K} \otimes F,$$

which is spanned by elements  $x \otimes \eta \otimes y$ , where  $x \in E$ ,  $\eta \in \mathfrak{K}$ ,  $y \in F$ , and the inner product is given by

$$\langle e \otimes \eta \otimes f | e' \otimes \eta' \otimes f' \rangle := \langle \eta | e^* e' f^* f'(\eta') \rangle = \langle \eta | f^* f' e^* e'(\eta') \rangle$$

for all  $e, e' \in E$ ,  $\eta, \eta' \in \mathfrak{K}$ ,  $f, f' \in F$ .

**Example 4.3.10.** If  $\mathcal{H}_E, \mathcal{K}_F$  are  $C^*$ - $\mathfrak{a}$ -modules, because  $E = |\mathcal{H}\rangle$  and  $F = |\mathcal{K}\rangle$ , then

$$\mathcal{H}_E \otimes_{\mathfrak{b}} \mathcal{K}_F \cong \mathcal{H} \otimes \mathcal{K}, \quad |\xi\rangle \otimes \lambda \otimes |\eta\rangle \mapsto \lambda\xi \otimes \eta = \xi \otimes \lambda\eta$$

**Remark 80.** We summarize here some observations on the relative tensor product of  $C^*$ -modules over  $C^*$ -bases.

(i) We have a unitary flip

$$\Sigma : \mathcal{H}_E \otimes_{\mathfrak{b}} \mathcal{K} \rightarrow \mathcal{K} \otimes_{\mathfrak{b}^\dagger} \mathcal{H}_E, \quad e \otimes \eta \otimes f \mapsto f \otimes \eta \otimes e$$

(ii) Using the isomorphisms of Lemma 4.3.6, we have the following isomorphisms

$$\begin{aligned} \mathcal{H} \otimes_{\rho_E} F &\cong \mathcal{H}_E \otimes_{\mathfrak{b}} \mathcal{K} \cong E \otimes_{\rho_F} \mathcal{K} \\ e(\eta) \otimes_{\rho_E} f &\mapsto e \otimes \eta \otimes f \mapsto e \otimes_{\rho_F} f(\eta) \end{aligned}$$

(iii) For each  $e \in E$  and  $f \in F$ , there exist bounded linear operators

$$\begin{aligned} |e\rangle_1 : \mathcal{K} &\rightarrow E \otimes_{\rho_F} \mathcal{K} \cong \mathcal{H}_E \otimes_{\mathfrak{b}} \mathcal{K}, & f'(\eta) &\mapsto e \otimes_{\rho_F} f'(\eta) \\ |f\rangle_2 : \mathcal{H} &\rightarrow \mathcal{H} \otimes_{\rho_E} F \cong \mathcal{H}_E \otimes_{\mathfrak{b}} \mathcal{K}, & e'(\eta) &\mapsto e'(\eta) \otimes_{\rho_F} f \end{aligned}$$

whose adjoints are given by

$$\begin{aligned} \langle e|_1 = |e\rangle_1^* : \mathcal{H}_E \otimes_{\mathfrak{b}} \mathcal{K} &\rightarrow \mathcal{K}, & e' \otimes_{\rho_F} f'(\eta) &\mapsto \rho_F(e^* e')(f'(\eta)) \\ \langle f|_2 = |f\rangle_2^* : \mathcal{H}_E \otimes_{\mathfrak{b}} \mathcal{K} &\rightarrow \mathcal{H}, & e'(\eta) \otimes_{\rho_E} f' &\mapsto \rho_E(f^* f')(e'(\eta)) \end{aligned}$$

(iv) For all  $S \in \rho_E(\mathfrak{B}^\dagger)'$  and  $T \in \rho_F(\mathfrak{B})'$ , the linear maps

$$\begin{aligned} S \otimes_{\rho_E} id : \mathcal{H} \otimes_{\rho_E} F &\rightarrow \mathcal{H} \otimes_{\rho_E} F, & e(\eta) \otimes_{\rho_E} f &\mapsto S(e(\eta)) \otimes_{\rho_E} f \\ id \otimes_{\rho_F} T : E \otimes_{\rho_F} \mathcal{K} &\rightarrow E \otimes_{\rho_F} \mathcal{K}, & e \otimes_{\rho_F} f(\eta) &\mapsto e \otimes_{\rho_F} T(f(\eta)) \end{aligned}$$

belongs to  $\mathcal{B}(\mathcal{H}_E \otimes_{\mathfrak{b}} \mathcal{K})$ . If these operator commute, we write

$$S \otimes_{\mathfrak{b}} T := (S \otimes_{\rho_E} id)(id \otimes_{\rho_F} T) = (id \otimes_{\rho_F} T)(S \otimes_{\rho_E} id).$$

**Notation 17.** Using the remark of the item (iii), we will use the notations

$$\begin{aligned} |E\rangle_1 := \{|e\rangle_1 : e \in E\} &\subseteq \mathcal{B}(\mathcal{K}, \mathcal{H}_E \otimes_{\mathfrak{b}} \mathcal{K}), & \langle E|_1 := \{\langle e|_1 : e \in E\} &\subseteq \mathcal{B}(\mathcal{H}_E \otimes_{\mathfrak{b}} \mathcal{K}, \mathcal{K}) \\ |F\rangle_2 := \{|f\rangle_2 : f \in F\} &\subseteq \mathcal{B}(\mathcal{H}, \mathcal{H}_E \otimes_{\mathfrak{b}} \mathcal{K}), & \langle F|_2 := \{\langle f|_2 : f \in F\} &\subseteq \mathcal{B}(\mathcal{H}_E \otimes_{\mathfrak{b}} \mathcal{K}, \mathcal{H}) \end{aligned}$$

Let  $\mathfrak{a} = (\mathfrak{H}, \mathfrak{A}, \mathfrak{A}^\dagger)$ ,  $\mathfrak{c} = (\mathfrak{L}, \mathfrak{C}, \mathfrak{C}^\dagger)$  be  $C^*$ -bases. The next proposition shows that the tensor product of bimodules over  $(\mathfrak{a}^\dagger, \mathfrak{b})$  and  $(\mathfrak{b}^\dagger, \mathfrak{c})$  is a bimodule over  $(\mathfrak{a}^\dagger, \mathfrak{c})$ .

**Proposition 4.3.11** (Proposition 2.13, [Tim12b]). *Let  $\mathcal{H} = {}_E \mathcal{H}_F$  be a  $C^*$ - $(\mathfrak{a}^\dagger, \mathfrak{b})$ -bimodule,  $\mathcal{K} = {}_M \mathcal{K}_N$  be a  $C^*$ - $(\mathfrak{b}^\dagger, \mathfrak{c})$ -bimodule, and*

$$E \triangleleft M := [|M\rangle_2 E] \subseteq \mathcal{B}(\mathfrak{H}, \mathcal{H}_F \otimes_{\mathfrak{b}} {}_M \mathcal{K}), \quad F \triangleright N := [|K\rangle_1 N] \subseteq \mathcal{B}(\mathfrak{L}, \mathcal{H}_F \otimes_{\mathfrak{b}} {}_M \mathcal{K}). \quad (4.3)$$

Then

$$\mathcal{H} \otimes_{\mathfrak{b}} \mathcal{K} := {}_{(E \triangleleft M)}(\mathcal{H}_F \otimes_{\mathfrak{b}} {}_M \mathcal{K})_{(F \triangleright N)}$$

is a  $C^*$ - $(\mathfrak{b}^\dagger, \mathfrak{c})$ -bimodule and

$$\rho_{E \triangleleft M}(a) = \rho_E(a) \otimes id, \quad \rho_{F \triangleright N}(c) = id \otimes \rho_N(c)$$

for all  $a \in (\mathfrak{A}^\dagger)' \subseteq \mathcal{B}(\mathfrak{H})$  and  $c \in \mathfrak{C}' \subseteq \mathcal{B}(\mathfrak{L})$ .

**Definition 4.3.12.** In the situation above, the  $C^*$ - $(\mathfrak{b}^\dagger, \mathfrak{c})$ -bimodule  $\mathcal{H} \otimes_{\mathfrak{b}} \mathcal{K}$  will be called the *relative tensor product* of the  $C^*$ -bimodules  $\mathcal{H}$  and  $\mathcal{K}$ .

**Remark 81.** We have the following commutative diagram of Hilbert spaces and closed spaces of operators between them

$$\begin{array}{ccccc}
 \mathfrak{H} & & \mathfrak{K} & & \mathfrak{L} \\
 \searrow E & & \swarrow F & & \swarrow M \\
 & \mathcal{H} & & & \mathcal{K} \\
 & \searrow |M\rangle_2 & & & \swarrow |F\rangle_1 \\
 & & \mathcal{H}_F \otimes_{\mathfrak{b}} {}_M \mathcal{K} & & \\
 \swarrow E \triangleleft M & & & & \swarrow F \triangleright N
 \end{array}$$

**Notation 18.** Given a  $C^*$ - $(\mathfrak{a}^\dagger, \mathfrak{b})$ -module  $\mathcal{H} = {}_E \mathcal{H}_F$  and a  $C^*$ - $(\mathfrak{b}^\dagger, \mathfrak{c})$ -bimodule  $\mathcal{K} = {}_M \mathcal{K}_N$ , we will use the following notations

$$\mathcal{H}_F \otimes_{\mathfrak{b}} {}_M \mathcal{K}_N := (\mathcal{H}_F \otimes_{\mathfrak{b}} {}_M \mathcal{K})_{(F \triangleright N)}, \quad {}_E \mathcal{H}_F \otimes_{\mathfrak{b}} {}_M \mathcal{K} := (\mathcal{H}_F \otimes_{\mathfrak{b}} {}_M \mathcal{K})_{(E \triangleleft M)},$$

$${}_E \mathcal{H}_F \otimes_{\mathfrak{b}} {}_M \mathcal{K}_N := {}_{(E \triangleleft M)}(\mathcal{H}_F \otimes_{\mathfrak{b}} {}_M \mathcal{K})_{(F \triangleright N)}.$$

#### 4.3.1.2 $C^*$ -pseudo-multiplicative unitaries

**Definition 4.3.13** (Definition 2.1, [Tim12a]). A  $C^*$ -pseudo-multiplicative unitary is a tuple  $(\mathfrak{b}, \mathcal{H}, \hat{F}, E, F, V)$  consisting of a  $C^*$ -base  $\mathfrak{b} = (\mathfrak{K}, \mathfrak{B}, \mathfrak{B}^\dagger)$ , a  $C^*$ - $(\mathfrak{b}^\dagger, \mathfrak{b}, \mathfrak{b}^\dagger)$ -module  $(\mathcal{H}, \hat{F}, E, F)$ , and a unitary

$$V : \mathcal{H}_{\hat{F}} \otimes_{\mathfrak{b}^\dagger} {}_E \mathcal{H} \rightarrow \mathcal{H}_E \otimes_{\mathfrak{b}} {}_F \mathcal{H}$$

such that

$$V(E \triangleleft E) = E \triangleright E, \quad V(\hat{F} \triangleright F) = \hat{F} \triangleleft F, \quad V(\hat{F} \triangleright \hat{F}) = E \triangleright \hat{F}, \quad V(F \triangleleft E) = F \triangleleft F \quad (4.4)$$

in  $\mathcal{B}(\mathfrak{K}, \mathcal{H}_E \otimes_F \mathcal{H})$  and the diagram

$$\begin{array}{ccccc}
 & & \mathcal{H}_E \otimes_F \mathcal{H}_{\hat{F}} \otimes_{\mathfrak{b}^\dagger} \mathcal{H} & & \\
 & \nearrow^{V \otimes id} & & \searrow^{id \otimes V} & \\
 \mathcal{H}_{\hat{F}} \otimes_{\mathfrak{b}^\dagger} \mathcal{H}_{\hat{F}} \otimes_{\mathfrak{b}^\dagger} \mathcal{H} & & & & \mathcal{H}_E \otimes_{\mathfrak{b}} \mathcal{H}_{\hat{F}} \otimes_{\mathfrak{b}} \mathcal{H} \\
 & \searrow^{id \otimes V} & & \nearrow^{V \otimes id} & \\
 & \mathcal{H}_{\hat{F}} \otimes_{\mathfrak{b}^\dagger} (E \triangleright E) (\mathcal{H}_E \otimes_F \mathcal{H}) & & & (\mathcal{H}_{\hat{F}} \otimes_{\mathfrak{b}^\dagger} \mathcal{H}) (E \triangleleft E) \otimes_{\mathfrak{b}} \mathcal{H} \\
 & \downarrow^{id \otimes \Sigma} & & \uparrow^{\Sigma_{[23]}} & \\
 \mathcal{H}_{\hat{F}} \otimes_{\mathfrak{b}^\dagger} \mathcal{H}_{\hat{F}} \otimes_{\mathfrak{b}^\dagger} \mathcal{H} & \xrightarrow{V \otimes id} & (\mathcal{H}_E \otimes_F \mathcal{H})_{(\hat{F} \triangleleft F)} \otimes_{\mathfrak{b}^\dagger} \mathcal{H} & & 
 \end{array}$$

is commutative. We frequently call  $V$  a  $C^*$ -pseudo-multiplicative unitary.

**Example 4.3.14.** If  $\mathfrak{t}$  is the trivial base  $(\mathbb{C}, \mathbb{C}, \mathbb{C})$ , then

$$\mathcal{H}_{\hat{F}} \otimes_{\mathfrak{t}^\dagger} \mathcal{H} \cong \mathcal{H} \otimes \mathcal{H} \cong \mathcal{H}_E \otimes_F \mathcal{H}$$

and a  $C^*$ -pseudo-multiplicative unitary is simply a multiplicative unitary in the sense of Baaĵ-Skandalis.

**Example 4.3.15.** Assume that  $\mathfrak{b}$  is the  $C^*$ -base associated with a proper faithful KMS-weight  $\mu$  on a  $C^*$ -algebra  $A$ . Then  $\mu$  extends to a normal, semi-finite and faithful weight  $\tilde{\mu}$  on  $\mathfrak{vN}(\pi_\mu(A))$ , and with respect to the canonical isomorphisms

$$\mathcal{H}_{\hat{F}} \otimes_{\mathfrak{b}^\dagger} \mathcal{H} \cong \mathcal{H}_{\tilde{\rho}_{\hat{F}}} \otimes_{\tilde{\mu}^\circ} \mathcal{H}, \quad \mathcal{H}_E \otimes_{\mathfrak{b}} \mathcal{H} \cong \mathcal{H}_{\tilde{\rho}_E} \otimes_{\tilde{\mu}} \mathcal{H},$$

the unitary  $V$  is a pseudo-multiplicative unitary on Hilbert spaces in the sense of Vallin [Val00].

**Remark 82.** The opposite of a  $C^*$ -pseudo-multiplicative unitary  $(\mathfrak{b}, \mathcal{H}, \hat{F}, E, F, V)$  is the  $C^*$ -pseudo-multiplicative unitary  $(\mathfrak{b}^\dagger, \mathcal{H}, F, E, \hat{F}, \hat{V})$ , where  $\hat{V}$  denotes the composition

$$\Sigma V^* \Sigma : \mathcal{H}_F \otimes_{\mathfrak{b}^\dagger} \mathcal{H} \xrightarrow{\Sigma} \mathcal{H}_E \otimes_{\mathfrak{b}} \mathcal{H} \xrightarrow{V^*} \mathcal{H}_{\hat{F}} \otimes_{\mathfrak{b}^\dagger} \mathcal{H} \xrightarrow{\Sigma} \mathcal{H}_E \otimes_{\mathfrak{b}} \mathcal{H}.$$

**Definition 4.3.16** (Definition 4.2, [Tim12a]). We will say that a  $C^*$ -pseudo-multiplicative unitary  $(\mathfrak{b}, \mathcal{H}, \hat{F}, E, F, V)$  is *regular* if

$$[\langle E |_1 V | E \rangle_2] = [EE^*] \subseteq \mathcal{B}(\mathcal{H}).$$

### 4.3.1.3 The fiber product of $C^*$ -algebras

Let  $\mathfrak{b} = (\mathfrak{K}, \mathfrak{B}, \mathfrak{B}^\dagger)$  be a  $C^*$ -base.

**Definition 4.3.17** (Definition 3.1 [Tim12b]). A *non-degenerate  $C^*$ - $\mathfrak{b}$ -algebra* is a pair  $A_{\mathcal{H}}^E := (\mathcal{H}_E, A)$ , where  $\mathcal{H}_E$  is a  $C^*$ - $\mathfrak{b}$ -module,  $A \subseteq \mathcal{B}(\mathcal{H})$  is a non-degenerate  $C^*$ -algebra, and  $\rho_E(\mathfrak{B}^\dagger)A \subseteq A$ . A *morphism between the  $C^*$ - $\mathfrak{b}$ -algebras  $A_{\mathcal{H}}^E, B_{\mathcal{K}}^F$*  is a  $*$ -homomorphism  $\pi : A \rightarrow \mathcal{M}(B)$  satisfying the condition  $F = [\mathcal{L}^\pi(\mathcal{H}_E, \mathcal{K}_F)E]$ , where

$$\mathcal{L}^\pi(\mathcal{H}_E, \mathcal{K}_F) := \{T \in \mathcal{L}(\mathcal{H}_E, \mathcal{K}_F) : Ta = \pi(a)T \text{ for all } a \in A\}.$$

**Remark 83.** We recall that if  $A \subseteq \mathcal{B}(\mathcal{H})$  is a non-degenerate  $C^*$ -algebra, then we can identify

$$\mathcal{M}(A) \cong \{T \in \mathcal{B}(\mathcal{H}) : Ta, aT \in A, \text{ for all } a \in A\} \subseteq \mathcal{B}(\mathcal{H}). \quad (4.5)$$

Then, if  $A_{\mathcal{H}}^E$  is a  $C^*$ - $\mathfrak{b}$ -algebra, the condition  $\rho_E(\mathfrak{B}^\dagger)A \subseteq A$  implies that

$$\rho_E(\mathfrak{B}^\dagger) \subseteq \mathcal{M}(A).$$

**Example 4.3.18.**

- If  $\mathcal{H}$  is a Hilbert space and  $A \subseteq \mathcal{B}(\mathcal{H})$  is a  $C^*$ -algebra, then the pair  $A_{\mathcal{H}}^{|\mathcal{H}\rangle} = (\mathcal{H}|_{\mathcal{H}}, A)$  is a  $C^*$ - $\mathfrak{t}$ -algebra.
- Let  $A_{\mathcal{H}}^E$  be a non-degenerate  $C^*$ - $\mathfrak{b}$ -algebra. Then, using the identification 4.5,  $\mathcal{M}(A)_{\mathcal{H}}^E$  becomes a  $C^*$ - $\mathfrak{b}$ -algebra.
- Let  $B$  be a  $C^*$ -algebra with KMS-weight  $\mu$  and associated  $C^*$ -base  $\mathfrak{b}$ , and let  $A$  be a  $C^*$ -algebra containing  $B$  with a conditional expectation  $\Phi : A \rightarrow B$ . Then,  $\pi_v(A)_{\mathcal{H}}^E$  is a non-degenerate  $C^*$ - $\mathfrak{b}$ -algebra because  $\rho_E(\mathfrak{B})\pi_v(A) = \pi_v(B)\pi_v(A) \subseteq \pi_v(A)$ , and similarly  $\pi_v^{op}(A^{op})_{\mathcal{H}}^F$  is a non-degenerate  $C^*$ - $\mathfrak{b}^\dagger$ -algebra.

**Lemma 4.3.19** (Lemma 2.2, [Tim07]). If  $\pi : A \rightarrow \mathcal{M}(B)$  is a morphism between the  $C^*$ - $\mathfrak{b}$ -algebras  $A_{\mathcal{H}}^E, B_{\mathcal{K}}^F$ , then

$$\pi(\rho_E(b^\dagger)a) = \rho_F(b^\dagger)\pi(a)$$

**Definition 4.3.20.** Let  $A_{\mathcal{H}}^F$  be a  $C^*$ - $\mathfrak{b}$ -algebra and  $B_{\mathcal{K}}^{\hat{F}}$  a  $C^*$ - $\mathfrak{b}^\dagger$ -algebra. The *fiber product* of  $A_{\mathcal{H}}^F$  and  $B_{\mathcal{K}}^{\hat{F}}$  is the  $C^*$ -algebra

$$A_F *_{\mathfrak{b}}^{\hat{F}} B := \{T \in \mathcal{B}(\mathcal{H}_F \otimes_{\mathfrak{b}}^{\hat{F}} \mathcal{K}) : T^*|\hat{F}\rangle_2 \subseteq [|\hat{F}\rangle_2 A] \text{ and } T^*|F\rangle_1 \subseteq [|F\rangle_1 B]\}.$$

The next diagram shows which spaces of operators are involved in this definition

$$\begin{array}{ccccc} \mathcal{H} & \xrightarrow{|\hat{F}\rangle_2} & \mathcal{H}_F \otimes_{\mathfrak{b}}^{\hat{F}} \mathcal{K} & \xleftarrow{|F\rangle_1} & \mathcal{K} \\ \downarrow A & & \downarrow A_F *_{\mathfrak{b}}^{\hat{F}} B & & \downarrow B \\ \mathcal{H} & \xrightarrow{|\hat{F}\rangle_2} & \mathcal{H}_F \otimes_{\mathfrak{b}}^{\hat{F}} \mathcal{K} & \xleftarrow{|F\rangle_1} & \mathcal{K} \end{array}$$

Let  $\mathfrak{a} = (\mathfrak{H}, \mathfrak{A}, \mathfrak{A}^\dagger)$  and  $\mathfrak{b} = (\mathfrak{K}, \mathfrak{B}, \mathfrak{B}^\dagger)$  be two  $C^*$ -bases.

**Definition 4.3.21** (Definition 3.9 [Tim12b]). A *non-degenerate  $C^*$ - $(\mathfrak{a}^\dagger, \mathfrak{b})$ -algebra* is a pair  $A_{\mathcal{H}}^{F,E} := ({}_F\mathcal{H}_E, A)$ , where  ${}_F\mathcal{H}_E$  is a  $C^*$ - $(\mathfrak{a}^\dagger, \mathfrak{b})$ -module,  $A_{\mathcal{H}}^F$  is a non-degenerate  $C^*$ - $\mathfrak{a}^\dagger$ -algebra and  $A_{\mathcal{H}}^E$  is a  $C^*$ - $\mathfrak{b}$ -algebra. A *morphism of  $C^*$ - $(\mathfrak{a}^\dagger, \mathfrak{b})$ -algebras*  $A_{\mathcal{H}}^{F,E}$  and  $B_{\mathcal{H}'}^{F',E'}$  is a  $*$ -homomorphism  $\pi : A \rightarrow B$  satisfying the conditions  $F' = [\mathcal{L}^\pi({}_F\mathcal{H}_E, {}_{F'}\mathcal{H}'_{E'})F]$  and  $E' = [\mathcal{L}^\pi({}_F\mathcal{H}_E, {}_{F'}\mathcal{H}'_{E'})E]$ , where

$$\mathcal{L}^\pi({}_F\mathcal{H}_E, {}_{F'}\mathcal{H}'_{E'}) := \{T \in \mathcal{L}({}_F\mathcal{H}_E, {}_{F'}\mathcal{H}'_{E'}) : Ta = \pi(a)T \text{ for all } a \in A\}.$$

Let  $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$  be  $C^*$ -bases.

**Proposition 4.3.22** (Proposition 3.15, [Tim12b]). Let  $A_{\mathcal{H}}^{F,E}$  be a  $C^*$ - $(\mathfrak{a}^\dagger, \mathfrak{b})$ -algebra and  $B_{\mathcal{K}}^{\hat{E},G}$  a  $C^*$ - $(\mathfrak{b}^\dagger, \mathfrak{c})$ -algebra. Then

$${}_FA_E *_b {}_{\hat{E}}B_G := ({}_F\mathcal{H}_E \otimes_b {}_{\hat{E}}\mathcal{K}_G, A_E *_b {}_{\hat{E}}B)$$

is a  $C^*$ - $(\mathfrak{a}^\dagger, \mathfrak{c})$ -algebra.

**Definition 4.3.23.** In the situation above, the  $C^*$ - $(\mathfrak{a}^\dagger, \mathfrak{c})$ -algebra  ${}_FA_E *_b {}_{\hat{E}}B_G$  is called the *fiber product of  $A_{\mathcal{H}}^{F,E}$  and  $B_{\mathcal{K}}^{\hat{E},G}$* .

**Remark 84.** Using the fiber product, we obtain also a  $C^*$ - $\mathfrak{a}^\dagger$ -algebra

$${}_FA_E *_b {}_{\hat{E}}B := ({}_F\mathcal{H}_E \otimes_b {}_{\hat{E}}\mathcal{K}, A_E *_b {}_{\hat{E}}B)$$

and a  $C^*$ - $\mathfrak{c}$ -algebra

$$A_E *_b {}_{\hat{E}}B_G := (\mathcal{H}_E \otimes_b {}_{\hat{E}}\mathcal{K}_G, A_E *_b {}_{\hat{E}}B).$$

**Theorem 4.3.24** (Theorem 3.19, [Tim12b]). Let  $\phi$  be a morphism of  $C^*$ - $(\mathfrak{a}, \mathfrak{b})$ -algebras  $A_{\mathcal{H}}^{F,E}$  and  $C_{\mathcal{H}'}^{F',E'}$ , and  $\psi$  a morphism of  $C^*$ - $(\mathfrak{b}^\dagger, \mathfrak{c})$ -algebras  $B_{\mathcal{K}}^{M,N}$  and  $D_{\mathcal{K}'}^{M',N'}$ . Then there exists a unique morphism of  $C^*$ - $(\mathfrak{a}, \mathfrak{c})$ -algebras

$$\phi *_b \psi : {}_FA_E *_b {}_M B_N \rightarrow {}_{F'}C_{E'} *_b {}_{M'} D_{N'}$$

such that

$$(\phi *_b \psi)(x)R = Rx$$

for all  $x \in A_E *_b {}_M B$  and  $R \in I_{\mathcal{K}'}J_{\mathcal{H}} + J_{\mathcal{H}'}I_{\mathcal{K}}$ , where

$$I_X := \mathcal{L}^\phi(\mathcal{H}_E, \mathcal{H}'_{E'}) \otimes_b id_X, \quad J_Y := id_Y \otimes_b \mathcal{L}^\psi(\mathcal{K}_M, \mathcal{K}'_{M'})$$

for  $X \in \{\mathcal{K}, \mathcal{K}'\}$  and  $Y \in \{\mathcal{H}, \mathcal{H}'\}$ .

### 4.3.2 Hopf $C^*$ -bimodules

Let  $\mathfrak{b} = (\mathfrak{K}, \mathfrak{B}, \mathfrak{B}^\dagger)$  be a  $C^*$ -base.

**Definition 4.3.25** (Definition 3.1, [Tim12a]). A *comultiplication on a  $C^*$ - $(\mathfrak{b}^\dagger, \mathfrak{b})$ -algebra  $A_{\mathcal{H}}^{F,E}$*  is a homomorphism

$$\Delta : A_{\mathcal{H}}^{F,E} \rightarrow A_{\mathcal{H}}^{F,E} *_b A_{\mathcal{H}}^{F,E}$$

that is coassociative:

$$(\Delta_E *_b id_A) \circ \Delta = (id_{A_E} *_b \Delta) \circ \Delta$$

as maps from  $A$  to  $\mathcal{B}(\mathcal{H}_E \otimes_b \mathcal{H}_E \otimes_b \mathcal{H})$ , i.e. the following diagram is commutative

$$\begin{array}{ccccc}
 & & A & & \\
 & \swarrow \Delta & & \searrow \Delta & \\
 A_E *_b F A & & & & A_E *_b F A \\
 \downarrow \Delta_E *_b F id_A & & & & \downarrow id_{A_E} *_b F \Delta \\
 (A_E *_b F A)_{(E \triangleleft E)} *_b F A & & & & A_E *_b (F \triangleright F) (A_E *_b F A) \\
 & \searrow & & \swarrow & \\
 & \mathcal{B}(\mathcal{H}_E \otimes_b \mathcal{H}_E \otimes_b \mathcal{H}) & & & 
 \end{array}$$

**Definition 4.3.26.** A *Hopf  $C^*$ -bimodule over  $\mathfrak{b}$*  is a  $C^*$ - $(\mathfrak{b}^\dagger, \mathfrak{b})$ -algebra with a comultiplication. A *morphism of Hopf  $C^*$ -bimodules  $(A_{\mathcal{H}}^{F,E}, \Delta_A)$ ,  $(B_{\mathcal{K}}^{N,M}, \Delta_B)$  over  $\mathfrak{b}$*  is a  $*$ -homomorphism  $\pi$  from  $A_{\mathcal{H}}^{F,E}$  to  $B_{\mathcal{K}}^{N,M}$  satisfying

$$\Delta_B \circ \pi = (\pi_E *_b F \pi) \circ \Delta_A.$$

**Definition 4.3.27.** Let  $(A_{\mathcal{H}}^{F,E}, \Delta)$  be a Hopf  $C^*$ -bimodule over  $\mathfrak{b}$ . A *co-involution for  $(A_{\mathcal{H}}^{F,E}, \Delta)$*  is an anti- $*$ -automorphism  $R : A \rightarrow A$  satisfying

- $R \circ R = id_A$ ,
- $R(\rho_F(b)) = \rho_E(b^\dagger)$  for every  $b \in \mathfrak{B}$ .
- $\Delta \circ R = \Sigma \circ (R_E *_b F R) \circ \Delta$

**Definition 4.3.28** (Definition 4.16, [Tim12a]). Let  $(A_{\mathcal{H}}^{F,E}, \Delta)$  be a Hopf  $C^*$ -bimodule over  $\mathfrak{b}$ .

- A *bounded left Haar weight for  $(A_{\mathcal{H}}^{F,E}, \Delta)$*  is a completely positive contraction  $\phi : A \rightarrow \mathfrak{B}$  satisfying

$$\phi(a\rho_F(b)) = \phi(a)b \quad \text{and} \quad \phi(\langle e | {}_1 \Delta(a) | e' \rangle_1) = e^* \rho_F(\phi(a)) e'$$

for all  $a \in A$ ,  $b \in \mathfrak{B}$ ,  $e, e' \in E$ .

- A *bounded right Haar weight* for  $(A_{\mathcal{H}}^{F,E}, \Delta)$  is a completely positive contraction  $\psi : A \rightarrow \mathfrak{B}^\dagger$  satisfying

$$\psi(a\rho_E(b^\dagger)) = \psi(a)b^\dagger \quad \text{and} \quad \psi(\langle f | {}_2\Delta(a) | f' \rangle_2) = f^*\rho_E(\psi(a))f'$$

for all  $a \in A, b^\dagger \in \mathfrak{B}^\dagger, f, f' \in F$ .

**Definition 4.3.29.** Let  $(A_{\mathcal{H}}^{F,E}, \Delta)$  be a Hopf  $C^*$ -bimodule over the  $C^*$ -base  $\mathfrak{b} = (\mathfrak{K}, \mathfrak{B}, \mathfrak{B}^\dagger)$ . An  $C^*$ -valued weight  $T : A_+ \rightarrow \rho_F(\mathfrak{B})$  is called *left-invariant* if

$$(id_E *_F T)\Delta(x) = T(x)_E \otimes_F 1$$

for all  $x \in \mathcal{M}_T^+$  and an  $C^*$ -valued weight  $T' : A \rightarrow \rho_E(\mathfrak{B}^\dagger)$  is called *right-invariant* if

$$(T'_E *_F id)\Delta(x) = 1_E \otimes_F T'(x)$$

for all  $x \in \mathcal{M}_T^+$ .





## Chapter N° 5

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### *C\*-algebraic quantum transformation groupoids*

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#### 5.1 The reduced $C^*$ -crossed product

Let  $\mathbb{G} = (A, \Delta, \varphi)$  be an algebraic quantum group and  $X$  be a  $\mathbb{G}^*$ -algebra with left reduced action  $\alpha : X \rightarrow \mathcal{M}(A \otimes X)$ . In this section, we assume that  $X$  is a non-degenerate idempotent  $*$ -algebra equipped with a positive faithful  $\alpha$ -invariant functional  $\mu : X \rightarrow \mathbb{C}$ . We fix a GNS-pair  $(L_\mu^2 X, \Lambda_\mu)$  for  $\mu$  and we assume that there exists a non-degenerate faithful  $*$ -representation of  $X$  associated with the GNS-pair  $(L_\mu^2 X, \Lambda_\mu)$ ,  $\pi_\mu : X \rightarrow \mathcal{B}(L_\mu^2 X)$ , defined by  $\pi_\mu(x)\Lambda_\mu(y) = \Lambda_\mu(xy)$  for  $x, y \in X$ .

**Notation 19.** Denote by  $X_r$  the norm closure of  $\pi_\mu(X)$  in  $\mathcal{B}(L_\mu^2 X)$ , i.e.

$$X_r := \overline{\pi_\mu(X)}^{\|\cdot\|} \subseteq \mathcal{B}(L_\mu^2 X).$$

So  $X_r$  is a non-degenerate sub- $C^*$ -algebra of  $\mathcal{B}(L_\mu^2 X)$ .

**Remark 85.** If  $\mu : X \rightarrow \mathbb{C}$  is a faithful positive functional which admits an analytic one-parameter group of automorphisms on  $X$ ,  $\sigma^\mu = (\sigma_t^\mu : X \rightarrow X)_{t \in \mathbb{C}}$  such that the antilinear map  $J_\mu : L_\mu^2 X \rightarrow L_\mu^2 X$ ,  $\Lambda_\mu(x) \mapsto \Lambda_\mu(\sigma_{i/2}^\mu(x)^*)$  is the modular conjugation related to the GNS-representation  $(L_\mu^2 X, \Lambda_\mu, \pi_\mu)$ . Then, we can consider  $(L_\mu^2 X, \Lambda_\mu^\circ, \pi_\mu^\circ)$ , where  $\Lambda_\mu^\circ : x^{op} \in X^{op} \mapsto J_\mu \Lambda_\mu(x^*) \in L_\mu^2 X$  and  $\pi_\mu^\circ : x^{op} \mapsto J_\mu \pi_\mu(x^*) J_\mu \in \mathcal{B}(L_\mu^2 X)$ , as a GNS-construction for the opposite functional  $\mu^\circ : X^{op} \rightarrow \mathbb{C}$ .

**Remark 86** (cf. [KV97]). We recall that there is a unique non-degenerate  $*$ -representation

$$\widehat{\pi} : \widehat{A} \rightarrow \mathcal{B}(L^2 \mathbb{G}), \quad \omega \mapsto \widehat{\pi}(\omega)$$

such that

$$\widehat{\pi}(\omega)(\Lambda(a)) = \Lambda((\widehat{S}^{-1}(\omega) \otimes id)\Delta(a))$$

for all  $a \in A$ ,  $\omega \in \widehat{A}$ .

**Lemma 5.1.1** (cf. [KV97]). For  $a, b \in A$ ,

$$\pi(a)\widehat{\pi}(\varphi_b) = \widehat{\pi}(\varphi_{a(1)b})\pi(a_{(2)})$$

in  $\mathcal{B}(L^2\mathbb{G})$ .

**Proposition 5.1.2.** *The map*

$$\pi^{H(\mathbb{G})} : H(\mathbb{G}) \rightarrow \mathcal{B}(L^2\mathbb{G}), \quad \omega a \mapsto \widehat{\pi}(\omega)\pi(a)$$

is a faithful non-degenerate  $*$ -representation.

*Proof* Follows from the lemma above. □

We construct now the faithful non-degenerate  $*$ -representation

$$\begin{aligned} \bar{\pi} := \pi^{H(\mathbb{G})} \otimes \pi_\mu : H(\mathbb{G}) \otimes X &\rightarrow \mathcal{B}(L^2\mathbb{G} \otimes L_\mu^2 X) \\ \omega a \otimes x &\mapsto \widehat{\pi}(\omega)\pi(a) \otimes \pi_\mu(x) \end{aligned}$$

which can be extended to a faithful non-degenerate  $*$ -representation

$$\bar{\pi} : \mathcal{M}(H(\mathbb{G}) \otimes X) \rightarrow \mathcal{B}(L^2\mathbb{G} \otimes L_\mu^2 X),$$

such that

$$\bar{\pi}(\alpha(x))(\Lambda(a) \otimes \Lambda_\mu(y)) = (\Lambda \otimes \Lambda_\mu)(\alpha(x)(a \otimes y)).$$

**Lemma 5.1.3.** *For all  $x \in X, a \in A$ ,*

$$\bar{\pi}(\alpha(x))(\widehat{\pi}(\varphi_a) \otimes 1_{\mathcal{B}(L_\mu^2 X)}) = (\widehat{\pi}(\varphi_{x_{[-1]}a}) \otimes 1_{\mathcal{B}(L_\mu^2 X)})\bar{\pi}(\alpha(x_{[0]}))$$

in  $\mathcal{B}(L^2\mathbb{G} \otimes L_\mu^2 X)$ .

*Proof* Fix  $\varphi_a \in \widehat{A}$  and  $x \in X$ , then

$$\begin{aligned} \bar{\pi}(\alpha(x))(\widehat{\pi}(\varphi_a) \otimes 1_{\mathcal{B}(L_\mu^2 X)})(\Lambda(b) \otimes \Lambda_\mu(y)) &= \pi(x_{[-1]})\widehat{\pi}(\varphi_a)(\Lambda(b)) \otimes \Lambda_\mu(x_{[0]}y) \\ &= \widehat{\pi}(\varphi_{x_{[-2]}a})\pi(x_{[-1]})(\Lambda(b)) \otimes \Lambda_\mu(x_{[0]}y) \\ &= (\widehat{\pi}(\varphi_{x_{[-2]}a}) \otimes 1_{\mathcal{B}(L_\mu^2 X)})(\pi(x_{[-1]})(\Lambda(b)) \otimes \Lambda_\mu(x_{[0]}y)) \\ &= (\widehat{\pi}(\varphi_{x_{[-1]}a}) \otimes 1_{\mathcal{B}(L_\mu^2 X)})\bar{\pi}(\alpha(x_{[0]}))(\Lambda(b) \otimes \Lambda_\mu(y)) \end{aligned}$$

for all  $b \in A$  and  $y \in X$ . □

**Definition 5.1.4.** The  $C^*$ -algebra

$$C_0(\mathbb{G} \rtimes_\alpha^{\text{red}} X) := \overline{\bar{\pi}(\mathbb{G} \rtimes_\alpha X)}^{\|\cdot\|} \subseteq \mathcal{B}(L^2\mathbb{G} \otimes L_\mu^2 X)$$

is called *the  $C^*$ -reduced crossed product of  $\mathbb{G}$  and  $X$* .

**Remark 87.** If we consider the  $*$ -homomorphism

$$\bar{\pi}|_{\alpha(X)} : \alpha(X) \rightarrow \mathcal{B}(L^2\mathbb{G} \otimes L_\mu^2 X), \quad \alpha(x) \mapsto \bar{\pi}(\alpha(x)) \quad .$$

Then, we have

$$\bar{\pi}(\alpha(X)) \subseteq \mathcal{M}(C_0(\mathbb{G} \rtimes_\alpha^{\text{red}} X)) .$$

Define the linear map

$$\begin{aligned} U_\mu^\alpha : L^2\mathbb{G} \otimes L_\mu^2 X &\rightarrow L^2\mathbb{G} \otimes L_\mu^2 X \\ \Lambda(a) \otimes \Lambda_\mu(x) &\mapsto (\Lambda \otimes \Lambda_\mu)(\alpha(x)(a \otimes 1_{\mathcal{M}(X)})) \end{aligned}$$

**Lemma 5.1.5.**  $U_\mu^\alpha$  is a unitary operator on  $L^2\mathbb{G} \otimes L_\mu^2 X$  and its adjoint is given by

$$\begin{aligned} (U_\mu^\alpha)^* : L^2\mathbb{G} \otimes L_\mu^2 X &\rightarrow L^2\mathbb{G} \otimes L_\mu^2 X \\ \Lambda(a) \otimes \Lambda_\mu(x) &\mapsto (\Lambda \otimes \Lambda_\mu)((S^{-1} \otimes id)(\alpha(x))(a \otimes 1_{\mathcal{M}(X)})) \end{aligned}$$

*Proof* The invariance of  $\mu$  implies that

$$\begin{aligned} \langle U_\mu^\alpha(\Lambda(a) \otimes \Lambda_\mu(x)) | U_\mu^\alpha(\Lambda(b) \otimes \Lambda_\mu(y)) \rangle_{\varphi \otimes \mu} &= \langle (\Lambda \otimes \Lambda_\mu)(\alpha(x)(a \otimes 1)) | (\Lambda \otimes \Lambda_\mu)(\alpha(y)(b \otimes 1)) \rangle_{\varphi \otimes \mu} \\ &= (\varphi \otimes \mu)((a^* \otimes 1)\alpha(x^*y)(b \otimes 1)) \\ &= \varphi(\mu(x^*y)a^*b) \\ &= \langle \Lambda(a) \otimes \Lambda_\mu(x) | \Lambda(b) \otimes \Lambda_\mu(y) \rangle_{\varphi \otimes \mu} \end{aligned}$$

for all  $a, b \in A$  and  $x, y \in X$ . Because  $\alpha(x)(A \otimes 1_{\mathcal{M}(X)}) = A \otimes X$ , then we can extend the relations below to all the Hilbert space  $L^2\mathbb{G} \otimes L_\mu^2 X$ . Now, if  $a \otimes x = \alpha(y)(b \otimes 1)$ , then it is not hard to show that  $S(b) \otimes y = (S(a) \otimes 1)\alpha(x)$ . This implies that

$$b \otimes y = (S^{-1} \otimes id)(\alpha(x))(a \otimes 1).$$

Now, we have

$$\begin{aligned} (U_\mu^\alpha)^*(\Lambda(a) \otimes \Lambda_\mu(x)) &= (U_\mu^\alpha)^*((\Lambda \otimes \Lambda_\mu)(\alpha(y)(b \otimes 1_{\mathcal{M}(X)}))) \\ &= (U_\mu^\alpha)^*U_\mu^\alpha(\Lambda(b) \otimes \Lambda_\mu(y)) \\ &= (\Lambda \otimes \Lambda_\mu)((S^{-1} \otimes id)(\alpha(x))(a \otimes 1_{\mathcal{M}(X)})). \end{aligned}$$

□

**Proposition 5.1.6.** Define the linear map

$$\alpha_r : X_r \rightarrow \mathcal{B}(L^2\mathbb{G} \otimes L_\mu^2 X), \quad x \mapsto U_\mu^\alpha(1 \otimes x)(U_\mu^\alpha)^*.$$

We have

- For all  $x, y \in X$  and  $a \in A$

$$(\pi_r \otimes \pi_\mu)(a \otimes x)\alpha_r(\pi_\mu(y)) = (\pi_r \otimes \pi_\mu)((a \otimes x)\alpha(y)),$$

$$\alpha_r(\pi_\mu(y))(\pi_r \otimes \pi_\mu)(a \otimes x) = (\pi_r \otimes \pi_\mu)(\alpha(y)(a \otimes x)),$$

In particular, for all  $a \in X$  and  $x \in X$ ,

$$(\pi_r(a) \otimes 1)\alpha_r(\pi_\mu(y)) = (\pi_r \otimes \pi_\mu)((a \otimes 1)\alpha(y)),$$

$$(1 \otimes \pi_\mu(x))\alpha_r(\pi_\mu(y)) = (\pi_r \otimes \pi_\mu)((1 \otimes x)\alpha(y)),$$

$$\alpha_r(\pi_\mu(y))(\pi_r(a) \otimes 1) = (\pi_r \otimes \pi_\mu)(\alpha(y)(a \otimes 1)),$$

$$\alpha_r(\pi_\mu(y))(1 \otimes \pi_\mu(x)) = (\pi_r \otimes \pi_\mu)(\alpha(y)(1 \otimes x)).$$

- $\alpha_r$  is a non-degenerate injective  $*$ -homomorphism from  $X_r$  into  $\mathcal{M}(A_r \otimes X_r)$  such that  $\alpha_r(X_r)(A_r \otimes 1)$  is a dense subsets of  $A_r \otimes X_r$ .
- $(id \otimes \alpha_r)\alpha_r = (\Delta \otimes id)\alpha_r$ .

*Proof*

- Fix  $x, y \in X$  and  $a \in A$ . Because

$$\begin{aligned} (\pi_r \otimes \pi_\mu)(x \otimes a)U_\mu^\alpha(1 \otimes \pi_\mu(y))(\Lambda(b) \otimes \Lambda_\mu(z)) &= (\pi_r \otimes \pi_\mu)(x \otimes a)U_\mu^\alpha(\Lambda(b) \otimes \Lambda_\mu(yz)) \\ &= (\pi_r \otimes \pi_\mu)(x \otimes a)(\Lambda \otimes \Lambda_\mu)(\alpha(yz)(b \otimes 1)) \\ &= (\Lambda \otimes \Lambda_\mu)((x \otimes a)\alpha(yz)(b \otimes 1)) \\ &= (\Lambda \otimes \Lambda_\mu)((x \otimes a)\alpha(y)\alpha(z)(b \otimes 1)) \\ &= (\pi_r \otimes \pi_\mu)((x \otimes a)\alpha(y))(\Lambda \otimes \Lambda_\mu)(\alpha(z)(b \otimes 1)) \\ &= (\pi_r \otimes \pi_\mu)((x \otimes a)\alpha(y))U_\mu^\alpha(\Lambda(b) \otimes \Lambda_\mu(z)) \end{aligned}$$

for all  $z \in X$  and  $b \in A$ , then

$$(\pi_r \otimes \pi_\mu)(x \otimes a)U_\mu^\alpha(1 \otimes \pi_\mu(y))U_\mu^\alpha = (\pi_r \otimes \pi_\mu)((x \otimes a)\alpha(y)).$$

The other equality can be prove in a similar way.

- Using the item above, the result follows because  $U_\mu^\alpha$  is unitary and  $(\pi_r \otimes \pi_\mu)(\alpha(X)(A \otimes X)) = \pi_r(A) \otimes \pi_\mu(X)$  is dense in  $A_r \otimes X_r$ .
- For all  $x, y \in X$  and  $a \in A$ , we have

$$\begin{aligned} (id \otimes \alpha_r)(\alpha_r(\pi_\mu(x)))(\pi_r(a) \otimes 1 \otimes \pi_\mu(y)) &= (id \otimes \alpha_r)(\alpha_r(\pi_\mu(x))(\pi_r(a) \otimes 1))(1 \otimes 1 \otimes \pi_\mu(y)) \\ &= (id \otimes \alpha_r)((\pi_r \otimes \pi_\mu)(\alpha(x)(a \otimes 1)))(1 \otimes 1 \otimes \pi_\mu(y)) \\ &= (\pi_r \otimes \pi_r \otimes \pi_\mu)((id \otimes \alpha)(\alpha(x)(a \otimes 1)))(1 \otimes 1 \otimes y) \\ &= (\pi_r \otimes \pi_r \otimes \pi_\mu)((\Delta \otimes id)(\alpha(x))(a \otimes 1 \otimes 1))(1 \otimes 1 \otimes y) \\ &= (\pi_r \otimes \pi_r \otimes \pi_\mu)((\Delta \otimes id)(\alpha(x)(1 \otimes y)))(a \otimes 1 \otimes 1) \\ &= (\Delta \otimes id)((\pi_r \otimes \pi_\mu)(\alpha(x)(1 \otimes y)))(\pi_r(a) \otimes 1 \otimes 1) \\ &= (\Delta \otimes id)\alpha_r(\pi_\mu(x))(\pi_r(a) \otimes 1 \otimes \pi_\mu(y)). \end{aligned}$$

The result follows from the non-degenerancy of  $\alpha_r$  and because  $A_r = \overline{\pi_r(A)}^{\|\cdot\|}$ .

□

**Remark 88.** If, we consider the action  $\alpha^\circ : X^{op} \rightarrow \mathcal{M}(A \otimes X^{op})$ . Then, because

$$R(\pi_r(a)) = \widehat{J}\pi_r(a)\widehat{J}, \quad \pi_\mu^\circ(x^{op}) = J_\mu\pi_\mu(x^*)J_\mu$$

for all  $a \in A$  and  $x \in X$ , we have

$$\alpha_r^\circ(\pi_\mu^\circ(x^{op})) = \text{Ad}(\widehat{J} \otimes J_\mu)(\alpha_r(\pi_\mu(x))^*).$$

for all  $x \in X$ .

**Proposition 5.1.7.** *Let  $(\mathcal{H}_{\hat{\mu}}, \Lambda_{\hat{\mu}})$  be the GNS-pair for the positive faithful dual functional  $\hat{\mu} : \mathbb{G} \rtimes_{\alpha} X \rightarrow \mathbb{C}$ ,  $(\omega \otimes 1_{\mathcal{M}(X)})\alpha(x) \mapsto \widehat{\varphi}(\omega)\mu(x)$ . Then, the linear map*

$$T : \begin{array}{ccc} L^2\mathbb{G} \otimes L_{\mu}^2 X & \rightarrow & \mathcal{H}_{\hat{\mu}} \\ \widehat{\Lambda}(\omega) \otimes \Lambda_{\mu}(x) & \mapsto & \Lambda_{\hat{\mu}}((\omega \otimes 1)\alpha(x)) \end{array}$$

is a unitary operator.

*Proof* For any  $\omega, \theta \in \widehat{A}$  and  $x, y \in X$ , we have

$$\begin{aligned} \langle T(\widehat{\Lambda}(\omega) \otimes \Lambda_{\mu}(x)) | T(\widehat{\Lambda}(\theta) \otimes \Lambda_{\mu}(y)) \rangle_{\hat{\mu}} &= \langle \Lambda_{\hat{\mu}}((\omega \otimes 1)\alpha(x)) | \Lambda_{\hat{\mu}}((\theta \otimes 1)\alpha(y)) \rangle_{\hat{\mu}} \\ &= \hat{\mu}(\alpha(x^*)(\omega^* \otimes 1)(\theta \otimes 1)\alpha(y)) \\ &= \mu \circ \alpha^{-1}(\alpha(x^*y))\widehat{\varphi}(\omega^*\theta) \\ &= \mu(x^*y)\widehat{\varphi}(\omega^*\theta) \\ &= \langle \widehat{\Lambda}(\omega) | \widehat{\Lambda}(\theta) \rangle_{\widehat{\varphi}} \langle \Lambda_{\mu}(x) | \Lambda_{\mu}(y) \rangle_{\mu} \\ &= \langle \widehat{\Lambda}(\omega) \otimes \Lambda_{\mu}(x) | \widehat{\Lambda}(\theta) \otimes \Lambda_{\mu}(y) \rangle_{\widehat{\varphi} \otimes \mu}. \end{aligned}$$

The result follows because  $\widehat{\Lambda}(\widehat{A})$  and  $\Lambda_{\mu}(X)$  are dense in  $L^2\mathbb{G}$  and  $L_{\mu}^2 X$ , respectively.  $\square$

## 5.2 The $C^*$ -completion of an algebraic quantum transformation groupoid

**Remark 89.** From now on, we will suppose that the actions  $\alpha$  and  $\widehat{\alpha}$  are reduced, and we will impose the following mild restriction on the faithful positive functional  $\mu : X \rightarrow \mathbb{C}$ : there exists an analytic one-parameter group of automorphisms on  $X$ ,  $\sigma^{\mu} = (\sigma_t^{\mu} : X \rightarrow X)_{t \in \mathbb{C}}$  such that  $J_{\mu} : L_{\mu}^2 X \rightarrow L_{\mu}^2 X$ ,  $\Lambda_{\mu}(x) \mapsto \Lambda_{\mu}(\sigma_{i/2}^{\mu}(x)^*)$  is the modular conjugation related to the GNS-representation  $(L_{\mu}^2 X, \Lambda_{\mu}, \pi_{\mu})$ . In particular, any KMS-weight on a  $C^*$ -algebra satisfies this requirement which is also satisfied by a class of functionals defined on an important class of  $*$ -algebras considered in [Ino98].

The goal of this section is to prove, step by step the following theorem.

**Theorem 5.2.1.** *Let  $(X, \alpha, \widehat{\alpha})$  be a braided commutative  $\mathbb{G}$ -Yetter-Drinfeld  $*$ -algebra and  $\mu : X \rightarrow \mathbb{C}$  be a *Yetter-Drinfeld integral*. Consider the  $C^*$ -base  $\mathfrak{b} = (L_{\mu}^2 X, \mathfrak{B}, \mathfrak{B}^{\dagger})$  where  $\mathfrak{B} := \overline{\pi_{\mu}(X)}^{\|\cdot\|}$  and  $\mathfrak{B}^{\dagger} := \overline{\pi_{\mu}^{\circ}(X^{op})}^{\|\cdot\|}$ , the Hilbert space  $\mathcal{H} := L^2\mathbb{G} \otimes L_{\mu}^2 X$  and the reduced  $C^*$ -completion of the algebraic crossed product  $\mathcal{A} := C_0(\mathbb{G} \rtimes_{\alpha}^{red} X)$ . Then, there exist closed subspaces  $E := E(\alpha, \widehat{\alpha})$ ,  $F := F(\alpha, \widehat{\alpha}) \in \mathcal{B}(L_{\mu}^2 X, L^2\mathbb{G} \otimes L_{\mu}^2 X)$ , a  $*$ -homomorphism*

$${}^E\Delta^F : \mathcal{A}_{\mathcal{H}}^{F,E} \rightarrow \mathcal{A}_{\mathcal{H}}^{F,E} \underset{\mathfrak{b}}{*} \mathcal{A}_{\mathcal{H}}^{F,E},$$

an anti-automorphism  ${}^E R^F : \mathcal{A} \rightarrow \mathcal{A}$  and a completely positive operator valued weight  ${}^E\Phi^F : \mathcal{A} \rightarrow \rho_F(\mathfrak{B})$  such that

- The pair  $(\mathcal{A}_{\mathcal{H}}^{F,E}, {}^E\Delta^F)$  is a Hopf  $C^*$ -bimodule over  $\mathfrak{b}$ .

- ${}^E R^F$  is a co-involution for the Hopf  $C^*$ -bimodule  $(\mathcal{A}_{\mathcal{H}}^{F,E}, {}^E \Delta^F)$ .
- ${}^E \Phi^F$  is a bounded left Haar weight for the Hopf  $C^*$ -bimodule  $(\mathcal{A}_{\mathcal{H}}^{F,E}, {}^E \Delta^F)$ .

**Remark 90.** We will use the same strategy as in [ET16] replacing systematically the von Neumann algebraic techniques by the techniques of  $C^*$ -modules over  $C^*$ -bases,  $C^*$ -algebras over  $C^*$ -bases and  $C^*$ -pseudo-multiplicative unitaries.

**Definition 5.2.2.** The tuple

$$(C_0(\mathbb{G} \rtimes_{\alpha} X), \mathfrak{b}, F, E, {}^E \Delta^F, {}^E R^F, {}^E \Phi^F)$$

is called a  $C^*$ -algebraic quantum transformation groupoid.

### 5.2.1 $C^*$ -bimodule structure

Let  $\mathfrak{B} := \overline{\pi_{\mu}(X)}^{\|\cdot\|} \subseteq \mathcal{B}(L_{\mu}^2 X)$  and take

$$\mathfrak{B}^{\dagger} := \overline{\pi_{\mu}^{\circ}(X^{op})}^{\|\cdot\|} = \overline{J_{\mu} \pi_{\mu}(X) J_{\mu}}^{\|\cdot\|} \subseteq \mathfrak{B}' \subseteq \mathcal{B}(L_{\mu}^2 X).$$

We have

**Lemma 5.2.3.** The triple  $\mathfrak{b} = (L_{\mu}^2 X, \mathfrak{B}, \mathfrak{B}^{\dagger})$  is a  $C^*$ -base.

*Proof* Follows directly from the definition. □

**Lemma 5.2.4.** For any  $a \in A$  and  $x \in X$ , the linear maps

$$E_{a,x} := U_{\mu}^{\alpha}(U_{\mu}^{\widehat{\alpha}})^* | \Lambda(a) \rangle_1 \pi_{\mu}(x), \quad F_{a,x} := U_{\mu}^{\alpha} | \Lambda(a) \rangle_1 J_{\mu} \pi_{\mu}(x) J_{\mu},$$

belongs to  $\mathcal{B}(L_{\mu}^2 X, L^2 \mathbb{G} \otimes L_{\mu}^2 X)$ .

*Proof* Follows directly. □

**Remark 91.** For any  $a \in A$ , we have the bounded operator

$$\begin{aligned} | \Lambda(a) \rangle_1 := | \Lambda(a) \rangle \otimes 1_{\mathcal{B}(L_{\mu}^2 X)} : L_{\mu}^2 X &\rightarrow L^2 \mathbb{G} \otimes L_{\mu}^2 X \\ \Lambda_{\mu}(x) &\mapsto \Lambda(a) \otimes \Lambda_{\mu}(x) \end{aligned}$$

with adjoint operator given by

$$\begin{aligned} \langle \Lambda(a) |_1 := \langle \Lambda(a) | \otimes 1_{\mathcal{B}(L_{\mu}^2 X)} : L^2 \mathbb{G} \otimes L_{\mu}^2 X &\rightarrow L_{\mu}^2 X \\ \Lambda(b) \otimes \Lambda_{\mu}(x) &\mapsto \langle \Lambda(a) | \Lambda(b) \rangle_{\varphi} \Lambda_{\mu}(x) \end{aligned}$$

Then, we can write more directly forms

$$\begin{aligned} E_{a,x} : L_{\mu}^2 X &\rightarrow L^2 \mathbb{G} \otimes L_{\mu}^2 X & F_{a,x} : L_{\mu}^2 X &\rightarrow L^2 \mathbb{G} \otimes L_{\mu}^2 X \\ \Lambda_{\mu}(y) &\mapsto U_{\mu}^{\alpha}(U_{\mu}^{\widehat{\alpha}})^*(\Lambda(a) \otimes \Lambda_{\mu}(xy)) & \Lambda_{\mu}(y) &\mapsto U_{\mu}^{\alpha}(\Lambda(a) \otimes \Lambda_{\mu}(y \sigma_{i/2}(x)^*)) \end{aligned}$$

Consider the Hilbert space  $\mathcal{H} := L^2\mathbb{G} \otimes L^2_\mu X$  and the vector subspaces

$$E_0 := \{E_{a,x} : a \in A, x \in X\} \subseteq \mathcal{B}(L^2_\mu X, L^2\mathbb{G} \otimes L^2_\mu X),$$

$$F_0 := \{F_{a,x} : a \in A, x \in X\} \subseteq \mathcal{B}(L^2_\mu X, L^2\mathbb{G} \otimes L^2_\mu X),$$

and denote by

$$E(\alpha, \hat{\alpha}) := \overline{E_0}^{\|\cdot\|}, F(\alpha, \hat{\alpha}) := \overline{F_0}^{\|\cdot\|} \subseteq \mathcal{B}(L^2_\mu X, L^2\mathbb{G} \otimes L^2_\mu X),$$

**Remark 92.** For any  $a \in A$  and  $x, y \in X$ , we have

$$\begin{aligned} E_{a,x} J_\mu(\Lambda_\mu(y)) &= U_\mu^\alpha (U_\mu^{\hat{\alpha}})^* | \Lambda(a) \rangle_1 \pi_\mu(x) J_\mu(\Lambda_\mu(y)) \\ &= U_\mu^\alpha (U_\mu^{\hat{\alpha}})^* (\Lambda(a) \otimes J_\mu \pi_\mu(y) J_\mu(\Lambda_\mu(x))) \\ &= U_\mu^\alpha (U_\mu^{\hat{\alpha}})^* (1 \otimes J_\mu \pi_\mu(y) J_\mu) (\Lambda(a) \otimes \Lambda_\mu(x)) \\ &= \beta_r(\pi_\mu(y^*)) U_\mu^\alpha (U_\mu^{\hat{\alpha}})^* (\Lambda(a) \otimes \Lambda_\mu(x)) \end{aligned}$$

$$\begin{aligned} F_{a,x}(\Lambda_\mu(y)) &= U_\mu^\alpha | \Lambda(a) \rangle_1 J_\mu \pi_\mu(x) J_\mu(\Lambda_\mu(y)) \\ &= U_\mu^\alpha (\Lambda(a) \otimes J_\mu \pi_\mu(x) J_\mu(\Lambda_\mu(y))) \\ &= U_\mu^\alpha (\Lambda(a) \otimes \pi_\mu(y) J_\mu(\Lambda_\mu(x))) \\ &= U_\mu^\alpha (1 \otimes \pi_\mu(y)) (\Lambda(a) \otimes J_\mu(\Lambda_\mu(x))) \\ &= \alpha_r(\pi_\mu(y)) U_\mu^\alpha (\Lambda(a) \otimes J_\mu(\Lambda_\mu(x))) \end{aligned}$$

This two equalities shows that the operators  $E_{a,x}$  and  $F_{a,x}$  are in relations with the actions  $\alpha_r$  and  $\beta_r$ . Later, we will use this relations.

**Proposition 5.2.5.**  $\mathcal{H}_E = (\mathcal{H}, E)$  is a  $C^*$ - $\mathfrak{b}$ -module.

*Proof*

- We have

$$E_{a,x}(\Lambda_\mu(y)) = U_\mu^\alpha (U_\mu^{\hat{\alpha}})^* (\Lambda(a) \otimes \Lambda_\mu(xy))$$

for every  $x, y \in X$  and  $a \in A$ . Because  $X$  is an idempotent  $*$ -algebra and  $U_\mu^\alpha U_\mu^{\hat{\alpha}}$  is an unitary operator, then  $[E(L^2_\mu X)] = L^2\mathbb{G} \otimes L^2_\mu X$ .

- We have

$$E_{a,x} \pi_\mu(y) = U_\mu^\alpha (U_\mu^{\hat{\alpha}})^* | \Lambda(a) \rangle_1 \pi_\mu(x) \pi_\mu(y) = E_{a,xy}$$

for every  $x, y \in X$  and  $x \in A$ . Because  $X$  is an idempotent  $*$ -algebra, then  $[E\mathfrak{B}] = E$ .

- We have

$$\begin{aligned} E_{a,x}^* E_{b,y} &= \pi_\mu(x^*) \langle \Lambda(a) |_1 U_\mu^{\hat{\alpha}} (U_\mu^\alpha)^* U_\mu^\alpha (U_\mu^{\hat{\alpha}})^* | \Lambda(b) \rangle_1 \pi_\mu(y) \\ &= \langle \Lambda(a) | \Lambda(b) \rangle_\varphi \pi_\mu(x^* y) \end{aligned}$$



for every  $x, y \in X$  and  $a, b \in A$ . Because  $X$  is an idempotent  $*$ -algebra, and there exists an element  $a \in A$  such that  $\varphi(a^*a) = 1$ , then  $[E^*E] = \mathfrak{B}$ .

□

**Proposition 5.2.6.**  $\mathcal{H}_F = (\mathcal{H}, F)$  is a  $C^*$ - $\mathfrak{b}^\dagger$ -module.

*Proof*

- We have

$$F_{a,x}(\Lambda_\mu(y)) = U_\mu^\alpha(\Lambda(a) \otimes \Lambda_\mu(y\sigma_{i/2}(x)^*))$$

for every  $x, y \in X$  and  $a \in A$ . Because  $X$  is an idempotent  $*$ -algebra,  $U_\mu^\alpha$  is an unitary operator and  $\sigma_{i/2}$  is an isomorphism then  $[F(L_\mu^2 X)] = L^2\mathbb{G} \otimes L_\mu^2 X$ .

- We have

$$F_{a,x}J_\mu\pi_\mu(y)J_\mu = U_\mu^\alpha|\Lambda(a)\rangle_1J_\mu\pi_\mu(x)\pi_\mu(y)J_\mu = F_{a,xy}$$

for every  $x, y \in X$  and  $a \in A$ . Because  $X$  is idempotent  $*$ -algebra, then  $[F\mathfrak{B}^\dagger] = F$ .

- We have

$$\begin{aligned} F_{a,x}^*F_{b,y} &= J_\mu\pi_\mu(x^*)J_\mu\langle\Lambda(a)|_1(U_\mu^\alpha)^*U_\mu^\alpha|\Lambda(b)\rangle_1J_\mu\pi_\mu(y)J_\mu \\ &= \langle\Lambda(a)|\Lambda(b)\rangle_\varphi J_\mu\pi_\mu(x^*y)J_\mu \end{aligned}$$

for every  $x, y \in X$  and  $a, b \in A$ . Because  $X$  is an idempotent  $*$ -algebra, and there exists an element  $a \in A$  such that  $\varphi(a^*a) = 1$ , then  $[F^*F] = \mathfrak{B}^\dagger$ . □

**Remark 93.** Because

$$\beta_r(\pi_\mu(x)) = U_\mu^\alpha(U_\mu^{\widehat{\alpha}})^*(1 \otimes J_\mu\pi_\mu(x)^*J_\mu)U_\mu^{\widehat{\alpha}}(U_\mu^\alpha)^*$$

for any  $x \in X$ , then

$$\beta_r(J_\mu\pi_\mu(x)J_\mu) = U_\mu^\alpha(U_\mu^{\widehat{\alpha}})^*(1 \otimes \pi_\mu(x^*))U_\mu^{\widehat{\alpha}}(U_\mu^\alpha)^*$$

for any  $x \in X$ .

**Remark 94.** Using Lemma 4.3.6 for the  $C^*$ -modules found in Proposition 5.2.5 and Proposition 5.2.6, we have two unique normal, unital and faithful  $*$ -representations

$$\rho_E : \mathfrak{B}^\dagger \rightarrow \mathcal{B}(\mathcal{H}), \quad \rho_F : \mathfrak{B} \rightarrow \mathcal{B}(\mathcal{H})$$

such that, by Remark 92,

$$\rho_E(J_\mu\pi_\mu(x)J_\mu) = \beta_r(J_\mu\pi_\mu(x)J_\mu), \quad \rho_F(\pi_\mu(x)) = \alpha_r(\pi_\mu(x)) = U_\mu^\alpha(1 \otimes \pi_\mu(x))(U_\mu^\alpha)^*.$$

**Proposition 5.2.7.**  $(\mathcal{H}, F, E)$  is a  $C^*$ - $(\mathfrak{b}^\dagger, \mathfrak{b})$ -module.

*Proof*

- Fix  $x \in X$  and  $a \in A$ . Take  $x'_i \in X$  and  $a'_i \in A$  such that  $\widehat{\alpha}(x)(\varphi_a \otimes 1) = \sum_i \varphi_{a'_i} \otimes x'_i$ , then

$$\begin{aligned}
 \rho_F(\pi_\mu(x))E_{a,y} &= U_\mu^\alpha(1 \otimes \pi_\mu(x))(U_\mu^\alpha)^*U_\mu^\alpha(U_\mu^{\widehat{\alpha}})^*|\Lambda(a)\rangle_1\pi_\mu(y) \\
 &= U_\mu^\alpha(1 \otimes \pi_\mu(x))(U_\mu^{\widehat{\alpha}})^*|\Lambda(a)\rangle_1\pi_\mu(y) \\
 &= U_\mu^\alpha(U_\mu^{\widehat{\alpha}})^*\widehat{\alpha}_r(\pi_\mu(x))|\Lambda(a)\rangle_1\pi_\mu(y) \\
 &= \sum_i U_\mu^\alpha(U_\mu^{\widehat{\alpha}})^*|\Lambda(a'_i)\rangle_1(1 \otimes \pi_\mu(x'_iy)) = \sum_i E_{a'_i,x'_iy}.
 \end{aligned}$$

This implies that  $[\rho_F(\mathfrak{B})E] = E$ , because  $\widehat{\alpha}(X)(\widehat{A} \otimes 1) = \widehat{A} \otimes X$  and  $X$  is an idempotent  $*$ -algebra.

- Fix  $x \in X$  and  $a \in A$ .

$$\begin{aligned}
 \rho_E(J_\mu\pi_\mu(x)J_\mu)F_{a,y} &= U_\mu^\alpha(U_\mu^{\widehat{\alpha}})^*(1 \otimes \pi_\mu(x^*))U_\mu^{\widehat{\alpha}}(U_\mu^\alpha)^*U_\mu^\alpha|\Lambda(a)\rangle_1J_\mu\pi_\mu(y)J_\mu \\
 &= U_\mu^\alpha(U_\mu^{\widehat{\alpha}})^*(1 \otimes \pi_\mu(x^*))U_\mu^{\widehat{\alpha}}|\Lambda(a)\rangle_1J_\mu\pi_\mu(y)J_\mu \\
 &= \sum_i U_\mu^\alpha|\Lambda(a'_i)\rangle_1J_\mu\pi_\mu(x'_iy)J_\mu = \sum_i F_{a'_i,x'_iy}.
 \end{aligned}$$

This implies that  $[\rho_E(\mathfrak{B}^\dagger)F] = F$ , because  $X$  is an idempotent  $*$ -algebra. □

Now, take  $\mathcal{A} := C_0(\mathbb{G} \ltimes_\alpha^{\text{red}} X) = \overline{\widehat{\pi}(\mathbb{G} \ltimes_\alpha X)}^{\|\cdot\|} \subseteq \mathcal{B}(L^2\mathbb{G} \otimes L^2X)$ ,

**Proposition 5.2.8.** *The triple  $\mathcal{A}_{\mathcal{H}}^{F,E} = (\mathcal{A}, \mathcal{H}_F, \mathcal{H}_E)$  is a  $C^*$ - $(\mathfrak{b}^\dagger, \mathfrak{b})$ -algebra.*

*Proof* We will prove this proposition in 2 steps:

- **Claim:  $(\mathcal{A}, \mathcal{H}_F)$  is a  $C^*$ - $\mathfrak{b}^\dagger$ -algebra.** It follows from Proposition 5.1.7 that  $\mathcal{A}$  is a non-degenerate  $C^*$ -algebra of  $\mathcal{B}(\mathcal{H})$ . Now, because

$$\rho_F(\pi_\mu(x))\widehat{\pi}(\alpha(y)(\omega \otimes 1)) = \alpha_r(\pi_\mu(x))\widehat{\pi}(\alpha(x)(\omega \otimes 1)) = \widehat{\pi}(\alpha(xy)(\omega \otimes 1))$$

and  $X$  is an idempotent  $*$ -algebra then  $[\rho_F(\mathfrak{B})\mathcal{A}] = \mathcal{A}$ .

- **Claim:  $(\mathcal{A}, \mathcal{H}_E)$  is a  $C^*$ - $\mathfrak{b}$ -algebra.** It follows from Proposition 5.1.7 that  $\mathcal{A}$  is a non-degenerate  $C^*$ -algebra of  $\mathcal{B}(\mathcal{H})$ . Now, because

$$\begin{aligned}
 \rho_E(J_\mu\pi_\mu(x)J_\mu)\widehat{\pi}(\alpha(y)(\omega \otimes 1)) &= \beta_r(J_\mu\pi_\mu(x)J_\mu)\alpha_r(\pi_\mu(y))(\widehat{\pi}(\omega) \otimes 1) \\
 &= \alpha_r(\pi_\mu(y))\beta_r(J_\mu\pi_\mu(x)J_\mu)(\widehat{\pi}(\omega) \otimes 1) \\
 &= \alpha_r(\pi_\mu(y))\alpha_r(\pi_\mu(x'))(\widehat{\pi}(\omega') \otimes 1)(\widehat{\pi}(\omega) \otimes 1) \\
 &= \alpha_r(\pi_\mu(yx'))(\widehat{\pi}(\omega' \cdot \omega) \otimes 1)
 \end{aligned}$$

and  $X$  and  $\widehat{A}$  are idempotent  $*$ -algebras then  $[\rho_E(\mathfrak{B}^\dagger)\mathcal{A}] = \mathcal{A}$ . Note that, we use the fact that  $\beta(X) \subseteq \mathcal{M}(\mathbb{G} \ltimes_\alpha X)$ . □

Let us summarize the results of this subsection as follows

**Theorem 5.2.9.** Consider the  $C^*$ -base  $\mathfrak{b} = (L_\mu^2 X, \mathfrak{B}, \mathfrak{B}^\dagger)$  where  $\mathfrak{B} = \overline{\pi_\mu(X)}^{\|\cdot\|}$ ,  $\mathfrak{B}^\dagger = \overline{\pi_\mu^\circ(X^{op})}^{\|\cdot\|}$ ,  $\mathcal{H} = L^2\mathbb{G} \otimes L_\mu^2 X$  and  $\mathcal{A} := C_0(\mathbb{G} \times_\alpha^{red} X)$  is the reduced  $C^*$ -completion of the algebraic crossed product. There exist closed vector subspaces  $E, F \in \mathcal{B}(L_\mu^2 X, L^2\mathbb{G} \otimes L_\mu^2 X)$  such that

$$\mathcal{A}_{\mathcal{H}}^{F,E} = (\mathcal{A}, \mathcal{H}_F, \mathcal{H}_E)$$

is a  $C^*$ - $(\mathfrak{b}^\dagger, \mathfrak{b})$ -algebra.

## 5.2.2 Hopf $C^*$ -bimodule structure

By the last subsection, we have

$$\begin{array}{ccccc} \mathcal{H} \otimes F & \cong & \mathcal{H}_E \otimes_{\mathfrak{b}} F \mathcal{H} & \cong & E \otimes \mathcal{H} \\ E_{a,x}(\Lambda_\mu(y)) \otimes_{\rho_E} F_{b,z} & \mapsto & E_{a,x} \otimes \Lambda_\mu(y) \otimes F_{b,z} & \mapsto & E_{a,x} \otimes_{\rho_F} F_{b,z}(\Lambda_\mu(y)) \end{array}$$

**Proposition 5.2.10.** There exists a unitary

$$V_1 : (L^2\mathbb{G} \otimes L_\mu^2 X)_{E \otimes_{\mathfrak{b}} F} (L^2\mathbb{G} \otimes L_\mu^2 X) \rightarrow L^2\mathbb{G} \otimes L^2\mathbb{G} \otimes L_\mu^2 X$$

such that for all  $a, b \in A, x, y, z \in X$

$$V_1(E_{a,x}(\Lambda_\mu(y)) \otimes_{\rho_E} F_{b,z}) = \Lambda(b) \otimes \beta_r(J_\mu \pi_\mu(z) J_\mu) E_{a,x}(\Lambda_\mu(y))$$

and

$$V_1(T \otimes_{\rho_E} id_F) = (1_{\mathcal{B}(L^2\mathbb{G})} \otimes T) V_1$$

for all  $T \in \rho_E(\mathfrak{B}^\dagger)'$ . Moreover, if  $\beta_r^\circ(J_\mu \pi_\mu(x^*) J_\mu) := \beta_r(\pi_\mu(x))$  for all  $x \in X$ , then

$$(id \otimes \beta_r^\circ)(\alpha_r^\circ(J_\mu \pi_\mu(x^*) J_\mu)) V_1 = V_1((1_{L^2\mathbb{G}} \otimes 1_{L_\mu^2 X})_{E \otimes_{\mathfrak{b}} F} (1_{\mathcal{B}(L^2\mathbb{G})} \otimes J_\mu \pi_\mu(x^*) J_\mu))$$

$$(id \otimes \beta_r^\circ)(\widehat{\alpha}_r^\circ(J_\mu \pi_\mu(x^*) J_\mu)) V_1 = V_1((1_{L^2\mathbb{G}} \otimes 1_{L_\mu^2 X})_{E \otimes_{\mathfrak{b}} F} \beta_r(\pi_\mu(x)))$$

for all  $x \in X$ .

*Proof* The linear operator  $V_1$  is well defined in  $E_0 \otimes \Lambda_\mu(X) \otimes F_0$  which is a dense subspace of  $(L^2\mathbb{G} \otimes L_\mu^2 X)_{E \otimes_{\mathfrak{b}} F} (L^2\mathbb{G} \otimes L_\mu^2 X)$ . Now, for all  $a, a', b, b' \in A$  and  $x, x', y, y', z, z' \in X$ , if we denote  $r = E_{a,x}(\Lambda_\mu(y)) \otimes_{\rho_E} F_{b,z}$  and  $r' = E_{a',x'}(\Lambda_\mu(y')) \otimes_{\rho_E} F_{b',z'}$ , we have

$$\begin{aligned} \langle V_1(r) | V_1(r') \rangle &= \langle \Lambda(b) \otimes \beta_r(J_\mu \pi_\mu(z) J_\mu) E_{a,x}(\Lambda_\mu(y)) | \Lambda(b') \otimes \beta_{r'}(J_\mu \pi_\mu(z') J_\mu) E_{a',x'}(\Lambda_\mu(y')) \rangle_{\varphi \otimes \varphi \otimes \mu} \\ &= \langle \Lambda(b) | \Lambda(b') \rangle_\varphi \langle \beta_r(J_\mu \pi_\mu(z) J_\mu) E_{a,x}(\Lambda_\mu(y)) | \beta_{r'}(J_\mu \pi_\mu(z') J_\mu) E_{a',x'}(\Lambda_\mu(y')) \rangle_{\varphi \otimes \mu} \\ &= \langle \Lambda(b) | \Lambda(b') \rangle_\varphi \langle E_{a,x}(\Lambda_\mu(y)) | \beta_r(J_\mu \pi_\mu(z^* z') J_\mu) E_{a',x'}(\Lambda_\mu(y')) \rangle_{\varphi \otimes \mu} \\ &= \langle E_{a,x}(\Lambda_\mu(y)) | \rho_E(F_{b,z}^* F_{b',z'}) E_{a',x'}(\Lambda_\mu(y')) \rangle_{\varphi \otimes \mu} \\ &= \langle E_{a,x}(\Lambda_\mu(y)) \otimes_{\rho_E} F_{b,z} | E_{a',x'}(\Lambda_\mu(y')) \otimes_{\rho_E} F_{b',z'} \rangle \\ &= \langle r | r' \rangle, \end{aligned}$$

so  $V_1$  is unitary. Let  $T \in \rho_E(\mathfrak{B}^\dagger)'$ ,  $a, b \in A$  and  $x, y, z \in X$  then

$$\begin{aligned}
V_1(T \otimes_{\rho_E} id_F)(E_{a,x}(\Lambda_\mu(y)) \otimes_{\rho_E} F_{b,z}) &= V_1(T E_{a,x}(\Lambda_\mu(y)) \otimes_{\rho_E} F_{b,z}) \\
&= \Lambda(b) \otimes \beta_r(J_\mu \pi_\mu(z) J_\mu) T E_{a,x}(\Lambda_\mu(y)) \\
&= \Lambda(b) \otimes T \beta_r(J_\mu \pi_\mu(z) J_\mu) E_{a,x}(\Lambda_\mu(y)) \\
&= (1_{\mathcal{B}(L^2\mathbb{G})} \otimes T)(\Lambda(b) \otimes \beta_r(\pi_\mu(z^*)) E_{a,x}(\Lambda_\mu(y))) \\
&= (1_{\mathcal{B}(L^2\mathbb{G})} \otimes T) V_1(E_{a,x}(\Lambda_\mu(y)) \otimes_{\rho_E} F_{b,z})
\end{aligned}$$

In particular, for all  $x \in X$

$$V_1(\alpha_r(\pi_\mu(x)) \otimes_{\rho_E} id_F) = (1_{\mathcal{B}(L^2\mathbb{G})} \otimes \alpha_r(\pi_\mu(x))) V_1.$$

□

**Proposition 5.2.11.** *There exists a unitary*

$$V_2 : (L^2\mathbb{G} \otimes L_\mu^2 X)_{E \otimes_F} \otimes_{\mathfrak{b}} (L^2\mathbb{G} \otimes L_\mu^2 X) \rightarrow L^2\mathbb{G} \otimes L^2\mathbb{G} \otimes L_\mu^2 X$$

such that for all  $a, b \in A$ ,  $x, y, z \in X$

$$V_2(E_{a,x} \otimes_{\rho_F} F_{b,z}(\Lambda_\mu(y))) = \Lambda(a) \otimes \alpha_r(\pi_\mu(x)) F_{b,z}(\Lambda_\mu(y))$$

and

$$V_2(id_E \otimes_{\rho_F} T) = (1_{\mathcal{B}(L^2\mathbb{G})} \otimes T) V_2$$

for all  $T \in \rho_F(\mathfrak{B})'$ , in particular, for  $T \in \beta_r(X_r)$ .

*Proof* The linear operator  $V_2$  is well defined in  $E_0 \otimes \Lambda_\mu(X) \otimes F_0$  which is a dense subspace of  $(L^2\mathbb{G} \otimes L_\mu^2 X)_{E \otimes_F} \otimes_{\mathfrak{b}} (L^2\mathbb{G} \otimes L_\mu^2 X)$ . Now, for all  $a, a', b, b' \in A$ ,  $x, x', y, y', z, z' \in X$ , if we denote  $r = E_{a,x} \otimes_{\rho_F} F_{b,z}(\Lambda_\mu(y))$  and  $r' = E_{a',x'} \otimes_{\rho_F} F_{b',z'}(\Lambda_\mu(y'))$ , we have

$$\begin{aligned}
\langle V_2(r) | V_2(r') \rangle &= \langle \Lambda(a) \otimes \alpha_r(\pi_\mu(x)) F_{b,z}(\Lambda_\mu(y)) | \Lambda(a') \otimes \alpha_r(\pi_\mu(x')) F_{b',z'}(\Lambda_\mu(y')) \rangle_{\varphi \otimes \varphi \otimes \mu} \\
&= \langle \Lambda(a) | \Lambda(a') \rangle_{\varphi} \langle \alpha_r(\pi_\mu(x)) F_{b,z}(\Lambda_\mu(y)) | \alpha_r(\pi_\mu(x')) F_{b',z'}(\Lambda_\mu(y')) \rangle_{\varphi \otimes \mu} \\
&= \langle \Lambda(a) | \Lambda(a') \rangle_{\varphi} \langle F_{b,z}(\Lambda_\mu(y)) | \alpha_r(\pi_\mu(x^* x')) F_{b',z'}(\Lambda_\mu(y')) \rangle_{\varphi \otimes \mu} \\
&= \langle F_{b,z}(\Lambda_\mu(y)) | \rho_F(E_{a,x}^* E_{a,x}) F_{b',z'}(\Lambda_\mu(y')) \rangle_{\varphi \otimes \mu} \\
&= \langle E_{a,x} \otimes_{\rho_F} F_{b,z}(\Lambda_\mu(y)) | E_{a',x'} \otimes_{\rho_F} F_{b',z'}(\Lambda_\mu(y')) \rangle \\
&= \langle r | r' \rangle.
\end{aligned}$$

Then  $V_2$  is unitary. Let  $T \in \rho_F(\mathfrak{B})'$ ,  $a, b \in A$  and  $x, y, z \in X$  then

$$\begin{aligned}
 V_2(id_F \otimes_{\rho_F} T)(E_{a,x} \otimes_{\rho_F} F_{b,z}(\Lambda_\mu(y))) &= V_2(E_{a,x} \otimes_{\rho_F} TF_{b,z}(\Lambda_\mu(y))) \\
 &= \Lambda(a) \otimes \alpha_r(\pi_\mu(x))TF_{b,z}(\Lambda_\mu(y)) \\
 &= \Lambda(a) \otimes T\alpha_r(\pi_\mu(x))F_{b,z}(\Lambda_\mu(y)) \\
 &= (1_{\mathcal{B}(L^2\mathbb{G})} \otimes T)(\Lambda(a) \otimes \alpha_r(\pi_\mu(x))F_{b,z}(\Lambda_\mu(y))) \\
 &= (1_{\mathcal{B}(L^2\mathbb{G})} \otimes T)V_2(E_{a,x} \otimes_{\rho_F} F_{b,z}(\Lambda_\mu(y))).
 \end{aligned}$$

□

**Lemma 5.2.12.** *We have the following relations*

(i)

$$V_2V_1^* = \sigma_{12}(U_\mu^\alpha)_{13}(U_\mu^{\hat{\alpha}})_{23}(U_\mu^\alpha)_{23}^* = \sigma_{12}U_{12}(U_\mu^{\hat{\alpha}})_{23}(U_\mu^\alpha)_{23}^*U_{12}^*,$$

(ii)

$$(id \otimes \sigma_{312}V_1) \circ (V_2 \otimes_E \otimes_F id) = (V_2 \otimes id) \circ (id \otimes_E \otimes_F \sigma_{312}V_1),$$

where

$$\sigma_{321} : L^2\mathbb{G} \otimes L^2\mathbb{G} \otimes L_\mu^2X \rightarrow L^2\mathbb{G} \otimes L_\mu^2X \otimes L^2\mathbb{G}, \quad \eta \otimes \eta' \otimes \xi \mapsto \eta' \otimes \xi \otimes \eta.$$

*Proof* Similar to the proved of item (iii) of Proposition 4.3 [ET16].

**Theorem 5.2.13.** *There exists a  $*$ -homomorphism*

$$\begin{aligned}
 {}^E\Delta^F : \mathcal{A} &\rightarrow \mathcal{A}_{E \otimes_F \mathcal{B}}^* \mathcal{A} \\
 P &\mapsto V_1^* \tilde{\alpha}_r(P) V_1
 \end{aligned}$$

such that

(i)

$${}^E\Delta^F(\alpha_r(\pi_\mu(x))) = \alpha_r(\pi_\mu(x))_{E \otimes_F \mathcal{B}} 1_{\mathcal{B}(L^2\mathbb{G} \otimes L_\mu^2X)},$$

$${}^E\Delta^F(\beta_r(\pi_\mu(x))) = 1_{\mathcal{B}(L^2\mathbb{G} \otimes L_\mu^2X)}_{E \otimes_F \mathcal{B}} \beta_r(\pi_\mu(x))$$

for all  $x \in X$  and

$${}^E\Delta^F(\widehat{\pi}(\omega) \otimes 1_{\mathcal{B}(L_\mu^2X)}) = V_1^*(\widehat{\Delta}^{cop}(\widehat{\pi}(\omega)) \otimes 1_{\mathcal{B}(L_\mu^2X)})V_1 = V_2^*(\widehat{\Delta}(\widehat{\pi}(\omega)) \otimes 1_{\mathcal{B}(L_\mu^2X)})V_2$$

for all  $\omega \in \widehat{A}$ ,

(ii)

$$({}^E\Delta^F_{E \otimes_F \mathcal{B}} id) \circ {}^E\Delta^F = (id_{E \otimes_F \mathcal{B}} {}^E\Delta^F) \circ {}^E\Delta^F$$

*Proof* By Proposition 5.2.10, the map given by

$$\begin{aligned}
 {}^F\Delta^E : \mathcal{A} &\rightarrow \mathcal{B}((L^2\mathbb{G} \otimes L_\mu^2X)_{E \otimes_F \mathcal{B}} (L^2\mathbb{G} \otimes L_\mu^2X)) \\
 P &\mapsto V_1^* \tilde{\alpha}_r(P) V_1
 \end{aligned}$$

is well defined and is a  $*$ -homomorphism.

(i) The inclusion  $\alpha_r(X_r) \subseteq \rho_E(\mathfrak{B}^\dagger)'$  implies that

$$\begin{aligned} {}^E\Delta^F(\alpha_r(\pi_\mu(x))) &= V_1^* \tilde{\alpha}_r(\alpha_r(\pi_\mu(x))) V_1 = V_1^*(1_{\mathcal{B}(L^2\mathbb{G})} \otimes \alpha_r(\pi_\mu(x))) V_1 \\ &= \alpha_r(\pi_\mu(x)) {}_E \otimes_F {}_F 1_{\mathcal{B}(L^2\mathbb{G} \otimes L_\mu^2 X)}, \end{aligned}$$

for all  $x \in X$ ; the relation  $\hat{U}_{12}^{\circ*}(U_\mu^\alpha)_{23} = (U_\mu^\alpha)_{23} \hat{U}_{12}^{\circ*}$  implies that

$$\begin{aligned} {}^E\Delta^F(\beta_r(\pi_\mu(x))) &= \text{Ad}(V_1^* \hat{U}_{12}^{\circ*})(1 \otimes \beta_r(\pi_\mu(x))) \\ &= \text{Ad}(V_1^* \hat{U}_{12}^{\circ*}(U_\mu^\alpha)_{23})(1 \otimes \hat{\alpha}_r(x^\circ)) \\ &= \text{Ad}(V_1^*(U_\mu^\alpha)_{23} \hat{U}_{12}^{\circ*})(1 \otimes \hat{\alpha}^\circ(x^\circ)) \\ &= \text{Ad}(V_1^*) \circ (id \otimes \beta^\circ)(\hat{\alpha}^\circ(x^\circ)) \\ &= 1_{\mathcal{B}(L^2\mathbb{G} \otimes L_\mu^2 X)} {}_E \otimes_F {}_F \beta_r(\pi_r(x)), \end{aligned}$$

for all  $x \in X$ , and by item (i) of Lemma 5.2.12

$$\begin{aligned} {}^E\Delta^F(\hat{\pi}(\omega) \otimes 1_{L_\mu^2 X}) &= \text{Ad}(V_1^*)(\tilde{\alpha}_r(\hat{\pi}(\omega) \otimes 1_{L_\mu^2 X})) \\ &= \text{Ad}(V_1^*)(\hat{\Delta}^{cop}(\hat{\pi}(\omega)) \otimes 1_{L_\mu^2 X}) \\ &= \text{Ad}(V_1^* U_{12})(\hat{\pi}(\omega) \otimes 1_{L_\mu^2 X} \otimes 1_{L_\mu^2 X}) \\ &= \text{Ad}(V_2^* \Sigma_{12} U_{12} (U_\mu^\alpha)_{23} (U_\mu^\alpha)_{23}^*)(\hat{\pi}(\omega) \otimes 1_{L_\mu^2 X} \otimes 1_{L_\mu^2 X}) \\ &= \text{Ad}(V_2^* \Sigma_{12} U_{12})(\hat{\pi}(\omega) \otimes 1_{L_\mu^2 X} \otimes 1_{L_\mu^2 X}) \\ &= \text{Ad}(V_2^*)(\hat{\Delta}(\hat{\pi}(\omega)) \otimes 1_{L_\mu^2 X}), \end{aligned}$$

for all  $\omega \in \hat{A}$ .

(ii) Fix  $x \in X$ , then easily

$$\begin{aligned} ({}^E\Delta^F {}_E * {}_F id) \circ {}^E\Delta^F(\alpha_r(\pi_r(x))) &= ({}^E\Delta^F {}_E * {}_F id)(\alpha_r(\pi_r(x)) {}_E \otimes_F (1_{L^2\mathbb{G}} \otimes 1_{L_\mu^2 X})) \\ &= \alpha_r(\pi_r(x)) {}_E \otimes_F (1_{L^2\mathbb{G}} \otimes 1_{L_\mu^2 X}) {}_E \otimes_F (1_{L^2\mathbb{G}} \otimes 1_{L_\mu^2 X}) \\ &= (id {}_E * {}_F {}^E\Delta^F)(\alpha_r(\pi_r(x)) {}_E \otimes_F (1_{L^2\mathbb{G}} \otimes 1_{L_\mu^2 X})) \\ &= (id {}_E * {}_F {}^E\Delta^F) \circ {}^E\Delta^F(\alpha_r(\pi_r(x))). \end{aligned}$$

Fix  $\omega \in \hat{A}$ , it follows from item (i) that

$$\begin{aligned} (id {}_E * {}_F {}^E\Delta^F) \circ {}^E\Delta^F(\hat{\pi}(\omega) \otimes 1_{L_\mu^2 X}) &= (id {}_E * {}_F {}^E\Delta^F) \circ \text{Ad}(V_2^*)(\hat{\Delta}(\hat{\pi}(\omega)) \otimes 1_{L_\mu^2 X}) \\ &= (\text{Ad}(V_2^*) {}_E * {}_F id) \circ (id \otimes {}^E\Delta^F)(\hat{\Delta}(\hat{\pi}(\omega)) \otimes 1_{L_\mu^2 X}), \end{aligned}$$

$$\begin{aligned} ({}^E\Delta^F {}_E * {}_F id) \circ {}^E\Delta^F(\hat{\pi}(\omega) \otimes 1_{L_\mu^2 X}) &= ({}^E\Delta^F {}_E * {}_F id) \circ \text{Ad}(V_1^* \sigma_{321}^*) \circ \Sigma_{23} \circ (\hat{\Delta}(\hat{\pi}(\omega)) \otimes 1_{L_\mu^2 X}) \\ &= (id {}_E * {}_F \text{Ad}(V_1^* \sigma_{321}^*)) \circ ({}^E\Delta^F \otimes id) \circ \Sigma_{23} \circ (\hat{\Delta}(\hat{\pi}(\omega)) \otimes 1_{L_\mu^2 X}), \end{aligned}$$

and by item (ii) of Lemma 5.2.12, the diagram

$$\begin{array}{ccccc}
 & & \widehat{A}_r \otimes 1_{L_\mu^2 X} & & \\
 & \swarrow \widehat{\Delta} \otimes id & & \searrow \Sigma_{23} \circ (\widehat{\Delta} \otimes id) & \\
 \mathcal{M}(\widehat{A}_r \otimes \widehat{A}_r \otimes 1_{L_\mu^2 X}) & & & & \mathcal{M}(\widehat{A}_r \otimes 1_{L_\mu^2 X} \otimes \widehat{A}_r) \\
 \downarrow id \otimes E \Delta^F & \searrow id \otimes \Sigma_{23} \circ (\widehat{\Delta} \otimes id) & & \swarrow \widehat{\Delta} \otimes id \otimes id & \downarrow E \Delta^F \otimes id \\
 & & \mathcal{M}(\widehat{A}_r \otimes \widehat{A}_r \otimes 1_{L_\mu^2 X} \otimes \widehat{A}_r) & & \\
 \downarrow id \otimes Ad(V_1^* \sigma_{312}^*) & & & \swarrow Ad(V_2^*) \otimes id & \\
 \mathcal{M}(\widehat{A}_r) \otimes \mathcal{A}_{E \ast_F \ast_b} \mathcal{A} & & & & \mathcal{A}_{E \ast_F \ast_b} \mathcal{A} \otimes \mathcal{M}(\widehat{A}_r) \\
 \downarrow Ad(V_2^*)_{E \ast_F \ast_b} id & & \swarrow id_{E \ast_F \ast_b} Ad(V_1^* \sigma_{312}^*) & & \\
 & & \mathcal{A}_{E \ast_F \ast_b} \mathcal{A}_{E \ast_F \ast_b} \mathcal{A} & & 
 \end{array}$$

is commutative, this implies

$$(id_{E \ast_F \ast_b} E \Delta^F) \circ E \Delta^F (\widehat{\pi}(\omega) \otimes 1_{L_\mu^2 X}) = (E \Delta^F_{E \ast_F \ast_b} id) \circ E \Delta^F (\widehat{\pi}(\omega) \otimes 1_{L_\mu^2 X}).$$

□

### 5.2.3 The co-involution and the left Haar weight

**Proposition 5.2.14.** Consider on the Hilbert space  $L^2 \mathbb{G} \otimes L_\mu^2 X$  the anti-linear operator

$$I = U_\mu^\alpha (J \otimes J_\mu) U_\mu^{\widehat{\alpha}} (U_\mu^\alpha)^* = U_\mu^\alpha (U_\mu^{\widehat{\alpha}})^* (J \otimes J_\mu) (U_\mu^\alpha)^* = U_\mu^\alpha (U_\mu^{\widehat{\alpha}})^* \widehat{J}_\mu U_\mu^{\widehat{\alpha}} (U_\mu^\alpha)^*,$$

where  $\widehat{\mu}$  denotes the dual functional of  $\mu$  on the crossed product  $\widehat{\mathbb{G}} \rtimes_{\widehat{\alpha}} X$ . Then

- (i)  $I$  is a bijective isometry and  $I^2 = 1_{L^2 \mathbb{G} \otimes L_\mu^2 X}$ .
- (ii)  $I \alpha_r(\pi_\mu(x^*)) I = \beta_r(\pi_\mu(x))$  and  $I \beta_r(\pi_\mu(x^*)) I = \alpha_r(\pi_r(x))$  for all  $x \in X$ .
- (iii)  $I(\omega^* \otimes 1_{L_\mu^2 X}) I = \widehat{R}(\omega) \otimes 1_{L_\mu^2 X}$  for all  $\omega \in \widehat{A}$ .
- (iv) If  $\chi : (L^2 \mathbb{G} \otimes L_\mu^2 X)_{E \otimes_b F} (L^2 \mathbb{G} \otimes L_\mu^2 X) \rightarrow (L^2 \mathbb{G} \otimes L_\mu^2 X)_{F \otimes_b^\dagger E} (L^2 \mathbb{G} \otimes L_\mu^2 X)$  denotes the flip, then

$$V_2 = (J \otimes I) V_1 (I_F \otimes_E I) \chi$$

*Proof* By the implementation of  $\widehat{\alpha}$ ,  $U_\mu^{\widehat{\alpha}} = \widehat{J}_\mu(J \otimes J_\mu)$ , we have

$$(J \otimes J_\mu)U_\mu^{\widehat{\alpha}} = (U_\mu^{\widehat{\alpha}})^*(J \otimes J_\mu) = (U_\mu^{\widehat{\alpha}})^*\widehat{J}_\mu U_\mu^{\widehat{\alpha}},$$

then

$$I := U_\mu^\alpha(J \otimes J_\mu)U_\mu^{\widehat{\alpha}}(U_\mu^\alpha)^* = U_\mu^\alpha(U_\mu^{\widehat{\alpha}})^*(J \otimes J_\mu)(U_\mu^\alpha)^* = U_\mu^\alpha(U_\mu^{\widehat{\alpha}})^*\widehat{J}_\mu U_\mu^{\widehat{\alpha}}(U_\mu^\alpha)^*.$$

(i) By the relation above, we have that  $I = I^*$  and  $I^*I = 1_{L^2\mathbb{G} \otimes L_\mu^2 X}$ .

(ii) Fix  $x \in X$ , then

$$\begin{aligned} I\alpha_r(\pi_\mu(x^*))I &= U_\mu^\alpha(U_\mu^{\widehat{\alpha}})^*(J \otimes J_\mu)(U_\mu^\alpha)^*\alpha_r(\pi_\mu(x^*))U_\mu^\alpha(J \otimes J_\mu)U_\mu^{\widehat{\alpha}}(U_\mu^\alpha)^* \\ &= U_\mu^\alpha(U_\mu^{\widehat{\alpha}})^*(J \otimes J_\mu)(1 \otimes \pi_\mu(x^*))(J \otimes J_\mu)U_\mu^{\widehat{\alpha}}(U_\mu^\alpha)^* \\ &= U_\mu^\alpha(U_\mu^{\widehat{\alpha}})^*(1 \otimes J_\mu\pi_\mu(x^*)J_\mu)U_\mu^{\widehat{\alpha}}(U_\mu^\alpha)^* = \beta_r(\pi_\mu(x)), \end{aligned}$$

then it is obvious also that  $I\beta_r(\pi_\mu(x))I = \alpha_r(\pi_\mu(x^*))$ .

(iii) It follows from item (ii) of Corollary 2.8.7, that

$$\begin{aligned} (\widehat{J} \otimes I)(U \otimes 1)(\widehat{J} \otimes I) &= \text{Ad}((U_\mu^\alpha)_{23}(\widehat{J} \otimes J \otimes J_\mu)(U_\mu^{\widehat{\alpha}})_{23}(U_\mu^\alpha)_{23}^*)(U_{12}) \\ &= \text{Ad}((U_\mu^\alpha)_{23}(\widehat{J} \otimes J \otimes J_\mu))((U_\mu^\alpha)_{13}^*U_{12}) \\ &= \text{Ad}((U_\mu^\alpha)_{23})((U_\mu^\alpha)_{13}U_{12}) = U_{12}^*. \end{aligned}$$

Then, for any  $a, b \in A$ , we have

$$\begin{aligned} I(\widehat{R}((\omega_{\Lambda(a), \Lambda(b)} \otimes id)(U)) \otimes 1_{L_\mu^2 X})I &= I(J(\omega_{\Lambda(a), \Lambda(b)} \otimes id)(U)^*J \otimes 1_{L_\mu^2 X})I \\ &= I((\omega_{\widehat{J}(\Lambda(a)), \widehat{J}(\Lambda(b))} \otimes id)(U) \otimes 1_{L_\mu^2 X})I \\ &= (\omega_{\Lambda(a), \Lambda(b)} \otimes id)(U)^* \otimes 1_{L_\mu^2 X} \end{aligned}$$

and this implies by item (ii) of Lemma 4.2.29 that

$$I(\omega^* \otimes 1_{L_\mu^2 X})I = \widehat{R}(\omega) \otimes 1_{L_\mu^2 X}.$$

(iv) By item (ii),

$$\begin{aligned} V_1(I_{F \otimes_{\mathfrak{b}^\dagger} E} I)\chi(U_\mu^\alpha(U_\mu^{\widehat{\alpha}})^*(\eta \otimes \Lambda_\mu(x))\beta \otimes_\mu \alpha E) &= V_1(I_{F \otimes_{\mathfrak{b}^\dagger} E} I)(E \otimes_{\mu^\circ} \beta U_\mu^\alpha(U_\mu^{\widehat{\alpha}})^*(\eta \otimes \Lambda_\mu(x))) \\ &= V_1(I(E) \otimes_{F \otimes_{\mathfrak{b}}} U_\mu^\alpha(J(\eta) \otimes J_\mu(\Lambda_\mu(x)))) \\ &= J(\eta) \otimes \beta_r(J_\mu\pi_\mu(x)J_\mu)I(E) \\ &= J(\eta) \otimes I\alpha_r(\pi_\mu(x))(E) \\ &= (J \otimes I)V_2(U_\mu^\alpha(U_\mu^{\widehat{\alpha}})^*(\eta \otimes \Lambda_\mu(x)) \otimes_{F \otimes_{\mathfrak{b}}} E), \end{aligned}$$

for all  $E \in L^2\mathbb{G} \otimes L_\mu^2 X$ ,  $\eta \in L^2\mathbb{G}$  and  $x \in X$ .

□



**Theorem 5.2.15.** *There exists an involutive anti- $*$ -isomorphism*

$$\begin{aligned} {}^E R^F : \mathcal{A} &\rightarrow \mathcal{A} \\ P &\mapsto IP^*I \end{aligned}$$

such that

- (i)  ${}^E R^F(\alpha_r(\pi_\mu(x))) = \beta_r(\pi_\mu(x))$  and  ${}^E R^F(\beta_r(\pi_\mu(x))) = \alpha_r(\pi_\mu(x))$  for all  $x \in X$ ;
- (ii)  ${}^E R^F(\widehat{\pi}(\omega) \otimes 1_{L_\mu^2 X}) = \widehat{\pi}(\widehat{R}(\omega)) \otimes 1_{L_\mu^2 X}$  for all  $\omega \in \widehat{A}$ ;
- (iii) if  $\Sigma : \mathcal{A}_F \underset{\mathfrak{b}^\dagger}{*} {}^E \mathcal{A} \rightarrow \mathcal{A}_E \underset{\mathfrak{b}}{*} {}^F \mathcal{A}$  is the flip map, then

$${}^E \Delta^F \circ {}^E R^F = \Sigma \circ ({}^E R^F \underset{\mathfrak{b}}{*} {}^F {}^E R^F) \circ {}^E \Delta^F.$$

*Proof* By item (i) of Proposition 5.2.14, the map given by

$$\begin{aligned} {}^E R^F : \mathcal{A} &\rightarrow \mathcal{B}(L^2 \mathbb{G} \otimes L_\mu^2 X) \\ P &\mapsto IP^*I \end{aligned}$$

is well defined and is an involutive injective anti- $*$ -homomorphism, and by item (ii) and item (iii) of Proposition 5.2.14, the image of  ${}^\beta R^\alpha$  is  $\mathcal{A}$ .

- (i) Follows from item (ii) of Proposition 5.2.14.
- (ii) Follows from item (iii) of Proposition 5.2.14.
- (iii) Follows from Theorem 5.2.13 that

$$\begin{aligned} \Sigma \circ ({}^E R^F \underset{\mathfrak{b}}{*} {}^F {}^E R^F) \circ {}^E \Delta^F(\alpha_r(\pi_\mu(x))) &= \Sigma \circ ({}^E R^F \underset{\mathfrak{b}}{*} {}^F {}^E R^F)(\alpha_r(\pi_\mu(x)) \underset{\mathfrak{b}}{\otimes} {}^F 1_{\mathcal{B}(L^2 \mathbb{G} \otimes L_\mu^2 X)}) \\ &= \Sigma(\beta_r(\pi_\mu(x)) \underset{\mathfrak{b}^\dagger}{\otimes} {}^E 1_{\mathcal{B}(L^2 \mathbb{G} \otimes L_\mu^2 X)}) \\ &= 1_{\mathcal{B}(L^2 \mathbb{G} \otimes L_\mu^2 X)} \underset{\mathfrak{b}}{\otimes} {}^F \beta_r(\pi_\mu(x)) \\ &= {}^E \Delta^F(\beta_r(\pi_\mu(x))) \\ &= {}^E \Delta^F \circ {}^E R^F(\alpha_r(\pi_\mu(x))) \end{aligned}$$

for all  $x \in X$  and

$$\begin{aligned} \Sigma \circ ({}^E R^F \underset{\mathfrak{b}}{*} {}^F {}^E R^F) \circ {}^E \Delta^F \circ {}^E R^F(\widehat{\pi}(\omega) \otimes 1_{L_\mu^2 X}) &= \text{Ad}(\chi(I \underset{\mathfrak{b}}{\otimes} {}^F I) V_2^*)(\widehat{\Delta}(\widehat{\pi}(\widehat{R}(\omega))) \otimes 1_{L_\mu^2 X}) \\ &= \text{Ad}(\chi(I \underset{\mathfrak{b}}{\otimes} {}^F I) V_2^*)(\widehat{\pi}(\widehat{R}(\omega_2)) \otimes \widehat{\pi}(\widehat{R}(\omega_1)) \otimes 1_{L_\mu^2 X}) \\ &= \text{Ad}(V_1^*(J \otimes I))(\widehat{\pi}(\widehat{R}(\omega_2)) \otimes \widehat{\pi}(\widehat{R}(\omega_1)) \otimes 1_{L_\mu^2 X}) \\ &= \text{Ad}(V_1^*)(\widehat{\Delta}^{cop}(\widehat{\pi}(\omega)) \otimes 1_{L_\mu^2 X}) \\ &= {}^E \Delta^F(\widehat{\pi}(\omega) \otimes 1_{L_\mu^2 X}) \end{aligned}$$

for all  $\omega \in \widehat{A}$ .

□

**Proposition 5.2.16.** *Define the map  $\Phi : \mathcal{A} \rightarrow \rho_F(\mathfrak{B})$  by*

$$\Phi(P) = (\widehat{\varphi} \circ \widehat{R} \otimes id)\tilde{\alpha}_r(P).$$

*We have*

$$(id_{E *_{\mathfrak{b}} F} \Phi)^E \Delta^F(P) = \Phi(P)_{E *_{\mathfrak{b}} F} (1_{L^2 \mathbb{G}} \otimes 1_{L^2_{\mu} X})$$

*and*

$$({}^E R^F \circ \Phi \circ {}^E R^F)_{E *_{\mathfrak{b}} F} id)^E \Delta^F(P) = 1_{E \otimes F} {}^E R^F \circ \Phi \circ {}^E R^F(P)$$

*for every  $P \in \mathcal{A}$ .*

*Proof* It is similar to the Theorem 5.4 of [ET16]. □

### 5.3 Example

Let  $\mathbb{G}$  be a compact quantum group and  $\mathbb{H}$  be a closed quantum subgroup of  $\mathbb{G}$ . We know using the theory of compact quantum group that there exist two dense Hopf  $*$ -algebras with integrals,  $(A, \Delta, \varphi)$  and  $(B, \Delta', \varphi')$ , inside  $\mathbb{G}$  and  $\mathbb{H}$ , respectively. Because  $\mathbb{H}$  is a closed quantum subgroup of  $\mathbb{G}$ , we have that  $(B, \Delta', \varphi')$  is a closed quantum subgroup of  $(A, \Delta, \varphi)$ , as in Definition 2.1.15. Then, making use of Proposition 3.3.2, if we consider  $I$  as the quotient type left coideal associated to this closed quantum subgroup, we obtain an algebraic quantum transformation groupoid and moreover our Yetter-Drinfeld integral satisfies the condition of Theorem 5.2.1 for the passage to the  $C^*$ -completion, this last statement follows from the theory of compact quantum groups.

### 5.4 Comment on the passage to the von Neumann algebraic setting

We want to make precise comments on the relationship between our  $C^*$ -algebraic construction and the construction of a Measured quantum transformation groupoid in the von Neumann framework.

#### 5.4.1 Principal ingredients

- Using the algebraic objects studied in Chapter 2, we have similar constructions as for von Neumann algebraic locally compact quantum groups. More precisely:
  - (1) The algebraic multiplicative unitary of an algebraic quantum group, Definition 2.4.1, replaces the multiplicative unitary of a locally compact quantum group, and has similar properties that allows to get the same algebraic relations between the comultiplication and its dual, and also serves as an important element in the construction of a Drinfeld double, Definition 2.5.5.
  - (2) We use the theory of actions of algebraic quantum groups in order to construct objects similar to those in [ET16]. For example the construction of the

crossed product of an action of an algebraic quantum group, Definition 2.7.4, the existence of the standard implementation when we equip our  $*$ -algebra with a positive functional invariant for the action, Definition 2.6.8, the construction of the dual action, Definition 2.7.7 which also gives us a way to construct the dual functional of the initial invariant functional, Definition 2.7.10.

- Similarly to that the construction made in [ET16], our principal ingredients are a braided commutative Yetter-Drinfeld algebra and an invariant functional on it, but in our case, over an algebraic quantum group.
  - (1) Our definition of a Yetter Drinfeld  $*$ -algebra over an algebraic quantum group, Definition 2.8.1 is similar to the one in section 2.4 of [ET16], but making use of actions of the algebraic quantum group and its dual and the algebraic multiplicative unitary of the algebraic quantum group. Also, we introduce the definition of a Yetter-Drinfeld integral, Definition 2.8.6, which allows us to construct the standard implementation of our actions and is an important object for the passage to the  $C^*$ -algebraic framework.
  - (2) Our definition of a braided commutative Yetter Drinfeld algebra, Definition 2.8.12, is equivalent to Definition 2.5.3 in [ET16] (Theorem 2.5.4, [ET16]).

## 5.4.2 The Hopf bimodule structure

In [ET16], the authors use the relative tensor products of Hilbert spaces and fiber products of von Neumann algebras in order to construct the objects necessary for the structure of a Hopf-von Neumann bimodule in the sense of Vallin [Val96]. Because our goal is the  $C^*$ -algebraic framework, we need to replace those objects by similar ones in the language of the  $C^*$ -algebras. For that we use the theory of  $C^*$ -algebras over  $C^*$ -bases developed by T. Timmermann [Tim09, Tim12b, Tim12a]. It was shown in section 2.3 of [Tim07] that using  $C^*$ -bases associated with KMS-weights, we can the relative tensor product over  $C^*$ -bases with the von Neumann algebraic relative tensor product of Hilbert spaces that involves the usage of bounded elements. More precisely:

- Theorem 5.2.9, Proposition 5.2.10 and Proposition 5.2.11 replace Proposition 4.3 [ET16] in order to construct a Hopf bimodule structure.
- Theorem 5.2.13 is proved in a similar way as Theorem 4.4 [ET16], in order to construct a comultiplication on the reduced  $C^*$ -completion of the crossed product which gives us a Hopf  $C^*$ -bimodule.
- Theorem 5.2.15 is proved in a similar way as Theorem 4.6 [ET16], for constructing a co-involution for our Hopf  $C^*$ -bimodule.

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## Appendix A

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### Tools to work with Multiplier Hopf<sup>\*</sup>-algebras

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#### A.1 Complete modules

Let  $X$  be a left  $A$ -module. We will write the left action of  $a$  on  $x$  as  $ax$ . So the associativity property we can write like  $a(a'x) = (aa')x$ . We will assume always that all modules are non-degenerate. And when necessary will use unital actions.

Define the vector space

$$Y = \{T : A \rightarrow X : T \text{ is a linear map and } T(aa') = aT(a') \text{ for all } a, a' \in A\}$$

For any  $x \in X$ , define also the linear map  $T_x : a \mapsto ax$ , then because the action of  $A$  on  $X$  is non-degenerate, we can see the vector space  $X$  as a vector subspace of  $Y$ , and because it follows that  $T_x \circ T \in Y$  for all  $x \in X$  and  $T \in Y$ , then  $Y$  is a left  $A$ -module containing  $X$  as a left sub- $A$ -module. Moreover, using the fact that  $A$  is an idempotent algebra, it follows that  $Y$  is a non-degenerate  $A$ -module.

**Definition A.1.1.** The vector space  $Y$  will be denoted by  $\overline{X}$  and it will be called *the completion of  $X$* .

**Example A.1.2.**

- If we consider a non-degenerate  $*$ -algebra  $A$ , we have  $\mathcal{M}(A) = \overline{{}_A A}$ .

#### A.2 The covering technique

Let  $A, B$  be algebras and  $X$  be a non-degenerate  $A$ - $B$ -bimodule  $X$ . We assume that  $A_A, {}_B B$  admit local units.

**Definition A.2.1** (Definition 2.2 - Definition 2.4, [Van08]). Let  $F : X \rightarrow V$  be a linear map from  $X$  to a vector space  $V$ . We say that the variable  $x$  in the expression  $F(x)$  is *covered from the left* if there exists an element  $e \in A$  such that  $F(ex) = F(x)$  for all  $x \in X$ . Similarly, we say that the variable  $x$  in the expression  $F(x)$  is *covered from the right* if there exists an element  $f \in B$  such that  $F(xf) = F(x)$  for all  $x \in X$ . We say that the variable  $x$  in the expression  $F(x)$  is *covered* if the variable is covered both from

the left and from the right, i.e. there exist  $e \in A$  and  $f \in B$  such that  $F(ex) = F(x)$  and  $F(xf) = F(x)$  for all  $x \in X$ .

**Remark 95.** It is important that  $e$  and  $f$  do not depend on  $x$ .

**Proposition A.2.2** (Proposition 2.5, [Van08]). *Let  $F : X \rightarrow V$  be a linear map from the left  $A$ -module  $X$  to a vector space  $V$  and assume that the variable is covered. Then there exists a unique extension of  $F$  to a linear map  $G : \overline{X} \rightarrow V$  such that*

$$G(y) = F(ey) = F(yf)$$

for all  $y \in \overline{X}$  and for any pair of elements  $e \in A$  and  $f \in B$  that satisfy  $F(e \cdot) = F(\cdot)$  and  $F(\cdot f) = F(\cdot)$ .

### A.3 The Sweedler type leg notation

Fix a multiplier Hopf  $*$ -algebra  $(A, \Delta)$ . Consider  $A \otimes A$  as a left  $A$ -module, where the action is given by the left multiplication in the first factor, i.e.  $a(b \otimes b') := ab \otimes b'$ , for all  $a, b, b' \in A$ .

**Notation 20.** Let  $F : A \otimes A \rightarrow V$  be a linear map from  $A \otimes A$  to the vector space  $V$ , and assume that the variable is covered from the left, as in Definition A.2.1. Then, by Proposition A.2.2,  $F$  can be extended to the completed module of  $A \otimes A$ . Because  $\Delta(A) \subseteq \overline{A \otimes A}$ , we can define  $F(\Delta(a))$  for all  $a \in A$ , then we will write

$$F(a_{(1)} \otimes a_{(2)}) := F(\Delta(a)) \in V$$

for all  $a \in A$ .

By definition, we have  $F(a_{(1)} \otimes a_{(2)}) = F(p \otimes q)$  where  $p \otimes q = (e \otimes 1)\Delta(a)$  and where  $e \in A$  is chosen so that  $F(e \cdot) = F(\cdot)$ .

**Definition A.3.1.** We say that, in the expression  $F(a_{(1)} \otimes a_{(2)})$ , the factor  $a_{(1)}$  is covered from the left.

## Appendix B

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### Yetter-Drinfeld $^*$ -algebras

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#### B.1 Yetter-Drinfeld $^*$ -algebras over Hopf $^*$ -algebras

Let  $(H, \Delta, S, \varepsilon)$  be a regular Hopf  $^*$ -algebra with integrals and  $X$  be a unital  $^*$ -algebra.

##### B.1.1 (Left-left) Yetter-Drinfeld $^*$ -algebras

Consider a left action  $\triangleright : H \otimes X \rightarrow X$  and a left coaction  $\delta : X \rightarrow H \otimes X$ , denoted by  $x \mapsto x_{[-1]} \otimes x_{[0]}$ .

**Definition B.1.1.** The  $^*$ -algebra  $X$  is called a (left-left) Yetter-Drinfeld  $^*$ -algebra if

$$h_{(1)}x_{[-1]} \otimes h_{(2)} \triangleright x_{[0]} = (h_{(1)} \triangleright x)_{[-1]}h_{(2)} \otimes (h_{(1)} \triangleright x)_{[0]} \quad (\text{B.1})$$

for all  $h \in H, x \in X$ .

The equality (B.1) is called the *Yetter-Drinfeld condition* for  $(X, \triangleright, \delta)$  and denoted as  $(X, \triangleright, \delta) \in \frac{H}{H}\mathcal{YD}$  for saying that  $X$  is a (left-left) Yetter-Drinfeld  $^*$ -algebra. We give another equivalent condition:

**Proposition B.1.2.** The  $^*$ -algebra  $X$  is a (left-left) Yetter-Drinfeld  $^*$ -algebra if and only if

$$\delta(h \triangleright x) = h_{(1)}x_{[-1]}S(h_{(3)}) \otimes h_{(2)} \triangleright x_{[0]} \quad (\text{B.2})$$

for all  $h \in H, x \in X$ .

*Proof*

$\Rightarrow$  Fix  $h \in H$  and  $x \in X$ , then

$$\begin{aligned} h_{(1)}x_{[-1]}S(h_{(3)}) \otimes h_{(2)} \triangleright x_{[0]} &= (m^\circ \otimes id)(S(h_{(3)}) \otimes h_{(1)}x_{[-1]} \otimes h_{(2)} \triangleright x_{[0]}) \\ &= (m^\circ \otimes id)(S(h_{(3)}) \otimes (h_{(1)} \triangleright x)_{[-1]}h_{(2)} \otimes (h_{(1)} \triangleright x)_{[0]}) \\ &= (h_{(1)} \triangleright x)_{[-1]}h_{(2)}S(h_{(3)}) \otimes (h_{(1)} \triangleright x)_{[0]} \\ &= \varepsilon(h_{(2)})\delta(h_{(1)} \triangleright x) \\ &= \delta(h \triangleright x) \end{aligned}$$

$\Leftarrow$  Fix  $h \in H$  and  $x \in X$ , then

$$\begin{aligned}
(h_{(1)} \triangleright x)_{[-1]} h_{(2)} \otimes (h_{(1)} \triangleright x)_{[0]} &= (m^\circ \otimes id)(h_{(2)} \otimes \delta(h_{(1)} \triangleright x)) \\
&= (m^\circ \otimes id)(h_{(2)} \otimes h_{(1)_{(1)}} x_{[-1]} S(h_{(1)_{(3)}}) \otimes h_{(1)_{(2)}} \triangleright x_{[0]}) \\
&= h_{(1)} x_{[-1]} S(h_{(3)_{(1)}}) h_{(3)_{(2)}} \otimes h_{(2)} \triangleright x_{[0]} \\
&= h_{(1)} x_{[-1]} \otimes \varepsilon(h_{(3)}) h_{(2)} \triangleright x_{[0]} \\
&= h_{(1)} x_{[-1]} \otimes h_{(2)} \triangleright x_{[0]}
\end{aligned}$$

□

Let  $(X, \triangleright, \delta) \in {}^H_H \mathcal{YD}$ , and consider the linear map

$$\tau_X : X \otimes X \rightarrow X \otimes X, \quad x \otimes y \mapsto (x_{[-1]} \triangleright y) \otimes x_{[0]}.$$

**Definition B.1.3.** A (left-left) Yetter-Drinfeld  $*$ -algebra  $(X, \triangleright, \delta)$  is called *braided commutative* if  $m_X \circ \tau_X = m_X$ , where  $m_X : X \otimes X \rightarrow X$  is the multiplication map of  $X$ , i.e. if

$$xy = (x_{[-1]} \triangleright y) x_{[0]}$$

for all  $x, y \in X$ .

**Proposition B.1.4.** A (left-left) Yetter-Drinfeld  $*$ -algebra  $(X, \triangleright, \delta)$  is *braided commutative* if and only if

$$xy = y_{[0]}(S^{-1}(y_{[-1]}) \triangleright x)$$

for all  $x, y \in X$

*Proof* Define the linear map

$$\rho_X : X \otimes X \rightarrow X \otimes X, \quad x \otimes y \mapsto y_{[0]} \otimes (S^{-1}(y_{[-1]}) \triangleright x),$$

then  $m_X \circ \rho_X = m_X$  if and only if

$$xy = y_{[0]}(S^{-1}(y_{[-1]}) \triangleright x)$$

for all  $x, y \in X$ . On the other hand, we have

$$\begin{aligned}
\rho_X \circ \tau_X(x \otimes y) &= \rho_X((x_{[-1]} \triangleright y) \otimes x_{[0]}) \\
&= x_{[0]_{[0]}} \otimes (S^{-1}(x_{[0]_{[-1]}}) \triangleright (x_{[-1]} \triangleright y)) \\
&= x_{[0]} \otimes (S^{-1}(x_{[-1]_{(2)}}) x_{[-1]_{(1)}} \triangleright y) \\
&= \varepsilon(x_{[-1]}) x_{[0]} \otimes y = x \otimes y
\end{aligned}$$

for all  $x, y \in X$ . Similarly,

$$\begin{aligned}
\tau_X \circ \rho_X(x \otimes y) &= \tau_X(y_{[0]} \otimes (S^{-1}(y_{[-1]}) \triangleright x)) \\
&= y_{[0]_{[-1]}} \triangleright (S^{-1}(y_{[-1]}) \triangleright x) \otimes y_{[0]_{[0]}} \\
&= \varepsilon(y_{[-1]}) x \otimes y_{[0]} = x \otimes y
\end{aligned}$$

for all  $x, y \in X$ . This implies that  $m_X \circ \tau_X = m_X$  if and only if  $m_X \circ \rho_X = m_X$ .  $\square$

### B.1.2 (Right-left) Yetter-Drinfeld \*-algebras

Consider a right action  $\triangleleft : X \otimes H \rightarrow X$  and a left coaction  $\delta : X \rightarrow H \otimes X$ , denoted by  $x \mapsto x_{[-1]} \otimes x_{[0]}$ .

**Definition B.1.5.** An \*-algebra  $X$  is called a (right-left) Yetter-Drinfeld \*-algebra if

$$x_{[-1]}h_{(1)} \otimes x_{[0]} \triangleleft h_{(2)} = h_{(2)}(x \triangleleft h_{(1)})_{[-1]} \otimes (x \triangleleft h_{(1)})_{[0]} \quad (\text{B.3})$$

for all  $h \in H, x \in X$ .

The equality (B.3) is called the *Yetter-Drinfeld condition* for  $(X, \triangleleft, \delta)$  and denoted as  $(X, \triangleright, \delta) \in {}^H\mathcal{YD}_H$  for saying that  $X$  is a (right-left) Yetter-Drinfeld \*-algebra. We give another equivalent condition:

**Proposition B.1.6.** The \*-algebra  $X$  is a (right-left) Yetter-Drinfeld \*-algebra if and only if

$$\delta(x \triangleleft h) = S^{-1}(h_{(3)})x_{[-1]}h_{(1)} \otimes x_{[0]} \triangleleft h_{(2)} \quad (\text{B.4})$$

for all  $h \in H, x \in X$ .

Let  $(X, \triangleright, \delta) \in {}^H\mathcal{YD}_H$ , and consider the linear map

$$\tau_X : X \otimes X \rightarrow X \otimes X, \quad x \otimes y \mapsto (y \triangleleft x_{[-1]}) \otimes x_{[0]}$$

**Definition B.1.7.** A (right-left) Yetter-Drinfeld \*-algebra  $(X, \triangleright, \delta)$  is called *braided commutative* if  $m_X \circ \tau_X = m_X$ , where  $m_X : X \otimes X \rightarrow X$  is the multiplication map of  $X$ , i.e. if

$$xy = (y \triangleleft x_{[-1]})x_{[0]}$$

for all  $x, y \in X$ .

**Proposition B.1.8.** A (right-left) Yetter-Drinfeld \*-algebra  $(X, \triangleright, \delta)$  is braided commutative if and only if

$$xy = y_{[0]}(x \triangleleft S(y_{[-1]})) \quad (\text{B.5})$$

for all  $x, y \in X$

*Proof* The proof is similar to the one of Proposition B.1.4.  $\square$

## B.2 Yetter-Drinfeld \*-algebras over multiplier Hopf \*-algebras

Let  $(H, \Delta)$  be a multiplier Hopf \*-algebra and  $X$  be a non-degenerate \*-algebra.

### B.2.1 (Left-left) Yetter-Drinfeld \*-algebras

Consider a unital left action  $\triangleright : H \otimes X \rightarrow X$  and a left coaction  $\delta : X \rightarrow \mathcal{M}(H \otimes X)$  denoted by  $x \mapsto x_{[-1]} \otimes x_{[0]}$  in the sense

$$x_{[-1]}h \otimes x_{[0]} = \delta(x)(h \otimes 1_{\mathcal{M}(X)}), \quad \text{or} \quad hx_{[-1]} \otimes x_{[0]} = (h \otimes 1_{\mathcal{M}(X)})\delta(x)$$

for all  $h \in H$  and  $x \in X$ .

**Definition B.2.1** (Definition 2.1, [Del07]). The \*-algebra  $X$  is called a (left-left) Yetter-Drinfeld \*-algebra if

$$h_{(1)}x_{[-1]}h' \otimes h_{(2)} \triangleright x_{[0]} = (h_{(1)} \triangleright x)_{[-1]}h_{(2)}h' \otimes (h_{(1)} \triangleright x)_{[0]} \quad (\text{B.6})$$

for all  $h, h' \in H, x \in X$ .

The equality (B.6) is called the *Yetter-Drinfeld condition* for  $(X, \triangleright, \delta)$  and denoted as  $(X, \triangleright, \delta) \in \frac{H}{H}\mathcal{YD}$  for saying that  $X$  is a (left-left) Yetter-Drinfeld \*-algebra. We give another equivalent condition:

**Proposition B.2.2.** *The \*-algebra  $X$  is a (left-left) Yetter-Drinfeld \*-algebra if and only if*

$$\delta(h \triangleright x)(h' \otimes 1_{\mathcal{M}(X)}) = h_{(1)}x_{[-1]}S(h_{(3)})h' \otimes h_{(2)} \triangleright x_{[0]} \quad (\text{B.7})$$

for all  $h, h' \in H, x \in X$ .

*Proof*

( $\Rightarrow$ ) Fix  $h, h' \in H$  and  $x \in X$ , because there is  $h_x \in H$  such that  $h_x \triangleright x = x$ , then

$$\begin{aligned} h_{(1)}x_{[-1]}S(h_{(3)})h' \otimes h_{(2)} \triangleright x_{[0]} &= (m^\circ \otimes id)(S(h_{(3)})h' \otimes h_{(1)}x_{[-1]} \otimes h_{(2)} \triangleright x_{[0]}) \\ &= (m^\circ \otimes id)(S(h_{(3)})h' \otimes (h_{(1)} \triangleright x)_{[-1]}h_{(2)} \otimes (h_{(1)} \triangleright x)_{[0]}) \\ &= (h_{(1)} \triangleright x)_{[-1]}h_{(2)}S(h_{(3)})h' \otimes (h_{(1)} \triangleright x)_{[0]} \\ &= \varepsilon(h_{(2)})\delta(h_{(1)} \triangleright x)(h' \otimes 1_{\mathcal{M}(X)}) \\ &= \delta(h_{(1)}h_x\varepsilon(h_{(2)} \triangleright x)(h' \otimes 1_{\mathcal{M}(X)}) \\ &= \delta(h \triangleright x)(h' \otimes 1_{\mathcal{M}(X)}) \end{aligned}$$

( $\Leftarrow$ ) Fix  $h, h' \in H$  and  $x \in X$ , because there are  $h_x \in H$  such that  $h_x \triangleright x = x$  and  $h'' \in H$  such that  $h''h_{(2)}h' = h_{(2)}h'$ , then

$$\begin{aligned} (h_{(1)} \triangleright x)_{[-1]}h_{(2)}h' \otimes (h_{(1)} \triangleright x)_{[0]} &= (h_{(1)}h_x \triangleright x)_{[-1]}h''h_{(2)}h' \otimes (h_{(1)}h_x \triangleright x)_{[0]} \\ &= (m^\circ \otimes id)(h_{(2)}h' \otimes \delta(h_{(1)} \triangleright x)(h'' \otimes 1_{\mathcal{M}(X)})) \\ &= (m^\circ \otimes id)(h_{(2)}h' \otimes h_{(1)(1)}x_{[-1]}S(h_{(1)(3)})h'' \otimes h_{(1)(2)} \triangleright x_{[0]}) \\ &= h_{(1)(1)}x_{[-1]}S(h_{(1)(3)})h_{(2)}h' \otimes h_{(1)(2)} \triangleright x_{[0]} \\ &= h_{(1)}x_{[-1]}S(h_{(3)(1)})h_{(3)(2)}h' \otimes h_{(2)} \triangleright x_{[0]} \\ &= h_{(1)}x_{[-1]}h' \otimes \varepsilon(h_{(3)})h_{(2)} \triangleright x_{[0]} \\ &= h_{(1)}x_{[-1]}h' \otimes h_{(2)} \triangleright x_{[0]} \end{aligned}$$

□

### B.2.2 (Right-left) Yetter-Drinfeld $*$ -algebras

Consider a unital right action  $\triangleleft : X \otimes H \rightarrow X$  and a left coaction  $\delta : X \rightarrow \mathcal{M}(H \otimes X)$ , denoted by  $x \mapsto x_{[-1]} \otimes x_{[0]}$ .

**Definition B.2.3.** The  $*$ -algebra  $X$  is called a (right-left) Yetter-Drinfeld  $*$ -algebra if

$$h'x_{[-1]}h_{(1)} \otimes x_{[0]} \triangleleft h_{(2)} = h'h_{(2)}(x \triangleleft h_{(1)})_{[-1]} \otimes (x \triangleleft h_{(1)})_{[0]} \quad (\text{B.8})$$

for all  $h, h' \in H, x \in X$ .

**Proposition B.2.4.** The  $*$ -algebra  $X$  is a (right-left) Yetter-Drinfeld  $*$ -algebra if and only if

$$\delta(x \triangleleft h)(h' \otimes 1_{\mathcal{M}(X)}) = S^{-1}(h_{(3)})x_{[-1]}h_{(1)}h' \otimes x_{[0]} \triangleleft h_{(2)} \quad (\text{B.9})$$

for all  $h, h' \in H, x \in X$ .

*Proof* The proof is similar to the one of Proposition B.2.2.  $\square$

### B.3 Yetter-Drinfeld $*$ -algebras over algebraic quantum groups

Let  $\mathbb{G} = (A, \Delta, \varphi)$  be an algebraic quantum group,  $\widehat{\mathbb{G}} = (\widehat{A}, \widehat{\Delta}^{cop}, \widehat{\psi})$  be its dual and  $U$  be the algebraic multiplicative unitary of  $\mathbb{G}$ . Consider a left coaction  $\alpha : X \rightarrow \mathcal{M}(A \otimes X)$  of  $\mathbb{G}$  on  $X$  denoted by  $x \mapsto x_{[-1]} \otimes x_{[0]}$ , and a left coaction  $\delta : X \rightarrow \mathcal{M}(\widehat{A} \otimes X)$  of  $\widehat{\mathbb{G}}$  on  $X$  denoted by  $x \mapsto x^{[-1]} \otimes x^{[0]}$ .

**Definition B.3.1.** We say that the triple  $(X, \alpha, \delta)$  is a  $\mathbb{G}$ -Yetter-Drinfeld  $*$ -algebra if

$$(id_{\mathcal{M}(\widehat{A})} \otimes \alpha) \circ \delta = (\Sigma \otimes id_X) \circ (\text{Ad}(U) \otimes id_X) \circ (id_{\mathcal{M}(A)} \otimes \delta) \circ \alpha. \quad (\text{B.10})$$

Indeed, the diagram

$$\begin{array}{ccccc} X & \xrightarrow{\delta} & \mathcal{M}(\widehat{A} \otimes X) & \xrightarrow{id_{\mathcal{M}(\widehat{A})} \otimes \alpha} & \mathcal{M}(\widehat{A} \otimes A \otimes X) \\ \alpha \downarrow & & & & \uparrow \Sigma \otimes id_X \\ \mathcal{M}(A \otimes X) & \xrightarrow{id_{\mathcal{M}(A)} \otimes \delta} & \mathcal{M}(A \otimes \widehat{A} \otimes X) & \xrightarrow{\text{Ad}(U) \otimes id_X} & \mathcal{M}(A \otimes \widehat{A} \otimes X) \end{array}$$

is commutative.

This definition is equivalent to the classical one.

**Proposition B.3.2.**  $(X, \alpha, \delta)$  is a  $\mathbb{G}$ -Yetter-Drinfeld  $*$ -algebra if and only if  $(X, \triangleleft_\delta, \alpha)$  is a (right-left) Yetter-Drinfeld  $*$ -algebra over the multiplier Hopf  $*$ -algebra  $(A^\circ, \Delta^\circ)$ . Here

$$\triangleleft_\delta : X \otimes A^{op} \rightarrow X, \quad x \otimes a^{op} \mapsto (a \otimes id)(\delta(x))$$



is the action of the multiplier Hopf\*-algebra  $(A^{op}, \Delta^{op})$  on  $X$  generated by  $\delta$  and we regard the left coaction  $\alpha$  as the map  $X \rightarrow \mathcal{M}((A^{op})^{op} \otimes X)$ ,  $x \mapsto x_{[1]}^{op} \otimes x_{[0]}$ .

*Proof* Fix  $a \in A$  and  $x \in X$ . It is obvious that

$$\alpha(x \triangleleft_{\delta} a^{op}) = ({}^{op} \otimes id \otimes id)((a^{op} \otimes id \otimes id)((id \otimes \alpha)\delta(x))) = (a \otimes id \otimes id)((id \otimes \alpha)\delta(x)).$$

Now, if we use the notation  $y \otimes z = \delta(x_{[0]})$ , then by Lemma 2.4.10

$$\begin{aligned} (S^{op})^{-1}(a_{(3)}^{op})x_{[-1]}^{op}a_{(1)}^{op} \otimes x_{[0]} \triangleleft_{\delta} a_{(2)}^{op} &= S(a_{(3)})^{op}x_{[-1]}^{op}a_{(1)}^{op} \otimes (a_{(2)} \otimes id)(y \otimes z) \\ &= S(a_{(3)})^{op}x_{[-1]}^{op}a_{(1)}^{op}a_{(2)}(y) \otimes z \\ &= S(a_{(3)})^{op}x_{[-1]}^{op}(id \otimes y)(\Delta^{op}(a_{(1)}^{op})) \otimes z \\ &= ((id \otimes y)(\Delta(a_{(1)}))x_{[-1]}S(a_{(2)}))^{op} \otimes z \\ &= ({}^{op} \otimes id)((id \otimes a)(U(x_{[-1]} \otimes y)U^{-1}) \otimes z) \\ &= ({}^{op} \otimes id)((id \otimes a)(Ad(U)(x_{[-1]} \otimes y)) \otimes z) \\ &= (a \otimes id \otimes id)(\Sigma \otimes id)(Ad(U) \otimes id)((id \otimes \delta)\alpha(x)). \end{aligned}$$

This implies that the equality

$$\alpha(x \triangleleft_{\delta} a^{op}) = (S^{op})^{-1}(a_{(3)}^{op})x_{[-1]}^{op}a_{(1)}^{op} \otimes x_{[0]} \triangleleft_{\delta} a_{(2)}^{op}$$

holds for all  $\omega \in \widehat{A}$  and  $x \in X$  if and only if

$$(id \otimes \alpha) \circ \delta = (\Sigma \otimes id) \circ (Ad(U) \otimes id) \circ (id \otimes \delta) \circ \alpha$$

□

In a similar way, we can define Yetter-Drinfeld \*-algebras over  $\widehat{\mathbb{G}}$ . Consider a left coaction  $\delta : X \rightarrow \mathcal{M}(\widehat{A} \otimes X)$  of  $\widehat{\mathbb{G}}$  on  $X$ , denoted by  $x \mapsto x^{[-1]} \otimes x^{[0]}$ , and a left coaction  $\alpha : X \rightarrow \mathcal{M}(A \otimes X)$  of  $\widehat{\mathbb{G}}$  on  $X$ , denoted by  $x \mapsto x_{[-1]} \otimes x_{[0]}$ .

**Definition B.3.3.** We say that the triple  $(X, \delta, \alpha)$  is a  $\widehat{\mathbb{G}}$ -Yetter-Drinfeld \*-algebra if

$$(id_{\mathcal{M}(\widehat{A})} \otimes \delta) \circ \alpha = (\Sigma \otimes id_X) \circ (Ad(\widehat{U}) \otimes id_X) \circ (id_{\mathcal{M}(A)} \otimes \alpha) \circ \delta. \quad (\text{B.11})$$

Indeed, the diagram

$$\begin{array}{ccccc} X & \xrightarrow{\alpha} & \mathcal{M}(A \otimes X) & \xrightarrow{id_{\mathcal{M}(A)} \otimes \delta} & \mathcal{M}(A \otimes \widehat{A} \otimes X) \\ \delta \downarrow & & & & \uparrow \Sigma \otimes id_X \\ \mathcal{M}(\widehat{A} \otimes X) & \xrightarrow{id_{\mathcal{M}(\widehat{A})} \otimes \alpha} & \mathcal{M}(\widehat{A} \otimes A \otimes X) & \xrightarrow{Ad(\widehat{U}) \otimes id_X} & \mathcal{M}(\widehat{A} \otimes A \otimes X) \end{array}$$

is commutative.

We show now that our new definition is equivalent to the classical one.

**Proposition B.3.4.**  $(X, \delta, \alpha)$  is a  $\widehat{\mathbb{G}}$ -Yetter-Drinfeld  $*$ -algebra if and only if  $(X, \triangleleft_\alpha, \bar{\delta})$  is a right-left Yetter-Drinfeld  $*$ -algebra over the multiplier Hopf  $*$ -algebra  $(\widehat{A}, \widehat{\Delta})$ . Here

$$\triangleleft_\alpha : X \otimes \widehat{A} \rightarrow X, \quad x \otimes \omega \mapsto (\omega \otimes id)(\alpha(x))$$

is the action of the multiplier Hopf  $*$ -algebra  $(\widehat{A}, \widehat{\Delta})$  on  $X$  generated by  $\alpha$  and

$$\bar{\delta} : X \rightarrow \mathcal{M}((\widehat{A})^\circ \otimes X), \quad x \mapsto (\widehat{S}^{-1} \otimes id)(\delta(x))$$

is the left coaction of the multiplier Hopf  $*$ -algebra  $((\widehat{A})^{op}, \widehat{\Delta}^{op})$  on  $X$ .

*Proof* Fix  $\omega \in \widehat{A}$  and  $x \in X$ . It is obvious that

$$\bar{\delta}(x \triangleleft_\alpha \omega) = (\omega \otimes \widehat{S}^{-1} \otimes id)((id \otimes \delta)\alpha(x)).$$

Now, if we use the notation  $y \otimes z = \alpha(x_{[0]})$ , then by Lemma 2.4.11

$$\begin{aligned} \widehat{S}^{-1}(\omega_{(3)}) \cdot \widehat{S}^{-1}(x^{[-1]}) \cdot \omega_{(1)} \otimes x_{[0]} \triangleleft_\alpha \omega_{(2)} &= \widehat{S}^{-1}(\omega_{(3)}) \cdot \widehat{S}^{-1}(x^{[-1]}) \cdot \omega_{(1)} \otimes (\omega_{(2)} \otimes id)(y \otimes z) \\ &= \widehat{S}^{-1}(\omega_{(3)})y(\omega_{(2)}) \cdot \widehat{S}^{-1}(x^{[-1]}) \cdot \omega_{(1)} \otimes z \\ &= \widehat{S}^{-1}((y \otimes id)(\widehat{\Delta}(\omega_{(2)}))) \cdot \widehat{S}^{-1}(x^{[-1]}) \cdot \omega_{(1)} \otimes z \\ &= \widehat{S}^{-1}(\widehat{S}(\omega_{(1)}) \cdot x^{[-1]} \cdot (y \otimes id)(\widehat{\Delta}(\omega_{(2)}))) \otimes z \\ &= \widehat{S}^{-1}((id \otimes \omega)(\widehat{U}(x^{[-1]} \otimes y)\widehat{U}^{-1})) \otimes z \\ &= (\widehat{S}^{-1} \otimes \omega)(Ad(\widehat{U})(x^{[-1]} \otimes y)) \otimes z \\ &= (\omega \otimes \widehat{S}^{-1} \otimes id)(\Sigma \otimes id)(Ad(\widehat{U}) \otimes id)((id \otimes \alpha)\delta(x)). \end{aligned}$$

This implies that the equality

$$\bar{\delta}(x \triangleleft_\alpha \omega) = \widehat{S}^{-1}(\omega_{(3)}) \cdot \widehat{S}^{-1}(x^{[-1]}) \cdot \omega_{(1)} \otimes x_{[0]} \triangleleft_\alpha \omega_{(2)}$$

holds for all  $a \in A$  and  $x \in X$  if and only if

$$(id \otimes \delta) \circ \alpha = (\Sigma \otimes id) \circ (Ad(\widehat{U}) \otimes id) \circ (id \otimes \alpha) \circ \delta$$

□



## Appendix C

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### Algebraic crossed product for actions of AQG

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#### C.1 The case of actions of $\mathbb{G}$

Let  $\alpha : X \rightarrow \mathcal{M}(A \otimes X)$  be a left coaction of  $\mathbb{G}$  on an algebra  $X$ . We know that the algebraic crossed product for  $\alpha$  is

$$\mathbb{G} \ltimes_{\alpha} X = \text{span}\{(\omega \otimes 1)\alpha(x) : x \in X, \omega \in \widehat{A}\} \subseteq \mathcal{M}(\widehat{A}A \otimes X).$$

Now, consider  $\alpha^{\circ} : X^{op} \rightarrow \mathcal{M}(A \otimes X^{op})$ , the left coaction of  $\mathbb{G}^{\circ} = (A, \Delta^{cop}, \varphi \circ S)$  on  $X^{op}$  (called *the opposite coaction of  $\alpha$* ) defined by  $\alpha^{\circ}(x^{op}) = (S \otimes {}^{op})\alpha(x)$ , its dual action is the right action of the regular multiplier Hopf algebra  $((\widehat{A})^{op}, \widehat{\Delta}^{op})$  on  $X^{op}$ :

$$\triangleleft_{\alpha^{\circ}} : X^{op} \otimes (\widehat{A})^{op} \rightarrow X^{op}, \quad x^{op} \otimes \omega^{op} \mapsto (\omega \otimes id)(\alpha^{\circ}(x^{op}))$$

Also, we can take the left action of the multiplier Hopf algebra  $(\widehat{A}, \widehat{\Delta})$  on  $X^{op}$  defined by

$$\triangleright_{\alpha^{\circ}} : \widehat{A} \otimes X^{op} \rightarrow X^{op}, \quad \omega \otimes x^{op} \mapsto (\omega \otimes id)(\alpha^{\circ}(x^{op}))$$

**Remark 96.** Observe that

$$\omega \triangleright_{\alpha^{\circ}} x^{op} = x^{op} \triangleleft_{\alpha^{\circ}} \omega^{op} = (x \triangleleft_{\alpha} \widehat{S}(\omega))^{op},$$

for all  $\omega \in \widehat{A}$  and  $x \in X$ .

The smash product of  $(\widehat{A})^{op}$  and  $X^{op}$  with respect to the left action  $\triangleleft_{\alpha^{\circ}}$  is defined by

$$(\widehat{A})^{op} \# X^{op} := \text{span}\{\omega^{op} \# x^{op} : (\omega^{op} \# x^{op})(\theta^{op} \# y^{op}) = \omega^{op} \cdot \theta_{(1)}^{op} \# (x \triangleleft_{\alpha} \widehat{S}(\theta_{(2)}))^{op} \cdot y^{op}\}$$

and the smash product of  $X^{op}$  and  $\widehat{A}$  with respect to the left action  $\triangleright_{\alpha^{\circ}}$  is defined by

$$X^{op} \# \widehat{A} := \text{span}\{x^{op} \# \omega : (x^{op} \# \omega)(y^{op} \# \theta) = x^{op}(y \triangleleft_{\alpha} \widehat{S}(\omega_{(1)}))^{op} \# \omega_{(2)} \cdot \theta\}.$$

Moreover, there is a relation between the smash products  $(\widehat{A})^{op} \# X^{op}$ ,  $X^{op} \# \widehat{A}$  and  $\widehat{A} \# X$ :

**Proposition C.1.1.** *The linear map*

$$\tau : X^{op} \# \widehat{A} \rightarrow (\widehat{A} \# X)^{op}, \quad x^{op} \# \omega \mapsto (\widehat{S}(\omega) \# x)^{op}$$

is an isomorphism.

*Proof* We have,

$$\begin{aligned} \tau((x^{op} \# \omega)(y^{op} \# \theta)) &= \tau(x^{op}(\omega_{(1)} \triangleright_{\alpha^\circ} y^{op}) \# \omega_{(2)} \cdot \theta) \\ &= \tau(x^{op}(y \triangleleft_{\alpha} \widehat{S}(\omega_{(1)}))^{op} \# \omega_{(2)} \cdot \theta) \\ &= (\widehat{S}(\theta) \cdot \widehat{S}(\omega_{(2)}) \# (y \triangleleft_{\alpha} \widehat{S}(\omega_{(1)}x))^{op}) \\ &= ((\widehat{S}(\theta) \# y)(\widehat{S}(\omega) \# x))^{op} \\ &= \tau(x^{op} \# \omega)\tau_2(y^{op} \# \theta) \end{aligned}$$

for all  $x, y \in X$  and  $\omega, \theta \in \widehat{A}$ . □

**Remark 97.** We will use the following

$$\begin{aligned} \triangleright : ((\widehat{A})^{op} A^{cop} \otimes X^{op}) \otimes (A \otimes X) &\rightarrow A \otimes X \\ (\omega^{op} a \otimes x^{op}) \otimes (b \otimes y) &\mapsto (id \otimes \widehat{S}^{-1}(\omega))(\Delta(b))a \otimes yx \end{aligned}$$

and

$$\begin{aligned} \triangleleft : (A \otimes X) \otimes ((\widehat{A})^{op} A^{cop} \otimes X^{op}) &\rightarrow A \otimes X \\ (b \otimes y) \otimes (\omega^{op} a \otimes x^{op}) &\mapsto (id \otimes \omega)(\Delta(b))a \otimes xy \end{aligned}$$

**Lemma C.1.2.** *We have*

$$\alpha^\circ(x^{op})(\omega^{op} \otimes 1) = (s_{(x_{[-1]})}\omega^{op} \otimes 1)\alpha^\circ(x_{[0]}^{op})$$

in  $\mathcal{M}((\widehat{A})^{op} A^{cop} \otimes X^\circ)$ .

*Proof*

$$\begin{aligned} (a \otimes y) \triangleleft \alpha^\circ(x^\circ)(\omega^\circ \otimes id) &= (aS(x_{[-1]}) \otimes x_{[0]}y) \triangleleft (\omega^\circ \otimes id) \\ &= (id \otimes \omega)(aS(x_{[-1]})) \otimes x_{[0]}y \\ &= a_{(1)}S(x_{[-1]_{(2)}})\omega(a_{(2)}S(x_{[-1]_{(1)}})) \otimes x_{[0]}y \\ &= a_{(1)}S(x_{[0]_{[-1]}})\omega(a_{(2)}S(x_{[-1]})) \otimes x_{[0]_{[0]}}y \\ &= (a_{(1)}s_{(x_{[-1]})}\omega(a_{(2)}) \otimes y) \triangleleft \alpha^\circ(x_{[0]}^\circ) \\ &= ((a \otimes y) \triangleleft (s_{(x_{[-1]})}\omega^\circ \otimes id)) \triangleleft \alpha^\circ(x_{[0]}^\circ) \\ &= (a \otimes y) \triangleleft ((s_{(x_{[-1]})}\omega^\circ \otimes id)\alpha^\circ(x_{[0]}^\circ)) \end{aligned}$$

□

Then, the algebraic crossed product is

$$\mathbb{G}^\circ \rtimes_{\alpha^\circ} X^{op} := \text{span}\{(\omega^{op} \otimes 1)\alpha^\circ(x^{op}) : x \in X, \omega \in \widehat{A}\} \subseteq \mathcal{M}((\widehat{A})^{op} A^{cop} \otimes X^{op}).$$

**Proposition C.1.3.** *We have the isomorphism*

$$\Theta : X^{op} \# \widehat{A} \rightarrow \mathbb{G}^\circ \rtimes_{\alpha^\circ} X^{op}, \quad x^{op} \# \omega \mapsto (\omega^{op} \otimes 1)\alpha^\circ(x^{op})$$

*Proof* Follows from the Lemma above.  $\square$

**Proposition C.1.4.** *We have the anti-isomorphism*

$$\begin{aligned} \mathbb{G} \rtimes_{\alpha} X &\longrightarrow \widehat{A} \# X \longrightarrow X^{op} \# \widehat{A} \longrightarrow \mathbb{G}^\circ \rtimes_{\alpha^\circ} X^{op} \\ (\omega \otimes 1)\alpha(x) &\longmapsto \omega \# x \longmapsto x^\circ \# \widehat{S}^{-1}(\omega) \longmapsto (\widehat{S}^{-1}(\omega)^\circ \otimes 1)\alpha^\circ(x^\circ) \end{aligned}$$

*Proof* Follows from Lemma C.1.1 and the Proposition above.  $\square$

## C.2 The case of actions of $\widehat{\mathbb{G}}$

Let  $\widehat{\alpha} : X \rightarrow \mathcal{M}(\widehat{A} \otimes X)$  be a left coaction of  $\widehat{\mathbb{G}}$  on an algebra  $X$ . Consider the left coaction  $\widehat{\alpha}^\circ : X^{op} \rightarrow \mathcal{M}(\widehat{A} \otimes X^{op})$  of  $(\widehat{\mathbb{G}})^\circ = (\widehat{A}, \widehat{\Delta}, \widehat{\varphi})$  on  $X^{op}$  defined by  $\widehat{\alpha}^\circ(x^{op}) = (\widehat{S}^{-1} \otimes {}^{op}) (\widehat{\alpha}(x))$ . If we denote  $\mathbb{H} := (\widehat{\mathbb{G}})^\circ = (B, \Delta_B, \varphi_B)$ ,  $\lambda := \widehat{\alpha}^\circ$  and  $Y := X^{op}$ , then  $\lambda : Y \rightarrow \mathcal{M}(B \otimes Y)$  is a left coaction of  $\mathbb{H}$  on  $Y$  and if we consider again the left coaction  $\lambda^\circ : Y^{op} \rightarrow \mathcal{M}(B \otimes Y^{op})$  of  $\mathbb{H}^\circ = \widehat{\mathbb{G}}$  on  $Y^\circ = X$ , we see that  $\lambda^\circ = \widehat{\alpha}$  because

$$\lambda^\circ(y^{op}) = (S_B \otimes {}^{op})(\lambda(y)) = (\widehat{S} \otimes {}^{op})(\widehat{S}^{-1} \otimes {}^{op})(\widehat{\alpha}(x)) = \widehat{\alpha}(x)$$

if  $x = y^\circ$ . Then, we can obtain

$$\begin{aligned} \widehat{\mathbb{G}} \rtimes_{\widehat{\alpha}} X &= \mathbb{H}^\circ \rtimes_{\lambda^\circ} Y^{op} = \text{span}\{(a^{op} \otimes 1)\widehat{\alpha}(x) : x \in X, a \in A\} \subseteq \mathcal{M}(A^{op}(\widehat{A})^{cop} \otimes X) \\ (\widehat{\mathbb{G}})^\circ \rtimes_{\widehat{\alpha}^\circ} X^{op} &= \mathbb{H} \rtimes_{\lambda} Y = \text{span}\{(a \otimes 1)\widehat{\alpha}^\circ(x^{op}) : x \in X, \omega \in \widehat{A}\} \subseteq \mathcal{M}(A\widehat{A} \otimes X^{op}). \end{aligned}$$

**Proposition C.2.1.** *We have the next important anti-isomorphism*

$$\begin{aligned} \widehat{\mathbb{G}} \rtimes_{\widehat{\alpha}} X &\longrightarrow A^\circ \# X \longrightarrow X^\circ \# A \longrightarrow \widehat{\mathbb{G}}^\circ \rtimes_{\widehat{\alpha}^\circ} X^\circ \\ (a^\circ \otimes 1)\widehat{\alpha}(x) &\longmapsto a^\circ \# x \longmapsto x^\circ \# S(a) \longmapsto (S(a) \otimes 1)\widehat{\alpha}^\circ(x^\circ) \end{aligned}$$

*Proof* Similar to Proposition C.1.4.  $\square$

**Remark 98.** Recall that  $(A^{op}, \Delta^{op})$  is the regular multiplier Hopf algebra with antipode  $S^{op}(a^{op}) := (S^{-1}(a))^{op}$ .



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## *List of symbols*

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$A^\vee$	Vector space of complex functionals on $A$ . . . . .	3
$A^{op}$	The opposite algebra of $A$ . . . . .	3
$M, N, \dots$	Modules . . . . .	3
$\otimes$	Algebraic tensor product of modules or algebras . . . . .	3
$\triangleright$	Notation for a left module map . . . . .	3
${}_A M$	$M$ is an left $A$ -module . . . . .	3
$A, B, \dots$	*-algebras . . . . .	4
$\mathcal{M}(A)$	The multiplier algebra of an *-algebra $A$ . . . . .	5
$\Sigma$	The flip map from $\mathcal{M}(A \otimes B)$ to $\mathcal{M}(B \otimes A)$ . . . . .	5
$\Sigma_P$	The map $\Sigma \circ \text{Ad}(P)$ . . . . .	6
$\text{Ad}(P)$	The map $Q \mapsto P Q P^{-1}$ . . . . .	6
$\chi_{A,B}$	The flip map from $A \otimes B$ to $B \otimes A$ . . . . .	5
$\text{Hom}(A, B)$	The set of non-degenerate homomorphism from $A$ to $\mathcal{M}(B)$ . . . . .	5
$\omega$	A non-zero faithful functional . . . . .	6
$\omega_a, {}_a \omega$	Notation for functionals construct from $\omega$ and $a$ . . . . .	6
$\sigma^\omega$	Modular automorphism associated withw the functional $\omega$ . . . . .	6
$S$	The antipode of a multiplier Hopf *-algebra $(A, \Delta)$ . . . . .	10
$\Delta$	The comultiplication of a multiplier Hopf *-algebra . . . . .	10
$\varepsilon$	The counit of a multiplier Hopf *-algebra $(A, \Delta)$ . . . . .	10
$G$	A group . . . . .	11
$U_g$	Formal element in $\mathbb{C}[G]$ associated with the element $g \in G$ . . . . .	11



$\mathbb{C}[G]$	The group $*$ -algebra of a group $G$ . . . . .	11
$\text{Fun}_f(G)$	The $*$ -algebra of complex finitely supported functions on $G$ . . . . .	11
$\psi$	Right integral of a multiplier Hopf $*$ -algebra $(A, \Delta)$ . . . . .	12
$\varphi$	Left integral of a multiplier Hopf $*$ -algebra $(A, \Delta)$ . . . . .	12
$\delta$	The modular element of a multiplier Hopf $*$ -algebra $(A, \Delta)$ . . . . .	13
$\rho$	The modular automorphism associated with the left integral $\varphi$ of a multiplier Hopf $*$ -algebra $(A, \Delta)$ . . . . .	13
$\rho'$	The modular automorphism associated with the right integral $\psi$ of a multiplier Hopf $*$ -algebra $(A, \Delta)$ . . . . .	13
$\widehat{A}$	The dual algebra construct for a multiplier Hopf $*$ -algebras with integrals $(A, \Delta, \varphi)$ . . . . .	14
$\widehat{S}$	The antipode of the dual multiplier Hopf $*$ -algebra of $(A, \Delta, \varphi)$ . . . . .	15
$\widehat{\Delta}$	The comultiplication of the dual multiplier Hopf $*$ -algebra of $(A, \Delta, \varphi)$ . . . . .	15
$\widehat{\varepsilon}$	The counit of the dual multiplier Hopf $*$ -algebra of $(A, \Delta, \varphi)$ . . . . .	15
$p_A$	The canonical pairing associated with the multiplier Hopf $*$ -algebras with integrals $(A, \Delta, \varphi)$ . . . . .	21
$\mathbb{G}$	The algebraic quantum group $(A, \Delta, \varphi)$ . . . . .	45
$\widehat{\mathbb{G}}$	The dual algebraic quantum group of $\mathbb{G}$ . . . . .	45
$\mathbb{G}^\circ$	The opposite algebraic quantum group of $\mathbb{G}$ . . . . .	46
$\mathbb{G}^c$	The commutant algebraic quantum group of $\mathbb{G}$ . . . . .	46
$H(\mathbb{G})$	The Heisenberg $*$ -algebra of the algebraic quantum group $\mathbb{G}$ . . . . .	49
$V^{\mathbb{G}}, V$	The duality of the algebraic quantum group $\mathbb{G}$ . . . . .	49
$U^{\mathbb{G}}, U$	The algebraic multiplicative unitary of $\mathbb{G}$ . . . . .	53
$D(\mathbb{G})$	The Drinfeld double of $\mathbb{G}$ . . . . .	62
$\triangleleft_\alpha$	The dual action associated with $\alpha$ . . . . .	63
$\text{Ad}_{\mathbb{G}}$	The left adjoint action of $G$ . . . . .	64
$U_\mu^\alpha$	The standard implementation of the left action $\alpha$ associated with the $\alpha$ -invariant functional $\mu$ . . . . .	66
$\alpha^\circ$	The opposite action of $\alpha$ . . . . .	66

$\mu^{op}$	The opposite functional of $\mu$ . . . . .	66
$\widehat{A} \# X$	The smash product of $\widehat{A}$ and $X$ . . . . .	67
$\mathbb{G} \rtimes_{\alpha} X$	The crossed product of $\mathbb{G}$ and $X$ using the action $\alpha$ . . . . .	69
$\tilde{\alpha}$	The dual action of $\alpha$ . . . . .	71
$\tilde{\mu}$	The dual functional of $\mu$ by the action $\alpha$ . . . . .	72
$(X, \alpha, \widehat{\alpha})$	A $\mathbb{G}$ -Yetter-Drinfeld $*$ -algebra . . . . .	73
$E, F, \dots$	Right Hilbert $C^*$ -modules . . . . .	106
$X'$	The commutant of $X$ . . . . .	105
$[Y]$	The closed linear span of $Y$ . . . . .	105
$ \cdot\rangle, \langle\cdot $	The ket-bra notation . . . . .	105
$\mathcal{H}, \mathcal{K}, \mathfrak{H}, \mathfrak{K}$	Hilbert spaces . . . . .	105
$\otimes$	Spatial tensor product when we work with $C^*$ -algebras or tensor product of Hilbert spaces . . . . .	105
$\otimes, \otimes$	Relative tensor products of right Hilbert $C^*$ -modules . . . . .	110
$(L^2\mathbb{G}, \Lambda)$	A GNS-pair for the left integral $\varphi$ of the algebraic quantum group $\mathbb{G}$ . . . . .	113
$A_r$	The $C^*$ -algebra of the $C^*$ -algebraic quantum group $G_r$ . . . . .	114
$\mathbb{G}_r$	The reduced $C^*$ -algebraic quantum group associated with $G$ . . . . .	114
$\Delta_r$	The comultiplication of the $C^*$ -algebraic quantum group $G_r$ . . . . .	114
$\pi_r$	The $*$ -representation of $\mathbb{G}$ associated with the GNS-pair $(L^2\mathbb{G}, \Lambda)$ . . . . .	114
$\mathbb{G}_u$	The universal $C^*$ -algebraic quantum group associated with $\mathbb{G}$ . . . . .	117
$\mathfrak{A}, \mathfrak{B}, \mathfrak{C}$	$C^*$ -algebras . . . . .	121
$\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$	$C^*$ -bases . . . . .	121
$\mathfrak{b}^\dagger$	The opposite $C^*$ -base of $\mathfrak{b}$ . . . . .	121
$\mathcal{H}_E$	A $C^*$ -module over a $C^*$ -base . . . . .	121
$\otimes_{\mathfrak{b}}$	Relative tensor product over the $C^*$ -base $\mathfrak{b}$ . . . . .	122
${}_E\mathcal{H}_F$	A $C^*$ -bimodule over $C^*$ -bases . . . . .	122
$*_{\mathfrak{b}}$	Fiber product over the $C^*$ -base $\mathfrak{b}$ . . . . .	126
$\mathcal{A}_{\mathcal{H}}^{F,E}$	A $C^*$ -algebra over $C^*$ -bases . . . . .	127
$(\mathcal{A}_{\mathcal{H}}^{F,E}, \Delta_A)$	Hilbert $C^*$ -module over a $C^*$ -base . . . . .	128





## Groupoïdes quantiques de transformations : une approche algébrique et analytique

**Résumé :** Cette thèse porte sur la construction d'une famille de groupoïdes quantiques de transformations qui dans le cadre algébrique sont des algébroïdes de Hopf de multiplicateurs mesurés au sens de Timmermann et Van Daele et qui dans le cadre des algèbres d'opérateurs sont des  $C^*$ -bimodules de Hopf sur une  $C^*$ -base au sens de Timmermann. Dans le contexte purement algébrique, nous définissons d'abord une algèbre involutive de Yetter-Drinfeld tressée commutative sur un groupe quantique algébrique au sens de Van Daele et une intégrale de Yetter-Drinfeld sur elle. En utilisant ces objets nous construisons après un algébroïde de Hopf de multiplicateurs involutif mesuré, ce nouvel objet nous l'appellons *groupoïde quantique algébrique de transformations*. Pour être capables de passer au cadre des algèbres d'opérateurs, nous donnons des conditions sur l'intégral de Yetter-Drinfeld qui vont nous permettre d'utiliser la construction Gelfand–Naimark–Segal pour étendre tous nos objets purement algébriques en des objets  $C^*$ -algébriques. Dans ce contexte, notre construction se fait d'une manière similaire à celle présentée dans le travail de Enock et Timmermann, un nouvel objet mathématique que nous appelons un *groupoïde quantique  $C^*$ -algébrique de transformations*, qui est défini en utilisant le langage des  $C^*$ -bimodules de Hopf sur une  $C^*$ -base.

**Mots-clés.** Groupoïde de transformations, Groupe quantique, Groupoïde quantique, Algébroïde de Hopf de multiplicateurs, Algèbre de Yetter-Drinfeld, Tressée commutative,  $C^*$ -bimodule de Hopf.

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## Quantum transformation groupoids: An algebraic and analytical approach

**Abstract:** This thesis is concerned with the construction of a family of quantum transformation groupoids in the algebraic framework in the form of the measured multiplier Hopf  $*$ -algebroids in the sense of Timmermann and Van Daele and also in the context of operator algebras in the form of Hopf  $C^*$ -bimodules on a  $C^*$ -base in the sense of Timmermann. In the purely algebraic context, we first give a definition of a braided commutative Yetter-Drinfeld  $*$ -algebra over an algebraic quantum group in the sense of Van Daele and a Yetter-Drinfeld integral on it. Then, using these objects we construct a measured multiplier Hopf  $*$ -algebroid, we call to this new object an *algebraic quantum transformation groupoid*. In order to pass to the operator algebra framework, we give some conditions on the Yetter-Drinfeld integral inspired by the properties of KMS-weights on  $C^*$ -algebras which will allow us to use the Gelfand–Naimark–Segal construction to extend all the purely algebraic objects to the  $C^*$ -algebraic level. At this level, we construct in a similar way to that used in the work of Enock and Timmermann, a new mathematical object that we call a  *$C^*$ -algebraic quantum transformation groupoid*, which is defined using the language of Hopf  $C^*$ -bimodules on  $C^*$ -bases.

**Keywords.** Transformation groupoid, Quantum group, Quantum groupoid, Multiplier Hopf algebroid, Yetter-Drinfeld algebra, Braided commutative, Hopf  $C^*$ -bimodule.